Centre Manifold Theory With an Application in Population Modelling

by

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Abstract

There are basically two types of variables in population modelling, global and local variables. The former describes the behavior of the entire population while the latter describes the behavior of individuals within this population. The description of the population using local variables is more detailed, but it is also computationally costly. In many cases to study the dynamics of this population, it is sufficient to focus only on global variables. In applied sciences, to achieve this, the method of aggregation of variables is used. One of methods used to mathematically justify variables aggregation is the centre manifold theory. In this dissertation we provide detailed proofs of basic results of the centre manifold theory and discuss some examples of applications in population modelling.

Preface and Declarations

The work described in this dissertation was carried out in the school of Mathematical Sciences, University of KwaZulu-Natal, Durban, from July 2007 to February 2009, under the supervision of Professor Jacek Banasiak.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Eddy KIMBA PHONGI

February 2009

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Publication 1

Centre Manifold Theory with Examples of Applications

Publication 2

Publication 3

etc.

Signed:

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Introduction

When we think of studying the stability of a dynamical system, we first look at certain points within the system where the vector field vanishes. In this work, we use terms *fixed points* or *equilibrium points* interchangeably to denote them.

In order to understand the behavior of solutions around a fixed point of a nonlinear system, we study an associated linear system called its linearization at the fixed point.

If the fixed point is hyperbolic *i.e* if none of the eigenvalues of the linearization has zero real part then we can easily deduce, based on the study of the linearization, the stability properties of the nonlinear system. On the other hand, if the fixed point is nonhyperbolic then the linearization does not provide any conclusive information.

In the latter case, there is a powerful mathematical technique that allows for a substantial progress. It is the *centre manifold theory*, whose main goal is to simplify the dynamical system by reducing its dimension. Moreover, since the interesting dynamics take place on the centre manifold, one can uniquely focus investigations on the centre manifold instead of studying the whole space.

In our work, based on Carr's monograph [1], we study and provide detailed proofs of basic results on the centre manifold theory in finite dimension. The main goal is to fill gaps and correct some errors in Carr's presentation. In further work we will extend the study to the theory of infinite dimensional invariant manifolds including stable, unstable and centre manifold. Then apply them to aggregation problems in singularly perturbed multi-structured population models.

This dissertation is divided into three chapters. In the first chapter we give definitions of manifolds, differentiable manifolds and invariant manifolds. We state without proofs the stable and centre manifolds theorems.

In the second chapter we present detailed proofs of the main results of the centre manifold theory namely: *the existence of the centre manifold, the reduction principle and the approximation of the centre manifold*. First a detailed proof of the existence of the centre manifold is given using the *contraction mapping principle*. Next we show in the reduction principle that knowing dynamics of the flow on the centre manifold, one can deduce the dynamics of the full system. Last we present a result that allows us to approximate the centre manifold to any degree of accuracy by a function of class C^2. We conclude the chapter with several examples which illustrate how the centre manifold theory may be used in order to study the behavior of nonlinear systems when linearization fails to provide enough information. We also give some interesting properties of the centre manifold theory.

In the third chapter we give an application of the centre manifold theory which focuses on population modelling. The model that we present consists of prey and predators living in two different patches. Prey can move between both patches but predators remain in their patch. We aggregate variables so that we can concentrate on global variables which contain information of the entire population instead of studying the behavior into individual patches. We perform some transformations to bring the model in an appropriate form so that we can use the centre manifold theory. We end with a system reduced to the centre manifold which consists only of global variables.

We note that we don't study the dynamics of the system reduced to the centre manifold since it requires elaborate techniques which are beyond the scope of this dissertation.

1. Preliminaries

The goal of this chapter is to introduce the notion of manifold, differentiable manifold and invariant manifold which will be used in the sequel of our work. We are not going to develop the theory of manifolds which can by itself be considered as the subject of a degree. Rather, we are going to define what we need from this huge theory.

1.1 Manifold

Definition 1.1.1

Let X be a non-empty set. A *metric* on X is a mapping d of $X \times X$ into $\mathbb{R}$ that satisfies the following conditions:

1. $d(x, y) \geq 0$ for all $x, y \in X$,
2. $d(x, y) = 0$ if and only if $x = y$,
3. $d(x, y) = d(y, x)$ for all $x, y \in X$,
4. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

A *metric space* is a pair (X, d) in which X is a non-empty set and d is a metric on X. A metric is also called a *distance function*.

Condition (3) expresses the fact that d is symmetric in x and y. Inequality (4) is usually called the *triangle inequality* and conditions (1-4) will sometimes be referred to as the *metric space axioms* [2].

Example

Let

$$d(x, y) = \left(\sum_{k=1}^{n} (x_k - y_k)^2 \right)^{1/2}$$

for all $x = (x_1, x_2, ..., x_n)$ and $y = (y_1, y_2, ..., y_n)$ in $\mathbb{R}^n$. Then d is a *metric* on $\mathbb{R}^n$ and is called *Euclidean metric*.

The metric space $(\mathbb{R}^n, d)$ is called *n-dimensional Euclidean Space* [2].

Definition 1.1.2

Let $n \in \mathbb{N}$. An n-manifold or a manifold of dimension n is a metric space M such that for every $x \in M$ there is a neighborhood U of x homeomorphic to an open subset of $\mathbb{R}^n$ [3].

Examples

The set $\mathbb{R}^n$ itself is the simplest example of a manifold. For each $x \in \mathbb{R}^n$ we can take U to be all of $\mathbb{R}^n$. We may define an homeomorphism between $\mathbb{R}^n$ supplied with the usual metric and $\mathbb{R}^n$ with a metric equivalent to the usual metric [3].

The second simplest example of a manifold is an open ball in $\mathbb{R}^n$. In this case we can take U to be the entire open ball since an open ball in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$ [3].

The next example derives directly from the second. Any open subset V of $\mathbb{R}^n$ is a manifold. For each $x \in V$ we can choose U to be some open ball with $x \in U \subset V$ [3].

Remark 1.1.1

From these few manifolds we can already construct many others by noting that if M_i are manifolds of dimension n_i $(i = 1, 2)$ then $M_1 \times M_2$ is an $(n_1 \times n_2)$-manifolds [3].

Example

The product $S^1 \times \cdots \times S^1$ of n 1-spheres is called an n-torus while $S^1 \times S^1$ is called the *torus*. It is homeomorphic to a subset of $\mathbb{R}^3$ which is obtained by revolving the circle

$$\left\{(0, y, z) \in \mathbb{R}^3 : (y-1)^2 + z^2 = 1/4\right\}$$

around the z-axis. This subset is what most of the people have in mind when they speak of a *torus* [3].

1.2 Differentiable Manifold

Definition 1.1.3

An n-dimensional differentiable manifold M (or a manifold of class C^k) is a connected metric space with an open covering $\{U_\alpha\}$ *i.e.*

$$M = \bigcup_\alpha U_\alpha$$

such that

1. for any α in M there exists an open neighborhood U_α homeomorphic to an open unit ball $B = \{x \in \mathbb{R}^n : |x| < 1\}$ in $\mathbb{R}^n$. *i.e.* for any α there exists a homeomorphism $h_\alpha : U_\alpha \to B$ of U_α onto B and

2. if $U_\alpha \cap U_\beta \neq \emptyset$ and $h_\alpha : U_\alpha \to B$, $h_\beta : U_\beta \to B$ are homeomorphisms then $h_\alpha(U_\alpha \cap U_\beta)$ and $h_\beta(U_\alpha \cap U_\beta)$ are subsets of $\mathbb{R}^n$ and the map

$$h_{\alpha\beta} \equiv h_\alpha \circ h_\beta^{-1} : h_\beta(U_\alpha \cap U_\beta) \to h_\alpha(U_\alpha \cap U_\beta)$$

is differentiable (or of class C^k), and for all $x \in h_\beta(U_\alpha \cap U_\beta)$ the Jacobian determinant $\det Dh(x) \neq 0$.

The manifold M is said to be *analytic* if maps $h_{\alpha\beta} \equiv h_\alpha \circ h_\beta^{-1}$ are analytic [4].

Remark 1.1.2

The pair (U_α, h_α) is called a *chart* for the manifold M and the set of all charts is called an *atlas* for M. The differentiable manifold M is called *orientable* if there is an atlas with $\det Dh_\alpha \circ h_\beta^{-1} > 0$ for all α, β and $x \in h_\beta\left(U_\alpha \cap U_\beta\right)$ [4].

1.3 Invariant Manifold

Let

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \quad \text{and} \quad f : \mathbb{R}^n \to \mathbb{R}^n \tag{1.1}$$

be a nonlinear system. Consider the initial value problem

$$\begin{aligned} \dot{x} &= f(x), \\ x(0) &= x_0. \end{aligned} \tag{1.2}$$

Definition 1.1.4

Let E be an open subset of $\mathbb{R}^n$ and let $f \in C^1(E)$. For $x_0 \in E$ let $I(x_0)$ denotes the maximal interval of existence of the solution $\phi(t, x_0)$ of (1.2). Then for $t \in I(x_0)$ the set of mappings $\{\phi_t\}_{t \in I(x_0)}$ of E onto E defined by

$$\phi_t(x_0) = \phi(t, x_0)$$

is called the flow of the differential equation (1.1) [4].

Definiton 1.1.5

Let E be an open subset of $\mathbb{R}^n$. Let $f \in C^1(E)$ and $\phi_t : E \to E$ be the flow of the nonlinear system (1.1) defined for all $t \in \mathbb{R}$.

A set $S \subset E$ is called *invariant* with respect to the flow ϕ_t if $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$. If we restrict the time to be positive (or negative) then we refer to S as positively (or negatively) invariant with respect to the flow ϕ_t [4].

Definition 1.1.6

An invariant set $S \subset \mathbb{R}^n$ is said to be C^k $(k \geq 1)$ invariant manifold of (1.1) if S has the structure of C^k differentiable manifold [5].

We will see now how some important invariant manifolds arise by studying an orbit structure near an equilibrium point.

Definition 1.1.7

A point $x^* \in \mathbb{R}^n$ is called an equilibrium point or a fixed point of (1.1) if $f(x^*) = 0$. An equilibrium point x^* is called a hyperbolic equilibrium point if each eigenvalue of $Df(x^*)$ has nonzero real part.

Note that here and throughout the dissertation Df denotes the Jacobian of f, where f is any differentiable function.

Let x^* be an equilibrium point of the nonlinear system (1.1). The linearization of (1.1) at x^* is given by

$$\dot{x} = Ax, \tag{1.3}$$

where $A \equiv Df(x^*)$ is a constant $n \times n$ matrix and $x \in \mathbb{R}^n$.

A solution of (1.3) through the point $x_0 \in \mathbb{R}^n$ is given by

$$x(t) = e^{At}x_0, \tag{1.4}$$

where

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + ... \tag{1.5}$$

and I is the $n \times n$ identity matrix.

It follows that $\mathbb{R}^n$ can be represented as a direct sum of the following subspaces: E^s, E^u and E^c known as stable, unstable and centre subspaces of the linear system (1.3) respectively, and they are defined as follows:

$$\begin{aligned} E^s &= \text{Span}\left\{v_1, \cdots, v_s\right\}, \\ E^u &= \text{Span}\left\{v_{s+1} \cdots, v_{s+u}\right\}, \\ E^c &= \text{Span}\left\{v_{s+u+1} \cdots, v_{s+u+c}\right\}, \end{aligned} \tag{1.6}$$

with $s + u + c = n$ and where $\{v_1, \cdots, v_s\}$ is a basis of (generalized) eigenvectors of A corresponding to the eigenvalues of A having negative real parts, $\{v_{s+1} \cdots, v_{s+u}\}$ is a basis of (generalized) eigenvectors of A corresponding to the eigenvalues of A having positive real parts and $\{v_{s+u+1} \cdots, v_{s+u+c}\}$ is a basis of (generalized) eigenvectors of A corresponding to the eigenvalues of A having zero real parts [5].

E^s, E^u and E^c are invariant subspaces since a solution of (1.3) with initial condition in one of these subspaces will remain there for all time [5].

Theorem 1.1.1 (The Stable and Unstable Manifold Theorem)

Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$ and ϕ_t the flow of the nonlinear system (1.1). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable stable manifold S tangent to the stable subspace E^s of the linear system (1.3) at 0 such that for all $t \geq 0$ $\phi_t(S) \subset S$ and for all $x_0 \in S$

$$\lim_{t\to\infty} \phi_t(x_0) = 0,$$

and there exists an $(n-k)$-dimensional differentiable unstable manifold U tangent to the unstable subspace E^u of (1.3) at 0 such that for all $t \leq 0$ $\phi_t(U) \subset U$ and for all $x_0 \in U$ [4]

$$\lim_{t\to-\infty} \phi_t(x_0) = 0.$$

Furthermore, S and U are of the same dimension as E^s and E^u respectively [4].

Example

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1,\\ \dot{x}_2 &= -x_2 + x_1^2,\\ \dot{x}_3 &= x_3 + x_1^2.\end{aligned} \tag{1.7}$$

We determine the stable and unstable manifolds, we verify the invariance property and we check the asymptotic behavior of solutions starting in those manifolds.

The only equilibrium point of the system (1.7) is at the origin. The linear system associated to (1.7) is given by

$$\dot{x} = Ax, \tag{1.8}$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, the stable and unstable subspaces E^s and E^u of (1.8) are the x_1x_2-plane and the x_3-axis respectively.

The solution of the nonlinear system (1.7) is given by

$$\begin{aligned} x_1(t) &= c_1 e^{-t},\\ x_2(t) &= c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}),\\ x_3(t) &= c_3 e^{t} + \frac{c_1^2}{3}(e^{t} - e^{-2t}),\end{aligned} \tag{1.9}$$

where $c = (c_1, c_2, c_3) = x(0)$. Therefore the flow of (1.7) is given by

$$\phi_t(c) = \phi(t,c) = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ c_3 e^t + \frac{c_1^2}{3}(e^t - e^{-2t}) \end{bmatrix}.$$

It follows that

$$\lim_{t\to\infty} \phi_t(c) = 0$$

if and only if $c_3 + \frac{c_1^2}{3} = 0$. Thus, the stable manifold of (1.7) is given by

$$S = \{c \in \mathbb{R}^3 | c_3 = \frac{-c_1^2}{3}\},$$

and for $c \in S$ we have

$$\phi_t(c) = \begin{bmatrix} c_1 e^{-t} \\ c_2 e^{-t} + c_1^2(e^{-t} - e^{-2t}) \\ -\frac{c_1^2}{3} e^{-2t} \end{bmatrix} \in S.$$

Hence, $\phi_t(S) \subset S$ for all $t \in \mathbb{R}$ so S is invariant under the flow ϕ_t.

Next

$$\lim_{t\to-\infty} \phi_t(c) = 0$$

if and only if $c_1 = c_2 = 0$. So, the unstable manifold of (1.7) is given by

$$U = \{c \in \mathbb{R}^3 | c_1 = c_2 = 0\}.$$

Then for $c \in U$ we have

$$\phi_t(c) = \begin{bmatrix} 0 \\ 0 \\ c_3 e^t + \frac{c_1^2}{3} e^t \end{bmatrix} \in U.$$

Hence, $\phi_t(U) \subset U$ for all $t \in \mathbb{R}$ so U is invariant under the flow ϕ_t [4]. □

The aim of studying the linear system (1.3) was to get information about the behavior of solutions around the equilibrium point $x = x^*$ of the nonlinear system (1.1). The stable manifold theorem will provide an answer to this question [5].

We need first to do some transformation of the fixed point $x = x^*$ to the origin by the translation $y = x - x^*$. Then since x^* is an equilibrium point $f(x^*) = 0$ and the nonlinear system (1.1) becomes

$$\dot{y} = f(x^* + y), \quad y \in \mathbb{R}^n. \tag{1.10}$$

The Taylor expansion of (1.10) around $x = x^*$ gives

$$\dot{y} = Df(x^*)y + R(y), \quad y \in \mathbb{R}^n, \tag{1.11}$$

where $R(y) = O(|y|^2)$.

The linear system (1.3) can be transformed into the following block diagonal form

$$\begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \end{bmatrix} = \begin{bmatrix} A_s & O & O \\ O & A_u & O \\ O & O & A_c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1.12}$$

by an appropriate linear transformation T, where $T^{-1}y \equiv (u_1, u_2, u_3) \in \mathbb{R}^s \times \mathbb{R}^u \times \mathbb{R}^c$ with $s + u + c = n$. A_s is an $s \times s$ matrix having eigenvalues with negative real parts, A_u is a $u \times u$ matrix having eigenvalues with positive real parts and A_c is a $c \times c$ matrix having eigenvalues with zero real parts and O denotes the appropriately sized block consisting all of zero's [5].

If we take $u = (u_1, u_2, u_3)$ then by the same linear transformation (1.10) becomes

$$\dot{u} = Au + G(u), \tag{1.13}$$

where

$$A = \begin{bmatrix} A_s & O & O \\ O & A_u & O \\ O & O & A_c \end{bmatrix} \tag{1.14}$$

and $G(u) = T^{-1}R(Tu)$. The linear system (1.12) has an s-dimensional invariant stable subspace, a u-dimensional invariant unstable subspace and a c-dimensional invariant centre subspace; all intersecting at the origin [5].

Remark 1.1.3

The stable and unstable manifolds of Theorem 1.1.2 are referred to as the local stable and unstable manifolds of (1.1) since they are only defined in a small neighborhood of the origin. The global stable and unstable manifolds are defined by flowing points in S backward in time and those in U forward in time.

If ϕ_t is the flow of the nonlinear system (1.1) then the global stable and unstable manifolds of (1.1) are defined as follows:

$$W^s = \bigcup_{t \leq 0} \phi_t(S)$$

and

$$W^u = \bigcup_{t \geq 0} \phi_t(U).$$

Note that, without loss of generality, we did not consider the unstable direction in our examples because the reduction principle may fail to work if we take that direction into account.

Another important result when studying the flow of the nonlinear system consists of establishing the existence of an invariant centre manifold $W^c(0)$ tangent to the centre subspace E^c of the linear system (1.3) at the origin [4].

Theorem 1.1.2 (The Centre Manifold Theorem)

Let $f \in C^k(E)$, where E is an open subset of $\mathbb{R}^n$ containing the origin and $k \geq 0$. Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real parts, j eigenvalues with positive real parts and $m = n - k - j$ eigenvalues with zero real parts. Then there exists an m-dimensional invariant centre manifold $W^c(0)$ of class C^k tangent to the centre subspace E^c of (1.3) at the origin [4].

The next chapter is devoted to the study of the centre manifold since near the origin all the interesting dynamic takes place on the centre manifold. So, instead of studying the flow through the whole space $\mathbb{R}^n$, one can reduce the study to the centre manifold [6].

2. Centre Manifold

The goal of this chapter is to present in detail, following Carr [1], some basic results on the centre manifold theory such as the *existence* of the centre manifold and the *reduction principle*. The latter result allows studying the flow of a nonlinear system through its restriction to the centre manifold which reduces the dimension of the problem and it is of major importance in applications [6].

Because in general it is impossible to solve the system for the centre manifold, we also present a result allowing the approximation of the centre manifold to any degree of accuracy.

We note that the results we present in this work are valid in finite dimension but some extensions to infinite dimension space are possible.

Let

$$\begin{aligned} \dot{x} &= Ax + f(x,y), \\ \dot{y} &= By + g(x,y), \end{aligned} \tag{2.1}$$

be a nonlinear system of ordinary differential equation, where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. A and B are constant matrices such that all eigenvalues of A have zero real parts and all eigenvalues of B have negative real parts. $f(x,y)$ and $g(x,y)$ are vectors functions of x and y of class C^2 with $f(0,0) = 0, Df(0,0) = 0, g(0,0) = 0, Dg(0,0) = 0$ [1].

Let us consider the case where f and g are identically zero. Then system (2.1) reduces to the following linear system:

$$\begin{aligned} \dot{x} &= Ax, \\ \dot{y} &= By, \end{aligned} \tag{2.2}$$

that we can write in the matrix form as follows:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{2.3}$$

If we analyse the above system in $\mathbb{R}^3$ with $n = 2$ and $m = 1$ then the x_1x_2-plane and the x_3-axis represent the centre manifold and the stable manifold respectively or more precisely the centre subspace and the stable subspace respectively.

In general the linear system (2.2) has two obvious invariant manifolds namely $x = 0$ that we refer as the *stable subspace* E^s and $y = 0$ as the *centre subspace* E^c [1].

