ITERATIVE APPROXIMATION OF SOLUTIONS OF SOME OPTIMIZATION PROBLEMS IN BANACH SPACES

by

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As the candidate's supervisor I have approved this dissertation for submission.

Dr. O.T. Mewomo

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Dedication

To my Teachers.

Abstract

Let C be a nonempty closed convex subset of a q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping. In this dissertation, we study the approximation of the zero of a strongly accretive operator $A : X \to X$ which is also a fixed point of a k-strictly pseudo-contractive self mapping T of C. Also, we introduce a U-mapping for finite family of mixed equilibrium problems involving $\mu - \alpha$ relaxed monotone operators. We prove a strong convergence theorem for finding a common solution of finite family of these equilibrium problems in a uniformly smooth and strictly convex Banach space. We present some applications of this theorem and a numerical example. Furthermore, due to the faster rate of convergence of inertial type algorithm, we propose an inertial type iterative algorithm and prove a weak convergence theorem of the scheme to a solution of split variational inclusion problems involving accretive operators in Banach spaces. We give some applications and a numerical example to show the relevance of our result. Our results in this dissertation extend and improve some recent results in the literature.

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Declaration

I declare that this dissertation, in part or in its entirety, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Oyewole Olawale Kazeem

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CHAPTER 1

Introduction

1.1 Background of Study

In mathematics, computer science and other fields of knowledge, an optimization problem is one where the values of a function $g : X \to \mathbb{R}$ are to be maximized or minimized over a given nonempty set of feasible alternatives $D \subset X$. The function g is called the objective function while the set $D \subset X$ is called the constraint set. Optimization problem can be modelled in the form of minimization problem, variational-inequality problem, equilibrium problem, convex minimization problem, linear programming, min-max problem, e.t.c. Some examples of optimization problem in real life include utility maximization, profit maximization, expenditure and cost minimization, portfolio choice, electricity units consumption minimization, replacement models among others.

Let X be a real Banach space, a point $x \in X$ is called a fixed point of a nonlinear operator $T : X \to X$ if

$$x = Tx. \tag{1.1.1}$$

If T is a set-valued mapping, then $x \in X$ is called a fixed point of T if $x \in Tx$. The theory of fixed point is a widely researched area in nonlinear analysis, it can be considered as the kernel of the modern nonlinear analysis for the role it plays in various mathematical models arising from optimization problems and differential equations. There have been enormous development of interesting and efficient techniques for computing fixed points, this has in turn increased the usefulness of the theory of fixed points and its applications. Therefore, the theory of fixed point is increasingly becoming a powerful, useful and effective tool in mathematics, engineering, physics, biology, economics, computer science etc. In several mathematical problems, the existence of solution of such problems is equivalent to the existence of fixed point of a suitable map. One of such instances is the so called zero of nonlinear operator in both finite and infinite dimensional spaces. The existence of fixed point is thus important in a wide range of mathematics and other sciences.

The concept of accretive operators introduced by Browder [29] in (1967) has proved to be very useful in partial differential equations. For example, consider an Initial Value Problem (IVP) of the form

$$\frac{dx}{dt} + Ax(t) = 0, \quad x(0) = x_0, \tag{1.1.2}$$

which describes an evolutional system where A is an accretive operator from a Banach space X onto itself. At equilibrium state, $\frac{dx}{dt} = 0$ and a solution of

$$Ax = 0, \tag{1.1.3}$$

describes the equilibrium state of the system (1.1.2).
Since A is nonlinear, there is no closed form solution of the equation (1.1.3). Rather the closer one can get is to find an approximation. To do this, Browder [29] considered the problem of solving the zeros of A in (1.1.3) as that of a fixed point of a certain $T = I - A$ which he referred to as being pseudo-contractive, where A is accretive and I is an identity map on X. He obtained therefore that a fixed point of T represents the zero of A. Thus finding a solution of (1.1.3) is equivalent to finding the fixed point of T ([29, 49]).

The equilibrium problem is one of the most important topics in nonlinear analysis and in other several applied fields [127]. The equilibrium problem has been widely studied in the context of optimization problem, fixed point problems, variational inequality problem whether monotone or otherwise (see [9, 42, 45]). It is also a special case of Nash equilibrium and some other problems ([7, 8, 89]).
The Equilibrium Problem (EP) is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C, \tag{1.1.4}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction and C is a nonempty, closed and convex subset of a Banach space X. The set of solutions of (1.1.4) will be denoted by EP(F).
The equilibrium problem was first investigated in 1961 by Ky Fan [63] in connection with the so-called "intersection theorems" (A result which proves that intersection of a certain family of sets is nonempty). Ky Fan considered what is called a Scalar Equilibrium Problem (SEP). The term "equilibrium problem" was however attributed to Oettli and Blum [18] who many believed introduced the problem. The existence of solution of equilibrium problem has thus been studied extensively since introduction (see [16, 13, 14, 84]). There have also been a lot of outstanding results in approximating solutions of equilibrium problem. There are also several problems of interest with various applications which generalizes the equilibrium problem, some of this problems are discussed in Chapter 2 and one of them solved in Chapter 3.

1.2 Research Motivation

Since the introduction of accretive operators in 1967 (see Browder [29]), there have been several attempts to find approximations of zeros of such operators in Banach spaces. Most especially using the connections between the theory of accretive operators and the fixed point theory. Xu [168], Kim and Xu [97] studied convergence of the sequence $\{x_n\} \subset X$ generated by $x_1 \in C$ for a nonempty, closed and convex subset C of X defined iteratively by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)J_{r_n}x_n, \quad n \geq 1. \tag{1.2.1}$$

They proved the strong convergence of (1.2.1) to the zero of an accretive operator A on X using the fixed point of $J_{r_n} := (I + r_n A)^{-1}$ called the reslovent of A, in the framework of a reflexive Banach space and uniformly smooth Banach space with a weakly sequentially continuous duality map respectively.
Qin and Su [133], proposed a sequence $\{x_n\}$ defined iteratively for an arbitrary $x_1 \in C$ by

$$\begin{cases} y_n = \beta_n x_n + (1-\beta_n)J_{r_n}x_n, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)y_n, \end{cases} \tag{1.2.2}$$

where $u \in C$ is an arbitrary but fixed element in C and the sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1]$ and $\{r_n\}$ a sequence of nonnegative real numbers. They proved under certain conditions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ that the sequence $\{x_n\}$ defined by (1.2.2) converges to zero point of A.
Very recently, Aoyoma and Toyoda [4], in their effort to approximate the zero of an m-accretive operator A on a Banach space X studied the sequence defined by (1.2.1). They obtained the strong convergence of the sequence $\{x_n\}$ to a zero of A using the resolvent operator defined on A.
Motivated by the above results, we introduce an algorithm which is a Halpern-type three step iterative scheme. Let $\{x_n\}$ be a sequence defined in the following manner: $x_1 \in C$, and

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)y_n, \quad n \geq 1. \end{cases} \tag{1.2.3}$$

We prove strong convergence of (1.2.3) a zero of a strongly accretive operator A which satisfies the range condition and fixed point of k-strictly pseudocontractive mapping T.

Let C be a nonempty, closed and convex susbset of a real Banach space X with a topological dual space X^*. The mixed equilibrium problem (see [165]) is to find $x \in C$ such that

$$F(x,y) + \langle Ax, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall\, y \in C, \tag{1.2.4}$$

where $F : C \times C \to \mathbb{R}$ is a bifunction, $A : C \to X^*$ is a nonlinear mapping and $\phi : C \times C \to R \cup \{+\infty\}$ is a proper function. There have been several iterative methods in the literature proposed for finding the solutions of fixed point problems and mixed equilibrium problems with relaxed monotone mappings in various settings (see [39, 67, 87, 91]). Wang et al [165] introduced the following iterative algorithm for finding a common element of the set of solutions of mixed equilibrium problem with relaxed monotone mapping and the set of fixed points of nonexpansive mappings in Hilbert space:

$$\begin{cases} x_1 \in C, \\ F(u_n, y) + \langle Ay_n, \mu(y, u_n)\rangle + \dfrac{1}{\lambda_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)\beta_n Sx_n + (1-\alpha_n)(1-\beta_n)u_n, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, D_n = \cap_{j=1}^n C_j, \\ x_{n+1} = P_{D_n}x_1, \quad n \geq 1, \end{cases} \tag{1.2.5}$$

where A is a relaxed $\mu - \alpha$ monotone mapping and $S : C \to C$ is a nonexpansive mapping. They obtained strong convergence of the scheme (1.2.5) to a common solution of mixed equilibrium problem (1.2.4) and the fixed point of S under appropriate conditions on the control sequences $\{\alpha_n\}, \{\beta_n\}$ and $\{\lambda_n\}$.

Chen et al [44] introduced a new algorithm for approximating solutions of the mixed equilibrium problem with a relaxed monotone mapping in the framework of a uniformly convex and uniformly smooth Banach spaces and proved the existence of solutions of the mixed equilibrium problem. In particular, they studied the following algorithm to find a common element of the set of solutions of the mixed equilibrium and the set of fixed points of a quasi-ϕ nonexpansive mapping $S : C \to C$.

$$\begin{cases} x_1 = x \in C, \\ y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \\ u_n \in C \ \text{ such that} \\ F(u_n, y) + \langle Au_n, \mu(y, u_n)\rangle + f(y) - f(u_n) + \dfrac{1}{r_n}\langle y - u_n, Ju_n - Jy_n\rangle \geq 0, \quad \forall \ y \in C, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ D_n = \cap_{j=1}^n C_j, \\ P_{D_n}x_1, \quad n \geq 1, \end{cases} \tag{1.2.6}$$

where $f : C \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous mapping and J is the normalized duality mapping. Under certain assumptions on the parameter sequences $\{\alpha_n\}$ and $\{r_n\}$, they obtained a strong convergence of (1.2.6) to a solution of the mixed equilibrium problem which is also the fixed point of S. On the hand, Atsushiba and Takahashi [6] introduced the following W-mapping for finite family of nonexpansive mappings. For $i = 1, 2 \cdots, N$, let $T_i : C \to C$ be a finite family of nonexpansive mappings such that $\cap_{i=1}^N F(T_i) \neq \emptyset$, the W-mapping is defined as follows:

$$\begin{aligned} U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\ U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})I, \\ &\vdots \\ U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})I, \\ W_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})I, \end{aligned}$$

where $\{\lambda_{n,i}\} \subset [0, 1]$ for $i = 1, 2 \ldots, N$ and C is a nonempty, closed and convex subset of a Banach space X. Takahashi and Shimoji [153] proved that if X is a strictly convex

Banach space, then $F(W_n) = \cap_{i=1}^N F(T_i)$, where $\{\lambda_{n,i}\} \subset (0,1)$ for $i = 1, 2\cdots, N$. Also, Kangtunyakarn and Suntai [94] introduced the K-mapping for finding a common fixed point of a finite family of nonexpansive mappings in a real Hilbert space. The K_n mapping is defined as follows:

$$\begin{aligned}
U_{n,1} &= \lambda_{n,1}T_1 + (1-\lambda_{n,1})I, \\
U_{n,2} &= \lambda_{n,2}T_2U_{n,1} + (1-\lambda_{n,2})U_{n,1}, \\
&\vdots \\
U_{n,N-1} &= \lambda_{n,N-1}T_{N-1}U_{n,N-2} + (1-\lambda_{n,N-1})U_{n,N-2}, \\
K_n &= U_{n,N} = \lambda_{n,N}T_NU_{n,N-1} + (1-\lambda_{n,N})U_{n,N-1}.
\end{aligned}$$

Motivated by the results in [6, 94, 153, 165], we propose a strong convergence theorem for finding the common solution of finite family of mixed equilibrium problems with $\mu-\alpha$ relaxed monotone mapping in the frame work of a uniformly smooth and strictly convex Banach space which also enjoys the Kadec-Klee property. First, we introduce the following mapping: Let C be a nonempty closed convex subset of a smooth and strictly convex Banach space X. For $i = 1, 2, \dots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, $A_i : C \to X^*$ be a finite family of μ hemicontinuous and relaxed $\mu-\alpha$ monotone mappings and $\phi_i : C \to \mathbb{R} \cup \{+\infty\}$ be a finite family of proper, convex and lower semicontinuous functions. For $i = 1, 2, \dots, N$ and $\{r_n\} \subset (0, \infty)$, the resolvent operator on F_i is defined in [44] as

$$K_{r_n}^i(x) := \{z \in X : F_i(z,y) + \langle A_iz, \mu(y,z)\rangle + \phi_i(y) - \phi_i(z) + \frac{1}{r_n}\langle y-z, Jz-Jx\rangle \geq 0.\ \forall\ y \in X\}.$$

However it has been proved in [44] that $K_{r_n}^i$ is single valued for each $i = 1, 2\dots, N$. (See Lemma 3.2.8). We define the mapping $U_n : C \to C$ as

$$\begin{cases}
S_{n,1} &= \lambda_{n,1}K_{r_n}^1 + (1-\lambda_{n,1})I, \\
S_{n,2} &= \lambda_{n,2}K_{r_n}^2S_{n,1} + (1-\lambda_{n,2})S_{n,1}, \\
\vdots & \\
S_{n,N-1} &= \lambda_{n,N-1}K_{r_n}^{N-1}S_{n,N-2} + (1-\lambda_{n,N-1})S_{n,N-2}, \\
U_n &= S_{n,N} = \lambda_{n,N}K_{r_n}^NS_{n,N-1} + (1-\lambda_{n,N})S_{n,N-1},
\end{cases} \tag{1.2.7}$$

where $0 \leq \lambda_{n,i} \leq 1$, for $i = 1, 2, \dots, N$. In addition, we present the following algorithm for finding a common solution of finite family of mixed equilibrium problems involving a relaxed monotone operator: For arbitrary $x_1 \in C$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n, \qquad \forall n \geq 1, \tag{1.2.8}$$

where U_n is as defined in (1.2.7) and f a contraction mapping from C to C. Furthermore, we obtain a strong convergence theorem with our proposed algorithm under appropriate conditions in a uniformly smooth and strictly convex Banach space which also enjoys Kadec-Klee property.

In [113], Moudafi introduced the Split Monotone Variational Inclusion Problem (SMVIP) : Find $x^* \in H_1$, such that

$$\begin{cases} 0 \in \{T_1(x^*) + S_1(x^*)\} \\ y^* = Ax^* \in H_2 : 0 \in \{T_2(y^*) + S_2(y^*)\}, \end{cases} \tag{1.2.9}$$

where $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ are two given single-valued operators and $A : H_1 \to H_2$ is a bounded linear operator. The SMVIP include as special cases, the split common fixed points problem, the split variational inequality problem, the split feasilbility problem and the split zero problem.
In solving the SMVIP, Moudafi [113] introduced the following iterative method:
Initialization: Let $\lambda > 0$ and $x_0 \in H_1$ be arbitrary:
Iterative step: $x_{k+1} = U(x_k + \gamma A^*(T - I)Ax_k)$, $k \in \mathbb{N}$, where $\gamma \in [0, \frac{1}{L}]$ with L the spectral radius of the operator A^*A, and $U := J_\lambda^{T_1}(I - \lambda S_1)$ and $T := J_\lambda^{T_2}(I - \lambda S_2)$ have been defined in the same way as can be found in the work Censor-Gibali-Reich [40].
In 2015, Takahashi [154] considered the split feasibility problem and split common null point problem in the setting of Banach spaces. By using the hybrid methods and Halpern's type methods under appropriate conditions, some strong and weak convergence theorems for such problems in the setting of one Hilbert space and one Banach space were obtained.
Tang et al [158], proved a weak convergence theorem and a strong convergence theorem for split common fixed point problem involving a quasi-strict-pseudo contractive mapping and an asymptotical nonexpansive mapping in the setting of two Banach spaces under some given conditions.
In [158], Tang et al, considered the so-called Split Common Fixed Point Problem (SCFPP) which is to find a point $q \in F(S)$ such that $Aq \in F(T)$, where $S : C \to C$ and $T : Q \to Q$ are two mappings, C and Q are nonempty subsets of Hilbert spaces H_1 and H_2 respectively and $A : H_1 \to H_2$ a bounded linear operator with F(S) and F(T) the set of fixed points of S and T respectively.
In solving the SCFPP, Tang et al [158] introduced the iterative sequence defined for $x_1 \in X_1$ by

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2(T - I)Ax_n, \\ x_{n+1} = (1 - \alpha_n)z_n + \alpha_n S^n z_n, \quad \forall \;\; n \geq 1. \end{cases} \tag{1.2.10}$$

In particular, they proved the strong convergence of (1.2.10) to the set of solutions of the SCFPP in the framework of two Banach spaces X_1 which is uniformly convex and 2-uniformly smooth and a real Banach space X_2.
Very recently, Zhang and Wang [173] proposed a new iterative scheme and proved weak and strong convergence of the scheme to a split common fixed point problem for nonexpansive semi-groups in Banach spaces under some suitable conditions. To be more precise, they proved the following theorem.

Theorem 1.2.1. *Let X_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, X_2 be a real Banach space, $A : X_1 \to X_2$ be a bounded linear operator and A^* be the adjoint* *of A. Let $\{S(t) : X_1 \to X_1,\ t \geq 0\}$ be a uniformly asymptotically regular nonexpansive semigroup with $\mathcal{C} := \cap_{t\geq 0} F(S(t)) \neq \emptyset$ and $\{T(t) : X_2 \to X_2,\ t \geq 0\}$ be a uniformly asymptotically regular nonexpansive semigroup with $\mathcal{Q} := \cap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by: $x_1 \in X_1$*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2(T(t_n) - I)Ax_n, \\ x_{n+1} = (1-\alpha_n)z_n + \alpha_n S(t_n)z_n, \quad \forall \quad n \geq 1, \end{cases} \tag{1.2.11}$$

where $\{t_n\}$ is sequence of real numbers, $\{\alpha_n\}$ a sequence in $(0,1)$ and γ is a positive constant satisfying

(1) $t_n > 0$ and $\lim\limits_{n\to\infty} t_n = \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $0 < \gamma < \frac{1-2k^2}{||A||^2}$.

(I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, then $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$.

(II) In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, and there is at least one $S(t) \in \{S(t) : t \geq 0\}$ which is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

On the other hand Polyak [132], proposed the inertial extrapolation as an acceleration process for solving smooth convex minimization problem. The inertial extrapolation term is known to accelerate the rate of convergence of iterative algorithms. Because of this increase in the speed of convergence rates of the iterative algorithms, there have been an increasing interest in the study of inertial type iterative schemes (see [19, 21, 43]).
Alvarez and Attouch [3], applied the idea of the heavy ball method to the setting of a general maximal monotone operator using the proximal point algorithm. They came up with the following inertial proximal point algorithm.

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n T)^{-1} y_n, \quad n \geq 1. \end{cases} \tag{1.2.12}$$

They proved a weak convergence theorem using (1.2.12) to a zero of the maximal monotone operator T with conditions that $\{r_n\}$ is nondecreasing and $\alpha_n \in (0,1)$ is such that $\sum_{n\geq 0} \alpha_n ||x_n - x_{n-1}||^2 < \infty$.
Moudafi and Oliny [115] improved on Algorithm (1.2.12) by introducing an additional single-valued co-coercive, and Lipschitz continuous operator S into the inertial proximal point algorithm as:

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}), \\ x_{n+1} = (I + r_n T)^{-1}(I - r_n S) y_n, \quad n \geq 1. \end{cases} \tag{1.2.13}$$

They obtained a weak convergence theorem for finding a zero of the sum $(S+T)$ provided that the conditions given on the parameters in (1.2.12) are satisfied. For more results involving the inertial extrapolation process see [10, 20, 58, 106, 131]. Inspired by the research presently going on in the direction of inertial extrapolation, we consider the following split variational inclusion problem involving accretive operators: Let X_1 and X_2 be Banach spaces. The split variational inclusion problem for accretive operators is given as: Find $x_1 \in X_1$ such that

$$\begin{cases} 0 \in X_1 : x^* \in (T_1 + S_1), \\ y^* = Ax^* \in X_2 : y^* \in (T_2 + S_2), \end{cases} \tag{1.2.14}$$

where $T_1 : X_1 \to 2^{X_1}, T_2 : X_2 \to 2^{X_2}$ are set-valued accretive operators, $S_1 : X_1 \to X_1$, $S_2 : X_2 \to X_2$ are inverse strongly accretive operators and $A : X_1 \to X_2$ is a bounded linear operator.
Furthermore, we introduce an inertial-type iterative scheme and prove a weak convergence theorem of the scheme to the solution of (1.2.14).

1.3 Objectives

The main objectives of this work are to:

(i) introduce a new algorithm for approximating a zero of a strongly accretive operator which is also a fixed point of a k-strictly pseudocontractive mapping:

(ii) introduce a mapping for a finite family of mixed equilibrium problems involving $\mu - \alpha$ relaxed monotone mapping and obtain a strong convergence of an algorithm which involves the mapping to the common solution of these finite family of mixed equilibrium problems:

(iii) propose an inertial type algorithm and prove its weak convergence to the solution of a split variational inclusion problem involving accretive operators in Banach spaces.

1.4 Organization of Study

This dissertation is organized as follows:

In Chapter 2, we give some preliminaries, which include basic definitions and important results which we found useful in achieving the objectives of the study. We also give detailed literature review on the problems considered in this disseartation from zero of accretive operator to mixed equilibrium problems and the split variational inclusion problems.

In Chapter 3, we consider the zero of a strongly accretive operator and fixed point of a k-strictly pseudo contractive operator. We introduce an iterative scheme for approximating the zero of the accretive operator and fixed point of the k-strictly pseudo contractive operator. We provide an application of the result to obtain the solution of an integral equation of Hammerstein type.

In Chapter 4, we introduce a mapping for a finite family of mixed equilibrium problem with $\mu - \alpha$ relaxed monotone mapping and introduce an iterative sequence for obtaining the common solution of these problems. We proved a strong convergence theorem for approximating these solutions in the framework of a uniformly smooth Banach space. We present a numerical example to show the importance of the result.

In Chapter 5, we considered the split variational inclusion problem with accretive operator in the framework of a Banach space. We study an inertial type iterative algorithm and obtain a weak convergence theorem for approximating the solution of the problem. We gave a numerical example and also use the Matlab version R2016a to show how the change in initial values affect the number of iterations. The graph also show the difference in the convergence rate of the inertial type and an unaccelerated scheme which does not include the inertials.

In Chapter 6, we present the conclusion, contributions to knowledge and some area of further and possible future studies.

CHAPTER 2

Literature Review

In this chapter, we provide definitions of basic terms and concepts that will be useful throughout this work. We also present some useful results and give detailed literature review of concepts that are relevant to the study.

2.1 Preliminaries and Definitions

Definition 2.1.1. Let C be a nonempty, closed and convex subset of a Banach Space X. A function $f : C \to \mathbb{R} \cup \{+\infty\}$ is said to be

(i) convex, if for any $x, y \in X$ and $\lambda \in [0, 1]$, we have

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y);$$

(ii) lower semi-continuous on C, if and only if for $\lambda \in \mathbb{R}$ the set $\{x \in C : f(x) > \lambda\}$, is open for all λ.

Definition 2.1.2. Let X be a normed linear space. A mapping $T : X \to X$ is said to be

(i) continuous at an arbitrary point $x_0 \in X$, if for each $\epsilon > 0$, there exists a $\delta > 0$ such that for $x \in X$

$$||x - x_0|| < \delta \quad \implies \quad ||Tx - Tx_0|| < \epsilon;$$

(ii) Lipschitz with a real constant $L > 0$ if,

$$||Tx - Ty|| \leq L||x - y||, \quad \forall \ x, y \in X,$$

in particular T is called L-Lipschitz;

(iii) a contraction, if it is Lipschitz with a Lipschitz constant $L \in [0, 1)$;

(iv) strict contractive, if it is Lipschitz with $L \in (0, 1)$.

Definition 2.1.3. Let X be a normed linear space and $T : X \to X$ be a nonlinear mapping. Then T is called

(i) monotone, if $\langle Tx - Ty, x - y\rangle \geq 0, \quad \forall \ \ x, y \in X$;

(ii) α-strongly monotone, if $\langle Tx - Ty, x - y\rangle \geq \alpha||x - y||^2 \ \forall \ \ x, y \in X$ and $\alpha > 0$;

(iii) β-inversely strongly monotone, if there exists a constant $\beta > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \beta||Tx - Ty||^2, \quad \forall \ \ x, y \in X;$$

(iv) firmly nonexpansive, if it is β-inversely strongly monotone with $\beta = 1$, that is

$$||Tx - Ty||^2 \leq \langle Tx - Ty, x - y\rangle.$$

We remark that every β-strongly monotone mapping is $\frac{1}{\beta}$-Lipschitz.

Definition 2.1.4. Let H be a Hilbert space. A multivalued mapping $Q : H \to 2^H$ is called monotone if for all $x, y \in H$ such that $u \in Qx$ and $v \in Qy$, then

$$\langle u - v, x - y\rangle \geq 0.$$

Definition 2.1.5. Let H be a Hilbert space. A multivalued monotone mapping $Q : H \to 2^H$ is said to be maximal if the graph of Q (denoted by G(Q)) is not properly contained in the graph of any other monotone mapping.

It is known that a multivalued mapping is maximal if and only if for $(x, u) \in H \times H$, $\langle u - v, x - y\rangle \geq 0$ for every $(y, v) \in G(Q)$ implies that $u \in Qx$.

Definition 2.1.6. [65] Let X be a Banach space and X^* be its dual space. A mapping $A : X \to X^*$ is said to be a relaxed $\mu{-}\alpha$ monotone if there exists a mapping $\mu : X \times X \to X$ and a function $\alpha : X \to \mathbb{R}$, with $\alpha(tz) = t^p\alpha(z)$ for all $t > 0$ and $x \in X$ such that

$$\langle Ax - Ay, \mu(x, y)\rangle \geq \alpha(x - y), \quad \forall \ \ x, y \in X, \tag{2.1.1}$$

where $p > 1$.

Remark 2.1.1. (i) If $\mu(x, y) = x - y$ for all $x, y \in X$, then (2.1.1) becomes $\langle Ax - Ay, x - y\rangle \geq \alpha(x - y), \quad \forall \ x, y \in X$ and A is said to be relaxed α monotone.

(ii) If $\mu(x, y) = x - y$ for all $x, y \in X$ and $\alpha(z) = k||z||^p$, where $p > 1$ and $k > 0$ is a constant, then (2.1.1) reduces to $\langle Ax - Ay, x - y\rangle \geq k||x - y||^p, \quad \forall \ x, y \in X$ and A is said to be p-monotone [69, 162, 163].

(iii) Every monotone mapping is relaxed $\mu - \alpha$ monotone with $\mu(x, y) = x - y$ for all $x, y \in X$ and $\alpha \equiv 0$.

Example 2.1.2. Let $X = (-\infty, +\infty)$, $Ax = -x^2$ and

$$\mu(x, y) = \begin{cases} -k(x^2 - y^2), & x \geq y, \\ k(x^2 - y^2), & x < y, \end{cases} \tag{2.1.2}$$

where $k > 0$ is a constant. Then, A is $\mu - \alpha$ relaxed monotone with

$$\alpha(z) = \begin{cases} kz^2, & z \geq 0, \\ -kz^2, & z < 0. \end{cases} \tag{2.1.3}$$

Definition 2.1.7. [172] Let C be a nonempty, closed and convex subset of a real Banach space X with dual space X^*. Let $A : C \to X^*$ and $\mu : C \times C \to X$ be two mappings. The mapping A is said to be μ-hemicontinuous if, for any fixed $x, y \in C$, the mapping $\phi : [0, 1] \to (-\infty, +\infty)$ defined by $\phi(t) = \langle A(x + t(y - x)), \mu(x, y)\rangle$ is continuous at 0^+.

