ASPECTS OF GRAPH VULNERABILITY

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To Dèsirée, Michelle and Nicole

Preface

The research on which this thesis was based was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, from January 1990 to January 1994, under the supervision of Professor Henda C. Swart and the co-supervision of Dr. Ortrud R. Oellermann.

This thesis represents original work by the author and has not been submitted in any other form to another university. Where use was made of the work of others, it has been duly acknowledged.

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Abstract

This dissertation details the results of an investigation into, primarily, three aspects of graph vulnerability namely, ℓ-connectivity, Steiner Distance hereditariness and functional isolation.

Following the introduction in Chapter one, Chapter two focusses on the ℓ-connectivity of graphs and introduces the concept of the strong ℓ-connectivity of digraphs. Bounds on this latter parameter are investigated and then the ℓ-connectivity function of particular types of graphs, namely caterpillars and complete multipartite graphs as well as the strong ℓ-connectivity function of digraphs, is explored. The chapter concludes with an examination of extremal graphs with a given ℓ-connectivity.

Chapter three investigates Steiner distance hereditary graphs. It is shown that if G is 2-Steiner distance hereditary, then G is k-Steiner distance hereditary for all $k \geq 2$. Further, it is shown that if G is k-Steiner distance hereditary ($k \geq 3$), then G need not be $(k-1)$-Steiner distance hereditary. An efficient algorithm for determining the Steiner distance of a set of k vertices in a k-Steiner distance hereditary graph is discussed and a characterization of 2-Steiner distance hereditary graphs is given which leads to an efficient algorithm for testing whether a graph is 2-Steiner distance hereditary. Some general properties about the cycle structure of k-Steiner distance hereditary graphs are established and are then used to characterize 3-Steiner distance hereditary graphs.

Chapter four contains an investigation of functional isolation sequences of supply graphs. The concept of the Ranked supply graph is introduced and both necessary and sufficient conditions for a sequence of positive non-decreasing integers to be a functional isolation sequence of a ranked supply graph are determined.

Contents

Chapter 1

1.1 Measures of graph vulnerability

Many parameters have been introduced to measure the extent of the damage and disruption caused in a communications network by the loss or failure of vertices or edges in the system.

Earliest investigations of such problems dealt with the connectivity, κ, and edge-connectivity, λ, of a graph which are of overriding importance in cases where the disconnection of the communications network due to vertex or edge failure is deemed to be catastrophic. These parameters have been studied since the twenties and thirties of this century (see [M2] and [W1]) and form the subject of an extensive literature of which we mention but some trends and highlights. Characterizations and properties of n-connected and n-edge connected graphs were obtained (see [M2], [W1], [FF1], [EFS1], [D1], [D2], [T1], [B3], [B1], [B4]).

Relations between κ, λ and other graph-theoretical parameters (p, q, δ, diam, degree sequences) were obtained and the existence of graphs having prescribed values of such parameters was established (in some cases with reference to special classes of graphs, such as line graphs, clique graphs and circulants) (see [W1], [CH1], [H3], [BS2], [BS3], [KQ1], [M3], [M4], [O3], [CS1], [CS2], [BT1], [Z1], [H4], [BT1], [B2], [BT2]). Minimally and

k-critically n-connected graphs were investigated initially in [CKL1], [ES1] and [MS2].

In dealing with practical problems such as the reliability of computer networks, we may be able to assess the probability that vertices or edges will remain operational and to model the network in terms of a probabilistic graph (see[C2] in which an extensive list of references is provided).

If a disconnected graph $G-S$ results when a set S of vertices or edges is removed from a graph G, it is possible that a sufficiently large component of $G-S$ exists to provide a viable communication system. The assessment of this situation requires the introduction of a new parameter which takes into account both the number of elements (vertices or edges) deleted from G and the maximum number of vertices between which communication is still possible:

1. In [BES1] and [BES2] the concepts of the integrity $I(G)$ and edge-integrity $I'(G)$ of a graph G were introduced and initially developed:

   $$I(G) = \min_{S \subset V(G)} \{|S| + m(G-S)\} \text{ and}$$

   $$I'(G) = \min_{S \subseteq E(G)} \{|S| + m(G-S)\},$$

   where $m(G)$ denotes the maximum order of a component of a graph G.

2. We shall not present a survey of the results on integrity and edge integrity that have appeared since 1987 and may be found in [BES1], [BES2], [G2], [GS1], [GS2], [GS3], [GS4], [BBLP1], [BBLP2], [BBLP3], [BBLPS1], [BBLPS2], [BBLP5], [BBLPS3], [BGL1], [BGL2], [CEF1], [FS1], [LSP1] as well as in the survey article [BBGLP1]. However, we

shall briefly introduce several related parameters designed to assess the degree to which other desirable properties are retained after the deletion of vertices or edges from a graph:

3. Let $S \subset V(G)$ and, for each $v \in G - S$, let $p_v(G - S)$ denote the order of the component of $G - S$ that contains v. The *mean integrity* of G, $J(G)$, was defined in [CKMO1]:

$$\begin{aligned} J(G) &= \min_{S \subset V(G)} \left\{ |S| + (1/p(G-S) \sum_{v \in V(G)-S} p_v(G-S) \right\} \\ &= \min_{S \subset V(G)} \left\{ |S| + \sum (p(H))^2 / \sum p(H) \right\}, \end{aligned}$$

where summation takes place over all components H of $G - S$.

4. The pure edge-integrity, $I_p(G)$, of G was defined in [BD1] to be

$$I_p(G) = \min_{S \subseteq E(G)} \{|S| + m_e(G - S)\},$$

where $m_e(G - S)$ denotes the maximum size of any component of $G - S$.

5. The *tenacity* of a graph G, $T(G)$, defined in [CMS1] as

$$T(G) = \min_S \left\{ \frac{|S| + m(G-S)}{k(G-S)} \right\},$$

where the minimum is taken over all vertex cutsets of G, is designed to be used if it is desirable that, after the loss of a cut set of vertices from G, $G - S$ should contain a component of large order and be easily reconnected by virtue of having few components. (See also [MS1] and [CMS1].)

6. The vertex-neighbourhood-integrity of a graph $G, VNI(G)$, is defined in [CW1] by

$$VNI(G) = \min_{S \subseteq V(G)} \{|S| + m(G - N[S])\}$$

(see also [WC1] and [WC2]).

7. Related to edge-integrity is the *honesty,* $h(G)$, of a graph G: A graph G is said to be *honest* if $I'(G) = p(G)$. The smallest number of edges in a subset of $E(\bar{G})$ whose addition to G yields an honest graph is defined to be $h(G)$ (see [BBLP4]).

8. The *toughness* of a graph G, $t(G)$, was defined in [C1] to be

$$t(G) = \min_S \{|S|/k(G - S)\},$$

where the minimum is taken over all cutsets S of G if G is non-complete and $t(K_p) = \infty$. (The definition was slightly altered in [G2] and [GS4], the minimum being taken over all $S \subset V(G)$ with $\kappa(G - S) = 0$, leading to the alteration of $t(K_n)$ from ∞ to $p(G) - 1$.) Toughness is of obvious use in assessing the extent of disruption caused by the removal of vertices from a graph in a situation where it is deemed desirable that the resulting disconnected graph should be easily reconnected or that the number of its components should be so small that the structures represented by them can economically be provided with essential services, etc. (cf. [BES1], [G2] and [GS4] in which relations between toughness and other measures of vulnerability are explored). However, many papers dealing with toughness have appeared since 1973, aimed mainly at establishing links between the toughness of a graph and its cycle structure, inspired by conjectures in [C1]: It was conjectured that a constant c exists such that $t(G) \geq c$ implies hamiltonicity (or pancyclicity) of G, that $t(G) \geq 3/2$ implies the existence of a two-factor in G and that, for any positive integer k such that $kp(G)$ is even, $t(G) \geq kp(G)$ implies the existence of a k-factor in G. Only the last of these conjectures has been proved [EJKS1] and it would be inappropriate to list all references to progress made in investigating the remaining conjectures. The names of authors currently most prominent in this field may be found in [BS1] and [GS4].

9. The binding number of a graph G, $b(G) = \min_S\{|N(S)|/|S|\}$, where the minimum is taken over all nonempty subsets S of $V(G)$ such that $N(S) \neq S$, was defined in [W2] and further investigated in [BES1], [G2], [GS2], [GS4], [KMH1], [WTL1], [W2], [G3] and [C3].

In the following chapters we shall explore further measures to assess the vulnerability of graphs and digraphs to disruption caused by the removal of vertices and edges.

1.2 Graph Theory Nomenclature

The basic text for the graph theory terminology and symbols used here is Chartrand and Lesniak's Graphs and Digraphs (second edition) [CL1]. However, certain clarification of our conventions is necessary.

All graphs considered are 'simple' graphs; i.e. undirected graphs without loops or multiple edges. Further, we use $p = p(G)$ and $q = q(G)$ to denote the order and size respectively of a graph G.

Recall that $G - S$ denotes the graph formed by the removal of a set of vertices S from G, while $\langle S \rangle$ denotes the vertex-induced subgraph of G with vertex set S.
For sets A and B, $[A, B]$ denotes the set of edges which have one end in A and one in B. We also speak of complete n-partite (or complete multipartite graphs) of the form $K_{p_1,p_2,\ldots,p_n} = K(p_1, p_2, \ldots, p_n)$, with the complete bipartite graphs of the form $K_{1,m}$ being called stars. The symbols $\beta(G)$ and $k(G)$ will denote the independence number and the number of components of G respectively.

The contraction of an edge $e = xy$ of a graph G yields the graph denoted $G \cdot e$, defined by removing e and identifying its ends, i.e. replacing x

and y by one vertex, w say, such that w is adjacent to $v \in V(G) - \{x, y\}$ if and only if xv or yv is an edge of G.

Further, we shall use the symbol $\subset$ to denote strict containment in the comparison of sets, $|S|$ to denote the cardinality of the set S, and $\lfloor x \rfloor$ and $\lceil x \rceil$ to denote the integer part and ceiling of x, respectively.

Chapter 2

$\mathcal{L}$-Connectivity

2.1 Introduction

The ℓ-connectivity and ℓ-edge connectivity of a graph G, was first introduced in 1984 by Chartrand, Kapoor, Lesniak and Lick [CKLL1] by generalising the concepts of the connectivity and edge-connectivity of a graph.

It is well known that the connectivity $\kappa(G)$ (edge-connectivity $\lambda(G)$) of a graph G is the minimum number of vertices (edges) whose deletion produces a graph with at least two components or the trivial graph. These two parameters have the advantage that they can be computed efficiently. However, there are situations where the connectivity (edge connectivity) is inadequate as a measure of vulnerability.

For example, the star $K_{1,m}$ and the path P_{m+1} ($m \geq 3$) are both graphs of order $m+1$ and size m that have connectivity 1, but the deletion of a cut vertex from $K_{1,m}$ produces m components whereas the deletion of a cut vertex from P_{m+1} always produces exactly two components. So in some sense $K_{1,m}$ is more vulnerable (or less reliable) than P_{m+1} (for $m \geq 3$). The ℓ-connectivity and ℓ-edge connectivity provide a differentiation between the vulnerability of these graphs.

In particular, for $\ell \geq 2$, the *ℓ-connectivity* $\kappa_\ell(G)$ (*ℓ-edge-connectivity* $\lambda_\ell(G)$) of a graph G of order $p \geq \ell - 1$ is defined as the minimum number of vertices (edges) that are required to be deleted from G to produce a graph with at least ℓ components or with fewer than ℓ vertices. So $\kappa_2(G) = \kappa(G)$ and $\lambda_2(G) = \lambda(G)$. Since the problem of determining whether the independence number $\beta(G)$ of a graph G, of order $p \geq \ell$, is at least ℓ is NP-complete and since $\beta(G) \geq \ell$ if and only if $\kappa_\ell(G) \neq p - \ell + 1$, it follows that the problem of determining whether $\kappa_\ell(G) \neq p-\ell+1$ is NP-complete. A graph is (n,ℓ)-*connected* if $\kappa_\ell(G) \geq n$. So n- connected graphs are the $(n,2)$-connected graphs.

Unfortunately there are no known efficient algorithms for computing $\kappa_\ell(G)$ or $\lambda_\ell(G)$ for a graph G. In [CKLL1] and [O1] sharp bounds for $\kappa_\ell(G)$ are established.

It is well-known that with the aid of Menger's Theorem, Whitney [W1] showed that a graph G is n-connected if and only if for every pair u, v of distinct vertices of G, there exist at least n-internally disjoint $u-v$ paths in G. It was pointed out in [M1] and [O1] that no analogous characterization of (n,ℓ)- connected graphs exists. It is well-known that if G is a graph of order p, and n is an integer such that $1 \leq n \leq p-1$, then if $\delta(G) \geq (p+n-1)/2$, the graph G is n- connected. So for such graphs G, Whitney's theorem implies, that for every pair u, v of vertices of G there exist at least n internally disjoint $u-v$ paths. Hedman [H1] actually showed that for such graphs G and every pair u, v of distinct vertices of G there exist at least n internally disjoint $u-v$ paths each of length at most 2. An analogue of this result is established in [O1]. For a set S of at least two vertices of a graph G an S-*path* is a path between a pair of vertices of S whose internal vertices do not belong to S. Two S- paths are *internally disjoint* if they have no internal vertices in common.

In [O1] it is shown that for a graph G of order $p \geq 2$, and integers $\ell \geq 3$ and n $(1 \leq n \leq p - l + 1)$, if

$$\delta(G) \geq \frac{p + (n-2)(\ell-1)}{\ell}$$

then for each set S of ℓ vertices of G there exist at least n internally disjoint

S-paths each of length at most 2.

In this chapter, we introduce and study the 'ℓ-connectivity' of a digraph making use of the concept of strong connectedness. We will then consider the ℓ-connectivity function of caterpillars and complete multipartite graphs, and then generalise this to define the strong ℓ-connectivity function of a digraph. Lastly, we consider minimal graphs of a given ℓ-connectivity.

2.2 The ℓ-Connectivity of a Digraph

A digraph D is *strongly connected* if for every two vertices u and v of D there exist both a $u-v$ path and a $v-u$ path in D. A *strong component* of a digraph is an induced subdigraph that is strongly connected and that is maximal with respect to this property. It is well-known that the strong components of a digraph partition its vertex set. The *strong independence number* $\beta_s(D)$ of a digraph D is the maximum cardinality of a set S of vertices of D so that the subdigraph $\langle S\rangle$ induced by S is acyclic, i.e., every strong component of $\langle S\rangle$ consists of a single vertex. Such a set S is called a strongly independent set. For example, if T is a transitive tournament of order p, then $\beta_s(T) = p$ and if C_p is a p-cycle, then $\beta_s(C_p) = p-1$ whereas the strong independence number of the complete symmetric digraph K_p^* is 1.

For an integer $\ell \geq 2$, and a digraph D of order p, the *strong ℓ − connectivity* $\kappa_\ell(D)$(*strong ℓ-arc connectivity* $\lambda_\ell(D)$) of D is the minimum number of vertices (arcs) whose deletion from D produces a digraph with at least ℓ strong components or a digraph with at most $\ell-1$ vertices. So $\kappa_\ell(C_p) = 1$ and $\kappa_\ell(K_p^*) = p-\ell+1$ if $p \geq \ell \geq 3$. Further, $\lambda_\ell(C_p) = 1$ and $\lambda_\ell(K_p^*) = (p-\ell+1)(\ell-1) + \binom{\ell-1}{2} = p(\ell-1) - \binom{\ell-1}{2}$ for $p \geq \ell$. Based on the work of Ford and Fulkerson [FF1], [FF2], efficient algorithms for computing the *connectivity*, i.e., $\kappa_2(D) = \kappa(D)$ and the *arc-connectivity* $\lambda(D) = \lambda_2(D)$ of a digraph D have been developed. However, in general no efficient algorithms for computing $\kappa_\ell(D)$ and $\lambda_\ell(D)$ exist. For an integer $n \geq 0$ we say that a digraph D is *strongly* (n, ℓ)*-connected* if $\kappa_\ell(D) \geq n$.

2.2.1 Bounds on the strong ℓ-connectivity of a digraph

Chartrand, Kapoor, Lesniak and Lick [CKLL1] provided the following sufficient condition for a graph to be (n, ℓ)-connected.

Theorem A Let G be a graph of order p with $\beta(G) \geq \ell \geq 2$. If for every vertex v of G

$$\deg v \geq \frac{p + (\ell - 1)(n - 2)}{\ell},$$

then G is (n, ℓ) connected.

This result can be extended to digraphs.

Theorem 2.2.1.1 Let D be a digraph of order $p \geq \ell + n - 1$ with $\beta_s(D) \geq \ell \geq 2$. If for every vertex v of D

$$\deg_D v = id_D v + od_D v > \frac{p(\ell + 1) + n(\ell - 1) - 3\ell + 1}{\ell},$$

then D is strongly (n, ℓ)-connected.

Proof Assume, to the contrary, that D is a digraph that satisfies the hypothesis of the theorem but that is not strongly (n, ℓ)-connected. Since $\beta_s(D) \geq \ell$, there exists a set S of $n - 1$ vertices of D such that $D - S$ has at least ℓ strong components. Thus $D - S$ has a strong component D_1 of order $p_1 \leq \frac{p-n+1}{\ell}$.

For any vertex v of D_1, we note that if v is adjacent to any vertex w in $V(D) - (V(D_1) \cup S)$, then v is not adjacent from w. Hence, since

$|V(D) - (V(D_1) \cup S)| = p - p_1 - n + 1$

$$\begin{aligned}\deg_D v &\leq 2(p_1 - 1) + 2(n - 1) + p - p_1 - n + 1 \\ &= p + p_1 + n - 3 \\ &\leq p + \tfrac{p-n+1}{\ell} + n - 3 \\ &= \tfrac{p(\ell+1)+n(\ell-1)-3\ell+1}{\ell}.\end{aligned}$$

This contradicts our assumption and therefore completes the proof. □

The result of Theorem 2.2.1.1 is best possible as we now show. Let $\ell \geq 2$ be an integer and let m and n be positive integers. For $i = 1, 2, ..., \ell$, let H_i be the complete symmetric digraph of order m. Let $H_{\ell+1}$ be a complete symmetric digraph of order $n - 1$ if $n \geq 2$. If $n = 1$, then let D be obtained from $H_1 \cup H_2 \cup ... \cup H_\ell$ by adding all arcs of the type (x, y) where $x \in V(H_i)$ and $y \in V(H_j)$ and $1 \leq i < j \leq \ell$. If $n \geq 2$, let D be obtained from $H_1 \cup H_2 \cup ... \cup H_{\ell+1}$ by adding every pair of arcs of the type (x, y) and (y, x) where $x \in V(H_{\ell+1})$ and $y \in H_i$ for $1 \leq i \leq \ell$ as well as all the arcs of the type (u, v) where $u \in H_i, v \in H_j$ and $1 \leq i < j \leq \ell$. Then $\kappa_\ell(D) = n - 1$ and since $p = m\ell + n - 1$,

$$\deg_D v \geq (\ell + 1)m + 2n - 4 = \frac{p(\ell + 1) + n(\ell - 1) - 3\ell + 1}{\ell}$$

for all $v \in V(D)$.

In [O1] another sufficient condition for a graph to be (n, ℓ)- connected is established.

Theorem B Let G be a graph of order $p \geq 2$, the degrees d_i of whose vertices satisfy $d_1 \leq d_2 \leq ... \leq d_p$. Suppose n and $\ell \geq 2$ are integers with $1 \leq n \leq p - \ell + 1$. If $d_k \leq k + n - 2 \Rightarrow d_{p-n+1} \geq p - k(\ell - 1)$ for each k such that $1 \leq k \leq \lfloor (p - n + 1)/\ell \rfloor$, then G is (n, ℓ)-connected.

We now provide an extension of Theorem B to digraphs.

Theorem 2.2.1.2 Let D be a digraph of order $p \geq 2$ and let the degrees d_i of the vertices of D satisfy $d_1 \leq d_2 \leq ... \leq d_p$. Suppose n and $\ell \geq 2$ are integers with $1 \leq n \leq p - \ell + 1$. If

$$d_k \leq p + k + n - 3 \Rightarrow d_{p-n+1} \geq 2p - k(\ell - 1) - 1$$

for each integer k such that $1 \leq k \leq \lfloor (p - n + 1)/\ell \rfloor$, then D is strongly (n, ℓ)-connected.

Proof Suppose the strong ℓ-connectivity of D is less than n. Then there is a set S of $n - 1$ vertices such that $D - S$ has either at least ℓ strong components or order less than ℓ. Since $|S| = n - 1 \leq p - \ell$, it follows that $D - S$ has at least ℓ vertices; so $D - S$ has at least ℓ strong components.

Let D_1 be a strong component of $D - S$ of minimum order k. Then $k \leq \lfloor (p - n + 1)/\ell \rfloor$ and so $p + k + n - 3 \leq 2p - k(\ell - 1) - 2 < 2p - k(\ell - 1) - 1$. Each vertex in D_1 has degree at most $p + k + n - 3$ in D; so $d_k \leq p + k + n - 3$. Hence, by the hypothesis, $d_{p-n+1} \geq 2p - k(\ell - 1) - 1$. Let $u \in V(D) - (S \cup V(D_1))$. Then u is non-adjacent to or from each vertex in at least $\ell - 1$ strong components of $D - S$, each of order at least k. Hence $\deg_D u \leq 2(p - 1) - k(\ell - 1) < 2p - k(\ell - 1) - 1$. It follows that S has at least n elements, contrary to our assumtion. $\square$

The digraph following Theorem 2.2.1.1 also serves to illustrate that Theorem 2.2.1.2 is best possible. Further, it is not difficult to see that Theorem 2.2.1.1 follows as a corollary to Theorem 2.2.1.2.

2.2.2 Strong connectivity sequences of digraphs

Let G be a graph of order p. Chartrand, Kapoor, Lesniak and Lick [CKLL1] defined the sequence of numbers $\kappa_2(G), \kappa_3(G), ..., \kappa_p(G)$ as the *sequence of connectivity numbers* of G. They characterized sequences of integers that are connectivity numbers of a graph in the following theorem.

Theorem C A sequence $b_2, b_3, ..., b_p$ of nonnegative integers is the connectivity sequence of a graph G of order p if and only if there exists an integer k such that $b_2 \le b_3 \le ... \le b_k \le b_{k+1}$ and $b_{k+i} = p - (k+i) + 1$ for $i = 1, 2, ..., p-k$. Moreover $k = \beta(G)$.

We now study the analogous concept for digraphs. Let D be a digraph of order p. Then the sequence $\kappa_2(D), \kappa_3(D), ..., \kappa_p(D)$ is called the *sequence of strong connectivity numbers* of D. The following lemma will be useful when characterizing these sequences.

Lemma 2.2.2.1 Let D be a digraph of order $p \ge 2$ and strong independence number $\beta_s(D) = \beta_s$. Then the sequence of strong connectivity numbers has a maximum value $p - \beta_s$ at $k = \beta_s + 1$, i.e. $\kappa_k(D) = p - \beta_s$.

Proof For $1 \le i \le p - \beta_s$ we have $\kappa_{\beta_s+i}(D) = p - (\beta_s + i) + 1$. Clearly the maximum of the subsequence

$$\kappa_{\beta_s+1}(D), \kappa_{\beta_s+2}(D), ..., \kappa_p(D) \text{ is } \kappa_{\beta_s+1}(D) = p - \beta_s.$$

Since the subsequence $\kappa_2(D), \kappa_3(D), ..., \kappa_{\beta_s}(D)$ of the sequence of connectivity numbers is nondecreasing, $\kappa_{\beta_s}(D)$ is the maximum value of this subsequence. Since $\kappa_{\beta_s}(D) \le p - \beta_s$, the lemma now follows. □

Theorem 2.2.2.1 A sequence $b_2, b_3, ..., b_p$ of nonnegative integers can be realized as the sequence of strong connectivity numbers of a digraph of order p if and only if there exists an integer k such that $b_2 \le b_3 \le ... \le b_k \le b_{k+1}$ and $b_{k+i} = p - (k+i) + 1$ for $i = 1, 2, ..., p-k$. Moreover $k = \beta_s(D)$.

Proof Let D be a digraph of order p. Let $b_i = \kappa_i(D)$ for $2 \le i \le p$ and let $k = \beta_s(D)$. Then, by the proof of Lemma 2.2.2.1, $b_2 \le b_3 \le ... \le b_k \le b_{k+1}$ and for $1 \le i \le p-k$, $b_{k+i} = p - (k+i) + 1$.

Suppose now that $b_2, b_3, ..., b_p$ is a sequence of nonnegative integers such that for some k the following conditions are satisfied:

(i) $0 \leq b_i \leq b_{i+1}$ for $2 \leq i \leq k$ and

(ii) $b_{k+i} = p - (k+i) + 1$ for $i = 1, 2, ..., p-k$.

Define a sequence $a_2, a_3, ..., a_{k+1}$ by $a_2 = b_2, a_3 = b_3 - b_2, a_4 = b_4 - b_3, ..., a_{k+1} = b_{k+1} - b_k$. For $2 \leq i \leq k+1$ let H_i be the complete symmetric digraph of order a_i if $a_i \geq 1$. For convenience we will assume that if $a_i = 0$, then H_i has no vertices and edges. Let K denote the complete symmetric digraph of order $p - k - \sum_{i=2}^{k+1} a_i = p - k - b_{k+1}$ and let H be the symmetric join of $H_2, H_3, ..., H_{k+1}$ and K. Now let $S = \{v_2, v_3, ..., v_{k+1}\}$ and construct a digraph D by joining each $v_r \in S$ by a symmetric pair of arcs to each vertex in $\cup_{i=2}^{r} H_i$. The order of D is p. Since S is a strongly independent set and since $H = \langle V(D) - S \rangle$ is a complete symmetric digraph of order $p - k$ and as each vertex of S is joined to at least one vertex of H by a symmetric pair of arcs, $\beta_s(D) = |S| = k$.

For $r = 2, 3, ..., k+1$, let $U_r = \cup_{i=2}^{r} V(H_i)$ and observe that the number of strong components of $D - U_r$ is at least r. Thus $\kappa_r(D) \leq |U_r| = \sum_{i=1}^{r} a_i = b_r$ for $2 \leq i \leq r+1$. By a straightforward inductive argument it can be shown if S is a set of vertices that does not contain all the vertices of U_r, then $D - S$ has at most $r-1$ strong components. So $\kappa_r(D) \geq |U_r| = b_r$. Thus $\kappa_i(D) = b_i$ for $i = 2, 3, .., k+1$. Further, $\kappa_{k+i}(D) = p - (k+i) + 1$ for $1 \leq i \leq p-k$. Hence $b_\ell = p - \ell + 1 = \kappa_\ell(D)$ for $\ell = k+1, k+2, ..., p$. Thus $b_2, b_3, ..., b_p$ is the sequence of strong connectivity numbers of D and $\beta_s(D) = k$. □

Even though the connectivity and arc-connectivity of a digraph are easily computable measures of reliability of a network the strong connectivity sequence of a digraph provides more information on the reliability of a network. In particular if D_1 and D_2 are two digraphs with the same strong connectivity and $k_i = \max\{\ell | \kappa_\ell(D_i) = \kappa(D_i)\}$, then D_1 can be considered to be more reliable than D_2 if $k_1 < k_2$.

2.3 The ℓ-connectivity function of certain classes of graphs

The problem of disconnecting a graph into at least two components by the deletion of both vertices and edges was first considered by Beineke and Harary [BH1]. These concepts were extended in [O2]. Let G be a graph with ℓ-connectivity $\kappa_\ell = \kappa_\ell(G)$. If $k \in \{0, 1, ..., \kappa_\ell(G)\}$, then let s_k be the minimum ℓ-edge-connectivity among all subgraphs obtained by removing k vertices from G. The *ℓ-connectivity function* of G is defined by $f_\ell(k) = s_k$ for $0 \leq k \leq \kappa_\ell(G)$. So for $\ell = 2$, the ℓ- connectivity function of a graph is its connectivity function, which has been characterized by Beineke and Harary [BH1]. For $\ell \geq 3$ no characterizations of the ℓ-connectivity function of a graph are known and it appears to be a difficult problem to characterize such functions. In [O2] several necessary conditions for a function to be an ℓ-connectivity function of a graph are established and the ℓ-connectivity function of the complete graph is derived. We study here the ℓ- connectivity function of certain types of trees and the complete n-partite graphs.

2.3.1 Caterpillars and complete Multipartite graphs

In [O2] the following formula for the ℓ-connectivity function of a complete graph is established.

Theorem D Let $p, \ell \geq 2$ be integers with $p \geq \ell$ and suppose that $G \cong K_p$. Then the ℓ- connectivity function of G is given by

$$f_\ell(k) = \begin{cases} 0 & \text{if } k = \kappa_\ell(G) \\ (\ell-1)(p-\ell-k+1) + \binom{\ell-1}{2} & \text{for } 0 \leq k < \kappa_\ell(G). \end{cases}$$

We now extend this result to complete n-partite graphs.

Theorem 2.3.1.1 Suppose $G \cong K_{m_1, m_2, ..., m_n}$ where $m_1 \leq m_2 \leq ... \leq m_n$ and $n \geq 2$. Let $p = \sum_{i=1}^{n} m_i$ and let k be an integer with $0 \leq k \leq \kappa_\ell(G)$.

If $s = \min\{m_{n-1}, \sum_{i=1}^{n-1} m_i - k\}$, then the ℓ-connectivity function of G is given by

$$f_\ell(k) = \begin{cases} 0 & \text{if } k = \kappa_\ell(G) \\ (\ell-1)(p-m_n-k) & \text{if } k \neq \kappa_\ell(G) \text{ and } \ell \leq m_n - s + 2 \\ (\ell-1)(p-m_n-k) - \binom{\ell-m_n+s-1}{2} & \text{if } k \neq \kappa_\ell(G) \text{ and } \ell > m_n - s + 2. \end{cases}$$

To prove this result we begin by establishing a series of lemmas.

Lemma 2.3.1.1 Let $G = K_{r_1,r_2,\ldots,r_t}$ be a complete t-partite graph $(t \geq 2)$ of order p and let ℓ be an integer, $2 \leq \ell \leq p$. There exists a set of $\lambda_\ell(G)$ edges of G, say E_ℓ, such that $G - E_\ell$ has ℓ components, at most one of which is non- trivial.

Proof: Let $V_1, V_2, \ldots, V_t$ be the partite sets of G with $|V_i| = r_i$ for $i = 1, 2, \ldots, t$. There exists a set F_ℓ of $\lambda_\ell(G)$ edges of G such that $G - F_\ell$ has ℓ components. Of all such sets F_ℓ let E_ℓ be one such that $G - E_\ell$ has as few non-trivial components as possible. We shall show that $G - E_\ell$ has at most one non-trivial component.

Assume, to the contrary, that $G - E_\ell$ has at least two non- trivial components, G_1 and G_2, with $V(G_1) = A$ and $V(G_2) = B$. For $i = 1, 2, \ldots, t$, let $A \cap V_i = A_i$, $B \cap V_i = B_i$, $|A_i| = a_i$ and $|B_i| = b_i$. Then there exist $i_1, i_2, j_1, j_2 \in \{1, 2, \ldots, t\}$ such that $i_1 \neq i_2, j_1 \neq j_2$ and $a_{i_1}, a_{i_2}, b_{j_1}, b_{j_2} \geq 1$. Letting $H = \langle A \cup B \rangle_G$, we note that for $v \in A_i \cup B_i$ $(i \in \{1, 2, \ldots, t\})$

$$\deg_H v = a + b - a_i - b_i \,. \tag{2.1}$$

Furthermore, the set $[A, B]$ of all edges in H with one end vertex in A, the other in B, has cardinality

$$|[A, B]| = \sum_{i=1}^{t} a_i(b - b_i) = \sum_{i=1}^{t} b_i(a - a_i). \tag{2.2}$$

It follows from our choice of E_ℓ that isolating a single vertex of H requires the removal of more edges than separating the components G_1 and G_2 in

H; i.e., for $v \in V(H), \deg_H v > |[A, B]|$. Hence, for every $i \in \{1, 2, ..., t\}$ such that $a_i + b_i \geq 1$,

$$2\sum_{\substack{j=1\\ j\neq i}}^{t}(a_j + b_j) = 2(a + b - a_i - b_i) > \sum_{j=1}^{t} a_j(b - b_j) + \sum_{j=1}^{t} b_j(a - a_j). \quad (2.3)$$

Assuming (without loss of generality) that $a_t + b_t \geq 1$, we obtain from (2.3) with $i = t$

$$\sum_{j=1}^{t-1} a_j(b - b_j - 2) + \sum_{j=1}^{t-1} b_j(a - a_j - 2) + a_t(b - b_t) + b_t(a - a_t) < 0. \quad (2.4)$$

Since $a - a_j, b - b_j \geq 1$ for all $j \in \{1, 2, ..., t\}$, it follows from (2.4) that there exists $j \in \{1, 2, ..., t-1\}$ such that $a_j \geq 1$ and $b - b_j - 2 < 0$ or $b_j \geq 1$ and $a - a_j - 2 < 0$; say $b_1 \geq 1$ and $a - a_1 < 2$. Then $a - a_1 = 1$ and there exists $m \in \{2, 3, ..., t\}$ such that $a_m = 1$ and $a_j = 0$ for all $j \in \{2, 3, ..., t\} - \{m\}$. We note that $a_1 \geq 1$.

Since $|[A, B]| < \deg_H v$ for $v \in A$, it follows from (2.1) and (2.2) that

$$a_1(b - b_1) + a_m(b - b_m) < a - a_1 + b - b_1 = 1 + b - b_1;$$

hence

$$(a_1 - 1)(b - b_1) + b - b_m < 0$$

which, with $a_1 - 1 \geq 0, b - b_1 \geq 1, b - b_m \geq 1$, yields a contradiction, thus establishing the validity of the lemma. $\square$

For a vertex v in a graph G, let the set of edges of G incident with v be denoted by $E_G(v)$.

Lemma 2.3.1.2 Let $G = K_{r_1, r_2, ..., r_t}$ with $r_1 \leq r_2 \leq ... \leq r_t$, $t \geq 2$, $p = p(G) = \sum_{i=1}^{t} r_i$ and $\ell \in \{2, 3, ..., p\}$. Let $V_1, V_2, ..., V_t$ be the partite sets of G with $|V_i| = r_i$. The following algorithm yields a set E_ℓ of edges of G such that $|E_\ell| = \lambda_\ell(G)$ and $G - E_\ell$ has ℓ components, at least $\ell - 1$ of which are trivial:

1. Let $H_1 = G$ and let v_1 be a vertex of minimum degree in H_1. (i.e., $v_1 \in V_t$). Let $E_2 = E_{H_1}(v_1)$ and $H_2 = H_1 - v_1$.

2. For $i \in \{2, ..., \ell - 1\}$, let v_i be a vertex of minimum degree in H_i and let $E_{i+1} = E_{H_i}(v_i) \cup E_i$, $H_{i+1} = H_i - v_i$.

Proof: The validity of the lemma for $\ell = 2$ is an immediate consequence of Lemma 2.3.1.1. Further, the lemma follows if $\ell = p$, in which case $|E_\ell| = q(G) = \lambda_p(G)$. Suppose that the lemma does not hold and let m be the smallest value of ℓ for which the algorithm yields a set E_ℓ that does not satisfy the requirements of the lemma; so $2 < m < p$. Since $G - E_m$ certainly contains m components, $m-1$ of which are trivial, it follows that $|E_m| > \lambda_m(G)$. Let F_m be a set of edges of G such that $|F_m| = \lambda_m(G)$, $G - F_m$ contains m components of which $m-1$ are trivial.

