UNIVERSITY OF KWAZULU-NATAL

New solutions for a radiating star

VUSI ZITHA

2018

UNIVERSITY OF KWAZULU-NATAL

New solutions for a radiating star

by

VUSI ZITHA

Submitted in fulfilment of the

academic requirements for the degree of

Doctor of Philosophy

in the

School of Mathematics, Statistics and Computer Science

University of KwaZulu-Natal

Durban

December 2018

As the candidate's supervisor I have approved this dissertation for submission.

Signed: .... Name: Prof S D Maharaj Date: ....

Signed: .... Name: Prof M Govender Date: ....

Abstract

Finding exact solutions to the Einstein field equations is important for applications in relativistic astrophysics. We model the collapse of a radiating star in general relativity. Firstly, we investigate an exact model of a radiating star undergoing expansion-free collapse. We demonstrate that it is possible to obtain new classes of exact solutions by integrating the nonlinear junction condition at the surface of the star. The gravitational potentials are given in terms of elementary functions. The solution makes it possible to consider the formation of a cavity in an exact model. Secondly, we rewrite the junction condition in the standard form of a Riccati equation. The boundary condition can be integrated to find new classes of solutions and extends the analysis of Thirukkanesh et al (2012). Thirdly, we investigate the radiating star when the interior expanding, shearing fluid particles are travelling in geodesic motion. Our analysis leads to new solutions and extends earlier results. A feature of our analysis is to investigate the role played by the cosmological constant in radiating stars with dissipation. We find that the presence of the cosmological constant affects the gravitational dynamics. The presence of this term in the field equations allows for positive pressure in particular situations leading to stable stellar configurations.

COLLEGE OF AGRICULTURE, ENGINEERING AND SCIENCE

DECLARATION 1 - PLAGIARISM

I, Vusi Zitha, student number: 206504126, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. The following thesis has not been submitted for any degree or examination at any other university.

3. The following thesis does not contain other persons data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. The following thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

    a. Their words have been re-written but the general information attributed to them has been referenced.

    b. Where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.

5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed: .... Vusi Zitha Date: ....

COLLEGE OF AGRICULTURE, ENGINEERING AND SCIENCE

DECLARATION 2 - PUBLICATIONS

Details of contribution to publications that form part of the research presented in this thesis.

Publication 1

Maharaj S D, Govender M, Zitha V and Tiwari A K, Expansion-free collapse - an exact model, *Gen. Relativ. Gravit.* submitted (2018)

Publication 2

Zitha V, Maharaj S D and Govender M, Dissipative collapse in the presence of shear, *Int. J. Mod. Phys. D* submitted (2018).

Publication 3

Zitha V, Maharaj S D and Govender M, Shearing dissipative collapse in the presence of λ, *Eur. Phys. J. Plus*, submitted (2019).

Acknowledgements

There are some important role players I would like to acknowledge for helping make this dissertation possible:

- I would like to thank my supervisor, Prof. S D Maharaj for his patience, guidance, encouragement and advice that he has provided throughout my time as his student.
- I further acknowledge the help, support and patience of my co-supervisor, Prof. M Govender.
- I want to thank the School of Mathematical Sciences for providing a good working environment, computer facilities and administrative assistance.
- I would like to thank the NRF and UKZN for providing the funding which allowed me to undertake this research.

Contents

Chapter 1

Introduction

It is well known that Isaac Newton discovered the inverse square law of gravitation leading to Newtonian gravity. Newtonian gravity describes the equilibrium of low mass stars, particularly the order of one solar mass and low mass stellar systems with particular type of sizes in parsecs. Neutron stars and black holes are relativistic compact objects that require a modification of Newtonian gravity to account for effects relating to observations and theoretical predictions (Glendenning (2000)). In 1914 Einstein extended the work on Newtonian gravity, which is now called general relativity (Misner et al (1973)). In general relativity, we consider the study of the modern theory of gravity. It is important to study the Einstein equations in general theory of relativity to fully understand astrophysics and cosmology. The Einstein equations lead us to investigate the cosmos, galaxies, stars, etc. In this thesis we focus on the study of radiating stars. Einstein equations allow us to estimate the state of the stars, such as the amount of mass, energy on spacetime, curvature of spacetime, energy momentum, pressure, density and heat flow that are distributed in stars. It is difficult to solve Einstein equations in general, hence we make various assumptions to the equations in order to carry out our investigations.

In astrophysics, Schwarzschild provided the first exact solution of the Einstein field equations and obtained the external gravitational field of a spherically symmetric, static mass distribution (Schwarzschild (1916)). We investigate gravitational collapse to obtain exact solutions of the Einstein equations, by studying the shear-free motion of the fluid and shearing spacetimes. Gravitational collapse was first investigated by Oppenheimer and Snyder (1939) in which they studied the contraction of a spherically symmetric dust cloud. In this work, we study and investigate solutions that contain shear because it plays a very important role in the study of gravitational collapse as Pinheiro and Chan (2008) mention. Maharaj et al (2011) studied the gravitational collapse of a radiating sphere evolving into a final static configuration described by the interior Schwarzschild solution. Thirukkanesh et al (2012) considered shearing models in a general setting for the first time. Their solutions extend particular results of earlier treatments. Maharaj et al (2013) showed that the junction conditions at the stellar surface may be generalized so that the pressure at the boundary depends on the interior heat flux and the exterior string density.

We investigate the behaviour of gravitational collapse of a radiating star, where the Vaidya spacetime describes the exterior geometry. In the radiating star the exterior spacetime is represented by the Vaidya solution for pure radiation and is no longer described by the exterior Schwarzschild solution. The Einstein field equations describing the atmosphere of a radiating star, with nonstatic geometry, was first found by Vaidya (1951). Vaidya investigated the outer field with an expanding inner zone of pure radiation extending the model of empty space decribed by Schwarzschild's static solution (Vaidya 1999). The Vaidya metric may be generalized so that the mass function depends on both time and the radial coordinate. The generalised Vaidya spacetime has physical significance and contains many known solutions of the Einstein field equations with spherical symmetry as shown by Maharaj et al (2013). The Santos boundary condition $p = qB$ is extended to include the external string density for the generalized Vaidya metric.

Shear-free spacetimes have been studied most often. The interior spacetime where the fluid is described is considered shear-free with is the slowest possible collapse. The exterior spacetime of course is given by the Vaidya (1953) metric which represents the radial flow of unpolarised radiation (Santos (1985)). The Santos junction conditions were generalised to include the effects of an electromaganetic field and shearing anistropic stresses during dissipative stellar collapse by de Oliveira et al (1985) and Maharaj and Govender (2000). The other physically important situation is geodesic motion. Thirukkanesh and Maharaj (2010) studied the behaviour of a radiating star when the interior expanding, shearing fluid particles are traveling in geodesic motion. They demonstrated that it is possible to obtain new classes of exact solutions in terms of elementary functions without assuming a separable form for the gravitational potentials or initially fixing the temporal evolution of the model unlike earlier treatments. In this thesis we extend their work using the same approach that enables us to write the junction condition as a Riccati equation which under particular conditions may be transformed into a separable equation.

In 1917 Einstein introduced the cosmological constant into his general relativistic theory of gravitation (Weinberg (1989)). The cosmological constant plays an important role in various cosmological scenarios (Misner et al (1973)). The cosmological constant is employed in several theories of the early universe (Maharaj and Beesham (1988)). We model a radiating star when the Einstein field equations include the cosmological constant. If the models with cosmological constant satisfy the field equations, then the pressures may be negative for a Kasner type solution and equations of state are formed

(Maharaj and Naidoo (1993)). Prokopec (2006) investigated that how an initially large and positive cosmological constant can be driven to zero by the gravitational backreaction induced by fermionic quantum fluctuations, when the effective cosmological term has a small value but nonvanishing . These physical applications of the cosmological constant in cosmology may be extended to radiating stars. In this thesis we extend their work by investigating shearing gravitational collapse when the Einstein field equations are generalised to include the consmological constant.

We also apply the Eckart and causal theories to study the thermodynamics of radiating stars (Tewari (2013), Sharma et al (2013), Pretel and da Silva (2019)). The evolution of the temperature profiles is investigated by employing a causal heat transport equation of the Maxwell-Cattaneo form. Both the Eckart and causal temperatures are enhanced by anisotropy at each interior point of the stellar configuration as shown by Govender et al (2016). Govender and Govinder (2002) showed that extended irreversible thermodynamics predict a higher temperature at all interior points of the stellar configuration compared to the Eckart theory. Relaxational effects on the temperature profile of radiating, spherically symmetric matter distributions, in both the shearing and shear-free cases, have been investigated by Govender and Govinder (2001), who demonstrated that relaxational effects lead to higher core temperatures and the monotonic decrease in the temperature, as a function of the radial coordinate, is much sharper than its noncausal counterpart.

This thesis is organised as follows:

- Chapter 1: Introduction
- Chapter 2: We show that it is possible to integrate the nonlinear junction condition at the surface of the star; the gravitational potentials are given in terms of elementary functions. The collapsing core dissipates energy in the form of

a radial heat flux. This exact model complements the perturbative models of dissipative collapse currently available in the literature. A physical and thermodynamical analysis of the model indicates that it is stable and approximates a realistic collapse scenario.

- Chapter 3: In this chapter we consider a collapsing stellar model dissipating energy in the form of a radial heat flux. We investigate the effect of the cosmological model on the collapse process. New families of exact solutions to the Einstein field equations are presented. Some of solutions can be written parametrically and correspond to both nonzero and vanishing cosmological contant. The underlying physics is studied for a particular model which is shown to be physically reasonable. A graphical analysis shows that the star has higher central temperatures and the energy conditions are satisfied. An interesting feature of our model is the effect on the cosmological constant which renders the radial pressure positive. The stellar model describes an isothermal sphere in the asymptotic limit with an equation of state.

- Chapter 4: In this chapter we present various classes of solutions describing dissipative gravitational collapse in the presence of shear and a nonzero cosmological constant. We model a spherically symmetric matter distribution undergoing gravitational collapse and radiating energy in the form of a radial heat flux to the exterior spacetime. The matching of the interior spacetime to Vaidya's outgoing solution leads to a nonlinear Riccati differential equation which encodes the temporal behaviour of the model. A physical analysis indicates that the energy conditions are satisfied. We solve this boundary condition and present various families of new solutions which generalise known shearing solutions to include the cosmological constant.

- Chapter 5: Conclusion

Chapter 2

Expansion-free collapse: an exact model

2.1 Introduction

Radiating matter in strong gravitational fields is an important area of present research in relativistic astrophysics. Models with radiating matter may be used to describe relativistic stars, investigate radiative processes in the external stellar atmosphere and study gravitational collapse. An interior spacetime, satisfying the field equations, has to match with the Vaidya exterior spacetime at the boundary of the star. Santos (1985) was the first to correctly complete the matching conditions for a spherically symmetric metric to smoothly match to the Vaidya exterior. An important consequence of the matching conditions is that the radial pressure for a radiating star is nonzero at the boundry; the radial pressure is proportional to the heat flux. Earlier attempts to generate exact models placed restrictions on the acceleration, shear, Weyl tensor or took the star to be initially static. A sample of research papers following these approaches is given in references by Kolassis et al (1988), Herrera et al (2002), Thirukkanesh and

Maharaj (2010), Misthry et al (2008), Tewari (2013, 2015), Herrera (2006), Maharaj and Govender (2005), Herrera et al (2006), Chan (1997), Pinheiro and Chan (2010), Tewari (2013) and Reddy et al (2015). It is also possible to produce analytic models in general relativity called Euclidean stars which are generalizations of Newtonian stars (Herrera (2010) , Govender et al (2010) and Abebe et al (2014)).

A systematic approach to study the junction condition, a nonlinear differential equation, is to use symmetry invariance; the Lie theory of differential equations produces infinitesimal point generators reducing the order of the boundary condition. With this approach new analytic solutions can be found as shown in Abebe et al (2015), Mohanlal et al (2016) and Mohanlal et al (2017). An unusual and interesting approach to find a model of a radiating star is to embed the spherical four-dimensional metric into a flat five-dimensional spacetime. Naidu et al (2018) recently employed the idea of embedding and smoothy matched the Vaidya outgoing solution to a well behaved stellar interior. We discuss the relevant model and background in Section 2.3 and Section 2.4 we give details of the interior spacetime structure and junction conditions at the stellar surface. The temporal evolution is discussed in Section 2.5 and an exact solution is generated in Section 2.6. The physical properties of the new solution are considered in Section 2.7 and Section 2.8. Concluding remarks are made in Section 2.9.

2.2 The model

Stellar radiating models with restrictions on the four-acceleration vector and the shear tensor have received considerable attention in the past. In contrast there has been much less attention on physical models restricting the expansion scalar. Expansion-free spherically symmetric fluids necessarily implies the appearance of a cavity inside the matter distribution. Consequently expansion-free models should be studied on physical grounds. The basic equations and construction of the model for expansion-free distributions, in spherical symmetry, were given in Herrera et al (2009), and such energy distributions must be inhomogeneous (Herrera et al (2009)). The evolution of expansion-free spherically symmetric fluids and some families of exact solutions are given in Prisco et al (2011), Kumar and Srivastava (2018a, 2018b). A study of dynamical instability in expansion-free structures shows that the range of instability is independent of stiffness, and depends on anisotropy and the radial profile of the energy density (Sharif and Azam (2012)). The effects of the electromagnetic field were included by Sharif (2012). Cylindrical symmetry, planar symmetry and axial symmetry in expansion-free distributions were analysed in the treatments of Sharif and Yousaf (2012), Sharif and Azam (2013), Sharif and Bhatti (2014). There have been attempts to study the physical behaviour of expansion-free collapse in alternate theories of gravity. Some results have found in f(R) gravity by Sharif and Nasir (2015), Sharif and Yousaf (2013), Sharif and Nasir (2015), and in f(R,T) gravity theories by Noureen and Zubair (2015), Zubair et al (2018).

As far as we are aware there has been no complete model of a radiating star found which satisfies all boundary conditions at the stellar surface with the expansion-free condition. In particular, no exact solution to the Santos boundary condition relating the radial pressure to the heat flux has been presented. In this chapter we perform a careful analysis of the Santos nonlinear differential equation and apply the expansion-free restriction. We show that it is possible to find a simple class of exact solutions in terms of elementary functions. A physical analysis of the resulting radiating model is undertaken.

2.3 Interior spacetime

The line element for the interior of the collapsing star is described by the general spherically symmetric, shearing metric in comoving coordinates (Bonnor et al (1989)),

$$ds^2 = -A^2dt^2 + B^2dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{2.3.1}$$

where $A = A(t,r)$, $B = B(t,r)$ and $R = R(t,r)$. The spacetime coordinates here are

$$(x^\alpha) = (t, r, \theta, \phi). \tag{2.3.2}$$

The interior matter content is that of an imperfect fluid given by

$$\begin{aligned} T_{\alpha\beta} &= (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta \\ &\quad + q_\alpha V_\beta + q_\beta V_\alpha\,, \end{aligned} \tag{2.3.3}$$

where μ is the energy density, P_r is the radial pressure, $P_\perp$ is the tangential pressure and q^α is the heat flux vector. The fluid four–velocity $\mathbf{V}$ is comoving and is given by

$$V^\alpha = \frac{1}{A}\delta^\alpha_0\,. \tag{2.3.4}$$

The heat flow is in the radial direction, and the heat flow vector takes the form

$$q^\alpha = (0, q^1, 0, 0)\,, \tag{2.3.5}$$

where $q^{\alpha}V_{\alpha} = 0$. We further have

$$\chi^{\alpha}\chi_{\alpha} = 1, \tag{2.3.6a}$$

$$\chi^{\alpha}V_{\alpha} = 0 \tag{2.3.6b}$$

The expansion scalar and the fluid four acceleration are given by

$$\Theta = V^{\alpha}{}_{;\alpha}, \tag{2.3.7a}$$

$$a_{\alpha} = V_{\alpha;\beta}V^{\beta}, \tag{2.3.7b}$$

and the shear tensor is

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_{\alpha}V_{\beta}). \tag{2.3.8}$$

For the comoving line element (2.3.1) the kinematical quantities take the following forms

$$a_1 = \frac{A'}{A}, \tag{2.3.9a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{R}}{R}\right), \tag{2.3.9b}$$

$$\sigma = \frac{1}{A}\left(\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right), \tag{2.3.9c}$$

where dots and primes denote differentiation with respect to t and r respectively.