Definition 2.1.8. [65] Let C be a nonempty, closed and convex subset of a real Banach space X with dual space X^*. Let $A : C \to X^*$ and $\mu : C \times C \to X$ be two mappings and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper function. A is said to be μ-coercive with respect to ϕ if there exists $x_0 \in C$ such that

$$[\langle Ax - Ax_0, \mu(x, x_0)\rangle + \phi(x) - \phi(x_0)]/||\mu(x, x_0)|| \to +\infty,$$

whenever $||x|| \to \infty$. If $\phi = \delta_C$, where δ_C is the indicator function of C, then Definition 2.1.8 coincides with the definition of μ-coercivity in the sense of [170].

Definition 2.1.9. [65] Let C be a nonempty, closed and convex subset of a real Banach space X with dual space X^*. A mapping $F : C \to 2^X$ is said to be a KKM mapping if, for any $\{x_1, x_2, \cdots, x_n\} \subset C$, $co\{x_1, x_2, \cdots, x_n\} \subset \cup_{i=1}^n F(x_i)$, where 2^X denotes the family of all the nonempty subsets of X.

Lemma 2.1.3. *[64] Let M be a nonempty subset of a Hausdorff topological vector space X and let $F : M \to 2^X$ be a KKM mapping. If $F(x)$ is closed in X for every $x \in C$ and compact for some $x \in C$, then $\cap_{x \in M} F(x) \neq \emptyset$.*

Theorem 2.1.4. *[65] Let C be a nonempty, closed and convex subset of a real Banach space X with dual space X^*. Let $A : C \to X^*$ be an μ-hemicontinuous and relaxed $\mu - \alpha$ monotone mapping; let $\phi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function.*
Assume that:

(i) $\mu(x, x) = 0$, for all $x \in C$;

(ii) for any $y, z \in C$, the mapping $x \mapsto \langle Az, \mu(x, y)\rangle$ is convex.

Then, the following problems

$$x \in C, \quad \langle Ax, \mu(y, x)\rangle + \phi(y) - \phi(x) \geq 0, \quad \forall y \in C; \tag{2.1.4}$$

and

$$x \in C, \quad \langle Ay, \mu(y, x)\rangle + \phi(y) - \phi(x) \geq \alpha(y - x), \quad \forall y \in C; \tag{2.1.5}$$

are equivalent.

Recall that a point $x \in H$ is said to be a fixed point of a mapping $T : H \to H$ if $x = Tx$ and $x \in Tx$, if $T : H \to 2^H$ is a multivalued mapping. The set of fixed points of T shall be denoted by $F(T)$ whether T is single-valued or multi-valued.

Example 2.1.5. (i) If $H = \mathbb{R}$ and $T(x) = x^2 + 3x$, then $F(T) = \{-2, 0\}$.

(ii) If $H = \mathbb{R}$ and $T(x) = x^2 + 5x + 4$, then $F(T) = \{-2\}$.

iii If $H = \mathbb{R}$ and $T(x) = x$, then $F(T) = \mathbb{R}$.

(iv) If $H = \mathbb{R}$ and $T(x) = x + \dfrac{1}{x}$, then $F(T) = \emptyset$.

2.2 Metric Projection

Definition 2.2.1. Let C be a nonempty closed convex subset of H. The metric (*or nearest point*) projection onto C is the mapping $P_C : H \to C$ which assigns to each $x \in H$ the unique point $P_C x \in C$ such that

$$||x - P_C x|| = \inf\{||x - y|| : y \in C\}. \tag{2.2.1}$$

Example 2.2.1. [85] Let $x \in \mathbb{R}$ and $C = [-\lambda, \lambda]$ for $\lambda > 0$, we define a map $P_C : \mathbb{R} \to C$ by

$$P_C x = \begin{cases} x, & if \quad x \in C, \\ \frac{\lambda x}{|x|}, & otherwise. \end{cases} \tag{2.2.2}$$

Then P_C is the metric projection from $\mathbb{R}$ onto C. Observe that if $x \in C$, then $|x - P_C x| = 0 \leq |x - y|, \ \forall \ \ y \in C$.
Also, observe that if $x \notin C$, then ($x > \lambda$ or $x < -\lambda$).
For $x < -\lambda$, we have

$$|x - P_C x| = \left| x - \frac{\lambda x}{|x|} \right| = |x - (-\lambda)|.$$

Observe that for any $y \in [-\lambda, \lambda]$, $x - (-\lambda) \geq x - y$.
Also, since $x - (-\lambda) < 0$, we have $x - y < 0$. Hence, $|x - (-\lambda)| \leq |x - y|$, which implies $|x - P_C x| \leq |x - y|, \quad \forall \ y \in [-\lambda, \lambda]$.
Similarly, for $x > \lambda$, we have that

$$|x - P_C x| = \left| x - \frac{\lambda x}{|x|} \right| = |x - \lambda| \leq |x - y|, \quad \forall \ y \in [-\lambda, \lambda].$$

Hence, we obtain that $|x - P_C x \leq |x - y|, \quad \forall \ y \in C, \ x \in \mathbb{R}$. Thus, P_C is the metric projection of $\mathbb{R}$ onto C.

Throughout the remaining sections, we shall denote the real Banach space by X, the norm and duality pairing between elements of X and its dual space X^* by $||.||$ and $\langle \cdot, \cdot \rangle$ respectively and C for a nonempty, closed and convex subset of X unless otherwise stated.

2.3 Some Geometric Properties of Banach Space

The Hilbert space is generally known to have the simplest and easy to work with geometric structures among all Banach spaces. Some of the geometric properties which characterize the Hilbert spaces make problem posed in them more manageable than those in the general Banach spaces (see [49]). These include, the availability of scalar(inner) product, the nonexpansity property of the nearest point map defined on a real Hilbert space H onto a closed convex subset C of H and the following two identities which holds for all $x, y \in H$ and $\lambda \in (0,1)$.

$$||x+y||^2 = ||x||^2 + 2\langle x, y\rangle + ||y||^2, \tag{2.3.1}$$

$$||\lambda x + (1-\lambda)y|| = \lambda||x||^2 + (1-\lambda)||y||^2 - \lambda(1-\lambda)||x-y||. \tag{2.3.2}$$

In trying to obtain the analogue of identity (2.3.1) in Banach spaces which is more general than the Hilbert space, one has to find a suitable and appropriate replacement for the inner product $\langle \cdot, \cdot \rangle$. The duality mapping provides such replacement of the inner product, it allows a pairing between elements of normed space X and elements of its dual space X^*. More details including the definition and properties of the duality mapping shall be given in a later section.

2.3.1 Uniformly Convex Banach Space

Definition 2.3.1. A normed linear space X is said to be uniformly convex if for each $\epsilon \in (0,2]$ there exists a $\delta(\epsilon) > 0$ such that for all $x, y \in X$ with $||x|| = ||y|| = 1$ and $||x-y|| \geq \epsilon$, then

$$\frac{||x+y||}{2} \leq 1 - \delta.$$

Equivalently, X is uniformly convex if for any $\epsilon \in (0,2]$, there exists $\delta(\epsilon) = \delta > 0$ such that if $x, y \in X$ with $||x|| \leq 1$, $||y|| \leq 1$ and $||x-y|| \geq \epsilon$, then

$$\frac{||x+y||}{2} \leq 1 - \delta.$$

Example 2.3.1. [49] The L_p spaces, $1 < p < \infty$ are uniformly convex.

Definition 2.3.2. Let $\dim X \geq 2$, then the modulus of convexity of a normed linear space X is the function

$$\delta_X : (0,2] \to [0,1],$$

defined by

$$\delta_X(\epsilon) = \inf\{1 - \frac{||x+y||}{2} : ||x|| = ||y|| = 1, ||x-y|| = \epsilon\}. \tag{2.3.3}$$

Definition 2.3.3. Let $p > 1$ be a real number. A normed linear space X is said to be p-uniformly convex if there exists $c_p > 0$ such that $\delta_X(\epsilon) \geq c_p\epsilon^p$, for any $\epsilon \in (0,2]$.

Definition 2.3.4. A normed linear space X is called strictly convex if for all $x, y \in X$, $x \neq y$, $||x|| = ||y|| = 1$, we have $||\lambda x + (1-\lambda)y|| < 1$, for all $\lambda \in (0, 1)$.

Theorem 2.3.2. *A normed linear space X is uniformly convex if and only if $\delta_X(\epsilon) > 0$, for all $\epsilon \in (0, 2]$.*

2.3.2 Uniformly Smooth Banach Space

Definition 2.3.5. A normed linear space X is called smooth if for every $x \in X$ with $||x|| = 1$, there exists $x^* \in X^*$ such that $||x^*|| = 1$ and $\langle x, x^* \rangle = 1$.

Definition 2.3.6. Let X be a real Banach space and $B_X = \{x \in X : ||x|| = 1\}$, then the norm of X is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{||x + ty|| - ||x||}{t} \tag{2.3.4}$$

exists for each $x, y \in B_X$. In this case X is said to be smooth.

Definition 2.3.7. Let X a real Banach space, the norm of X is said to be Fréchet differentiable if, for each $x \in X$, the limit (2.3.4) exists and is attained uniformly for all y such that $||y|| = 1$.

Definition 2.3.8. Let X be a real normed space with dim ≥ 2, then the modulus of smoothness of X is the function $\rho_X : [0, \infty) \to [0, \infty)$, defined by

$$\rho_X(t) := \sup\{\frac{1}{2}(||x + y|| - ||x - y||) - 1 : ||x|| \leq 1, ||y|| \leq t\}. \tag{2.3.5}$$

Definition 2.3.9. [49] A normed linear space X is uniformly smooth if and only if

$$\lim_{t \to 0} \frac{\rho_X(t)}{t} = 0. \tag{2.3.6}$$

Example 2.3.3. [49] The L_p spaces, $1 < p < \infty$ are uniformly smooth.

Definition 2.3.10. For $q > 1$, a Banach space X is said to be q-uniformly smooth if there exists $c_q > 0$ such that $\rho_X(t) \leq c_q t^q$ for any $t > 0$.

2.3.3 Reflexive Banach Space

Definition 2.3.11. Let X^* and X^{**} be the dual and the bidual of a Banach space X respectively. Then there exists a canonical (or canonical embedding) mapping $\mathbf{J} : X \to X^{**}$ defined, for each $x \in X$ by $\mathbf{J}(x) = \Phi_x \in X^{**}$, where $\Phi_x : X^* \to \mathbb{R}$ is defined by $\langle \Phi_x, f \rangle = \langle f, x \rangle$, for each $f \in X^*$.

Thus, $\langle \mathbf{J}(x), f \rangle = \langle f, x \rangle$ for each $f \in X^*$. If the canonical mapping $\mathbf{J}$ is an onto mapping, then X is called reflexive. Thus, a reflexive Banach space is one in which the canonical embedding is onto.

Remark 2.3.4. [47] The canonical mapping $\mathbf{J}$ defined above have the following properties.

(i) $\mathbf{J}$ is linear.

(ii) $\mathbf{J}$ is an isometry, i.e $||\mathbf{J}x|| = ||x||, \forall\, x \in X$.

Remark 2.3.5. [49]

(i) Every uniformly convex space is strictly convex.

(ii) Every uniformly convex space is reflexive.

(iii) Every uniformly smooth space is smooth.

(iv) X is uniformly smooth if and only if X^* is uniformly convex.

(v) If the dual space X^* is reflexive, then X is reflexive.

Combinning Remarks 2.3.5(ii), 2.3.5(iv) and 2.3.5(v), we have the following remark.

Remark 2.3.6. Every uniformly smooth space is reflexive.

2.3.4 More Properties of Banach Space

A normed linear space $(X, ||\cdot||_X)$ is said to have the Kadec-Klee property (also called the Radon Riesz property or H property) if and only if sequential convergence on the unit sphere coincides with norm convergence. This property was first studied by Radon [135] and subsequently by Riesz [139] who showed that the classical L_p-spaces $1 < p < \infty$ have the Kadec-Klee property. In other world, a Banach space X has the Kadec-Klee property if for any sequence $\{x_n\} \subset X$ and $x \in X$ with $x_n \rightharpoonup x$ and $||x_n|| \to ||x||$, then $||x_n - x|| \to 0$ as $n \to \infty$. It is well known that if X is a uniformly convex Banach space, then X enjoys the Kadec-Klee Banach space property. For more on the Kadec-Klee property (see [55]) and the references therein.

A Banach space X is said to satisfy Opial's property if, for any sequence $\{x_n\} \subset X$,

$$x_n \rightharpoonup x \implies \lim ||x_n - x|| < \lim ||x_n - y||, \quad \forall \;\; y \in X, \;\; y \neq x. \tag{2.3.7}$$

Banach spaces satisfying Opial's property includes all Hilbert spaces, and l^p for $1 \leq p < \infty$, L^p fails to satisfy this property unless $p = 2$. This property plays an important role in determination of weak convergence of sequences in Banach spaces. The property implies the fixed point property for nonexpansive mappings, that is if C is a nonempty weakly compact convex subset of X with Opial's property then every mapping $T : C \to C$ admits a fixed point whenever T is nonexpansive.

2.4 Generalized Duality Mapping

We now give the definition and properties of the duality mapping.

Let X be a real Banach space with X^* its dual space, let $\varphi : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing function such that $\varphi(0) = 0$, $\varphi(t) \to \infty$ as $t \to \infty$. The function so defined is called a gauge function. The generalized duality mapping $J_\varphi : X \to 2^{X^*}$ associated with φ is defined by

$$J_\varphi(x) = \{x^* \in X^* : \langle x, x^* \rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}.$$

If $\varphi(t) = t$, then $J_\varphi = J$, where J is called the normalized duality mapping. Notice that, in Hilbert space, the generalized duality mapping with a guage function is φ defined as (see [75]);

$$J_\varphi(x) = \begin{cases} 0 & \text{if } \ x = 0, \\ \frac{\varphi(||x||)}{||x||}x & \text{if } \ x \neq 0. \end{cases}$$

Also, for $p > 1$, if $\varphi(t) = t^{p-1}$ then the generalized duality mapping with a gauge function φ becomes $J_\varphi = J_p$ defined by

$$J_p(x) = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^p, ||x^*|| = ||x||^{p-1}\}.$$

For $p = 2$, then $J_p = J_2 = J$ and is the normalized duality mapping on X.

Definition 2.4.1. The duality mapping is J_p said to be weak-to-weak continuous, if $x_n \rightharpoonup x \implies \langle J_p x_n, y \rangle \to \langle J_p x, y \rangle$ holds for any $y \in X$.

We note that l_p $(p > 1)$ spaces possesses this property, but L_p $(p > 2)$ does not posses this property. For $1 < q \leq 2 \leq p$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have the following important remark.

Remark 2.4.1. [2] It is known that X is p-uniformly convex and uniformly smooth if and only if X^* is q-uniformly smooth and uniformly convex. In this case, the duality mapping J_p is one-to-one, single valued and satisfies $J_p^{-1} = J_q^*$, where J_q^* is the duality mapping of X^*.

We recall the following properties of the normalized duality mappings in different Banach spaces (see [155, 161]).

(i) For any $x \in X$, $J(x)$ is nonempty, bounded and convex.

(ii) J is a homogenous in arbitrary Banach space X, that is for any $x \in X$ and a real α, $J(\alpha x) = \alpha J(x)$.

(iii) J is a monotone operator in arbitrary Banach space X, if for any $x, y \in X$, $u \in J(x)$ and $v \in J(y)$,
$$\langle u - v, x - y \rangle \geq 0.$$

(iv) If X is smooth, then J is a single-valued mapping.

(v) If X is reflexive, then J is a mapping of X onto X^*.

(vi) If X is strictly convex, then J is one-to-one, that is $x \neq y \implies J(x) \cap J(y) = \emptyset$.

(vii) J is a continuous operator in smooth Banach spaces.

(viii) J is the identity operator in Hilbert spaces.

(ix) For any $x, y \in X$ and $j \in J(y)$,

$$||x||^2 - ||y||^2 \geq 2\langle j(y), x - y\rangle.$$

The following properties of the generalized duality mapping (Remark 2.4.3, Propositions 2.4.4,2.4.5,2.4.6 and 2.4.7) correspond to the above properties of the normalized mapping.

Proposition 2.4.2. *[75]*

$$J_\varphi(x) = \begin{cases} J(x) & \text{if } \; x = 0, \\ \frac{\varphi(||x||)}{||x||} J(x) & \text{if } \; x \neq 0. \end{cases}$$

Remark 2.4.3. [75] From Proposition 2.4.2, the following are easily observed:

(a) For any $x \in X$, $J_\varphi(x)$ is nonempty, bounded, closed and convex.

(b) If X is smooth, then J_φ is single-valued mapping.

(c) If X is smooth, then J_φ is a continuous operator.

(d) If X has a uniform Gâteaux differentiable norm, then J_φ is norm-to-weak* uniformly continuous on bounded subsets of X.

Proposition 2.4.4. *[75] If X is reflexive, then J_φ is a mapping of X onto X^*.*

Proposition 2.4.5. *[75] Let φ be a gauge function and $\alpha \geq 0$. For any $x \in X$ we have*

$$J_\varphi(\alpha x) = \alpha J_\varphi(x).$$

Proposition 2.4.6. *[75] J_φ is a monotone mapping in an arbitrary Banach space X.*

Proposition 2.4.7. *[75] Suppose that X is a smooth Banach space. Then for any $x, y \in X$.*

$$||x||^2 - ||y||\varphi(||y||) \geq 2\langle J_\varphi(y), x - y\rangle.$$

2.4.1 Retraction Mappings

Let D be a nonempty subset of a set C. Let $P : C \to D$ be a map, then P is said to be

(1) sunny if for each $x \in C$ and $t \in (0, 1)$, we have $P(tx + (1 - t)Px) = Px$;

(2) a retraction, if $P^2 = P$;

(3) a sunny nonexpansive retraction if P is sunny, nonexpansive and a retraction.

D is said to be a nonexpansive retract of C if there exists a nonexpansive retraction from C onto D. We refer the reader to the results established in [35, 68, 137], which describes a characterization of sunny nonexpansive retractions on a smooth Banach space. Let X be a smooth Banach space and let C be a nonempty subset of X. Let $P : X \to C$ be a retraction and J be the duality mapping on X. Then the following are equivalent:

(1) P is sunny and nonexpansive;

(2) $\langle x - Px, J(y - Px)\rangle \leq 0, \quad \forall \;\; x \in X, \quad y \in C$;

(3) $||Px - Py||^2 \leq \langle x - y, j(Px - Py)\rangle, \quad \forall \;\; x, y \in X$.

It is widely known that if $X = H$ is a Hilbert space, then the sunny nonexpansive retraction P is coincident with the metric projection from X to C. Let C be a nonempty closed subset of a smooth Banach space X, let $x \in X$, and let $x_0 \in C$. Then we have that $x_0 = Px$ if and only if $\langle x - x_0, j(y - x_0)\rangle \leq 0$ for all $y \in C$, where P is sunny nonexpansive retraction from X to C. For more on nonexpansive retracts (see [99]) and the references there in.

2.5 Accretive Operators

Definition 2.5.1. Let X be a real normed space with dual X^*. A mapping with domain D(A) and range R(A) in X is called accretive if and only if for all $x, y \in D(A) \subset X$, the following inequality is satisfied

$$||x - y|| \leq ||x - y + s(Ax - Ay)|| \;\; \forall \;\; s > 0. \tag{2.5.1}$$

Equivalently, by Lemma (1.1) of Kato ([93]), we have that A is accretive if and only if, for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Ax - Ay, j(x - y)\rangle \geq 0.$$

However, if A is a multivalued mapping, then A is accretive if and only if

$$\langle u - v, j(x - y)\rangle \geq 0, \quad u \in Ax, \quad v \in Ay.$$

Definition 2.5.2. A multi-valued mapping $A : X \to 2^{X^*}$ is said to be α-inverse strongly accretive (α-isa) of order q if, for each $x, y \in D(A) \subset X$, there exists $j_q(x - y) \in J_q(x - y)$ such that

$$\langle u - v, j_q(x - y)\rangle \geq \alpha||u - v||^q, \quad u \in Ax, \quad v \in Ay. \tag{2.5.2}$$

In particular when $q = 2$, we simply say α-isa, that is A is α-isa, if for each $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle u - v, j(x - y)\rangle \geq \alpha||u - v||^2, \quad u \in Ax, \quad v \in Ay. \tag{2.5.3}$$

Remark 2.5.1. We note that for $q = 2$; the class of inverse strongly accretive q coincides with that of inverse strongly accretive.

Thus, in the case of $q \neq 2$, 2.5.2 is not equivalent to 2.5.3. Furthermore, for $q < 2$, inverse strongly accretive operators of order q do represent a subclass of inverse strongly accretive operators ([146]).

We denote by $N(A)$ the zeros of an accretive operator A (i.e $N(A) = \{x \in D(A) : Ax = 0\} = A^{-1}(0)$).

Definition 2.5.3. Let I denote the identity operator on X. An accretive operator A on a Banach space X is said to be "m-accretive" if $R(I + \lambda A) = X$ for all $\lambda \in (0, \infty)$. A is said to satisfy the range condition if $\overline{D(A)} \subset R(I + \lambda A)$ for all $\lambda > 0$, where $\overline{D(A)}$ denotes the closure of the domain of A.

An m-accretive operator on X is a maximal element within the class of accretive operators on X ordered by inclusion (see [12]).

For an "m-accretive operator" A, the "resolvent operator" $J_\lambda : R(I + \lambda A) \to D(A)$ of A is defined by $J_\lambda = (I + \lambda A)^{-1}$ for all $\lambda \in (0, \infty)$. Some of the properties of J_λ are: (see [155])

(1) J_λ is single valued.

(2) $||J_\lambda x - J_\lambda y|| \leq ||x - y||, \quad \forall\ x, y \in R(I + \lambda A)$.

(3) $F(J_\lambda) = A^{-1}(0)$.

2.5.1 Zeros of Accretive Operator

The accretive operators were introduced independently in 1967 by Browder [29] and Kato [93]. The theory of this class of operators have been widely studied in nonlinear analysis. Interest in such mappings stems mainly from their firm connection with the existence theory for nonlinear evolutional equations in Banach spaces. It is widely known that many physically significant problems can be modelled in terms of an initial value problem of the form

$$\frac{dx}{dt} + Ax = 0, \quad x(0) = x_0, \tag{2.5.4}$$

where A is an accretive operator on a suitable and appropriate Banach space. Typical examples of such evolutions are found in models involving the heat, wave or shrödinger equations (see e.g [31]). Especially, an early fundamental results in the theory of accretive operators, which is due to Browder [31], states that equation (2.5.4) is solvable if A is locally Lipschitzian and accretive on the Banach space X. Utilizing the existence result for (2.5.4), Browder [29] proved that if A is locally Lipsichitzian and accretive on X then A is m-accretive. A clear consequence of this is that the equation $x + Tx = h$ for a given $h \in X$, where $T = I - A$ has a solution. In 1979, Ray [136] gave an elementary proof of the result by Browder by using the fixed point theorem of Caristi [37]. Martin [110, 111],

proved that (2.5.4) is solvable if A is continuous and accretive on X, utilizing this result he further proved that if A is continuous and accretive, then A is m-accretive. Other existence theorems for zeros of accretive operators can be found in Browder [27, 28, 30, 32]. If $x(t)$ is independent of t, then (2.5.4) reduces to $Ax = 0$ whose solutions correspond to the equilibrium points of system (2.5.4). Consequently, considerable research effects have been devoted, especially within the past 40 years or so, to iterative methods for approximating these equilibrium points.
In recent years, some iterative methods have been developed for finding zeros of accretive operators and related fixed points problems, (see for example [39, 53, 78, 79, 90, 91, 134, 144, 148, 149, 175]) and the references therein.
In 1974, Bruck [34] introduced an iteration process and proved the convergence of the process to a zero of a maximal monotone operator in the setting of Hilbert space. In 1979, Reich [138] extended this result to uniformly smooth Banach spaces provided that the operator is m-accretive. In 2003 Benavides et al [11], inspired by the proximal point algorithm of Rockafellar technique [140] and the iterative methods of Halpern [73], studied the following Halpern type iterative scheme to find a zero of an m-accretive operator A in a uniformly smooth Banach space with a weakly continuous J_φ with gauge function φ by virtue of the resolvent J_{λ_n} of A :

$$x_{n+1} = \alpha_n u + (1-\alpha_n) J_{\lambda_n} x_n \quad n \geq 1. \tag{2.5.5}$$

2.6 Equilibrium Problem

The so called equilibrium problems is one of the most important topics in nonlinear analysis and in other several applied fields (see [127]). The equilibrium problem has found extensive study in recent years (see for example [15, 41, 70, 82, 92, 98]). The equilibrium problem has been widely studied in the context of optimization problem, fixed point problems, variational inequality problem whether monotone or otherwise (see [9, 42, 45]). It is also a special case of Nash equilibrium and some other problems ([7, 8, 89]).

2.6.1 Existence of Results for the Equilibrium Problems

Let $F : C \times C \to \mathbb{R}$ be a given bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium with respect to F and C in the sense of Blum and Oettli [18] is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall\, y \in C. \tag{2.6.1}$$

As we have mentioned earlier in our introduction, the introduction of this class of problem was done by K Fan [63], but the term "equilibrium problem" was attributed to Blum and Oettli [18] some 20 years after. The existence of solution to this problem has been discussed extensively in literature (see [16, 83, 84]). Other recent existence results can be found in ([13, 14, 125]) and the references therein.
In solving the Equlibrium Problem (EP), it is assumed that the bifunction F satisfies the following conditions:

(F_1) $F(x,x) = 0,\ \forall\, x \in C,$

(F_2) F is monotone in the sense that $F(x,y) + F(y,x) \leq 0,\ \forall\, x, y \in C,$

(F_3) $\lim_{t \downarrow 0} F(tz + (1-t)x, y) \leq F(x,y)$ for each $x, y, z \in C,$

(F_4) For each $x \in C$, the function $y \mapsto F(x,y)$ is convex, lower semicontinuous.

2.6.2 The Variational Inequality Problems

In solving various number of mathematical problems which includes the optimization problem, equilibrium problems, boundary value problem among others, the theory of Variational Inequality Problem (VIP) is widely known to be very useful. The VIPs are known to generalize problem like the boundary value problem, minimization problem, equilibrium problems (see [56]).
Given a nonempty closed and convex subset C of X, the variational inequality problem consists of finding $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C, \tag{2.6.2}$$

where $A : X \to X^*$ is a nonlinear operator.

2.6.3 Existence of Solution

The VIPs have been studied extensively in finite dimensional space or otherwise. For the finite dimensional spaces, the work of Dafermos [56] stands out and represents a major breakthrough in the study of VIPs. Dafermos worked with a traffic network equilibrium which had the structure of the variational inequalities under the assumption of monotonicity. In this work, Dafermo employed the technique of the theory of variational inequalities to establish the existence of traffic pattern. He proposed an iterative scheme for the construction of pattern and derived estimates on the rate of convergence of the iterative algorithm. This work therefore became a springboard as it attracted the interest of many other researchers. For more works in finite dimensional space (see for example [57, 61, 101, 122]).
Following the research which are finite dimensional space based on VIPs, the study was extended to spaces of infinite dimensions. The existence and uniqueness of solutions to the VIP in this type of space was established by Stampacchia [150] under the assumption that A is a coercive and linear operator from a Hilbert space H to its topological space H^*. The case where A is positive or semicoercive was further considered by Lions and Stampacchia [103]. Hartman and Stampacchia [77] worked on partial differential equations using the VIP as a tool, with application to problems in mechanics. They proved the existence and uniqueness theorem to the solution of the VIP in the framework of a reflexive Banach space where the operator A is assumed to be a monotone hemicontinuous. In particular, they obtain and prove the following result.

