

UNIVERSITY OF KWAZULU-NATAL

EXACT SOLUTIONS for
PERFECT FLUIDS CONFORMAL
to a PETROV TYPE D
SPACETIME

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Exact Solutions for Perfect Fluids Conformal to a Petrov Type D Spacetime

by

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Submitted in fulfilment of the
requirements for the degree of
Master of Science

in the

School of Mathematical Sciences
University of KwaZulu-Natal

As the candidate's supervisor, I have approved this dissertation for submission

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To
Nikoshia and Nikhil

Declaration

The research done in this dissertation is original work and has not been previously submitted to any other institution. Where reference to the work of other researchers was made, it has been duly acknowledged.

This thesis was completed under the supervision of Dr S. Hansraj and was co-supervised by Professor KS Govinder.

Narenee Mewalal

Acknowledgements

I offer my sincere thanks and appreciation to the following people and organisations for their contribution in helping me see this dissertation through to a purposeful end:

- Dr S Hansraj and the University of KwaZulu-Natal for affording me the opportunity to pursue a master's program in mathematics. Special thanks to Dr Hansraj who has been instrumental in guiding my progress to its successful completion. His insight into the field of relativity certainly helped me to interpret the demands of the project.
- Professor KS Govinder for his expert guidance on Lie group analysis and partial differential equations. His masterful teaching style left me in awe of the fact that even I was able to utilise programmes such as MATHEMATICA and PROGRAM LIE. His willingness to avail himself, often at short notice, has been very much appreciated.
- Durban High School, in particular, Mrs MA Vermaak and Mr DA Magner for their encouragement, support and continued interest throughout the duration of this work.
- The many authors of textbooks and articles whose ideas are contained in this dissertation.
- The National Research Foundation for financial assistance.
- Lastly, but most importantly, my husband Arvin and children Nikoshia and Nikhil without whose patience and understanding this work would never have come to fruition.

Abstract

Exact Solutions for Perfect Fluids Conformal to a Petrov Type D Spacetime

We find new exact solutions of the Einstein field equations for a perfect fluid metric conformal to a spacetime of type D in the Petrov classification scheme. The initial work of Coley *et al* (1992) was extended to a wider class of solutions by Hansraj *et al* (2006) using the method of Lie group analysis. We take up a claim in the latter work that even wider classes of solutions exist. Whereas Hansraj *et al* confined their attention to the case where the conformal factor was of the form $\bar{U} = \bar{U}(t, x)$, our work investigates the complete situation $U = U(t, x, y, z)$ as well as an auxiliary integrability condition. New classes of solutions are generated for certain symmetry generators. Finally, we analyse our solutions for physical plausibility. Since the solutions are four dimensional we investigate slices of the solution space graphically. In particular we obtain expressions for the energy density and pressure and check these for positivity.

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Chapter 1

Introduction

Albert Einstein's greatest achievement, his general theory of relativity has, since it was propounded in 1916, impacted hugely on our notion of space, time and their interaction with matter. Whilst the initial view of most physicists was that general relativity had very little practical significance, this attitude began to change towards the late 1950's. The theory slowly began to gain recognition as an improvement on Newtonian physics.

In general relativity, spacetime is represented by a four-dimensional manifold M endowed with a metric g_{ab} . The coupling of the geometry with the matter distribution generates the Einstein field equations. The field equations are a set of ten nonlinear partial differential equations which are very difficult to solve in general. Exact solutions of the Einstein field equations play an important role in the development of models of realistic celestial phenomena (Hawking and Israel 1979). Of course, numerical solutions may also be found when exact solutions cannot be achieved, however these solutions suffer the drawback that they can only convey a reasonable understanding of the evolution of the matter under stringent conditions. Exact solutions, in contrast, allow for a more holistic study of the distribution.

In generating solutions to the Einstein field equations, different approaches have been employed. One approach involves the imposition of symmetry conditions on the metric tensor.

resulting in the reduction of the ten field equations to a system of fewer equations. The caveat with this method is that the resulting system is not necessarily easier to solve. For example, in static spherically symmetric fluid spheres, the ten field equations reduce to four and new solutions are still being sought for such spheres. The earliest reported useful exact solution is due to Schwarzschild (1916a) who found the unique solution for the exterior gravitational field of a spherical object. Promptly thereafter Schwarzschild (1916b) found an interior solution by assuming that the sphere had a constant energy density and is consequently an incompressible fluid. Since then, many interior solutions have been obtained. A comprehensive collection may be found in Finch and Skea (1998) and Stephani *et al* (2003). The effects of the electromagnetic field were then considered and the Reissner-N rdstrom (1916, 1918) solution which describes the exterior gravitational field of a charged star, was found. Subsequently, a variety of solutions have appeared in the literature describing a charged relativistic sphere (Ivanov 2002).

Another approach in finding exact solutions involves imposing algebraic conditions on the Weyl tensor. These restrictions give rise to the Petrov (Petrov 1954) classification of the possible symmetries of the Weyl tensor. For a nonzero Weyl tensor C_{abcd} , there are four roots of the associated eigenvalue equation. According to the multiplicity of these roots, the Petrov classification may be described as follows (MacCullum 2006, Stephani 1990): type I (four distinct roots), type II (one pair of roots coincide), type D (two pairs of roots coincide), type III (three roots coincide), type N (all four roots coincide) and type O (the Weyl tensor vanishes). It should be noted that some types may degenerate into other types under certain conditions. Classes of exact solutions, most of which are of type D found in this way are detailed in Stephani *et al* (2003) . Such Type D solutions include the well-studied spherical geometry.

Additionally, solution generating theorems have made an appearance of late by observations of the field equations written in certain coordinate systems. For example, see the work of Martin

and Visser (2004) and Boonserm *et al* (2005) where the Einstein field equations for a static fluid sphere are written in Schwarzschild coordinates. The resulting master field equation is first order and of the Ricatti type. Ostensibly all solutions for a static fluid sphere are found, however, only up to integration of the Ricatti equation. In practice, the prescriptions provided are difficult to employ and fortuitous choices for one of the variables have to be made (Lake 2003). An attack using computer software systems is likely to yield new exact solutions from these algorithms. Closely allied to this approach, is the method of utilising purely mathematical techniques to solve the field equations and then undertaking an analysis of the resulting solution for physical plausibility. Conformal transformations on metric spaces offer such a route. One can take, for example, a vacuum solution and conformally map it onto a perfect fluid solution. In addition, it is possible to commence with a perfect fluid solution with unphysical behaviour and, by using a conformal transformation, generate a new perfect fluid with desirable properties. This is precisely the direction we pursue.

In our work we use the method of Lie group analysis of ordinary differential equations to obtain new group invariant solutions to a conformally related perfect fluid spacetime of Petrov type D. The analysis is greatly simplified by applying a theorem due to Defrise-Carter (1975) which asserts the following: Suppose that a manifold (M, \mathbf{g}) is neither conformally flat nor conformally related to a generalised plane wave. Then a Lie algebra of conformal Killing vectors on M with respect to \mathbf{g} can be regarded as a Lie algebra of Killing vectors with regard to some metric on M conformally related to \mathbf{g} (Hall and Steele 1991). We exploit this theorem with the selected Type D spacetime. The problem that arises on performing a conformal transformation, is that the field equations governing the conformal factor are extremely difficult to integrate. This is where the Lie group analysis approach is useful.

Lie groups, named after Sophus Lie who laid the foundation for the theory of continuous transformation groups, are indispensable tools for many aspects of mathematics and physics. Since a Lie group is a smooth manifold, it can be studied using differential calculus. Sophus

Lie demonstrated that an n th order ordinary differential equation which is invariant under a one-parameter group of point transformations may be constructively reduced to an ordinary differential equation of order $(n - 1)$. So an important use of symmetries is the reduction of order of an equation. When Lie's method is applied to a system of partial differential equations which are invariant under a Lie group of point transformations, one may constructively obtain solutions that are invariant under some subgroup or the entire group admitted by the system. A few basic examples of Lie groups, some of which we encounter in this thesis, include translations, rotations and scalings. Lie group analysis has its applications in a wide variety of fields such as algebraic topology, differential geometry, relativity and numerical analysis. to mention but a few (Bluman and Kumei 1989), Ibragimov (1995) and Anderson and Ibragimov (1979).

The main objectives of this thesis are:

- to implement the technique of Lie group analysis in seeking new exact solutions of the Einstein field equations for a perfect fluid metric conformal to a Petrov type D spacetime
- to generate a new class of complete solutions for the conformal factor $U = U(t, x, y, z)$ using the solutions found in this work together with previously determined results from the work of Hansraj *et al* (2006)
- to conduct an analysis of our solutions for physical plausibility.

The outline of the work is as follows:

- Chapter 1: Introduction
- Chapter 2: Mathematical Preliminaries

We discuss the mathematical concepts, notations and definitions relevant to our work. These include, amongst others, spacetime geometry, kinematical and dynamical quantities, Lie algebras, conformal transformations and Lie group analysis. We present the

methodology that underpins Lie group analysis and illustrate this with the example of the Robinson-Trautman equation.

- Chapter 3: Group Invariant Solutions

A perfect fluid metric conformally related to a Petrov type D spacetime is studied. We make reference to the initial work of Castejon and Coley (1992) and the subsequent work of Hansraj *et al* (2006) in which the initial solutions were extended to a wider class of solutions. We make a claim that it is possible to obtain new solutions with even wider latitude in behaviour. We honour this claim by generating new invariant solutions via two of our four generators. We employ the method of Lie group analysis to achieve this end.

- Chapter 4: Physical Analysis

We present full solutions for the conformally related metric, the energy density, and pressure using our results and previously obtained results of Hansraj *et al* (2006) and Msomi (2003). Where necessary, numerical methods are employed when the reduced equations prove intractable. We examine our solutions for physical admissability by investigating graphical representations of slices of the solution space.

- Chapter 5: Conclusion

Chapter 2

Mathematical Preliminaries

2.1 Notations and Definitions

We consider a spacetime (M, g) where M is a 4-dimensional differentiable manifold with respect to a symmetric, nonsingular metric field g . Points in M are labelled by real coordinates, three of which are spacelike and one timelike. The points are represented as $(x^a) = (x^0, x^1, x^2, x^3)$ where x^0 is timelike and x^1, x^2, x^3 are spacelike.

The invariant distance between neighbouring points of a curve in M is defined by the fundamental metric form

$$ds^2 = g_{ab} dx^a dx^b$$

where g_{ab} is referred to as the metric tensor. It is symmetric and possesses an inverse $g^{ab} = \frac{1}{g_{ab}}$.

The Christoffel symbol Γ is a metric connection that preserves inner products under parallel transport. The coefficients of Γ are calculated by

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (g_{cd,b} + g_{db,c} - g_{bc,d}) \quad (2.1)$$

where commas denote partial differentiation.

The Riemann curvature tensor (also known as Riemann-Christoffel tensor) is a (1,3) tensor field whose coordinate components are given in terms of the coordinate components of the connection as follows:

$$R_{bcd}^a = \Gamma_{bd,c}^a - \Gamma_{bc,d}^a + \Gamma_{ec}^a \Gamma_{bd}^e - \Gamma_{ed}^a \Gamma_{bc}^e. \quad (2.2)$$

The Ricci tensor R_{ab} , obtained by contraction of the Riemann curvature tensor, is given by

$$R_{ab} = \Gamma_{ab,d}^d - \Gamma_{ad,b}^d + \Gamma_{ab}^e \Gamma_{ed}^d - \Gamma_{ad}^e \Gamma_{eb}^d. \quad (2.3)$$

Upon contraction of the Ricci tensor, we obtain the Ricci scalar R , given by

$$R = g^{ab} R_{ab}. \quad (2.4)$$

The Einstein tensor G_{ab} is obtained from the Ricci tensor and the Ricci scalar in the following way:

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab}. \quad (2.5)$$

For matter in its neutral state, the energy-momentum tensor T_{ab} is given by

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \quad (2.6a)$$

where μ is the energy density, p is the isotropic pressure, q_a is the heat flow vector, π_{ab} is the stress tensor and u_a is the velocity field vector.