The nonzero components of the Einstein field equations for the line element (2.3.1) and the energy momentum (2.3.3) are

$$\mu = -\frac{1}{B^2}\left[2\frac{R''}{R} + \left(\frac{R'}{R}\right)^2 - 2\frac{B'}{B}\frac{R'}{R}\left(\frac{B}{R}\right)^2\right]$$

$$+\frac{1}{A^2}\left(2\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}, \tag{2.3.10a}$$

$$P_r = -\frac{1}{A^2}\left[2\frac{\ddot{R}}{R} - \left(2\frac{\dot{A}}{A} - \frac{\dot{R}}{R}\right)\frac{\dot{R}}{R}\right] + \frac{1}{B^2}\left(2\frac{A'}{A} + \frac{R'}{R}\right)\frac{R'}{R}$$

$$-\frac{1}{R^2}, \tag{2.3.10b}$$

$$P_\perp = -\frac{1}{A^2}\left[\frac{\ddot{B}}{B} + \frac{\ddot{R}}{R} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{R}}{R}\right) + \frac{\dot{B}}{B}\frac{\dot{R}}{R}\right]$$

$$+\frac{1}{B^2}\left[\frac{A''}{A} + \frac{R''}{R} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{R'}{R}\right], \tag{2.3.10c}$$

$$q = \frac{2}{AB}\left(\frac{\dot{R}'}{R} - \frac{\dot{B}}{B}\frac{R'}{R} - \frac{\dot{R}}{R}\frac{A'}{A}\right), \tag{2.3.10d}$$

where $q = Bq^1$. This is an underdetermined system of four coupled partial differential equations in seven unknowns, viz., $A, B, R, \mu, P_r, P_\perp$ and q.

2.4 Exterior spacetime and junction conditions

The exterior spacetime is taken to be the Vaidya solution given by

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m(v)}{\mathcal{R}}\right)dv^2 - 2dvd\mathcal{R} \\ &\quad +\mathcal{R}^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \end{aligned} \tag{2.4.1}$$

where $m(v)$ represents the Newtonian mass of the gravitating body as measured by an observer at infinity. The spacetime coordinates here are

$$(x^\alpha) = (v, \mathcal{R}, \theta, \phi). \tag{2.4.2}$$

The necessary conditions for the smooth matching of the interior spacetime (2.3.1) to the exterior spacetime (2.4.1) have been extensively investigated. We present the main results that are necessary for modeling a radiating star. The continuity of the intrinsic and extrinsic curvature components of the interior and exterior spacetimes across a timelike boundary gives

$$m(v)_\Sigma = \left\{ \frac{R}{2} \left[\left(\frac{\dot{R}}{A} \right)^2 - \left(\frac{R'}{B} \right)^2 + 1 \right] \right\}_\Sigma , \tag{2.4.3}$$

$$(P_r)_\Sigma = q_\Sigma. \tag{2.4.4}$$

Relation (2.4.4) determines the temporal evolution of the collapsing star across the boundary Σ. Equation (2.4.4) is a highly nonlinear differential equation. As far as we are aware it has not been solved exactly for expansion-free collapsing stars with null radiation in the exterior atmosphere.

2.5 Temporal evolution

The junction condition $(P_r)_\Sigma = q_\Sigma$ yields

$$\dot{B} = \left(\frac{R}{2AR'}\right)\left[2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 - 2\frac{\dot{A}}{A}\frac{\dot{R}}{R} + \frac{A^2}{R^2}\right]B^2$$

$$+ \left[\frac{\dot{R}'}{R'} - \frac{A'}{A}\frac{\dot{R}}{R'}\right]B - \frac{A}{2}\left[\frac{R'}{R} + 2\frac{A'}{A}\right], \qquad (2.5.1)$$

which is of the form

$$\dot{B} = \mathcal{C}_0(t)B^2 + \mathcal{C}_1(t)B + \mathcal{C}_2(t). \qquad (2.5.2)$$

In general this is a Riccati equation. Thirukkanesh et al (2012) have provided various classes of exact solutions to (2.5.1) based on ad hoc assumptions.

In seeking solutions to (2.5.1) we are going to focus on the socalled expansion-free collapse scenario. The framework and motivation for studying expansion-free collapse models were well motivated by Herrera and coworkers. The vanishing of the expansion scalar leads to

$$B(r,t) = \frac{g(r)}{R^2(r,t)}, \qquad (2.5.3)$$

where $g(r)$ is an arbitrary function. With the assumption (2.5.3) the boundary condition (2.5.1) reduces to

$$\frac{\ddot{R}}{R} + \frac{1}{2}\left(\frac{\dot{R}}{R}\right)^2 - \frac{\dot{A}}{A}\frac{\dot{R}}{R} - \left(\frac{AR^2R'}{g} + \dot{R}\right)\frac{A'R}{g}$$

$$- \left(\frac{R^2R'^2}{2g^2} - \frac{1}{2R^2}\right)A^2 + \left(2\dot{R}R' + R\dot{R}'\right)\frac{A}{g} = 0, \qquad (2.5.4)$$

which was first obtained by Herrera et al (2009). This equation helps to provide the physical meaning of expansion-free collapse.

2.6 Exact model

It is necessary to integrate (2.5.4). In spite of its nonlinearity exact solutions do exist. By inspection it is possible to identify a family of solutions that solves (2.5.4) exactly. A simple family of exact solutions is given by

$$A(r,t) = \frac{2^{2/3}C_2^{1/3}(-1+4C_2^{1/3}-3C_2^2)^{1/3}t}{(2C_1-3r^2+3t^2)^{2/3}}, \tag{2.6.1a}$$

$$B(r,t) = \frac{2^{2/3}(-1+4C_2-3C_2^2)^{1/3}r}{C_2^{2/3}(2C_1-3r^2+3t^2)^{2/3}}, \tag{2.6.1b}$$

$$\begin{aligned} R(r,t) &= (C_2^{1/3}(-1+4C_2-3C_2^2)^{5/6}(C_1^2+6C_1(-\frac{r^2}{2}+\frac{t^2}{2}) \\ &\quad +9(-\frac{r^2}{2}+\frac{t^2}{2})^2)^{1/6}) \times \left(1-C_2+3C_2^2\right)^{-1}, \end{aligned} \tag{2.6.1c}$$

where C_1, C_2, C_3 are constants.

With the above gravitational potentials the Einstein field equations (2.3.10a)–(2.3.10d) reduce to

$$\mu = -\frac{4 \times 2^{2/3}(1 - C_2 + C_2^2)}{C_2^{2/3}(-1 + 4C_2 - 3C_2^2)^{2/3}(2C_1 - 3r^2 + 3t^2)^{2/3}}, \tag{2.6.2a}$$

$$P_r = -\frac{4 \times 2^{2/3} C_2^{1/3}}{(-1 + (4 - 3C_2)C_2)^{2/3}(2C_1 - 3r^2 + 3t^2)^{2/3}}, \tag{2.6.2b}$$

$$P_\perp = \frac{2^{2/3} C_2^{-2/3}(-4r^2 + C_2^2(2C_1 + 5r^2 + 3t^2))}{(-1 + (4 - 3C_2)C_2)^{2/3} r^2 (2C_1 - 3r^2 + 3t^2)^{2/3}}, \tag{2.6.2c}$$

$$q = -\frac{4 \times 2^{2/3} C_2^{1/3}}{(-1 + 4C_2 - 3C_2^2)^{2/3}(2C_1 - 3r^2 + 3t^2)^{2/3}}. \tag{2.6.2d}$$

Therefore equation (2.6.2a)-(2.6.2d) are an exact solution to the Einstein field equations (2.3.10a)-(2.3.10d) wich may be utilised to model the interior of a spherically symmetric star with heat flow.

2.7 Energy conditions

We will now examine the physical viability of our stellar model. First, we require that the thermodynamical quantities be positive within the star

$$\mu \geq 0, \tag{2.7.1a}$$

$$p_r \geq 0, \tag{2.7.1b}$$

$$p_\perp \geq 0. \tag{2.7.1c}$$

Next, the energy density and radial pressure must decrease outwards from the centre of the star to its surface

$$\mu' \ < \ 0, \tag{2.7.2a}$$

$$p_r' \ < \ 0. \tag{2.7.2b}$$

Figure 2.1 and Figure 2.2 confirm these trends in the energy density and radial pressure profiles of the model respectively. We also observe that the energy density and both the radial and tangential pressures (Figure 2.3) increase with time. This is expected since as the collapse proceeds more matter gets squeezed into smaller volumes leading to a more compact core. The profile of the heat flux, given in Figure 2.4, we observe that it is an increasing function of the temporal coordinate. As the core collapses, its density increases thus squeezing more atoms together. This initiates a larger number of fusion reactions which produce vast amounts of energy. Figure 2.5 and Figure 2.6 show that the energy conditions

$$z_1 \ = \ (\mu + p_r)^2 - 4q^2 > 0, \quad (E1) \tag{2.7.3a}$$

$$z_2 \ = \ \mu - p_r - 2p_\perp + \left[(\mu + p_r)^2 - 4q^2\right]^{1/2} > 0, \quad (E2) \tag{2.7.3b}$$

always hold within the stellar interior. Thus, the weak, strong and dominant energy conditions are satisfied throughout the interior of the star.

2.8 Thermal behaviour

Extended irreversible thermodynamics has been widely used in the context of dissipative collapse. The effect of the relaxation time on the temperature and luminosity profiles has been exhibited in several studies. Within an astrophysical context, it was shown that relaxational effects predict higher core temperatures and enhanced cooling at the surface of the radiating star. In this regard see the works of Govender at al (1998), Govender at al (1999), Govender and Govinder (2001) and Maharaj and Govender (2005). In order to explore the contributions from relaxational effects as the fluid exits from hydrostatic equilibrium we will employ a causal heat transport equation of the Maxwell-Cattaneo form (Martinez (1996)). The truncated causal transport equation in the absence of rotation and viscous-heat coupling is given by

$$\tau_r {h_\alpha}^\beta \dot{q}_\beta + q_\alpha = -\kappa\big({h_\alpha}^\beta \nabla_\beta \mathcal{T} + \mathcal{T}\dot{u}_\alpha\big), \tag{2.8.1}$$

where $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the projection tensor, $\mathcal{T}(t,r)$ is the local equilibrium temperature, $\kappa(\geq 0)$ is the thermal conductivity, and $\tau_r(\geq 0)$ is the relaxation time-scale over which causal, stable behaviour is achieved. The noncausal Fourier heat transport equation is obtained by setting the relaxation time $\tau_r = 0$ in (2.8.1). With the aid of the metric (2.3.1), equation (2.8.1) becomes

$$\tau_r (qB)^{\dot{}} + A(qB) = -\kappa \frac{(A\mathcal{T})'}{B}. \tag{2.8.2}$$

The thermodynamic coefficients associated with radiative transfer are well motivated by Govender et al (1998, 1999). In order to obtain the causal temperature profile from

(2.8.2) we take

$$\kappa = \gamma \mathcal{T}^3 \tau_{\rm c}, \tag{2.8.3a}$$

$$\tau_{\rm c} = \left(\frac{\xi}{\gamma}\right) \mathcal{T}^{-\omega}, \tag{2.8.3b}$$

where τ_c is the mean collision time, ξ, γ and ω are positive constants. The relaxation time is taken to be of the order of the mean collision time

$$\tau_r = \left(\frac{\psi\gamma}{\xi}\right) \tau_{\rm c}, \tag{2.8.4}$$

where ψ (≥ 0) is a constant. Employing the definitions for τ_r and κ, it can be shown that equation (2.8.2) takes the form

$$\psi \dot{(qB)} \mathcal{T}^{-\omega} + A(qB) = -\xi \frac{\mathcal{T}^{3-\omega}(A\mathcal{T})'}{B}, \tag{2.8.5}$$

where ψ can be considered to be a 'causality index', that enables us to quantify the impact of relaxation effects on the system. The noncausal case is obtained when $\psi = 0$. Figure 6 shows both the causal and noncausal temperatures from the centre of the star towards its surface. The noncausal temperature is approximately constant throughout the stellar fluid. This behaviour is not a true representation of the temperature profile of a star close to hydrostatic equilibrium. We expect the temperature to be highest at the centre of the star and to gradually decrease towards the surface layers. The core is dense and we expect heat generation (in our case thermally generated neutrinos) to be highest in regions of high densities. This pathological behaviour of the noncausal temperature is due to the absence of relaxational effects. The causal temperature shows that the core temperature starts off at a maximum at the centre and drops off sharply towards the boundary. This enhanced cooling has been observed in various models of dissipative collapse.

2.9 Cavity

We have presented an exact model of dissipative collapse in which the interior is expansion-free. This is a first exact model of expansion-free collapse. An analysis of the density, radial pressure, tangential pressure and heat flow show that all these quantities are well behaved. Our model obeys the energy conditions rendering it a physically viable description of dissipative collapse. The noncausal temperature is approximately constant. The causal temperature dominates the noncausal temperature up to a finite radius r_0, for $r_0 < r \leq r_b$. For $r > r_0$ the noncausal temperature is greater than causal temperature. This is the first time that we observe such a phenomenon. Could this be the presence of a cavity forming within the stellar fluid for $r \leq r_0$ or is it the consequence of the adoption of a noncausal heat transport equation?. This is an isuue that requires further investigation.

Herrera and coworkers (Herrera et al (2012)) have shown that expansion-free collapse in both the nondissipative and dissipative cases lead to the formation of a void within the stellar fluid. In order to appreciate the formation of the cavity consider the expansion scalar given by

$$\Theta = \frac{D_T(\delta l)}{\delta l} + \frac{2D_T R}{R} = \frac{D_T(\delta l)}{\delta l} + \frac{2U}{R} \tag{2.9.1}$$

The first term represents the relative velocity between neighouring fluid layers. The circumferential velocity is described by the second term. These two velocities need not necessarily be equal in magnitude. In the case of expansion-free collapse, a decrease in the perimeter of a comoving sphere is balanced by an increase in the radial separation distance between two neighbouring particles (Herrera et al 2010). The formation of the cavity may alter the relaxation time within the different regions within the collapsing body which gives rise to the peculiar behaviour of the temperature profiles.

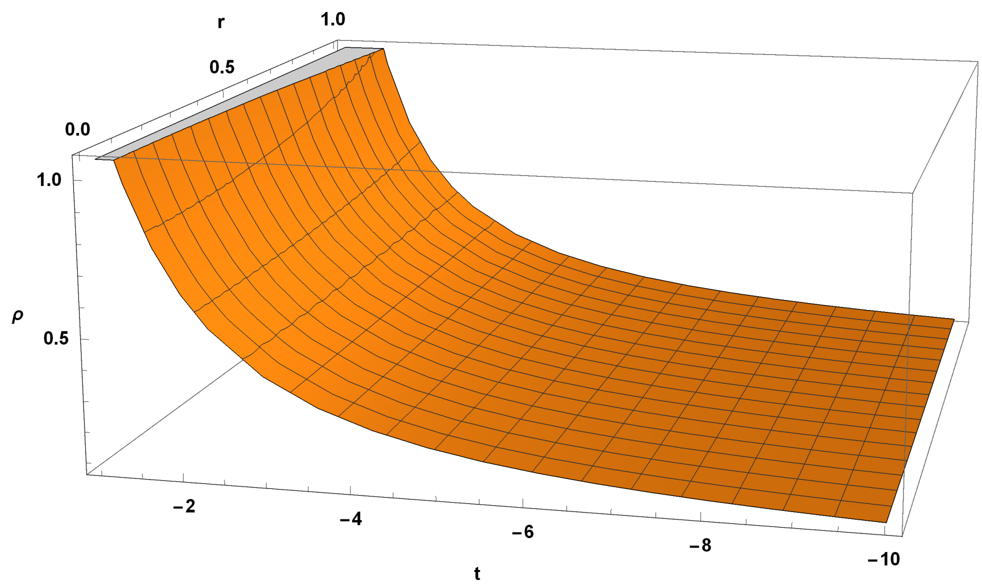

Figure 2.1: Density as a function of the radial r and temporal t coordinates

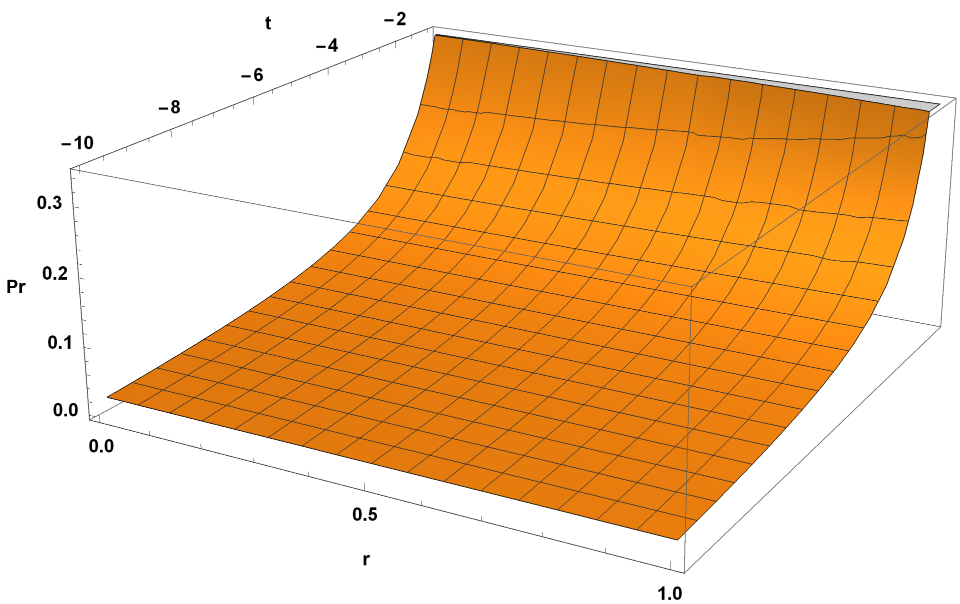

Figure 2.2: Radial pressure as a function of the radial coordinate r and temporal coordinate t.

