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NEW CLASS OF LRS SPACETIMES WITH SIMULTANEOUS ROTATION AND SPATIAL TWIST

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New Class of LRS Spacetimes with Simultaneous Rotation and Spatial Twist

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"No problem can be solved from the same level of Consciousness that created it."

~ Albert Einstein★

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Parts of this thesis is based on the following publication:

- S. Singh, G. F. R. Ellis, R. Goswami, and S. D. Maharaj, New Class of LRS Spacetimes with Simultaneous Rotation and Spatial Twist, Phys. Rev. D., **94**, 104040 (2016).

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Sayuri Singh

Abstract

In this thesis we study Locally Rotationally Symmetric (LRS) spacetimes in which there exists a unique preferred spatial direction at each point. The conventional 1+3 decomposition of spacetime is extended to a 1+1+2 decomposition which is a natural setting in LRS models. We establish the existence and find the necessary and sufficient conditions for a new class of solutions of LRS spacetimes that have non-vanishing rotation and spatial twist simultaneously. In this study there are three key questions. By relaxing the condition of a perfect fluid, that is by introducing pressure anisotropy and heat flux, is it possible to have dynamical solutions with non-zero rotation and non-zero twist? If yes, can these solutions be physical? What are the local geometrical properties of such solutions? We investigate these questions in detail by using the semi-tetrad 1+1+2 covariant formalism. It is transparently shown that the existence of such solutions demand non-vanishing and bounded heat flux and these solutions are self-similar. We provide a brief algorithm indicating how to solve the system of field equations with the given Cauchy data on an initial spacelike Cauchy surface. We indicate that these solutions can be used as a first approximation from spherical symmetry to study rotating, inhomogeneous, dynamic and radiating astrophysical stars.

Contents

Chapter 1

Introduction

In the study of the dynamics and geometry of relativistic cosmological models, there are two main streams. The first focusses on the spacetime (Killing) symmetries of these models as in Kramer *et al* (1980), Krasiński (1993). The second focusses on their covariant properties which arise from a 1+3 "threading" decomposition of the spacetime manifold with respect to an invariantly defined normalised timelike congruence (Ehlers (1961) and Ellis (1971)). It is a natural tendency to determine the covariant properties of models when the main aim of analysis is imposed spacetime symmetries. However, the converse case has not been systematically addressed.

The spacetimes that are *Locally Rotationally Symmetric* (LRS) have been studied in detail and discussed many times in the literature in the cosmological context, i.e. with a fluid matter source (see for example Ellis (1967), Stewart and Ellis (1968), Elst and Ellis (1996), and the references therein). For spacetimes of this kind, there exist a continuous isotropy group at each point and hence, there is a multiply-transitive isometry group acting on the spacetime manifold. The isotropies around a point in a spacetime with a fluid can occur as a one-dimensional or three-dimensional subgroup of the full group of isometries, leaving the normalised 4-velocity of the matter flow invariant. A three-dimensional group of isotropies at each point implies that the spacetime is isotropic at every point and gives rise to the homogeneous and isotropic Friedmann-

Lemaître-Robertson-Walker (FLRW) models. A one-dimensional group of isotropies at each point corresponds to anisotropic, and in general, spatially inhomogeneous models (generically with one or two centres where the isotropy group is three-dimensional). However, it includes also some spatially homogeneous (Bianchi and Kantowski-Sachs) models (see Ellis and MacCallum (1969), King and Ellis (1973)).

When the spacetime is LRS, there exists a unique preferred spatial direction at each point. This preferred direction is covariantly defined by, for example, either a vorticity vector field, an eigendirection of a rate of shear tensor field, or a non-vanishing non-gravitational acceleration of the matter fluid elements. LRS spacetimes with a perfect fluid matter source have been analysed and classified by Stewart and Ellis (1968) using tetrad methods, and by Elst and Ellis (1996) using a 1+3 covariant approach. Using a semi-tetrad covariant formalism, it was shown that the Einstein field equations can be written as a set of first order equations of geometrical scalars as shown by Elst and Ellis (1996) and Clarkson (2007). By analysing the consistency conditions of the field equations, it was proven that a perfect fluid LRS spacetime cannot have simultaneous fluid rotation and spatial twist of the preferred spatial direction. Based on this, the perfect fluid LRS spacetimes can be divided into three distinct classes. *Class I* spacetimes are those where the rotation is non-zero but the twist vanishes. This class was shown to be non-expanding, non-distorting and stationary and the solutions generalise the well known Gödel solution. For *Class II* spacetimes, both the rotation and the twist vanish. These consist of the spherical, hyper-spherical, and plane symmetric (cylindrical) solutions. The *Class III* spacetimes have no rotation or acceleration but have non-zero twist of the preferred spatial direction. These spacetimes are spatially homogeneous.

Though all these classes are of interest, and LRS-II solutions have been used extensively to study spherically symmetric astrophysical objects, none of them are suitable for modelling a dynamical rotating star (gravitational collapse of a rotating star, for example). For LRS-I, the rotation is non-zero but the spacetime is stationary, yet the other two LRS classes allow dynamical solutions with vanishing rotation. In this study the three key questions are: *By relaxing the condition of a perfect fluid, that is by introducing pressure anisotropy and heat flux, is it possible to have dynamical solutions with non-zero rotation and non-zero twist? If yes, can these solutions be physical? What are the local geometrical properties of such solutions?*

In this dissertation we investigate in detail the above questions by using the semi-tetrad 1+1+2 covariant formalism (Clarkson and Barrett (2003), Betschart and Clarkson (2004), Clarkson (2007)). First, we establish the existence of such solutions and then find the constraints on the thermodynamic quantities of matter that generate such solutions. We then demonstrate that there exists physically realistic solutions where the matter satisfies physically reasonable energy conditions.

The dissertation is organised as follows: In the next two chapters we describe briefly the basic concepts of the local semi-tetrad 1+3 and 1+1+2 covariant formalisms. In the subsequent chapters we discuss the various properties of LRS spacetimes and the field equations written in terms of the 1+1+2 geometrical variables. In chapter 5, we then proceed to show the existence of dynamic solutions for imperfect fluids (with pressure anisotropy and heat flux) with non-zero rotation and spatial twist. We also transparently investigate the constraints that the thermodynamic quantities of the matter must satisfy for such solutions to exist. A brief algorithm is provided indicating how to solve the system of field equations with the given initial data. Finally, we briefly discuss how these solutions can be used as a first approximation to spherical symmetry in order to study rotating, inhomogeneous and dynamic astrophysical objects.

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) and $(-,+,+,+)$ signature throughout this study. The symbol ∇ represents the usual covariant derivative. The Riemann tensor is defined by

$$R^{a}{}_{bcd} = \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} + \Gamma^{e}{}_{bd}\Gamma^{a}{}_{ce} - \Gamma^{e}{}_{bc}\Gamma^{a}{}_{de} \ , \tag{1.0.1}$$

and the Ricci tensor is obtained by contracting the *first* and *third* indices

$$R_{ab} = g^{cd}R_{cadb} \ . \tag{1.0.2}$$

The Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - 2\Lambda - 2\mathcal{L}_m\right] \ , \tag{1.0.3}$$

where $\mathcal{L}_m$ is the matter contribution and Λ is the cosmological contant. Then variation of $\mathcal{S}$ gives the Einstein field equations as

$$G_{ab} + \Lambda g_{ab} = T_{ab} \ , \tag{1.0.4}$$

where

$$G_{ab} = R_{ab} - \frac{1}{2}Rg_{ab} \ , \tag{1.0.5}$$

is the Einstein tensor and

$$R = R^{a}{}_{a} \ , \tag{1.0.6}$$

is the Ricci scalar.

Chapter 2

1+3 decomposition of spacetime

2.1 Introduction

The 1+3 approach is helpful for understanding various physical and geometrical aspects of relativistic fluid flows, either in the gauge invariant, covariant perturbation formalism or in non-linear GR studies as shown by Ellis *et al* (2007). It provides a covariant description of the spacetime in terms of 3-vectors, scalars and the projected symmetric trace-free (PSTF) 3-tensors.

2.2 Kinematics

With respect to a timelike congruence, the spacetime can be locally decomposed into time and space parts. One natural way to define such a timelike congruence would be along the matter flow, with the *four-velocity* defined as

$$u^a = \frac{dx^a}{d\tau}, \quad \text{with} \quad u^a u_a = -1, \tag{2.2.1}$$

where τ is the proper time. Then the timelike unit vector u^a is split in the form $R \otimes V$, where V is the 3-space which is perpendicular to u^a and R represents the timeline along the unit vector u^a. This vector field u^a provides a timelike threading for the spacetime.

Given the *four-velocity* u^a, we have the unique *projection tensors*

$$U^a{}_b = -u^a u_b, \tag{2.2.2}$$

$$h^a{}_b = g^a{}_b + u^a u_b, \tag{2.2.3}$$

where $h^a{}_b$ is the projection tensor that projects any 4D vector or tensor onto the local 3-space orthogonal to u^a. It follows that

$$U^a{}_c U^c{}_b = -U^a{}_b, \qquad U^a{}_b u^b = u^a, \qquad U^a{}_a = 1,$$

$$h_{ab} u^b = 0, \qquad h^a{}_c h^c{}_b = h^a{}_b, \qquad h^a{}_a = 3.$$

With the choice of this timelike vector, we have two well defined directional derivatives. We have the vector u^a which is used to define the *covariant time derivative* (denoted by a dot) for any tensor $S^{a..b}{}_{c..d}$, given by

$$\dot{S}^{a..b}{}_{c..d} = u^e \nabla_e S^{a..b}{}_{c..d}, \tag{2.2.4}$$

and we have the tensor h_{ab} which is used to define the fully orthogonally *projected covariant derivative* D for any tensor $S^{a..b}{}_{c..d}$:

$$D_e S^{a..b}{}_{c..d} = h^a{}_f h^p{}_c ... h^b{}_g h^q{}_d h^r{}_e \nabla_r S^{f..g}{}_{p..q}\ , \tag{2.2.5}$$

with total projection on all the free indices. The splitting of the spacetime gives a 3-volume element

$$\epsilon_{abc} = \eta_{abcd} u^d, \ \text{ where } \ \epsilon_{abc} = \epsilon_{[abc]} \ \text{ and } \ \epsilon_{abc} u^c = 0. \tag{2.2.6}$$

Since η_{abcd} is the four-dimensional volume element, i.e., $\eta_{abcd} = \sqrt{|\det g|}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}$, we have

$$\eta_{abcd} = 2u_{[a}\epsilon_{bcd]}. \tag{2.2.7}$$

Since η_{abcd} is skew-symmetric, the following contractions hold

$$\epsilon_{abc}\epsilon^{def} = 3!h^d{}_{[a}h^e{}_b h^f{}_{c]}, \tag{2.2.8}$$

$$\epsilon_{abc}\epsilon^{dec} = 2h^d{}_{[a}h^e{}_{b]}, \tag{2.2.9}$$

$$\epsilon_{abc}\epsilon^{dbc} = 2h^d{}_a, \tag{2.2.10}$$

$$\epsilon_{abc}\epsilon^{abc} = 3. \tag{2.2.11}$$

The covariant derivative of u^a can be decomposed as

$$\nabla_a u_b = -u_a A_b + D_a u_b, \tag{2.2.12}$$

where D_a totally projects derivatives onto the 3-space. $D_a u_b$ can be decomposed into the trace part, the trace-free symmetric part and the trace-free anti-symmetric part, i.e.,

$$\nabla_a u_b = -u_a A_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \epsilon_{abc}\omega^c, \tag{2.2.13}$$

where $A_b = \dot{u}_b$ is the acceleration, $\Theta = D_a u^a$ represents the expansion of u_a, $\sigma_{ab} = \left(h^c{}_{(a}h^d{}_{b)} - \frac{1}{3}h_{ab}h^{cd}\right) D_c u_d$ is the shear tensor that denotes the distortion and ω^c is the vorticity vector denoting the rotation. The *Weyl curvature tensor* C_{abcd}, which gives the locally free gravitational field, is defined by the equation

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}R^{b]}{}_{d]} + \tfrac{1}{3}Rg^{[a}{}_{[c}g^{b]}{}_{d]}. \tag{2.2.14}$$

Since the Weyl tensor is trace-free on all its indices ($C^c{}_{acb} = 0$), the Ricci tensor R_{ab} is the trace of R_{abcd}, and C_{abcd} is the trace-free part. The Weyl tensor can be split relative to u^a into the *electric* and *magnetic Weyl curvature* parts as

$$E_{ab} = C_{abcd}u^b u^d \tag{2.2.15}$$

$$\Rightarrow \quad E^a{}_a = 0, \quad E_{ab} = E_{\langle ab\rangle}, \quad E_{ab}u^b = 0, \tag{2.2.16}$$

and

$$H_{ab} = \tfrac{1}{2}\epsilon_{ade}C^{de}{}_{bc}u^c \tag{2.2.17}$$

$$\Rightarrow \quad H^a{}_a = 0, \quad H_{ab} = H_{\langle ab\rangle}, \quad H_{ab}u^b = 0. \tag{2.2.18}$$

The energy momentum tensor of matter can be decomposed similarly as

$$T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab}\ , \tag{2.2.19}$$

where $p = (1/3)h^{ab}T_{ab}$ is the isotropic pressure, $\mu = T_{ab}u^a u^b$ is the energy density, $q_a = q_{\langle a\rangle} = -h^c{}_a T_{cd}u^d$ is the 3-vector that defines the heat flux, and $\pi_{ab} = \pi_{\langle ab\rangle}$ is the anisotropic stress.