Theorem 2.6.1. *Let X be a Banach space and X^* be its dual. Let $A : X \to X^*$ be a monotone hemicontinuous and C a bounded convex subset of X, then there exists at least one solution of the problem $\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall \quad x \in C$.*

Variational like inequalities with relaxed $\mu - \alpha$ monotone operator

The variational like inequality problem introduced by Fang and Huang [65] is to find $x \in C$ such that

$$\langle Ax, \mu(y, x) \rangle + \phi(y) - \phi(x) \geq 0 \quad \forall \, y \in C; \tag{2.6.3}$$

where A is a $\mu - \alpha$ relaxed motone mapping from C to X^*, $\mu : C \times C :\to X$ is a nonlinear mapping, $\alpha : X \to \mathbb{R}$ is a function and $\phi : C \to \mathbb{R}$ is a proper function.

Generalized Equilibrium Problem (GEP)

In 1999, Moudafi and Thera [116] introduced the GEP which is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0, y \in C, \tag{2.6.4}$$

where $A : C \to H$ is a nonlinear mapping. The set of solution of (2.6.4) is denoted by GEP(F).

Mixed Equlibrium Problem (MEP)

This class of equilibrium problem was introduced in 2008 by Ceng and Yao [39], the problem consists of finding $x \in C$ such that

$$F(x, y) + \phi(y) - \phi(x) \geq 0, \ \forall \, y \in C,$$

where $\phi : C \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. The set of solutions of the MEP is denoted by MEP(F,ϕ).

Generalized Mixed Equlibrium Problem (GMEP)

The GMEP is also another class of the equilibrium problem which was first studied by Peng and Yao [130]. The GMEP is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle + \phi(y) - \phi(x) \geq 0, \ \forall \, y \in C, \tag{2.6.5}$$

where $A : C \to H$ is a nonlinear mapping and $\phi : C \to \mathbb{R} \cup \{+\infty\}$ is a nonlinear functional. The set of solution of (2.6.5) is denoted by GMEP(F,A,ϕ).

We note here that the GMEP generalizes the MEP and the GEP. For instance by setting $\phi \equiv 0$, in the GMEP one obtains the GEP. Also, by setting $A \equiv 0$ we obtain the MEP from the GMEP. And we obtain the EP if we set both A and ϕ to zero in (2.6.5).

In this study, we considered a GMEP with A a $\mu - \alpha$ relaxed monotone operator defined in the preliminaries above. This problem generalizes the GMEP introduced by Peng and Yao [130].

2.7 Monotone Inclusion Problem

The Monotone Inclusion Problem (MIP), like the VIP's discussed above can be used to model many mathematical problems such as the Optimization problem, Saddle point problems, fixed points problems, e.t.c. The MIP has been widely studied due to its various applications both in the finite and infinite dimensional spaces. The introduction of the MIP can be attributed to Rockafellar [140], he defined the problem as one which consists of finding a point $x \in H$ such that

$$0 \in A(x), \tag{2.7.1}$$

where H is a real Hilbert space and $A : H \to 2^H$ is a maximal monotone operator. The solution set of the problem (2.7.1) is denoted by $A^{-1}(0)$. The solution set is known to be closed and convex, see [140].

Split Monotone Inclusion Problem

The Split Monotone Inclusion Problem (SMIP) is to find $x \in H_1$ such that

$$0 \in \cap_{i=1}^{p} T_i(x) \tag{2.7.2}$$

and $y_j = A_j x \in H_2$ with

$$0 \in \cap_{j=1}^{r} S_j(x), \tag{2.7.3}$$

where $T_i : H_1 \to 2^{H_1}$ for $1 \leq i \leq p$ and $S_j : H_2 \to 2^{H_2}$ for $1 \leq j \leq r$ are maximal monotone operators and $A : H_1 \to H_2$ is a bounded linear operator. This class of inclusion problem was introduced by [36], they denote the problem (2.7.2) and (2.7.3) by SCNPP(p,r) to emphasize the multiplicity of the mappings T_i and S_j. Bryne [36] observed that at $p = r = 1$, then the SCNPP(p,r) structurally becomes the split variational inequality problem introduced and studied by Censor et al [40].

Monotone Variational Inclusion Problem

The Monotone Variational Inclusion Problem is to find $x \in H$ such that

$$0 \in P(x) + S(x), \tag{2.7.4}$$

where 0 is the zero vector in H, $P : H \to H$ is a single valued mapping and $S : H \to 2^H$ is a set-valued mapping. The set of solutions of the MVIP is denoted by I(P,S). The Variational inclusion problem is a special case of the MVIP which is obtained by setting $P = 0$ in (2.7.4). The monotone variational inclusion problem generalizes the zero problem for nonlinear mapping and the classical variational inclusion problem. For existence of solution and more studies on the monotone variational inclusion problem (see for example [33, 72, 95, 88, 114, 118, 119, 120, 121, 143, 145]) and the references therein.

Split Monotone Variational Inclusion Problem

The Split Monotone Variational Inclusion Problem (SMVIP) introduced in 2011 by Moudafi [113] is to find $x^* \in H_1$ such that

$$0 \in [T_1(x^*) + S_1(x^*)] \tag{2.7.5}$$

and

$$y^* = Ax^* \in \ H_2 : 0 \in [T_2(y^*) + S_2(y^*)], \tag{2.7.6}$$

where $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ are single valued monotone mappings and $A : H_1 \to H_2$ is a bounded linear operator. The split monotone variational inclusion problem generalizes other optimization problem such as the split common fixed points problem, the split variational inequality problem, the split feasibility problem among others. Since its introduction in 2011, the split monotone variational inclusion problem has undergone several studies and researchers have proposed several iterative schemes to obtaining the solutions of the SMVIP in the Hilbert spaces and other spaces of interest.

2.8 Some Important Iterative Schemes

In this section, we will take a look at the iterative schemes that have been employed in approximating the solutions of problems considered in this study.

2.8.1 Picard Iterative Scheme

We first give the much celebrated and widely known Banach contraction mapping principle.

Theorem 2.8.1. *(Contraction Mapping Principle).*
Let (X, ρ) be a complete metric space and $T : X \to X$ be a contraction map. Then

(a) T has a unique fixed point, say $\bar{x} \in X$,

(b) the sequence $\{x_n\}_{n=0}^{\infty}$ in X defined by $x_0 \in X$,

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \cdots, \tag{2.8.1}$$

converges to $\bar{x} \in X$.

Theorem 2.8.1 represents the most important theorem in the theory of fixed point. The sequence of the recursion formula (2.8.1) is refered to as the Picard sequence.

2.8.2 Krasnoselskij Iterative Scheme

Suppose we replace the recursion formula in the Picard iteration by the following sequence: For $x_0 \in C$,

$$x_{n+1} = \frac{(I+T)}{2} x_n, \ n \geq 0, \tag{2.8.2}$$

then the iterative scheme (2.8.2) converges to the unique fixed point of any self map T of a set X. In general, if X is a normed linear space and T is a nonexpansive mapping, the Krasnoselskij iteration is, for an initial $x_0 \in C$,

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, \ \ \geq 0, \tag{2.8.3}$$

where $\lambda \in (0,1)$. The recursion formula (2.8.3) generalizes (2.8.2) and has proved successful in the approximation of a fixed point of T when it exists, see Schaefer [142].

Remark 2.8.2. (i) The Krasnoselskij iteration (2.8.2) is the Picard iteration of the averaged operator $T_\lambda = (I-\lambda)T + \lambda T$, where I is the identity operator and $\lambda = \frac{1}{2}$.

(ii) At $\lambda = 1$, the Krasnoselskij iteration reduces to the Picard iteration.

2.8.3 Mann Iterative Scheme

The Mann iteration formula due to Mann [108] represents the most general iterative scheme for approximation of fixed points of nonexpansive mapping. The algorithm is defined iteratively for $x_0 \in C$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \ n \geq 0, \tag{2.8.4}$$

where $\{\alpha_n\}$ is a sequence in (0,1) satisfying:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$,

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 2.8.3. (i) It is easy to see that if $\alpha_n = \lambda$ (constant), then the Mann iteration becomes a Krasnoselkij.

(ii) Also, if $\alpha_n = 1$ then Mann iteration reduces to the Picard formula.

2.8.4 Ishikawa Iterative Scheme

In 1974, Ishikawa [80] offered an enlargement and improvement to the iterative scheme by Mann. He proposed a new iterative scheme generated by $x_0 \in C$ and defined iteratively by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n], \tag{2.8.5}$$

where

(i) $0 \leq \alpha_n \leq \beta_n < 1$.

(ii) $\lim\limits_{n\to} \beta_n = 0$.

(iii) $\sum_{n\geq 1} \alpha_n\beta_n = \infty$.

The scheme (2.8.5), if written in the form:

$$\begin{cases} y_n = (1-\beta_n)x_n + \beta_n x_n \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \end{cases} \tag{2.8.6}$$

then it is regarded as a two step Mann iteration with two different control sequences. The Mann and Ishikawa iterative scheme have been employed by different authors in the last three decades to approximate fixed points of various classes of nonlinear operators in different spaces. We note, that the Mann iteration may fail to converge while the Ishikawa iterative scheme can still converge for a Lipschitz pseudo-contractive in Hilbert space. Additional conditions such as (compactness) may however be needed to obtain strong convergence of the Mann iteration to a fixed point of k-strictly contractive maps. The conditions can be placed on the operator T or the subset C.

2.8.5 Proximal Point Algorithm

Another fundamental algorithm for solving monotone inclusion problem is the proximal point algorithm. The algorithm is based on the fact that for each $x \in H$ and $\lambda > 0$ there is a unique $z \in H$ such that $x - z \in \lambda B(x)$, (see Minty [112]). In other words $x \in (I + \lambda B)(z)$. The operator $J_\lambda : (I + \lambda T)^{-1}$ called the resolvent is a single valued and nonexpansive map whenever T is accretive. The proximal point algorithm generates for any starting point $x_0 \in X$ by the approximate rule

$$x_{n+1} = (I + \lambda T)^{-1} x_n, \tag{2.8.7}$$

where λ is a positive real number, (see [140]).

2.8.6 Implicit Iteration

For solving the problem of approximating a fixed point of a mapping say T which may have multiple solutions, an implicit iteration becomes very useful. The problem is replaced by a family of perturbed problems admitting a unique solution. A particular solution is obtained as the limit of the perturbed solutions of the perturbation vanishes. For instance, given a nonempty closed and convex $C \subseteq H$, $T : C \to C$, $z \in C$ and $t \in (0, 1)$, Browder [25] studied the approximating sequence $\{u_t\}$ defined by

$$u_t = tz + (1-t)Tu_t,$$

that is u_t is the unique fixed point of the contraction $tz + (1-t)T$. He proved under the framework of a Hilbert space that $\{u_t\}$ converges to a fixed point of T closest to z as $t \to 0$.

2.8.7 Halpern Iteration

Halpern [73] introduced the explicit iterative sequence defined for an arbitrary point $x_0 \in C$ by

$$x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \quad n \geq 0,$$

for finding a fixed point of a nonexpansive self mapping T of C, where $u \in C$ is fixed and $\{\alpha_n\}$ is a sequence in (0,1).

2.8.8 Viscosity Approximation Method

Moudafi [117], presented a generalization of the results of Browder [25] and Halpern [73] as follows:

$$u_t = tf(x_t) + (1-t)Tx_t \tag{2.8.8}$$

and

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \tag{2.8.9}$$

where $\{\alpha_n\} \subset [0,1)$, $t \in (0,1)$, $f : C \to C$ is a contraction and $T : C \to C$ is a nonexpansive mapping. He proved that if $F(T) \neq \emptyset$, then the recursions (2.8.8) and (2.8.9) converge strongly to the fixed point of T which also doubles as the solution of the variational inequality:

$$\langle (I-f)x^*, x - x^* \rangle \geq 0, \quad \forall\, x \in F(T),$$

where I is the identity operator. Other generalization of this method can be found in Takahashi and Takahashi [157] and Xu [169].

2.9 Inertial Iteration

In [132], Polyak introduced the so-called heavy ball method, a two-step iterative method for minimizing a smooth convex function f. The algorithm is of the form:

$$\begin{cases} y_k = x_k + \theta_k(x_k - x_{k-1}), \\ x_{k+1} = y_k - \lambda_k \nabla f(x_k), \end{cases} \tag{2.9.1}$$

where $\theta_k \in [0,1)$ is an extrapolation factor which in conjuction with the iterates x_k, x_{k-1} improves the speed of convergence and λ_k is another step size parameter which has to be chosen sufficiently small. The difference compared to a standard gradient method is found in each iteration in which the extrapolated y_k is used instead of x_k. Remarkably, this minor change greatly improves the performance of the scheme. In actual fact, its efficiency estimate [132] on strongly convex functions is equivalent to the known lower complexity bounds of first order methods [124] and therefore the heavy-ball method can be likened to an optimal method. This acceleration of the inertial type scheme can be explained by the fact that the new iterate is given by taking a step which is a combination of the direction $x_k - x_{k-1}$ and the current anti gradient $-\nabla f(x_k)$. Another interpretation of the heavy-ball method is an expilicit finite differences discretization of the dynamical system,

$$\frac{dx^2}{dt^2} + \theta_1 \frac{dx}{dt} + \theta_2 \nabla f(x,t) = 0, \tag{2.9.2}$$

where $\theta_1, \theta_2 > 0$ are free model parameters of the equation. Equation (2.9.2) describes the motion of a heavy body in a potential field f and hence the system is termed the heavy-ball with friction dynamical system.

Alvarez and Attouch [3], translated the idea of the heavy-ball method to the setting of a general maximal monotone operator using the framework of proximal point algorithm. They called the resulting algorithm the inertial proximal point algorithm and is written as

$$\begin{cases} y_k = x_k + \theta_k(x_k - x_{k-1}), \\ x_{k+1} = (I + \lambda_k T)^{-1} y_k. \end{cases} \tag{2.9.3}$$

They showed that under certain conditions on the parameters θ_k and λ_k, the algorithm converges weakly to a zero of T. In fact, the algorithm converges if $\{\lambda_k\}$ is non-decreasing and $\theta_k \in [0,1)$ is such that

$$\sum_k \theta_k ||x_k - x_{k-1}||^2 < \infty, \tag{2.9.4}$$

which can be achieved by choosing θ_k with respect to a simple on-line rule which ensures summability or in particular it is also true for $\theta_k < \frac{1}{3}$. With the inertial type algorithm getting more attention, Moudafi and Oliny [115] in a subsequent paper introduced an additional single-valued and Lipschitz operator B into the inertial proximal point algorithm:

$$\begin{cases} x_{k+1} = x_k + \theta_k (I + \lambda T)^{-1}(y_k - \lambda_k B(x_k)). \end{cases} \tag{2.9.5}$$

The algorithm was shown to converge so long $\lambda_k < \frac{2}{L}$, where L is the Lipschitz constant of B and $\sum\limits_k \theta_k ||x_k - x_{k-1}||^2 < \infty$.

In other to improve the convergence rate on smooth convex functions, Nesterov [123] proposed a modification of the heavy-ball method. While the heavy ball method evaluates the gradient in each iterate at the point x_k, the idea of Nesterov was to use the extrapolated point y_k also for evaluating the gradient. In addition, the extrapolation parameter θ_k is computed according to some special rule that allows to prove optimal convergence rates of this scheme. The scheme is defined as follows:

$$\begin{cases} y_k = x_k + \theta_k(x_k - x_{k-1}), \\ x_{k+1} = y_k - \lambda_k \nabla f(y_k), \end{cases} \tag{2.9.6}$$

where $\lambda_k = \frac{1}{L}$. There are however several choices to define an optimal sequence $\{\alpha_k\}$ (see [10, 123, 124, 159]). It has been shown in [124] that the efficiency of the estimate of the scheme above is up to some constant factor equivalent to the lower complexity bounds of first-methods for the class of μ-strongly convex functions, $\mu \geq 0$, with L-Lipschitz gradient.

CHAPTER 3

Iterative Approximation of Common Solution of the Zero of an Accretive Operator and a Fixed Point of k-strictly Pseudo-contractive Mapping

3.1 Introduction

Let C be a nonempty, closed and convex subset of a real q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. In this chapter, we study the approximation of the zero of a strongly accretive operator $A : X \to X$ which is also the fixed point of a k strictly pseudo-contractive self mapping on C. We introduce a new algorithm and prove its strong convergence to the zero of A and fixed point of T. The obtained result is applied to obtain solution of nonlinear integral equation of the Hammerstein type.

Since its introduction by Browder [29] in 1967, several authors (see e.g, [126, 149]) have studied different types of iterative scheme in approximating the solutions of accretive operators. Xu [168], Kim and Xu [97] studied the sequence $\{x_n\}$ generated by the algorithm

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{rn} x_n, \quad n \geq 0, \tag{3.1.1}$$

for an arbitrary element $x_1 \in C$, and proved the strong convergence of (3.1.1) in the frame work of reflexive Banach spaces and uniformly smooth Banach spaces which has a weak continuous duality map, respectively.

Qin and Su [133] studied the sequence $\{x_n\}$ defined iteratively for an arbitrary $x_1 \in C$ by

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n) J_{rn} x_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases} \tag{3.1.2}$$

where $u \in C$ is an arbitrary but fixed element in C and sequences $\{\alpha_n\} \subset (0,1)$, $\{\beta_n\} \subset [0,1]$. They proved under some certain conditions on the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{r_n\}$ that $\{x_n\}$ defined by (3.1.2) converges to a zero point of A.

Very recently, Aoyama and Toyoda [4] in their attempt to approximate the zero of an m-accretive operator A in a Banach space X studied the sequence defined iteratively by (3.1.1). They obtained the strong convergence of the sequence $\{x_n\}$ to the zero of A using the resolvent operator defined on A.

Motivated by the above results, we introduce an algorithm which is a Halpern-type three step iterative scheme. Let $\{x_n\}$ be a sequence defined in the following manner: $x_1 \in C$, and

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)y_n, \quad n \geq 0. \end{cases} \tag{3.1.3}$$

We apply the resolvent operator to obtain the zero of a strongly accretive operator A which satisfies the range condition and a k-strictly pseudocontractive mapping T. We prove under appropriate conditions on $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\lambda_n\}$ that the sequence (3.1.3) converges strongly to the zero of A which is also the fixed point of T. Our result is established in the framework of a q-uniformly smooth Banach space with a weak continuous duality map. It extends and compliments some existing results in literature.

3.2 Preliminaries

In this section, we give some lemmas which will be used to prove and establish our result in this chapter. We denote the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$.

Lemma 3.2.1. *[133] A Banach space X is uniformly smooth if and only if the duality map J is single valued and norm-to-norm uniformly continuous on bounded subsets of X.*

Lemma 3.2.2. *[171] Let $q > 1$, be a given real number and X be a real normed linear space. Then, for any given $x, y \in X$, the following inequality holds:*

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x+y)\rangle, \ \forall \ j_q(x+y) \in J_q(x+y).$$

Lemma 3.2.3. *[167] Let $q > 1$ be a given real number and X be a smooth Banach space. Then the following are equivalent.*

i. X is q-uniformly smooth.

ii. There is a constant $c > 0$ such that for every $x, y \in X$ the following inequality holds:

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + c_q||y||^q.$$

iii. There is a constant $c_1 > 0$ such that for every $x, y \in X$ the following the inequality holds:

$$\langle x - y, j_q(x) - j_q(y)\rangle \leq c_1||x - y||^q.$$

Lemma 3.2.4. *[149] Let C be a nonempty and closed subset of a uniformly smooth real Banach space X. Let $T : C \to C$ be a k-strict pseudo-contraction. For $\beta \in (0, 1)$, define $T_\beta x = (1 - \beta)x + \beta Tx$. Then, for $\beta \in (0, a)$, $a = min\{1, (\frac{qk}{c_q})^{\frac{1}{q-1}}\}$, $T_\beta : C \to C$ is nonexpansive and $F(T_{\beta)}) = F(T)$.*

Lemma 3.2.5. *[151](Demiclosedness principle): Let C be a non empty closed convex subset of a q-uniformly smooth Banach space X which admits a weakly sequential generalized duality mapping j_p, from X into X^*. Let $T : C \to C$ be a nonexpansive mapping. Then, for all $\{x_n\} \subset C$, if $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x = Tx$.*

Lemma 3.2.6. *[169] Let X be a uniformly smooth Banach space, C a closed nonempty subset of X, $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ a contraction mapping. For each $t \in (0, 1)$, define $u_t = tf(u_t) + (1 - t)Tu_t$, then $\{u_t\}$ converges strongly to the unique fixed point $\bar{x}$ of T as $t \to 0$, where $\bar{x} = Q_{F(T)}f(\bar{x})$ and $Q_{F(T)} : C \to F(T)$ is the sunny nonexpansive retraction from C onto $F(T)$.*

Lemma 3.2.7. *[166] Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n, n \geq 0,$$

where $\{\gamma_n\}_{n=0}^\infty \subset (0, 1)$ and $\{\sigma_n\}_{n=0}^\infty$ such that

i. $\lim\limits_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^\infty \gamma_n = \infty$,

ii. either $\lim\limits_{n\to\infty} sup\ \sigma_n \leq 0$ or $\sum_{n=0}^\infty |\gamma_n\sigma_n| < \infty$.

Then $\lim\limits_{n\to\infty} \alpha_n = 0$.

Lemma 3.2.8. *[76] Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n\tau_n,$$

and

$$a_{n+1} \leq a_n - \eta_n + \rho_n, \quad \forall n \geq 1,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\eta_n\}$ is a sequence of nonnegative real numbers, $\{\tau_n\}$ and $\{\rho_n\}$ are real sequences such that

i. $\sum_{n=1}^\infty \gamma_n = \infty$

ii. $\lim\limits_{n\to\infty} \rho_n = 0$

*iii. $\lim\limits_{k\to\infty} \eta_{n_k} = 0$ implies $\lim\limits_{k\to\infty} sup\ \eta_{n_k} \leq 0$, for any subsequence $\{n_k\} \subset \{n\}$.
Then, $\lim\limits_{n\to\infty} a_n = 0$.*

Lemma 3.2.9. *[107] Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_k}\}$ of $\{\Gamma_n\}$ such that*

$$\Gamma_{n_k} < \Gamma_{n_{k+1}} \quad for \quad all \quad k \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}$ defined by

$$\tau(n) = \max\{k \leq n; \Gamma_k \leq \Gamma_{k+1}\}.$$

Then $\tau(n)$ is a non decreasing sequence satisfying

$$\lim_{n\to\infty} \tau(n) = +\infty,$$

and, for all $n \geq 0$, the following two estimates hold

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1} \quad and \quad \Gamma_n \leq \Gamma_{\tau(n)+1}. \tag{3.2.1}$$

3.3 Main Result

The following lemma is useful in establishing our main result.

Lemma 3.3.1. *For $q > 1$, C be a nonempty closed convex subset of a real q-uniformly smooth Banach X space which admits a weakly sequentially continuous generalized duality mapping j_q. Let $T : C \to C$ be a k-strictly pseudo-contractive mapping, f a contraction mapping on C with constant $\gamma \in (0,1)$ and A be a strongly accretive mapping on X which satisfies the range condition with $D(A) \subset C$, and $\Omega := A^{-1}(0) \cap F(T) \neq \emptyset$. Given a fixed but arbitrary element x_0 in C, sequences $\{\lambda_n\}, \{\beta_n\}, \{\alpha_n\} \subset (0,1)$, and $\{r_n\}$ is a sequence of positive real numbers, define the sequence $\{x_n\}$ iteratively by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J^A_{r_n} x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n) y_n, \quad n \geq 0. \end{cases} \tag{3.3.1}$$

Then $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ are bounded.

Proof. Fix $p \in \Omega$, then $p = Tp$ and $Ap = 0$. From (3.3.1) we have the following estimate:

$$\begin{aligned} ||z_n - p|| &= ||(1-\alpha_n)x_n + \alpha_n J^A_{r_n} x_n - p|| \\ &= ||(1-\alpha_n)(x_n - p) + \alpha_n (J^A_{r_n} x_n - p)|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||J^A_{r_n} x_n - p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n ||x_n - p|| \\ &\leq ||x_n - p||. \end{aligned} \tag{3.3.2}$$

Also from (3.3.1), Lemma 3.2.4 and (3.3.2), we have

$$\begin{aligned}
||x_{n+1}-p|| &= ||\lambda_n f(x_n)+(1-\lambda_n)[(1-\beta_n)z_n+\beta_n Tz_n]-p|| \\
&\leq ||\lambda_n f(x_n)+(1-\lambda_n)T_{\beta_n}z_n-p|| \\
&\leq \lambda_n||f(x_n)-p||+(1-\lambda_n)||T_{\beta_n}z_n-p|| \\
&= \lambda_n[||f(x_n)-f(p)+f(p)-p||]+(1-\lambda_n)||T_{\beta_n}z_n-p|| \\
&\leq \lambda_n||f(x_n)-f(p)||+\lambda_n||f(p)-p||+(1-\lambda_n)||z_n-p|| \\
&\leq \lambda_n\gamma||x_n-p||+\lambda_n||f(p)-p||+(1-\lambda_n)||x_n-p|| \\
&= (1-\lambda_n(1-\gamma))||x_n-p||+\lambda_n||f(p)-p|| \\
&\leq \max\left\{||x_n-p||,\frac{1}{1-\gamma}||f(p)-p||\right\} \\
&\leq \vdots \\
&\leq \max\left\{||x_0-p||,\frac{1}{1-\gamma}||f(p)-p||\right\}, \forall n\geq 0, \qquad (3.3.3)
\end{aligned}$$

which implies that the sequences $\{x_n\}$, $\{z_n\}$ and $\{y_n\}$ are bounded. □

Theorem 3.3.2. *For $q>1$, let C be a nonempty closed convex subset of a real q-uniformly smooth Banach space X which admits a weakly sequentially continuous generalized duality mapping j_q. Let $T:C\to C$ be a k-strictly pseudo-contractive mapping, f a contraction mapping on C with constant $\gamma\in(0,1)$ and A be a strongly accretive mapping on X which satisfies the range condition with $D(A)\subset C$ and $\Omega:=A^{-1}(0)\cap F(T)\neq\emptyset$. Given a fixed but arbitrary element x_0 in C, sequences $\{\lambda_n\},\{\beta_n\}$ and $\{\alpha_n\}\subset(0,1)$ and $\{r_n\}$ is a sequence of positive real numbers, satisfying the following conditions*

i. $\sum_{n=0}^{\infty}\lambda_n=\infty, \lambda_n\to 0,$

ii. $\{\beta_n\}\subset[0,k),$

iii. $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$ *and* $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty$,

define the sequence $\{x_n\}$ iteratively by

$$\begin{cases} z_n=(1-\alpha_n)x_n+\alpha_n J_{r_n}^A x_n, \\ y_n=(1-\beta_n)z_n+\beta_n Tz_n, \\ x_{n+1}=\lambda_n f(x_n)+(1-\lambda_n)y_n, \quad n\geq 0. \end{cases} \qquad (3.3.4)$$

Then $\{x_n\}$ converges strongly to $Q_\Omega f(p)$ where $p\in\Omega$ and Q_Ω is the sunny nonexpansive retraction of C onto Ω.