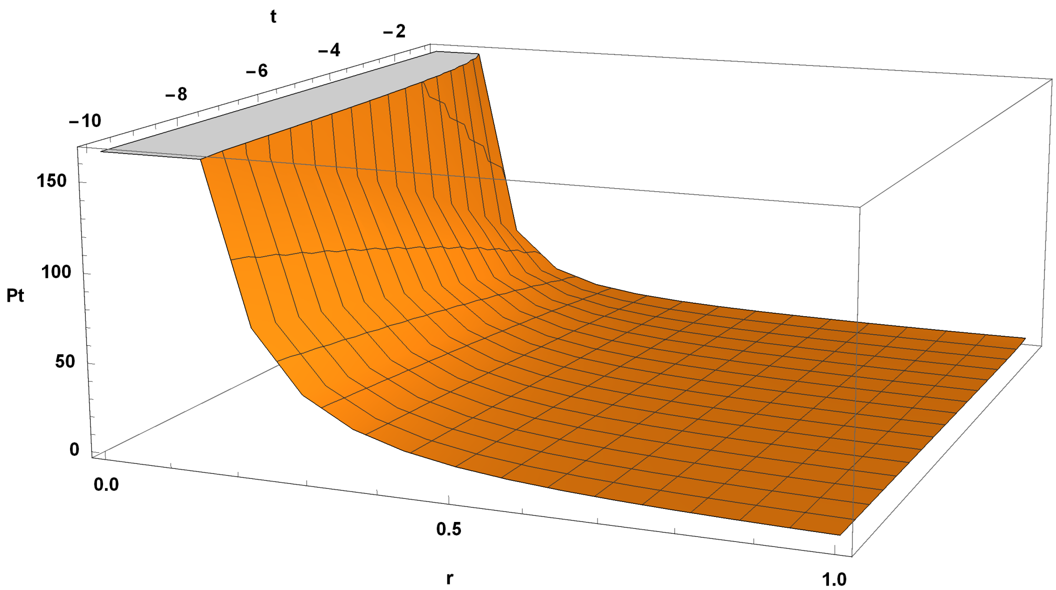

Figure 2.3: Tangential pressure as a function of the radial r and temporal t coordinates

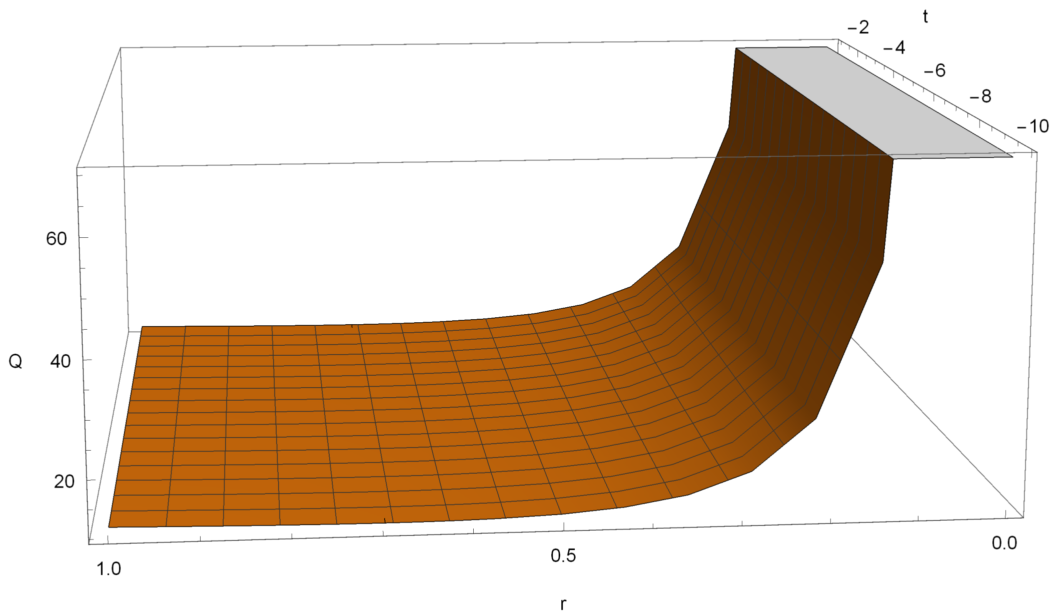

Figure 2.4: Heat flow as a function of the radial r and temporal t coordinates

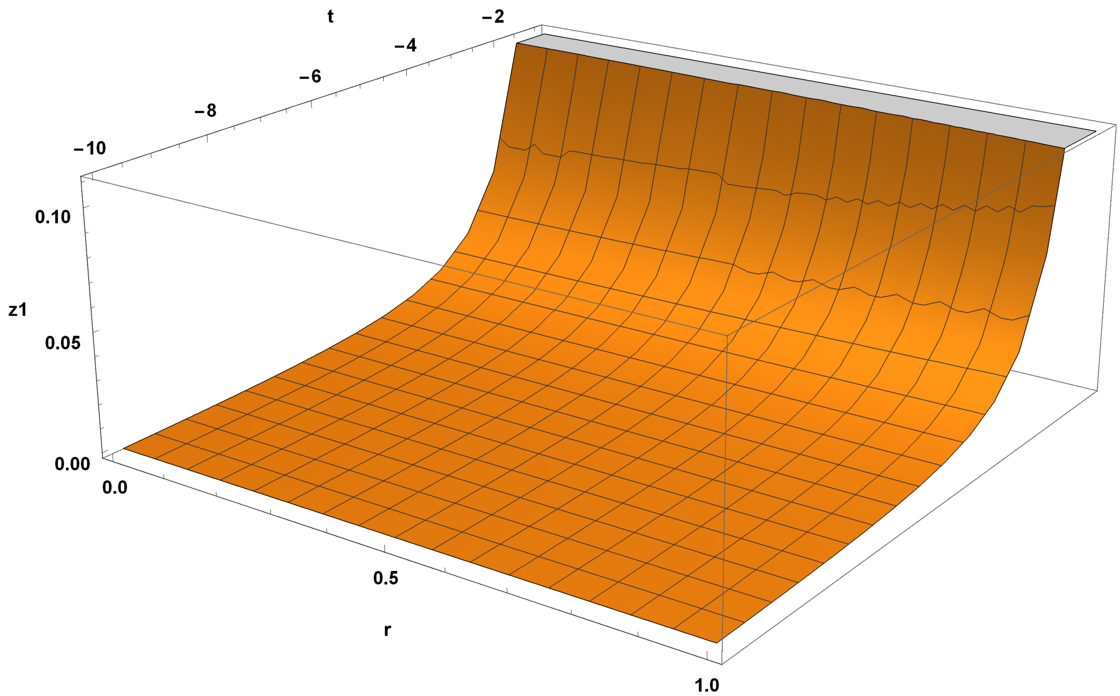

Figure 2.5: Energy conditions as a function of the radial r and temporal t coordinates

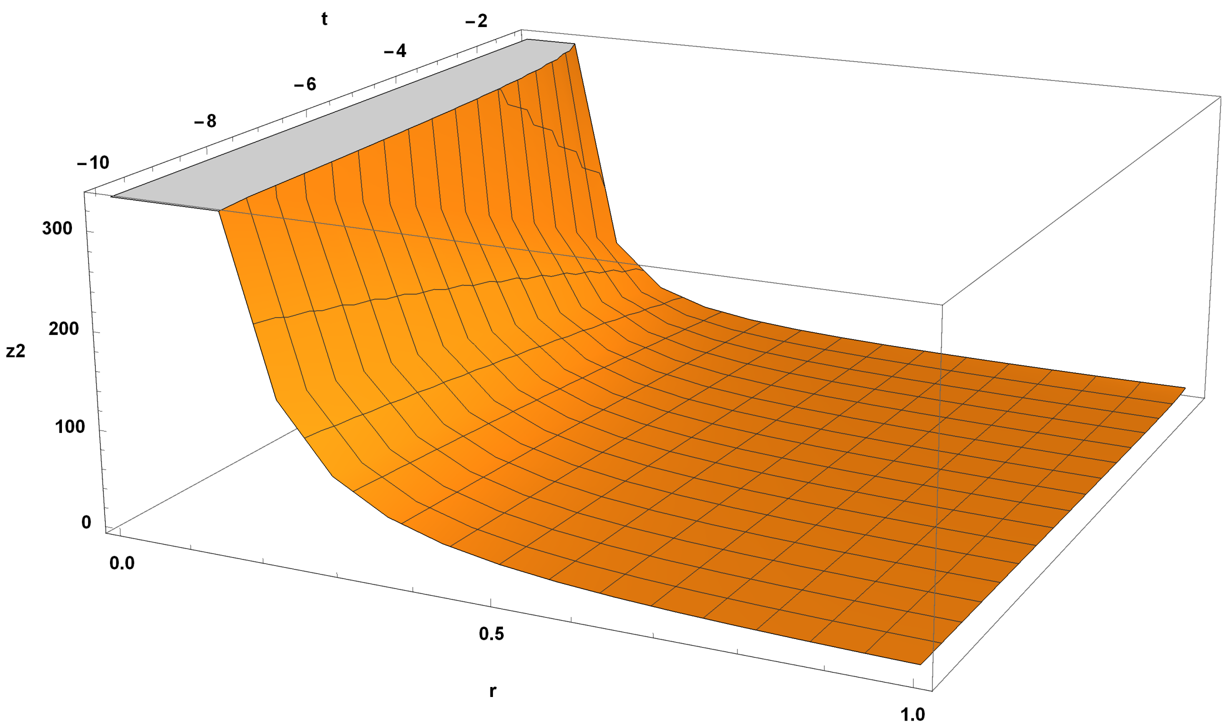

Figure 2.6: Energy conditions as a function of the radial r and temporal t coordinates

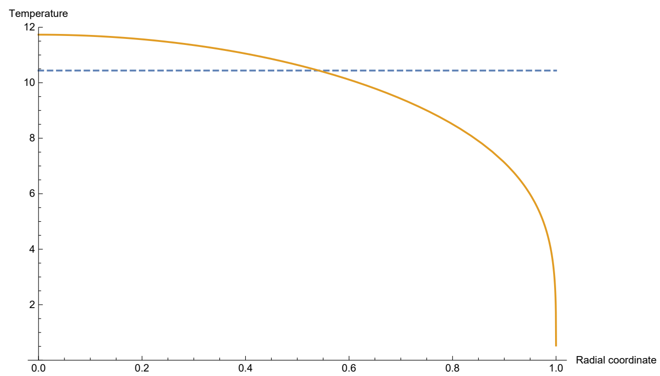

Figure 2.7: Causal (solid line) and noncausal (dashed line) as functions of the radial coordinate

Chapter 3

Dissipative collapse in the presence of shear

3.1 Introduction

Radiating stars, with the external atmosphere comprising outgoing null radiation, are an important area of research in general relativity. It is important to find exact solutions to the field equations and the boundary condition to describe the physics in strong gravity. There have been several exact models found in recent times that have been applied to gravitational collapse, stability of stars, dissipative effects, thermodynamics in the Eckart theory and more general causal theories, and other important astrophysical processes. Some examples of recent investigations addressing these issues are given in the works by Sharma and Tikekar (2012), Tewari (2012, 2013), Thirukkanesh and Govender (2013), Sharma et al (2015), and Ivanov (2012). A recent approach to study radiating stellar system involves the application of Lie symmetry infinitesimal generators as outlined in the investigation of Mohanlal et al (2017). Also the embedding of a curved metric in four dimensions into a metric in five dimesions, which is flat with vanishing Riemann tensor, has been analysed by Naidu et al (2018) giving a radiating stellar model.

The general case of expanding, shearing and accelerating stellar interiors was first considered by Thirukkanesh et al (2012) in spherically symmetric gravitational fluids. The special case of geodesic flows leads to considerable simplification of the underlying nonlinear differential equations and consequently has received much attention. A comprehensive treatment of the dissipative effects for gravitating geodesic stars was undertaken by Kolassis et al (1988). Radiating bodies with neutrino flux were analysed by Grammenos and Kolassis (1992). Particle trajectories which are geodesic with separable metric potentials are restrictive and require anisotropic pressure (Govender and Maharaj (2009)). Models of geodesic stars have been found by Thirukkanesh and Maharaj (2009, 2010) by transforming the boundary condition to Bernoulli equations, Riccati equations and confluent hypergeometric equations. Abebe et al (2014) found several new families of exact solutions using the geometric properties of Lie groups; travelling wave solutions and self-similar solutions arise in the Lie symmetry approach. Ivanov (2016) found a generating function for geodesic stars undergoing anisotropic collapse with shear and outgoing null radiation. This allows for the introduction of the horizon function with is related to the redshift and the formation of horizons. Recently several new farmilies of exact solutions with elementary functions for geodesic radiating systems were found by Tiwari and Maharaj (2017) with physically reasonable profiles.

The cosmological constant is often included in studies because of obsevations relating to type Ia supernovae, high redshifts and baryon acoustic oscillations. The relevant data point to a small positive cosmological constant which should be included in models describing strong gravity scenarios. In particular it is important to determine the degree to which a nonzero cosmological constant influences the collapse of a general relativistic star. Some studies involving the cosmological constant include decreased bending of light (Rindler and Ishak (2007)), modification of the Buchdahl compactness ratio (Andreasson and Boehmer (2009)), formation of gravastars (Chan et al (2009)), and strange stars with null strange quark fluid (Ghosh and Dadhich (2003)). Deshingkar et al (2001) and Bohmer and Harko (2005) modelled stars and collapsing matter distributions showing that the cosmological constant is connected to formation of both black holes and naked singularities. Govender and Thirukkanesh (2009) considered the effect of the nonzero cosmological constant on the star's temperature profile in causal thermodynamics; relaxational effects lead to larger temperature gradients in the core of the star. These results were confirmed in the subsequent treatment of Thirukkanesh et al (2012). Recently Bhatti (2018) considered shear-free matter distributions dominated by the cosmological constant and found solutions satisfying the Darmois matching conditions.

In this chapter we consider the situation of a collapsing star in spherical symmetry both in the absence and the presence of the cosmological constant. We perform an analysis of the underlying boundary condition and find new exact solutions. In Section 3.2 we present the model of a radiating star, the field equations and the boundary condition at the surface. In Section 3.3 a transformation due to Thirukkanesh and Maharaj (2010) is discussed. Exact solutions with both nonzero and vanishing cosmological constant are presented in Sections 3.4 and 3.5 respectively. A detailed physical analysis is performed in Section 3.6. Concluding remarks are made in Section 3.7.

3.2 The model

The line element for the interior goemetry of a radiating star for a general spherically symmetric, spacetime can be written as

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{3.2.1}$$

where $A = A(t,r)$, $B = B(t,r)$ and $Y = Y(t,r)$ are potentials. The energy momentum tensor for the interior matter distribution is given by

$$\begin{aligned} T_{\alpha\beta} &= (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta \\ &\quad + q_\alpha V_\beta + q_\beta V_\alpha\,, \end{aligned} \tag{3.2.2}$$

where μ is the energy density, P_r is the radial pressure, $P_\perp$ is the tangential pressure, and q^α is the heat flow vector. The fluid four–velocity $\mathbf{V}$ is comoving and has the form given by

$$V^\alpha = \frac{1}{A}\delta^\alpha_0\,. \tag{3.2.3}$$

The heat flow vector is outgoing and spacelike. It is given in the form

$$q^\alpha = (0, q^1, 0, 0)\,, \tag{3.2.4}$$

where $q^\alpha V_\alpha = 0$. In addition we have

$$\chi^\alpha\chi_\alpha = 1, \tag{3.2.5a}$$

$$\chi^\alpha V_\alpha = 0. \tag{3.2.5b}$$

The expansion scalar and the fluid four acceleration are defined by

$$\Theta = V^\alpha{}_{;\alpha}, \tag{3.2.6a}$$

$$a_\alpha = V_{\alpha;\beta}V^\beta, \tag{3.2.6b}$$

and the expression

$$\sigma_{\alpha_\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta), \tag{3.2.7}$$

gives the shear tensor. For the comoving line element (2.3.1) these kinematical quantities can be written as

$$a_1 = \frac{A'}{A}, \tag{3.2.8a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right), \tag{3.2.8b}$$

$$\sigma = \frac{1}{A}\left(\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right), \tag{3.2.8c}$$

In terms of the potentials A, B and Y. In the above dots and primes denote differentiation with respect to t and r respectively.