For a perfect fluid, the energy-momentum tensor reduces to

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}. \quad (2.6b)$$

The energy-momentum tensor (2.6b) and the Einstein tensor (2.5) are related via the Einstein field equations in the following way:

$$G_{ab} = T_{ab}. \quad (2.7)$$

This is a system of 10 partial differential equations which are highly nonlinear and hence difficult to integrate in general.

It is also of importance to our work to mention the Weyl conformal tensor which has physical relevance in expressing the tidal force a body experiences when moving along a geodesic. In fact the Weyl Tensor is the traceless component of the Riemann tensor, and is given by

$$\begin{aligned}
C_{abcd} = & R_{abcd} + \frac{1}{(n-1)(n-2)} R(g_{ac}g_{bd} - g_{ad}g_{bc}) \\
& - \frac{1}{n-2} (g_{ac}R_{bd} - g_{bc}R_{ad} + g_{bd}R_{ac} - g_{ad}R_{bc}) .
\end{aligned} \tag{2.8}$$

The commutator or Lie bracket for two first order linear differential operators \mathbf{X} and \mathbf{Y} is defined as

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X}. \tag{2.9}$$

A Lie algebra is a finite dimensional vector space \mathbf{V} having the bilinear product $[\mathbf{X}, \mathbf{Y}]$ as defined in (2.9) above. The operation has the following properties: (Olver 1993)

(i) it is skew-symmetric. i.e.

$$[\mathbf{X}, \mathbf{Y}] = -[\mathbf{Y}, \mathbf{X}]$$

(ii) it satisfies the Jacobi identity

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0$$

for all $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbf{V}$.

The Lie bracket is related to the Lie derivative by the identities

$$\begin{aligned}
\mathcal{L}_{\mathbf{X}}\mathbf{Y} &= [\mathbf{X}, \mathbf{Y}] \\
\mathcal{L}_{[\mathbf{X}, \mathbf{Y}]} &= [\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}]
\end{aligned}$$

for any vector fields \mathbf{X} and \mathbf{Y} . There exists a close relationship between Lie algebras and Lie groups: every Lie algebra defines a unique simply connected Lie group. The properties of a

Lie group are largely determined by those of its associated Lie algebra.

If g is a Lie algebra and $X \in g$, the adjoint operator, $\text{ad } X$ that maps Y to $[X, Y]$ is a linear transformation of g into itself (Sattering and Weaver 1986).

We may develop the adjoint representation by summing up the Lie series as follows:

$$\begin{aligned} \text{Ad}(\exp(\epsilon X))Y &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} (\text{Ad}(X))^n(Y) \\ &= Y - \epsilon[X, Y] + \frac{1}{2}\epsilon^2[X, [X, Y]] \\ &\quad - \frac{1}{6}\epsilon^3[X, [X, [X, Y]]] + \dots \end{aligned} \tag{2.10}$$

where we have employed an analogue of the Taylor series expansion.

A variety of symmetries may be defined on the manifold by the action of $\mathcal{L}_{\mathbf{X}}$ on the metric tensor and associated quantities (Katzin and Levine 1972, Katzin *et al* 1969). Of the various symmetries that are possible we are primarily concerned with conformal motions. A conformal Killing vector \mathbf{X} is defined by the action of $\mathcal{L}_{\mathbf{X}}$ on the metric tensor field \mathbf{g} so that

$$\mathcal{L}_{\mathbf{X}}g_{ab} = 2\psi g_{ab} \tag{2.11}$$

where $\psi(x^a)$ is the conformal factor. There are four special cases associated with (2.11), viz

1. $\psi = 0$	\mathbf{X} is a Killing vector
2. $\psi_{;a} = 0 \neq \psi$	\mathbf{X} is a homothetic Killing vector
3. $\psi_{;ab} = 0 \neq \psi_{;a}$	\mathbf{X} is a special conformal Killing vector
4. $\psi_{;ab} \neq 0$	\mathbf{X} is a nonspecial conformal Killing vector

The spanning set $\{\mathbf{X}_A\} = \{\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_r\}$ of all the conformal Killing vectors of a spacetime generates a Lie algebra. The elements of this basis are related by

$$\begin{aligned}
[\mathbf{X}_A, \mathbf{X}_B] &= \mathbf{X}_A \mathbf{X}_B - \mathbf{X}_B \mathbf{X}_A \\
&= C^D{}_{AB} \mathbf{X}_D
\end{aligned}$$

where the quantities $C^D{}_{AB}$ are the structure constants of the group. The structure constants have the property of being independent of the coordinate system but do depend on the choice of the basis. The structure constants are skew-symmetric so that

$$C^D{}_{AB} = -C^D{}_{BA}$$

and satisfy the identity

$$C^E{}_{AD} C^D{}_{BC} + C^E{}_{BD} C^D{}_{CA} + C^E{}_{CD} C^D{}_{AB} = 0.$$

The integrability condition for the existence of a conformal symmetry is given by

$$\mathcal{L}_{\mathbf{X}} C^a{}_{bcd} = 0. \tag{2.12}$$

Suppose that we are given a spacetime (M, \mathbf{g}) with line element

$$ds^2 = g_{ab} dx^a dx^b$$

and a related spacetime $(M, \bar{\mathbf{g}})$ with the line element

$$d\bar{s}^2 = \bar{g}_{ab} dx^a dx^b . \quad (2.13)$$

Then the above two line elements are said to be conformally related if

$$\bar{g}_{ab} = e^{2U} g_{ab} \text{ and } \bar{g}^{ab} = e^{-2U} g^{ab} , \quad (2.14)$$

where $U(x^c)$ is a nonzero, real-valued function of the coordinates on M . This transformation between $\bar{\mathbf{g}}$ and \mathbf{g} is called a conformal transformation and is a special type of mapping between metric spaces given by dilatation (or contraction) of all lengths by a common factor which varies from point to point. If \mathbf{X} is a conformal Killing vector in the (M, \mathbf{g}) spacetime so that (2.11) holds, i.e.

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\psi g_{ab} .$$

then \mathbf{X} is also a conformal Killing vector in the related spacetime $(M, \bar{\mathbf{g}})$, and we have

$$\mathcal{L}_{\mathbf{X}} \bar{g}_{ab} = 2\sigma \bar{g}_{ab} . \quad (2.15)$$

With the help of (2.11), (2.14) and (2.15), we can show that the factors σ and ψ are related by

$$\sigma = \left[\frac{\mathcal{L}_{\mathbf{X}}(e^{2U})}{2e^{2U}} + \psi \right] . \quad (2.16)$$

From this we can conclude that conformal transformations map conformal Killing vectors to conformal Killing vectors, the conformal factors being different. However, conformal transformations do not map Killing vectors or homothetic Killing vectors to their respective counterparts in the conformally related spacetime. Note that in the trivial case where e^{2U} is constant, we have $\sigma = \psi$. However, it is important to note that the converse does not hold. If $\sigma = \psi$,

then (2.16) implies that $\mathcal{L}_{\mathbf{X}}(e^{2U}) = 0$. This means that U is constant along the integral curves of the vector \mathbf{X} , but it may vary elsewhere on the manifold.

2.2 Conformal Transformations

It should be noted that the connection coefficients, Riemann curvature tensor, Ricci tensor and Ricci scalar for the metric g_{ab} are related to those of the metric $\bar{g}_{ab} = e^{2U}g_{ab}$ by the following formulae which are given in de Felice and Clarke (1990):

$$\bar{\Gamma}^a_{bc} = \Gamma^a_{bc} + \frac{1}{2} [\delta^a_b \phi_c + \delta^a_c \phi_b - g_{bc} \phi^a] \quad (2.17a)$$

$$\bar{R}^a_{bcd} = R^a_{bcd} + \delta^a_{[d} \nabla_{c]} \phi_b + g_{b[c} \nabla_{d]} \phi^a \quad (2.17b)$$

$$+ \frac{1}{2} \delta^a_{[c} \phi_{d]} \phi_b - \frac{1}{2} g_{b[c} \phi_{d]} \phi^a - \frac{1}{2} \delta^a_{[c} g_{d]b} \phi^e \phi_e \quad (2.17c)$$

$$\bar{R}_{bd} = R_{bd} - \frac{1}{2} [2\phi_{b;d} - \phi_a \phi_b + g_{ab} \phi^e \phi_e] - \frac{1}{2} g_{bd} \phi^a_{;a} \quad (2.17d)$$

$$\bar{R} = \frac{1}{\Omega} \left[R - 3\phi^e_{;e} - \frac{3}{2} \phi_e \phi^e \right], \quad (2.17d)$$

where we have defined $\phi_a = \partial_a(\ln \Omega)$ and $\Omega = e^{2U}$. Additionally the conformal Einstein tensor $\bar{\mathbf{G}}$ is given by

$$\bar{G}_{ab} = G_{ab} + 2 \left(U_a U_b - \frac{1}{2} U^c U_c g_{ab} \right) + 2(U^c_{;c} + U^c U_c) g_{ab} - 2U_{a;b}, \quad (2.18)$$

where the covariant derivatives and contractions are calculated on the original metric g_{ab} . Note that we may also write the conformal Einstein tensor (2.18) as

$$\bar{G}_{ab} = G_{ab} - 2U_a U_b - 2U_{a;b} + (2U^c_{;c} - U^c U_c) \bar{g}_{ab}$$

where the geometric quantities are now evaluated using the conformally related metric \bar{g}_{ab} (Tupper 1990).

An important characteristic of conformal mappings is that the Weyl tensor C is invariant under the transformation, i.e.

$$\bar{C}_{abcd} = C_{abcd}$$

and consequently a conformal transformation is sometimes referred to as a Weyl rescaling in the literature. A necessary and sufficient condition that a spacetime is conformally flat is that the Weyl tensor C vanishes.

The covariant derivative of the timelike fluid 4-vector field \mathbf{u} , can be decomposed as follows

$$u_{a;b} = \sigma_{ab} + \frac{1}{3}\Theta h_{ab} - \dot{u}_a u_b + \omega_{ab}$$

where $h_{ab} = g_{ab} + u_a u_b$ is the projection tensor. In the above we have defined

$$\dot{u}_a = u_{a;b} u^b \quad (2.19a)$$

$$\omega_{ab} = u_{[a;b]} + \dot{u}_{[a} u_{b]} \quad (2.19b)$$

$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a} u_{b)} - \frac{1}{3}\Theta h_{ab} \quad (2.19c)$$

$$\Theta = u^a{}_{;a} \quad (2.19d)$$

where \dot{u}_a is the acceleration vector ($\dot{u}^a u_a = 0$), ω_{ab} is the skew-symmetric vorticity tensor ($\omega_{ab} u^a = 0$), σ_{ab} is the symmetric shear tensor ($\sigma_{ab} u^b = 0 = \sigma^a{}_a$) and Θ is the rate of expansion.

Under a conformal transformation $\bar{g}_{ab} = e^{2U} g_{ab}$, the world lines are the same and the velocity field transforms as

$$\bar{u}_a = e^U u_a$$

and we obtain

$$\bar{u}_a = e^U \left(\dot{u}_a + u_a u_b U^{,b} + U_{,a} \right) \quad (2.20a)$$

$$\bar{\Theta} = e^{-U} \Theta - 3u^a \left(e^{-U} \right)_{,a} \quad (2.20b)$$

$$\bar{\omega}_{ab} = e^U \omega_{ab} \quad (2.20c)$$

$$\bar{\sigma}_{ab} = e^U \sigma_{ab} \quad (2.20d)$$

for the transformed kinematical quantities listed in (2.19). The quantities (2.20) will be useful in studying the physical behaviour of the models generated by a conformal transformation (Hansraj *et al* 2006).

2.3 Lie Group Analysis

2.3.1 Methodology

We consider a k th order partial differential equation given by (Bluman and Kumei, 1989)

$$F(x, u, u^{(1)}, u^{(2)}, \dots, u^{(k)}) = 0 \quad (2.21)$$

where

$x = (x_1, x_2, \dots, x_n)$ denotes n independent variables

u denotes the coordinate corresponding to the dependent variable, and

$u^{(j)}$ denotes the set of coordinates corresponding to all j th order partial derivatives of u with respect to x .