Definition 2.1.1

A centre manifold for (2.1) is an invariant differentiable manifold tangent to the centre subspace E^c of $\mathbb{R}^n$ at the origin.

A centre manifold for (2.1) can be represented as follows:

$$W^c(0) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | y = h(x), |x| < \delta, h(0) = 0, Dh(0) = 0\},$$

for δ sufficiently small [5]. Here and throughout the dissertation $|.|$ denotes the euclidean norm.

Remark 2.1.1

Conditions $h(0) = 0$ and $Dh(0) = 0$ are conditions of tangency of the centre manifold $W^c(0)$ to the centre subspace E^c of $\mathbb{R}^n$ at the origin.

Remark 2.1.2

If f and g are identically zero then all solutions of (2.1) tend exponentially fast, as $t \to \infty$, to solutions of

$$\dot{x} = Ax. \tag{2.4}$$

In other words, all solutions will rapidly decay to the centre subspace along the stable subspace. That is, we can determine the asymptotic behavior of solutions of a nonlinear system once we know the equation that determines the behavior of small solutions on the centre manifold [1].

Now if f and g are non-zero then there are similar results for the system (2.1) that we give in the sequel of this chapter.

2.1 Existence of the Centre Manifold

First we give an example to illustrate the method used to prove the existence of the centre manifold.

Example

Let us consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= 0, \\ \dot{y} &= -y + g(x_1, x_2), \end{aligned} \tag{2.5}$$

where g is smooth and $g(0, 0) = 0$, $Dg(0, 0) = 0$.

Take $G(x_1, x_2) = \psi(x_1, x_2)g(x_1, x_2)$, where $\psi : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ function with compact support such that $\psi(x_1, x_2) = 1$ for (x_1, x_2) sufficiently small. We show that the new system of equations

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= 0, \\ \dot{y} &= -y + G(x_1, x_2), \end{aligned} \tag{2.6}$$

has a centre manifold $y = h(x_1, x_2)$ with $(x_1, x_2) \in \mathbb{R}^2$. Since $G(x_1, x_2) = g(x_1, x_2)$ for (x_1, x_2) sufficiently small then $y = h(x_1, x_2)$ with $x_1^2 + x_2^2 < \delta$ for some δ, is a *local centre manifold* for (2.5).

The first two equations in (2.6) have $x_1(t) = z_1 + z_2 t$ and $x_2(t) = z_2$, where $x_i(0) = z_i$ for $i = 1, 2$, as solutions.

Assume the solution of the third equation in (2.6) be given by $y(t) = h(x_1(t), x_2(t))$. Therefore

$$\frac{d}{dt} h(z_1 + z_2 t, z_2) = -h(z_1 + z_2 t, z_2) + G(z_1 + z_2 t, z_2). \tag{2.7}$$

To compute a centre manifold for (2.6) we select a special solution of (2.7). Solutions of (2.7) have the following form:

$$h(z_1 + z_2 t, z_2) = e^{-t} h(z_1, z_2) + \int_0^t e^{s-t} G(z_1 + z_2 s, z_2) ds. \tag{2.8}$$

We see that (2.8) has a term e^{-t}. Since the centre manifold should be tangent to the z_1z_2-plane at the origin and the component that contains e^{-t} tends to the origin along the stable manifold perpendicular to the z_1z_2-plane as $t \to \infty$, we must eliminate it. To do so, we include in (2.8) the condition

$$\lim_{t \to -\infty} h(z_1 + z_2 t, z_2) e^t = 0.$$

Thus, multiplying (2.8) by e^t and solving the limit as $t \to -\infty$, we get

$$\lim_{t \to -\infty} h(z_1 + z_2 t, z_2) e^t = \lim_{t \to -\infty} h(z_1, z_2) - \lim_{t \to -\infty} \int_t^0 e^s G(z_1 + z_2 s, z_2) ds,$$

$$0 = h(z_1, z_2) - \int_{-\infty}^0 e^s G(z_1 + z_2 s, z_2) ds.$$

Hence,

$$h(z_1, z_2) = \int_{-\infty}^0 e^s G(z_1 + z_2 s, z_2) ds. \tag{2.9}$$

For h, defined by equation (2.9), to be a centre manifold for (2.8) we must show that it is an invariant manifold. Replacing z_1 by $z_1 + z_2 t$ in equation (2.9) gives

$$h(z_1 + z_2 t, z_2) = \int_{-\infty}^0 e^s G(z_1 + z_2(s + t), z_2) ds. \tag{2.10}$$

We must show that (2.10) satisfies the third equation in (2.6). That is, we must compute its derivative with respect to t. Let $\xi_1 = z_1 + z_2(s + t)$ and $\xi_2 = z_2$ therefore,

$$\begin{aligned}
\frac{d}{dt}h(z_1+z_2t,z_2) &= \int_{-\infty}^{0} e^s[G_{.1}z_2+G_{.2}0]ds \\
&= \int_{-\infty}^{0} e^s[G_{.1}z_2]ds \\
&= \int_{-\infty}^{0} e^s\left[\frac{\partial G}{\partial s}\right]ds \quad \left(\text{since } \frac{\partial G}{\partial s}=\frac{\partial G}{\partial \xi_1}z_2\right) \qquad (2.11)\\
&= [e^sG(z_1+z_2(t+s),z_2)]_{-\infty}^{0} - \int_{-\infty}^{0} e^sG(z_1+z_2(t+s),z_2)ds \\
&= G(z_1+z_2t,z_2) - \int_{-\infty}^{0} e^sG(z_1+z_2(t+s),z_2)ds,
\end{aligned}$$

where $G_{.1}$ denotes the partial derivative of G with respect to ξ_1 and $G_{.2}$ the partial derivative of G with respect to ξ_2. It follows, using (2.10), that

$$\dot{h}(z_1+z_2t,z_2) = -h(z_1+z_2t,z_2)+G(z_1+z_2t,z_2). \qquad (2.12)$$

We have shown that the solution of equation (2.6) through $(z_1,z_2,h(z_1,z_2))$ lies on the curve $y(t)=h(z_1+z_2t,z_2)$. This shows $y=h(z_1,z_2)$ is an invariant manifold for (2.6). Since G is C^∞, $G(0,0)=0$ and $DG(0,0)=0$ it follows that h is C^∞, $h(0,0)=0$ and $Dh(0,0)=0$. So, h is a centre manifold for (2.6). As an example we consider $G(z_1,z_2)=z_1z_2$. Then $G(z_1+z_2t,z_2)=z_2(z_1+z_2t)$. Therefore

$$\begin{aligned}
h(z_1,z_2) &= z_2\int_{-\infty}^{0} e^s(z_1+z_2s)ds \\
&= z_1z_2 - z_2^2.
\end{aligned} \qquad (2.13)$$

Replacing z_1 by z_1+z_2t in equation (2.13) gives

$$\begin{aligned}
h(z_1+z_2t,z_2) &= z_2(z_1+z_2t)-z_2^2 \\
&= z^1z^2+z_2^2t-z_2^2,
\end{aligned} \qquad (2.14)$$

then

$$\begin{aligned}
\frac{d}{dt}h(z_1+z_2t,z_2) &= z_2^2 \\
&= -z_1z_2-z_2^2t+z_2^2+z_1z_2+z_2^2t \\
&= -(z_1z_2+z_2^2t-z_2^2)+z_1z_2+z_2^2t \\
&= -h(z_1+z_2t,z_2)+G(z_1+z_2t,z_2).
\end{aligned} \qquad (2.15)$$

$\square$

After this example we return to the general case.

Definition 2.1.2

Let X be a metric space with metric d. A mapping T of X into itself is said to be a *contraction mapping* if and only if there is a real α with $0 < \alpha < 1$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$. To simplify notation, we will write Tx for $T(x)$ [2].

Theorem 2.1.1 (Contraction Mapping Principle)

Let T be a contraction mapping of a complete metric space X into itself. Then there is a unique point $u \in X$ such that $Tu = u$.

Lemma 2.1.1

Let us define a set X of Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}^m$. With the Lipschitz constant p_1, $|h(x)| \leq p$ for $x \in \mathbb{R}^n$ and $h(0) = 0$ for given $p, p_1 > 0$. With the metric induced by the supremum norm $\|.\|$ from the space $C(\mathbb{R}^n, \mathbb{R}^m)$ of continuous functions of $\mathbb{R}^n$ into $\mathbb{R}^m$, X is a complete metric space.

Proof

Indeed, X is a subspace of $C(\mathbb{R}^n, \mathbb{R}^m)$ and we show that X is closed in $C(\mathbb{R}^n, \mathbb{R}^m)$ so that its completeness follows. Let $h \in \bar{X}$, the closure of X. Then there are $(h_n) \in X$ such that $h_n \to h$ as $n \to \infty$. Hence, given an $\epsilon > 0$. Then there is an integer N such that for $n \geq N$ we have

$$\|h_n - h\| = \sup\{|h_n(x) - h(x)| : x \in \mathbb{R}^n\} < \epsilon.$$

Hence, for any fixed $x_0 \in \mathbb{R}^n$ $|h_n(x_0) - h(x_0)| < \epsilon$ for $n \geq N$. This shows that $h_n(x_0) \to h(x_0)$ as $n \to \infty$.

We have to show that $h \in X$. Since $h_n \in X$ for $n = 1, 2, ...$ we have $|h_n(x) - h_n(y)| \leq p_1|x - y|$ for $x, y \in \mathbb{R}^n$ and $|h_n(x)| \leq p$ with $p_1, p > 0$. Letting $n \to \infty$ yields $|h(x) - h(y)| \leq p_1|x - y|$ for $x, y \in \mathbb{R}^n$ and $|h(x)| \leq p$. This shows that $h \in X$. Since $h \in \bar{X}$ was arbitrary, this proves the closedness of X in $C(\mathbb{R}^n, \mathbb{R}^m)$ and so its completeness as a metric space with metric induced from $C(\mathbb{R}^n, \mathbb{R}^m)$. $\square$

Next we formulate and prove the main result of this section.

Theorem 2.1.2

There exists a C^2 centre manifold $y = h(x)$, $|x| < \delta$ for (2.1).

Proof

The proof of the existence of the centre manifold uses the *contraction mapping principle*.

As in the previous example, we define from (2.1) a new system

$$\begin{aligned} \dot{x} &= Ax + F(x, y), \\ \dot{y} &= By + G(x, y), \end{aligned} \tag{2.16}$$

where $F(x,y) = f\left(x\psi\left(\frac{x}{\epsilon}\right), y\right)$, $G(x,y) = g\left(x\psi\left(\frac{x}{\epsilon}\right), y\right)$ and $\psi : \mathbb{R}^n \to [0,1]$ is a C^∞ function with $\psi\left(\frac{x}{\epsilon}\right) = 0$ when $|x| \geq 2\epsilon$ and $\psi\left(\frac{x}{\epsilon}\right) \leq 1$ when $|x| \leq 2\epsilon$.

We prove that (2.16) has a centre manifold $y = h(x)$, $x \in \mathbb{R}^n$, for ϵ small enough. Using the fact that $F = f$ and $G = g$ in a neighborhood of the origin, we show that there exists a local centre manifold for (2.1) [1].

We assume that we can solve the first equation in (2.16) and substitute its solution into the second equation to solve it to get the centre manifold. That is, for $h \in X$ and $x_0 \in \mathbb{R}^n$ let $x(t, x_0; h)$ be the solution of

$$\dot{x} = Ax + F(x, h(x)), \quad \text{with} \quad x(0, x_0; h) = x_0. \tag{2.17}$$

The solution of (2.17) is given by

$$x(t, x_0; h) = e^{At}x_0 + e^{At}\int_0^t e^{-As}F(x(s, x_0, h), h(x(s, x_0; h)))ds, \tag{2.18}$$

where $y(t) = h(x(t, x_0, h))$ is the solution of the second equation in (2.17). Therefore

$$\frac{d}{dt}h(x(t, x_0; h)) = Bh(x(t, x_0; h)) + G(x(t, x_0; h), h(x(t, x_0; h))). \tag{2.19}$$

As in the example, to compute a centre manifold for (2.16) we must single out a special solution of (2.19). Solutions of (2.19) have the following form:

$$h(x(t, x_0; h)) = e^{Bt}h(x_0) + e^{Bt}\int_0^t e^{-Bs}G(x(s, x_0, h), h(x(s, x_0; h)))ds. \tag{2.20}$$

We must eliminate the component which decays to the origin along the stable manifold since the centre manifold should be tangent to the invariant manifold $y = 0$. That is, we must include the condition $\lim_{t\to-\infty} e^{-Bt}h(x(t, x_0; h)) = 0$ in (2.20). Thus, multiplying (2.20) by e^{-Bt} and using the limit as $t \to -\infty$ gives

$$0 = \lim_{t\to-\infty} e^{-Bt}h(x(t, x_0; h)) = \lim_{t\to-\infty} h(x_0) - \lim_{t\to-\infty}\int_t^0 e^{-Bs}G(x(s, x_0; h), h(x(s, x_0; h)))ds.$$

Hence,

$$0 = h(x_0) - \int_{-\infty}^0 e^{-Bs}G(x(s, x_0; h), h(x(s, x_0; h)))ds$$

so that

$$h(x_0) = \int_{-\infty}^0 e^{-Bs}G(x(s, x_0; h), h(x(s, x_0; h)))ds. \tag{2.21}$$

We use the contraction mapping principle to show that under suitable conditions (2.21) has a unique solution in certain subset U of $\mathbb{R}^n$ containing x_0. That is, let define $T : X \to X$ by

$$(Th)(x_0) = \int_{-\infty}^{0} e^{-Bs} G(x(s, x_0, h), h(x(s, x_0; h)))ds, \tag{2.22}$$

where $h \in X$ and $x_0 \in \mathbb{R}^n$.

We have to prove that T is a contraction on X. Therefore if h is a fixed point of (2.22) then by (2.21), it is the centre manifold for (2.16). To do so, we need the following auxiliary estimates:

Lemma 2.1.2

There exists a continuous function $k(\epsilon)$ with $k(0) = 0$ such that

$$\begin{aligned} &|F(x, y)| + |G(x, y)| \leq \epsilon k(\epsilon), \\ &|F(x, y) - F(x^{'}, y^{'})| \leq k(\epsilon)[|x - x^{'}| + |y - y^{'}|], \\ &|G(x, y) - G(x^{'}, y^{'})| \leq k(\epsilon)[|x - x^{'}| + |y - y^{'}|], \end{aligned} \tag{2.23}$$

for all $x, x^{'} \in \mathbb{R}^n$ and $y, y^{'} \in \mathbb{R}^m$ with $|y|, |y^{'}| < \epsilon$ [1].

Proof

Given a real-valued C^k-function f defined on an open convex subset U of $\mathbb{R}^n$. The multivariate Taylor expansion of f at $a \in U$ is given by

$$f(a + h) = \sum_{r=0}^{k} \frac{D_h^r f(a)}{r!} + R_k(h), \tag{2.24}$$

where

$$R_k(h) = \frac{D_h^{k+1} f(\xi)}{(k+1)!}$$

is the k^{th} degree remainder of f at a and ξ is on the segment L joining a with $a + h$. Note that

$$\lim_{h \to 0} \frac{R_k(h)}{|h|^k} = 0$$

and

$$\begin{aligned} D_h^k f &= (h_1 D_1 + \cdots + h_n D_n)^k f \\ &= \sum_{j_1 + \cdots + j_n = k} \binom{k,}{j_1 \cdots j_n} h_1^{j_1} \cdots h_n^{j_n} D_1^{j_1} \cdots D_n^{j_n} f, \end{aligned} \tag{2.25}$$

is the iterated directional derivative with respect to $h = (h_1 \cdots h_n)$. For instance, if $n = 2$ and $k = 2$, we have

$$\begin{aligned} D_h^2 f &= (h_1 D_1 + h_2 D_2)^2 f \\ &= \left(h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} \right)^2 f \\ &= h_1^2 \frac{\partial^2 f}{\partial x^2} + 2h_1 h_2 \frac{\partial^2 f}{\partial x \partial y} + h_2^2 \frac{\partial^2 f}{\partial y^2}. \end{aligned} \tag{2.26}$$

Using (2.24) with $h = (z, y)$ and $a = (0, 0)$, the Taylor formula for f up to the second order at $(0, 0)$ is given by

$$f((0,0) + (z,y)) = f(0,0) + D_h f(0,0) + R_1(h),$$

where $R_1(h) = \dfrac{1}{2!} \left[D_h^2 f\right] (\theta_z, \theta_y)$; with $0 < |\theta_z| < |z|$ and $0 < |\theta_y| < |y|$.

Since $f(0,0) = 0$ and $Df(0,0) = 0$ we have $f(z,y) = R_1(h)$ with $\lim_{h \to 0} \dfrac{R_1(h)}{|h|} = 0$.

Define

$$\tilde{k}(\epsilon) = \sup_{|h| \leq \epsilon} \frac{|R_1(h)|}{|h|}.$$

Thus, $\tilde{k}(\epsilon) \to 0$ as $\epsilon \to 0$ for $|h| \leq \epsilon$.

Therefore

$$\frac{|R_1(h)|}{|h|} \leq \tilde{k}(\epsilon) \quad \Leftrightarrow \quad |R_1(h)| \leq \tilde{k}(\epsilon)|h|.$$

Since

$$\psi(x) \leq 1 \quad \text{if} \quad |x| < 2 \quad \text{and} \quad \psi(x) = 0 \quad \text{if} \quad |x| \geq 2,$$

we have

$$\left| x\psi\left(\frac{x}{\epsilon}\right) \right|^2 \leq (2\epsilon)^2 \quad \text{if} \quad |x| < 2\epsilon \quad \text{and} \quad \left| x\psi\left(\frac{x}{\epsilon}\right) \right|^2 = 0 \quad \text{if} \quad |x| \geq 2\epsilon.$$

Which implies that

$$\left| x\psi\left(\frac{x}{\epsilon}\right) \right|^2 \leq 4\epsilon^2 \quad \text{for all} \quad x$$

and

$$\begin{aligned} |h| &= \sqrt{|z|^2 + |y|^2}, \\ &= \sqrt{\left| x\psi\left(\frac{x}{\epsilon}\right) \right|^2 + |y|^2}, \\ &\leq \sqrt{4\epsilon^2 + \epsilon^2}, \\ &= \sqrt{5}\epsilon. \end{aligned}$$

It follows that

$$|F(x,y)| \leq \frac{\epsilon}{2} k(\epsilon),$$

where $k(\epsilon) = 2\sqrt{5}\tilde{k}(\epsilon)$.

Using the same reasoning, we find the estimate of $G(x,y)$ by $|G(x,y)| \leq \dfrac{\epsilon}{2}k(\epsilon)$.

Hence,

$$|F(x,y)| + |G(x,y)| \leq \epsilon k(\epsilon).$$

Next we define

$$u(t) = F(tx + (1-t)x^{'}, ty + (1-t)y^{'}). \tag{2.27}$$

Then $u(0) = F(x^{'}, y^{'})$, $u(1) = F(x,y)$ and the *Taylor's formula* of u up to the first order in one variable on the interval $[0,1]$ gives

$$u(1) = u(0) + u^{(1)}(c), \tag{2.28}$$

for some $0 < c < 1$.

Therefore we have from (2.28)

$$F(x,y) - F(x^{'}, y^{'}) = F_{.1}(\theta_x, \theta_y)(x - x^{'}) + F_{.2}(\theta_x, \theta_y)(y - y^{'}), \tag{2.29}$$

where $|x^{'}| < |\theta_x| < |x|$, $|y^{'}| < |\theta_y| < |y|$,

$$F_{.1}(\theta_x, \theta_y) = \left[\left(\frac{\partial F_i}{\partial x_j}\right)_{|(\theta_x,\theta_y)}\right]_{1\leq i,j\leq n} \quad \text{and} \quad F_{.2}(\theta_x, \theta_y) = \left[\left(\frac{\partial F_i}{\partial y_k}\right)_{|(\theta_x,\theta_y)}\right]_{1\leq i\leq n,\ 1\leq k\leq m}.$$

Then

$$|F(x,y) - F(x^{'}, y^{'})| \leq |F_{.1}(\theta_x, \theta_y)||(x - x^{'})| + |F_{.2}(\theta_x, \theta_y)||(y - y^{'})|. \tag{2.30}$$

Now, if $|x|, |x^{'}| < 2\epsilon$ and $|y|, |y^{'}| < \epsilon$ then $|\theta_x| < 2\epsilon$ and $|\theta_y| < \epsilon$. Therefore, derivatives can be estimated by their supremum with respect to (x,y) over balls $|x| < 2\epsilon, |y| < \epsilon$.