The Einstein field equations with cosmological constant λ has the form

$$G_{\alpha_\beta} + \lambda g_{\alpha_\beta} = T_{\alpha_\beta}. \tag{3.2.9}$$

For the metric (3.2.1) and the matter distribution (3.2.2), the equations (3.2.9) can be written as

$$\mu = -\frac{1}{B^2}\left[2\frac{Y''}{Y} + \left(\frac{Y'}{Y}\right)^2 - 2\frac{B'}{B}\frac{Y'}{Y}\right]$$

$$+\frac{1}{A^2}\left(2\frac{\dot{B}}{B} + \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y} + \frac{1}{Y^2} - \lambda, \tag{3.2.10a}$$

$$P_r = -\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y} - \left(2\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] - \frac{1}{Y^2}$$

$$+\frac{1}{B^2}\left(2\frac{A'}{A} + \frac{Y'}{Y}\right)\frac{Y'}{Y} + \lambda, \tag{3.2.10b}$$

$$P_\perp = -\frac{1}{A}\left[\frac{\ddot{B}}{B} + \frac{\ddot{Y}}{Y} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{Y}}{Y}\right) + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right]$$

$$+\frac{1}{B^2}\left[\frac{A''}{A} + \frac{Y''}{Y} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{Y'}{Y}\right] + \lambda, \tag{3.2.10c}$$

$$q = \frac{2}{AB}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\frac{A'}{A}\right), \tag{3.2.10d}$$

where $q = Bq^1$. This is a nonlinear system of partial differential equations.

At the boundary of the star there has to be matching of the intrinsic and extrinsic curvature components connecting the interior and exterior spacetimes. The matching of the spacetimes is at the timelike boundary Σ. We generate the condition

$$(P_r)_\Sigma = q_\Sigma, \tag{3.2.11}$$

which indicates that the radial pressure is not vanishing across the boundary of the star. With the help of (3.2.10b), (3.2.10d) and (3.2.11) we find that the junction condition becomes

$$\frac{2}{AB}\left(\frac{\dot{Y}'}{Y}-\frac{\dot{B}}{B}\frac{Y'}{Y}-\frac{\dot{Y}}{Y}\frac{A'}{A}\right)=-\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y}-\left(2\frac{\dot{A}}{A}-\frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right]$$

$$-\frac{1}{Y^2}+\frac{1}{B^2}\left(2\frac{A'}{A}+\frac{Y'}{Y}\right)\frac{Y'}{Y}+\lambda, \tag{3.2.12}$$

which governs the temporal evolution of the radiating star across the stellar surface Σ. This equation can be rewritten in the form

$$\dot{B}=\left[\frac{\ddot{Y}}{AY'}-\frac{\dot{A}\dot{Y}}{A^2Y'}+\frac{\dot{Y}^2}{2AYY'}+\frac{A}{2YY'}-\frac{\lambda AY}{2Y'}\right]B^2+\left(\frac{\dot{Y}'}{Y'}-\frac{\dot{Y}}{Y'}\frac{A'}{A}\right)B$$

$$-\frac{A}{2}\left(\frac{2A'}{A}+\frac{Y'}{Y}\right), \tag{3.2.13}$$

which has the form of a first order equation in the potential B. When $\lambda=0$ we regain the boundary condition studied by Thirukkanesh et al (2012).

3.3 A transformation

Equation (3.2.13) may be viewed as a Riccati equation in B. Particular solutions have been found using various methods. In the case of geodesic flows Thirukkanesh and Maharaj (2009) introduced the transformation

$$B=ZY', \tag{3.3.1}$$

which leads to a new family of solutions. We now apply (3.3.1) to the general case including nongeodesic motion with cosmological constant. In this general case we find that (3.2.13) and (3.3.1) lead to the result

$$\dot{Z} = \left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right) Z^2 - \frac{A'\dot{Y}}{AY'}Z - \left(\frac{A'}{Y'} + \frac{A}{2Y}\right), \tag{3.3.2}$$

which is a first order equation in Z.

If we set $A = 1$ for geodesic flows with cosmological constant then equation (3.3.2) gives

$$\dot{Z} = \frac{1}{2Y}[FZ^2 - 1], \tag{3.3.3}$$

where we have set

$$F = 2Y\ddot{Y} + \dot{Y}^2 - \lambda Y^2 + 1. \tag{3.3.4}$$

When $A = 1$ and $\lambda = 0$ then

$$F = 2Y\ddot{Y} + \dot{Y}^2 + 1, \tag{3.3.5}$$

which in the simplest case first studied by Thirukkanesh and Maharaj (2010).

Equation (3.3.2) is nonlinear and difficult to integrate in general. However exact solutions exist. A simpler class of solutions can be found if we let

$$\dot{Y} = 0, \tag{3.3.6a}$$

$$\frac{A'}{Y'} + \frac{A}{2Y} = 0. \tag{3.3.6b}$$

Then (3.3.2) becomes

$$\dot{Z} = \left(\frac{A}{2Y} - \frac{\lambda A Y}{2} \right) Z^2. \tag{3.3.7}$$

We can integrate (3.3.6a), (3.3.6b) and (3.3.7) to find the potentials

$$A = f(t) Y^{-1/2},, \tag{3.3.8a}$$

$$B = 2Y' \left[\left(\lambda Y^{1/2} - Y^{-3/2} \right) \int f(t) dt + g(r) \right]^{-1}, \tag{3.3.8b}$$

$$Y = Y(r). \tag{3.3.8c}$$

in terms of the arbitrary functions $f(t)$ and $g(r)$ which arise from the integration process. With these potentials the line element (3.2.1) becomes

$$ds^2 = -f^2(t) Y^{-1} dt^2 + 4Y'^2 \left[\left(\lambda Y^{1/2} - Y^{-3/2} \right) \int f(t) dt + g(r) \right]^{-2} dr^2$$

$$+ Y^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{3.3.9}$$

containing the cosmological constant λ and the function $Y(r)$. It is interesting to observe that the spacetime related to (3.3.9) is expanding, accelerating and shearing. When $\lambda = 0$ then (3.3.9) is related to one of the metrics in the treatment of Thirukkanesh et al (2009).

3.4 Geodesic motion with $\lambda \neq 0$

As shown in Section 3.3, exact solutions to equation (3.3.2) exist when $A \neq 1$. We now consider the simpler case of geodesic motion ($A = 1$) in the presence of the cosmological constent ($\lambda \neq 0$). We have to solve (3.3.3) subject to (3.3.4). We can find two classes of simple solutions.

3.4.1 $\lambda \neq 0$, $F = 1$

If $\lambda \neq 0$ and $F = 1$, then (3.3.4) becomes

$$2Y\ddot{Y} + \dot{Y}^2 = \lambda Y^2. \tag{3.4.1}$$

This equation can be written as

$$Y\frac{d\dot{Y}^2}{dY} + \dot{Y}^2 = \lambda Y^2. \tag{3.4.2}$$

Integrating (3.4.2) gives

$$\dot{Y}^2 = \frac{\lambda Y^2}{3} + \frac{c(r)}{Y}, \tag{3.4.3}$$

where $c(r)$ is a function of integrating. We can express (3.4.3) in the form

$$\frac{\sqrt{Y}dY}{\sqrt{\frac{\lambda Y^3}{3} + c(r)}} = dt, \tag{3.4.4}$$

and the variables Y and t have separated.

Equation (3.4.4) can be simplified if we define

$$Y^{3/2} = \sqrt{\frac{3c(r)}{\lambda}} \sinh v, \tag{3.4.5}$$

in terms of the new variable v. Then (3.4.4) is simply $\frac{2}{\sqrt{3\lambda}} dv = dt$. Finally integrating yields

$$v = \frac{\sqrt{3\lambda}}{2}(t + C). \tag{3.4.6}$$

With $F = 1$ we observe that (3.3.3) is a separable equation; integrating gives

$$Z = \left(\frac{1 + \tilde{C} e^{\int 1/Y dt}}{1 - \tilde{C} e^{\int 1/Y dt}} \right), \tag{3.4.7}$$

where C and $\tilde{C}$ are new constants. Therefore

$$A = 1, \tag{3.4.8a}$$

$$B = Y' \left(\frac{1 + C_3 e^{\int 1/Y dt}}{1 - C_3 e^{\int 1/Y dt}} \right), \tag{3.4.8b}$$

$$Y = \left[\sqrt{\frac{3c(r)}{\lambda}} \sinh \left[\frac{\sqrt{3\lambda}}{2}(t + C) \right] \right]^{2/3}, \tag{3.4.8c}$$

gives a new solution to the boundary condition with $\lambda \neq 0$.

3.4.2 $\lambda \neq 0$, $F = 1 + a(r)$

If $\lambda \neq 0$ and $F = 1 + a(r)$, then (3.3.4) gives

$$2Y\ddot{Y} + \dot{Y}^2 = \lambda Y^2 + a(r). \tag{3.4.9}$$

Equation (3.4.9) can be written as

$$Y\frac{d\dot{Y}^2}{dY}+\dot{Y}^2=\lambda Y^2+a(r), \tag{3.4.10}$$

which is a first order in $\dot{Y}^2$. Integrating gives

$$\dot{Y}^2=\frac{\lambda Y^2}{3}+a(r)+\frac{c(r)}{Y}. \tag{3.4.11}$$

This can be written as

$$\frac{dY}{\sqrt{\frac{\lambda Y^2}{3}+a(r)+\frac{c(r)}{Y}}}=dt, \tag{3.4.12}$$

with separated variables.

It is difficult to find Y from the above. We can achieve some simplification if we let

$$c(r) \;=\; 0, \tag{3.4.13a}$$

$$Y \;=\; \sqrt{\frac{3a(r)}{\lambda}}\sinh v. \tag{3.4.13b}$$

Then (3.4.12) is simply $\sqrt{\frac{3}{\lambda}}dv=dt$. Then integrating yields

$$\sqrt{\frac{3}{\lambda}}v=t+C. \tag{3.4.14}$$

We can integrate (3.3.3) with $F=1+a(r)$. Note that equation (3.3.3) becomes

$$\dot{Z}=\frac{1}{2Y}\left[\left(1+a(r)\right)Z^2-1\right]. \tag{3.4.15}$$

We can rewrite equation (3.4.15) as

$$\int \frac{dZ}{Z - \frac{1}{\sqrt{1+a(r)}}} + \int \frac{dZ}{Z + \frac{1}{\sqrt{1+a(r)}}} = \int \frac{1}{Y} dt, \tag{3.4.16}$$

which is separable. Integrating (3.4.16) gives

$$Z = \frac{1}{(1+a(r))^{1/2}} \left(\frac{1 + Ce^{\int \frac{1}{Y} dt}}{1 - Ce^{\int \frac{1}{Y} dt}} \right). \tag{3.4.17}$$

Therefore

$$A = 1, \tag{3.4.18a}$$

$$B = \frac{Y'}{(1+a(r))^{1/2}} \left(\frac{1 + Ce^{\int \frac{1}{Y} dt}}{1 - Ce^{\int \frac{1}{Y} dt}} \right), \tag{3.4.18b}$$

$$Y = \sqrt{\frac{3a(r)}{\lambda}} \sinh \left[\sqrt{\frac{\lambda}{3}} (t + C) \right]. \tag{3.4.18c}$$

gives another new solution to the boundary condition with $\lambda \neq 0$.

3.5 Geodesic motion with $\lambda = 0$

The case of geodesic motion with vanishing cosmological constant was first analysed by Thirukkanesh and Maharaj (2010). In this case $A = 1$ and $\lambda = 0$. It is possible to find new solutions with $\lambda = 0$ that are not contained in the analysis of Thirukkanesh and Maharaj (2010). We have to solve (3.3.3) and (3.3.5).

In this case equation (3.3.3) becomes

$$\dot{Z} = \frac{1}{2Y}\left[(1 + a(r))Z^2 - 1\right]. \tag{3.5.1}$$

This is the same as equation (3.4.15). Hence the general solution is given by

$$Z = \frac{1}{(1 + a(r))^{1/2}}\left(\frac{1 + Ce^{\int \frac{1}{Y}dt}}{1 - Ce^{\int \frac{1}{Y}dt}}\right), \tag{3.5.2}$$

as in Section 3.4. Then the potential B is given by

$$B = \frac{Y'}{(1 + a(r))^{1/2}}\left(\frac{1 + Ce^{\int \frac{1}{Y}dt}}{1 - Ce^{\int \frac{1}{Y}dt}}\right). \tag{3.5.3}$$

If $\lambda = 0$ and $F = 1 + a(r)$, then (3.3.4) can be written as

$$Y\frac{d\dot{Y}^2}{dY} + \dot{Y}^2 = a(r), \tag{3.5.4}$$

which is a first order in $\dot{Y}^2$. Integrating yields

$$\dot{Y}^2 = \frac{a(r)Y + \epsilon b(r)}{Y}, \tag{3.5.5}$$

where $\epsilon = 0, \pm 1$ and $b(r) > 0$, and the arbitrary functions $b(r)$ and ϵ arise from the integration process. Three cases arise which we consider separately below.

3.5.1 $\epsilon = 0$

Equation (3.5.5) become

$$\dot{Y}^2 = a(r), \tag{3.5.6}$$

which can be written as

$$dY = \sqrt{a(r)}dt. \tag{3.5.7}$$

Then integrating (3.5.7) we obtain

$$Y = \sqrt{a(r)}t + k(r), \tag{3.5.8}$$

where $k(r)$ is a new arbitrary function. This results is the same as the potential in Thirukkanesh and Maharaj (2010) (see their equation (21)).

Hence the exact solution is given by

$$A = 1, \tag{3.5.9a}$$

$$B = \frac{Y'}{(1+a(r))^{1/2}} \left(\frac{1 + C\left(\sqrt{a(r)}t + k(r)\right)/\sqrt{a(r)}}{1 - C\left(\sqrt{a(r)} + k(r)\right)/\sqrt{a(r)}} \right), \tag{3.5.9b}$$

$$Y = \sqrt{a(r)}t + k(r). \tag{3.5.9c}$$

3.5.2 $\epsilon = 1$

Equation (3.5.5) becomes

$$\frac{\sqrt{Y}}{\sqrt{a(r)Y + b(r)}}dY = dt. \tag{3.5.10}$$

We now set $a(r)Y + b(r) = b(r)\sinh^2 v$. Then (3.5.10) becomes

$$\frac{2b(r)}{a(r)^{2/3}}\sqrt{\sinh^2 v - 1}\cosh v dv = dt. \tag{3.5.11}$$

Integrating gives

$$\frac{2b(r)}{a(r)^{2/3}}\left[v + \frac{1}{2}\sinh 2v\right] = t + C, \tag{3.5.12}$$

where C is constant.

We have the following new class of solution

$$A = 1, \tag{3.5.13a}$$

$$B = \frac{Y'}{(1+a(r))^{1/2}}\left(\frac{1+Ce^{\int \frac{1}{Y}dt}}{1-Ce^{\int \frac{1}{Y}dt}}\right), \tag{3.5.13b}$$

$$Y = \frac{b(r)}{a(r)}\sinh^2 v - \frac{b(r)}{a(r)}, \tag{3.5.13c}$$

$$v + \frac{1}{2}\sinh 2v = \frac{a(r)^{2/3}}{2d(r)}[t + C], \tag{3.5.13d}$$

in parametric form. This is a new solution for the boundary condition.

3.5.3 $\epsilon = -1$

Equation (3.5.5) becomes

$$\frac{\sqrt{Y}}{\sqrt{a(r)Y - b(r)}}dY = dt. \tag{3.5.14}$$

We now set $a(r)Y + b(r) = b(r)\cosh^2 v$. Then equation (3.5.14) becomes

$$\frac{2b(r)}{a(r)^{2/3}} \int \sqrt{\cosh^2 v + 1} \cosh v dv = dt. \tag{3.5.15}$$

Integrating gives

$$\frac{2b(r)}{a(r)^{2/3}} \left[v - \frac{1}{2} \sinh 2v \right] = t + C \tag{3.5.16}$$

where C is constant.

