UNIVERSITY OF KWAZULU-NATAL

EPIDEMIOLOGICAL MODELLING OF FOOT AND MOUTH DISEASE CONTROL IN CATTLE INCORPORATING TIME AND SPATIAL SPREAD OF DISEASE DYNAMICS

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Epidemiological Modelling of Foot and Mouth Disease Control in Cattle Incorporating Time and Spatial Spread of Disease Dynamics

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By

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COLLEGE OF AGRICULTURE, ENGINEERING AND SCIENCE

DECLARATION

This dissertation is submitted in fulfilment of the academic requirements for the degree of Doctor of Philosophy in Applied Mathematics to the School of Mathematics, Statistics and Computer Science; College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Pietermaritzburg.

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Signed .....

Kassahun Mengist Tessema

Date: 28/10/2019

As the candidate's supervisor(s), I/we have approved this dissertation for submission.

Signed .....

Dr. Faraimunashe Chirove

Date: 28/10/2019

Signed .....

Prof. Precious Sibanda

Date: 30/10/2019

DECLARATION 1 - PLAGIARISM

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DEDICATION

This thesis is dedicated to the Almighty God

and

To my darling wife, Firehywot Engida Mogess.

Firehywot thank you for your unconditional love, support, encouragement, and prayers!!!

ABSTRACT

Foot and mouth disease (FMD) is a contagious animal viral infection that can spread rapidly if the disease is not monitored and controlled. Therefore, protecting livestock and controlling foot and mouth disease is important for preventing economic losses. Much of the global burden of economic losses due to foot and mouth disease falls on the worlds poorest countries that mostly depend upon the health of their livestock. In these countries, the availability of FMD also has an impact on the overall herd fertility, modifying the herd structure and affecting the selection of breeds. Modelling the dynamics of FMD using mathematical analysis and simulations can assist to monitor and control the spread of the disease. In this thesis, we develop, study, and analyse models of foot and mouth disease in cattle by incorporate vaccination that does not induce rapid protection, time delays, both time and spatial spread with different control strategies. The results show that even though vaccines may not induce rapid protection the combining of a high rate of vaccination and low loss of vaccine protection rate may be successful in reducing the foot and mouth burden provided critical vaccination thresholds are taken into consideration. The results also show that control strategies play a significant role in moving the animals into protected routes of infection than leaving more animals into the unprotected route of infection. We also capture the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment, reactive culling and the effects of time delay. The results of foot and mouth disease with two-time delays show that the burden of infection decreases significantly when unprotected animals delay maximally their time to show clinical symptoms, and at the same time by increasing the effectiveness of the control strategies. The study also explores the effects of spatial diffusion, quarantine of clinically infected animals and shedding of foot and mouth disease virus into the environment. Analysis of foot and mouth disease control models suggests that implementing of an effective combination of control strategies, limiting the movement of susceptible animals and the shedding of FMDV protects animals from foot and mouth disease burden.

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LIST OF ABBREVIATIONS

FMD	Foot and mouth disease
FMDV	Foot and mouth disease virus
HMD	Hoof-and-mouth disease
HFMD	Hand foot and mouth disease
SAT1	Southern African Territories type 1
SAT2	Southern African Territories type 2
SAT3	Southern African Territories type 3
ODE	Ordinary differential equation
DDE	Delay differential equation
PDE	Partial differential equation

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CHAPTER 1

INTRODUCTION

1.1 Background and Motivation

For many poor and agriculture dependent countries, a major limitation on economic growth, poverty reduction and food security is livestock diseases [1, 2]. One of these diseases which can damage the economy and trade is foot and mouth disease (FMD) [3, 4, 5, 6]. Foot and mouth disease is a major threat to maintaining a healthy domestic and wildlife livestock industry [6]. FMD may quickly and suddenly spread in a country or across national boundaries, when the FMD virus is introduced into disease-free herds, areas or countries [7]. Foot and mouth disease is not fatal in adult animals, and although infected animals recover from foot and mouth disease, the disease leaves them weakened and causes a reduction in milk and meat production and a loss of weight [8]. The disease also causes fever and blister-like lesions followed by erosion on the tongue and lips, teats, and between the animal hooves. In the long run affected animals develop permanent hoof damage and chronic mastitis [9]. Control of foot and mouth disease transmission and spreading of virus from an infected animals into disease-free herds could be achieved through measures such as restricting the movement of infected animals and foot and mouth disease virus (FMDV), preventing movement of infected animal products or by reducing the number of susceptibles using vaccination [6, 10, 11]. Quarantining or culling of infected animals is also another means of controlling foot and mouth disease burden [8]. Antiviral drugs may also be used to protect animals from disease and reduce the risk of transmitting foot and mouth disease [12, 13]. An outbreak of foot-and-mouth disease could lead to severe trade restrictions with severe economic loses on affected countries. The consequences include reduction in productivity and restrictions in movement of livestock products [6, 14]. Under-

standing the transmission characteristics of infectious diseases can lead to better approaches in reducing the transmission of such diseases [15, 16]. This work builds on and is motivated by mathematical models developed by Maidana and Yang [17] and deterministic and stochastic models developed by Keeling [5, 10]. Maidana and Yang [17] used a partial differential equation that incorporated only the diffusion of healthy animals. Their compartmental model took into account susceptible, subclinical infections, clinical infections and recovered subpopulations. By adding additional compartments and necessary modifications to their model we formulate deterministic ordinary, delay and partial differential equations to study the dynamics of the foot and mouth disease. The studies [5, 10, 15, 16, 17] do not include important compartments such as the exposed and vaccinated carrier animal classes and a vaccine that does not induce rapid protection, the rate of vaccination, quarantining clinically infected animals and aerosol atmospheric environment parameters. In this study we incorporate these compartments and parameters in our system of equations.

1.2 Biology of Foot and Mouth Disease

Foot-and-mouth disease (FMD) or hoof-and-mouth disease (HMD) is a fatal viral disease which affects cloven-hoofed animals such as domestic and wild bovids [14, 18, 19, 20, 21, 22]. The disease is transmitted by viral particles in the air that are transported by the wind as well as through direct contact with infected animals [23, 24]. The foot and mouth disease causes a high fever followed by blisters inside the mouth and on the feet for two to six days [19, 20]. Most animals fully recover from foot and mouth disease infection after three weeks [19]. The cause of FMD was shown to be viral by Friedrich Loeffler in 1897 [25]. The virus that causes the foot and mouth disease is an *Aphthovirus* which is a member of the *Picornaviridae* family [6]. The word *Aphthovirus* comes from the Greek word *Aphtha* which means vesicles in the mouth [25, 26, 27, 28, 29]. According to immunological classification, there are seven distinct types

of FMD viruses, $A, O, C, SAT1, SAT2, SAT3$ and $Asia1$ [30, 31, 32, 33]. Research showed that the FMD virus can survive in infectious form up to 12 years in the soil and at least for a year in a cell culture medium [6]. FMD is commonly found in parts of Asia, Africa, the Middle East, and South America and serotype O virus is present in all continents where FMD is reported [30]. Of the seven serotypes, six occur in Africa ($SAT1, SAT2, SAT3, A, O$ and C) [31, 34, 35], four in Asia $(A, C, O, Asia1)$, and only three in South America (A, O, C). Serotypes $SAT1$ and $SAT2$ have been occasionally reported in the Middle East [35]. Historically, 48% of FMD outbreaks in domestic animals in Southern Africa are caused by serotype $SAT2$ [31, 34]. Immunization from the infection of one serotype does not guarantee immunity against other serotypes [32, 35]. In the American hemisphere, outbreaks of FMD epidemic were recorded in 1870 at the same time in the United States of America (USA), Argentina and Uruguay, and later at some years in Paraguay [35].

1.3 Mathematical Models of Foot and Mouth Disease in Cattle Population

In this section, we present the structure of specific models that have been utilized for foot and mouth disease research. Studies of foot and mouth disease using mathematical models have become popular [11, 36]. In general, it is important to study infectious diseases using mathematical models and in particular, it is also important to study foot and mouth disease. A lot of epidemic model structures have been developed over the years to address a variety of questions on foot and mouth disease including the use of deterministic mathematical models [10, 11, 37, 38, 39, 40], continuous-time Markov chain stochastic model, an explicit stochastic simulation model [40], ecological models [41, 42], age-structured difference equations [43, 44, 45], matrix-based models [46, 47] and hybrid mathematical techniques [48, 49]. Although vaccination of foot and mouth disease has a positive impact to minimize the disease

risk and reducing cumulative FMD cases when an outbreak occurs but vaccination alone may not be sufficient to eradicate the foot and mouth disease [50, 51]. The impact of vaccination and culling on controlling foot and mouth disease can be evaluated using mathematical modelling [14]. Therefore, the transmission dynamics of FMD modeling is an important and interesting topic for a lot of researchers [5, 10, 14, 52]. In a country, normally free of FMD, previous mathematical models of foot and mouth disease from farm-to-farm transmission have explored the impacts of control measures such as culling and vaccination during a single outbreak [53]. A few mathematical models for foot and mouth disease that incorporate time and spatial dynamics have however not considered the significant effects of the combined control strategies [11, 17, 36]. Guiding to develop the control policies in UK during 2001 using mathematical models played an important role in guiding the foot and mouth epidemic [39]. Mathematical models that incorporate ecological interactions are an essential tool in predicting the behavior of complex model systems. Several studies explore predator-prey interactions, resource selection, population growth, and dynamics of disease transmission models [54]. They provide an effective way to test new management and control strategies without resorting to empirical testing that is often costly, time-consuming, and impractical [54]. Foot and mouth disease mathematical model has been studied using by construction the animal population into the susceptible, latent, infectious, quarantine susceptible and quarantine latent classes [40]. Recent construction of mathematical model used a system of reaction-diffusion equations and determined the wave speed of foot and mouth disease as a function of the diffusion coefficient [17]. We developed the mathematical models that incorporate vaccination that does not induce rapid protection, time and spatial diffusion effects to capture the real-life epidemics with the effects of diffusion of FMDV and aerosol atmospheric environment using reaction-diffusion equations.

1.4 Overview of Ordinary Differential Equation Models

In this section, we present an overview of models based on ordinary differential equations. A differential equation that involves functions of only one independent variable and one or more of their derivatives with respect to that variable is an ordinary differential equation (ODE) [55, 56]. Formulation of models as ordinary differential equations are the most prevalent in dynamical systems [55, 56]. A number of studies use ordinary differential equation models to study the foot and mouth disease dynamics [5, 10, 50, 52]. A limitation or shortcoming of the ODE models is that they are not able to capture the spatial spread of most infectious diseases. Most foot and mouth disease researchers have used ordinary differential equations that incorporate control strategies such as vaccination and culling, but do not include the use of a vaccine that does not induce rapid protection [57]. This study includes the effects of the vaccine that does not induce rapid protection and different control strategies using systems of ordinary differential equations. Mushayabasa *et al.* [14] formulated an ODE model to study the impact of vaccination and culling on controlling of foot and mouth disease. Using the system of ordinary differential equation Mushayabasa *et al.* showed that vaccination alone may not be sufficient to eradicate the foot and mouth disease [50]. The dynamics of foot and mouth disease in a contaminated environment using the ordinary differential equations was studied by Mugabi *et al.* [58]. There study showed that the best option of a control strategy of FMD that involves optimal use of vaccination has to be implemented with combination of vaccination and environmental decontamination. Hence, we study the foot and mouth disease by incorporating different control strategies together, such as vaccinating before (prophylactic vaccination) and vaccinating after (reactive vaccination) the endemic of the disease, treating using different drugs and quarantining clinically infected animals.

1.5 Overview of Delay Differential Equation Models

In this section, we provide an overview of the delay differential equation models. A delay differential equation (DDE) involves some ordinary derivatives with of time delay terms [59, 60]. The unique solution for the ordinary differential system is determined by an initial point at an initial time t_0. For a delay differential system, one requires information on the entire interval $[t_0 - \tau, t_0]$, where, t_0 and τ represent as initial time and delay time respectively [60]. There are many systems in biology, medicine, chemistry, physics, engineering, economics, whose analysis involves time delays. Therefore, ignoring the time delay is to ignore reality [61]. A delayed epidemic model with stage-structure and impulses was studied by Zhang *et al.* [62] and study suggested that the time lag plays a very important role in the stability of system. Several models with time delays have been used that incorporate the exposed class and/or infective classes by Busenberg and Cooke [63], Hethcote *et al.* [64], and Gao [65]. A dynamical model of hand foot and mouth disease (HFMD) become infectious studied by considering varying total population size, saturation incidence rate and discrete-time delay [66]. The effects of time delay on predator-prey systems were studied using systems of delay differential equations on the dynamics of the generalized Gausetype predator-prey models [67]. Mathematical models also used to formulate the growth of tumors incorporating time delays [68]. The impacts of delayed detection of foot-and-mouth disease of epidemic and economic was studied by Carpenter *et al.* [69]. There findings underlined that the critical importance of early effective detection in place before an introduction of FMDV to avoid dramatic losses to both livestock and the economy. A number of delay differential equations have been used to model other infections with one time delay [61, 70, 71] but few studies use time delay to study foot and mouth disease. Hence, we formulate, develop and study the foot and mouth disease control by incorporating the time delays of susceptible and clinically infected animals and different control strategies.

1.6 Overview of Partial Differential Equation Models

In this section, we present an overview of partial differential equation models. A differential equation that contains multi-variable functions and their derivatives is a partial differential equation (PDE). PDEs are formulations of problems involving functions of several variables [72]. Partial differential equations have been used in the literature to model the dynamics of epidemics. The solutions of the partial differential equations are essential in understanding epidemiological models [73, 74]. Partial differential equations have also been used to model different ecological phenomenon such as ecological invasion, critical patch size, dispersal of mediated coexistence, and diffusion that drive spatial patterning [74, 75, 76]. A reaction-diffusion epidemic model is formulated in terms of a spatial dispersal of species depending on both time and space [77, 78]. A lot of researchers use mathematical modelling by implementing partial differential equations to study the control strategies of foot and mouth disease burden [17, 51]. However, they used control strategies of vaccination with culling or with the quarantine of infected animals but in our studies we include combined control strategies such as prophylactic and reactive vaccination, quarantining of clinically infected animals by including the vaccine that does not induce rapid protection and aerosol atmospheric environment effects. Maidana *et al.* [17] incorporated the spatial spread of FMD without any control strategies. The control strategies for foot and mouth disease are of paramount importance to animal health and for this reason, it is imperative to consider the effects of control strategies available when modelling the dynamics of FMD. In addition, the previous studies did not capture the effects of FMDV infection spread through aerosols. Aerosol transmission of FMDV has been found to be an important route for FMD transmission [12] and its inclusion in the models has a potential to alter the prediction of the infection progress. In areas where there is an outbreak of FMD, quarantining of infected animals is also an effective control measure to reduce contact between infected and healthy animals [52]. However, because quarantine animals can still shed the virus into the atmosphere through aerosols, this group of animals, together with the subclinically infected

animals, clinically infected animals and vaccinated carrier animals can contribute towards the transmission of FMDV to healthy animals [12]. This, therefore, suggests that studies modelling FMD should capture the effects of aerosol transmission by including the concentration of FMDV in the atmospheric environment.

Using the studies [12, 17, 79, 80, 81] as building blocks, we develop a model with spatial spread of FMD incorporating vaccination that does not induce rapid protection, quarantine and shedding the FMDV into the atmospheric environment through aerosols.

1.7 Statement of the Problem

The management and control of the foot and mouth disease is difficult because it can be transmitted through direct and indirect contact with infected animals, as well as by the atmospheric aerosol dispersal of FMDV in the environment. As a consequence of these and other complexities surrounding the disease dynamics, the disease is a serious concern in farming communities [9]. Studies exist investigate the effects of vaccination but the current challenge is most of these vaccines do not induce rapid protection, which means the vaccine that does not protect the disease as soon as the vaccination is administrated. Other studies have investigated a combination of control strategies, such as vaccination and culling, vaccination and quarantining, and vaccination and drug treatment [14]. Vaccination as a strategy involves the use of prophylactic and reactive vaccines which can also be use a combined strategy. Treatment of FMD infected animals is generally not available but there is evidence of prophylactic treatment that can also be used in combination with other control strategies [12, 13]. Most of the studies on foot and mouth epidemics have focused mainly on time-dependence, but the dynamics of FDM are also been influenced by spatial expansion. Therefore, focusing on the time-variable only presents a limited view of the predictive power and projections of mathematical modeling. In addition,

the transmission of foot and mouth disease is also promoted through contact and aerosol sprays in the environment. Hence, the environmental contributions in the transmission of FMD may complicate to understand the dynamics of FMD. Various mathematical models can be used to capture the change in status of cattle over time and space. These include the use of systems of ordinary, delay and partial differential equations to capture the dynamics of FMD at a specific point in time, a specific interval and space. These mathematical models have increasing become available and their use can go a long way in exposing potential FMD threats and pointing to potential measures. This study seeks to formulate mathematical models that incorporate vaccination that does not induce rapid protection and control strategies, such as vaccination, culling, quarantine and treatment. Various transmission routes inclusive of the effects of aerosol transmission of FMBV are investigated.

1.8 Objectives of the Study

The aim of this study is to formulate mathematical epidemiological models using ordinary, delay and partial differential equation to understand the dynamics of foot and mouth disease and to use the data to inform preventive and control decisions.

The specific objectives of this study are:

(i) To formulate dynamical models of the foot-and-mouth disease in animals and to analyze the model capturing the effects of control parameters and loss of vaccination protection.

(ii) To evaluate the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment, reactive culling and the effects of time delay on foot and mouth disease. and

(iii) To highlight the impact of vaccination, quarantining of clinically infected animals and en-

vironmental effects on the transmission of disease viruses using a reaction-diffusion model that captures the spread of foot and mouth disease in both space and time.

1.9 Significance of the Study

Various studies have been conducted with the aim of improving the surveillance, control of foot and mouth disease and providing more information on the complex biology of the virus. This study uses models that incorporate time and spatial effects and evaluates the use of combined control strategies in controlling the FMD dynamics. The study explores the role of vaccination, using a vaccine that does not induce rapid protection, vaccinating carriers, quarantining animals and the environmental effects in the aerosol transmission of FMDV. The findings in this study will be useful to a variety of stakeholders, such as livestock farmers and policy-makers when developing intervention strategies to prevent, control and eliminate foot and mouth disease. The findings will highlight specific areas that need attention and effort if the battle against foot and mouth disease is to be won. For example, the findings provide a guide on whether a combination of control strategies are effective in reducing or eliminating the foot and mouth disease.

The use of reaction-diffusion epidemiological models that incorporate the use of combined control strategies in FMD studies has not been fully explored in the literature. This study will potentially serve as a reference for future studies that apply the reaction-diffusion equation and combined control strategies models in FMD research. The findings and models provide valuable information on reaction-diffusion models for many infections that spread in both time and space in biology and ecology.

1.10 Outline of the Thesis

This thesis contains five chapters. Chapter 1 provided the background and motivation information on this research. We presented the biology of foot and mouth disease, objectives, statement of the problem and purpose or significance of the study. We presented an overview of studies that used mathematical models to study the dynamics of the foot and mouth disease in cattle populations. This included an overview of ordinary, delay and partial differential equation models used in this study.

In Chapter 2, we formulate and analyze a deterministic model for the transmission dynamics of foot and mouth disease in a cattle population. We study the impact of vaccination that does not induce rapid protection from the foot and mouth disease.

Chapter 3 focuses on the development and analysis of control of foot and mouth disease with two time delays. We incorporate control strategies based on prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling of animals when there is a time delay.

In Chapter 4, we extend the model formulated in Chapter 2 by using spatial diffusion partial differential equations. We use a reaction-diffusion model with environmental effects to investigate the effects of vaccination, quarantining of clinically infected animals, shedding of foot and mouth disease virus into the environment and rates of movement of animals and virus. We generalize the formula for computing the basic reproduction number. Numerical simulations are performed using a reaction-diffusion model incorporating environmental effects to capture different rates.