The coordinate of $u^{(j)}$ corresponding to $\frac{\partial^j u}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_j}}$ is denoted by

$$u_{i_1 i_2 \dots i_j}, \quad i_j = 1, 2, \dots, n \quad \text{for } j = 1, 2, \dots, k.$$

We seek the one-parameter Lie group of transformations

$$x_i^* = \psi_i(x, u, \epsilon) \tag{2.22a}$$

$$u^* = \phi(x, u, \epsilon) \tag{2.22b}$$

that leaves (2.21) invariant.

In general, it is difficult to calculate (2.22a)-(2.22b) directly.

To proceed in a meaningful way, we take Taylor expansions of these transformations about $\epsilon = 0$ to obtain

$$x_i^* = x_i + \zeta \left. \frac{d\psi_i}{d\epsilon} \right|_{\epsilon=0} + O(\epsilon^2)$$

$$y^* = y + \zeta \left. \frac{d\phi}{d\epsilon} \right|_{\epsilon=0} + O(\epsilon^2),$$

where we have used the group properties of (2.22a)-(2.22b).

We now define (Bluman and Kumei 1989)

$$\begin{aligned} G &= \eta_1(x, u) \frac{\partial}{\partial x_1} + \eta_2(x, u) \frac{\partial}{\partial x_2} + \cdots + \eta_n(x, u) \frac{\partial}{\partial x_n} \\ &\quad + \zeta(x, u) \frac{\partial}{\partial u} \\ &= \eta_i(x, u) \frac{\partial}{\partial x_i} + \zeta(x, u) \frac{\partial}{\partial u} \end{aligned} \tag{2.23}$$

where

$$\eta_i(x, u) = \left. \frac{d\psi_i}{d\epsilon} \right|_{\epsilon=0}$$

$$\zeta(x, u) = \left. \frac{d\phi}{d\epsilon} \right|_{\epsilon=0}$$

as the infinitesimal generator of (2.22a) and (2.22b).

The k th extension of G needed to transform derivatives in (2.21) is (Bluman and Kumei 1989)

$$\begin{aligned} G^{[k]} &= \eta_i(x, u) \frac{\partial}{\partial x_i} + \zeta(x, u) \frac{\partial}{\partial u} \\ &+ \zeta_i^{(1)}(x, u, u^{(1)}) \frac{\partial}{\partial u_1} + \cdots + \\ &\zeta_{i_1 i_2 \dots i_k}^{(k)}(x, u, u^{(1)}, u^{(2)}, \dots, u^{(k)}) \frac{\partial}{\partial u_{i_1 i_2 \dots i_k}}, \end{aligned} \quad (2.24)$$

where

$$\zeta_i^{(1)} = D_i \zeta - (D_i \eta_j) u_j, \quad i = 1, 2, \dots, n$$

and

$$\zeta_{i_1 i_2 \dots i_k}^{(j)} = D_{i_k} \zeta_{i_1 i_2 \dots i_{k-1}}^{(j-1)} - (D_{i_k} \eta_j) u_{i_1 i_2 \dots i_{k-1} j},$$

$$i_j = 1, 2, \dots, n \text{ for } j = 1, 2, \dots, k.$$

Then the partial differential equation (2.21) admits (2.22a) and (2.22b) if and only if

$$G^{[k]} F \Big|_{F=0} = 0. \quad (2.25)$$

Equation (2.25) is an identity in powers of the partial derivatives of the dependent variables. However, the η_i and ζ only depend on x and u , not derivatives. As a result, by the principle

of undetermined coefficients, a system of partial differential equations is generated in η and ζ . Solving this system explicitly yields the symmetry G . Then a characteristic system of equations is solved to yield new variables. When these new variables are substituted into the original system of partial differential equations in n variables, these equations are reduced to a system of partial differential equations in $n - 1$ variables. This process may be repeated until a system of ordinary differential equations is obtained. This new system may then be solved to yield solutions which can be transformed into solutions of the original partial differential equations. It should be noted, however, that there is no guarantee of solving the resulting system of ordinary differential equations. A more detailed stepwise construction of group-invariant solutions may be found in Olver (1993).

It should be pointed out that since the process of implementing the Lie analysis is algorithmic, the calculation of the symmetries are invariably completed via some computer software package *e.g.* PROGRAM LIE (Head 1993).

We illustrate these ideas with an example in the next section.

2.3.2 Robinson-Trautman Equation

In a vacuum, the Einstein tensor vanishes in view of the absence of matter and the Einstein field equations reduce to a system of ten second order differential equations. For the Robinson-Trautman solutions of Petrov Type III the pivotal equation is given by (Stephani 1989)

$$P(P_{xx} + P_{yy}) - (P_x^2 + P_y^2) - 3x\sqrt{2} = 0. \quad (2.26)$$

This is a nonlinear partial differential equation in one dependent variable, P , and two independent variables, x and y .

We operate on (2.26) with

$$\begin{aligned}
G^{[2]} = & \eta_1 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial P} \\
& + \zeta_1^{(1)} \frac{\partial}{\partial P_x} + \zeta_2^{(1)} \frac{\partial}{\partial P_y} \\
& + \zeta_{11}^{(2)} \frac{\partial}{\partial P_{xx}} + \zeta_{12}^{(2)} \frac{\partial}{\partial P_{xy}} + \zeta_{22}^{(2)} \frac{\partial}{\partial P_{yy}}.
\end{aligned} \tag{2.27}$$

Invoking (2.25) yields

$$\begin{aligned}
& \eta_1(3\sqrt{2}) + \zeta \left\{ \frac{1}{P}(P_x^2 + P_y^2 - 3x\sqrt{2}) - P_{yy} + P_{yy} \right\} \\
& - 2 \left\{ \frac{\partial \zeta}{\partial x} P_x + \left(\frac{\partial \zeta}{\partial P} - \frac{\partial \eta_1}{\partial x} \right) P_x^2 - \frac{\partial \eta_2}{\partial x} P_x P_y - \frac{\partial \eta_1}{\partial P} P_x^3 - \frac{\partial \eta_2}{\partial P} P_x^2 P_y \right\} \\
& - 2 \left\{ \frac{\partial \zeta}{\partial x} P_y + \left(\frac{\partial \zeta}{\partial P} - \frac{\partial \eta_2}{\partial y} \right) P_y^2 - \frac{\partial \eta_1}{\partial y} P_x P_y - \frac{\partial \eta_2}{\partial P} P_y^3 - \frac{\partial \eta_1}{\partial P} P_x P_y^2 \right\} \\
& + \left\{ \frac{\partial^2 \zeta}{\partial x^2} P + \left(2 \frac{\partial^2 \zeta}{\partial x \partial P} - \frac{\partial^2 \eta_1}{\partial x^2} \right) P P_x - \frac{\partial^2 \eta_2}{\partial x^2} P P_y \right. \\
& + \left(\frac{\partial \zeta}{\partial P} - 2 \frac{\partial \eta_1}{\partial x} \right) (P_x^2 + P_y^2 - 3x\sqrt{2} - P P_{yy}) - 2 \frac{\partial \eta_2}{\partial x} P P_{xy} \\
& + \left(\frac{\partial^2 \eta}{\partial P^2} - 2 \frac{\partial^2 \eta_1}{\partial x \partial P} \right) P P_x^2 \\
& - 2 \frac{\partial^2 \eta_2}{\partial x \partial P} P P_x P_y - \frac{\partial^2 \eta_1}{\partial P^2} P P_x^3 - \frac{\partial^2 \eta_2}{\partial P^2} P P_x^2 P_y \\
& - \& 3 \frac{\partial \eta_1}{2P} (P_x^3 + P_x P_y^2 - 3x\sqrt{2} P_x - P P_x P_{yy}) \\
& - \left. \frac{\partial \eta_2}{\partial P} (P_x^2 P_y + P_y^3 - 3x\sqrt{2} P_y - P P_y P_{yy}) - 2 \frac{\partial \eta_2}{\partial P} P P_x P_{xy} \right\} \\
& + \left\{ \frac{\partial^2 \zeta}{\partial y^2} P + \left(2 \frac{\partial^2 \zeta}{\partial y \partial P} - \frac{\partial^2 \eta_2}{\partial y^2} \right) P P_y - \frac{\partial^2 \eta_1}{\partial y^2} P P_x + \left(\frac{\partial \zeta}{\partial P} - 2 \frac{\partial \eta_2}{\partial y} \right) P P_{yy} \right. \\
& - 2 \frac{\partial \eta_1}{\partial y} P P_{xy} + \left(\frac{\partial^2 \zeta}{\partial P^2} - 2 \frac{\partial^2 \eta_2}{\partial y \partial P} \right) P P_y^2 \\
& - 2 \frac{\partial^2 \eta_1}{\partial y \partial P} P P_x P_y - \frac{\partial^2 \eta_2}{\partial P^2} P P_y^3 - \frac{\partial^2 \eta_1}{\partial P^2} P P_x P_y^2 - 3 \frac{\partial \eta_2}{\partial P} P P_y P_{yy} \\
& - \left. 2 \frac{\partial \eta_1}{\partial P} P P_y P_{xy} \right\} = 0. \tag{2.28}
\end{aligned}$$

Separating by coefficients of powers of derivatives of P yields the system

$$P_x^3 \quad : \quad 2 \frac{\partial \eta_1}{\partial P} - \frac{\partial^2 \eta_1}{\partial P^2} P - 3 \frac{\partial \eta_1}{\partial P} \left(\frac{1}{P} \right) = 0$$

$$P_x^2 \quad : \quad \zeta \left(\frac{1}{P} \right) - \frac{\partial \zeta}{\partial P} + \frac{\partial^2 \eta_1}{\partial P^2} P - 2 \frac{\partial^2 \eta_1}{\partial x \partial P} P = 0$$

$$P_x^2 P_y \quad : \quad \frac{\partial \eta_2}{\partial P} - \frac{\partial^2 \eta_2}{\partial P^2} P = 0$$

$$P_x \quad : \quad -2 \frac{\partial \zeta}{\partial x} + 2 \frac{\partial^2 \zeta}{\partial x \partial P} P - \frac{\partial^2 \eta_1}{\partial x^2} P - \frac{\partial^2 \eta_1}{\partial y^2} P - 3x\sqrt{2} = 0$$

$$P_x P_y \quad : \quad 2 \frac{\partial \eta_2}{\partial x} + 2 \frac{\partial \eta_1}{\partial y} - 2 \frac{\partial^2 \eta_2}{\partial x \partial P} P - 2 \frac{\partial^2 \eta_1}{\partial y \partial P} P = 0$$

$$P_x P_y^2 \quad : \quad -\frac{\partial \eta_1}{\partial P} - \frac{\partial^2 \eta_1}{\partial P^2} P = 0$$

$$P_x P_{yy} \quad : \quad 3 \frac{\partial \eta_1}{\partial P} = 0$$

$$P_x P_{xy} \quad : \quad -2 \frac{\partial \eta_2}{\partial P} P = 0$$

$$P_x^0 \quad : \quad \eta_1(3\sqrt{2}) + \left(2 \frac{\partial \eta_1}{\partial x} - \frac{\partial \zeta}{\partial P} - \zeta \right) 3x\sqrt{2} + \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right) P = 0$$

$$P_y \quad : \quad \frac{\partial \eta_2}{\partial P} 3x\sqrt{2} - 2 \frac{\partial \zeta}{\partial x} + \left(2 \frac{\partial^2 \zeta}{\partial y \partial P} - \frac{\partial^2 \eta_2}{\partial x^2} - \frac{\partial^2 \eta_2}{\partial y^2} \right) P = 0$$

$$P_y^2 \quad : \quad \zeta \left(\frac{1}{P} \right) + 2 \frac{\partial \eta_2}{\partial y} - 2 \frac{\partial \eta}{\partial x} - \frac{\partial \zeta}{\partial P} + \left(\frac{\partial^2 \zeta}{\partial P^2} - 2 \frac{\partial^2 \eta_2}{\partial y \partial P} \right) P = 0$$

$$\begin{aligned}
P_y^3 & : \quad \frac{\partial \eta_2}{\partial P} - \frac{\partial^2 \eta_2}{\partial P^2} = 0 \\
P_{yy} & : \quad 2 \left(\frac{\partial \eta_1}{\partial x} - \frac{\partial \eta_2}{\partial y} \right) P = 0 \\
P_{xy} & : \quad -2 \left(\frac{\partial \eta_2}{\partial x} + \frac{\partial \eta_1}{\partial y} \right) P = 0 \\
P_y P_{xy} & : \quad -2 \frac{\partial \eta_1}{\partial P} P = 0 \\
P_y P_{yy} & : \quad -2 \frac{\partial \eta_2}{\partial P} P = 0
\end{aligned} \tag{2.29}$$