We have the following new class of solution

$$A = 1, \tag{3.5.17a}$$

$$B = \frac{Y'}{(1 + a(r))^{1/2}} \left(\frac{1 + Ce^{\int \frac{1}{Y} dt}}{1 - Ce^{\int \frac{1}{Y} dt}} \right), \tag{3.5.17b}$$

$$Y = \frac{b(r)}{a(r)} \cosh^2 v + \frac{b(r)}{a(r)}, \tag{3.5.17c}$$

$$v - \frac{1}{2} \sinh 2v = \frac{a(r)^{2/3}}{2b(r)} [t + C], \tag{3.5.17d}$$

in parametric form. This is a new solution for the boundary condition.

3.6 Physical analysis

In order to carry out a physical analysis of our solutions, we use the line element (3.3.9) to describe the interior of a radiating star. For simplicity we choose $Y(r) = r$ and $f(t) = a$ where a is a positive constant. The gravitational potentials A and B follow from (3.3.8a) and (3.3.8b) for this solution. The thermodynamical variables are

$$\mu = -\frac{a^2t^2}{2r^5} - \lambda + \frac{1}{r^2} - \frac{\lambda^2}{2r}, \tag{3.6.1a}$$

$$P_r = \lambda - \frac{1}{r^2}, \tag{3.6.1b}$$

$$P_\perp = -\frac{(5at + 3\lambda r^2)(at - \lambda r^2)^3 - 16\lambda r^5 (at - \lambda r^2)^2 + 32r^6}{16r^5 (at - \lambda r^2)^2}, \tag{3.6.1c}$$

$$Q = qB = \frac{1}{r^2}. \tag{3.6.1d}$$

Figure 3.1 shows the heat flow as a function of the radial and temporal coordinates. Bearing in mind that the collapse proceeds from $t = -\infty$ to $t \rightarrow> 0$, we observe that the heat flow is an increasing function of time. This is expected as the collapse proceeds the core becomes more dense leading to higher central temperatures. The increase in temperature leads to higher heat generation within the collapsing core. Figure 3.2 displays the behaviour of the density. The density is positive at each interior point of the stellar configuration and decreases monotonically as one approaches the stellar surface. The radial pressure is portrayed in Figure 3.3. Our model displays a very interesting feature. In the case of vanishing cosmology constant ($\lambda = 0$) the radial pressure is always negative. The inclusion of the cosmological constant ($\lambda \neq 0$) allows for $P_r > 0$. We also study the behaviour of energy conditions as given by

$$Z_1 \;=\; (\mu + p_r)^2 - 4q^2 > 0, \quad (E1) \tag{3.6.2a}$$

$$Z_2 \;=\; \mu - p_r - 2p_\perp + \left[(\mu + p_r)^2 - 4q^2\right]^{1/2} > 0, \quad (E2) \tag{3.6.2b}$$

which ensure that the weak, strong and dominant energy conditions are satisfied throughout the interior of the star. From Figure 3.4 and Figure 3.5 we observe that the energy conditions are satisfied indicating that our model is physically viable. We further observe that as the collapse proceeds ($t \to 0$) our model obeys an equation of state of the form

$$p = -\alpha\mu, \tag{3.6.3}$$

where $\mu \approx \frac{1}{r^2}$. This density distribution is that of an isothermal sphere and the corresponding general relativistic case was studied by Saslaw et al (1996).

3.7 Discussion

We have presented a new family of solutions to the classical Einstein field equations describing radiating spheres in the presence of a cosmological constant. This extends the results of earlier treatments. We employed a transformation which allowed us to obtain several exact solutions of the boundary condition. In the first class of solutions we obtain new models with a cosmological constant present. These solutions can be written parametrically. In the second case we find new solutions with vanishing cosmological constant. This class extends the category of models found by Thirukkanesh and Maharaj (2010). This allowed us to fully specify the gravitational behaviour of the stellar model; an analysis of the density, pressure, heat flux and energy conditions reveals that this model adequately describes a collapsing sphere. We note that the presence of the cosmological constant affects the gravitational dynamics. An interesting feature of our model is the impact of the cosmological constant on the pressure profile. In the absence of λ the radial presure is always negative. The presence of λ allows for positive radial presure. In essence we can view λ as having a repulsive effect within the stellar core, thus making $P_r > 0$ and stabilising the stellar configuration.

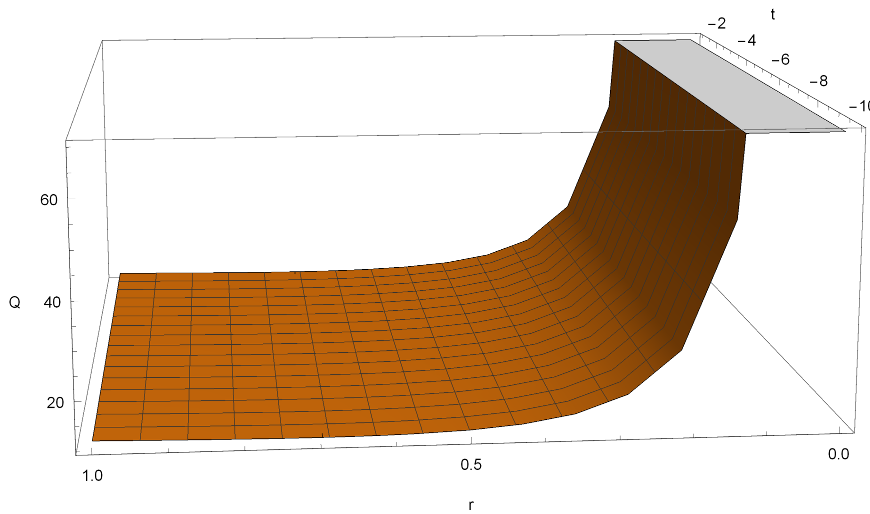

Figure 3.1: Heat flow as a function of the radial r and temporal t coordinates

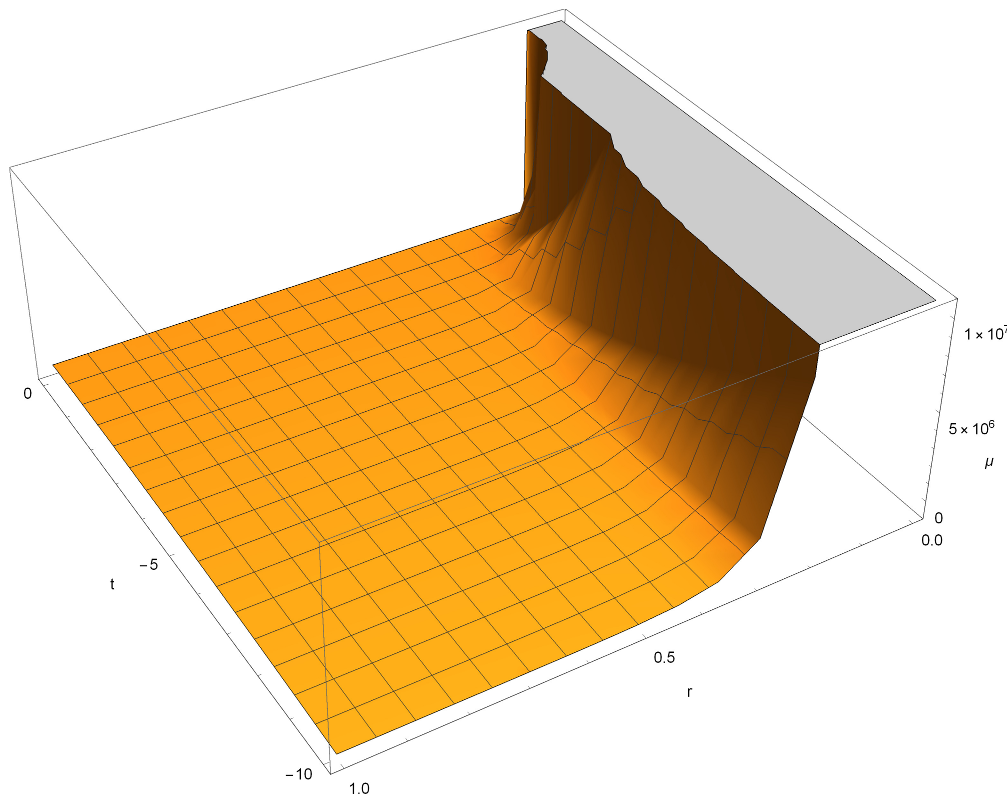

Figure 3.2: Density as a function of the radial coordinate r and temporal coordinate t.

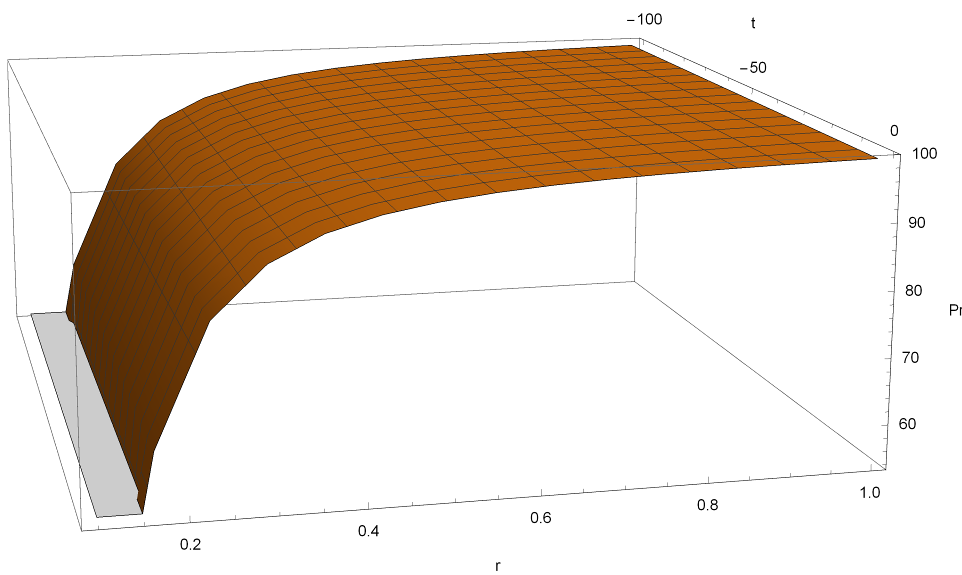

Figure 3.3: Radial pressure as a function of the radial coordinate r and temporal coordinate t.

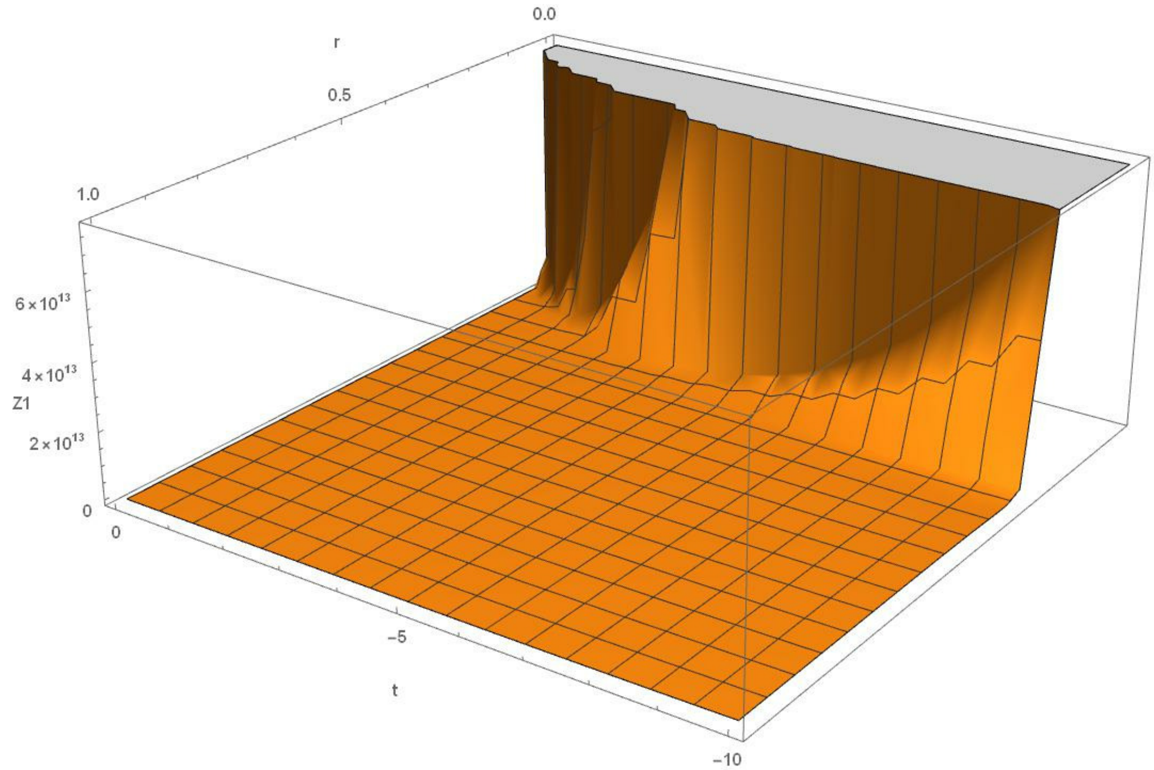

Figure 3.4: Energy conditions as a function of the radial r and temporal t coordinate

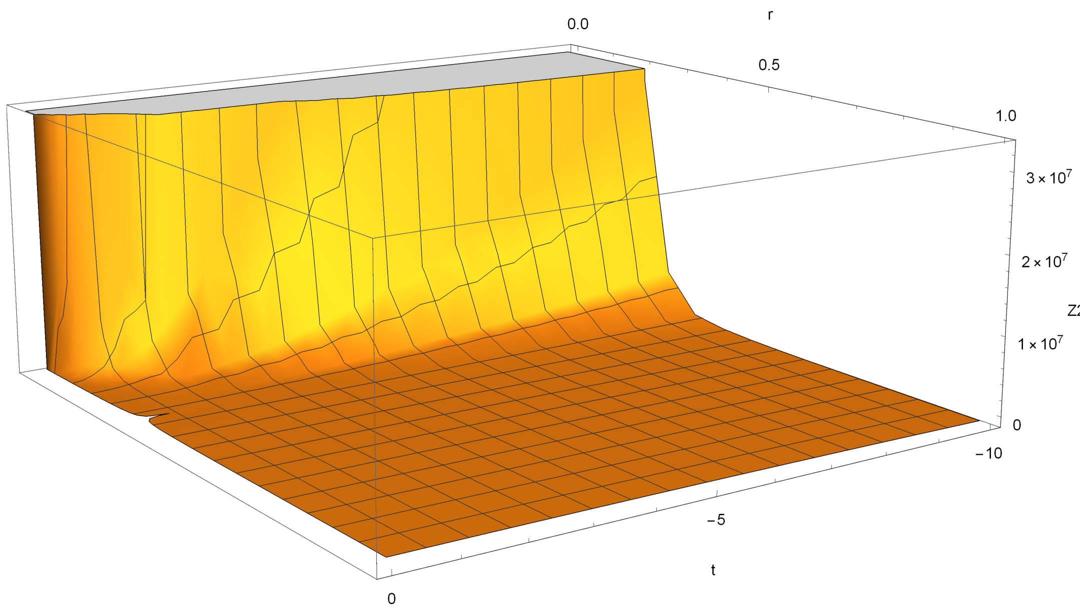

Figure 3.5: Energy conditions as a function of the radial r and temporal t coordinates

Chapter 4

Shearing dissipative collapse in the presence of λ

4.1 Introduction

In this chapter we extend earlier work done to include the presence of the cosmological constant. It is well know that the gravitational collapse in general relativity, in which the interior matter distribution is dust with the exterior spacetime being Schwarzchild, was first investigated by Oppenheimer and Snyder (1939). This can be extended to radiating matter by using the junction conditions of Santos (1935). Recently Tewari et al (2016) found a relativistic model for a spherically symmetric anisotropic fluid to investigate and analyse the various factors of physical and thermal phenomena during the evolution of a collapsing star dissipating energy in the form of radial heat flow. It is important to study radiative gravitational collapse where the collapsing core radiates energy to the exterior spacetime, as shown in the treatment of Naidu et al (2006). There are recent models that investigate gravitational collapse, the stability of the star and dissipative processes. In addition there are models that apply the Eckart theory and causal theores to investigate the thermodynamics (Tewari (2013), Sharma et al (2013), Pretel (2019)). Mohanlal et al (2017) investigated radiating stellar systems that involves the application of Lie symmetry infinitesimal generators. Thirukkanesh et al (2012) demonstrated solutions to a Riccati boundary equation governing the gravitational behaviour for a radiating star. It appears that Riccati equations are generic to the description of radiating stars which have a Vaidya exterior. In this paper, we extend their work by introducing a particular transformation to the Riccati equation and applying a different approach to generate new exact solutions.

