Locally Conformal Almost Kenmotsu Manifolds

Submitted in fulfillment of the academic requirements for the degree of Masters in Mathematics in the School of Mathematics, Statistics and Computer Sciences, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban, South Africa

By

Snethemba Hlobisile Maduna [1]
(213508761)

Supervisor: **Prof. Fortuné Massamba**

June 2019

[1]The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Declaration 1 - Plagiarism

I, Snethemba Hlobisile Maduna, declare that

1. The research reported in this dissertation, except where otherwise indicated, is my research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged or referenced as being sourced from other persons.

4. This thesis does not contain any other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

   (a) Their words have been rewritten but the general information attributed to them has been referenced;

   (b) Where their exact words have been used, then they oath to have been referenced.

5. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed ....

I, Prof. Fortuné Massamba, certify that, as a supervisor of the candidate's Masters project, the above statements are true to my knowledge.
Signed

....

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the authors and are not necessarily to be attributed to the NRF.

Dedication

First and most to Our Beloved Father who art in Heaven, Creator of Heaven, Earth and all that dwells in it

and

To my loving parents, Mrs. Nonkululeko and Mr. William Maduna
(May Our Lord bless them)

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Preface

The work described in this dissertation was carried out in the School of Mathematics, Statistics, and Computer Sciences, University of KwaZulu-Natal, Durban,South Africa from February 2018 to June 2019, under the supervision of Prof. Fortuné Massamba .This study represents original work by the Student and has not otherwise been submitted in any form for any degree or diploma to any other tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

Miss Snethemba Hlobisile Maduna

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Acknowledgments

First and foremost, I am deeply grateful to my supervisor, Prof. Fortuné Massamba, for his continual guidance and advice. Without his help and direction, this work would not have been possibly.

I would also like to thank my graduate class of University of KwaZulu-Natal, mostly I extend my gratitude to Blessing Khuzwayo and Njabulo Mbanjwa whom have been very supportive. The support and conversation about mathematics, research and life have been mostly beneficial.

I am grateful to my loving parents and to my family at large. Without them my success would feel pointless. I thank Our God for this family, they are indeed a precious gift. Always supporting and inspired me throughout my journey. I would also like to acknowledge the University of KwaZulu-Natal, where this research was prepared, and the financial support from National Research Foundation. I am also grateful to the referees for valuable comments and suggestions. Now last but not the least, to everyone I have mistakenly forgotten but oath to have been mentioned, thank you.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Abstract

In this dissertation, we introduce and investigate locally conformal almost Kenmotsu structures. We show that, under some conditions, these structures contain the classes of $C(\lambda)$-structures. We prove that its contact distributions admit foliations whose leaves are locally conformal almost Kählerian with mean curvature vector proportional to the characteristic vector field. A locally conformal Kenmotsu manifold is a locally warped product space. We also prove that integral manifolds immersed in the CR-submanifolds of the locally conformal Kenmotsu which are locally conformal Kählerian which, under some conditions cannot be minimal.

Keywords : Kenmotsu manifold, locally conformal almost Kenmotsu manifold, Foliation.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Contents

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Chapter One

Introduction

The first study on locally conformal Kähler manifolds was done by Libermann in 1955 [24]. Vaisman in [40] introduced a geometric condition for locally conformal Kähler manifolds to be Kähler and in 1982 Tricerri [37] gave different examples. In 2001, Banaru [3] succeeded in classifying the sixteen classes of almost Hermitian manifolds by using the so-called Kirichenko tensors. Abood studied the properties of these tensors in [1]. The class of locally conformal Kähler manifolds is one of the sixteen classes of almost Hermitian manifolds (see, for example, [18]).

In 1972, K. Kenmotsu [21] studied a class of almost contact Riemannian manifold and introduced a new class of almost contact Riemannian manifold which are called Kenmotsu manifold. Kenmotsu manifolds are normal almost contact Riemannian manifolds. Kenmotsu [21] investigated fundamental properties on local structure of such manifolds . A Kenmotsu manifold is always an almost Kenmotsu manifold, but the converse is not necessarily true.

In general, the contact and almost contact structures are two of the most interesting examples of differential geometric structures. Their theory is a natural generalization of so-called contact geometry, which has important applications in quantum and classical mechanics. Their study as differential geometry structures dates form the work of Gererdo [36] and Sasaki [33]. Almost contact metric structures are an odd-dimensional analogue of almost Hermitian structures and there exist many important connections between these two classes. Sasakian manifolds are manifolds of positive or zero curvature. But Kenmotsu manifolds are manifolds of negative curvature. Tanno in [34] classified the connected almost-contact metric manifolds whose automorphism groups has the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. Then there are three classes:

(i) Homogeneous normal contact Riemannian manifolds with $c > 0$,

(ii) Global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$,

(iii) Warped product space $L \times_f F$ if $c < 0$.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

In [21], the author proved that a locally kenmotsu manifold is a warped product $L \times_f F$ of an interval L and Kähler manifold N with mapping function $f(t) = ce^t$, where c is a non-zero constant and that a Kenmotsu manifold of constant ϕ-sectional curvature is a space of constant curvature -1 and so it is locally hyperbolic space. He also proved that if Kenmotsu manifold satisfies the condition $R(X,Y).R = 0$, then it is of constant negative curvature -1. Kenmotsu manifolds were studied by many authors such as G. Pitis [13], De and Pathak [9], Jun, De and Pathak [10], Binh, Tamassy, De and Tarafdar [8], Bagewadi and collaborators [5], [6], [7], Ozgur [11],[14] and many others. Since then, up to our knowledge, a systematic study of locally conformal almost Kenmotsu manifolds has not been undertaken yet.

The aim of this dissertation is to study, thus, providing some technical apparatus needed for further investigations. Therefore, we consider the class of almost contact metric manifolds called locally conformal almost Kenmotsu manifolds.

We asked ourself "can a locally conformal almost Kenmotsu manifold admit a 1-form ω that is proportional to η". We shall try to prove why this is true; where is the connection?

This dissertation covers foundations of almost contact geometry in modern language. Throughout the dissertation, we provide theorems. Almost contact metric structure is given by a pair (η, Φ), where η is a 1-form, Φ is a 2-form and $\eta \wedge \Phi$ is a volume element. It is well known that then there exists a unique vector field ξ, called the characteristic (Reeb) vector field such that $\eta(\xi) = 1$.

The dissertation is organized as follows: In Chapter 2, we recall some preliminary definitions on almost Kenmotsu, Kenmotsu and locally conformal (l.c.) almost Kenmotsu structures. We characterize l.c. almost Kenmotsu manifolds. We prove that, under a certain condition, the class of locally conformal almost Kenmotsu structures belong to the class of β-Kenmotsu structures. Furthermore, we give an example of a three-dimensional conformal Kenmotsu manifold that is not Kenmotsu. We also prove that the contact distribution is always integrable and admits foliations whose leaves are almost Kähler with mean curvature vector supported by the line bundle spanned by the characteristic vector field. Chapter 3 gives some preliminary lemmas on submanifolds of l.c. almost Kenmotsu manifolds. By adapting the concept of contact CR-submanifolds given in [41] and under some conditions, there exist foliations whose leaves are l.c. Kähler manifolds which cannot be minimal. We finally end the thesis by a concluding remark and some perspectives.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Chapter Two

Locally Conformal Almost Kenmostu Manifolds

In this chapter, we recall some general definitions and basic properties of contact metric structures and (almost) Kenmotsu manifolds with particular attention to locally conforml almost Kenmotsu manifolds. For more information and details, we recommend the reference [30]. We assume (unless otherwise stated) that all manifolds in this dissertation are smooth and paracompact.

2.1 Almost Kenmotsu and Kenmotsu structures

Let M be a $(2n+1)$-dimensional manifold endowed with an almost contact structure (ϕ, ξ, η), i.e. ϕ is a tensor field of type $(1,1)$, ξ is a vector field, and η is a 1-form satisfying

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi,\ \eta(\xi) = 1,\ \eta \circ \phi = 0 \text{ and } \phi\xi = 0. \tag{2.1}$$

Then (ϕ, ξ, η, g) is called an almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure on M and g is a Riemannian metric on M such that, for any vector field X, Y on M [10]:

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\,\eta(Y). \tag{2.2}$$

for any vector field X on M,

$$\eta(X) = g(X, \xi). \tag{2.3}$$

Let consider the manifold $M \times \mathbb{R}$. We denote a vector field on $M \times \mathbb{R}$ by $(X, f\frac{d}{dt})$, where X is tangent to M, t the coordinate on $\mathbb{R}$, and f a C^∞ function on $M \times \mathbb{R}$. Define an almost complex structure J on $M \times \mathbb{R}$ by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi X - f\xi, \eta(X)\frac{d}{dt}\right);$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

where $J^2 = \mathbb{I}$. If now J is integrable, we say that the almost contact structure (ϕ, ξ, η) is *normal* (Sasaki and Hatakeyama [32]). Since the vanishing of the Nijenhuis torsion of J is a necessary and sufficient condition for integrability, we seek to express the condition of normality in terms of the Nijenhuis torsion of ϕ. Since $[J, J]$ is a tensor field of type $(1, 2)$, it suffices to compute $[J, J]((X, 0), (Y, 0))$ and $[J, J]((X, 0), (0, \frac{d}{dt}))$ for vector fields X and Y on M:

$$\begin{aligned}
[J,J]((X,0),(Y,0)) &= ([X,Y],0) + \left[\left(\phi X, \eta(X)\frac{d}{dt}\right), \left(\phi Y, \eta(Y)\frac{d}{dt}\right)\right] \\
&- J\left[\left(\phi X, \eta(X)\frac{d}{dt}\right), (Y,0)\right] - J\left[(X,0), (\phi Y, \eta(Y)\frac{d}{dt})\right] \\
&= \left(\phi^2[X,Y] - \eta([X,Y])\xi, 0\right) + \left([\phi X, \phi Y], (\phi X\eta(Y) - \phi Y\eta(X))\frac{d}{dt}\right) \\
&- \left(\phi[\phi X, Y] + (Y\eta(X)\xi, \eta([\phi X, Y])\frac{d}{dt}\right) \\
&- \left(\phi[X, \phi Y] + (X\eta(Y)\xi, \eta([X, \phi Y])\frac{d}{dt}\right) \\
&= \left([\phi,\phi](X,Y) + 2d\eta(X,Y)\xi, ((\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X))\frac{d}{dt}\right),
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
[J,J]((X,0),(0,\frac{d}{dt})) &= \left[\left(\phi X, \eta(X)\frac{d}{dt}\right), (-\xi, 0)\right] - J\left[\left(\phi X, \eta(X)\frac{d}{dt}\right), \left(0, \frac{d}{dt}\right)\right] \\
&- J\left[(X,0), (-\xi,0)\right] \\
&= \left(-[\phi X, \xi], (\xi\eta(X))\frac{d}{dt}\right) + \left(\phi[X,\xi], \eta([X,\xi])\frac{d}{dt}\right) \\
&= ((\mathcal{L}_\xi\phi)X, (\mathcal{L}_\xi\eta)(X)).
\end{aligned} \tag{2.5}$$

Here we call $\mathcal{L}_X$ the *Lie derivative* with respect to the vector field X.

We are thus led to define four tensors $N^{(1)}$, $N^{(2)}$, $N^{(3)}$, $N^{(4)}$ by

$$N^{(1)}(X,Y) = [\phi,\phi](X,Y) + 2d\eta(X,Y)\xi, \tag{2.6}$$
$$N^{(2)}(X,Y) = (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X), \tag{2.7}$$
$$N^{(3)} = (\mathcal{L}_\xi\phi)X, \tag{2.8}$$
$$N^{(4)} = (\mathcal{L}_\xi\eta)(X). \tag{2.9}$$

The almost contact structure (ϕ, ξ, η) is normal if and only if these four tensors vanish. However, the vanishing of $N^{(1)}$ implies the vanishing of $N^{(2)}$, $N^{(3)}$ and $N^{(4)}$, so that the normality condition is simply

$$[\phi,\phi](X,Y) + 2d\eta(X,Y)\xi = 0, \tag{2.10}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] - \phi[\phi X, Y] - \phi[X, \phi Y] + [\phi X, \phi Y].$$

The fundamental 2-form of M is defined by

$$\Phi(X, Y) = g(X, \phi Y),$$

for any vector fields X and Y on M.

If, moreover,

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.11}$$

where ∇ is the Levi-Civita connection for the Riemannian metric g, we call M a *Kenmotsu manifold.*

Moreover, Kenmotsu proved that such a manifold M^{2n+1} is locally a warped product $(-\epsilon, \epsilon) \times_f N^{2n}$, N^{2n} being a Kähler manifold and $f^2 = ce^{2t}$, c a positive onstant.

More recently in [15], [22] and [27], almost contact metric manifolds such that

$$d\eta = 0 \quad \text{and} \quad d\Phi = 2\eta \wedge \Phi, \tag{2.12}$$

are studied and they are called almost Kenmotsu. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold.

2.2 Locally conformal almost Kenmotsu

Let $(M^{(2m+1)}, \phi, \xi, \eta, g)$ be an almost contact metric manifold. Such a manifold is said to be *locally conformal (l.c.) almost Kenmotsu*, if M has an open covering $\{U_t\}_{t\in I}$ endowed with smooth function $\sigma_t : U_t \longrightarrow \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \quad \xi_t = \exp(\sigma_t)\xi, \quad \eta_t = \exp(-\sigma_t)\eta, \quad g_t = \exp(-2\sigma_t)g, \tag{2.13}$$

is almost Kenmotsu.