Since $F(x,y) = f\left(x\psi\left(\dfrac{x}{\epsilon}\right), y\right)$ for $|x^{'}| < |\theta_x| < |x|$, $|y^{'}| < |\theta_y| < |y|$ and $z = x\psi\left(\dfrac{x}{\epsilon}\right)$ we have

$$F_{.1}(\theta_x, \theta_y) = \left[\left(\frac{\partial f_i}{\partial z_j}\right)_{|(\theta_x\psi(\frac{\theta_x}{\epsilon}),\theta_y)} \cdot \left[\psi\left(\frac{\theta_x}{\epsilon}\right)\underline{e}_j + \frac{\theta_x}{\epsilon}\left(\frac{\partial \psi}{\partial x_j}\right)_{|(\frac{\theta_x}{\epsilon})}\right]\right]_{1\leq i,j\leq n}$$

and

$$F_{.2}(\theta_x,\theta_y)=\left[\left(\frac{\partial f_i}{\partial y_k}\right)_{|(\theta_x\psi(\frac{\theta_x}{\epsilon}),\theta_y)}\right]_{1\le i\le n,\ 1\le k\le m}.$$

Define

$$\tilde{k}(\epsilon)=\sup_{|x|<2\epsilon,|y|<\epsilon}\left\{\left|\left(\frac{\partial f_i}{\partial z_j}\right)_{|(\theta_x\psi(\frac{\theta_x}{\epsilon}),\theta_y)}\right|,\left|\left(\frac{\partial f_i}{\partial y_k}\right)_{|(\theta_x\psi(\frac{\theta_x}{\epsilon}),\theta_y)}\right| : 1\le i,j\le n \quad \text{and} \quad 1\le k\le m\right\},$$

so that $\tilde{k}(\epsilon)\to 0$ as $\epsilon\to 0$.

Let us consider the following different cases:

Case I: $|x^{'}|,|x|<\epsilon \quad \Rightarrow \quad |\theta_x|<\epsilon, \quad \psi\left(\frac{\theta_x}{\epsilon}\right)=1$ and $\left(\frac{\partial\psi}{\partial x_j}\right)_{|(\frac{\theta_x}{\epsilon})}=0$ for $1\le j\le n$.

Therefore

$$F_{.1}(\theta_x,\theta_y)=f_{.1}(\theta_x,\theta_y) \quad \Rightarrow \quad |F_{.1}(\theta_x,\theta_y)|\le\tilde{k}(\epsilon)$$

and

$$F_{.2}(\theta_x,\theta_y)=f_{.2}(\theta_x,\theta_y) \quad \Rightarrow \quad |F_{.2}(\theta_x,\theta_y)|\le\tilde{k}(\epsilon),$$

where

$$f_{.1}(\theta_x,\theta_y)=\left[\left(\frac{\partial f_i}{\partial z_j}\right)_{|(\theta_x,\theta_y)}\right]_{1\le i,j\le n} \quad \text{and} \quad f_{.2}(\theta_x,\theta_y)=\left[\left(\frac{\partial f_i}{\partial y_k}\right)_{|(\theta_x,\theta_y)}\right]_{1\le i\le n,\ 1\le k\le m}.$$

Case II: $\epsilon\le|x^{'}|,|x|<2\epsilon \quad \Rightarrow \quad |\theta_x|<2\epsilon$ implies

$$\left|\theta_x\psi\left(\frac{\theta_x}{\epsilon}\right)\right|<2\epsilon, \quad \left|\left(\frac{\partial\psi}{\partial x_j}\right)_{|(\frac{\theta_x}{\epsilon})}\right|\le K \quad \text{and} \quad \left|\psi\left(\frac{\theta_x}{\epsilon}\right)\underline{e}_j+\frac{\theta_x}{\epsilon}\left(\frac{\partial\psi}{\partial x_j}\right)_{|(\frac{\theta_x}{\epsilon})}\right|\le(1+2K)$$

for $1\le j\le n$ therefore

$$|F_{.1}(\theta_x,\theta_y)|\le(1+2K)\tilde{k}(\epsilon)$$

and

$$|F_{.2}(\theta_x,\theta_y)|\le\tilde{k}(\epsilon).$$

Case III: $|x^{'}|<2\epsilon$ and $|x|\ge 2\epsilon$.

In which case we may have two different situations

(1)

$$|\theta_x|\ge 2\epsilon \quad \Rightarrow \quad \psi\left(\frac{\theta_x}{\epsilon}\right)=0 \quad \text{and} \quad \left(\frac{\partial\psi}{\partial x_j}\right)_{|(\frac{\theta_x}{\epsilon})}=0$$

for $1\le j\le n$. Therefore

$$F_{.1}(\theta_x,\theta_y)=0 \quad \Rightarrow \quad |F_{.1}(\theta_x,\theta_y)|\le C\tilde{k}(\epsilon)$$

and

$$F_{.2}(\theta_x, \theta_y) = f_{.2}(0, \theta_y) \quad \Rightarrow \quad |F_{.2}(\theta_x, \theta_y)| \leq \tilde{k}(\epsilon).$$

(2)We have either $|\theta_x| < \epsilon$ which is similar to case I or $\epsilon \leq |\theta_x| < 2\epsilon$ which is similar to case II above.

Case IV: $|x^{'}|, |x| \geq 2\epsilon$.

In this case, we may also have two different situations:

(1)The segment joining $x^{'}$ to x does not cross the ball. This one is similar to case III(1)

(2)The segment joining $x^{'}$ to x pass through the ball. Then we have either $|\theta_x| < \epsilon$ we are in case I or $\epsilon \leq |\theta_x| < 2\epsilon$ we are in case II or $|\theta_x| \geq 2\epsilon$ we are case III(1).

Note that $|y^{'}|, |y| < \epsilon$ always so is $|\theta_y| < \epsilon$, and in any case $F_{.1}(\theta_x, \theta_y), F_{.2}(\theta_x, \theta_y)$ may be estimated by their suprema with respect to (x, y).

Taking $k(\epsilon) = \text{Constant}\tilde{k}(\epsilon)$ which approaches 0 as $\epsilon \to 0$. We have

$$|F(x, y) - F(x^{'}, y^{'})| \leq k(\epsilon)[|x - x^{'}| + |y - y^{'}|].$$

Since $G(x, y) = g\left(x\psi\left(\frac{x}{\epsilon}\right), y\right)$, using the same reasoning as for f, we compute the estimate of $G(x, y) - G(x^{'}, y^{'})$ which is given by

$$|G(x, y) - G(x^{'}, y^{'})| \leq k(\epsilon)[|x - x^{'}| + |y - y^{'}|].$$

□

Lemma 2.1.3

If eigenvalues of B all have negative real parts then there exist positive constants β, C such that for $s \leq 0$ and $y \in \mathbb{R}^m$ [1]

$$|e^{-Bs}y| \leq Ce^{\beta s}|y|. \tag{2.31}$$

If eigenvalues of A all have zero real parts then for each $r > 0$ there is a constant $M(r)$ with in general $M(r) \to \infty$ as $r \to 0$, such that for $x \in \mathbb{R}^n$ and $s \in \mathbb{R}$ [1]

$$|e^{As}x| \leq M(r)e^{r|s|}|x|. \tag{2.32}$$

Proof

Given an $m \times m$ Jordan canonical form of a real matrix M having real eigenvalues $\lambda_j,\ j = 1, \cdots, k$, and complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j,\ j = k+1, \cdots, m$.

$$B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_r \end{bmatrix},$$

where the elementary Jordan blocks $B_j,\ j = 1, \cdots, r$ are either of the form

$$B_j = \begin{bmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \cdots & & & & \\ 0 & \cdots & & \lambda_j & 1 \\ 0 & \cdots & & 0 & \lambda_j \end{bmatrix},$$

for λ_j one of the real eigenvalues of M or of the form

$$B_j = \begin{bmatrix} a_j & -b_j & 1 & 0 & 0 & & 0 & \cdots & 0 \\ b_j & a_j & 0 & 1 & 0 & & 0 & \cdots & 0 \\ 0 & 0 & a_j & -b_j & 1 & 0 & & \cdots & 0 \\ 0 & 0 & b_j & a_j & 0 & 1 & & \cdots & 0 \\ \cdots & & & & & & & & \\ 0 & \cdots & & & & a_j & -b_j & 1 & 0 \\ 0 & \cdots & & & & b_j & a_j & 0 & 1 \\ 0 & \cdots & & & & & & a_j & -b_j \\ 0 & \cdots & & & & & & b_j & a_j \end{bmatrix},$$

for $\lambda_j = a_j + ib_j$ one of the complex eigenvalues of M.

Using the fact that Jordan's canonical form splits space into invariant subspaces, one may work with each block separately and the final result is valid for the full matrix. Let us consider a real eigenvalue λ_j of M. Then

$$e^{-B_j s} y = \begin{bmatrix} y_1 e^{-\lambda_j s} + s y_2 e^{-\lambda_j s} + \frac{s^2}{2} y_3 e^{-\lambda_j s} \cdots + \dfrac{s^{k-1}}{(k-1)!} y_k e^{-\lambda_j s} \\ y_2 e^{-\lambda_j s} + s y_3 e^{-\lambda_j s} + \cdots + \dfrac{s^{k-2}}{(k-2)!} y_k e^{-\lambda_j s} \\ y_3 e^{-\lambda_j s} + \cdots + \dfrac{s^{k-3}}{(k-3)!} y_k e^{-\lambda_j s} \\ \cdots \\ \cdots \\ y_{k-1} e^{-\lambda_j s} + s y_k e^{-\lambda_j s} \\ y_k e^{-\lambda_j s} \end{bmatrix}.$$

Hence,

$$
\begin{aligned}
|e^{-B_j s}y|^2 &= e^{-2\lambda_j s}\left[\left(y_1 + sy_2 + \cdots + \frac{s^{k-1}}{(k-1)!}y_k\right)^2 + \left(y_2 + sy_3 + \cdots + \frac{s^{k-2}}{(k-2)!}y_k\right)^2 + \cdots + y_k^2\right] \\
&= e^{2\lambda_j s}(y_1^2 + y_2^2 \cdots + \frac{s^{2(k-1)}}{[(k-1)!]^2}y_k^2 + \cdots + 2y_1y_k\frac{s^{k-1}}{(k-1)!} + \cdots + 2y_2y_k\frac{s^{k-2}}{(k-2)!} + \cdots + y_k^2) \\
&\le e^{-2\lambda_j s}|y|^2 P_{2(k-1)}(s) \\
&= e^{-2\lambda_j s}e^{2\beta_j s}e^{-2\beta_j s}|y|^2 P_{2(k-1)}(s) \\
&= e^{2\beta_j s}|y|^2 e^{-2(\beta_j+\lambda_j)s}P_{2(k-1)}(s),
\end{aligned}
$$

where $P_{2(k-1)}(s)$ is a polynomial of degree $2(k-1)$ in s and $\lambda_j < -\beta_j < 0$ for $j = 1, \cdots, k$. Since $-\beta_j - \lambda_j > 0$ and $s < 0$ the expression $e^{-2(\beta_j+\lambda_j)s}P_{2(k-1)}(s)$ is bounded. It follows that

$$
|e^{-B_j s}y| \le C_j e^{-\beta_j s}|y|, \ j = 1 \cdots, k,
$$

and for $\lambda_j = a_j + ib_j$ one of the complex eigenvalues of M we have

$$
e^{-B_j s}y = \begin{bmatrix}
e^{-a_j s}(y_{k+1}\sin b_1 s - y_{k+2}\cos b_j s) + \cdots + \frac{s^{m-1}}{(m-1)!}e^{-a_j s}(y_{m-1}\sin b_j s - y_m \cos b_j s) \\
e^{-a_j s}(y_{k+1}\cos b_1 s + y_{k+2}\sin b_j s) + \cdots + \frac{s^{m-1}}{(m-1)!}e^{-a_j s}(y_{m-1}\cos b_j s + y_m \sin b_j s) \\
\cdots \\
\cdots \\
y_{m-1}e^{-a_j s}\sin b_j s - y_m e^{-a_j s}\cos b_j s \\
y_{m-1}e^{-a_j s}\cos b_j s + y_m e^{-a_j s}\sin b_j s
\end{bmatrix}.
$$

Hence,

$$
\begin{aligned}
|e^{-B_j s}y|^2 &= e^{-2a_j s}((y_{k+1}\sin b_j s - y_{k+2}\cos b_j s) + \cdots + \frac{s^{m-1}}{(m-1)!}(y_{m-1}\sin b_j s - y_m \cos b_j s) \\
&\quad + (y_{k+1}\cos b_j s + y_{k+2}\sin b_j s) + \cdots + \frac{s^{m-1}}{(m-1)!}(y_{m-1}\cos b_j s + y_n \sin b_j s) \\
&\quad + \cdots + (y_{m-1}\sin b_j s - y_m \cos b_j s) + (y_{m-1}\cos b_j s + y_m \sin b_j s))^2 \\
&= e^{2a_1 s}(y_{k+1}^2 + y_{k+2}^2 + \cdots + \frac{s^{2(m-1)}}{[(m-1)!]^2}(y_{m-1}^2 + y_m^2) + \cdots + y_{m-1}^2 + y_m^2) \\
&\le e^{-2a_j s}|y|^2 P_{2(n-1)}(s) \\
&= e^{-2a_j s}e^{-2\beta_j s}e^{2\beta_j s}|y|^2 P_{2(m-1)}(s) \\
&= e^{2\beta_j s}e^{-2(\beta_j+a_j)s}P_{2(m-1)}(s)|y|^2,
\end{aligned}
$$

where $a_j < -\beta_j < 0$ and $P_{2(m-1)}(s)$ is a polynomial of degree $2(m-1)$ in s. Since $-\beta_j - a_j > 0,\ j = k+1, \cdots, m$ and $s < 0$ the expression $e^{-2(\beta_j + a_j)s}P_{2(m-1)}(s)$ is bounded. It follows that

$$|e^{-B_j s}y| \leq C_j e^{\beta_j s}|y|,\ j = k+1, \cdots, m.$$

Therefore for the full matrix we have

$$|e^{-Bs}y| \leq Ce^{\beta s}|y| \text{ for } \mathrm{Re}\lambda_j < -\beta < 0,\ j = 1, \cdots, m.$$

Now we estimate $e^{As}x$ where A is an $n \times n$ Jordan canonical form of a real matrix M whose eigenvalues have zero real parts and $x \in \mathbb{R}^n$. We derive the estimate of $e^{As}x$ from the estimate of $e^{-Bs}y$ by setting the real parts of the latter to zero. Therefore for $\lambda_k,\ k = 1, \cdots, q$, one of the eigenvalues of M we have

$$\begin{aligned} |e^{A_k s}x|^2 &\leq |x|^2 P_{2(q-1)}(s) \\ &= e^{2r|s|}e^{-2r|s|}|x|^2 P_{2(q-1)}(s) \\ &= e^{2r|s|}e^{-2r|s|}P_{2(q-1)}(s)|x|^2, \end{aligned}$$

where $r > 0$ and $P_{2(q-1)}(s)$ is a polynomial of degree $2(q-1)$ in s. The expression $e^{-2r|s|}P_{2(q-1)}(s)$ is bounded by a constant $M(r)$ depending on r such that $M(r) \to 0$ as $r \to \infty$ and vice versa. Hence,

$$|e^{A_k s}x| \leq M(r)e^{2r|s|}|x|.$$

For $\lambda_k = a_k + ib_k,\ k = q+1, \cdots, n$ we have

$$\begin{aligned} |e^{A_k s}x|^2 &\leq |x|^2 P_{2(n-1)}(s) \\ &= e^{2r|s|}e^{-2r|s|}|x|^2 P_{2(n-1)}(s) \\ &= e^{2r|s|}e^{-2r|s|}P_{2(n-1)}(s)|x|^2, \end{aligned}$$

where $r > 0$ and $P_{2(n-1)}(s)$ is a polynomial of degree $2(n-1)$ in s. The expression $e^{-2r|s|}P_{2(n-1)}(s)$ is bounded by a constant $M(r)$ depending on r as above. Hence,

$$|e^{A_k s}x| \leq M(r)e^{2r|s|}|x|.$$

It follows, for the full matrix, that we have

$$|e^{As}x| \leq M(r)e^{r|s|}|x|.$$

□

The next lemma is of great importance in the proof of the existence of the centre manifold.

Lemma 2.1.4 (Gronwall's inequality)

Let $f(t)$ and $g(t)$ be continuous nonnegative real valued functions and

$$f(t) \leq C + \int_{t_0}^{t} f(s)g(s)ds \tag{2.33}$$

for all $t \in [t_0, t_0 + \alpha]$, where $\alpha > 0$ and $C > 0$. It then follows that for all $t \in [t_0, t_0 + \alpha]$

$$f(t) \leq C \exp\left(\int_{t_0}^{t} g(s)ds\right). \tag{2.34}$$

Proof

Define $F(t) = C + \int_{t_0}^{t} f(s)g(s)ds$ on $[t_0, t_0 + \alpha]$. Then $F(t) \geq f(t)$ and $F(t) > 0$ on $[t_0, t_0 + \alpha]$. Differentiating F with respect to t we get $F^{'}(t) = f(t)g(t)$. Therefore $\dfrac{F^{'}(t)}{F(t)} = \dfrac{f(t)g(t)}{F(t)} \leq \dfrac{f(t)g(t)}{f(t)}$. It follows that $\dfrac{F^{'}(t)}{F(t)} = \dfrac{d}{dt} \log F(t) \leq g(t)$. Integrating the latter expression, we get

$$\log F(t) - \log F(t_0) \leq \int_{t_0}^{t} g(s)ds,$$

$$F(t) \leq F(t_0) \exp\left(\int_{t_0}^{t} g(s)ds\right).$$

Taking $C = F(t_0)$ we obtain

$$F(t) \leq C \exp\left(\int_{t_0}^{t} g(s)ds\right).$$

But, since we observed $f(t) \leq F(t)$ we have

$$f(t) \leq C \exp\left(\int_{t_0}^{t} g(s)ds\right)$$

for all $t \in [t_0, t]$. □

Now, using above results, we prove that T is a contraction on X. In what follows we assume $p < \epsilon$.

Let $x_0 \in \mathbb{R}^n$. From (2.22), by using estimates on G and h and the inequality (2.31) we have [1]

$$\begin{aligned}
|(Th)(x_0)| &\leq \int_{-\infty}^{0} |e^{-Bs}||G(x(s, x_0; h), h(x(s, x_0; h)))|ds \\
&\leq \int_{-\infty}^{0} Ce^{\beta s}|G(x(s, x_0; h), h(x(s, x_0; h)))|ds \\
&\leq C\epsilon k(\epsilon) \int_{-\infty}^{0} e^{\beta s} ds \\
&= C\beta^{-1}\epsilon k(\epsilon).
\end{aligned}$$

We choose an appropriate ϵ so that $k(\epsilon) \leq C^{-1}\beta$. Therefore

$$|(Th)(x_0)| \leq \epsilon, \tag{2.35}$$

for $x_0 \in \mathbb{R}^n$.