Let $W = \{w_1, w_2, ..., w_{m-1}\}$ denote the set of $m-1$ isolated vertices in $G - F_m$ and, for $w_k \in W$ let $G_k = G - (W - \{w_k\})$. Let $i = i(F_m)$ be such that $v_1, ..., v_{i-1} \in W$ and $v_i \notin W$. Choose F_m such that $i(F_m)$ is as large as possible. Suppose $v_s = w_s$ for $1 \leq s \leq i-1$. Let $W' = W - \{v_1, ..., v_{i-1}\}$ and let $v_i \in V_j$; then $V_j \cap W' = \emptyset$, since otherwise, if $w_k \in V_j \cap W'$, the set of edges of F_m incident with w_k in G_k, namely $E_{G_k}(w_k)$, may be replaced by $E_{G_k}(v_i)$ to yield a set F'_m of edges of G with $|F'_m| = \lambda_m(G)$ such that $G - F'_m$ has m components, $m-1$ of which are trivial and $i(F'_m) > i(F_m)$, contrary to our choice of F_m. Hence the only vertices which are adjacent to v_i in H_i and not to v_i in G_{m-1} are those in $W' - \{w_{m-1}\}$. Consequently $\deg_{G_{m-1}} v_i = \deg_{H_i} v_i - (m-2-i+1)$. Furthermore, $\deg_{G_{m-1}} w_{m-1} \geq \deg_{H_i} w_{m-1} - (m-2-i+1)$; so, since $\deg_{H_i} w_{m-1} \geq \deg_{H_i} v_i$, it follows that $\deg_{G_{m-1}} w_{m-1} \geq \deg_{G_{m-1}} v_i$. Hence, replacing the subset $E_{G_{m-1}}(w_{m-1})$ of F_m by $E_{G_{m-1}}(v_i)$, we obtain a set F''_m of edges of G with $|F''_m| \leq |F_m| = \lambda_m(G)$ such that $G - F''_m$ has m components, $m-1$ of which are trivial, and $i(F''_m) > i(F_m)$.

Thus the validity of the lemma is established. □

Let $G = K_{m_1,m_2,...,m_n}$ with $m_1 \leq m_2 \leq ... \leq m_n (n \geq 2)$ and partite sets $V_1, ..., V_n$, where $|V_i| = m_i$ for $i = 1, 2, ..., n; p = \sum_{i=1}^{n} m_i$. Let S be a proper subset of $V(G)$ such that $|S| = k \in \{0, 1, ..., \kappa_\ell(G)\}$ and $k < p - m_n$, where we note that

$$\kappa_\ell(G) = \begin{cases} p - m_n = \sum_{i=1}^{n-1} m_i & \text{if } \ell \leq \beta(G) = m_n \\ p - \ell + 1 & \text{if } \ell > m_n, \end{cases};$$

then $G - S$ is a complete multipartite graph, say $K_{r_1,...,r_t}$.It is an immediate consequence of Lemma 2.3.1.2 that S may be chosen to yield $G - S$ of minimum ℓ-edge connectivity, namely $\lambda_\ell(G - S) = f_\ell(k)$, by letting S consist of k vertices of maximum degree in G, i.e., for some $j \in \{1, 2, ..., m - 1\}$, $S = \cup_{i=1}^{j} V_i'$, where $V_i' = V_i$ if $i < j$ and $V_j' \subseteq V_j$. Then $E_\ell \subseteq E(G - S)$ may be obtained as prescribed by Lemma 2.3.1.2 to produce $G - S - E_\ell$ containing ℓ components, $\ell - 1$ of which are trivial.

If $\ell > m_n$ or $k = p - m_n$, then $f_\ell(k) = 0$, obviously. Hence we have the following lemma:

Lemma 2.3.1.3 If $G = K_{m_1,...,m_n}$ with $m_1 \leq m_2 \leq ... \leq m_n (n \geq 2)$, and partite sets $V_1, ..., V_n$ such that $|V_i| = m_i$ for $i = 1, ..., n$, then, for $2 \leq \ell \leq p$ and $0 \leq k \leq \kappa_\ell$, there exist $S \subseteq V(G)$ and $E_\ell \subseteq E(G - S)$ such that $|S| = k, |E_\ell| = f_\ell(k)$, and such that $G - S - E_\ell$ contains at least ℓ components, at least $\ell - 1$ of which are trivial and, for some $j \in \{1, ..., n\}, S = \cup_{i=1}^{j} V_i'$, where $V_i' = V_i$ for $i < j$ and $V_j' \subseteq V_j$.

Proof of Theorem 2.3.1.1 Clearly if $k = \kappa_\ell(G)$, then $f_\ell(k) = 0$. If $\ell \leq m_n - s + 2$ then, since the degrees of vertices in V_{n-1} exceed those of vertices in V_n by $m_n - s$ in $G - S$, the $\ell - 1$ vertices isolated in $G - S - E_\ell$ occur in V_n. (We note that, for $i \in \{1, ..., \ell - 2\}$, if $w \in V_{n-1} - S$ and $z \in V_n$, then, in $G - S - \{v_1, ..., v_i\}$, deg $w \geq$ deg z.) In this case it is obvious that $|E_\ell| = (\ell - 1)(p - m_n - k)$.

If $\ell > m_n - s + 2$ then, applying the algorithm in Lemma 2.3.1.2 to $G - S$, we note that $v_1, \dots, v_{m_n - s+1}$ may be chosen from V_n and that their isolation requires the removal of $(p - m_n - k)(m_n - s + 1)$ edges. The isolation of $v_{m_n-s+2}, \dots, v_{\ell-1}$ requires the removal, successively of $p - m_n - k - 1, p - m_n - k - 2, \dots, [(p - m_n - k) - (\ell - m_n + s - 2)]$ edges. Hence, in this case,

$$\begin{aligned} |E_\ell| &= (p - m_n - k)(m_n - s + 1) + \sum_{i=1}^{\ell - m_n + s - 2} (p - m_n - k - i) \\ &= (\ell - 1)(p - m_n - k) - \binom{\ell - m_n + s - 1}{2} \text{ if } k \neq \kappa_\ell(G). \end{aligned}$$

□

It is not difficult to see that Theorem D follows as a corollary to Theorem 2.3.1.1.

We next turn our attention to the ℓ-connectivity function of caterpillars. Recall that a caterpillar is a tree that is either isomorphic to K_1 or K_2 or has the property that if its end- vertices are deleted, then a path is produced. For a graph G of order p and an integer k, $0 \leq k < p$, let $c_k(G)$ be the maximum number of components that are produced when k vertices are deleted from G. Note that if $\ell \geq 2$ is an integer and T is a tree with independence number $\beta(T) \geq \ell$, then $f_\ell(k) = (\ell - 1) - c_k(T)$ for $0 \leq k < \kappa_\ell(T)$. Let $\delta_\beta(T) = \min\{k | c_k(T) = \beta(T)\}$. The following algorithm finds for a given caterpillar T and every k, $0 \leq k \leq \delta_\beta(T)$, a set V_k of k vertices such that $k(T - V_k) = c_k$.

Algorithm 1 Let $T \ncong K_1$, K_2 be a caterpillar.

1. (a) $F_0 \leftarrow T$.
   (b) $V_0 \leftarrow \emptyset$.
   (c) $S_0 \leftarrow \{v \in V(F_0) | \deg_{F_0} v = \Delta(F_0)\}$
   (d) $H_0 \leftarrow \langle S_0 \rangle_{F_0}$
   (e) $n \leftarrow 0$
   (f) Let $P : u_1, u_2, \dots, u_r$ be the path produced by deleting the end-

vertices of T.

2. Let $T_1, T_2, ..., T_s$ be the components of H_n and $a_i = \left\lceil \frac{p(T_i)}{2} \right\rceil$.
Let $U_n = \{w_1^n, w_2^n, ..., w_{\beta_n}^n\}$ be a maximum independent set of vertices of H_n (with $\beta_n = \beta(H_n)$) chosen as follows: The vertices $w_1^n, w_2^n, ..., w_{a_1}^n$ belong to T_1. If $s > 1$, then for $i = 2, .., s$, the vertices $w_{a_1+...+a_{i-1}+1}^n, ...,$ $w_{a_1+....+a_{i-1}+a_i}^n$ belong to T_i and if $w_m^n = u_{i_1}$ and $w_r^n = u_{i_2}$ belong to some T_i and $m < r$, then $i_1 < i_2$. Further, $w_{a_1+...+a_{i-1}+1}^n$ is an end-vertex of T_i for $2 \leq i \leq s$ and w_1^n is an end-vertex of T_1.

3. (a) $F_{n+1} \leftarrow F_n - U_n$.
(b) $n \leftarrow n + 1$
(c) $S_n \leftarrow \{v \in V(F_n) | \deg_{F_n} v = \Delta(F_n)\}$
(d) $H_n \leftarrow \langle S_n \rangle_{F_n}$
(e) If $\Delta(F_n) > 1$, return to Step 2; otherwise let $\delta_\beta \leftarrow \sum_{i=1}^{n-1} |U_i|$ and continue.

4. For $k = 1, 2, ..., \delta_\beta$ let $v_1, v_2, ..., v_k$ denote, in order, the first k vertices in the sequence $w_1^1, w_2^1, ..., w_{a_1}^1, w_1^2, ..., w_{a_2}^2, ...,$ and define $V_k = \{v_1, v_2, ..., v_k\}$.

Theorem 2.3.1.2 Suppose Algorithm 1 is applied to a caterpillar $T \ncong K_1$ or K_2. Then

$$k(T - V_k) = c_k(T) \qquad \text{for } 0 \leq k \leq \delta_\beta.$$

Proof: Suppose the theorem does not hold. Let k be the smallest integer such that $k(T - V_k) < c_k$. Let $Z = \{z_1, z_2, ..., z_k\} \subseteq V(T)$ such that $k(T - Z) = c_k$. If $v_1 \in Z$, let j be the smallest integer such that $v_{j+1} \notin Z$, otherwise let $j = 0$. Among all sets $Z \subseteq V(T)$ satisfying $k(T - Z) = c_k$, choose Z such that j is as large as possible. For $i = 1, 2, ..., k$, let $Z_i = Z - \{z_i\}$ and suppose the vertices of Z have been labelled in such a way that if $j \geq 1$, then $z_s = v_s$ for $1 \leq s \leq j$. By our choice of Z, it follows that for $i = j+1, j+2, ..., k$ the vertex z_i cannot be replaced by v_{j+1} in Z to form $Z_i' = Z_i \cup \{v_{j+1}\}$ with $k(T - Z_i') = c_k$. Hence

$$\deg_{T-Z_i} v_{j+1} < \deg_{T-Z_i'} z_i.$$

However,

$$\deg_{T-\{v_1,v_2,...,v_j\}} v_{j+1} \geq \deg_{T-\{v_1,v_2,...,v_j\}} z_i.$$

Therefore v_{j+1} has a neighbour in $\{z_{j+1}, ..., z_k\} - \{z_i\}$, say z_m is such a neighbour. Similarly, v_{j+1} has a neighbour in $\{z_{j+1}, ..., z_k\} - \{z_m\}$; say z_n.

Note that every vertex of $Z \cup V_k$ lies on the path P described in Step 1 (f). Let a and b be neighbours different from v_{j+1} of z_m and z_n, respectively. We show next that $a, b \notin Z$. Suppose $v_{j+1} \in U_t$. Then $\deg_{F_t} v_{j+1} \geq \deg_{F_t} z_m$. Suppose $a \in Z$. Then a lies on P. Therefore

$$k(T - (Z_m \cup \{v_{j+1}\})) \geq k(T - Z) = c_k,$$

which contradicts our choice of Z. So $a \notin Z$, and similarly $b \notin Z$.

Suppose $\deg_{F_t} z_m < \deg_{F_t} v_{j+1} = \triangle(F_t)$. Then once again it follows that

$$k(T - (Z_m \cup \{v_{j+1}\})) \geq k(T - Z),$$

which contradicts our choice of Z. Hence $\deg_{F_t} z_m = \triangle(F_t)$. Similarly $\deg_{F_t} z_n = \triangle(F_t)$. If $\deg_{F_t} a$ and $\deg_{F_t} b$ are less than $\triangle(F_t)$, then z_m and z_n are end vertices of a component of H_t, which contradicts our choice of V_k. Hence $\deg_{F_t} a = \triangle(F_t)$. If $\deg_{F_t} b < \triangle(F_t)$, then by the choice of V_k it follows since z_n is an end vertex of a component of H_t, not in V_k, a must be v_j. This is impossible since $a \notin Z$. Otherwise, if $\deg_{F_t} b = \triangle(F_t)$, then a or b is v_j which once again produces a contradiction. This completes the proof of the validity of Algorithm 1. □

With the aid of Algorithm 1 and Theorem 2.3.1.2 we are now able, in the next two theorems, to characterize the ℓ-connectivity functions of caterpillars.

Theorem 2.3.1.3 For an integer $\ell \geq 2$, a function $f_\ell : \{0, ..., \kappa_\ell\} \to \mathbf{N} \cup \{0\}$ is the ℓ-connectivity function of a caterpillar with independence number at least ℓ if and only if

(i) f_ℓ is decreasing,

(ii) $f_\ell(0) = \ell - 1$, and $f_\ell(\kappa_\ell) = 0$, and

(iii) if $\kappa_\ell \geq 2$, then $f_\ell(k) - f_\ell(k+1) \geq f_\ell(k+1) - f_\ell(k+2)$ for $0 \leq k < \kappa_\ell - 2$.

Proof: Suppose first that f_ℓ is the ℓ-connectivity function of a caterpillar T. Then $f_\ell(k) = \ell - c_k(T)$ for $0 \leq k < \kappa_\ell(G) = \kappa_\ell$. Since $c_k(T) < c_{k+1}(T)$ for $0 \leq k < \kappa_\ell$, it follows that f_ℓ is decreasing. Since every edge of a tree is a bridge, $\ell - 1$ edges must be deleted from a tree to produce ℓ components. Hence $f_\ell(0) = \ell - 1$. Since $\ell \leq \beta(T)$, it follows that there exists a set of $\kappa_\ell(T)$ vertices whose deletion produces a graph with at least ℓ components. Hence $f_\ell(\kappa_\ell) = 0$. Hence (ii) holds.

Observe that if $\kappa_\ell \geq 2$, then $f_\ell(k) - f_\ell(k+1) = c_{k+1}(T) - c_k(T)$ and $f_\ell(k+1) - f_\ell(k+2) = c_{k+2}(T) - c_{k+1}(T)$. Let $v_1, v_2, ...$ be as in Step 4 of Algorithm 1. Suppose $v_{k+1} \in U_r$ and $v_{k+2} \in U_s$. Then $r \leq s \leq r+1$ and $\deg_{F_r} v_{k+1} \geq \deg_{F_s} v_{k+2}$. Since $c_{k+1}(T) - c_k(T) = \deg_{F_r} v_{k+1} - 1$ and $c_{k+2}(T) - c_{k+1}(T) = \deg_{F_s} v_{k+2} - 1$, condition (iii) follows.

For the converse suppose that $f_\ell : \{0, ..., \kappa_\ell\} \to \mathbf{N} \cup \{0\}$ is a function that satisfies conditions (i), (ii) and (iii) of Theorem 2.3.1.3. Construct a caterpillar T as follows. Begin with a path $v_1, u_1, v_2, u_2, ..., u_{\kappa_\ell - 1}, v_{\kappa_\ell}$. Next join $f_\ell(0) - f_\ell(1)$ new vertices to v_1 and for $2 \leq i \leq v_{\kappa_\ell - 1}$ join $f_\ell(i-1) - f_\ell(i) - 1$ new vertices to v_i. Finally join $f_\ell(\kappa_\ell - 1) - f_\ell(\kappa_\ell)$ new vertices to v_{κ_ℓ}. Let T be the resulting caterpillar. Then it can be shown that T has independence number at least ℓ and its ℓ-connectivity function is f_ℓ. □

The next result characterizes ℓ-connectivity functions of caterpillars whose independence numbers are less than ℓ.

Theorem 2.3.1.4 For an integer $\ell \geq 2$ a function $f_\ell : \{0, 1, ..., \kappa_\ell\} \rightarrow \mathbb{N} \cup \{0\}$ is the ℓ-connectivity function of a caterpillar T of order $p \geq \ell$, independence number $\beta = \beta(T) < \ell$ and $m = \delta_\beta(T)$ if and only if

(i) $f_\ell(0) = \ell - 1,\ f_\ell(\kappa_\ell) = 0,$

(ii) $f_\ell(k+1) < f_\ell(k)$ for $0 \leq k \leq m-1$ and $f_\ell(m) = f_\ell(m+1) = ... = f_\ell(\kappa_\ell - 1) = \ell - \beta$.

(iii) $f_\ell(k) - f_\ell(k+1) \geq f_\ell(k+1) - f_\ell(k+2)$ for $0 \leq k < \kappa_\ell - 2$,

(iv) (a) if $f_\ell(m-1) - f_\ell(m) > 1$, then $m < \kappa_\ell \leq 2m - f_\ell(m) + 2$, otherwise (b) let s be the largest positive integer such that $f_\ell(t) - f_\ell(t+1) = 1$ for $m - s \leq t \leq m-1$, then $m < \kappa_\ell \leq 2m - f_\ell(m) - s + 2$.

Proof: Suppose f_ℓ is the ℓ-connectivity function of a caterpillar with independence number $\beta = \beta(T)$ and $m = \delta_\beta(T)$. Then condition (i) clearly holds. As in Theorem 2.3.1.3 $f_\ell(k) = \ell - c_k(T)$ for $0 \leq k < \kappa_\ell$. Since $c_k(T) < c_{k+1}(T)$ for $0 \leq k < \delta_\beta(T) = m$ it follows that $f_\ell(k+1) < f_\ell(k)$ for $0 \leq k \leq m-1$. Since $c_k(T) = \beta$ for $m = \delta_\beta(T) \leq k \leq \kappa_\ell - 1$, $f_\ell(m) = f_\ell(m+1) = ... = f_\ell(\kappa_\ell - 1) = \ell - \beta$. Hence condition (ii) holds.

Since $f_\ell(k+1) - f_\ell(k+2) = 0$ and $f_\ell(k) - f_\ell(k+1) \geq 0$ for $m - 1 \leq k < \kappa_\ell - 2$, condition (iii) holds for $m - 1 \leq k < \kappa_\ell - 2$. Suppose now that $0 \leq k \leq m-2$. Then, as in the proof of Theorem 2.3.1.3, $f_\ell(k) - f_\ell(k+1) \geq f_\ell(k+1) - f_\ell(k+2)$. Thus condition (iii) holds.

Let m_1 be the smallest integer so that if S consists of the first m_1 vertices selected by Algorithm 1, then the components of $T - S$ are all paths. (Note possibly $m_1 = m$.) For each of the $m - m_1$ vertices $v_i \in \{v_{m_1+1}, ..., v_m\}$ removed next by the algorithm there exists a vertex w_i isolated by the removal of v_i. Let P be a longest path in T. Let $T_0 = T$ and for $i = 1, 2, ..., m_1 - 1$ let $T_i = T - \{v_1, ..., v_i\}$. Observe that if vertex v_j is deleted from T_{j-1} $(1 \leq j \leq m_1)$, the number of components is increased by $f_\ell(j-1)-f_\ell(j)$. Hence at least $f_\ell(j-1)-f_\ell(j)-1$ vertices not on P are isolated in the process. Let there be k vertices v_j for which $f_\ell(j-1)-f_\ell(j)$ vertices not on P are isolated when v_j is deleted from T_{j-1}. Then v_j is adjacent with a vertex from the set $\{v_1, v_2, ..., v_{j-1}\}$. Thus there are exactly $\sum_{j=1}^{m_1}(f_\ell(j-1)-f_\ell(j)-1)+k = f_\ell(0)-f_\ell(m_1)-m_1+k$ vertices of T not on P. Let S_1 denote the set of these vertices and $S_2 = \{v_1, v_2, ..., v_{m_1}\}$. Further, let $S_3 = \{v_{m_1+1}, v_{m_1+2}, ..., v_m\} \cup \{w_{m_1+1}, w_{m_1+2}, ..., w_m\}$. Note that each component of $T_m = T - \{v_1, v_2, ..., v_m\}$ is isomorphic to K_1 or K_2. Let S_4 be the set of vertices that belong to components isomorphic to K_2 in T_m. Then $|S_4| \leq 2(m_1+1-k)$. To see this note that the deletion of the vertices of S_2 from T produces a tree with at most m_1+1-k nontrivial components. If Algorithm 1 is now applied to $T-S_2$ to delete the next $m-m_1$ vertices and thus to produce T_m, each of the nontrivial components of $T-S_2$ corresponds to at most one K_2 of T_m. Thus

$$\begin{aligned} p &= |S_1|+|S_2|+|S_3|+|S_4| \\ &\leq f_\ell(0)-f_\ell(m_1)-m_1+k+m_1+2(m-m_1)+2(m_1+1-k) \\ &= 2m-f_\ell(m_1)+2. \end{aligned}$$

Since $\kappa_\ell = p-\ell+1 = p-f_\ell(0)$, it follows that $\kappa_\ell \leq 2m-f_\ell(m_1)+2$. Clearly $m<\kappa_\ell$. Now if $f_\ell(m-1)-f_\ell(m)>1$, then $m_1=m$ so that (iv) (a) follows. Otherwise, $s=m-m_1$ and $f_\ell(m_1)=f_\ell(m)+s$. Hence, in this case, $\kappa_\ell \leq 2m-f_\ell(m)-s+2$; thus (iv) (b) follows.

For the converse suppose $f_\ell : \{0, 1, ..., \kappa_\ell\} \to \mathbf{N} \cup \{0\}$ is a function that satisfies conditions (i) - (iv). Let $p = \kappa_\ell + f_\ell(0)$. Let $P : u_1, v_1, u_2, v_2, ..., u_m, v_m, u_{m+1}$. Join v_i to $f_\ell(i-1)-f_\ell(i)-1$ new vertices for $1 \leq i \leq m$ and let T' be the resulting caterpillar. Observe that the caterpillar constructed thus far has order $f_\ell(0)-f_\ell(m)+m+1$. Since $f_\ell(m) \geq 1$ it follows by (iv) that $p' = p-(f_\ell(0)-f_\ell(m)+m+1) = \kappa_\ell - m + f_\ell(m) - 1 \geq 0$. If $p'=0$, then it can be shown that $T=T'$ has f_ℓ as its ℓ-connectivity function and independence number β and $\delta_\beta(T)=m$. If $p'>0$, then $p' \leq m+1$ if $f_\ell(m-1)-f_\ell(m)>1$ and $p' \leq m-s+1$ if $f_\ell(m-1)-f_\ell(m)=1$. Suppose first that $f_\ell(m-1) - f_\ell(m) > 1$. In this case, if $p' \leq m$, subdivide the edges u_iv_i exactly once for $1 \leq i \leq p'$ to obtain T; otherwise subdivide the edges u_iv_i for $1 \leq i \leq m$ and the edge v_mu_{m+1} exactly once to obtain T. Suppose now that $f_\ell(m-1) - f_\ell(m) = 1$. Now subdivide the edges u_iv_i exactly once ($1 \leq i \leq p'$) to obtain T. In both cases it can be seen that the corresponding f_ℓ is the ℓ-connectivity function of T. □

The complex characterizations of the ℓ-connectivity functions of caterpillars given in Theorems 2.3.1.3 and 2.3.1.4 lead one to believe that the problem of characterizing the ℓ- connectivity functions of trees in general is a difficult task. It also remains an open problem to characterize the ℓ-connectivity functions of the n-cube.

2.4 The strong ℓ-connectivity function of a digraph

Let G be a graph with connectivity κ. The function $f : \{0, 1, ..., \kappa\} \to \mathbf{N} \cup \{0\}$ defined by $f(r) = \ell_r$ where ℓ_r is the minimum edge-connectivity among all subgraphs of G obtained by deleting r vertices, $0 \leq r \leq \kappa$, is called the *connectivity function* of G. Beineke and Harary [BH1] characterized the connectivity functions of graphs in the following theorem.

2.4.1 Generalisation from Graphs

Theorem E Let κ be a positive integer. A function $f : \{0, 1, ..., \kappa\} \to \mathbf{N} \cup \{0\}$ is the connectivity function of a graph with connectivity $\kappa \geq 1$ if and only if $f(\kappa) = 0$ and f is decreasing.

For a digraph D with strong connectivity κ, the function $f : \{0, 1, ..., \kappa\} \to \mathbf{N} \cup \{0\}$ defined by $f(r) = \ell_r$ where ℓ_r is the minimum arc-connectivity among all subdigraphs of D obtained by deleting r vertices, $0 \leq r \leq \kappa$, from D is called the *connectivity function* of D. Theorem E has an immediate extension to digraphs.

Theorem 2.4.1.1 Let κ be a positive integer. A function $f : \{0, 1, ..., \kappa\} \rightarrow \mathbf{N} \cup \{0\}$ is the connectivity function of a digraph with connectivity $\kappa \geq 1$ if and only if $f(\kappa) = 0$ and f is decreasing.

Proof: Suppose f is the connectivity function of a digraph with connectivity κ. Then $f(\kappa) = 0$. Suppose $0 \leq k < \kappa$ and that $f(k) = \ell_k$. Then D contains a set S of k vertices such that $\lambda(D - S) = \ell_k$. Let E be a set of ℓ_k edges of $D - S$ so that $D' = D - S - E$ has at least two strong components. If $D - S - E$ has a nontrivial strong component D_1, then D_1 contains a vertex v that is incident with an edge of E. Hence $S' = S \cup \{v\}$ is a set of $k + 1$ vertices so that $\lambda(D - S') \leq |E| - 1 = \ell_k - 1$. If every strong component of $D - S - E$ is trivial, then $D - S - E$ consists of exactly two vertices. Thus D contains a set of $k + 1$ vertices and $\ell_k - 1$ edges whose deletion produces the trivial graph. Therefore in either case $f(k + 1) \leq \ell_k - 1$. Hence f is decreasing.

Suppose now that $f : \{0, 1, .., \kappa\} \rightarrow \mathbf{N} \cup \{0\}$ is a decreasing function such that $f(\kappa) = 0$. Let $\lambda = f(0)$. Let $H_0, H_1, ..., H_\kappa$ be $\kappa + 1$ disjoint copies of the complete symmetric digraph K_λ^*. Denote the vertices of H_k by $v_{k,j}$ for $j = 1, 2, ..., \lambda$. Add a vertex u_0 and join it by a symmetric pair of arcs to every vertex of H_0. For $0 < k \leq \kappa$, add vertices $u_{k,1}, u_{k,2}, ..., u_{k,k}$ and join each of these vertices to every vertex of $V(H_{k-1}) \cup V(H_k)$ by a symmetric pair of arcs. Finally, join $v_{k-1,i}$ and $v_{k,i}$ by a symmetric pair of arcs for $i = 1, 2, ..., f(k)$. Let D be the resulting digraph. It can now be shown that for each $k = 1, 2, ..., \kappa$, the minimum arc-connectivity of a subdigraph obtained by deleting k vertices from D is $\lambda(D - \{u_{k,1}, u_{k,2}, ..., u_{k,k}\}) = |\{(v_{k-1,i}, v_{k,i}) : 1 \leq i \leq f(k)\}| = f(k)$. So D has f as its connectivity function. $\square$

The digraph D constructed in the proof of Theorem 2.4.1.1 is a symmetric digraph, i.e., if $(u, v) \in E(D)$, then $(v, u) \in E(D)$. Since many digraphs are obtained by assigning directions to the edges of a graph it is natural to consider the connectivity function of an asymmetric digraph. We now characterize the connectivity functions of these digraphs.

Theorem 2.4.1.2 Let κ be a positive integer. A function $f : \{0, 1, ..., \kappa\} \rightarrow \mathbf{N} \cup \{0\}$ is the connectivity function of an asymmetric digraph with connectivity $\kappa \geq 0$ if and only if $f(\kappa) = 0$ and f is decreasing.

Proof: The necessity of the theorem follows as in Theorem 2.4.1.1. Suppose now that $f : \{0, 1, ..., \kappa\} \rightarrow \mathbf{N} \cup \{0\}$ is a decreasing function with $f(\kappa) = 0$.

Let $\lambda = f(0)$. Consider $K_{2\lambda+1}$. It is well- known that the edge set of this complete graph can be decomposed into λ hamiltonian cycles $G_1, G_2, ..., G_\lambda$. Direct the edges of G_i $(1 \leq i \leq \lambda)$ in such a way that a directed cycle G'_i is produced. Let T be the tournament of order $2\lambda + 1$ whose arc set is $\cup\{E(G'_i)|1 \leq i \leq \lambda\}$. Then T has strong connectivity and strong arc connectivity λ. Let $H_0, H_1, ..., H_\kappa$ be $\kappa + 1$ disjoint copies of T. Denote the vertices of H_k by $v_{k,1}, v_{k,2}, ..., v_{k,\lambda}, w_{k,1}, w_{k,2}, ..., w_{k,\lambda+1}$. Add a vertex u_0 and the arcs $(u_0, v_{0,i})$ for $1 \leq i \leq \lambda$, as well as the arcs $(w_{0,j}, u_0)$ for $1 \leq j \leq \lambda + 1$. For $0 < k \leq \kappa$ add vertices $u_{k,1}, u_{k,2}, ..., u_{k,k}$ and the arcs $\{(v_{k-1,i}, u_{k,j})|1 \leq i \leq \lambda, 1 \leq j \leq k\} \cup \{(u_{k,j}, v_{k,i})|1 \leq i \leq \lambda, 1 \leq j \leq k\} \cup \{(u_{k,j}, w_{k-1,i})|1 \leq j \leq k, 1 \leq i \leq \lambda + 1\} \cup \{(w_{k,i}, u_{k,j})|1 \leq j \leq k, 1 \leq i \leq \lambda + 1\}$. Finally, add the arcs $(v_{k-1,i}, v_{k,i})$ for $1 \leq i \leq f(k)$ and $(w_{k,j}, w_{k-1,j})$ for $1 \leq j \leq f(k)$. Let D be the resulting asymmetric digraph. As in Theorem 2.4.1.1 it can be shown that f is the connectivity function of this asymmetric digraph. $\square$

We turn to the problem of 'disconnecting' a digraph into more than two strong components by the deletion of vertices and arcs.

We have already, in this chapter, dealt with the concept of the ℓ-connectivity function of a graph and have established some properties of this function relating to specific graphs. We now investigate this concept for digraphs. Let D be a digraph of order $p \geq \ell - 1 \geq 1$ having strong ℓ-connectivity $\kappa_\ell(D) = \kappa_\ell$. Then the *strong ℓ-connectivity function* f_ℓ of D is defined as follows: $f_\ell : \{0, 1, ..., \kappa_\ell\} \rightarrow \mathrm{N} \cup \{0\}$ and for $0 \leq k \leq \kappa_\ell$, $f_\ell(k) = s_k$ where s_k is the minimum strong ℓ-arc- connectivity among all subdigraphs of D obtained by deleting k vertices from D. Then $f_\ell(\kappa_\ell) = 0$ and $f_\ell(k) > 0$ for $0 \leq k < \kappa_\ell$. Further, f_ℓ is a non-increasing function; for suppose $0 \leq k < \kappa_\ell$ and that $f_\ell(k) = s_k$. Then there exists a set V_k of k vertices of D and a set E_k of s_k edges of D such that $D_k = D - V_k - E_k$ has at least ℓ strong components. If D_k has at least $\ell + 1$ vertices, then there exists a vertex v of D_k such that $D_k - \{v\}$ still has at least ℓ strong components. So $V_{k+1} = V_k \cup \{v\}$ is a set of $k + 1$ vertices such that the number of strong components of $D - V_{k+1} - E_k$ is at least ℓ. So in this case $f_\ell(k + 1) \leq |E_k| = f_\ell(k)$. If D_k has exactly ℓ vertices then $k + 1 = \kappa_\ell = p - \ell + 1$. So in this case $\mathrm{f}_\ell(k + 1) = 0 < f_\ell(k)$.

While the strong 2-connectivity function of a digraph is strictly decreasing, this is no longer the case for the strong ℓ- connectivity functions of digraphs for $\ell \geq 3$. For example, if $D \cong K^*_{2,2} \cup K^*_2$, then $\kappa_3(D) = 2$ and $\{(0,1), (1,1), (2,0)\}$ is the strong 3-connectivity function of D and is thus not strictly decreasing.

Recall (theorem D, section 2.3.1), it was stated that if $p \geq \ell$, then the ℓ- connectivity function of K_p is given by

$$f_\ell(k) = \begin{cases} (\ell - 1)(p - \ell - k + 1) + \binom{\ell-1}{2} & , \text{ for } 0 \leq k < \kappa_\ell(K_p) \\ 0 & , \text{ for } k = \kappa_\ell(K_p) = p - \ell + 1. \end{cases} \qquad 2.4.1$$

Using arguments similar to those employed in [O2] it can be shown that the strong ℓ-connectivity function of K^*_p is also given by the function defined in (2.4.1).

For an integer $\ell \geq 2$ it was shown in [O2] that if G is a graph and $f_\ell(k) = f_\ell(k+1)$ for some $k, 0 \leq k < \kappa_\ell(G)$, then $f_\ell(k) \leq \binom{\ell-1}{2}$. We now establish an analogue for digraphs.

Theorem 2.4.1.3. Let $\ell \geq 2$ and suppose D is a digraph with strong ℓ-connectivity function f_ℓ and $\kappa_\ell(D) = n$. If $f_\ell(k) = f_\ell(k+1)$ for some k, $0 \leq k < n$, then $f_\ell(k) \leq \binom{\ell-1}{2}$.

Proof: Let $f_\ell(k) = s_k$. Then there exists a set V_k of k vertices of D and a set E_k of s_k arcs of D such that $D_k = D - V_k - E_k$ has at least ℓ strong components. If $e = (u, v) \in E_k$, then the strong component of D_k containing u (and the one containing v) consists of a single vertex, namely u (respectively, v). To see this suppose, to the contrary, that u, say, belongs to a nontrivial strong component of D_k. Then $D_k - u$ has at least ℓ strong components, that is, $D - (V_k \cup \{u\}) - (E_k - \{e\})$ has at least ℓ strong components. However, then $f_\ell(k+1) \leq |E_k - \{e\}| \leq s_k - 1 = f_\ell(k) - 1$, which contradicts our assumption. Now since $f_\ell(k+1) > 0$, it follows that $p(D) - (k+1) \geq \ell$, i.e. $p(D) - k \geq \ell + 1$. Hence D_k contains at least $\ell + 1$ vertices. So since $D_k + e$ has at most $\ell - 1$ strong components $D_k + e$ has at least $\ell - 2$ strong components that consist of a single vertex. Further, u and v belong to a nontrivial strong component of $D_k + e$. If D_k has more than ℓ strong components, then $D_k - u$ has at least ℓ strong components so that $D - (V_k \cup \{u\}) - (E_k - e)$ has at least ℓ strong components. However, then $f_\ell(k+1) < f_\ell(k)$, which contradicts the hypothesis. Hence D_k has exactly ℓ strong components. This implies that D_k has at most $\ell - 1$ trivial strong components. Further, from an earlier argument, no arc of D_k joins a vertex from a trivial strong component of D_k with a nontrivial strong component of D_k. Hence the arcs of D_k are only between vertices that correspond to trivial components of D_k. Since at most $\binom{\ell-1}{2}$ arcs need to be deleted from any digraph on at most $\ell-1$ vertices to produce an acyclic digraph, it follows that $|E_k| \leq \binom{\ell-1}{2}$, i.e. $f_\ell(k) \leq \binom{\ell-1}{2}$. $\square$

From Theorem 2.4.1.3 we know that if D is a digraph such that $f_\ell(k) = f_\ell(k+1)$ for some $k(0 \leq k < \kappa_\ell(D) - 1)$, then $f_\ell(k) \leq \binom{\ell-1}{2}$. The next result shows that if in addition $f_\ell(k) > \binom{\ell-1}{2}$, then $f_\ell(j)$ for $0 \leq j < k$ cannot be arbitrarily large.

Theorem 2.4.1.4 Let D be a digraph with $\kappa_\ell(D) = n$, where $\ell \geq 3$ and let f_ℓ be as in Theorem 2.4.1.3. If $f_\ell(k) = f_\ell(k+1)$, for some k, $0 \leq k < n$, and $\binom{\ell-2}{2} < f_\ell(k) \leq \binom{\ell-1}{2}$, then $f_\ell(j) \leq (k-j)(\ell-1) + f_\ell(k)$ for $0 \leq j \leq k-1$.