Lastly, in Chapter 5, we present a general conclusion to the thesis. Potential areas for further research are highlighted.

1.11 List of Publications

This thesis is built around the following papers which constitute Chapters 2, 3, and 4.

Chapter 2

- K. M Tessema, F Chirove, and P Sibanda, (2019), Modelling the effects of vaccination that does not induce rapid protection on foot and mouth disease, *Indian Journal of Pure and Applied Mathematics*, The submission Id is: IJPA-D-20-00143, (Under review).

Chapter 3

- K. M Tessema, F Chirove, and P Sibanda, Modelling control of foot and mouth disease with two time delays, *International Journal of Biomathematics* **12**, **04**, *1930001* (*2019*), https://doi.org/10.1142/S179352451930001X.

Chapter 4

- K. M Tessema, F Chirove, and P Sibanda, (2019), Analysis of foot and mouth disease control using a reaction-diffusion model incorporating environmental effects, The submission Id is: BMAB-D-20-00019, *Bulletin of Mathematical Biology*, (Under review).

CHAPTER 2

MODELLING THE EFFECTS OF VACCINATION THAT DOES NOT INDUCE RAPID PROTECTION ON FOOT AND MOUTH DISEASE

In this Chapter we present a mathematical model of foot and mouth disease incorporating vaccination that does not induce rapid protection due to prolonged susceptibility of vaccinated animals before the induction of the vaccine induces an adaptive immune response.

Modelling the effects of vaccination that does not induce rapid protection on foot and mouth disease.

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Abstract

We developed a mathematical model incorporating vaccination that does not induce rapid protection due to prolonged susceptibility of vaccinated animals before the introduction of the vaccine that induces an adaptive immune response. The vaccinated animals can become long-term carriers following contact with the foot and mouth disease virus. We investigated the effects of the vaccination coverage and loss of vaccination protection on animal population dynamics. Important thresholds were derived and used to determine the progression levels of the foot and mouth disease. Findings suggest that a high vaccination rate and low loss of protection from the vaccine have better benefits in decreasing of foot and mouth disease burden. On the other hand, low protection from the vaccine and low loss of vaccination rate leads to an increase in the foot and mouth disease burden. The increase in the long-term vaccinated carrier population was associated with the increase in the foot and mouth burden whilst the decrease in the vaccinated long-term carriers led to a decrease in the foot and mouth burden.

Keywords: Foot and Mouth disease; Vaccination; Carriers; Stability.

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1 Introduction:

Foot and mouth disease (FMD) is a viral disease which affects cloven-hoofed animals such as domestic and wild bovids in Asia, Africa, Europe and North America [1, 2, 3]. The disease is transmitted by viral particles in the air which are transported by the wind, animal bodies as well as direct contact through primarily infected epithelial cells [4, 5].

Foot and mouth disease is not fatal in adult animals but it causes a decrease in milk and meat products as well as a reduction in the weight of the animals. Affected animals ultimately develop permanent hoof damage and chronic mastitis. Severe trade restrictions can be imposed which have economic impacts on affected countries. FMD is a major challenge for industrialized countries and with agricultural export-dependent economies [3]. Understanding the transmission dynamics of infectious diseases in communities, regions, and countries can lead to better approaches to reducing the transmission of infectious diseases [6, 7].

Some attempts of modelling foot and mouth disease transmission which focused on airborne transmission between farms have been done [6, 8, 9]. Foot and mouth disease airborne transmission models were developed after the outbreak of 1966 FMD epidemic in the UK when the airborne transmission considered the major driver of FMD spread. Hagenaars et al. [6] reviewed various modelling approaches and discussed parameter values used and how estimates for these parameters were obtained. The study considered a system of ordinary differential equation (ODE) models by taking into account three subpopulations; the susceptible, infected and recovery animals.

Keeling [10] reviewed the studies on foot and mouth disease focusing on the three foot-and-mouth disease models used in 2001 for UK outbreak [10], namely the InterSpread model [11], the Cambridge-Edinburgh model [12] and the imperial model [13]. To capture the between farms transmission, the InterSpread model applied a stochastic model to calculate the probability of spread depending on number and type of livestock and the distance between farms. The InterSpread model can simulate a number of models with different complexities. The Cambridge-Edinburgh model is a spatial model which used data from all the farms in UK and their livestock recorded at the last census and the starting criteria are the same as InterSpread model but Cambridge-Edinburgh had more simple and transparent mechanisms and had fewer parameters and easier parameterizations. The Imperial model was based on the strategy of SIR differential equations incorporating the number of farms and pairs of locally connected farms.

A study by Ortiz et al. [14] presented a model on foot and mouth disease by dividing the animal population into the susceptible, latent, infectious, quarantine susceptible and quarantine latent classes of the system. They used three different approaches to model the FMD spread, which are a deterministic model, a continuous-time Markov chain stochastic model, and an explicit stochastic simulation model. The study showed that for effective control of foot and mouth disease, screening and quarantine of infected animals before they affect the susceptible group of animals was essential.

Maidana and Yang proposed a PDE model which divided the population into susceptibles,

subclinical infectives, clinical infectives and recovered subpopulations [15]. Their results showed that direct transmission is an important route in the disease spreading due to the high rate of direct contact by animals.

Mushayabasa et al. [3] studied the impact of vaccination and culling on controlling foot and mouth disease by splitting the population into susceptible, vaccinated, latently infected and infectious animals. Their result showed for essential control of FMD both vaccination and culling should be implemented.

Ringa and Bauch [16] developed a model by dividing the animal population into susceptible, exposed, infectious, recovered and vaccinated. Their objective was to investigate the impacts of prophylactic and ring vaccination, vaccine waning, loss of natural immunity and disease re-introduction from an external source. The study showed that the dependence of disease control effectiveness on loss of natural immunity and vaccine waning are enough to model any mathematical model of FMD transmission and control in endemic countries.

Current vaccination has limitations on foot and mouth disease control programs and the studies revealed similar challenges affect foot and mouth disease overall efficiencies [3, 16]. The limitation are driven by factors such as establishment of high containment facilities, contamination of the viral protein during preparation of the vaccine and susceptibility of the vaccinated animals prior to the induction of the adaptive immune. As a result, the vaccine will not induce rapid protection and some vaccinated animals become long term carriers [17].

In this study we shall investigate the effects of vaccination that does not induce rapid protection on the progression of foot and mouth disease. We design a dynamical model which seeks to reveal the extent of the damage caused by such vaccines. In the next section we formulate the model and analyze it in section 3. In section 4, we present parameter estimation and numerical simulations. Discussion and conclusion will be presented in section 5.

2 Model Formulation

We present a model by subdividing the total population into susceptible animals $S(t)$, exposed animals $E(t)$, subclinical infectious animals $I_s(t)$, clinical infectious animals $I_c(t)$, recovered animals $R(t)$, vaccinated animals $V_v(t)$ and vaccinated carrier animals $V_{ca}(t)$. Susceptible animals are animals which are free of foot and mouth disease virus (FMDV), exposed animals are animals which are not yet infectious but have the virus. Infectious animals are divided into two subgroups namely the subclinical and clinical infective animals. A subclinical animal is an infectious animal that is nearly or completely asymptomatic with no signs or symptoms of infection [18]. The clinical infective animal is an infectious animal with clinically diagnosed signs or symptoms. The vaccinated animals are animals which are protected from the disease by a vaccine and vaccinated carrier animals are animals which are vaccinated but get infected because the vaccine does not induce rapid and complete protection. The removed animals are either recovered or immune to the infection. The immunity may

wane with time and the recovered animals become susceptible again [3, 19, 20].

We use the following assumptions for our model formulation: a susceptible animal contract FMD infection and become exposed then an exposed animal become subclinical. A subclinical animal is capable of transmitting the infection. The subclinical animal later shows signs and symptoms of FMD and become highly infectious. The clinically infected animal develops temporary immunity and moves to the recovered class and the recovered animal by losing immunity becomes susceptible again [21]. We split the vaccinated animals into vaccinated and vaccinated carrier animals. The vaccinated animals' progress to vaccinated carrier animals which then progress to the recovered animals. The vaccinated animals' progression to the vaccinated carrier class is due to the fact that a vaccine that does not induce rapid protection [17].

The parameter ρ is the rate of vaccination and $0 \leq \rho \leq 1$. We assume that the recruitment of animals is given by bN where b is the rate of recruitment of susceptible animals through birth and immigration and N is the total population. The force of infection for susceptible animals is given by $\beta(I_s + \eta_1 I_c + \eta_2 V_{ca})/N$. The new infections arise by successive contacts between susceptible and infectious animals, where the contact occurs at a rate of β. $\eta_1 > 1$ is an amplification to show that I_c is more infectious than I_s. $\eta_2 < 1$ is an amplification to show that V_{ca} is less infectious compared to I_s and I_c. The exposed animals E progress from E to I_s at the rate ϵ, I_s, progress to I_c at rate of α_1 and I_c develop immunity at the rate of α_2. The recovered animals R progress from R to S at the rate of ω. The vaccinated V_v progress to V_{ca} by the force of infection, $\beta\,\phi\,(I_s + \eta_1\,I_c + \eta_2\,V_{ca})/N$ and V_{ca} progress to R at the rate of α_3. ϕ is the rate of protection loss due to the vaccination and $0 \leq \phi \leq 1$. Animals die due to natural mortality at a rate of μ and γ is the density-dependent death rate. The flow diagram for the model is presented in Figure 1.

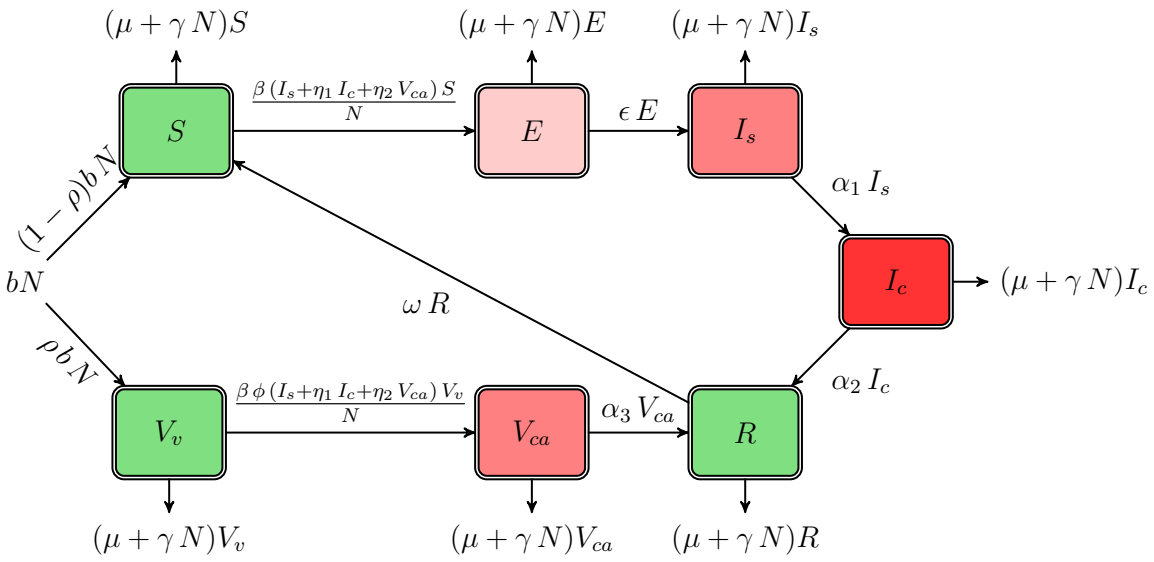

Figure 1: Flow diagram for the model

The resultant FMD model is governed by the system of ordinary differential equations,

$$\frac{dS}{dt} = (1-\rho)\, b\, N - \left(\frac{\beta\, (I_s + \eta_1\, I_c + \eta_2\, V_{ca})}{N} + \mu + \gamma N\right) S + \omega\, R, \tag{1}$$

$$\frac{dE}{dt} = \left(\frac{\beta\, (I_s + \eta_1\, I_c + \eta_2\, V_{ca})}{N}\right) S - (\epsilon + \mu + \gamma N)\, E, \tag{2}$$

$$\frac{dI_s}{dt} = \epsilon\, E - (\alpha_1 + \mu + \gamma N)\, I_s, \tag{3}$$

$$\frac{dI_c}{dt} = \alpha_1\, I_s - (\alpha_2 + \mu + \gamma N)\, I_c, \tag{4}$$

$$\frac{dR}{dt} = \alpha_2\, I_c - (\omega + \mu + \gamma N)\, R + \alpha_3\, V_{ca}, \tag{5}$$

$$\frac{dV_v}{dt} = \rho\, b\, N - \left(\frac{\beta\, \phi\, (I_s + \eta_1\, I_c + \eta_2\, V_{ca})}{N} + \mu + \gamma N\right) V_v, \tag{6}$$

$$\frac{dV_{ca}}{dt} = \left(\frac{\beta\, \phi\, (I_s + \eta_1\, I_c + \eta_2\, V_{ca})}{N}\right) V_v - (\alpha_3 + \mu + \gamma N)\, V_{ca}, \tag{7}$$

subject to the initial conditions

$$\begin{aligned} &S(0) = S_0 \geq 0 \;\; E(0) = E_0 \geq 0, \;\; I_s(0) = I_{s\,0} \geq 0, \;\; I_c(0) = I_{c\,0} \geq 0, \\ &R(0) = R_0 \geq 0, \;\; V_v(0) = V_{v0} \geq 0, \;\; V_c(0) = V_{ca0} \geq 0. \end{aligned} \tag{8}$$

Our model exhibits logistic growth, which is realistic in capturing density-dependent regulatory mechanisms and a kind of compensating the effect of overcrowding of the population dynamics [22]. Adding equations of the model (1)-(7), leads to a logistic differential equation.

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \text{ where } \; K = \frac{r}{\gamma}, \quad r = b - \mu,$$

whose solution is

$$N\,(t) = \frac{K}{1 + \psi \mathrm{e}^{-rt}}, \text{ where}, \psi = \left(\frac{K}{N\,(0)} - 1\right) \text{and } N\,(0) \leq K. \tag{9}$$

K is the carrying capacity of animals, r is the linear growth rate. If $r < 0$, then N approaches 0 as $t \to \infty$. If $r = 0$, then the population becomes constant as $t \to \infty$ but when the parameter $r > 0$ then

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{K}{1 + \psi \mathrm{e}^{-rt}} = K.$$

we henceforth assume that $r > 0$.

3 Model analysis

3.1 Feasible region

For the model to be biologically meaningful we require that, all the variables and parameters be non negative values.

Theorem 1. *Let the system of equations (1) - (7) have initial conditions (8). Then the region* Γ *defined by (10) is positively invariant and attracting.*

$$\Gamma = \{(S(t), E(t), I_s(t), I_c(s), R(t), V_v(t), V_{ca}(t)) \in \Re_+^7 \mid N(t) \leq K\}, \tag{10}$$

Proof. Assume for $t > 0$, $N(0) \geq 0$, $S(0) \geq 0$, $E(0) \geq 0$, $I_s(0) \geq 0$, $I_c(0) \geq 0$ $R(0) \geq 0$, $V_v(0) \geq 0$ and $V_{ca}(0) \geq 0$. From equation (7) we get

$$\frac{d}{dt}V_{ca}\,(t) \;>\; -(\alpha_3 + \mu + \gamma N\,(s))V_{ca}\,(t)\,, \tag{11}$$

Integrating the expression, we get

$$V_{ca}(t) \;\geq\; V_{ca}(0)e^{-\left((\alpha_3+\mu\,)\,t + \int_0^t \gamma N(s)ds\right)} \;\geq\; 0. \tag{12}$$

Hence, $V_{ca} \;\geq\; 0$, this implies that at any finite time, V_{ca} is non-negative. A similar analysis holds for equation (1) - (6) where,

$$\begin{aligned}
R\,(t) &\geq R(0)e^{-\left((\omega+\mu\,)\,t + \int_0^t \gamma N(s)ds\right)} \;\geq\; 0,\\
I_s(t) &\geq I_s(0)e^{-\left((\alpha_1+\mu,\,)\,t + \int_0^t \gamma N\,(s)\,ds)\right)} \;\geq\; 0,\\
I_c(t) &\geq I_c(0)e^{-\left((\alpha_2+\mu\,)\,t + \int_0^t \gamma N\,(s)\,ds)\right)} \;\geq\; 0,\\
S\,(t) &\geq S(0)\,\mathrm{e}^{-\left(\int_0^t \left(\frac{\beta\,(I_s\,(s) + \eta_1\,I_c\,(s) + \eta_2\,V_{ca}\,(s))}{N\,(s)} + \gamma N\,(s)\right)ds + \mu\,t\right)} \;\geq\; 0,\\
E\,(t) &\geq E(0)\mathrm{e}^{-\left((\epsilon+\mu\,)\,t + \int_0^t \gamma N\,(s)\,ds\right)} \;\geq\; 0,\\
V_v\,(t) &\geq V_v(0)\,\mathrm{e}^{-\left(\int_0^t \left(\frac{\beta\,\phi\,(I_s\,(s) + \eta_1\,I_c(s) + \eta_2\,V_{ca}(s))}{N(s)} + \gamma N(s)\right)ds + \mu\,t\right)} \;\geq\; 0.
\end{aligned}$$

Therefore, the solutions of the model with non-negative initial conditions remains non-negative for all all $0 \leq t < \infty$. Since $0 \leq (S(t),\, E(t),\, I_s(t),\, I_c(t),\, R(t),\, V_v(t),\, V_{ca}(t)) \leq ((1-\rho)K, 0, 0, 0, 0, \rho K, 0)$, all variables are bounded in $[0, K]$. This shows that for initial conditions (8) the region Γ is positively invariant and attracting. □

3.2 The control reproduction ratio for the model

The control reproduction ratio is calculated using the next-generation matrix method [23, 24]. We take only the exposed and the infected classes of the model to calculate the control

reproduction ratio. At the disease free equilibrium point, $E = I_s = I_c = R = V_{ca} = 0$, $S = (1-\rho)\,K$ and $V_v = \rho K$. The control reproduction number is given by

$$\mathcal{R}_c = \frac{\beta\,\epsilon\,(\alpha_2 + b + \eta_1\alpha_1)\,(1-\rho)}{(\alpha_1 + b)\,(\epsilon + b)\,(\alpha_2 + b)} + \frac{\beta\,\phi\,\eta_2\rho}{\alpha_3 + b},$$

where $b = (\mu + \gamma K)$.

The basic reproduction number ([25]) is obtained when there is no vaccination i.e when $\rho = 0$,

$$\mathcal{R}_0 = \frac{\beta\,\epsilon\,(\alpha_2 + b + \eta_1\alpha_1)}{(\alpha_1 + b)\,(\epsilon + b)\,(\alpha_2 + b)}.$$

The control reproduction ratio in terms of basic reproduction number is

$$\mathcal{R}_c = \mathcal{R}_0\,(1 - \rho\,U)\,, \tag{13}$$

where U is the impact of vaccination and $U = 1 - \frac{\beta\,\phi\,\eta_2}{\mathcal{R}_0\,(\alpha_3+b)}$. The critical vaccination coverage $\rho_c = \frac{1}{U}(1 - \frac{1}{\mathcal{R}_0})$. The critical vaccinations coverage is a function of the basic reproductive rate $\mathcal{R}_0$ and the vaccine impact U. The critical vaccinations coverage (ρ_c) is greater or equals to the vaccination rate ρ (see [26]).