Now, let $x_0, x_1 \in \mathbb{R}^n$. From the inequality (2.32), estimates on F and h, and the equation (2.17), we have for $r > 0$ and $t \leq 0$

$$\begin{aligned}
x(t, x_0; h) &= e^{-At}x_0 + \int_t^0 e^{A(s-t)}F(x(s, x_0; h), h(x(s, x_0; h)))ds, \\
x(t, x_1; h) &= e^{-At}x_1 + \int_t^0 e^{A(s-t)}F(x(s, x_1; h), h(x(s, x_1; h)))ds.
\end{aligned}$$

Then writing X_0 for $x(t, x_0; h)$ and X_1 for $x(t, x_1; h)$ we have

$$\begin{aligned}
|X_0 - X_1| &\leq |e^{-At}(x_0 - x_1)| + \int_t^0 |e^{A(s-t)}[F(X_0, h(X_0)) - F(X_1, h(X_1))]|ds \\
&\leq M(r)e^{-rt}|x_1 - x_0| + M(r)\int_t^0 e^{r(s-t)}k(\epsilon)[|F(X_0, h(X_0)) - F(X_1, h(X_1)))|ds \\
&\leq M(r)e^{-rt}|x_1 - x_0| + k(\epsilon)M(r)\int_t^0 e^{r(s-t)}[|X_0 - X_1| + |h(X_0) - h(X_1)|]ds \\
&\leq M(r)e^{-rt}|x_1 - x_0| + k(\epsilon)M(r)\int_t^0 e^{r(s-t)}[|X_0 - X_1| + p_1|X_0 - X_1|]ds \\
&= M(r)e^{-rt}|x_1 - x_0| + k(\epsilon)M(r)\int_t^0 e^{r(s-t)}(1 + p_1)|X_0 - X_1|ds \\
&= M(r)e^{-rt}|x_1 - x_0| + (1 + p_1)k(\epsilon)M(r)\int_t^0 e^{r(s-t)}|X_0 - X_1|ds.
\end{aligned}$$

Using Gronwall's inequality for $t \leq 0$ we have

$$|X_0 - X_1| \leq M(r)|x_1 - x_0|e^{-[r+(1+p_1)k(\epsilon)M(r)]t}.$$

Taking $\gamma = r + (1 + p_1)k(\epsilon)M(r)$, and rewriting $x(t, x_0; h)$ for X_0 and $x(t, x_1; h)$ for X_1 we obtain

$$|x(t, x_0; h) - x(t, x_1; h)| \leq M(r)|x_1 - x_0|e^{-\gamma t}. \tag{2.36}$$

From (2.36), the bounds on G, h and (2.22), we have

$$\begin{aligned}
|(Th)(x_0)-(Th)(x_1)| &\le \int_{-\infty}^{0} |e^{-Bs}[G(X_0,h(X_0))-G(X_1,h(X_1))]|ds \\
&\le \int_{-\infty}^{0} Ce^{\beta s}k(\epsilon)[|X_0-X_1|+|h(X_0)-h(X_1)|]ds \\
&= Ck(\epsilon)\int_{-\infty}^{0} e^{\beta s}(1+p_1)|X_0-X_1|ds \\
&\le (1+p_1)Ck(\epsilon)\int_{-\infty}^{0} e^{\beta s}M(r)|x_1-x_0|e^{-\gamma t}ds \\
&= Ck(\epsilon)M(r)(1+p_1)(\beta-\gamma)^{-1}|x_1-x_0|,
\end{aligned}$$

where h is Lipschitz with Lipschitz constant p_1. We choose appropriate ϵ and r so that $\beta-\gamma>0$ and $Ck(\epsilon)M(r)(1+p_1)(\beta-\gamma)^{-1}<p_1$. Hence,

$$|(Th)(x_0)-(Th)(x_1)| \le p_1|x_1-x_0|. \tag{2.37}$$

From (2.35) and (2.37) we have shown that $Th:\mathbb{R}^n\to\mathbb{R}^m$ is a bounded Lipschitz function with same bounds as h. Thus, $T:X\to X$.

Similarly, let $h_1,h_2\in X$ and $x_0\in\mathbb{R}^n$. Writing H_1 for $x(t,x_0;h_1)$ and H_2 for $x(t,x_0;h_2)$ we have

$$\begin{aligned}
|h_1(H_1)-h_2(H_2)| &= |h_1(H_1)-h_1(H_2)+h_1(H_2)-h_2(H_2| \\
&\le |h_1(H_1)-h_1(H_2)|+|h_1(H_2)-h_2(H_2)| \\
&\le p_1|H_1-H_2|+|(h_1-h_2)(H_2)| \\
&\le p_1|H_1-H_2|+\|h_1-h_2\|,
\end{aligned}$$

where $\|h_1-h_2\| = \text{Sup}\{|h_1(x_0)-h_2(x_0)| : x_0\in\mathbb{R}^n\}$ and

$$\begin{aligned}
|H_1-H_2| &= |\int_t^0 e^{A(s-t)}[F(H_1,h_1(H_1))-F(H_2,h_2(H_2))]ds| \\
&\le \int_t^0 M(r)e^{r(s-t)}|F(H_1,h_1(H_1))-F(H_2,h_2(H_2))|ds \\
&\le \int_t^0 M(r)e^{r(s-t)}k(\epsilon)[|H_1-H_2|+|h_1(H_1)-h_2(H_2)|]ds \\
&\le \int_t^0 M(r)e^{r(s-t)}k(\epsilon)[|H_1-H_2|+p_1|H_1-H_2|+\|h_1-h_2\|]ds \\
&= M(r)e^{-rt}k(\epsilon)(1+p_1)\int_t^0 e^{rs}|H_1-H_2|ds + M(r)e^{-rt}k(\epsilon)\|h_1-h_2\|\int_t^0 e^{rs}ds \\
&\le M(r)k(\epsilon)\|h_1-h_2\|r^{-1} + M(r)k(\epsilon)(1+p_1)\int_t^0 e^{r(s-t)}|H_1-H_2|ds.
\end{aligned}$$

Hence by Gronwall's inequality,

$$|H_1 - H_2| \leq M(r)k(\epsilon)r^{-1}\|h_1 - h_2\|e^{-[r+(1+p_1)k(\epsilon)M(r)]t}.$$

Rewritting $x(t, x_0; h_1)$ for H_1 and $x(t, x_0; h_2)$ for H_2, and taking $\gamma = r + (1 + p_1)k(\epsilon)M(r)$ we have

$$|x(t, x_0; h_1) - x(t, x_0; h_2)| \leq M(r)k(\epsilon)r^{-1}\|h_1 - h_2\|e^{-\gamma t}.$$

Therefore

$$\begin{aligned}
|(Th_1)(x_0) - (Th_2)(x_0)| &= |\int_{-\infty}^{0} e^{-Bs}G(H_1, h_1(H_1))ds - \int_{-\infty}^{0} e^{-Bs}G(H_2, h_2(H_2))ds| \\
&= |\int_{-\infty}^{0} e^{-Bs}[G(H_1, h_1(H_1)) - G(H_2, h_2(H_2))]ds| \\
&\leq \int_{-\infty}^{0} |e^{-Bs}||G(H_1, h_1(H_1)) - G(H_2, h_2(H_2))|ds \\
&\leq \int_{-\infty}^{0} Ce^{\beta s}k(\epsilon)[|H_1 - H_2| + |h_1(H_1) - h_2(H_2)|]ds \\
&\leq \int_{-\infty}^{0} Ce^{\beta s}k(\epsilon)[(1 + p_1)|H_1 - H_2| + \|h_1 - h_2\|]ds \\
&\leq Ck(\epsilon)[\int_{-\infty}^{0} e^{\beta s}ds + (1 + p_1)M(r)k(\epsilon)r^{-1}\int_{-\infty}^{0} e^{(\beta-\gamma)s}ds]\|h_1 - h_2\| \\
&= Ck(\epsilon)[\beta^{-1} + (1 + p_1)M(r)k(\epsilon)r^{-1}(\beta - \gamma)^{-1}]\|h_1 - h_2\|.
\end{aligned}$$

Hence,

$$\|(Th_1) - (Th_2)\| \leq Ck(\epsilon)[\beta^{-1} + (1 + p_1)M(r)k(\epsilon)r^{-1}(\beta - \gamma)^{-1}]\|h_1 - h_2\|, \tag{2.38}$$

where $\|(Th_1) - (Th_2)\| = \sup\{|(Th_1)(x_0) - (Th_2)(x_0)| : x_0 \in \mathbb{R}^n\}$.

If we choose p_1, ϵ small enough and r big enough then

$$0 < Ck(\epsilon)[\beta^{-1} + (1 + p_1)M(r)k(\epsilon)r^{-1}(\beta - \gamma)^{-1}] < 1.$$

□

Hence, T is a contraction on X therefore. There exists therefore a *Lipschitz centre manifold* for (2.16) and hence, there exists a local centre manifold for (2.1).

To prove that the centre manifold is C^1 we have to show that T is a contraction on a subset of X. The details are similar to proofs given above. To prove that the centre manifold is C^2 we refer to the proof of Theorem 4.2 in Coddington and Levinson [7, p333].

2.2 Reduction Principle

In this section we show that we can restrict the study of the flow of (2.1) to the centre manifold.

The flow on the centre manifold is determined by the n-dimensional system

$$\dot{u} = Au + f(u, h(u)). \tag{2.39}$$

Theorem 2.2.1

(a)Suppose that the zero solution of (2.39) is stable (asymptotically stable)(unstable). Then the zero solution of (2.1) is stable (asymptotically stable)(unstable).

(b)Suppose that the zero solution of (2.39) is stable. Let $(x(t), y(t))$ be the solution of (2.1) with $(x(0), y(0))$ sufficiently small. Then there exists a solution $u(t)$ of (2.39) such that as $t \to \infty$

$$\begin{aligned} x(t) &= u(t) + O(e^{-\gamma t}), \\ y(t) &= h(u(t)) + O(e^{-\gamma t}), \end{aligned} \tag{2.40}$$

where $\gamma > 0$ is a constant.

Let $(x_0, h(x_0))$ be on the centre manifold. Then by invariance solutions $(x(t), y(t))$ of (2.1) through $(x_0, h(x_0))$ are on the centre manifold. That is $y(t) = h(x(t))$. Differentiating $y(t)$ with respect to t, we get $\dot{y} = Dh(x)\dot{x}$. Replacing $\dot{y}$ and $\dot{x}$ by their values in (2.16), and y by its value we get

$$Dh(x)\left[Ax + F(x, h(x))\right] = Bh(x) + G(x, h(x)), \tag{2.41}$$

which, together with conditions $h(0) = 0$ and $Dh(0) = 0$, is the system to be solved to compute the centre manifold [1].

Before we give the proof of this theorem 2.2.1, we first prove a stability property of the centre manifold [1].

Lemma 2.2.1

Let $(x(t), y(t))$ be a solution of (2.16) with $|(x(0), y(0))|$ sufficiently small. Then there exist positive constants C_1 and μ such that

$$|y(t) - h(x(t))| \leq C_1 e^{-\mu t} |y(0) - h(x(0))|$$

for all $t \geq 0$.

This lemma means that a trajectory which starts close enough to the origin will decay exponentially fast to the centre manifold.

Proof

Let $(x(t), y(t))$ be a solution of (2.16) with $(x(0), y(0))$ sufficiently small and let $z(t) = y(t) - h(x(t))$. Then

$$\begin{aligned} \dot{z} &= \dot{y} - Dh(x)\dot{x} \\ &= By + G(x,y) - Dh(x)[Ax + F(x,y)] \\ &= B[z + h(x)] + G(x, z + h(x)) - Dh(x)[Ax + F(x, z + h(x))] \\ &= Bz + Bh(x) + G(x, z + h(x)) - Dh(x)[Ax + F(x, z + h(x))], \end{aligned} \tag{2.42}$$

where $y = z + h(x)$.

From (2.41) we have

$$Bh(x) = Dh(x)[Ax + F(x, h(x))] - G(x, h(x)). \tag{2.43}$$

Substituting equation (2.43) into (2.42) gives

$$\begin{aligned} \dot{z} &= Bz + Dh(x)[Ax + F(x, h(x))] - G(x, h(x)) + G(x, z + h(x)) - Dh(x)[Ax + F(x, z + h(x))] \\ &= Bz + Dh(x)[Ax + F(x, h(x)) - Ax - F(x, z + h(x))] + G(x, z + h(x)) - G(x, h(x)) \\ &= Bz + Dh(x)[F(x, h(x)) - F(x, z + h(x))] + G(x, z + h(x)) - G(x, h(x)). \end{aligned}$$

If we define $N(x,z) = Dh(x)[F(x,h(x)) - F(x,z+h(x)))] + G(x,z+h(x)) - G(x,h(x))$ then we have

$$\dot{z} = Bz + N(x, z). \tag{2.44}$$

The estimate of $N(x, z)$ is given by

$$\begin{aligned} |N(x,z)| &= |Dh(x)[F(x, h(x)) - F(x, z + h(x))] + G(x, z + h(x)) - G(x, h(x))| \\ &\leq |Dh(x)||F(x, h(x)) - F(x, z + h(x))| + |G(x, z + h(x)) - G(x, h(x))| \\ &\leq p_1 k(\epsilon)[|x - x| + |h(x) - z - h(x)|] + k(\epsilon)[|x - x| + |z + h(x) - h(x)|] \\ &= p_1 k(\epsilon)|z| + k(\epsilon)|z| \\ &= (p_1 + 1)k(\epsilon)|z|, \end{aligned}$$

where $|Dh(x)| \leq p_1$.

Taking $\delta(\epsilon) = (p_1 + 1)k(\epsilon)$, clearly $\delta(\epsilon)$ is continuous function such that $\delta(0) = 0$. Hence,

$$|N(x,z)| \leq \delta(\epsilon)|z|. \tag{2.45}$$

From (2.44) we have

$$z(t) = e^{Bt}z(0) + e^{Bt}\int_0^t e^{-Bs}N(x(s), z(s))ds.$$

Using (2.31) we obtain

$$\begin{aligned}
|z(t)| &= |e^{Bt}z(0) + \int_0^t e^{B(t-s)}N(x(s), z(s))ds| \\
&\leq |e^{Bt}z(0)| + |\int_0^t e^{B(t-s)}N(x(s), z(s))ds| \\
&\leq Ce^{-\beta t}|z(0)| + \int_0^t Ce^{-\beta(t-s)}\delta(\epsilon)|z(s)|ds \\
&\leq Ce^{-\beta t}|z(0)| + C\delta(\epsilon)\int_0^t e^{-\beta(t-s)}|z(s)|ds.
\end{aligned}$$

By the Gronwall inequality we have

$$\begin{aligned}
|z(t)| &\leq Ce^{-\beta t}|z(0)|\exp(C\delta(\epsilon)\int_0^t ds) \\
&\leq Ce^{-\beta t}|z(0)|e^{C\delta(\epsilon)t} \\
&= Ce^{-(\beta - C\delta(\epsilon))t}|z(0)|.
\end{aligned}$$

Taking ϵ small enough so that $\beta - C\delta(\epsilon) > 0$, and $C_1 = C$ gives

$$|z(t)| \leq C_1 e^{-\mu t}|z(0)|,$$

where $\mu = \beta - C\delta(\epsilon)$. Hence,

$$|y(t) - h(x(t))| \leq C_1 e^{-\mu t}|y(0) - h(x(0))|.$$

□

This complete the proof of the Lemma 2.2.1.

Lemma 2.2.2

Let A be a real matrix whose eigenvalues have zero real parts. By a change of basis A can be put in the form $A_1 + A_2$, where A_2 is nilpotent,

$$|e^{A_1 t}\underline{X}| = |\underline{X}| \tag{2.46}$$

and

$$|A_2\underline{X}| \leq (\beta/4)|\underline{X}|. \tag{2.47}$$

β is defined in (2.31).

Proof

Suppose that A has real eigenvalues $\lambda_j,\ j = 1, \cdots, k$, and complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j,\ j = k+1, \cdots, n$. Then by the Jordan canonical form theorem there exists a basis of generalized eigenvectors or a linear transformation T that converts A into the canonical form. For simplicity, we consider A with one Jordan block corresponding to a multiple real eigenvalue and one Jordan block corresponding to a multiple complex eigenvalue. This two blocks capture all possible behavior relevant to this lemma.

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & \cdots & 0 & 0 \\ & & \ddots & 1 & & & \\ \vdots & \vdots & & 0 & & & \\ & & & & B & I_2 & O \\ O & & \cdots & & & \ddots & I_2 \\ O & & \cdots & & & O & B \end{bmatrix}$$

with

$$B = \begin{bmatrix} 0 & -b \\ b & 0 \end{bmatrix}, \qquad O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then A may be split into

$$A_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & \cdots & 0 & 0 \\ & & \ddots & 0 & & & \\ \vdots & \vdots & & 0 & & & \\ & & & & B & & O \\ O & & \cdots & & & \ddots & O \\ O & & \cdots & & & O & B \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & \cdots & 0 & 0 \\ & & \ddots & 1 & & & \\ \vdots & \vdots & & 0 & & & \\ & & & & O & I_2 & O \\ O & & \cdots & & & \ddots & I_2 \\ O & & \cdots & & & O & O \end{bmatrix},$$

such that $A = A_1 + A_2$, where A_2 is a nilpotent matrix. It follows that

$$e^{A_1 t}\underline{X} = \begin{bmatrix} x_1 \\ \vdots \\ x_k \\ x_{k+1}\cos bt - x_{k+2}\sin bt \\ x_{k+1}\sin bt + x_{k+2}\cos bt \\ \vdots \\ x_{n-1}\cos bt - x_n \sin bt \\ x_{n-1}\sin bt + x_n\cos bt \end{bmatrix}$$

and

$$\begin{aligned} |e^{A_1 t}\underline{X}| &= [x_1^2 + \cdots + x_k^2 + (x_{k+1}\cos bt - x_{k+2}\sin bt)^2 + \cdots + (x_{n-1}\sin bt + x_n\cos bt)^2]^{1/2} \\ &= [x_1^2 + \cdots + x_k^2 + x_{k+1}^2\cos^2 bt + x_{k+1}^2\sin^2 bt + \cdots + x_n^2\cos^2 bt + x_n^2\sin^2 bt]^{1/2} \\ &= [x_1^2 + \cdots + x_k^2 + x_{k+1}^2(\cos^2 bt + \sin^2 bt) + \cdots + x_n^2(\cos^2 bt + \sin^2 bt)]^{1/2} \\ &= [x_1^2 + x_2^2 + \cdots + x_k^2 + x_{k+1}^2 \cdots + x_{n-1}^2 + x_n^2]^{1/2} \\ &= |\underline{X}|. \end{aligned}$$

□

Next consider A_2. In the standard basis of $\mathbb{R}^n$

$$\underline{X} = (x_1, \cdots, x_n)$$

and

$$\underline{Y} = A_2\underline{X} = (x_2, \cdots, x_{k_1}, x_{k_1+2}, \cdots, x_n, 0, 0).$$

Let us introduce a new basis in which vector $\underline{X}$ is given by

$$\underline{X}' = \begin{cases} x_1' = \alpha_1 x_1, \\ \quad \vdots \\ x_n' = \alpha_n x_n, \end{cases}$$

and $\underline{Y}$ is given by

$$\underline{Y}' = \begin{cases} y_1' = \alpha_1 y_1, \\ \vdots \\ y_{k-1}' = \alpha_{k-1} y_{k-1}, \\ y_k' = \alpha_k y_k, \\ y_{k+1}' = \alpha_{k+1} y_{k+1}, \\ \vdots \\ y_{n-2}' = \alpha_{n-2} y_{n-2}, \\ y_{n-1}' = \alpha_{n-1} y_{n-1}, \\ y_n' = \alpha_n y_n, \end{cases} = \begin{cases} y_1' = \alpha_1 x_2, \\ \vdots \\ y_{k-1}' = \alpha_{k-1} x_k, \\ y_k' = \alpha_k x_{k+2}, \\ y_{k+1}' = \alpha_{k+1} x_{k+3}, \\ \vdots \\ y_{n-2}' = \alpha_{n-2} x_n, \\ y_{n-1}' = 0, \\ y_n' = 0. \end{cases}$$

Therefore

$$\begin{aligned} |\underline{Y}'| &= [y_1'^2 + \cdots + y_{k-1}'^2 + y_k'^2 + y_{k+1}'^2 + \cdots + y_{n-2}'^2 + y_{n-1}'^2 + y_n'^2]^{1/2} \\ &= [\alpha_1^1 x_2^2 + \cdots + \alpha_{k-1}^2 x_k^2 + \alpha_k^2 x_{k+2}^2 + \alpha_{k+1}^2 x_{k+3}^2 + \cdots + \alpha_{n-2}^2 x_n^2]^{1/2} \\ &= [\frac{\alpha_1^2}{\alpha_2^2} x_2'^2 + \cdots + \frac{\alpha_{k-1}^2}{\alpha_k^2} x_k'^2 + \frac{\alpha_k^2}{\alpha_{k+2}^2} x_{k+2}'^2 + \frac{\alpha_{k+1}^2}{\alpha_{k+3}^2} x_{k+3}'^2 + \cdots + \frac{\alpha_{n-2}^2}{\alpha_n^2} x_n'^2]^{1/2}. \end{aligned}$$