We consider exact solutions by introducing a cosmological constant in the Einstein field equations. Some recent studies include the cosmological constant for observations relating to type Ia supernovae, high redshifts and baryon acoustic oscillations. It is important to investigate the degree in which the nonzero cosmological constant affects the collapse of a general relativistic star. Rindler and Ishak (2007) investigate the role played by the cosmological constant on the bending of light. Andreasson and Boehmer (2009) extend the work of the Buchdahl compactness ratio to include the cosmological constant. Note that the cosmological constant imposes a limit to the gravastar formation by Chan (2009), and constrains strange stars with null strange quark fluid by Ghosh (2003). Deshingkar et al (2009) and Bohmer and Harko (2005) demonstrated that the cosmological constant is connected to formation of both black holes and visible singularities in gravitational collapse. Govender and Thirukkanesh (2009) studied the presence of a nonzero cosmological constant on temperature profiles in causal thermodynamics; relaxational effects produce larger temperature gradients in the core regions of the star. These results were confirmed in the study of Thirukkanesh et al (2012). Recently Bhatti (2018) investigated shear-free matter distributions in the presence of the cosmological constant and generated exact solutions for a radiating star. Zitha et al (2019) found exact solutions satisfying the condition for a collapsing star in spherical symmetry both in the absence and the presence of the cosmological constant.

In this chapter we consider the presence of cosmological constant in a collapsing model. In Section 4.2 we present the model of a radiating star, the Einstein field equations, the boundary conditions, and the transformed equation that needs to be solved and analysed. In Sections 4.3, 4.4 and 4.5 we find exact solutions with nonzero cosmological constant under various assumptions. We complete our model by showing a physical analysis in Section 4.6. We summarise our findings in Section 4.7.

4.2 The model

We investigate the dynamics of a relativistic star. The interior geometry of the radiating star for the general spherically symmetric line element, is given by

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{4.2.1}$$

where $A = A(t,r)$, $B = B(t,r)$ and $Y = Y(t,r)$ are the metric functions and control the process of the collapse. The stellar model is represented by the interior matter distribution with energy momentum tensor $\mathbf{T}$ and is given by

$$\begin{aligned} T_{\alpha\beta} &= (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta \\ &\quad + q_\alpha V_\beta + q_\beta V_\alpha\,, \end{aligned} \tag{4.2.2}$$

where μ is the energy density, P_r is the radial pressure, $P_\perp$ is the tangential pressure, and $\mathbf{q}$ is the heat flow vector. These quantities are measured relative to a timelike fluid four-velocity. The fluid four–velocity $\mathbf{V}$ is comoving and has the particular form

$$V^\alpha = \frac{1}{A}\delta^\alpha_0 \,. \tag{4.2.3}$$

The heat flow vector $\mathbf{q}$ represents heat loss, across the boundary to the outside of the star, and is a spacelike vector. It has the form

$$q^\alpha = (0, q^1, 0, 0)\,, \tag{4.2.4}$$

and it satisfies $q^\alpha V_\alpha = 0$. In addition the vectors χ and $\mathbf{V}$ satisfy

$$\chi^\alpha \chi_\alpha \;=\; 1, \tag{4.2.5a}$$

$$\chi^\alpha V_\alpha \;=\; 0. \tag{4.2.5b}$$

The expansion scalar and the fluid four-acceleration are defined by the equations

$$\Theta \;=\; V^\alpha{}_{;\alpha}, \tag{4.2.6a}$$

$$a_\alpha \;=\; V_{\alpha;\beta}V^\beta. \tag{4.2.6b}$$

If the star is collapsing then we have $\Theta < 0$. The following expression represents the shear tensor

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta). \tag{4.2.7}$$

The kinematical quantities can be written explicitly for the comoving line element (4.2.1). These kinematical quantities are given by

$$a_1 = \frac{A'}{A}, \tag{4.2.8a}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right), \tag{4.2.8b}$$

$$\sigma = \frac{1}{A}\left(\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right), \tag{4.2.8c}$$

where dots and primes represent differentiation with respect to t and r respectively.

The Einstein field equations have the form

$$R_{\alpha_\beta} - \frac{1}{2}Rg_{\alpha_\beta} + \lambda g_{\alpha_\beta} = T_{\alpha_\beta}, \tag{4.2.9}$$

in the presence of the cosmological constant λ. For the spherical metric (4.2.1) and the general matter distribution (4.2.2), the equations (4.2.9) become

$$
\begin{aligned}
\rho &= -\frac{1}{B^2}\left[2\frac{Y''}{Y} + \left(\frac{Y'}{Y}\right)^2 - 2\frac{B'}{B}\frac{Y'}{Y}\right] \\
&\quad + \frac{1}{A^2}\left(2\frac{\dot{B}}{B} + \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y} + \frac{1}{Y^2} - \lambda, \qquad (4.2.10a)
\end{aligned}
$$

$$
\begin{aligned}
P_r &= -\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y} - \left(2\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] - \frac{1}{Y^2} \\
&\quad + \frac{1}{B^2}\left(2\frac{A'}{A} + \frac{Y'}{Y}\right)\frac{Y'}{Y} + \lambda, \qquad (4.2.10b)
\end{aligned}
$$

$$
\begin{aligned}
P_\perp &= -\frac{1}{A}\left[\frac{\ddot{B}}{B} + \frac{\ddot{Y}}{Y} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{Y}}{Y}\right) + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right] \\
&\quad + \frac{1}{B^2}\left[\frac{A''}{A} + \frac{Y''}{Y} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{Y'}{Y}\right] + \lambda, \qquad (4.2.10c)
\end{aligned}
$$

$$
q = \frac{2}{AB}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\frac{A'}{A}\right), \qquad (4.2.10d)
$$

where we have set $q = Bq^1$. The system (4.2.10a)-(4.2.10d) is nonlinear relating ρ, P_r, $P_\perp$, q, λ, A, B and Y.

At the boundary of the star the intrinsic and extrinsic curvature components, related to the interior and exterior spacetimes, have to match. The matching of the interior and exterior spacetimes is at the boundary Σ which is timelike. This yields the boundary condition

$$
(p_r)_\Sigma = q_\Sigma. \qquad (4.2.11)
$$

Consequently the radial pressure is not zero across the boundary of a radiating star. This condition was first stablished by Santos (1985). Using equations (4.2.10b) and

(4.2.10d) we find that the junction condition (4.2.11) can be written in the form

$$
\begin{aligned}
\frac{2}{AB}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\frac{A'}{A}\right) &= -\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y} - \left(2\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] - \frac{1}{Y^2} \\
&\quad + \frac{1}{B^2}\left(2\frac{A'}{A} + \frac{Y'}{Y}\right)\frac{Y'}{Y} + \lambda. \qquad (4.2.12)
\end{aligned}
$$

This condition determines the temporal evolution of the radiating body across the stellar boundary Σ. In the original form (4.2.12) the equation is complicated but can be rewritten in the form

$$
\begin{aligned}
\dot{B} &= \left[\frac{\ddot{Y}}{AY'} - \frac{\dot{A}\dot{Y}}{A^2Y'} + \frac{\dot{Y}^2}{2AYY'} + \frac{A}{2YY'} - \frac{\lambda AY}{2Y'}\right]B^2 \\
&\quad + \left(\frac{\dot{Y}'}{Y'} - \frac{\dot{Y}}{Y'}\frac{A'}{A}\right)B - \frac{A}{2}\left(\frac{2A'}{A} + \frac{Y'}{Y}\right). \qquad (4.2.13)
\end{aligned}
$$

This representation is simpler as it may be considered as a first order equation in the metric function B.

Equation (4.2.13) is assentially a Riccati equation in B and some solutions have been found utilizing ad hoc methods. A further simplification of (4.2.13) is possible if we introduce the transformation

$$B = ZY'. \qquad (4.2.14)$$

The transformation of (4.2.14) was used by Thirukkanesh et al (2012) and Ivanov (2012) for geodesic fluids in the absence of λ. Zitha et al (2019) showed that the transformation may be used to study the effect of the cosmological contant for a collapsing star for a nonaccelerating boby. It is also possible to consider the general case including nongeodesic motion with cosmological constant. We observe that (4.2.13) and (4.2.14) produce the result

$$\dot{Z} = \left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right) Z^2 - \frac{A'\dot{Y}}{AY'} Z - \left(\frac{A'}{Y'} + \frac{A}{2Y}\right). \tag{4.2.15}$$

Equation (4.2.15) is a first order equation in the new variable Z. It is possible to integrate (4.2.15) and several simple families of exact solutions arise. These families are presented in the subsequent sections.

4.3 Class A models

We investigate the case when equation (4.2.15) becomes a linear equation in Z. We set

$$\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2} = 0. \tag{4.3.1}$$

Equivalently (4.3.1) can be written as

$$\dot{A} - \left(\frac{\ddot{Y}}{\dot{Y}} + \frac{\dot{Y}}{2Y}\right) A = -\left(\frac{\lambda Y}{2\dot{Y}} - \frac{1}{2Y\dot{Y}}\right) A^3, \tag{4.3.2}$$

which can be considered as a Bernoulli equation in A if Y is known.

Then equation (4.2.15) becomes

$$\dot{Z} + \frac{A'\dot{Y}}{AY'} Z = -\left(\frac{A'}{Y'} + \frac{A}{2Y}\right), \tag{4.3.3}$$

which is linear in Z. Equation (4.3.3) can be integrated and obtain

$$Z = -\frac{\int \left(\frac{A'}{Y'} + \frac{A}{2Y}\right) \exp[\int (\ln A)' \frac{\dot{Y}}{Y'} dt] dt}{\exp[\int (\ln A)' \frac{\dot{Y}}{Y'} dt]}. \tag{4.3.4}$$

To complete the solution we need to integrate (4.3.2). It is not possible to solve (4.3.2) in general. However there are particular cases which are solvable which we present below.

4.3.1 Solution 1 : $\dot{A} = 0$

First set

$$\dot{A} = 0, \tag{4.3.5}$$

which gives

$$A = f(r). \tag{4.3.6}$$

Equation (4.3.2) becomes

$$2Y\ddot{Y} + \dot{Y}^2 + \left(1 - \lambda Y^2\right) A^2 = 0, \tag{4.3.7}$$

which can be written as

$$\frac{d}{dY}(\dot{Y}^2 Y) = \left(\lambda Y^2 - 1\right) A^2. \tag{4.3.8}$$

Integrating gives the results

$$\dot{Y}^2 Y = \left(\frac{\lambda Y^3}{3} - Y\right) A^2 + F, \tag{4.3.9}$$

where F is a arbitrary function. We cannot solve to get a form for Y in general. If we set $F = 0$ then

$$\dot{Y} = \sqrt{Y^2 - \frac{3}{\lambda}}\sqrt{\frac{\lambda}{3}}A, \tag{4.3.10}$$

this is separable integral

$$\frac{dY}{\sqrt{Y^2 - \frac{3}{\lambda}}} = \sqrt{\frac{\lambda}{3}}Adt. \tag{4.3.11}$$

If set $Y = \sqrt{\frac{3}{\lambda}}\sec v$, then equation (4.3.11) becomes

$$\int \sec v dv = \int Adt. \tag{4.3.12}$$

Integrating we obtain

$$\ln(\sec v + \tan v) = At + c, \tag{4.3.13}$$

so that

$$\ln\left(\sqrt{\frac{\lambda}{3}}Y + \sqrt{\frac{\lambda Y^2}{3} - 1}\right) = At + c. \tag{4.3.14}$$

Therefore this class of exact solution is given by

$$B = -\frac{\int (\frac{A'}{Y'} + \frac{A}{2Y}) \exp[\int (\ln A)' \frac{\dot{Y}}{Y'} dt] dt}{\exp[\int (\ln A)' \frac{\dot{Y}}{Y'} dt]} Y', \quad (4.3.15a)$$

$$\ln\left(\sqrt{\frac{\lambda}{3}} Y + \sqrt{\frac{\lambda Y^2}{3} - 1}\right) = At + c, \quad (4.3.15b)$$

$$A = f(r). \quad (4.3.15c)$$

For this category of solution we can make any choice for the function $f(r)$.

4.3.2 Solution 2 : $\dot{A} \neq 0$

If $\dot{A} \neq 0$ then we can set

$$U = A^{-1}. \quad (4.3.16)$$

Equation (4.3.2) is the transformed to the linear equation

$$\dot{U} + 2\left(\frac{\ddot{Y}}{\dot{Y}} + \frac{\dot{Y}}{2Y}\right) U = \left(\frac{\lambda Y}{\dot{Y}} - \frac{1}{Y\dot{Y}}\right). \quad (4.3.17)$$

Integrating equation (4.3.17) we obtain

$$U = \frac{\lambda Y^2 - 3}{3\dot{Y}^2} + \frac{f(r)}{\dot{Y}^2 Y}, \quad (4.3.18)$$

where $f(r)$ is a function of integration. Therefore

$$A = \frac{\sqrt{3Y}\dot{Y}}{\sqrt{\lambda Y^3 - 3Y + 3f(r)\dot{Y}^2}}. \tag{4.3.19}$$

Therefore this class of exact solution is given by

$$B = -\frac{\int (\frac{A'}{Y'} + \frac{A}{2Y}) \exp[\int (\ln A)' \frac{\dot{Y}}{Y'} dt] dt}{\exp[\int (\ln A)' \frac{\dot{Y}}{Y'} dt]} Y', \tag{4.3.20a}$$

$$A = \frac{\sqrt{3Y}\dot{Y}}{\sqrt{\lambda Y^3 - 3Y + 3f(r)\dot{Y}^2}} \tag{4.3.20b}$$

$$Y = Y(r,t) \tag{4.3.20c}$$

For this category of solution we can make any choice for the function of $Y(r,t)$.

4.4 Class B models

We now consider the case when equation (4.2.15) becomes a Bernoulli equation in Z. In this case we set

$$\frac{A'}{Y'} + \frac{A}{2Y} = 0, \tag{4.4.1}$$

which gives

$$Y = \frac{f(t)}{A^2}. \tag{4.4.2}$$

Then equation (4.2.15) becomes the Bernoulli

$$\dot{Z} + \frac{A'\dot{Y}}{AY'}Z = \left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right)Z^2. \tag{4.4.3}$$

Integrating this equation we get

$$Z = -\left[\exp[\int (\ln A)'\frac{\dot{Y}}{Y'}dt]\int \frac{\left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right)}{\exp[\int (\ln A)'\frac{\dot{Y}}{Y'}dt]}dt\right]^{-1}, \tag{4.4.4}$$

The general solution is given by

$$B = -Y'\left[\exp[\int (\ln A)'\frac{\dot{Y}}{Y'}dt]\int \frac{\left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right)}{\exp[\int (\ln A)'\frac{\dot{Y}}{Y'}dt]}dt\right]^{-1}, \tag{4.4.5a}$$

$$A^2 = \frac{f(t)}{Y}, \tag{4.4.5b}$$

$$Y = Y(r,t). \tag{4.4.5c}$$

In this category of solution any choice can be made for the potential Y. Clearly $a_1 \neq 0$, $\Theta \neq 0$ and $\sigma \neq 0$ in (4.2.8a)-(4.2.8c) for the kinematical quantities. Some simplification is possible when we set $\dot{Y} = 0$, giving $Y = s(r)$. Then Z in (4.4.4) is reduced to the simple expression

$$Z = -\frac{1}{\left(\frac{1}{2Y} - \frac{\lambda Y}{2}\right)\int Adt}, \tag{4.4.6}$$

and $A^2 = \frac{f(t)}{Y}$ with $Y = s(r)$. In this special case $\dot{Y} = 0$ but $\dot{A} \neq 0$ so the particles are not geodesic and remain shearing.

4.5 Class C models

Other solutions of (4.2.15) are possible but not easy to find. It is possible to convert (4.2.15) to an algebraic equation in Z. To achieve this we set

$$\dot{Z} = 0, \tag{4.5.1}$$

which gives

$$Z = k(r). \tag{4.5.2}$$

In terms of the potential B we have

$$B = k(r)Y'. \tag{4.5.3}$$

Then equation (4.2.15) becomes the quadratic

$$\left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right)Z^2 + \frac{A'\dot{Y}}{AY'}Z + \left(\frac{A'}{Y'} + \frac{A}{2Y}\right) = 0. \tag{4.5.4}$$

Equation (4.5.4) can be solved by placing restrictions on its coefficients. We demonstrate this with an example.