The system (2.29) is solved step by step to obtain

$$\eta_1 = ax + bxy \tag{2.30a}$$

$$\eta_2 = ay + \frac{1}{2} b(y^2 - x^2) + c \tag{2.30b}$$

$$\zeta = \frac{3}{2} aP + \frac{3}{2} by P. \tag{2.30c}$$

These functions give the three parameter symmetry

$$\begin{aligned}
X &= (ax + bxy) \partial_x + \left[ay + \frac{1}{2} b(y^2 - x^2) + c \right] \partial_y \\
&+ \left(\frac{3}{2} aP + \frac{3}{2} by P \right) \partial_P.
\end{aligned} \tag{2.31}$$

If we take $a = 1$, $b = c = 0$, we obtain

$$X = x \partial_x + y \partial_y + \frac{3}{2} P \partial_P. \tag{2.32}$$

The symmetry (2.32) defines the reduction variables via

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dP}{\frac{3}{2}P}$$

to be

$$u = \frac{x}{y}, \quad v = P^{-\frac{3}{2}}x.$$

The partial differential equation (2.26) now reduces to the ordinary differential equation

$$-3\sqrt{2}u - \frac{3}{2}uq^2 - 2u^4qq_u + (u^3 + u^5)qq_{uu} - (u^3 - u^5)q_u^2 = 0. \quad (2.33)$$

Solutions to (2.33) will now provide group invariant (in this case, scaling) solutions to (2.26).

2.3.3 Optimal Group Invariant Reductions

For the three parameter symmetry (2.31) of the Robinson-Trautman equation (2.26), we set

$$(i) \quad a = 1, \quad b = c = 0$$

$$(ii) \quad b = 1, \quad a = c = 0$$

$$(iii) \quad c = 1, \quad a = b = 0.$$

in turn to obtain the following three symmetries:

$$G_1 = x \partial x + y \partial y + \frac{3}{2} P \partial P \quad (2.34a)$$

$$G_2 = xy \partial x + \frac{1}{2}(y^2 - x^2)\partial y + \frac{3}{2} y P \partial P \quad (2.34b)$$

$$G_3 = \partial y. \quad (2.34c)$$

In Section 2.3.2 we reduced the partial differential equation (2.26) to the ordinary differential equation (2.33) using only G_1 . However, as there are three symmetries of the Robinson-Trautman equation, we have other options. We can establish the optimal combination of these symmetries necessary for the reduction of the partial differential equation to an ordinary differential equation. We achieve this by the method of optimal subgroups which requires the

concepts of a commutation table and an adjoint table. The commutation table is constructed by invoking (2.9) for all the symmetries.

Instead of reducing the equation via all possible linear combinations of the symmetries we can proceed in a more systematic manner. The complete commutation table is represented in Table 2.1.

	G_1	G_2	G_3
G_1	0	G_2	$-G_3$
G_2	$-G_2$	0	$-G_1$
G_3	G_3	G_1	0

Table 2.1: Commutation Table for Vector Fields $G_1 - G_3$

To construct the adjoint table, we need to utilize (2.10)

In the case of G_1 and G_2 , we have

$$\begin{aligned}
\text{Ad}(\exp(\epsilon G_1))G_2 &= G_2 - \epsilon[G_1, G_2] + \frac{1}{2}\epsilon^2[G_1, [G_1, G_2]] \\
&\quad - \frac{1}{6}\epsilon^3[G_1, [G_1, [G_1, G_2]]] + \dots \\
&= e^{-\epsilon}G_2.
\end{aligned}$$

Continuing in this manner, we have the adjoint table given in Table 2.2.

Ad	G_1	G_2	G_3
G_1	G_1	$e^{-\epsilon}G_2$	G_3e^ϵ
G_2	$G_1 + \epsilon G_2$	G_2	$G_3 + \epsilon G_1 + \frac{1}{2}\epsilon^2 G_2$
G_3	$G_1 - \epsilon G_3$	$G_2 - \epsilon G_1 + \frac{1}{2}\epsilon^2 G_3$	G_3

Table 2.2: Table of Adjoint Operators for $G_1 - G_3$

We consider the most general linear combination of the symmetries given by

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3. \quad (2.35)$$

We attempt to simplify as many as possible of the coefficients a_i of G through the application of adjoint maps to G .

We begin by assuming that $a_3 \neq 0$, and we set $a_3 = 1$ in general.

Thus we have

$$G = a_1 G_1 + a_2 G_2 + G_3. \quad (2.36)$$

Acting on G in (2.36) by $\text{Ad}(\exp(\epsilon G_2))$ we have

$$\text{Ad}(\exp(\epsilon G_2))(a_1 G_1 + a_2 G_2 + G_3) = a_1 G_1 + a_1 \epsilon G_2 + a_2 G_2 + G_3 + \epsilon G_1 + \frac{1}{2} \epsilon^2 G_2.$$

Setting $\epsilon = -a_1$, we obtain

$$\begin{aligned} G &= a_1 \epsilon G_2 + a_2 G_2 + G_3 + \frac{1}{2} \epsilon^2 G_2 \\ &= \left(a_1 \epsilon + a_2 + \frac{1}{2} \epsilon^2 \right) G_2 + G_3 \end{aligned}$$

which we write as

$$G = \alpha G_2 + G_3 \tag{2.37}$$

where

$$\alpha = a_1 \epsilon + a_2 + \frac{1}{2} \epsilon^2.$$

We cannot simplify (2.37) any further by adjoint maps and so it is one element of our optimal system.

Next we set $a_3 = 0$ and $a_2 = 1$.

Then (2.36) becomes

$$G = a_1 G_1 + G_2. \tag{2.38}$$

Acting on G in (2.38) by $\text{Ad}(\exp(\epsilon G_2))$ we have

$$\text{Ad}(\exp(\epsilon G_2))(a_1 G_1 + G_2) = a_1 G_1 + a_1 \epsilon G_2 + G_2.$$

Setting $\epsilon = -\frac{1}{a_1}$, $a_1 \neq 0$, we obtain

$$G = a_1 G_1$$

and so

$$G = G_1 \tag{2.39}$$

is an element of our optimal system.

Lastly, we set $a_3 = a_2 = 0$ and $a_1 = 1$ which leaves us with

$$G = G_1$$

which is just (2.39) again.

Ultimately we have the following linear combinations:

$$G = G_1 \tag{2.40a}$$

$$G = \alpha G_2 + G_3 \quad , \quad \alpha = 0, \pm 1. \tag{2.40b}$$

We also observe that (2.26) is invariant under the following involutions:

$$P \rightarrow -P \tag{2.41a}$$

$$y \rightarrow -y. \tag{2.41b}$$

These involutions may reduce our optimal system further. Using (2.41) we obtain the optimal combination of the symmetries (2.34) to be

$$G_1 = x \partial x + y \partial y + \frac{3}{2} P \partial p \tag{2.42a}$$

$$G_3 = \partial y \tag{2.42b}$$

$$G_2 + G_3 = xy \partial x + \left(\frac{1}{2}(y^2 - x^2) + 1 \right) \partial y + \frac{3}{2} y P \partial p \tag{2.42c}$$

$$G_3 - G_2 = -xy \partial x + \left(1 - \frac{1}{2}(y^2 - x^2) \right) \partial y - \frac{3}{2} y P \partial p. \tag{2.42d}.$$

These symmetries can be used separately to reduce (2.26) to ordinary differential equations which hopefully can be solved.

We have now set the framework necessary to find group invariant solutions of some interesting partial differential equations in relativity.

Chapter 3

Group Invariant Solutions

3.1 Introduction

Exact solutions of Einstein's field equations may serve as models of realistic matter configurations such as fluid planets or neutron stars. Therefore it is important to seek exact solutions for the conformally related spacetime under investigation in this thesis. The seed metric under review possesses undesirable physical properties and our intention is to utilise its geometry and construct a new metric, using conformal transformations, which indeed do exhibit properties associated with realistic matter distributions. It is noteworthy that conformal structures play an important role in understanding the deep underlying mathematics of general relativity. For instance conformal transformations map null rays to null rays and light cones to light cones. that is, the structure of the null cone is preserved in four dimensions. Additionally, the existence of conformal symmetries has the effect of simplifying the resulting system of partial differential equations and thus possibly making the integration process simpler. Only a relatively small number of attempts have been made in this direction. For example the conformal symmetries of locally rotationally symmetric metrics were investigated by Moodley (1992) and spherically symmetric spacetimes were intensively studied by Moopanar (2010) and Maharaj *et al* (1995). Additionally, the presence of conformal symmetries have allowed researchers to classify known solutions according to their conformal structure (Coley and Tupper 1994).

3.2 Spacetime Geometry

We consider the metric

$$ds^2 = -dt^2 + dx^2 + e^{2\nu(y,z)}(dy^2 + dz^2) \quad (3.1)$$

which is of Petrov type D and Szekeres form.

The spacetime (3.1) admits three Killing vectors, namely

$$X_1 = \partial_t \quad (3.2a)$$

$$X_2 = \partial_x \quad (3.2b)$$

$$X_3 = x \partial_t + t \partial_x \quad (3.2c)$$

with the Lie bracket relations

$$[X_1, X_2] = 0 \quad (3.3a)$$

$$[X_1, X_3] = X_2 \quad (3.3b)$$

$$[X_2, X_3] = X_1. \quad (3.3c)$$

The conformally related analogue is given by

$$ds^2 = e^{2U(t,x,y,z)}[-dt^2 + dx^2 + e^{2\nu(y,z)}(dy^2 + dz^2)]. \quad (3.4)$$

By the Defrise-Carter (1975) theorem, the Killing vectors (3.2) are now conformal Killing vectors of (3.4), given by

$$Y_1 = U_t \quad (3.5a)$$

$$Y_2 = U_x \quad (3.5b)$$

$$Y_3 = x U_t + t U_x. \quad (3.5c)$$

Note that $\nu_{yy} + \nu_{zz}$ is not zero. To simplify calculations we follow the treatment of Castejon-Amendo and Coley (1992), and invoke the prescription

$$\nu_{yy} + \nu_{zz} = -2ke^{2\nu} \quad (3.6)$$

where k is a nonzero constant. This allows us to solve the system of partial differential equations and constitutes an integrability condition for this method. We undertake a complete analysis of this condition later in this thesis.