Taking

$$\alpha_1 = \frac{1}{4},\ \alpha_2 = \frac{1}{\beta},\ \alpha_3 = \frac{4}{\beta^2},\ \cdots,\ \alpha_{k-1} = \frac{4^{k-3}}{\beta^{k-2}},\ \alpha_k = \frac{4^{k-2}}{\beta^{k-1}}$$

and

$$\alpha_{k+1} = \alpha_{k+2} = \frac{4^{k+1}}{\beta^{k+2}}, \quad \alpha_{k+3} = \alpha_{k+4} = \frac{4^{k+2}}{\beta^{k+3}} \cdots,$$

$$\alpha_{n-2} = \alpha_{n-3} = \frac{4^{n-2}}{\beta^{n-1}}, \quad \alpha_{n-1} = \alpha_n = \frac{4^{n-1}}{\beta^n}.$$

We have

$$\begin{aligned} |\underline{Y}'| = |A_2 \underline{X}'| &= \frac{\beta}{4}[x_2'^2 + \cdots + x_k'^2 + x_{k+2}'^2 + \cdots + x_n'^2]^{1/2}, \\ &\le \frac{\beta}{4} |\underline{X}'| \end{aligned} \tag{2.48}$$

and still

$$
\begin{aligned}
|e^{A_1 t}\underline{X}'| &= [\alpha_1^2 x_1^2 + \cdots + \alpha_k^2 x_k^2 + \alpha_{k+1}^2 (x_{k+1}\cos bt - x_{k+2}\sin bt)^2 + \cdots \\
&+ \alpha_n^2 (x_{n-1}\sin bt + x_n \cos bt)^2]^{1/2} \\
&= [\frac{\alpha_1^2}{\alpha_1^2} x_1^{'2} + \cdots + \frac{\alpha_k^2}{\alpha_k^2} x_k^{'2} + \alpha_{k+1}^2 (\frac{1}{\alpha_{k+1}} x_{k+1}^{'} \cos bt - \frac{1}{\alpha_{k+2}} x_{k+2}^{'} \sin bt)^2 + \\
&\cdots + \alpha_n^2 (\frac{1}{\alpha_{n-1}} x_{n-1}^{'} \sin bt + \frac{1}{\alpha_n} x_n^{'} \cos bt)^2]^{1/2} \\
&= [x_1^{'2} + \cdots + x_k^{'2} + \frac{\alpha_{k+1}^2}{\alpha_{k+1}^2}(x_{k+1}^{'}\cos bt - x_{k+2}^{'}\sin bt)^2 + \qquad (2.49) \\
&\cdots + \frac{\alpha_n^2}{\alpha_n^2}(x_{n-1}^{'}\sin bt + x_n^{'}\cos bt)^2]^{1/2} \\
&= [x_1^{'2} + \cdots + x_k^{'2} + x_{k+1}^{'2}(\cos^2 bt + \sin^2 bt) + \cdots + x_n^{'2}(\cos^2 bt + \sin^2 bt)]^{1/2} \\
&= [x_1^{'1} + \cdots + x_k^{'2} + x_{k+1}^{'2} + \cdots + x_n^{'2}]^{1/2} \\
&= |\underline{X}'|,
\end{aligned}
$$

since $\alpha_{k+1} = \alpha_{k+2}$ and $\alpha_{n-1} = \alpha_n$. □

Proof of theorem 2.2.1

(a)Suppose that the zero solution of (2.39) is unstable. Let $u(t)$ be a solution of (2.39). Then there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists u_0 such that $\|u_0\| < \delta$ and there is t_δ such that $\|u(t_\delta)\| \geq \epsilon$. Let $(x(t), y(t))$ be a solution of (2.1) such that $x(t) = u(t, u_0)$ and $y(t) = h(u(t, u_0))$ for $t > 0$ with $u_0 = u(0)$. Then given $\delta > 0$. We have

$$
\begin{aligned}
\|(x(t_\delta), y(t_\delta))\|^2 &= (u(t_\delta))^2 + (h(u(t_\delta)))^2 \\
&\geq \epsilon^2 + (h(u(t_\delta)))^2 \\
&\geq \epsilon^2.
\end{aligned}
$$

Thus, there exists an $\epsilon > 0$ such that for every $\delta > 0$ there exists t_δ such that $\|(x(t_\delta), y(t_\delta))\| \geq \epsilon$ whenever $\|(x_0, y_0)\| < \delta$, with $x(0) = x_0$ and $y(0) = y_0$. It follows that the zero solution of (2.1) is unstable.

(b)We assume that the zero solution of (2.39) is stable and prove that (2.40) holds where $(x(t), y(t))$ is a solution of (2.16) with $|(x(0), y(0))|$ sufficiently small. Since F and G are equal to f and g in a neighborhood of the origin, this holds for (2.1) and proves Theorem 2.2.1 [1].

The proof is divided into two steps. In step 1, given a solution $u(t)$ of (2.39). We prove the existence of a solution $(x(t), y(t))$ of (2.16) exponentially close to $u(t)$.

In step 2, using the Invariance of Domain Theorm [8, 9], we show that the mapping which relate small solutions of (2.39) to that of (2.16) is a homeomorphism and hence we show the reverse of step 1 from which (2.40) follows.

Step 1

Let $u_0 \in \mathbb{R}^n$ and $z_0 \in \mathbb{R}^m$ with $|(u_0, z_0)|$ sufficiently small. If $u(t)$ is a solution of (2.39), with $u(0) = u_0$ then we have to prove the existence of a solution $(x(t), y(t))$ of (2.16) such that $y(0) - h(x_0) = z_0$, where $x_0 = x(0)$, and $x(t) - u(t)$, $y(t) - h(u(t))$ are exponentially small as $t \to \infty$.

Indeed, let $(x(t), y(t))$ be a solution of (2.16) and $u(t)$ a solution (2.39). Note that if $u(0)$ is sufficiently small then

$$\dot{u} = Au + F(u, h(u)), \tag{2.50}$$

since F is equal to f in the neighborhood of the origin. Let $z(t) = y(t) - h(x(t))$ and $\phi(t) = x(t) - u(t)$ then using the proof of Lemma 2.2.1 we have

$$\dot{z} = Bz + N(\phi + u, z) \tag{2.51}$$

and

$$\begin{aligned}
\dot{\phi} &= \dot{x} - \dot{u} \\
&= Ax + F(x, y) - Au - F(u, h(u)) \\
&= A[\phi + u] + F(\phi + u, z + h(\phi + u)) - Au - F(u, h(u)) \\
&= A\phi + Au + F(\phi + u, z + h(\phi + u)) - Au - F(u, h(u)) \\
&= A\phi + F(\phi + u, z + h(\phi + u)) - F(u, h(u)),
\end{aligned}$$

where $x = \phi + u$ and $y = z + h(\phi + u)$.

Defining $R(\phi, z) = F(\phi + u, z + h(\phi + u)) - F(u, h(u))$, we can write

$$\dot{\phi} = A\phi + R(\phi, z). \tag{2.52}$$

We formulate (2.51) and (2.52) as a fixed point problem. For $a > 0$, $K > 0$ let X be the set of continuous function $\phi : [0, \infty) \to \mathbb{R}^n$ with $|\phi(t)e^{at}| \leq K$ for all $t \geq 0$.

If we define $\|\phi\| = \sup\{|\phi(t)e^{at}| : t \geq 0\}$ then X is a complete space (the proof is similar to that of the set of Lipschitz functions in theorem 2.1.2). Let (u_0, z_0) be sufficiently small and let $u(t)$ be the solution of (2.50) with $u(0) = u_0$. Given $\phi \in X$. Let $z(t)$ be the solution of (2.51) with $z(0) = z_0$.

Using the fact that A can be put in the form $A = A_1 + A_2$, where A_1, A_2 satisfy (2.46) and (2.47) respectively, we obtain

$$\begin{aligned}
\dot{\phi} &= A\phi + R(\phi, z) \\
&= (A_1 + A_2)\phi + R(\phi, z) \\
&= A_1\phi + A_2\phi + R(\phi, z),
\end{aligned}$$

so that

$$\phi(t) = e^{A_1 t}\phi(0) + e^{A_1 t}\int_0^t e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds \tag{2.53}$$

is a solution of (2.52).

Since $|\phi(t)e^{at}| \leq K \Leftrightarrow |\phi(t)| \leq Ke^{-at}$, we have $|\phi(t)| \to 0$ as $t \to \infty$. Hence, taking the limit as $t \to \infty$ in (2.53) gives

$$\begin{aligned}
\lim_{t\to\infty}\phi(t) &= \lim_{t\to\infty} e^{A_1 t}\left[\phi(0) + \lim_{t\to\infty}\int_0^t e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds\right],\\
0 &= \lim_{t\to\infty} e^{A_1 t}\left[\phi(0) + \int_0^\infty e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds\right],\\
0 &= \phi(0) + \int_0^\infty e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds,
\end{aligned}$$

thus,

$$\phi(0) = -\int_0^\infty e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds. \tag{2.54}$$

Substituting (2.54) into (2.53) gives

$$\begin{aligned}
\phi(t) &= -e^{A_1 t}\int_0^\infty e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds + e^{A_1 t}\int_0^t e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds\\
&= -e^{A_1 t}\int_t^\infty e^{-A_1 s}[A_2\phi(s) + R(\phi(s), z(s))]ds\\
&= -\int_t^\infty e^{-A_1(t-s)}[A_2\phi(s) + R(\phi(s), z(s))]ds.
\end{aligned}$$

Let $T : X \to X$ be defined by

$$(T\phi)(t) = -\int_t^\infty e^{-A_1(t-s)}[A_2\phi(s) + R(\phi(s), z(s))]ds. \tag{2.55}$$

We solve (2.55) by means of the contraction mapping principle. If ϕ is a fixed point of T then $x(t) = u(t) + \phi(t)$, $y(t) = z(t) + h(x(t))$ is a solution of (2.16).

By appropriately changing the neighborhood on which we operate we may fix $K = 1$ and $2a = \beta$ [1].

Lemma 2.2.3

There exist a constant C and a continuous function $\delta(\epsilon)$, with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$, such that if $\phi_1, \phi_2 \in \mathbb{R}^n$ with $|\phi_i| \leq 1$ for $i = 1, 2$ and $z_1, z_2 \in \mathbb{R}^m$ with $|z_i| < \epsilon$ for $i = 1, 2$ then

$$|N(\phi_1, z_1) - N(\phi_2, z_2)| \leq C[|z_1||\phi_1 - \phi_2| + \delta(\epsilon)|z_1 - z_2|]. \tag{2.56}$$

Proof

Using the bounds on F, G and h, let us consider

$$N(\phi, z) = h'(\phi)[\Phi(\phi, z)] + \Psi(\phi, z),$$

where $\Phi(\phi, z) = F(\phi, h(\phi)) - F(\phi, z + h(\phi))$ and $\Psi(\phi, z) = G(\phi, z + h(\phi)) - G(\phi, h(\phi))$. Therefore for $i = 1, 2$, writing Φ_i for $\Phi(\phi_i, z_i)$ and Ψ_i for $\Psi(\phi_i, z_i)$ we have

$$\begin{aligned} N(\phi_1, z_1) - N(\phi_2, z_2) &= h'(\phi_1)\Phi_1 - h'(\phi_2)\Phi_2 + \Psi_1 - \Psi_2 \\ &= (h'(\phi_1) - h'(\phi_2))\Phi_1 + h'(\phi_2)[\Phi_1 - \Phi_2] + \Psi_1 - \Psi_2. \end{aligned}$$

It follows that

$$|N(\phi_1, z_1) - N(\phi_2, z_2)| \leq |h'(\phi_1) - h'(\phi_2)||\Phi_1| + |h'(\phi_2)||\Phi_1 - \Phi_2| + |\Psi_1 - \Psi_2|.$$

Since h is a Lipschitz function with Lipschitz constant p_1 and $|h(\phi)| < \epsilon$ we have

$$|h'(\phi_2)| \leq p_1.$$

h' is of class C^1. Therefore by the mean value Theorem, we have

$$h'(\phi_1) - h'(\phi_2) \leq h''(\xi)(\phi_1 - \phi_2).$$

Hence, for $|\phi_1|, |\phi_2| \leq 1$

$$|h'(\phi_1) - h'(\phi_2)| \leq C_1|\phi_1 - \phi_2|.$$

Where

$$C_1 = \sup_{|\xi| \leq 1} |h^{''}(\xi)|.$$

The estimate of Φ_1 is given by

$$\begin{aligned} |\Phi_1| \equiv |\Phi(\phi_1, z_1)| &= |F(\phi_1, h(\phi_1)) - F(\phi_1, z_1 + h(\phi_1))| \\ &\leq k(\epsilon)|z_1|. \end{aligned}$$

Let us give an alternative expression of $\Psi(\phi, z)$.

$$\begin{aligned} \Psi(\phi, z) &= \int_0^1 \frac{d}{dt} G(\phi, tz + h(\phi))dt \\ &= z\int_0^1 G_{.2}(\phi, tz + h(\phi))dt, \end{aligned}$$

where $G_{.2}$ is defined the same way as $F_{.2}$ in (2.29). Then

$$\begin{aligned}\Psi_1 - \Psi_2 &= z_1 \int_0^1 G_{.2}(\phi_1, tz_1 + h(\phi_1))dt - z_2 \int_0^1 G_{.2}(\phi_2, tz_2 + h(\phi_2))dt \\ &= z_1 \int_0^1 \left(G_{.2}(\phi_1, tz_1 + h(\phi_1)) - G_{.2}(\phi_2, tz_2 + h(\phi_2))\right) dt + (z_1 - z_2) \int_0^1 G_{.2}(\phi_2, tz_2 + h(\phi_2))dt.\end{aligned}$$

Hence,

$$|\Psi_1 - \Psi_2| \leq |z_1| \int_0^1 |G_{.2}(\phi_1, tz_1 + h(\phi_1)) - G_{.2}(\phi_2, tz_2 + h(\phi_2))|dt + |z_1 - z_2| \int_0^1 |G_{.2}(\phi_2, tz_2 + h(\phi_2))|dt.$$

$G_{.2}$ can be expanded in Taylor's formula up to one term as follows:

$$G_{.2}(\phi_1, tz_1 + h(\phi_1)) - G_{.2}(\phi_2, tz_2 + h(\phi_2)) = G_{.21}(\zeta_1, \zeta_2)(\phi_1 - \phi_2) + G_{.22}(\zeta_1, \zeta_2)(tz_1 + h(\phi_1) - tz_2 - h(\phi_2)),$$

where

$$G_{.21}(\zeta_1, \zeta_2) = \left[\left(\frac{\partial^2 G_i}{\partial u_k \partial \phi_j} \right)_{|(\zeta_1, \zeta_2)} \right] \quad \text{and} \quad G_{.22}(\zeta_1, \zeta_2) = \left[\left(\frac{\partial^2 G_i}{\partial u_k \partial u_l} \right)_{|(\zeta_1, \zeta_2)} \right],$$

with $1 \leq i, j \leq n$, $1 \leq k, l \leq m$ and $u = tz + h(\phi)$. If $|\phi| \leq 1$ and $|u| < 2\epsilon$ then the second derivatives may be estimated by their suprema with respect to (ϕ, u).

Let

$$C_2 = \sup_{|\phi| \leq 1,\ |u| < 2\epsilon} \left\{ \left| \left(\frac{\partial^2 G_i}{\partial u_k \partial \phi_j} \right)_{|(\zeta_1, \zeta_2)} \right|,\ \left| \left(\frac{\partial^2 G_i}{\partial u_k \partial u_l} \right)_{|(\zeta_1, \zeta_2)} \right| : 1 \leq i, j \leq n \text{ and } 1 \leq k, l \leq m \right\}.$$

It follows that

$$\begin{aligned}|G_{.2}(\phi_1, tz_1 + h(\phi_1)) - G_{.2}(\phi_2, tz_2 + h(\phi_2))| &\leq C_2[|\phi_1 - \phi_2| + |tz_1 + h(\phi_1) - tz_2 - h(\phi_2)|] \\ &\leq C_2[|\phi_1 - \phi_2| + |tz_1 - tz_2| + |h(\phi_1) - h(\phi_2)|] \\ &\leq C_2[|\phi_1 - \phi_2| + t|z_1 - z_2| + p_1|\phi_1 - \phi_2|] \\ &\leq C_2[(1 + p_1)|\phi_1 - \phi_2| + |z_1 - z_2|],\end{aligned}$$

since $0 \leq t \leq 1$ and h is Lipschitz with Lipschitz constant p_1. From the proof of (2.23) and using the fact that G is defined the same way as F, we have

$$|G_{.2}(\phi_2, tz_2 + h(\phi_2))| \leq k(\epsilon).$$

Hence,

$$\begin{aligned}|\Psi_1 - \Psi_2| &\le |z_1|C_2[(1+p_1)|\phi_1 - \phi_2| + |z_1 - z_2|] + k(\epsilon)|z_1 - z_2| \\ &= C_2(1+p_1)|z_1||\phi_1 - \phi_2| + (C_2|z_1| + k(\epsilon))|z_1 - z_2| \\ &= C|z_1||\phi_1 - \phi_2| + \delta(\epsilon)|z_1 - z_2|,\end{aligned}$$

where C is some positive constant and $|z_1| < \epsilon$ and $\delta(\epsilon) \to 0$ as $\epsilon \to 0$.

Since F and G are defined the same way, following similar reasoning as for $\Psi(\phi, z)$ we can estimate $\Phi_1 - \Phi_2$ by the same quantity as $\Psi_1 - \Psi_2$. We therefore have

$$|\Phi_1 - \Phi_2| \le C|z_1||\phi_1 - \phi_2| + \delta(\epsilon)|z_1 - z_2|$$

and hence,

$$\begin{aligned}|N(\phi_1, z_1) - N(\phi_2, z_2)| &\le |h'(\phi_1) - h'(\phi_2)||\Psi_1| + |h'(\phi_2)||\Psi_1 - \Psi_2| + |\Psi_1 - \Psi_2| \\ &\le C_1 k(\epsilon)|z_1||\phi_1 - \phi_2| + (p_1 + 1)C[|z_1||\phi_1 - \phi_2| + \delta(\epsilon)|z_1 - z_2|] \\ &\le C[|z_1||\phi_1 - \phi_2| + \delta(\epsilon)|z_1 - z_2|].\end{aligned}$$

Note that here, for the convenience, h' denotes the Jacobian of h $\square$.

Lemma 2.2.4

There exists a continuous function $\delta(\epsilon)$ with $\delta(0) = 0$ such that if $\phi_1, \phi_2 \in \mathbb{R}^n$, with $|\phi_i| \le 1$ for $i = 1, 2$ and $z_1, z_2 \in \mathbb{R}^m$ with $|z_i| < \epsilon$ for $i = 1, 2$, then

$$|R(\phi_1, z_1) - R(\phi_2, z_2)| \le \delta(\epsilon)[|z_1 - z_2| + |\phi_1 - \phi_2|]. \tag{2.57}$$

Defining $R(\phi, z) = F(u + \phi, z + h(u + \phi)) - F(u, h(u))$ we get

$$\begin{aligned}|R(\phi, z)| &= |F(u + \phi, z + h(u + \phi)) - F(u, h(u))| \\ &\le k(\epsilon)[|u + \phi - u| + |z + h(u + \phi) - h(u)|] \\ &\le k(\epsilon)[|\phi| + |z| + |h(u + \phi) - h(u)|] \\ &\le k(\epsilon)[|\phi| + |z| + p_1|u + \phi - u|] \\ &= k(\epsilon)[|\phi| + |z| + p_1|\phi|] \\ &= k(\epsilon)[(1 + p_1)|\phi| + |z|] \\ &\le k(\epsilon)[(1 + p_1)|\phi| + (1 + p_1)|z|] \\ &= (1 + p_1)k(\epsilon)[|\phi| + |z|] \\ &\le \delta(\epsilon)[|\phi| + |z|].\end{aligned}$$

It follows that

$$
\begin{aligned}
|R(\phi_1, z_1) - R(\phi_2, z_2)| &= |F(u+\phi_1, z_1 + h(u+\phi_1)) - F(u+\phi_2, z_2 + h(u+\phi_2))| \\
&\le k(\epsilon)[|u+\phi_1 - u - \phi_2| + |z_1 + h(u+\phi_1) - z_2 - h(u+\phi_2)|] \\
&= k(\epsilon)[|\phi_1 - \phi_2| + |z_1 - z_2| + |h(u+\phi_1) - h(u+\phi_2)|] \\
&\le k(\epsilon)[|\phi_1 - \phi_2| + |z_1 - z_2| + p_1|u+\phi_1 - u - \phi_2|] \\
&= k(\epsilon)[|\phi_1 - \phi_2| + |z_1 - z_2| + p_1|\phi_1 - \phi_2|] \\
&= k(\epsilon)[(1+p_1)|\phi_1 - \phi_2| + |z_1 - z_2|] \\
&\le k(\epsilon)[(1+p_1)|\phi_1 - \phi_2| + (1+p_1)|z_1 - z_2|] \\
&= (1+p_1)k(\epsilon)[|\phi_1 - \phi_2| + |z_1 - z_2|] \\
&\le \delta(\epsilon)[|\phi_1 - \phi_2| + |z_1 - z_2|].
\end{aligned}
$$

□

After the proof of Lemma 2.2.4, we come back to the proof of Theorem 2.2.1. From (2.51) we have

$$z(t) = e^{Bt}z_0 + e^{Bt}\int_0^t e^{-Bs}N(\phi(s), z(s))ds.$$

Then

$$
\begin{aligned}
|z(t)| &\le |e^{Bt}z_0| + \int_0^t |e^{B(t-s)}||N(\phi(s), z(s))|ds \\
&\le C|z_0|e^{-\beta t} + C\delta(\epsilon)\int_0^t e^{-\beta(t-s)}|z(s)|ds,
\end{aligned}
$$

where we have used (2.31) and (2.56). By Gronwall's inequality

$$|z(t)| \le C|z_0|e^{-\beta_1 t}, \tag{2.58}$$

where $\beta_1 = \beta - C\delta(\epsilon)$.