Proof. For $p\in\Omega$, from (3.3.4) and Lemma 3.2.2, we have that

$$\begin{aligned}
||x_{n+1}-p||^q &= ||\lambda_n f(x_n)+(1-\lambda_n)T_{\beta_n}z_n-p||^q \\
&\leq ||\lambda_n(f(x_n)-p)+(1-\lambda_n)(T_{\beta_n}z_n-p)||^q \\
&\leq (1-\lambda_n)||T_{\beta_n}z_n-p||^q+q\lambda_n\langle f(x_n)-p,j_q(x_{n+1}-p)\rangle. \qquad (3.3.5)
\end{aligned}$$

But from Lemma 3.2.3 and (3.3.2), we have

$$\begin{aligned}
||T_{\beta_n} z_n - p||^q &= ||(1-\beta_n)z_n + \beta_n T z_n - p||^q \\
&= ||z_n - p + \beta_n(Tz_n - z_n)||^q \\
&\leq ||z_n - p||^q + q\beta_n\langle Tz_n - z_n, j_q(z_n - p)\rangle + \beta_n^q c_q ||Tz_n - z_n||^q \\
&\leq ||z_n - p||^q - q\beta_n k||z_n - Tz_n||^q + c_q\beta_n^q||z_n - Tz_n||^q \\
&\leq ||z_n - p||^q - \beta_n(qk - c_q\beta_n^{q-1})||z_n - Tz_n||^q \\
&\leq ||x_n - p||^q - \beta_n(qk - c_q\beta_n^{q-1})||z_n - Tz_n||^q. \qquad (3.3.6)
\end{aligned}$$

Also

$$\begin{aligned}
q\lambda_n\langle f(x_n) - p, j_q(x_{n+1} - p)\rangle &= q\lambda_n\langle f(x_n) - f(p), j_q(x_{n+1} - p)\rangle + q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle \\
&\leq q\lambda_n\gamma(||x_n - p||.||x_{n+1} - p||^{q-1}) + q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle \\
&\leq q\lambda_n\gamma(\frac{1}{q}||x_n - p||^q + \frac{q-1}{q}||x_{n+1} - p||^{q(\frac{q-1}{q-1})}) + \\
&\quad q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle \\
&\leq \lambda_n\gamma(||x_n - p||^q + (q-1)||x_{n+1} - p||^q) + \\
&\quad q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle. \qquad (3.3.7)
\end{aligned}$$

Substituting (3.3.6) and (3.3.7) into (3.3.5), we have

$$\begin{aligned}
||x_{n+1} - p||^q &\leq (1-\lambda_n)||x_n - p||^q - \beta_n(1-\lambda_n)(qk - c_q\beta_n^{q-1})||z_n - Tz_n||^q + \lambda_n\gamma||x_n - p||^q \\
&\quad + \lambda_n\gamma(q-1)||x_{n+1} - p||^q + q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle.
\end{aligned}$$

Therefore, it follows that

$$\begin{aligned}
(1 - \lambda_n\gamma(q-1))||x_{n+1} - p||^q &\leq (1 - \lambda_n + \lambda_n\gamma)||x_n - p||^q - \beta_n(1-\lambda_n)(qk - c_q\beta_n^{q-1})||z_n - Tz_n||^q \\
&\quad + q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle.
\end{aligned}$$

Hence

$$\begin{aligned}
||x_{n+1} - p||^q &\leq \left[\frac{1 - \lambda_n(1-\gamma)}{(1 - \lambda_n\gamma(q-1))}\right]||x_n - p||^q - \frac{\beta_n(1-\lambda_n)(qk - c_q\beta_n^{q-1})}{(1 - \lambda_n\gamma(q-1))}||z_n - Tz_n||^q \\
&\quad + \frac{q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle}{(1 - \lambda_n\gamma(q-1))}. \qquad (3.3.8)
\end{aligned}$$

The rest of the proof will be divided into two cases:

Case 1. Suppose $\{||x_n - p||^q\}$ is monotonically nonincreasing, then$\{||x_n - p||^q\}$ converges and thus

$$||x_n - p||^q - ||x_{n+1} - p||^q \to 0. \qquad (3.3.9)$$

Observe that (3.3.8) can be written as

$$\begin{aligned}
||x_{n+1} - p||^q &\leq \left[1 - \frac{\lambda_n(1-\gamma) - \gamma\lambda_n(q-1)}{(1 - \lambda_n\gamma(q-1))}\right]||x_n - p||^q - \frac{\beta_n(1-\lambda_n)(qk - c_q\beta_n^{q-1})}{(1 - \lambda_n\gamma(q-1))}||z_n - Tz_n||^q \\
&\quad + \frac{q\lambda_n\langle f(p) - p, j_q(x_{n+1} - p)\rangle}{(1 - \lambda_n\gamma(q-1))}. \qquad (3.3.10)
\end{aligned}$$

Therefore from (3.3.9), (3.3.10) and the fact that $\lambda_n \to 0$ as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \beta_n(qk - c_q\beta_n^{q-1})||z_n - Tz_n||^q = 0. \tag{3.3.11}$$

Since $\{\beta_n\} \subset (0, k)$, thus we have

$$\lim_{n\to\infty} ||z_n - Tz_n|| = 0. \tag{3.3.12}$$

Next, we claim that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0. \tag{3.3.13}$$

To see this, from (3.3.4), we have

$$\begin{aligned} z_n - z_{n-1} &= (1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n - \big((1-\alpha_{n-1})x_{n-1} + \alpha_{n-1}J_{r_n}^A x_{n-1}\big) \\ &= (1-\alpha_n)(x_n - x_{n-1}) + \alpha_n(J_{r_n}^A x_n - J_{r_n}^A x_{n-1}) + (\alpha_n - \alpha_{n-1})(J_{r_n}^A x_{n-1} - x_{n-1}), \end{aligned}$$

hence,

$$\begin{aligned} ||z_n - z_{n-1}|| &\leq (1-\alpha_n)||x_n - x_{n-1}|| + \alpha_n||J_{r_n}^A x_n - J_{r_n}^A x_{n-1}|| + (\alpha_n - \alpha_{n-1})||J_{r_n}^A x_{n-1} - x_{n-1}|| \\ &\leq ||x_n - x_{n-1}|| + (\alpha_n - \alpha_{n-1})||J_{r_n}^A x_{n-1} - x_{n-1}|| \\ &\leq ||x_n - x_{n-1}|| + (\alpha_n - \alpha_{n-1})M_1, \end{aligned} \tag{3.3.14}$$

where $M_1 > ||J_{r_n}^A x_{n-1} - x_{n-1}|| \ \forall n \in \mathbb{N}$.

Also from (3.3.4), we have

$$\begin{aligned} ||y_n - y_{n-1}|| &= (1-\beta_n)||z_n - z_{n-1}|| + \beta_n||Tz_n - Tz_{n-1}|| + (\beta_n - \beta_{n-1})||Tz_{n-1} - z_{n-1}|| \\ &\leq (1-\beta_n + \beta_n L)||z_n - z_{n-1}|| + (\beta_n - \beta_{n-1})||Tz_{n-1} - z_{n-1}|| \\ &\leq ||z_n - z_{n-1}|| + (\beta_n - \beta_{n-1})||Tz_{n-1} - z_{n-1}||. \end{aligned} \tag{3.3.15}$$

Substituting (3.3.14) into (3.3.15), we have

$$\begin{aligned} ||y_n - y_{n-1}|| &\leq ||x_n - x_{n-1}|| + M_1(\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})||Tz_{n-1} - z_{n-1}|| \\ &\leq ||x_n - x_{n-1}|| + M_2\big((\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1})\big), \end{aligned} \tag{3.3.16}$$

where M_2 is a constant such that $M_2 > \max\{||Tz_{n-1} - z_{n-1}||, M_1\} \ \ \forall \ \ n \in \mathbb{N}$.

Again from (3.3.4), we have

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq (1-\lambda_n)||y_n - y_{n-1}|| + \lambda_n||f(x_n) - f(x_{n-1})|| + (\lambda_n - \lambda_{n-1})||f(x_{n-1}) - y_{n-1}|| \\ &\leq (1-\lambda_n)||y_n - y_{n-1}|| + \gamma\lambda_n||x_n - x_{n-1}|| + \\ &\quad (\lambda_n - \lambda_{n-1})||f(x_{n-1}) - y_{n-1}||. \end{aligned} \tag{3.3.17}$$

Substituting (3.3.16) into (3.3.17), we have

$$\begin{aligned} ||x_{n+1} - x_n|| &\leq (1-\lambda_n)||x_n - x_{n-1}|| + M_3\big((\alpha_n - \alpha_{n-1}) + (\beta_n - \beta_{n-1}) + (\lambda_n - \lambda_{n-1})\big) + \\ &\quad \gamma\lambda_n||x_n - x_{n-1}||, \end{aligned} \tag{3.3.18}$$

where $M_3 > \max\ \{M_2, ||f(x_{n-1}) - y_{n-1}||\}\ \ \forall\ \ n \in \mathbb{N}$.

By applying conditions (i)-(iii) of the hypothesis, we have that $\sum_{n=0}^{\infty} \lambda_n = \infty$, $\lim\limits_{n\to\infty} \lambda_n = 0$ and

$$\sum_{n=1}^{\infty} \left(|\alpha_n - \alpha_{n-1}|\right) + |\lambda_n - \lambda_{n-1}| + |\beta_n - \beta_{n-1}| < \infty.$$

Therefore, by applying Lemma 3.2.7 to (3.3.18), we have that $\lim\limits_{n\to\infty} ||x_{n+1} - x_n|| = 0$.

Furthermore, we have from (3.3.4) that

$$||y_n - z_n|| \ = \ \beta_n ||Tz_n - z_n|| \to 0,$$

and

$$||x_{n+1} - z_n|| \ \leq \ \lambda_n ||f(x_n) - z_n|| + (1 - \lambda_n)||y_n - z_n|| \to 0. \tag{3.3.19}$$

Moreover

$$\begin{aligned} ||z_n - J_{r_n}^A x_n|| &= ||(1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n - J_{r_n}^A x_n|| \\ &= ||(1-\alpha_n)(x_n - J_{r_n}^A x_n) + \alpha_n (J_{r_n}^A x_n - J_{r_n}^A x_n)|| \\ &\leq (1-\alpha_n)||J_{r_n}^A x_n - x_n||, \end{aligned} \tag{3.3.20}$$

but

$$\begin{aligned} ||J_{r_n}^A x_n - x_n|| &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - z_n|| + ||z_n - J_{r_n}^A x_n|| \\ &\leq ||x_n - x_{n+1}|| + ||x_{n+1} - z_n|| + (1-\alpha_n)||J_{r_n}^A x_n - x_n||. \end{aligned}$$

Therefore from (3.3.13) and (3.3.19), we have

$$\alpha_n ||J_{r_n}^A x_n - x_n|| \ \leq \ ||x_n - x_{n+1}|| + ||x_{n+1} - z_n|| \to 0, \quad n \to \infty.$$

Hence,

$$||J_{r_n}^A x_n - x_n|| \to 0. \tag{3.3.21}$$

It follows from (3.3.20) that $||z_n - x_n|| \to 0,\ \ n \to \infty$. Noticing that,

$$||y_n - x_n|| \ \leq \ ||y_n - z_n|| + ||z_n - x_n|| \to 0 \text{ as } n \to \infty. \tag{3.3.22}$$

Now, since $\{x_n\}$ is bounded by Lemma 3.3.1, and X is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup x^* \in C$. Since $||z_{n_i} - x_{n_i}|| \to 0,\ \ i \to \infty$, it implies that $z_{n_i} \rightharpoonup x^*$. By the demiclosednesss principle Lemma 3.2.5 and (3.3.12), we have $x^* \in F(T)$. Also since $||J_{r_n}^A x_{n_i} - x_{n_i}|| \to 0$, then $x^* \in F(J_{r_n}^A)$. Hence $x^* \in \Omega := F(T) \cap A^{-1}(0)$.

We now show that $\{x_n\}$ converges strongly to $Q_\Omega f(x^*)$. Recall from (3.3.8) that

$$\begin{aligned} ||x_{n+1} - p||^q \ \leq\ & \left[1 - \frac{\lambda_n(1-\gamma) - \gamma\lambda_n(q-1)}{(1-\lambda_n\gamma(q-1))}\right]||x_n - p||^q + \frac{q\lambda_n}{(1-\lambda_n\gamma(q-1))}\langle f(p) - p, j_q(x_{n+1} - p)\rangle \\ & - \frac{\beta_n(1-\lambda_n)(qk - c_q\beta_n^{q-1})}{(1-\lambda_n\gamma(q-1))}||z_n - Tz_n||^q. \end{aligned} \tag{3.3.23}$$

In view of Lemma 3.2.8, we set

$$\begin{aligned}
a_n &= ||x_n - p||^q, \qquad \theta_n = \frac{\lambda_n(1-\gamma) - \gamma\lambda_n(q-1)}{(1-\lambda_n\gamma(q-1))}, \\
\delta_n &= \frac{q\lambda_n}{\lambda_n(1-\gamma) - \gamma\lambda_n(q-1)}\rangle, \\
g_n &= \frac{\beta_n(1-\lambda_n)(qk - c_q\beta_n^{q-1})}{(1-\lambda_n\gamma(q-1))}||z_n - Tz_n||^q, \\
k_n &= \frac{q\lambda_n}{(1-\lambda_n\gamma(q-1))}\langle f(p) - p, j_q(x_{n+1} - p)\rangle.
\end{aligned}$$

Then

$$a_{n+1} \le (1-\theta_n)a_n + \theta_n\delta_n \quad \text{and} \quad a_{n+1} \le a_n - g_n + k_n.$$

Since $\{\lambda_n\} \subset (0,1)$, $\lambda_n \to 0$ and $\sum_{n=0}^{\infty}\lambda_n = \infty$, we have that $\theta_n \in (0,1)$, $\sum_{n=0}^{\infty}\theta_n = \infty$ and $\lim_{n\to\infty} g_n = 0$. In order to show that $a_n \to 0$, by Lemma 3.2.8, it suffices to show that for any subsequence $\{n_k\} \subset \{n\}$, if $\lim_{k\to\infty} g_{n_k} = 0$ then $\limsup_{k\to\infty} \delta_{n_k} \le 0$. To see this we show that

$$\limsup_{k\to\infty}\langle p - f(p), j_q(p - x_{n_k+1})\rangle \le 0. \tag{3.3.24}$$

Equivalently (should $||x_n - p|| \neq 0$), we need to show that

$$\limsup_{k\to\infty}\langle p - f(p), j(p - x_{n_k+1})\rangle \le 0. \tag{3.3.25}$$

For any $t \in (0,1)$, set $u_t = tf(u_t) + (1-t)T_s\beta_n u_t$. Then we have

$$\begin{aligned}
||u_t - x_{n_k}||^2 &= ||tf(u_t) + (1-t)T_{\beta_n}u_t - x_{n_k}||^2 \\
&\le (1-t)^2||T_{\beta_n}u_t - x_{n_k}||^2 + 2t\langle f(u_t) - u_t, j(u_t - x_{n_k})\rangle + 2t\langle u_t - x_{n_k}, j(u_t - x_{n_k})\rangle \\
&\le (1-t)^2(||T_{\beta_n}u_t - T_{\beta_n}x_{n_k} + T_{\beta_n}x_{n_k})^2 + 2t\langle f(u_t) - u_t, j(u_t - x_{n_k})\rangle 2t||u_t - x_{n_k}||^2 \\
&\le (1-t)^2(||T_{\beta_n}u_t - T_{\beta_n}x_{n_k}|| + ||T_{\beta_n}x_{n_k}x_{n_k} - x_{n_k}||)^2 + 2t\langle f(u_t) - u_t, j(u_t - x_{n_k})\rangle + \\
&\quad 2t||u_t - x_{n_k}||^2 \\
&\le (1-t)^2(||u_t - x_{n_k}||^2 + 2||u_t - x_{n_k}||||T_{\beta_n}x_{n_k}x_{n_k} - x_{n_k}|| + ||T_{\beta_n}x_{n_k} - x_{n_k}||^2) + \\
&\quad 2t\langle f(u_t) - u_t, j(u_t - x_{n_k})\rangle + 2t||u_t - x_{n_k}||^2 \\
&\le (1+t^2)||u_t - x_{n_k}|| + (1-t)^2[2||u_t - x_{n_k}|| + ||T_{\beta_n}x_{n_k}||]||T_{\beta_n}x_{n_k} - x_{n_k}|| + \\
&\quad 2t\langle f(u_t) - u_t, j(u_t - x_{n_k})\rangle
\end{aligned}$$

$$\begin{aligned}
2t\langle u_t - f(u_t), j(u_t - x_{n_k})\rangle &\le t^2||u_t - x_{n_k}||^2 + f_{n_k}(t) \\
\langle u_t - f(u_t), j(u_t - x_{n_k})\rangle &\le \frac{t}{2}||u_t - x_{n_k}||^2 + \frac{1}{2t}f_{n_k}(t),
\end{aligned} \tag{3.3.26}$$

where $f_{n_k}(t) = (1-t)^2(||u_t - x_{n_k}|| + ||T_{\beta_n}x_{n_k} - x_{n_k}||)||T_{\beta_n}x_{n_k} - x_{n_k}|| \to 0 \;\text{ as }\; k \to \infty$.

Let $k \to \infty$ in (3.3.26), we obtain that

$$\lim_{k\to\infty} \langle u_t - f(u_t), j(u_t - x_{n_k})\rangle \leq \frac{1}{2}M, \tag{3.3.27}$$

where $M > 0$ is a constant such that $M \geq ||u_t - x_{n_k}||^2$ for all $t \in (0,1)$ and $n \geq 1$. Let $t \to 0$ in (3.3.27), we obtain

$$\limsup_{t\to 0}\limsup_{k\to\infty}\langle u_t - f(u_1), j(u_t - x_{n_k})\rangle \leq 0. \tag{3.3.28}$$

On the other hand, we have that

$$\begin{aligned}
\langle p - f(p), j(p - x_{n_k})\rangle &= \langle p - f(p), j(p - x_{n_k})\rangle - \langle p - f(p), j(u_t - x_{n_k})\rangle \\
&\quad + \langle p - f(p), j(u_t - x_{n_k})\rangle - \langle u_t - f(p), j(u_t - x_{n_k})\rangle \\
&\quad + \langle u_t - f(p), j(x_t - x_{n_k})\rangle - \langle u_t - f(u_t), j(u_t - x_{n_k}) \\
&\quad + \langle u_t - f(u_t), j(u_t - x_{n_k})\rangle \\
&= \langle p - f(p), j(p - x_{n_k}) - j(u_t - x_{n_k})\rangle + \langle p - u_t, j(u_t - x_{n_k})\rangle + \\
&\quad \langle f(u_t) - f(p), j(u_t - x_{n_k})\rangle + \langle u_t - f(u_t), j(u_t - x_{n_k})\rangle.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\limsup_{k\to\infty}\langle p - f(p), j(p - x_{n_k}) \leq \\
&\limsup_{k\to\infty}\langle p - f(p), j(p - x_{n_k}) - j(u_t - x_{n_k})\rangle + ||u_t - p||\limsup_{k\to\infty}||x_{n_k} - u_t|| \\
&\quad + \gamma||u_t - p||\limsup_{k\to\infty}||x_{n_k} - u_t|| + \limsup_{k\to\infty}\langle u_t - f(u_t), j(u_t - x_{n_k})\rangle.
\end{aligned} \tag{3.3.29}$$

Since j is norm-to-norm continuous on every bounded subsets of C, it follows from (3.3.28) and $\lim_{t\to 0} u_t = p = Q_{F(T)}f(p)$ that,

$$\limsup_{k\to\infty}\langle p - f(p), j(p - x_{n_k})\rangle = \limsup_{t\to 0}\limsup_{k\to\infty}\langle p - f(p), j(p - x_{n_k})\rangle \leq 0. \tag{3.3.30}$$

It therefore follows from Lemma 3.2.8 that $||x_n - p|| \to 0$ as $n \to \infty$.

Case 2. Suppose $\{||x_n - p||^q\}$ is not a monotonically decreasing sequence. Let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n > n_1$ (for some n_1 large enough) defined by

$$\tau(n) = \ max\ \{k \in \mathbb{N} : k \leq n, \tau_k \leq \tau_{k+1}\}.$$

τ_k is non decreasing and

$$0 \leq ||x_{\tau(n)+1} - p|||^q - ||x_{\tau(n)} - p||^q, \quad \forall n \geq 1.$$

Following similar analysis as in case 1, we obtain
$||z_{\tau(n)} - Tz_{\tau(n)+1}|| \to 0$, $||x_{\tau(n)} - J^A_{r_n}x_{\tau(n)}|| \to 0$ and $||x_{\tau(n)+1} - x_{\tau(n)}|| \to 0$ as $n \to \infty$. We also have that

$$\lim_{n\to\infty}\ sup\ \langle p - f(p), j_q(p - x_{\tau(n)+1})\rangle \leq 0.$$

From (3.3.8), it is true that

$$||x_{\tau(n)+1}-p||^q \leq \left[1-\frac{\lambda_{\tau(n)}(1-\gamma)-\gamma\lambda_{\tau(n)}(q-1)}{(1-\lambda_{\tau(n)}(q-1))}\right]||x_{\tau(n)}-p||^q \\ +\frac{q\lambda_{\tau(n)}}{(1-\lambda_{\tau(n)}(q-1))}\langle f(p)-p, j_q(x_{\tau(n)+1}-p)\rangle. \tag{3.3.31}$$

$$\frac{\lambda_{\tau(n)}(1-\gamma)-\gamma\lambda_{\tau(n)}(q-1)}{(1-\gamma\lambda_{\tau(n)}(q-1))}||x_{\tau(n)}-p||^q \leq \frac{q\lambda_{\tau(n)}}{(1-\gamma\lambda_{\tau(n)}(q-1))}\langle f(p)-p, j_q(x_{\tau(n)+1}-p)\rangle,$$

which implies

$$(1-\gamma q))||x_{\tau(n)}-p||^q \quad \leq \quad q\langle f(p)-p, j_q(x_{\tau(n)+1}-p)\rangle. \tag{3.3.32}$$

Hence,

$$\lim_{n\to\infty}||x_{\tau(n)}-p||^q \to 0, \quad as\ n\to\infty. \tag{3.3.33}$$

That completes the proof. $\square$

The following results are obtained as a consequence of Theorem 3.3.2.

Suppose $f(x_n)$ is replaced in (3.3.4) by a fixed but arbitrary $u \in X$, then the following is obtained as a corollary.

Corollary 3.3.3. *For $q>1$, let C be a closed convex subset of a real q-uniformly smooth Banach X space which admits a weakly sequentially continuous generalized duality mapping j_q. Let $T: C\to C$ be a Lipschitz, k-strictly pseudo-contractive mapping with Lipschitz constant $L\in(0,1)$, and A be a strongly accretive mapping on X which satisfies the range condition with $D(A)\subset C$ and $\Omega := A^{-1}(0)\cap F(T)\neq\emptyset$. Given a fixed but arbitrary element x_0 in C and sequences $\{\lambda_n\},\{\beta_n\},\{\alpha_n\}\subset(0,1)$ and $\{r_n\}>0$ satisfying the following conditions*

i. $\sum_{n=0}^{\infty}\lambda_n=\infty, \lambda_n\to 0,$

ii. $\{\beta_n\}\subset[0,k),$

iii. $\sum_{n=0}^{\infty}|\alpha_{n+1}-\alpha_n|<\infty$, $\sum_{n=0}^{\infty}|\beta_{n+1}-\beta_n|<\infty$ *and* $\sum_{n=0}^{\infty}|\lambda_{n+1}-\lambda_n|<\infty,$

define the sequence $\{x_n\}$ iteratively by

$$\begin{cases} z_n=(1-\alpha_n)x_n+\alpha_n J_{r_n}^A x_n, \\ y_n=(1-\beta_n)z_n+\beta_n T z_n, \\ x_{n+1}=\lambda_n u+(1-\lambda_n)y_n, \quad n\geq 0. \end{cases} \tag{3.3.34}$$

Then $\{x_n\}$ converges strongly to $p\in\Omega$ where $p=Q_\Omega u$ and Q_Ω is the sunny nonexpansive retraction of C onto Ω.

For $\beta_n = 0$ in (3.3.4), we have the following corollary from Theorem 3.3.2.

Corollary 3.3.4. *For $q > 1$, let C be a closed convex subset of a real q-uniformly smooth Banach X space which admits a weakly sequentially continuous generalized duality mapping j_q. Let A be a strongly accretive mapping on X which satisfies the range condition with $D(A) \subset C$ and $A^{-1}(0) \neq \emptyset$, f a contraction mapping on K with constant $\gamma \in (0,1)$. Given a fixed but arbitrary element $x_0 \in C$ sequences $\{\alpha_n\}, \{\lambda_n\} \subset (0,1)$ and $\{r_n\} > 0$ satisfying the following conditions*

i. $\sum_{n=0}^{\infty} \lambda_n = \infty, \lambda_n \to 0$

ii. $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

define the sequence $\{x_n\}$ iteratively by

$$\begin{cases} y_n = (1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)y_n, n \geq 0. \end{cases} \tag{3.3.35}$$

Then $\{x_n\}$ converges to $x^* \in A^{-1}(0)$.

Corollary 3.3.5. *Let C be a non empty closed convex subset of a real Hilbert space H. Let $T: C \to C$ be a Lipschitz, k-strictly pseudo-contractive mapping with Lipschitz constant $L \in (0,1)$, f a contraction mapping on X with constant $\gamma \in (0,1)$ and A be a strongly monotone mapping in X which satisfies the range condition with $D(A) \subset C$ and $\Omega := A^{-1}(0) \cap F(T) \neq \emptyset$. Given a fixed but arbitrary element x_0 in C sequences $\{\lambda_n\}, \{\beta_n\}, \{\alpha_n\} \subset (0,1)$ and $\{r_n\} > 0$ satisfying the following conditions*

i. $\sum_{n=0}^{\infty} \lambda_n = \infty, \lambda_n \to 0$,

ii. $\{\beta_n\} \subset [0, k)$,

iii. $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

define the sequence $\{x_n\}$ iteratively by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)y_n, \quad n \geq 0. \end{cases} \tag{3.3.36}$$

Then $\{x_n\}$ converges strongly to $Q_\Omega f(p)$ where $p \in \Omega$ and Q_Ω is the metric projection of C onto Ω.

3.4 Application to Hammerstein Equation

In this section, we shall present a strong convergence theorem which applies our result obtained in Theorem 3.3.2 to approximating the solution of a nonlinear integral equation of the Hammerstein type. A nonlinear equation of the Hammerstein type (see e.g [74]) is one of the form

$$u(x) + \int_{\Omega} K(x,y)f(y,u(y))dy = h(x), \tag{3.4.1}$$

where dy is a σ-finite measure on the measure space Ω, the real kernel K is defined on $\Omega \times \Omega$, f is a real valued function defined on $\Omega \times \mathbb{R}$, and is, in general nonlinear and h is a given function on Ω. If we now define a mapping K by

$$Kv(x) := \int_{\Omega} K(x,y)v(y)dy : x \in \Omega,$$

and the so-called *superpostion* or *Nemytskii* operator by $Fu(y) := f(y,u(y))$, then, the integral equation (3.4.1) can be put in the operator theoretic form as follows:

$$u + KFu = 0, \tag{3.4.2}$$

where without loss of generality, we have taken $h \equiv 0$. We note that if K is an arbitrary accretive operator not necessarily the identity, then $A := I + KF$ need not be accretive. Interest in (3.4.2) stems mainly from the fact that several problems that arises in differential equations, for instance, elliptic boundary problems whose linear part admit Green functions can, as a rule, be transformed into the form (3.4.2) (see, [128] Chapter IV). Equations of Hammerstein type play a crucial role in the theory of feedback control systems [59]. Several existence and uniqueness theorems have been proved for solutions of nonlinear integral equations of Hammerstein type (see [66, 22, 23]) for instance and the reference contained therein. In general, equations of Hammerstein type (3.4.2) are nonlinear and there is no known standard method of finding solutions to them. Consequently, methods of approximating solutions of such equations have been studied [1] and also constitute a flourishing area of research in the theory of fixed points. We therefore, remark that for the iterative approximation of solutions of the equation $Ax = 0$, the monotonicity or accretivity of the operator A is crucial and that a very useful iterative scheme has been employed. Chidume [49] defined an auxiliary operator T in terms of K and F in q-uniformly Banach space which under certain condition is (strongly) accretive whenever K and F are. The zeroes represent the solution of (3.4.2).