This means that if ∇^t is the Levi-Civita connection associated with g_t, then (see [15] for more details and references therein)

$$\begin{aligned} 2g_t((\nabla^t_X \phi_t)Y, Z) &= 2\eta_t(Z)g_t(\phi X, Y) - 2\eta_t(Y)g_t(\phi X, Z) \\ &\quad + g_t(N_t^{(1)}(Y, Z), \phi X), \end{aligned} \tag{2.14}$$

for any vector fields X, Y and Z. Therefore, we have the following.

Theorem 2.2.1. *An almost contact manifold M is l.c. almost Kenmotsu manifold if and only if there exists a closed 1-form ω such that*

$$d\eta = \omega \wedge \eta \ \text{ and } \ d\Phi = 2\{\exp(-\sigma_t)\eta + \omega\} \wedge \Phi, \tag{2.15}$$

with $\omega = d\sigma_t$.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Proof. Since $\eta_t = \exp(-\sigma_t)\eta$ on U_t, we obtain,

$$0 = d\eta_t = -\exp(-\sigma_t)d\sigma_t \wedge \eta + \exp(-\sigma_t)d\eta. \tag{2.16}$$

That is

$$d\eta = d\sigma_t \wedge \eta. \tag{2.17}$$

Likewise,

$$\Phi_t = \exp(-2\sigma_t)\Phi.$$

Then its differential is given by

$$\begin{aligned} d\Phi_t &= -2\exp(-2\sigma_t)d\sigma_t \wedge \Phi + \exp(-2\sigma_t)d\Phi \\ &= \exp(-2\sigma_t)\left\{-2d\sigma_t \wedge \Phi + d\Phi\right\}. \end{aligned} \tag{2.18}$$

On the other hand,

$$\begin{aligned} d\Phi_t &= 2\eta_t \wedge \Phi_t \\ &= 2\exp(-3\sigma_t)\eta \wedge \Phi. \end{aligned} \tag{2.19}$$

From (2.18) and (2.19), we have

$$2\exp(-3\sigma_t)\eta \wedge \Phi = \exp(-2\sigma_t)\left\{-2d\sigma_t \wedge \Phi + d\Phi\right\}. \tag{2.20}$$

That is,

$$2\exp(-\sigma_t)\eta \wedge \Phi = -2d\sigma_t \wedge \Phi + d\Phi, \tag{2.21}$$

which completes the proof with $\omega = d\sigma_t$. □

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g(X,[Y,Z]) \\ &\quad + g(Y,[Z,X]) + g(Z,[X,Y]), \end{aligned} \tag{2.22}$$

which is known as the *Koszul formula.* Let ∇ and ∇^t be Levi-Civita connections associated with the metrics g and g_t, respectively. We set

$$\sigma_X Y = \nabla_X Y - \nabla^t_X Y, \tag{2.23}$$

for any vector fields X and Y on M. Then σ is symmetric, that is,

$$\sigma_X Y = \sigma_Y X.$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Lemma 2.2.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Let ∇ and ∇^t be Levi-Civita connections associated with the metrics g and g_t, respectively. Then, for any vector fields X and Y on M,*

$$\nabla^t_X Y = \nabla_X Y - \omega(X)Y - \omega(Y)X + g(X,Y)B, \tag{2.24}$$

where B is the vector field defined by $\omega(X) = g(X, B)$.

Proof. For any $X, Y, Z \in \Gamma(TM)$,

$$0 = (\nabla^t_X)g_t(Y,Z) = X(g_t(Y,Z)) - g_t(\nabla^t_X Y, Z) - g_t(Y, \nabla^t_X Z). \tag{2.25}$$

Since $g_t = \exp(-2\sigma_t)g$, we have that $g_t(Y,Z) = \exp(-2\sigma_t)g(Y,Z)$. Similarly,

$$\begin{aligned} g_t(\nabla^t_X Y, Z) &= \exp(-2\sigma_t)g(\nabla^t_X Y, Z), \\ g_t(Y, \nabla^t_X Z) &= \exp(-2\sigma_t)g(Y, \nabla^t_X Z), \end{aligned} \tag{2.26}$$

$$\begin{aligned} \text{and} \quad X(g_t(Y,Z)) &= X(\exp(-2\sigma_t)g(Y,Z)) + \exp(-2\sigma_t)X(g(Y,Z)) \\ &= -2\exp(-2\sigma_t)X(\sigma_t)g(Y,Z) + \exp(-2\sigma_t)X(g(Y,Z)) \\ &= \exp(-2\sigma_t)\{-2X(\sigma_t)g(Y,Z) + X(g(Y,Z))\}. \end{aligned} \tag{2.27}$$

Substituting equations (2.26) and (2.27) into (2.25) gives

$$\begin{aligned} 0 &= \exp(-2\sigma_t)\{-2X(\sigma_t)g(Y,Z) + X(g(Y,Z))\} - \exp(-2\sigma_t)g(\nabla^t_X Y, Z) \\ &\quad - \exp(-2\sigma_t)g(Y, \nabla^t_X Z) \\ &= -2X(\sigma_t)g(Y,Z) + X(g(Y,Z)) - g(\nabla^t_X Y, Z) - g(Y, \nabla^t_X Z) \\ &= -2X(\sigma_t)g(Y,Z) + X(g(Y,Z)) - g(\nabla_X Y - \sigma_X Y, Z) - g(Y, \nabla_X Z - \sigma_X Z) \\ &= -2X(\sigma_t)g(Y,Z) + X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) + g(\sigma_X Y, Z) \\ &\quad + g(Y, \sigma_X Z) \\ &= -2X(\sigma_t)g(Y,Z) + (\nabla_X g)(Y,Z) + g(\sigma_X Y, Z) + g(Y, \sigma_X Z) \\ &= -2X(\sigma_t)g(Y,Z) + g(\sigma_X Y, Z) + g(Y, \sigma_X Z) \\ &= -2d\sigma_t(X)g(Y,Z) + g(\sigma_X Y, Z) + g(Y, \sigma_X Z) \\ &= -2\omega(X)g(Y,Z) + g(\sigma_X Y, Z) + g(Y, \sigma_X Z). \end{aligned} \tag{2.28}$$

Rearranging relation (2.28), yields

$$g(\sigma_X Y, Z) + g(Y, \sigma_X Z) = 2\omega(X)g(Y,Z). \tag{2.29}$$

The circular permutation in (2.29) gives

$$g(\sigma_Y Z, X) + g(Z, \sigma_Y X) = 2\omega(Y)g(Z,X). \tag{2.30}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

$$g(\sigma_Z X, Y) + g(X, \sigma_Z Y) = 2\omega(Z)g(X, Y), \tag{2.31}$$

and using the operation (2.29)-(2.31)+(2.30), we get

$$\begin{aligned} g(\sigma_X Y, Z) - g(X, \sigma_Z Y) + g(\sigma_Y Z, X) + g(Z, \sigma_Y X) = 2\omega(X)g(Y, Z) \\ - 2\omega(Z)g(X, Y) + 2\omega(Y)g(Z, X), \end{aligned} \tag{2.32}$$

which is simplified as

$$\begin{aligned} g(\sigma_X Y, Z) &= g(\omega(X)Y, Z) + g(\omega(Y)X, Z) - \omega(Z)g(X, Y) \\ &= g(\omega(X)Y, Z) + g(\omega(Y)X, Z) - g(B, Z)g(X, Y) \\ &= g(\omega(X)Y + \omega(Y)X - g(X, Y)B, Z), \end{aligned} \tag{2.33}$$

which implies that

$$\sigma_X Y = \omega(X)Y + \omega(Y)X - g(X, Y)B, \tag{2.34}$$

where B is the vector field defined by $g(B, Z) = \omega(Z)$ and this ends the proof. □

Now consider the tensors $N^{(1)}$ and $N^{(2)}$ given in (2.6) and (2.7). Clearly the almost contact structure (ϕ, ξ, η) is normal if and only if $N^{(1)}, N^2$ vanishes. However the vanishing of $N^{(1)}$ implies the vanishing of N^2. We now prove the tensors for the structure $(\phi_t, \xi_t, \eta_t, g_t)$ on U_t in the following lemma.

Lemma 2.2.2. *If the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are almost Kenmotsu, we have that, on U_t,*

$$N_t^{(1)}(X, Y) = N_1(X, Y) - 2d\eta(X, Y)\xi, \tag{2.35}$$

$$N_t^{(2)}(X, Y) = \exp(-\sigma_t)\{\omega(\phi Y)\eta(X) - \omega(\phi X)\eta(Y) + N_2(X, Y)\}. \tag{2.36}$$

Proof. From equation (2.6) we have that

$$\begin{aligned} N_t^{(1)}(X, Y) &= [\phi_t, \phi_t](X, Y) + 2d\eta_t(X, Y)\xi_t \\ &= [\phi, \phi](X, Y) + 2d\eta_t(X, Y)\xi_t, \end{aligned} \tag{2.37}$$

where $[\phi, \phi]$ is the Nijenhuis torsion. Making $[\phi, \phi](X, Y)$ the subject of the formula in equation (2.6) and substituting what you get into equation (2.37) gives

$$N_t^{(1)}(X, Y) = N_1(X, Y) - 2d\eta(X, Y)\xi + 2d\eta_t(X, Y)\xi_t. \tag{2.38}$$

Since the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are almost Kenmotsu, $d\eta_t = 0$, equation (2.38) becomes

$$N_t^{(1)}(X, Y) = N_1(X, Y) - 2d\eta(X, Y)\xi. \tag{2.39}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

from (2.7), one gets that

$$N_t^{(2)}(X,Y) = (\mathcal{L}_{\phi_t X}\eta_t)Y - (\mathcal{L}_{\phi_t Y}\eta_t)X, \tag{2.40}$$

where $\mathcal{L}$ is the Lie derivative. Note that

$$\begin{aligned}(\mathcal{L}_{\phi_t X}\eta_t)Y &= \phi X(\exp(-\sigma_t)\eta(Y)) - \exp(-\sigma_t)\eta([\phi X, Y])\\ &= \phi X(\exp(-\sigma_t)\eta(Y)) + \exp(-\sigma_t)\phi X(\eta(Y)) - \exp(-\sigma_t)\eta([\phi X, Y])\\ &= \phi X(\sigma_t)\exp(-\sigma_t)\eta(Y) + \exp(-\sigma_t)\phi X(\eta(Y)) - \exp(-\sigma_t)\eta([\phi X, Y])\\ &= \exp(-\sigma_t)\{-\omega(\phi X)\eta(Y) + (\mathcal{L}_{\phi X}\eta)Y\} \end{aligned}\tag{2.41}$$

and

$$\begin{aligned}(\mathcal{L}_{\phi_t Y}\eta_t)X &= \phi Y(\exp(-\sigma_t)\eta(X)) - \exp(-\sigma_t)\eta([\phi Y, X])\\ &= \phi Y(\exp(-\sigma_t)\eta(X)) + \exp(-\sigma_t)\phi Y(\eta(X)) - \exp(-\sigma_t)\eta([\phi Y, X])\\ &= \phi Y(\sigma_t)\exp(-\sigma_t)\eta(X) + \exp(-\sigma_t)\phi Y(\eta(X)) - \exp(-\sigma_t)\eta([\phi Y, X])\\ &= \exp(-\sigma_t)\{-\omega(\phi Y)\eta(X) + (\mathcal{L}_{\phi Y}\eta)X\}. \end{aligned}\tag{2.42}$$

Now substituting equation (2.41) and (2.42) into equation (2.40), gives

$$\begin{aligned}N_t^{(2)}(X,Y) &= \exp(-\sigma_t)\{-\omega(\phi X)\eta(Y) + (\mathcal{L}_{\phi X}\eta)Y\} - \exp(-\sigma_t)\{-\omega(\phi Y)\eta(X)\\ &\quad + (\mathcal{L}_{\phi Y}\eta)X\}\\ &= \exp(-\sigma_t)\{\omega(\phi Y)\eta(X) - \omega(\phi X)\eta(Y) + (\mathcal{L}_{\phi X}\eta)Y - (\mathcal{L}_{\phi Y}\eta)X\}\\ &= \exp(-\sigma_t)\{\omega(\phi Y)\eta(X) - \omega(\phi X)\eta(Y) + N_2(X,Y)\}, \end{aligned}\tag{2.43}$$

and this ends the proof. □

On U_t, if the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are almost Kenmotsu, the relation (2.14) leads to

$$\begin{aligned}2\exp(-2\sigma_t)g((\nabla^t_X\phi_t)Y, Z) &= 2\exp(-3\sigma_t)\{\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z)\}\\ &\quad + \exp(-2\sigma_t)g(N^{(1)}(Y,Z), \phi X). \end{aligned}\tag{2.44}$$

That is,

$$\begin{aligned}2g((\nabla^t_X\phi_t)Y, Z) &= 2\exp(-\sigma_t)\{\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z)\}\\ &\quad + g(N^{(1)}(Y,Z), \phi X). \end{aligned}\tag{2.45}$$

The covariant derivatives $\nabla^t\phi_t$ and $\nabla\phi$ are related as follows. For any vector fields

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

X, Y on M and using (2.24), we have

$$\begin{aligned}(\nabla^t_X\phi_t)Y &= \nabla^t_X\phi_tY - \phi_t(\nabla^t_XY)\\ &= \nabla^t_X\phi Y - \phi(\nabla^t_XY)\\ &= \nabla_X\phi Y - \omega(X)\phi Y - \omega(\phi Y)X + g(X,\phi Y)B - \phi\nabla_XY + \omega(X)\phi Y\\ &+ \omega(Y)\phi X - g(X,Y)\phi B\\ &= \nabla_X\phi Y - \omega(\phi Y)X + g(X,\phi Y)B - \phi\nabla_XY + \omega(Y)\phi X - g(X,Y)\phi B\\ &= (\nabla_X\phi)Y - \omega(\phi Y)X + \omega(Y)\phi X + g(X,\phi Y)B - g(X,Y)\phi B. \qquad (2.46)\end{aligned}$$

Now, using the relation (2.46) into (2.45), one has

$$\begin{aligned}&2\exp(-\sigma_t)\left\{\eta(Z)g(\phi X,Y) - \eta(Y)g(\phi X,Z)\right\} + g(N^{(1)}(Y,Z),\phi X)\\ &= 2g((\nabla_X\phi)Y,Z) - 2\omega(\phi Y)g(X,Z) + 2\omega(Y)g(\phi X,Z) + 2g(X,\phi Y)\omega(Z)\\ &- 2g(X,Y)g(\phi B,Z). \qquad (2.47)\end{aligned}$$

Therefore, we have the following.