We can now write all the Ricci identities and doubly contracted Bianchi identities for these geometrical variables (see Elst and Ellis (1973)).

2.3 Ricci identities

2.3.1 Time derivative equations

$$\begin{aligned}
\dot{\Theta} - D_a\dot{u}^a &= -\tfrac{1}{3}\Theta^2 + (\dot{u}_a\dot{u}^a) - \left(\Sigma_{ab}\Sigma^{ab}\right) + 2\left(\Omega_a\Omega^a\right) \\
&\quad -\tfrac{1}{2}\left(\mu + 3p\right) + \Lambda, && (2.3.1)\\
\dot{\Sigma}^{\langle ab\rangle} - D^{\langle a}\dot{u}^{b\rangle} &= -\tfrac{2}{3}\Theta\Sigma^{ab} + \dot{u}^{\langle a}\dot{u}^{b\rangle} - \Sigma^{\langle a}{}_c\Sigma^{b\rangle c} - \Omega^{\langle a}\Omega^{b\rangle} \\
&\quad - \left(\mathcal{E}^{ab} - \tfrac{1}{2}\pi^{ab}\right), && (2.3.2)\\
\dot{\Omega}^{\langle a\rangle} - \tfrac{1}{2}\epsilon^{abc}D_b\dot{u}_c &= -\tfrac{2}{3}\Theta\Omega^a + \Sigma^a{}_b\Omega^b. && (2.3.3)
\end{aligned}$$

2.3.2 Constraint equations

$$
\begin{aligned}
0 \;=\; (C_1)^a &= D_b\Sigma^{ab} - \tfrac{2}{3}D^a\Theta + \epsilon^{abc}\left[D_b\Omega_c + 2\dot{u}_b\Omega_c\right] + q^a, && (2.3.4)\\
0 \;=\; (C_2) &= D_a\Omega^a - \left(\dot{u}_a\Omega^a\right), && (2.3.5)\\
0 \;=\; (C_3)^{ab} &= \mathcal{H}^{ab} + 2\dot{u}^{\langle a}\Omega^{b\rangle} + D^{\langle a}\Omega^{b\rangle} - \epsilon^{cd\langle a}D_c\Sigma^{b\rangle}{}_d\,. && (2.3.6)
\end{aligned}
$$

2.4 (Doubly) contracted Bianchi identities

2.4.1 Time derivative equations

$$
\begin{aligned}
\left(\dot{\mathcal{E}}^{\langle ab\rangle} + \tfrac{1}{2}\dot{\pi}^{\langle ab\rangle}\right) - \epsilon^{cd\langle a}D_c\mathcal{H}^{b\rangle}{}_d + \tfrac{1}{2}D^{\langle a}q^{b\rangle} &= -\tfrac{1}{2}(\mu+p)\Sigma^{ab} - \Theta\left(\mathcal{E}^{ab} + \tfrac{1}{6}\pi^{ab}\right)\\
&\quad +3\Sigma^{\langle a}{}_c\left(\mathcal{E}^{b\rangle c} - \tfrac{1}{6}\pi^{b\rangle c}\right) - \dot{u}^{\langle a}q^{b\rangle}\\
&\quad \epsilon^{cd\langle a}\Big[2\dot{u}_c\mathcal{H}^{b\rangle}{}_d + \Omega_c\Big(\mathcal{E}^{b\rangle}{}_d\\
&\quad + \tfrac{1}{2}\pi^{b\rangle}{}_d\Big)\Big], && (2.4.1)\\
\dot{\mathcal{H}}^{\langle ab\rangle} + \epsilon^{cd\langle a}D_c\left(\mathcal{E}^{b\rangle}{}_d - \tfrac{1}{2}\pi^{b\rangle}{}_d\right) &= -\Theta\mathcal{H}^{ab} + 3\Sigma^{\langle a}{}_c\mathcal{H}^{b\rangle c} + \tfrac{3}{2}\Omega^{\langle a}q^{b\rangle}\\
&\quad -\epsilon^{cd\langle a}\Big[2\dot{u}_c\mathcal{E}^{b\rangle}{}_d - \tfrac{1}{2}\Sigma^{b\rangle}{}_c q_d\\
&\quad -\Omega_c\mathcal{H}^{b\rangle}{}_d\Big], && (2.4.2)\\
\dot{q}^{\langle a\rangle} + D^a p + D_b\pi^{ab} &= \tfrac{4}{3}\Theta q^a - \Sigma^a{}_b q^b - (\mu+p)\dot{u}^a\\
&\quad -\dot{u}_b\pi^{ab} - \epsilon^{abc}\Omega_b q_c, && (2.4.3)\\
\dot{\mu} + D_a q^a &= -\Theta(\mu+p) - 2(\dot{u}_a q^a)\\
&\quad -(\Sigma_{ab}\pi^{ab}). && (2.4.4)
\end{aligned}
$$

2.4.2 Constraint equations

$$\begin{aligned} 0 \;=\; (C_4)^a = \;& D_b\left(\mathcal{E}^{ab}+\tfrac{1}{2}\pi^{ab}\right)-\tfrac{1}{3}D^a\mu+\tfrac{1}{3}\Theta q^a-\tfrac{1}{2}\Sigma^a{}_b-3\Omega_b\mathcal{H}^{ab} \\ & -\epsilon^{abc}\left[\Sigma_{bd}\mathcal{H}_c{}^d-\tfrac{3}{2}\Omega_b q_c\right], \end{aligned} \tag{2.4.5}$$

$$\begin{aligned} 0 \;=\; (C_5)^a = \;& D_b\mathcal{H}^{ab}+(\mu+p)\Omega^a+\epsilon^{abc}\left[\tfrac{1}{2}D_b q_c+\Sigma_{bd}\left(\mathcal{E}_c{}^d+\tfrac{1}{2}\pi_c{}^d\right)\right] \\ & +3\Omega_b\left(\mathcal{E}^{ab}-\tfrac{1}{6}\pi^{ab}\right). \end{aligned} \tag{2.4.6}$$

Chapter 3

1+1+2 decomposition of spacetime

3.1 Introduction

The recently developed 1+1+2 decomposition by Clarkson and Barrett (2003) is a natural extension of the 1+3 decomposition. This approach is optimised for spacetimes having a preferred spatial direction. In this approach, the 3-space is further decomposed with respect to this given spatial direction, i.e., we now have another split along a preferred spatial direction.

3.2 Kinematics

In the 1+3 approach, the timelike unit vector u^a is split in the form $R \otimes V$. In this split, V is the 3-space which is perpendicular to u^a and R represents the timeline along the unit vector u^a. In the 1+1+2 approach, the 3-space V is split furthur by introducing a unit vector e^a that is orthogonal to u^a. We choose this spacelike vector field e^a such that

$$u^a e_a = 0 \quad \text{and} \quad e^a e_a = 1. \tag{3.2.1}$$

By combining e^a with the 1+3 projection tensor $h_a{}^b \equiv g_a{}^b + u_a u^b$, we get the new projection tensor $N_a{}^b$ which is given by

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b \ . \tag{3.2.2}$$

This tensor projects vectors orthogonal to e^a and u^a onto local 2-spaces, defined as *sheets* (note that these are not subspaces of the 3-space if the twist of e^a is non-zero). Thus

$$e^a N_{ab} = 0 = u^a N_{ab}, \quad N^a{}_a = 2. \tag{3.2.3}$$

The volume element of this sheet is then the Levi-Civita 2-tensor

$$\varepsilon_{ab} \equiv \epsilon_{abc} e^c = u^d \eta_{dabc} e^c \ . \tag{3.2.4}$$

Using the definitions of ε_{ab} and N_{ab}, we have the following conditions

$$\begin{aligned}
\varepsilon_{ab} e^b &= 0 = \varepsilon_{(ab)}, && (3.2.5)\\
\varepsilon_{abc} &= e_a \varepsilon_{bc} - e_b \varepsilon_{ac} + e_c \varepsilon_{ab}, && (3.2.6)\\
\varepsilon_{ab} \varepsilon^{cd} &= N_a{}^c N_b{}^d - N_a{}^d N_b{}^c, && (3.2.7)\\
\varepsilon_a{}^c \varepsilon_{bc} &= N_{ab}, && (3.2.8)\\
\varepsilon^{ab} \varepsilon_{ab} &= 2. && (3.2.9)
\end{aligned}$$

Any 3-vector ψ^a can now be irreducibly split into a scalar, Ψ, which is the vector component parallel to e^a, and a vector, Ψ^a that lies in the sheet by

$$\begin{aligned}
\psi^a &= \Psi e^a + \Psi^a, \quad \text{where} \quad \Psi \equiv \psi_a e^a \ , \\
&\quad \text{and} \quad \Psi^a \equiv N^{ab} \psi_b \equiv \psi^{\bar{a}}, && (3.2.10)
\end{aligned}$$

where the bar over the index denotes projection with N_{ab}. Similarly, the same decomposition can be done for any 3-tensor, ψ_{ab} as follows,

$$\psi_{ab} = \psi_{\langle ab \rangle} = \Psi \left(e_a e_b - \tfrac{1}{2} N_{ab} \right) + 2\Psi_{(a} e_{b)} + \Psi_{ab} \ , \tag{3.2.11}$$

where

$$\begin{aligned} \Psi &\equiv e^a e^b \psi_{ab} = -N^{ab} \psi_{ab} \ , \\ \Psi_a &\equiv N_a{}^b e^c \psi_{bc} = \Psi_{\bar{a}} \ , \\ \Psi_{ab} &\equiv \left(N_{(a}{}^c N_{b)}{}^d - \tfrac{1}{2} N_{ab} N^{cd} \right) \psi_{cd} \equiv \Psi_{\{ab\}} \ . \end{aligned} \tag{3.2.12}$$

In the above, the curly brackets denote the PSTF tensors on the 2-sheet, i.e. the part of the tensor which is PSTF with respect to e^a. Apart from the '*time*' (dot) derivative of an object (such as a scalar, vector or tensor), we introduce two new derivatives, which for any tensor $\psi_{a...b}{}^{c...d}$:

$$\hat{\psi}_{a..b}{}^{c..d} \equiv e^f D_f \psi_{a..b}{}^{c..d} \ , \tag{3.2.13}$$

$$\delta_f \psi_{a..b}{}^{c..d} \equiv N_a{}^f ... N_b{}^g N_h{}^c .. N_i{}^d N_f{}^j D_j \psi_{f..g}{}^{i..j} \ . \tag{3.2.14}$$

The derivative along the e^a vector-field in the surfaces orthogonal to u^a is called the hat-derivative, while the derivative projected onto the sheet is called the δ -derivative. This projection is on every free index. We also have

$$h_{ab} = 0 = N_{ab}, \qquad N_{\langle ab \rangle} = -e_{\langle a} e_{b \rangle} = N_{ab} - \tfrac{2}{3} h_{ab}. \tag{3.2.15}$$

By taking e^a to be arbitrary and using (3.2.10) and (3.2.11), the usual 1+3 kinematical and Weyl quantities can now be split into the irreducible set

$$\mathcal{D}_1 = \{\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \mathcal{A}^a, \Sigma^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}.$$

The 4-acceleration, vorticity and shear split as

$$\begin{aligned}
\dot{u}^a &= \mathcal{A}e^a + \mathcal{A}^a, && (3.2.16)\\
\omega^a &= \Omega e^a + \Omega^a, && (3.2.17)\\
\sigma_{ab} &= \Sigma\left(e_a e_b - \tfrac{1}{2}N_{ab}\right) + 2\Sigma_{(a}e_{b)} + \Sigma_{ab}. && (3.2.18)
\end{aligned}$$

For the electric and magnetic Weyl tensors we get

$$\begin{aligned}
E_{ab} &= \mathcal{E}\left(e_a e_b - \tfrac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}e_{b)} + \mathcal{E}_{ab}, && (3.2.19)\\
H_{ab} &= \mathcal{H}\left(e_a e_b - \tfrac{1}{2}N_{ab}\right) + 2\mathcal{H}_{(a}e_{b)} + \mathcal{H}_{ab}. && (3.2.20)
\end{aligned}$$

Similarly the fluid variables, q^a and π_{ab}, may be split as follows

$$\begin{aligned}
q^a &= Qe^a + Q^a, && (3.2.21)\\
\pi_{ab} &= \Pi\left(e_a e_b - \tfrac{1}{2}N_{ab}\right) + 2\Pi_{(a}e_{b)} + \Pi_{ab}. && (3.2.22)
\end{aligned}$$

By decomposing the covariant derivative of e^a in the direction orthogonal to u^a into it's irreducible parts, i.e., the spatial derivative of e^a, we get

$$D_a e_b = e_a a_b + \frac{1}{2}\phi N_{ab} + \xi\epsilon_{ab} + \zeta_{ab} \ , \qquad (3.2.23)$$

where

$$\begin{aligned}
a_a &\equiv e^c D_c e_a = \hat{e}_a \ , && (3.2.24)\\
\phi &\equiv \delta_a e^a \ , && (3.2.25)\\
\xi &\equiv \frac{1}{2}\epsilon^{ab}\delta_a e_b \ , && (3.2.26)\\
\zeta_{ab} &\equiv \delta_{\{a}e_{b\}} \ . && (3.2.27)
\end{aligned}$$

Here, ϕ represents the *expansion of the sheet*, ζ_{ab} is the *shear*, i.e., the distortion of the sheet, a^a its *acceleration* and ξ is its (spatial) *vorticity*, i.e. the "twisting" or rotation of the sheet.