Very recently Tufa et al [160], constructed a new explicit iterative scheme and prove its strong convergence to the solution of the generalized Hammerstein type equation in a real Hilbert space. They proved the following result

Theorem 3.4.1. *Let H be a real Hilbert space. Let $F, K : H \to H$ be Lipschitz monotone mappings with Lipschitz constants L_1 and L_2, respectively. Suppose that the equation $0 = u + KFu$ has a solution in H. Let $\bar{u}, \bar{v} \in H$ and the sequences $\{u_n\}, \{v_n\} \subset H$ be generated from arbitrary $u_0, v_0 \in H$ by*

$$\begin{cases} u_{n+1} = \alpha_n \bar{u} + (1-\alpha_n)(a_n u_n + (1-a_n)t_n), \\ v_{n+1} = \alpha_n \bar{v} + (1-\alpha_n)(a_n v_n + (1-a_n)s_n), \end{cases} \tag{3.4.3}$$

where $\{\gamma_n\} \subset [a,b] \subset (0, \frac{1}{L})$, *for* $L := \sqrt{2}\max\{\sqrt{L_1^2+1}, \sqrt{L_2^2+1}\}$, $\{a_n\} \subset (0,r] \subset (0,1)$ *and* $\{\alpha_n\} \subset (0,c] \subset (0,1)$, *for all* $n \geq 0$ *satisfies* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum \alpha_n = \infty$. *Then the sequences* $\{u_n\}$ *and* $\{v_n\}$ *converge strongly to* u^* *and* v^* *respectively in* H, *where* u^* *is the solution of* $0 = u + KFu$ *and* $v^* = Fu^*$.

With t_n and s_n defined as $t_n = u_n - \gamma_n[u_n' - v_n + \gamma_n(Kv_n + u_n)]$, $s_n = v_n - \gamma_n[v_n' + u_n - \gamma_n(Fu_n - v_n)]$.

We first state the following results which are essential for the application of our result in 3.3.2.

Theorem 3.4.2. *[167] Let* $q > 1$ *and* X *be a real Banach space. Then the following are equivalent.*

i X *is q-uniformly smooth;*

ii There exists a constant $d_q > 0$ *such that for all* $x, y \in X$,

$$||x+y||^q \leq ||x||^q + q\langle y, j_q(x)\rangle + d_q||y||^q;$$

iii There exists a constant $c_q > 0$ *such that for all* $x, y \in X$ *and* $\lambda \in [0,1]$,

$$||(1-\lambda)x + \lambda y||^q \geq (1-\lambda)||x||^q + ||y|| - w_q(\lambda)c_q||x-y||^q,$$

where $w_q(\lambda) := \lambda^q(1-\lambda) + \lambda(1-\lambda)^q$.

Lemma 3.4.3. *[174]Let* X *be real uniformly smooth Banach space. Let* $E = X \times X$ *with the norm* $||x||_E^q = ||u||_X^q + ||v||_X^q$, *for arbitrary* $x = [u,v] \in E$. *Let* $E^* = X^* \times X^*$ *denote the dual space of* E. *For* $z = [z_1, z_2] \in E$, *define the map* $j_q^E : E \to E^*$ *by*

$$j_q^E(z) := [j_q^X(z_1), j_q^X(z_2)],$$

so that for arbitrary $x_1 = [u_1, v_1]$, $x_2 = [u_2, v_2]$ *in* E, *the duality pairing* $\langle \cdot, \cdot \rangle$ *is given by*

$$\langle x_1, j_q^E(x_2)\rangle := \langle u_1, j_q^X(u_2)\rangle + \langle v_1, j_q^X(v_2)\rangle.$$

Then

(a) E *is uniformly smooth and convex,*

(b) j_q^E *is single-valued duality mapping on* E.

Lemma 3.4.4. *[49] Let* X *be a real q-uniformly smooth Banach space. Let* $F, K : X \to X$ *be maps with* $D(K) = R(F) = X$ *such that the following conditions hold:*

i For each $u_1, u_2 \in X$, *there exists* $\alpha > 0$ *such that*

$$\langle Fu_1 - Fu_2, j_q(u_1 - u_2) \geq \alpha||u_1 - u_2||^q;$$

ii *For each* $u_1, u_2 \in X$, *there exists* $\beta > 0$ *such that*

$$\langle Ku_1 - Ku_2, j_q(u_1 - u_2) \geq \beta||u_1 - u_2||^q;$$

iii $(1 + d_q)(1 + c_q) \geq 2^q$, *where* c_q *and* d_q *are the constants appearing in inequalities (ii) and (iii), respectively, of Theorem 3.4.2. Let* $E = X \times X$, *define* $Ax = A[u, v] = [u + Kv, Fu - v]$ $\forall (u, v) \in E$, *then, for* $x_1, x_2 \in E$, *the following inequality hold:*

$$\langle Ax_1 - Ax_2, j_q^E(x_1 - x_2)\rangle \geq \left[\gamma - \frac{1}{q}\left[\frac{(1 + d_q)(1 + c_q) - 2^q}{(1 + c_q)}\right]\right]||x_1 - x_2||^q,$$

where we assume $\gamma := min\ \{\alpha, \beta\} > \frac{1}{q}\left[\frac{(1+d_q)(1+c_q)-2^q}{1+c_q}\right]$.

Theorem 3.4.5. *For* $q > 1$, *let* C *be a closed convex subset of a real q-uniformly smooth Banach* X *space which admits a weakly sequentially continuous generalized duality mapping* J_q. *Let* $F, K : C \to C$ *be strongly accretive mappings with constants* α *and* $\beta \in (0, 1)$ *respectively, satisfying the range condition. Let* $E := X \times X$, *define* $A : E \to E$ *by* $Ax = [u, v] = [Fu - v, u + Kv]$, $\forall\ (u, v) \in E$. *Let* T *be a k-strictly pseudo-contractive mapping with Lipschitz constant* $L \in (0, 1)$. *Assume that the equation* $0 = u + KFu$ *has a solution in* X. *Given a fixed but arbitrary element* x_0 *in* C *sequences* $\{\lambda_n\}, \{\beta_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{r_n\} > 0$ *satisfying the following conditions*

i. $\sum_{n=0}^{\infty} \lambda_n = \infty, \lambda_n \to 0,$

ii. $\{\beta_n\} \subset [0, k)$,

iii. $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

define the sequence $\{x_n\}$ *iteratively by*

$$\begin{cases} z_n = (1 - \alpha_n)x_n + \alpha_n J_{r_n}^A x_n, \\ y_n = (1 - \beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \lambda_n f(x_n) + (1 - \lambda_n)y_n, \quad n \geq 0. \end{cases} \tag{3.4.4}$$

Then $\{x_n\} = [u_n, v_n]$ *converges to* $x^* = [u^*, v^*]$, *where* $v^* = Fu^*$ *and* u^* *is the solution of* $u + KFu = 0$.

As a consequence to Theorem 3.4.5, we obtain the following corollary.

Corollary 3.4.6. *For* $q > 1$, *let* C *be a closed convex subset of a real q-uniformly smooth Banach* X *space which admits a weakly sequentially continuous generalized duality mapping* J_q. *Let* $F, K : C \to C$ *be accretive mappings satisfying the range condition. Let* $E := X \times X$, *define* $A : E \to E$ *by* $Ax = [u, v] = [Fu - v, u + Kv]$, $\forall\ (u, v) \in E$. *Let* T *be a k-strictly pseudo-contractive mapping with Lipschitz constant* $L \in (0, 1)$. *Assume that the equation* $0 = u + KFu$ *has a solution in* X. *Given a fixed but arbitrary element* x_0 *in* C *sequences* $\{\lambda_n\}, \{\beta_n\}, \{\alpha_n\} \subset (0, 1)$ *and* $\{r_n\} > 0$ *satisfying the following conditions*

i. $\sum_{n=0}^{\infty} \lambda_n = \infty, \lambda_n \to 0,$

ii. $\{\beta_n\} \subset [0, k)$,

iii. $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$,

define the sequence $\{x_n\}$ *iteratively by*

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n J_{r_n}^A x_n, \\ y_n = (1-\beta_n)z_n + \beta_n T z_n, \\ x_{n+1} = \lambda_n f(x_n) + (1-\lambda_n)y_n, \quad n \geq 0. \end{cases} \tag{3.4.5}$$

Then $\{x_n\} = [u_n, v_n]$ *converges to* $x^* = [u^*, v^*]$, *where* $v^* = Fu^*$ *and* u^* *is the solution of* $u + KFu = 0$.

CHAPTER 4

Iterative Approximation of Common Solution of a Finite Family of Mixed Equilibrium Problem with Relaxed $\mu - \alpha$ Monotone Mapping

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4.1 Introduction

In this chapter, we introduce a U-mapping for finite family of mixed equilibrium problems involving $\mu - \alpha$-relaxed monotone operator. We prove a strong convergence theorem for finding the common solution of finite family of these equilibrium problems in a uniformly smooth and strictly convex Banach space which also enjoys Kadec-Klee property. Furthermore, we give some applications of our result and numerical example to show its relevance.

Let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem with respect to F and C in the sense of Blum and Oettli(1994) is to find $x \in C$ such that

$$F(x, y) \geq 0 \ \forall y \in C. \tag{4.1.1}$$

Fang and Huang [65] introduced the concept of relaxed $\mu - \alpha$ monotone mapping for solving a mixed equilibrium problem. A mapping $A : C \to X^*$ is said to be relaxed $\mu - \alpha$ monotone [147], if there exists a mapping $\mu : C \times C \to X$ and a function $\alpha : X \to \mathbb{R}$ with $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in X$, where $p > 1$ such that

$$\langle Ax - Ay, \mu(x, y) \rangle \geq \alpha(x - y).$$

In particular if $\mu(x, y) = x - y, \quad \forall x, y \in C$ and $\alpha(z) = k||z||^p$, where $p > 1$ and $k > 1$ are constants, then A is called p monotone [163].

Fang and Huang [65] proved that under some appropriate conditions, the following variational inequality is solvable; find $x \in C$ such that

$$\langle Ax, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq 0 \ \forall y \in C, \tag{4.1.2}$$

where $\phi : C \to \mathbb{R} \cup \{\infty\}$ is a nonlinear mapping. They also proved that the following inequality is equivalent to the variational inequality (4.1.2) : find $x \in C$ such that

$$\langle Ax, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq \alpha(y-x) \ \forall y \in C. \tag{4.1.3}$$

The mixed equilibrium problem (see e.g [165]) is to find $x \in C$ such that

$$F(x,y) + \langle Ax, \mu(y,x)\rangle + \phi(y) - \phi(x) \geq 0, \ \forall y \in C. \tag{4.1.4}$$

4.2 Preliminaries

In this section, we give some lemmas which will be used to prove and establish our result in this chapter. We denote the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$.

Lemma 4.2.1. *[171] Let X be a real Banach space. Then for all $x, y \in X$ and $j(x+y) \in J(x+y)$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle. \tag{4.2.1}$$

Lemma 4.2.2. *[152] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X such that*

$$x_{n+1} = \beta_n x_n + (1-\beta_n)y_n, \quad n \geq 0,$$

where $\{\beta_n\}$ is a sequence in $(0,1)$ such that $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Assume that

$$\limsup_{n\to\infty}(||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \leq 0.$$

Then $\lim_{n\to\infty} ||y_n - x_n|| = 0$.

Lemma 4.2.3. *[166] Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the condition*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n\sigma_n, n \geq 0,$$

where $\{\gamma_n\} \subset (0,1)$ and $\{\sigma_n\}$ is a sequence in $\mathbb{R}$ such that

i. $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$,

ii. either $\lim_{n\to\infty} sup\ \sigma_n \leq 0$ or $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 4.2.4. *[166] Let X be a uniformly smooth Banach space and C be a nonepmty, closed and convex subset of X. Let $U : C \to C$ be a nonexpansive mapping such that $F(U) \neq \emptyset$ and $f : C \to C$ be a contraction mapping. For each t in $(0,1)$, define $z_t = tf(z_t) + (1-t)Uz_t$, then $\{z_t\}$ converges strongly to the unique fixed point q of U as $t \to 0$, where $q = P_{F(U)}f(q)$ and $P_{F(U)} : C \to F(U)$ is the sunny nonexpansive retraction from C to $F(U)$.*

The following result is proved by Chen et.al [44] for the resolvent operator of mixed equilibrium problem with relaxed $\mu - \alpha$ mapping.

Lemma 4.2.5. *[44] Let X be a uniformly smooth, strictly convex Banach space with the dual space X^* and let C be a nonempty, closed, convex and bounded subset of X. Let $A : C \to X^*$ be a μ-hemicontinuous and relaxed $\mu - \alpha$ monotone mapping, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(F1) - (F4)$ and $\phi : C \times C \to \mathbb{R} \cup \{+\infty\}$. Let $r > 0$ and define a mapping $K_r : X \to C$ as follows:*

$$K_r(x) = \{z \in C : F(z,y) + \langle Az, \mu(y,z)\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \ \ \forall \ \ y \in C\}$$

for all $x \in X$. Assume that

- *(i)* $\mu(x,y) + \mu(y,x) = 0 \quad \forall \ \ x,y \in C$;
- *(ii) for any fixed $u, v \in C$, the mapping $x \mapsto \langle Av, \mu(x,u)\rangle$ is convex and lower semicontinuous;*
- *(iii) $\alpha : X \to \mathbb{R}$ is weakly lower semicontinuous; that is, for any net $\{x_\beta\}$, $\{x_\beta\}$ converges to x in $\sigma(X, X^*)$ implying that $\alpha(x) \leq \liminf \alpha(x_\beta)$;*
- *(iv) for any $x, y \in C$, $\alpha(x-y) + \alpha(y-x) \geq 0$;*
- *(v)* $\langle A(tz_1 + (1-t)z_2), \mu(y, tz_1 + (1-t)z_2)\rangle \geq t\langle Az_1, \mu(y,z_1)\rangle + (1-t)\langle Az_2, \mu(y,z_2)\rangle$ *for any $z_1, z_2, y \in C$ and $t \in [0,1]$.*

Then the following hold:

- *(1) K_r is single-valued;*
- *(2) K_r is a firmly nonexpansive type mapping; that is, for all $x, y \in X$, $\langle K_r x - K_r y, JK_r x - JK_r y\rangle \leq \langle K_r x - K_r y, Jx - Jy\rangle$;*
- *(3) $F(K_r) = EP(F, A)$;*
- *(4) $EP(F, A)$ is closed and convex.*

Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, $A_i : C \to X^*$ be a finite family of μ hemicontinuous relaxed $\mu - \alpha$ monotone mappings and $\phi_i : C \times C \to \mathbb{R} \cup \{+\infty\}$ be a finite family of proper, convex and lower semicontinuous functions. Let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for all $i = 1, 2 \ldots, N$. We define a mapping $U : C \to C$ as follows:

$$\begin{aligned}
S_1 &= \lambda_1 K_r^1 + (1 - \lambda_1) I, \\
S_2 &= \lambda_2 K_r^2 S_1 + (1 - \lambda_2) S_1, \\
&\vdots \\
S_{N-1} &= \lambda_{N-1} K_r^{N-1} S_{N-2} + (1 - \lambda_{N-1}) S_{N-2}, \\
U &= S_N = \lambda_N K_r^N S_{N-1} + (1 - \lambda_N) S_{N-1}. \qquad (4.2.2)
\end{aligned}$$

The mapping so defined above is called U-mapping generated by $K_r^1, K_r^2, \ldots, K_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$.

4.3 Main Result

In this section, we present our main results.

Lemma 4.3.1. *Let X be a uniformly smooth, strictly convex Banach space with the dual space X^* and let C be a nonempty, closed and convex subset of X. Let $A : C \to X^*$ be a relaxed $\mu - \alpha$ monotone mapping, let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying $(F2)$ and $\phi : C \times C \to \mathbb{R} \cup \{+\infty\}$. Assume that*

(i) $\mu(x, y) + \mu(y, x) = 0 \quad \forall \;\; x, y \in C$;

(ii) for any $x, y \in C$, $\alpha(x - y) + \alpha(y - x) \geq 0$.

For $s > 0$ and $r > 0$, then $||K_s x - K_r x|| \leq \left|1 - \frac{r}{s}\right| ||x - K_s x||$.

Proof. Let $z = K_r(x)$ and $w = K_s(x)$, from the definition of K_r, we have

$$F(z, y) + \langle Az, \mu(y, z)\rangle + \phi(y) - \phi(z) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0 \;\; \forall \;\; y \in C.$$

In particular, we have

$$F(z, w) + \langle Az, \mu(w, z)\rangle + \phi(w) - \phi(z) + \frac{1}{r}\langle w - z, Jz - Jx\rangle \geq 0. \qquad (4.3.1)$$

Similarly, we obtain

$$F(w, z) + \langle Aw, \mu(z, w)\rangle + \phi(z) - \phi(w) + \frac{1}{s}\langle z - w, Jw - Jx\rangle \geq 0. \qquad (4.3.2)$$

Adding equation (4.3.1) and (4.3.2), we obtain from (i) that

$$F(z, w) + F(w, z) + \langle Az - Aw, \mu(w, z)\rangle + \frac{1}{r}\langle w - z, Jz - Jx\rangle + \langle z - w, Jw - Jx\rangle \geq 0.$$

Using condition $(F2)$, we have

$$\frac{1}{r}\langle w-z, Jz-Jx\rangle + \frac{1}{s}\langle z-w, Jw-Jx\rangle \geq \langle Aw-Az, \mu(w,z)\rangle \geq \alpha(w-z), \quad (4.3.3)$$

interchanging the roles of w and z in (4.3.3) we obtain

$$\frac{1}{s}\langle z-w, Jw-Jx\rangle + \frac{1}{r}\langle w-z, Jz-Jx\rangle \geq \alpha(z-w). \quad (4.3.4)$$

Adding (4.3.3) and (4.3.4), and using condition (ii), we have

$$\frac{1}{r}\langle w-z, Jz-Jx\rangle + \frac{1}{s}\langle z-w, Jw-Jx\rangle \geq 0,$$

which implies that

$$\langle w-z, Jz-Jx\rangle - \langle w-z, \frac{rJw-rJx}{s}\rangle \geq 0.$$

That is,

$$\langle w-z, \frac{rJw-rJx}{s} - (Jz-Jx)\rangle \leq 0,$$

which implies

$$\langle w-z, \frac{rJw-rJx-sJz+sJw-sJw+sJx}{s}\rangle \leq 0. \quad (4.3.5)$$

This further implies that

$$\begin{aligned} ||w-z||^2 &\leq \langle w-z, \frac{r-s}{s}(Jx-Jw)\rangle \\ &\leq \left|1-\frac{r}{s}\right| ||w-z|.||x-w||. \end{aligned}$$

That is,

$$||w-z|| \leq \left|1-\frac{r}{s}\right| ||x-w||. \quad (4.3.6)$$

Hence,

$$||K_s x - K_r x|| \leq \left|1-\frac{r}{s}\right| ||x-K_s x||. \quad (4.3.7)$$

□

Proposition 4.3.2. *Let C be a nonempty closed convex subset of a uniformly smooth and strictly convex Banach space X. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2\ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, $A_i : C \to X^*$ be a finite family of μ-hemicontinuous relaxed $\mu - \alpha$ monotone mapping and $\phi_i : C \to \mathbb{R} \cup \{+\infty\}$ be a finite family of proper convex lower semicontinuous mapping. Let $\lambda_1, \lambda_n, \ldots, \lambda_N$ be real numbers such that $0 \leq \lambda_i \leq 1$ for all $i = 1, 2, \ldots, N$. Let U be the U-mapping defined in (1.2.7). Then $S_1, S_2, \ldots, S_{N-1}$ and U are nonexpansive. Also, $F(U) = \cap_{i=1}^N EP(F_i, A_i)$.*

Proof. By the nonexpansivity property of K_r^i, for $i = 1, 2, \ldots, N$, it follows that $S_1, S_2, \ldots, S_N = U$ are nonexpansive mappings. Since $\cap_{i=1}^N F(K_r^i) = \cap_{i=1}^N EP(F_i, A_i)$, then it suffices to show that $F(U) = \cap_{i=1}^N F(K_r^i)$. To show that $F(U) = \cap_{i=1}^N F(K_r^i)$, we have to show that $\cap_{i=1}^N F(K_r^i) \subseteq F(U)$ and $F(U) \subseteq \cap_{i=1}^N F(K_r^i)$. It is easily observed that the first part is obvious. Next we show that $F(U) \subseteq \cap_{i=1}^N F(K_r^i)$. Let $a \in F(U)$ and $b \in \cap_{i=1}^N F(K_r^i)$. Using the definition of U, we have

$$\begin{aligned}
||a-b|| &= ||Ua-b|| \\
&= ||\lambda_N K_r^N S_{N-1}a + (1-\lambda_N)S_{N-1}a - b|| \\
&\leq \lambda_N ||K_r^N S_{N-1}a - b|| + (1-\lambda_N)||S_{N-1}a - b|| \\
&\leq ||S_{N-1}a - b||, \\
&= ||\lambda_{N-1}(K_r^{N-1}S_{N-2}a - b) + (1-\lambda_{N-1})(S_{N-2}a - b)|| \\
&\leq \lambda_{N-1}||K_r^{N-1}S_{N-2}a - b|| + (1-\lambda_{N-1})||S_{N-2}a - b|| \\
&\leq ||S_{N-2}a - b|| \\
&\vdots \\
&\leq ||S_1 a - b|| \\
&= ||\lambda_1 K_r^1 a + (1-\lambda_1)a - b|| \\
&\leq \lambda_1 ||K_r^1 a - b|| + (1-\lambda_1)||a-b|| \\
&\leq ||a-b||. \qquad (4.3.8)
\end{aligned}$$

It follows that

$$||a-b|| = ||\lambda_1(K_r^1 a - b) + (1-\lambda_1)(a-b)||$$

and

$$||a-b|| = \lambda_1 ||K_r^1 a - b|| + (1-\lambda_1)||a-b||,$$

that is $||a-b|| = ||K_r^1 a - b||$. Using the strict convexity of X, we obtain $K_r^1 a = a$, which implies that $a \in F(K_r^1)$.

Hence $S_1 a = a$. Again from (4.3.8) and the fact that $S_1 a = a$, we have

$$||a-b|| = ||\lambda_2(K_r^2 S_1 a - b) + (1-\lambda_2)(a-b)||$$

and

$$||a-b|| = \lambda_2 ||K_r^2 a - b|| + (1-\lambda_2)||a-b||,$$

that is $||a-b|| = ||K_r^2 a - b||$. Using the strict convexity of X, we obtain $K_r^2 a = a$, which implies that $a \in F(K_r^2)$. From which we obtain $S_2 a = a$. Proceeding the same way, we obtain $a = K_r^1 a = K_r^2 a = \cdots = K_r^{N-1} a$ and $a = S_1 a = S_2 a = \cdots = S_{N-1} a$.

Since $a \in F(U) = F(S_N)$ and $S_{N-1}a = a$, then $a = \lambda_N K_r^N a + (1-\lambda_N)a$. This implies that $a = K_r^N a$. Hence $F(U) \subset F(K_r^i)$ for $i = 1, 2, \ldots, N$ and thus $F(U) \subset \cap_{i=1}^N F(K_r^i)$. Therefore, $F(U) = \cap_{i=1}^N F(K_r^i) = \cap_{i=1}^N EP(F_i, A_i)$.
The proof is complete. □

Proposition 4.3.3. *Let X be a uniformly smooth and strictly convex Banach space. For $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$, let U_n be a U-mapping defined in (1.2.7). Let $\{x_n\}$ be a bounded sequence in X, then the following inequality is satisfied.*

$$||U_{n+1}x_n - U_n x_n|| \leq ||x_{n+1} - x_n|| + M_N \sum_{i=1}^{N} \big|\lambda_{n+1,i} - \lambda_{n,i}\big|. \tag{4.3.9}$$

Proof. Using the fact that $K^i_{r_n}$ and $S_{n,i}$ for $i = 1, 2 \ldots, N$ are nonexpansive with Lemma 4.3.1, we obtain the following estimates:

$$\begin{aligned}
&||U_{n+1}x_n - U_n x_n|| \\
&= ||\lambda_{n+1,N}K^N_{r_{n+1}}S_{n+1,N-1}x_n + (1-\lambda_{n+1,N})S_{n+1,N-1}x_n - [\lambda_{n,N}K^N_{r_n}S_{n,N-1}x_n + (1-\lambda_{n,N})S_{n,N-1}x_n]|| \\
&= ||\lambda_{n+1,N}(K^N_{r_{n+1}}S_{n+1,N-1}x_n - K^N_{r_{n+1}}S_{n,N-1}x_n) + (S_{n+1,N-1}x_n - S_{n,N-1}x_n) + \\
&\quad (\lambda_{n+1,N})(S_{n,N-1}x_n - S_{n+1,N-1}x_n) + (\lambda_{n,N} - \lambda_{n+1,N})(S_{n,N-1}x_n) + \\
&\quad (\lambda_{n+1,N})(K^N_{r_{n+1}}S_{n,N-1}x_n - K^N_{r_n}S_{n,N-1}x_n) + (\lambda_{n+1,N} - \lambda_{n,N})(K^N_{r_n}S_{n,N-1}x_n)|| \\
&\leq \lambda_{n+1,N}||K^N_{r_{n+1}}S_{n+1,N-1}x_n - K^N_{r_{n+1}}S_{n,N-1}x_n|| + (1-\lambda_{n+1,N})||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| + \\
&\quad |\lambda_{n+1,N} - \lambda_{n,N}|.||K^N_{r_n}S_{n,N-1}x_n - S_{n,N-1}x_n|| + \lambda_{n+1,N}||K^N_{r_{n+1}}S_{n,N-1}x_n - K^N_{r_n}S_{n,N-1}x_n|| \\
&\leq \lambda_{n+1,N}||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| + (1-\lambda_{n+1,N})||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| + \\
&\quad |\lambda_{n+1,N} - \lambda_{n,N}|.||K^N_{r_n}S_{n,N-1}x_n - S_{n,N-1}x_n|| + \lambda_{n+1,N}||K^N_{r_{n+1}}S_{n,N-1}x_n - K^N_{r_n}S_{n,N-1}x_n|| \\
&\leq ||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|.||K^N_{r_n}S_{n,N-1}x_n - S_{n,N-1}x_n|| + \\
&\quad ||K^N_{r_{n+1}}S_{n,N-1}x_n - K^N_{r_n}S_{n,N-1}x_n|| \\
&\leq ||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| + |\lambda_{n+1,N} - \lambda_{n,N}|.||K^N_{r_n}S_{n,N-1}x_n - S_{n,N-1}x_n|| + \left|1 - \frac{r_{n+1}}{r_n}\right| ||S_{n,N-1}x_n|| \\
&\leq ||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| + M_1\left(|\lambda_{n+1,N} - \lambda_{n,N}|, \left|1 - \frac{r_{n+1}}{r_n}\right|\right),
\end{aligned} \tag{4.3.10}$$

where M_1 is a constant such that $M_1 \geq \max\{||K^N_{r_n}S_{n,N-1}x_n - S_{n,N-1}x_n||, ||S_{n,N-1}x_n||\}$. Furthermore,

$$\begin{aligned}
&||S_{n+1,N-1}x_n - S_{n,N-1}x_n|| \\
&= ||\lambda_{n+1,N-1}K^{N-1}_{r_{n+1}}S_{n+1,N-2}x_n + (1-\lambda_{n+1,N-1})S_{n+1,N-2}x_n - \\
&\quad [\lambda_{n,N-1}K^{N-1}_{r_n}S_{n,N-2}x_n + (1-\lambda_{n,N-1})S_{n,N-2}x_n]|| \\
&= ||\lambda_{n+1,N-1}(K^{N-1}_{r_{n+1}}S_{n+1,N-2}x_n - K^{N-1}_{r_{n+1}}S_{n,N-2}x_n) + (1-\lambda_{n+1,N-1})(S_{n+1,N-2}x_n - S_{n,N-2}x_n) + \\
&\quad (\lambda_{n+1,N-1} - \lambda_{n,N-1})(K^{N-1}_{r_n}S_{n,N-2}x_n - S_{n,N-2}x_n) + \lambda_{n+1,N-1}(K^{N-1}_{r_{n+1}}S_{n+1,N-2}x_n - K^{N-1}_{r_n}S_{n,N-2}x_n)|| \\
&\leq ||S_{n+1,N-2}x_n - S_{n,N-2}x_n|| + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|.||K^{N-1}_{r_n}S_{n,N-2}x_n - S_{n,N-2}x_n|| + \\
&\quad \left|1 - \frac{r_{n+1}}{r_n}\right| ||S_{n,N-2}x_n||,
\end{aligned} \tag{4.3.11}$$

substituting (4.3.11) into (4.3.10), we obtain

$$
\begin{aligned}
&||U_{n+1}x_n - U_nx_n|| \\
&\leq M_1\left(\left|1-\frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,N}-\lambda_{n,N}|\right) + ||S_{n+1,N-2}x_n - S_{n,N-2}x_n|| + \\
&\qquad |\lambda_{n+1,N-1}-\lambda_{n,N-1}|.||K^{N-1}_{r_n}S_{n,N-2}x_n - S_{n,N-2}x_n|| + \left|1-\frac{r_{n+1}}{r_n}\right| ||S_{n,N-2}x_n|| \\
&\leq M_2\left(2\left|1-\frac{r_{n+1}}{r_n}\right| + |\lambda_{n+1,N}-\lambda_{n,N}| + |\lambda_{n+1,N-1}-\lambda_{n,N-1}|\right) + ||S_{n+1,N-2}x_n - S_{n,N-2}x_n||,
\end{aligned}
\tag{4.3.12}
$$

where $M_2 \geq \max\{M_1, ||K^{N-1}_{r_n}S_{n,N-2}x_n - S_{n,N-2}x_n||, ||S_{n,N-2}x_n||\}$.
Proceeding the same way as above, we obtain

$$
\begin{aligned}
||U_{n+1}x_n - U_nx_n|| \leq M_{N-1}\left((N-1)\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|\right) + \\
||S_{n+1,1}x_n - S_{n,1}x_n||,
\end{aligned}
\tag{4.3.13}
$$

where $M_{N-1} \geq \max\{M_{N-2}, ||K^2_{r_n}S_{n,1}x_n - S_{n,1}x_n||, ||S_{n,1}x_n||\}$.
Hence,

$$
\begin{aligned}
||U_{n+1}x_n - U_nx_n|| &\leq M_{N-1}\left((N-1)\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|\right) + ||S_{n+1,1}x_n - S_{n+1,1}|| \\
&= ||\lambda_{n+1,1}K^1_r + (1-\lambda_{n+1,1})x_n - \lambda_{n,1}K^1_r - (1-\lambda_{n,1})x_n|| + \\
&\quad M_{N-1}\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&= |\lambda_{n+1,1}-\lambda_{n,1}.||K^1_rx_n - x_n|| + M_{N-1}\sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&= ||\lambda_{n+1,1}K^1_{r_{n+1}}x_n + (1-\lambda_{n+1,1}x_n) - \lambda_{n,1}K^1_{r_n}x_n - (1-\lambda_{n,1})x_n|| + \\
&\qquad M_{N-1}\left((N-1)\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=2}^{N-1}|\lambda_{n+1,i}-\lambda_{n,i}|\right) \\
&\leq M_N\left(N\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right),
\end{aligned}
\tag{4.3.14}
$$

where $M_N > \max\{M_{N-1}, ||K^1_rx_n - x_n||, ||x_n||\}$.
But,

$$
\begin{aligned}
||U_{n+1}x_{n+1} - U_nx_n|| &\leq ||U_{n+1}x_{n+1} - U_{n+1}x_n|| + ||U_{n+1}x_n - U_nx_n|| \\
&\leq ||x_{n+1} - x_n|| + M_N\left(N\left|1-\frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|\right).
\end{aligned}
$$