Theorem 2.2.2. *An almost contact metric manifold M is l.c. almost Kenmotsu if and only if there exists a 1-form ω on M such that $d\omega = 0$ and*

$$\begin{aligned}2g((\nabla_X\phi)Y,Z) &= 2\exp(-\sigma_t)\left\{\eta(Z)g(\phi X,Y) - \eta(Y)g(\phi X,Z)\right\} + g(N^{(1)}(Y,Z),\phi X)\\ &+ 2\omega(\phi Y)g(X,Z) - 2\omega(Y)g(\phi X,Z) - 2\omega(Z)g(X,\phi Y)\\ &- 2\omega(\phi Z)g(X,Y), \qquad (2.48)\end{aligned}$$

for any vector fields X, Y and Z on M.

As a consequence, we have the following result.

Theorem 2.2.3. *An almost contact metric manifold M is l.c. almost Kenmotsu if and only if there exists a 1-form ω on M such that $d\omega = 0$ and*

$$\begin{aligned}2g((\nabla_X\phi)Y,Z) &= 2\exp(-\sigma_t)\left\{\eta(Z)g(\phi X,Y) - \eta(Y)g(\phi X,Z)\right\} + g(N^{(1)}(Y,Z),\phi X)\\ &+ 2\omega(\phi Y)g(X,Z) - 2\omega(Y)g(\phi X,Z) - 2\omega(Z)g(X,\phi Y)\\ &- 2\omega(\phi Z)g(X,Y), \qquad (2.49)\end{aligned}$$

for any vector fields X, Y and Z on M.

For an l.c. Kenmotsu manifold, the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are normal. Therefore, we have the following theorem.

Theorem 2.2.4. *An almost contact metric manifold M is l.c. Kenmotsu if and only if there exists a 1-form ω on M such that $d\omega = 0$ and*

$$\begin{aligned}(\nabla_X\phi)Y &= \exp(-\sigma_t)\{g(\phi X,Y)\xi - \eta(Y)\phi X\} + \omega(\phi Y)X - \omega(Y)\phi X\\ &- g(X,\phi Y)B + g(X,Y)\phi B, \qquad (2.50)\end{aligned}$$

for any vector fields X, Y and Z on M.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Proof. The proof follows from Theorem 2.2.3 and the fact if M is an l.c. Kenmotsu manifold, then the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are Kenmotsu, that is, they are normal almost Kenmotsu. □

If M is an l.c. almost Kenmotsu manifold and the smooth 1-form ω is proportional to η, that is, $\omega = \alpha\eta$, $\alpha = const \neq 0$, the relation (2.48) becomes

$$\begin{aligned} 2g((\nabla_X\phi)Y, Z) = 2\{\exp(-\sigma_t) + \alpha\}\,\{\eta(Z)g(\phi X, Y) - \eta(Y)g(\phi X, Z)\} \\ + g(N^{(1)}(Y, Z), \phi X). \end{aligned} \tag{2.51}$$

This means that an l.c. almost Kenmotsu manifold with $\omega = \alpha\eta$ is an almost β-Kenmotsu manifold with $\beta = \alpha + \exp(-\sigma_t)$. If the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are normal, from (2.48), one has

$$\begin{aligned} (\nabla_X\phi)Y &= \exp(-\sigma_t)\{g(\phi X, Y)\xi - \eta(Y)\phi X\} - \alpha\eta(Y)\phi X \\ &\quad - \alpha g(X, \phi Y)\xi \\ &= (\alpha + \exp(-\sigma_t))\{g(\phi X, Y)\xi - \eta(Y)\phi X\}. \end{aligned} \tag{2.52}$$

Therefore, we have the following result.

Theorem 2.2.5. *An l.c. Kenmotsu manifold with $\omega = \alpha\eta$ is β-Kenmotsu manifold with $\beta = \alpha + \exp(-\sigma_t)$. Moreover, the structure $(\phi, \xi, \eta,\ g)$ satisfies*

$$d\eta = 0 \ \ \text{and} \ \ d\Phi = 2\beta\eta \wedge \Phi. \tag{2.53}$$

Note that an l.c. Kenmotsu manifold which is not Kenmotsu.

2.3 Example of l.c. Kenmotsu manifolds

We construct an example of an l.c. Kenmotsu manifold which is not Kenmotsu. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ with the linearly independent vector fields

$$e_1 = -2x\frac{\partial}{\partial z}, \quad e_2 = -2x\frac{\partial}{\partial y}, \quad e_3 = -2\exp(-x)x\frac{\partial}{\partial x}$$

Let g be the Riemannian metric defined by

$$g(e_i, e_i) = \exp(2x), \text{ for } \ i = 1, 2, \ g(e_3, e_3) = 1, \ g(e_i, e_j) = 0, \text{ for } \ i \neq j, i, j = 1, 2, 3.$$

Let η be the 1-form defined by

$$\eta(e_1) = \eta(e_2) = 0 \ \text{ and } \ \eta(e_3) = 1.$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

We define the $(1,1)$-tensor field ϕ as $\phi e_1 = e_2$, $\phi e_2 = -e_1$, and $\phi e_3 = 0$. Then using the linearity of ϕ and g, we have

$$\phi^2 X = -X + \eta(X)e_3, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for all X and Y on M. Thus for $e_3 = \xi$, in view of the definition of almost contact metric manifold, (ϕ, ξ, η, g) yields an almost contact metric structure on M.

$$[e_1, e_2] = 0, \quad [e_1, e_3] = 2\exp(-x)e_1, \quad [e_2, e_3] = 2\exp(-x)e_2.$$

By using the Koszul formula (2.22), we obtain

$$\begin{aligned}
&\nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -2\exp(-x)(-x+1)e_1, \quad \nabla_{e_1} e_1 = 2\exp(x)(-x+1)e_3,\\
&\nabla_{e_2} e_3 = -2\exp(-x)(-x+1)e_2, \quad \nabla_{e_2} e_2 = 2\exp(x)(-x+1)e_3, \quad \nabla_{e_2} e_1 = 0,\\
&\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = 2\exp(-x)xe_2, \quad \nabla_{e_3} e_1 = 2\exp(-x)xe_1.
\end{aligned}$$

By the following contact transformation

$$\widetilde{g} = \exp(-2x)g, \quad \widetilde{\xi} = \exp(x)\xi, \quad \widetilde{\eta} = \exp(-x)\eta, \quad \widetilde{\phi} = \phi,$$

$(M, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ is a Kenmotsu manifold (see [21]). Hence, (M, ϕ, ξ, η, g) is a conformal Kenmotsu manifold but is not Kenmotsu because we have

$$(\nabla_X \phi)Y \neq g(\phi X, Y)\xi - \eta(Y)\phi X,$$

for all X and Y on M (for instance, $(\nabla_{e_2}\phi)e_1 \neq g(\phi e_2, e_1)\xi - \eta(e_1)\phi e_2$). By using the above results, we can easily see that

$$\begin{aligned}
&R(e_1, e_2)e_2 = -4(-x+1)^2 e_1, \quad R(e_2, e_3)e_3 = -4\exp(-2x)e_2,\\
&R(e_1, e_2)e_3 = 0, \quad R(e_1, e_3)e_3 = -4\exp(-2x)e_1, \quad R(e_3, e_1)e_1 = -4e_3,\\
&R(e_3, e_2)e_1 = 0 \quad R(e_3, e_1)e_2 = 0, \quad R(e_2, e_1)e_1 = -4(-x+2)^2 e_2, \quad R(e_3, e_2)e_2 = -4e_3.
\end{aligned}$$

In view of the above relations, we conclude that

$$K(X, e_3) = -\exp(-2x), \quad K(X,Y) = -\exp(-2x)(-x+1)^2,$$

for all X and Y orthogonal to e_3. Note that $(M, \widetilde{\phi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ is a Kenmotsu manifold of constant ϕ-holomorphic sectional curvature -1 (see [21]).

2.4 Classes of l.c. almost Kenmotsu structures

Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Now, we investigate the Olszak and Blair $(1,1)$-tensor fields defined, respectively, by

$$hX = \nabla_X \xi - \omega(\xi)X + \eta(X)B, \tag{2.54}$$

$$\text{and} \quad \widehat{h}X = \frac{1}{2}(\mathcal{L}_\xi \phi)X, \tag{2.55}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

for any vector field X on M. The relation (2.54) leads to

$$\nabla_{\xi}\xi = -B + \omega(\xi)\xi. \tag{2.56}$$

Using (2.13) and (2.24), we can see that on each U_t,

$$\exp(-\sigma_t)\{-X + \eta(X)\xi\} = hX. \tag{2.57}$$

The linear operator h has the following properties [28]:

$$h\phi + \phi h = 0, \quad h\xi = 0, \quad \text{trace}(h) = 0, \;\; g(hX, Y) = g(hY, X). \tag{2.58}$$

For the Blair $(1,1)$-tensor field, we have the following:

$$\begin{aligned} 2\widehat{h}X &= (\mathcal{L}_{\xi}\phi)X \\ &= [\xi, \phi X] - \phi[\xi, X] \\ &= \nabla_{\xi}\phi X - \nabla_{\phi X}\xi - \phi(\nabla_{\xi}X - \nabla_{X}\xi) \\ &= \nabla_{\xi}\phi X - \nabla_{\phi X}\xi - \phi\nabla_{\xi}X + \phi\nabla_{X}\xi \\ &= (\nabla_{\xi}\phi)X - \nabla_{\phi X}\xi + \phi\nabla_{X}\xi, \end{aligned} \tag{2.59}$$

for any vector field X on M. Using (2.54), the relation (2.59) becomes

$$\begin{aligned} 2\widehat{h}X = (\nabla_{\xi}\phi)X - h\phi X - \omega(\xi)\phi X + \phi hX + \omega(\xi)\phi X \\ - \eta(X)\phi B. \end{aligned} \tag{2.60}$$

Since

$$(\nabla_{\xi}\phi)X = \omega(\phi Y)\xi + \eta(Y)\phi B, \tag{2.61}$$

one obtains

$$\begin{aligned} 2\widehat{h}X &= \omega(\phi Y)\xi + \eta(Y)\phi B - h\phi X - \omega(\xi)\phi X + \phi hX + \omega(\xi)\phi X \\ &\quad - \eta(X)\phi B \\ &= \omega(\phi Y)\xi + 2\phi hX. \end{aligned} \tag{2.62}$$

From trace$(\widehat{h}) = 0$, $\widehat{h}\xi = 0$ and

$$\begin{aligned} \widehat{h}\phi X + \phi\widehat{h}X &= \frac{1}{2}\{\omega(\xi)\eta(X) - \omega(X)\}\xi, \\ g(\widehat{h}X, Y) - g(X, \widehat{h}Y) &= \frac{1}{2}\{\omega(\phi X)\eta(Y) - \omega(\phi Y)\eta(X)\}. \end{aligned} \tag{2.63}$$

Therefore, we have the following:

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Lemma 2.4.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. The Olszak and Blair $(1,1)$-tensor fields h and $\widehat{h}$ are related as*

$$\widehat{h} = \frac{1}{2}(\omega \circ \phi) \otimes \xi + \phi \circ h.$$

Moreover, $\widehat{h}$ is not a symmetric operator, $\widehat{h}$ does not anticommute with ϕ,

$$trace(\widehat{h}) = 0 \ \ and \ \ \nabla_X \xi = \omega(\xi)X - \eta(X)B + \phi\widehat{h}X,$$

for any vector field X on M.

We have the following:

Lemma 2.4.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Then $\widehat{h}\phi + \phi\widehat{h} = 0$ and $\widehat{h}$ is a symmetric operator if and only if there exists a smooth function f on M such that $\omega = f\eta$ with $df \wedge \eta = 0$ and $\widehat{h} = \phi h$.*

Another condition in which an l.c. almost Kenmotsu manifold admits a 1-form ω that is proportional to η is as follows.

Let $D := \ker \eta$ be the contact distribution and $D^{\perp}$ be the distribution spanned the structure vector field ξ. Then, we have the following decomposition

$$TM = D \oplus D^{\perp}, \tag{2.64}$$

where $\oplus$ denotes the orthogonal direct sum. By the decomposition (2.64), any $X \in \Gamma(TM)$ is written as

$$X = QX + Q^{\perp}X, \tag{2.65}$$

where Q and $Q^{\perp}$ are the projection morphisms of TM into D and $D^{\perp}$, respectively. Then, $Q^{\perp}X = \eta(X)\xi$ and $X = QX + \eta(X)\xi$.