Proof Let $f_\ell(m) = s_m$ for $0 \leq m \leq n$. Since $f_\ell(k) = f_\ell(k+1)$ it follows from the proof of Theorem 2.4.1.3, that there is a set V_k of k vertices and a set E_k of s_k arcs of D such that $D_k = D - V_k - E_k$ has exactly ℓ components and where every arc of E_k is incident, in D, with a pair of vertices that each belong to a trivial strong component of D_k. Since $s_k > \binom{\ell-2}{2}$, D_k contains more than $\ell - 2$ trivial strong components. Since $f_\ell(k) = f_\ell(k+1) = s_{k+1} > 0$, we have $p(D_k) \geq \ell + 1$. Hence D_k has at least one nontrivial strong component. Thus D_k has exactly $\ell - 1$ trivial strong components, with vertices say $v_1, v_2, ..., v_{\ell-1}$. Let $V = \{v_1, v_2, ..., v_{\ell-1}\}$. Now every vertex of V_k is adjacent to at most $\ell - 1$ vertices of V. Hence any set of $k-j$ vertices of $V_k, 0 \leq j \leq k-1$, is joined by at most $(k-j)(\ell-1)$ arcs to vertices of V. Let $V_k = \{u_1, u_2, ..., u_k\}$. Then there are at most $(k-j)(\ell-1)$ arcs denoted by E'_j that join vertices of $\{u_{j+1}, u_{j+2}, ..., u_k\}$ to vertices of $V(0 \leq j \leq k-1)$. Hence $D - (E'_0 \cup E_k)$ is disconnected with at least ℓ strong components and for $1 \leq j \leq k-1$, the digraph $D - \{u_1, u_2, ..., u_j\} - (E'_j \cup E_k)$ has at least ℓ strong components. So $s_j \leq (k-j)(\ell-1) + f_\ell(k)$ for $0 \leq j \leq k-1$. $\square$

It appears to be a difficult problem to characterize all ℓ-connectivity functions of digraphs. However, the following result provides sufficient conditions for a function to be the ℓ-connectivity function of some digraph.

Theorem 2.4.1.5 Let $\ell \geq 2$ be an integer. If f is a decreasing function from $\{0, 1, ..., \kappa\}$, $\kappa \geq 1$, to the nonnegative integers such that $f(\kappa) = 0$, then f is the strong ℓ-connectivity function of some digraph.

Proof If $\ell = 2$, then the result follows from Theorem 2.4.1.1. Let D be the digraph having f as strong 2-connectivity function. Then for $\ell \geq 3$, $D \cup \bar{K}_{\ell-2}$ is a digraph with ℓ-connectivity funciton f, that is, $f_\ell(k) = f(k)$ for $0 \leq k \leq \kappa$. □

2.5 Maximal and Minimal Graphs of given ℓ-connectivity

2.5.1 Maximal Graphs

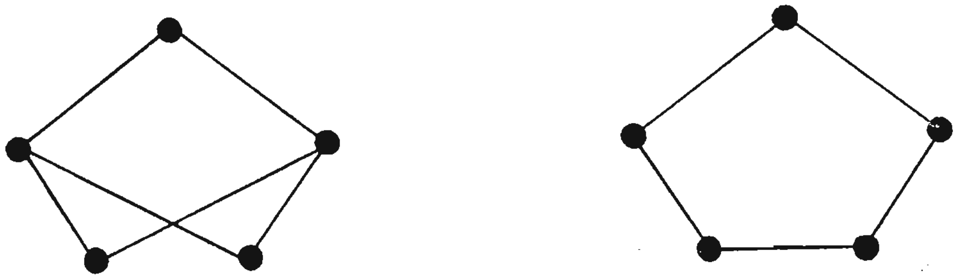
A connected graph G is (n, ℓ)-maximal if G is not complete, $\kappa_\ell(G) = n$ and $\kappa_\ell(G + e) > n$ for every edge $e \in E(\bar{G})$.

The largest integer q for which there exists a connected (p, q) graph G of given order p, such that $\kappa_\ell(G) = n$ is denoted by $Q_{n,\ell}(p)$.

A graph $G = G(p, q)$ with $q = Q_{n,\ell}(p)$ and $\kappa_\ell(G) = n$ is called an (n, ℓ)-maximum graph.

The graph $G = K_n + (K_{p_1} \cup K_{p_2} \cup ... \cup K_{p_\ell})$, where $p = p_1 + p_2 + ... + p_\ell + n$, is clearly (n, ℓ)-maximal.

That every (n, ℓ)-maximum graph is of this form may be seen as follows: Let S be a set of n vertices of G such that the number of components



(a) (b)

Figure 2.5.2.1

of $G - S$ is $k(G - S) = m \geq \ell$. Let $G_1, G_2, .., G_m$ be the components of $G - S$, of orders $p_1, p_2, ..., p_m$ respectively. Then $\langle S \rangle$ is complete (otherwise, if $e \in \langle S \rangle_{\bar{G}}$, then $k((G + e) - S) = m \geq \ell$, and $|S| = n$. Similarly it follows that $G_1, G_2, ..., G_m$ are complete. Furthermore, $m = \ell$ (otherwise, if $m > \ell$, let e join a vertex of G_ℓ to a vertex of $G_{\ell+1}$ and note that $k((G + e) - S) = m - 1 \geq \ell$). Also, every vertex in S is adjacent to every vertex in $V(G) - S$. So $G = \langle S \rangle + (G_1 \cup ... \cup G_m)$ where $S \cong K_m$ and $G_i \cong K_{p_i}$ (with $p_i = p(G_i)$).

It follows that in order to obtain an (n, ℓ)-maximum graph of order p, we should choose $p_1 = p_2 = ... = p_{\ell-1} = 1$ and $p_\ell = p-n-(\ell-1) = p-n-\ell+1$.

Thus $Q_{n,\ell}(p) = \frac{1}{2}\{(p - n - \ell + 1)(p - n - \ell) + n(n - 1)\} + n(p - n)$.

2.5.2 Minimal Graphs

Let $n, \ell, p \in \mathbb{N}$ with $\ell \geq 2$ and $p \geq \ell + n$. A graph G is (n, ℓ)-minimal if $\kappa_\ell(G) = n$ and $\kappa_\ell(G - e) < n$ for every edge $e \in E(G)$. The smallest integer q for which there exists a (p, q) graph G of given order p, such that $\kappa_\ell(G) = n$ is denoted by $q_{n,\ell}(p)$.

A graph $G = G(p, q)$ with $q = q_{n,\ell}(p)$ and $\kappa_\ell(G) = n$ is called an (n, ℓ)-minimum graph and will be denoted by $G_{n,\ell}(p)$
The class of (n, ℓ)-minimum graphs will be denoted by $\mathcal{G}_{n,\ell}$, and $\mathcal{G}_{n,\ell}(p)$ denotes the set of all graphs in $\mathcal{G}_{n,\ell}$ of order p.

By definition (n, ℓ)-minimum graphs are (n, ℓ)-minimal. However, the converse is not true, as can be seen in figure 2.5.2.1 where both graphs are (2,2)-minimal.

The characterisation of graphs of $\mathcal{G}_{n,\ell}$ proves to be more difficult than that of the (n, ℓ)-maximum graphs characterised above. However, the graphs of $\mathcal{G}_{n,\ell}$ could be useful in designing a network which is deemed to fail if it splinters into ℓ or more components after the simultaneous failure of some of its centres, or if at least n centres fail simultaneously. The graph, with the minimum possible number of links, which represents such a network, will belong to $\mathcal{G}_{n,\ell}$.

We first prove two general results.

Theorem 2.5.2.1 If $p \geq n+3 \geq 5$ and G is an $(n, 3)$-minimal graph of order p, then G contains an edge e for which $\kappa_3(G-e) = n-1$ or G contains at least n vertices b such that $G-b$ is $(n-1, 3)$-minimal.

Proof: Let $T \subset V(G)$ such that $k(G-T) \geq 3$ and $|T| = n$. For $e \in E(G)$, $\kappa_3(G-e) \leq n-1$; say $\kappa_3(G-e) = m_e \leq n-1$ and denote by S_e an m_e-set of vertices of G such that $k(G-e-S_e) \geq 3$. We note that $k(G-e-S_e) = 3$, otherwise, if $k(G-e-S_e) \geq 4$, it follows that $k(G-S_e) \geq 3$, contrary to the assumption that $\kappa_3(G) = n$. Furthermore, e is a bridge of $G-S_e$; so either the component (G_1, say) of $G-S_e$ that contains e is isomorphic to K_2 or $p(G_1) \geq 3$ and the $(m+1)$-set $S'_e = S_e \cup \{u\}$ (where u is the endvertex of e in a nontrivial component of $G-e-S_e$) satisfies $k(G-S') \geq 3$ which implies that $m_e + 1 \geq n$. Hence, in this latter case, $\kappa_3(G-e) = n-1$ (and so, if G is $(n, 3)$-minimum, $q_{n-1,3}(p) \leq q(G-e) \leq q_{n,3}(p) - 1$).

We now assume that the statement of the theorem is invalid and that n is the smallest integer ($n \geq 2$) for which G provides a counter-example to the theorem. Then $m_e \leq n-2$ for each $e \in E(G)$. Let $b \in T$; then $\kappa_3(G-b) \leq n-1$ (as $k(G-b-(T-\{b\})) \geq 3$) and $\kappa_3(G-b) \geq n-1$ (as $\kappa_3(G) \geq n$); hence $\kappa_3(G-b) = n-1$. For $e \in E(G-b)$ we note that $\kappa_3(G-e) = m_e \leq n-2$ and e joins two trivial components of $G-e-S_e$.

Furthermore, either $k(G - b - e - S_e) \geq 3$ or $\langle b \rangle$ is a trivial component of $G - e - S_e$ and $k(G - e - S_e) = 3$. In the latter case it follows that $G - e - S_e$ has (only) three trivial components, whence $p = m_e + 3 < p$, a contradiction. Hence $k(G - b - e - S_e) \geq 3$, $|S_e| \leq n - 2$ for each $e \in E(G - b)$ and $G - b$ is an $(n - 1, 3)$ graph of order $p - 1$. □

Theorem 2.5.2.2

$$q_{n,\ell}(p) < q_{n,\ell-1}(p).$$

Proof: Obviously $q_{n,\ell}(p) \leq q_{n,\ell-1}(p)$. Now assume that $q_{n,\ell}(p) = q_{n,\ell-1}(p)$ and let $G \in \mathcal{G}_{n,\ell}(p)$. Let $S \subset V(G)$ such that $|S| = n$ and $k(G - S) \geq \ell > \ell - 1$. Then, since G has size $q_{n,\ell}(p)$ it follows that $G \in \mathcal{G}_{n,\ell-1}$. Now let $e \in E(G)$ and let $S' \subset V(G - e)$ such that $|S'| = m = \kappa_\ell(G - e)$ and $k(G - e - S') \geq \ell$. Then, as we have seen in the above theorem, $G - e - S'$ has exactly ℓ components; so $G - S'$ has ℓ or $\ell - 1$ components. However, $k(G - S') \neq \ell$ since $\kappa_\ell(G) = n > |S'|$ and $k(G - S') \neq \ell - 1$, since $\kappa_{\ell-1}(G) = n > |S'|$. This contradiction yields the desired result. □

Graphs of $\mathcal{G}_{n,2}$

If $\ell = 2$, $\kappa_2(G) = \kappa(G)$ and so $q_{n,\ell}(p) = q_{n,2}(p)$ is the smallest size of a graph of order p and connectivity n. Harary [H3] has shown that $q_{n,2}(p) = \lceil \frac{pn}{2} \rceil$ and has provided the following associated $(p, \lceil \frac{pn}{2} \rceil)$ graphs of connectivity n, $G_{n,2}(p) = H_{n,p}$.

In all cases, let $V = V(H_{m,p}) = \{0, 1, ..., p - 1\}$, $p \geq m + 2 \geq 4$.

Case 1: If m is even, say $m = 2r$, then, for $i, j \in V$, $i, j \in E(H_{m,p})$ iff $|i - j| \leq r$ (addition modulo p). Hence, denoting by C_p the cycle $0, 1, ..., p - 2, p - 1, 0$, we note that $H_{2r,p} \cong C_p^r$.

Case 2: If m is odd, say $m = 2r + 1$ and p is even, say $p = 2a$, then $H_{m,p}$ is obtained from $H_{2r,p}$ by the insertion of the a edges $i(i + a)$ for $0 \leq i \leq a - 1$.

Case 3: If m is odd, say $m = 2r + 1$ and p is odd, say $p = 2a + 1$, then $H_{m,p}$ is obtained from $H_{2r,p}$ by the insertion of the $a+1$ edges $0a, 0(a+1)$ and $i(i+a+1)$ for $1 \le i \le a-1$.

We note that $q(H_{m,p}) = \lceil \frac{mp}{2} \rceil$ in all cases and next determine the ℓ-connectivity of $H_{m,p}$ in each case, where $\ell \ge 2$ and the above notation is retained.

Proposition 2.5.2.3 In Case 1 $\kappa_\ell(H_{2r,p}) = \begin{cases} \ell r & \text{if } p \ge \ell(r+1) \\ p - \ell + 1 & \text{if } \ell \le p < \ell(r+1). \end{cases}$

Proof: Let $p \ge \ell(r+1)$ and $H = H_{2r,p}$. That $\kappa_\ell(H) \le \ell r$ follows from the observation that $S = \bigcup_{j=0}^{\ell-1} \{j(r+1)+1,\ j(r+1)+2, ..., j(r+1)+r\}$ is such that $|S| = \ell r$ and $k(H-S) = \ell$, the components of $H-S$ having vertex sets $\{r+1\}, \{2(r+1)\}, ..., \{(\ell-1)(r+1)\}$ and $\{\ell(r+1), \ell(r+1)+1, ..., p-1, 0\}$.

To show that $\kappa_\ell(H) \ge \ell r$, we assume to the contrary that $\kappa_\ell(H) < \ell r$ and let $S \subset V(H)$ such that $|S| < \ell r$ and let $k(H-S) \ge \ell$. Let $i_0, i_1, ..., i_{\ell-1}$ be vertices from ℓ distinct components of $H-S$, labelled so that $0 \le i_0 < i_0+1 < i_1 < i_1+1 < ... < i_{\ell-1} < p-1$. For $j = 0, 1, ..., \ell-1$, let $S_j = \{i_j, i_j+1, ..., i_{j+1}\}$ (all addition modulo p) and $T_j = S_j \cap S$. We note that $i_j, i_{j+1} \notin T_j$; hence, since $\left|\bigcup_{j=0}^{\ell-1} T_j\right| = |S| < \ell r$, there exists $j \in \{0, 1, ..., \ell-1\}$ such that $|T_j| < r$. Consequently there exist vertices $i_j = a_1, a_2, ..., a_s = i_{j+1}$ in $S_j - T_j$ such that $a_1 < a_2 < ... < a_s$ and $a_{t+1} - a_t \le r$ for $t = 1, ..., s-1$. So $a_1a_2...a_s$ is an $i_j - i_{j+1}$ path in $H_{2r,p} - S$, contradicting our assumption that i_j and i_{j+1} are in distinct components of $H-S$, whence it follows that $\kappa_\ell(H) \ge \ell r$ and so $\kappa_\ell(H) = \ell r$. That $\kappa_\ell(H_{2r,p}) = p - (\ell-1)$ if $\ell \le p < \ell(r+1)$ follows immediately from the observation that $\beta(H_{2r,p}) \le \left\lfloor \frac{p}{r+1} \right\rfloor$. □

Propositon 2.5.2.4 In case 2, if $\ell \geq 3$ and $p \geq 2r\ell$ (where p is even) then

$$\text{if } r \geq \ell,\ \kappa_\ell(H_{2r+1,p}) = \begin{cases} \ell r + \ell - 1 & \text{if} \quad p \geq 2r\ell + 2\ell \\ \frac{1}{2}p & \text{if} \quad 2r\ell \leq p \leq 2r\ell + 2\ell - 2 \text{ and } \ell \text{ is odd} \\ & \text{or} \quad 2r\ell + 2 \leq p \leq 2r\ell + 2\ell - 2 \text{ and } \ell \text{ is even} \\ r\ell + 1 & \text{if} \quad p = 2r\ell \text{ and } \ell \text{ is even.} \end{cases}$$

$$\text{if } r < \ell,\ \kappa_\ell(H_{2r+1,p}) = \begin{cases} \frac{1}{2}p & \text{if} \quad 2r\ell \leq p \leq 2r\ell + 2r - 2 \text{ and } \ell \text{ is odd} \\ & \text{or} \quad 2r\ell + 2 \leq p \leq 2r\ell + 2r - 2 \text{ and } \ell \text{ is even} \\ r\ell + 1 & \text{if} \quad p = 2r\ell \text{ and } \ell \text{ is even} \\ \ell r + r & \text{if} \quad p \geq 2r\ell + 2r. \end{cases}$$

Proof: Let $S = S' \cup S'' \subset V(H_{2r+1,p})$, where $p \geq \ell r + r + \ell$

$$S' = \bigcup_{j=0}^{\lceil \frac{\ell-1}{2} \rceil - 1} \{j(r+1)+1, ..., j(r+1)+r\} \textit{ and}$$

$$S'' = \bigcup_{j=0}^{\lfloor \frac{\ell-1}{2} \rfloor + 1} \{a + j(r+1), a + j(r+1) + 1, ..., a + j(r+1) + r - 1\}$$

(addition modulo p).

Then $|S| = (\ell+1)r$ and $H_{2r+1,p}$ contains ℓ components, namely $\ell-1$ isolated components induced by the vertices $r+1, 2(r+1), ..., \left(\lceil \frac{\ell-1}{2} \rceil - 1\right)(r+1), r+a, r+a+(r+1), ..., r+a+\lfloor \frac{\ell-1}{2} \rfloor (r+1)$, and a component which is nontrivial if $p > \ell r + r + \ell$, containing the vertices in $A \cup B$, where $A = \left\{\lceil \frac{\ell-1}{2} \rceil (r+1), \lceil \frac{\ell-1}{2} \rceil (r+1) + 1, .., a-1\right\}$ and

$$B = \left\{a + r + \left(\lfloor \tfrac{\ell-1}{2} \rfloor + 1\right)(r+1), a + r + \left(\lfloor \tfrac{\ell-1}{2} \rfloor + 1\right)(r+1) + 1, ..., p-1, 0\right\}$$

if $a + r + \left(\lfloor \frac{\ell-1}{2} \rfloor + 1\right)(r+1) \leq p$ and $B = \emptyset$ otherwise.
(We note that certainly $r + a + \lfloor \frac{\ell-1}{2} \rfloor (r+1) \leq p$ and $\lceil \frac{\ell-1}{2} \rceil (r+1) \leq a$ as $p \geq \ell r + r + \ell$.) So $\kappa_\ell(H_{2r+1,p}) \leq (\ell+1)r$ if $p \geq \ell r + r + \ell$ and p is even. We next investigate conditions under which $\kappa_\ell(H_{2r+1,p}) < (\ell+1)r$ for (even)

values of $p \geq 2r\ell \geq \ell r + r + \ell$.

Denoting $H_{2r+1,p}$ by G and $H_{2r,p}$ by H, let $S \subset V(G)$ with $k(G-S) \geq \ell$ and suppose that $|S| < (\ell+1)r$. Then it follows from the proof of the preceding lemma that $k(H-S) < \ell+1$. Hence, as $k(H-S) \geq k(G-S) \geq \ell$, $k(H-S) = k(G-S) = \ell$. We recall that $H = C_p^r$, where the vertices of C_p are labelled consecutively $0, 1, ..., p-1$ in, say, the clockwise sense. Denote the consecutive components of $G-S$ by $G_1, ..., G_\ell$ and let u_i, v_i be the first and last vertices in G_i so that all vertices in G_i are contained in $\{u_i, u_{i+1}, ..., v_i\}$ $(i = 1, 2, .., \ell)$ where addition is modulo p. Let $p_i = p(G_i)$ and $p_1 = \max\{p_i | i = 1, ..., \ell\}$.

We consider two cases:

Case (i): If $p_1 = a + c > a$, let $u_1 = a - c, v_1 = 2a - 1$; then, as every pair of consecutive components of $G - S$ are separated by at least r vertices on C_p and $u_1 \leq i + a \leq v_1$ for each $i \in \bigcup_{j=2}^{\ell} V(G_j)$, it follows that $|S| \geq \ell r + \sum_{i=2}^{\ell} p_i \geq \ell r + \ell - 1$, with equality iff $p_i = 1$ for $i = 2, ..., \ell$. Furthermore, $a - c = \ell r + \sum_{i=2}^{\ell} p_i \geq \ell r + \ell - 1$, whence

$$\begin{aligned} p = 2a &\geq 2(\ell r + \ell - 1 + c) \\ &\geq 2\ell(r+1) \end{aligned}$$

In this case, as $|S| \leq \ell r + r - 1$, it follows that $r \geq \ell$. The bound $|S| = \ell r + \ell - 1$ can be attained by letting $V(G_i) = r + (i-2)(r+1)$ for $i = 2, ..., \ell$ and $V(G_1) = \{r\ell+\ell-1, r\ell+\ell, ..., 2a-1\} - \bigcup_{i=2}^{\ell}(r+(i-2)(r+1)+a)$.

Case (ii): If $p_1 \leq a$, then, as $i+a \in V-S$ for each $i \in V-S$, $|S| \geq |V|-|S|$ and so $|S| \geq a = \frac{1}{2}p$. Consequently $p \leq 2|S| \leq 2\ell r + 2r - 2$. We have to consider two subcases, where the notation of Case (i) is retained throughout.

Subcase (ii)(a): If $2r\ell \leq p \leq 2rl + 2r - 2$ and ℓ *is odd*, the bound $|S| = a = \frac{1}{2}p$ may be attained (where $a = \ell r + m$, $0 \leq m \leq r - 1$) by letting S consist of all integers in the following intervals: $[1, r], [2r + 1, 3r], ..., [\frac{r}{2}(\ell - 1) + 1, \frac{r}{2}(\ell + 1]), [\frac{r}{2}(\ell + 3) + 1, \frac{r}{2}(\ell + 5) + m], [\frac{r}{2}(\ell + 7) + m + 1, \frac{r}{2}(\ell + 9) + m], ..., [p - 2r - m + 1, p - r - m]$. If $r \geq \ell$, this bound is as good as or an improvement of the bound obtained in case (i) iff $p \leq 2r\ell + 2\ell - 2$.

Subcase (ii)(b): Let $2r\ell \leq p \leq 2r\ell + 2r - 2$, where ℓ *is even*. We shall first show that, if $p = 2r\ell$, then $|S| > \frac{1}{2}p = r\ell$: Suppose, to the contrary, that $|S| = \frac{1}{2}p = r\ell$; then exactly r vertices of S are contained in the interval (v_i, u_{i+1}) and no vertex of S is contained in $[u_i, v_i]$, where $i = 1, 2, .., \ell$ (additon modulo p). Hence $p_1 = p_2 = ... = p_\ell = r$. Letting $V(G_1) = \{0, 1, ..., r - 1\}$, we note that $V(G_i) = \{2ri, ..., 2ri + r - 1\}$; hence $V(G_{\frac{1}{2}\ell}) = \{\ell r, ..., \ell r + r - 1\}$, but, as $a = \ell r$ and $V(G_1) = \{0, 1, ..., r - 1\}$, it follows from the preceding argument that $\{\ell r, ..., \ell r + r - 1\} \subset S$, a contradiction.

However, the value $|S| = \frac{1}{2}p + 1$ may be attained as follows: Let
$V(G_i) = \{(2i - 1)r + 1, ..., 2ir\}$ for $i = 1, ..., \frac{1}{2}\ell - 1$;
$V(G_{\frac{1}{2}\ell}) = \{\ell r - r + 1, ..., \ell r + r\} - \{\ell r\}$;
$V(G_{\frac{1}{2}\ell + i}) = \{\ell r + 2ir + 1, ..., \ell r + (2i + 1)r\}$ for $i = 1, ..., \frac{1}{2}\ell - 2$;
$V(G_{\ell-1}) = \{2\ell r - 2r + 1, ..., 2\ell r - r - 1\}$, $V(G_\ell) = \{0\}$.

We note that $p_i = r$ for $i = 1, ..., \frac{1}{2}\ell - 1, \frac{1}{2}\ell + 1, ..., \ell - 2$, while $p_{\frac{1}{2}\ell} = 2r - 1, p_{\ell-1} = r - 1$ and $p_\ell = 1$; so $\sum_{i=1}^{\ell} p_i = \ell r - 1$ and $|S| = \ell r + 1$.

If $p = 2r\ell + 2m$, where $1 \leq m \leq r - 1$, the bound $|S| = \frac{1}{2}p$ may be attained as follows:
Let

$$V(G_i) = \{(2i - 1)r + 1, ..., 2ir\} \text{ for } i = 1, 2, ..., \frac{1}{2}\ell;$$

$$\begin{aligned}
V(G_{\frac{1}{2}\ell+1}) &= \{\ell r + r + 1, ..., \ell r + m + r\}, \\
V(G_{\frac{1}{2}\ell+i}) &= \{\ell r + m + (2i-1)r - r + 1, ..., \ell r + m + (2i-1)r\} \\
&\qquad\qquad \text{for } i = 1, 2, ..., \frac{\ell}{2} - 1. \\
V(G_\ell) &= \{p - 2r + 1, p - 2r + 2, ..., p - 1, 0\}
\end{aligned}$$

with $p_i = r$ for $i = 1, 2, ..., \frac{1}{2}\ell, \frac{1}{2}\ell + 1, ..., \ell - 1$,
$p_{\frac{1}{2}\ell+1} = m, p_\ell = 2r$, hence $\sum_{i=1}^{\ell} p_i = \ell r + m$.

As in subcase (ii)(a) above, we note that the bound $|S| = \frac{1}{2}p$ attained in (ii)(b) if $2r\ell + 2 \leq p \leq 2r\ell + 2r - 2$ is as good as or an improvement on the bound $|S| = \ell r + \ell - 1$ attained in case (i) iff $p \leq 2r\ell + 2\ell - 2$. □

Similar techniques suffice to prove the following proposition.

Proposition 2.5.2.5 In case 3, if $\ell \geq 3$ and $p \geq 2r\ell + 1$ (where p is odd), then

$$\text{if } r \geq \ell, \quad \kappa_\ell\,(H_{2r+1,p}) = \begin{cases} \ell r + \ell - 1 & \text{if} \quad p \geq 2r\ell + 2\ell + 1 \\ \frac{1}{2}(p-1) & \text{if} \quad 2r\ell + 1 \leq p \leq 2r\ell + 2\ell - 1 \\ & \qquad \text{and } \ell \text{ is odd} \\ & \text{or} \quad 2r\ell + 3 \leq p \leq 2r\ell + 2\ell - 1 \\ & \qquad \text{and } \ell \text{ is even} \\ \frac{1}{2}(p+1) & \text{if} \quad p = 2r\ell + 1 \text{ and } \ell \text{ is even} \end{cases}$$

and

$$\text{if } r < \ell, \quad \kappa_\ell(H_{2r+1,p}) = \begin{cases} \frac{1}{2}(p-1) & \text{if} \quad 2r\ell + 1 \leq p \leq 2r\ell + 2r - 1 \\ & \qquad \text{and } \ell \text{ is odd} \\ & \text{or} \quad \text{if } 2r\ell + 3 \leq p \leq 2r\ell + 2r - 1 \\ & \qquad \text{and } \ell \text{ is even} \\ \frac{1}{2}(p+1) & \text{if} \quad p = 2r\ell + 1 \text{ and } \ell \text{ is even} \\ \ell r + r & \text{if} \quad p \geq 2r\ell + 2r + 1. \end{cases}$$

Graphs of $\mathcal{G}_{n,3}$

a) If $n = 1$ and $p \geq 4$ then $q_{1,3}(p) = p - 2$ and $\mathcal{G}_{1,3}$ consists of all unions of two trees, $T_1 \cup T_2$, with $p(T_1) + p(T_2) = p$. We note that if G is a connected graph of order p with $\kappa_3(G) = 1$, then $q(G) \geq p-1$, where equality is attained by all trees G of order p with $\Delta(G) \geq 3$.

b) If $n = 2$ and $p \geq 5$, then $q_{2,3}(p) = p - 1$ and $\mathcal{G}_{2,3}$ consists of the path P_p and of the (disjoint) union of a cycle and a trivial graph,$C_{p-1} \cup K_1$ and of the (disjoint) union of a cycle and a complete graph on two vertices, $C_{p-2} \cup K_2$.

c) If $n = 3$ and $p \geq 6$, then $q_{3,3}(p) = p$ and the cycle $G_{3,3}(p) = C_p$ belongs to $\mathcal{G}_{3,3}$

Theorem 2.5.2.6

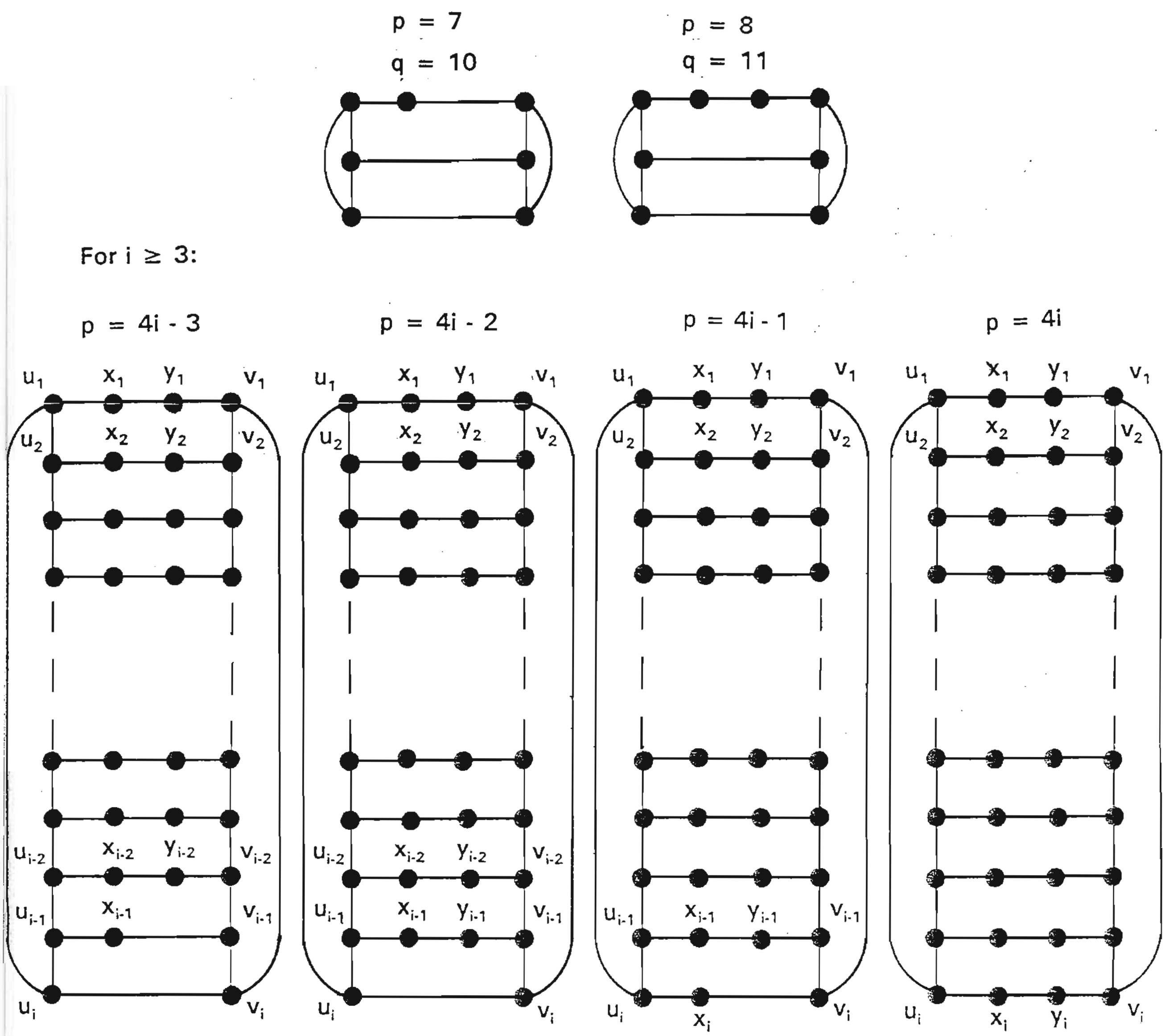
If $G \in \mathcal{G}_{3,3}$, then G is connected and is unicyclic.

Proof: Suppose $G = G_1 \cup G_2 \in \mathcal{G}_{3,3}$ with $p = p(G_1) + p(G_2)$, $q(G_1) + q(G_2) \leq p$ and $\kappa_3(G) = 3$, then for each G_i $(i \in \{1, 2\})$ we have that either G_i is complete or $\kappa_2(G_i) = 3$. However, since $\kappa_3(G) = 3$ and $p(G) \geq 6, G_1$ and G_2 cannot both be complete.

So, say G_1 has $p_1 = p(G_1) \in \{5, ..., p - 1\}$ and $\kappa_2(G_1) = 3$ and G_2 is connected. Hence,

$$\begin{aligned}
q(G) = q(G_1) + q(G_2) &\geq \frac{3p_1}{2} + p - p_1 - 1 \\
&= p + \frac{p_1}{2} - 1 \\
&\geq p + \frac{3}{2} \\
&> q(C_p).
\end{aligned}$$

Thus it follows that, for $p \geq 6, q_{3,3}(p)$ is realized by a connected graph.



$p = 7$
$q = 10$
$p = 8$
$q = 11$
For $i \geq 3$:
$p = 4i - 3$
u_1 x_1 y_1 v_1
u_2 x_2 y_2 v_2
u_{i-2} x_{i-2} y_{i-2} v_{i-2}
u_{i-1} x_{i-1} v_{i-1}
u_i v_i
$p = 4i - 2$
u_1 x_1 y_1 v_1
u_2 x_2 y_2 v_2
u_{i-2} x_{i-2} y_{i-2} v_{i-2}
u_{i-1} x_{i-1} y_{i-1} v_{i-1}
u_i v_i
$p = 4i - 1$
u_1 x_1 y_1 v_1
u_2 x_2 y_2 v_2
u_{i-1} x_{i-1} y_{i-1} v_{i-1}
u_i x_i v_i
$p = 4i$
u_1 x_1 y_1 v_1
u_2 x_2 y_2 v_2
u_i x_i y_i v_i

Figure 2.5.2.2

By (a) and (b) above G cannot be a tree and $q(G) \leq p$. Thus G is unicyclic. In fact C_p is the only unicyclic graph G with $\kappa_3(G) = 3$.

d) If $n = 4$ and $p \geq 7$, the graphs of figure 2.5.2.2 can be shown to have 3-connectivity equal to 4, and thus give upper bounds for $q_{4,3}(p)$. It will be seen that these graphs belong to $\mathcal{G}_{4,3}$.

For $p \geq 9$, the construction of a graph G in figure 2.5.2.2 may be described as follows:-

Let $p = 4i - j$ $(i, j \in \mathbf{N}, i \geq 3; j \in \{0, 1, 2, 3\})$ and let C', C'' be two disjoint i-cycles $C' = u_1, u_2, .., u_i, u_1$ and $C'' = v_1, v_2, ..., v_i, v_1$ with $V(C') = \mathcal{U}$ and $V(C'') = \mathcal{V}$. G is obtained from $C' \cup C''$ by connecting

i) u_m to v_m by a path $P_4 = u_m x_m y_m v_m$ for $m = 1, ..., i-2$.

ii) u_{i-1} to v_{i-1} by $\begin{cases} u_{i-1}x_{i-1}y_{i-1}v_{i-1} & \text{if } j \leq 2 \\ u_{i-1}x_{i-1}v_{i-1} & \text{if } j = 3 \end{cases}$

iii) u_i to v_i by $\begin{cases} u_i v_i & \text{if } j \geq 2 \\ u_i x_i v_i & \text{if } j = 1 \\ u_i x_i y_i v_i & \text{if } j = 0 \end{cases}$

Theorem 2.5.2.7 If G is a graph constructed as above, then $\kappa_3(G) = 4$.

Proof: For $p \in \{7; 8; 9\}$ it is easy to show the theorem true.
For $p \geq 9$, since $k(G - \{u_1, u_2, v_1, v_2\} = 3$, it follows that $\kappa_3(G) \leq 4$.

Suppose now that $\kappa_3(G) \leq 3$, then there exists $S = \{s_1, s_2, s_3\} \subset V(G)$ such that $k(G - S) \geq 3$. Clearly $|S \cap \mathcal{U}| \leq 1$ or $|S \cap \mathcal{V}| \leq 1$. Suppose $x_m \in S$, then either u_m is in a trivial component of $G - S$ (and $S = \{x_m, u_{m-1}, u_{m+1}\}$ which is impossible if $k(G - S) \geq 3$) or $S' = (S - \{x_m\}) \cup u_m$ is such that $k(G - S') \geq 3$.