3.3 Equilibrium points and stability analysis

The equilibria of this model are calculated by setting the right-hand side of equations (1) - (7) to zero. The disease-free equilibrium point of the system of equation is given by

$$E_0 = (\hat{S}, \hat{E}, \hat{I}_s, \hat{I}_c, \hat{R}) = ((1-\rho)\,K, 0, 0, 0, 0, \rho K, 0)\,.$$

The force of infection at the equilibrium point is

$$\lambda_e^* = \frac{\beta\,(I_s + \eta_1\,I_c + \eta_2\,V_{ca})}{K}. \tag{14}$$

The endemic equilibrium of the system is given by

$$E_1 = (S^*, E^*, I_s^*, I_c^*, R^*, V_v^*, V_{ca}^*)$$

$$
\begin{aligned}
S^* &= \frac{K\,b\chi\,((1-\rho)\,\nu\,(b+\phi\,\lambda_e^*)+\lambda_e^*\,\omega\,\alpha_3\phi\,\rho)}{(\alpha_3+b)\,(b+\phi\,\lambda_e^*)\,(b\,\chi\,(b+\omega)+\Pi\,\lambda_e^*)},\\
E^* &= \frac{K\,b\,\lambda_e^*\,(b+\alpha_1)\,(b+\alpha_2)\,((1-\rho)\,\nu\,(b+\phi\,\lambda_e^*)+\lambda_e^*\omega\,\alpha_3\phi\,(1-\rho)\,\rho)}{(b+\alpha_3)\,(b+\phi\,\lambda_e^*)\,(b\,\chi\,(\omega+b)+\Pi\,\lambda_e^*)},\\
I_s^* &= \frac{K\,\epsilon\,b\,\lambda_e^*\,(b+\alpha_2)\,((1-\rho)\,\nu\,(b+\phi\,\lambda_e^*)+\lambda_e^*\,\omega\,\alpha_3\phi\,\rho)}{(\alpha_3+b)\,(b+\phi\,\lambda_e^*)\,(b\chi\,(\omega+b)+\Pi\,\lambda_e^*)},\\
I_c^* &= \frac{K\,\lambda_e^*\,\alpha_1\epsilon\,b\,((1-\rho)\,\nu\,(b+\phi\,\lambda_e^*)+\lambda_e^*\omega\,\alpha_3\phi\,\rho)}{(\alpha_3+b)\,(b+\phi\,\lambda_e^*)\,(b\,\chi\,(b+\omega)+\Pi\,\lambda_e^*)},\\
R^* &= \frac{K\,b\,\lambda_e^*\,(\phi\,\alpha_3\,(\lambda_e^*+b)\,\chi\,\rho+(b+\alpha_3)\,(1-\rho)\,\alpha_2\,\alpha_1\epsilon\,(b+\phi\,\lambda_e^*))}{(b+\alpha_3)\,(b+\phi\,\lambda_e^*)\,(b\,\chi\,(\omega+b)+\Pi\,\lambda_e^*)},\\
V_v^* &= \frac{b\,K\rho}{b+\phi\,\lambda_e^*},\quad V_{ca}^* = \frac{\phi\,\lambda_e^* b\,K\rho}{(b+\alpha_3)\,(b+\phi\,\lambda_e^*)},
\end{aligned}
\tag{15}
$$

If I_s, I_c^* and V_{ca}^* are substituted into (14), we obtain the equation in terms of λ_e^*:

$$
\lambda_e^*(A\,\lambda_e^{*2}+B\,\lambda_e^*+C)=0, \tag{16}
$$

where

$$
\begin{aligned}
\Pi &= b\,(b+\epsilon)\,(b+\omega)\,(b+\alpha_2+\alpha_1)+b\,\alpha_1\,\alpha_2\,(b+\epsilon+\omega)\,,\\
\chi &= (b+\epsilon)\,(\alpha_1+b)\,(b+\alpha_2)\,,\quad \nu=(\alpha_3+b)\,(b+\omega)\,,\\
A &= \phi\,(\alpha_3+b)\,((\alpha_1+\alpha_2+b)\,\epsilon\,\omega+(\alpha_2+b)\,(\alpha_1+b)\,\epsilon+(\alpha_2+b)\,(\alpha_1+b)\,(b+\omega))\,,\\
B &= b\,(\alpha_3+b)\,(b+\alpha_2)\,(b+\alpha_1)\,(b+\omega)\,(1-\mathcal{R}_c)+b\,(\alpha_3+b)\,(b+\alpha_2)\,(b+\alpha_1)\,\epsilon\,(1-\mathcal{R}_c)\\
&\quad+b\,(\alpha_3+b)\,(\alpha_1+b+\alpha_2)\,\omega\,\epsilon\,(1-\mathcal{R}_c)+(\alpha_3+b)\,(b+\omega)\,(b+\alpha_2)\,(b+\alpha_1)\,(\epsilon+b)\,\phi\\
&\quad+\beta\,b\,\phi\,\rho\,\epsilon\,(b+\alpha_2+\eta_1\,\alpha_1)\,(b+\omega+\alpha_3)+\frac{\beta\,\epsilon^2\,(b+\alpha_2+\eta_1\,\alpha_1)^2\,(1-\rho)\,(\alpha_3+b)\,(b+\omega)}{\eta_2\,\rho\,(\epsilon+b)\,(b+\alpha_1)\,(b+\alpha_2)}\\
&\quad+\frac{\beta\;b\,\epsilon^2\,(b+\alpha_2+\eta_1\,\alpha_1)\,(1-\rho)\,(b^2+b\,\omega+b\,\alpha_1+\alpha_1\,\omega+\alpha_1\,\alpha_2+b\alpha_2+\alpha_2\,\omega)}{(\epsilon+b)\,(b+\alpha_1)\,(b+\alpha_2)}\\
&\quad+\frac{\beta\,\epsilon\,b\;(b+\alpha_2+\eta_1\,\alpha_1)\,(1-\rho)\,(b+\omega)}{\epsilon+b}-\frac{\mathcal{R}_c\,(\alpha_3+b)^2\,\epsilon\;(b+\alpha_2+\eta_1\,\alpha_1)\,(b+\omega)}{\eta_2\,\rho},\\
C &= b\,(\alpha_3+b)\,(\alpha_2+b)\,(\alpha_1+b)\,(b+\omega)\,(\epsilon+b)\,(1-\mathcal{R}_c)\,.
\end{aligned}
$$

The roots of equation (16) are $\lambda_e^*=0$ which corresponds to the disease free equilibrium point and

$$
\lambda_e^*=\frac{-B\pm\sqrt{B^2-4\,A\,C}}{2\,A}. \tag{17}
$$

The condition $A>0$ and $C<0$ for any values of B when $\mathcal{R}_c>1$ is satisfied to ensures positivity of λ_e^* and subsequently the positivity of E_1. If $A>0$, $B<0$, $C>0$ and $B^2-4AC>0$ when $\mathcal{R}_c<1$, then there is a possibility of existence of two real positive solutions. Since these two positive solutions exist when $\mathcal{R}_c<1$, then there is a possibility of existence of a backward bifurcation.

3.4 Stability analysis

In this section we study the stability of disease-free (E_0) and endemic (E_1) equilibria of the system of ordinary differential model (1)-(7) with the initial condition (8).

Theorem 2. *The disease-free equilibrium point E_0 exist for all $\mathcal{R}_c$ and endemic equilibrium point exists only for $\mathcal{R}_c > 1$.*
Theorem 3. *The disease-free equilibrium point is locally asymptotically stable when $\mathcal{R}_c < 1$ and unstable when $\mathcal{R}_c > 1$.*

Proof. Evaluating the Jacobian matrix of equations (1) - (7) at the disease-free equilibrium point E_0, yields the eigenvalues

$$\lambda_1 = -b < 0, \quad \lambda_2 = -(b+\omega) < 0 \quad \lambda_3 = -\gamma K < 0,$$

and the remaining eigenvalues are roots of

$$P(\lambda) = \lambda^4 + A_1\lambda^3 + A_2\lambda^2 + A_3\lambda + A_4 = 0, \tag{18}$$

where

$$\begin{aligned}
A_1 &= \frac{\beta\,\epsilon\,(\alpha_2 + b + \eta_1\alpha_1)(\alpha_3 + b)(1-\rho)}{(b+\epsilon)(b+\alpha_1)(\alpha_2+b)} + 3\,b + \epsilon + \alpha_2 + \alpha_1 \\
&\quad + (\alpha_3 + b)(1 - \mathcal{R}_c), \\
A_2 &= \frac{\beta\,\epsilon\,(\alpha_2 + b + \eta_1\alpha_1)(1-\rho)(\alpha_3+b)(3\,b+\epsilon+\alpha_1+\alpha_2)}{(b+\epsilon)(b+\alpha_1)(\alpha_2+b)} \\
&\quad + \frac{\beta\,\rho\,\phi\,\eta_2\,(b+\epsilon)(b+\alpha_1)(\alpha_2+b)}{(\alpha_3+b)(\alpha_2+b+\eta_1\alpha_1)} + \frac{(\alpha_2{}^2 + \eta_1\alpha_1\alpha_2 + b^2 + \eta_1\alpha_1{}^2 + 2\,\eta_1\alpha_1 b + 2\,b\alpha_2)\,\epsilon}{\alpha_2+b+\eta_1\alpha_1} \\
&\quad + \frac{\alpha_1\,(2\,\alpha_1 b + \alpha_2\alpha_1 + 3\,b^2 + 2\,b\alpha_2)\,\eta_1}{\alpha_2+b+\eta_1\alpha_1} + \frac{(\alpha_2+b)^2\,(2\,b+\alpha_1)}{\alpha_2+b+\eta_1\alpha_1} \\
&\quad + (1-\mathcal{R}_c)\left((\alpha_3+b)(3\,b+\epsilon+\alpha_1+\alpha_2) + \frac{(b+\epsilon)(b+\alpha_1)(\alpha_2+b)}{\alpha_2+b+\eta_1\alpha_1}\right), \\
A_3 &= (1-\mathcal{R}_c)(\alpha_3+b)\left(\alpha_2\epsilon + 3\,b^2 + 2\,b\epsilon + 2\,\alpha_1 b + 2\,b\alpha_2 + \alpha_1\epsilon + \alpha_2\alpha_1\right) \\
&\quad + (1-\mathcal{R}_c)(b+\alpha_1)(b+\epsilon)(\alpha_2+b) + \frac{\beta\,\epsilon\,(1-\rho)(\alpha_3+b)(\alpha_2+b)(2\,b+\epsilon+\alpha_1)}{(b+\alpha_1)(b+\epsilon)} \\
&\quad + \frac{\beta\,\epsilon\,\eta_1\,(1-\rho)(\alpha_3+b)(\alpha_2\alpha_1\epsilon + 2\,\alpha_1\epsilon\,b + 2\,b\alpha_2\alpha_1 + \alpha_1{}^2\alpha_2 + \alpha_1{}^2\epsilon + 3\,\alpha_1 b^2 + 2\,\alpha_1{}^2 b)}{(b+\alpha_1)(b+\epsilon)(\alpha_2+b)} \\
&\quad + \frac{\beta\,\phi\,\eta_2\rho\,(b+\alpha_1)(b+\epsilon)(\alpha_2+b)}{\alpha_3+b}, \\
A_4 &= (\alpha_3+b)(b+\alpha_2)(b+\alpha_1)(b+\epsilon)(1-\mathcal{R}_c).
\end{aligned}$$

We use Descartes's rule of signs to determine the number of positive roots of equation (18) [22].

In our case equation (18) has positive coefficients and there is no sign changes in the sequence of coefficients when $\mathcal{R}_c < 1$ and so there are zero positive roots. If we now set $\lambda = -\omega$, the equation becomes

$$\omega^4 - A_1\omega^3 + A_2\omega^2 - A_3\omega + A_4 = 0, \tag{19}$$

A_1, A_2, A_3 and A_4 are positive for $\mathcal{R}_c < 1$ and the polynomial equation (19) has four sign changes in the sequence, and so there is at most four real positive root ω. This means, there are either four or two or zero negative roots (see Table (1)).

Table 1: Table for roots

	1	A_1	A_2	A_3	A_4	$\mathcal{R}_c < 1$	Positive roots	Negative roots
$\omega = \lambda$	+	+	+	+	+	0 sign change	0 positive roots	four negative λ
$\omega = -\lambda$	+	-	+	-	+	4 sign changes	at most 4 positive roots ω	4 negative λ

From Table 1 we can see that the roots are real and negative roots and so the disease-free equilibrium point is asymptotically stable for $\mathcal{R}_c < 1$.

Table 2: Table for roots

1	A_1	A_2	A_3	A_4	$\mathcal{R}_c > 1$	roots	roots λ
+	+	+	+	-	1 sign changes	1 positive root	3 negative roots
+	+	+	-	-	1 sign changes	1 positive root	3 negative roots
+	+	-	+	-	3 sign changes	3 or 1 positive roots	1 or 3 negative roots
+	+	-	-	-	1 sign changes	1 positive root	3 negative roots
+	-	+	+	-	3 sign changes	3 or 1 positive roots	3 or 1 negative roots
+	-	+	-	-	3 sign changes	3 or 1 positive roots	3 or 1 negative roots
+	-	-	+	-	3 sign changes	3 or 1 positive roots	3 or 1 negative roots
+	-	-	-	-	1 sign changes	1 positive root	3 negative roots

The case when $\mathcal{R}_c > 1$ is described in Table 2 where there exists at least one positive root of (18) when $\mathcal{R}_c > 1$. This implies that the disease free equilibrium point is a saddle point when $\mathcal{R}_c > 1$ and hence is unstable. □

Theorem 4. *The endemic equilibrium point is locally asymptotically stable for* $\mathcal{R}_c > 1$ *and* $\phi \geq \frac{(1-\rho)b}{b+\epsilon}$.

Proof. To prove this theorem we use the center manifold theory which states that the stability of the steady state under the initial system is determined by its stability under the restriction of the system to the center manifold [27, 28, 29]. Introducing new variables $x_1 = S$, $x_2 = E$, $x_3 = I_s$, $x_4 = I_c$, $x_5 = R$ $x_6 = V_v$, $x_7 = V_{ca}$ and rewriting the system of equations (1)- (7),

$$\frac{dx_1}{dt} = (1-\rho)\, b\, N^* - \left(\frac{\beta\,(x_3 + \eta_1\, x_4 + \eta_2\, x_7)}{N^*} + \mu + \gamma N^*\right) x_1 + \omega\, x_5 = f_1, \tag{20}$$

$$\frac{dx_2}{dt} = \frac{\beta\, x_1\,(x_3 + \eta_1\, x_4 + \eta_2\, x_7)}{N^*} - (\epsilon + \mu + \gamma N^*)\, x_2 = f_2, \tag{21}$$

$$\frac{dx_3}{dt} = \epsilon\, x_2 - (\alpha_1 + \mu + \gamma N^*)\, x_3 = f_3, \tag{22}$$

$$\frac{dx_4}{dt} = \alpha_1\, x_3 - (\alpha_2 + \mu + \gamma N^*)\, x_4 = f_4, \tag{23}$$

$$\frac{dx_5}{dt} = \alpha_2\, x_4 - (\omega + \mu + \gamma N^*)\, x_5 = f_5, \tag{24}$$

$$\frac{dx_6}{dt} = \rho\, b\, N^* - \left(\frac{\beta\,\phi\,(x_3 + \eta_1\, x_4 + \eta_2\, x_7)}{N^*} + \mu + \gamma N^*\right) x_6 = f_6, \tag{25}$$

$$\frac{dx_7}{dt} = \frac{\beta\,\phi\,\,(x_3 + \eta_1\, x_4 + \eta_2\, x_7)\, x_6}{N^*} - (\alpha_3 + \mu + \gamma N^*) x_7 = f_7, \tag{26}$$

Choosing ϕ as a bifurcation parameter when $\mathcal{R}_c = 1$, we get

$$\phi = \phi^* = \frac{(\alpha_3 + b)}{\beta\,\eta_2\,\rho} - \frac{\epsilon\,(\alpha_2 + b + \eta_1\alpha_1)\,(\alpha_3 + b)\,(1-\rho)}{\eta_2\,\rho\,(\epsilon + b)\,(\alpha_1 + b)\,(\alpha_2 + b)},$$

The first four eigenvalues of the Jacobian matrix obtained when $\mathcal{R}_c = 1$ are $\lambda_{11} = 0$, $\lambda_{12} = -b$, $\lambda_{13} = -b-\omega$ and $\lambda_{14} = -\gamma\,K$. The other three eigenvalues are calculated using the following cubic equation:

$$P_2\,(\lambda_1) = \lambda_1^3 + B_1\lambda_1^2 + B_2\lambda_1 + B_3 = 0, \tag{27}$$

where

$$\begin{aligned}
B_1 &= \frac{\beta\,\epsilon\,(1-\rho)\,(b+\alpha_3)\,(b+\alpha_2+\eta_1\alpha_1)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + 3\,b + \epsilon + \alpha_1 + \alpha_2,\\
B_2 &= \frac{\beta\,\phi^*\,\rho\,\eta_2\,(b+\alpha_2)\,\epsilon\,\alpha_1}{(b+\alpha_2+\eta_1\,\alpha_1)\,(b+\alpha_3)} + \frac{(2\,b^2 + \eta_1\,\alpha_2\,\alpha_1 + {\alpha_1}^2\eta_1 + {\alpha_2}^2)\,\epsilon}{b+\alpha_2+\eta_1\alpha_1}\\
&\quad + \frac{(2\,\eta_1\alpha_1 + 3\,\alpha_2)\,b\,\epsilon}{b+\alpha_2+\eta_1\,\alpha_1} + 3\,b^2 + 2\,b\,\alpha_1 + 2\,b\,\alpha_2 + \alpha_1\alpha_2 + \frac{\beta\,\alpha_3\epsilon^2\,(b+\alpha_2+\eta_1\alpha_1)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)}\\
&\quad + \frac{\beta\,\alpha_3\,\epsilon\,(\alpha_2+\alpha_1+3\,b)\,(b+\alpha_2+\eta_1\,\alpha_1)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + \frac{\beta\,\epsilon\,b\,\eta_1\,(3\,b+\epsilon+\alpha_2)\,\alpha_1\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)}\\
&\quad + \frac{\beta\,\epsilon\,b\,({\alpha_1}^2\eta_1 + 2\,b^2 + 3\,b\alpha_2 + {\alpha_2}^2)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)},
\end{aligned}$$

$$\begin{aligned}
B_3 &= \frac{\beta\,\epsilon^2\,b\,\alpha_3\,(1-\rho)\,(2\,\eta_1\alpha_1 + 2\,\alpha_2)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + \frac{\beta\,\epsilon^2\,(1-\rho)\,b^2\,\eta_1\,\alpha_1}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)}\\
&\quad + \frac{\beta\,\epsilon^2\,\alpha_3\,(1-\rho)\,(\eta_1\,\alpha_2\alpha_1 + b^2 + {\alpha_2}^2 + {\alpha_1}^2\,\eta_1)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + \frac{\beta\,\alpha_3\eta_1\epsilon\,\alpha_1\,(3\,b^2 + \alpha_2\,\alpha_1)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)}\\
&\quad + \frac{2\,\beta\,b\,\alpha_3\,\eta_1\epsilon\,\alpha_1\,(\alpha_2+\alpha_1)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + \frac{\beta\,\alpha_3\epsilon\,({\alpha_2}^2 + b^2)\,(2\,b+\alpha_1)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)}\\
&\quad + \frac{2\,\beta\,\alpha_3\epsilon\,b\,(2\,b+\alpha_1)\,\alpha_2\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + \frac{\beta\,\epsilon\,b^2\,(2\,b+\eta_1\alpha_1)\,\alpha_2\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)}\\
&\quad + \frac{\beta\,\epsilon\,b^2\,({\alpha_2}^2 + 2\,b\,\eta_1\,\alpha_1 + {\alpha_1}^2\,\eta_1 + b^2)\,(1-\rho)}{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)} + \frac{\beta\,\phi^*\,\rho\,\eta_2\,\epsilon\,\alpha_1\,\left(\eta_1\alpha_2\,\alpha_1 + (b+\alpha_2)^2\right)}{(b+\alpha_2+\eta_1\,\alpha_1)\,(b+\alpha_3)}\\
&\quad + \frac{b\,\eta_1\,(b+\epsilon+\alpha_2)\,{\alpha_1}^2}{b+\alpha_2+\eta_1\,\alpha_1} + \frac{b\,(b+\alpha_2)\,(b\,\eta_1 + b + \epsilon\,\eta_1 + \alpha_2)\,\alpha_1}{b+\alpha_2+\eta_1\,\alpha_1} + \frac{b\,(b+\alpha_2)^2\,(b+\epsilon)}{b+\alpha_2+\eta_1\,\alpha_1}.
\end{aligned}$$