The inequality (2.58) means that $z(t)$ is exponentially small. Therefore also by Lemma 2.2.1 $y(t) - h(u(t))$ is exponentially small as in the second equation in (2.40).

From (2.55), if ϵ is sufficiently small we show that T is a mapping of X into X. Indeed,

$$
\begin{aligned}
|(T\phi)(t)| &\leq \int_t^\infty |e^{A_1(t-s)}[A_2\phi(s) + R(\phi(s), z(s))]|ds \\
&\leq \int_t^\infty |A_2\phi(s) + R(\phi(s), z(s))|ds \\
&\leq \int_t^\infty (|A_2\phi(s)| + |R(\phi(s), z(s))|)ds \\
&\leq \int_t^\infty |A_2\phi(s)|ds + \int_t^\infty |R(\phi(s), z(s))|ds \\
&\leq \int_t^\infty \frac{\beta}{4}|\phi(s)|ds + \int_t^\infty k(\epsilon)[|\phi(s)| + |z(s))|]ds \\
&\leq \frac{\beta}{4}\int_t^\infty e^{-as}ds + k(\epsilon)\int_t^\infty e^{-as}ds + k(\epsilon)\int_t^\infty C|z_0|e^{-\beta_1 s}ds \\
&\leq \frac{e^{-at}}{2} + k(\epsilon)\int_t^\infty (e^{-as} + C|z_0|e^{-\beta_1 s})ds \\
&\leq \frac{e^{-at}}{2} + \frac{k(\epsilon)}{a}e^{-at} + C\frac{k(\epsilon)}{\beta_1}|z_0|e^{-\beta_1 t} \\
&\leq \frac{e^{-at}}{2} + \frac{k(\epsilon)}{a}e^{-at} + C\frac{k(\epsilon)}{\beta_1}|z_0|e^{-2at}e^{Ck(\epsilon)t} \\
&\leq \frac{e^{-at}}{2} + \frac{e^{-at}}{2}\left[\frac{2k(\epsilon)}{a} + \frac{2k(\epsilon)|z_0|}{\beta_1 e^{at}}e^{Ck(\epsilon)t}\right].
\end{aligned}
\tag{2.59}
$$

For sufficiently small ϵ the expression in the bracket is less than 1. Therefore

$$|(T\phi)(t)| \leq e^{-at}. \tag{2.60}$$

Where we have used (2.46),(2.47),(2.56),(2.57) and (2.58). Hence T maps X into X. Now we show that T is a contraction mapping on X.

Let $\phi_1, \phi_2 \in X$ and let z_1, z_2 be the corresponding solutions of (2.51) with $z_i(0) = z_0$ for $i = 1, 2$. We first estimate $W(t) = z_1(t) - z_2(t)$.

From (2.51) and (2.56) we have

$$W(t) = \int_0^t e^{B(t-s)}(N(\phi_1(s), z_1(s)) - N(\phi_2(s), z_2(s)))ds.$$

Therefore

$$
\begin{aligned}
|W(t)| &\leq \int_0^t |e^{B(t-s)}||(N(\phi_1(s), z_1(s)) - N(\phi_2(s), z_2(s)))|ds \\
&\leq \int_0^t Ce^{-\beta(t-s)}[C|z_1(s)||\phi_1(s) - \phi_2(s)| + \delta(\epsilon)|z_1(s) - z_2(s)|]ds \\
&\leq \tilde{C} \int_0^t e^{-\beta(t-s)}|z_0|e^{-\beta_1 s}|\phi_1(s) - \phi_2(s)|ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&\leq \tilde{C}|z_0| \int_0^t e^{-\beta(t-s)}e^{\beta_1 s}\|\phi_1 - \phi_2\|e^{-as}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \tilde{C}|z_0|\|\phi_1 - \phi_2\| \int_0^t e^{-\beta(t-s)}e^{\beta_1 s}e^{-as}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \tilde{C}|z_0|\|\phi_1 - \phi_2\|e^{-\beta t} \int_0^t e^{\beta s}e^{\beta_1 s}e^{-as}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \tilde{C}|z_0|\|\phi_1 - \phi_2\|e^{-\beta t} \int_0^t e^{\beta s}e^{-\beta s}e^{C\delta(\epsilon)}e^{-as}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \tilde{C}|z_0|\|\phi_1 - \phi_2\|e^{-\beta t} \int_0^t e^{-(a-C\delta(\epsilon))s}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \tilde{C}|z_0|\|\phi_1 - \phi_2\|e^{-\beta t} \int_0^t e^{-\tau_\epsilon s}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&\leq \tilde{C}|z_0|\|\phi_1 - \phi_2\|e^{-\beta t} \int_0^\infty e^{-\tau_\epsilon s}ds + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \tilde{C}|z_0|\|\phi_1 - \phi_2\|e^{-\beta t}\frac{1}{\tau_\epsilon} + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= \frac{\tilde{C}|z_0|}{\tau_\epsilon}\|\phi_1 - \phi_2\|e^{-\beta t} + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds \\
&= C_1\|\phi_1 - \phi_2\|e^{-\beta t} + C\delta(\epsilon) \int_0^t e^{-\beta(t-s)}|W(s)|ds,
\end{aligned}
$$

where $C_1 = \dfrac{\tilde{C}|z_0|}{\tau_\epsilon}$ is a constant and $\tau_\epsilon = a - Ck(\epsilon)$.

Then by Gronwall's inequality

$$
\begin{aligned}
|W(t)| = |z_1(t) - z_2(t)| &\leq C_1\|\phi_1 - \phi_2\|e^{-\beta t}e^{C\delta(\epsilon)t} \\
&= C_1\|\phi_1 - \phi_2\|e^{-(\beta - C\delta(\epsilon))t} \\
&= C_1\|\phi_1 - \phi_2\|e^{-\beta_1 t}.
\end{aligned} \tag{2.61}
$$

Using (2.31) and (2.61) for ϵ sufficiently small, we have from (2.55)

$$
(T\phi_1)(t) - (T\phi_2)(t) = \int_t^\infty e^{A_1(t-s)}[A_2(\phi_2(s) - \phi_1(s)) + R(\phi_2(s), z_2(s)) - R(\phi_1(s), z_1(s))]ds.
$$

Then

$$\begin{aligned}
|(T\phi_1)(t)-(T\phi_2)(t)| &\le \int_t^\infty |e^{A_1(t-s)}[A_2(\phi_2(s)-\phi_1(s))+R(\phi_2(s),z_2(s))-R(\phi_1(s),z_1(s))]|ds \\
&\le \int_t^\infty |A_2(\phi_2(s)-\phi_1(s))|ds+\int_t^\infty |R(\phi_2(s),z_2(s))-R(\phi_1(s),z_1(s))|ds \\
&\le \int_t^\infty \frac{\beta}{4}|\phi_2(s)-\phi_1(s)|ds+\int_t^\infty \delta(\epsilon)[|\phi_1(s)-\phi_2(s)|+|z_1(s)-z_2(s)|]ds \\
&\le \frac{\beta}{4}\|\phi_1-\phi_2\|\int_t^\infty e^{-as}ds+\delta(\epsilon)\left(\int_t^\infty (|\phi_1(s)-\phi_2(s)|ds+|W(s)|)\,ds\right) \\
&\le \|\phi_1-\phi_2\|\left(\frac{e^{-at}}{2}+\delta(\epsilon)\left(\int_t^\infty e^{-as}ds+C_1\int_t^\infty e^{-\beta_1 s}ds\right)\right) \\
&= \|\phi_1-\phi_2\|\left[\frac{e^{-at}}{2}+\delta(\epsilon)\int_t^\infty (e^{-as}+C_1e^{-\beta_1 s})ds\right], \\
&= \|\phi_1-\phi_2\|e^{-at}\left[\frac{1}{2}+\delta(\epsilon)\left(\frac{1}{a}+\frac{C_1}{\beta_1}e^{-(a-C\delta(\epsilon))t}\right)\right].
\end{aligned}$$

It follows that

$$|(T\phi_1)(t)-(T\phi_2)(t)|e^{at}\le \|\phi_1-\phi_2\|\left[\frac{1}{2}+\delta(\epsilon)\left(\frac{1}{a}+\frac{C_1}{\beta_1}e^{-(a-C\delta(\epsilon))t}\right)\right].$$

Hence,

$$\|T\phi_1-T\phi_2\|\le \alpha\|\phi_1-\phi_2\|,$$

where

$$\|T\phi_1-T\phi_2\|=\sup\left\{|(T\phi_1)(t)-(T\phi_2)(t)|e^{at}:t\ge 0\right\}$$

and

$$\alpha=\left[\frac{1}{2}+\delta(\epsilon)\left(\frac{1}{a}+\frac{C_1}{\beta_1}\delta(\epsilon)e^{-(a-C\delta(\epsilon))t}\right)\right]<1,$$

provided ϵ sufficiently small.

This shows that T is a contraction on X. Therefore it has a fixed point. Suppose that $\phi(t)$ is the fixed point of T, then $|\phi(t)|\le e^{-at}$.

Next we have to prove the converse. That is for any (x_0,y_0) sufficiently small there is $u(t)$ solution on the centre manifold such that (2.40) holds. For this we use in step 2 the Invariance of Domain Theorem [9] or [8].

Step 2

Let U be an open neighborhood of the origin in $\mathbb{R}^{n+m}$ and $(u_0,z_0)\in U$, where $u_0=u(0)$ and $z_0=y(0)-h(x(0))$. We define a mapping S of U into $\mathbb{R}^{n+m}$ by $S(u_0,z_0)=(x_0,z_0)$ where $x_0=u_0+\phi(0)$. We show that S is a homeomorphism. Repeating the above analysis, one can show that $T:X\times U\to X$ is a continuous uniform contraction. This proves that the fixed point depends continuously on u_0 and z_0. Since ϕ depends continuously on u_0 and z_0, S is continuous [1].

We now show that S is one-to-one, so that by *Invariance of Domain Theorem* [8, 9], S is an open mapping. Hence, a homeomorphism. $S(0,0) = 0$ proves that S is a full neighborhood of the origin.

Indeed, given (u_0, z_0) and (u_1, z_1) in U we prove that if $S(u_0, z_0) = S(u_1, z_1)$ then $(u_0, z_0) = (u_1, z_1)$. In other words, we have to prove that if $u_0 + \phi_0(0) = u_1 + \phi_1(0)$ then $u_0 = u_1$ and $\phi_0(0) = \phi_1(0)$.

Let $u_0 + \phi_0(0) = u_1 + \phi_1(0)$. Then the initial values for x and y are the same. Using the fact that solution $(x(t), y(t))$ of (2.16) is unique then $u_0(t) + \phi_0(t) = u_1(t) + \phi_1(t)$ for all $t \geq 0$, where $u_i(t)$ is the solution of (2.50) with $u_i(0) = u_i$ [1].

Hence,

$$u_0(t) - u_1(t) = \phi_1(t) - \phi_0(t).$$

From (2.50) we have

$$\dot{u} = A_1 u + A_2 u + F(u, h(u)). \tag{2.62}$$

Taking

$$u = e^{A_1 t} v \;\Rightarrow\; |u| = |e^{A_1 t} v| = |v| \text{ and } \dot{u} = A_1 u + e^{A_1 t}\dot{v}. \tag{2.63}$$

Using (2.63), (2.62) gives

$$\dot{v} = A_2 v + e^{-A_1 t} F(e^{A_1 t} v, h(e^{A_1 t} v)). \tag{2.64}$$

Then

$$\frac{d}{dt}(v_1 - v_0) = A_2(v_1 - v_0) + e^{-A_1 t}[F(e^{A_1 t} v_1, h(e^{A_1 t} v_1)) - F(e^{A_1 t} v_0, h(e^{A_1 t} v_0))]$$

$$\langle \frac{d}{dt}(v_1 - v_0), v_1 - v_0 \rangle = \langle A_2(v_1 - v_0), v_1 - v_0 \rangle + \langle e^{-A_1 t}[F(e^{A_1 t} v_1, h(e^{A_1 t} v_1)) - F(e^{A_1 t} v_0, h(e^{A_1 t} v_0))], v_1 - v_0 \rangle$$

$$\frac{1}{2}\frac{d}{dt}\langle v_1 - v_0, v_1 - v_0 \rangle = \langle A_2(v_1 - v_0), v_1 - v_0 \rangle + \langle e^{-A_1 t}[F(e^{A_1 t} v_1, h(e^{A_1 t} v_1)) - F(e^{A_1 t} v_0, h(e^{A_1 t} v_0))], v_1 - v_0 \rangle$$

$$\frac{1}{2}\frac{d}{dt}|v_1 - v_0|^2 \geq -|A_2(v_1 - v_0)||v_1 - v_0| - |e^{-A_1 t}[F(e^{A_1 t} v_1, h(e^{A_1 t} v_1)) - F(e^{A_1 t} v_0, h(e^{A_1 t} v_0))]||v_1 - v_0|$$

$$\frac{1}{2}\frac{d}{dt}|v_1 - v_0|^2 \geq -\frac{\beta}{4}|v_1 - v_0|^2 - k(\epsilon)[|v_1 - v_0| + p_1|v_1 - v_0|]|v_1 - v_0|$$

$$\frac{1}{2}\frac{d}{dt}|v_1 - v_0|^2 \geq -\left(\frac{\beta}{4} + k(\epsilon)(1 + p_1)\right)|v_1 - v_0|^2$$

$$\frac{1}{2}\frac{d}{dt}|v_1 - v_0|^2 \geq -\left(\frac{\beta}{4} + \delta(\epsilon)\right)|v_1 - v_0|^2$$
$$\frac{1}{2}\frac{d}{dt}\ln|v_1 - v_0|^2 \geq -\left(\frac{\beta}{4} + \delta(\epsilon)\right)$$
$$\frac{d}{dt}\ln|v_1 - v_0| \geq -\left(\frac{\beta}{4} + \delta(\epsilon)\right)$$
$$\ln\frac{|v_1(t) - v_0(t)|}{|v_1 - v_0|} \geq -\left(\frac{\beta}{4} + \delta(\epsilon)\right)t$$
$$|v_1(t) - v_0(t)| \geq |v_1 - v_0|e^{-\left(\frac{\beta}{4}+\delta(\epsilon)\right)t}$$
$$|v_1(t) - v_0(t)|e^{\left(\frac{\beta}{4}+\delta(\epsilon)\right)t} \geq |v_1 - v_0|.$$

Now using the first implication in (2.63) we have

$$|u_1(t) - u_0(t)|e^{\left(\frac{\beta}{4}+\delta(\epsilon)\right)t} \geq |u_1 - u_0|,$$

since we assumed that $u_1(t) - u_0(t) = \phi_1(t) - \phi_0(t)$, we have

$$|\phi_1(t) - \phi_0(t)|e^{\left(\frac{\beta}{4}+\delta(\epsilon)\right)t} \geq |u_1 - u_0|.$$

It follows that

$$|\phi_1(t) - \phi_0(t)|e^{\frac{\beta}{2}t} \geq |u_1 - u_0|e^{\left(\frac{\beta}{4}-\delta(\epsilon)\right)t}.$$

Then from (2.53), $|\phi_1(t) - \phi_0(t)|e^{\frac{\beta}{2}t}$ is bounded whereas $|u_1 - u_0|e^{\left(\frac{\beta}{4}-\delta(\epsilon)\right)t}$ is diverging for ϵ small enough. Therefore we have a contraction, unless $u_0 = u_1$.

2.3 Approximation of the Centre Manifold

The computation of the centre manifold requires solving the following system

$$\begin{aligned} Dh(x)[Ax + f(x, h(x))] - Bh(x) - g(x, h(x)) &= 0, \\ h(0) &= 0, \\ Dh(0) &= 0. \end{aligned} \tag{2.65}$$

System (2.65) is in general impossible to solve. In the next Theorem we present a method of approximation of the centre manifold to any degree of accuracy by a function of class C^2.

Given C^2-functions $\phi : \mathbb{R}^n \to \mathbb{R}^m$ in the neighborhood of the origin and using (2.41) we define

$$(M\phi)(x) = D\phi(x)[Ax + f(x, \phi(x))] - B\phi(x) - g(x, \phi(x)).$$

Theorem 2.3.1

Suppose that $\phi(0) = 0$, $D\phi(0) = 0$ and that $(M\phi)(x) = O(|x|^q)$ as $x \to 0$, where $q > 1$. Then as $x \to 0$

$$|h(x) - \phi(x)| = O(|x|^q).$$

Proof

Let $\theta : \mathbb{R}^n \to \mathbb{R}^m$ be a continuously differentiable function with compact support such that $\theta(x) = \phi(x)$ for $|x|$ small. Consider

$$N(x) = D\theta(x)\left[Ax + F(x, \theta(x))\right] - B\theta(x) - G(x, \theta(x)), \tag{2.66}$$

where F and G are defined in theorem 2.1.2. Note that $N(x) = O(|x|^q)$ as $x \to 0$ since $\theta(x) = \phi(x)$ for $|x|$ small.

For a given $K > 0$, let $Y_K = \{z : z + \theta \in X,\ |z(x)| \leq K|x|^q \text{ for all } x \in \mathbb{R}^n\}$, where X is the set of Lipschitz functions $h : \mathbb{R}^n \to \mathbb{R}^m$ defined in lemma 2.1.1. With the sup norm and the same Lipschtiz constant Y_K is a subset of $C(\mathbb{R}^n, \mathbb{R}^m)$ which is closed. Indeed, let $z \in \bar{Y_K}$, the closure of Y_K. There are $(z_n) \in Y_K$ such that $z_n \to z$ as $n \to \infty$. Hence, given an $\epsilon > 0$. There is a natural number N such that for $n \geq N$ we have

$$\|z_n - z\| = \sup\{|z_n(x) - z(x)| : x \in \mathbb{R}^n\} < \epsilon.$$

Now we have to show that $z \in Y_K$. Since $z_n \in Y_K$ for $n = 1, 2, \cdots$ we have $|z_n(x)| \leq K|x|^q$ for $x \in \mathbb{R}^n$ and $q > 1$. Letting $n \to \infty$ gives $|z(x)| \leq K|x|^q$ for $x \in \mathbb{R}^n$ and $q > 1$. This shows that $z \in Y_K$. Since $z \in \bar{Y_K}$ was arbitrary, this proves that Y_K is closed in $C(\mathbb{R}^n, \mathbb{R}^m)$.