Lemma 2.4.3. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Then the contact distribution D defines on M a foliation $\mathcal{F}$ of codimension 1.*

Proof. Let X, $Y \in \Gamma(D)$. Then, $\eta(X) = \eta(Y) = 0$ and

$$\eta([X, Y]) = -2(\omega \wedge \eta)(X, Y) = \omega(Y)\eta(X) - \omega(X)\eta(Y) = 0.$$

This means that $[X, Y] \in \Gamma(D)$, i.e, the contact distribution D is integrable. □

Let $\mathcal{F}$ be a foliation on an l.c. almost Kenmotsu manifold (M, ϕ, ξ, η, g) of codimension 1. The metric g is said to be *bundle-like* for the foliation $\mathcal{F}$ if the induced metric on the transversal distribution $D^{\perp}$ is parallel with respect to the intrinsic connection on $D^{\perp}$. This is true if and only if the Levi-Civita connection ∇ of (M, ϕ, ξ, η, g) satisfies (see [7] and [35] for more details):

$$g(\nabla_{Q^{\perp}Y} QX, Q^{\perp}Z) + g(\nabla_{Q^{\perp}Z} QX, Q^{\perp}Y) = 0, \tag{2.66}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

for any X, Y, $Z \in \Gamma(TM)$. If for a given foliation $\mathcal{F}$, the Riemannian metric g on M is bundle-like for $\mathcal{F}$, then we say that $\mathcal{F}$ is a *Riemannian foliation on* (M, ϕ, ξ, η, g).

Let $\mathcal{F}^{\perp}$ be the orthogonal complementary foliation generated by ξ. Now we provide necessary and sufficient conditions for the metric on an l.c. almost Kenmotsu manifold to be bundle-like for foliations $\mathcal{F}$ and $\mathcal{F}^{\perp}$. Therefore, we have the following results.

Theorem 2.4.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. Then the following assertions are equivalent:*

(i) The metric g on M is bundle-like for the foliation $\mathcal{F}$.

(ii) The dual vector field B of ω has a no components along D.

Proof. Using (2.56), for any $X, Y, Z \in \Gamma(TM)$, we have $Q^{\perp}Y = \eta(Y)\xi$, $Q^{\perp}Z = \eta(Z)\xi$ and the left-hand side of (2.66) gives

$$g(\nabla_{Q^{\perp}Y}QX, Q^{\perp}Z) + g(\nabla_{Q^{\perp}Z}QX, Q^{\perp}Y) = 2\eta(Y)\eta(Z)\omega(QX).$$

□

Next, we investigate the *torsion tensor* τ for an l.c. almost cosymplectic manifold. This tensor was introduced by Chern and Hamilton [16] and is defined by

$$g(\tau X, Y) = (\mathcal{L}_{\xi}g)(X, Y),$$

for vector fields X, Y on a contact metric manifold (see [17] for details).

The Lie derivative $\mathcal{L}_{\xi}$ of g with respect to the vector field ξ is given by

$$(\mathcal{L}_{\xi}g)(X, Y) = 2g(hX, Y) + 2\omega(\xi)g(X, Y) - \omega(X)\eta(Y) - \omega(Y)\eta(X), \tag{2.67}$$

where h is given by (2.54). Thus, an l.c. almost cosymplectic manifold M has a tensor τ such that $g(\tau X, Y) = (\mathcal{L}_{\xi}g)(X, Y)$, $\forall\, X, Y \in \Gamma(TM)$. By (2.67), have

$$\tau X = 2hX + 2\omega(\xi)X - \omega(X)\xi - \eta(X)B. \tag{2.68}$$

Lemma 2.4.4. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Then, the following assertions are equivalent:*

(a) The structure vector field ξ is Killing.

(b) The differential 1-form ω and the operator h vanish.

Proof. Suppose the structure vector field ξ is Killing. Then, the Lie derivative

$$(\mathcal{L}_{\xi}g)(X, Y) = 0,$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

for any vector fields X and Y on M. The latter implies that the Chern-Hamilton tensor τ vanishes identically on M. Its trace, with respect to an adapted frame $\{e_i\}_{1\leq i\leq 2n+1}$ in TM, gives

$$\begin{aligned}0 &= 2\sum_{i=1}^{2n+1} g(he_i, e_i) + 2\omega(\xi)\sum_{i=1}^{2n+1} g(e_i, e_i) - \sum_{i=1}^{2n+1}\omega(e_i)g(\xi, e_i) - \sum_{i=1}^{2n+1}\eta(e_i)g(B, e_i)\\ &= 2\mathrm{trace}(h) + 2(2n+1)\omega(\xi) - 2\omega(\xi)\\ &= 2\mathrm{trace}(h) + 4n\omega(\xi). \end{aligned} \tag{2.69}$$

Since $\mathrm{trace}(h) = 0$ and $n \geq 1$, we have $\omega(\xi) = 0$. Also, for $X = \xi$ in (2.68) with $\tau = 0$, we have $B = 0$. Hence $\omega = 0$ and $h = 0$. This means that (a) implies (b), and the conserve is obvious. □

For any X, Y, $Z \in \Gamma(TM)$, using (2.67) and the fact that h is symmetric and

$$g(\nabla_{QY}Q^{\perp}X, QZ) = \eta(X)\{g(hQY, QZ) + \omega(\xi)g(QY, QZ)\}, \tag{2.70}$$

we have

$$\begin{aligned}g(\nabla_{QY}Q^{\perp}X, QZ) + g(\nabla_{QZ}Q^{\perp}X, QY) = 2\eta(X)\{g(hQY, QZ)\\ + 2\omega(\xi)g(QY, QZ).\end{aligned} \tag{2.71}$$

Using the Lie derivative in (2.67), one obtains

$$g(\nabla_{QY}Q^{\perp}X, QZ) + g(\nabla_{QZ}Q^{\perp}X, QY) = 2\eta(X)(\mathcal{L}_{\xi}g)(QY, QZ). \tag{2.72}$$

We have therefore the following.

Theorem 2.4.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. Then the following assertions are equivalent:*

(a) The metric g on M is bundle-like for the canonical totally real foliation $\mathcal{F}^{\perp}$.

(b) The structure vector field ξ is D-Killing (i.e. $D^{\perp}$ is D-Killing distribution).

Let M' be a leaf of the distribution D. Since M' is a submanifold of M and for any X, $Y \in \Gamma(TM')$, we have

$$\nabla_X Y = \nabla'_X Y + \alpha(X, Y), \tag{2.73}$$

$$\nabla_X \xi = -A_{\xi}X + \nabla'^{\perp}_X \xi, \tag{2.74}$$

where ∇' and α are the Levi-Civita connection and the second fundamental form of M', respectively. On the other hand, since ξ is a unit normal vector field, we have $g(\nabla_X\xi, \xi) = 0$, hence $\nabla'^{\perp}_X\xi = 0$, for any $X \in \Gamma(TM')$. Therefore, the Weingartem formula (2.74) becomes

$$\nabla_X\xi = -A_{\xi}X.$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Proposition 2.4.1. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Then, integral manifolds of the distribution D in (2.64) are l.c. almost Kähler manifolds with mean curvature vector field $H' = -\omega(\xi)\xi$. They are totally umbilical submanifolds of M if and only if the operator h vanishes.*

Proof. Let M' be an integral manifold of D. The tensor fields ϕ_t and g_t induce an almost complex structure $J_t = J$ and a Hermitian metric g'_t on M'. Then, for any X, $Y \in \Gamma(TM')$, we have $\Phi'_t(X, Y) = g'_t(X, J_tY) = g_t(X, \phi_tY) = \Phi_t(X, Y)$ and $d\Phi'_t = (d\Phi_t)_{|M'} = 0$, so M' is an l.c. almost Kähler. Using (2.73), the second fundamental form of M' gives

$$\alpha(X, Y) = g(A_\xi X, Y)\xi = -g(hX, Y)\xi - \omega(\xi)g(X, Y)\xi. \tag{2.75}$$

Fixing a local orthonormal frame $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n\}$ in TM' and applying the properties on h, one has,

$$H = \frac{1}{\mathrm{rank}(D)}\{\sum_{i=1}^{n}\alpha(e_i, e_i) + \sum_{i=1}^{n}\alpha(\phi e_i, \phi e_i)\} = -\omega(\xi)\xi.$$

The last assertion follows and this completes the proof. □

This result can be extended to the foliation $\mathcal{F}^\perp$. That is, if $h = \omega(\xi) = 0$, $g(\nabla_X Y, \xi) = 0$. This means that the foliation $\mathcal{F}^\perp$ is Riemannian. Therefore, $h = 0$, the leaves of $\mathcal{F}$ are totally geodesic if and only if the orthogonal complementary foliation $\mathcal{F}^\perp$ generated by ξ is Riemannian.

On each $U_t \cap M'$, the Gauss and Weingartem formulas are given by

$$\nabla^t_X Y = \nabla'^t_X Y + \alpha^t(X, Y), \tag{2.76}$$

$$\nabla^t_X \xi_t = -A_{\xi_t}X, \tag{2.77}$$

where $g_t(\alpha^t(X, Y), \xi_t) = g_t(A_{\xi_t}X, Y)$, that is, $\alpha^t(X, Y) = g_t(A_{\xi_t}X, Y)\xi_t$. However,

$$\alpha^t(X, Y) = g_t(A_{\xi_t}X, Y)\xi_t = g(A_\xi X, Y)\xi = \alpha(X, Y). \tag{2.78}$$

For any X, $Y \in \Gamma(TM')$, and using (2.24) and (2.73), we have

$$\begin{aligned}(\nabla'^t_X J)Y &= \nabla'^t_X JY - J(\nabla'^t_X Y) = \nabla^t_X \phi Y - \alpha(X, \phi Y) - \phi(\nabla^t_X Y)\\ &= (\nabla_X \phi)Y - \omega(\phi Y)X + \omega(Y)\phi X + g(X, \phi Y)B\\ &\quad - g(X, Y)\phi B - g(A_\xi X, \phi Y)\xi.\end{aligned} \tag{2.79}$$

If the integral manifold M' is an l.c. Kähler, then, $(\nabla'^t_X J)Y = 0$ and we have

$$\begin{aligned}(\nabla_X \phi)Y &= \omega(\phi Y)X - \omega(Y)\phi X - g(X, \phi Y)B + g(X, Y)\phi B\\ &\quad + g(A_\xi X, \phi Y)\xi,\end{aligned} \tag{2.80}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

for any X, $Y \in \Gamma(TM')$. Therefore, if the foliation $\mathcal{F}$ has locally conformal Kähler leaves, then for any X, $Y \in \Gamma(TM)$, the vector fields $X - \eta(X)\xi$, $Y - \eta(Y)\xi$ and $B - \eta(B)\xi$ belong to D and using (2.56) and (2.61), we have

$$\begin{aligned}(\nabla_{X-\eta(X)\xi}\phi)(Y - \eta(Y)\xi) &= (\nabla_X\phi)Y - \eta(Y)\phi A_\xi X - \eta(X)\omega(\phi Y)\xi,\\ g(A_\xi(X - \eta(X)\xi), \phi Y) &= g(A_\xi X, \phi Y) - \eta(X)\omega(\phi Y).\end{aligned}$$

Putting these pieces into (2.80) and taking into account the following relations

$$\begin{aligned}\omega(\phi(Y - \eta(Y)\xi))(X - \eta(X)\xi) &= \omega(\phi Y)X - \eta(X)\omega(\phi Y)\xi,\\ \omega(Y - \eta(Y)\xi)\phi(X - \eta(X)\xi) &= \omega(Y)\phi X - \eta(Y)\omega(\xi)\phi X,\\ g(X - \eta(X)\xi, \phi(Y - \eta(Y)\xi)) &= g(X, \phi Y),\\ g(X - \eta(X)\xi, Y - \eta(Y)\xi) &= g(X, Y) - \eta(X)\eta(Y),\end{aligned}$$

one obtains,

$$\begin{aligned}(\nabla_X\phi)Y = &-g(\phi A_\xi X, Y)\xi + \eta(Y)\phi A_\xi X + \omega(\phi Y)X + \{\eta(Y)\omega(\xi) - \omega(Y)\}\phi X\\ &- g(X, \phi Y)\{B - \omega(\xi)\xi\} - \eta(X)\omega(\phi Y)\xi.\end{aligned}$$

Proposition 2.4.2. *Let (M, ϕ, ξ, η, g) be an l.c. almost Kenmotsu manifold. Then the distribution D in (2.64) has locally conformal Kähler leaves if and only if*

$$\begin{aligned}(\nabla_X\phi)Y = &-g(\phi A_\xi X, Y)\xi + \eta(Y)\phi A_\xi X + \omega(\phi Y)X + \{\eta(Y)\omega(\xi) - \omega(Y)\}\phi X\\ &- g(X, \phi Y)\{B - \omega(\xi)\xi\} - \eta(X)\omega(\phi Y)\xi, \qquad (2.81)\end{aligned}$$

for any X, $Y \in \Gamma(TM)$.

When the differential 1-form ω is reduced to $\omega = f\eta$, where f is a function such that $df \wedge \eta = 0$, then M becomes β-Kenmotsu manifold with $\beta = \alpha + \exp(-\sigma_t)$ and the relation (2.81) for any leaves of M to be Kählerian becomes

$$(\nabla_X\phi)Y = -g(\phi A_\xi X, Y)\xi + \eta(Y)\phi A_\xi X,$$

for any X, $Y \in \Gamma(TM)$. The latter relation is exactly the one found by Aktan *et al* in [2, Proposition 6] but in the case of f-cosymplectic. We have the following.