We use Descartes's rule of signs [22] to determine the sign of the roots of (27). There are exactly 3 negative real roots for equation (3.24) (see Table 3). The Jacobian matrix has a

Table 3: Table for roots

	1	B_1	B_2	B_3	$\mathcal{R}_c = 1$	roots	roots
$\omega = \lambda$	+	+	+	+	0 sign change	0 positive roots λ	3 negative roots λ
$\omega = -\lambda$	-	+	-	+	3 sign changes	at most 3 positive ω	3 negative roots λ

simple zero eigenvalue with corresponding right and left eigenvectors respectively given by, $(w_1, w_2, w_3, w_4, w_5, w_6, w_7)$, and

$$(v_1, v_2, v_3, v_4, v_5, v_6, v_7) = \left(0, 1, \frac{b+\epsilon}{\epsilon}, \frac{(\alpha_1+b)\,(\epsilon+b)\,\eta_1}{(\alpha_2+b+\eta_1\alpha_1)\,\epsilon}, 0, 0, v_7\right),$$

where,

$$w_1 = -\omega\ (1-\rho)\,\beta\,\epsilon\,b\eta_2\,(b+\alpha_2+\alpha_1)\,(\alpha_1+b)\left(\mathcal{R}_c - \frac{\phi\,\rho\,\alpha_3\,(b+\alpha_2)\,(\alpha_1+b)\,(b+\epsilon)}{(1-\rho)\,\epsilon\,b\,(b+\alpha_2+\alpha_1)\,(\alpha_3+b)}\right)$$
$$-\,\beta\,\eta_2 b\,(b+\alpha_2)\,(\alpha_1+b)\,(b+\epsilon+\omega)\,(1-\rho) < 0,$$
$$w_2 = \beta\,\eta_2\,b\ (\alpha_2+b)\,(b+\alpha_1)\,(b+\omega)\,(1-\rho) > 0,\quad w_3 = \beta\,\eta_2\epsilon\,b\ (1-\rho)\,(\alpha_2+b)\,(b+\omega) > 0,$$
$$w_4 = \beta\,\eta_2\epsilon\,b\,\alpha_1\ (1-\rho)\,(b+\omega) > 0,$$
$$w_5 = b\alpha_3\,(\alpha_2+b)\,(\alpha_1+b)\,(b+\epsilon)\left(\mathcal{R}_c - \frac{(1-\rho)\,\beta\,\epsilon\ (\alpha_2+b+\eta_1\alpha_1)}{(\alpha_2+b)\,(\alpha_1+b)\,(b+\epsilon)}\right)$$
$$+\,b\,\epsilon\ (1-\rho)\,\alpha_1\eta_2\alpha_2\beta > 0,$$
$$w_6 = -\,(\alpha_3+b)\,(\omega+b)\,(\alpha_2+b)\,(\alpha_1+b)\,(b+\epsilon)\left(\mathcal{R}_c - \frac{(1-\rho)\,\beta\,\epsilon\ (\alpha_2+b+\eta_1\alpha_1)}{(\alpha_2+b)\,(\alpha_1+b)\,(b+\epsilon)}\right) < 0,$$
$$w_7 = (\omega+b)\,(\alpha_2+b)\,(\alpha_1+b)\,(b+\epsilon)\,b\left(\mathcal{R}_c - \frac{(1-\rho)\,\beta\,\epsilon\ (\alpha_2+b+\eta_1\alpha_1)}{(\alpha_2+b)\,(\alpha_1+b)\,(b+\epsilon)}\right),$$
$$v_7 = \frac{(b+\alpha_2)\,(b+\alpha_1)\,(b+\epsilon)\,\eta_2}{(\alpha_3+b)\,(\alpha_2+\alpha_1\eta_1+b)\,\epsilon} > 0.$$

The nonzero second order partial derivatives of f_i in equations (20) - (24), where $i = 1,2,3,4,5$ are given by

$$\frac{\partial^2 f_2}{\partial x_1\partial x_2} = \frac{\partial^2 f_2}{\partial x_2\partial x_5} = \frac{\partial^2 f_2}{\partial x_2\partial x_6} = \frac{\partial^2 f_3}{\partial x_3\partial x_1} = \frac{\partial^2 f_3}{\partial x_3\partial x_2} = \frac{\partial^2 f_3}{\partial x_3\partial x_4} = \frac{\partial^2 f_3}{\partial x_3\partial x_5} = -\gamma$$
$$\frac{\partial^2 f_3}{\partial x_6\partial x_3} = \frac{\partial^2 f_3}{\partial x_7\partial x_3} = \frac{\partial^2 f_4}{\partial x_1\partial x_4} = \frac{\partial^2 f_4}{\partial x_2\partial x_4} = \frac{\partial^2 f_4}{\partial x_3\partial x_4} = \frac{\partial^2 f_4}{\partial x_5\partial x_4} = \frac{\partial^2 f_4}{\partial x_6\partial x_4} = \frac{\partial^2 f_4}{\partial x_7\partial x_4} = -\gamma,$$
$$\frac{\partial^2 f_2}{\partial x_1\partial x_3} = \frac{\beta\,\rho}{K},\quad \frac{\partial^2 f_2}{\partial x_3\partial x_5} = \frac{\partial^2 f_2}{\partial x_6\partial x_3} = -\frac{\beta\phi^*\,(1-\rho)}{K},\quad \frac{\partial^2 f_2}{\partial x_1\partial x_4} = \frac{\beta\eta_1\,\rho}{K},$$
$$\frac{\partial^2 f_2}{\partial x_7\partial x_3} = \frac{-\beta\,(1-\rho)(\eta_2+1)}{K},\quad \frac{\partial^2 f_2}{\partial x_2\partial x_2} = \frac{\partial^2 f_3}{\partial x_3\partial x_3} = \frac{\partial^2 f_4}{\partial x_4\partial x_4} = -2\gamma,$$
$$\frac{\partial^2 f_2}{\partial x_4\partial x_4} = \frac{-2\beta\eta_1\,(1-\rho)}{K},\quad \frac{\partial^2 f_2}{\partial x_5\partial x_4} = \frac{\partial^2 f_2}{\partial x_6\partial x_4} = \frac{-\beta\eta_1\rho}{K},\quad \frac{\partial^2 f_2}{\partial x_3\partial x_3} = \frac{-2\beta\,(1-\rho)}{K},$$
$$\frac{\partial^2 f_2}{\partial x_4\partial x_3} = \frac{-\beta\,(1-\rho)(\eta_1+1)}{K},\quad \frac{\partial^2 f_2}{\partial x_7\partial x_1} = \frac{\beta\eta_2\,\rho}{K},\quad \frac{\partial^2 f_2}{\partial x_3\partial x_2} = \frac{-\beta(1-\rho)}{K} - \gamma,$$
$$\frac{\partial^2 f_2}{\partial x_4\partial x_2} = \frac{-\beta\eta_1(1-\rho)}{K} - \gamma,\ \frac{\partial^2 f_2}{\partial x_7\partial x_2} = \frac{-\beta\eta_2(1-\rho)}{K} - \gamma,$$
$$\frac{\partial^2 f_2}{\partial x_7\partial x_5} = \frac{\partial^2 f_2}{\partial x_7\partial x_6} = \frac{-\beta(1-\rho)\,\eta_2}{K},\quad \frac{\partial^2 f_2}{\partial x_7\partial x_7} = \frac{-2\beta(1-\rho)\,\eta_2}{K},$$
$$\frac{\partial^2 f_7}{\partial x_3\partial x_1} = \frac{\partial^2 f_7}{\partial x_3\partial x_2} = \frac{-\beta\phi^*\,\rho}{K},\quad \frac{\partial^2 f_2}{\partial x_7\partial x_4} = \frac{-\beta(1-\rho)(\eta_1+\eta_2)}{K},$$
$$\frac{\partial^2 f_7}{\partial x_4\partial x_1} = \frac{\partial^2 f_7}{\partial x_4\partial x_2} = \frac{-\beta\phi^*\,\eta_1\,\rho}{K},\quad \frac{\partial^2 f_7}{\partial x_7\partial x_1} = \frac{-\beta\phi^*\,\eta_2\,\rho}{K} - \gamma,$$
$$\frac{\partial^2 f_7}{\partial x_7\partial x_2} = \frac{-\beta\phi^*\,\eta_2\,\rho}{K} - \gamma,\quad \frac{\partial^2 f_7}{\partial x_3\partial x_3} = \frac{-2\beta\phi^*\,\rho}{K},\quad \frac{\partial^2 f_7}{\partial x_4\partial x_3} = \frac{-\beta\phi^*\,\rho(\eta_1+1)}{K},$$
$$\frac{\partial^2 f_7}{\partial x_5\partial x_3} = \frac{-\beta\phi^*\,\rho}{K},\quad \frac{\partial^2 f_7}{\partial x_6\partial x_3} = \frac{\beta\phi^*(1-\rho)}{K},\quad \frac{\partial^2 f_7}{\partial x_7\partial x_3} = \frac{-\beta\phi^*\,\rho(\eta_2+1)}{K} - \gamma,$$
$$\frac{\partial^2 f_7}{\partial x_4\partial x_4} = \frac{-2\beta\phi^*\,\eta_1\,\rho}{K},\quad \frac{\partial^2 f_7}{\partial x_5\partial x_4} = \frac{-\beta\phi^*\,\eta_1\,\rho}{K},\quad \frac{\partial^2 f_7}{\partial x_6\partial x_4} = \frac{\beta\phi^*\,\eta_1\,(1-\rho)}{K},$$

$$\frac{\partial^2 f_7}{\partial x_7 \partial x_4} = \frac{-\beta\phi^*(\eta_1+\eta_2)\rho}{K} - \gamma, \quad \frac{\partial^2 f_7}{\partial x_7 \partial\, \phi^*} = \beta\,\eta_2\,\rho, \quad \frac{\partial^2 f_7}{\partial x_3 \partial\, \phi^*} = \beta\,\rho,$$
$$\frac{\partial^2 f_7}{\partial x_7 \partial x_7} = \frac{-2\beta\phi^*\,\eta_2\,\rho}{K} - 2\gamma, \quad \frac{\partial^2 f_7}{\partial x_4 \partial\, \phi^*} = \beta\,\eta_1\,\rho, \quad \frac{\partial^2 f_7}{\partial x_7 \partial x_5} = \frac{-\beta\phi^*\,\eta_2\,\rho}{K} - \gamma.$$

The expressions for a and c are given by:

$$a = \sum_{i,\,j,\,k=1}^{7} v_k\, w_i\, w_j\, \frac{\partial^2 f_k}{\partial x_i \partial x_j} = M < 0, \tag{28}$$

$$c = \sum_{i,\,k=1}^{5} v_k\, w_i\, \frac{\partial^2 f_k}{\partial x_i \partial \phi^*} = \frac{(\alpha_2+b)^2\,(\alpha_1+b)^2\,(b+\epsilon)^2\,{\eta_2}^2\beta\,b\,(\omega+b)\,\rho}{(\alpha_3+b)\,(\alpha_2+\alpha_1\eta_1+b)\,\epsilon} > 0, \tag{29}$$

where $w_1+w_2+w_3+w_4+w_5+w_6+w_7=0$ and

$$M = -\frac{2\,\gamma\,w_2\,(b+\alpha_2)\,b\beta\,\eta_2\,(b+\epsilon)^2\,\phi\,\rho\,\alpha_3}{\alpha_3+b}\left(\mathcal{R}_c - \frac{(\alpha_1+b)\,(b+\omega)}{\alpha_3\,(b+\epsilon)}\right)$$
$$-\frac{2\,\gamma\,w_2\,(b+\epsilon)\,b\beta\,\eta_2\,(1-\rho)\,\epsilon\,\alpha_1\alpha_2}{\alpha_1+b} - \frac{2\,w_2 w_3\beta\,v_7\phi\,\,(b+\epsilon)\,\rho}{(1-\rho)\,b}\left(\phi - \frac{(1-\rho)\,b}{b+\epsilon}\right)$$
$$+\frac{2\,w_1 w_4\beta\,\eta_1}{K} + \frac{2\,w_1 w_7\beta\,\eta_2}{K} + \frac{2\,w_1 w_3\beta}{K} + \frac{2\,v_7 w_6 w_7\beta\,\phi\,\eta_2}{K} + \frac{2\,v_7 w_4 w_6\beta\,\phi\,\eta_1}{K} < 0.$$

we know that $w_1, w_6 < 0$ and $w_2, w_3, w_4, w_5, w_7 > 0$, and if $\phi \geq \frac{(1-\rho)b}{b+\epsilon}$, then $a < 0$ and $c > 0$ and by Castillo-Chavez et al [28], (1)–(7) exhibits a transcritical bifurcation which is a supercritical (forward) bifurcation. In scenario of a supercritical bifurcation, the stability exchange between the infected and uninfected equilibrium points guarantees that the infected equilibrium point is locally asymptotically stable whenever $\mathcal{R}_c > 1$. This suggest that, on one hand when $\mathcal{R}_c > 1$, FMD infection continues in the cattle population. On the other hand, if the condition on $\mathcal{R}_c$ is reversed to $\mathcal{R}_c < 1$ then the disease-free equilibrium point is the only equilibrium point in existence [29]. A backward bifurcation happens when $\mathcal{R}_c < 1$ for a small positive unstable equilibrium appears while the disease-free equilibrium and a larger positive equilibrium are locally asymptotically stable [28]. The direction of the bifurcation system using (28) and (29) at $\mathcal{R}_c = 1$ corresponds to a negative unstable equilibrium becoming positive and locally asymptotically stable for $\mathcal{R}_c > 1$. □

4 Parameter estimation and numerical simulation

4.1 Parameter estimation

This section contains parameter estimates for the model based upon findings that are available within the relevant literature and estimating the values of the parameters. We use the initial number of susceptible animals of 200 per km^2 and one infected animal [15]. Natural birth and immigration constant rate of b and the natural death and emigration rate μ are 0.4 and 0.0324 respectively [30]. The rate of progression from exposed to subclinically infected animals is ϵ and the value is 0.5 [31]. The rate of progress from I_s to I_c is α_1 and the recovery

rate α_2 are estimated as 0.1 and 0.17 respectively [15]. η_1 is the amplification constant for I_c and it is estimated that 1.4 since $\eta_1 > 1$. η_2 is the amplification constant for V_c and it is estimated that 0.4 since η_2 must be less than one. The loss of immunity rate ω is 0.011 [21]. The per-capita death rate γ is estimated using the carrying capacity of animal population, natural birth and immigration rate and the natural death and emigration rate, $\gamma = (b - \mu)/200$ and it is given by $\gamma = 0.0053$. ρ is the rate of vaccination, $0 \leq \rho \leq 1$ and $\beta = 1.5$ [32, 33].

4.2 Numerical simulations of the model

In this section, simulations are carried out to show the effects of long-term carrier resulting from vaccination that does not induce rapid protection of foot and mouth disease. We shall achieve this by investigating the effects of rate of protection loss by the vaccination (ϕ) and the vaccination rate (ρ) on the control reproduction number ($\mathcal{R}_c$), the effects of increasing long-term vaccinated carrier on the foot and mouth disease burden, the effects of low vaccination and high loss of protection due to vaccination rates, high vaccination and low loss of protection due to vaccination rates, low vaccination and low loss of protection due to vaccination rates and high vaccination and high loss of protection due to vaccination rates on the foot and mouth disease burdens, the effect on increasing the rate of recovery of vaccination carrier (α_3), the effects of early and late vaccination, and effects of critical vaccination (ρ_c) and basic reproductive ratio ($\mathcal{R}_0$) on the impact of vaccination (U).

Figure 2 shows the effects of the rate of protection loss by the vaccine ϕ and the vaccination rate ρ on the control reproduction ratio. From the graph we observe that the control reproduction ratio $\mathcal{R}_c < 1$ in region Q when $\phi < 0.7$, $\rho > 0.65$ and

$$\phi \leq \left(1 - \frac{\beta\,\varepsilon\,(\eta_1\alpha_1 + b + \alpha_2)\,(1-\rho)}{(\varepsilon + b)\,(\alpha_1 + b)\,(\alpha_2 + b)}\right)\left(\frac{\alpha_3 + b}{\beta\,\eta_2\rho}\right).$$

This means more animals should be vaccinated and the rate of loss of protection should be reduced to maintain $\mathcal{R}_c < 1$. Figure 3 shows that increasing the number of long-term vaccinated carriers results in the decrease of the susceptible and vaccinated groups of animals and increasing the infectious of animals. Figure 4 shows the effects of high loss of protection and low vaccination rate. There is a reduction in the infection classes and increase in the vaccinated, vaccinated carrier and removed animals. This scenario leads to a high flow of animals into the vaccination route of infection but the low flow of animals into the unprotected route of infection. Figure 5 shows the effects of low loss of protection from vaccination and high vaccination rates. There is a reduction in the infection classes, vaccinated carrier and removed animals and an increase in vaccinated animals. This leads to high flow of animals into the protected route of infection with more animals locked into the vaccinated class. Figure 6 presents graphs showing the effects of low loss of protection due to vaccination and low vaccination rate. There is a reduction in the infection classes and recovered animals but increase on the vaccinated and vaccinated carrier classes. This leads to high flow of animals into the vaccination route of infection but low flow into the unprotected route of infection with more animals locked in the vaccinated and vaccinated carrier compartment. Figure 7 shows the effects of high loss of protection due to vaccination and high vaccination rates. From this graph we observe some reduction in infection classes and increase in vaccinated animals but a slight increase in vaccinated carrier and recovered animals. This leads to high

flow of animals into the protected route of infection but low flow into the unprotected route of infection. Figure 8 shows that increasing the rate of recovery of vaccinated carriers results in slight reduction of infected classes and slight increase in vaccinated but high decrease in vaccinated carrier and increasing in recovery animals. This scenario leads more animals going back to the susceptible class. Figure 9 shows the effects of vaccination rate when vaccination is administered on the first day on detection and after 100 days of detection. The results show that vaccination is more effective in decreasing the infection when it is administered early than when administered later.

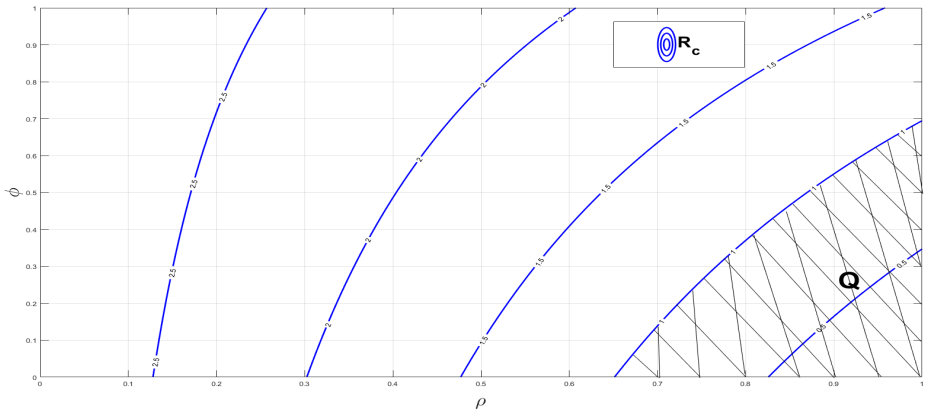

Figure 2: Graph showing the effects of rate of protection loss by the vaccination (ϕ) and the vaccination rate (ρ) on the control reproduction number.