We set

$$\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2} = 0, \tag{4.5.5a}$$

$$\dot{A} = 0. \tag{4.5.5b}$$

Then equation (4.5.4) becomes

$$Z = -\frac{A'\dot{Y}}{AY'\left(\frac{A'}{Y'} + \frac{A}{2Y}\right)} = k(r). \tag{4.5.6}$$

Equation (4.5.6) can be written in the from

$$\dot{Y} = \frac{(2A' + A(\ln Y)')\,k(r)}{-2(\ln A)'}, \tag{4.5.7}$$

which relates the temporal and spatial derivatives of Y. Equation (4.5.7) is then a consistency condition for the existence of the exact solution.

Since $\dot{A} = 0$ equation (4.5.5a) becomes

$$\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\lambda AY}{2} = 0, \tag{4.5.8}$$

which can be written as

$$\frac{d}{dY}(\dot{Y}^2 Y) = \lambda A^2 Y^2 - A^2. \tag{4.5.9}$$

Integrating (4.5.9) gives

$$\dot{Y}^2 Y = \frac{\lambda A^2 Y^3}{3} - A^2 Y + G, \tag{4.5.10}$$

where G is a function from integration. It is difficult to integrate (4.5.10) in general; we therefore set $G = 0$, so that we can express equation (4.5.10) as

$$\frac{\sqrt{3}}{\sqrt{\lambda}} \frac{dY}{\sqrt{Y^2 - \frac{3}{\lambda}}} = Adt. \tag{4.5.11}$$

This can be integrated if we set $Y = \sqrt{\frac{3}{\lambda}} \sec v$. Equation (4.5.11) becomes

$$\int \sec v dv = \int dt. \tag{4.5.12}$$

Then integrating we obtain

$$\ln(\sec v + \tan v) = At + c. \tag{4.5.13}$$

In terms of the original variables

$$\ln\left(\sqrt{\frac{\lambda}{3}} Y + \sqrt{\frac{\lambda Y^2}{3} - 1}\right) = At + c, \tag{4.5.14}$$

whish gives the potential Y.

4.6 Physical analysis

In order to perfom our physical analysis of the radiating star, we simple considered our solution from Section 4.4. We choose

$$Y = 4 - \frac{C}{r},$$

then

$$A = \frac{1}{\sqrt{4 - \frac{C}{r}}},$$

and

$$B = -\frac{2C\sqrt{4 - \frac{C}{r}}(4r - C)}{rt\left(C^2\lambda - 8C\lambda r + (16\lambda - 1)r^2\right)},$$

where C is a positive constant. The Einstein field equations (4.2.10a)-(4.2.10d) yield the following expressions for the matter variables.

$$\rho = \frac{\left(-2C^3 + C^2 r\left(\lambda t^2 + 24\right) - 8Cr^2\left(\lambda t^2 + 12\right)\right)}{2(C-4r)^5\left(C^2\lambda - 8C\lambda r + (16\lambda - 1)r^2\right)^{-1}},$$

$$+\frac{r^3\left((16\lambda+1)t^2 + 128\right)}{2(C-4r)^5\left(C^2\lambda - 8C\lambda r + (16\lambda - 1)r^2\right)^{-1}} \tag{4.6.1a}$$

$$p_r = \lambda - \frac{r^2}{(C-4r)^2}, \tag{4.6.1b}$$

$$p_t = \frac{2\lambda r t^2 (C-4r)^2\left(r^3 t^2(16r - 5C) + 8C(C-4r)^3\right)}{16Crt^2(C-4r)^5}$$

$$+\frac{\lambda^2 r^2 t^4 (C-4r)^4 (C-16r)}{16Crt^2(C-4r)^5}$$

$$+\frac{32C^7 - 768C^6 r + 7680C^5 r^2 - 40960C^4 r^3 + 122880C^3 r^4 - 196608C^2 r^5}{16Crt^2(C-4r)^5}$$

$$+\frac{9Cr^6t^4 + 131072Cr^6}{16Crt^2(C-4r)^5} - \frac{16r^7t^4}{16Crt^2(C-4r)^5}, \tag{4.6.1c}$$

$$q = \frac{C^2\lambda - 8C\lambda r + (16\lambda - 1)r^2}{(C-4r)^2}. \tag{4.6.1d}$$

Figure 1 shows that the heat flux decreases monotonically from the centre of the star towards the stellar surface. We expect this behaviour as the core of the star is more dense and hotter than the surface layers, hence energy production at the centre is much higher. Figure 2 displays the behaviour of the density. We observe that the density is positive at each interior point of the star. The radial pressure is plotted in

Figure 3. Observation of the radial pressure indicates that it decreases monotonically towards the boundary of the star. This behaviour mimics the trend in the heat flux. Higher energy output at the centre of the stellar configuration is accompanied by higher pressures. It is also important to note from (4.6.1b) that in the case of vanishing Λ, the radial pressure is negative throughout the fluid interior. Figure 4 indicates that the tangential pressure is positive everywhere within the stellar interior. The tangential pressure decreases radially outwards from the centre of the star. We also observe that the tangential pressure increases as the collapse proceeds. We expect this trend as the core collapses, matter is squeezed into shells of smaller volume. We can think of the radial pressure as the surface tension acting on each shell. As this surface tension increases it enhances the collapse of the core. The energy conditions

$$Z_1 = (\rho + p_r)^2 - 4q^2 > 0, \quad (E1) \tag{4.6.2a}$$

$$Z_2 = \rho - p_r - 2p_t + \left[(\rho + p_r)^2 - 4q^2\right]^{1/2} > 0, \quad (E2) \tag{4.6.2b}$$

were investigated. The strong, weak and dominant energy conditions are all satisfied within the stellar interior thus indicating that our model is stable and is physically viable.

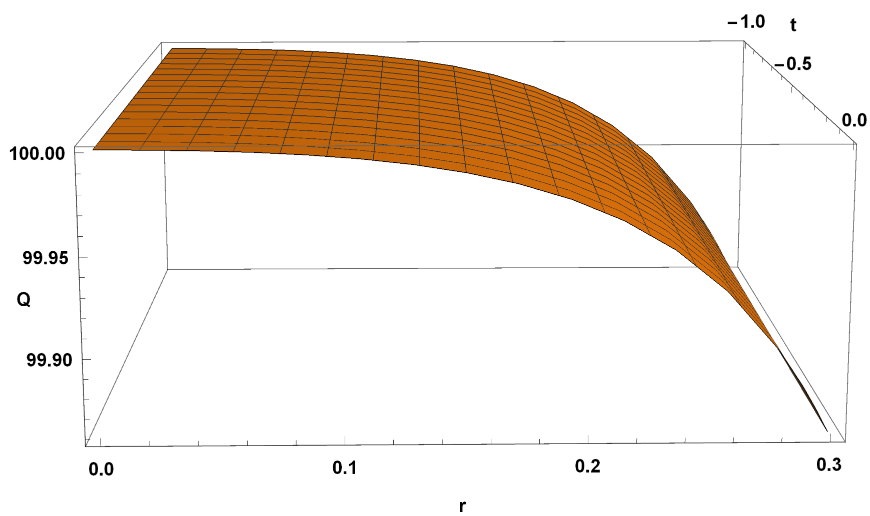

Figure 4.1: Heat flow as a function of the radial r and temporal t coordinates.

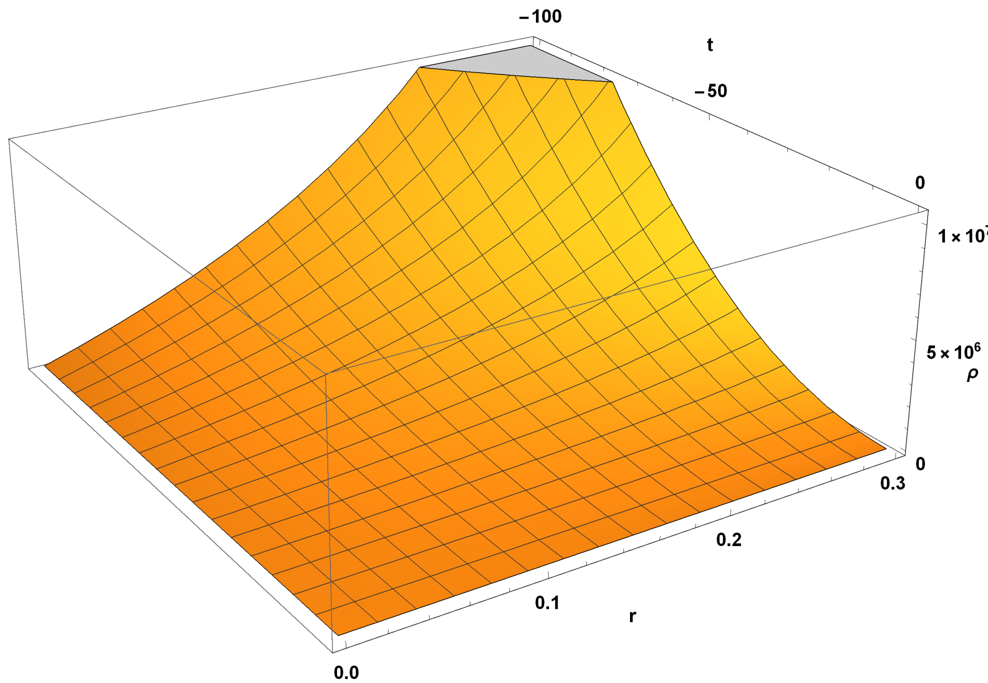

Figure 4.2: Density as a function of the radial coordinate r and temporal coordinate t.

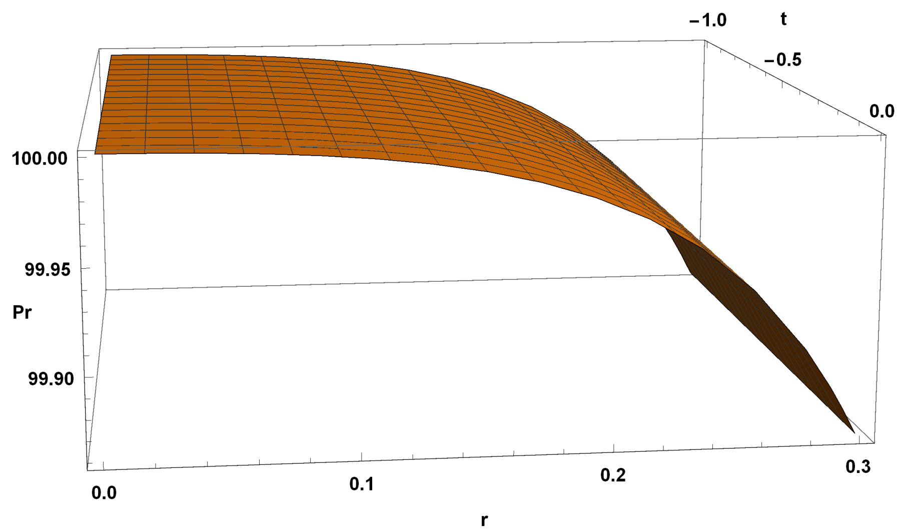

Figure 4.3: Radial pressure as a function of the radial coordinate r and temporal coordinate t.

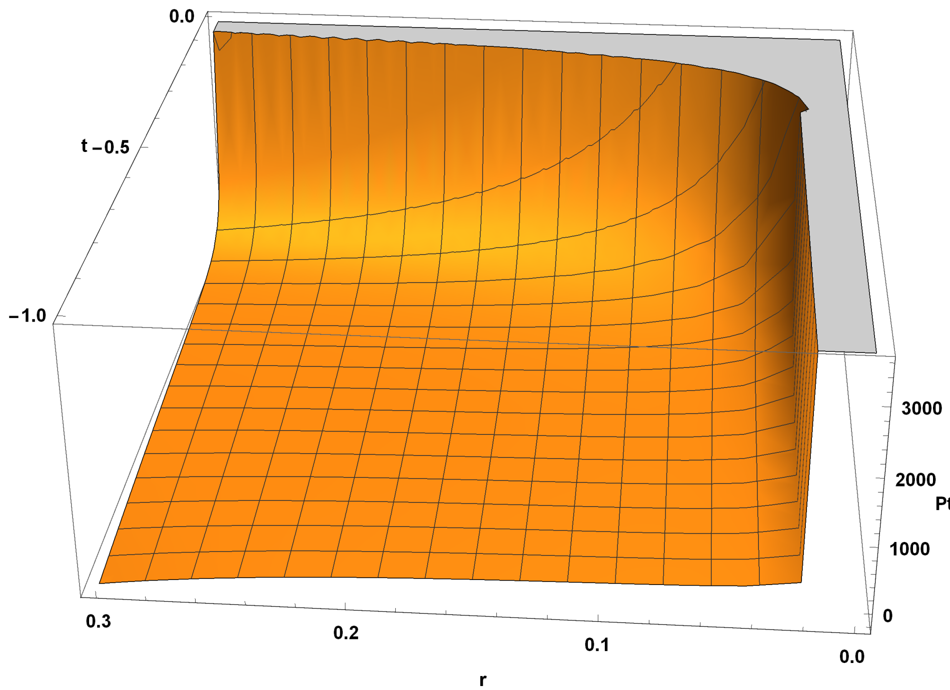

Figure 4.4: Tangential pressure pressure as a function of the radial coordinate r and temporal coordinate t.

4.7 Discussion

In this work we presented a new family of solutions describing a radiating shearing sphere undergoing gravitational collapse in the presence of the cosmological constant. Our solutions generalise previously known shearing, radiating models of gravitational collapse. Although the junction condition which encodes the temporal behaviour of the model is nonlinear we are able to obtain exact solutions thus completely specifying the gravitational potentials. The inclusion of the cosmological constant is not a simple addition of a constant term to the dynamical boundary condition as is the case with the density and pressures given in (4.2.10a)–(4.2.10c). The inclusion of the cosmological constant in the boundary condition increases the nonlinearity of the temporal evolution of the model. It is remarkable that we are able to solve the boundary condition for nonvanishing λ. A study of the physics of a specific model indicates that our solutions can be utilised to model a radiating star undergoing shearing collapse. The effect of the cosmological constant was brought out in (4.6.1b) which shows that the radial pressure becomes negative when λ vanishes. Negative pressures arise in models incorporating dark energy. In our model the presence of the cosmological constant distinguishes the matter content of the interior: $\lambda = 0$, dark energy or nonzero λ, baryonic matter.

Chapter 5

Conclusion

The major theme of this dissertation was to generalise the junction conditions at the surface of the body to model a radiating star with shearing fluid. We also studied the effect of the cosmological constant on the evolution of the star. We presented an exact model of a radiating star undergoing expansion-free collapse. We showed that the junction condition can be written in the standard form of a Riccati equation. We transformed the standard equation into particular Riccati equations. We generated several new classes of solutions to the field equations. We able to construct new models for radiating relativistic stars, in the form of geodesic motion and demonstrated the role of cosmological constant in Einstein equations. We studied the behaviour of the thermodynamical variables in the interior matter distribution. The role of shear was studied in relation to dissipative effects and new families of exact solutions were found.

We now provide an overview of our research and findings obtained during the course of our investigations:

- Chapter 2: We obtained exact solutions by studying a Riccati equation, where we set $B = \frac{g(r)}{R^2(r,t)}$. The fundamental equation, we solved is given by

$$
\frac{\ddot{R}}{R}+\frac{1}{2}\left(\frac{\dot{R}}{R}\right)^{2}-\frac{\dot{A}}{A}\frac{\dot{R}}{R}-\left(\frac{AR^{2}R'}{g}+\dot{R}\right)\frac{A'R}{g}
$$

$$
-\left(\frac{R^{2}R'^{2}}{2g^{2}}-\frac{1}{2R^{2}}\right)A^{2}+\left(2\dot{R}R'+R\dot{R}'\right)\frac{A}{g}=0.
$$

We demonstrated that it is possible to integrate the nonlinear junction condition at the surface of the star; the gravitational potentials are given in terms of elementary functions. A physical and thermodynamical analysis of the model indicates that it is stable and approximates a realistic collapse scenario. As far as we are aware the results of this chapter represent the first exact solutions for expansion-free collapse. This solution may be used to model the formation of a cavity in gravitational collapse. We have shown that relaxational effects lead to higher core temperatures and monotonically decrease at the surface of the radiating star. The energy density and the radial pressure are decreasing outward from the centre to the stellar surface. All energy conditions are satisfied at each interior point of the star. The causal temperature is higher than its noncausal counterpart. The causal temperature is decreasing as the fluid collapses and the noncausal temperature does not change its form.