Chapter 4

LRS spacetimes

4.1 Introduction

For a spacetime manifold $(\mathcal{M}, g)$, if every point $p \in (\mathcal{M}, g)$ has a continuous non-trivial isotropy group then the manifold is called *locally isotropic*. When this isotropy group consists of spatial rotations, the spacetime is called *locally rotationally symmetric* or LRS (Elst and Ellis (1996)). In LRS spacetimes, there exists a unique, preferred spatial direction at each point and this preferred direction is covariantly defined. This direction creates a local axis of symmetry, i.e., all observations are identical under rotations about it. In particular, they are the same in all spatial directions that are perpendicular to that direction. Hence the 1+1+2 decomposition described in the previous section is ideally suited for the study of LRS spacetimes.

We can immediately see that if we choose the spacelike unit vector e^a along the preferred spatial direction of the spacetime, then by symmetry all the sheet vectors and tensors vanish identically. Thus, all the non-zero 1+1+2 variables are covariantly defined scalars. The geometrical scalar variables that fully describe LRS spacetimes are

$$\mathcal{D}_2 = \{\mathcal{A}, \Theta, \phi, \xi, \Sigma, \Omega, \mathcal{E}, \mathcal{H}, \mu, p, \Pi, Q\}\,.$$

Decomposing the Ricci identities for u^a and e^a and the doubly contracted Bianchi identities, we now get the following field equations for LRS spacetimes:

Evolution:

$$\dot{\phi} = \left(\tfrac{2}{3}\Theta - \Sigma\right)\left(\mathcal{A} - \tfrac{1}{2}\phi\right) + 2\xi\Omega + Q \ , \tag{4.1.1}$$

$$\begin{aligned}\dot{\xi} &= \left(\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta\right)\xi + \left(\mathcal{A} - \tfrac{1}{2}\phi\right)\Omega \\ &\quad + \tfrac{1}{2}\mathcal{H},\end{aligned} \tag{4.1.2}$$

$$\dot{\Omega} = \mathcal{A}\xi + \Omega\left(\Sigma - \tfrac{2}{3}\Theta\right), \tag{4.1.3}$$

$$\begin{aligned}\dot{\mathcal{H}} &= -3\xi\mathcal{E} + \left(\tfrac{3}{2}\Sigma - \Theta\right)\mathcal{H} + \Omega Q \\ &\quad + \tfrac{3}{2}\xi\Pi.\end{aligned} \tag{4.1.4}$$

Propagation:

$$\begin{aligned}\hat{\phi} &= -\tfrac{1}{2}\phi^2 + \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(\tfrac{2}{3}\Theta - \Sigma\right) \\ &\quad + 2\xi^2 - \tfrac{2}{3}\left(\mu + \Lambda\right) - \mathcal{E} - \tfrac{1}{2}\Pi,\end{aligned} \tag{4.1.5}$$

$$\hat{\xi} = -\phi\xi + \left(\tfrac{1}{3}\Theta + \Sigma\right)\Omega, \tag{4.1.6}$$

$$\hat{\Sigma} - \tfrac{2}{3}\hat{\Theta} = -\tfrac{3}{2}\phi\Sigma - 2\xi\Omega - Q \ , \tag{4.1.7}$$

$$\hat{\Omega} = \left(\mathcal{A} - \phi\right)\Omega, \tag{4.1.8}$$

$$\begin{aligned}\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} + \tfrac{1}{2}\hat{\Pi} &= -\tfrac{3}{2}\phi\left(\mathcal{E} + \tfrac{1}{2}\Pi\right) + 3\Omega\mathcal{H} \\ &\quad + \left(\tfrac{1}{2}\Sigma - \tfrac{1}{3}\Theta\right)Q,\end{aligned} \tag{4.1.9}$$

$$\begin{aligned}\hat{\mathcal{H}} &= -\left(3\mathcal{E} + \mu + p - \tfrac{1}{2}\Pi\right)\Omega \\ &\quad - \tfrac{3}{2}\phi\mathcal{H} - Q\xi.\end{aligned} \tag{4.1.10}$$

Propagation/evolution:

$$
\begin{aligned}
\hat{\mathcal{A}} - \dot{\Theta} &= -\left(\mathcal{A} + \phi\right)\mathcal{A} + \tfrac{1}{3}\Theta^2 + \tfrac{3}{2}\Sigma^2 \\
&\quad -2\Omega^2 + \tfrac{1}{2}\left(\mu + 3p - 2\Lambda\right)\ , && (4.1.11)\\
\dot{\mu} + \hat{Q} &= -\Theta\left(\mu + p\right) - \left(\phi + 2\mathcal{A}\right)Q \\
&\quad -\tfrac{3}{2}\Sigma\Pi, && (4.1.12)\\
\dot{Q} + \hat{p} + \hat{\Pi} &= -\left(\tfrac{3}{2}\phi + \mathcal{A}\right)\Pi - \left(\tfrac{4}{3}\Theta + \Sigma\right)Q \\
&\quad -\left(\mu + p\right)\mathcal{A}\ , && (4.1.13)\\
\dot{\Sigma} - \tfrac{2}{3}\hat{\mathcal{A}} &= \tfrac{1}{3}\left(2\mathcal{A} - \phi\right)\mathcal{A} - \left(\tfrac{2}{3}\Theta + \tfrac{1}{2}\Sigma\right)\Sigma \\
&\quad -\tfrac{2}{3}\Omega^2 - \mathcal{E} + \tfrac{1}{2}\Pi\,, && (4.1.14)\\
\dot{\mathcal{E}} + \tfrac{1}{2}\dot{\Pi} + \tfrac{1}{3}\hat{Q} &= +\left(\tfrac{3}{2}\Sigma - \Theta\right)\mathcal{E} - \tfrac{1}{2}\left(\mu + p\right)\Sigma \\
&\quad -\tfrac{1}{2}\left(\tfrac{1}{3}\Theta + \tfrac{1}{2}\Sigma\right)\Pi + 3\xi\mathcal{H} \\
&\quad +\tfrac{1}{3}\left(\tfrac{1}{2}\phi - 2\mathcal{A}\right)Q. && (4.1.15)
\end{aligned}
$$

Constraint:

$$\mathcal{H} = 3\xi\Sigma - \left(2\mathcal{A} - \phi\right)\Omega. \tag{4.1.16}$$

Also we give the commutation relation for the *dot* and *hat* derivatives, for LRS spacetimes:

$$\hat{\dot{\Psi}} - \dot{\hat{\Psi}} = -\mathcal{A}\dot{\Psi} + \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\Psi}, \tag{4.1.17}$$

which holds true for any scalar Ψ.

4.2 Perfect fluids

Considering a perfect fluid with $Q = \Pi = 0$, we can write the full covariant derivatives of the vectors u^a and e^a in terms of the LRS scalars in the following way:

$$\nabla_a u_b = -\mathcal{A} u_a e_b + \left(\tfrac{1}{3}\Theta + \Sigma\right) e_a e_b + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right) N_{ab} + \Omega \varepsilon_{ab}, \tag{4.2.1}$$

$$\nabla_a e_b = -\mathcal{A} u_a u_b + \left(\tfrac{1}{3}\Theta + \Sigma\right) e_a u_b + \tfrac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab}. \tag{4.2.2}$$

Here, the alternating Levi-Cevita 2-tensor ε_{ab} is defined in the following way from η^{dabc} as $\varepsilon_{ab} \equiv \eta^{dabc} u_d$. By antisymmetry

$$\begin{aligned} \varepsilon^{abc} u_a &= 0, \quad \varepsilon^{abc}\varepsilon_{abc} = 6, \\ \Rightarrow \quad \varepsilon^{ab} u_a &= \varepsilon^{ab} e_a = 0, \quad \varepsilon^{ab}\varepsilon_{ab} = 2. \end{aligned} \tag{4.2.3}$$

From equations (4.2.1), (4.2.2) and (4.2.3), we get an important result:

$$\Omega = \tfrac{1}{2}\varepsilon^{ab}\nabla_a u_b, \quad \xi = \tfrac{1}{2}\varepsilon^{ab}\nabla_a e_b. \tag{4.2.4}$$

Now for any scalar f we have

$$\nabla_b f = -\dot{f} u_b + \hat{f} e_b. \tag{4.2.5}$$

Differentiating again, we get

$$\nabla_a \nabla_b f = -\left(\nabla_a \dot{f}\right) u_b - \dot{f}\left(\nabla_a u_b\right) + \left(\nabla_a \hat{f}\right) e_b + \hat{f}\left(\nabla_a e_b\right). \tag{4.2.6}$$

By contracting with ε^{ab}, the LHS becomes zero. Using (4.2.3) and (4.2.4), the RHS becomes

$$\dot{f}\Omega = \hat{f}\xi. \tag{4.2.7}$$

Now, letting $f = \Omega$ we have

$$\dot{\Omega}\Omega = \hat{\Omega}\xi,$$

and using (4.1.3) and (4.1.8) we get

$$\Omega\left[\tfrac{2}{3}\Theta\Omega - \Sigma\Omega - \phi\xi\right] = 0. \tag{4.2.8}$$

Now letting $f = \xi$ and using (4.1.2), (4.1.6) and (4.1.16) we arrive at

$$\xi\left[\tfrac{2}{3}\Theta\Omega - \Sigma\Omega - \phi\xi\right] = 0. \tag{4.2.9}$$

For the LRS equation to be consistent, we can see from (4.2.8) and (4.2.9) that we can have either

$$\Omega = \xi = 0, \tag{4.2.10}$$

or

$$\tfrac{2}{3}\Theta\Omega - \Sigma\Omega - \phi\xi = 0. \tag{4.2.11}$$

We note here that (4.2.10) $\Rightarrow$ (4.2.11) but the converse is not true. Now letting $f = p$ we get $\dot{p}\Omega = \hat{p}\xi$. From (4.1.13) we have

$$\hat{p} = -\left(\mu + p\right)\mathcal{A}, \tag{4.2.12}$$

and if $p = p\left(\mu\right)$, then $\dot{p} = \frac{\partial p}{\partial\mu}\dot{\mu}$. We also have, from (4.1.12),

$$\dot{\mu} = -\Theta\left(\mu + p\right). \tag{4.2.13}$$

Putting equations (4.2.12) and (4.2.13) together results in

$$\tfrac{\partial p}{\partial\mu}\Theta\Omega = \mathcal{A}\xi. \tag{4.2.14}$$

Now from (4.1.9), the constraint equation for the Weyl scalar $\mathcal{E}$ is given by

$$\hat{\mathcal{E}} - \tfrac{1}{3}\hat{\mu} = -\tfrac{3}{2}\phi\mathcal{E} + 3\Omega\mathcal{H}. \tag{4.2.15}$$

Since this is true for all epochs, we can take the time derivative

$$\dot{\hat{\mathcal{E}}} - \tfrac{1}{3}\dot{\hat{\mu}} = -\tfrac{3}{2}\mathcal{E}\dot{\phi} - \tfrac{3}{2}\phi\dot{\mathcal{E}} + 3\dot{\Omega}\mathcal{H} + 3\Omega\dot{\mathcal{H}}. \tag{4.2.16}$$

Now using our commutation relation $\hat{\dot{\Psi}} - \dot{\hat{\Psi}}$ given by (4.1.17) we get

$$\begin{aligned} \dot{\hat{\mathcal{E}}} &= \hat{\dot{\mathcal{E}}} + \mathcal{A}\dot{\mathcal{E}} - \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\mathcal{E}}, \\ \dot{\hat{\mu}} &= \hat{\dot{\mu}} + \mathcal{A}\dot{\mu} - \left(\tfrac{1}{3}\Theta + \Sigma\right)\hat{\mu}. \end{aligned}$$

Hence

$$\dot{\hat{\mathcal{E}}} - \tfrac{1}{3}\dot{\hat{\mu}} = \hat{\dot{\mathcal{E}}} - \tfrac{1}{3}\hat{\dot{\mu}} + \mathcal{A}\left(\dot{\mathcal{E}} - \tfrac{1}{3}\dot{\mu}\right) - \left(\tfrac{1}{3}\Theta + \Sigma\right)\left(-\tfrac{3}{2}\phi\mathcal{E} + 3\Omega\mathcal{H}\right). \tag{4.2.17}$$

Now from (4.1.15) and (4.2.13) we have

$$\begin{aligned} \dot{\mathcal{E}} &= \left(\tfrac{3}{2}\Sigma - \Theta\right)\mathcal{E} + 3\xi\mathcal{H} - \tfrac{1}{2}(\mu + p)\Sigma, & (4.2.18) \\ \dot{\mu} &= -\Theta(\mu + p). & (4.2.19) \end{aligned}$$

Therefore we get

$$\dot{\mathcal{E}} - \tfrac{1}{3}\dot{\mu} = \left(\Sigma - \tfrac{2}{3}\Theta\right)\left[\tfrac{3}{2}\left(\mathcal{E} - \tfrac{1}{3}\mu\right) - \frac{p}{2}\right] + 3\xi\mathcal{H}. \tag{4.2.20}$$

From (4.1.7) and (4.1.10) we have

$$\begin{aligned} \hat{\Sigma} - \tfrac{2}{3}\hat{\Theta} &= -\tfrac{3}{2}\phi\Sigma - 2\xi\Omega, & (4.2.21) \\ \hat{\mathcal{H}} &= -\tfrac{3}{2}\phi\mathcal{H} - (3\mathcal{E} + \mu + p)\Omega. & (4.2.22) \end{aligned}$$