So we may assume, without loss of generality, that $S \cap X = \emptyset, S \cap Y = \emptyset$ and $|S \cap \mathcal{U}| = 1, |S \cap \mathcal{V}| = 2$, where $X = \{x_1, x_2, ..., x_i\}$ and

$Y = \{y_1, y_2, ..., y_i\}$. Let $S \cap \mathcal{U} = \{u_m\}$; then in $G - S$, all vertices in $(\mathcal{U} - \{u_m\}) \cup (X - \{x_m\}) \cup (Y - \{y_m\}) \cup (V - S)$ are in a single component and x_m, y_m are in another component which can be $\langle\{x_m, y_m\}\rangle$ if $v_m \in S$ or $\langle\{x_m, y_m, v_m\}\rangle$ if $\{v_{m-1}, v_{m+1}\} \subset S$.

However, $G - \{u_m, v_m\}$ has as components $\langle\{x_m, y_m\}\rangle \cong K_2$ and a 2-connected component $G - \{u_m, x_m, y_m, v_m\}$, so $\{u_m, v_m\} \not\subset S$.

Hence $S = \{u_m, v_{m-1}, v_{m+1}\}$ and $k(G - S) = 2$, again a contradiction. Thus $\kappa_3(G) = 4$.

It follows that

$$q_{4,3}(p) \leq q(G) = \begin{cases} p+3 & \text{if} \quad p \in \{7; 8\} \\ p+i & \text{if} \quad p \in \{4i-3; ...; 4i\}, \qquad i \geq 3, i \in \mathbf{N} \end{cases}$$

Now $p \in \{4i-3, ..., 4i\}$ if and only if $4i \in \{p, p+1, p+2, p+3\} \cap \mathbf{N}$ if and only if $i = \lceil \frac{p}{4} \rceil$.

Hence

$$q_{4,3}(p) \leq q(G) = \begin{cases} p+3 \text{ if } p \in \{7, 8\} \\ \\ p + \lceil \frac{p}{4} \rceil = \lceil \frac{5p}{4} \rceil \text{ if } p \geq 9 \end{cases}$$

Theorem 2.5.2.8

For $G \in \mathcal{G}_{4,3}$, $\quad \kappa(G) = 2$.

Proof: We show first that G has no cut vertices. Suppose to the contrary that $k(G-v) \geq 2$ for some $v \in V(G)$. In fact since $G \in \mathcal{G}_{4,3}$; $k(G-v) = 2$. Let H_1 and H_2 be the two components of $G - v$, then clearly $\kappa(H_i) \geq 3$, $i \in \{1, 2\}$. If p_i is the order of H_i, then since $q_{3,2}(p) \geq \frac{3p}{2}$ it follows that

$$\begin{aligned} q_{4,3}(p) \geq q(H_1) + q(H_2) + 2 \quad &\geq \quad \frac{3p_1}{2} + \frac{3p_2}{2} + 2 \\ &= \quad \frac{3(p-1)}{2} + 2 \end{aligned}$$

$$> \begin{cases} p+3 & \text{if} \quad p \in \{7,8\} \\ \frac{5p+3}{4} & \text{if} \quad p \geq 9 \end{cases}$$

Thus $\kappa(G) \geq 2$.

Suppose $\kappa(G) \geq 3$, then $q_{4,3}(p) \geq q_{3,2}(p) \geq \frac{3p}{2}$.

Thus $\kappa(G) = 2$. □

Let G have p_2 vertices of degree 2 and p_3 vertices of degree at least 3 forming sets V_2 and V_3 respectively. We note that $\langle V_2 \rangle$ cannot contain a path $P_3 : v_1v_2v_3$, otherwise G would contain either a $P_5 : v_0v_1v_2v_3v_4$ or a $C_4 : v_0v_1v_2v_3v_0$ as an induced subgraph. This would imply that $\kappa_3(G) < 4$.

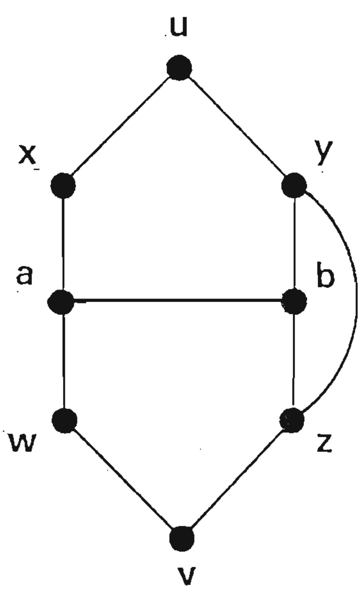
Theorem 2.5.2.9

$$q_{4,3} = \begin{cases} p+3 & \text{if} \quad p \in \{7,8\} \\ \lceil \frac{5p}{4} \rceil & \text{if} \quad p \geq 9 \end{cases}$$

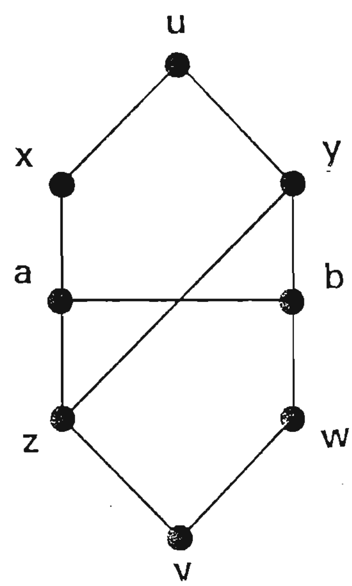
Proof: If $p = 7$, then $2p_2 + 3(7 - p_2) \leq 20$, hence $p_2 \geq 1$. Let $v \in V(G)$ with $\deg v = 2, N(v) = \{x, y\}$. Since $P_3 \not\subset \langle V_2 \rangle$, at least one vertex in $\{x, y\}$ is also in V_3; say $\deg y \geq 3$. Note that $\kappa_3(G - y) \geq 3$ and $p(G - y) = 6$. Then $q(G - y) \geq q_{3,3}(6) = 6$ and hence $q(G) \geq 6 + 3 = 9$ with equality if and only if $q(G - y) = 6$ and $\deg y = 3$.

Suppose $q = 9$. If $\deg x = 2$ and $xy \in E(G)$, then y is a cut vertex which is impossible since $\kappa(G) = 2$. So either $\deg x \geq 3$ or $xy \notin E(G)$, hence $q(G) = 9 \geq q(G - \{x, y, v\}) + 5$ and so $H = G - \{x, y, v\}$ has $p(H) = 4$, $q(H) \leq 4$, and $\kappa(H) \geq 2$ since H is a component of $G - \{x, y\}$ and $\kappa_3(G) \geq 4$. So $q(H) \geq \frac{2(4)}{2} = 4$ implying that $q(H) = 4$ and H is a 4-cycle. Furthermore, x is adjacent to exactly one vertex (say w) of H, otherwise $q(G) \geq q(H) + 6 \geq 10$.

Now $G - \{w, y\}$ has two components, namely $\langle \{x, y\} \rangle$ and $H - w$ which is a P_3 with connectivity 1. So $\kappa_3(G) = 3 < 4$ which is a contradiction. Thus $q_{4,3}(7) = 10$.



G_1



G_2

If $p = 8$, then $2p_2 + 3(8 - p_2) \leq 22$, hence $p_2 \geq 2$ and $p_3 \leq 6$.
Let $u, v \in V_2$ with $N(u) = \{x, y\}$, $N(v) = \{w, z\}$ where (since $P_3 \not\subset \langle V_2 \rangle$), $\deg y \geq 3$ and $\deg z \geq 3$ say. Since $\kappa_3(G - y) \geq 3$, $q(G - y) \geq q_{3,3}(7) = 7$ and so $q(G) \geq 10$.
Suppose $q(G) = 10$. Then $2p_2 + 3(8 - p_2) \leq 20$ and hence $p_2 \geq 4$ and so there exist $u, v \in V_2$ with $uv \notin E(G)$. Since $\kappa_3(G) = 4$, it follows that $|N(u) \cup N(v)| \geq 4$ and so x, y, w and z are distinct.

If $H = G - \{u, x, y\}$ then, by a similar argument to that used in the case $p = 7$, we can show that $q(H) = 5$ and $G - \{x, y, u\}$ is a 5-cycle. Similarly it follows that $G - \{v, w, z\} \cong C_5$. Thus two possibilities (G_1 and G_2) for G exist, both of which satisfy $\kappa_3(G) = 3 < 4$. Hence $q_{4,3}(8) \geq 11$ and so $q_{4,3}(8) = 11$.

If $p \geq 9$, we note first that

$$2p_2 + 3p_3 \leq 2q \leq 2\lceil \tfrac{5p}{4} \rceil$$

$$\Rightarrow \quad 2p_2 + 3(p - p_2) \leq 2p + 2\lceil \tfrac{p}{4} \rceil.$$

Hence $\qquad p_2 \geq p - 2\lceil \tfrac{p}{4} \rceil.$

Suppose $\qquad q \leq \lceil \frac{5p}{4} \rceil - 1 = p + \lceil \frac{p}{4} \rceil - 1$, then

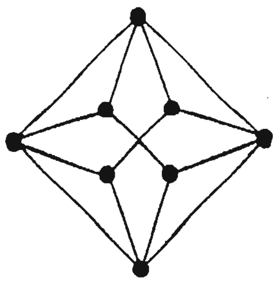
$$2p_2 + 3(p - p_2) \leq 2p + 2\lceil \tfrac{p}{4} \rceil - 2 \text{ and so}$$

$$p_2 \geq p + 2 - 2\lceil \tfrac{p}{4} \rceil \text{ and } p_3 \leq 2\lceil \tfrac{p}{4} \rceil - 2.$$

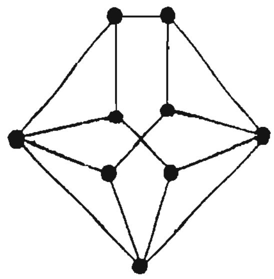
Thus $\qquad p_3 < p_2.$

Now, every vertex in V_2 is adjacent to at least one vertex in V_3. Furthermore, any vertex in V_3 is adjacent to at most one vertex in V_2, otherwise if $u, v \in V_2$ with $xu, xv \in E(G)$, then if $uv \in E(G)$, x is a cut vertex of G contradicting $\kappa(G) = 2$. Whereas, if $uv \notin E(G)$, then $|N(u) \cup N(v)| < 4$ and $\kappa_3(G) \leq 3$ contrary to assumption. Hence $p_3 \geq p_2$ which again is a contradiciton. Thus $q_{4,3}(p) = \lceil \frac{5p}{4} \rceil$. □

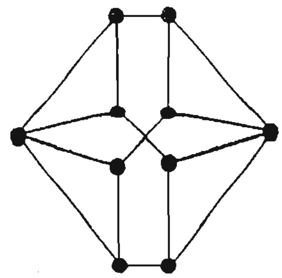
p = 8
q = 14



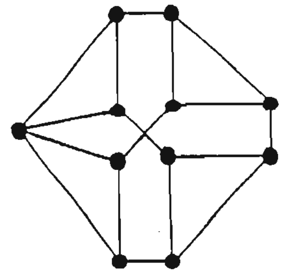
p = 9
q = 15



p = 10
q = 16

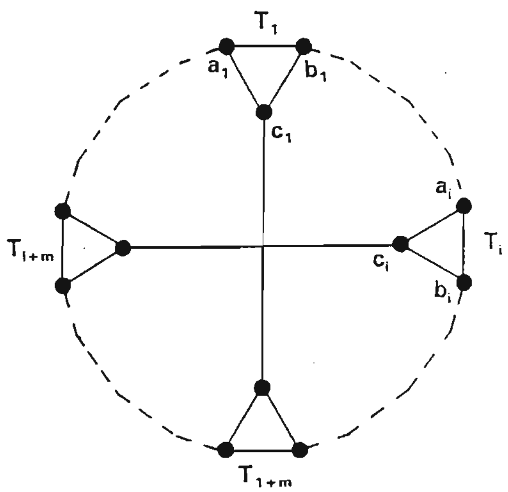


p = 11
q = 17

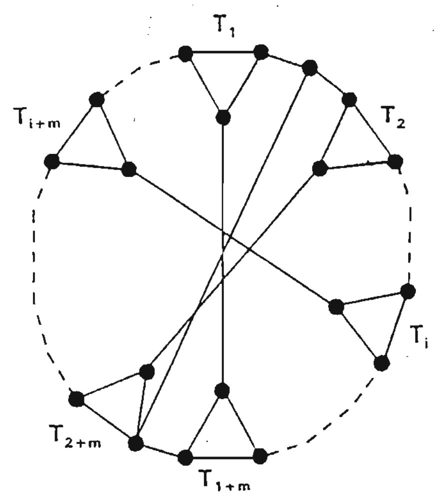


For m ≥ 2:

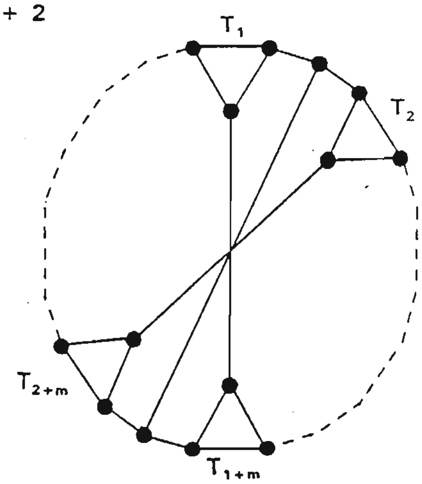
= 6m



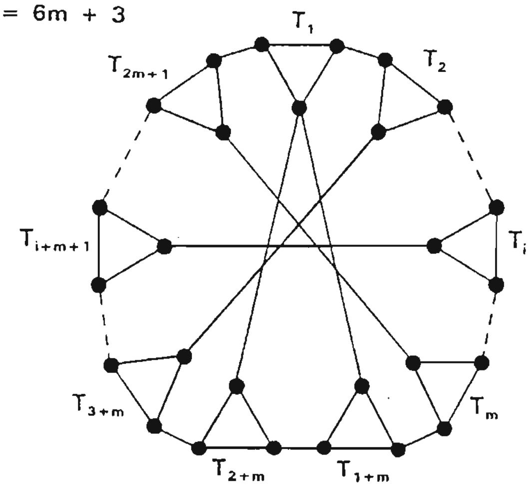
p = 6m + 1



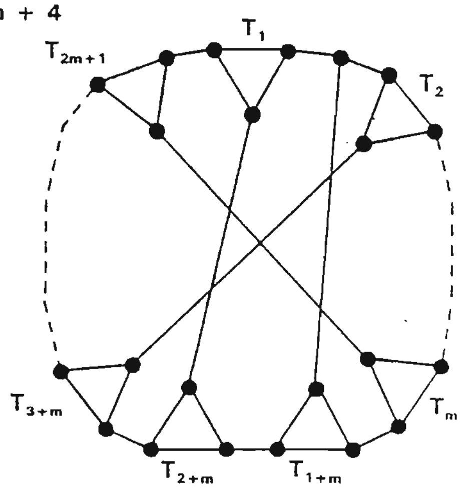
= 6m + 2



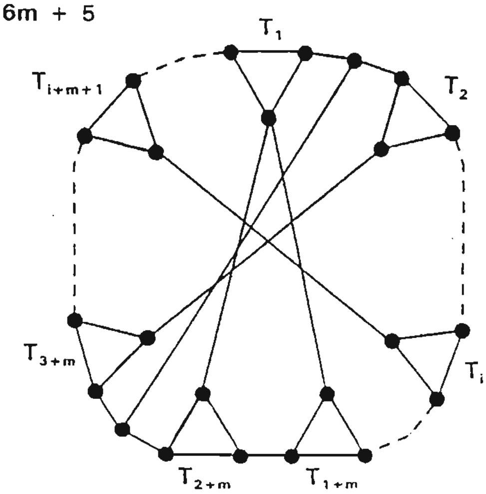
p = 6m + 3



= 6m + 4



p = 6m + 5



Clearly the graphs of figure 2.5.2.2 are (4,3)-minimum graphs and hence belong to $\mathcal{G}_{4,3}(p)$.

e) If $n = 5$

Theorem 2.5.2.10 The graphs of figure 2.5.2.3 have $\kappa_3(G) = 5$.

Proof: For $p \in \{8, 9, 10, 11\}$ it is not difficult to see that $\kappa_3(G) = 5$.

For $m \in \mathbf{N}, m \geq 2$, consider the cubic graph G, obtained from the $6m$-cycle $C_{6m} : a_0c_0b_0a_1c_1b_1...a_{2m-1}c_{2m-1}b_{2m-1}a_0$ by the insertion of the edges in the set $\{a_ib_i, c_jc_{j+m} | i = 0, 1, ..., 2m-1;\ j = 0, 1, ..., m-1\}$. Let T_i denote the triangle $a_ib_ic_i$ $(i = 0, ..., 2m-1)$ and let $C = \{c_i | i = 0, ..., 2m-i\}$. That $\kappa_3(G) \leq 5$ follows from the observation that $k(G - \{a_0, b_0, c_m, b_{m-1}, b_m\}) = 3$.

To prove that $\kappa_3(G) = 5$, we assume the existence of a set $S \subset V(G)$ such that $|S| \leq 4$ and $k(G-S) = 3$. Let G_1, G_2, G_3 be components of $G-S$ and, without loss of generality, assume that G_1 contains at least one vertex of T_0 and none of T_1. Note that $H : a_0b_{2m-1}c_{2m-1}a_{2m-1}...b_{m+3}c_{m+3}a_{m+3}$ $b_{m+2}a_{m+2}c_{m+2}c_2a_2b_2a_3c_3b_c...a_mb_mc_mc_0b_0a_0$ is a hamiltonian cycle of $G - (V(T_1) \cup V(T_{m+1}))$ whence it follows that $V(T_1) \not\subset S$. (Otherwise, if $V(T_1) \subset S$, $G - S$ contains at least two components which contain no vertex of $V(T_1) \cup V(T_{m+1})$ and are separated on H by at least two vertices of S; so $|S| \geq 5$, a contradiction.) Hence T_1 contains a vertex of $G - (S \cup V(G_1))$, say a vertex of G_2. Since $b_0a_1 \in E(G)$ and T_1 is complete, $\{b_0, a_1\} \cap S \neq \emptyset$.

Let i be the smallest index such that $V(G_2) \cap V(T_{i+1}) = \emptyset$; then it follows as above that $V(T_{i+1}) \cap V(G_r) \neq \emptyset$ for $r = 3$ or 1. Let j be the smallest index such that $V(G_r) \cap V(T_{j+1}) = \emptyset$; then $V(T_{j+1}) \cap V(G_s) \neq \emptyset$ for some $s \in \{1, 2, 3\} - \{r\}$.

It follows as above that $\{b_i, a_{i+1}\} \cap S \neq \emptyset$ and $\{b_j, a_{j+1}\} \cap S \neq \emptyset$, where $0 < i < j < 2m$. Hence $|S \cap C| \leq 1$.

We note that, as $|S| \leq 4$, the graph $C_{6m} - S$ contains at most four components, with vertex sets (say, without loss of generality) either $V(G_1), V(G_2), V(G_3)$ or $V(G_1), V(G_2), V(G_4), V(G_5)$, where $V(G_4) \cup V(G_5) = V(G_3)$. In the latter case, let k be the smallest index following j such that $V(G_5) \cap V(T_{k+1}) = \emptyset$, then $\{b_k, a_{k+1}\} \cap S \neq \emptyset$ and $j < k < 2m$, so $S \cap C = \emptyset$. However, it then follows from $c_0 \in V(G_1)$ and $C \cap S = \emptyset$, as G_1 is a component of $C_{6m} - S$ that G_1 contains all vertices in the set $\{c_m, b_m, a_{m+1}, c_{m+1}, b_{m+1}, \ldots, a_{2m-1}, c_{2m-1}, b_{2m-1}, a_0, c_0\}$; however, from $c_1 \in V(G_2)$ it follows similarly that $c_{m+1} \in V(G_2)$, a contradiction.

So $k(C_{6m} - S) = 3$ and G_1, G_2, G_3 are the three components of $C_{6m} - S$ (so $r = 3$ and $s = 1$). As $|S \cap C| \leq 1$, at most one of the vertices c_0, c_1 and c_{i+1} are contained in S; say $c_0 \in V(G_1)$ and $c_1 \in V(G_2)$; then a contradiction follows as above. Hence $|S| > 4$; i.e. $\kappa_3(G) = 5$. □

Similar methods suffice to prove that each of the other graphs shown in Figure 2.5.2.3 have $\kappa_3(G) = 5$.

From the above theorem and the graphs of figure 2.5.2.3 it follows that

$$q_{5,3}(p) \leq \begin{cases} p+6 \text{ if } p \in \{8,9,10\} \\ \lceil \frac{6p}{4} \rceil = \lceil \frac{3p}{2} \rceil \text{ if } p \geq 11. \end{cases}$$

Theorem 2.5.2.11

$$q_{5,3}(p) = \begin{cases} p+6 \text{ if } p \in \{8,9,10\} \\ \lceil \frac{3p}{2} \rceil \text{ if } p \geq 11 \end{cases}$$

Proof:

1. Let $p = 8$ and suppose that $q_{5,3}(8) \leq 13$. If v is a vertex of maximum degree in G, say deg $v = \Delta(G) \geq 3$, let $H = G - v$. Then $\kappa_3(G) \geq 5$

implies that $\kappa_3(H) \geq 4$ and so $q(H) \geq q_{4,3}(7) = 10$. But $q(G) \geq q(H) + \Delta(G) \geq 13$. So $q(G) = 13$ and hence $\Delta(G) = 3$. However, if $q(G) = 13$ and $p(G) = 8$ with $\Delta(G) = 3$, then $q(G) \leq \frac{8(3)}{2} = 12$,a contradiction. Hence $q_{5,3}(8) = 14$.

2. Suppose next that $q_{5,3}(9) \leq 14$ and let deg $v = \Delta(G) \geq 3$. Then $H = G - v$ has $p(H) = 8, \kappa_3(H) \geq 4$, and hence $q(H) \geq q_{4,3}(8) = 11$. Consequently $q(G) \geq 11 + 3 = 14$, with equality only if $\Delta(G) = 3$. Hence $q(G) = 14$, $\Delta(G) = 3$ and G has at most 8 vertices of degree 3, one of degree 2 (since $\Delta(G) = 3$) yielding $q(G) = \frac{1}{2}[8(3) + 1(2)] = 13 < 14$. Hence $q_{5,3}(9) = 15$.

3. Assume that there exists a (10,15) graph G with $\kappa_3(G) = 5$. Let $V(G) = \{v_1, ..., v_{10}\}$. Note that if $\Delta(G) \geq 14$ with (say) deg $v_1 \geq 4$, then $H = G - v_1$ has $p(H) = 9, q(H) \leq 11$ and $\kappa_3(H) \geq 4$, contradicting the fact that $q_{4,3}(9) = \left\lceil \frac{5(9)}{4} \right\rceil = 12$.
Hence, G is a 3-regular graph.

Let $N(v_1) = \{v_2, v_3, v_4\}$ and note that $\Delta(\langle\{v_2, v_3, v_4\}\rangle) \leq 1$, otherwise, if (say) $v_2v_3, v_3v_4 \in E(G)$, then $G - \{v_2, v_4\}$ has two components : $L_1 = \langle\{v_1, v_3\}\rangle \cong K_2$ and L_2, where L_2 is a (6,8)-graph with $\kappa_2(L_2) \geq 3$; hence $\delta(L_2) \geq 3$ and $q(L_2) \geq 9$, a contradiction. Since $|N(\{v_1, v_j\})| \geq 5$ for $j \in \{5, 6, 7, 8, 9, 10\}$, it follows that each vertex v_j ($j \in \{5, ..., 10\}$) is adjacent to at most one vertex in $\{v_2, v_3, v_4\}$ and to at least two vertices in $\{v_5, ..., v_{10}\}$. Without loss of generality, let $N(v_2) = \{v_1, v_5, v_6\}$ and suppose that $v_3v_4 \in E(G)$. Then, as $|N(\{v_2, v_3\})| \geq 5$, v_3 is non-adjacent to v_5 and v_6, as is v_4. Let $N(v_3) = \{v_1, v_4, v_7\}$; then $v_7v_4 \notin E(G)$, so let $N(v_4) = \{v_1, v_4, v_8\}$. If $v_7v_8 \notin E(G)$, then $G - \{v_2, v_7, v_8\}$ has two components, $L_3 = \langle\{v_1, v_3, v_4\}\rangle$ and $L_4 = \langle\{v_5, v_6, v_9, v_{10}\}\rangle$, where $p(L_4) = 4$, $q(L_4) = 15 - 12 = 3$; so $\kappa_2(L_4) \leq 1$ and $\kappa_3(G) \leq 4$, a contradiction. So $v_7v_8 \in E(G)$, which again produces a contradiction since if we follow the same argument for v_3 as for v_1 above, then v_8 can be adjacent to at most one neighbour of v_3. Thus $v_3v_4 \notin E(G)$.

Hence $N(v_3) = \{v_1, v_7, v_8\}$ and $N(v_4) = \{v_1, v_9, v_{10}\}$ (say) and we note that the girth of G is at least 5. Recall that an n-cage is a 3-regular graph with girth n and smallest possible order (viz. $f(3,n)$; see [CL1] in which it is shown that $f(3,5) = 3^2 + 1 = 10 = p(G)$).

It is also known that the Petersen graph is the unique 5-cage ([CL1], p. 42, Th. 2.9) and so our graph G must be the Petersen graph P for which $\kappa_3(P) = 4$, producing a contradiction. Thus $q_{5,3}(10) \geq 16$, which together with the above theorem gives $q_{5,3}(10) = 16$.

4. For $p \geq 11$ note that if $\delta(G) \geq 3$ then $q_{5,3}(p) = q(G) \geq \frac{3p}{2}$; hence $q(G) \geq \lceil \frac{3p}{2} \rceil$ if $\delta(G) \geq 3$.

Suppose $q(G) < \frac{3p}{2}$; then $\delta(G) \in \{1, 2\}$. If $\delta(G) = 1$, let deg $u = 1$, $N(u) = \{v\}$ and note that $H = G - \{u, v\}$ has $\kappa(H) \geq 4$, hence $\delta(H) \geq 4$ and $q(H) \geq 2(p-2) = 2p - 4$; so $q(G) \geq 2p - 4 +$ deg $v \geq 2p - 2$. From $q(G) < \frac{3p}{2}$ we obtain $\frac{p}{2} < 2$, whence $p < 4$, contrary to assumption.

If $\delta(G) = 2$, let deg $u = 2$, $N(u) = \{v_1, v_2\}$ and $J = G - \{u, v_1, v_2\}$. From $|N(\{u, w\})| \geq 5$ if $w \notin N[u]$, we obtain deg $w \geq 3$ if $w \in V(G) - N[\{v_1, v_2\}]$ and deg $w \geq 4$ if $w \in N[\{v_1, v_2\}] - \{u, v_1, v_2\}$; furthermore, since $\kappa(J) \geq 3$, we have $\delta(J) \geq 3$ and so $q(J) \geq \frac{3}{2}(p-3)$. Now $q(G) \geq \frac{3}{2}(p-3) +$ deg $v_1 +$ deg $v_2 - \epsilon$, where

$$\epsilon = \begin{cases} 0 & \text{if} \quad v_1v_2 \notin E(G) \\ 1 & \text{if} \quad v_1v_2 \in E(G). \end{cases}$$

So, if $q(G) < \frac{3p}{2}$, it follows that $\frac{3p}{2} > \frac{3p}{2} - \frac{9}{2} + \deg v_1 + \deg v_2 - \epsilon$, whence $\deg v_1 + \deg v_2 < \frac{9}{2} + \epsilon$, i.e. $\deg v_1 + \deg v_2 \leq 4 + \epsilon$. But if $v_1v_2 \notin E(G)$ (so $\epsilon = 0$), then $\deg v_1 + \deg v_2 \geq 5$, so $v_1v_2 \in E(G)$. Let $|N(v_1) \cap V(J)| = a_i \quad (i = 1, 2)$; then $a_1 + a_2 + 4 = \deg v_1 + \deg v_2 \leq 4+1$ and so $a_1 + a_2 \leq 1$, but G is connected, so $a_1 + a_2 \geq 1$; hence (say) $a_1 = 1$ and $a_2 = 0$, with $N(v_1) = \{u, v_2, z\}$.

Let $L = G - \{u, v_1, v_2, z\}$; then $\deg z \geq 4$ (since $|N(\{u, z\})| \geq 5$). Now $\kappa(L) \geq 4$, so $\delta(L) \geq 4$ and $q(L) \geq 2(p-4)$, whence $q(G) \geq 2p - 4 + 3 + 4 \geq 2p - 3$. So, as $q(G) < \frac{3p}{2}$, we have $\frac{3p}{2} > 2p - 3$ and so $p < 6$, a contradiction.

So $q_{5,3}(p) \geq \lceil \frac{3p}{2} \rceil$ which together with the above theorem gives $q_{5,3}(p) = \lceil \frac{3p}{2} \rceil$ for $p \geq 11$. □

Clearly, from the above, the graphs of figure 2.5.2.3 belong to $\mathcal{G}_{5,3}$.

f) $n = 6$.

We first make the observation that, for $G \in \mathcal{G}_{6,3}$, if $uv \in E(\bar{G})$, then $|N(u) \cup N(v)| \geq 6$.

Also it is known that $\kappa_3(C_p^2) = 6$ for $p \geq 9$; so $q_{6,3}(p) \leq 2p$ for $p \geq 9$.

Theorem 2.5.2.12 If $G \in \mathcal{G}_{6,3}$, then $\delta(G) \geq 3$.

Proof: It is clear that $\delta(G) \geq 2$, otherwise, if $\deg_G u = 1$ and $uv \in E(G)$, then $\kappa(G - \{u, v\}) \geq 5$ and so $q(G - \{u, v\}) \geq \frac{5}{2}(p-2)$. But $q(G - \{u, v\} \leq 2p - 2$; so $p \leq 6$, a contradiction.

Suppose there exists $u \in V(G)$ with $N(u) = \{v_1, v_2\}$, and let $H = G - \{u, v_1, v_2\}$. Then $\kappa(H) \geq 4$; hence $\delta(H) \geq 4$ and $q(H) \geq 2(p-3) = 2p - 6$.

So the number of edges covered by $\{v_1, v_2\}$ is at most 6, whence it follows that $v_1v_2 \in E(G)$, otherwise, if $v_1v_2 \in E(\bar{G})$, then, $|N(v_1) \cup N(v_2) - \{u\}| \geq 5$ and so $\{v_1, v_2\}$ covers at least 7 edges, a contradiction. Thus it follows that either $\deg_G v_1 \leq 3$ or $\deg_G v_2 \leq 3$ (say the former).

If $\deg_G v_1 = 2$, then $G - v_2$ has two components $\langle\{u, v_1\}\rangle \cong K_2$ and H, with $\kappa(H) \geq 5$, hence $q(H) \geq \frac{5}{2}(p-3)$ and so $q(G) \geq \frac{5}{2}(p-3) + 4$, but $q(G) \leq 2p$, whence it follows that $p \leq 7$, a contradiction.

So $\deg_G v_1 = 3$ and $\deg_G v_2 \leq 4$. Let $N(v_1) = \{u, v_2, w\}$; then $G - \{w, v_2\}$ has 2 components, $\langle\{v_1, u\}\rangle \cong K_2$ and (say) J, where $\kappa(J) \geq 4$ and so $\delta(J) \geq 4$,whence $q(J) \geq 2(p-4) = 2p-8$; but, as $\delta(H) \geq 4$, w is adjacent to at least 4 vertices in H and so $q(G) \geq q(J) + 3 + |[v_2, V(J)]| +$ deg $w \geq 2p - 8 + 3 + 1 + 5 > 2p$, a contradiction. So $\delta(G) \geq 3$. □

Let $G_3 = \langle\{v \in V(G) |\ \deg v = 3; G \in \mathcal{G}_{6,3}\}\rangle$ and let p_i denote the number of vertices in G of degree $i (i \geq 3)$.

Theorem 2.5.2.13 If H is a component of G_3, then $|N(V(H)) - V(H)| \geq 3$.

Proof: We note first that each component of G_3 is complete, otherwise G_3 contains two vertices, v_1 and v_2, with $d_G(v_1, v_2) = 2$ and $|N(v_1) \cup N(v_2)| \leq 5$. So each component of G_3 is isomorphic to K_1, K_2 or K_3.

A similar argument shows that if $w \in V(G) - V(G_3)$, then all vertices in $N(w) \cap V(G_3)$ are contained in a single component of G_3 (or $N(w) \cap V(G_3) = \emptyset$).

If $V(H) = \{v_1, v_2, v_3\}$ and $N(V(H)) - V(H) = \{w\}$, then $G - w$ has two components, viz. H and (say) L_1 where $\kappa(L_1) \geq 5$ and so $q(L_1) \geq \frac{5}{2}p(L_1) = \frac{5}{2}(p-4)$; but $q(L_1) \leq q(G) - 7 \leq 2p - 7$ whence $p \leq 6$, a contradiction.

If $N(V(H)) - V(H) = \{w_1, w_2\}$, where (say) $v_1w_1, v_2w_1 \in E(G)$ and

$v_3w_2 \in E(G)$; but $v_3w_1 \in E(\bar{G})$, so (as $\deg_G v_3 = 3$ and $6 \leq |N(v_3) \cup N(w_1)| = \deg_G v_3 + \deg_G w_1 - |N(v_3) \cap N(w_1)| = \deg_G w_1 + 1$), it follows that $\deg_G w_1 \geq 5$. Now $G - \{w_1, w_2\}$ has two components, viz. H and (say) L_2, where $\kappa(L_2) \geq 4$ and so $q(L_2) \geq 2(p-5) = 2p - 10$. But $q(L_2) \leq q - 11 \leq 2p - 11$, a contradiction.

So, if H is a component of G_3 of order 3, then $|N(V(H)) - V(H)| \geq 3$.

If H is a trivial component of G_3, then obviously $|N(V(H)) - V(H)| = |N(V(H))| = 3$.

Finally, if $H \cong K_2$ with $V(H) = \{v_1, v_2\}$ then, if $|N(V(H)) - V(H)| = 2$, (say $N(V(H)) - V(H) = \{w_1, w_2\}$), then $G - \{w_1, w_2\}$ has two components viz. H and (say) L_3, where $\kappa(L_3) \geq 4$, so $\delta(L_3) \geq 4$ and consequently $q(L_3) \geq 2p(L_3) = 2(p-4) = 2p - 8$.

However, $q(L_3) \leq q(G) - 8 \leq 2p - 8$, with equality if and only if $q = 2p$, deg w_1 = deg $w_2 = 4$ and $w_1w_2 \in E(G)$, which must therefore be valid in this case.

If $|N(\{w_1, w_2\}) \cap V(L_3)| = 1$, let $\{a\} = N(\{w_1, w_2\}) \cap V(L_3)$ and note that $G - a$ has two components, $\langle\{v_1, v_2, w_1, w_2\}\rangle$ and (say) L_4, where $\kappa(L_4) \geq 5, q(L_4) \geq \frac{5}{2}(p-5)$ and $q(L_4) \leq q - 10 \leq 2p - 10$; whence $p \leq 5$, a contradiction. If $N(\{w_1, w_2\}) \cap V(L_3) = \{a_1, a_2\}$, then deg $a_i \geq 4$ for $i = 1, 2$ and $G - \{a_1, a_2\}$ has two components, $\langle\{v_1, v_2, w_1, w_2\}\rangle$ and (say) L_5 where $\kappa(L_5) \geq 4$, $q(L_5) \geq 2(p-6) = 2p - 12$ and $q(L_5) \leq q(G) - 13 \leq 2p - 13$, a contradiction.

So $|N(V(H)) - V(H)| \geq 3$. □

Theorem 2.5.2.14 For $p \geq 9$, $\lceil \frac{7p}{4} \rceil \leq q_{6,3}(p) \leq 2p$.