Thus completing the proof. □

Proposition 4.3.4. *Let C be a nonempty, closed and convex subset of a uniformly smooth and strictly convex Banach space X. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2 \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions, let $A_i : C \to X^*$ be a relaxed $\mu - \alpha$ monotone mappings. Let $\phi_i : C \to \mathbb{R} \cup \{\infty\}$ be a finite family of proper convex lower semicontinuous mappings. For $i = 1, 2 \ldots N$, let $\{\lambda_{n,i}\}$ and $\{\lambda_i\}$ be sequences in $[0, 1]$ such that $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$ and $\{r_n\}$ be a sequence in $(0, \infty)$ such that $r_n \to r$ as $n \to \infty$ with $r > 0$. Suppose U is the mapping generated by $K_r^1, K_r^2, \ldots, K_r^N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. For $n \in \mathbb{N}$, let U_n be the mapping generated by $K_{r_n}^1, K_{r_n}^2, \ldots, K_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Assuming the conditions of Lemma 4.3.1 are satisfied, then for each $x \in C$, we have*

$$\lim_{n\to\infty} ||U_n x - Ux|| = 0. \tag{4.3.15}$$

Proof. Let $x \in C$, using Lemma 4.3.1, we have

$$\begin{aligned}
||S_{n,1}x - S_1x|| &= ||\lambda_{n,1}K_{r_n}^1 x + (1-\lambda_{n,1})x - \lambda_1 K_r^1 x - (1-\lambda_1)x|| \\
&= ||\lambda_{n,1}(K_{r_n}^1 - K_r^1 x) + (\lambda_{n,1} - \lambda_1)(K_r^1 x - x)|| \\
&\leq \left|1 - \frac{r_n}{r}\right| ||K_r^1 x - x|| + |\lambda_{n,1} - \lambda_1|.||K_r^1 x - x|| \\
&\leq \left(\left|1 - \frac{r_n}{r}\right| + |\lambda_{n,1} - \lambda_1|\right)||K_r^1 x - x||.
\end{aligned}$$

Using the same argument as above, for each $i = 1, 2, \ldots, N$, we obtain

$$\begin{aligned}
||S_{n,N}x - S_Nx|| &= ||\lambda_{n,N}K_{r_n}^N S_{n,N-1}x + (1-\lambda_{n,N})x - \lambda_N K_r^N S_{N-1}x - (1-\lambda_N)x|| \\
&\leq \lambda_{n,N}||K_{r_n}^N S_{n,N-1}x - K_{r_n}^N S_{N-1}x|| + \lambda_{n,N}||K_{r_n}^N S_{N-1}x - K_r^N S_{N-1}x|| + \\
&\quad |\lambda_{n,1} - \lambda_1|.||K_r^N S_{N-1}x - x|| \\
&\leq ||S_{n,N-1}x - S_{N-1}x|| + \left|1 - \frac{r_n}{r}\right| ||K_r^N S_{N-1}x - S_{N-1}x|| + \\
&\quad |\lambda_{n,1} - \lambda_1|.||K_r^N S_{N-1}x - x||.
\end{aligned}$$

It follows that

$$\begin{aligned}
||U_n x - Ux|| &= ||S_{n,N}x - S_Nx|| \\
&\leq ||S_{n,1}x - S_1x|| + \sum_{i=1}^{N} |\lambda_{n,i} - \lambda_i|.||K_r^i S_{i-1}x - S_{i-1}x|| \\
&\leq \left(\left|1 - \frac{r_n}{r}\right| + |\lambda_{n,1} - \lambda_1|\right)||K_r^1 x - x|| + \sum_{i=1}^{N} |\lambda_{n,i} - \lambda_i|.||K_r^i S_{i-1}x - S_{i-1}x||.
\end{aligned}$$

Since $r_n \to r$ and $\lambda_{n,i} \to \lambda_i$ as $n \to \infty$, then

$$\lim_{n\to\infty} ||U_n x - Ux|| = 0.$$

□

Theorem 4.3.5. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying conditions $(F1) - (F4)$, $A_i : C \to X^*$ be a finite family of μ-hemicontinuous relaxed $\mu - \alpha$ monotone mapping and $\phi_i : C \to \mathbb{R} \cup \{\infty\}$ be a finite family of proper convex lower semicontinuous functions. Let $K_r^1, K_r^2, \ldots, K_r^N$ be a finite family of resolvent operators for mixed equilibrium problems with relaxed $\mu - \alpha$ mappings on C such that $\cap_{i=1}^N F(K_r^i) \neq \emptyset$. Let $f : C \to C$ be a contraction with constant $\theta \in (0,1)$, let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ be real numbers satisfying $0 \leq, \lambda_{n,i} \leq 1$ such that $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_i| = 0$ for $i = 1, 2, \ldots, N$ with $0 \leq \lambda_i \leq 0$. For $n \in \mathbb{N}$, let U_n be a U-mapping generated by $K_{r_n}^1, K_{r_n}^2, \ldots K_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, r is a positive parameter and $\{r_n\}$ is a sequence in $(0,\infty)$. Assume that the conditions (i)-(v) of Lemma 4.2.5 and the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;

(iii) $r_n \to r$ *as* $n \to \infty$.

(iv) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$.

For a given $x_1 \in C$, let $\{x_n\}$ be the sequence defined iteratively by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n, \quad \forall n \geq 1. \tag{4.3.16}$$

Then $\{x_n\}$ converges to $P_\Gamma f(q)$ where $\Gamma = \cap_{i=1}^N EP(F_i, A_i)$ and P_Γ is the sunny nonexpansive retraction of C onto Γ.

Proof. The proof of this theorem will be divided into several steps.

Step 1: $\{x_n\}$ is bounded. To see this, fix $q \in \Gamma$. We have,

$$\begin{aligned}
||x_{n+1} - q|| &= ||\alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n - q|| \\
&= ||\alpha_n (f(x_n) - q) + \beta_n (x_n - q) + \gamma_n (U_n x_n - q)|| \\
&= ||\alpha_n (f(x_n) - f(q) + f(q) - q) + \beta_n (x_n - q) + \gamma_n (U_n x_n - q)|| \\
&\leq \alpha_n ||f(x_n) - f(q)|| + \alpha_n ||f(q) - q|| + \beta_n ||x_n - q|| + \gamma_n ||U_n x_n - q|| \\
&\leq \theta\alpha_n ||x_n - q|| + \alpha_n ||f(q) - q|| + \beta_n ||x_n - q|| + \gamma_n ||x_n - q|| \\
&\leq \theta\alpha_n ||x_n - q|| + (1 - \alpha_n) ||x_n - q|| + \alpha_n ||f(q) - q|| \\
&\leq [1 - \alpha_n (1 - \theta)] ||x_n - q|| + \alpha_n ||f(q) - q|| \\
&\leq \max\left\{||x_n - q||, \frac{1}{1-\theta} ||f(q) - q||\right\} \\
&\vdots \\
&\leq \max\left\{||x_1 - q||, \frac{1}{1-\theta} ||f(q) - q||\right\}, \quad \forall n \geq 1.
\end{aligned} \tag{4.3.17}$$

Therefore, the sequences $\{x_n\}$ and $\{U_n x_n\}$ are bounded.

Step 2: We show that

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = 0 \tag{4.3.18}$$

Putting

$$y_n = \frac{\alpha_n f(x_n) + \gamma_n U_n x_n}{1-\beta_n},$$

then (4.3.16) becomes

$$x_{n+1} = \beta_n x_n + (1-\beta_n) y_n.$$

Since U_n is nonexpansive, $\{x_n\}$ and $\{U_n x_n\}$ are bounded, we get that $\{y_n\}$ is also bounded. Now,

$$\begin{aligned} y_{n+1} - y_n &= \frac{\alpha_{n+1} f(x_{n+1}) + \gamma_{n+1} U_{n+1} x_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n U_n x_n}{1-\beta_n} \\ &= \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(f(x_{n+1}) - f(x_n)) + \left(\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right)(f(x_n) - U_n x_n) \\ &\quad + \left(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right)(U_{n+1} x_{n+1} - U_n x_n), \end{aligned}$$

hence, using Proposition 4.3.3 and the fact that $\theta \in (0,1)$, we have

$$\begin{aligned} ||y_{n+1} - y_n|| &\le \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| ||f(x_{n+1}) - f(x_n)|| + \left|1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| ||U_{n+1} x_{n+1} - U_n x_n|| \\ &\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| ||f(x_n) - U_n x_n|| \\ &\le \frac{\theta \alpha_{n+1}}{1-\beta_{n+1}} ||x_{n+1} - x_n|| + \left|1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\right| ||x_{n+1} - x_n|| + \\ &\quad M_N\left(N\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|\right) \\ &\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| ||f(x_n) - U_n x_n|| \\ &\le ||x_{n+1} - x_n|| + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| ||f(x_n) - U_n x_n|| + \\ &\quad M_N\left(N\left|1 - \frac{r_{n+1}}{r_n}\right| + \sum_{i=1}^{N} |\lambda_{n+1,i} - \lambda_{n,i}|\right). \end{aligned}$$

This together with $\alpha_n \to 0$, $r_{n+1} \to r_n$ and $|\lambda_{n+1,i} - \lambda_{n,i}| \to 0$ as $n \to \infty$ implies that

$$\limsup_{n\to\infty} (||y_{n+1} - y_n|| - ||x_{n+1} - x_n||) \le 0.$$

Hence by Lemma 4.2.2, we obtain $||y_n - x_n|| \to 0$ as $n \to \infty$.
Consequently,

$$\lim_{n\to\infty} ||x_{n+1} - x_n|| = \lim_{n\to\infty} (1 - \beta_n)||y_n - x_n|| = 0.$$

Step 3: Next, we show that

$$\lim_{n\to\infty} ||x_n - Ux_n|| = 0. \tag{4.3.19}$$

We note that,

$$\begin{aligned}
||x_{n+1} - U_n x_n|| &= ||\alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n - U_n x_n|| \\
&\leq \alpha_n ||f(x_n) - U_n x_n|| + \beta_n ||x_n - U_n x_n|| \\
&\leq \alpha_n ||f(x_n) - U_n x_n|| + \beta_n ||x_{n+1} - x_{n+1} + x_n - U_n x_n|| \\
&\leq \frac{\alpha_n}{1-\beta_n} ||f(x_n) - U_n x_n|| + \frac{\beta_n}{1-\beta_n} ||x_n - x_{n+1}||.
\end{aligned}$$

From conditions (1),(2) and step 2, we have that $\lim_{n\to\infty} ||x_{n+1} - U_n x_n|| = 0$.

Also,

$$||x_n - U_n x_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - U_n x_n|| \to 0, \quad as \quad n \to \infty. \tag{4.3.20}$$

Note also that,

$$\begin{aligned}
||x_n - Ux_n|| &\leq ||x_n - U_n x_n|| + ||U_n x_n - Ux_n|| \\
&\leq ||x_n - U_n x_n|| + \sup_{x\in C} ||U_n x - Ux||.
\end{aligned} \tag{4.3.21}$$

Therefore from (4.3.20) and Proposition 4.3.4, we have that

$$\lim_{n\to\infty} ||x_n - Ux_n|| = 0.$$

Step 4: We show that

$$\lim_{n\to\infty} \langle f(q) - q, j(q - x_n)\rangle \leq 0. \tag{4.3.22}$$

For any $t \in (0,1)$, set $z_t = tf(z_t) + (1-t)Uz_t$. Then we have,

$$\begin{aligned}
||z_t - x_n||^2 &= ||t(f(z_t) - x_n) + (1-t)(Uz_t - x_n)||^2 \\
&\leq (1-t)^2 ||Uz_t - x_n||^2 + 2t\langle f(z_t) - x_n, j(z_t - x_n)\rangle \\
&\leq (1-t)^2 \big[||Uz_t - Ux_n|| + ||Ux_n - x_n||\big]^2 + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle \\
&\quad +2t\langle z_t - x_n, j(z_t - x_n)\rangle \\
&\leq (1-t)^2 \big[||z_t - x_n|| + ||Ux_n - x_n||\big]^2 + 2t||z_t - x_n||^2 + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle, \\
&\leq (1-t)^2 ||z_t - x_n||^2 + g_n(t) + 2t\langle f(z_t) - z_t, j(z_t - x_n)\rangle + 2t||z_t - x_n||^2,
\end{aligned}$$

where

$$g_n(t) = (1-t)^2(2||z_t - x_n|| + ||x_n - U_n x_n||)||x_n - U_n x_n|| \to 0 \quad as \quad n \to \infty. \quad (4.3.23)$$

It follows that

$$\langle z_t - f(z_t), j(z_t - x_n) \quad \leq \quad \frac{t}{2}||z_t - x_n||^2 + \frac{1}{2t} g_n(t). \quad (4.3.24)$$

Letting $n \to \infty$ in (4.3.24) and noting (4.3.23), we obtain

$$\langle z_t - f(z_t), j(z_t - x_n)\rangle \leq \frac{t}{2} M^*,$$

where $M^* = \limsup_{n\to\infty} ||z_t - x_n||^2$. Clearly $\frac{t}{2}M^* \to 0$ as $t \to 0$ from which we obtain

$$\limsup_{t\to 0} \limsup_{n\to\infty} \langle z_t - f(z_t), j(z_t - x_n)\rangle \leq 0.$$

Since j is norm-to-norm continuous on bounded subset of X and by Lemma 4.2.4, $z_t \to q$, where $q = P_\Gamma f(q)$, we have

$$||j(z_t - x_n) - j(q - x_n)|| \to 0$$

as $n \to \infty$.

Observe that

$$\begin{aligned}
&\Big|\langle z_t - f(z_t), j(z_t - x_n)\rangle - \langle q - f(z_t), j(q - x_n)\rangle\Big| \\
&\qquad \leq \Big|\langle z_t - q, j(z_t - x_n)\rangle + \langle q - f(z_t), j(z_t - x_n)\rangle - \langle q - f(z_t), j(q - x_n)\rangle\Big| \\
&\qquad \leq \langle z_t - x_n, j(z_t - x_n)\rangle + \langle q - f(z_t), j(z_t - x_n) - j(q - x_n)\rangle \\
&\qquad \leq ||z_t - q||.||z_t - x_n|| + ||q - f(z_t)||.||j(z_t - x_n) - j(q - x_n)|| \to 0, \ as \ n \to \infty,
\end{aligned}$$

therefore,

$$\langle z_t - f(z_t), j(z_t - x_n)\rangle \to \langle q - f(q), j(q - x_n)\rangle. \quad (4.3.25)$$

Hence,

$$\lim_{n\to\infty} \langle q - f(q), j(q - x_n)\rangle \leq 0. \quad (4.3.26)$$

Step 5: Finally, we show that $x_n \to q$ as $n \to \infty$. From Lemma 5.4.1 and step 1, we have

$$\begin{aligned}
||x_{n+1} - q||^2 &= ||\alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n - q||^2 \\
&= ||\alpha_n(f(x_n) - q) + \beta_n(x_n - q) + \gamma_n(U_n x_n - q)||^2 \\
&\leq ||\beta_n(x_n - q) + \gamma_n(U_n x_n - q)||^2 + 2\alpha_n\langle f(x_n) - q, j(x_{n+1} - q)\rangle \\
&\leq \{\beta_n||x_n - q|| + \gamma_n||x_n - q||\}^2 + 2\alpha_n\langle f(x_n) - f(q), j(x_{n+1} - q)\rangle + \\
&\quad 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq (1-\alpha)^2||x_n - q||^2 + 2\theta\alpha_n(||x_n - q||.||x_{n+1} - q||) + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq (1-\alpha)^2||x_n - q||^2 + \theta\alpha_n||x_n - q||^2 + \theta\alpha_n||x_{n+1} - q||^2 + 2\alpha_n\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq \frac{(1-\alpha_n)^2 + \theta\alpha_n}{1-\theta\alpha_n}||x_n - q||^2 + \frac{2\alpha_n}{1-\theta\alpha_n}\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&= \frac{1 - 2\alpha_n + \theta\alpha_n}{1-\theta\alpha_n}||x_n - q||^2 + \frac{\alpha_n^2}{1-\theta\alpha_n}||x_n - q||^2 + \frac{2\alpha_n}{1-\theta\alpha_n}\langle f(q) - q, j(x_{n+1} - q)\rangle \\
&\leq \left\{1 - \frac{2(1-\theta)\alpha_n}{1-\theta\alpha_n}\right\}||x_n - q||^2 + \frac{2(1-\theta)\alpha_n}{1-\theta\alpha_n}\left\{\frac{M^{**}\alpha_n}{2(1-\theta\alpha_n)} + \right. \\
&\quad \left. \frac{1}{1-\theta}\langle f(q) - q, j(x_{n+1} - q)\rangle\right\}.
\end{aligned} \tag{4.3.27}$$

Observe that the conditions of Lemma 4.2.3 are satisfied with $\gamma_n = \frac{2(1-\theta)\alpha_n}{1-\theta\alpha_n}$ and $\sigma_n = \left\{\frac{M^{**}\alpha_n}{2(1-\theta\alpha_n)} + \frac{1}{1-\theta}\langle f(q) - q, j(x_{n+1} - q)\rangle\right\}$. By Lemma 4.2.3 and (4.3.26), it follows that $||x_n - q|| \to 0$ as $n \to 0$. Therefore $\{x_n\}$ converges strongly to $q = P_\Gamma f(q)$. This completes the proof. □

We obtain the following as consequences of Theorem 4.3.5.

Suppose $A_i = 0$, in Theorem 4.3.5, the mixed equilibrium problem with $\mu - \alpha$ monotone mapping reduces to the following classical mixed equilibrium problem:

$$F_i(z, y) + \phi_i(y) - \phi_i(z) \geq 0.$$

We thus obtain the following result:

Corollary 4.3.6. *Let C be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space X which also enjoys the Kadec-Klee property. Let $\mu : C \times C \to X$ be a nonlinear mapping. For $i = 1, 2, \ldots, N$, let $F_i : C \times C \to \mathbb{R}$ be a finite family of bifunctions satisfying conditions $(F1) - (F4)$, and let $\phi_i : C \to \mathbb{R} \cup \{\infty\}$ be a finite family of proper convex lower semicontinuous functions. Let $K_r^1, K_r^2, \ldots, K_r^N$ be a finite family of resolvent operators for mixed equilibrium problems on C such that $\cap_{i=1}^N F(K_r^i) \neq \emptyset$. Let $f : C \to C$ be a contraction with constant $\theta \in (0, 1)$, let $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ be real numbers satisfying $0 \leq, \lambda_{n,i} \leq 1$ such that $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_i| = 0$ for all $i = 1, 2, \ldots, N$. For all $n \in \mathbb{N}$, let U_n be a U-mapping generated by $K_{r_n}^1, K_{r_n}^2, \ldots K_{r_n}^N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Suppose $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ with $\alpha_n + \beta_n + \gamma_n = 1$, r is a positive parameter and $\{r_n\}$ is a sequence in $(0, \infty)$. Assume that the conditions of Lemma 3.2.8 and the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $r_n \to r$ *as* $n \to \infty$;

(iv) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$.

For a given $x_1 \in C$*, let* $\{x_n\}$ *be the sequence defined iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n, \quad \forall n \ge 1. \tag{4.3.28}$$

Then $\{x_n\}$ *converges to* $P_\Gamma f(q)$ *where* $\Gamma = \cap_{i=1}^{N} EPF_i$ *and* P_Γ *is the sunny nonexpansive retraction of* C *onto* Γ.

For $F_i(x, y) = 0$, in Theorem 4.3.5, the mixed equilibrium problem reduces to the following variational inequality

$$\langle A_i z, \mu(y, z)\rangle + \phi_i(y) - \phi_i(z) \ge 0.$$

We obtain a result which solves the finite family of variational inequalities as follows:

Corollary 4.3.7. *Let* C *be a nonempty closed and convex subset of a uniformly smooth and strictly convex Banach space* X *which also enjoys the Kadec-Klee property. Let* $\mu : C \times C \to \mathbb{R}$ *be a nonlinear mapping. For* $i = 1, 2, \ldots, N$*, let* $A_i : C \to X^*$ *be a finite family of* μ*-hemicontinuous relaxed* $\mu - \alpha$ *monotone mapping and let* $\phi_i : C \to \mathbb{R}$ *be a finite family of proper convex lower semicontinuous functions. Let* $K_r^1, K_r^2, \ldots, K_r^N$ *be a finite family of resolvent operators for variational inequalities with relaxed* $\mu - \alpha$ *mappings on* C *such that* $\cap_{i=1}^{N} F(K_r^i) \neq \emptyset$*. Let* $f : C \to C$ *be a contraction with constant* $\theta \in (0, 1)$*, let* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$ *be real numbers satisfying* $0 \le, \lambda_{n,i} \le 1$ *such that* $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_i| = 0$ *for all* $i = 1, 2, \ldots, N$*. For all* $n \in \mathbb{N}$*, let* U_n *be a* U*-mapping generated by* $K_{r_n}^1, K_{r_n}^2, \ldots K_{r_n}^N$ *and* $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$*. Suppose* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0, 1)$ *with* $\alpha_n + \beta_n + \gamma_n = 1$*,* r *is a positive parameter and* $\{r_n\}$ *is a sequence in* $(0, \infty)$*. Assume that the conditions of Lemma 4.2.5 and the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$;

(iii) $r_n \to r$ *as* $n \to \infty$;

(iv) $\lim_{n\to\infty} |\lambda_{n+1,i} - \lambda_{n,i}| = 0$.

For a given $x_1 \in C$*, let* $\{x_n\}$ *be the sequence defined iteratively by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n U_n x_n, \quad \forall n \ge 1. \tag{4.3.29}$$

Then $\{x_n\}$ *converges to* $P_\Gamma f(q)$ *where* $\Gamma = \cap_{i=1}^{N} VIA_i$ *and* P_Γ *is the sunny nonexpansive retraction of* C *onto* Γ.