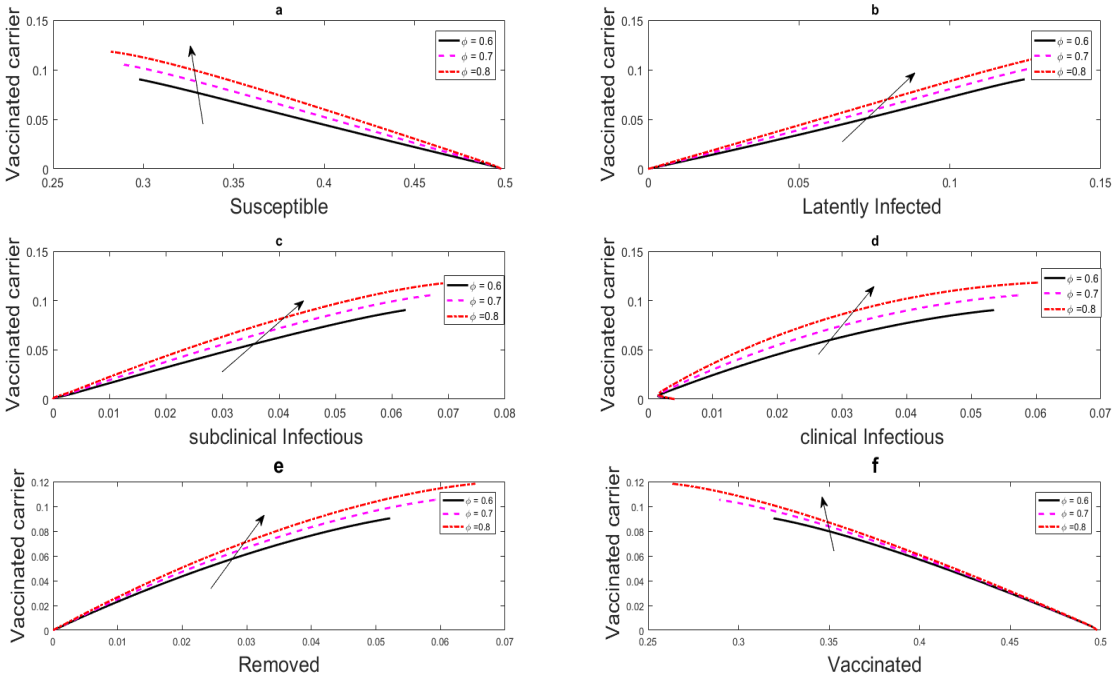

Figure 3: The phase portraits analysis showing the effects of long-term carrier resulting from vaccination.

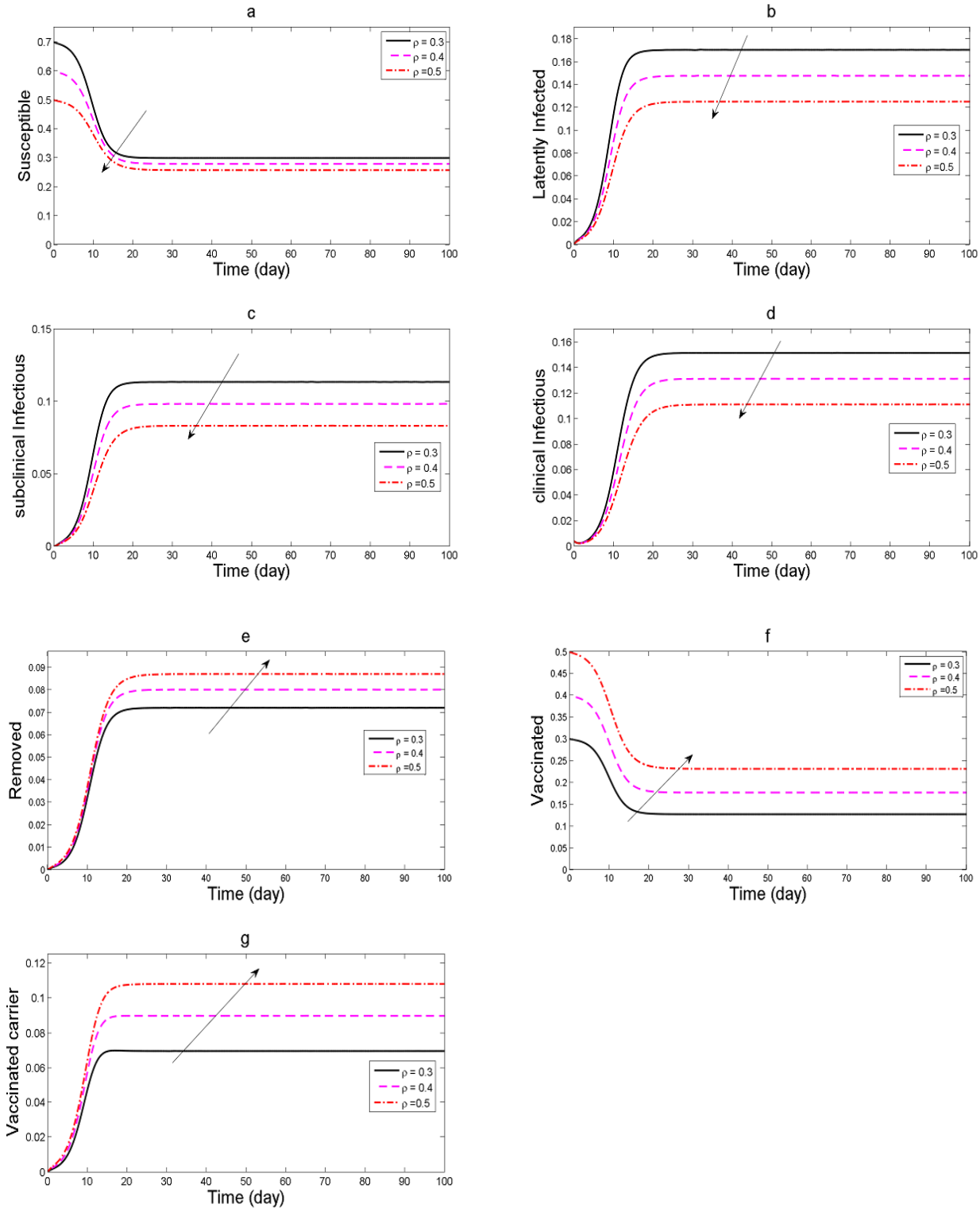

Figure 4: The graphs showing the effects of low vaccination rates and high loss of protection due to vaccination ($\phi = 0.8$).

Figure 10 shows the proportion of critical vaccination (ρ_c) for eradication and impact of vaccine (U). Vaccines with low impact have critical vaccination proportion for eradication

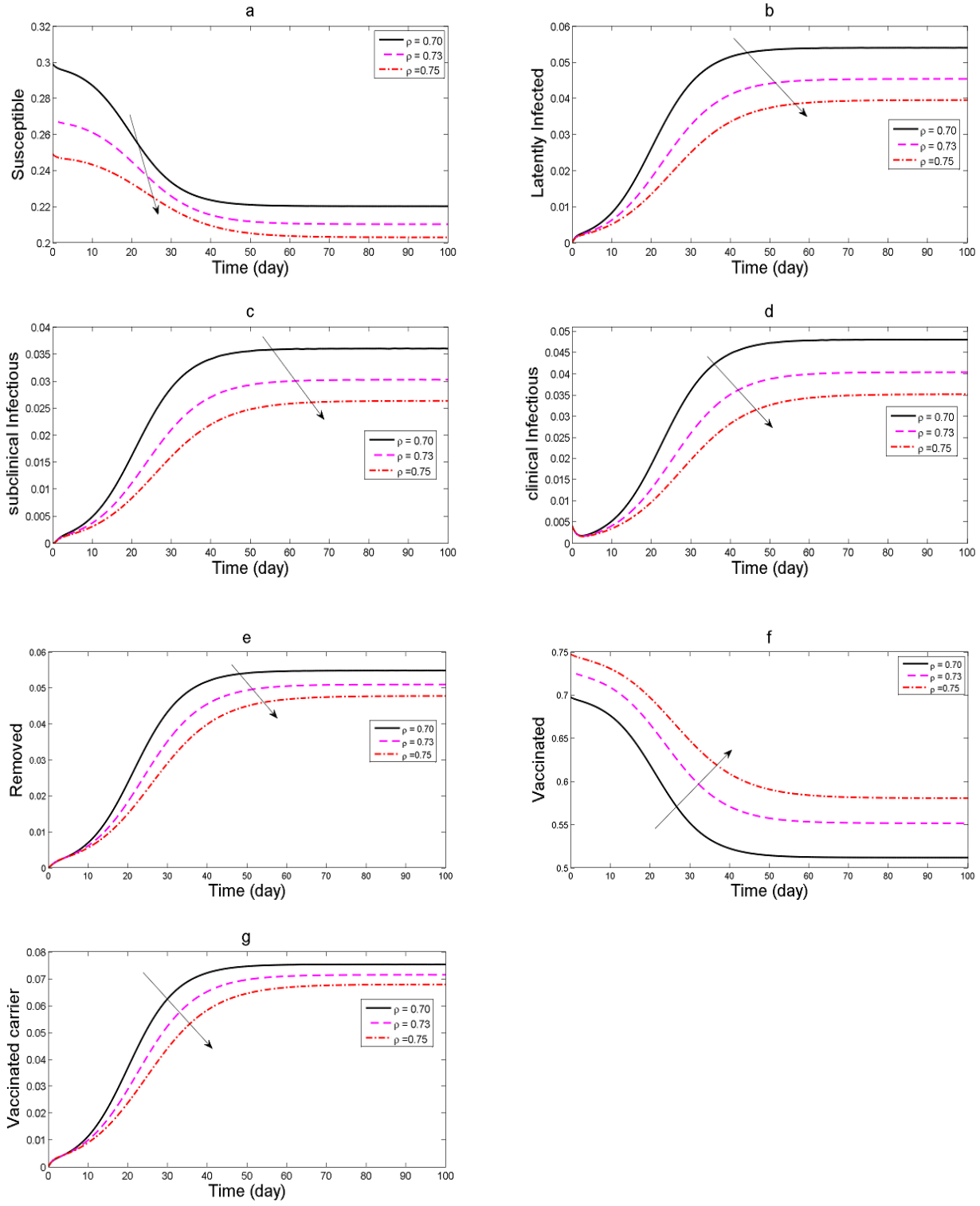

Figure 5: The graphs showing the effects of high vaccination rates and low loss of protection due to vaccination ($\phi = 0.2$).

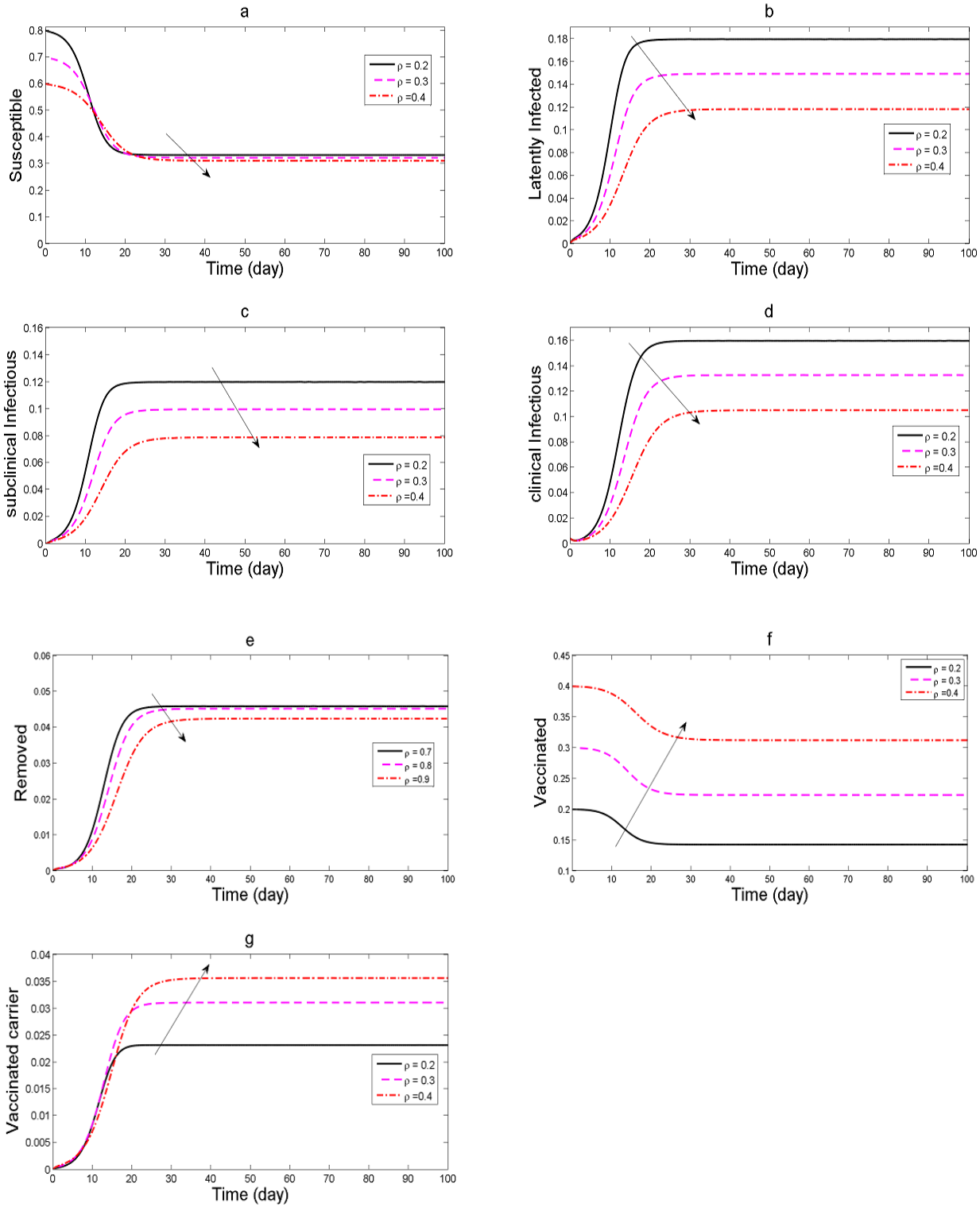

Figure 6: The graphs showing the effects of low vaccination rates and low loss of protection due to vaccination ($\phi = 0.2$).

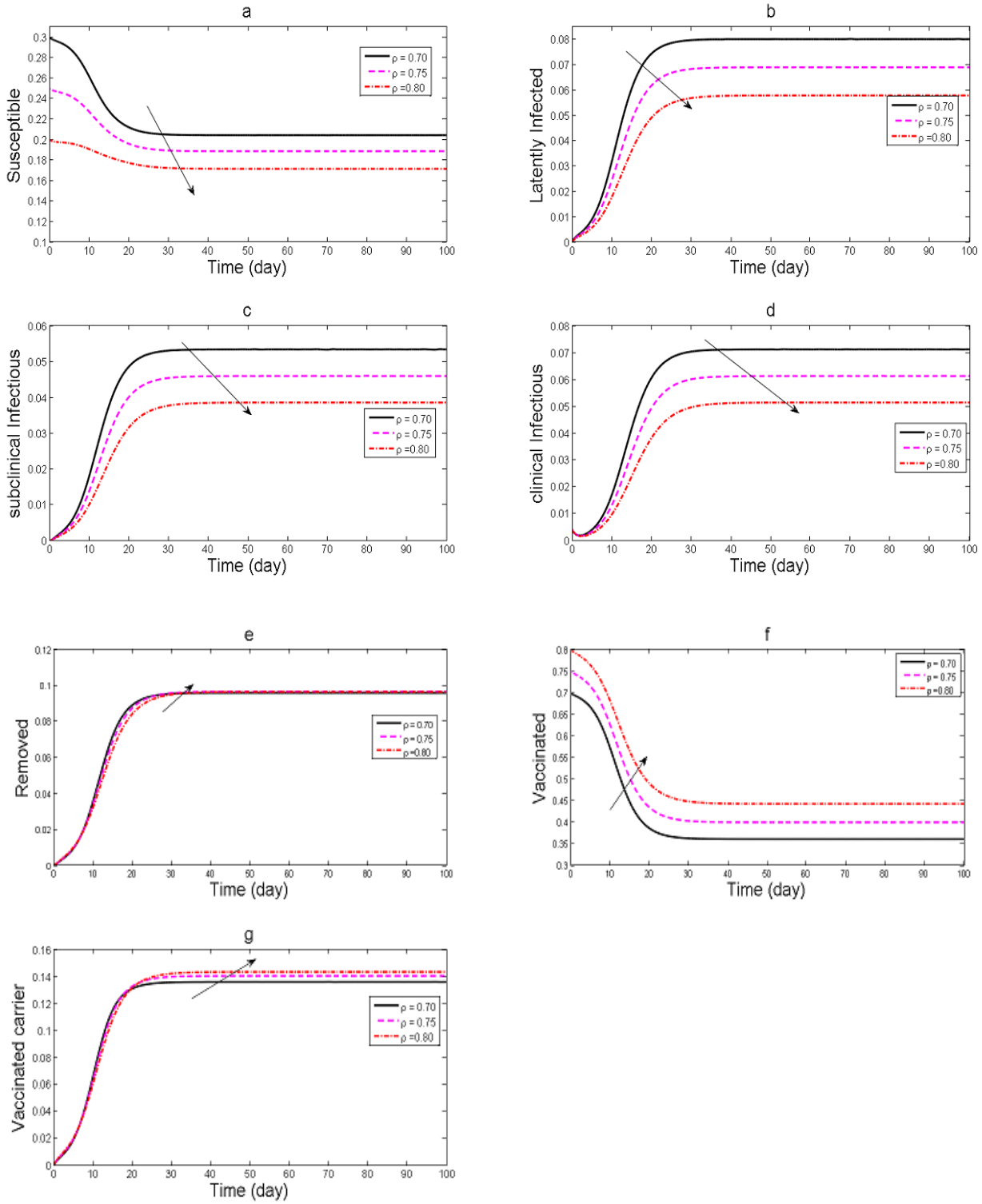

Figure 7: The graphs showing the effects of high vaccination rates and high loss of protection due to vaccination ($\phi = 0.8$).