Now using (4.2.12), (4.2.15), (4.2.21) and (4.2.22) we get

$$\begin{aligned}\hat{\dot{\mathcal{E}}}-\tfrac{1}{3}\hat{\dot{\mu}} &= \left(-\tfrac{3}{2}\phi\Sigma-2\xi\Omega\right)\left[\tfrac{3}{2}\left(\mathcal{E}-\tfrac{1}{3}\mu\right)-\frac{p}{2}\right]+3\Omega\mathcal{H}\left(\tfrac{1}{3}\Theta+\Sigma\right)\\ &\quad+\left(\Sigma-\tfrac{2}{3}\Theta\right)\left[\tfrac{3}{2}\left(-\tfrac{3}{2}\phi\mathcal{E}+3\Omega\mathcal{H}\right)+\tfrac{1}{2}\left(\mu+p\right)\mathcal{A}\right]\\ &\quad-\tfrac{15}{2}\phi\mathcal{H}\xi-3\xi\Omega\left(3\mathcal{E}+\mu+p\right). \end{aligned} \tag{4.2.23}$$

Using (4.2.15), (4.2.20) and (4.2.23) we obtain

$$\begin{aligned}\dot{\hat{\mathcal{E}}}-\tfrac{1}{3}\dot{\hat{\mu}} &= \left(-\tfrac{3}{2}\phi\Sigma-2\xi\Omega\right)\left[\tfrac{3}{2}\mathcal{E}-\tfrac{1}{2}\left(\mu+p\right)\right]-\tfrac{15}{2}\phi\mathcal{H}\xi-3\xi\Omega\left(3\mathcal{E}+\mu+p\right)\\ &\quad+\left(\Sigma-\tfrac{2}{3}\Theta\right)\left[-\tfrac{9}{4}\phi\mathcal{E}+\tfrac{9}{2}\Omega\mathcal{H}+\tfrac{1}{2}\left(\mu+p\right)\mathcal{A}\right]+3\Omega\mathcal{H}\left(\tfrac{1}{3}\Theta+\Sigma\right)\\ &\quad\mathcal{A}\left[\left(\Sigma-\tfrac{2}{3}\Theta\right)\left(\tfrac{3}{2}\mathcal{E}-\tfrac{1}{2}\left(\mu+p\right)+3\xi\mathcal{H}\right)-\left(\tfrac{1}{3}\Theta+\Sigma\right)\left(-\tfrac{3}{2}\phi\mathcal{E}\right)\right.\\ &\quad\left.+3\Omega\mathcal{H}\right]. \end{aligned} \tag{4.2.24}$$

Similarly, by taking the dot derivative of the RHS of (4.2.15) and using (4.1.3) and (4.2.18) along with

$$\begin{aligned}\dot{\phi} &= \left(\tfrac{2}{3}\Theta-\Sigma\right)\left(\mathcal{A}-\tfrac{1}{2}\phi\right)-2\xi\Omega,\\ \dot{\mathcal{H}} &= -3\xi\mathcal{E}\left(\tfrac{3}{2}\Sigma-\Theta\right)\mathcal{H},\end{aligned}$$

we finally arrive at

$$\begin{aligned}\dot{\hat{\mathcal{E}}}-\tfrac{1}{3}\dot{\hat{\mu}} &= -\tfrac{3}{2}\mathcal{E}\left[\left(\tfrac{2}{3}\Theta-\Sigma\right)\left(\mathcal{A}-\tfrac{1}{2}\phi\right)+2\xi\Omega\right]\\ &\quad-\tfrac{3}{2}\phi\left[\left(\tfrac{3}{2}\Sigma-\Theta\right)\mathcal{E}+3\xi\mathcal{H}-\tfrac{1}{2}\left(\mu+p\right)\Sigma\right]\\ &\quad+3\mathcal{H}\left[\mathcal{A}\xi+\Omega\left(\Sigma-\tfrac{2}{3}\Theta\right)\right]\\ &\quad+3\Omega\left[-3\xi\mathcal{E}+\left(\tfrac{3}{2}\Sigma-\Theta\right)\mathcal{H}\right]. \end{aligned} \tag{4.2.25}$$

Equating (4.2.24) to (4.2.25) and simplifying we get

$$3\mathcal{H}\left(\tfrac{2}{3}\Omega\Theta=\Sigma\Omega-\phi\xi\right)-2\Omega\xi\left(\mu+p\right)=0. \tag{4.2.26}$$

Using (4.2.8), we see that the first term of (4.2.26) vanishes, hence we have

$$\Omega\xi\left(\mu+p\right)=0. \tag{4.2.27}$$

For a perfect fluid obeying the Weak Energy Condition (WEC), we have $\mu+p>0$

$$\Rightarrow \quad \Omega\xi=0. \tag{4.2.28}$$

Therefore, for a perfect fluid LRS spacetime, either u^a or e^a must be hypersurface orthogonal. Due to this condition, the spacetime is divided into 3 distinct subclasses, i.e., LRS class I ($\Omega\neq 0$), LRS class II ($\xi=0=\Omega$) and LRS class III ($\xi\neq 0$).

4.3 LRS class I: $\Omega\neq 0$

In this class e^a is hypersurface orthogonal and u^a is twisting. When $\Omega\neq 0$, we see that $\Rightarrow \xi=0=\Theta=\Sigma$, $\dot{f}=0$. Therefore models with LRS class I solutions can neither expand nor distort. These models are stationary as the dot of all the scalar quantities vanish, i.e., ∂_t is a timelike Killing vector field and all spacetimes within this LRS class will be stationary. Thus, there exists a G_4 multiply-transitive group on timelike 3-surfaces. The non-zero quantities in general are μ, p, $\mathcal{A}$, ϕ, $\mathcal{E}$ and $\mathcal{H}$. The set of equations needed to be solved in this LRS class is given by

$$\begin{aligned}
\hat{\mathcal{A}} &= -\phi\mathcal{A}-\mathcal{A}^2-2\Omega^2+\tfrac{1}{2}\left(\mu+3p\right), && (4.3.1)\\
\hat{\Omega} &= -\phi\Omega+\mathcal{A}\Omega, && (4.3.2)\\
\mathcal{A} &= -\tfrac{\partial p}{\partial \mu}\hat{\mu}/\left(\mu+p\right), && (4.3.3)\\
\hat{\phi} &= -\tfrac{\phi^2}{2}+\phi\mathcal{A}+2\Omega^2-\left(\mu+p\right). && (4.3.4)
\end{aligned}$$

The magnitudes of the "electric" and "magnetic" parts of the Weyl curvature tensor in these models are given by

$$\mathcal{E} = -\tfrac{\sqrt{3}}{2}\phi\mathcal{A} - \sqrt{3}\Omega^2 + \tfrac{1}{2\sqrt{3}}\left(\mu + 3p\right), \tag{4.3.5}$$

$$\mathcal{H} = -\sqrt{3}\mathcal{A}\Omega + \tfrac{\sqrt{3}}{2}\phi\Omega. \tag{4.3.6}$$

When the acceleration is non-zero, a coupled second order ordinary differential equation for the spatial distribution of the energy density, can be derived by combining equations (4.3.1) and (4.3.3), which is

$$\begin{aligned} \tfrac{\partial p}{\partial \mu}\hat{\hat{\mu}} + \tfrac{\partial^2 p}{\partial \mu^2}\hat{\mu}^2 + \phi\tfrac{\partial p}{\partial \mu}\hat{\mu} - \left(2\tfrac{\partial p}{\partial \mu} + 1\right)\tfrac{\partial p}{\partial \mu}\hat{\mu}^2/\left(\mu + p\right) \\ -2\left(\mu + p\right)\Omega^2 + \tfrac{1}{2}\left(\mu + p\right)\left(\mu + 3p\right) &= 0. \end{aligned} \tag{4.3.7}$$

4.3.1 Solutions with $\phi = 0$

This subclass of solutions does exist, but for consistency, these solutions demand an equation of state of the form $p\left(\mu\right) = -\frac{1}{3}\mu + \text{const}$. This is usually dismissed as unphysical for conventional matter distributions. Thus one equation remains to be solved. This is an ordinary differential equation describing the spatial distribution of the total energy density μ. It follows from equation (4.3.1) and is given by

$$\hat{\hat{\mu}} - \tfrac{1}{3}\hat{\mu}/\left(\mu + p\right) + \tfrac{3}{2}\left(\mu^2 - p^2\right) = 0. \tag{4.3.8}$$

For the remaining non-zero quantities, we have the following expressions:

$$\begin{aligned} \mathcal{A} &= \tfrac{1}{3}\hat{\mu}/\left(\mu + p\right), \\ \Omega &= \left(\tfrac{1}{2}\right)^{\frac{1}{2}}\left(\mu + p\right)^{\frac{1}{2}}, \\ \mathcal{E} &= -\tfrac{1}{\sqrt{3}}\mu, \\ \mathcal{H} &= -\left(\tfrac{1}{6}\right)^{\frac{1}{2}}\hat{\mu}/\left(\mu + p\right)^{\frac{1}{2}}. \end{aligned} \tag{4.3.9}$$

Models of this kind are considered as non-physical generalisations of the Gödel LRS case.

4.3.2 Solutions with $p = 0$

In this subclass of dust models, we have $p = 0 \Leftrightarrow \mathcal{A} = 0$. Again, one ordinary differential equation for μ remains to be solved. Following from equation (4.3.4), it is given by

$$\hat{\hat{\mu}} - \tfrac{5}{4}\hat{\mu}^2 - \mu^2 = 0. \tag{4.3.10}$$

It follows that

$$\begin{aligned} \phi &= -\tfrac{1}{2}\hat{\mu}/\mu, \\ \Omega &= \left(\tfrac{1}{4}\right)^{\frac{1}{2}} \mu^{\frac{1}{2}}, \\ \mathcal{E} &= -\tfrac{1}{4\sqrt{3}}\mu, \\ \mathcal{H} &= -\tfrac{\sqrt{3}}{4}\left(\tfrac{1}{4}\right)^{\frac{1}{2}} \hat{\mu}/\mu^{\frac{1}{2}}. \end{aligned} \tag{4.3.11}$$

Models of this kind are also considered as generalisations of the Gödel LRS case. Now if we were to impose the dynamical restriction $\mathcal{A} = 0$ instead of $p = 0$, then equation (4.3.3) would be solved by either $\frac{\partial p}{\partial \mu} = 0 \Rightarrow p = \text{const}$, or $\hat{\mu} = 0$. These lead to either a slight generalisation of (4.3.10) or $\hat{f} = 0$ (Gödel's LRS case).

4.3.3 Solutions with $\mathcal{H} = 0$

From (4.3.6), if we apply the dynamical restriction $\mathcal{H} = 0$ then $\Rightarrow \phi = 2\mathcal{A}$. Then it follows from equations (4.3.1) and (4.3.4) that

$$\Omega^2 = -\mathcal{A}^2 + \tfrac{1}{3}\left(\mu + 2p\right). \tag{4.3.12}$$

We see that, on using (4.3.3), consistency of (4.3.2) demands that $\frac{\partial p}{\partial \mu} = 1$ $\Rightarrow p\left(\mu\right) = \mu + \text{const}$.

4.3.4 Solutions with $\mathcal{E} = 0$

From equation (4.3.5), if we apply the dynamical restriction $\mathcal{E} = 0$, then

$$\Omega^2 = -\tfrac{\phi}{2}\mathcal{A} + \tfrac{1}{6}\left(\mu + 3p\right). \tag{4.3.13}$$

In order to obtain a purely "magnetic" solution, consistency of equation (4.3.2) requires that

$$\mathcal{A}^2 - \left[\tfrac{\phi}{2} + \tfrac{2}{3\phi}p + \tfrac{1}{3\phi}\left[\tfrac{1}{3}\left(\tfrac{\partial p}{\partial \mu}\right)^{-1} + 1\right]\left(\mu + p\right)\right]\mathcal{A} + \tfrac{1}{6}\left(\mu + 3p\right) = 0. \tag{4.3.14}$$

4.3.5 Gödel's rotating model of the Universe: $\hat{f} = 0$

By imposing the condition $\hat{f} = 0$, the symmetry group becomes a G_5 multiply-transitive group on the full spacetime manifold which, consequently, is now homogeneous. Since the spatial derivatives of *all* non-zero scalar quantities f vanishes, we have following from equation (4.3.3), that the matter moves geodesically: $\mathcal{A} = 0$. On the other hand, since *no* spacelike 3-surfaces of constant f exist (group orbits of a simply-transitive G_3), models of this LRS subclass are not spatially homogeneous. Equation (4.3.2) shows that $\phi = 0$. From equation (4.3.6), we find that the magnitude of the "magnetic" part of the Weyl tensor is zero, i.e. $\mathcal{H} = 0$. All the scalar quantities f are constant on the spacetime manifold. Therefore, the remaining equations are purely algebraic. From equations (4.3.1), (4.3.4) and (4.3.5) we find

$$2\Omega^2 = \tfrac{1}{2}\left(\mu + 3p\right) = \left(\mu + p\right), \tag{4.3.15}$$

$$\mathcal{E} = -\sqrt{3}\Omega^2 + \tfrac{1}{2\sqrt{3}}\left(\mu + 3p\right). \tag{4.3.16}$$

From the above, it can be seen that this system of algebraic equations is consistent, as long as the equation of state is $p\left(\mu\right) = \mu$, that is for stiff matter. An equivalent formation would be a combination of dust matter ($p = 0$) with a negative cosmological constant ($\Lambda = 0$) as shown by Stephani (1991). This equivalent configuration gives Gödel's model of the Universe (Gödel (1949)).