4.4 Numerical Example

Let $X = \mathbb{R} \times \mathbb{R}$ and $C = [-1,1] \times [-1,1]$. Define a mapping $A : C \to \mathbb{R} \times \mathbb{R}$ by $A(x_1, x_2) = (x_1, x_2)$ for all $(x_1, x_2) \in C$, $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $\alpha((x_1, x_2)) = 3x_1^2 + 3x_2^2$ for all $(x_1, x_2) \in X$ and $\mu : C \times C \to \mathbb{R} \times \mathbb{R}$ by $\mu((x_1, x_2), (y_1, y_2)) = (2(x_1 - y_1), 2(x_2 - y_2))$ for all $(x_1, x_2) \times (y_1, y_2) \in C \times C$.
Then the mapping A is a relaxed $\mu - \alpha$ monotone mapping. Indeed, for all $x = (x_1, x_2), y = (y_1, y_2) \in C$, we have

$$\begin{aligned} \langle Ax - Ay, \mu(x,y)\rangle &= ((x_1 - y_1), (x_2 - y_2)), (2(x_1 - y_1), 2(x_2 - y_2)) \\ &= 4[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &\geq 3[(x_1 - y_1)^2 + (x_2 - y_2)^2] \\ &= \alpha(x - y). \end{aligned} \tag{4.4.1}$$

Hence A is a relaxed $\mu - \alpha$ monotone mapping.
Let $\bar{z} = (z_1, z_2)$, $\bar{y} = (y_1, y_2)$ and $\bar{x} = (x_1, x_2)$. Define

$$F_i(\bar{z}, \bar{y}) = -3i\bar{z}^2 + 2i\bar{z}\bar{y} + i\bar{y}^2, \quad A_i(\bar{z}) = i\bar{z}, \quad \text{and} \quad \phi_i(\bar{z}) = i\bar{z}^2.$$

Lemma 4.2.5 ensures that there exist $\bar{x} \in \mathbb{R}^2$ such that

$$F_i(\bar{z}, \bar{y}) + \langle A_i\bar{z}, \mu(\bar{y}, \bar{z})\rangle + \phi_i(\bar{y}) - \phi_i(\bar{z}) + \frac{1}{r_n}\langle \bar{y} - \bar{z}, \bar{z} - \bar{x}\rangle \geq 0 \quad \forall \quad \bar{y} \in \mathbb{R}^2$$

$$\begin{aligned} &\iff -3i\bar{z}^2 + 2i\bar{z}\bar{y} + i\bar{y}^2 + i\bar{z}(2(\bar{y} - \bar{z})) + (i\bar{y}^2) - (i\bar{z}^2) + \frac{1}{r_n}(\bar{y} - \bar{z}) \times (\bar{z} - \bar{x}) \geq 0 \\ &\iff -3i\bar{z}^2 + 2i\bar{z}\bar{y} + i\bar{y}^2 + 2i\bar{y}\bar{z} - 2i\bar{z}^2 + (i\bar{y}^2) - (i\bar{z}^2) + \frac{1}{r_n}(\bar{y}\bar{z} - \bar{y}\bar{x} - \bar{z}^2 + \bar{z}\bar{x}) \geq 0 \\ &\iff -3ir_n\bar{z}^2 + 2r_ni\bar{z}\bar{y} + ir_n\bar{y}^2 + 2r_ni\bar{y}\bar{z} - 2r_ni\bar{z}^2 + r_ni\bar{y}^2 - r_ni\bar{z}^2 + \bar{y}\bar{z} - \bar{y}\bar{x} - \bar{z}^2 + \bar{z}\bar{x} \geq 0 \\ &\iff 2ir_n\bar{y}^2 + (4ir_n\bar{z} + \bar{z} - \bar{x})\bar{y} + \bar{z}\bar{x} - \bar{z}^2 - 6ir_n\bar{z}^2 \geq 0. \end{aligned}$$

Let $H(\bar{y}) = 2ir_n\bar{y}^2 + (4ir_n\bar{z} + \bar{z} - \bar{x})\bar{y} + \bar{z}\bar{x} - \bar{z}^2 - 6ir_n\bar{z}^2$, then $H(\bar{y})$ is a quadratic equation in $\bar{y}$.
With $a = 2ir_n$, $b = 4ir_n\bar{z} + \bar{z} - \bar{x}$ and $c = -6ir_n\bar{z}^2 - \bar{z}^2 + \bar{z}\bar{x}$.
We obtain the discriminant Δ of $H(\bar{y})$ as follows:

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (4ir_n\bar{z} + \bar{z} - \bar{x})^2 - 4(2ir_n)(-6ir_n\bar{z}^2 - \bar{z}^2 + \bar{z}\bar{x}) \\ &= \bar{x}^2 + 64i^2r_n^2\bar{z}^2 + 16ir_n\bar{z} + \bar{z}^2 - 16ir_n\bar{x}\bar{z} - 2\bar{x}\bar{z} \\ &= \bar{x}^2 + (8ir_n\bar{z} + \bar{z})^2 - 2\bar{x}\bar{z} - 16ir_n\bar{x}\bar{z} \\ &= \bar{x}^2 - 2(8ir_n\bar{z} + \bar{z})\bar{x} + (8ir_n\bar{z} + \bar{z})^2 \\ &= (\bar{x} - (8r_n\bar{z} + \bar{z})) \geq 0. \end{aligned}$$

Hence,

$$\bar{z} = \frac{\bar{x}}{8ir_n + 1}. \tag{4.4.2}$$

This implies

$$\bar{z} = \left(\frac{x_1}{8ir_n + 1}, \frac{x_2}{8ir + n - 1} \right)$$

and thus

$$K_{r_n}^i(\bar{x}) = \left(\frac{x_1}{8ir_n + 1}, \frac{x_2}{8ir_n + 1} \right). \tag{4.4.3}$$

Assume that $\lambda_{n,i} = \dfrac{1}{in + 2}$ and $S_{n,0}\bar{x} = \bar{x}$. Using (1.2.7) and (4.4.3), we have

$$S_{n,i}\bar{x} = \frac{1}{in + 2} \times \frac{1}{8ir_n + 1} S_{n,i-1}\bar{x} + \frac{in + 1}{in + 2} S_{n,i-1}\bar{x}, \quad \text{for} \ \ i = 1, 2, \ldots, 100, \tag{4.4.4}$$

and $U_n = S_{n,100}$. Choosing $\alpha_n = \frac{1}{n+1}$, $\beta_n = \frac{8}{8n-1}$, $\gamma_n = \frac{16n-7}{8n^2+7n-1}$ and $r_n = \frac{n-1}{2n+1}$. Let $f(\bar{x}) = \frac{1}{10}\bar{x}$, then our iterative algorithm (4.3.28) becomes

$$\bar{x}_{n+1} = \frac{\bar{x}_n}{10(n+1)} + \frac{8\bar{x}_n}{8n - 1} + \frac{16n - 7}{8n^2 + 7n - 1} U_n \bar{x}_n, \quad \forall \quad n \geq 1.$$

We make different choices of our initial value as follow:

$$(1) \ \ \bar{x}_1 = 0.25, \quad (2) \ \ \bar{x}_1 = -0.5 \quad \text{and} \quad (3) \ \ \bar{x}_1 = 0.05.$$

We also vary the stopping criterion as:

$$(a) \ \ \frac{|\bar{x}_{n+1} - \bar{x}_n|}{|\bar{x}_2 - \bar{x}_1|} < 10^{-6} \quad \text{and} \quad (b) \ \ \frac{|\bar{x}_{n+1} - \bar{x}_n|}{|\bar{x}_2 - \bar{x}_1|} < 10^{-12}.$$

Matlab version 2014a is used to obtain the graphs of errors against the number of iterations.

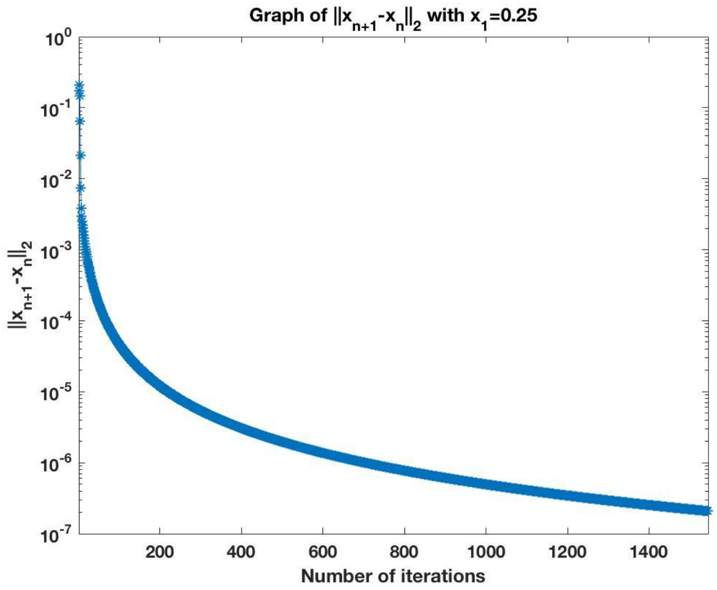

Figure 4.1: Errors vs number of iterations for initial value **1**.

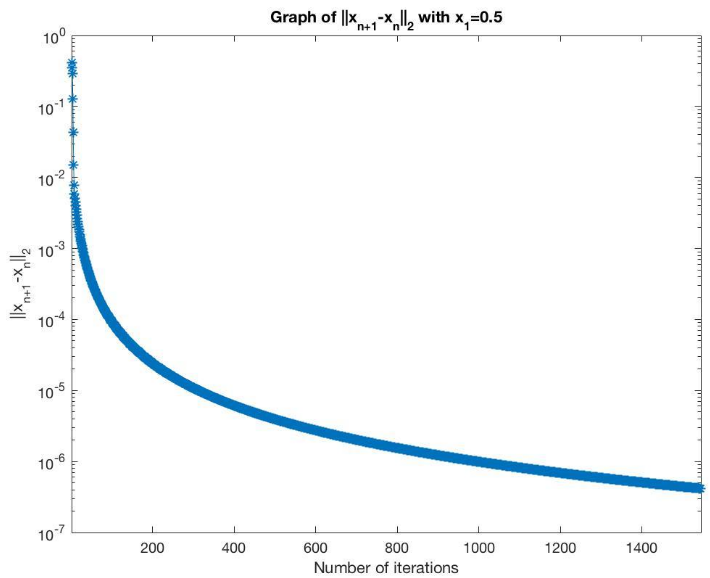

Figure 4.2: Errors vs number of iterations for initial value **2**.

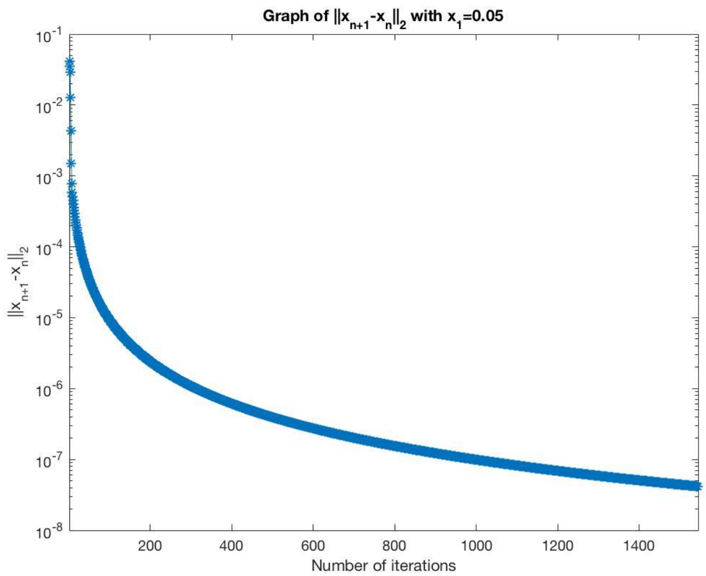

Figure 4.3: Errors vs number of iterations for initial value **3**.

CHAPTER 5

Iterative Approximation of Solution of a Split Variational Inclusion Problem Involving Accretive Operators in Banach Spaces

5.1 Introduction

In recent years, inertial type algorithm has gained a lot more attention due to its speedy rate of convergence. Most especially several works have been done in the Hilbert spaces and few in Banach spaces. To this end, we propose an inertial type iterative algorithm and prove a weak convergence theorem of the scheme to the solution of a split variational inclusion problem involving accrective operators in a Banach space. We present some applications and a numerical example to show the relevance of our result.

Splitting Method for Sum of Accretive Mappings

Splitting method have received more attention recently due to the fact that many nonlinear problems arising in applied areas such as image recovery, machine learning and signal processing can be mathematically modelled as a nonlinear operator equation, which in turn can be further decomposed into the sum of possibly simpler nonlinear operators. Splitting method for linear equations were introduced by Peaceman and Rachford [129] and Douglas and Rachford [60]. Extension to Hilbert spaces were carried out by Kellog [96], Lions and Mercier [102]. The defining problem is to iteratively find a zero of the sum of two monotone operators T_1 and T_2 in Hilbert space H, that is the solution to the inclusion problem

$$0 \in (T_1 + T_2)x. \tag{5.1.1}$$

Many problems in real life can be formulated as (5.1.1). A prominent example is the stationary solution to the initial value problem of the evolution

$$\frac{\partial u}{\partial t} + Fu \ni 0, \quad u(0) = u_0,$$

where the governing maximal monotone F is of the form $T_1 + T_2$. This problem models the optimization problem

$$\min_{x \in H}\{f(x) + gT(x)\}, \tag{5.1.2}$$

where f, g are proper lower semicontinuous functions from H to the extended real line $\mathbb{R} = \{-\infty, +\infty\}$ and T is a bounded linear operator on H. The minimization problem (5.1.2) is widely used in image recovery, machine learning and signal processing. A splitting method for solving (5.1.1) involves an iterative algorithm for which each iteration involves only with the individual operators T_1 and T_2, but not the sum $T_1 + T_2$ concurrently. To solve (5.1.1), Lions and Mercier [102] introduced the nonlinear Peaceman-Rachford and Douglas-Rachford which generate a sequence $\{x_n\}$ by the recursion formula $x_{n+1} = (2J_\lambda^{T_1} - I)(2J_\lambda^{T_2} - I)x_n$ and a sequence $\{x_n\}$ generated by $x_{n+1} = J_\lambda^{T_1}(2J_\lambda^{T_2} - I)x_n + (I - J_\lambda^{T_2})x_n$,

where $J_\lambda^{T_1}$ denotes the resolvent of the monotone operator T_1. Of the two recursion formula, the Douglas-Rachford algorithm always converges in the weak topology to a point y^* and $y^* = J_\lambda^{T_2}x$ is a solution of (5.1.1), since the generating operator $J_\lambda^{T_1}(2J_\lambda^{T_2} - I) + (I - J_\lambda^{T_2})$ for this algorithm is firmly nonexpansive. The Peaceman-Rachford algorithm however fails to converge even in the weak topology in the infinite dimensional settings. There have been several iterative methods since Douglas-Rachford, introduced their scheme for solving sum of monotone inclusions albeit in Hilbert spaces. In recent years, most of these works are being extended in Banach spaces which are more general than the Hilbert spaces; (see [146]).

Split Monotone Variational Inclusion

In 2011, Moudafi [113] introduced the split monotone variational inclusion problem: Find $x^* \in H_1$, such that

$$\begin{cases} 0 \in (T_1(x^*) + S_1(x^*)) \\ y^* = Ax^* \in H_2 : 0 \in (T_2(y^*) + S_2(y^*)), \end{cases} \tag{5.1.3}$$

where $T_1 : H_1 \to 2^{H_1}$ and $T_2 : H_2 \to 2^{H_2}$ are set-valued maximal monotone mappings, $S_1 : H_1 \to H_1$ and $S_2 : H_2 \to H_2$ are single valued monotone operators and $A : H_1 \to H_2$ is a bounded linear operator. In [113], Moudafi obtained a weak convergence theorem of the split monotone variational inclusion problem in the framework of Hilbert spaces.
The split monotone variational inclusion problem includes as special cases; the split common fixed points problem, the split variational inequality problem, the split feasibility problem and the split zero problem. All of which have been extensively studied in literature; (see [154]). Since its introduction in 2011, there has been various iterative method devised to obtaining the solution of the split monotone variational inclusion problem and

related problem in Hilbert spaces and spaces more general.
In 2015, Takahashi [154] considered the split feasibility problem and split common null point problem in the setting of Banach spaces. By using the hybrid methods and Hapern's type methods under appropriate conditions, some strong and weak convergence theorems for such problems in the setting of one Hilbert space and one Banach space were obtained. Tang et al [158], proved a weak convergence theorem and a strong convergence theorem for split common fixed point problem involving a quasi strict pseudo contractive mapping and an asymptotical nonexpansive mapping in the setting of two Banach spaces under some given conditions.
Very recently, Zhang and Wang [173] proposed a new iterative scheme and proved that the scheme converges weakly and strongly to a split common fixed point problem for nonexpansive semi groups in Banach spaces under some suitable conditions. To be more precise they proved the following theorem.

Theorem 5.1.1. *[173] Let X_1 be a real uniformly convex and 2-uniformly smooth Banach space satisfying Opial's condition and with the best smoothness constant k satisfying $0 < k < \frac{1}{\sqrt{2}}$, X_2 be a real Banach space, $A : X_1 \to X_2$ be a bounded linear operator, and A^* be the adjoint of A. Let $\{S(t) : t \geq 0\} : X_1 \to X_1$ be a uniformly asymptotically regular nonexpansive semigroup with $\mathcal{C} := \cap_{t\geq 0} F(S(t)) \neq \emptyset$ and $\{T(t) : t \geq 0\} : X_2 \to X_2$ be a uniformly asymptotically regular nonexpansive semigroup with $\mathcal{Q} := \cap_{t\geq 0} F(T(t)) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by: $x_1 \in X_1$*

$$\begin{cases} z_n = x_n + \gamma J_1^{-1} A^* J_2 (T(t_n) - I) A x_n, \\ x_{n+1} = (1-\alpha_n) z_n + \alpha_n S(t_n) z_n, \quad \forall \ \ n \geq 1, \end{cases} \tag{5.1.4}$$

where $\{t_n\}$ is sequence of real numbers, $\{\alpha_n\}$ a sequence in $(0,1)$ and γ is a positive constant satisfying

(1) $t_n > 0$ and $\lim\limits_{n\to\infty} t_n = \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$ and $0 < \gamma < \frac{1-2k^2}{||A||^2}$.

(I) If $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, then $\{x_n\}$ converges weakly to a split common fixed point $x^* \in \Gamma$.

(II) In addition, if $\Gamma = \{p \in C : Ap \in Q\} \neq \emptyset$, and there is at least one $S(t) \in \{S(t) : t \geq 0\}$ which is semi-compact, then $\{x_n\}$ converges strongly to a split common fixed point $x^* \in \Gamma$.

We consider the following split variational inclusion problem involving accretive operators:

Let X_1 and X_2 be Banach spaces. The split variational inclusion problem for accretive operators is given as: Find $x_1 \in X_1$ such that

$$\begin{cases} 0 \in X_1 : x^* \in (T_1 + S_1), \\ y^* = Ax^* \in X_2 : y^* \in (T_2 + S_2), \end{cases} \tag{5.1.5}$$

where $T_1 : X_1 \to 2^{X_1}, T_2 : X_2 \to 2^{X_2}$ are set-valued accretive operators, $S_1 : X_1 \to X_1$, $S_2 : X_2 \to X_2$ are inverse strongly accretive operators and $A : X_1 \to X_2$ is a bounded linear operator.
Furthermore, we introduce an inertial-type iterative scheme and prove a weak convergence theorem of the scheme to the solution of (5.1.5).

5.2 Preliminaries

In the section, we shall give some lemmas and restate some definitions which we find useful to obtain our result. We denote the strong convergence of $\{x_n\}$ to x by $x_n \to x$ and the weak convergence of $\{x_n\}$ to x by $x_n \rightharpoonup x$.

Lemma 5.2.1. *[106] Let $\{\phi_n\} \subset [0, \infty)$ and $\{\delta_n\} \subset [0, \infty)$ satisfying:*

(1) $\phi_{n+1} - \phi_n \leq \theta_n(\phi_n - \phi_{n-1}) + \delta_n$,

(2) $\sum \delta_n < \infty$,

(3) $\theta_n \subset [0, \theta]$, *where* $\theta \in [0, 1)$.

Then ϕ_n is a convergent sequence and $\sum[\phi_{n+1} - \phi_n]_+ < \infty$, where $[t]_+ := \max\{t, 0\}$ for any $t \in \mathbb{R}$.

Lemma 5.2.2. *[167] Given a number $r > 0$. A real Banach space X is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1 - \lambda)y||^2 \leq \lambda||x||^2 + (1 - \lambda)||y||^2 - \lambda(1 - \lambda)g(||x - y||),$$

for all $x, y \in X$, $\lambda \in [0, 1]$, with $||x|| < r$ and $||y|| < r$.

Recall that a Banach space X is said to satisfy the Opial's condition, if whenever $\{x_n\}$ is a sequence in X which converges weakly to x as $n \to \infty$, then

$$\limsup_{n\to\infty} ||x_n - x|| < \limsup_{n\to\infty} ||x_n - y||, \quad \forall \ y \in X, \quad y \neq x. \tag{5.2.1}$$

Lemma 5.2.3. *[71] Let X be a uniformly convex Banach space, let C be a nonempty closed convex subset of X and let $T : X \to X$ be a nonexpansive mapping. Then $(I - T)$ is demiclosed at zero.*

Lemma 5.2.4. *[54] Let X be a real Banach space with Fréchet differentiable norm. For $x \in X$, let β^* be defined for $t \in (0, \infty)$ by*

$$\beta^*(t) = \sup\left\{\left|\frac{||x + ty||^2 - ||x||^2}{t} - 2\langle y, j(x)\rangle\right| : ||y|| = 1\right\}.$$

Then, $\lim_{t\to 0^+} \beta^*(t) = 0$ and

$$||x + h||^2 \leq ||x||^2 + 2\langle h, j(x)\rangle + ||h||\beta^*||h|| \tag{5.2.2}$$

for all $h \in X - \{0\}$.

Remark 5.2.5. In Lemma 5.2.4 we will assume $\beta^*(t) \leq ct$, $t > 0$ for some $c > 1$. It is easy therefore to obtain the following estimate

$$2\langle h, j(x)\rangle \leq ||x||^2 + c||h||^2 - ||x-h||^2, \tag{5.2.3}$$

by replacing h in (5.2.2) by $-h$.

Lemma 5.2.6. *[104] Let X be a real Banach space. Let $T_1 : X \to 2^X$ be an m-accretive operator and $S_1 : X \to X$ be an α-inverse strongly accretive mapping on X. Then we have*

(i) for $\lambda > 0, F(Q_\lambda) = (T_1 + S_1)^{-1}(0)$,

(ii) for $0 < \lambda < \mu$ and $x \in X$, $||x - Q_\lambda x|| \leq 2||x - Q_\mu x||$,

where $Q_\lambda = J_\lambda^{T_1}(I - \lambda S_1) = (I + \lambda T_1)^{-1}(I - \lambda S_1)$.

Lemma 5.2.7. *[167] Let X be a 2-uniformly smooth Banach space with the best of smoothness constants $k > 0$. Then the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle j(x), y\rangle + 2k^2||y||^2. \quad \forall \quad x, y \in X. \tag{5.2.4}$$

In this sequel we shall use the following notations $P_{\lambda_n} := J_{\lambda_n}^{T_2}(I - \lambda_n S_2) = (I + \lambda_n T_2)^{-1}(I - \lambda_n S_2)$ and $Q_{\lambda_n} := J_{\lambda_n}^{T_1}(I - \lambda_n S_1) = (I + \lambda_n T_1)^{-1}(I - \lambda_n S_1)$, where T_1, S_1, T_2 and S_2 are as defined in (5.1.5).

5.3 Main Result

In this section, we give our main results.

Lemma 5.3.1. *Let X_1 be real uniformly convex and 2-uniformly smooth Banach space, X_2 a real Banach space with Fréchet differentiable norm, $A : X_1 \to X_2$ a bounded linear operator and A^* the adjoint of A. Let $T_1 : X_1 \to 2^{X_1}, T_2 : X_2 \to 2^{X_2}$ be set-valued accretive operators and $S_1 : X_1 \to X_1, S_2 : X_2 \to X_2$ be α-inverse strongly accretive operators. Assume $\Gamma := \{q \in (T_1 + S_1)^{-1}(0) : Aq \in (T_2 + S_2)^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in X_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma J_1^{-1} A^* J_2(P_{\lambda_n} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)Q_{\lambda_n} y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{5.3.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, γ is a positive constant and $\theta_n \subset [0, \theta]$ where $\theta \in [0,1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $0 < \gamma < \frac{1-2k^2}{||A||^2}$, where k is the smoothness constant satisfying $0 < k^2 < \frac{1}{2}$;

(3) $\lambda_n \in (0, \frac{2\alpha}{c}), \quad \forall \quad n \geq 1, \quad c > 1.$

Then $\{x_n\}$ *is bounded.*

Proof. For each $n \geq 1$, let $Q_{\lambda_n}^{T_1} := J_{\lambda_n}^{T_1}(I - \lambda_n S_1)$ and fix $q \in \Gamma$, then $q \in (T_1 + S_1)$ and $Aq \in (T_2 + S_2)$. For all $x, y \in X_1$, using the nonexpansivity of J_{λ_n} and Lemma 5.2.4, we have

$$\begin{aligned}
||Q_{\lambda_n}x - Q_{\lambda_n}y||^2 &= ||J_{\lambda_n}^{T_1}(I - \lambda_n S_1)x - J_{\lambda_n}^{T_1}(I - \lambda_n S_1)y||^2 \\
&\leq ||x - y - \lambda_n(S_1x - S_1y)||^2 \\
&\leq ||x - y|| - 2\lambda_n\langle S_1x - S_1y, j(x - y)\rangle + c\lambda_n^2||S_1x - S_1y||^2 \\
&\leq ||x - y||^2 - \lambda_n(2\alpha - c\lambda_n)||S_1x - S_2y||^2 \\
&\leq ||x - y||^2. \qquad (5.3.2)
\end{aligned}$$

Thus, Q_{λ_n} is nonexpansive for all $n \geq 1$. Similarly, P_{λ_n} is nonexpansive.

So, it follows from (5.3.1) and Lemma 5.2.2, that

$$\begin{aligned}
||x_{n+1} - q||^2 &= ||\alpha_n(y_n - p) + (1 - \alpha_n)(Q_{\lambda_n}y_n - q)||^2 \\
&\leq \alpha_n||y_n - q||^2 + (1 - \alpha_n)||Q_{\lambda_n}y_n - q||^2 - \alpha_n(1 - \alpha_n)g(||y_n - Q_{\lambda_n}y_n||) \\
&\leq \alpha_n||y_n - q||^2 + (1 - \alpha_n)||y_n - q||^2 - \alpha_n(1 - \alpha_n)g(||y_n - Q_{\lambda_n}y_n||) \\
&\leq ||y_n - q||^2 - \alpha_n(1 - \alpha_n)g(||y_n - Q_{\lambda_n}y_n||). \qquad (5.3.3)
\end{aligned}$$

Again, from (5.3.1) and Lemma 5.2.7, we have

$$\begin{aligned}
||y_n - q||^2 &= ||(u_n - q) + \gamma J_1^{-1}A^*J_2(P_{\lambda_n} - I)Au_n||^2 \\
&\leq ||\gamma J_1^{-1}A^*J_2(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle u_n - q, A^*J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\
&\leq \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle u_n - q, A^*J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\
&\leq \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle Au_n - Ap, J_2(P_{\lambda_n} - I)Au_n\rangle + 2k^2||u_n - q||^2 \\
&= \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle Au_n - P_{\lambda_n}Au_n + P_{\lambda_n}Au_n - Ap, J_2(P_{\lambda_n} - I)Au_n\rangle + \\
&\quad\; 2k^2||u_n - q||^2 \\
&= \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 + 2\gamma\langle P_{\lambda_n}Au_n - P_{\lambda_n}Ap, J_2(P_{\lambda_n} - I)Au_n\rangle - \\
&\quad\; 2\gamma||Au_n - P_{\lambda_n}Au_n||^2 + 2k^2||u_n - q||^2 \\
&\leq \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 - 2\gamma||Au_n - P_{\lambda_n}Au_n||^2 + \gamma[||P_{\lambda_n}Au_n - P_{\lambda_n}Ap||^2 + \\
&\quad\; ||(P_{\lambda_n} - I)Au_n||^2] + 2k^2||u_n - q||^2 \\
&\leq \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 - \gamma||Au_n - P_{\lambda_n}Au_n||^2 + 2k^2||u_n - q||^2 + \gamma||P_{\lambda_n}Au_n - P_{\lambda_n}Ap||^2 \\
&\leq \gamma^2||A||^2||(P_{\lambda_n} - I)Au_n||^2 - \gamma||Au_n - P_{\lambda_n}Au_n||^2 + 2k^2||u_n - q|| + \gamma||A||^2||u_n - q||^2 \\
&\leq \gamma(\gamma||A||^2 - 1)||(P_{\lambda_n} - I)Au_n||^2 + (\gamma||A||^2 + 2k^2)||u_n - q||^2
\end{aligned}$$

Furthermore, from (5.3.1), Lemma 5.2.4 and Remark 5.2.5, we have

$$\begin{aligned}
||u_n - q||^2 &= ||(x_n - q) + \theta_n(x_n - x_n - 1)||^2 \\
&\leq ||x_n - q||^2 + 2\theta_n\langle x_n - x_{n-1}, j(x_n - q)\rangle + c\theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - q||^2 + \theta_n[||x_n - q||^2 + c||x_n - x_{n-1}||^2 - ||x_{n-1} - q||^2] + c\theta_n^2||x_n - x_{n-1}||^2 \\
&\leq ||x_n - q||^2 + \theta_n[||x_n - q||^2 - ||x_{n-1} - q||^2] + 2c\theta_n||x_n - x_{n-1}||^2, \qquad (5.3.4)
\end{aligned}$$

which together with (**??**), implies

$$\begin{aligned} ||y_n - q||^2 &\leq (\gamma||A||^2 + 2k^2)[||x_n - q||^2 + \theta_n(||x_n - q||^2 - ||x_{n-1} - q||^2) + 2c\theta_n||x_n - x_{n-1}||^2] - \\ &\quad \gamma(1 - \gamma||A||^2)||(P_{\lambda_n} - I)Au_n||^2. \end{aligned} \tag{5.3.5}$$