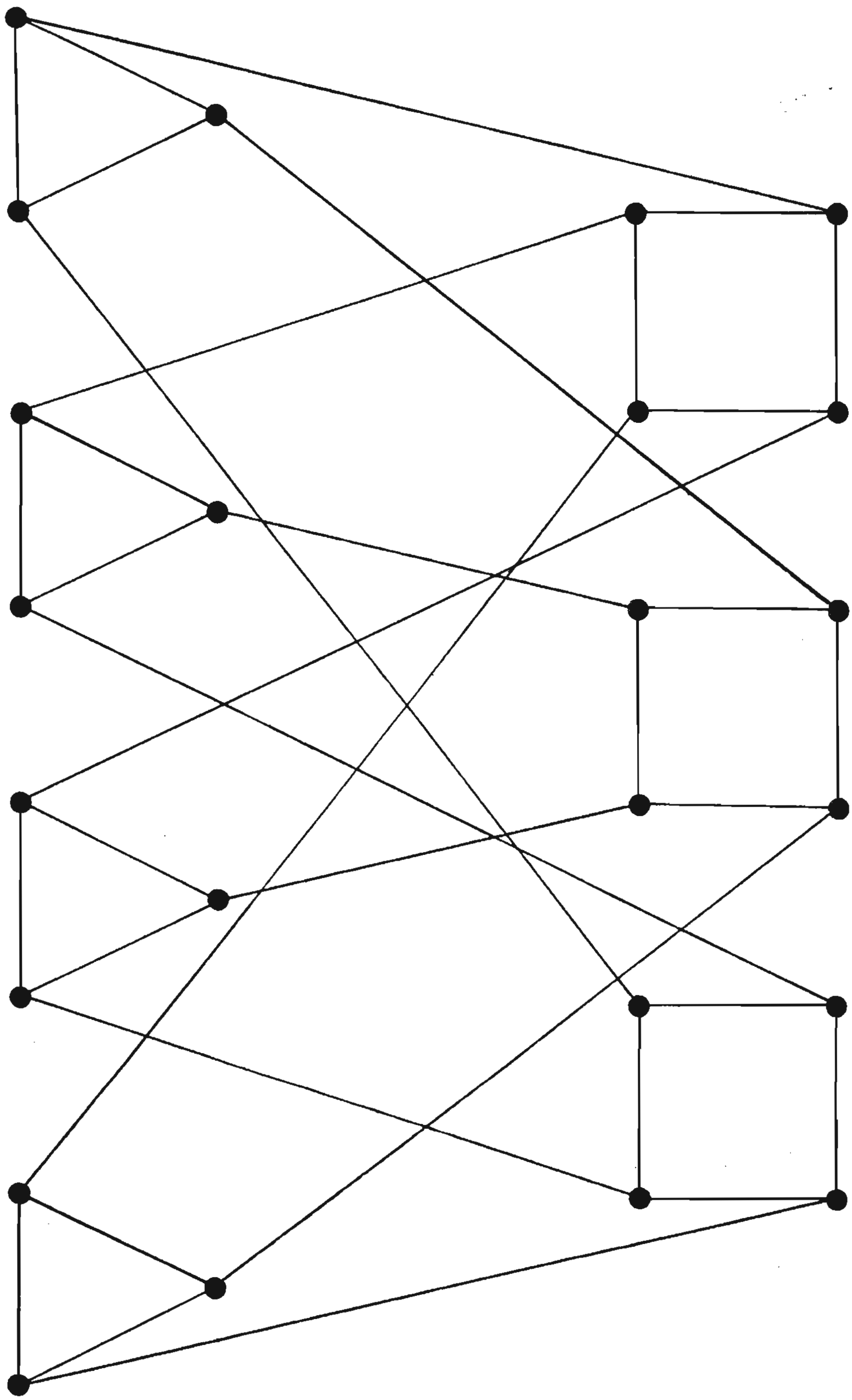


Figure 2.5.2.4

Proof: It follows from the above theorem that if $G \in \mathcal{G}_{6,3}$, then $p - p_3 = |V(G) - V(G_3)| \geq 3k(G_3) = 3n_1 + 3n_2 + 3n_3$ and $p_3 = 3n_3 + 2n_2 + n_1$, where n_i is the number of components of order i in G_3 $(i = 1, 2, 3)$.

So $p_3 \leq p/2, p - p_3 \geq p/2$ and $2q \geq (3+4)\frac{p}{2}$.
Thus $q \geq \frac{7p}{4}$. $\square$

For the case $p = 9$, $q_{6,3}(9) \leq q(C_9^2) = 18$. Furthermore, if $S \subset V(G)$ such that $|S| = 6$ and $k(G - S) = 3$, then $K_3(\langle S \rangle) \geq 3$, hence $q(\langle S \rangle) \geq 6$ and each vertex in S is adjacent to at least 2 vertices in $V - S$; hence $|[S, V - S]| \geq 12$ and $q \geq 6 + 12 = 18$. Thus $q_{6,3}(9) = q(C_9^2) = 18$.

Figure 2.5.2.4 shows a graph on 24 vertices and $\left\lceil \frac{7(24)}{4} \right\rceil = 42$ edges which is easily seen to have 3-connectivity equal to 6.

At this stage it remains open to discover whether or not $q_{6,3}(p) = \lceil \frac{7p}{4} \rceil$ for all $p \geq t > 9$ and to establish the values of $q_{6,3}(p)$ for $10 \leq p < t$.

Finally we conjecture that, for $p \geq n + \ell$ and $n, \ell \geq 2$, both $q_{n-1,\ell}(p) < q_{n,\ell}(p)$ and $q_{n,\ell}(p-1) < q_{n,\ell}(p)$. It should be noted that the validity of these statements in the case where $\ell = 2$ follows from our knowledge of the exact value of $q_{n,2}(p)$ $\left(= \lceil \frac{pn}{2} \rceil\right)$ and that the proofs of the above conjectures (if true) may be dependent on the establishment of a corresponding value of $q_{n,\ell}(p)$ for $\ell \geq 3$.

Chapter 3

Steiner Distance Hereditary Graphs

3.1 Introduction

The *distance* $d_G(u,v)$ between two vertices, u, v of a connected graph G is the length of a shortest $u-v$ path of G. The *eccentricity* $e(v)$ of a vertex v is $\max\{d(v,u) \mid u \in V(G)\}$. If G is a connected graph and $S \subseteq V(G)$, then the *Steiner distance* $d_G(S)$ is the size of a smallest connected subgraph of G that contains S. Such a subgraph is obviously a tree and is called a *Steiner tree* for S. If T is a tree then a vertex of degree 1 in T is an *end-vertex* whilst all other vertices of T are called *internal* vertices of T.

Howorka [H2] in 1977 defined a graph G to be *distance- hereditary* if each connected induced subgraph F of G has the property that $d_F(u,v) = d_G(u,v)$ for each $u, v \in V(F)$. In order to state the characterizations of distance hereditary graphs given by Howorka [H2], we need the following terminology. An *induced path* of G is a path which is an induced subgraph of G. Let $u, v \in V(G)$. Then a *u-v geodesic* is a shortest $u-v$ path. Let C be a cycle of G. A path P is an *essential part* of C if P is a subgraph of C and $\frac{1}{2}|E(C)| < |E(P)| < |E(C)|$. An edge of G that joins two vertices of C that are not adjacent in C is called a *diagonal* of C. We say that two diag-

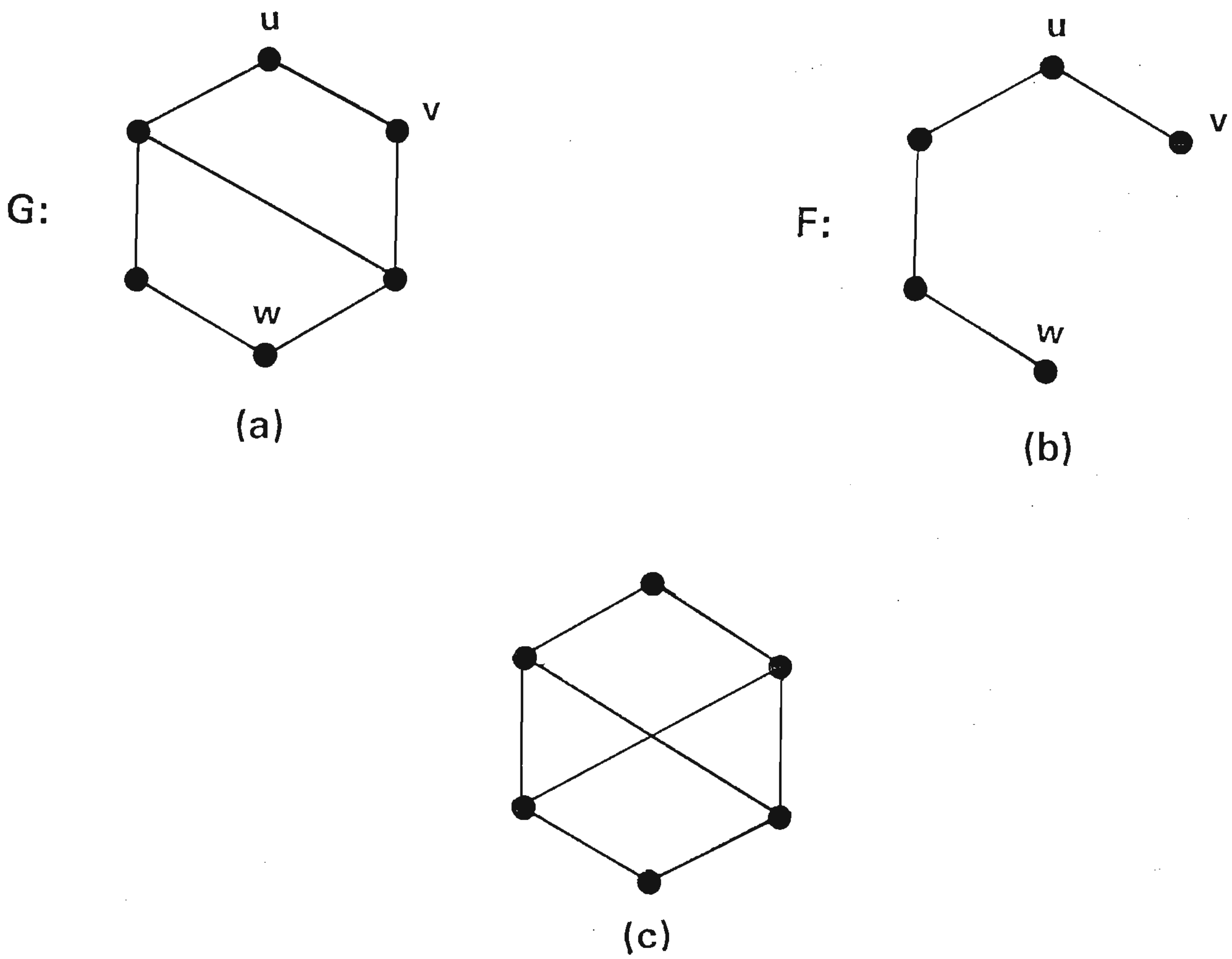


Figure 3.2

onals e_1, e_2 of C are *skew diagonals*, if $C + e_1 + e_2$ is homeomorphic with K_4.

Theorem F (Howorka)

The following are equivalent:

(i) G is distance-hereditary;

(ii) every induced path of G is a geodesic;

(iii) no essential part of a cycle is induced;

(iv) each cycle of length at least 5 has at least two diagonals and each 5-cycle has a pair of skew diagonals.

(v) Each cycle of G of length at least 5 has a pair of skew diagonals.

The definition of the Steiner distance of a set of vertices together with the concept of distance-hereditary graphs suggests a generalization to Steiner distance hereditary graphs. In this chapter we first consider this generalization and then characterize the 3-Steiner Distance Hereditary Graphs.

3.2 Generalization of Steiner Distance Hereditary Graphs

A connected graph is *k-Steiner distance hereditary,* $k \geq 2$, if for every connected induced subgraph H of G of order at least k and set S of k vertices of H, $d_H(S) = d_G(S)$. Thus 2-Steiner distance hereditary graphs are distance hereditary. Figure 3.2(a) shows a graph G that is not 3-Steiner distance hereditary since $d_F(\{u, v, w\}) \neq d_G(\{u, v, w\})$ where F is the induced subgraph of G shown in Figure 3.2(b). However, it is not difficult to show that the graph of Figure 3.2(c) is 3-Steiner distance hereditary.

The problem of determining the Steiner distance of a set of vertices in a graph appears to be difficult. In fact the following related decision problem π is NP-complete (see [GJ1 p. 208]).

π : Suppose G is a weighted graph whose edges have positive integer weights. Let $S \subseteq V(G)$ and suppose B is a positive integer. Does there exist a subtree T of G that includes S and is such that the sum of the weights of the edges of T is no more than B?

Furthermore, the problem remains NP-complete even if G is a graph. This suggests solving the problem in certain special cases. If it is known that a graph is k-Steiner distance hereditary, then $d_G(S)$ can easily be determined for every set S of $k \geq 2$ vertices of G as follows:

Let the vertices of $G-S$ be denoted by $v_1, v_2, ..., v_{p-k}$. Let $G_0 = G$. For each $i(1 \leq i \leq p-k)$, if the vertices of S belong to the same component of $G_{i-1} - v_i$, then G_i is defined to be $G_{i-1} - v_i$, otherwise, let G_i be G_{i-1}. Thus G_{p-k} is a connected induced subgraph of G that contains S. Therefore $d_{G_{p-k}}(S) = d_G(S)$. However, since the deletion of any vertex of G_{p-k} separates at least two vertices of S, no subgraph with fewer vertices than $p(G_{p-k})$ contains S and is connected. Thus G_{p-k} is a connected subgraph of smallest order that contains S. Hence any spanning tree of G_{p-k} is a Steiner tree for S.

Our first result shows that if G is a connected distance hereditary graph, then $d_G(S)$ can be determined by the above procedure for any set $S \subseteq V(G)$ of at least two vertices.

Theorem 3.2.1 If G is 2-Steiner distance hereditary, then G is k-Steiner distance hereditary for all $k \geq 3$.

Proof Suppose, to the contrary, there exists a graph G which is 2-Steiner distance hereditary, but not k- Steiner distance hereditary for some $k \geq 3$. Let k be as small as possible and let H be a connected induced subgraph of G of smallest order, n say, for which there is a set S of k vertices of H such that $d_H(S) > d_G(S)$. Let $S = \{x_1, x_2, ..., x_k\}$. If $|V(H)| = k$, then there exists exactly one set of k vertices in H, namely $V(H)$. However, then every spanning tree of H is a Steiner tree for $V(H)$ in H and has size $k-1$. Since $d_G(V(H)) \geq k-1$, it follows that $d_G(V(H)) = d_H(V(H))$ in this case. This contradicts our choice of H. Hence $|V(H)| \geq k+1$. If $d_H(S) \leq k-2$ let T be a Steiner tree for S in H and let $H' = \langle V(T) \rangle_G$. Then $d_{H'}(S) = d_H(S) > d_G(S)$ and $|V(H')| < |V(H)|$ which contradicts our choice of H. Hence $d_H(S) = n-1$, i.e. a Steiner tree for S in H must contain all the vertices of H. By our choice of $k, d_H(S - \{x_i\}) = d_G(S - \{x_i\})$ for all $i(1 \leq i \leq k)$.

We now show that no Steiner tree T' for S in G contains any x_i $(1 \leq i \leq k)$ as an internal vertex. Suppose T' contains some x_i as internal vertex. Let $T_1, T_2, ..., T_m$ be the components of $T' - x_i$. Let T_1' be the subgraph of T' induced by $V(T_1) \cup \{x_i\}$ and let T_2' be the subgraph of T' induced by $(\bigcup_{j=2}^{m} V(T_j)) \cup \{x_i\}$. Let $S_1 = S \cap V(T_1')$ and $S_2 = S \cap V(T_2')$. Since $2 \leq |S_i| < k$ for $i = 1, 2$, it follows that $d_H(S_i) = d_G(S_i)$ for $i = 1, 2$. Further, $|E(T_i')| = d_G(S_i)$ for $i = 1, 2$, otherwise we can find a tree with fewer than $q(T') = d_G(S)$ edges that contains S. This is not possible. Let T_i be a Steiner tree for S_i in $H(i = 1, 2)$. Then $d_H(S) \leq d_H(S_1) + d_H(S_2) = |E(T_1')| + |E(T_2')| = d_G(S)$. This again produces a contradiction to the choice of S. Hence, every Steiner tree for S in G has k end-vertices which are precisely the vertices of S. Thus $d_G(S - \{x_i\}) < d_G(S)$ for all $i(1 \leq i \leq k)$.

We prove next that every vertex of S has degree 1 in H and is therefore an end-vertex of every Steiner tree for S in H.

Let $x_i \in S$ and note that every Steiner tree for $S - \{x_i\}$ in H does not contain x_i; otherwise $d_H(S) = d_H(S - \{x_i\}) = d_G(S - \{x_i\}) < d_G(S)$

which contradicts the fact that $d_H(S) > d_G(S)$. Let T_i be a Steiner tree for $S - \{x_i\}$ in H. Denote by P_i a shortest path in H from x_i to $V(T_i)$ and note that every vertex in H occurs in $V(T_i) \cup V(P_i)$ for $1 \le i \le k$, since $d_H(S) = |V(H)| - 1$. So P_i contains at least one edge. If P_i contains an internal vertex, w say, and $\deg_H x_i \ge 2$, then x_i has a neighbour y in H which is contained in $V(T_i)$ and $y \notin V(P_i)$, which produces a contradiction as x_i, y is a path from x_i to $V(T_i)$, which is shorter than P_i. Hence if $\deg_H x_i \ge 2$, then P_i has length 1. Therefore

$$\begin{aligned} d_H(S) &= d_H(S - \{x_i\}) + 1 \\ &= d_G(S - \{x_i\}) + 1 \\ &\le d_G(S); \end{aligned}$$

contrary to our assumption. Hence every $x_i \in S$ has degree 1 in H. Therefore every Steiner tree for S in H has k end- vertices.

Next consider T, a Steiner tree for S in H. Let ℓ_i be the length of a shortest path Q_i (in H) from x_i to a vertex v_i of degree at least 3 in T for $i = 1, 2, \ldots, k$. Let $w_{i,1}$ be the vertex that precedes v_i on Q_i and observe that except for possibly $w_{i,1}$ no internal vertex of Q_i has degree exceeding 2 in H. We now show that

$$d_H(S) = \begin{cases} d_H(S - \{x_i\}) + \ell_i \text{ if } v_i \in V(T_i) \\ d_H(S - \{x_i\}) + \ell_i + 1 \text{ if } v_i \notin V(T_i), \end{cases} \tag{3.2.1}$$

where T_i is a Steiner tree on $S - \{x_i\}$ and where in the latter case $w_{i,1}$ has degree 2 in H.

We show first that $d_H(S - \{x_i\}) \ge d_H(S) - (\ell_i + 1)$. If this is not the case, then $d_H(S - \{x_i\}) \le d_H(S) - \ell_i - 2$ and neither v_i nor any of its neighbours in T belongs to T_i. Let $w_{i,2}$ and $w_{i,3}$ be two vertices distinct from $w_{i,1}$ that are adjacent with v_i in T. Then $T - v_i w_{i,2}$ must contain x_i and $w_{i,3}$ in the same component and thus some vertex $x_j \ne x_i$ such that the $x_i - x_j$ path P' in T contains $w_{i,3}$. Then P' together with T_i produces a connected subgraph of H that contains S but not $w_{i,2}$. However, then $d_H(S) < p(H) - 1$, a contradiction. Hence, $d_H(S - \{x_i\}) \ge d_H(S) - (\ell_i + 1)$.

If $v_i \in V(T_i)$, then the length of a shortest path from x_i to T_i is at most ℓ_i. On the other hand we know that it is at least ℓ_i. Hence it is exactly ℓ_i. So $d_H(S) = d_H(S - \{x_i\}) + \ell_i$ in this case. If $v_i \notin V(T_i)$, then some neighbour of v_i distinct from $w_{i,1}$ must belong to T_i. Further, v_i must be on a shortest path from x_i to T_i. Therefore $w_{i,1}$ has degree 2 in H. Hence $d_H(S) = d_H(S - \{x_i\}) + \ell_i + 1$ in this case.

Let T' be a Steiner tree for S in G and let $H' = \langle V(T')\rangle_G$. Since T' has k end-vertices there is some pair x_i, x_j of vertices of S for which the $x_i - x_j$ path in T' contains exactly one vertex of degree at least 3 in T', say y. Without loss of generality we may assume $x_i = x_1$ and $x_j = x_2$. Let $\ell_1' = d_{T'}(x_1, y)$ and $\ell_2' = d_{T'}(x_2, y)$. Observe that $d_G(x_1, x_2) \leq \ell_1' + \ell_2'$ and that $d_H(x_1, x_2) \geq \ell_1 + \ell_2 - 1$. Hence $d_G(x_1, x_2) \geq \ell_1 + \ell_2 - 1$. We now consider two cases.

Case 1 Suppose $d_H(x_1, x_2) = \ell_1 + \ell_2 - 1$. Then $w_{1,1}$ and $w_{2,1}$ must be adjacent in H and further, v_i must belong to T_i for $i = 1, 2$, by (3.2.1). Thus $d_H(S) = d_H(S - \{x_i\}) + \ell_i > d_G(S) \geq d_G(S - \{x_i\}) + \ell_i'$ for $i = 1, 2$. Therefore $\ell_i \geq \ell_i' + 1$ for $i = 1, 2$. Hence

$$d_H(x_1, x_2) = \ell_1 + \ell_2 - 1 \geq \ell_1' + \ell_2' + 1 > d_G(x_1, x_2),$$

a contradiction, since G is 2-Steiner distance hereditary and because H is a connected induced subgraph of G.

Case 2 Suppose $d_H(x_1, x_2) \geq \ell_1 + \ell_2$. Suppose first that $d_H(x_1, x_2) \geq \ell_1 + \ell_2 + 1$. Since $d_H(S - \{x_i\}) + \ell_i + 1 \geq d_H(S) > d_G(S) \geq d_G(S - \{x_i\}) + \ell_i'$, it follows that $\ell_i \geq \ell_i'$ for $i = 1, 2$. Hence $d_H(x_1, x_2) \geq \ell_1 + \ell_2 + 1 > \ell_1' + \ell_2'$ $\geq d_G(x_1, x_2)$. This again contradicts the fact that G is 2-Steiner distance hereditary.

Suppose thus that $d_H(x_1, x_2) = \ell_1 + \ell_2$. Then $w_{1,1}$ and $w_{2,1}$ are not adjacent in H. If $d_H(S - \{x_i\}) + \ell_i = d_H(S)$ for $i = 1, 2$, then, by (3.2.1), v_i is in the vertex set of T_i. Suppose $d_H(S - \{x_1\}) + \ell_1 = d_H(S)$. Then, as before $\ell_1 \geq \ell_1' + 1$, and $\ell_2 \geq \ell_2'$. Hence $d_H(x_1, x_2) =$

$\ell_1+\ell_2 > \ell_1'+\ell_2' \geq d_G(x_1,x_2)$. This is not possible since G is 2-Steiner distance hereditary.

So we may assume $d_H(S) = d_H(S-\{x_i\})+\ell_i+1$ for $i = 1,2$. Thus by (3.2.1), $v_i \notin V(T_i)$ for $i = 1,2$. We show next that $w_{1,1}$ and $w_{2,1}$ both have degree 2 in H. Suppose $w_{1,1}$ has degree at least 3 in H. Let w be a vertex adjacent with $w_{1,1}$ that does not belong to Q_1. Then there is a path P in H from x_1 to T_1 that passes through w but does not contain v_1. Thus T_1 together with P produce a connected subgraph of H that contains all the vertices S but not v_1. Thus $d_H(S) < p(H) - 1$, a contradiction. Therefore $w_{1,1}$ and $w_{2,1}$ both have degree 2 in H. Thus $v_1 = v_2$. However, then necessarily $v_1(= v_2)$ must belong to T_1, so that $d_H(S) = d_H(S - \{x_1\}) + \ell_1$, which we have already shown cannot happen. □

Observe that, for $k \geq 3$, the $(k+2)$-cycle C_{k+2}, is $(k+2)-, (k+1)-$ and k- Steiner distance hereditary but not $(k-1)$-Steiner distance hereditary. Thus the converse of Theorem 3.2.1 does not hold.

Several characterizations of distance hereditary graphs which yield polynomial algorithms that test whether a graph is distance hereditary have been established. In order to state some of these characterizations we define an *isolated vertex* to be a vertex having degree 0, and two vertices v and v' are *twins* if they have the same neighbourhood or the same closed neighbourhood.

The following characterization of distance hereditary graphs was discovered independently by Bandelt and Mulder [BM1], D'Atri and Moscarini [DM1] and Hammer and Maffray [HM1].

Theorem G A graph G is distance hereditary if and only if every induced subgraph of G contains an isolated vertex, an end-vertex or a pair of twins.

The result we establish next is another characterization of 2- Steiner distance hereditary graphs and also suggests an efficient algorithm for determining whether a connected graph is 2-Steiner distance hereditary. This result is also a direct consequence of a characterization of distance hereditary graphs obtained independently by Bandelt and Mulder [BM1] and D'Atri and Moscarini [DM1]. We will need the following terminology. Suppose G is a connected graph and $u \in V(G)$. Let $V_{u,i} = \{x \in V(G) | d_G(u,x) = i\}$ for $0 \leq i \leq e_G(u)$ where $e_G(u)$ is the eccentricity of u in G and let $N_{i-1}(u,v) = N(v) \cap V_{u,i-1}$ for $1 \leq i \leq e_G(u)$.

Theorem 3.2.2

A connected graph G contains an induced path that is not a geodesic, if and only if there exists a vertex u and an integer $i \geq 2$ such that for some pair x, y of vertices in $V_{u,i}$,

(1) $xy \in E(G)$ and $N_{i-1}(u,x) \neq N_{i-1}(u,y)$ or 3.2.2.1

(2) $xy \notin E(G)$, $N_{i-1}(u,v) \neq N_{i-1}(u,y)$ and x and y are both adjacent with some vertex z in $V_{u,i+1}$. 3.2.2.2

Proof Suppose there is some vertex u and an integer $i \geq 2$ such that for some pair $x, y \in V_{u,i}$, 3.2.2.1 or 3.2.2.2 holds. Suppose first 3.2.2.1 holds. Since $N_{i-1}(u,x) \neq N_{i-1}(u,y)$, so $N_{i-1}(u,x) - N_{i-1}(u,y) \neq \emptyset$ or $N_{i-1}(u,y) - N_{i-1}(u,x) \neq \emptyset$. Suppose that the former holds. Let $x_1 \in N_{i-1}(u,x) - N_{i-1}(u,y)$. Let P_1 be a shortest $u-x$ path that passes through x_1 and let P_2 be a shortest $u-y$ path. Let a be the last vertex that P_1 and P_2 have in common (possibly $a = u$). Then the vertices on the $a-x$ subpath of P_1 together with y induce an $a-y$ path P that is longer than the $a-y$ subpath of P_2. Hence G contains an induced path that is not a geodesic.

Suppose now that 3.2.2.2 holds. We may again assume that there exists a vertex $x_1 \in N_{i-1}(u,x) - N_{i-1}(u,y)$. Clearly $x_1 y \notin E(G)$ and $x_1 z \notin E(G)$.

As above, let P_1 be a shortest $u-x$ path that contains x_1 and P_2 a shortest $u-y$ path, and a the last vertex that P_1 and P_2 have in common. The vertices on the $a-x$ subpath of P_1 together with z and y induce a path that has length two bigger than the $a-y$ subpath of P_2 (which is a geodesic). Hence G contains an induced subpath that is not a geodesic.

For the converse suppose G contains an induced path P (say a $u-v$ path) which is not a geodesic. Then $d_G(u,v) > 1$. Among the induced paths that are not geodesics let P be one that is as short as possible. We show that P has length at most $d_G(u,v)+2$. Suppose $|E(P)| > d_G(u,v)+2$. Let $P : u = u_1, u_2, ..., u_n = v$. Then $d_G(u, u_{n-1}) \leq d_G(u,v) + 1$ and $P' : u_1, u_2, ..., u_{n-1}$ is a path of length at least $|E(P)| - 1 \geq d_G(u,v) + 2 > d_G(u, u_{n-1})$. However, then P' is an induced path that is not a geodesic but has length less than P. This contradicts our choice of P. Hence P has length $d_G(u,v) + 1$ or $d_G(u,v) + 2$. Note that the $u - u_{n-1}$ subpath of P must be a geodesic, otherwise we have a contradiction to our choice of P.

Thus if $|E(P)| = d_G(u,v) + 1$, then $d_G(u, u_{n-1}) = d_G(u,v)$. Since the vertex that precedes u_{n-1} on P is not adjacent with v, $N_{i-1}(u, u_{n-1}) \neq N_{i-1}(u,v)$ where $i = d_G(u,v)$. If we let $x = u_{n-1}$ and $y = v$, then it follows that 3.2.2.1 holds.

Suppose now that $|E(P)| = d_G(u,v)+2$. Then $d_G(u, u_{n-1}) = d_G(u, u_{n-1})+1$. Let x be u_{n-2} and let $y = v$. Then $xy \notin E(G)$ and the vertex that precedes x on P is not adjacent with y. Thus if $i = d_G(u,v)$, then $N_{i-1}(u,x) \neq N_{i-1}(u,y)$. If we now let $z = u_{n-1}$, then $z \in V_{u,i+1}$ and z is adjacent with both x and y. Thus 3.2.2.2 holds. □

This result suggests a polynomial algorithm, using a breadth first search technique which has complexity $O(|V(G)|^4)$, for determining whether a (connected) graph is 2-Steiner distance hereditary. Spinrad [S1] has developed an algorithm based on this characterization which has complexity $0(|V(G)|^2)$. Once this is done and the graph has been found to be 2-Steiner distance hereditary, we can efficiently determine, by Theorem 3.2.1, the Steiner distance of any set of vertices, which was also shown independently in [DM1].

We conjecture here that whenever G is $k-$Steiner distance hereditary, then G is $(k+1)$-Steiner distance hereditary for $k \geq 3$.

3.3 The Characterization of 3-Steiner Distance Hereditary Graphs

Before proving this characterization we establish some useful properties about the cycle structure of k-Steiner distance hereditary graphs.

Proposition 3.3.1 If G is k-Steiner distance hereditary, then no cycle C of length $\ell \geq k+3$ has two adjacent vertices neither of which is incident with a diagonal of C.

Proof Suppose $C : v_1, v_2, ..., v_\ell, v_1$ is a cycle of length $\ell \geq k+3$ that has two adjacent vertices neither of which is incident with a diagonal of C. We may assume v_1 and v_2 are not incident with diagonals of C. Let $S = \{v_2, v_4, v_5, ..., v_{k+2}\}$. Then $d_G(S) = k$ since $\langle S \rangle_G$ is not connected and since $\langle S \cup \{v_3\} \rangle_G$ is connected. Let $H = \langle V(C) - \{v_3\} \rangle_G$. Then H contains S and $d_H(S) > k$ since a Steiner tree for S in H must contain v_1 and v_ℓ, neither of which belongs to S. Thus $d_H(S) > d_G(S)$, contrary to hypothesis that G is k-Steiner distance hereditary. □

Proposition 3.3.2 If G is k-Steiner distance hereditary, then every cycle $C : v_1, v_2, ..., v_\ell, v_1$ of length $\ell \geq k+3$ has at least two diagonals not all of which are incident with a single vertex.

Proof Suppose G has a cycle as described in the statement of the proposition and assume, to the contrary, that all the diagonals of C are incident with the same vertex, say v_1. Let $S = \{v_2, ..., v_k, v_\ell\}$. Then $\langle S \rangle_G$ is not connected. Thus $d_G(S) \geq k$. However, since $\langle S \cup \{v_1\} \rangle$ is connected, $d_G(S) \leq k$. Therefore $d_G(S) = k$. If $H = \langle V(C) - \{v_1\} \rangle$, then it follows since every diagonal of C is incident with v_1 and since $\ell \geq k+3$ that $d_H(S) \geq k+1$. This contradicts the fact that G is k-Steiner distance hereditary. □

Proposition 3.3.3 If G is k-Steiner distance hereditary, then every cycle $C : v_1, v_2, ..., v_\ell, v_1$ of length $\ell \geq k+3$ has at least two skew diagonals, or if $\ell = k+3$ and k is odd, then $v_1, v_3, ..., v_{k+2}, v_1$ or $v_2, v_4, ..., v_{k+3}, v_2$ is a cycle.

Proof Suppose first that $\ell = k+3$ and that C does not have skew diagonals. Then there exists a vertex of C not incident with a diagonal. We show that if v_i is incident with a diagonal, then v_iv_{i+2} is a diagonal where subscripts are expressed modulo $(k+3)$. Suppose that this is not the case. Then there exists a v_i which is incident with a diagonal $v_iv_{i+n} (n \geq 3)$ but v_iv_{i+j} is not a diagonal for $2 \leq j \leq n$. Let $S = V(C) - \{v_i, v_{i+1}, v_{i+n}\}$.Then $d_G(S) = k$. Let $H = \langle V(C) - \{v_{i+n}\} \rangle$. Then every connected subgraph of H that contains S must contain v_i and v_{i+1}. So $d_H(S) = k+1 > d_G(S)$, contrary to the fact that G is k-Steiner distance hereditary. Since C has no skew diagonals, it follows that k is odd and that either $v_1, v_3, ..., v_{k+2}, v_1$ or $v_2, v_4, ..., v_{k+3}, v_2$ is a cycle.

Suppose now that $\ell \geq k+4$ and that C does not have skew diagonals. We show again if v_i is incident with a diagonal of C, then v_iv_{i+2} is a diagonal. Suppose that this is not the case. Then there is a v_i such that v_iv_{i+n} is a diagonal where $n \geq 3$ and v_iv_{i+j} is not a diagonal for

$2 \leq j < n$. If $n - 1 \geq k$, let $S = \{v_{i+n+1}, v_{i+n-1}, v_{i+n+2}, ..., v_{i+n-(k-1)}\}$. Otherwise let $S = \{v_{i+2}, v_{i+3}, ..., v_{i+n-1}\} \cup \{v_{i+n+1}, v_{i+n+2}, ...v_{i+k+2}\}$. Then $d_G(S) = k$, since $\langle S \rangle$ is disconnected, but $\langle S \cup \{v_{i+n}\} \rangle$ is connected. Let $H = \langle V(C) - \{v_{i+n}\} \rangle$. Then v_i and v_{i+1} must both belong to a Steiner tree for S in H. So $d_H(S) \geq k + 1$ contrary to the fact that G is k-Steiner distance hereditary. Thus if v_i is incident with a diagonal, then $v_i v_{i+2}$ is a diagonal (subscripts expressed modulo ℓ). By Proposition 3.3.2, C has diagonals. Thus ℓ must be even and either $v_1, v_3, v_5, ..., v_{\ell-1}, v_1$ or $v_2, v_4, ..., v_\ell, v_2$ is a cycle, suppose the former. Let $S = \{v_2, v_4, v_5, ..., v_{k+2}\}$. Then $\langle S \rangle_G$ is disconnected but $\langle S \cup \{v_3\} \rangle_G$ is connected. Hence $d_G(S) = k$.

Let $H = \langle V(C) - \{v_3\} \rangle_G$. Since G is k- Steiner distance hereditary, $d_H(S) = k = d_G(S)$. Thus $v_{k+2}v_1$ is an edge. Now let $S' = \{v_\ell, v_2, v_3, ..., v_k\}$. Then it is not difficult to see that $d_G(S') = k$. Let $F = \langle V(C) - \{v_1\} \rangle_G$. Since G is k-Steiner distance hereditary, $d_F(S') = k$. So $v_k v_{\ell-1}$ must be an edge of G. Thus C has two skew diagonals. $\square$

We have already mentioned that, if G is 2- Steiner distance hereditary, then G is 3-Steiner distance hereditary, but that the converse of this statement does not hold. The next result shows that if a graph is 3-Steiner distance hereditary but not 2-Steiner distance hereditary, then G has short cycles without skew diagonals.

Proposition 3.3.4 If G is 3-Steiner distance hereditary, but not 2-Steiner distance hereditary, then there exists a 5-cycle in G which does not possess two skew diagonals.