It is observed that LRS class I of perfect fluid spacetime geometries, with the equation of state $p = p(\mu)$, comprises of a variety of stationary solutions. We note that, apart from the Gödel model, these solutions are differentially rotating. However, most of them are of little interest for astrophysical and cosmological purposes. This is due to the fact that it is difficult to see how, given non-zero vorticity, the geometry of a star model could correspond to exact rotational symmetry about every point.

4.4 LRS class II: $\xi = 0 = \Omega$

Here, *both* e^a and u^a are hypersurface orthogonal. When $\xi = 0 = \Omega$, there exist 3-surfaces orthogonal to the fluid flow in which there acts a G_3 multiply-transitive groups on spacelike 2-surfaces orthogonal to e^a. All models in this dynamic and spatially inhomogeneous LRS class have vanishing "magnetic part" of the Weyl curvature tensor. It follows from constraint equation (4.1.16) that

$$\mathcal{H} = 0. \tag{4.4.1}$$

The non-zero quantities in this case are μ, p, Θ, Σ, $\mathcal{A}$, ϕ and $\mathcal{E}$.

From the Gauss equation for e^a and the 3-Ricci identities (see Stephani (1991)), the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a is determined to be

$$^3R_{ab} = -\tfrac{1}{3}\left[\hat{\phi} + 2K\right]e_{ab} - \tfrac{1}{3}\left[2\hat{\phi} + \tfrac{3}{2}\phi^2 - 2K\right]h_{ab}, \tag{4.4.2}$$

where K denotes the constant Gaussian curvature ${}^2R := 2K$ of the 2D spacelike group. Another constraint is the generalised Friedmann equation given by

$$\begin{aligned} {}^3R &= -\left[2\hat{\phi} + \tfrac{3}{2}\phi^2 - 2K\right] \\ &= 2\mu - \tfrac{2}{3}\Theta^2 + 2\Sigma^2. \end{aligned} \tag{4.4.3}$$

From the above, we obtain an expression for $\hat{\phi}$ as

$$\hat{\phi} = -\tfrac{3}{4}\phi^2 + K - \mu + \tfrac{1}{3}\Theta^2 - \Sigma^2. \tag{4.4.4}$$

The purely algebraic expression for $\mathcal{E}$ is

$$\mathcal{E} = \tfrac{1}{2\sqrt{3}}\mu + \tfrac{\sqrt{3}}{8}\phi^2 - \tfrac{\sqrt{3}}{2}K - \tfrac{1}{6\sqrt{3}}\left[\Theta - \sqrt{3}\Sigma\right]^2. \tag{4.4.5}$$

From this, the set of equations defining this LRS class is

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 + \hat{\mathcal{A}} + \phi\mathcal{A} + \mathcal{A}^2 - 2\Sigma^2 - \tfrac{1}{2}\left(\mu + 3p\right), \tag{4.4.6}$$

$$\begin{aligned} \dot{\Sigma} &= \tfrac{1}{\sqrt{3}}\hat{\mathcal{A}} - \tfrac{\phi}{2\sqrt{3}}\mathcal{A} + \tfrac{1}{\sqrt{3}}\mathcal{A}^2 + \tfrac{1}{6\sqrt{3}}\Theta^2 - \Theta\Sigma - \tfrac{1}{2\sqrt{3}}\Sigma^2 \\ &\quad -\tfrac{1}{2\sqrt{3}}\mu - \tfrac{\sqrt{3}}{8}\phi^2 + \tfrac{\sqrt{3}}{2}K, \end{aligned} \tag{4.4.7}$$

$$\dot{\mu} = -\left(\mu + p\right)\Theta, \tag{4.4.8}$$

$$\dot{\phi} = -\tfrac{\phi}{3}\Theta + \tfrac{\phi}{\sqrt{3}}\Sigma + \tfrac{2}{3}\Theta\mathcal{A} - \tfrac{2}{\sqrt{3}}\Sigma\mathcal{A}, \tag{4.4.9}$$

$$\dot{K} = -\tfrac{2}{3}K\Theta + \tfrac{2}{\sqrt{3}}K\Sigma, \tag{4.4.10}$$

$$\hat{\Sigma} = \tfrac{1}{\sqrt{3}}\hat{\Theta} - \tfrac{3}{2}\phi\Sigma, \tag{4.4.11}$$

$$\mathcal{A} = -\tfrac{\partial p}{\partial \mu}\hat{\mu}/\left(\mu + p\right), \tag{4.4.12}$$

$$\hat{\phi} = -\tfrac{3}{4}\phi^2 + K - \mu + \tfrac{1}{3}\Theta^2\Sigma^2, \tag{4.4.13}$$

$$\hat{K} = -\phi K. \tag{4.4.14}$$

The evolution equation (4.4.10) arises from demanding conservation of the time along the matter flow lines of the constraint equation (4.4.4). However, equation (4.4.14) is a result of constraint (4.2.15). The evolution equation for the 3-Ricci scalar equation (4.4.3) is

$$\begin{aligned} {}^3R &= -\tfrac{1}{3}\left[2\Theta+\sqrt{3}\Sigma\right]\left[{}^3R+2\phi\mathcal{A}\right]+2\sqrt{3}\left[K-\tfrac{\phi^2}{4}\right]\Sigma \\ &= -\tfrac{4}{3}\left[\Theta-\sqrt{3}\Sigma\right]\left[\hat{\mathcal{A}}+\mathcal{A}^2\right]. \end{aligned} \tag{4.4.15}$$

Alternatively, if we use the set $S_4=\{\mu,\Theta,\Sigma,\mathcal{E}\}$ as the four generalised coordinates for the study of functional dependencies in models that have a non-zero expansion rate, then one can solve for K obtaining

$$K=\tfrac{1}{3}\mu+\tfrac{\phi^2}{4}-\tfrac{2}{\sqrt{3}}\mathcal{E}-\tfrac{1}{9}\left[\Theta-\sqrt{3}\Sigma\right]^2. \tag{4.4.16}$$

The relevant dynamical equations are

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2+\hat{\mathcal{A}}+\phi\mathcal{A}+\mathcal{A}^2-2\Sigma^2-\tfrac{1}{2}\left(\mu+3p\right), \tag{4.4.17}$$

$$\dot{\Sigma} = \tfrac{1}{\sqrt{3}}\hat{\mathcal{A}}-\tfrac{\phi}{2\sqrt{3}}\mathcal{A}+\tfrac{1}{\sqrt{3}}\mathcal{A}^2-\tfrac{2}{3}\Theta\Sigma-\tfrac{1}{\sqrt{3}}\Sigma^2-\mathcal{E}, \tag{4.4.18}$$

$$\dot{\mathcal{E}} = -\tfrac{1}{2}\left(\mu+p\right)\Sigma+\sqrt{3}\mathcal{E}\Sigma-\Theta\mathcal{E}, \tag{4.4.19}$$

$$\dot{\mu} = -\left(\mu+p\right)\Theta, \tag{4.4.20}$$

$$\dot{\phi} = -\tfrac{\phi}{3}\Theta+\tfrac{\phi}{\sqrt{3}}\Sigma+\tfrac{2}{3}\Theta\mathcal{A}-\tfrac{2}{\sqrt{3}}\Sigma\mathcal{A}, \tag{4.4.21}$$

$$\hat{\Sigma} = \tfrac{1}{\sqrt{3}}\hat{\Theta}-\tfrac{3}{2}\phi\Sigma, \tag{4.4.22}$$

$$\hat{\mathcal{E}} = -\tfrac{3}{2}\phi\mathcal{E}+\tfrac{1}{2\sqrt{3}}\hat{\mu}, \tag{4.4.23}$$

$$\mathcal{A} = -\tfrac{\partial p}{\partial\mu}\hat{\mu}/\left(\mu+p\right), \tag{4.4.24}$$

$$\hat{\phi} = \tfrac{2}{9}\Theta^2+\tfrac{2}{3\sqrt{3}}\Theta\Sigma-\tfrac{4}{3}\Sigma^2-\tfrac{2}{\sqrt{3}}\mathcal{E}-\tfrac{\phi^2}{2}-\tfrac{2}{3}\mu. \tag{4.4.25}$$

If local comoving coordinates are chosen, then the line element has the following form (cf. equation (2.8) of Stewart and Ellis (1968), and equation (13.2) of Kramer *et al* (1980))

$$ds^2=X^2(x,t)dx^2+Y^2(x,t)\left[dy^2+\Sigma^2(y)dz^2\right]-F^{-2}(x,t)dt^2. \tag{4.4.26}$$

Then, for scalars ϕ, Θ, Σ and $\mathcal{A}$, one obtains the following relations:

$$
\begin{aligned}
\phi &= \tfrac{2}{X}\tfrac{\hat{Y}}{Y}, && (4.4.27)\\
\Theta &= F\left(\tfrac{\dot{X}}{X}+2\tfrac{\dot{Y}}{Y}\right), && (4.4.28)\\
\Sigma &= \tfrac{F}{\sqrt{3}}\left(\tfrac{\dot{X}}{X}-\tfrac{\dot{Y}}{Y}\right), && (4.4.29)\\
\mathcal{A} &= -\tfrac{1}{X}\tfrac{\hat{F}}{F}. && (4.4.30)
\end{aligned}
$$

Integrating equations (4.4.10) and (4.4.14) gives

$$K = \tfrac{C_1}{Y^2}, \qquad (4.4.31)$$

where C_1 is an integration constant. Then, from equation (4.4.3) we obtain the differential equation

$$^3R = -\tfrac{2}{X^2}\left[2\tfrac{\hat{\hat{Y}}}{Y} - 2\tfrac{\hat{X}}{X}\tfrac{\hat{Y}}{Y} + \left(\tfrac{\hat{Y}}{Y}\right)^2\right] + 2\tfrac{C_1}{Y^2}. \qquad (4.4.32)$$

Exact solutions to the dynamical equations within the LRS class, for the spherically symmetric case $K > 0$, have been discussed by Kramer *et al* (1980).

Using our relation

$$\dot{\hat{f}} = \hat{\dot{f}} + \mathcal{A}\dot{f} - \tfrac{2}{\sqrt{3}}\Sigma\hat{f} - \tfrac{1}{3}\Theta\hat{f}, \qquad (4.4.33)$$

from the set of equations (4.4.17)-(4.4.25), the following relations showing the non-linear growth of spatial inhomogeneities within the LRS class can be derived (cf. Ellis and Bruni (1989), Bruni *et al* (1992))

$$
\begin{aligned}
\dot{\hat{\mu}} &= -\tfrac{2}{3}\left[2\Theta+\sqrt{3}\Sigma\right]\hat{\mu} - (\mu+p)\,\hat{\Theta}, && (4.4.34)\\
\dot{\hat{\Theta}} &= -\left[\Theta+2\sqrt{3}\Sigma\right]\hat{\Theta} + \hat{\hat{\mathcal{A}}} + \phi\hat{\mathcal{A}} + 3\mathcal{A}\hat{\mathcal{A}} + 6\phi\Sigma^2 - \mathcal{A}\Big[\tfrac{1}{9}\Theta^2\\
&\quad -\tfrac{2}{3\sqrt{3}}\Theta\Sigma + \tfrac{10}{3}\Sigma^2 + \tfrac{2}{\sqrt{3}}\mathcal{E} + \tfrac{\phi^2}{2} - \tfrac{1}{3}\mu - \tfrac{1}{2}(\mu+p)\left(\tfrac{\partial p}{\partial\mu}\right)^{-1}\\
&\quad - \phi\mathcal{A} - \mathcal{A}^2\Big], && (4.4.35)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\hat{\Sigma}} &= -\tfrac{1}{\sqrt{3}}\left[\Theta+2\sqrt{3}\Sigma\right]\hat{\Theta}+\tfrac{1}{\sqrt{3}}\hat{\hat{\mathcal{A}}}-\tfrac{\phi}{2\sqrt{3}}\hat{\mathcal{A}}+\sqrt{3}\mathcal{A}\hat{\mathcal{A}}+2\sqrt{3}\phi\Sigma^2 \\
&\quad +\tfrac{3}{2}\phi\Theta\Sigma+\tfrac{3}{2}\phi\mathcal{E}-\tfrac{1}{\sqrt{3}}\mathcal{A}\Big[\tfrac{1}{9}\Theta^2+\tfrac{7}{3\sqrt{3}}\Theta\Sigma+\tfrac{1}{3}\Sigma^2+\tfrac{2}{\sqrt{3}}\mathcal{E}-\tfrac{\phi^2}{4} \\
&\quad -\tfrac{1}{3}\mu-\tfrac{1}{2}\left(\mu+p\right)\left(\tfrac{\partial p}{\partial\mu}\right)^{-1}+\tfrac{\phi}{2}\mathcal{A}-\mathcal{A}^2\Big], \qquad (4.4.36) \\
\dot{\hat{\mathcal{E}}} &= -\tfrac{1}{3\sqrt{3}}\left[2\Theta+\sqrt{3}\Sigma\right]\hat{\mu}-\tfrac{1}{2\sqrt{3}}\left(\mu+p\right)\hat{\Theta}+\tfrac{3}{4}\phi\Sigma\left(\mu+p\right) \\
&\quad \left[2\phi-\mathcal{A}\right]\left[\Theta-\sqrt{3}\Sigma\right]\mathcal{E}, \qquad (4.4.37) \\
\dot{\hat{\phi}} &= \tfrac{2}{3}\left[\Theta-\sqrt{3}\Sigma\right]\hat{\mathcal{A}}-\tfrac{4}{27}\Theta^3-\tfrac{2}{3\sqrt{3}}\Theta^2\Sigma+\tfrac{2}{3}\Theta\Sigma^2+\tfrac{4}{3\sqrt{3}}\Sigma^3 \\
&\quad +\tfrac{\phi^2}{3}\left[\Theta-\sqrt{3}\Sigma\right]+\tfrac{2}{3\sqrt{3}}\left[2\Theta+\sqrt{3}\Sigma\right]\left[\mathcal{E}+\tfrac{1}{\sqrt{3}}\mu\right] \\
&\quad -\mathcal{A}\left[\tfrac{1}{3}\phi\Theta-\tfrac{4}{\sqrt{3}}\phi\Sigma-\tfrac{2}{3}\Theta\mathcal{A}+\tfrac{2}{\sqrt{3}}\Sigma\mathcal{A}\right], \qquad (4.4.38) \\
{}^3\dot{\hat{R}} &= -2\left[\Theta+\sqrt{3}\left(1+\tfrac{\partial p}{\partial\mu}\right)\Sigma\right]\hat{\mu}+2\phi\Sigma\left[\Theta\Sigma+5\sqrt{3}\Sigma^2+6\mathcal{E}\right] \\
&\quad +\left[\tfrac{16}{9}\Theta^2+\tfrac{4}{3\sqrt{3}}\Theta\Sigma-\tfrac{20}{3}\Sigma^2-2\phi\mathcal{A}-\tfrac{4}{\sqrt{3}}\mathcal{E}-\tfrac{4}{3}\mu\right]\hat{\Theta} \\
&\quad -\tfrac{4}{3}\left[\Theta-\sqrt{3}\Sigma\right]\hat{\hat{\mathcal{A}}}-\tfrac{4}{3}\phi\left[\Theta+2\sqrt{3}\Sigma\right]\hat{\mathcal{A}}-4\left[\Theta-\sqrt{3}\Sigma\right]\mathcal{A}\hat{\mathcal{A}} \\
&\quad -\mathcal{A}\Big[-\tfrac{4}{27}\Theta^3+\tfrac{4}{3\sqrt{3}}\Theta^2\Sigma-\tfrac{4}{3}\Theta\Sigma^2+\tfrac{4}{3\sqrt{3}}\Sigma^3-\tfrac{8}{3\sqrt{3}}\left[\Theta-\sqrt{3}\Sigma\right]\mathcal{E} \\
&\quad \tfrac{4}{3}\left[\Theta+2\sqrt{3}\Sigma\right]\phi\mathcal{A}+\tfrac{2}{3\sqrt{3}}\left(7\mu+9p\right)\Sigma+\tfrac{4}{9}\mu\Theta-\tfrac{2}{3}\phi^2\left[\Theta+2\sqrt{3}\Sigma\right] \\
&\quad +\tfrac{4}{3}\left[\Theta-\sqrt{3}\Sigma\right]\mathcal{A}^2\Big]. \qquad (4.4.39)
\end{aligned}
$$