For the line element (3.1) the nonvanishing coefficients of the metric connection Γ are given by

$$\begin{aligned}\Gamma^2_{22} &= -\Gamma^2_{33} = \Gamma^3_{32} = \Gamma^3_{23} = \nu_y \\ \Gamma^2_{23} &= -\Gamma^3_{22} = \Gamma^2_{32} = \Gamma^3_{33} = \nu_z.\end{aligned}$$

It is now possible to evaluate the Ricci tensor (2.3), for the line element (3.1), by using the above connection coefficients. The nonvanishing components are

$$R_{22} = -(\nu_{yy} + \nu_{zz}) \quad (3.7a)$$

$$R_{33} = -(\nu_{yy} + \nu_{zz}) \quad (3.7b)$$

and therefore the Ricci scalar (2.4) is given by

$$R = -2e^{-2\nu}(\nu_{yy} + \nu_{zz}). \quad (3.8)$$

The Einstein tensor (2.5) is given by

$$G_{00} = -e^{-2\nu} (\nu_{yy} + \nu_{zz}) \quad (3.9a)$$

$$G_{11} = e^{-2\nu} (\nu_{yy} + \nu_{zz}). \quad (3.9b)$$

Then on using (3.7)-(3.9), as well as the formulae (2.17), we may evaluate the geometric quantities for the conformally related metric (3.4). The conformal Ricci tensor (2.17c) has the

form

$$\bar{R}_{00} = -3U_{tt} + 2U_x^2 + e^{-2\nu} (2U_y^2 + 2U_z^2 + U_{yy} + U_{zz}) \quad (3.10a)$$

$$\bar{R}_{01} = 2U_t U_x - 2U_{tx} \quad (3.10b)$$

$$\bar{R}_{02} = 2U_t U_y - 2U_{ty} \quad (3.10c)$$

$$\bar{R}_{03} = 2U_t U_z - 2U_{tz} \quad (3.10d)$$

$$\bar{R}_{11} = U_{tt} - 3U_{xx} + 2U_t^2 - e^{-2\nu} (2U_y^2 + 2U_z^2 + U_{yy} + U_{zz}) \quad (3.10e)$$

$$\bar{R}_{12} = 2U_x U_y - 2U_{xy} \quad (3.10f)$$

$$\bar{R}_{13} = 2U_x U_z - 2U_{xz} \quad (3.10g)$$

$$\bar{R}_{22} = -(\nu_{yy} + \nu_{zz}) - 3U_{yy} - U_{zz} + 2\nu_y U_y - 2\nu_z U_z - 2U_z^2 \quad (3.10h)$$

$$+ e^{2\nu} (U_{tt} - U_{xx} + 2U_t^2 - 2U_x^2)$$

$$\bar{R}_{23} = 2U_y U_z - 2U_{yz} + 2\nu_z U_y + 2\nu_y U_z \quad (3.10i)$$

$$\bar{R}_{33} = -(\nu_{yy} + \nu_{zz}) - 3U_{zz} - U_{yy} - 2\nu_y U_y + 2\nu_z U_z - 2U_y^2 \quad (3.10j)$$

$$+ e^{2\nu} (U_{tt} - U_{xx} + 2U_t^2 - 2U_x^2).$$

The conformal Ricci scalar (2.17d) is given by

$$\begin{aligned}\bar{R} = & e^{-2U} [6 (U_{tt} - U_{xx} + U_t^2 - U_x^2) \\ & - 6e^{-2\nu} (U_{yy} + U_{zz} + U_y^2 + U_z^2) - 2e^{-2\nu} (\nu_{yy} + \nu_{zz})] .\end{aligned}\quad (3.11)$$

The conformal Einstein tensor may be evaluated by using the relationship (2.17d), (3.10) and (3.11). It has the form

$$\bar{G}_{00} = -2U_{xx} - U_x^2 + 3U_t^2 - e^{-2\nu} (2U_{yy} + 2U_{zz} + U_y^2 + U_z^2 + \nu_{yy} + \nu_{zz}) \quad (3.12a)$$

$$\bar{G}_{01} = 2U_t U_x - 2U_{tx} \quad (3.12b)$$

$$\bar{G}_{02} = 2U_t U_y - 2U_{ty} \quad (3.12c)$$

$$\bar{G}_{03} = 2U_t U_z - 2U_{tz} \quad (3.12d)$$

$$\bar{G}_{11} = -2U_{tt} - U_t^2 + 3U_x^2 + e^{-2\nu} (2U_{yy} + 2U_{zz} + U_y^2 + U_z^2 + \nu_{yy} + \nu_{zz}) \quad (3.12e)$$

$$\bar{G}_{12} = 2U_x U_y - 2U_{xy} \quad (3.12f)$$

$$\bar{G}_{13} = 2U_x U_z - 2U_{xz} \quad (3.12g)$$

$$\begin{aligned}\bar{G}_{22} = & 2U_{zz} + U_z^2 + 3U_y^2 + 2\nu_y U_y - 2\nu_z U_z \\ & + e^{2\nu} (2U_{xx} - 2U_{tt} + U_x^2 - U_t^2)\end{aligned}\quad (3.12h)$$

$$\bar{G}_{23} = 2U_y U_z - 2U_{yz} + 2\nu_z U_y + 2\nu_y U_z \quad (3.12i).$$

$$\begin{aligned}
\bar{G}_{33} &= 2U_{yy} + U_y^2 + 3U_z^2 - 2\nu_y U_y + 2\nu_z U_z \\
&\quad + e^{2\nu} (2U_{xx} - 2U_{tt} + U_x^2 - U_t^2)
\end{aligned}
\tag{3.12j}$$

for the line element (3.4). The Weyl conformal tensor is given by

$$\begin{aligned}
3e^{2\nu} \bar{C}_{0101} &= -6\bar{C}_{0202} = -6\bar{C}_{0303} = 6\bar{C}_{1212} \\
&= 6\bar{C}_{1313} = \frac{3}{5}e^{-2\nu} \bar{C}_{2323} = \frac{3}{5}e^{-2\nu} \bar{C}_{3232} \\
&= \nu_{yy} + \nu_{zz}.
\end{aligned}$$

Clearly this form for the conformal tensor indicates that the metric (3.4) is not conformally flat and motivates the choice (3.6).

To determine the perfect fluid energy–momentum tensor, we select a fluid 4–velocity vector \mathbf{u} that is noncomoving with the form

$$u^a = e^{-U} (\cosh v \partial_0^a + \sinh v \partial_1^a) \tag{3.13}$$

where $v = v(t, x)$. Note that a trivial calculation reveals that when ν is a constant, the perfect fluid Einstein field equations would imply conformal flatness and this case is not of immense interest. All conformally flat solutions have been found. They are either generalised Schwarzschild metrics or, in the case of expansion, Stephani spacetimes (Stephani 1967). Additionally, Lake (1989) demonstrated a necessary and sufficient condition for a spherically symmetric spacetime to be conformally flat. The energy–momentum tensor (2.6b) has the

nonzero components

$$\bar{T}_{00} = (\mu + p)e^{2U} \cosh^2 v - pe^{2U} \quad (3.14a)$$

$$\bar{T}_{01} = -(\mu + p)e^{2U} \cosh v \sinh v \quad (3.14b)$$

$$\bar{T}_{11} = (\mu + p)e^{2U} \sinh^2 v + pe^{2U} \quad (3.14c)$$

$$\bar{T}_{22} = pe^{2\nu+2U} \quad (3.14d)$$

$$\bar{T}_{33} = \bar{T}_{22}. \quad (3.14e)$$

We are now in a position to generate the Einstein field equations using (3.12) and (3.14). These are given by

$$U_t U_y - U_{ty} = 0 \quad (3.15a)$$

$$U_t U_z - U_{tz} = 0 \quad (3.15b)$$

$$U_x U_y - U_{xy} = 0 \quad (3.15c)$$

$$U_x U_z - U_{xz} = 0 \quad (3.15d)$$

$$U_t U_x - U_{tx} = -\frac{1}{4}(\mu + p)e^{2U} \sinh 2v \quad (3.15e)$$

$$U_y U_z - U_{yz} + \nu_z U_y + \nu_y U_z = 0 \quad (3.15f)$$

$$\begin{aligned} & -2U_{xx} - U_x^2 + 3U_t^2 - e^{-2\nu} (2U_{yy} + 2U_{zz} + U_y^2 + U_z^2 + \nu_{yy} + \nu_{zz}) \\ & = (\mu + p)e^{2U} \cosh^2 v - pe^{2U} \end{aligned} \quad (3.15g)$$

$$\begin{aligned}
& - 2U_{tt} - U_t^2 + 3U_x^2 + e^{-2\nu} (2U_{yy} + 2U_{zz} + U_y^2 + U_z^2 + \nu_{yy} + \nu_{zz}) \\
& = (\mu + p)e^{2U} \sinh^2 v + pe^{2U}
\end{aligned} \tag{3.15h}$$

$$\begin{aligned}
& 2U_{zz} + U_z^2 + 3U_y^2 + 2\nu_y U_y - 2\nu_z U_z + e^{2\nu} (2U_{xx} - 2U_{tt} + U_x^2 - U_t^2) \\
& = pe^{2\nu+2U}
\end{aligned} \tag{3.15i}$$

$$\begin{aligned}
& 2U_{yy} + U_y^2 + 3U_z^2 - 2\nu_y U_y + 2\nu_z U_z + e^{2\nu} (2U_{xx} - 2U_{tt} + U_x^2 - U_t^2) \\
& = pe^{2\nu+2U}
\end{aligned} \tag{3.15j}$$

for the line element (3.4).

3.3 Reduction of the Field Equations

The field equations (3.15) may be transformed to a variety of forms to achieve their integration. An immediate consequence of equations (3.15a) - (3.15d) is the functional form

$$e^{-U} = f(t, x) + h(y, z) \tag{3.16}$$

where f and h are arbitrary functions and $U = U(t, x, y, z)$.

The reduction to a simpler form of the remaining field equations (3.15e)-(3.15j) has been fully dealt with in the paper by Hansraj *et al* (2006)

The resulting system is given by

$$\mu = 3(f_t^2 - f_x^2) + (f + h)(4kf - 2kh + 3\alpha) - 3e^{-2\nu} (h_y^2 + h_z^2) \quad (3.17a)$$

$$p = -3(f_t^2 - f_x^2) + (f + h)(2f_{tt} - 2f_{xx} + 2kh - \alpha) + 3e^{-2\nu} (h_y^2 + h_z^2) \quad (3.17b)$$

$$\tanh^2 v = \frac{2f_{xx} - 2kf - \alpha}{2f_{tt} + 2kf + \alpha} \quad (3.17c)$$

$$f_{tx}^2 = \frac{1}{4} (2f_{xx} - 2kf - \alpha) (2f_{tt} + 2kf + \alpha) \quad (3.17d)$$

$$h_{yz} = \nu_z h_y + \nu_y h_z \quad (3.17e)$$

$$h_{yy} - h_{zz} = 2\nu_y h_y + 2\nu_z h_z. \quad (3.17f)$$

The system (3.17), subject to condition (3.6), viz. $\nu_{yy} + \nu_{zz} = -2ke^{2\nu}$, must be solved in order to generate a conformally related perfect fluid model.

It should be observed that Castejon and Coley (1992) incorrectly inferred that equations (3.17e) and (3.17f) necessitate h to be a constant. Hansraj *et al* (2006) demonstrated by way of a counter-example that this is not necessarily the case. Additionally, they extended the results of Castejon and Coley (1992) to more general classes of solutions and in addition obtained new solutions for the case $h = \text{constant}$ by using the methods of Lie group analysis.

Consequently we elect NOT to consider the case $h = \text{constant}$ only, but instead to allow for the variation in all spatial directions. This represents the major point of departure from the work of Hansraj *et al* (2006). We seek solutions to the conformal Einstein equations with a wider latitude in behaviour.

3.4 Group Invariant Solutions

The field equations (3.17a)–(3.17c) may be taken as definitions for the energy density μ , the pressure p and the velocity vector angle ν . Equation (3.17d) depends only on f . A full analysis thereof can be found in Hansraj *et al* (2006). We therefore now focus our attention on equations (3.17e)–(3.17f) which constitute a master system of equations which are free of f . Solutions of this system may be substituted into (3.17a)–(3.17c) to yield the quantities μ, p, ν .

We implement the method of Lie group analysis in our attempt to obtain these solutions. With the aid of the computer package PROGRAM LIE (Head 1993) we find that the symmetry algebra for (3.17e) - (3.17f) is spanned by the following vector fields:

$$G_1 = \partial_h \tag{3.18a}$$

$$G_2 = \partial_z \tag{3.18b}$$

$$G_3 = \partial_y \tag{3.18c}$$

$$G_4 = y \partial_y + z \partial_z \tag{3.18d}$$

$$G_5 = \partial_\nu \tag{3.18e}$$

$$G_6 = h \partial_h. \tag{3.18f}$$

We now attempt to find group invariant solutions of (3.17e)–(3.17f) using these symmetries. Instead of proceeding in an *ad hoc* manner, we first classify (3.18a) – (3.18f) into an optimal system using the method described in section (2.3.3). We note that the equations (3.17e) –

(3.17f) are also invariant under the following involutions

$$h \rightarrow -h \quad (3.19a)$$

$$y \rightarrow -y \quad (3.19b)$$

$$z \rightarrow -z. \quad (3.19c)$$

Under these transformations we have

$$G_1 \rightarrow -G_1 \quad (3.20a)$$

$$G_2 \rightarrow -G_2 \quad (3.20b)$$

$$G_3 \rightarrow -G_3. \quad (3.20c)$$

Using this method and taking into account (3.20) we only need to consider the following linear combinations when reducing (3.17e)-(3.17f):

$$Z_1 = G_4 + G_6 = y \partial_y + z \partial_z + h \partial_h \quad (3.21a)$$

$$Z_2 = G_4 = y \partial_y + z \partial_z \quad (3.21b)$$

$$Z_3 = G_4 - G_6 = y \partial_y + z \partial_z - h \partial_h \quad (3.21c)$$

$$\begin{aligned} Z_4 &= a_1 G_4 + G_5 + a_2 G_6 \\ &= a_1 y \partial_y + a_1 z \partial_z + \partial_v + a_2 h \partial_h \end{aligned} \quad (3.21d)$$

where a_1, a_2 are constants, $a_1 \neq 0, a_2 \neq 0$.