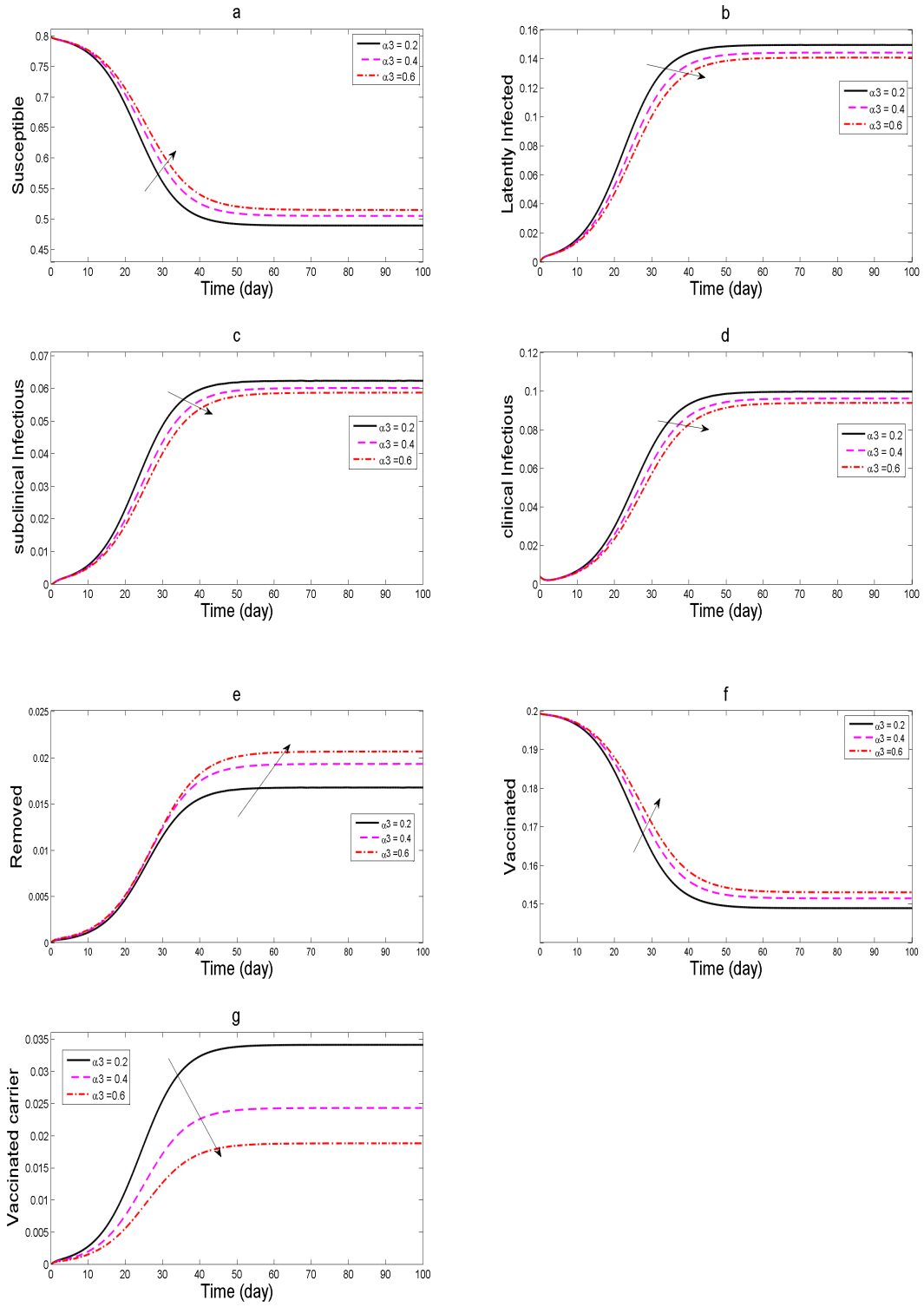

Figure 8: The graphs showing the effects rate of recovery of vaccination carrier.

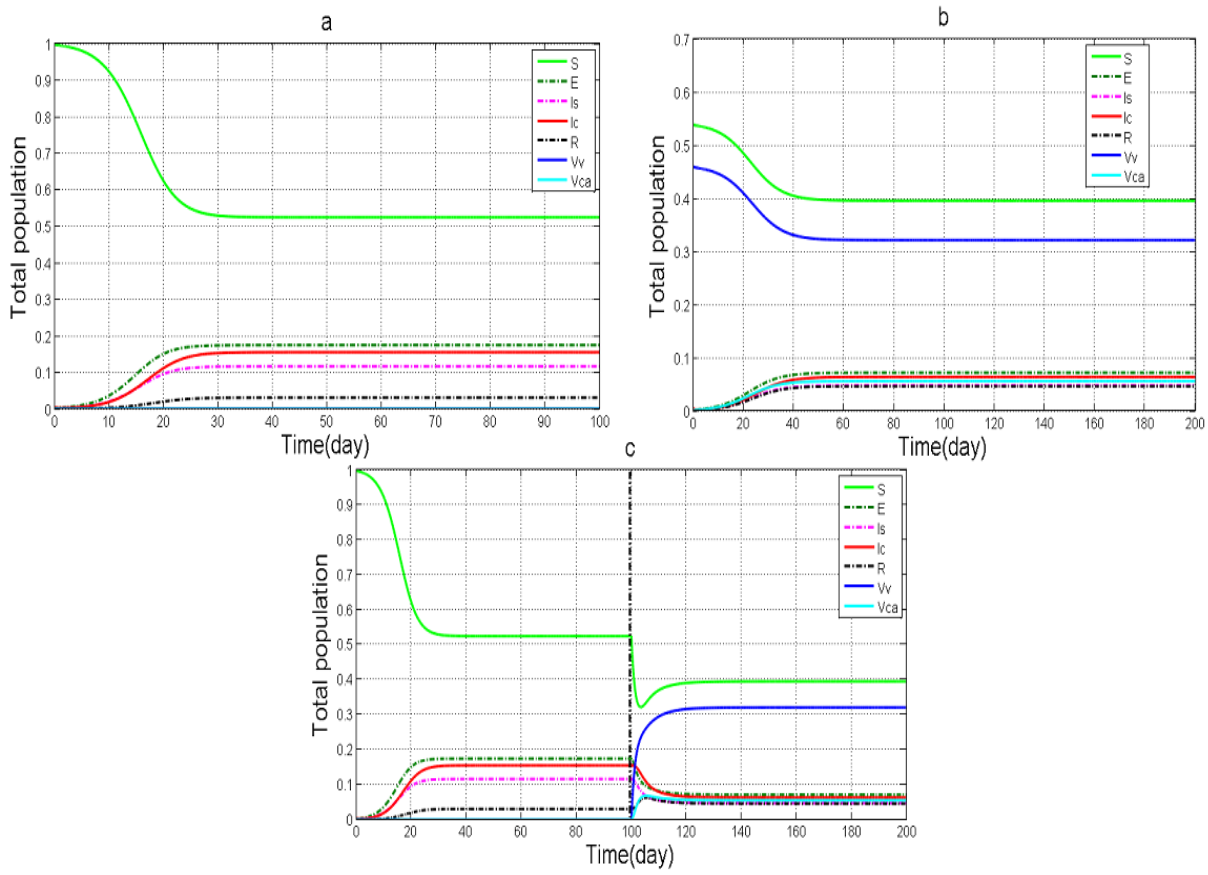

Figure 9: Graph showing the effects of vaccination initiated at time $t = 0$ and at $t = 100$.

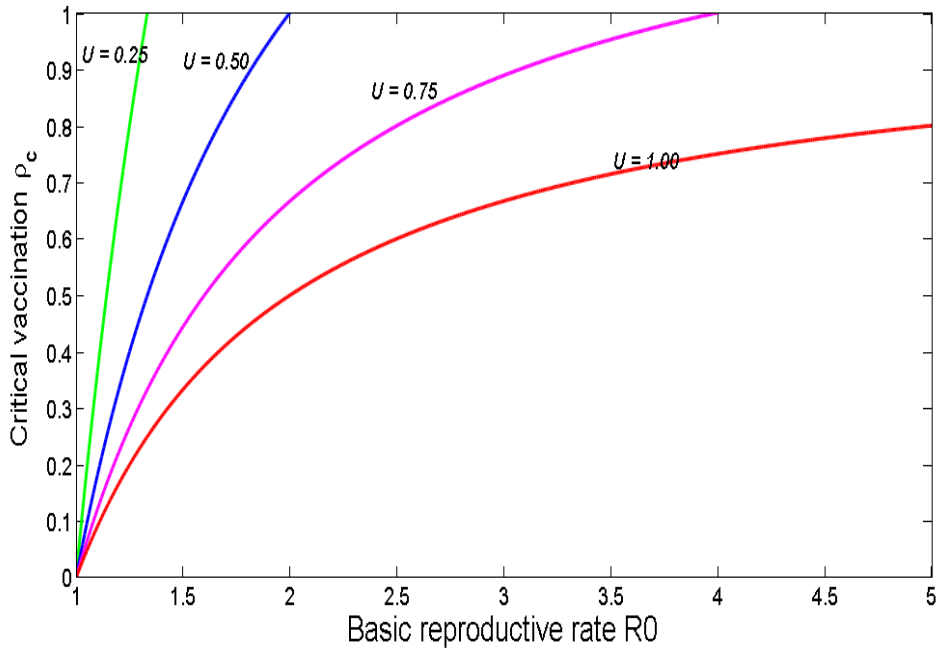

Figure 10: Proportion of critical vaccination (ρ_c) for eradication and impact of vaccine (U).

beyond one. The coverage of critical vaccination that leads to eradicating the foot and mouth disease is the function of the impact of vaccination U and the basic reproductive rate $\mathcal{R}_0$ see (13). Since the vaccine under consideration does not induce rapid protection, the vaccine impact is less than one $(U < 1)$ and the basic reproduction ratio $(\mathcal{R}_0)$ for the critical vaccination (ρ_c) is greater than one $(\mathcal{R}_0 > 1)$. Thus, this indicates that even if all the animals are vaccinated, FMD will not be eradicated.

5 Discussion and conclusion

In this study, we presented the dynamics of the foot and mouth disease (FMD) using a system of differential equations. We used the vaccination strategy to study the impact of vaccination that does not induce rapid protection of foot and mouth disease. Foot and mouth disease vaccine is important to keep livestock productivity and food security in the country. However, the use of vaccination programs has a limitation due to a vaccine which does not induce rapid protection of FMD. Hence, in our study, we have included the vaccination that does not induce rapid protection of FMD. The positivity of solutions of the model and the stability of the equilibrium were presented.

Our results showed that the disease-free equilibrium is stable when $\mathcal{R}_c < 1$ and the endemic equilibrium also locally stable when $\mathcal{R}_c > 1$. We investigated the effects of the vaccination coverage and loss of vaccination protection on both the vaccinated and unvaccinated animal population. High vaccination rate and low loss of protection was the best strategy that reduced the foot and mouth disease burden, followed by high vaccination rate and high loss of protection. Low vaccination rate and low loss of protection is the least strategy to protect the foot and mouth disease. Low vaccination rate and high loss of protection is the worst strategy for the foot and mouth disease protection because in this strategy the flow of animals is high into the unprotected route of infection but flow into the vaccination route of infection is low.

Increasing the rate of recovery from vaccinated carrier V_{ca} increases the recovered class and the recovered animals go back to the susceptible class with a slight decrease in infectious classes and high decrease in vaccination carrier. To decrease the foot and mouth disease burden, it is better to introduce vaccination as soon as foot and mouth disease is detected than to wait and administer after a few days.

As a significance, we propose that any control measure to reduce or eliminate the foot and mouth disease suggest that even though vaccines may not induce rapid protection high rate of vaccination and low loss of vaccine protection rate may be successful in reducing the foot and mouth burden, provided critical vaccination thresholds are taken into consideration. Early detection mechanisms should be in place and vaccination should be implemented as soon as the infection is detected in the animals. Strategies targeting the vaccinated carriers are also recommended for our results revealed that the increase in numbers of these animals is associated with increasing the foot and mouth disease burden. Strategies that promote increases of animals that are not vaccinated or vaccinated careers should be avoided at all

costs this is agree with Mushayabasa et al. [3].

This study is not all encompassing and can be improved by incorporating the spread of foot and mouth disease in both space and time. Optimization strategies need to be employed to make the best cost-effective.

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CHAPTER 3

MODELLING FOOT AND MOUTH DISEASE INCORPORATING AN INCOMPETENT VACCINE WITH TWO TIME DELAY

In this Chapter we develop a delay ordinary differential equation model that captures the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling on the spread of foot and mouth disease with time delays.

Modelling control of foot and mouth disease with two time delays

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Abstract

We develop a delay ordinary differential equation model that captures the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling on the spread of foot and mouth disease with time delays. Simulation results from the study suggest that increasing time delay whilst increasing the control strategies decreases the burden of foot and mouth disease. Further, the results reveal, that decreasing time delay whilst decreasing the control strategies increases the burden of foot and mouth disease. The intermediate scenarios of either (i) increasing time delay whilst decreasing control or (ii) decreasing time delay whilst increasing control have intermediate effects on burden reduction. Thus, the implementation of effective control strategies combination can play an important role in mitigating against the foot and mouth disease burden.

Keywords: Foot and Mouth; Vaccination; Prophylactic vaccination; Treatment; Culling; Time delay.

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1 Introduction:

One of the most common contagious animal viral diseases that can cause devastating economic, social, and environmental damages is foot and mouth disease (FMD) [1]. FMD affects cloven-hoofed animals [2, 3, 4] and is communicated through viral particles in the air transported by the wind and through direct and indirect contact. Susceptible animals that are exposed to FMD remain exposed for 2 to 4 days and subsequently proceed to a subclinical state. A clinically infected animal can recover by developing natural immunity [5, 6, 7] which can wane with time making the animal susceptible again [3, 8, 9, 10]. The vaccinated animals may become carriers and transmit the infection subclinically. The animal may recover by developing natural immunity [6, 11, 12, 13].

The spread of foot and mouth disease may be controlled by vaccination and treatment of different animal groups. Vaccination can be administered as either a prophylactic vaccination or as a reactive vaccination strategy [12, 14, 15]. Prophylactic vaccination is carried out before the introduction of the disease whereas reactive vaccination is carried out after the outbreak [14, 15, 16, 17]. Vaccination alone is inadequate to completely protect the animals from FMD [16]. Evidence available shows the occurrence of several outbreaks of FMD in places where vaccination was administered [6, 15]. Prophylactic treatment has also been used in the control of FMD. The treatment uses antiviral drugs to protect animals from infection and to reduce the risk of carrier animals spreading the virus during emergency vaccination programs [6, 18]. Reactive culling is another measure that has been used in combination with other control measures such as reactive vaccination [14, 15]. It is administered after the outbreak of FMD. Prophylactic and reactive vaccination are to some extent, effective in preventing FMD. However, it remains to be determined how effective the combination of prophylactic and reactive vaccination is. It is also important to investigate the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling.

Differential equations have been used to model the dynamics of a number of diseases [19, 20, 21]. These equations include delay differential equations that can be used to describe epidemiological phenomena at a certain time in terms of the values of a given function at previous times [14]. A number of delay differential equations have been used to model other infections with one time delay [22, 23, 24].

In this study, we investigate the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling on the dynamics of FMD infection using a two-time delay model. We formulate the mathematical model in section 2, provide some model analysis in section 3, present parameter estimation and numerical simulations in section 4 and give a discussion of the results in section 5.

2 Model incorporating two time delays

We propose a model of time delay differential equations for the spread of foot and mouth disease in animals by subdividing the total population into susceptible animals $S(t)$, "treated and vaccinated" animals $T_v(t)$, clinically infectious animals $I_c(t)$, recovered animals $R(t)$, vaccinated animals $V_v(t)$ and vaccinated carrier animals $V_{ca}(t)$. Susceptible animals are free of the foot and mouth disease virus (FMDV), treated and vaccinated animals are animals which are treated with prophylactic drugs and vaccinated with prophylactic reactive vaccines, the clinically infectious animals are those with clinically diagnosed symptoms. The vaccinated animals are protected from the disease by a prophylactic vaccine as well as prophylactic reactive vaccines, and vaccinated carrier animals may get infected because the prophylactic vaccine does not induce complete protection. The removed animals are either recovered or immune to the infection. The immunity may wane with time and the recovered animals become susceptible again [3, 8, 9, 10].

The susceptible animals are recruited at the rate $bN(t)$ where b is the per-capita birth rate and $N(t)$ is the total population. A proportion of new birth is given a prophylactic vaccine at a constant rate of ρ where $0 \leq \rho \leq 1$, and hence, the net recruitment of the susceptible animal is given by $(1-\rho)bN(t)$. Susceptible animals are given either reactive vaccines at a constant rate ρ_2 where $0 \leq \rho_2 \leq 1$ or given prophylactic treatment [18]. ρ_1 is a rate of treating susceptible animals and subsequently vaccinating them so that they end up in the $T_v(t)$ class. Prophylactic drugs are preventive drugs that are administered to animals that are free of infection. Prophylactic drugs have been administered successfully for other viruses and diseases, for instance, HIV and malaria infections [25, 26]. Susceptible animals which are given a reactive vaccine move to the vaccinated class $V_v(t)$. Prophylactically treated susceptible animals will in principle move to the treated susceptible class and when subsequently vaccinated will move ultimately to the treated and vaccinated class $T_v(t)$. We introduce a time delay $\tau_1 > 0$ to replace the treated susceptible class and capture the time required to move the treated and subsequently vaccinated animals to the treated and vaccinated class $T_v(t)$. However, because the treated susceptible class is not immune to infection, they may be infected and move in principle to the sub-clinically infected animals which we replace by the time delay $\tau_2 > 0$ and allow movement of infected animals into the clinically infected animals $I_c(t)$. We capture the force of infection in the treated susceptible animals by $\beta(1-\epsilon)(I_c(t-\tau_2)+\eta V_{ca}(t))S(t-\tau_1)/N(t)$ with ϵ the rate of treating susceptible animals where $0 \leq \epsilon \leq 1$ and $(1-\epsilon)$ capturing the treatment protection failure. The new infections through successive contacts between susceptible and infected animals occur at a rate β. Since V_{ca} animals are less infectious as compared to I_c we introduce an amplification factor $\eta < 1$.

Susceptible animals which are not treated or vaccinated may get infected and move in principle to the sub-clinically infected class and progress to the clinically infected class $I_c(t)$. As in the treated susceptible animals, we replace the sub-clinically infected animals by the time delay τ_2 and capture the force of infection by $\beta(I_c(t-\tau_2)+\eta V_{ca}(t))S(t)/N(t)$. Susceptible animals are also subjected to natural death at a rate of μ. They also suffer from density-dependent death rate that comes due to crowding and we capture the combined death rate by the term $\mu+\gamma N$, where γ is the per-capita density-dependent death rate. The density-dependent death rate has the effect of inducing the logistic growth in the total population

which is one of the realistic ways of capturing the growth of populations.

The clinically infected class are recruited from infection of susceptible animals and from infection of treated susceptible animals. They may recover naturally and move to the recovered class $R(t)$ at a rate α_2. They are also subjected to the combined death rate above. As a control measure, we introduce culling at a rate of δ and move animals out of $I_c(t)$ through this control measure. We assume that $I_c(t)$ animals cannot move to $T_v(t)$, due to the fact that (i) there is no known post-infection treatment for FMD and also that (ii) even if treatment existed, they would need to be subsequently vaccinated to be able to join $T_v(t)$ class. A vaccine cannot be administered to sick animals. Hence, the natural recovery and culling of sick animals are assumed to be sufficient to capture all the dynamics of $I_c(t)$ animals.

Vaccinated animals are recruited through prophylactic vaccination at a rate $\rho b N(t)$ and also from reactive vaccination of susceptible animals at a rate ρ_2. Since the vaccination do not induce rapid protection [6] and is not perfect, some vaccinated animals are infected and become vaccinated carriers class $V_{ca}(t)$ with a force of infection $\beta\,\phi\,(I_c(t-\tau_2)+\eta V_{ca}(t))S(t)/N(t)$, where ϕ is the rate of protection loss by the vaccination where $0 \leq \phi \leq 1$. Some vaccinated animals can be given prophylactic treatment at a rate ϵ and move to the treated and vaccinated class $T_v(t)$. They are also subjected to the combined death rate. The vaccinated carrier animals are recruited from infection of vaccinated animals, they recover naturally at a rate α_3, removed through culling at a rate δ and as well as through combined death rate.

The treated and vaccinated animals are recruited from the treating of vaccinated animals and from treated and subsequently vaccinated susceptible animals. They recover at a rate α_4 and die due to combined death. The recovered animals are recruited from the recovery of clinically infected animals, vaccinated carrier animals and, treated and vaccinated animals. The immunity wanes at a rate ω which moves the recovered animals back to the susceptible class otherwise the recovered animal are removed from their class through combined death. Figure 1 shows the flow diagram for the model proposed.