Theorem 2.4.3. *Let (M, ϕ, ξ, η, g) be an l.c. Kenmotsu manifold and let $\mathcal{F}$ be a foliation on M of codimension 1. If the metric g on M is bundle-like for the foliation $\mathcal{F}$, then the leaves of $\mathcal{F}$ are Kähler and totally umbilical.*

Proof. Let $\mathcal{F}$ be a foliation on an l.c. Kenmotsu manifold M of codimension 1. If the metric g on M is bundle-like for the foliation $\mathcal{F}$, then, by Theorem 2.4.1, the dual vector field B of ω is proportional to ξ, that is, $B = \omega(\xi)\xi$. This means that M

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

becomes β-Kenmotsu manifold with $\beta = \alpha + \exp(-\sigma_t)$ and the leaves of $\mathcal{F}$ are almost Kähler.

Since the structures $(\phi_t, \xi_t, \eta_t, g_t)$ are normal, then the tensor $N^{(1)} = N_t^{(1)} = 0$ in (2.35) vanishes. By the equality

$$\begin{aligned} N^{(1)}(X,Y) &= [\phi,\phi](X,Y) = \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y] \\ &= [J,J](X,Y), \end{aligned}$$

and Proposition 2.4.1, we complete the proof. □

Let R^t and R denote the curvature tensors of $(M, \phi_t, \eta_t, \xi_t, g_t)$ and (M, ϕ, η, ξ, g), respectively. Then the relation between R_t and R is given by

$$\begin{aligned} R^t(X,Y)Z &= \nabla^t_X \nabla^t_Y Z - \nabla^t_Y \nabla^t_X Z - \nabla^t_{[X,Y]} Z \\ &= \nabla^t_X \left(\nabla_Y Z - \omega(Y)Z - \omega(Z)Y + g(Y,Z)B\right) \\ &- \nabla^t_Y \left(\nabla_X Z - \omega(X)Z - \omega(Z)X + g(X,Z)B\right) \\ &- \nabla_{[X,Y]} Z + \omega([X,Y])Z + \omega(Z)[X,Y] - g([X,Y],Z)B. \end{aligned} \tag{2.82}$$

Since

$$\begin{aligned} \nabla^t_X \left(\nabla_Y Z - \omega(Y)Z - \omega(Z)Y + g(Y,Z)B\right) &= \nabla_X \nabla_Y Z - \omega(X)\nabla_Y Z - \omega(\nabla_Y Z)X \\ &+ g(X, \nabla_Y Z)B - X(\omega(Y))Z - \omega(Y)\nabla_X Z + \omega(X)\omega(Y)Z + 2\omega(Y)\omega(Z)X \\ &- \omega(Y)g(X,Z)B - X(\omega(Z))Y - \omega(Z)\nabla_X Y + \omega(X)\omega(Z)Y - \omega(Z)g(X,Y)B \\ &+ X(g(Y,Z))B + g(Y,Z)\nabla_X B - \omega(B)g(Y,Z)X \end{aligned} \tag{2.83}$$

and

$$\begin{aligned} \nabla^t_Y \left(\nabla_X Z - \omega(X)Z - \omega(Z)X + g(X,Z)B\right) &= \nabla_Y \nabla_X Z - \omega(Y)\nabla_X Z - \omega(\nabla_X Z)Y \\ &+ g(Y, \nabla_X Z)B - Y(\omega(X))Z - \omega(X)\nabla_Y Z + \omega(Y)\omega(X)Z + 2\omega(X)\omega(Z)Y \\ &- \omega(X)g(Y,Z)B - Y(\omega(Z))X - \omega(Z)\nabla_Y X + \omega(Y)\omega(Z)X - \omega(Z)g(X,Y)B \\ &+ Y(g(X,Z))B + g(X,Z)\nabla_Y B - \omega(B)g(X,Z)Y, \end{aligned} \tag{2.84}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

and using the fact that ω is closed, we have

$$
\begin{aligned}
R^t(X,Y)Z &= R(X,Y)Z - \omega(X)\nabla_Y Z - \omega(\nabla_Y Z)X + g(X,\nabla_Y Z)B - X(\omega(Y))Z \\
&- \omega(Y)\nabla_X Z + \omega(X)\omega(Y)Z + 2\omega(Y)\omega(Z)X - \omega(Y)g(X,Z)B \\
&- X(\omega(Z))Y - \omega(Z)\nabla_X Y + \omega(X)\omega(Z)Y - \omega(Z)g(X,Y)B \\
&+ X(g(Y,Z))B + g(Y,Z)\nabla_X B - \omega(B)g(Y,Z)X + \omega(Y)\nabla_X Z \\
&+ \omega(\nabla_X Z)Y - g(Y,\nabla_X Z)B + Y(\omega(X))Z + \omega(X)\nabla_Y Z - \omega(Y)\omega(X)Z \\
&- 2\omega(X)\omega(Z)Y + \omega(X)g(Y,Z)B + Y(\omega(Z))X + \omega(Z)\nabla_Y X - \omega(Y)\omega(Z)X \\
&+ \omega(Z)g(X,Y)B - Y(g(X,Z))B - g(X,Z)\nabla_Y B + \omega(B)g(X,Z)Y \\
&+ \omega([X,Y])Z + \omega(Z)[X,Y] - g([X,Y],Z)B \\
&= R(X,Y)Z + ((\nabla_Y\omega)Z)X + \omega(Y)\omega(Z)X - \omega(Y)g(X,Z)B \\
&- (\nabla_X\omega)Z)Y - \omega(X)\omega(Z)Y + g(Y,Z)\nabla_X B - \omega(B)g(Y,Z)X \\
&+ \omega(X)g(Y,Z)B - g(X,Z)\nabla_Y B + \omega(B)g(X,Z)Y. \qquad (2.85)
\end{aligned}
$$

Therefore, we have, for all X, Y, Z, and W on M,

$$
\begin{aligned}
& g(R^t(X,Y)Z,W) = g(R(X,Y)Z,W) + \{(\nabla_Y\omega)Z + \omega(Y)\omega(Z)\}g(X,W) \\
& - \omega(Y)\omega(W)g(X,Z) - \{(\nabla_X\omega)Z + \omega(X)\omega(Z)\}g(Y,W) \\
& + g(Y,Z)g(\nabla_X B,W) - \omega(B)g(Y,Z)g(X,W) \\
& + \omega(X)\omega(W)g(Y,Z) - g(X,Z)g(\nabla_Y B,W) \\
& + \omega(B)g(X,Z)g(Y,W) \\
& = g(R(X,Y)Z,W) + \{(\nabla_Y\omega)Z + \omega(Y)\omega(Z)\}g(X,W) \\
& - \{(\nabla_X\omega)Z + \omega(X)\omega(Z)\}g(Y,W) \\
& - \omega(B)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\
& + \{(\nabla_X\omega)W + \omega(X)\omega(W)\}g(Y,Z) - \{(\nabla_Y\omega)W + \omega(Y)\omega(W)\}g(X,Z). \qquad (2.86)
\end{aligned}
$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

This leads, on the arbitrary U_t, to

$$
\begin{aligned}
\exp(2\sigma_t)g_t(R(X,Y)Z,W) &= g(R(X,Y)Z,W) + \{(\nabla_Y\omega)Z + \omega(Y)\omega(Z)\}g(X,W)\\
&- \{(\nabla_X\omega)Z + \omega(X)\omega(Z)\}g(Y,W)\\
&- \omega(B)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}\\
&+ \{(\nabla_X\omega)W + \omega(X)\omega(W)\}g(Y,Z) - \{(\nabla_Y\omega)W + \omega(Y)\omega(W)\}g(X,Z)\\
&= g(R(X,Y)Z,W) + \{(\nabla_Y\omega)Z + \omega(Y)\omega(Z) - \frac{1}{2}\omega(B)g(Y,Z)\}g(X,W)\\
&- \{(\nabla_X\omega)Z + \omega(X)\omega(Z) - \frac{1}{2}\omega(B)g(X,Z)\}g(Y,W)\\
&+ \{(\nabla_X\omega)W + \omega(X)\omega(W) - \frac{1}{2}\omega(B)g(X,W)\}g(Y,Z)\\
&- \{(\nabla_Y\omega)W + \omega(Y)\omega(W) - \frac{1}{2}\omega(B)Y,W)\}g(X,Z)\\
&= g(R(X,Y)Z,W) + g(X,W)P(Y,Z) - g(Y,W)P(X,Z)\\
&+ g(Y,Z)P(X,W) - g(X,Z)P(Y,W),
\end{aligned}
\tag{2.87}
$$

where

$$P := \nabla\omega + \omega\otimes\omega - \frac{1}{2}||B||^2 g. \tag{2.88}$$

Note also that, with the help of (2.56), from (2.88), it can be derived

$$P(\xi,\xi) = \xi(\omega(\xi)) + \frac{1}{2}||B||^2, \ \ \mathrm{trace}(P) = \mathrm{div}B - \frac{1}{2}(2n-1)||B||^2. \tag{2.89}$$

Lemma 2.4.5. *The operator P in (2.88) is symmetric.*

Proof. For any vector fields X and Y on M

$$
\begin{aligned}
P(X,Y) - P(Y,X) &= (\nabla_X\omega)Y - (\nabla_Y\omega)X\\
&= X(\omega(Y)) - \omega(\nabla_X Y) - Y(\omega(X)) + \omega(\nabla_Y X)\\
&= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]) = 2d\omega(X,Y) = 0,
\end{aligned}
$$

which completes the proof. □

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Chapter Three

Submanifolds of l.c. Almost Kenmotsu Manifolds

This chapter is focused on submanifolds of l.c. almost Kenmotsu manifolds, We define the contact CR-submanifolds in these settings and prove that they contain distributions which, under some conditions, admit foliations that are l.c. (almost) Kähler.

3.1 Gauss and Weingarten formulas

Let $(\widetilde{M}, \widetilde{g})$ be an m-dimensional submanifold of a $(2n+1)$-dimensional l.c. almost Kenmotsu manifold (M, g). The Gauss and Weingarten formulas are given by

$$\nabla_X Y = \widetilde{\nabla}_X Y + \widetilde{h}(X, Y), \tag{3.1}$$

$$\nabla_X N = -A_N X + \widetilde{\nabla}^{\perp}_X N, \tag{3.2}$$

for all vector fields X and Y tangent to $\widetilde{M}$ and the normal vector field N on $\widetilde{M}$, where $\widetilde{\nabla}$ is the Riemannian connection on $\widetilde{M}$ determined by the induced metric $\widetilde{g}$, $\widetilde{\nabla}^{\perp}$ is the normal connection on $TM^{\perp}$ of $\widetilde{M}$ and $\widetilde{h}$ is the second fundamental form of $\widetilde{M}$. It is known that

$$g(\widetilde{h}(X, Y), N) = g(A_N X, Y) = \widetilde{g}(A_N X, Y), \tag{3.3}$$

where A is called the *shape operator* of $\widetilde{M}$ with respect to the unit normal vector field N.

If we assume that ξ is tangent to $\widetilde{M}$ and if X is a vector field in $\widetilde{M}$, then by (2.57), hX is tangent to $\widetilde{M}$, that is, $hX \in T\widetilde{M}$.

Lemma 3.1.1. *Let $\widetilde{M}$ be a submanifold of a conformal Kenmotsu manifold M tan-*

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

gent to ξ and normal to B. Then

$$P(X,Y) = -\omega(h(X,Y)) - \frac{1}{2}||B||^2 g(X,Y), \tag{3.4}$$

$$\widetilde{h}(X,\xi) = -\eta(X)B, \tag{3.5}$$

$$\widetilde{\nabla}_X \xi = hX + \omega(\xi)X, \tag{3.6}$$

for all vector fields X and Y tangent to $\widetilde{M}$.

Proof. From (2.88), we have

$$\begin{aligned} P(X,Y) &= (\nabla_X \omega)Y + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y) \\ &= \nabla_X(\omega(Y)) - \omega(\nabla_X Y) + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y), \end{aligned}$$

for all vector fields X and Y tangent to $\widetilde{M}$. Since B is normal to $\widetilde{M}$, the above equation can be written as

$$P(X,Y) = -\omega(\nabla_X Y) - \frac{1}{2}||B||^2 g(X,Y),$$

for all X and Y on $\widetilde{M}$. Then we obtain (3.4) using the Gauss formula.

Taking $Y = \xi$ in the Gauss formula and using (2.54), we have

$$\widetilde{\nabla}_X \xi + \widetilde{h}(X,\xi) = \nabla_X \xi = hX + \omega(\xi)X - \eta(X)B \tag{3.7}$$

for each vector field X tangent to $\widetilde{M}$. Since B is normal to $\widetilde{M}$, we compare the tangential part and the normal part in the above equation. Then we obtain (3.5) and (3.6). □

Lemma 3.1.2. *Let $\widetilde{M}$ be a submanifold of a conformal Kenmotsu manifold M tangent to both ξ and B. Then*

$$P(X,Y) = g(\nabla_X B, Y) + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y), \tag{3.8}$$

$$\widetilde{h}(X,\xi) = 0, \tag{3.9}$$

$$\widetilde{\nabla}_X \xi = hX + \omega(\xi)X - \eta(X)B, \tag{3.10}$$

for all vector fields X and Y tangent to $\widetilde{M}$.