We now consider, in turn, the reduced form of the ordinary differential equation for each of $Z_1 - Z_4$.

3.4.1 Invariance under Z_1

The associated Lagrange's system for

$$Z_1 = y \partial_y + z \partial_z + h \partial_h$$

is

$$\frac{dy}{y} = \frac{dz}{z} = \frac{dh}{h} = \frac{d\nu}{0}. \quad (3.22)$$

From the above system, we obtain the reduction variables

$$p(u) = \nu \quad (3.23a)$$

$$q(u) = \frac{h}{y} \quad (3.23b)$$

$$u = \frac{z}{y}. \quad (3.23c)$$

In terms of these variables equations (3.17e) - (3.17f) become

$$-uq_{uu} = qp_u - 2up_uq_u \quad (3.24a)$$

$$(u^2 - 1)q_{uu} = 2up_uq + 2u^2p_uq_u + 2p_uq_u \quad (3.24b)$$

with solutions

$$p(u) = \frac{u^2}{16} + \log \left[u^{\frac{3}{8}} (u^2 - 1)^{\frac{1}{4}} \right] + c_2 \quad (3.25a)$$

$$q(u) = \exp \left(\frac{1}{8}u^2 + \frac{1}{4} \log u + c_1 \right) \quad (3.25b)$$

where c_1, c_2 are arbitrary constants of integration.

Using the solutions (3.25) together with the system (3.23) we obtain

$$\nu = \frac{z^2}{16y^2} + \log \left[\left(\frac{z}{y} \right)^{\frac{3}{8}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{4}} \right] + c_2 \quad (3.26a)$$

$$h = z^{\frac{1}{4}} y^{\frac{3}{4}} e^{c_1 + \frac{z^2}{8y^2}}. \quad (3.26b)$$

3.4.2 Invariance under Z_2

For the generator

$$Z_2 = y \partial_y + z \partial_z$$

the associated characteristic system is

$$\frac{dy}{y} = \frac{dz}{z} = \frac{d\nu}{0} = \frac{dh}{0}, \quad (3.27)$$

from which we have the following reduction variables:

$$p(u) = \nu \quad (3.28a)$$

$$q(u) = h \quad (3.28b)$$

$$u = \frac{z}{y}. \quad (3.28c)$$

In terms of the variable u , in the field equations (3.17e) - (3.17f) assume the form

$$uq_{uu} = (2up_u - 1)q_u \quad (3.29a)$$

$$(u^2 - 1)q_{uu} = 2(u^2p_u + p_u - u)q_u. \quad (3.29b)$$

On solving (3.29a) and (3.29b) simultaneously we obtain

$$p(u) = \frac{1}{8}u^2 + \frac{1}{4}\log u + c_1 \quad (3.30a)$$

$$q(u) = \frac{\sqrt{u} \Gamma\left[\frac{1}{4}, -\frac{u^2}{4}\right] + c_2}{\sqrt{2}(-u^2)^{\frac{1}{4}}} + c_2 \quad (3.30b)$$

where c_1, c_2 are arbitrary constants of integration and Γ is defined by

$$\Gamma[s, \theta] = \int_{\theta}^{\infty} t^{s-1} e^{-t} dt.$$

Since $\nu = p(u)$ and $h = q(u)$ with $u = \frac{z}{y}$, we thus have

$$\nu = \frac{z^2}{8y^2} + \log \left(\frac{z}{y} \right)^{\frac{1}{4}} + c_1 \quad (3.31a)$$

$$h = - \frac{\sqrt{\frac{z}{y}} \Gamma \left[\frac{1}{4}, -\frac{z^2}{4y^2} \right]}{\sqrt{2} \left(-\frac{z^2}{y^2} \right)^{\frac{1}{4}}} + c_2 \quad (3.31b)$$

as solutions to (3.17e)–(3.17f).

3.4.3 Invariance under Z_3

$$Z_3 = y \partial_y + z \partial_z - h \partial_h$$

has associated characteristic system of the form

$$\frac{dy}{y} = \frac{dz}{z} = \frac{dh}{-h} = \frac{d\nu}{0} \quad (3.32)$$

with invariants given by

$$p(u) = \nu \quad (3.33a)$$

$$q(u) = hy \quad (3.33b)$$

$$u = \frac{z}{y} \quad (3.33c)$$

in terms of which the field equations (3.17e) - (3.17f) reduce to

$$uq_{uu} = p_u q + 2q_u(p_u - 1) \quad (3.34a)$$

$$(u^2 - 1)q_{uu} = 2q(up_u - 1) + 2q_u(u^2 p_u + p_u - 2u). \quad (3.34b)$$

We solve the system (3.34) simultaneously to obtain

$$q_{uu} = \frac{(2uq + 2(u^2 + 1)q_u)^2 - (4uq_u + 2q)(4uq_u + (u^2 + 1)q)}{(4uq_u + (u^2 + 1)q)(u^2 - 1)} \quad (3.35)$$

which is an ordinary differential equation that is difficult to integrate.

We reduce the order of (3.35) in an attempt to facilitate integration.

Equation (3.35) has just one symmetry, viz

$$G = q \partial_q. \quad (3.36)$$

The first extension of G in (3.36) is given by

$$G^{[1]} = q \partial_q + q_u \partial_{q_u}. \quad (3.37)$$

The associated characteristic system for (3.37) is

$$\frac{dq}{q} = \frac{dq_u}{q_u} = \frac{du}{0}$$

from which we obtain the reduction variables

$$r = u \quad (3.38a)$$

$$s = \frac{q_u}{q}. \quad (3.38b)$$

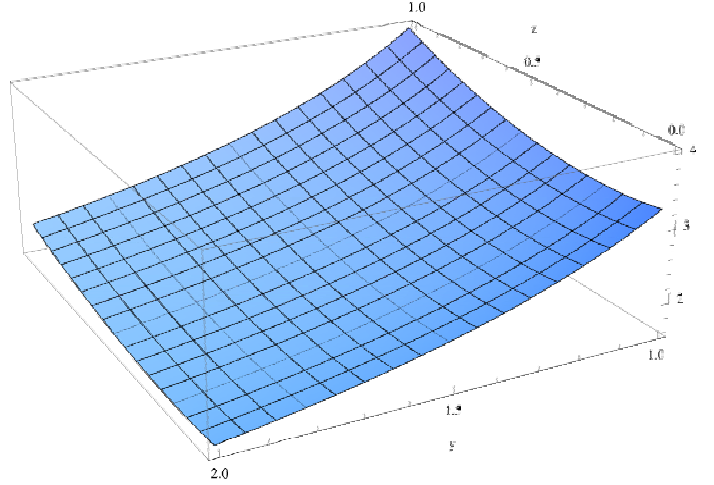


Figure 3.1: Graphical Solution for for Z_3

On substituting (3.38) into (3.35), we obtain

$$s_r = \frac{(2r + 2(r^2 + 1)s)^2 - (4rs + 2)(4rs + (r^2 + 1))}{(r^2 - 1)(4rs + (r^2 + 1))} - s^2. \quad (3.39)$$

Unfortunately we find that this first order equation is also difficult to integrate. Hence we are unable to find an analytic solution via Z_3 and must therefore resort to a numerical solution. A graphical representation of Z_3 , generated by MATHEMATICA 7.0 (Wolfram Inc. 2009), is given in Figure 3.1.

3.4.4 Invariance under Z_4

The associated characteristic system for

$$Z_4 = a_1 y \partial y + a_1 z \partial z + \partial \nu + a_2 h \partial h$$

where a_1, a_2 are constants, is of the form

$$\frac{dy}{a_1 y} = \frac{dz}{a_1 z} = \frac{d\nu}{1} = \frac{dh}{a_2 h}, \quad (3.40)$$

for which we obtain the following reduction variables:

$$p(u) = h y^{-a_2/a_1} \quad (3.41a)$$

$$q(u) = a_1 \nu - \log y \quad (3.41b)$$

$$u = \frac{z}{y}. \quad (3.41c)$$

Under the transformation (3.41) the field equations (3.17e) - (3.17f) reduce to

$$u p_{uu} = \left(\frac{a_2 - a_1 - 1}{a_1} \right) p_u - \frac{a_2}{a_1^2} p q_u \quad (3.42a)$$

$$+ \frac{2}{a_1} u p_u q_u$$

$$\begin{aligned} (u^2 - 1) p_{uu} &= 2 \left(\frac{a_2 - a_1 - 1}{a_1} \right) u p_u - \frac{2a_2}{a_1^2} u p q_u \\ &+ \frac{2}{a_1} (u^2 + 1) p_u q_u + \left(\frac{2a_2 - a_2^2 + a_1 a_2}{a_1^2} \right) p. \end{aligned} \quad (3.42b)$$

Solving (3.42a) - (3.42b) simultaneously yields the equation

$$\begin{aligned} &\left(\frac{a_2 - a_1 - 1}{a_1} \right) (u^2 + 1) p_u - \frac{a_2}{a_1^2} (u^2 + 1) p q_u \\ &+ \frac{4}{a_1} u p_u q_u + \left(\frac{2a_2 - a_2^2 + a_1 a_2}{a_1^2} \right) u p = 0 \end{aligned} \quad (3.43)$$

from which we obtain

$$q_u = \frac{\left(\frac{a_2 - a_1 - 1}{a_1}\right)(u^2 + 1)p_u + \left(\frac{2a_2 - a_2^2 + a_1a_2}{a_1^2}\right)up}{\frac{a_2}{a_1^2}(u^2 + 1)p - \frac{4}{a_1}up_u}. \quad (3.44)$$

Substituting (3.44) into (3.42a) yields

$$\begin{aligned} up_{uu} = & \left(\frac{a_2 - a_1 - 1}{a_1}\right)p_u \\ & + \left(\frac{2}{a_1}up_u - \frac{a_2}{a_1^2}p\right) \left[\frac{\left(\frac{a_2 - a_1 - 1}{a_1}\right)(u^2 + 1)p_u + \left(\frac{2a_2 - a_2^2 + a_1a_2}{a_1^2}\right)up}{\frac{a_2}{a_1^2}(u^2 + 1)p - \frac{4}{a_1}up_u} \right]. \end{aligned} \quad (3.45)$$

Once again we find that we have in (3.45) an ordinary differential equation that is highly non-linear and hence difficult to integrate. In addition, an attempt at reducing (3.45) leads nowhere.

Having explored all our options, we again finally resort to a numerical solution. With the aid of MATHEMATICA 7.0 (Wolfram Inc. 2009), we generate the plot for Z_4 , illustrated in Figure 3.2.

3.5 Summary

We have managed, for the first time, to systematically find solutions to the equations (3.17e) – (3.17f). Where possible exact solutions were obtained. In two cases we had to resort to numerical solutions. These results, taken together with those of Hansraj *et al* (2006) constitute a complete group analysis of the Einstein field equations (3.17).