Proof Let G be 3-Steiner distance hereditary, but not 2-Steiner distance hereditary. Then, by Theorem F, G contains a cycle C_ℓ of length $\ell \geq 5$ which does not have two skew diagonals. Let $C_\ell = v_1, v_2, ..., v_\ell, v_1$. If $\ell \geq 6$, then certainly C_ℓ contains at least one diagonal, otherwise $C_\ell - v_1$ is an induced path and $d_{C_\ell - v_1}(\{v_2, v_3, v_\ell\}) > d_G(\{v_2, v_3, v_\ell\})$, contradicting the assumption that G is 3-Steiner distance hereditary. Furthermore, if $\ell \geq 6$ and all the diagonals of C_ℓ are incident with a single vertex v_1 (say), then $C_\ell - v_1$ is an induced path and once again a contradiction arises to the fact that G is 3-Steiner distance hereditary. Hence $\ell = 5$ or C_ℓ contains two diagonals which are independent, but not skew. Suppose the statement of the proposition is false and let C_ℓ $(\ell \geq 5)$ be a shortest cycle of length $\ell \geq 5$ in G which does not have two skew diagonals. Certainly $\ell \geq 6$ and C_ℓ has two nonadjacent diagonals, say $v_1 v_i$ and $v_j v_k$, where $3 \leq i < j$ and $j + 2 \leq k \leq \ell$. Then $C_m : v_1, v_i, v_{i+1}, ..., v_j, ..., v_k, ..., v_\ell, v_1$ is a cycle of length $m \geq 5$ (where $m = \ell + 2 - i$) without two skew diagonals. We note $m \neq 5$ (by assumption), but $m < \ell$, which contradicts our choice of ℓ. The validity of the proposition now follows. □

We are now in a position to characterize the graphs that are 3-Steiner distance hereditary.

Theorem 3.3.1 A graph G is 3-Steiner distance hereditary if and only if it is 2-Steiner distance hereditary or the following conditions hold.

3.3.1.1 Every cycle $C : v_1, v_2, ..., v_\ell, v_1$ of length $\ell \geq 6$

(a) has at least two skew diagonals, or, if $\ell = 6$, then v_1, v_3, v_5, v_1 or v_2, v_4, v_6, v_2 is a cycle in $\langle V(C) \rangle$; and

(b) has no two adjacent vertices neither of which is on a diagonal of C.

3.3.1.2 G does not contain an induced subgraph isomorphic to any of the graphs shown in Figure 3.3.1 (any subset of dotted edges may be included in the graph).

Proof Suppose first that G is 3-Steiner distance hereditary but not 2-Steiner distance hereditary. Then conditions 3.3.1.1(a) and (b) follow from

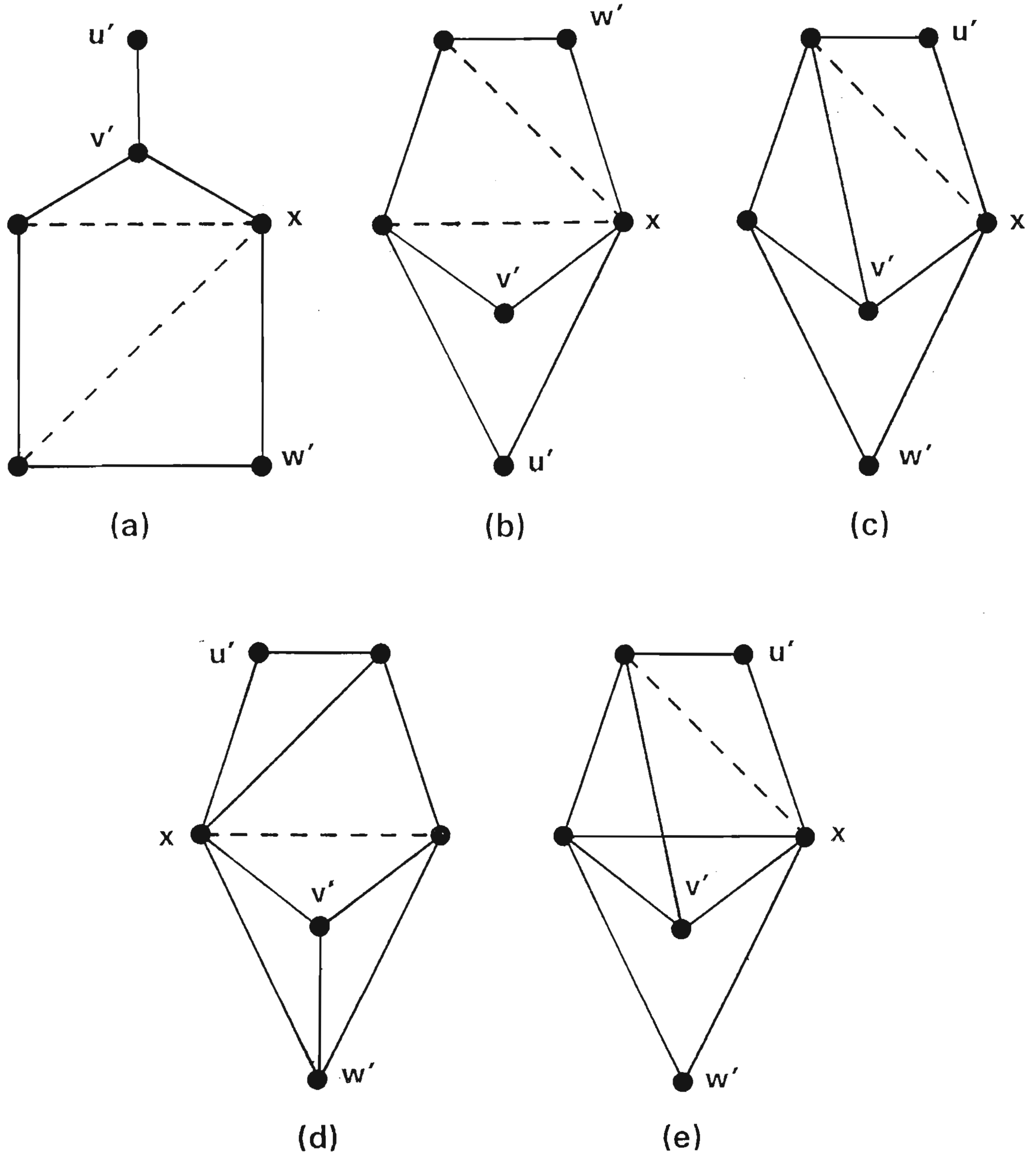


Figure 3.3.1

Propositions 3.3.1 and 3.3.3. Suppose now that G contains one of the subgraphs shown in Figure 3.3.1 as induced subgraphs. Let $S = \{u', v', w'\}$. Since $\langle S \rangle$ is not connected, $d_G(S) \geq 3$. Since $\langle \{u', v', x, w'\} \rangle$ is a connected graph in all cases, $d_G(S) = 3$. If we now delete x from any one of the subgraphs shown in Figure 3.3.1 we obtain a connected induced subgraph of G that contains S; but the distance of S in each of these induced subgraphs is $4 > d_G(S)$. This contradicts the fact that G is 3-Steiner distance hereditary. Thus G does not contain any of the graphs shown in Figure 3.3.1 as induced subgraphs.

For the converse, we already know from Theorem 3.3.1, that if G is 2-Steiner distance hereditary, then G is 3-Steiner distance hereditary. Suppose thus that G is not 2-Steiner distance hereditary and that G satisfies conditions 3.3.1.1 and 3.3.1.2 of the theorem. Suppose G is not 3-Steiner distance hereditary. Then G contains an induced subgraph H and a set $S = \{u, v, w\}$ such that $d_H(S) > d_G(S)$. Choose H in such a way that $|V(H)|$ is as small as possible. Let T_S and T'_S be Steiner trees for $S = \{u, v, w\}$ in H and G, respectively. By our choice of H, $|V(H)| = |V(T_S)|$. Moreover $\langle S \rangle$ is not connected.

We now consider several cases.

Case 1 Suppose T_S and T'_S are both paths, but that they do not have the same end-vertices; say T_S is a $v - w$ path and T'_S a $u - w$ path and that no Steiner tree for S in G is a $v - w$ path. Let P and P' be the $u - v$ sections in T_S and T'_S, respectively. If the lengths of P and P' are the same we may assume $P = P'$. Let the length of P' be a' and of P be a. Note that u cannot be adjacent with w; otherwise there would be a Steiner tree in G that is a $v - w$ path. So the vertex x adjacent with u on the $u - w$ section of T_S is distinct from w. Thus if $a > a'$, then $d_{H-w}(\{x, u, v\}) > d_G(\{x, u, v\})$ and so we have a contradiction to our choice of H. Thus $a \leq a'$. Clearly

$a' \leq a$. Thus $a = a'$. By our choice of H, $a = a' = 1$.

Let $u = x_0, x_1, ..., x_m = w$ be the $u-w$ section of T_S and $v = y_0, y_1, ..., y_n = w$ the $v - w$ section in T'_S. Then $n \geq 2$ and $m \geq 3$. Observe that the $u - w$ section of T_S and the $v - w$ section of T'_S have no vertex except w in common; otherwise we have a contradiction to our choice of H. Hence, the $u - w$ section of T_S and the $v - w$ section of T'_S and the edge uv form a cycle C of length at least 6.

If $n = 2$ and $m = 3$, so that C has length 6, then neither x_1, w, v, x_1 nor x_2, y_1, u, x_2 is a cycle since x_1w and ux_2 are not edges of G. Thus C must have two skew diagonals whether or not C has length 6, i.e., x_iy_k and x_jy_ℓ are edges where $1 \leq i < j < m$ and $1 \leq \ell < k < n$. By our choice of H, $d_G(\{x_j, u, v\}) = d_H(\{x_j, u, v\})$; hence $1+j = d_H(\{x_j, u, v\}) = d_G(\{x_j, u, v\}) \leq \ell + 2$. So

$$j \leq \ell + 1 \leq k. \qquad 3.3.1.3$$

But $n + 1 = d_G(u, v, w) \leq 1 + i + 1 + (n - k)$. So

$$k \leq 1 + i \leq j. \qquad 3.3.1.4$$

From 3.3.1.3 and 3.3.1.4 it follows that $k = j = 1 + i = \ell + 1$. Now $d_H(\{u, x_i, w\}) = m$, whereas $d_G(\{u, x_i, w\}) \leq i + 1 + n - k = n < m$; so $d_{H-v}(\{u, x_i, w\}) > d_G(\{u, x_i, w\})$, contradicting our choice of H. Hence Case 1 cannot occur.

Case 2 Suppose T_S and T'_S are both paths with the same end-vertices, say they are both $u - w$ paths. Then either uv or vw is an edge, otherwise we have a contradiction to our choice of H. Suppose $uv \in E(G)$. Observe then that the $v - w$ sections of T_S and T'_S are internally disjoint otherwise we have a contradiction to our choice of H. So the $v - w$ sections of T_S and T'_S form a cycle C. Suppose $T_S : u, v = v_0, v_1, ..., v_m = w$ and $T'_S : u, v = w_0, w_1, ..., w_n = w$. Then $n < m$. If C has length 6, then $m = 4$ and $n = 2$. Clearly neither v_1, v_3, w_1, v_1 nor v, v_2, w, v is a cycle.

Thus if C has length at least 6, then C has a pair of skew diagonals. As in Case 1 we obtain a contradiction. Thus we may assume C has length 5. Note $d_H(\{u,v,w\}) = 4$ and $d_G(\{u,v,w\}) = 3$. So $C : v, w_1, w, v_2, v_1, v$. Since v is not adjacent with any vertices of C except v_1 and w_1 and since the graphs of Figure 3.3.1(a) are not induced subgraphs of G, it follows that u must be adjacent with at least one vertex in $\{w_1, v_1, v_2\}$. (Note that $uw \notin E(G)$, since otherwise $d_H(\{u,v,w\}) = 2 = d_G(\{u,v,w\})$.) If $uv_2 \in E(G)$, then $d_H(\{u,v,w\}) = 3 < 4$. So uw_1 or uv_1 is an edge of G. If $uw_1 \notin E(G)$, the 6-cycle u, v, w_1, w, v_2, v_1 cannot have two skew diagonals since vv_1, w_1v_1 and w_1v_2 are the only possible diagonals. If $uv_1 \notin E(G)$, the 6- cycle $u, v, v_1, v_2, w, w_1, u$ also does not have two skew diagonals. So both uv_1 and $uw_1 \in E(G)$. Further, $w_1v_2 \in E(G)$; otherwise w and v_2 are two consecutive vertices on the 6-cycle w, v_2, v_1, u, v, w, that are not incident with any diagonal. However, then $\langle V(T_S) \cup V(T'_S)\rangle$ is isomorphic to one of the graphs in Figure 3.3.1(d) which is impossible. Thus Case 2 cannot occur either.

Case 3 Suppose both T_S and T'_S have three end-vertices. Let y and z be the vertices of degree 3 in T_S and T'_S, respectively.

We show first that $V(T_S) \cap V(T'_S) = \{u, v, w\}$. Suppose there exists $x \in V(T_S) \cap V(T'_S) - \{u, v, w\}$. Without loss of generality we may assume that x is on the $v - y$ path in T_S. If x also belongs to the $v - z$ path of T'_S, then we obtain a contradiction to our choice of H. So we may assume that x belongs to the $w - z$ or $u - z$ path of T'_S, say the former. Also $x \neq y$ otherwise we again have a contradiction to our choice of H.

Let $d_{T'_S}(u, z) = \ell_1, d_{T'_S}(z, v) = \ell_2, d_{T'_S}(z, x) = \ell_3, d_{T'_S}(x, w) = \ell_4, d_{T_S}(u, y)$ $= d_1, d_{T_S}(y, x) = d_2, d_{T_S}(x, v) = d_3$ and $d_{T_S}(y, w) = d_4$. By our choice of H, $d_H(\{u, x, v\}) = d_G(\{u, x, v\})$. So $\ell_1 + \ell_2 + \ell_3 = d_1 + d_2 + d_3$ or $d_1 + d_2 + d_3 - 1$.

Suppose first that $\ell_1 + \ell_2 + \ell_3 = d_1 + d_2 + d_3$. Since $d_G(S) < d_H(S)$

it follows that $\ell_1 + \ell_2 + \ell_3 + \ell_4 < d_1 + d_2 + d_3 + d_4$. Thus $\ell_4 < d_4$. But $d_H(\{v, w, y\}) = d_G(\{v, w, y\})$. So $d_2 + d_3 + d_4 \leq d_2 + d_3 + \ell_4$, i.e. $d_4 \leq \ell_4$. This produces a contradiction. Hence we may assume that $\ell_1 + \ell_2 + \ell_3 = d_1 + d_2 + d_3 - 1$. Since $d_G(S) < d_H(S)$, it follows that $\ell_4 \leq d_4$. As above it follows that $d_4 \leq \ell_4$. So $\ell_4 = d_4$. Let u_1, v_1 and w_1 be the neighbours of y in T_S that lie on the $y - u$, $y - v$ and $y - w$ paths, respectively. Since $d_H(\{u, x, v\}) = d_1 + d_2 + d_3 - 1$, it follows that $u_1v_1 \in E(G)$. So $v_1w_1 \notin E(G)$. Hence $d_H(x, w) = d_2 + d_4$. Since $x \neq y, d_2 \geq 1$. Hence $d_H(x, w) > d_4 = \ell_4 \geq d_G(x, w)$. So if x' is the neighbour of x on the $x - z$ path in T'_S, and if H' is the subgraph induced by x' and the vertices on the $x - w$ path in H, then $d_{H'}(\{x', x, w\}) > \ell_4 + 1 \geq d_G(\{x', x, w\})$. But $p(H') < p(H)$, so we have a contradiction to our choice of H. Thus we may conclude that

$$V(T_S) \cap V(T'_S) = \{u, v, w\}.$$

Note that $d_H(S) \leq d_G(S) + 2$, otherwise let $a \neq y$ be a vertex on one of the paths from y to $\{u, v, w\}$ in T_S, say on the $y - u$ path such that $ua \in E(G)$, and observe that $d_G(\{v, w, a\}) \leq d_G(\{v, w, u\}) + 1 < d_H(\{v, w, u\}) - 1 = d_{H-\{u\}}(\{v, w, a\})$, contrary to our choice of H. Let P_u, P_v and P_w be the $y - u, y - v$ and $y - w$ paths in T_S, respectively and suppose Q_u, Q_v and Q_w are the $z - u, z - v$ and $z - w$ paths in T'_S, respectively. We say that P_u corresponds to Q_u, P_v to Q_v and P_w to Q_w in T'_S. Then, by the above observation, at most two of the paths P_u, P_v and P_w are longer than their corresponding paths in T'_S. We consider several subcases.

Subcase 3.1 Suppose exactly two of the paths in $\{P_u, P_v, P_w\}$ are longer than their corresponding paths in T'_S. Suppose P_v and P_w are two such paths. Let C_{vw} be the cycle produced by P_v, P_w, Q_v and Q_w.

Assume first that the neighbours of y on the P_v and P_w paths in T_S are not adjacent. By our assumption, no vertex of P_v is joined to a vertex of P_w. Since G satisfies condition 3.3.1.1(b) of the theorem, it follows, if we consider C_{vw}, that some vertex $y' \neq v, w$ on the $v - w$ path of T_S is joined to some vertex $y'' \neq v, w$ on the $v - w$ path of T'_S. But then $d_{H-\{u\}}(\{y', v, w\}) > d_G(\{y', v, w\})$. This contradicts our choice of H.

Assume thus that the neighbours of y on P_v and P_w are adjacent. If y is adjacent with any vertex on the $v - w$ path in T'_S (different from v and w), then $d_{H-\{u\}}(\{v, w, y\}) > d_G(\{v, w, y\})$, again a contradiction to our choice of H. Note that $q(P_v) = q(Q_v) + 1$ and $q(P_w) = q(Q_w) + 1$ otherwise we again have a contradiction to our choice of H.

If Q_v or Q_w has length at least 2, say Q_w, then P_w has length at least 3.

Let v' and w' be the vertices adjacent with y on P_v and P_w, respectively. Let C'_{vw} be the cycle obtained from C_{vw} by deleting y and adding the edge $v'w'$. Since Q_w has length at least 2, C'_{vw} has length at least 7. We show next that C'_{vw} does not have a pair of skew diagonals, thereby producing a contradiction to condition 3.3.1.1(a).

Observe first that every diagonal of C'_{vw} must join a vertex of $V(P_v) \cup V(P_w) - \{y\}$ with a vertex in $V(Q_v) \cup V(Q_w)$. Let x be a vertex of $P_w - \{y\}$ that is incident with a diagonal $e = xx'$. Let $d_H(x, w) = d$. Then $d_{T'_S}(w, x') \leq d$; otherwise

$$\begin{aligned} d_G(\{v, w, x\}) &\leq q(Q_v) + q(Q_w) \\ &< q(P_v) + q(P_w) - 1 \\ &= d_{H-\{u\}}(v, w, x) \end{aligned}$$

which contradicts our choice of H. Similarly if x is a vertex of $P_v - \{y\}$ that is incident with a diagonal $e = xx'$, then $d_H(v, x) \geq d_{T'_S}(v, x')$. So the only diagonals of C'_{vw} join vertices of $P_w - \{y\}$ and Q_w or vertices of $P_v - \{y\}$ and Q_v.

Suppose aa' and bb' are skew diagonals with $a, b \in V(P_w - \{y\})$ and $a', b' \in V(Q_w)$. Suppose $d_H(a, w) > d_H(b, w)$. Then necessarily $d = d_{T'_S}(b', w) > d_{T'_S}(a', w)$. From the earlier observation it follows that $d_H(a, w) >$

$d_H(b, w) \geq d$; i.e., $d_H(a, w) \geq d + 1$. If we now take the shortest $v - a$ path in H, together with the edge aa' and the $a' - w$ path in T'_S we obtain a tree of size at most $d_H(v, a) + d$ that contains v, w and a. So $d_G(\{v, w, a\}) < d_H(v, a) + d$. This contradicts our choice of H. So C'_{vw} cannot have two skew diagonals each of which is incident with a vertex of $P_w - \{y\}$. Similarly C'_{vw} cannot have two skew diagonals each of which is incident with a vertex of $P_v - \{y\}$.

Thus Q_v and Q_w each have length 1, so that P_v and P_w each have length 2. Hence C_{vw} has length 6. We already know that $v'w'$ is an edge of G where v' and w' are the neighbours of y on P_v and P_w, respectively. Note that $vw \notin E(G)$. Also $wv' \notin E(G)$ and $vw' \notin E(G)$, otherwise $d_H(\{u, v, w\}) < p(H) - 1$. So C_{vw} does not have crossing diagonals. Thus v', z, w', v' must be a cycle. Let u' be the neighbour of y on P_u. Note that u' is not adjacent with v', w', v or w, otherwise we obtain a contradiction to our choice of H. If $u'z \notin E(G)$, then $\langle\{u', y, v', w', v, z\}\rangle_G$ is isomorphic to one of the graphs shown in Figure 3.3.1a. Since this is not possible, $u'z \in E(G)$. However, then $\langle\{u', y, w', w, v', z\}\rangle_G$ is isomorphic to one of the graphs of Figure 3.3.1(e), which is again impossible.

Subcase 3.2 So exactly one of P_u, P_v and P_w has length greater than their corresponding paths. Suppose P_w is longer than Q_w.

Subcase 3.2.1 Suppose $q(P_w) = q(Q_w) + 1$. Then necessarily $q(Q_v) = q(P_v)$ and $q(Q_u) = q(P_u)$. Let C_{uv} be the cycle induced by the edges of P_u, P_v, Q_u and Q_v. Then C_{uv} has even length. Suppose first that C_{uv} has length exceeding 6. Then, by condition 3.3.1.1(a) of the theorem, C_{uv} has a pair of skew diagonals. We show next that the only possible such skew diagonals are yz and $u'v'$ (where u' and v' are adjacent with y on P_u and P_v, respectively) or yz and $u''v''$ (where u'' and v'' are adjacent with z on Q_u and Q_v, respectively).

Suppose that P_v (and hence Q_v) has an internal vertex and that some vertex a of P_v is adjacent with some vertex a' of Q_v. Then $d_{P_v}(v,a) = d_{Q_v}(v,a')$ as we now see.

If $d_{P_v}(v,a) < d_{Q_v}(v,a')$, then the edges of Q_u together with those of Q_w, the edge aa', the edges of the $a-v$ path of P_v and those of the $z-a'$ path of Q_v induce a tree T of size at most $q(T'_S)$ that contains u, v and w. Thus T is a Steiner tree for u, v and w. This is impossible since T contains vertices of H other than u, v and w which we have shown is impossible.

Suppose now that $d_{P_v}(v,a) > d_{Q_v}(v,a')$. Let T be the tree induced by the edges of P_u, P_w, the edge aa', the edges of the $a-y$ path of P_v and those of the $a'-v$ path of Q_v. If T is a Steiner tree for u, v and w, then T and T_S have more vertices in common than only u, v and w. But this is not possible as we have shown.

Thus $q(T) = q(T_S)$. If $H' = \langle V(T) \rangle$ contains a Steiner tree for $\{u,v,w\}$, then we again have a contradiction since such a Steiner tree has vertices other than u, v and w in common with T_S. So $d_{H'}(\{u,v,w\}) = d_H(\{u,v,w\})$, i.e., H' is an induced subgraph of G of the same order as H for which $d_{H'}(\{u,v,w\}) > d_G(\{u,v,w\})$. Once again a Steiner tree for $\{u,v,w\}$ in H' contains vertices of T'_S other than u, v and w which is impossible.

Therefore if aa' is a diagonal of C_{uv} where a lies on P_v and a' on Q_v, then $d_{P_v}(v,a) = d_{Q_v}(v,a')$. It can be shown similarly if aa' is a diagonal of C_{uv} such that a lies on P_u and a' on Q_u, then $d_{P_u}(u,a) = d_{Q_u}(u,a')$.

It remains to show that no internal vertex of $P_v(P_u)$ is adjacent with an internal vertex of $Q_u(Q_v)$. Suppose aa' is an edge of G where a is an internal vertex of P_v and a' is an internal vertex of Q_u. Then let T be the tree induced by the edges of Q_u, Q_w, the $a-v$ path in P_v and the edge aa'. Then $q(T) \leq q(T'_S)$. So T must be a Steiner tree for $\{u,v,w\}$. But T has vertices other than u, v and w in common with T_S, which we have shown is impossible. Therefore no internal vertex of P_v is adjacent with an internal vertex of Q_u. Similarly no internal vertex of P_u is adjacent with an internal vertex of Q_v.

Hence the only possible candidates for crossing diagonals are yz and $u'v'$ or yz and $u''v''$. If $u'v'$ is a diagonal, then let C'_{uv} be the cycle obtained from C_{uv} by deleting y and adding the edge $u'v'$. Observe that C'_{uv} is a cycle of length at least 7 without skew diagonals, contrary to the hypothesis. If $u''v''$ is a diagonal of C_{uv}, then let C''_{uv} be the cycle obtained from C_{uv} by deleting z and adding the edge $u''v''$. Once again it follows that C''_{uv} is a cycle of length at least 7 without skew diagonals, contrary to hypothesis.

Therefore we may assume C_{uv} has length at most 6. Assume first that C_{uv} has length 6. We may, without loss of generality, assume that P_v and Q_v each have length 2. Let C_{uv} be y, v', v, v'', z, u, y. Then it can be shown in a straightforward manner that zv, yv, uv and zv' are not diagonals of C_{uv}.

Suppose uv'' is a diagonal of C_{uv}. Then the subgraph H' induced by u, y, v'', v and the vertices of P_w either contains a Steiner tree for u, v and w that has more vertices in common with T_S than u, v and w or $d_{H'}(\{u, v, w\}) = d_H(\{u, v, w\})$ and a Steiner tree for u, v and w in H' has more vertices in common with T'_S than only u, v and w, which is again impossible. Hence $uv'' \notin E(G)$. By condition 3.3.1.1(a) it now follows, since neither y, v, z, y nor v', v'', u, v' is a cycle, that C_{uv} must contain skew diagonals. However, then $uv' \in E(G)$. This is again not possible since $uv', uz, v'v$ and the edges of Q_w induce a Steiner tree for u, v and w that has more vertices in common with T_S than only u, v and w.

Hence C_{uv} must have length exactly 4. Let w' be the vertex adjacent with y on P_w. If vw' is an edge, then $T_S - yv + vw'$ is a Steiner tree for $\{u, v, w\}$ in H. Using the argument that was used to show that C_{uv} has length 4 it can be shown that ww' and wz are edges of G. The induced subgraph $\langle\{u, v, w, y, z, w'\}\rangle$ thus has edge set $\{uy, uz, yv, vz, vw', yw', w'w, wz\} \cup E$ where E is any subset of $\{zy, zw'\}$. However, then G has as induced subgraph one of the graphs of Figure 3.3.1(c) or (e) which is impossible.

Assume thus that $vw' \notin E(G)$. Similarly we may assume that $uw' \notin E(G)$. We show next that $zw \in E(G)$. If this is not the case, then the cycle C_{vw} induced by the edges of P_v, P_w, Q_v and Q_w has length at least 7. Thus by condition 3.3.1.1(a) C_{vw} must have skew diagonals. Using arguments similar to those used before, we can show that the only diagonals of C_{vw} of the type aa' where $a \in V(P_w - y), a' \in V(Q_w)$ are those where $d_{P_w}(w, a) = d_{Q_w}(w, a')$. It is not difficult to see that the only diagonal with which y may be incident is yz. Since w cannot be incident with any diagonals of C_{vw}, it now follows that $vw'' \in E(G)$ where w'' is adjacent with z on Q_w. If C is the cycle obtained from C_{vw} by deleting z and adding the edge vw'', then C has length at least 6 and contains adjacent vertices namely y and w', neither of which is incident with a diagonal. This contradicts condition 3.3.1.1(b). Hence wz must be an edge of G. Thus the induced subgraph $\langle\{u, v, y, z, w, w'\}\rangle$ has edge set $\{uy, uz, vy, vz, wz, ww', w'y\} \cup E$ where E is any subset of $\{zy, zw'\}$. But then G contains any one of the subgraphs of Figure 3.3.1(b) as induced subgraph which is not possible. Hence subcase 3.2.1 cannot occur.

Subcase 3.2.2 Suppose $q(P_w) \geq q(Q_w) + 2$. We have already shown that $q(T_S) = q(T'_S) + k$ where $k = 1$ or 2. Furthermore we are assuming that $q(P_u) \leq q(Q_v)$ and $q(P_v) \leq q(Q_v)$. Suppose $q(P_u) = q(Q_u) - \ell$ where ℓ is a non-zero integer. We show now that $k = 1$ and that $\ell \leq 1$. Further if $k = 1$ and $\ell = 1$, then the neighbours of y on P_v and P_w are adjacent and $q(P_v) = q(Q_v)$.

To see this let w'' be the neighbour of w on P_w and note that

$$\begin{aligned} d_H(\{v,w,w''\}) &\geq q(T_S) - q(P_u) - 1 \\ &= q(T_S) - q(Q_u) + \ell - 1 \\ \text{and} \quad d_G(\{v,w,w''\}) &\leq q(T'_S) - q(Q_u) + 1 \\ &= q(T_S) - k - q(Q_u) + 1. \\ \text{Hence} \quad d_G(\{v,w,w''\}) &< d_H(\{v,w,w''\}) \text{ if } k \geq 2 \text{ or if } \ell \geq 2. \end{aligned}$$

This contradicts our choice of H. Hence $k = 1$ and $\ell \leq 1$. So $d_G(\{v,w,w''\}) \leq q(T_S) - q(Q_u)$ and $d_H\{(v,w,w''\}) \geq q(T_S) - q(Q_u) + \ell - 1$. However, $d_H(\{v,w,w''\}) = q(T_S) - q(P_u) - 1$ only if the neighbours of y on P_v and P_w are adjacent. Since $d_G(\{v,w,w''\}) = d_H(\{v,w,w''\})$, otherwise we get a contradiction to our choice of H, it follows if $\ell = 1$ that the neighbours of y on P_v and P_w are adjacent and, since $k = 1$, this implies that $q(P_v) = q(Q_v)$. Suppose $\ell = 1$. Let v' and w' be the neighbours of y on P_v and P_w. Let $T = T_S + v'w' - v'y$. Then T is a Steiner tree for $\{u,v,w\}$ in H that was considered in Subcase 3.2.1.

So we may assume $\ell = 0$ so that $q(P_u) = q(Q_u)$. We may also assume $q(P_v) = q(Q_v)$, otherwise we may repeat the above arguments with P_v instead of P_u and arrive at a contradiction. In addition we may assume that w', the neighbour of y on P_w, is not adjacent with the neighbour of y on either P_u or P_v. Otherwise we can easily arrive at a case we have already considered. But this contradicts the fact that $k = 1$.

Case 4 Suppose T_S has three end-vertices, namely u, v and w and that T'_S is a path, say a $u - w$ path. Let y be the vertex of degree 3 in T_S and P_u, P_v and P_w the $u-y, v-y$ and $w-y$ paths in T_S. Observe that u, v and w are pairwise nonadjacent. Therefore the $u - v$ path Q_u and the $v - w$ path Q_w in T'_S must both have length at least 2.

We show first that $V(T_S) \cap V(T'_S) = \{u,v,w\}$. Suppose this is not the case. Then we may assume that some internal vertex of Q_u also belongs to T_S. Let x be the first such vertex on Q_u after u that also belongs to T_S.

Suppose first that x belongs to P_u. Let $d_{P_u}(u,x) = d$ and $d_{Q_u}(u,x) = \ell$. Note first that $d \not< \ell$; otherwise there exists a tree of size less than that of T'_S which contains u, v and w, which is impossible. Hence $d \geq \ell$. If $d = \ell$, then we obtain a contradiction to our choice of H as follows: We may assume that the $u - x$ subpaths of P_u and Q_u are the same. If u' follows u on these subpaths, then $d_{H-\{u\}}(\{u', v, w\}) > d_G(\{u', v, w\})$, contrary to our choice of H. Thus we may assume $d > \ell$. If x is an internal vertex of P_u, then let x' be a vertex of P_u that is adjacent with x but does not lie on the $u - x$ subpath of P_u. Then $d_{H-w}(\{u, x, x'\}) = d + 1 > \ell + 1 \geq d_G(\{u, x, x'\})$, contrary to our choice of H. If $x = y$, then let z be a vertex on P_v adjacent with y and observe that $d_{H-w}(\{u, x, z\}) > d_G(\{u, x, z\})$ contrary to our choice of H. Hence x cannot be on P_u.

Suppose next that x is an internal vertex of P_v. Let $d_H(u,x) = d$ and $d_{Q_u}(u,x) = \ell$. Then $d \not< \ell$ since otherwise T'_S is not a Steiner tree for u, v and w. If $d > \ell$ we have, as before, a contradiction to our choice of H. Suppose thus that $d = \ell$.

If the shortest $u - x$ path in H contains vertices of Q_u distinct from u, then we can replace the $u - x$ subpath of Q_u with the $u - x$ subpath in H and obtain, as above, a contradiction. Suppose then that the shortest $u - x$ path in H contains no vertex of Q_u. Then necessarily u must be adjacent with x and x and u must be neighbours of y. But then

$$\begin{aligned} d_{H-u}(\{x, v, w\}) = d_H(\{u, v, w\}) - 1 &> d_G(\{u, v, w\}) - 1 \\ &= d_G(\{x, v, w\}), \end{aligned}$$

contrary to our choice of H. Thus x is not an internal vertex of P_v.

Suppose now that x is an internal vertex of P_w. As before, let $d = d_H(u,x)$ and let $\ell = d_{Q_u}(u,x)$. Then it can be shown, using arguments that were used before, that $d \not> \ell$ and that $d \not< \ell$, so that $d = \ell$. But again, as in the previous paragraph, it can be argued that in this case necessarily $d_{H-u}(\{x, v, w\}) > d_G(\{x, v, w\})$ which produces a contradiction to our choice of H. Hence $V(T_S) \cap V(T'_S) = \{u, v, w\}$.

We may assume that $d_H(v,w) > d_G(v,w) = d_{Q_w}(v,w)$ and that $d_H(u,v) > d_G(u,v) = d_{Q_u}(u,v)$; otherwise there either exists a Steiner tree for u,v and w that has vertices other than u,v and w in common with T_S, which we have shown is impossible, or there exists a tree with fewer edges than T'_S which contains u,v and w, which is again impossible.

Thus $d_H(v,w) \geq d_G(v,w)+1$. We show next that $d_H(v,w) = d_G(v,w)+1$. Suppose this is not the case, then $d_H(v,w) \geq d_G(v,w)+2$. Let v' be the vertex adjacent with v on a shortest $v-w$ path in H. Then $d_{H-u}(\{v,v',w\}) \geq d_G(v,w)+2 > d_G(v,w)+1 \geq d_G(\{v,v',w\})$, contrary to our choice of H. Hence $d_H(v,w) = d_G(v,w)+1$. Similarly $d_H(u,v) = d_G(u,v)+1$. Let $d_1 = d_H(u,y), d_2 = d_H(v,y), d_3 = d_H(w,y), \ell_1 = d_G(u,v)$ and $\ell_2 = d_G(v,w)$. Then $d_1+d_2+d_3 = d_H(\{u,v,w\}) > d_G(\{u,v,w\}) = \ell_1+\ell_2$, i.e.

$$d_1 + d_2 + d_3 \geq \ell_1 + \ell_2 + 1 \qquad 3.3.1.5$$

Since at most one pair of the three pairs of neighbours of y in T_S are adjacent,

$$d_1 + 2d_2 + d_3 - 1 \leq \ell_1 + \ell_2 + 2. \qquad 3.3.1.6$$

So $d_2 \leq 2$. Let u', v' and w' be the vertices adjacent with y on P_u, P_v and P_w, respectively.

We show first that we need only consider the case where $d_2 = 1$. Note that d_2 is the length of the $v-y$ path in T_S. If $d_2 = 2$, then it follows from 3.3.1.5 and 3.3.1.6 that either $u'v'$ or $v'w'$ must be an edge of H. Of course, since T_S is a Steiner tree for $\{u,v,w\}$ in H, at most one of these two edges belongs to H. Hence exactly one of these edges belongs to H, say $v'w'$. But then $T = T_S + v'w' - yw'$ is a Steiner tree for u,v and w in H for which the distance from v to the vertex of degree 3 in T is 1.

Suppose thus that $d_2 = 1$. Then $u'v' = u'v$ and $v'w' = vw'$ are not edges of H, otherwise there is a Steiner tree for u,v and w in H which is a $u-w$ path. But this situation was already considered in Case 2 and shown to be impossible. However, then $d_1 = \ell_1 + 1$ and $d_2 = \ell_2 + 1$.