The model representing the dynamics of foot and mouth and disease infection is represented as a system of delay differential equations (DDEs) as follows:

$$\begin{aligned}
\frac{dS(t)}{dt} &= (1-\rho)\,b\,N(t) - \lambda\,S(t) - (1-\epsilon)\;\lambda\,S\,(t-\tau_1) - \rho_1\,S(t-\tau_1) \qquad (1)\\
&\quad - (\rho_2+\mu+\gamma\,N(t))\,S(t) + \omega\,R(t),\\
\frac{dT_v(t)}{dt} &= \rho_1\,S(t-\tau_1) + \epsilon\,V_v(t) - (\alpha_4+\mu+\gamma\,N(t))\,T_v(t), \qquad (2)\\
\frac{dI_c(t)}{dt} &= \lambda\,S(t) + (1-\epsilon)\;\lambda\,S\,(t-\tau_1) - (\delta+\alpha_2+\mu+\gamma\,N(t))\,Ic(t), \qquad (3)\\
\frac{dR(t)}{dt} &= \alpha_2 Ic(t) - (\omega+\mu+\gamma\,N(t))\,R(t) + \alpha_3 V_{ca}(t) + \alpha_4\,T_v(t), \qquad (4)\\
\frac{dV_v(t)}{dt} &= \rho\,b\,N(t) + \rho_2\,S(t) - \phi\,\lambda V_v(t) - (\epsilon+\mu+\gamma\,N(t))\,V_v(t), \qquad (5)\\
\frac{dV_{ca}(t)}{dt} &= \phi\,\lambda V_v(t) - (\delta+\alpha_3+\mu+\gamma\,N(t))\,V_{ca}(t), \qquad (6)
\end{aligned}$$

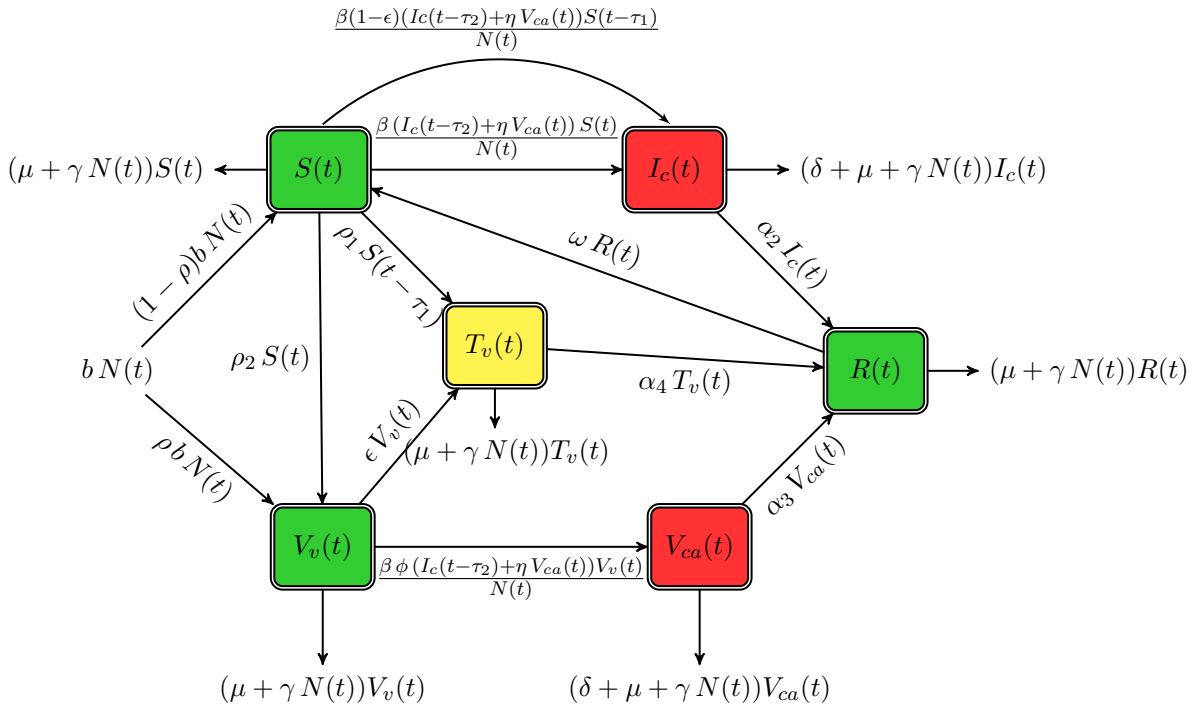

Figure 1: Flow diagram for the FMD model with two time delay.

where $\lambda = \beta\ (I_c\,(t-\tau_2) + \eta\, V_{ca}(t))\,/N(t)$ and subject to the initial conditions

$$\begin{aligned} &S(0) = S_0 \geq 0,\ T_v(0) = T_{v0} \geq 0,\ I_c(0) = I_{c0} \geq 0, R(0) = R_0 \geq 0, \\ &V_v(0) = V_{v0} \geq 0,\ V_c(0) = V_{ca0} \geq 0,\ t \in [-\tau_i, 0],\ \text{where } \tau_i \text{ is the maximum delay.} \end{aligned} \tag{7}$$

Adding equations of the model (2)-(6), leads to a logistic differential equation

$$\frac{dN(t)}{dt} = r\,N(t)\left(1 - \frac{\gamma\,N(t)}{r}\right) - \delta\ (I_c(t) + V_{ca}(t))\,. \tag{8}$$

From 8 we note that

$$\frac{dN(t)}{dt} \leq r\,N(t)\left(1 - \frac{N(t)}{K}\right),$$

where $K = \dfrac{r}{\gamma}$ is the carrying capacity and $r = b - \mu > 0,$ is the growth rate.

The solution satisfies

$$\begin{aligned} &N(t) \leq \frac{K}{1 + \psi \mathrm{e}^{-rt}}, \\ &\text{where}, \psi = \left(\frac{K}{N(0)} - 1\right),\ \ N(0) \leq K,\ \ \text{and} \\ &\lim_{t\to\infty} N(t) \leq \lim_{t\to\infty} \frac{K}{1 + \psi \mathrm{e}^{-rt}} = K. \end{aligned}$$

3 Model analysis

In this section, we define a feasible region for the model (2) - (6), where, Γ is positively invariant and attracting. We also calculate the reproduction ratio and determine the equilibrium points and their stability.

3.1 Feasible region

All the variables and parameters are assumed to be non-negative for the model to be biologically meaningful.

Theorem 1. *Let the system of equations (2) - (6) have initial conditions (7). Then, the region* Γ *defined by (9) is positively invariant and attracting where*

$$\Gamma = \{(S(t), T_v(t), I_c(s), R(t), V_v(t), V_{ca}(t)) \in \Re^6_+ \mid N(t) \leq K\}. \tag{9}$$

Proof. Assume for $t > 0$, $N(0) \geq 0$, $S(0) \geq 0$, $T_v(0) \geq 0$, $I_c(0) \geq 0$ $R(0) \geq 0$, $V_v(0) \geq 0$ and $V_{ca}(0) \geq 0$. From equation (6) we get

$$\frac{d}{dt}V_{ca}(t) = \frac{\beta\,\phi\,\left(I_c\left(t-\tau_2\right)+\eta\,V_{ca}\left(t\right)\right)V_v(t)}{N\left(t\right)} \tag{10}$$

$$-\left(\delta+\alpha_3+\mu+\gamma\,N\left(t\right)\right)V_{ca}(t). \tag{11}$$

Integrating equation (10) and using a differential inequality, we get

$$V_{ca}(t) \geq V_{ca}(0)\exp\left(-(\delta+\alpha_3+\mu)t-\gamma\int_0^t N(s)ds\right) \geq 0,$$

Hence, $V_{ca} \geq 0$, as $t \to 0$ and this implies that at any finite time, V_{ca} is non-negative. A similar analysis holds for equations (2) - (6) where,

$$R(t) \geq R\left(0\right)\exp\left(-\left(\omega+\mu\right)t-\gamma\int_0^t N\left(s\right)ds\right) \geq 0,$$

$$I_c(t) \geq I_c\left(0\right)\exp\left(-\left(\delta+\alpha_2+\mu\right)t-\gamma\int_0^t N\left(s\right)ds\right) \geq 0,$$

$$T_v(t) \geq T_v(0)\exp\left(-(\alpha_4+\mu)\,t-\gamma\int_0^t N\left(s\right)ds\right) \geq 0,$$

$$V_v\left(t\right) \geq V_v\left(0\right)\exp\left(-\left(\epsilon+\mu\right)\,t-\int_0^t\left(\frac{\beta\,\phi\,\left(I_c\left(s-\tau_1\right)+\eta\,V_{ca}\left(s\right)\right)}{N\left(s\right)}+\gamma\,N\left(s\right)\right)ds\right) \geq 0,$$

$$S\left(t\right) \geq S(0)\exp\left(-\left(\rho_1+\rho_2+\mu\right)\,t-\int_0^t\left(\frac{\beta\,\left(2-\epsilon\right)\,I_c\left(s-\tau_1\right)+\eta\,V_{ca}\left(s\right)\right)}{N\left(s\right)}+\gamma N\left(s\right)\right)ds\right) \geq 0.$$

Therefore, the solutions of the model with non-negative initial conditions remain non-negative for all $0 \leq t < \infty$. Since $0 \leq (S(t),\, T_v,\, I_c(t),\, R(t),\, V_v(t),\, V_{ca}(t)) \leq (S_0, T_{v0}, 0, R_0, V_{v0}, 0)$, all

variables are bounded in $[0, K]$,
where

$$S_0 = \frac{((1-\rho)\,(b^3 + (\omega+\epsilon+\alpha_4)\,b^2 + ((\omega+\alpha_4)\,\epsilon + \omega\,\alpha_4)\,b) + \epsilon\,\omega\,\alpha_4)\,K}{M_1},$$
$$T_{v0} = \frac{(b+\omega)\,K\,(\epsilon\,(b\rho+\rho_1+\rho_2) + b\rho_1\,(1-\rho))}{M_1},$$
$$R_0 = \frac{K\alpha_4\,(\epsilon\,(b\rho+\rho_1+\rho_2) + b\rho_1\,(1-\rho))}{M_1},$$

$$V_{v0} = \frac{Kb\,(b+\omega+\alpha_4)\,\rho\,\rho_1 + K\,(b+\alpha_4)\,(b+\omega)\,(b\rho+\rho_2)}{M_1},$$
$$M_1 = b^3 + (\omega+\epsilon+\rho_1+\rho_2+\alpha_4)\,b^2 + ((\omega+\rho_1+\rho_2+\alpha_4)\,\epsilon + (\omega+\rho_1+\rho_2)\,\alpha_4 + \omega\,(\rho_1+\rho_2))\,b$$
$$+ ((\omega+\rho_1+\rho_2)\,\alpha_4 + \omega\,(\rho_1+\rho_2))\,\epsilon + \omega\,\alpha_4\rho_2.$$

This shows that for initial conditions (7), the region Γ is positively invariant and attracting and therefore the region Γ is a feasible region for the model (2) - (6). □

3.2 The control reproduction ratio for the model

The control reproduction ratio is calculated using the next generation matrix method [27, 28]. We take only the infected classes of the model to calculate the control reproduction ratio. At the disease free equilibrium point, $I_c = V_{ca} = 0$, S_0, R_0 and V_{v0}. The control reproduction number is given by

$$\mathcal{R}_c = \frac{\beta\,\phi\,\eta\,\rho\,b}{(\epsilon+b)\,(\alpha_3+b+\delta)} + \frac{\beta\,(2-\epsilon)\,(1-\rho)\,b}{(b+\rho_1)\,(b+\delta+\alpha_2)},$$

where $b = (\mu + \gamma K)$.

To test the parameters that significantly affect the transmission dynamics of foot and mouth disease in cattle, sensitivity analysis on $\mathcal{R}_c$ was carried out through differentiating $\mathcal{R}_c$ with respect to parameters of the model. The following results were obtained:

$$\frac{\partial\mathcal{R}_c}{\partial\epsilon} = -\frac{\beta\,\phi\,\eta\,b\rho}{(\epsilon+b)^2\,(\alpha_3+b+\delta)} - \frac{\beta\,(1-\rho)\,b}{(b+\rho_1)\,(b+\delta+\alpha_2)} < 0,$$
$$\frac{\partial\mathcal{R}_c}{\partial\delta} = -\frac{\beta\,\phi\,\eta\,b\rho}{(\epsilon+b)\,(\alpha_3+b+\delta)^2} - \frac{\beta\,(2-\epsilon)\,(1-\rho)\,b}{(b+\rho_1)\,(b+\delta+\alpha_2)^2} < 0,$$
$$\frac{\partial\mathcal{R}_c}{\partial\rho_1} = -\frac{\beta\,(2-\epsilon)\,(1-\rho)\,b}{(b+\rho_1)^2\,(b+\delta+\alpha_2)} < 0,$$
$$\frac{\partial\mathcal{R}_c}{\partial\rho} = \frac{\beta\,\phi\,\eta\,b}{(\epsilon+b)\,(\alpha_3+b+\delta)} - \frac{\beta\,(2-\epsilon)\,b}{(b+\rho_1)\,(b+\delta+\alpha_2)}$$
$$= \frac{\beta\,b\eta\,(\phi-\phi_{crit})}{(\epsilon+b)\,(\alpha_3+b+\delta)} < 0,$$

where $\phi_{crit} = ((2-\epsilon)(\epsilon+b)(\alpha_3+b+\delta))/(\eta(b+\rho_1)(b+\delta+\alpha_2))$. $\partial\mathcal{R}_c/\partial\rho > 0$, $\phi > \phi_{crit}$ and $\partial\mathcal{R}_c/\partial\rho < 0$ when $\phi < \phi_{crit}$ and $\partial\mathcal{R}_c/\partial\rho = 0$ when $\phi = \phi_{crit}$. For vaccination to be effective $\mathcal{R}_c < 1$, the loss of protection from vaccination should be less than the critical value of loss of protection from the vaccine, that is, $\phi < \phi_{crit}$.

The sensitivity analysis showed that the derivatives of $\mathcal{R}_c$ with respect to the rate of vaccination, the rate of treating susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and vaccinated carrier animals are all less than zero.

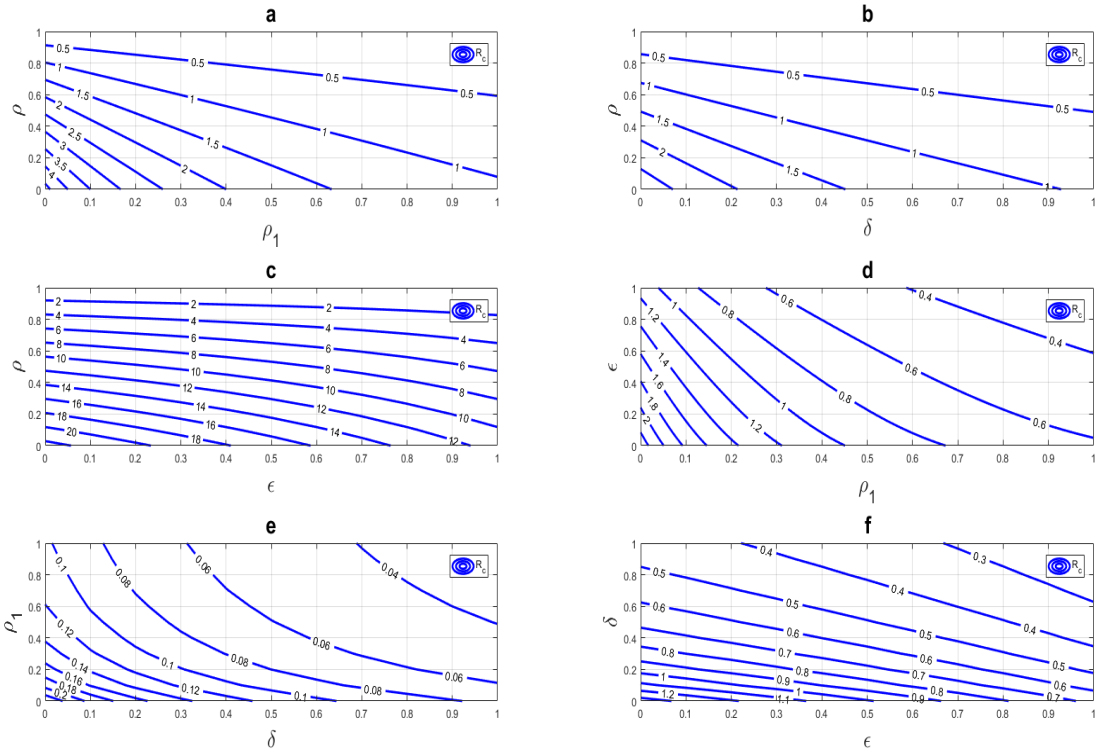

Figure 2: The effects of the rate of vaccination (ρ), the rate of treating susceptible animals (ρ_1), the rate of treating vaccinated animals (ϵ) and the rate of culling infected and vaccinated carrier animals (δ) on the control reproduction number.

Figure 2 shows the effects of vaccination, treating and vaccinating susceptible animals, treating vaccinated animals and culling infected and vaccinated carrier animals on $\mathcal{R}_c$. Figs. 2 (a) and (b) show that $\mathcal{R}_c$ is reduced and less than one as the parameters, ρ, ρ_1 and δ increase. The graph of ϵ versus ρ shows that increasing both ϵ and ρ reduces $\mathcal{R}_c$ to a value still above unity in Figure 2 (c). The graph in Figure 2 (d) illustrates that $\mathcal{R}_c$ reduces as both ρ_1 and ϵ increase. In addition, we note that $\mathcal{R}_c$ is reduced to a value less than one as the parameters δ, ρ_1 and ϵ are increased as shown in Figure 2 (e) and (f). Reducing the control reproduction number ($\mathcal{R}_c$) corresponds to decreasing the number of newly infected animals leading to low shedding of foot and mouth disease virus to other animals and subsequently decreasing the foot and mouth disease burden.