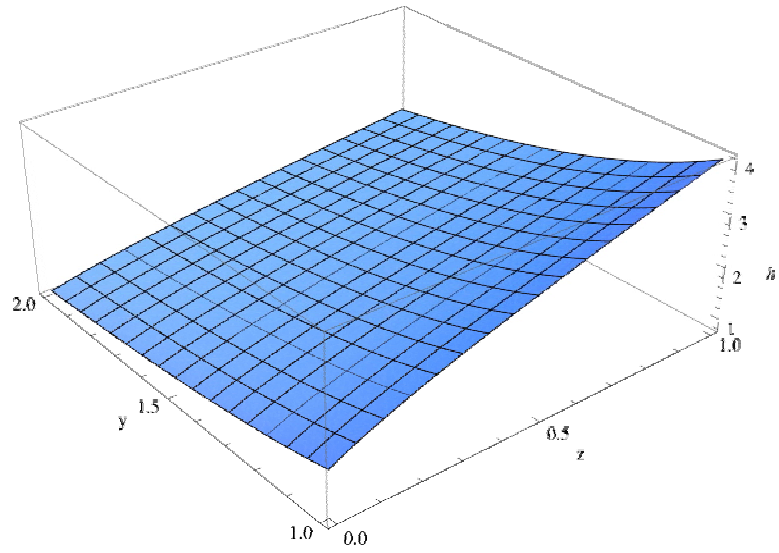


Figure 3.2: Graphical Solution for Z_4 with $a_1 = -1$ and $a_2 = 1$

In the next chapter we present the complete solutions and embark on a physical analysis of some of the solutions.

Chapter 4

Complete Solutions and Physical Analysis

4.1 Introduction

Arising out of the success of the Lie group analysis method of Chapter 3 we are able to provide the complete solution for the Einstein field equations for a conformally related perfect fluid. We reiterate that, as a result of the complicated nature of the system of partial differential equations, very limited success has been achieved by other researchers in this area. The work of Castejon and Coley (1992) was based on the assumption that the conformal factor was firstly of the form $U(t, x)$ and secondly that the solutions were then separable. Our approach, in contrast, utilising the Lie group analysis method, has yielded a rich class of new solutions. These solutions continue to be of Petrov type D as the Petrov type is preserved under conformal mappings.

Our work has considered the full blown possibility that the conformal factor has the form $e^{2U} = f(t, x) + h(y, z)$. The functional forms $h(y, z)$ which we have established, and $f(t, x)$ which were obtained by Hansraj *et al* (2006) must be substituted into expressions (3.17a) and (3.17b) to establish the dynamical quantities, energy density (μ) and pressure (p). Thereafter

it needs to be verified that the pressure and density profiles satisfy certain conditions for physical acceptability which we enumerate in section 4.3. Finally we study a particular solution against the conditions referred to in section 4.3. A rigorous analytical treatment is prohibitive in general, in view of the complexity of the resultant expressions. Furthermore, for graphical purposes, it is necessary to consider a simplified situation such as foliations of the distribution in terms of the temporal and one space variable.

4.2 Complete Solutions

4.2.1 Solutions for Z_1

It was established in Chapter 3 that the function h had the form

$$h(y, z) = z^{\frac{1}{4}} y^{\frac{3}{4}} e^{c_1 + \frac{z^2}{8y^2}}$$

while the function ν can be written as

$$e^{2\nu} = e^{2c_2 + \frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}}$$

where c_1, c_2 are arbitrary constants of integration in (3.25).

In accordance with these solutions, the metric (3.1) is expressed as

$$ds^2 = -dt^2 + dx^2 + c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}} (dy^2 + dz^2)$$

and the conformally related metric (3.4) for a perfect fluid is given by

$$ds^2 = \frac{1}{\left(f + e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}} z^{\frac{1}{4}} \right)^2} \left(-dt^2 + dx^2 + c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}} (dy^2 + dz^2) \right)$$

where the function f may be chosen from the analysis of Hansraj *et al* (2006), since f and h are governed by independent equations.

The energy density for this solution is given by

$$\begin{aligned}\mu = & -3 \left(c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}} \right)^{-1} \left[\left(\frac{e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}}}{4z^{\frac{3}{4}}} + \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{5}{4}}}{4y^{\frac{5}{4}}} \right)^2 + \left(\frac{3e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{1}{4}}}{4y^{\frac{1}{4}}} - \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{9}{4}}}{4y^{\frac{9}{4}}} \right)^2 \right] \\ & + \left(f + e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}} z^{\frac{1}{4}} \right) \left(4fk - 2e^{c_1 + \frac{z^2}{8y^2}} k y^{\frac{3}{4}} z^{\frac{1}{4}} + 3\alpha \right) + 3(f_t^2 - f_x^2)\end{aligned}$$

and the pressure has the form

$$\begin{aligned}p = & 3 \left(c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}} \right)^{-1} \left[\left(\frac{e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}}}{4z^{\frac{3}{4}}} + \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{5}{4}}}{4y^{\frac{5}{4}}} \right)^2 + \left(\frac{3e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{1}{4}}}{4y^{\frac{1}{4}}} - \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{9}{4}}}{4y^{\frac{9}{4}}} \right)^2 \right] \\ & + \left(f + e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}} z^{\frac{1}{4}} \right) \left(2e^{c_1 + \frac{z^2}{8y^2}} k y^{\frac{3}{4}} z^{\frac{1}{4}} - \alpha + 2f_{tt} - 2f_{xx} \right) - 3(f_t^2 - f_x^2).\end{aligned}$$

4.2.2 Solutions for Z_2

For this generator we obtained the solution for h as follows

$$h(y, z) = c_2 - \frac{\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{\sqrt{2} \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}$$

and the solution for ν can be written as

$$e^{2\nu} = e^{2c_1} e^{\frac{z^2}{4y^2}} \left(\frac{z}{y} \right)^{\frac{1}{2}}.$$

As before c_1 and c_2 are arbitrary constants of integration.

The metric (3.1) is accordingly given by

$$ds^2 = -dt^2 + dx^2 + c_3 e^{\frac{z^2}{4y^2}} \left(\frac{z}{y} \right)^{\frac{1}{2}} (dy^2 + dz^2)$$

while the conformally related metric (3.4) for a perfect fluid is of the form

$$ds^2 = \frac{1}{\left(f + c_2 - \frac{\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{\sqrt{2}\left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}}\right)^2} \left(-dt^2 + dx^2 + c_3 e^{\frac{z^2}{4y^2}} \left(\frac{z}{y}\right)^{\frac{1}{2}} (dy^2 + dz^2)\right).$$

The energy density, expressed in terms of the solutions obtained via Z_2 , is written as

$$\begin{aligned} \mu = & \left(f + c_2 - \frac{\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{\sqrt{2}\left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right) \left(4kf + 3\alpha - 2k \left(c_2 - \frac{\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{\sqrt{2}\left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right)\right) \\ & - 3 \left(\frac{1}{c_3} e^{-\frac{z^2}{4y^2}} \left(\frac{y}{z}\right)^{\frac{1}{2}}\right) \left(\left(-\frac{z\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^2 \left(-\frac{z^2}{y^2}\right)^{\frac{5}{4}}} - \frac{\Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y\sqrt{\frac{z}{y}} \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}} + \frac{z\sqrt{\frac{z}{y}} \Gamma^{(0,1)}\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^2 \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right)^2 \right. \\ & \left. + \left(\frac{z^2\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^3 \left(-\frac{z^2}{y^2}\right)^{\frac{5}{4}}} + \frac{z\Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^2\sqrt{\frac{z}{y}} \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}} - \frac{z^2\sqrt{\frac{z}{y}} \Gamma^{(0,1)}\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^3 \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right)^2\right) + 3(f_t^2 - f_x^2) \end{aligned}$$

while the pressure is given by

$$\begin{aligned} p = & \left(f + c_2 - \frac{\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{\sqrt{2}\left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right) \left(-\alpha + 2k \left(c_2 - \frac{\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{\sqrt{2}\left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right) + 2f_{tt} - 2f_{xx}\right) \\ & + 3 \left(\frac{1}{c_3} e^{-\frac{z^2}{4y^2}} \left(\frac{y}{z}\right)^{\frac{1}{2}}\right) \left(\left(-\frac{z\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^2 \left(-\frac{z^2}{y^2}\right)^{\frac{5}{4}}} - \frac{\Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y\sqrt{\frac{z}{y}} \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}} + \frac{z\sqrt{\frac{z}{y}} \Gamma^{(0,1)}\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^2 \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right)^2 \right. \\ & \left. + \left(\frac{z^2\sqrt{\frac{z}{y}} \Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^3 \left(-\frac{z^2}{y^2}\right)^{\frac{5}{4}}} + \frac{z\Gamma\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^2\sqrt{\frac{z}{y}} \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}} - \frac{z^2\sqrt{\frac{z}{y}} \Gamma^{(0,1)}\left[\frac{1}{4}, -\frac{z^2}{4y^2}\right]}{2\sqrt{2}y^3 \left(-\frac{z^2}{y^2}\right)^{\frac{1}{4}}}\right)^2\right) - 3(f_t^2 - f_x^2) \end{aligned}$$

where, again, we may elect to use the function f from the treatment of Hansraj *et al* (2006).

4.3 Conditions for Physical Admissability

Generally for physically acceptable relativistic models of perfect fluids, it is demanded that the following conditions are satisfied:

4.3.1 the pressure and energy density must be positive and finite everywhere in the interior of the star, including the origin and boundary, i.e.

$$\begin{aligned} 0 &\leq p < \infty \\ 0 &< \mu < \infty. \end{aligned}$$

4.3.2 the pressure and energy density are usually decreasing monotonic functions of the coordinate r . A pressure-free hypersurface should exist to define the boundary of the matter distribution for applications to stellar objects. The absence of a vanishing pressure hypersurface is taken to be indicative of a cosmological scenario.

4.3.3 continuity of gravitational potential across the boundary hypersurface: the metric has to be matched with an appropriate exterior vacuum solution.

4.3.4 the speed of sound should be less than the speed of light everywhere in the interior, that is the principle of causality must be satisfied

$$0 < \frac{dp}{d\mu} < 1.$$

4.3.5 The metric potentials must be everywhere positive.

A detailed motivation of these requirements is found in Lake (1998) and Knutsen (1992). It must be observed that the most elementary requirement is the positivity of pressure and energy density. The remaining conditions are subject to dispute by researchers. For example, the well established causality principle (4.3.4) has been discussed by Macorra and Vucetich (2004) in the context of scalar fields as candidates for dark energy models. It is argued here

that the speed of sound remains less than the speed of light irrespective of the ratios $\frac{p}{\mu}$ or $\frac{dp}{d\mu}$. In addition, condition (4.3.2) requiring monotonic decrease of p and μ is also viewed as unduly restrictive since the thermodynamical process within a physical distribution is practically impossible to determine (Rhodes and Ruffini 1974).