Consider the cycle C induced by the edges of P_v, P_w and Q_w. This cycle has odd length, namely length $2\ell_2 + 1$. Suppose first that C has length at least 7. Then C must have a pair of skew diagonals $z'z$ and $x'x$ where $z', x' \in V(Q_w)$ and $z, x \in V(P_w)$. Note $w \notin \{x, x', z, z'\}$. Suppose $d_G(w, z') > d_G(w, x')$. We show first that $d_H(w, x) \geq d_G(w, x')$, otherwise we can replace the $x' - w$ subpath of T'_S by the edge $x'x$ followed by the $x - w$ subpath of P_w and obtain either a Steiner tree for u, v and w that has more vertices in common with T_S than u, v and w or a tree containing u, v and w with fewer edges than T'_S. Both situations cannot occur. Hence $d_H(w, x) \geq d_G(w, x')$. Similarly $d_H(w, z) \geq d_G(w, z')$. However, then $d_H(w, x) \geq d_G(w, x') + 2$. Let T' be the tree induced by the edges of P_u, the edges of the $y - x$ subpath of P_w, the edge xx', the edges of the $x' - w$ subpath of Q_w and the edge of P_v. Then T' is a tree that contains u, v, w, and has at most $q(T'_S)$ edges and contains vertices of T_S other than u, v and w. This cannot happen as we have seen before. Hence C cannot have skew diagonals. This contradicts condition 3.3.1.1(a) of the hypothesis. Hence C must have length 5. Suppose $C : y, w', w, w'', v, y$. Using arguments similar to those employed earlier, it can be shown that the only potential diagonals of C are $w''y$ and $w''w'$.

Let u' be the vertex adjacent with v on Q_u. Then $u'w \notin E(G)$; otherwise there exists a tree with fewer edges than T'_S that contains u, v and w. This is impossible. We may also assume that $u'w''$ is not an edge of G otherwise $T'_S + u'w'' - u'v$ is a Steiner tree for u, v and w having three end vertices. This is a situation already considered in Case 3. We show next that $u'w' \notin E(G)$. If $u'w' \in E(G)$, then $T'_S - w''$ together with w' and the edges $u'w'$ and $w'w$ is a Steiner tree for u, v and w having three end vertices, again a case we have already considered. If $u'y \notin E(G)$, then $\langle\{u', y, v, w, w', w''\}\rangle_G$ is one of the subgraphs shown in Figure 3.3.1(a). This contradicts the hypothesis. So we may assume $u'y \in E(G)$. However, then u', y, w', w, w'', v, u' is a 6-cycle that does not satisfy condition

3.3.1.1(a), contrary to the hypothesis. This completes the proof of Case 4.

Case 5 Suppose that no Steiner tree for S in H has three end-vertices and that every Steiner tree for S in G has three end-vertices. Suppose T_S has u and w as end-vertices. Observe that T_S must be an induced path. Let P_u be the $u-v$ subpath of T_S and P_w the $w-v$ subpath of T_S. Let z be the vertex of degree 3 in T'_S and let Q_u, Q_v and Q_w be the $u-z$ subpath, the $v-z$ subpath and the $w-z$ subpath of T'_S, respectively. Suppose T'_S has been chosen in such a way that Q_v is as short as possible. With this choice of T'_S it follows that the vertex adjacent with z on Q_v is not adjacent with the vertices adjacent with z on Q_u or Q_w. Let $\ell_1 = q(Q_u), \ell_2 = q(Q_w), \ell_3 = q(Q_v), d_1 = q(P_u)$ and $d_2 = q(P_w)$. Observe that $d_1 \geq \ell_1 + 1$ and $d_2 \geq \ell_2 + 1$, otherwise there is a Steiner tree for u, v and w that is a path, contrary to our assumption.

We now show that P_u and Q_u have no vertex in common except u. Suppose this is not the case. Let x be the first vertex after u on Q_u that also lies on P_u. The $u-x$ subpath of P_u cannot be longer than the $u-x$ subpath of Q_u otherwise, by choosing u, x and the vertex x' following x on P_u, we see that $d_G(\{u, x, x'\}) < d_{H-w}(\{u, x, x'\})$ contrary to our choice of H. Further, the length of the $u-x$ subpath of P_u cannot be less than the length of the $u-x$ subpath of Q_u, otherwise we could replace the $u-x$ subpath in T'_S with the $u-x$ subpath in T_S to obtain a tree containing u, v and w which has fewer edges than T'_S and this is impossible. Hence the $u-x$ subpaths of P_u and Q_u have the same length. However, then $d_G(\{x, v, w\}) < d_{H-u}(\{x, v, w\})$ contrary to our choice of H. Thus P_u and Q_u are vertex disjoint except for u. Similarly P_w and Q_w are disjoint except for w.

We show next that P_u and Q_v have at most one vertex distinct from v in common. Suppose x is the first vertex after v on Q_v that belongs to both P_u and Q_v. Clearly $x \neq u$. Then the length of the $v-x$ subpath of P_u cannot exceed the length of the $v - x$ subpath of Q_v; otherwise we obtain a contradiction to our choice of H. Further, the length of the $v - x$ subpath in P_u cannot be less than the length of the $v - x$ subpath in Q_v otherwise we can find a tree of size less than that of T'_S which contains u, v and w. So $d_{P_u}(v, x) = d_{Q_v}(v, x)$. If $d_{P_u}(v, x) \geq 2$, then

$$\begin{aligned} d_H(\{u, x, w\}) &= d_H(\{u, v, w\}) \\ &\geq d_G(\{u, v, w\}) + 1 \\ &\geq d_G(\{u, x, w\}) + 3 \end{aligned}$$

which is impossible. Hence $d_{P_u}(v, x) = 1$.

So the only vertex x of P_u distinct from v that can possibly belong to both P_u and Q_v is the neighbour of v on P_u and in this case $d_H(\{u, v, w\}) = d_G(\{u, v, w\}) + 1$, otherwise $d_H(\{u, x, w\}) = d_G(\{u, x, w\}) + 3$, which is not possible.

We assume first that P_u and Q_v have no vertex other than v in common. Consider the cycle C_u induced by the edges of P_u, Q_u and Q_v.

Suppose first that C_u has length at least 6. We show that C_u cannot possess two skew diagonals. Since u is incident with no diagonals of C_u, it follows from condition 3.3.1.1(b) that the vertex x_1 adjacent with u on P_u must be incident with a diagonal of C_u. We use this to show that $d_H(\{u, v, w\}) = d_1 + d_2 = d_G(\{u, v, w\}) + 1 = \ell_1 + \ell_2 + \ell_3 + 1$. If this is not the case, then $d_H(\{u, v, w\}) = d_G(\{u, v, w\}) + 2$. Let $x_1 x'_1$ be a diagonal incident with x_1. If x'_1 belongs to Q_u, let T be the tree induced by the edges of Q_w, Q_v, the $x'_1 - z$ path of Q_u and $x_1 x'_1$. Then $q(T) \leq q(T'_S)$. Also $d_{H-u}(\{x_1, v, w\}) = d_H(\{u, v, w\}) - 1 = d_G(\{u, v, w\}) + 1 > q(T) \geq d_G(\{x_1, v, w\})$. This contradicts our choice of H. So $d_H(\{u, v, w\}) = d_G(\{u, v, w\}) + 1$.

Observe that the diagonals of C_u must join internal vertices of P_u with vertices of Q_u or Q_v. Since condition 3.3.1.1(a) holds, it follows that Q_u or Q_v must have length at least 2.

Suppose first that Q_u has length at least 2. Suppose xx' is a diagonal of C_u where x is an internal vertex of P_u and x' is a vertex of Q_u. If $d_{Q_u}(u,x') > d_H(u,x)$, then the $u-x$ subpath of P_u together with the edge xx', the $x'-z$ subpath of Q_u, Q_v and Q_w induces a tree T of size at most $d_G(\{u,v,w\})$ that contains u,v and w. Since T cannot have size less than $d_G(\{u,v,w\})$, it follows that $d_{Q_u}(u,x') = d_H(u,x)+1$. However, then $d_{H-u}(\{x,v,w\}) > d_G(\{x,v,w\})$, contrary to our choice of H. Hence $d_{Q_u}(u,x') \le d_H(u,x)$. But $d_{Q_u}(u,x')+1 \ge d_H(u,x)$. To see this, suppose $d_H(u,x) \ge d_{Q_u}(u,x')+2$. Let T be the tree induced by the edges of the $u-x'$ subpath of Q_u, the edge $x'x$,the $x-v$ subpath of P_u and the edges of P_w. Then T has size (at least) one less than T_S and T contains u,v and w. Hence T is a Steiner tree for u,v and w. However, this gives rise to a situation already shown to be impossible in Case 2. Thus if xx' is a diagonal of C_u where $x \neq u$ is on P_u and x' on Q_u, then $d_H(u,x) = d_{Q_u}(u,x')$ or $d_{Q_u}(u,x')+1$. Consequently there is no pair of skew diagonals for which two end-vertices are on Q_u and the other two on P_u.

Suppose now that Q_v has length at least 2. Suppose xx' is a diagonal of C_u where x is on P_u and x' on Q_v. Then $d_H(x,v) \ge d_{Q_v}(v,x')$ unless $d_{Q_v}(v,x') = 2$, in which case $d_H(v,x)$ may be 1. Suppose $d_H(x,v) = d < d_{Q_v}(v,x') = \ell$ where $\ell \ge 2$. Then the edges of Q_u, Q_w, those on the $z-x'$ subpath of Q_v and the edge $x'x$ induce a tree of size $\ell_1+\ell_2+\ell_3-\ell+1$ which contains u,x and w. However, $d_H(\{u,x,w\}) = \ell_1+\ell_2+\ell_3+1$. So $d_H(\{u,x,w\})-3 \ge d_G(\{u,x,w\})$ if $\ell \ge 3$, which is impossible. So $\ell = 2$ if $d_H(v,x) < d_{Q_v}(v,x')$ and in this case $d_H(v,x) = d_{Q_v}(v,x')-1 = 1$.

We show next that $d_H(x,v) \le d_{Q_v}(v,x')+1$. Suppose $d_H(x,v) \ge d_{Q_v}(v,x')+2$. Then the edges of the $u-x$ subpath of P_u, together with the edge xx', the edges of the $x'-v$ subpath of Q_v and P_w induce a tree of size $d_1+d_2-1 = d_G(\{u,v,w\})$ which is a path and contains u,v and w. This again leads to a situation we have considered in Case 2 and shown to be impossible. So if there are skew diagonals xx', yy' of C_u such that x

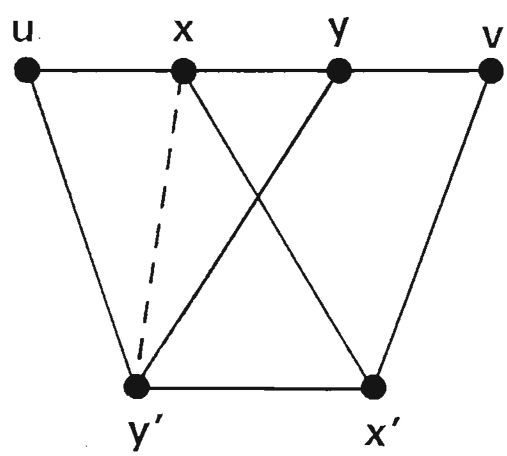


Figure 3.3.2

and y are on P_u and x' and y' are on Q_v, then we may assume $d_{Q_v}(v, x') = 1, d_{P_u}(v, x) = 2, d_{Q_v}(v, y') = 2$ and $d_{P_u}(v, y) = 1$. If Q_u has length at least 2, then necessarily the neighbour x_1 of u on P_u must be incident with the diagonal whose other end-vertex is the neighbour of u on Q_u. However, then $d_{H-u}(\{x_1, y, w\}) = d_1 + d_2 - 1$ and $d_G(\{x_1, y, w\}) \leq d_G(\{u, v, w\}) - 1$. This produces a contradiction to our choice of H. If Q_u has length 1, then the neighbour x_1 of u on P_u must be incident with a diagonal whose other end-vertex x_1' is on Q_v. If $x_1 \neq x$ and so $x_1' \neq x'$, then we still obtain a contradiction to our choice of H. So we may assume $x_1' = x'$. However, then $d_1 = 3$. So since $\ell_3 \geq 2$, and since $d_1 \geq \ell_3 + 1$, it follows that $\ell_3 = 2$. So the subgraph induced by the vertices of C_u is as shown in Figure 3.3.2. (The dotted line may or may not be in the subgraph.) Let v' be the neighbour of v on P_w. Then $v' \neq x'$ since xx' is an edge but xv' is not an edge. Also $v' \neq y'$ since yy' is an edge but $v'y$ is not an edge. If $v'x'$ is an edge, then $d_G(\{u, x, v'\}) = 3$, and $d_{H-w}(\{u, x, v'\}) = 4$. This contradicts our choice of H. Hence $v'x' \notin E(G)$. If $v'y' \in E(G)$, then $d_G(\{u, y, v'\}) = 3$ and $d_{H-w}(\{u, y, v'\}) = 4$. Once again this contradicts our choice of H. Hence $v'y' \notin E(G)$.

Note that neither x nor y is on Q_w. Also, as it was shown that P_u and Q_u have no vertices in common, it can be shown that P_w and Q_w have no vertex in common. Also since $v' \neq x'$, P_w and Q_v have no vertex in common.

Thus the edges of T_S, Q_u and Q_w induce some cycle C of length at least 7. If $y'w$ is an edge of G, i.e., if Q_w is a path of length 1, then C has no skew diagonals contrary to condition 3.3.1.1(a). Hence we may assume Q_w has length at least 2. Since $\ell_2 = q(Q_w) > q(P_w)$, the cycle C_w induced by the edges of Q_w, Q_v and P_w has length at least 7 and hence by condition 3.3.1.1(a) has skew diagonals. Furthermore, these skew diagonals have two end-vertices that belong to P_w and the other two are on Q_v or Q_w. As was argued in the case of C_u, they cannot both be on Q_w. Also they cannot both be on Q_v, since in this case C_w would have length 6 which it does not. Hence one end-vertex is on Q_v and the other one on Q_w. Next consider C. Since C has length at least 8, it must have a pair of skew diagonals. One end-vertex from each of these two skew diagonals must be on T_S (call them a and b) and the other two (one from each of the pair of skew diagonals (call them a' and b') must be on Q_w. Say aa' and bb' are edges. Note that a and b cannot both be on P_w by an earlier observation. So a, say, is an internal vertex of P_u. Now $a \neq x$; otherwise the edges of P_u, the edge aa' and the edges of the $a' - w$ subpath of Q_w induce a tree of size at most T'_S. However, then $d_{H-u}(\{x, v, w\}) > d_G(\{x, v, w\})$, contrary to our choice of H. Since u is not incident with a diagonal of C, x must be incident with a diagonal, which therefore is xy'. We show next that the only diagonal of C with which y can be adjacent is yy'. Suppose yc is a diagonal of C where c is an internal vertex of Q_w. Then P_u together with the edge yc and the $c - w$ subpath of Q_w produces a tree T of size at most $q(T'_S) = d_G(S)$ which contains u, v and w. So T is a Steiner tree for S. But the distance from v to a vertex of degree 3 in T has length less than $\ell_3 = d_{T'_S}(v, y')$, which contradicts our choice of T'_S. Hence the only diagonal of C with which y can be incident is yy'. But then C has no skew diagonals.

So we may assume that C_u has no skew diagonals with two end- vertices on P_u and the other two on Q_v. Suppose now that C_u has two skew diagonals xx', yy' where x, y are on P_u and x precedes y on P_u, x' is on Q_v and y' on Q_u. By the previous cases x' and y' must be internal vertices of Q_v and Q_u, respectively. Note that $d_H(u, x) \geq \ell_1$: Suppose $d_H(u, x) < \ell_1$. If $d_H(u, x) \leq \ell_1 - 2$, then the edges of the $u - x$ subpath of P_u, the edge xx' and the edges of Q_v and Q_w induce a tree that contains u, v and w but has fewer edges than T'_S, which is impossible. If $d_H(u, x) = \ell_1 - 1$, then $d_{H-u}(\{x, v, w\}) = d_1 + d_2 - \ell_1 + 1$ and $d_G(\{x, v, w\}) = \ell_2 + \ell_3 + 1 = d_1 + d_2 - \ell_1$; contrary to our choice of H. Hence $d_H(u, x) \geq \ell_1$.

If $d_{P_u}(v, y) \leq \ell_3 - 2$, then there is a tree of size less than $d_G(\{u, v, w\})$

which contains u, v and w. Since this is not possible we may assume $d_{P_u}(v, y) \geq \ell_3 - 1$. If $d_{P_u}(v, y) = \ell_3 - 1$, then $d_1 \geq \ell_1 + \ell_3$. Observe that $d_2 \geq \ell_2 + 1$, otherwise we can either find a Steiner tree for u, v and w that is a $u - w$ path (a situation considered in Case 2) or a tree of size less than $d_G(\{u, v, w\})$ that contains u, v and w, which is again impossible. However, since $d_1 + d_2 = \ell_1 + \ell_2 + \ell_3 + 1$, it follows that $d_2 = \ell_2 + 1$ and that $d_1 = \ell_1 + \ell_3$. If we now take the $u - y'$ subpath of Q_u together with the edge $y'y$, the $y - v$ subpath of P_u and P_w, we obtain a path of length at most $\ell_1 + \ell_2 + \ell_3$ which contains u, v and w. Again this is a situation considered in Case 2. Hence $d_{P_u}(v, y) \geq \ell_3$. However, then $d_1 \geq \ell_1 + \ell_3 + 1$ and since $d_2 \geq \ell_2 + 1$, it follows that $d_1 + d_2 \geq \ell_1 + \ell_2 + \ell_3 + 2$. This contradicts the fact that $d_1 + d_2 = \ell_1 + \ell_2 + \ell_3 + 1$. So C_u cannot contain skew diagonals. However, C_u cannot be a 6-cycle that satisfies the remaining conditions in 3.3.1.1(a) either.

Hence C_u is a 5-cycle or a 4-cycle. Suppose C_u is a 5-cycle. Since $d_1 \geq \ell_1 + 1$ and $d_1 \geq \ell_3 + 1$, it follows that uz and vz are edges and that the path P_u has length 3. Suppose $C_u : u, u_1, u_2, v, z, u$. Let v' be the neighbour of v on P_w. Then $v' \neq z$ since zu is an edge but $v'u$ is not an edge. If $v'z \notin E(G)$, then the subgraph induced by $V(C_u) \cup \{v'\}$ is one of the forbidden subgraphs shown in Figure 3.3.1(a). If $v'z \in E(G)$, then u, u_1, u_2, v, v', z, u is a 6-cycle that does not satisfy the conditions in 3.3.1.1(a).

Thus we may assume that C_u is a 4-cycle,say u, u_1, v, z, u. Clearly z does not lie on P_w. By considering the cycle C_w induced by the edges of Q_v, Q_w and P_w and applying the arguments similar to those applied to C_u, it can be shown that C_w must be a 4-cycle, say v, v_1, w, z, v. However, then u, u_1, v, v_1, w, z, u is a 6-cycle that does not satisfy the conditions in 3.3.1.1(a).

If P_u and Q_v have a vertex in common, then we have shown that it must be the vertex x that precedes v on P_u. Since $d_G(\{u, x, w\}) \leq$

$d_G(\{u,v,w\}) - 1$ and $d_H(\{u,x,w\}) = d_H(\{u,v,w\})$, it follows that $d_G(\{u,x,w\}) = d_H(\{u,x,w\}) - 2$. Also Q_u, Q_w and the $z - x$ subpath of Q_v (call it Q_x) form a Steiner tree for u, x and w. If we now replace Q_v in the preceding arguments with Q_x, we once again arrive at a contradiction. Hence Case 5 cannot occur either.

Thus G must be 3-Steiner distance hereditary. $\square$

Chapter 4

Functional Isolation Sequences

The concept of a Supply Graph was introduced by Goldsmith [G1] and was defined to be a connected graph with its vertex set $V(G)$ partitioned into two non-empty subsets $P = P(G)$ and $C = C(G)$, called the sets of producer and consumer vertices respectively. We denote a supply graph by $G = G(P, C)$.

Such a graph could represent a network in which the vertices of P represent producers of commodities or services (e.g. power stations, supply depots, computers with data storage facilities etc.) and the vertices in $V(G) - P(G) = C(G)$ represent consumers of the commodities produced (e.g. users of power, dealers, computers processing data, radio receivers, military outposts, etc.).

Further, Goldsmith [G1] defined the k^{th}-order functional edge-connectivity $(\lambda_f^{(k)}(G))$ of $G = G(P, C)$ to be the smallest number of edges of G whose removal from G yields a graph with k functionally isolated components (i.e. components containing consumer vertices only).

We introduce here, the parameter $\mu_f^{(k)}(G)$ which we define to represent the minimum number of edges in G whose removal ensures that at least k vertices are functionally isolated.

Clearly $\lambda_f^{(k)}(G) = \mu_f^{(k)}(G)$ whenever the k functionally isolated components existing, after the removal of $\lambda_f^{(k)}(G) = \mu_f^{(k)}(G)$ edges, consist of single vertices.

4.1 μ_f-sequences

Let $G = G(P,C)$ be a supply graph with $|C| = m$; then the sequence $\mu_f^{(1)}(G), \mu_f^{(2)}(G), ..., \mu_f^{(m)}(G)$ will be called the μ_f-sequence of G.

A non-decreasing sequence of positive integers $A : a_1, a_2, ..., a_m$ is a μ_f-sequence if there exists a supply graph $G = G(P,C)$, with $|C| = m$, which has A as its μ_f-sequence.

In this chapter we will characterize the μ_f-sequence of a Ranked Supply Graph (yet to be defined), and give both necessary and sufficient conditions for a non-decreasing sequence of positive integers to be the μ_f-sequence of a Ranked Supply Graph.

First, some general examples:

1. $A : 1, 2, ..., m$ is the μ_f-sequence of $K_{1,m}$ where P consists of the central vertex if $m \geq 2$ or of either vertex if $m = 1$.

2. The sequence A where $a_1 = ... = a_\ell = 1$, $a_{\ell+1} = ... = a_{2\ell} = 2; ...; a_{(n-1)\ell+1} = ... = a_{n\ell} = n$ is the μ_f-sequence of the supply graph obtained by $\ell - 1$ subdivisions of each edge of $K_{1,n}$ with P consisting of the central vertex.

3. The sequence A where $a_1 = ... = a_\ell = 2$; $a_{\ell+1} = ... = a_{2\ell} = 4; ...; a_{(n-1)\ell+1} = ... = a_{n\ell} = 2n$ is the μ_f-sequence of the supply

graph obtained from the graph in (2) above by the introduction of a new vertex made adjacent to all end-vertices.

4. If $i_1, i_2, ..., i_n$ are positive integers with $i_1 > i_2 > ... > i_n \geq 2$, let G be the graph obtained from the union of the disjoint stars $K_{1,i_1-1}; K_{1,i_2-1}; ...; K_{1,i_n-1}$ (with centres $x_1, ..., x_n$ respectively) by the introduction of a new vertex, x_0, adjacent to $x_1, x_2, ..., x_n$ where $P = \{x_0\}$. The μ_f-sequence of G is $\mu_f^{(1)} = \mu_f^{(2)} = ... = \mu_f^{(i_1)} = 1; \mu_f^{(i_1+1)} = ... = \mu_f^{(i_2)} = 2; ...; \mu_f^{(i_{n-1}+1)} = \mu_f^{(i_n)} = n$.

5. If $G = G(P, C) \cong K_{m+k}$, where $|P| = k$ and $|C| = m$, then with $p = m + k$

$$\begin{aligned} \mu_f^{(\ell)}(G) &= \min\{k(p-k),\ \ell(p-\ell)\} \\ &= \begin{cases} \ell(p-\ell) & \text{for } 1 \leq \ell \leq k \\ k(p-k) & \text{for } k \leq \ell \leq |C| = m = p-k. \end{cases} \end{aligned}$$

4.2 The Ranked Supply Graph

Let $A : a_1, ..., a_m$ be a sequence of positive integers such that

$a_1 = ... = a_{i_1} = b_1 < a_{i_1+1} = ... = a_{i_1+i_2} = b_2 < ... < a_{i_1+i_2+...+i_{j-2}+1} = ... = a_{i_1+..+i_{j-1}} = b_{j-1} < a_{i_1+...+i_{j-1}+1} = ... = a_{i_1+..+i_j} = b_j$ (where $m = i_1 + ...i_j$). 4.1

Consider the supply graph $G_R = G_R(P, C)$ (with $|P| = k, |C| = m$) such that 4.1 holds as a μ_f-sequence. Let C be partitioned into (disjoint subsets $V_1, V_2, ..., V_j$ such that for $t \in \{1, ..., j-1\}$, $|V_t| = i_t$ and $V_1 \cup V_2 \cup .. \cup V_t$ can be functionally isolated by the removal of a set E_t of $b_t = \mu_f^{(s)}(G_R)$ edges where $s = i_1 + i_2 + ... + i_t$ and $E_t \subseteq [V_1 \cup ... \cup V_t, P \cup V_{t+1} \cup ... \cup V_j]$.

Furthermore, for $1 \leq r < j$ the functional isolation of a set of $i_1 + .. + i_r$ vertices containing at least one element from some $V_n (n > r)$ requires the removal of more than b_r edges.

We will call G_R a Ranked Supply Graph.

For $\ell, m \in \{1, 2, ..., j\}$ let $s_\ell = |[V_\ell, P]|$ and for $\ell < m$, let $r_{\ell m} = |[V_\ell, V_m]|$.

For $i = 1, 2, ..., j-1$ let

$$b_i = \sum_{\ell=1}^{i} s_\ell + \sum_{\substack{\ell=1,\dots,i \\ m=i+1,\dots,j}} r_{\ell m} \quad \text{and}$$

$$b_j = \sum_{\ell=1}^{j} s_\ell .$$

It follows from the above definition that $A_{t-1} : a_1, a_2, ..., a_{i_1+i_2+...+i_{t-1}}$ is the μ_f-sequence of the ranked supply graph $G_R^{(t-1)}(P^{(t-1)}, C^{(t-1)})$, where $P^{(t-1)} = P \cup V_t \cup V_{t+1} \cup ... \cup V_j$; $C^{(t-1)} = V_1 \cup V_2 \cup ... \cup V_{t-1}$ and $E(G_R^{(t-1)}) = E(G_R)$.

The consumer vertices in a ranked supply graph could represent consumers which have been ranked according to strategic importance with i_1 vertices of V_1 being the least important and the i_j vertices of V_j being the most important. The values of the b_i would give an indication of the relative importance of the vertices in V_i.

Lemma 4.2.1

If A is the $\mu_f^{(k)}$-sequence of the ranked supply graph $G_R = G_R(P,C)$ then A is also the $\mu_f^{(k)}$-sequence of the ranked supply graph $H_R = H_R(P,C)$ obtained from G_R by joining every pair of non-adjacent vertices in $\langle V_\ell \rangle_{G_R}$ for $\ell = 1, 2, ..., j$.

For $1 \leq \ell < m \leq j$, the (ℓ, m)-deficiency of $G_R, \tau_{\ell m}(G_R)$ and the ℓ-deficiency of $G_R, \tau_\ell(G_R)$, are defined as follows:

$\tau_{\ell m}(G_R) = i_\ell i_m - r_{\ell m}$ and $\tau_\ell(G_R) = \sum_{m=\ell+1}^{j} \tau_{\ell m}(G_R)$. (So $\tau_{\ell m}(G_R)$ is the number of edges in $[V_\ell, V_m]_{\bar{G}_R}$ and $\tau_\ell(G_R)$ is the number of edges in $[V_\ell, V_{\ell+1} \cup ... \cup V_j]_{\bar{G}_R}$.)

We introduce now, a ranked supply graph $J_R = J_R(P', C)$ obtained from G_R as follows:-

Define J_1 as follows if $j \geq 2$:

a) If $s_1 = 0$ or $\tau_1(G_R) = 0$, let $J_1 = G_R$.

b) If $s_1, \tau_1(G_R) \geq 1$, we distinguish between two cases:

(i) If $s_1 \leq \tau_1(G_R)$, let m be the largest integer such that $2 \leq m \leq j$ and $s_1 \leq \sum_{\ell=m}^{j} \tau_{1\ell}(G_R)$. Let $s_1 = t_m + t_{m+1} + .. + t_j$, where $t_\ell = \tau_{1\ell}(G_R)$ if $m < \ell \leq j$ (and $t_m \leq \tau_{1m}(G_R)$, obviously). Replace the s_1 edges in $[V_1, P]$ by s_1 edges in $[V_1, V_m \cup ... \cup V_j]$, assigning t_ℓ edges to $[V_1, V_\ell]$, $(\ell = m, .., j)$ and keeping the degrees of all vertices in V_1 fixed (i.e. $\deg_{J_1} v = \deg_{G_R} v \; \forall v \in V_1$). Finally, for each "new" edge vw inserted above between a vertex $v \in V_1$ and a vertex $w \in V_\ell$ $(m \leq \ell \leq j)$, insert another new edge wx, with $x \in P'$ (where P' is a superset of P, containing new vertices, not in $V(G_R)$ as required). Hence t_ℓ new edges are inserted into $[V_\ell, P']$, $m \leq \ell \leq j$.

(ii) If $s_1 > \tau_1(G_R)$, let $s_1 - \tau_1(G_R)$ edges of $[V_1, P]_{G_R}$ be retained and replace the remaining $\tau_1(G_R)$ edges in $[V_1, P]_{G_R}$ by $2\tau_1(G_R)$ edges as indicated in (i), with $\tau_1(G_R)$ replacing s_1 in (i).

If $j \geq 3$ and $J_1, ..., J_{t-1}$ have been defined, we introduce J_t as follows ($t \in \{2, ..., j-1\}$).

a) If $s_t = 0$ and $\tau_t(J_{t-1}) = 0$, let $J_t = J_{t-1}$.

b) If $s_t > 0$ and $\tau_t(J_{t-1}) > 0$, we distinguish between two cases:

(i) If $s_t \leq \tau_t(J_{t-1})$, let m be the largest integer such that $t+1 \leq m \leq j$ and $s_t \leq \sum_{\ell=m}^{j} \tau_{t\ell}(J_{t-1})$. Let $s_t = t_m + ... + t_j$, where $t_\ell = \tau_{t\ell}(J_{t-1})$ for $m < \ell \leq j$ (if $j > m$) (and $t_m \leq \tau_{tm}(J_{t-1})$, obviously).

Replace the s_t edges in $[V_t, P]$ by s_t edges in $[V_t, V_{t+1} \cup ... \cup V_j]$ by assigning t_ℓ new edges to $[V_t, V_\ell]$ ($\ell = m, ..., j$), keeping the degrees of vertices in V_t unchanged from their values in J_{t-1} and finally inserting t_ℓ new edges into $[V_\ell, P']$, for each edge $vw \in [V_\ell, V_m]$ inserted above, introducing a new edge wx with $x \in P'$, where P' is again a superset of P, if necessary.

(ii) If $s_t > \tau_t(J_{t-1})$, retain $s_t - \tau_t(J_{t-1})$ edges of $[V_t, P]$ and replace the remaining $\tau_t(J_{t-1})$ edges in $[V_t, P]$ by $\tau_t(J_{t-1})$ edges in each of $[V_t, V_{t+1} \cup ... \cup V_j]$ and $[V_t, P']$ as above.

Finally, denote J_{j-1} by $J_R = J_R(P', C)$

Lemma 4.2.2

$J_R(P', C)$ is a ranked supply graph (with P' and C as sets of producers and consumers, respectively) and $J_R(P', C)$ has A as its μ_f-sequence.

Proof: Denote the μ_f-sequence of J_R by $\mathcal{C} : c_1 \leq ... \leq c_m$. Suppose that, for some $i \in \{1, .., m\}$, $c_i \neq a_i$. Let $a_i = b_r$; then $a_i = \sum_{\ell=1}^{r} s_\ell + \sum_{\substack{\ell=1,\dots,r \\ m=r+1,\dots,j}} r_{\ell m}$ and $r \neq j$.

Observe that $V_1 \cup ... \cup V_r$ is functionally isolated in G_R by the removal of S, the set of $\sum_{\ell=1}^{r} s_\ell$ edges from $[V_1 \cup ... \cup V_r, P]_{G_R}$ and R, the set of $\sum_{\substack{\ell=1,\dots,r \\ m=r+1,\dots,j}} r_{\ell m}$ edges in $[V_1 \cup ... \cup V_r, V_{r+1} \cup ... \cup V_j]_{G_R}$.

In J_R the vertices in $V_1 \cup ... \cup V_r$ can be functionally isolated by the removal of the set of edges R above as well as the set S' of edges in $[V_1 \cup ... \cup V_r, P']_{J_R}$ and S'', the set of all "new" edges incident with a vertex in $V_1 \cup ... \cup V_r$ and a vertex in $V_{r+1} \cup ... \cup V_j$ in J_R, i.e. $S'' = [V_1 \cup ... V_r, V_{r+1} \cup ... \cup V_j]_{J_R} - [V_1 \cup ... \cup V_r, V_{r+1} \cup ... \cup V_j]_{G_R}$. By the definition of J_R, if $\ell < m < r$ and $t_{\ell m}$ edges from $[V_\ell, P]$ are replaced by $t_{\ell m}$ edges in $[V_\ell, V_m]$ together with $t_{\ell m}$ edges in $[V_m, P']$, then eventually, when J_m is defined, each element of the latter set of $t_{\ell m}$ edges in $[V_m, P']_{J_\ell}$ is either left unchanged in $[V_m, P']_{J_m}$ or is replaced by an edge in $[V_m, V_{m+1} \cup .. \cup V_j]_{J_m}$ and an edge in $[V_{m+1} \cup ... \cup V_j; P']_{J_m}$; consequently $|S' \cup S''| = \sum_{i=1}^{r} s_i$ and so $V_1 \cup ... \cup V_r$ is functionally isolated in J_R by the removal of a_i edges. It follows that $c_i < a$.

Let B be a largest set of vertices in J_R functionally isolated by the removal of c_i edges from $E(J_R)$, and let $F = [B, V - B]_{J_R}$. We note that, by the maximality of B, $|F| = c_i$. Let $F_1 = F \cap E(G_R)$ and $F_2 = F - F_1$. As $c_i < a_i$, B is not functionally isolated in $G_R - F_1$.
Hence, there exists at least one edge $e = v_1w_1 \in [B, V - B]_{G_R}$ with $v_1 \in B$ (say $v_1 \in V_{\ell_1}$) and $w_1 \in V - B$ such that $v_1w_1 \notin F_1$; so $w_1 \in P$. Furthermore, in the construction of J_R, v_1w_1 was replaced by a sequence of edges, say v_1w_1 by v_1v_2 and v_2w_2 ($v_2 \in V_{\ell_2}, w_1 \in P'$) in the construction

of J_{ℓ_1}, v_2w_1 by v_2v_3 and v_3w_3 ($v_3 \in V_{\ell_3}, w_3 \in P'_r$) in the construction of J_{ℓ_2}, ... etc. until, in J_R, the single edge v_1w_1 has been replaced by the edges of a path $v_1v_2v_3...v_n$ and an edge v_nw_n, where $v_i \in V_{\ell_i}$, $w_n \in P'$ and $\ell_1 < \ell_2 < ... < \ell_n$.

If $v_1, v_2, ..., v_n \in B$, then $v_nw_n \in F_2$ and, if $v_\ell \notin B$ for some $\ell \in \{2, ..., n\}$ with ℓ as small as possible, then $\ell \geq 2$ and $v_{\ell-1}v_\ell \in F_2$ (denote by e' the appropriate edge v_nw_n or $v_{\ell-1}v_\ell$).

Obviously, if $e_1 \neq e_2$ in $[B, V-B]_{G_R}$, then $e'_1 \neq e'_2$ and so $|F_2| \geq |[B, V-B]_{G_R} - F_1|$. So $|F_1 \cup F_2| \geq |[B, V-B]_{G_R}| \geq a_i$, a contradiction.

Hence $a_i = c_i$ for $i = 1, ..., m$. □

We return now to the ranked supply graph G_R and derive some necessary conditions for a non-decreasing sequence of positive integers to be a μ_f-sequence for G_R.