Let Y_K be the domain of a mapping S defined by $Sz = T(z + \theta) - \theta$, where $T : X \to X$ is the contraction mapping with h as fixed point defined in theorem 2.1.2. We see that $Sz + \theta = T(z + \theta) \in X$. So it is enough to show that $|Sz(x)| \leq K|x|^q$. That is, to prove theorem 2.3.1, we have to show that $Sz \in Y_K$. In other words, we have to exhibit a positive K such that S maps Y_K into Y_K. We first show that S is a contraction on Y_K. Using the fact that T is a contraction mapping on X. For $x_0 \in \mathbb{R}^n$, z_1 and $z_2 \in Y_K$ we have

$$\begin{aligned}
|(Sz_1)(x_0) - (Sz_2)(x_0)| &= |T(z_1 + \theta)(x_0) - \theta(x_0) - T(z_2 + \theta)(x_0) + \theta(x_0)| \\
&= |T(z_1 + \theta)(x_0) - T(z_2 + \theta)(x_0)| \\
&\leq q|(z_1 + \theta)(x_0) - (z_2 + \theta)(x_0)| \\
&= q|z_1(x_0) + \theta(x_0) - z_2(x_0) - \theta(x_0)| \\
&= q|z_1(x_0) - z_2(x_0)| \\
&\leq q\|z_1 - z_2\|.
\end{aligned}$$

Consequently

$$\|Sz_1 - Sz_2\| = \sup\{|(Sz_1)(x_0) - (Sz_2)(x_0)| : x_0 \in \mathbb{R}^n\} \leq q\|z_1 - z_2\|.$$

Hence, S is a contraction mapping on Y_K. Therefore there exists a unique point $\bar{z} \in Y_K$ such that $S\bar{z} = \bar{z}$. It follows that $T(\bar{z} + \theta) - \theta = \bar{z}$ which implies $T(\bar{z} + \theta) = \bar{z} + \theta$. Therefore $\bar{z} + \theta$ is a fixed point of T. Since T has a unique fixed point h on X and $\bar{z} + \theta \in X$ we have $h = \bar{z} + \theta$.

To show the existence of a positive K such that S maps Y_K into Y_K, we first give another formulation of S. For $z \in Y_K$ let $x(t, x_0)$ be the solution of

$$\dot{x} = Ax + F(x, z(x) + \theta(x)), \quad x(0, x_0) = x_0. \tag{2.67}$$

Using (2.17) and (2.22) we define

$$(T(z+\theta))(x_0) = \int_{-\infty}^{0} e^{-Bs} G(x(s,x_0), z(x(s,x_0)) + \theta(x(s,x_0)))ds.$$

From the definition of $N(x)$ we draw $\dot{\theta}(x) = B\theta(x) + G(x, \theta(x))$. Then, arguing as in (2.21) we have

$$\begin{aligned}\theta(x_0) &= \int_{-\infty}^{0} e^{-Bs} G(x(s,x_0), \theta(x(s,x_0)))ds \\ &= -\int_{-\infty}^{0} e^{-Bs}[B\theta(x(s,x_0)) - \frac{d}{ds}\theta(x(s,x_0))]ds.\end{aligned}$$

From (2.66) and (2.67), writing x for $x(s, x_0)$ we have

$$\begin{aligned}B\theta(x) - \frac{d}{ds}\theta(x) &= B\theta(x) - D\theta(x)\dot{x} \\ &= B\theta(x) - D\theta(x)[Ax + F(x, z(x) + \theta(x))] \\ &= D\theta(x)[Ax + F(x,\theta(x))] - N(x) - G(x,\theta(x)) - D\theta(x)[Ax + F(x, z(x) + \theta(x))] \\ &= -N(x) - G(x, \theta(x)) + D\theta(x)[F(x,\theta(x)) - F(x, z(x) + \theta(x))].\end{aligned}$$

Therefore

$$\theta(x_0) = \int_{-\infty}^{0} e^{-Bs}[N(x) + G(x,\theta(x)) - D\theta(x)[F(x,\theta(x)) - F(x, z(x) + \theta(x))]]ds.$$

Hence,

$$\begin{aligned}(Sz)(x_0) &= T(z+\theta)(x_0) - \theta(x_0) \\ &= \int_{-\infty}^{0} e^{-Bs}\left\{G(x, z+\theta) - N(x) - G(x,\theta) + \theta^{'}(x)[F(x,\theta(x)) - F(x, z(x) + \theta(x))]\right\} ds.\end{aligned}$$

Taking $Q(x,z) = G(x, z+\theta) - N(x) - G(x,\theta) + \theta^{'}(x)[F(x,\theta(x)) - F(x, z(x) + \theta(x))]$ and writing $x(s, x_0)$ for x we have

$$(Sz)(x_0) = \int_{-\infty}^{0} e^{-Bs} Q(x(s,x_0), z(x(s,x_0)))ds,$$

where $x(t, x_0)$ is the solution of (2.67).

We now show the existence of $K > 0$ such that S maps Y_K into Y_K. We assume that θ was chosen so that $|\theta(x)| \leq \epsilon$ for all $x \in \mathbb{R}^n$. Since $N(x) = O(|x|^q)$ as $x \to 0$ there exists a constant $C_1 > 0$ such that

$$|N(x)| \leq C_1|x|^q, \ x \in \mathbb{R}^n \tag{2.68}$$

and

$$\begin{aligned} |Q(x,z)| &= |Q(x,0) + Q(x,z) - Q(x,0)| \\ &\leq |Q(x,0)| + |Q(x,z) - Q(x,0)| \\ &= |N(x)| + |Q(x,z) - Q(x,0)|. \end{aligned} \tag{2.69}$$

The estimate of $|Q(x,z) - Q(x,0)|$, using (2.23), is given by

$$\begin{aligned} |Q(x,z) - Q(x,0)| &= |G(x,z+\theta) - G(x,\theta) + D\theta(x)[F(x,\theta(x)) - F(x,z(x)+\theta(x))]| \\ &\leq |G(x,z+\theta) - G(x,\theta)| + |D\theta(x)||F(x,\theta(x)) - F(x,z(x)+\theta(x))| \\ &\leq k(\epsilon)|z| + p_1 k(\epsilon)|z| \\ &\leq (1+p_1)k(\epsilon)|z| = \delta(\epsilon)|z|. \end{aligned} \tag{2.70}$$

Using (2.68), (2.69) and (2.70) for $z \in Y$ and $x \in \mathbb{R}^n$ we have

$$\begin{aligned} |Q(x,z)| &\leq C_1|x|^q + \delta(\epsilon)|z(x)| \\ &\leq C_1|x|^q + K\delta(\epsilon)|x|^q \\ &= (C_1 + K\delta(\epsilon))|x|^q. \end{aligned} \tag{2.71}$$

Using (2.31) and (2.71) for $z \in Y_K$ we have

$$\begin{aligned} |(Sz)(x_0)| &= |\int_{-\infty}^{0} e^{-Bs} Q(x(s,x_0), z(x(s,x_0)))ds| \\ &\leq \int_{-\infty}^{0} |e^{-Bs}||Q(x(s,x_0), z(x(s,x_0)))|ds \\ &\leq \int_{-\infty}^{0} Ce^{\beta s}(C_1 + K\delta(\epsilon))|x(s,x_0)|^q ds \\ &\leq C(C_1 + K\delta(\epsilon)) \int_{-\infty}^{0} e^{\beta s}(M(r))^q |x_0|^q e^{-q\gamma s} ds \\ &= C(C_1 + K\delta(\epsilon))(M(r))^q |x_0|^q \int_{-\infty}^{0} e^{(\beta - q\gamma)s} ds \\ &= C(C_1 + K\delta(\epsilon))(M(r))^q (\beta - q\gamma)^{-1} |x_0|^q, \end{aligned}$$

where $x(t, x_0)$ is the solution of (2.67), $\gamma = r + 2M(r)k(\epsilon)$ and $|x(t, x_0)| \leq M(r)|x_0|e^{-\gamma t}$. $M(r)$ was defined in (2.32).

It follows that

$$|(Sz)(x_0)| \leq C(C_1 + 2Kk(\epsilon))(M(r))^q(\beta - q\gamma)^{-1}|x_0|^q, \tag{2.72}$$

provided ϵ and r are small enough so that $\beta - q\gamma > 0$ [1].

We choose K large enough and ϵ small enough so that $C(C_1 + 2Kk(\epsilon))(M(r))^q(\beta - q\gamma)^{-1} \leq K$. Therefore

$$|(Sz)(x_0)| \leq K|x_0|^q.$$

This shows that $Sz \in Y_K$ and since $Sz + \theta = T(z + \theta) \in X$ we have $|h(x) - \theta(x)| = O(|x|^q)$.

Since $\theta(x) = \phi(x)$ for $|x|$ small we have $|h(x) - \phi(x)| = O(|x|^q)$ as $|x| \to 0$, and this completes the proof of the Theorem. □

2.4 Examples

In this section we give few examples to illustrate how the above theorems may be applied to non linear systems.

Lemma 2.4.1

Let us consider the following equation

$$\dot{y} = ay^\alpha + o(|y|^q), \tag{2.73}$$

where $y \in \mathbb{R},\ \alpha \in \mathbb{N}$ and $q > \alpha$. If $a < 0$ and α is odd then (2.73) is asymptotically stable.

Proof

Let $\dot{y} = y^\alpha(a + o(|1|))$. As $a < 0$ there exists a neighborhood N_0 of the origin such that $a + o(1) < 0$. Let $y_0 \in N_0$ be an initial condition of (2.73).

If $y_0 > 0$ then $\dot{y}(t, y_0) < 0 \Rightarrow y(t, y_0) \to 0$ as $t \to \infty$ and if $y_0 < 0$ then $\dot{y}(t, y_0) > 0 \Rightarrow y(t, y_0) \to 0$ as $t \to \infty$.

Hence in any case, $y(t, y_0) \to 0$ as $t \to \infty$. □

Example 2.4.1

Let us consider the following system:

$$\begin{aligned} \dot{x} &= xy + ax^3 + by^2x \equiv f(x, y) \\ \dot{y} &= -y + cx^2 + dx^2y \equiv g(x, y). \end{aligned} \tag{2.74}$$

We first put (2.74) in an appropriate form for the application of the theorems. The linearization of (2.74) about the origin is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} xy + ax^3 + bxy^2 \\ cx^2 + dx^2 y \end{bmatrix}.$$

Since the eigenalues of the linear part are 0 and -1 then by Theorem 2.1.2, (2.74) has a local centre manifold $y = h(x)$. To approximate the centre manifold we take

$$h(x) = a_1 x^2 + a_2 x^3 + o(x^4).$$

We set

$$(Mh)(x) = 0 \;\Leftrightarrow\; h'(x)[xh(x) + ax^3 + bxh^2(x)] + h(x) - cx^2 - dx^2 h(x) = 0.$$

Hence,

$$(2a_1x+3a_2x^2+\cdots)[a_1x^3+a_2x^4+ax^3+ba_1^2x^5+ba_2^2x^7+2a_1a_2bx^6]+a_1x^2+a_2x^3-cx^2-da_1x^4-da_2x^5 = 0,$$

so that

$$(a_1 - c)x^2 + a_2 x^3 + o(x^4) = 0 \;\Leftrightarrow\; a_1 = c, \text{ and } a_2 = 0.$$

Hence,

$$h(x) = cx^2 + o(x^4).$$

By theorem 2.2.1, the equation which determines the stability of the zero solution of (2.74) is given by

$$\begin{aligned} \dot{u} &= uh(u) + au^3 + buh^2(u) \\ &= u(cu^2 + o(u^4)) + au^3 + bu(cu^2 + o(u^4))^2 \\ &= cu^3 + au^3 + bc^2u^5 + \cdots \\ &= (a+c)u^3 + o(u^5). \end{aligned}$$

Thus, the zero solution of (2.74) is asymptotically stable if $a + c < 0$ and unstable if $a + c > 0$. We cannot say anything about the stability when $a + c = 0$. In this case we have to obtain a better approximation of the centre manifold.

Suppose $a + c = 0$ and let $h(x) = cx^2 + \psi(x)$, where $\psi(x) = o(x^4)$. Therefore

$$(Mh)(x) = 0 \;\Leftrightarrow\; (2cx+\psi'(x))[x\psi(x)+bc^2x^5+2bcx^3\psi(x)+bx\psi^2(x)]+\psi(x)-cdx^4-dx^2\psi(x) = 0.$$

Let $\psi(x) = wx^4$. Then

$$(Mh)(x) = 0 \;\Leftrightarrow\; (2cx+4wx^3+\cdots)[wx^5+bc^2x^5+2bcwx^7+bw^2x^9]+wx^4-cdx^4-dwx^6 = 0,$$

$$(w - cd)x^4 + o(x^6) = 0 \;\Leftrightarrow\; \psi(x) - cdx^4 = o(x^6).$$

Hence, $\psi(x) = cdx^4 + o(x^6)$, and then

$$h(x) = cx^2 + cdx^4 + o(x^6).$$

By theorem 2.2.1, the equation which determines the stability of the zero solution of (2.74) is given by

$$\begin{aligned}\dot{u} &= uh(u) + au^3 + buh^2(u)\\ &= u(cu^2 + cdu^4 + o(u^6)) + au^3 + bu(cu^2 + cdu^4 + o(u^6))^2\\ &= (cd + bc^2)u^5 + o(u^7).\end{aligned}$$

Hence, in the case $a + c = 0$, the zero solution of (2.74) is asymptotically stable if $cd + bc^2 < 0$ and unstable if $cd + bc^2 > 0$. We cannot conclude anything about stability when $cd + bc^2 = 0$. Therefore we should still get a better approximation of the centre manifold. That is, suppose $a + c = cd + bc^2 = 0$ and let $h(x) = cx^2 + cdx^4 + \delta(x)$ where $\delta(x) = o(x^6)$ then $(Mh)(x) = 0$ which is equivalent to

$$\begin{aligned}(2cx + 4cdx^3 + \delta'(x))[x\delta(x) + 2bc^2dx^7 + bc^2d^2x^9 + 2bcdx^5\delta(x) + 2bcx^3\delta(x) + bx\delta^2(x)]\\ + \delta(x) - cd^2x^6 + dx^2\delta(x) = 0.\end{aligned}$$

Let $\delta(x) = qx^6$. Then we have

$$(2cx+4cdx^3+6qx^5+\cdots)[qx^7+2bc^2dx^7+bc^2d^2x^9+2bcdqx^{11}+2bcqx^9+bq^2x^{13}]+qx^6-cd^2x^6+dqx^8 = 0.$$

It follows that

$$(q - cd^2)x^6 + o(x^8) = 0 \;\Leftrightarrow\; \delta(x) - cd^2x^6 = o(x^8).$$

Hence, $\delta(x) = cd^2x^6 + o(x^8)$ and then

$$h(x) = cx^2 + cdx^4 + cd^2x^6 + o(x^8).$$

By theorem 2.2.1, the equation which determines the stability of the zero solution of (2.74) when $a + c = cd + bc^2 = 0$ is given by

$$\begin{aligned}\dot{u} &= uh(u) + au^3 + buh^2(u)\\ &= cd^2u^7 + 2bc^2du^7 + o(u^9).\end{aligned}$$

Since $bc^2 = -cd$ we have

$$\dot{u} = -cd^2u^7 + o(u^9).$$

Hence, the zero solution of (2.74) is asymptotically stable if $c > 0$ and unstable if $c < 0$.

Example 2.4.2

Let us consider the following system:

$$\begin{aligned}\dot{x} &= \epsilon y - x^3 + xy \equiv f(x,y) \\ \dot{y} &= -y + y^2 - x^2 \equiv g(x,y),\end{aligned} \tag{2.75}$$

where ϵ is a real parameter. We study the zero solution of (2.75) for ϵ sufficiently small.

The linearization of (2.75) about the origin is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \epsilon & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ x \end{bmatrix} + \begin{bmatrix} xy - x^3 \\ y^2 - x^2 \end{bmatrix}.$$

The linear part has ϵ and -1 as eigenvalues. Therefore the results from the centre manifold theory cannot directly be applied at this level. System (2.75) can be rewritten in the following equivalent form

$$\begin{aligned}\dot{x} &= \epsilon x - x^3 + xy \equiv f(x,y,\epsilon) \\ \dot{y} &= -y + y^2 - x^2 \equiv g(x,y,\epsilon) \\ \dot{\epsilon} &= 0 \equiv h(x,y,\epsilon).\end{aligned} \tag{2.76}$$

The linearization of (2.76) about the origin is given by

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\epsilon} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ \epsilon \end{bmatrix} + \begin{bmatrix} \epsilon x - x^3 + xy \\ y^2 - x^2 \\ 0 \end{bmatrix}.$$

Since the eigenvalues of the linear part are 0, -1 and 0 then by theorem 2.1.2, (2.76) has a two dimenional centre manifold $y = h(x,\epsilon)$ where x and ϵ are sufficiently small. To approximate the centre manifold we take

$$h(x,\epsilon) = a_1 x^2 + a_2 x\epsilon + a_3\epsilon^2 + o(\nu),$$

where ν denotes a cubic in $x,\ \epsilon$, and we set

$$(Mh)(x,\epsilon) = 0 \;\Leftrightarrow\; h_x(x,\epsilon)[\epsilon x - x^3 + xh(x,\epsilon)] + h_\epsilon(x,\epsilon)\dot{\epsilon} + h(x,\epsilon) - h^2(x,\epsilon) + x^2 = 0.$$

Hence,

$$\begin{aligned}(2a_1x + a_2\epsilon + \cdots)[\epsilon x - x^3 + a_1x^3 + a_2x^2\epsilon + a_3x\epsilon^2] + a_1x^2 + a_2x\epsilon + a_3\epsilon^2 \\ -a_1^2x^4 - a_2^2x^2\epsilon^2 - a_3^2\epsilon^4 - a_1a_2x^3\epsilon - a_1a_3x^2\epsilon^2 - a_2a_3x\epsilon^3 + x^2 = 0,\end{aligned}$$

so that

$$(a_1+1)x^2 + (2a_1 + a_2)x\epsilon + a_3\epsilon^2 + o(c(x,\epsilon)) = 0 \;\Leftrightarrow\; a_1 = -1,\ a_2 = 2,\ a_3 = 0.$$

Hence,

$$h(x,\epsilon) = -x^2 + 2x\epsilon + o(\nu(x,\epsilon)).$$

By Theorem 2.2.1, the equation which determines the stability of small solutions of (2.75) is given by

$$
\begin{aligned}
\dot{u} &= \epsilon u - 2u^3 + uh(u,\epsilon)\\
&= \epsilon u - 2u^3 + u(-u^2 + 2u\epsilon + o(\nu(u,\epsilon)))\\
&= \epsilon u - 3u^3 + o(\nu(u,\epsilon))\\
\dot{\epsilon} &= 0.
\end{aligned}
\tag{2.77}
$$

Neglecting the highest term in the first equation reduces (2.77) to

$$
\dot{u} = \epsilon u - 3u^3 \equiv f(u). \tag{2.78}
$$

If $\epsilon < 0$ then the solution $u = 0$ of (2.78) is asymtotically stable. Then by Theorem 2.2.1, the zero solution of (2.75) is asymptotically stable [1].

If $\epsilon > 0$ then (2.78) has two fixed points namely $u = 0$ and $u = \pm\sqrt{\frac{\epsilon}{3}}$, and $f'(u) = \epsilon - 9u^2$. Then $f'(0) = \epsilon$ and $f'(\pm\sqrt{\frac{\epsilon}{3}}) = -2\epsilon$ show that $u = 0$ is an unstable fixed point and $u = \pm\sqrt{\frac{\epsilon}{3}}$ are stable fixed points. The zero solution of (2.75) is a saddle.

Example 2.4.3 (Singular pertubation-the linear case)

Let us consider the following system:

$$
\begin{aligned}
\dot{x} &= ax + by,\\
\epsilon\dot{y} &= cx + dy.
\end{aligned}
\tag{2.79}
$$

Taking $t = \epsilon\tau$ changes equation (2.79) to

$$
\begin{aligned}
x' &= \epsilon ax + \epsilon by,\\
y' &= cx + dy,
\end{aligned}
\tag{2.80}
$$

where $'$ denotes the derivative with respect to τ, while $\cdot$ denotes the derivative with respect to t.