- Chapter 3: A collapsing star dissipating energy in the form of heat flux satisfies the fundamental equation

$$
\dot{Z}=\left(\frac{\ddot{Y}}{A}+\frac{\dot{Y}^{2}}{2AY}+\frac{A}{2Y}-\frac{\dot{A}\dot{Y}}{A^{2}}-\frac{\lambda AY}{2}\right)Z^{2}-\frac{A'\dot{Y}}{AY'}Z-\left(\frac{A'}{Y'}+\frac{A}{2Y}\right),
$$

in the presence of the cosmological constant. We first found solutions by assuming that $A = 1$ and $\lambda = 0$, leading to four different types of models. A particular model leads to a spacetime that is expanding, accelerating and shearing. We obtained three solutions by integrating the following equation

$$\dot{Y}^2 = \frac{a(r)Y + \epsilon b(r)}{Y},$$

where $\epsilon = 0, \pm 1$ and $b(r) > 0$. For the case $A = 1$ and $\lambda \neq 0$ we have generated solutions using the following equation

$$F = 2Y\ddot{Y} + \dot{Y}^2 - \lambda Y^2 + 1,$$

by setting $F = 1$ and $F = 1 + a(r)$. A graphical analysis shows that the energy density is decreasing outwards from the centre to the stellar surface. The radial pressure increases as the collapse proceeds with the outermost shells being squeezed greater than the inner core. The energy conditions are satisfied at each interior point of the star, and the physical analysis indicates that the weak, strong, and dominant energy conditions are satisfied in interior points away from the centre. In the asymptotic limit we regain the equation of state $P_r \approx -\alpha\mu$.

- Chapter 4: Gravitational collapse in the presence of shear and the cosmological constant is govered by the nonlinear differential equation

$$\dot{Z} = \left(\frac{\ddot{Y}}{A} + \frac{\dot{Y}^2}{2AY} + \frac{A}{2Y} - \frac{\dot{A}\dot{Y}}{A^2} - \frac{\lambda AY}{2}\right) Z^2 - \frac{A'\dot{Y}}{AY'}Z - \left(\frac{A'}{Y'} + \frac{A}{2Y}\right).$$

We made assumptions to transform this equation to a Riccati equation in A leading to two types of solutions. Other assumptions lead to a Bernoulli equation in Z. Several families of solution are possible. As an axample if we set $\dot{Y} = 0$. Then we get

$$\begin{aligned}
B &= -\frac{Y'}{\left(\frac{1}{2Y} - \frac{\lambda Y}{2}\right)\int A dt}, \\
A^2 &= \frac{f(t)}{Y}, \\
Y &= s(r),
\end{aligned}$$

as a simple exact solution. To perform a physical analysis we chose $Y = 4 - \frac{C}{r}$. This leads to the result $p_r = -\alpha\rho$ (which is a barotropic equation of state). Therefore this shows that the radial presure is related to the energy density via the equation of state. A particular assumption leads to a quadratic equation in Z. A more detailed investigation is still needed for this case. The energy density, as a function of the radial coordinate, decreases rapidly as approaches the boundary. The radial pressure is a decreasing function outwards from the centre to the stellar surface. Higher energy output at the centre of the star is accompanied by higher pressures.

The approach that we have taken in the thesis is to write the boundary condition as a Riccati equation. The role of the cosmological constant in relation to dissipation in the presence of shear was studied. Exact models were generated which appear to be physically reasonable. More work needs to be done in studying the thermodynamics of these exact solutions. In future work, we intend to investigate the effects of the causal temperature, in the presence of cosmological constant in collapsing bodies.

Bibliography

[1] Abebe G Z, Maharaj S D and Govinder K S, Generalized Euclidean stars with equation of state, Gen. Relativ. Gravit. **46**, 1733 (2014).

[2] Abebe G Z, Maharaj S D and Govinder K S, Geodesic models generated by Lie symmetries, Gen. Relativ. Gravit. **46**, 1650 (2014).

[3] Abebe G Z, Maharaj S D and Govinder K S, A group thoeretic approach to shear-free radiating stars, Eur. Phys. J. C **75**, 496 (2015).

[4] Andreasson H and Boehmer C G, Bounds on M/R for static objects with a positive cosmological constant, Class. Quantum. Grav. **26**, 195007 (2009).

[5] Bonnor W B, de Oliveira A K G and Santos N O, Radiating spherical collapse, Phys. Rep. **181**, 269 (1989).

[6] Bohmer C G and Harko T, Dynamical instability of fluid spheres in the presence of a cosmological constant, Phys. Rev. D, **71**, 084026 (2005).

[7] Bhatti M J, Shear-free stellar solutions in Λ dominated era, Eur. Phys. J. Plus **131**, 428 (2016).

[8] Chan R, da Silva M F A and Rocha P, How the cosmological constant affects gravastar formation, J. Cosmol. Astropart. Phys. **12**, 017 (2009).

[9] Chan R, Collpse of a radiating star with shear, Mon. Not. R. Astron. Soc. **288**, 589 (1997).

[10] de Olivier A K G, Santos N O and Kolassis C A, Collapse of radiating star, Mon. Not. R. Astron. Soc, **216**, 1001 (1985).

[11] Di Prisco A, Herrera L, Ospino J, Santos N O and Vina-Cervantes V M, Expansion-free cavity evolution: Some exact analytical models, Int. J. Mod. Phys. D **20**, 2351 (2011).

[12] Deshingkar S S, Jhingan S, Chamorro A and Joshi P S, Gravitational collapse and the cosmological constant, Phys. Rev. D **63**, 124005 (2001).

[13] Govender G, Bogadi R S and Lortan D B, Radiating collapse in the presence of anisotropic stresses, Int. J. Mod. Phys. D **25**, 03 (2016).

[14] Govender G, Brassel B P and Maharaj S D, A relativistic heat conducting model, Eur. Phys. J. **133**, 478 (2018).

[15] Govender G, Govender M and Govinder K S, Thermal behaviour of Euclidean stars, Int. J. Mod. Phys. D **19**, 1773 (2010).

[16] Govender G and Govinder K S, Temperature evolution during radiative gravitational collapse, Int. J. Theo. Phys. D **41**, 1979 (2002).

[17] Govender G and Govinder K S, Causal solutions for radiating stellar collapse, Phys. Lett. A **283**, 71 (2002).

[18] Govender M, Maartens R and Maharaj S D, Relaxational effects in radiating stellar collapse, Mon. Not. R. Astron. Soc. **310**, 557 (1999).

[19] Govender M, Maharaj S D and Maartens R, A causal model of radiating star stellar collapse, Class. Quantum Grav. **15**, 323 (1998).

[20] Govender M and Thirukkanesh S, Dissipative collapse in the presence of Λ, Int. J. Theor. Phys. **48**, 3558 (2009).

[21] Ghosh S G and Dadhich N K, Gravitational collapse of null strange quark fluid and cosmic censorship, Gen. Ralativ. Gravit. **35**, 359 (2003).

[22] Glendenning N K, Compact stars, nuclear physics, particle physics, and general relativity (New York: Springer) (2000).

[23] Grammenos T and Kolassis C A, An analytic model of radiating spherical gravitational collapse, Phys. Lett. A **169**, 589 (1992).

[24] Herrera L, Di Prisco A and Ospino J, Some analytical models of radiating collapsing spheres, Phys. Rev. D **74**, 044001 (2006).

[25] Herrera L, Le Denmat G, Santos N O and Wang A, Shear-free radiating collapse and conformal flatness, Int. J. Mod. Phys. D **13**, 583 (2004).

[26] Herrera L, Le Denmat G and Santos N O, Expansion-free evolving spheres must have inhomogeneous energy density distributions, Phys. Rev. D, **79**, 087505 (2009).

[27] Herrera L, Santos N O and Wang A, Shearing expansion-free spherical anisotropic fluid evolution, Phys. Rev. D, **78**, 084026 (2009).

[28] Herrera L, Le Denmat G and Santos N O, Dynamical instability and the expansion-free condition, Gen. Relativ. Gravit. **44**, 1143 (2012).

[29] Herrera L, Le Denmat G, Santos N O, Cavity evolution in relativistic self-gravitatiny fluids, Class. Quantum Grav. **27**, 135017 (2010).

[30] Herrera L, Martin J and Ospino J, Anisotropic geodesic fluid spheres in general relativity, J. Math. Phys. **43**, 4889 (2002).

[31] Herrera L and Santos N O, Collapsing spheres satisflying an “Euclidean condition”, Gen. Relativ. Gravit. **42**, 2383 (2010).

[32] Ivanov B V, Collapsing shear-free perfect fluid spheres with heat flow, Gen. Relativ. Gravit. **44**, 1835 (2012).

[33] Ivanov B V, All solutions for geodesic anisotropic spherical collapse with shear and heat radiation, Astrophys. Space Sci. **361**, 18 (2016).

[34] Kolassis C A, Santos N O, and Tsoubelis D, Friedman-like collapsing model of a radiating sphere with heat flow, Astrophys. J. **327**, 755 (1988).

[35] Kumar R and Srivastava S K, Evolution of expansion-free spherically symmetric self-gravitating nondissipative fluids and some analytical solutions, Int. J. Geom. Methods Mod. Phys. **15**, 1850058 (2018a).

[36] Kumar R and Srivastava S K, Expansion-free self-gravitating dust dissipative fluids, Gen. Relativ. Gravit. **50**, 95 (2018b).

[37] Maharaj S D and Beesham A, On the scalar-tensor theory of Lau and Prokhovnik, Astrophys. J. **9** 67 (1988).

[38] Maharaj S D and Govender M, Spherical gravitational of anisotropic radiating fluid sphere, Int. J. Mod. Phys. D **14**, 667 (2005).

[39] Maharaj S D and Govender M, Collapse of a charged radiating star with shear, Prama **54**, 715 (2000).

[40] Maharaj S D, Govender G and Govender M , Temperature evolution during dissipative collapse, Pramana **77**, 469 (2011).

[41] Maharaj S D, Govender G and Govender M, Radiating stars with generalised Vaidya atmospheres, Gen. Relativ. Gravit. **44**, 4 (2013).

[42] Maharaj S D and Naidoo R, Solutions to the field equations and the deceleration parameter, Astrophys. Space Sci. **208**, 261 (1993).

[43] Martinez J, Transport processes in the gravitational collapse of an anisotropic fluid, Phys. Rev. D **53**, 6921(1996).

[44] Misner C W, Thorne K S and Wheeler J A, Gravitation (San Francisco: W H Freeman) (1973).

[45] Misthry S S, Maharaj S.D and Leach P G L, Nonlinear shear-free radiating collapse, Math. Meth. Appl. Sci. **31**, 363 (2008).

[46] Mohanlal R, Narain R and Maharaj S D, Nonlinear equations in radiating stellar structures, J. Math. Phys. **58**, 072503 (2017).

[47] Mohanlal R, Maharaj S D, Tiwari A K and Narain R, Radiating stars with exponetial Lie symmetries, Gen. Relativ. Gravit. **48**, 87 (2016).

[48] Naidu N F, Govender M and Govinder K S, Thermal evolution of a radiating anisotropic star with shear, Int. J. Mod. Phys. **15**, 1053 (2006).

[49] Naidu N F, Govender M and Maharaj S D, Radiating star with a time-dependent Karmarkar condition, Eur. Phys. J. C **78**, 48 (2018).

[50] Noureen I and Zubair M, Dynamical instability and expansion-free condition in f(R,T) gravity, Eur. Phys. J. C **75**, 62 (2015).

[51] Naidu N F, Govender M and Maharaj S D, Radiating star with a time-dependent Karmarkar condition, Eur. Phys. J. C **58**, 48 (2018).

[52] Oppenheimer J R and Syder H, Continued gravitational contraction, Phys. Rev. **56**, 455 (1939).

[53] Pinheiro G and Chan R, Radiating gravitational collapse with shearing motion and bulk viscosity revisted, Int. J. Mod. Phys. D **19**, 1797 (2010).

[54] Pretel J M Z and da Silva M F A, Causal thermodynamics of a gravitational collapse model for an anisotropic fluid with dissipative flows, Gen. Relativ. Gravit. **51**, 3 (2019)

[55] Prokopec T, General relativity and quantum cosmology, https://arxiv.org/abs/gr-qc/0603088 (2006).

[56] Reddy K P, Govender M and Maharaj S D, Impact of anisotropic stresses during dissipative gravitational collapse, Gen. Relativ. Gravit. **47**, 35 (2015).

[57] Rindler W and Ishak I, The contribution of the cosmological constant to the relativistic bending of light revisited, Phys. Rev. D **76**, 043006 (2007).

[58] Santos N O, Spherical gravitational collapse of anisotropic radiating fluid sphere, Mon. Not. R. Astron. Soc. **216**, 403 (1985).

[59] Saslaw W C, Maharaj S D and Dadhich N K, An isothermal universe, Astrophys. J. **47**, 571 (1996).

[60] Schwarzschild K, Uber das gravitational eins massenpunktes nach der Einsteinschen theorie, Sitz. Deut. Akad. Wiss Berlin, KL. Math. Phys. 189 (1916).

[61] Sharif M and Bhatti Z Z, Astrophys. Space Sci. **355**, 2174 (2014).

[62] Sharif M and Azam M, Effects of electromagnetic field on the dynamical instability of expansion-free gravitational collapse, Gen. Relativ. Gravit. **44**, 1181 (2012).

[63] Sharif M and Azam M, Stability of anisotropic cylinder with zero expansion, Mon. Not. R. Astron. Soc. **430**, 3048 (2013).

[64] Sharif M and Nasir Z, Evolution of dissipative anisotropic expansion-free arial fluids, Commun. Theor. Phys. **64**, 139 (2015).

[65] Sharif M and Nasir Z, Expansion-free axial dissipative fluids in f(R) gravity, Astrophys. Space Sci. **357**, 89 (2015).

[66] Sharif M and Yousaf Z, Expansion-free cylindrically symmetric models, Can. J. Phys. **90**, 865 (2012).

[67] Sharif M and Yousaf Z, Dynamical instability of the changed expansion-free spherical collapse in f(R) gravity, Phys. Rev. D **88**, 024020 (2013).

[68] Sharma R, Das S and Tikekar R, A class of conformally flat solutions for systems undergoing radiative gravitational collapse, Gen. Relativ. Gravit. **47**, 25 (2015).

[69] Sharma R and Tikekar R, Spacetime inhomogeneity, anisotropy and gravitational collapse, Gen. Relativ. Gravit. **44**, 2503 (2012).

[70] Tewari B C, Spherical gravitational collapse of anisotropic radiating fluid sphere, Astrophys. Space Sci. **342**, 73 (2012).

[71] Tewari B C, Collapsing shear-free radiating fluid spheres, Gen. Relativ. Gravit. **45**, 1547 (2013).

[72] Tewari B C and Charan K, Horizon free eternally collapsing anisotropic radiating star, Astrophys. Space Sci. **357**, 107 (2015).

[73] Tewari B C, Charan K and Rani J, Spherical Gravitational Collapse of Anisotropic Radiating Fluid Sphere, International Journal of Astronomy and Astrophysics, **6**, 155 (2016).

[74] Tiwari A K and Maharaj S D, New class of geodesic radiating systems, Eur. Phys. Plus **132**, 493 (2017).

[75] Thirukkanesh S and Govender M, The role of the electromagnetic field in dissipative collapse, Int. J. Mod. Phys. D **22**, 1350087 (2013).

[76] Thirukkanesh S, Misthry S S and Maharaj S D, Shearing radiative collapse with expansion and acceleration, J. Math. Phys. **58**, 032506 (2012).

[77] Thirukkanesh S and Maharaj S D, Radiating relativistic matter in geodesic motion, J. Math. Phys. **50**, 022502 (2009).

[78] Thirukkanesh S and Maharaj S D, Mixed potentials in radiative stellar collapse, J. Math. Phys. **51**, 072502 (2010).

[79] Thirukkanesh S, Moopanar S and Govender M, The final outcome of dissipative collapse in the presence of Λ, Pramana-J. Phys. **79**, 223 (2012).

[80] Thirukkanesh S, Rajah S S and Maharaj S D, Shearing radiative collapse with expansion and acceleration, J. Math. Phys. **53**, 032506 (2012).

[81] Vaidya P C, The gravitational field of a radiating star, Physical Sciences. **33**, 264 (1951).

[82] Vaidya P C, Newtonian time in general relativity, Nature. **171**, 260 (1953).

[83] Vaidya P C, The External field of a radiating Star in General Relativity, Gen. Relativ. Gravit. **31**, 119 (1999).

[84] Weinberg S, The cosmological constant problem, Rev. Mod. Phys. **61**, 1 (1989).

[85] Zitha V, Maharaj S D and Govender M, Dissipative collapse in the presence of shear, Int. J. Mod. Phys. D submitted (2019).

[86] Zubair M, Azmet H and Noureen I, Anisotropic stellar filaments evolving under expansion-free condition in f(R,T) gravity, Int. J. Mod. Phys. D **27**, 1850047 (2018).