4.4 Physical Properties of a Particular Solution

In 4.2.1 we expressed the conformally related metric (3.4), energy density (μ) and pressure (p) in terms of f as well as h and ν obtained via Z_1 . We now present a complete solution of the metric (3.4) and the dynamical quantities using our results of Z_1 and the following solution for f from the work of Hansraj *et al* (2006)

$$f(x, t) = A \sinh \sqrt{k(t^2 - x^2)} + B \cosh \sqrt{k(t^2 - x^2)}.$$

The conformally related metric (3.4) may now be given by

$$ds^2 = \frac{-dt^2 + dx^2 + c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y}\right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1\right)^{\frac{1}{2}} (dy^2 + dz^2)}{\left(e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}} z^{\frac{1}{4}} + B \cosh \left[\sqrt{k(t^2 - x^2)}\right] + A \sinh \left[\sqrt{k(t^2 - x^2)}\right]\right)^2}$$

and the energy density takes the form

$$\begin{aligned}
\mu = & -3 \left(c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}} \right)^{-1} \left(\left(\frac{e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}}}{4z^{\frac{3}{4}}} + \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{5}{4}}}{4y^{\frac{5}{4}}} \right)^2 + \left(\frac{3e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{1}{4}}}{4y^{\frac{1}{4}}} - \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{9}{4}}}{4y^{\frac{9}{4}}} \right)^2 \right) \\
& + \left(e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}} z^{\frac{1}{4}} + B \cosh \left[\sqrt{k(t^2 - x^2)} \right] + A \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right) \\
& \left(-2e^{c_1 + \frac{z^2}{8y^2}} k y^{\frac{3}{4}} z^{\frac{1}{4}} - 3\alpha + 4k \left(B \cosh \left[\sqrt{k(t^2 - x^2)} \right] + A \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right) \right) \\
& + 3 \left(\left(\frac{A k t \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} + \frac{B k t \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right)^2 \right. \\
& \left. - \left(-\frac{A k x \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} - \frac{B k x \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right)^2 \right) .
\end{aligned}$$

while the pressure is given by

$$\begin{aligned}
p = & 3 \left(c_1 e^{\frac{z^2}{8y^2}} \left(\frac{z}{y} \right)^{\frac{3}{4}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{2}} \right)^{-1} \left(\left(\frac{e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}}}{4z^{\frac{3}{4}}} + \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{5}{4}}}{4y^{\frac{5}{4}}} \right)^2 + \left(\frac{3e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{1}{4}}}{4y^{\frac{1}{4}}} - \frac{e^{c_1 + \frac{z^2}{8y^2}} z^{\frac{9}{4}}}{4y^{\frac{9}{4}}} \right)^2 \right) \\
& - \frac{k(3t^2 + x^2)}{t^2 - x^2} \left(A \cosh \left[\sqrt{k(t^2 - x^2)} \right] + B \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right)^2 \\
& + \left(e^{c_1 + \frac{z^2}{8y^2}} y^{\frac{3}{4}} z^{\frac{1}{4}} + B \cosh \left[\sqrt{k(t^2 - x^2)} \right] + A \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right) \left(-\alpha + 2 \left(e^{c_1 + \frac{z^2}{8y^2}} k y^{\frac{3}{4}} z^{\frac{1}{4}} \right. \right. \\
& + A \left(-\frac{k^2 t^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{\frac{3}{2}}} + \frac{k \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} + \frac{k t^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} \right) \\
& - A \left(-\frac{k^2 x^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{\frac{3}{2}}} - \frac{k \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} + \frac{k x^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} \right) \\
& - B \left(\frac{k x^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} - \frac{k^2 x^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{\frac{3}{2}}} - \frac{k \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right) \\
& \left. \left. + B \left(-\frac{k t^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} - \frac{k^2 t^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{\frac{3}{2}}} + \frac{k \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right) \right) \right).
\end{aligned}$$

To investigate the plausibility of our solutions analytically is quite intricate given the complicated forms for the metric and the dynamical variables. Moreover, we are dealing with a four-dimensional solution and so even graphical plots are impossible.

In order to obtain an indication of the model's feasibility to represent a realistic distribution, we elect to make a graphical study of a particular solution. We consider (t, x) slices from the result obtained in the analysis of Hansraj *et al* (2006) where

$$f(t, x) = A \sinh \sqrt{k(t^2 - x^2)} + B \cosh \sqrt{k(t^2 - x^2)}$$

and $h = 0$. It should be observed that the aforementioned authors neglected to analyse the physical properties of their solutions.

The energy density and pressure are given by

$$\begin{aligned}\mu &= 4k \left(B \cosh \left[\sqrt{k(t^2 - x^2)} \right] + A \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right)^2 \\ &\quad - 3 \left(\left(\frac{Akt \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} + \frac{Bkt \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right)^2 \right. \\ &\quad \left. - \left(\frac{Akx \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} - \frac{Bkx \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right)^2 \right)\end{aligned}$$

and

$$\begin{aligned}p &= 2 \left(B \cosh \left[\sqrt{k(t^2 - x^2)} \right] + A \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right) \\ &\quad \left(A \left(-\frac{k^2 t^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{3/2}} + \frac{k \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} + \frac{kt^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} \right) \right. \\ &\quad - A \left(-\frac{k^2 x^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{3/2}} - \frac{k \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} + \frac{kx^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} \right) \\ &\quad - B \left(\frac{kx^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} - \frac{k^2 x^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{3/2}} - \frac{k \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right) \\ &\quad \left. + B \left(\frac{kt^2 \cosh \left[\sqrt{k(t^2 - x^2)} \right]}{t^2 - x^2} - \frac{k^2 t^2 \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{(k(t^2 - x^2))^{3/2}} + \frac{k \sinh \left[\sqrt{k(t^2 - x^2)} \right]}{\sqrt{k(t^2 - x^2)}} \right) \right) \\ &\quad - \frac{k(3t^2 + x^2)}{t^2 - x^2} \left(A \cosh \left[\sqrt{k(t^2 - x^2)} \right] + B \sinh \left[\sqrt{k(t^2 - x^2)} \right] \right)^2.\end{aligned}$$

We need to select appropriate values for the parameters A, B, k to finalise the model. It

should be observed at this stage that, in many situations, it is possible to obtain bounds for the constants by examining the central conditions $p(t, 0, 0, 0)$ and $\mu(t, 0, 0, 0)$. We then require that all the physical conditions are satisfied at the centre. This is not fruitful in the present context. Furthermore, the approach in fixing integration constants in most problems in general relativity involves matching of the perfect fluid solution with its corresponding vacuum solution. For example, if we were devising a model of a static neutral sphere, then integration constants can be fixed by demanding continuity of the potentials at the boundary hypersurface. In addition, for such solutions, the pressure vanishes at the boundary and this determines the size of the sphere. In the case of radiating spheres, this pressure-free interface does not exist and this condition is modified to include the effects of radiation and the junction conditions must be established via other means. The difficulty in our analysis is that we have no known vacuum solution. Consequently our selection of the constants is completely random. We are fortunate that available computing technology is able to quickly check the efficacy of our choices for the integration constants (See Fig. 4.1 and Fig. 4.2).

These plots, generated via Mathematica (Wolfram Inc 2009), demonstrate many pleasing features. For example, it is evident that in the region chosen, the pressure and energy density are both positive. These are the most basic requirements for models to serve as candidates for realistic celestial phenomena.

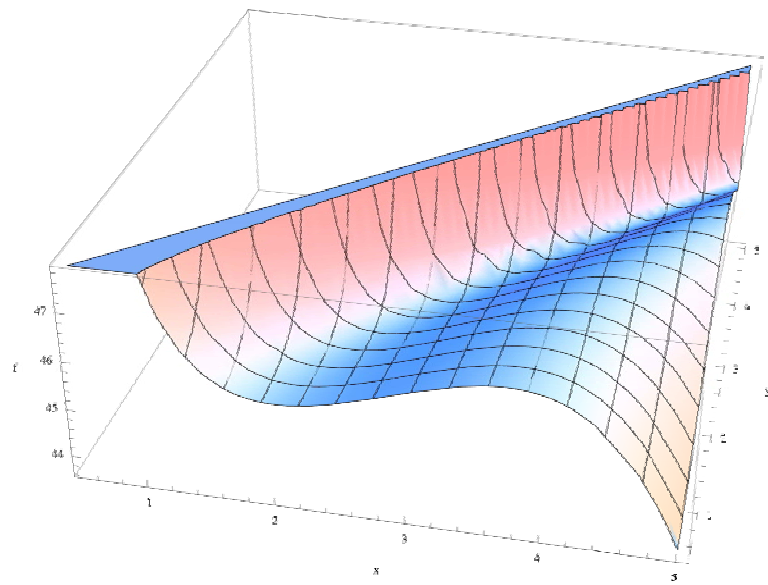


Figure 4.1: Plot of Pressure versus t and x with $A = 1$, $B = 10$, $k = 0.1$

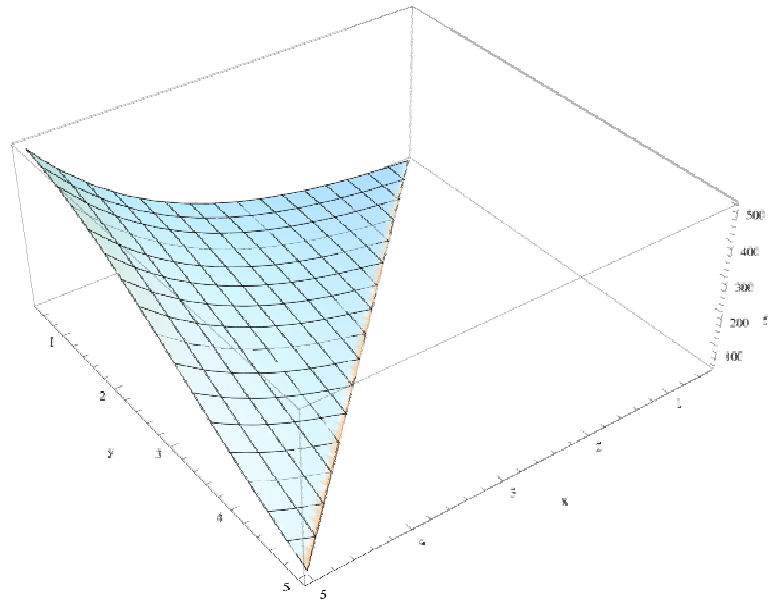


Figure 4.2: Plot of Energy Density versus t and x with $A = 1$, $B = 10$, $k = 0.1$

Chapter 5

Conclusion

Our main objective in this thesis was to obtain new exact solutions of the Einstein Field equations for a perfect fluid metric conformal to a spacetime of Petrov type D, and to generate a new class of complete solutions for $U(t, x, y, z)$. To this end, we employed the method of Lie group analysis.

In chapter 2 we discussed the mathematical concepts that were necessary for later chapters of the thesis. In particular, the technique of Lie group analysis was outlined. Using the example of the Robinson-Trautman equation, we illustrated Lie's method of reducing a second order nonlinear partial differential equation to an ordinary differential equation, the solutions of which would provide group invariant solutions to the Robinson-Trautman equation. In addition, we also discussed and illustrated the systematic method of optimal subgroups.

In chapter 3 we considered a perfect fluid metric conformally related to a Petrov type D spacetime. Using the relevant aspects of differential geometry discussed in chapter 2, we derived the appropriate Einstein field equations. We then considered the reduction to a simpler form of these field equations. In giving particular attention to the equations

$$h_{yz} = \nu_z h_y + \nu_y h_z$$

and

$$h_{yy} - h_{zz} = 2\nu_y h_y + 2\nu_z h_z$$

we claimed that new exact solutions for these equations could be found with no restrictions having to be placed on h . Our claim was in contradiction to Castejon and Coley (1992) who proposed that the system of field equations could be solved only on condition that h is a constant. Using Lie's method we obtained four generators $Z_1 - Z_4$. We successfully found exact solutions for two of these generators. For Z_1 our solution was

$$h = z^{\frac{1}{4}} y^{\frac{3}{4}} e^{c_1 + \frac{z^2}{8y^2}}$$

and

$$\nu = \frac{z^2}{16y^2} + \log \left[\left(\frac{z}{y} \right)^{\frac{3}{8}} \left(\frac{z^2}{y^2} - 1 \right)^{\frac{1}{4}} \right] + c_2$$

and for Z_2 we obtained the solution

$$h = \frac{-\sqrt{\frac{z}{y}} \Gamma \left[\frac{1}{4}, -\frac{z^2}{4y^2} \right]}{\sqrt{2} \left(-\frac{z^2}{y^2} \right)^{\frac{1}{4}}} + c_2$$

and

$$\nu = \frac{z^2}{8y^2} + \frac{1}{4} \log \left(\frac{z}{y} \right) + c_1.$$

The remaining generators, Z_3 and Z_4 , gave rise to ordinary differential equations that were difficult to integrate. We resorted to numerical solutions for these cases.

In chapter 4 we presented complete solutions and analysed our solutions for physical plausibility. We listed the metric (3.1), its conformally related counterpart (3.4), the energy density (μ) and the pressure (p) in terms of our solutions for Z_1 as well as for Z_2 . Graphical plots in the remaining cases were exhibited using numerical methods. We then presented a particular exact solution for (3.4), μ and p in terms of t, x, y and z using a solution for f from the work

of Hansraj *et al* (2006) and our solution for Z_1 . In addition we generated plots of (t, x) slices of the solution.

This work adds to the growing body of applications of group theory to equations arising in general relativity. As a result, complicated problems now have a realistic chance of being resolved. We intend utilising group analysis to investigate extensions of this model as well as to other Petrov type seed solutions.

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