To apply the results from the centre manifold theory we have to rewrite (2.80) in the following equivalent form:

$$
\begin{aligned}
x' &= \epsilon ax + \epsilon by,\\
y' &= cx + dy,\\
\epsilon' &= 0.
\end{aligned}
\tag{2.81}
$$

Hence, the linearization of (2.81) about the origin gives

$$
L = \begin{bmatrix} 0 & 0 & 0\\ c & d & 0\\ 0 & 0 & 0 \end{bmatrix}.
$$

which is not in canonical form. To have L in canonical form, we change variables as follows: taking $v = y + \frac{c}{d}x \;\Rightarrow\; y = v - \frac{c}{d}x$, (2.81) changes into

$$\begin{aligned} x' &= \epsilon ax + \epsilon bv - \epsilon b\frac{c}{d}x, \\ v' &= dv + \epsilon a\frac{c}{d}x + \epsilon b\frac{c}{d}v - \epsilon b\frac{c^2}{d^2}x, \\ \epsilon' &= 0. \end{aligned} \tag{2.82}$$

Then the linear part of (2.82) about the origin is given by

$$Li = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of Li are 0, d and 0. Hence (2.82) has a two dimensional centre manifold $v = h(x, \epsilon)$. To approximate the centre manifold we take

$$h(x, \epsilon) = a_1x^2 + a_2x\epsilon + a_3\epsilon^2 + o(\nu).$$

We set $(Mh)(x, \epsilon) = 0$ which is equivalent to

$$h_x(x, \epsilon)[\epsilon ax + \epsilon bh(x, \epsilon) - \epsilon b\frac{c}{d}x] + h_\epsilon(x, \epsilon)\epsilon' - dh(x, \epsilon) - \epsilon a\frac{c}{d}x - \epsilon b\frac{c}{d}h(x, \epsilon) + \epsilon b\frac{c^2}{d^2}x = 0.$$

Hence,

$$(2a_1x + a_2\epsilon + \cdots)[a\epsilon x + a_1b\epsilon x^2 + a_2b\epsilon^2x + a_3b\epsilon^3 - b\frac{c}{d}\epsilon x] - a_1dx^2 - a_2d\epsilon x - a_3d\epsilon^2 - a\frac{c}{d}\epsilon x$$

$$-a_1b\frac{c}{d}\epsilon x^2 - a_2b\frac{c}{d}\epsilon^2x - a_3b\frac{c}{d}\epsilon^3 + b\frac{c^2}{d^2}\epsilon x = 0,$$

so that

$$a_1dx^2 + (a_2d + a\frac{c}{d} - b\frac{c^2}{d^2})x\epsilon + a_3d\epsilon^2 + o(|\epsilon|^3 + c(x, \epsilon)) \Leftrightarrow a_1 = 0,\ a_2 = \frac{c}{d^2}\left(b\frac{c}{d} - a\right),\ a_3 = 0.$$

Hence,

$$h(x, \epsilon) = \frac{c}{d^2}\left(b\frac{c}{d} - a\right)x\epsilon + o(|\epsilon|^3 + \nu(x, \epsilon))$$

is the centre subspace for (2.82). The equation that determines the stability of small solutions of (2.82) is given by

$$u' = \epsilon\left(a - b\frac{c}{d}\right)u + o(\epsilon^2).$$

In the original time scale we have

$$\dot{u} = \left(a - b\frac{c}{d}\right)u + o(\epsilon). \tag{2.83}$$

Therefore by the second part of Theorem 2.2.1, for ϵ and $(x(0), v(0))$ sufficiently small there exists solution $u(t)$ of (2.83) such that for $\mu > 0$ we have

$$\begin{aligned} x(t) &= u(t) + o(e^{-\mu t/\epsilon}) \\ v(t) &= h(u(t), \epsilon) + o(e^{-\mu t/\epsilon}). \end{aligned} \tag{2.84}$$

Replacing $v(t)$ and $h(u, \epsilon)$ by their respective values in (2.84) gives

$$\begin{aligned} x(t) &= u(t) + o(e^{-\mu t/\epsilon}) \\ y(t) &= \left(-\frac{c}{d} + o(\epsilon)\right) u(t) + o(e^{-\mu t/\epsilon}) \text{ and } t > 0. \end{aligned} \tag{2.85}$$

We have shown that using the centre manifold theory for ϵ sufficiently small solutions of (2.79) are close to solutions of

$$\begin{cases} y = -\dfrac{c}{d}x \\ \dot{x} = \left(a - b\dfrac{c}{d}\right) x, \end{cases}$$

computed by setting $\epsilon = 0$ in (2.79).

Example 2.4.4

Let us consider the following system:

$$\begin{aligned} \dot{y} &= -y + (y + c)z \\ \epsilon\dot{z} &= y - (y + 1)z, \end{aligned} \tag{2.86}$$

where $\epsilon > 0$ is small and $0 < c < 1$.

Letting $\epsilon = 0$ we have

$$z = \frac{y}{y + 1}. \tag{2.87}$$

Inserting (2.87) into the first equation of (2.86) gives

$$\dot{y} = \frac{-\lambda y}{y + 1}, \tag{2.88}$$

where $\lambda = 1 - c$. Using the centre manifold theory, we show that solutions of (2.86) are indeed close to solutions of (2.87) and (2.88) for ϵ small enough.

Considering a new time scale $t = \epsilon\tau$ in (2.86) gives

$$\begin{aligned} y' &= -\epsilon y + \epsilon(y+c)z \\ z' &= y - (y+1)z \end{aligned} \tag{2.89}$$

where $'$ denotes the derivative with respect to τ while $\cdot$ denotes the derivative with respect to t. The linear part of (2.89) is given by

$$l = \begin{bmatrix} -\epsilon & \epsilon c \\ 1 & -1 \end{bmatrix}.$$

Therefore the centre manifold theory results can not be applied at this level directly. We need some transformations. That is, we write (2.89) in the following equivalent form

$$\begin{aligned} y' &= -\epsilon y + \epsilon(y+c)z \\ z' &= y - (y+1)z \\ \epsilon' &= 0 \end{aligned} \tag{2.90}$$

The linear part of (2.90) is given by

$$li = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which is not in canonical form. Taking $w = y - z$ into (2.90) yields

$$\begin{aligned} y' &= -\lambda\epsilon y + \epsilon y^2 - \epsilon y w - c\epsilon w \\ w' &= -w - \lambda\epsilon y + y^2 - yw + \epsilon y^2 - \epsilon y w - c\epsilon w \\ \epsilon' &= 0 \end{aligned} \tag{2.91}$$

whose linear part

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is indeed in canonical form.

Since the eigenvalues of L are $0, -1$ and 0 we can apply the results from the centre manifold theory. It follows, by Theorem 2.1.2, that (2.91) has a two dimensional centre manifold

$$w = h(y, \epsilon).$$

To approximate the centre manifold we take

$$h(y, \epsilon) = b_1 y^2 + b_2 y\epsilon + b_3\epsilon^2 + o(\nu),$$

and we set

$$(Mh)(y,\epsilon) = 0$$

which is equivalent to

$$h_y(y,\epsilon)[-\lambda\epsilon y + \epsilon y^2 - \epsilon yh - c\epsilon h] + h_\epsilon(y,\epsilon)\epsilon' + h + \lambda\epsilon y - y^2 + yh - \epsilon y^2 + \epsilon yh + c\epsilon h = 0.$$

Hence,

$$(2b_1 y + b_2\epsilon + \cdots)[-\lambda\epsilon y + \epsilon y^2 - b_1\epsilon y^3 - b_2\epsilon^2 y^2 - b_3\epsilon^3 y - b_1 c\epsilon y^2 - b_2 c\epsilon^2 y - b_3 c\epsilon^3]$$

$$+b_1y^2+b_2\epsilon y+b_3\epsilon^2+\lambda\epsilon y-y^2+b_1y^3+b_2\epsilon y^2+b_3\epsilon^2 y-\epsilon y^2+b_1\epsilon y^3+b_2\epsilon^2y^2+b_3\epsilon^3 y+b_1c\epsilon y^2+b_2c\epsilon^2 y+b_3c\epsilon^3 = 0,$$

where we have written h for $h(y,\epsilon)$. Then

$$(b_1-1)y^2 + (b_2-\lambda)y\epsilon + b_3\epsilon^2 + o(c) = 0 \Leftrightarrow b_1 = 1,\ b_2 = -\lambda,\ b_3 = 0.$$

Hence,

$$h(y,\epsilon) = y^2 - \lambda y\epsilon + o(\nu).$$

By Theorem 2.2.1, the equation which determines the stability of small solutions of (2.91) is given by

$$u' = -\lambda\epsilon u + \lambda\epsilon u^2 - \epsilon u^3 + \lambda\epsilon^2 u^2 + \lambda c\epsilon^2 u \tag{2.92}$$

Then in the original time scale we have

$$\dot{u} = -\lambda u + \lambda u^2 + o(|u|^3 + |\epsilon u|). \tag{2.93}$$

Hence, since λ is positive small solutions of (2.86) are asymptotically stable. Therefore by the second part of Theorem 2.2.1, for ϵ and $(y(0), z(0))$ sufficiently small there exists solution $u(t)$ of (2.93) such that for a constant $\gamma > 0$ we have

$$\begin{aligned} y(t) &= u(t) + o(e^{-\gamma t/\epsilon}) \\ w(t) &= h(y(t),\epsilon) + o(e^{-\gamma t/\epsilon}) \end{aligned}. \tag{2.94}$$

Replacing $w(t)$ by $y(t) - z(t)$ system (2.94) becomes

$$\begin{aligned} y(t) &= u(t) + o(e^{-\gamma t/\epsilon}) \\ z(t) &= y(t) - h(y(t),\epsilon) + o(e^{-\gamma t/\epsilon}). \end{aligned} \tag{2.95}$$

Consider the Taylor expansion

$$\frac{1}{1+y} = 1 - y + y^2 + \cdots .$$

We have from (2.88)

$$\dot{y} = -\lambda y + \lambda y^2,$$

which is approximately close to (2.93).

Replacing $h(y(t), \epsilon)$ by its value in the second equation of (2.95) gives

$$z(t) \simeq y(t) - y^2(t),$$

and from (2.87) using the above Taylor expansion we have

$$z \simeq y - y^2.$$

Therefore (2.87) is approximately correct [1].

2.5 Properties of the Centre Manifold

In this section we give few properties of the centre manifolds.

(1)The centre manifold for (2.1) is not unique. We can see this by considering the following system

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y \end{aligned} \tag{2.96}$$

where $(x, y) \in \mathbb{R}^2$. $x = 0$ is an invariant stable manifold for (2.96) and $y = 0$ is an invariant centre manifold for (2.96). But, we can find other centre manifolds for (2.96). That is, eliminating the independent variable t in (2.96) gives

$$\frac{dy}{dx} = \frac{y}{x^3}, \tag{2.97}$$

and solving (2.97) we get for $x \neq 0$

$$y(x) = C \exp(-\frac{1}{2}x^{-2}),$$

where C is any real constant. Hence, (2.96) has one parameter family of centre manifolds of $(x, y) = (0, 0)$ given by

$$W_0^c = \left\{ (x, y) \in \mathbb{R}^2 | y = C \exp(-\frac{1}{2}x^{-2}) \text{ for } x \neq 0,\ y = 0 \text{ for } x = 0 \right\},$$

where $(x, y) = (0, 0)$ is a fixed point of (2.96).

If h_1 and h_2 are two centre manifolds of a given fixed point of (2.1) then by Theorem 2.3.1 we have $|h_1(x) - h_2(x)| = O(|x|^q)$ as $x \to 0$ where $q > 1$ [1].

(2)If f and g are C^k, $k \geq 2$ then h is C^k. If f and g are analytic then in general (2.1) does not have an analytic centre manifold. For example consider the following system

$$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= -y + x^2 \end{aligned} \tag{2.98}$$

The linear part of (2.98) is given by

$$L = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}.$$

By Theorem 2.1.2, (2.98) has a local centre manifold $y = h(x)$. Suppose that h is analytic at $x = 0$. Then

$$h(x) = \sum_{n=2}^{\infty} a_n x^n$$

for small x. To approximate h we set

$$(Mh)(x) = 0 \;\Leftrightarrow\; h'(x)\dot{x} - \dot{y} = 0$$

which is equivalent to

$$(\sum_{n=2}^{\infty} n a_n x^{n-1})(-x^3) + \sum_{n=2}^{\infty} a_n x^n - x^2 = 0,$$

$$-\sum_{n=2}^{\infty} n a_n x^{n+2} + \sum_{n=2}^{\infty} a_n x^n - x^2 = 0.$$

Using one-to-one correspondence we have

$$(a_2 - 1)x^2 + a_3 x^3 + \sum_{n=2}^{\infty} (a_{n+2} - n a_n) x^{n+2} = 0,$$

and this is equivalent to $a_2 = 1$, $a_3 = 0$ and $n a_n = a_{n+2}$ for all n. Since $a_3 = 0$ we have $a_{2n+1} = 0$ for all n and $n a_n = a_{n+2}$ for $n = 2, 4, 6 \cdots$

As we can notice, coefficients a_{2n} of $h(x)$ are increasing as $2^{n-1}(n-1)!$ so that its radius of convergence approaches zero. Therefore it is not analytic. Hence, (2.98) does not have an analytic centre manifold.

3. An application in Mathematical Biology

In this chapter we show how the centre manifold theory can be used to study the dynamic that governs systems in a prey-predator model. We consider a model in which prey and predators are living in two different patches but, prey can move between both patches while predators remain on patch 1. The patch 2 is a refuge for prey [10].

The model we are investigating is due to Poggiale and Auger [10], and it is given by

$$\begin{aligned} \frac{dn_1}{dt} &= R(m_2n_2 - m_1n_1) + n_1(r_1 - ap), \\ \frac{dn_2}{dt} &= R(m_1n_1 - m_2n_2) + n_2r_2, \\ \frac{dp}{dt} &= p(bn_1 - d), \end{aligned} \tag{3.1}$$

where for $i = 1, 2$ n_i denotes prey density in patch i, m_i denote proportions of prey populations leaving patch i per unit time, r_i is the prey population growth rate on patch i, d is the predator population death rate, a is the predator rate on patch 1, p denotes predator density, and bn_1 is the per capita predator growth rate [10].

Expressions on the right hand side of equation (3.1) are divided into two terms. The first one, which contains R, is called the faster term and the second one the slower term.

Taking $R = \dfrac{1}{\epsilon}$ and $t = \epsilon\tau$ in (3.1) gives the following corresponding system in new coordinates with another time scale

$$\begin{aligned} \frac{dn_1}{d\tau} &= m_2n_2 - m_1n_1 + \epsilon n_1(r_1 - ap), \\ \frac{dn_2}{d\tau} &= m_1n_1 - m_2n_2 + \epsilon n_2r_2, \\ \frac{dp}{d\tau} &= \epsilon p(bn_1 - d), \\ \frac{d\epsilon}{d\tau} &= 0. \end{aligned} \tag{3.2}$$

System (3.2) can be written in the following standard form:

$$\begin{aligned} \frac{dx}{d\tau} &= Bx + \epsilon f(x, y), \\ \frac{dy}{d\tau} &= Ay + \epsilon g(x, y), \\ \frac{d\epsilon}{d\tau} &= 0, \end{aligned} \tag{3.3}$$

where $x = (n_1, n_2) \in \mathbb{R}^2$, $y = p \in \mathbb{R}$ and $\epsilon \in \mathbb{R}$.

$$B = \begin{bmatrix} -m_1 & m_2 \\ m_1 & -m_2 \end{bmatrix}, \; A = 0, \; f(x, y) = f(n_1, n_2, p) = \begin{bmatrix} n_1(r_1 - ap) \\ n_2 r_2 \end{bmatrix},$$

$g(x, y) = g(n_1, n_2, p) = p(bn_1 - d)$.

First, we shall put matrix B into canonical form. The eigenvalues of B are $0, -(m_1 + m_2)$, with eigenvectors

$$\begin{bmatrix} \frac{m_2}{m_1} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Using the above eigenbasis, we have

$$\begin{bmatrix} n_1 \\ n_2 \end{bmatrix} = \begin{bmatrix} \frac{m_2}{m_1} & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

with inverse

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{m_1}{m_1 + m_2} \begin{bmatrix} 1 & 1 \\ 1 & -\frac{m_2}{m_1} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \end{bmatrix},$$

which changes (3.2) into

$$\begin{aligned} \begin{bmatrix} u_1' \\ u_2' \\ p' \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -(m_1 + m_2) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ p \end{bmatrix} + \epsilon U \begin{bmatrix} (r_1 - r_2 - ap)u_2 + (r_1 m + r_2 - apm)u_1 \\ (r_1 + r_2 m - ap)u_2 + m(r_1 - r_2 - ap)u_1 \\ pbu_2 + pbmu_1 - pd \end{bmatrix} \\ \epsilon' &= 0, \end{aligned} \tag{3.4}$$

where $U = \dfrac{m_1}{m_1 + m_2}$ and $m = \dfrac{m_2}{m_1}$. By Theorem 2.1.2, (3.4) has a local centre manifold

$$W^c(0) = \left\{(u_1, u_2, p, \epsilon) \in \mathbb{R}^4 : u_2 = h(u_1, p, \epsilon), h(0) = 0, Dh(0) = 0\right\}$$

for u_1, p and ϵ sufficiently small. Next we use Theorem 4 of [1] which states that $h = O(\epsilon)$ uniformly in other variables so that we can write

$$u_2 = h(u_1, p, \epsilon) = \epsilon w_1(u_1, p) + \epsilon^2 w_2(u_1, p) + \dots. \tag{3.5}$$

Inserting this into the second equation of (3.4) we get

$$\frac{du_2}{d\tau} = -\epsilon(m_1 + m_2)w_1(u_1, p) + \epsilon(1 - U)(r_1 - r_2 - ap)u_1 + O(\epsilon^2). \tag{3.6}$$

On the other hand, differentiating (3.5) and using the fact that $\dfrac{dp}{dt}, \dfrac{du_1}{dt}$ are bounded as $\epsilon \to 0$ we get

$$\frac{du_2}{d\tau} = \epsilon \left(\frac{\partial w_1}{\partial p} \frac{dp}{d\tau} + \frac{\partial w_1}{\partial u_1} \frac{du_1}{d\tau} \right) + \ldots = \epsilon^2 \left(\frac{\partial w_1}{\partial p} \frac{dp}{dt} + \frac{\partial w_1}{\partial u_1} \frac{du_1}{dt} \right) + \ldots = O(\epsilon^2). \tag{3.7}$$

Hence, comparing (3.7) and (3.6) we get

$$w_1(u_1, p) = \frac{(1-U)(r_1 - r_2 - ap)}{m_1 + m_2} u_1. \tag{3.8}$$

It follows that

$$\begin{aligned} \frac{dn}{dt} &= n(r - a^1 p) + \epsilon n \frac{U(1-U)(r_1 - r_2 - ap)(r_1 - r_2 - ap)}{m_1 + m_2} + O(\epsilon^2), \\ \frac{dp}{dt} &= p\left(b^1 n - d\right) + \epsilon p b n \frac{U(1-U)(r_1 - r_2 - ap)}{m_1 + m_2} + O(\epsilon^2), \end{aligned} \tag{3.9}$$

where $t = \epsilon\tau,\ a^1 = a(1-U),\ b^1 = b(1-U),\ n = n_1 + n_2,\ nU = u_1$. System (3.9) is the reduction to the centre manifold of system (3.4).

By numerical simulation, it is shown in [10] that the solutions to (3.1) decay to zero as $t \to \infty$. However, the zero order approximation to (3.1), which is the Lotka-Voltera system ($\epsilon = 0$ in (3.9)), has periodic solutions and it cannot provide satisfactory approximation to the original system. This necessitates introduction of ϵ order correction in (3.9) to understand the dynamics of the global variables.

Conclusion

In this dissertation we have studied the centre manifold theory in finite dimension based on the monograph of Carr. The main goal of the work was to acquire a working understanding of the theory and to fill several gaps in the proofs of Carr [1] as well as to correct mistakes.

We have provided proofs of several technical results, which were skipped in [1], such as inequalities $(2.3.5), (2.3.6), (2.3.7), (2.3.8)$ on p. 18, inequalities $(2.4.3), (2.4.4)$ on pp. 20 and 21. We have noticed that the first inequality in $(2.4.10)$ was incorrect and we provided a correct version proving that it is sufficient to carry out the proofs. Also the statement $\|u_1(t) - u_0(t)\|e^{\epsilon t} \to \infty$ as $t \to 0$ in [1, p25] is imprecise. We gave a precise version of it which was shown to be sufficient to complete the proof of the relevant theorem.

Furthermore, we have provided several examples in order to make the use of the centre manifold theory easier to understand.

In the main example of application in Chapter 3, contrary to the examples in paragraph 2.4, we were limited to verify the existence of the centre manifold and to reduce the system to it. This reduced the dimension of the system. However, since the dimension of the centre manifold in this case is bigger than 1, further analytic study was impossible and one has to resort to numerical techniques. This is a subject of further studies which will consists in performing numerical simulations and compare phase portraits of the original system and that of the system reduced to the centre manifold.

Our further work will also consist of studying the theory for infinite dimensional problems and investigate techniques which allow us to study the dynamics of the system reduced to the centre manifold.

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