The set of equations (4.4.34)-(4.4.39), in their FLRW-linearised form simplify considerably. This could be used (as an example) to investigate covariant and gauge-invariant spherically symmetric perturbations of an FLRW spacetime (see Ellis and Bruni (1989), Bruni *et al* (1992)). The description of the scalar, or total energy density, perturbations would be completely covered by only two equations from this set and the equations underlying the evolution of the FLRW background spacetime geometry. The scalar perturbations are the only ones contributing at linear order.

4.4.1 Spatially inhomogeneous LRS dust models

When imposing the condition $p = 0 \Leftrightarrow \mathcal{A} = 0$, the cosmological models within this LRS class are the Lemaître-Tolman-Bondi spherically symmetric solutions as shown by Lemaître (1933), Tolman (1934) and Bondi (1947). These spatially inhomogeneous dust models have been discussed by Bruni *et al* (1995) and are known as the "silent" class. The relevant set of equations are

$$\dot{\Theta} = -\tfrac{1}{3}\Theta^2 - 2\Sigma^2 - \tfrac{1}{2}\mu, \tag{4.4.40}$$

$$\dot{\Sigma} = -\tfrac{1}{\sqrt{3}}\Sigma^2 - \tfrac{2}{3}\Theta\Sigma - \mathcal{E}, \tag{4.4.41}$$

$$\dot{\mathcal{E}} = -\tfrac{1}{2}\mu\Sigma + \sqrt{3}\mathcal{E}\Sigma - \Theta\mathcal{E}, \tag{4.4.42}$$

$$\dot{\mu} = -\mu\Theta, \tag{4.4.43}$$

$$\dot{\phi} = -\tfrac{\phi}{3}\Theta + \tfrac{\phi}{\sqrt{3}}\Sigma, \tag{4.4.44}$$

$$\hat{\Sigma} = \tfrac{1}{\sqrt{3}}\hat{\Theta} - \tfrac{3}{2}\phi\Sigma, \tag{4.4.45}$$

$$\hat{\mathcal{E}} = -\tfrac{3}{2}\phi\mathcal{E} + \tfrac{1}{2\sqrt{3}}\hat{\mu}. \tag{4.4.46}$$

$$\hat{\phi} = \tfrac{2}{9}\Theta^2 + \tfrac{2}{3\sqrt{3}}\Theta\Sigma - \tfrac{4}{3}\Sigma^2 - \tfrac{2}{\sqrt{3}}\mathcal{E} - \tfrac{\phi^2}{2} - \tfrac{2}{3}\mu. \tag{4.4.47}$$

For the spherically symmetric cas $K > 0$, exact solutions to the dynamical equations within this LRS class have been discussed by Lemaître (1933), Tolman (1934), Bondi (1947) and Kramer *et al* (1980). The more general case has been discussed by Ellis (1967). In local comoving coordinates of equation (4.4.26), we have from equation (4.4.30) that $F = 1$, for dust. The set of equations (4.4.34)-(4.4.39) simplify to become

$$\dot{\hat{\mathcal{A}}} = -\tfrac{2}{3}\left[2\Theta + \sqrt{3}\Sigma\right]\hat{\mu} - \mu\hat{\Theta}, \tag{4.4.48}$$

$$\dot{\hat{\Theta}} = -\tfrac{1}{2}\hat{\mu} - \left[\Theta + 2\sqrt{3}\Sigma\right]\hat{\Theta} + 6\phi\Sigma^2, \tag{4.4.49}$$

$$\dot{\hat{\Sigma}} = -\tfrac{1}{2\sqrt{3}}\hat{\mu} - \tfrac{1}{\sqrt{3}}\left[\Theta + 2\sqrt{3}\Sigma\right]\hat{\Theta} + 2\sqrt{3}\phi\Sigma^2 + \tfrac{3}{2}\phi\Theta\Sigma + \tfrac{3}{2}\phi\mathcal{E}, \tag{4.4.50}$$

$$\dot{\hat{\mathcal{E}}} = -\tfrac{1}{3\sqrt{3}}\left[2\Theta + \sqrt{3}\Sigma\right]\mu - \tfrac{1}{2\sqrt{3}}\mu\hat{\Theta} + \tfrac{3}{4}\mu\phi\Sigma + 2\phi\mathcal{E}\left[\Theta - \sqrt{3}\Sigma\right], \tag{4.4.51}$$

$$\begin{aligned}\dot{\hat{\phi}} &= -\tfrac{4}{27}\Theta^3 - \tfrac{2}{3\sqrt{3}}\Theta^2\Sigma + \tfrac{2}{3}\Theta\Sigma^2 + \tfrac{4}{3\sqrt{3}}\Sigma^3 + \tfrac{\phi^2}{3}\left[\Theta - \sqrt{3}\Sigma\right] \\ &\quad + \tfrac{2}{3\sqrt{3}}\left[2\Theta - \sqrt{3}\Sigma\right]\left[\mathcal{E} + \tfrac{1}{\sqrt{3}}\mu\right],\end{aligned} \tag{4.4.52}$$

$$\begin{aligned}{}^3\dot{\hat{R}} &= -2\left[\Theta + \sqrt{3}\Sigma\right]\hat{\mu} + \left[\tfrac{16}{9}\Theta^2 + \tfrac{4}{3\sqrt{3}}\Theta\Sigma - \tfrac{20}{3}\Sigma^2 - \tfrac{4}{\sqrt{3}}\mathcal{E} - \tfrac{4}{3}\mu\right]\hat{\Theta} \\ &\quad + 2\phi\Sigma\left[\Theta\Sigma + 5\sqrt{3}\Sigma^2 + 6\mathcal{E}\right].\end{aligned} \tag{4.4.53}$$

The linearised set of equations that corresponds to Ellis and Bruni (1989) follows easily.

4.4.2 The shear-free subcase: $\Sigma = 0$

If we demand $\Sigma = 0$, then an interesting subcase arises. From equation (4.4.18) we obtain the magnitude of the "electric" part of the Weyl conformal curvature tensor as

$$\mathcal{E} = \tfrac{1}{\sqrt{3}}\hat{\mathcal{A}} - \tfrac{\phi}{2\sqrt{3}}\mathcal{A} + \tfrac{1}{\sqrt{3}}\mathcal{A}^2. \tag{4.4.54}$$

Hence, to obtain a solution with $\mathcal{E} \neq 0$, we must have the fluid acceleration to be non-zero. Also, we assume a rate of expansion of the matter fluid. From (4.4.22) it follows that $\hat{\Theta} = 0$, i.e. the spatial distribution of the expansion rate is homogeneous. Therefore, it is constant on the spacelike 3-surfaces orthogonal to u^a. Using equations (4.4.12) and (4.4.54), the conditions arising from equations (4.4.35) and (4.4.36) for $\Sigma = 0$ are equivalent and they provide the expression

$$\hat{\hat{\mathcal{A}}} = \tfrac{1}{2}\hat{\mu} - \phi\hat{\mathcal{A}} - \tfrac{7}{3}\mathcal{A}\hat{\mathcal{A}} + \mathcal{A}\left[\tfrac{1}{9}\Theta^2 + \tfrac{\phi^2}{2} - \tfrac{1}{3}\mu - \tfrac{4}{3}\phi\mathcal{A} - \tfrac{1}{3}\mathcal{A}^2\right]. \tag{4.4.55}$$

This ensures that (4.4.23) is solved. If we use equation (4.4.12) then equation (4.4.55) is a constraint equation for the spatial distribution of the total energy density μ of third order. Coupling this with constraint equation (4.4.25) gives the spatial distribution of the spatial divergence ϕ. In local comoving coordinates of equation (4.4.26), we have

$$\Sigma = 0 \Rightarrow F\tfrac{\dot{X}}{X} = F\tfrac{\dot{Y}}{Y} \Rightarrow \Theta = 3F\tfrac{\dot{X}}{X}. \tag{4.4.56}$$

Upon integration of (4.4.34) we find

$$\hat{\mu} = \tfrac{C_2}{X^4}, \tag{4.4.57}$$

where C_2 is a constant of integration. An expression for $\frac{\partial p}{\partial \mu}$ can be found using this result along with equations (4.4.12) and (4.4.30). Exact solutions to the dynamical equations for this LRS subclass for $K > 0$ and $p = p(\mu)$ are discussed by Kramer *et al* (1980).

4.4.3 The non-diverging subcase: $\phi = 0$

When $\phi = 0$, the normals to the spacelike 2-surfaces that are spanned by the isometry group are non-diverging. For this dynamical restriction to be consistent, equation (4.4.21) demands that

$$\left[\Theta - \sqrt{3}\Sigma\right] \mathcal{A} = 0, \tag{4.4.58}$$

whereas equation (4.4.47) gives $\mathcal{E}$ algebraically. If, on one hand, we have $\Theta = \sqrt{3}\Sigma$, then inserting $\mathcal{E}$ into the constraint equation (4.4.46) requires $\hat{\mu} = 0$ and equations (4.4.18) and (4.4.19) yield $(\mu + p) = 0$ and $(\mu + p)\Theta = 0$ respectively. Therefore, this case can be discarded. If, on the other hand, we have $\mathcal{A} = 0$, for $\hat{f} \neq 0$ we must have $p = 0$. The equation (4.4.46) gives the condition

$$\hat{\Theta} = \tfrac{3}{2}\left[\Theta - \sqrt{3}\Sigma\right]^{-1} \hat{\mu}. \tag{4.4.59}$$

The covariant time derivatives along the matter fluid flow lines of the constraint equations (4.4.45), (4.4.46) and (4.4.47) vanish. In other words, an appropriate matter source for spatially inhomogeneous models of LRS class II (with non-diverging isotropy generator) is provided by dust *only.* These models are a further specialisation of the "silent" class (see Bruni *et al* (1995)).

The spatially inhomogeneous LRS class II comprises the greatest class of solutions of perfect fluid spacetime geometries with the equation of state $p = p(\mu)$. In general, the models are time-dependent. Previously, they have been helpful for the numerical and theoretical description of star and galaxy processes as well as supernova explosions. One can refer to Misner *et al* (1973), Shapiro and Teukolsky (1983), Longair (1994) and Stephani (1991) for further information. The relevant dynamical equations become simple and easily tractable as they have highly idealised spacetime symmetry properties. However, an isentropic matter fluid flow, which resulted from a barotropic equation of state, will be too restrictive to realistically model (as an example) the explosion of stars during the late stages of their evolution.