Proof. From (2.88), we have

$$\begin{aligned} P(X,Y) &= (\nabla_X \omega)Y + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y) \\ &= \nabla_X(\omega(Y)) - \omega(\nabla_X Y) + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y), \end{aligned}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

for all vector fields X and Y tangent to $\widetilde{M}$. Since $\omega(X) = g(X, B)$, the above equation can be written as

$$\begin{aligned} P(X,Y) &= \nabla_X(g(Y,B)) - \omega(g(\nabla_X Y, B)) + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y), \\ &= g(\nabla_X B, Y) + \omega(X)\omega(Y) - \frac{1}{2}||B||^2 g(X,Y), \end{aligned}$$

for all X and Y on $\widetilde{M}$. Then we obtain (3.8) using the Gauss formula.

Taking $Y = \xi$ in the Gauss formula and using (2.54), we have

$$\widetilde{\nabla}_X \xi + \widetilde{h}(X,\xi) = \nabla_X \xi = hX + \omega(\xi)X - \eta(X)B$$

Substitution of (2.54) into the above equation, we get

$$\begin{aligned} &hX + \omega(\xi)X - \eta(X)B + h(X,\xi) = hX + \omega(\xi)X - \eta(X)\xi \\ &h(X,\xi) = 0 \end{aligned}$$

for each vector field X tangent to $\widetilde{M}$. □

3.2 Contact CR-submanifolds

In this section, we introduce the concept of contact CR-submanifold of an l.c. almost Kenmotsu manifold.

Let $(\widetilde{M}, \widetilde{g})$ be an immersed submanifold of a $(2n+1)$-dimensional l.c. almost Kenmotsu manifold M. Then we have the following definition adapted from the one given in [41].

Definition 3.2.1. *A submanifold $\widetilde{M}$ in an l.c. almost Kenmotsu manifold M is called contact CR-submanifold if ξ is tangent to $\widetilde{M}$ and there exists on $\widetilde{M}$ a smooth distribution $\widetilde{D} : x \longmapsto \widetilde{D}_x \subset T_x M$ such that*

(i) $\widetilde{D}_x$ is invariant under ϕ (i.e. $\phi\widetilde{D}_x \subset \widetilde{D}_x$) for each $x \in \widetilde{M}$;

(ii) the orthogonal complementary distribution $\widetilde{D}^{\perp} : x \longmapsto \widetilde{D}^{\perp}_x \subset T_x\widetilde{M}$ of the distribution $\widetilde{D}$ is totally real (i.e. $\phi\widetilde{D}^{\perp} \subset T_x\widetilde{M}^{\perp}$);

(iii) $T\widetilde{M} = \widetilde{D} \oplus \widetilde{D}^{\perp} \oplus \{\xi\}$, where $T_x\widetilde{M}$ and $T_x\widetilde{M}^{\perp}$ are the tangent space and the normal space of $\widetilde{M}$ at x, respectively, $\oplus$ denotes the orthogonal direct sum and $\{\xi\}$ is the line bundle spanned by ξ.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

The projections of $T\widetilde{M}$ to $\widetilde{D}$ and $\widetilde{D}^{\perp}$ are denoted by h and v, respectively, i.e for any $X \in \Gamma(T\widetilde{M})$,

$$X = X^h + X^v + \eta(X)\xi. \tag{3.11}$$

Applying ϕ to X, one gets

$$\phi X = FX + NX, \quad \forall X \in \Gamma(TM), \tag{3.12}$$

where $FX = \phi hX$ and $NX = \phi\nu X$ are tangential and normal components of ϕX, respectively. The normal bundle to M has the following decomposition

$$T\widetilde{M}^{\perp} = \phi\widetilde{D}^{\perp} \oplus \nu, \tag{3.13}$$

where ν denotes the orthogonal complementary distribution of $\phi\widetilde{D}^{\perp}$ and is an invariant normal subbundle of $T\widetilde{M}^{\perp}$ under ϕ. For any $V \in TM^{\perp}$, we have

$$V = pV + qV, \tag{3.14}$$

where $pV \in \phi\widetilde{D}^{\perp}$, $qV \in \nu$. Applying ϕ to the above equation yield

$$\phi V = fV + nV, \ \ \forall V \in TM^{\perp}, \tag{3.15}$$

where $fV = \phi pV \in \widetilde{D}^{\perp}$ and $nV = \phi qV \in \nu$.

Comparing tangential and normal components in (2.1), we obtain the following lemmas.

Lemma 3.2.1. *For a contact CR-submanifold $\widetilde{M}$ of an l.c. almost Kenmotsu manifold M tangent to ξ, the following identities hold:*

$$F^2 + fN = -\mathbb{I} + \eta \oplus \xi, \tag{3.16}$$

$$NF + nN = 0, \tag{3.17}$$

$$Ff + fn = 0, \tag{3.18}$$

$$n^2 + Nf = -\mathbb{I}. \tag{3.19}$$

Proof. Let $X \in \Gamma(T\widetilde{M})$. Then

$$\phi X = FX + NX. \tag{3.20}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Multiplying equation (3.20) by ϕ, gives

$$\begin{aligned}\phi^2 X &= \phi(\phi X)\\ &= \phi(FX + NX)\\ &= \phi FX + \phi NX\\ &= F(FX) + N(FX) + fNX + nNX\\ &= F^2X + NFX + fNX + nNX\\ &= F^2X + fNX + FNX + nNX. \end{aligned} \tag{3.21}$$

Since $\phi^2 X = -X + \eta(X)\xi$, we have

$$-X + \eta(X)\xi = F^2X + fNX + NFX + nNX, \tag{3.22}$$

and by comparing the tangential and normal part in (3.22), we get

$$F^2X + fNX = -X + \eta(X)\xi, \tag{3.23}$$

$$NFX + nNX = 0. \tag{3.24}$$

Therefore,

$$F^2 + fN = -\mathbb{I} + \eta \oplus \xi, \tag{3.25}$$

$$NF + nN = 0. \tag{3.26}$$

Similarly, let $V \in \Gamma(T\widetilde{M}^{\perp})$. Then $\phi V = fV + nV$. Applying ϕ, we have,

$$\begin{aligned}\phi^2 V &= \phi(\phi V)\\ \phi^2 V &= \phi(fV + nV)\\ -V &= \phi fV + \phi nV\\ -V &= FfV + NfV + fnV + nnV \end{aligned} \tag{3.27}$$

Therefore,

$$\begin{aligned} Ff + fN &= = 0\\ Nf + n^2 &= -\mathbb{I}, \end{aligned} \tag{3.28}$$

and this ends the proof. □

By letting $Y = \xi$ into (3.1), one has,

$$\begin{aligned}\widetilde{\nabla}_X \xi + \widetilde{h}(X,\xi) &= \nabla_X \xi\\ &= hX + \omega(\xi)X - \eta(X)B. \end{aligned} \tag{3.29}$$

We have the following.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Lemma 3.2.2. *Let $\widetilde{M}$ be a contact CR-submanifold of an l.c. almost Kenmotsu M tangent to ξ. Then, we have the following identities:*

$$\begin{aligned} \widetilde{\nabla}_X \xi &= \omega(\xi)X - \eta(X)B^T + hX, \\ \widetilde{h}(X,\xi) &= -\eta(X)B^N, \end{aligned} \tag{3.30}$$

for all vector fields X and Y tangent to $\widetilde{M}$.

Proof. If M is an l.c. almost Kenmotsu manifold, then we have

$$\nabla_X \xi = \omega(\xi)X - \eta(X)B + hX. \tag{3.31}$$

Then

$$\widetilde{\nabla}_X \xi + \widetilde{h}(X,\xi) = hX + \omega(\xi)X - \eta(X)B \tag{3.32}$$
$$= (h^N + h^T)X + \omega(\xi)X - \eta(X)(B^N + B^T). \tag{3.33}$$

Thus,

$$\widetilde{\nabla}_X \xi = \omega(\xi)X - \eta(X)B^T + h^T X, \tag{3.34}$$
$$\widetilde{h}(X,\xi) = -\eta(X)B^N + h^N X, \tag{3.35}$$

and this ends the proof. □

We define the covariant derivatives of F and N by

$$(\widetilde{\nabla}_X F)Y = \widetilde{\nabla}_X FY - F\widetilde{\nabla}_X Y \tag{3.36}$$

and

$$(\widetilde{\nabla}_X N)Y = \widetilde{\nabla}^{\perp}_X NY - N\widetilde{\nabla}_X Y, \tag{3.37}$$

respectively, for all vector fields X and Y tangent to $\widetilde{M}$. We have

$$\begin{aligned} (\nabla_X \phi)Y &= \nabla_X \phi Y - \phi(\nabla_X Y) \\ &= \nabla_X FY + \nabla_X NY - \phi(\nabla_X Y + \widetilde{h}(X,Y)) \\ &= \widetilde{\nabla}_X FY + \widetilde{h}(X,FY) - A_{NY}X + \widetilde{\nabla}^{\perp}_X NY - \phi\widetilde{\nabla}_X Y - \phi\widetilde{h}(X,Y) \\ &= \widetilde{\nabla}_X FY + \widetilde{h}(X,FY) - A_{NY}X + \widetilde{\nabla}^{\perp}_X NY - F\widetilde{\nabla}_X Y - N\widetilde{\nabla}_X Y + \\ &- f\widetilde{h}(X,Y) - n\widetilde{h}(X,Y) \\ &= (\widetilde{\nabla}_X F)Y + \widetilde{h}(X,FY) - A_{NY}X + (\widetilde{\nabla}_X N)Y - f\widetilde{h}(X,Y) + \\ &- n\widetilde{h}(X,Y). \end{aligned} \tag{3.38}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

If $\widetilde{M}$ is a contact CR-submanifold of an l.c. Kenmotsu M tangent to ξ, then equating (2.50) and (3.38), gives

$$\begin{aligned}\exp(-\sigma_t)\{\widetilde{g}(\phi X, Y)\xi - \eta(Y)\phi X\} + \omega(\phi Y)X - \omega(Y)\phi X \\ - \widetilde{g}(X, \phi Y)B + \widetilde{g}(X, Y)\phi B = (\widetilde{\nabla}_X F)Y + \widetilde{h}(X, FY) - A_{NY}X \\ + (\widetilde{\nabla}_X N)Y - f\widetilde{h}(X, Y) - n\widetilde{h}(X, Y). \qquad (3.39)\end{aligned}$$

We know that $\phi B = FB + NB$ and $B = B^T + B^N$, $\phi X = FX + NX$. Comparing the tangential and normal components in (3.39), one gets

$$\begin{aligned}\exp(-\sigma_t)\{\widetilde{g}(\phi X, Y)\xi - \eta(Y)FX\} + \omega(\phi Y)X - \omega(Y)FX - \widetilde{g}(X, \phi Y)B^T \\ + \widetilde{g}(X, Y)FB = (\widetilde{\nabla}_X F)Y - A_{NY}X - f\widetilde{h}(X, Y), \qquad (3.40)\end{aligned}$$

and

$$\begin{aligned}-\exp(-\sigma_t)\eta(Y)NX - \omega(Y)NX - \widetilde{g}(X, \phi Y)B^N + \widetilde{g}(X, Y)NB = \widetilde{h}(X, FY) \\ + (\widetilde{\nabla}_X N)Y - n\widetilde{h}(X, Y). \qquad (3.41)\end{aligned}$$

Therefore,

$$\begin{aligned}(\widetilde{\nabla}_X F)Y = \exp(-\sigma_t)\{\widetilde{g}(\phi X, Y)\xi - \eta(Y)FX\} + \omega(\phi Y)X - \omega(Y)FX \\ - \widetilde{g}(X, \phi Y)B^T + \widetilde{g}(X, Y)FB + A_{NY}X + f\widetilde{h}(X, Y), \qquad (3.42)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\nabla}_X N)Y = -\exp(-\sigma_t)\eta(Y)NX + n\widetilde{h}(X, Y) - \widetilde{h}(X, FY) - \omega(Y)NX \\ - \widetilde{g}(X, \phi Y)B^N + \widetilde{g}(X, Y)NB. \qquad (3.43)\end{aligned}$$

Therefore, we have the following.