Thus, from (5.3.3), we obtain

$$\begin{aligned} ||x_{n+1} - q||^2 &\leq (\gamma||A||^2 + 2k^2)[||x_n - q||^2 + \theta_n(||x_n - q||^2 - ||x_{n-1} - q||^2) + 2c\theta_n||x_n - x_{n-1}||^2] - \\ &\quad \gamma(1 - \gamma||A||^2)||(||(P_{\lambda_n} - I)Au_n||^2) - \alpha_n(1 - \alpha_n)g(||y_n - Q_{\lambda_n}y_n||). \end{aligned} \tag{5.3.6}$$

Since $0 < \gamma||A||^2 + 2k^2 < 1$, we obtain

$$\begin{aligned} ||x_{n+1} - q||^2 &\leq ||x_n - q||^2 + \theta_n[||x_n - q||^2 - ||x_{n-1} - q||^2] + 2c\theta_n||x_n - x_{n-1}||^2 - \\ &\quad \gamma(1 - \gamma||A||^2)||(||(P_{\lambda_n} - I)Au_n||^2 - \\ &\quad \alpha_n(1 - \alpha_n)g(||y_n - Q_{\lambda_n}y_n||). \end{aligned} \tag{5.3.7}$$

That is,

$$\begin{aligned} ||x_{n+1} - q||^2 &\leq ||x_n - q||^2 + \theta_n(||x_n - q||^2 - ||x_{n-1} - q||^2) + \\ &\quad 2c\theta_n||x_n - x_{n-1}||^2. \end{aligned} \tag{5.3.8}$$

Since $\sum_{n\geq 0} \theta_n||x_n - x_{n-1}||^2 < \infty$, and $\theta_n \subset [0, \theta]$, $[\theta \in (0, 1)$, we obtain from Lemma 5.2.1 that the sequence $\{||x_n - q||\}$ is convergent, hence bounded. Consequently, the sequence $\{||y_n - q||\}$ is bounded. □

Theorem 5.3.2. *Let X_1 be a real uniformly convex Banach space and 2-uniformly smooth satisfying Opial's condition, X_2 a real Banach space with Fréchet differentiable norm, $A : X_1 \to X_2$ a bounded linear operator and A^* the adjoint of A. Let $T_1 : X_1 \to 2^{X_1}, T_2 : X_2 \to 2^{X_2}$ be set-valued accretive operators and $S_1 : X_1 \to X_1, S_2 : X_2 \to X_2$ be α-inverse strongly accretive operators. Assume $\Gamma := \{q \in (T_1 + S_1)^{-1}(0) : Aq \in (T_2 + S_2)^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in X_1$, $\{x_n\}$ be the sequence given by (5.3.1) where α_n is a sequence in $(0, 1)$, γ is a positive constant and $\theta_n \subset [0, \theta)$ where $\theta \in [0, 1)$ satisfying the following conditions:*

(1) $\sum_{n\geq 0} \theta_n||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$;

(3) $0 < \gamma < \frac{1-2k^2}{||A||^2}$, *where k is the smoothness constant satisfying* $0 < k^2 < \frac{1}{2}$;

(4) $0 < \lambda \leq \lambda_n \leq b < \frac{2\alpha}{c}, \quad \forall\, n \geq 1, \quad c > 1.$

Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

Proof. Let $q \in \Gamma$, then by Lemma 5.2.1 and (5.3.8), we obtain $\sum_{n\geq 0}[||x_n - q||^2 - ||x_{n-1} - q||^2]_+ < \infty$, also from (5.3.7), we have

$$\begin{aligned} \gamma(1 - \gamma||A||^2)||(P_{\lambda_n} - I)Au_n||^2 + \alpha_n(1 - \alpha_n)g(||y_n - Q_{\lambda_n}y_n||) &\leq ||x_{n+1} - q||^2 - ||x_n - q||^2 + \\ &\quad \theta_n[||x_n - q||^2 - ||x_{n-1} - q||^2]_+ \\ &\quad +2c\theta_n||x_n - x_{n-1}||^2. \end{aligned} \tag{5.3.9}$$

Hence, we obtain

$$\sum_{n\geq 0}[\gamma(1-\gamma||A||^2)||(P_{\lambda_n}-I)Au_n||^2+\alpha_n(1-\alpha_n)g(||y_n-Q_{\lambda_n}y_n||)] \quad < \quad \infty. \tag{5.3.10}$$

This implies that

$$\lim_{n\to\infty}||(P_{\lambda_n}-I)Au_n||=0. \tag{5.3.11}$$

Also, condition (2) and the property of the function g in Lemma 5.2.2, we obtain

$$\lim_{n\to\infty}||y_n-Q_{\lambda_n}y_n||=0. \tag{5.3.12}$$

From Condition (4) we have $\lambda_n>0, \quad \forall\ n\geq 1$ therefore, there exists $\epsilon>0$ such that $\lambda_n\geq\epsilon$ for all $n\geq 1$. Then, by Lemma 5.2.6,

$$\lim_{n\to\infty}||Q_\epsilon x_n-x_n||\leq 2\lim_{n\to\infty}||Q_{\lambda_n}x_n-x_n||=0. \tag{5.3.13}$$

Since, Q_ϵ is nonexpansive, we have $F(Q_\epsilon)=(T_1+S_1)^{-1}(0)\neq\emptyset$.

Same argument holds for P_ϵ, hence, P_ϵ is nonexpansive and $F(P_\epsilon)=(T_2+S_2)^{-1}(0)\neq\emptyset$.

From condition (1), we have $\sum_{n\geq 0}\theta_n||x_n-x_{n-1}||^2<\infty$, which implies $\theta_n||x_n-x_{n-1}||\to 0$ as $n\to\infty$.

Observe that

$$||u_n-x_n||=||(x_n-x_n)+\theta_n(x_n-x_{n-1})||\to 0.$$

Hence,

$$\lim_{n\to\infty}||u_n-x_n||=0. \tag{5.3.14}$$

Also,

$$\begin{aligned}||y_n-x_n|| &= ||(u_n-x_n)+\gamma J_1^{-1}A^*J_2(P_{\lambda_n}-I)Au_n||\\ &\leq ||u_n-x_n||+||\gamma J_1^{-1}A^*J_2(P_{\lambda_n}-I)Au_n||.\end{aligned} \tag{5.3.15}$$

Using (5.3.11) and (5.3.14), we obtain

$$\lim_{n\to\infty}||y_n-x_n||=0. \tag{5.3.16}$$

By Lemma 5.3.1, $\{x_n\}$ is bounded and by the reflexivity of the Banach space X_1, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to x^*. Using (5.3.14), we have $\{u_{n_j}\}$ of $\{u_n\}$ converges weakly to x^*. (5.3.16), also implies that $\{y_{n_j}\}$ of $\{y_n\}$ converges weakly to x^*. From (5.3.12), we have that $||y_{n_j}-Q_{\lambda_{n_j}}y_{n_j}||\to 0$, as $j\to\infty$. Since Q_{λ_n} is nonexpansive, then by Lemma 5.2.3 and Lemma 5.2.6(i) we have that $x^*\in(T_1+S_1)^{-1}(0)$.

Furthermore, since the operator A is linear and bounded, we know that $\{Ax_{n_j}\}$ converges weakly to Ax^*. It follows from (5.3.11) and the fact that P_{λ_n} is demiclosed at zero that $Ax^*\in(T_2+S_2)^{-1}(0)$. Hence, x^* belongs to Γ.
Now, suppose there exists another subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges to $y^*\in X_1$, we know by (5.3.15) and previous analysis that $y^*\in(T_2+S_2)^{-1}(0)$. Applying the Opial's condition on the space X_1, we conclude that $\{x_n\}$ converges weakly to x^*. $\square$

The following results are easily obtained as corollaries to our main result.

Corollary 5.3.3. *Let H_1 and H_2 be real Hilbert spaces with H_1 satisfying the Opial's condition, and $A : H_1 \to H_2$ a bounded linear operator A^* the adjoint of A. Let $T_1 : H_1 \to 2^{H_1}, T_2 : H_2 \to 2^{H_2}$ be set-valued monotone operators and $S_1 : H_1 \to H_1, S_2 : H_2 \to H_2$ be α-inverse strongly monotone. Assume $\Gamma := \{q \in (T_1+S_1)^{-1}(0) : Aq \in (T_2+S_2)^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in H_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma A^*(P_{\lambda_n} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)Q_{\lambda_n} y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{5.3.17}$$

where α_n is a sequence in $(0,1)$, γ is a positive constant and $\theta_n \subset [0,\theta)$, where $\theta \in [0,1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0} \theta_n ||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(3) $0 < \gamma < \frac{1}{||A||^2}$;

(4) $0 < \lambda \leq \lambda_n \leq b < \frac{2\alpha}{c}, \quad \forall\ n \geq 1, \quad c > 1.$

Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

Suppose $S_1 \equiv 0$ and $S_2 \equiv 0$ in (5.1.5), then the split accretive variational inclusion problem (5.1.5) reduces to split variational inclusion problem: Find $x^* \in X_1$ such that

$$\begin{cases} 0 \in T_1(x^*) \\ y^* = Ax^* \in X_2 : 0 \in T_2(y^*). \end{cases} \tag{5.3.18}$$

Therefore, we obtain the following corollary.

Corollary 5.3.4. *Let X_1 be a real uniformly convex Banach space and 2-uniformly smooth satisfying Opial's condition, X_2 a real Banach space with Féchet differentiable norm, $A : X_1 \to X_2$ a bounded linear operator and A^* the adjoint of A. Let $T_1 : X_1 \to 2^{X_1}$ and $T_2 : X_2 \to 2^{X_2}$ multi-valued maximal accretive operators. Assume $\Gamma := \{q \in T_1^{-1}(0) : Aq \in T_2^{-1}(0)\} \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in X_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma J_1^{-1} A^* J_2(J_{\lambda_n}^{T_2} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)J_{\lambda_n}^{T_1} y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{5.3.19}$$

where α_n is a sequence in $(0,1)$, γ is a positive constant and $\theta_n \subset [0,\theta)$ where $\theta \in [0,1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0}\theta_n||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(3) $0 < \gamma < \frac{1-2k^2}{||A||^2}$, *where k is the smoothness constant satisfying* $0 < k^2 < \frac{1}{2}$.

Then $\{x_n\}$ converges weakly to $x^* \in \Gamma$.

5.4 Application and Numerical Example.

5.4.1 Application

Recall that the concept of accretivity in Banach space coincides with the concept of monotonicity in Hilbert space. Thus, we apply our result to solve convex minimization problem which is an example of optimization problem. Suppose H is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $||.||$.

Let $M : H \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function and $N : H \to \mathbb{R}$ be convex and continuously differentiable function . Then the subdifferential of M denoted ∂M is maximal monotone and the gradient ∇N of N is monotone and continuous (see [141]). Moreover ,

$$M(x^*) + N(x^*) = \min_{x\in X}[M(x) + N(x)] \Longleftrightarrow 0 \in \partial(M(x^*) + \nabla N(x^*)). \tag{5.4.1}$$

We consider the following Split Convex Minimization Problem (SCMP): Find $x^* \in H_1$, such that

$$\begin{cases} M_1(x^*) + N_1(x^*) = \min\limits_{x\in H_1}[M_1(x) + N_1(x)], \\ y^* = Ax^* \in H_2 : M_2(y^*) + N_2(y^*) = \min\limits_{y\in H_2}[M_2(y) + N_2(y)], \end{cases} \tag{5.4.2}$$

where $A : H_1 \to H_2$ is a bounded linear operator M_1, M_2 are proper convex lower semi-continuous functions and N_1, N_2 are convex and differentiable functions. Suppose the solution set of (5.4.2), is denoted by Γ. By setting $S_1 = \partial N_1, S_2 = \partial N_1, T_1 = \nabla M_1$ and $T_2 = \nabla M_2$ in Corollary 5.3.3, we obtain the following result for solving SCMP (5.4.2):

Theorem 5.4.1. *Let H_1 and H_2 be real Hilbert spaces with H_1 and $A : H_1 \to H_2$ a bounded linear operator A^* the adjoint of A. Let $M_1 : H_1 \to (-\infty, +\infty], M_2 : H_2(-\infty, +\infty]$ be proper convex and continuously differentiable function and $N_1 : H_1 \to \mathbb{R}, N_2 : H_2 \to \mathbb{R}$ be convex and continuously differentiable function such that ∇N_i is $\frac{1}{\alpha}$-Lipschitz for $i = 1, 2$. Assume $\Gamma \neq \emptyset$, for let $\{\lambda_n\}$ be a sequence of non-negative real numbers, for $x_1 \in H_1$, let $\{x_n\}$ be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma A^*((I + \lambda_n\partial M_2)^{-1})(I - \lambda_n\nabla N_2) - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)(I + \lambda_n\partial M_1)^{-1}(I - \lambda_n\nabla N_1)y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{5.4.3}$$

where α_n is a sequence in $(0, 1)$, γ is a positive constant and $\theta_n \subset [0, \theta)$ where $\theta \in [0, 1)$ satisfying the following conditions:

(1) $\sum_{n\geq 0}\theta_n||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(3) $0 < \gamma < \frac{1}{||A||^2}$;

(4) $0 < \lambda \leq \lambda_n \leq b < \frac{2\alpha}{c}, \quad \forall\ n \geq 1, \quad c > 1$.

Then $\{x_n\}$ *converges weakly to* $x^* \in \Gamma$.

Let A, M_1 and M_2 be defined as above, we define the Convex Minimization Problem (CMP) as follows: Find $x^* \in H_1$ such that

$$\begin{cases} M_1(x^*) = \min\limits_{x\in H_1} M_1(x), \\ y^* = Ax^* \in H_2 : \min\limits_{y\in H_2} M_2(y). \end{cases} \tag{5.4.4}$$

Suppose that the solution set of the (CMP) (5.4.4) is denoted by Γ. By setting $T_1 = \partial M_1$ and $T_2 = \partial M_2$ in Theorem 5.3.2, with $S_1 = S_2 \equiv 0$, we obtain the following result:

Corollary 5.4.2. *Let* H_1 *and* H_2 *be real Hilbert spaces with* H_1 *and* $A : H_1 \to H_2$ *a bounded linear operator,* A^* *the adjoint of* A. *Let* $M_1 : H_1 \to (-\infty, +\infty], M_2 : H_2(-\infty, +\infty]$ *be proper convex and continuously differentiable function. Assume* $\Gamma \neq \emptyset$, *let* $\{\lambda_n\}$ *be a sequence of non-negative real numbers, for* $x_1 \in H_1$, *let* $\{x_n\}$ *be a sequence given by*

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma A^*(J_{\lambda_n}^{\partial M_2} - I)Au_n, \\ x_{n+1} = \alpha_n y_n + (1-\alpha_n)J_{\lambda_n}^{\partial M_1} y_n, \quad \forall \quad n \geq 1, \end{cases} \tag{5.4.5}$$

where α_n *is a sequence in* $(0,1)$, γ *is a positive constant and* $\theta_n \subset [0,\theta)$, *where* $\theta \in [0,1)$ *satisfying the following conditions:*

(1) $\sum_{n\geq 0}\theta_n||x_n - x_{n-1}||^2 < \infty$;

(2) $\liminf\limits_{n\to\infty} \alpha_n(1-\alpha_n) > 0$;

(3) $0 < \gamma < \frac{1}{||A||^2}$.

Then $\{x_n\}$ *converges weakly to* $x^* \in \Gamma$.

5.4.2 Numerical Example

Here we present a numerical example in $(\mathbb{R}^2, ||.||_2)$ to our result Theorem 5.3.2.
Let $X_1 = X_2 = \mathbb{R}^2$, we define $A(x) : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A(x) = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ then, } A^*(x) = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Let $T_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T_1(\bar{x}) = (-x_1 - x_2, x_1 + x_2)$ and $T_2(\bar{x}) = (x_1, x_2)$.
We obtain the resolvent mappings associated with T_1 and T_2 as follows:

$$\begin{aligned} J_{\lambda_n}^{T_1}(\bar{x}) &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -\lambda_n & -\lambda_n \\ \lambda_n & \lambda_n \end{pmatrix}\right]^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda_n & -\lambda \\ \lambda_n & 1+\lambda_n \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} 1+\lambda_n & \lambda_n \\ -\lambda_n & 1-\lambda_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \big((1+\lambda_n)x_1 + \lambda_n x_2, (1-\lambda_n)x_2 - \lambda_n x_1\big). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} J_{\lambda_n}^{T_2}(\bar{x}) &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda_n & 0 \\ 0 & \lambda_n \end{pmatrix}\right]^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \left(\frac{1}{1+\lambda_n}x_1, \frac{1}{1+\lambda_n}x_2\right). \end{aligned}$$

Let $S_1 : \mathbb{R}^2 \to \mathbb{R}^2$ respectively $S_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $S_1(\bar{x}) = (2x_1, -2x_2)$ and $S_2(\bar{x}) = (x_1, -x_2)$.

Let $\alpha_n = \frac{n}{2n+1}$, $r = \frac{1-4k^2}{||A||^2}$, $k = \frac{1}{2}$. Then, $\lambda_n = \frac{n+1}{10n+70}$. Hence, our Algorithm 5.3.1 becomes: for $x_0, x_1 \in \mathbb{R}^2$

$$\begin{cases} u_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = u_n + \gamma J_1^{-1} A^* J_2 \left[\begin{pmatrix} \frac{1-\lambda_n}{1+\lambda_n} & 0 \\ 0 & 1 \end{pmatrix} - I\right] A u_n \quad n \geq 0, \\ x_{n+1} = \left(\frac{n}{2n+1}\right) y_n + \left(\frac{n+1}{2n+1}\right) \begin{pmatrix} (1+\lambda_n)(1-2\lambda_n) & \lambda_n(1-2\lambda_n) \\ \lambda_n(2\lambda_n - 1) & (1-\lambda_n)(1-2\lambda_n) \end{pmatrix} y_n, \quad n \geq 1. \end{cases} \tag{5.4.6}$$

Case I: $\bar{x}_0 = (0.1,\ 0.01)^T$, $\bar{x}_1 = (1,\ 2)^T$ and $\theta_n = \frac{n}{4n^5+1}$.

Case II: $\bar{x}_0 = (1,\ 2)^T$, $\bar{x}_1 = (0.1,\ 0.01)^T$ and $\theta_n = \frac{n}{2n^2+1}$.

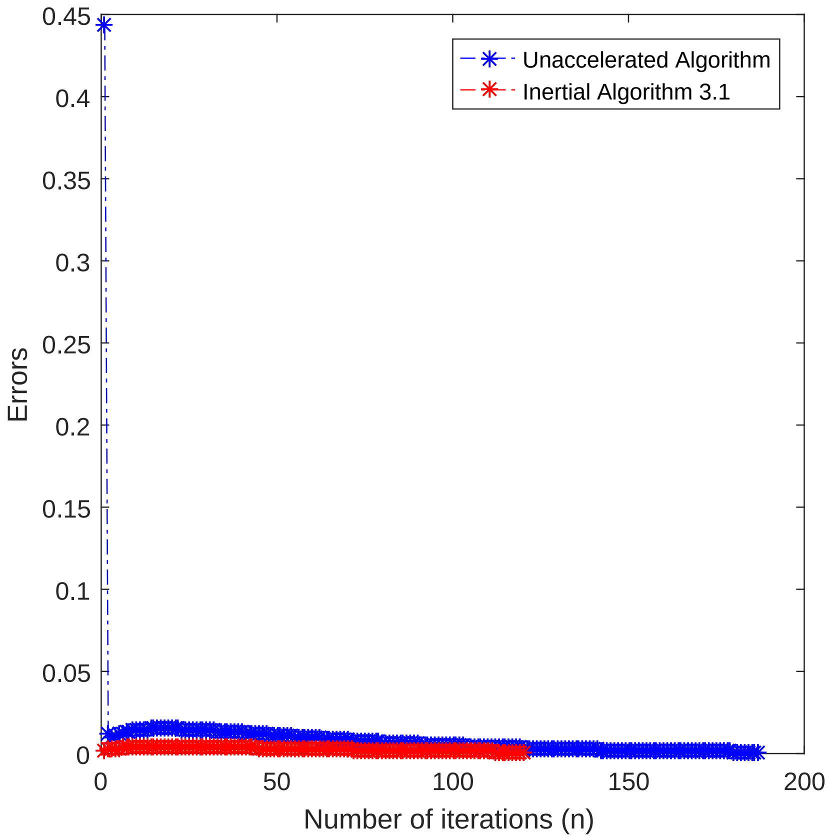

Figure 5.1: Errors vs number of iterations for **Case I**.

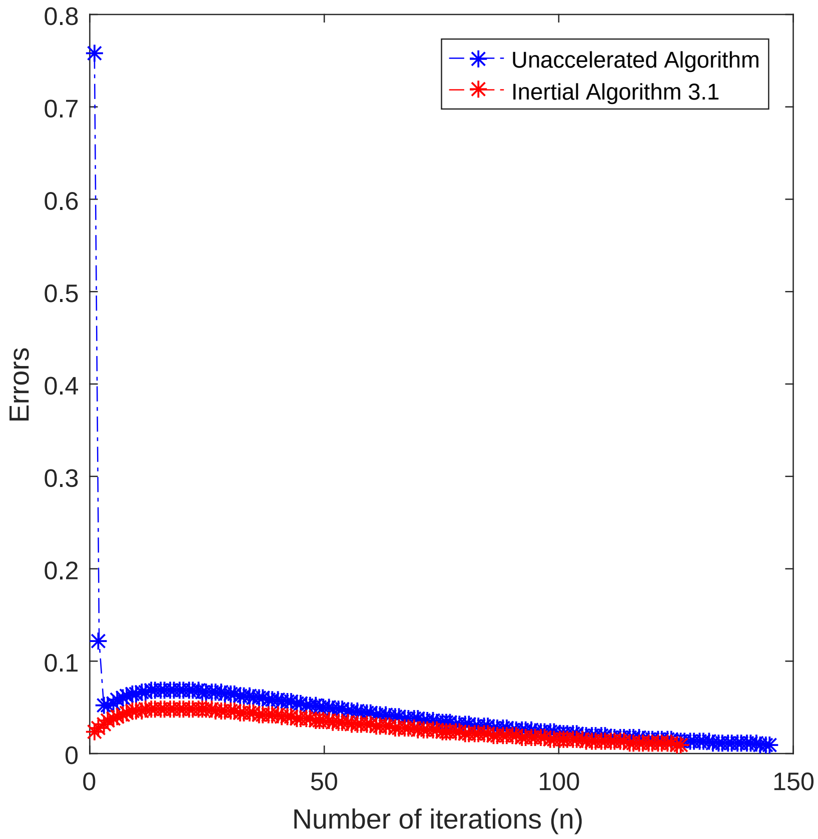

Figure 5.2: Errors vs number of iterations for **Case I**.

Table 5.1: Showing numerical results for **Case I**.

No. of iterations	Accelerated Algorithm 3.1	Unaccelerated Algorithm
1		
2	0.0021	0.4435
3	0.0025	0.0120
4	0.0029	0.0100
5	0.0032	0.0110
6	0.0034	0.0119
7	0.0036	0.0125
8	0.0038	0.0131
9	0.0039	0.0136
10	0.0041	0.0140
11	0.0041	0.0143
12	0.0042	0.0145
13	0.0043	0.0147
14	0.0043	0.0149
15	0.0044	0.0150
16	0.0044	0.0151
17	0.0044	0.0151
18	0.0044	0.0152
19	0.0044	0.0152
20	0.0044	0.0152

Table 5.2: Showing numerical results for **Case II**.

No. of iterations	Accelerated Algorithm 3.1	Unaccelerated Algorithm
1		
2	0.0236	0.7576
3	0.0283	0.1216
4	0.0323	0.0515
5	0.0356	0.0516
6	0.0383	0.0548
7	0.0405	0.0577
8	0.0423	0.0601
9	0.0438	0.0621
10	0.0451	0.0637
11	0.0461	0.0651
12	0.0469	0.0662
13	0.0476	0.0671
14	0.0481	0.0677
15	0.0485	0.0682
16	0.0487	0.0686
17	0.0489	0.0688
18	0.0490	0.0689
19	0.0490	0.0690
20	0.0490	0.0689
21	0.0489	0.0687
22	0.0487	0.0685
23	0.0485	0.0682
24	0.0483	0.0679
25	0.0480	0.0675
26	0.0477	0.0670
27	0.0473	0.0665
28	0.0470	0.0660
29	0.0466	0.0655
30	0.0462	0.0649
31	0.0458	0.0643

CHAPTER 6

Conclusion, Contribution to knowledge and Future Research.

6.1 Conclusion

In summary, we gave a little but explicit introduction to the content of this work in Chapter 1, we gave a definition of Optimization problem and also some introduction to the various problems under consideration in the study.
In Chapter 2, we reviewed some basic definitions and some important results as necessary and important in establishing our main results in the study. We also gave a review of some iterative scheme required in the study, we make remarks and give examples where necessary.
Our result in Theorem 3.3.2 improves other previous results in existence in the literature. For instance the iterative sequence in [133] by Qin and Su which was used to obtain the solution of an accretive operator. The result offers an improvement to the result of Aoyoma and Toyoda [4] which proves the strong convergence result for approximating the zero of an accretive operator. Theorem 3.3.2 also solves the fixed point of a k-strictly pseudo contractive operator.
In Chapter 4, we proposed a new mapping and an iterative algorithm for approximating the solution of a finite family of generalized mixed equilibrium problems with $\mu-\alpha$ relaxed monotone mapping in a Banach space. The result we obtained in Theorem 4.3.5 generalizes and extends previous results in the same direction in the literature. It extends the result of Wang et al [165] from a Hilbert space to a more general Banach space. It also extends the work of Chen et al [44] from a single mapping to a finite family of mappings.
We proposed an inertial type algorithm and prove a weak convergence due to this algorithm to a solution of a split variational inclusion problem in the framework of Banach spaces. Our result in Theorem 5.3.2 improves the result due to Tang et al [158] , Zhang and Wang [173] due to the introduction of an inertial extrapolation which improves the rate of convergence of our proposed algorithm.

6.2 Contribution to Knowledge

The following represents the authors original contribution to knowledge:

(1) We gave an example of a $\mu - \alpha$ relaxed monotone mapping in Chapter 2.

(2) Our Theorem 3.3.2 further shows the relationship between the theory of accretive operators and fixed point theory. We make use of a three step algorithm which improves other algorithms for approximating the zeros of accretive operator. The result also generalizes other results in literature.

(3) In Chapter 4, we introduce a mapping for a finite family of mixed equilibrium problem with $\mu - \alpha$ relaxed monotone mapping. Making use of the nonexpansivity property of the resolvent mappings of these problems. The mapping thus offers a generalization to other mappings of its type for instance (see [6, 94]). Also, the result extend the result of [165] and [44] and many other results in many ways.

(4) Our result in Chapter 5 is based on the inertial type iterative algorithm, and it is new as there are few result of this type in Banach spaces in literature. Though, we obtained a weak convergence theorem, the result improve and extend many other which exist in literature. For instance, the problem we considered in this Chapter involves accretive operators which generalizes the Hilbert space as studied by [113] who introduced the problem.

The following are research articles submitted for publication from this work.

(1) O.K. Oyewole, L.O. Jolaoso, M.O. Aibinu and O.T. Mewomo, Strong convergence theorem for approximating zero of accretive operators and application to Hammerstein equation.

(2) O.K. Oyewole, L.O. Jolaoso, C. Izuchukwu and O.T. Mewomo, On approximation of common solution of finite family of mixed equilibrium problems involving $\mu - \alpha$ relaxed monotone mapping in a Banach space.

(3) O.K. Oyewole, C. Izuchukwu, C.C. Okeke and O.T. Mewomo, Inertial approximation method for split variational inclusion problem in Banach spaces.

6.3 Future Research

In Chapter 3, we obtained our result in a q-uniformly smooth Banach space, we will like to extend the study to a more general space. We obtained a weak convergence result in Chapter 5 to a solution of split variational inclusion problem with an inertial type algorithm. We will like to obtain a strong convergence to the solution of this problem in the future.

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