4.5 LRS class III: $\xi \neq 0$

In this case, e^a is twisting and u^a is hypersurface orthogonal. When $\xi \neq 0 \Rightarrow \Omega = \phi = \mathcal{A} = 0$, *all* spatial derivatives vanish. Since u^a is normal and geodesic, all scalars f are spatially homogeneous and there exists a G_4 of isometries multipy-transitive on spacelike 3-surfaces orthogonal to u^a. This means that the spacetimes themselves are orthogonally spatially homogeneous or OSH (Ellis and MacCallum (1969)). The non-zero quantities in this case are μ, p, Θ, Σ, $\mathcal{E}$ and $\mathcal{H}$. From the 3-Ricci identities and the Gauss equation for e^a as shown by Stephani (1991), the 3-Ricci curvature tensor of the spacelike 3-surfaces orthogonal to u^a is determined to be

$$ {}^3R_{ab} = \tfrac{2}{\sqrt{3}}\left[\mathcal{E} - \tfrac{1}{3}\Theta\Sigma + \tfrac{1}{\sqrt{3}}\Sigma^2\right] e_{ab} + \tfrac{1}{3}\left[6\xi^2 - 2\sqrt{3}\mathcal{E} + \tfrac{2}{\sqrt{3}}\Theta\Sigma - 2\Sigma^2\right] h_{ab}. \tag{4.5.1} $$

The trace of (4.5.1) yields the generalised Friedmann equation

$$ \begin{aligned} {}^3R &= 6\xi^2 - 2\sqrt{3}\mathcal{E} + \tfrac{2}{\sqrt{3}}\Theta\Sigma - 2\Sigma^2 \\ &= 2\mu - \tfrac{2}{3}\Theta^2 + 2\Sigma^2. \end{aligned} \tag{4.5.2} $$

The magnitudes of the "electric" and "magnetic" parts of the Weyl conformal curvature tensor are

$$ \mathcal{E} = \tfrac{1}{3\sqrt{3}}\Theta^2 + \tfrac{1}{3}\Theta\Sigma - \tfrac{2}{\sqrt{3}}\Sigma^2 + \sqrt{3}\xi^2 - \tfrac{1}{\sqrt{3}}\mu, \tag{4.5.3} $$

$$ \mathcal{H} = 3\xi\Sigma. \tag{4.5.4} $$

The set of dynamical equations are

$$
\begin{aligned}
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - 2\Sigma^2 - \tfrac{1}{2}\left(\mu + 3p\right), &(4.5.5)\\
\dot{\Sigma} &= -\tfrac{1}{3\sqrt{3}}\Theta^2 - \Theta\Sigma + \tfrac{1}{\sqrt{3}}\Sigma^2 - \sqrt{3}\xi^2 + \tfrac{1}{\sqrt{3}}\mu, &(4.5.6)\\
\dot{\mu} &= -\left(\mu + p\right)\Theta, &(4.5.7)\\
\dot{\xi} &= -\tfrac{2}{3}\xi\Theta + \tfrac{8}{\sqrt{3}}\xi\Sigma. &(4.5.8)
\end{aligned}
$$

An evolution equation for the 3-Ricci scalar (4.5.2) can be derived as

$$
{}^3\dot{R} = -\tfrac{2}{3}\Theta^3R + \tfrac{2}{\sqrt{3}}{}^3R - 4\sqrt{3}\xi^2\Sigma. \tag{4.5.9}
$$

On the other hand, if we use the set $S_4 = \{\mu, \Theta, \Sigma, \mathcal{E}\}$ as the four generalised coordinates for the study of functional dependencies in models that have a non-zero expansion rate, then one can solve for ξ^2 obtaining

$$
\xi^2 = -\tfrac{1}{9}\Theta^2 - \tfrac{1}{3\sqrt{3}}\Theta\Sigma + \tfrac{2}{3}\Sigma^2 + \tfrac{1}{\sqrt{3}}\mathcal{E} + \tfrac{1}{3}\mu. \tag{4.5.10}
$$

Since $\xi^2 > 0$ is demanded, this results in an algebraic restriction on the set S_4. Thus the set of dynamical equations is

$$
\begin{aligned}
\dot{\Theta} &= -\tfrac{1}{3}\Theta^2 - 2\Sigma^2 - \tfrac{1}{2}\left(\mu + 3p\right), &(4.5.11)\\
\dot{\Sigma} &= -\tfrac{1}{\sqrt{3}}\Sigma^2 - \tfrac{2}{3}\Theta\Sigma - \mathcal{E}, &(4.5.12)\\
\dot{\mu} &= -\left(\mu + p\right)\Theta, &(4.5.13)\\
\dot{\mathcal{E}} &= \tfrac{1}{2}\left(5\mu - p\right)\Sigma + 4\sqrt{3}\mathcal{E} - \Theta\mathcal{E} - \Theta^2\Sigma - \sqrt{3}\Theta\Sigma^2 + 6\Sigma^3. &(4.5.14)
\end{aligned}
$$

Consistency of this set of equations with the evolution equation (4.5.8) has been tested.

4.5.1 Dust solutions

Equivalent forms of equations (4.5.2), (4.5.6) and (4.5.7) are given by

$$
\begin{aligned}
2\tfrac{\ddot{Y}}{Y}+\left(\tfrac{\dot{Y}}{Y}\right)^2+\tfrac{C_2}{Y^2}-\tfrac{3{C_1}^2}{4}\tfrac{X^2}{Y^4} &= -p, &(4.5.15)\\
2\tfrac{\dot{X}}{X}\tfrac{\dot{Y}}{Y}+\left(\tfrac{\dot{Y}}{Y}\right)^2+\tfrac{C_2}{Y^2}-\tfrac{{C_1}^2}{4}\tfrac{X^2}{Y^4} &= \mu, &(4.5.16)\\
\tfrac{\ddot{X}}{X}+\tfrac{\dot{X}}{X}\tfrac{\dot{Y}}{Y}+\tfrac{\ddot{Y}}{Y}+\tfrac{{C_1}^2}{4}\tfrac{X^2}{Y^4} &= -p. &(4.5.17)
\end{aligned}
$$

This set of equations is provided by Ellis (1967) for $p = 0$.

Because the fluid flow is geodesic, the dust subcases are simply prominent by simpler evolution equations, i.e. $p = 0$ in equations (4.5.7), (4.5.15) and (4.5.17).

4.5.2 Solutions with $\mathcal{E} = 0$ (Pure magnetic)

Imposing the dynamical restriction $\mathcal{E} = 0$ on equation (4.5.14) results in, for $\Sigma \neq 0$, the algebraic condition

$$\Theta^2 = \tfrac{1}{2}\left(5\mu - p\right) - \sqrt{3}\Theta\Sigma + 6\Sigma^2, \tag{4.5.18}$$

which, in order to satisfy equation (4.5.11), restricts the equation of state through the algebraic condition for $\frac{\partial p}{\partial \mu}$. Using equation (4.5.10, the value of ξ^2 is determined to be

$$\xi^2 = \tfrac{1}{18}\left(\mu + p\right), \tag{4.5.19}$$

which, clearly, is consistent with the condition $\xi^2 > 0$. Thus, from equation (4.5.4), with $\Sigma \neq 0$, we have $\mathcal{H} \neq 0$. Solutions of this nature are of purely “magnetic” character. An example if given in the self-similar Bianchi Type-II OSH LRS solutions of Collins and Stewart (1971). These have a linear barotropic equation of state of the form $p(\mu) = (\gamma - 1)\,\mu$ and one obtains $\mathcal{E} = 0$ for $\gamma = 6/5$ as shown by Hsu and Wainwright (1986).

4.5.3 The FLRW subcase

If, for $\Sigma \neq 0$, we demand that the spacelike 3-surfaces orthogonal to u^a be of *constant* curvature (from equation (4.5.1))

$$\mathcal{E} = \tfrac{1}{3}\left[\Theta - \sqrt{3}\Sigma\right]\Sigma, \tag{4.5.20}$$

then one obtains, from (4.5.14), an algebraic expression for Θ^2. Substituting this algebraic expression in equation (4.5.10) gives the result $\xi^2 = 0$, thus violating the definition of this specific LRS class. Therefore, there does not exist any shearing solutions in this LRS class with isotropic ${}^3R_{ab}$. Following from equation (4.5.2), the ${}^3R > 0$ - FLRW models with are the only LRS models with $\xi \neq 0$ and have 3-spaces of constant curvature. These are invariant under a G_6 of isometries multiply-transitive on spacelike 3-surfaces and there exists a 3D family of rotational symmetries rather than one. In this special case, the spacelike unit vector field e^a is not uniquely defined. Also, there is no covariant trait that picks out a preferred spatial direction. Despite this, local coordinate or orthogonal frame bases can still be found as before. When $\Theta = 0$, we have the special FLRW case known as the Einstein static model. Having $\Theta = 0$ leads to $\xi^2 = \frac{1}{3}\mu > 0$ from equation (4.5.10). Then the spacetime manifold becomes invariant under a multiply-transitive G_7 of isometries and, therefore, is homogeneous. From equation (4.5.11), we must have an equation of state of the form $p(\mu) = -1/3\mu =$ const in order to have consistency. This can be interpreted as a dust model ($p = 0$) with a *positive* cosmological constant, i.e., $\Lambda > 0$.

The general OSH LRS class III perfect fluid spacetimes with equation of state $p = p(\mu)$ are characterised by the existence of a G_4 isometry group multiply-transitive on spacelike 3-surfaces. As well as a simply-transitive subgroup G_3 belonging to one of the numerous Bianchi types. The ${}^3R > 0$ - FLRW models are the special cases. The possibilty of constructing simple cosmological models with purely "magnetic" Weyl conformal curvature tensor helps in studying the underlying physical mechanisms. This could produce solutions of this eccentric kind.

Chapter 5

A new class of LRS with $\Omega, \xi \neq 0$

5.1 Introduction

By analysing the consistency conditions of the field equations, it was rigorously proved that a perfect fluid LRS spacetime cannot have simultaneous fluid rotation and spatial twist of the preferred spatial direction. Though all these classes are of interest, none of them are suitable for modelling a dynamical rotating star (gravitational collapse of a rotating star, for example). In this study the three key questions are: By relaxing the condition of a perfect fluid, that is by introducing pressure anisotropy and heat flux, is it possible to have dynamical solutions with non-zero rotation and non-zero twist? If yes, can these solutions be physical? What are the local geometrical properties of such solutions?

5.2 Extra symmetry of these spacetimes

We would like to relax the perfect fluid condition, that is we introduce pressure anisotropy and heat flux in the matter, and search for the existence of solutions that have both rotation and twist of the preferred direction. To do this, let us first derive an important result for LRS spacetimes. We can write the full covariant derivatives of the vectors u^a and e^a in terms of the LRS scalars in the following way:

$$\begin{aligned}\nabla_a u_b &= -\mathcal{A} u_a e_b + \left(\tfrac{1}{3}\Theta + \Sigma\right) e_a e_b \\ &\quad + \left(\tfrac{1}{3}\Theta - \tfrac{1}{2}\Sigma\right) N_{ab} + \Omega \varepsilon_{ab}, \qquad (5.2.1)\\ \nabla_a e_b &= -\mathcal{A} u_a u_b + \left(\tfrac{1}{3}\Theta + \Sigma\right) e_a u_b \\ &\quad + \tfrac{1}{2}\phi N_{ab} + \xi \varepsilon_{ab}. \qquad (5.2.2)\end{aligned}$$

Contracting the above with ε_{ab} and using

$$\varepsilon^{ab} u_a = \varepsilon^{ab} e_a = 0, \quad \varepsilon^{ab}\varepsilon_{ab} = 2, \qquad (5.2.3)$$

we get

$$\Omega = \tfrac{1}{2}\varepsilon^{ab}\nabla_a u_b, \quad \xi = \tfrac{1}{2}\varepsilon^{ab}\nabla_a e_b. \qquad (5.2.4)$$

Now, for any scalar function 'Ψ', we have

$$\nabla_b \Psi = -\dot{\Psi} u_b + \hat{\Psi} e_b. \qquad (5.2.5)$$

Differentiating again we have

$$\begin{aligned}\nabla_a \nabla_b \Psi &= -\left(\nabla_a \dot{\Psi}\right) u_b - \dot{\Psi}\left(\nabla_a u_b\right) \\ &\quad + \left(\nabla_a \hat{\Psi}\right) e_b + \hat{\Psi}\left(\nabla_a e_b\right). \qquad (5.2.6)\end{aligned}$$

Contracting with ε^{ab}, and noting that $\nabla_a \nabla_b \Psi$ is symmetric in a and b, we see that the LHS of (5.2.6) vanishes. Using equations (5.2.3) and (5.2.4) we get an important result:

$$\forall \Psi, \;\; \dot{\Psi}\Omega = \hat{\Psi}\xi. \qquad (5.2.7)$$

This equation implies self-similarity, for it applies to all scalars. It is also unchanged under the transformation $t \to at$, $r \to ar$, where t and r are the curve parameters of the integral curves of u and e.