Lemma 4.2.3

If A is the μ_f-sequence of G_R,then

$$\left\lceil \frac{(i_j-1)}{i_j}(b_j - b_{j-1}) \right\rceil \leq i_j - 1.$$

Proof: For $w \in V_j$, all the vertices in $V_1 \cup ... \cup V_{j-1} \cup \{w\}$ can be functionally isolated by removing from G_R all edges in the set $E' = [V_1 \cup ... \cup V_{j-1} \cup \{w\}, P \cup (V_j - \{w\})] = [V_1 \cup ... \cup V_{j-1} \cup \{w\}, P] \cup [V_1 \cup ... \cup V_{j-1}, V_j - \{w\}] \cup [\{w\}, V_j - \{w\}]$ 4.2.3.1

Furthermore $\quad |E'| \geq b_j$ 4.2.3.2

and $\quad b_{j-1} = \sum_{\ell=1}^{j-1}(s_\ell + r_{\ell j})$ 4.2.3.3

Also

$$\begin{aligned}
\sum_{w \in V_j} & (|[\{w\}, P]| - |[\{w\}, V_1 \cup ... \cup V_{j-1}]|) \\
= & \quad |[V_j, P]| - |[V_j, V_1 \cup ... \cup V_{j-1}]| \\
= & \quad s_j - \textstyle\sum_{\ell=1}^{j-1} r_{\ell j}; \text{ hence } w \text{ can be chosen so that}
\end{aligned}$$

$$|[\{w\}, P]| - |[\{w\}, V_1 \cup ... \cup V_{j-1}]| \leq \left\lfloor \frac{1}{i_j}\left(s_j - \sum_{\ell=1}^{j-1} r_{\ell j}\right)\right\rfloor \qquad 4.2.3.4$$

Thus from 4.2.3.1; 4.2.3.2; 4.2.3.3 and 4.2.3.4 we obtain

$$b_j \leq |E'| \leq \left(\sum_{\ell=1}^{j-1} s_\ell + |[\{w\}, P]|\right) + \left(\sum_{\ell=1}^{j-1} r_{\ell j} - |[\{w\}, V_1 \cup ... \cup V_{j-1}]|\right) + (i_j - 1)$$

$$= b_{j-1} + (i_j - 1) + \left\lfloor \frac{1}{i_j}\left(s_j - \sum_{\ell=1}^{j-1} r_{\ell j}\right)\right\rfloor \qquad 4.2.3.5$$

Since $b_j - b_{j-1} = \sum_{\ell-1}^{j} s_j - \sum_{\ell-1}^{j-1}(s_\ell - r_{\ell j}) = s_j - \sum_{\ell=1}^{j-1} r_{\ell j}$, it follows from 4.2.3.5 that

$$b_j \leq b_{j-1} + (i_j - 1) + \left\lfloor \frac{b_j - b_{j-1}}{i_j} \right\rfloor$$

from which it follows that

$$\left\lceil \frac{(i_j - 1)}{i_j}(b_j - b_{j-1}) \right\rceil \leq i_j - 1. \qquad \square$$

By applying lemma 4.2.3 to the μ_f-sequence of $G_R(P \cup V_{\ell+1} \cup ... \cup V_j, C - (V_{\ell+1} \cup ... \cup V_j))$ we obtain the following corollary.

Corollary 4.2.3

If A is the μ_f-sequence of $G_R(P, C)$, then for $2 \leq \ell \leq j$, $\left\lceil \frac{(i_\ell - 1)}{i_\ell}(b_\ell - b_{\ell-1}) \right\rceil \leq i_\ell - 1$.

Lemma 4.2.4

If A is the μ_f-sequence of $G_R(P, C)$, then for $\ell < j$

$$b_{\ell+1} - b_\ell \leq \left\lfloor \frac{b_{j-1} + b_j}{i_j} \right\rfloor + i_j - 1$$

Proof: For $w \in V_j$ and $\ell < j$, the vertices in $V_1 \cup ... \cup V_\ell \cup \{w\}$ can be functionally isolated by removing from G_R all edges in the set $E' = [V_1 \cup ... \cup V_\ell \cup \{w\}, P \cup V_{\ell+1} \cup ... \cup (V_j - \{w\})] = [V_1 \cup ... \cup V_\ell, V_{\ell+1} \cup ... \cup V_j \cup P] + [\{w\}, V_{\ell+1} \cup ... \cup V_{j-1} \cup (V_j - \{w\} \cup P] - [\{w\}, V_1 \cup ... \cup V_\ell]$.

Thus

$$\begin{aligned} b_{\ell+1} &\leq \sum_{\substack{i \leq \ell \\ t \geq \ell+1}} r_{it} + \sum_{i \leq \ell} s_i + |[\{w\}, V_{\ell+1} \cup ... \cup V_{j-1} \cup (V_j - \{w\}) \cup P]| \\ &\qquad - |[\{w\}, V_1 \cup ... \cup V_\ell]| \\ &= b_\ell + |[\{w\}, V_1 \cup ... \cup V_{j-1} \cup (V_j - \{w\}) \cup P]| - 2|[\{w\}, V_1 \cup ... \cup V_\ell]| \\ &\leq b_\ell + |[\{w\}, V_1 \cup ... \cup V_{j-1}]| + i_j - 1 + |[\{w\}, P]|. \end{aligned}$$

Now summing over all $w \in V_j$ gives

$$i_j(b_{\ell+1} - b_\ell) \leq |[V_j, V_1 \cup ... \cup V_{j-1}]| + i_j(i_j - 1) + s_j$$

and since $b_{j-1} = |[V_j, V_1 \cup ... \cup V_{j-1}]| + \sum_{i=1}^{j-1} s_i$ it follows that

$$i_j(b_{\ell+1} - b_\ell) \leq b_{j-1} - \sum_{i=1}^{j-1} s_i + i_j(i_j - 1) + b_j - \sum_{i=1}^{j-1} s_i$$

Hence
$$\begin{aligned} i_j(b_{\ell+1} - b_\ell) &\leq b_{j-1} + i_j(i_j - 1) - 2\sum_{i=1}^{j-1} s_i + b_j \\ &\leq b_{j-1} + i_j(i_j - 1) + b_j. \end{aligned}$$

Thus $\quad b_{\ell+1} - b_\ell \leq \left\lfloor \frac{b_{j-1}+b_j}{i_j} \right\rfloor + i_j - 1.$ □

Corollary 4.2.4

If A is the μ_f-sequence of $G_R(P, C)$, then for $1 \leq t \leq q - 1 \leq j - 1$,

$$b_t - b_{t-1} \leq i_q - 1 + \left\lfloor \frac{b_q + b_{q-1}}{i_q} \right\rfloor$$

Lemma 4.2.5

If A is the μ_f-sequence of $G_R(P, C)$ for which $j = 1$, then $m = 1$ or $b_1 \leq m$.

Proof: Let $A : a_1 = a_2 = ... = a_m = b_1$ be the μ_f-sequence of G_R and let $v \in C$. Since v can be functionally isolated by the removal of the edges incident with v and since there exists $v^* \in C$ for which $\|[\{v^*\}, P]\| \leq \frac{\|[C,P]\|}{m} = \frac{b_1}{m}$, it follows that

$$b_1 \leq \deg v^* \leq m - 1 + \frac{b_1}{m};$$

hence $b_1(m-1) \leq (m-1)m$.
Consequently $m = 1$ or $b_1 \leq m$. □

For the ranked supply graph $G_R = G_R(P, C)$ with A as μ_f-sequence with $j \geq 2$, lemma 4.2.5 together with the fact that $a_1, a_2, ..., a_{i_1}$ is the μ_f-sequence of $G_R(P \cup V_2 \cup ... \cup V_j, V_1)$ leads to the following corollary.

Corollary 4.2.5

For the μ_f-sequence of the ranked supply graph $G_R(P, C)$, $b_1 \leq i_1$ or $i_1 = 1$.

Lemma 4.2.6

If A is the μ_f-sequence of $G_R(P, C)$, then

$$b_{j-1} + b_j \geq (b_1 - i_j + 1)i_j$$

Proof: Let $w \in V_j$ be a vertex of smallest degree in $G_R - E(\langle V_j \rangle)$, then $\deg_{G_R} w \leq i_j - 1 + \left\lfloor \frac{\sum_{i=1}^{j-1} r_{\ell j} + s_j}{i_j} \right\rfloor$. Hence

$$b_1 \leq i_j - 1 + \left\lfloor \frac{\sum_{\ell=1}^{j-1} r_{\ell j} + s_j}{i_j} \right\rfloor \leq i_j - 1 + \left\lfloor \frac{b_{j-1} + b_j}{i_j} \right\rfloor$$

and consequently $(b_1 - i_j + 1)i_j \leq b_{j-1} + b_j$. □

This condition is clearly satisfied if $i_j \leq 2$ and since $a_1, a_2, ..., a_{i_1+...+i_\ell}$ is the μ_f-sequence of $G_R(P \cup V_{\ell+1} \cup ... \cup V_j, V_1 \cup ... \cup V_\ell)$ for $\ell \in \{2, ..., j\}$ the next corollary follows.

Corollary 4.2.6

If A is the μ_f-sequence of $G_R(P, C)$ then $b_{\ell-1} + b_\ell \geq (b_1 - i_\ell + 1)i_\ell$ for $\ell \in \{2, ..., j\}$ and $i_\ell \geq 3$.

Lemma 4.2.7

If A is the μ_f-sequence of $G_R(P, C)$, let $2 \leq t + 2 \leq n \leq j$ and $a_m = b_r$ for $m = \begin{cases} i_1 + ... + i_t + i_n + ... + i_j = |V_1 \cup ... \cup V_t \cup V_n \cup ... \cup V_j| \text{ if } t \geq 1 \\ i_n + ... + i_j = |V_n \cup ... \cup V_j| \text{ if } t = 0, \end{cases}$ then

$b_r - b_t < b_j - b_{n-1} + 2i_{n-1}i_n$ if $t \geq 1$ and

$b_r < b_j - b_{n-1} + 2i_{n-1}i_n$ if $t = 0$.

Proof: If G is a ranked supply graph, the functional isolation of $V_1 \cup V_2 \cup ... \cup V_r$ requires fewer edge removals than the functional isolation of

$$S = \begin{cases} V_1 \cup ... \cup V_t \cup V_n \cup ... \cup V_j & \text{if} \quad t \geq 1 \\ V_n \cup ... \cup V_t & \text{if} \quad t = 0. \end{cases}$$

Consequently, if $t \geq 1$

$$b_r < b_t + b_j - b_{n-1} + 2|[V_{n-1}, V_n]|,$$

whence we obtain

$$b_r - b_t < b_j - b_{n-1} + 2i_{n-1}i_n$$

and, if $t = 0$,

$$b_r < b_j - b_{n-1} + 2i_{n-1}i_n. \qquad \square$$

Corollary 4.2.7

If A is the μ_f-sequence of $G_R(P,C)$ then for $n \le q \le j$,

$$b_r - b_t < b_q - b_{n-1} + 2i_{n-1}i_n \quad \text{if} \quad t \ge 1 \text{ and}$$
$$b_r < b_q - b_{n-1} + 2i_{n-1}i_n \quad \text{if} \quad t = 0.$$

The necessary conditions for a sequence A, of positive non-decreasing integers, to be the μ_f-sequence of a ranked supply graph can be summarized as follows:-

1) $b_1 \le i_1$ or $i_1 = 1$.

2) $\left\lceil \frac{(i_\ell - 1)}{i_\ell}(b_\ell - b_{\ell-1}) \right\rceil \le i_\ell - 1 \qquad 2 \le \ell \le j$

3) $b_t - b_{t-1} \le i_q - 1 + \left\lfloor \frac{b_q + b_{q-1}}{i_q} \right\rfloor \qquad i \le t \le q-1 \le j-1$

4) $b_{\ell-1} + b_\ell \ge (b_1 - i_\ell + 1)i_\ell \qquad 2 \le \ell \le j$

5) $b_{\ell-1} - b_{\ell-2} \le b_\ell - b_{\ell-1} + 2i_\ell i_{\ell-1} \qquad 3 \le \ell \le j$

That these conditions are independent can be shown by the following sequences

A_1 : 1,1,1,3,3,5,5

A_2 : 3,3,5,5,7,7

A_3 : 1,1,4,4,7,7

$A_4 : a_1 = 1, a_2 = ... = a_{13} = 10, a_{14} = ... = a_{18} = 11$

$A_5 : a_1 = ... = a_{10} = 10; a_{11} = ... = a_{13} = 11; a_{14} = ... = a_{16} = 12$

A_6 : 2,2,10, 11

A_1 satisfies all conditions, A_2 satisfies all but condition 1, A_3 satisfies all but condition 2, A_4 satisfies all but condition 3, A_5 satisfies all but condition 4 and A_6 satisfies all but condition 5.

Theorem 4.2.8 If A is a non-decreasing sequence of m positive numbers, $A : a_1, a_2, ..., a_m$, where $a_1 = ... = a_{i_1} = b_1 < a_{i_1+1} = ... = a_{i_1+i_2} =$

$b_2 < \ldots < a_{i_1+\ldots i_{j-1}+1} = \ldots = a_{i_1+\ldots+i_{j-1}+i_j} = b_j, m = i_1 + i_2 + \ldots + i_j$, and A satisfies the following conditions

$$b_1 \leq i_1 \text{ or } i_1 = 1 \tag{1}$$

$$\lceil (i_\ell - 1)(b_\ell - b_{\ell-1})/i_\ell \rceil \leq i_\ell - 1 \text{ for } 2 \leq \ell \leq j \tag{2}$$

$$b_{\ell-1} + b_\ell \geq (b_1 - i_\ell + 1)i_\ell \text{ for } 2 \leq \ell \leq j \tag{3}$$

$$b_t - b_{t-1} \leq i_q - 1 + \lfloor (b_q + b_{q-1})/i_q \rfloor \text{ for } 2 \leq t \leq q-1 \leq j-1 \tag{4}$$

$$b_{q-1} - b_{q-2} \leq b_q - b_{q-1} + 2i_q i_{q-1} \text{ for } 3 \leq q \leq j, \tag{5}$$

then there exists a ranked supply graph $G_R = G_R(P, C)$ with $|C| = m$ which has A as its μ_f-sequence.

Proof: We construct a sequence of supply graphs, $G_1, G_2, \ldots, G_j$ as follows:

$G_1 = G_1(P_1, C_1)$ is a graph with $C_1 = V_1$ (say), $|V_1| = i_1$, $\langle V_1 \rangle \cong K_{i_1}$ and each vertex in V_1 is adjacent to $\lfloor b_1/i_1 \rfloor$ or $\lceil b_1/i_1 \rceil$ vertices in P_1, so that $|[V_1, P_1]| = b_1$. (The only restriction on P_1 is that $|P_1| \geq \lceil b_1/i_1 \rceil$. Let $\langle P_1 \rangle$ be empty.)

Suppose that $G_1, \ldots, G_\ell$ have been defined $(1 \leq \ell \leq j-1)$ where, for $1 \leq h \leq \ell$, $G_h = G_h(P_h, C_h)$, $C_h = V_1 \cup \ldots \cup V_h$, $\langle V_1 \rangle_{G_h} \cong K_{i_1}, \ldots, \langle V_h \rangle_{G_h} \cong K_{i_h}$, and, for $s_h = |[V_h, P_h]|$, each vertex in V_h is on $\lceil s_h/i_h \rceil$ or $\lfloor s_h/i_h \rfloor$ edges of $[V_h, P_\ell]$ in G_ℓ. Furthermore, $b_\ell - \sum_{i=1}^{\ell=1} s_i = s_\ell$.

We next define $G_{\ell+1} = G_{\ell+1}(P_{\ell+1}, C_{\ell+1})$; $C_{\ell+1} = V_1 \cup \ldots \cup V_\ell \cup V_{\ell+1}$, where $|V_{\ell+1}| = i_\ell$ and $\langle V_{\ell+1} \rangle_{G_{\ell+1}} \cong K_{i_{\ell+1}}$. The edge set $E(G_{\ell+1})$ consists of all the edges in $E(G_\ell) - [V_\ell, P_\ell]$ together with $E(\langle V_{\ell+1} \rangle)$ and a set F_ℓ defined as follows:

Case (a): If $s_\ell \leq i_\ell i_{\ell+1}$, then for each edge $e = uv \in [V_\ell, P_\ell]$ with $u \in V_\ell$, $v \in P_\ell$, F_ℓ contains the edges $e' = uw$ and $e'' = wv$ (with

$w \in V_{\ell+1}$), assigned so that each vertex w in $V_{\ell+1}$ is on $\lceil s_\ell/i_{\ell+1}\rceil$ or $\lfloor s_\ell/i_{\ell+1}\rfloor$ such edges of $[V_\ell, V_{\ell+1}]$. Finally a further set of $b_{\ell+1} - b_\ell$ edges is inserted into $[V_{\ell+1}, P_{\ell+1}]$ in F_ℓ and $|[V_{\ell+1}, P_{\ell+1}]|$ is denoted by $s_{\ell+1}$, where the edges of $[V_{\ell+1}, P_{\ell+1}]$ are chosen so that each vertex in $V_{\ell+1}$ is on $\lceil s_{\ell+1}/i_{\ell+1}\rceil$ or $\lfloor s_{\ell+1}/i_{\ell+1}\rfloor$ edges of $[V_{\ell+1}, P_{\ell+1}]$. In this case in $G_{\ell+1}$, $[V_\ell, P_{\ell+1}] = \emptyset$ and (in $G_{\ell+1}$) $s_\ell = 0$ (whereas $s_\ell = b_\ell - \sum_{i=1}^{\ell-1} s_i$ in G_ℓ), so $s_{\ell+1} = b_{\ell+1} - b_\ell = b_{\ell+1} - \sum_{i=1}^{\ell-1} s_i = b_{\ell+1} - \sum_{i=1}^{\ell} s_i$. $P_{\ell+1}$ is equal to P_ℓ if $\lceil s_{\ell+1}/i_\ell\rceil \le |P_\ell|$ and is a superset of P_ℓ of size $\lceil s_{\ell+1}/i_{\ell+1}\rceil$ otherwise, inducing an empty subgraph in $G_{\ell+1}$.

Case (b): If $s_\ell > i_\ell i_{\ell+1}$, we recall that each vertex in V_ℓ is on $\lfloor s_\ell/i_\ell\rfloor$ or $\lceil s_\ell/i_\ell\rceil$ edges of $[V_\ell, P_\ell]$, hence on at least $i_{\ell+1}$ edges of $[V_\ell, P_\ell]$ in G_ℓ. For each $u \in V_\ell$, insert into F_ℓ, $i_{\ell+1}$ edges in $[\{u\}, V_{\ell+1}]$ (so $\langle V_\ell \cup V_{\ell+1}\rangle_{G_{\ell+1}} \cong K_{i_\ell+i_{\ell+1}}$) as well as the set of $\lceil s_\ell/i_\ell\rceil - i_{\ell+1}$ or $\lfloor s_\ell/i_\ell\rfloor - i_{\ell+1}$ edges of $[\{u\}, P_\ell]$ remaining after the removal of any edges from $[\{u\}, P_\ell]_{G_\ell}$. Finally $s_{\ell+1} = b_{\ell+1} - b_\ell + i_\ell i_{\ell+1}$ edges of $[V_{\ell+1}, P_{\ell+1}]$ are inserted into F_ℓ, so that each vertex of $V_{\ell+1}$ is on $\lfloor s_{\ell+1}/i_{\ell+1}\rfloor$ or $\lceil s_{\ell+1}/i_{\ell+1}\rceil$ of them. The set $P_{\ell+1}$ equals P_ℓ if $\lceil s_{\ell+1}/i_{\ell+1}\rceil \le |P_\ell|$ and is a superset of P_ℓ with $|P_{\ell+1}| = \lceil s_{\ell+1}/i_{\ell+1}\rceil$ otherwise, inducing an empty subgraph in $G_{\ell+1}$. Finally the symbol s_ℓ is changed to denote $|[V_\ell, P_{\ell+1}]_{G_{\ell+1}}|$ (i.e. s_ℓ in G_ℓ is reduced by $i_\ell i_{\ell+1}$ to s_ℓ in $G_{\ell+1}$). Thereafter $s_{\ell+1} = b_{\ell+1} - b_\ell + i_\ell i_{\ell+1} = b_{\ell+1} - (b_\ell - i_\ell i_{\ell+1}) = b_{\ell+1} - \sum_{i=1}^{\ell} s_i$. Note that in $G_{\ell+1}$, for $1 \le h \le \ell+1$, each vertex of V_h is on $\lceil s_h/i_h\rceil$ or $\lfloor s_h/i_h\rfloor$ edges in $[V_h, P_{\ell+1}]$, as required in the inductive definition.

Note that in this case, as in (a), it may be said that each edge $e = uv \in [V_\ell, P_\ell]_{G_\ell} - [V_\ell, P_{\ell+1}]_{G_{\ell+1}}$ ($u \in V_\ell$, $v \in P_\ell$) is replaced by $e' = uw \in [V_\ell, V_{\ell+1}]_{G_{\ell+1}}$ and $e'' = wv \in [V_{\ell+1}, P_{\ell+1}]_{G_{\ell+1}}$ and we shall say that e', e'' correspond to e.

Now let $G_R(P, C) = G_j(P_j, C_j)$ and denote the μ_f-sequence of $G_\ell(P_\ell, C_\ell)$

by $D^\ell : d_1^\ell, d_2^\ell, ..., d_{i_1+...+i_\ell}^\ell$. We shall prove by induction on ℓ that D^ℓ is $a_1, a_2, ..., a_{i_1+...+i_\ell}$:

Let $\ell = 1$. The i_1 vertices in V_1 can be functionally isolated in $G_1(P_1, C_1)$ by the removal of the b_1 edges in $[V_1, P_1]$. If $i_1 \geq 2$, then the fundamental isolation of exactly k vertices of V_1 in $G_1(P_1, C_1)$ (for $1 \leq k < i_1$) requires the removal of at least $n_k = k(i_1 - k) + k\lfloor b_1/i_1 \rfloor$ edges. We recall that in this case, by condition (1), $b_1 \leq i_1$. However, the use of elementary calculus yields $n_k \geq i_1 - 1 + \lfloor b_1/i_1 \rfloor$. Hence $n_k \geq b_1$ for $k = 1, ..., i_1 - 1$ and so $d_1^1 = ... = d_{i_1}^1 = b_1$, as required.

We now assume that D^n is $a_1, a_2, ..., a_{i_1+...+i_r}$ for all integers r satisfying $1 \leq r \leq \ell$ and that $V_1 \cup ... \cup V_r$ can be isolated by the removal of b_r edges. To show that $D^{\ell+1}$ is $a_1, a_2, ..., a_{i_1+..+i_\ell+i_{\ell+1}}$, we establish a few lemmas.

Lemma 4.2.9 If $\ell \in \{1, 2, ..., j-1\}$ and $S \subseteq C_\ell$, the minimum number of edges required to be removed in G_ℓ and in $G_{\ell+1}$ for the functional isolation of S are equal.

Proof: Let the minimum number of edges whose removal from G_ℓ (or $G_{\ell+1}$) functionally isolates S be α (or β, respectively) and let $F' \subseteq E(G_\ell), F'' \subseteq E(G_{\ell+1})$ such that $|F'| = \alpha$, $|F''| = \beta$ and S is functionally isolated in both $G_\ell - F'$ and $G_{\ell+1} - F''$. Then, by replacing each edge e in $F' \cap [V_\ell, P_\ell] - E(G_{\ell+1})$ by a corresponding edge e' in $[V_\ell, V_{\ell+1}]$, we obtain from F' a set $F''' \subseteq E(G_{\ell+1})$ with $|F'''| = |F'| = \alpha$ such that S is functionally isolated in $G_{\ell+1}$. Hence $\beta \leq \alpha$. Conversely, by replacing every edge $e' \in F'' \cap [V_\ell, V_{\ell+1}]$ or $e'' \in F'' \cap [V_{\ell+1}, P_{\ell+1}]$ by the corresponding edge e in $[V_\ell, P_\ell]$, we obtain a set F^{IV} from F'' with $|F^{IV}| \leq |F''|$ such that S is functionally isolated in $G_\ell - F^{IV}$. So $\alpha \leq |F^{IV}| \leq |F''| = \beta$ and hence $\alpha = \beta$. □

We now suppose that $D^{\ell+1} \neq a_1, a_2, ..., a_{i_1+...+i_{\ell+1}}$ and let i be the largest index for which $a_i \neq d_i^{\ell+1}$; say $a_i = b_r$. Let $S \subseteq C_{\ell+1}$ such that $|S| = i$ and S can be functionally isolated in $G_{\ell+1}$ by the removal of a set F of $d_i^{\ell+1}$ edges of $G_{\ell+1}$. For $i = 1, 2, ..., \ell+1$, let $V_i \cap S = S_i$ and $V_i - S_i = T_i$.

Lemma 4.2.10

a) $i < i_1 + ... + i_{\ell+1} = |C_{\ell+1}|$,

b) $a_i < d_i^{\ell+1}$,

c) $d_i^{\ell+1} < d_{i+1}^{\ell+1}$,

d) $S \cap V_{\ell+1} \neq \emptyset$.

Proof: That (a) holds is obvious, as the functional isolation of all vertices in $C_{\ell+1}$ requires removal of the $\sum_{i=1}^{\ell+1} s_i$ $(= b_{\ell+1})$ edges in $[C_{\ell+1}, P_{\ell+1}]$. Furthermore, as the set $V_1 \cup ... \cup V_r$ of $i_1 + ... + i_r$ $(\geq i)$ vertices can be functionally isolated by the removal from $G_{\ell+1}$ of the b_r edges in $[V_1 \cup .. \cup V_r, V_{\ell+1} \cup P_{\ell+1}]$, it follows that $d_i^{\ell+1} \leq b_r = a_i$ and so $d_i^{\ell+1} < a_i$. From our choice of i it follows that $d_{i+1}^{\ell+1} = a_{i+1}$ and so $d_i^{\ell+1} < a_i \leq a_{i+1} = d_{i+1}^{\ell+1}$, whence (c) follows. If $S \cap V_{\ell+1} = \emptyset$, then $S \subseteq C_\ell$ and so, by Lemma 4.2.9 and the inductive hypothesis, $d_i^{\ell+1} = d_i^\ell = a_i$, a contradiction. □

Lemma 4.2.11 If $S \cap V_1 \neq \emptyset$, then $V_1 \subseteq S$, i.e., $V_1 = S_1$.

Proof: If $i_1 = 1$, this is obvious. So assume that $i_1 \geq 2$. Note that $i_1 \geq b_1$ and let F' be the set of edges obtained from F by replacing the set of all edges in F covered by V_1 by $[V_1, V_2]_{G_{\ell+1}}$. Then, if $S_1 \neq V_1, |F'| \leq |F|$. (This is obvious if $b_1 < i_1$, whereas, if $b_1 = i_1$, each vertex of V_1 is on an edge in $[V_1, V_2]$ and so at least one of the b_1 edges in $[V_1, V_2]$ is contained in F, at most $b_1 - 1$ in $F' - F$, whereas $F - F'$ contains at least $i_1 - 1 (\geq b_1 - 1)$ edges.) However, the set of vertices functionally isolated in $G_{\ell+1} - F'$ is a proper superset of S, viz. $S \cup V_1$, which contradicts Lemma 4.2.10(c). So, if $S \cap V_1 \neq \emptyset$, then $V_1 \subseteq S$. $\square$

Lemma 4.2.12 If t is the smallest index for which $V_t \neq S_t$, then $S_t = \emptyset$.

Proof: The statement is obviously true if $i_t = 1$. So assume that $i_t \geq 2$ and suppose that $\emptyset \neq S_t \neq V_t$; then $t \geq 2$. Furthermore, if $t = \ell + 1$, then $V_1 \cup ... \cup V_\ell \subset S$ and it is a consequence of condition (2) and the maximality of i that $S_t = V_t$, a contradiction. So $2 \leq t \leq \ell$.

Let F_t denote the subset of F covered by $V_1 \cup ... \cup V_{t-1} \cup S_t$; then $V_1 \cup .. \cup V_{t-1} \cup S_t$ is functionally isolated in $G_{\ell+1} - F_t$ by the removal of $h_t = |F_t|$ edges. By Lemma 4.2.9, $V_1 \cup ... \cup V_{t-1} \cup S_t$ can be functionally isolated in G_t by the removal of $h'_t \leq h_t$ edges. By the inductive hypothesis all vertices in $V_1 \cup .. \cup V_{t-1} \cup V_t$ can be functionally isolated in G_ℓ (hence, by Lemma 4.2.9, in $G_{\ell+1}$) by the removal of b_t edges and $b_t \leq h'_t \leq h_t = |F_t|$. So $S \cup T_t$ can be functionally isolated in $G_{\ell+1}$ by the removal of the $d_i^{\ell+1} - h_t + b_t (\leq d_i)$ edges in $(F - F_t) \cup [V_1 \cup, ..., \cup V_t, V_{t+1} \cup P_{\ell+1}]$. But $S \cup T_t$ is a proper superset of S and so Lemma 4.2.10(c) is contradicted. Hence it follows that $S_t = \emptyset$. $\square$

For the following Lemma t is defined as above, so $t \leq \ell$, and we note that, as $S_{\ell+1} \neq \emptyset$, there exists some $q \geq t+1$ for which $S_q \neq \emptyset$.

Lemma 4.2.13 If q is the smallest index such that $q \geq t+1$ and $S_q \neq \emptyset$, then $S_q = V_q$.

Proof: Denote by n_1 and n_2 the smallest number of edges of G_q the removal of which functionally isolates the sets $V_1 \cup ... \cup V_{t-1} \cup S_q$ and $V_1 \cup .. \cup V_{t-1} \cup V_q$. Suppose that $S_q \neq V_q$; then $i_q \geq 2$ and, by Lemma 4.2.10(c), $n_1 < n_2$. We note that in G_q, $|[V_q, V_{q-1}]| = b_{q-1} - \sum_{i=1}^{q-1} s_i$ and $|[V_q, P_q]| = b_q - \sum_{i=1}^{q-1} s_i$; so, for

$x \in S_q$, $|[S_q, V_{q-1} \cup P_q]| \geq |[\{x\}, V_{q-1} \cup P_q]| \geq \left\lfloor \left(b_{q-1} + b_q - 2\sum_{i=1}^{q-1} s_i \right) / i_q \right\rfloor$.

$$
\begin{aligned}
\text{Now} \quad n_1 &= b_{t-1} + |[S_q, V_{q-1} \cup T_q \cup P_q]| \\
&= b_{t-1} + |S_q|\,|T_q| + |[S_q, V_{q-1} \cup P_q]| \\
&\geq b_{t-1} + i_q - 1 + \left\lfloor \left(b_{q-1} + b_q - 2\sum_{i=1}^{q-1} s_i \right) / i_q \right\rfloor,
\end{aligned}
$$

and

$$
\begin{aligned}
n_2 &= b_{t-1} + |[V_{q-1}, V_q \cup P_q]| \\
&= b_{t-1} + b_{q-1} + b_q - 2\sum_{i=1}^{q-1} s_i.
\end{aligned}
$$

So, as $n_2 \geq n_1 + 1$,

$$
\begin{aligned}
b_q + b_{q-1} - 2\sum_{i=1}^{q-1} s_i &\geq i_q + \left\lfloor (b_{q-1} + b_q - 2\sum_{i=1}^{q-1} s_i)/i_q \right\rfloor \\
&> i_q - 1 + (b_{q-1} + b_q - 2\sum_{i=1}^{q-1} s_i)/i_q.
\end{aligned}
\qquad \text{...(4.2.13.1)}
$$

But, by condition (4),

$$
b_q + b_{q-1} \leq i_q - 1 + (b_{q-1} + b_q - 2\sum_{i=1}^{q-1} s_i)/i_q. \qquad \text{...(4.2.13.2)}
$$

Now (4.2.13.2)-(4.2.13.1) yields $\sum_{i=1}^{q-1} s_i < \left(\sum_{i=1}^{q-1} s_i \right) / i_q$, a contradiction, from which it follows that $S_q = V_q$. □

By applying the conditions (1) to (5) and the techniques used in the proofs of Lemma 4.2.12 and Lemma 4.2.13 in the obvious manner, we obtain the following result:

Lemma 4.2.14 If, for $i \in \{1, .., \ell + 1\}$, $S_i \neq \emptyset$, then $S_i = V_i$.

Lemma 4.2.15 If S is chosen to yield the largest possible value of t, then $S = V_1 \cup ... \cup V_x \cup V_y \cup ... \cup V_{\ell+1}$ for some indices x, y satisfying $1 \leq x < x + 2 \leq y \leq \ell + 1$.

Proof: Let y be the largest number in $\{1, ..., \ell+1\}$ for which $S_{y-1} \neq V_{y-1}$ (i.e., $S_{y-1} = \emptyset$). Then $V_y \cup .. \cup V_{\ell+1} \subset S$ and we note that, as observed in the proof of Lemma 4.2.12, $V_1 \cup ... \cup V_\ell \not\subset S$, so y exists and, by Lemma 4.2.10, $y \geq t+1$.
Let $S' = S - (V_x \cup ... \cup V_{\ell+1}), |S'| = i'$ and $a_{i'} = b_x$, then $|V_1 \cup ... \cup V_x| \geq |S'| = i'$.

Denote by f, f', f'' and f''' the smallest numbers of edges whose removal from $G_{\ell+1}$ functionally isolates S, S', $V_1 \cup ... \cup V_x$ and $V_1 \cup ... \cup V_x \cup V_y \cup ... \cup V_{\ell+1}$, respectively. Then $f = f' + f''$ and $f''' = f'' + b_x$.

Note that $f' \geq a_{i'} = b_x$ (by the inductive hypothesis and Lemma 4.2.9). By our choice of S, $f \leq f'''$ and, if $S' \neq V_1 \cup ... \cup V_x$, then $f < f'''$; hence $f' < b_x$, a contradiction. So $S' = V_1 \cup ... \cup V_x$. □

Lemma 4.2.16 $S = V_1 \cup ... \cup V_t \cup V_n \cup ... \cup V_{\ell+1}$, where $1 \leq t < n \leq \ell+1$.

Proof: Let $S'' = S - (V_n \cup ... \cup V_{\ell+1})$ and $i'' = |S''|$; then $S'' \subset C_\ell$ and so, by Lemma 4.2.9 and the inductive hypothesis, the functional isolation of S'' in G_ℓ (and in $G_{\ell+1}$) requires the removal of at least $a_{i''}$ edges. Let $a_{i''} = b_m$ and let $f'' = |[S'', V(G_\ell) - S'']|$. Then $f'' \geq b_m$ with equality attained if and only if $S'' = V_1 \cup ... \cup V_m$ (as $m \geq |S''| = i''$ and i is maximal).

Furthermore, as $|V_1 \cup ... \cup V_m \cup V_n \cup ... \cup V_{\ell+1}| \geq i$ and the functional isolation of $V_1 \cup ... \cup V_m \cup V_n \cup ... \cup V_{\ell+1}$ is accomplished by the removal of $b_m + r_{n-1} + \sum_{i=n}^{\ell+1} s_i$ edges, it follows that $d_i^{\ell+1} \leq b_m + r_{n-1} + \sum_{i=n}^{\ell+1} s_i$. However, $d_i^{\ell+1} = f'' + r_{n-1} + \sum_{i=n}^{\ell+1} s_i$; so $f'' < b_m$ and consequently $f'' = b_m$. It follows that $S'' = V_1 \cup ... \cup V_m$ and $m = t$, as required. □

We are now able to complete the proof of the theorem:

If $r_{n-1} = i_{n-1}i_n$, then $d_i^{\ell+1} = b_t + i_{n-1}i_n + \sum_{i=n}^{\ell+1} s_i = b_t + b_{\ell+1} - b_{n-1} + 2i_{n-1}i_n$; so $b_r > d_i^{\ell+1} = b_t + b_{\ell+1} - b_{n-1} + 2i_{n-1}i_n$, which contradicts condition (5).

Hence $r_{n-1} < i_{n-1}i_n$ and (by the definition of $G_{\ell+1}$) $s_{n-1} = 0$. We note that $1 \leq b_{n-1} - b_{n-2} = r_{n-1} + s_{n-1} - r_{n-2} = r_{n-1} - r_{n-2}$ and so $r_{n-1} > r_{n-2}$. It now follows that the functional isolation of $S \cup V_{n-1} = V_1 \cup \ldots \cup V_t \cup V_{n-1} \cup V_n \cup \ldots \cup V_{\ell+1}$ in $G_{\ell+1}$ may be accomplished by the removal of $b_t + r_{n-2} + \sum_{i=n}^{\ell+1} s_i < b_t + r_{n-1} + \sum_{i=n}^{\ell+1} s_i = b_i^{\ell+1}$ edges, which is impossible, as $|S \cup V_{n-1}| > |S| = i$ and $b_{i+1}^{\ell+1} > b_i^{\ell+1}$. This contradiction completes the proof of the theorem. □

We conclude this chapter with the conjecture:

Every supply graph is a ranked supply graph.

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