Lemma 3.2.3. *Let $\widetilde{M}$ be a contact CR-submanifold of an l.c. Kenmotsu M tangent to ξ. Then,*

$$\begin{aligned}(\widetilde{\nabla}_X F)Y = \exp(-\sigma_t)\{\widetilde{g}(\phi X, Y)\xi - \eta(Y)FX\} + \omega(\phi Y)X - \omega(Y)FX \\ - \widetilde{g}(X, \phi Y)B^T + \widetilde{g}(X, Y)FB + A_{NY}X + f\widetilde{h}(X, Y), \qquad (3.44)\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\nabla}_X N)Y = -\exp(-\sigma_t)\eta(Y)NX + n\widetilde{h}(X, Y) - \widetilde{h}(X, FY) - \omega(Y)NX \\ - \widetilde{g}(X, \phi Y)B^N + \widetilde{g}(X, Y)NB, \qquad (3.45)\end{aligned}$$

for all vector fields X and Y tangent to $\widetilde{M}$.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Similarly, for any vector fields X tangent to $\widetilde{M}$ and any vector field V normal to $\widetilde{M}$, we have

$$\begin{aligned}(\nabla_X\phi)V &= \nabla_X\phi V - \phi(\nabla_X V)\\ &= \nabla_X fV + \nabla_X nV - \phi(-A_V X + \widetilde{\nabla}^{\perp}_X V)\\ &= \widetilde{\nabla}_X fV + \widetilde{h}(X, fV) - A_{nV}X + \widetilde{\nabla}^T_X nV - \phi(-A_V X) - \phi\widetilde{\nabla}^T_X V\\ &= \widetilde{\nabla}_X fV + \widetilde{h}(X, fV) - A_{nV}X + \widetilde{\nabla}^T_X nV + fA_V X + nA_V X\\ &- f\widetilde{\nabla}^T_X V - n\widetilde{\nabla}^T_X V\\ &= (\widetilde{\nabla}_X f)V + \widetilde{h}(X, fV) - A_{nV}X + (\widetilde{\nabla}_X n)V + fA_V X + nA_n X.\end{aligned}$$

That is,

$$\begin{aligned}\exp(-\sigma_t)\widetilde{g}(\phi X, V)\xi + \omega(\phi V)X - \omega(V)\phi X - \widetilde{g}(X, \phi V)B = (\widetilde{\nabla}_X f)V + \widetilde{h}(X, fV)\\ - A_{nV}X + (\widetilde{\nabla}_X n)V + fA_V X + nA_n X. \qquad (3.46)\end{aligned}$$

Since $\phi X = FX + NX$, (3.46) becomes

$$\begin{aligned}&\exp(-\sigma_t)\widetilde{g}(\phi X, V)\xi + \omega(\phi V)X - \omega(V)(FX + NX) - \widetilde{g}(X, \phi V)(B^T + B^N)\\ &= (\widetilde{\nabla}_X f)V + \widetilde{h}(X, fV) - A_{nV}X + (\widetilde{\nabla}_X n)V + fA_V X + nA_n X. \qquad (3.47)\end{aligned}$$

Lemma 3.2.4. *Let $\widetilde{M}$ be a contact CR-submanifold of an l.c. Kenmotsu M tangent to ξ. Then,*

$$\begin{aligned}(\widetilde{\nabla}_X f)V &= \exp(-\sigma_t)\widetilde{g}(\phi X, V)\xi + A_{nV}X - fA_V X + \omega(\phi V)X - \omega(V)FX\\ &- \widetilde{g}(X, \phi V)B^T, \qquad (3.48)\end{aligned}$$

$$\text{and} \quad (\widetilde{\nabla}_X n)V = -nA_n X - \widetilde{h}(X, fV) - \omega(V)NX - \widetilde{g}(X, \phi V)B^N, \qquad (3.49)$$

for all vector field X tangent to $\widetilde{M}$.

3.3 Integrability of distributions in CR-submanifolds

In this subsection we study the integrability of distributions of $\widetilde{D}$ and $\widetilde{D}^{\perp}$ of the contact CR-submanifold $\widetilde{M}$ of an l.c. Kenmotsu manifold M. Let $X, Y \in \widetilde{D}^{\perp}$. Then $FX = 0$, and hence

$$\begin{aligned}\widetilde{g}((\widetilde{\nabla}_Z F)X, Y) &= \widetilde{g}(\widetilde{\nabla}_Z(FX), Y) - \widetilde{g}(F\widetilde{\nabla}_Z X, Y)\\ &= -\widetilde{g}(\phi\widetilde{\nabla}_Z X - N\widetilde{\nabla}_Z X, Y)\\ &= -\widetilde{g}(\phi\widetilde{\nabla}_Z X, Y) + \widetilde{g}(N\widetilde{\nabla}_Z X, Y)\\ &= -\widetilde{g}(\phi\widetilde{\nabla}_Z X, Y)\\ &= \widetilde{g}(\widetilde{\nabla}_Z X, \phi Y)\\ &= 0 \qquad (3.50)\end{aligned}$$

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

for any vector field Z tangent to $\widetilde{M}$. Since

$$(\widetilde{\nabla}_X F)Y = \exp(-\sigma_t)\{\widetilde{g}(\phi X, Y)\xi - \eta(Y)FX\} + \omega(\phi Y)X - \omega(Y)FX \\ - \widetilde{g}(X, \phi Y)B^T + \widetilde{g}(X, Y)FB + A_{NY}X + f\widetilde{h}(X, Y), \tag{3.51}$$

then, for any $X, Y \in \Gamma(\widetilde{D}^{\perp})$ and $Z \in \Gamma(T\widetilde{M})$, we have that

$$0 = \widetilde{g}((\widetilde{\nabla}_Z F)X, Y) = \exp(-\sigma_t)\{\widetilde{g}(\phi Z, X)\eta(Y) - \eta(X)\widetilde{g}(FZ, Y)\} + \omega(\phi X)\widetilde{g}(Z, Y) \\ - \omega(X)\widetilde{g}(FZ, Y) - \widetilde{g}(Z, \phi X)\widetilde{g}(B^T, Y) + \widetilde{g}(Z, X)\widetilde{g}(FB, Y) \\ + \widetilde{g}(A_{NX}Z, Y) + \widetilde{g}(f\widetilde{h}(Z, X), Y). \tag{3.52}$$

since $\eta(X) = \eta(Y) = 0$. Equation (3.52) becomes

$$0 = \widetilde{g}((\widetilde{\nabla}_Z F)X, Y) = \omega(\phi X)\widetilde{g}(Z, Y) - \omega(X)\widetilde{g}(FZ, Y) - \widetilde{g}(Z, \phi X)\widetilde{g}(B^T, Y) \\ + \widetilde{g}(Z, X)\widetilde{g}(FB, Y) + \widetilde{g}(A_{NX}Z, Y) + \widetilde{g}(f\widetilde{h}(Z, X), Y). \tag{3.53}$$

On the other hand, $\widetilde{g}(FZ, Y) = 0$, $\widetilde{g}(Z, \phi X) = 0$, $\widetilde{g}(FB, Y) = -\omega(\phi Y)$ and

$$\begin{aligned} \widetilde{g}(f\widetilde{h}(Z, X), Y) &= \widetilde{g}(\phi\widetilde{h}(Z, X), Y) \\ &= -\widetilde{g}(\widetilde{h}(Z, X), \phi Y) \\ &= -\widetilde{g}(\widetilde{h}(Z, X), NY) \\ &= -\widetilde{g}(A_{NY}Z, X). \end{aligned} \tag{3.54}$$

Substituting (3.54) into (3.53) gives

$$0 = \omega(\phi X)\widetilde{g}(Z, Y) - \omega(\phi Y)\widetilde{g}(Z, X) + \widetilde{g}(A_{NX}Z, Y) - \widetilde{g}(A_{NY}Z, X). \tag{3.55}$$

That is,

$$\widetilde{g}(A_{NX}Y, Z) - \widetilde{g}(A_{NY}X, Z) = -\omega(\phi X)\widetilde{g}(Z, Y) + \omega(\phi Y)\widetilde{g}(Z, X). \tag{3.56}$$

Thus, we have the following Lemma.

Lemma 3.3.1. *Let $\widetilde{M}$ be a contact CR-submanifold of an l.c. Kenmotsu M tangent to ξ. Then,*

$$A_{NX}Y - A_{NY}X = \omega(\phi Y)X - \omega(\phi X)Y \tag{3.57}$$

for all vector fields X and Y tangent to $\widetilde{M}$.

Theorem 3.3.1. *Let $\widetilde{M}$ be a contact CR-submanifold of a $(2n+1)$-dimensional l.c. Kenmotsu manifold M tangent to ξ. The distribution $\widetilde{D}^{\perp}$ is completely integrable and its maximal integral submanifold is a finite-dimensional anti-invariant submanifold of M normal to ξ.*

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Proof. Let $X, Y \in \widetilde{D}^{\perp}$. Then we have that

$$\begin{aligned}\phi[X,Y] &= F[X,Y] + N[X,Y] \\ &= F\widetilde{\nabla}_X Y - F\widetilde{\nabla}_Y X + N[X,Y] \\ &= \widetilde{\nabla}_X FY - (\widetilde{\nabla}_X F)Y - \widetilde{\nabla}_Y FX + (\widetilde{\nabla}_Y F)X + N[X,Y] \\ &= -(\widetilde{\nabla}_X F)Y + (\widetilde{\nabla}_Y F)X + N[X,Y]. \end{aligned} \tag{3.58}$$

Substituting (3.51) into (3.58), gives

$$\phi[X,Y] = \omega(\phi X)Y - \omega(\phi Y)X + A_{NX}Y - A_{NY}X + N[X,Y], \tag{3.59}$$

and by using (3.57), equation (3.59) simplifies to

$$\phi[X,Y] = N[X,Y]. \tag{3.60}$$

Therefore, $[X,Y] \in \Gamma(\widetilde{D}^{\perp})$. □

Theorem 3.3.2. *Let $\widetilde{M}$ be a contact CR-submanifold of a $(2n+1)$-dimensional l.c. Kenmotsu manifold M tangent to ξ. Then the distribution $\widetilde{D} \oplus \{\xi\}$ is integrable if and only if*

$$\widetilde{h}(X, FY) - \widetilde{h}(Y, FX) = 2g(Y, \phi X)B^N, \tag{3.61}$$

for any $X, Y \in \Gamma(\widetilde{D} \oplus \{\xi\})$.

Proof. Let $X, Y \in \Gamma(\widetilde{D} \oplus \{\xi\})$. Then, $NX = NY = 0$ and

$$\begin{aligned}\phi[X,Y] &= F[X,Y] + N[X,Y] \\ &= F[X,Y] + N(\widetilde{\nabla}_X Y) - N(\widetilde{\nabla}_Y)X \\ &= F[X,Y] + \widetilde{\nabla}_X NY - (\widetilde{\nabla}_X N)Y - \widetilde{\nabla}_Y NX + (\widetilde{\nabla}_Y N)X \\ &= F[X,Y] + (\widetilde{\nabla}_Y N)X - (\widetilde{\nabla}_X N)Y. \end{aligned} \tag{3.62}$$

Substituting (3.45) into (3.62), gives

$$\begin{aligned}\phi[X,Y] = F[X,Y] &+ n\widetilde{h}(Y,X) - \widetilde{h}(Y,FX) - \omega(X)NY - \widetilde{g}(Y,\phi X)B^N \\ &+ \widetilde{g}(Y,X)NB - n\widetilde{h}(X,Y) + \widetilde{h}(X,FY) + \omega(Y)NX + \widetilde{g}(X,\phi Y)B^N \\ &- \widetilde{g}(X,Y)NB, \end{aligned} \tag{3.63}$$

which becomes

$$\begin{aligned}\phi[X,Y] = F[X,Y] &- \widetilde{h}(Y,FX) - 2\widetilde{g}(Y,\phi X)B^N \\ &+ \widetilde{h}(X,FY). \end{aligned} \tag{3.64}$$

Thus, we see that $[X,Y] \in \Gamma(\widetilde{D} \oplus \{\xi\})$ iff $\widetilde{h}(X,FY) - \widetilde{h}(Y,FX) = 2\widetilde{g}(Y,\phi X)B^N$. □

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Proposition 3.3.1. *Let $\widetilde{M}$ be a contact CR-submanifold of a $(2n+1)$-dimensional l.c. Kenmotsu manifold M tangent to ξ. Then the distribution $\widetilde{D} \oplus \{\xi\}$ is integrable if and only if*

$$\widetilde{h}(X, FY) - \widetilde{h}(Y, FX) = 2\widetilde{g}(Y, \phi X)B^N, \tag{3.65}$$

for any $X, Y \in \Gamma(\widetilde{D} \oplus \{\xi\})$.

Proof. Let $X, Y \in \Gamma(\widetilde{D})$. Then, $NX = NY = 0$ and

$$\begin{aligned}\phi[X, Y] &= F[X, Y] + N[X, Y] \\ &= F[X, Y] + N(\widetilde{\nabla}_X Y) - N(\widetilde{\nabla}_Y)X \\ &= F[X, Y] + \widetilde{\nabla}_X NY - (\widetilde{\nabla}_X N)Y - \widetilde{\nabla}_Y NX + (\widetilde{\nabla}_Y N)X \\ &= F[X, Y] + (\widetilde{\nabla}_Y N)X - (\widetilde{\nabla}_X N)Y. \end{aligned} \tag{3.66}$$

Substituting (3.45) into (3.66), gives

$$\begin{aligned}\phi[X, Y] = F[X, Y] &+ n\widetilde{h}(Y, X) - \widetilde{h}(Y, FX) - \omega(X)NY - \widetilde{g}(Y, \phi X)B^N \\ &+ \widetilde{g}(Y, X)NB - n\widetilde{h}(X, Y) + \widetilde{h}(X, FY) + \omega(Y)NX + \widetilde{g}(X, \phi Y)B^N \\ &- \widetilde{g}(X, Y)NB, \end{aligned} \tag{3.67}$$

which becomes

$$\phi[X, Y] = F[X, Y] - \widetilde{h}(Y, FX) - 2\widetilde{g}(Y, \phi X)B^N + \widetilde{h}(X, FY), \tag{3.68}$$

Thus, we see that $[X, Y] \in \Gamma(\widetilde{D} \oplus \{\xi\})$ if and only if

$$\widetilde{h}(X, FY) - \widetilde{h}(Y, FX) = 2\widetilde{g}(Y, \phi X)B^N,$$

and that completes the proof. ∎

The relation (3.65) in the Theorem 3.3.2 is also equivalent to the integrability of the distribution $\widetilde{D}$.

Assume that $\widetilde{D}$ is integrable. Then, the relation (3.65) can be rewritten as,

$$\widetilde{h}(FX, FY) = \widetilde{h}(X, Y) - 2\widetilde{g}(X, Y)B^N, \tag{3.69}$$

for any $X, Y \in \Gamma(\widetilde{D})$.