From the above equation it is clear that if $\Omega \neq 0$, $\xi = 0$, the dot derivatives of all the scalars vanish, making the spacetime stationary. On the other hand if $\Omega = 0$, $\xi \neq 0$, then the hat derivatives of all scalars vanish, making the spacetime spatially homogeneous. Thus we arrive at an important result:

Theorem 1. *For LRS spacetimes with non-zero rotation and spatial twist, there always exists a conformal Killing vector in the $[u, e]$ plane. When one of these quantities vanish then the conformal Killing vector becomes a Killing vector. This Killing vector is timelike for vanishing spatial twist and it is spacelike for vanishing rotation. However when both the rotation and spatial twist vanish no such symmetry is guaranteed.*

Another important point to be noted here is that Ω and ξ do not evolve independently. Supposing that both are not equal to zero, first considering $\Psi = \Omega$ in equation (5.2.7) and then $\Psi = \xi$ and using the field equations (4.1.2), (4.1.3), (4.1.6), (4.1.8) and (4.1.16) we get the constraint

$$\frac{\Omega}{\xi} = -\frac{\phi}{\Sigma - \frac{2}{3}\Theta}\,. \tag{5.2.8}$$

Now to establish the existence of solutions with non-zero rotation and spatial twist, we state and prove the following theorem:

Theorem 2. *Evolution of all the independent geometrical scalars of LRS spacetimes that have non-zero rotation and spatial twist, obey a common second order linear hyperbolic partial differential equation and the existence of a initial spacelike Cauchy surface is guaranteed. Subject to the initial Cauchy data on this surface these geometrical scalars can be uniquely determined, and hence unique solutions of the field equations exist.*

Proof. Taking the *dot* derivative and then the *hat* derivative of the equation (5.2.7), subtracting them and then using the commutation relation (4.1.17) and the field equations (4.1.2), (4.1.3), (4.1.6), (4.1.8) and (4.1.16), we obtain the following equation $\forall\Psi$,

$$\begin{aligned} -\Omega^2\ddot{\Psi} + \xi^2\hat{\hat{\Psi}} &- \dot{\Psi}\Omega\left[\xi(\mathcal{A}-\phi) + \Omega(\Sigma - \tfrac{2}{3}\Theta)\right] \\ &+ \hat{\Psi}\xi\left[2\Omega\Sigma - \tfrac{1}{3}\Omega\Theta - \phi\xi\right] = 0. \end{aligned} \tag{5.2.9}$$

It is evident that the above equation is a hyperbolic (wave-like) second order linear partial differential equation for $\Omega, \xi \neq 0$, that governs the evolution of all independent geometrical scalars that describe a LRS spacetime. By the properties of hyperbolic partial differential equations, there exists a unique solution subject to Cauchy initial data on a spacelike Cauchy surface. In order to check whether such a three-dimensional spacelike surface exists, we consider the Lie derivative of the tensor N^{ab} with respect to the spacelike vector e^a. We know that

$$(\mathcal{L}_{\boldsymbol{e}}N)^{ab} = e^c\nabla_c N^{ab} - N^{cb}\nabla_c e^a - N^{ac}\nabla_c e^b\ . \tag{5.2.10}$$

Using (3.2.2), (3.2.3), (5.2.1), (5.2.2) and (5.2.3) we see that

$$(\mathcal{L}_{\boldsymbol{e}}N)^{ab} = \phi N^{ab}\ , \tag{5.2.11}$$

which implies

$$(\mathcal{L}_{\boldsymbol{e}}N)^{ab}u_a = (\mathcal{L}_{\boldsymbol{e}}N)^{ab}u_b = 0\ , \tag{5.2.12}$$

that is neither the vector e^a and the tensor N^{ab}, nor the Lie derivative of N^{ab} with respect to e^a has any component along the timelike vector u^a. It is clear that the tensor product of e^a and N^{ab} indeed spans a spacelike 3-surface where we can specify the Cauchy initial data to obtain a unique solution of (5.2.9) for all the independent geometrical and thermodynamic scalars of the LRS spacetime. □

Furthermore, the hyperbolic nature of the above equation dictates the existence of two families of characteristics. In analogy with incoming and outgoing waves, these characteristics describe the expanding/collapsing branches of the solutions.

5.3 Constraints on thermodynamic variables

We will now describe the constraints on the thermodynamic variables for the energy momentum tensor of the matter field. These generate LRS solutions with non-zero rotation and spatial twist. First we observe that the common wave-like equation (5.2.9) was obtained by the Ricci identities of the timelike vector u^a and spacelike vector e^a. In order to obtain the constraints on the matter variables, we need to observe the consistencies of doubly contracted Bianchi identities carefully. We state and prove the following theorem here:

Theorem 3. *The necessary condition for a LRS spacetime to have non-zero rotation and spatial twist simultaneously is non-zero heat flux which is bounded from both sides.*

Proof. Taking the *time-like* derivative for the equation (4.1.9) and using (4.1.17) and the field equations, we get

$$\Omega\xi\left(\mu+p+\Pi\right)+Q\left(\Omega^2+\xi^2\right)=0.$$

Simplifying the above equation we get

$$\frac{\frac{\Omega}{\xi}}{1+\left(\frac{\Omega}{\xi}\right)^2}=\frac{-Q}{\mu+p+\Pi}. \tag{5.3.1}$$

It is apparent from the above equation that if we demand both Ω and ξ are well defined and non-zero, and all the energy conditions to be satisfied we must have $Q\neq 0$. Also it is interesting to note the the ratio of the rotation and spatial twist can be described in terms of the thermodynamic quantities only. Now using (5.3.1) to solve for $\frac{\Omega}{\xi}$ gives

$$\frac{\Omega}{\xi}=\frac{-\left(\mu+p+\Pi\right)\mp\sqrt{\left(\mu+p+\Pi\right)^2-4Q^2}}{2Q}. \tag{5.3.2}$$

From the above equation it is clear that for the rotation and spatial twist to be well defined, real and non-zero, we must have $(\mu+p+\Pi)^2 > 4Q^2$. Thus, the thermodynamic quantities must satisfy the following constraint

$$-\frac{1}{2}(\mu+p+\Pi) < Q < \frac{1}{2}(\mu+p+\Pi)\ ;\ Q \neq 0. \tag{5.3.3}$$

$\square$

From Kolassis *et al* (1988) and Chan (2003) we can see that the above constraints are consistent with the Dominant Energy Conditions (DEC) for the matter field. Hence we do have matter that obeys the physically reasonable energy conditions that can generate a LRS spacetime with non-zero rotation and spatial twist. Furthermore, the rest of the propagation equations evolve identically in time and yield no new constraints.

5.4 Other constraints and solution finding algorithm

Let us now try to reduce the number of independent geometrical scalars of an LRS spacetime by using equation (5.2.7). Inserting the scalar variables ϕ, $\left(\Sigma - \frac{2}{3}\Theta\right)$ and $\mathcal{H}$, and using the field equations, we get the following set of equations:

$$\begin{aligned}
\Omega Q - 2\xi^3 + 2\Omega^2\xi - \tfrac{1}{3}\xi\Theta\Sigma + \xi\Sigma^2 + \tfrac{2}{3}\xi\mu& \\
-\tfrac{2}{9}\xi\Theta^2 + \xi\mathcal{E} + \tfrac{1}{2}\xi\Pi + A\phi\xi &= 0, \qquad (5.4.1)\\
-\Omega A\phi - \tfrac{1}{3}\Omega\Theta\Sigma + \tfrac{2}{9}\Omega\Theta^2 - 2\Omega^3 - \Omega E& \\
+\tfrac{1}{2}\Omega\Pi + \tfrac{1}{3}\Omega\mu + \Omega p + \xi\phi\Sigma + 2\Omega\xi^2 + Q\xi &= 0, \qquad (5.4.2)\\
\tfrac{9}{2}\Omega\xi\Sigma^2 - 3\Omega\xi\Theta\Sigma + \Omega^2 Q + \Omega\xi\Pi + \tfrac{9}{2}\xi^2\phi\Sigma& \\
+\xi\Omega\mu + \xi\Omega p + Q\xi^2 &= 0. \qquad (5.4.3)
\end{aligned}$$

Solving the above system of equations for $\mathcal{E}$, p and ϕ, we get

$$p = -\frac{\Omega^2 Q + \xi\Omega\mu + \Omega\xi\Pi + Q\xi^2}{\Omega\xi}, \tag{5.4.4}$$

$$\phi = -\frac{(3\Sigma - 2\Theta)\,\Omega}{3\xi}, \tag{5.4.5}$$

which are same as (5.3.1) and (5.2.8). Also, we get a new algebraic relation for $\mathcal{E}$

$$\begin{aligned}\mathcal{E} &= \frac{\Omega}{\xi}\mathcal{A}(\Sigma - \frac{2}{3}\Theta) - \Sigma^2 + \frac{1}{3}\Theta\Sigma + \frac{2}{9}\Theta^2 \\ &\quad +2(\xi^2 - \Omega^2) - \frac{\Omega}{\xi}Q - \frac{1}{2}\Pi - \frac{2}{3}\mu\,. \end{aligned} \tag{5.4.6}$$

The above equation, along with equation (4.1.16), completely describes the Weyl tensor in LRS spacetimes.

Now taking into account the results for perfect fluid LRS spacetimes from Elst and Ellis (1996), we see that the above phenomenon is true for any LRS spacetime and we can state this interesting theorem:

Theorem 4. *The symmetry of LRS spacetimes makes the Weyl tensor obey an algebraic constraint with other 1+1+2 geometrical variables. Hence the doubly contracted Bianchi identities that describe the propagation and evolution of the Weyl tensor become redundant.*

Now we can see that the number of independent geometrical 1+1+2 scalars that describe a LRS spacetime is reduced considerably. For example, specifying

$$\mathcal{D}_3 = \{\mathcal{A}, \Theta, \xi, \Sigma, \mu, p(\mu, \Pi, Q), \Pi, Q\}\,,$$

will automatically specify

$$\mathcal{D}_4 = \{\Omega, \phi, \mathcal{E}, \mathcal{H}\}\,,$$

via the constraint equations (5.2.8), (5.3.2), (5.4.6) and (4.1.16). Hence, we can give the initial Cauchy data on any spacelike Cauchy surface for the independent variables using any suitable chosen equation of state $p(\mu, \Pi, Q)$,. Then we can determine their evolution via equation (5.2.9), which applies equally to all the variables in $\mathcal{D}_3$. This will then provide us with a unique self-similar dynamical solution for the LRS spacetimes with non-zero rotation and spatial twist. The nature of the matter required for such solutions to exist follows from Theorem 2, where there is a non-trivial condition on the presence of heat flux. There are no other conditions on the density, pressure or pressure anisotropy. Nonetheless, for physically realistic solutions these must obey the Dominant Energy Conditions (DEC).

As described in detail in Ellis (1968), if a spacetime exhibits local rotational symmetry in an open neighbourhood of a point P, then the coordinate freedom can be used to describe the local metric in the neighbourhood in (t, r, x, y) coordinates in the following way:

$$\begin{aligned} ds^2 &= -F^2(t,r)dt^2 + X^2(t,r)dr^2 \\ &\quad + Y^2(t,r)[dx^2 + D(x)dy^2] \\ &\quad + g(x)F^2(t,r)[2dt - g(x)dy]dy \\ &\quad - h(x)X^2(t,r)[2dr - h(x)dy]dy. \end{aligned} \tag{5.4.7}$$

We can immediately see that $g(x) = h(x) = 0$ and $D(x) = \sin^2 x$ gives a general spherically symmetric metric which is of LRS class II. However, we have already determined that LRS spacetimes with non-vanishing rotation and spatial twist must be self-similar. As a result thereof, the functions F, X and Y can be written in terms of a single variable $z \equiv t/r$. Hence only self-similar spherically symmetric solutions can be obtained in the limit $g(x) \to 0, h(x) \to 0$ or equivalently $\Omega \to 0, \xi \to 0$. Therefore to study the interior of a rotating, radiating and inhomogeneous star as a first approximation from spherical symmetry, we can start with a self-similar spherically symmetric spacetime and add sufficiently small $g(x)$ and $h(x)$, with respect to some covariant scale in the problem (the Misner Sharp mass of the spherical star for example). Then we can solve the field equations with the matter source that obeys all the restrictions as imposed by Theorem 2 and the energy conditions.

Chapter 6

Discussion

We showed that it is possible to have a Locally Rotationally Symmetric spacetime with non-zero rotation and spatial twist simultaneously if we allow for non-zero and bounded heat flux. We investigated in detail all the covariant geometrical properties of such spacetimes and proved an interesting result: the evolution of all the covariant scalars obeys a single common hyperbolic linear second order partial differential equation. The existence of a spacelike Cauchy surface, where initial Cauchy data can be provided, is guaranteed. It was also shown that these solutions are self-similar as they possess a conformal Killing vector in the $[u, e]$ plane. For an example of spacetimes with conformal Killing vectors, refer to the analysis of Moopanar and Maharaj (2013).

In general, these solutions are neither stationary nor spatially homogeneous. Therefore with suitable equations of state, such as the temperature T as an internal variable in the equations of state for P, Π, and Q, they have the potential to give exact general relativistic models for rotating, dynamic and radiating stellar structures as they definitely have non-zero heat flux in the interior. The radiative heat flux can be prominently seen for neutron stars: in a newly formed neutron star the core temperature is of the order of $10^{11}K - 10^{12}K$, that rapidly drops to $10^{6}K$ within a few years (see Shapiro and Teukolsky (1983) and Miller (2016) for more information). This huge amount of heat transfer from the core to the surface of the neutron stars is a direct result of several processes such as neutrino transmission, electron sound waves coupled with electromagnetic radiation in the superfluid stellar core (Svidzinsky (2003)). Even for the main sequence stars (like the sun), the radiative heat transfer from the core to the convection zone is always present, albeit with an extremely high opacity as indicated by Mitalas and Sills (1992). Therefore while studying the interior of realistic astrophysical stars, we can see that heat flux does play a very important role and assuming a perfect fluid form of matter in these cases may lead to over-simplification. Hence any solution to the Einstein field equations incorporating rotation, spatial twist and heat flux simultaneously is definitely a better candidate to provide a relativistic description of a rotating stellar interior with quadrupole and other higher multipole moments. These may account for physical features of stars that cannot be explained by Newtonian dynamics.

Also, since these spacetimes are shown to be necessarily self-similar, all the field equations can be recasted as ordinary differential equations with the variable $z = t/r$. In that case, the field equations become much simpler. Thus we can perform a dynamical system analysis to find out different properties of these spacetimes. This is an area of ongoing research which is likely to produce new results.

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