Let M' be a leaf of $\widetilde{D}$. Then M' is a maximal integral submanifold immersed in M. For any X, $Y \in \Gamma(TM')$,

$$\nabla_X Y = \nabla'_X Y + h'(X, Y), \tag{3.70}$$

where

$$h' = \widetilde{h} + \sigma + \{-\omega(\xi)g + g \circ h\}\xi, \tag{3.71}$$

is the second fundamental form of M', immersed as a submanifold in M with σ a vector field in $\widetilde{D}^{\perp}$, and ∇' the Levi-Civita connection on M'.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Theorem 3.3.3. *Let $\widetilde{M}$ be a contact CR-submanifold of a $(2n+1)$-dimensional l.c. Kenmotsu manifold M tangent to ξ. Assume that the distribution $\widetilde{D}$ is integrable. Then the integral manifolds of the distribution $\widetilde{D}$ are l.c. Kähler manifolds with mean curvature vector field given by*

$$H' = \frac{1}{\mathrm{rank}(\widetilde{D})}\{ \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{h}(e_i, e_i) + \mathrm{trace}_{|_{M'}}\sigma\} - 4B^N - 2\omega(\xi)\xi\}. \tag{3.72}$$

Proof. Assume that the distribution $\widetilde{D}$ is integrable. Let M' be an integral manifold of $\widetilde{D}$. The tensor fields ϕ_t and g_t induce an almost complex structure $J_t = J$ and a Hermitian metric g'_t on M'. Then, for any $X \in \Gamma(TM')$, we have $\phi'^2_t X = -X + \eta'_t X \xi_t = -X = J_t^2 X = J^2 X$ and $d\Phi_t = 0$, so M' is an l.c. Kähler. Using (3.71), the second fundamental form of M' is explicitly given by

$$\widetilde{h}'(X,Y) = \widetilde{h}(X,Y) + \sigma(X,Y) + \{-\omega(\xi)\widetilde{g}(X,Y) + \widetilde{g}(hX,Y)\}\xi. \tag{3.73}$$

Fixing a local orthonormal frame $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n\}$ in TM' and applying the properties on h, one has,

$$\begin{aligned}
H' &= \frac{1}{\mathrm{rank}(\widetilde{D})}\{ \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{h}'(e_i, e_i) + \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{h}'(\phi e_i, \phi e_i)\} \\
&= \frac{1}{\mathrm{rank}(\widetilde{D})}\{\mathrm{trace}_{|_{M'}}\widetilde{h} + \mathrm{trace}_{|_{M'}}\sigma + \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \left(-\omega(\xi)\widetilde{g}(e_i, e_i) + \widetilde{g}(he_i, e_i)\right)\xi \\
&+ \sum_{i=1}^{\mathrm{rank}D)} \left(-\omega(\xi)\widetilde{g}(\phi e_i, \phi e_i) + \widetilde{g}(h\phi e_i, \phi e_i)\right)\xi\} \\
&= \frac{1}{\mathrm{rank}(\widetilde{D})}\{\mathrm{trace}_{|_{M'}}\widetilde{h} + \mathrm{trace}_{|_{M'}}\sigma - 2\omega(\xi) \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{g}(e_i, e_i)\xi \\
&= \frac{1}{\mathrm{rank}(\widetilde{D})}\{\mathrm{trace}_{|_{M'}}\widetilde{h} + \mathrm{trace}_{|_{M'}}\sigma - 2\mathrm{rank}(\widetilde{D})\omega(\xi)\xi\}.
\end{aligned} \tag{3.74}$$

Now, using the identity (3.16) and (3.69), the trace of $\widetilde{h}$ on M' is given,

$$\begin{aligned}
\mathrm{trace}_{|_{M'}}\widetilde{h} &= \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{h}(e_i, e_i) + \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{h}(\phi e_i, \phi e_i) \\
&= \sum_{i=1}^{\mathrm{rank}(\widetilde{D})} \widetilde{h}(e_i, e_i) - 4\mathrm{rank}(\widetilde{D})B^N.
\end{aligned} \tag{3.75}$$

Putting the relation (3.75) into (3.74) completes the proof. □

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Corollary 3.3.1. *Let $\widetilde{M}$ be a contact CR-submanifold of a $(2n+1)$-dimensional l.c. Kenmotsu manifold M tangent to ξ. Assume that the distribution $\widetilde{D}$ is integrable with a non-vanishing normal component of the vector field B (or a non-vanishing function $\omega(\xi)$), then the integral manifolds of the distribution $\widetilde{D}$ cannot be minimal.*

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

CHAPTER FOUR

CONCLUSION AND PERSPECTIVES

We introduced a new concept of almost contact structures, namely, l.c. almost Kenmotsu structures, supported by an example. The latter was characterized by the existence of a closed 1-form ω such that

$$d\eta = \omega \wedge \eta \text{ and } d\Phi = 2\left\{\exp(-\sigma_t)\eta + \omega\right\} \wedge \Phi,$$

with $\omega = d\sigma_t$. We proved that a locally conformal almost Kenmotsu manifold with the smooth 1-form, ω, that is proportional to the contact structure, η, i.e. $\omega = \alpha\eta$, is a β-Kenmotsu manifold with $\beta = \alpha + \exp(-\sigma_t)$.

In addition, the integrability of distributions was studied and the results have shown that the contact distribution $D = \ker\eta$ admits foliations whose leaves are l.c. almost Kählerian with mean curvature vector field $H' = -\omega(\xi)\xi$. We also proved that there exist classes of almost contact structures that admit foliations whose leaves are Kählerian and umbilical.

We investigated CR-submanifolds of l.c. almost Kenmotsu manifold and paid attention of distributions $\widetilde{D}$, $\widetilde{D}^{\perp}$ and $\widetilde{D} \oplus \{\xi\}$. As a result, we concluded that the distribution $\widehat{D}^{\perp}$ is completely integrable and its maximal integral submanifold is a finite-dimensional anti-invariant submanifold of M normal to ξ. Furthermore, $\widetilde{D}$ and $\widetilde{D} \oplus \{\xi\}$ are integrable if and only if

$$\widetilde{h}(X, FY) - \widetilde{h}(Y, FX) = 2\widetilde{g}(Y, \phi X)B^N.$$

The latter can then be extended to the foliation on the ambient manifolds M. That is, if the metric on an l.c. (almost) Kenmotsu manifold is bundle-like for the foliation $\mathcal{F}$, as one of the perspectives, we would like to know whether the leaves of $\mathcal{F}$ are (almost) Kähler or admit another geometric structure.

However, studies indicated that we are only at the beginning of the first study of locally conformal (almost) Kenmotsu manifolds. A lot can still be done in an l.c. (almost) Kenmotsu manifold. One may also look at which properties can be preserved in an l.c. (almost) Kenmotsu manifold with reference to other types of manifolds.

Our study revealed that l.c. almost Kenmotsu manifolds are not necessary Kenmotsu and that the vector field ξ is not a Killing vector field for l.c. almost Kenmotsu

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

manifolds. Among other orientations, we shall study the topology of l..c almost Kenmotsu geometry. It is known that there are more than four thousand almost contact structures, so we are planing to pay attention to some of them and see how we can extract their even-dimensional structure in order to see the light of proving the Golberg Conjecture which says that: *Any Einstein symplectic structure is Kähler.*

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

Bibliography

[1] H. M. Abood, Holomorphic-Geodesic Transformations of Almost Hermitian Manifold, PhD. Thesis, Moscow State Pedagogical University, Moscow (2002).

[2] N. Aktan, M. Yildirim and C. Murathan, Almost f-cosymplectic manifolds, Mediterr. J. Math., Vol. 11 no. 2 (2014), 775-787.

[3] M. Banaru, A new characterization of the Gray:Hervella classes of almost Hermitian manifolds, in: 8th International Conference on Differential Geometry and Its Applications, Opava, Czech Republic, 2001, 27-31.

[4] C. S. Bagewadi and Venkatesha, On projective recurrent Kenmotsu manifolds, MSJ, Bangalore 4(2005), 15-21.

[5] C. S. Bagewadi and Venkatesha, On pseudo projective ϕ-recurrent Kenmotsu manifolds, Soochow J. of Mathematics 32(2006), 1-7.

[6] C. S. Bagewadi and Venkatesha, Some curvature tensor on a Kenmotsu manfold, Tensor, 68(2007), 140-147.

[7] A. Bejancu and H. R. Farran, Foliations and Geometric Structures, Mathematics and its Applications, Springer, 2006.

[8] T. Q. Binh, L. Tamassy,U. C. De and M. Tarafder, Some Remarks on almost Kenmotsu manifolds, Math Pannon. 13(2002), 31-39.

[9] V. Bonanzinga and K. Matsumoto, Warped product of CR-submanifolds in locally conformal Kaehler manifolds, Period. Math. Hung., 48(2004), no. 1?2, 207-221 .

[10] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds.* Progress in Mathematics 203, Birkhauser Boston, Inc., Boston, MA, 2002.

[11] D. E. Blair, The theory of quasi-Sasakian structure J.D. Geo., 1 (1967), 331-345.

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

[12] S. S. Chern and R. S. Hamilton, On Riemannian metrics adopted to three dimensional contact manifolds. Let. Note Math.1111,279-308(1985)

[13] U. C. De, On ϕ-symmetric Kenmotsu manifolds. Int. Electronic J. Geom. 1, 1(2008), 33-38.

[14] U. C. De and G. Pathak, On 3-dimensional Kenmotsu manifolds. Ind. J. Pure Applied Math. 35(2004) 159-165.

[15] G. Dileo and A. M. Pastore, Almost Kenmotsu manifolds and local symmetry, Bull. Belg. Math. Soc. Simon Stevin 14 (2007) 343-354.

[16] S. S. Chern, R. S. Hamilton, On Riemannian metrics adapted to three dimensional contact manifolds, Lecture Note in Mathematics, Vol. 1111 (1985), 279-308.

[17] A. Ghosh, R. Sharma and J. T. Cho, Contact metric manifolds with η-parallel torsion tensor, Ann. Global Anal. Geom., Vol. 34 no. 3 (2008), 287-299.

[18] A. Gray and L. M. Hervella, The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., 123 (1980), no. 4, 35-58.

[19] J. B. Jun, U. C. De and G. Pathak, On Kenmotsu manifolds, J. Korean Math. Soc.42(2005),435-445.

[20] D. L. Johson, L. B. WHITT,: Totally Geodesic foliation. J.Differential geometry.15(1980)225-235.

[21] K. Kenmotsu: A class of almost contact Riemannian manifolds. Tohoku Math.J. 24, 93-103(1972).

[22] T. W. Kim and H. K. Pak, Canonical foliations of certain classes of almost contact metric structures, Acta Math. Sin. (Engl. Ser.) 21 No. 4 (2005) 841-846.

[23] M. KON, Invariant submanifolds in Sasakian manifolds, Mathematische Annalen 219(1976), 277-290.

[24] P. Libermann, Sur les structures presque complexes et autres structures infinitésimales régulières. Bull. Soc. Math. France, Vol. 83, 195-224 (1955).

[25] F. Massamba and A. Maloko Mavambou, A class of locally conformal almost cosymplectic manifolds. Bull. Malays. Math. Sci. soc. DOI 10.1007/s4840-016-03009-3, 2016

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.

[26] R. S. Mishra, Structures on differentiable manifold and their applications, Chandrama Prakashan, 50-A, Bairampur house, Allahabad, (1984).

[27] Z. Olszak, Locally conformal almost cosymplectic manifolds, Colloq. Math. 57 (1989), 73-87.

[28] Z. Olszak, On almost cosymplectic manifolds. Kodai Math.J.4(2),239-250(1981)

[29] C. Ozgur, On generalized recurrent Kenmotsu manifolds, World Applied Sciences Journal 2(2007), 29-33.

[30] C. Ozgur, , Differ. Geom. Dyn. Syt.8(2006), 204-209.

[31] G. Pitis, A remark on Kenmotsu manifolds. Buletinul Universitatii Din Brasov. Vol XXX. 1988.

[32] S. Sasaki and Y. Hatakeyama, On differentiable manifolds with certain structures which are closely related to almost contact structures II, Tohoku Math. J., 13 (1961), 281-294.

[33] S. Sasaki, *On differentiable manifolds with certain structures which are closely related to almost contact structure*, Tohoku Math.J, 12 (1960), 459-476.

[34] S. Tanno, The automorphism groups of almost contact Riemannian manifolds, Tohoku Math. J. 21 (1969), 21-38.

[35] P. Tondeur, Geometry of Foliations, Monographs in Mathematics, Birkhäuser, vol. 90, 1997.

[36] G.F. Torress del Castillo, *Differential manifolds: A Theoretical Physics Approach*, Springer Science+Business Media, New York, (2010).

[37] F. Tricerri, Some examples of locally conformal Kähler manifolds, Rend. Sem. Mat. TÃ´rino, 40 (1982), 81-92.

[38] T. Vanli, A. Ramazan and S. Ramazan, 2016). Generalized Kenmotsu Manifolds.

[39] I. Vaisman, Conformal Changes of Almost Contact Metric structures. Lecture Notes in Mathematics, Vol. 792, pp. 435-443. Springer, Berlin(1980).

[40] I. Vaisman, A geometric condition for an l.c.K. manifold to be Kähler, Geom. Dedicata, Vol. 10, 129?134 (1981).

[41] K. Yano and M. Kon, *CR-submanifolds of Kaehlerian and Sasakian manifolds*, Birkhauser, Vol.30, U.S.A (1983).

The financial assistance of the National Research Foundation (NRF) towards this research is hereby acknowledged. Opinions expressed and conclusions arrived at, are those of the student and are not necessarily to be attributed to the NRF.