3.3 Equilibrium points of the system

The disease-free equilibrium point of the system of equation is given by

$$E_0 = (S_0, T_{v0}, 0, R_0, V_{v0}, 0)\,. \tag{12}$$

where

$$\begin{aligned}
S_0 &= \frac{\left((1-\rho)\left(b^3+(\omega+\epsilon+\alpha_4)\,b^2+((\omega+\alpha_4)\,\epsilon+\omega\,\alpha_4)\,b\right)+\epsilon\,\omega\,\alpha_4\right)K}{M_1},\\
T_{v0} &= \frac{(b+\omega)\,K\left(\epsilon\,\left(b\rho+\rho_1+\rho_2\right)+b\rho_1\left(1-\rho\right)\right)}{M_1},\\
R_0 &= \frac{K\alpha_4\left(\epsilon\,\left(b\rho+\rho_1+\rho_2\right)+b\rho_1\left(1-\rho\right)\right)}{M_1},
\end{aligned}$$

$$\begin{aligned}
V_{v0} &= \frac{Kb\left(b+\omega+\alpha_4\right)\rho\,\rho_1+K\left(b+\alpha_4\right)\left(b+\omega\right)\left(b\rho+\rho_2\right)}{M_1},\\
M_1 &= b^3+\left(\omega+\epsilon+\rho_1+\rho_2+\alpha_4\right)b^2+\left(\left(\omega+\rho_1+\rho_2+\alpha_4\right)\epsilon+\left(\omega+\rho_1+\rho_2\right)\alpha_4+\omega\,\left(\rho_1+\rho_2\right)\right)b\\
&\quad+\left(\left(\omega+\rho_1+\rho_2\right)\alpha_4+\omega\,\left(\rho_1+\rho_2\right)\right)\epsilon+\omega\,\alpha_4\rho_2,
\end{aligned}$$

The force of infection at the equilibrium point is

$$\lambda^* = \frac{\beta(I_c^* + \eta V_{ca}^*)}{K}. \tag{13}$$

The endemic equilibrium E_1 of the model is given in terms of λ^* and R^* with

$$E_1 = (S^*, T_v^*, I_c^*, R^*, V_v^*, V_{ca}^*), \tag{14}$$

where

$$\begin{aligned}
S^* &= \frac{(1-\rho)\,bK+\omega\,R^*}{(2-\epsilon)\,\lambda^*+b+\rho_1+\rho_2},\\
T_v^* &= \frac{\rho_1\left((1-\rho)\,bK+\omega\,R\right)}{A_4}+\frac{\epsilon\,Kb\rho}{(b+\alpha_4)\,(\lambda^*\,\phi+b+\epsilon)}+\frac{\epsilon\,\rho_2\left((1-\rho)\,b\,K+\omega\,R^*\right)}{(\lambda^*\,\phi+b+\epsilon)\,A_4},\\
I_c^* &= \frac{(2-\epsilon)\,\lambda^*\,\left((1-\rho)\,bK+\omega\,R^*\right)}{(\delta+b+\alpha_2)\left((2-\epsilon)\,\lambda^*+b+\rho_1+\rho_2\right)},\\
R^* &= \frac{1}{1-A_2}\left(\frac{A_3}{(b+\omega)\,(\delta+b+\alpha_2)\,A_4}+A_5\,\left(\frac{\rho}{(b+\alpha_4)\,(\delta+b+\alpha_3)}+\frac{\rho_2\,(1-\rho)}{(\delta+b+\alpha_3)\,A_4}\right)\right),\\
V_v^* &= \frac{Kb\rho}{\lambda^*\,\phi+b+\epsilon}+\frac{\rho_2\left((1-\rho)\,bK+\omega\,R^*\right)}{(\lambda^*\,\phi+b+\epsilon)\left((2-\epsilon)\,\lambda^*+b+\rho_1+\rho_2\right)},\\
V_{ca}^* &= \frac{\lambda^*\,\phi}{\delta+b+\alpha_3}\left(\frac{Kb\rho}{\lambda^*\,\phi+b+\epsilon}+\frac{\rho_2\left((1-\rho)\,bK+\omega\,R^*\right)}{(\lambda^*\,\phi+b+\epsilon)\left((2-\epsilon)\,\lambda^*+b+\rho_1+\rho_2\right)}\right),
\end{aligned}$$

the parameters are given by

$$
\begin{aligned}
A_1 &= (b+\omega)\left((2-\epsilon)\,\lambda^* + b + \rho_1 + \rho_2\right),\\
A_2 &= \frac{\omega}{A_1}\left(\frac{\alpha_2\,(2-\epsilon)\,\lambda^*}{\delta+b+\alpha_2} + \frac{\alpha_4\left(\lambda^*\,\phi\,\rho_1 + (b+\epsilon)\,\rho_1 + \epsilon\,\rho_2\right)}{(b+\alpha_4)\left(\lambda^*\,\phi + b + \epsilon\right)} + \frac{\alpha_3\phi\,\lambda^*\,\rho_2}{(\delta+b+\alpha_3)\left(\lambda^*\,\phi+b+\epsilon\right)}\right),\\
A_3 &= (1-\rho)\,bK\left((2-\epsilon)\,\lambda^*\,(b+\alpha_4)\,\alpha_2 + \alpha_4\rho_1\,(\delta+b+\alpha_2)\right),\\
A_4 &= \left((2-\epsilon)\,\lambda^* + b + \rho_1 + \rho_2\right)(b+\alpha_4),\\
A_5 &= \frac{bK\left(\alpha_3\phi\,\lambda^*\,(b+\alpha_4) + \epsilon\,\alpha_4\,(\delta+b+\alpha_3)\right)}{(b+\omega)\left(\lambda^*\,\phi+b+\epsilon\right)},\\
A_6 &= (b+\epsilon)\,(b+\rho_1)\,(\alpha_3+b+\delta)\,(b+\delta+\alpha_2)\,K,
\end{aligned}
$$

If I_c^* and V_{ca}^* are substituted into (13), we obtain the equation in terms of λ^*:

$$\lambda^*(B_1\lambda^{*2} + B_2\lambda^* + B_3) = 0, \tag{15}$$

where

$$
\begin{aligned}
B_1 &= \phi\,(2-\epsilon)\,(\alpha_3+b+\delta)\,(b+\delta+\alpha_2)\,K,\\
B_2 &= \left((b+\epsilon)\,(2-\epsilon) + \phi\,(b+\rho_1+\rho_2)\right)(\delta+b+\alpha_3)\,(\delta+b+\alpha_2)\,K\\
&\quad - \beta\,\left(K\,b\,(2-\epsilon)\,\phi\,\left(\rho\,(\delta+b+\alpha_2)\,\eta + (1-\rho)\,(\delta+b+\alpha_3)\right)\right.\\
&\quad \left. + R^*\,(2-\epsilon)\,\omega\,\phi\,(\delta+b+\alpha_3)\right),\\
B_3 &= (b+\epsilon)\,\rho_2\,(\alpha_3+b+\delta)\,(b+\delta+\alpha_2)\,K - (b+\delta+\alpha_2)\,K\beta\,b\,\eta\,\phi\,\rho_2\\
&\quad - \beta\,\omega\,R^*\left(\eta\,\phi\,\rho_2\,(b+\delta+\alpha_2) + (b+\epsilon)\,(\alpha_3+b+\delta)\,(2-\epsilon)\right) + A_6\,(1-\mathcal{R}_c).
\end{aligned}
$$

The roots of equation (15) are $\lambda^* = 0$ which corresponds to the disease free equilibrium point and

$$\lambda^* = \frac{-B_2 \pm \sqrt{{B_2}^2 - 4\,B_1\,B_3}}{2\,B_1}. \tag{16}$$

The condition $B_1 > 0$ and $B_3 < 0$ for any values of B_2 when $\mathcal{R}_c > 1$ is satisfied to ensures positivity of λ^* and subsequently the positivity of E_1. If $B_1 > 0$, $B_2 < 0$, $B_3 > 0$ and $B_2^2 - 4B_1B_3 > 0$ when $\mathcal{R}_c < 1$, then there is a possibility of existence of two real positive solutions. Since these two positive solutions exist when $\mathcal{R}_c < 1$, then there is a possibility of existence of a backward bifurcation.

3.4 Stability analysis

In this section we study the stability of disease-free (E_0) and endemic (E_1) equilibria of the system of delay differential model (2) - (6) with the initial condition (7).

Theorem 2. *The disease-free equilibrium point E_0 of system (2) - (7) is locally asymptotically stable for $\mathcal{R}_c < 1$ and unstable for $\mathcal{R}_c > 1$.*

Proof. The characteristic equation of the Jacobian matrix at the disease-free equilibrium E_0

of system (2) - (7) takes the form

$$P(\lambda)\left(\lambda^2+\frac{1}{P_1}\left(C_4+C_5\,(1-\mathcal{R}_c)+C_6+C_7\,\left(\mathrm{e}^{-\lambda\tau_2}\,(\epsilon-1)-1\right)\mathrm{e}^{-\lambda\tau_1}\right)\lambda\right.$$
$$\left.+\frac{1}{P_1}\left(C_8\,(1-\mathcal{R}_c)+C_9+D_1+D_2\,\left(\mathrm{e}^{-\lambda\tau_2}\epsilon-\mathrm{e}^{-\lambda\tau_2}-1\right)\mathrm{e}^{-\lambda\tau_1}\right)\right)=0, \tag{17}$$

where

$$P(\lambda)=(K\gamma+\lambda)\left(\lambda^3+C_1\,\lambda^2+C_2\,\lambda+C_3\right),$$
$$P_1=(b+\omega+\rho_1)\,(b+\epsilon)\,\alpha_4+\rho_2\,(b+\epsilon+\omega)\,\alpha_4+(b+\rho_1+\rho_2)\,(b+\omega)\,(b+\epsilon),$$
$$C_1=3\,b+\epsilon+\omega+\alpha_4+\rho_1+\rho_2,$$
$$C_2=(2\,b+\epsilon+\omega+\rho_1+\rho_2)\,\alpha_4+(2\,b+\epsilon+\omega)\,\rho_2+3\,b^2+2\,b\epsilon+2\,b\omega+2\,b\rho_1+\epsilon\,\omega+\epsilon\,\rho_1+\omega\,\rho_1,$$
$$C_3=(b+\epsilon+\omega)\,\alpha_4\rho_2+(b+\epsilon)\,(b+\omega+\rho_1)\,\alpha_4+(b+\omega)\,(b+\epsilon)\,\rho_2+(b+\rho_1)\,(b+\omega)\,(b+\epsilon),$$
$$C_4=(\epsilon+b+\omega)\,(2\,b+2\,\delta+\alpha_2+\alpha_3)\,\alpha_4\rho_2+(b+\omega+\rho_1)\,(\epsilon+b)\,(b+\delta+\alpha_2)\,\alpha_4$$
$$+(b+\omega)\,(\epsilon+b)\left(b^2+b\delta+b\alpha_2+b\rho_1+2\,b\rho_2+\delta\,\rho_1+2\,\delta\,\rho_2+\alpha_2\rho_1+\alpha_2\rho_2+\alpha_3\rho_2\right),$$
$$C_5=(b+\alpha_3+\delta)\,(\epsilon+b)\,(b+\omega)\,(b+\rho_1)+\alpha_4\,(b+\omega+\rho_1)\,(b+\alpha_3+\delta)\,(\epsilon+b),$$
$$C_6=\frac{(\epsilon+b)\,b\beta\,(\epsilon-2)\,(b+\alpha_3+\delta)\,(\rho-1)\,(b^2+b\omega+b\alpha_4+b\rho_1+\omega\,\alpha_4+\omega\,\rho_1+\alpha_4\rho_1)}{(b+\delta+\alpha_2)\,(b+\rho_1)}$$
$$-\eta\,\phi\,b\,(b+\omega)\,\rho_2\beta-\eta\,\phi\,\,(b+\omega)\,\beta\,\alpha_4\rho_2,$$
$$C_7=\beta\,\left(b\,(1-\rho)\,(\epsilon+b+\omega)\,\alpha_4+\epsilon\,\omega\,\alpha_4+b\,(1-\rho)\,(b+\omega)\,(\epsilon+b)\right),$$
$$C_8=(\epsilon+b+\omega)\,(b+\alpha_3+\delta)\,\alpha_4\rho_2+\alpha_4\,(b+\omega+\rho_1)\,(b+\alpha_3+\delta)\,(\epsilon+b)$$
$$+(b+\rho_1+\rho_2)\,(b+\omega)\,(\epsilon+b)\,(b+\alpha_3+\delta),$$
$$C_9=(b+\delta+\alpha_2)\,(b+\alpha_3+\delta)\,\rho_2\left(b^2+b\epsilon+b\omega+b\alpha_4+\epsilon\,\omega+\epsilon\,\alpha_4+\omega\,\alpha_4\right)$$
$$+\frac{(b+\alpha_3+\delta)\,(\epsilon+b)\,(b+\omega+\rho_1)\,\alpha_4 b\,(1-\rho)\,(2-\epsilon)\,\beta}{b+\rho_1},$$
$$D_1=(b+\alpha_3+\delta)\,(\epsilon+b)\,(b+\omega)\,b\,(1-\rho)\,(2-\epsilon)\,\beta-(b+\delta+\alpha_2)\,\eta\,\phi\,\,(b+\omega)\,\beta\,\rho_2\,(\alpha_4+b),$$
$$D_2=(b+\alpha_3+\delta)\,(\beta\,b\,(1-\rho)\,(b+\omega+\alpha_4)\,(\epsilon+b)+\beta\,\left(-b\omega\,\rho\,\alpha_4+b\omega\,\alpha_4+\epsilon\,\omega\,\alpha_4\right)).$$

The characteristic equation (17) has clearly one negative real root ($\lambda_4=-\gamma\,K$,) and since $C_1C_2-C_3>0$ the other three negative real valued roots are granted by Routh- Hurwitz criterion.

The remaining roots are given by the roots of equation (18)

$$g(\lambda)\equiv\lambda^2+\frac{1}{P_1}\left(C_4+C_5\,(1-\mathcal{R}_c)+C_6+C_7\,\left(\mathrm{e}^{-\lambda\tau_2}\,(\epsilon-1)-1\right)\mathrm{e}^{-\lambda\tau_1}\right)\lambda$$
$$+\frac{1}{P_1}\left(C_8\,(1-\mathcal{R}_c)+C_9+D_1+D_2\,\left(\mathrm{e}^{-\lambda\tau_2}\epsilon-\mathrm{e}^{-\lambda\tau_2}-1\right)\mathrm{e}^{-\lambda\tau_1}\right)=0. \tag{18}$$

If $\mathcal{R}_c>1$, we can get the real λ,

$$g(0)=E_1+E_2+E_3\,(1-\mathcal{R}_c)<0,\qquad\lim_{\lambda\to\infty}g(\lambda)=+\infty.$$

where

$$E_1 = \left(b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\,\omega + \epsilon\,\alpha_4 + \omega\,\alpha_4\right)\left(2\,b + 2\,\delta + \alpha_2 + \alpha_3\right)\rho_2 + \omega\,\alpha_4\left(\epsilon - \rho_1\right)\left(b + \delta + \alpha_2\right) + \frac{\left(b+\epsilon\right) b\beta\ \left(2-\epsilon\right)\left(b+\alpha_3+\delta\right)\left(1-\rho\right)\left(b^2 + b\omega + b\alpha_4 + b\rho_1 + \omega\,\alpha_4 + \omega\,\rho_1 + \alpha_4\rho_1\right)}{\left(b+\delta+\alpha_2\right)\left(b+\rho_1\right)},$$

$$E_2 = \frac{\left(b+\delta+\alpha_2\right)\left(b+\rho_1\right) b\eta\,\phi\,\rho\,\beta\ \left(b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\,\omega + \epsilon\,\alpha_4 + \omega\,\alpha_4\right)}{\left(b+\alpha_3+\delta\right)\left(b+\epsilon\right)},$$

$$E_3 = \left(b+\alpha_3+\delta\right)\left(b+\epsilon\right)\left(b+\omega\right)\left(b+\rho_1\right) + \alpha_4\left(b+\omega+\rho_1\right)\left(b+\alpha_3+\delta\right)\left(b+\epsilon\right) + \left(b+\delta+\alpha_2\right)\left(b+\rho_1\right)\left(b+\epsilon+\omega\right)\alpha_4 + \left(b+\delta+\alpha_2\right)\left(b+\rho_1\right)\left(b+\omega\right)\left(b+\epsilon\right),$$

Therefore, equation (18) has positive real roots for $\mathcal{R}_c > 1$, hence E_0 of (2) - (7) is unstable for $\mathcal{R}_c > 1$.

For $\mathcal{R}_c < 1$ and $\tau_1 = \tau_2 = 0$, equation (18) becomes

$$g\left(\lambda\right) = \lambda^2 + \frac{\left(E_1 + E_2 + E_3\ \left(1-\mathcal{R}_c\right)\right)\lambda}{P_1} + \frac{E_5 + E_6 + E_4\ \left(1-\mathcal{R}_c\right)}{P_1} = 0, \tag{19}$$

where

$$E_4 = \left(b+\alpha_3+\delta\right)\left(b+\epsilon\right)\left(b+\delta+\alpha_2\right)\left(b+\omega+\rho_1\right)\alpha_4 + \left(b+\delta+\alpha_2\right)\left(b+\rho_1\right)\left(b+\alpha_3+\delta\right)\left(b+\epsilon\right)\left(b+\omega\right) + \left(b+\rho_1\right)\left(b+\delta+\alpha_2\right)\left(b+\alpha_3+\delta\right)\left(b+\omega+\alpha_4\right)\left(b+\epsilon\right),$$

$$E_5 = \left(b+\delta+\alpha_2\right)\left(b+\alpha_3+\delta\right)\rho_2\left(b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\,\omega + \epsilon\,\alpha_4 + \omega\,\alpha_4\right) + \left(b+\alpha_3+\delta\right)\left(b+\epsilon\right)\left(b+\omega\right) b\left(1-\rho\right)\left(2-\epsilon\right)\beta + \left(b+\rho_1\right)\left(b+\delta+\alpha_2\right)\left(b+\omega+\alpha_4\right) b\eta\,\phi\,\rho\,\beta,$$

$$E_6 = \frac{\beta\ \left(2-\epsilon\right)\left(b+\alpha_3+\delta\right)\alpha_4\left(b\left(1-\rho\right)\left(b+\rho_1\right)\left(b+\epsilon\right) + b\omega\,\rho\,\rho_1\right)}{b+\rho_1},$$

Equation (19) is quadratic and for $\mathcal{R}_c < 1$

$$E_1 + E_2 + E_3\ \left(1-\mathcal{R}_c\right) > 0, \qquad E_5 + E_6 + E_4\ \left(1-\mathcal{R}_c\right) > 0,$$

Hence, by the RouthHurwitz criterion, the roots of equation (19) have negative real parts for $\mathcal{R}_c < 1$. Therefore, when $\tau_1 = \tau_2 = 0$, the disease-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_c < 1$ and it is unstable if $\mathcal{R}_c > 1$.

For the general non zero delay values ($\tau_1 \neq 0$, $\tau_2 \neq 0$), we first rearrange equation (18) in the following form

$$\lambda^2 + \frac{1}{P_1}\left(C_4 + C_5\ \left(1-\mathcal{R}_c\right) + C_6 + C_7\ \left(\mathrm{e}^{-\lambda\tau_2}\left(\epsilon-1\right) - 1\right)\mathrm{e}^{-\lambda\tau_1}\right)\lambda = -\frac{1}{P_1}\left(C_8\ \left(1-\mathcal{R}_c\right) + C_9 + D_1 + D_2\ \left(\mathrm{e}^{-\lambda\tau_2}\epsilon - \mathrm{e}^{-\lambda\tau_2} - 1\right)\mathrm{e}^{-\lambda\tau_1}\right). \tag{20}$$

Suppose in equation (20) λ is real and denote the left hand side by $L(\lambda)$ and the right hand side by $H(\lambda)$. We can see that $L(0) = 0$ and as $\lambda \to \infty$ the value $L(\lambda)$ approaches infinity, so the left hand side of equation (20) is an increasing function. On the other hand $H(\lambda)$ is a decreasing function as the value of λ increases and

$$H\left(0\right) = \frac{1}{P_1}\left(C_8\ \left(\mathcal{R}_c - 1\right) + C_9 + D_1 + D_2\ \left(\mathrm{e}^{-\lambda\tau_2}\epsilon - \mathrm{e}^{-\lambda\tau_2} - 1\right)\mathrm{e}^{-\lambda\tau_1}\right) > 0. \tag{21}$$

The two functions should intersect at a positive value λ^* which is greater than zero. Hence, equation (21) has a positive real root. Therefore, the disease-free equilibrium point, E_0 is unstable for $\mathcal{R}_c > 1$. For $\mathcal{R}_c < 1$ $L(\lambda)$ is increasing and $H(\lambda)$ decreasing but $H(0) > 0$. Thus, equation (21) has negative real roots and therefore E_0 is unstable for $\mathcal{R}_c > 1$ and stable for $\mathcal{R}_c < 1$.

For positive delays ($\tau_1 \neq 0$, $\tau_2 \neq 0$), assume that $\lambda = i\,\sigma$ without loss of generality where $\sigma > 0$ is a root of equation (18). Substituting into equation (18) shows that

$$\begin{aligned} -\sigma^2 + \frac{1}{P_1}\left(i\left(C_5\ (1-\mathcal{R}_c) + C_7\,F_1 + C_4 + C_6\right)\sigma\right) \\ + \frac{1}{P_1}\left(C_8\ (1-\mathcal{R}_c) + D_2\,F_2 + C_9 + D_1\right) = 0, \end{aligned} \tag{22}$$

where

$$\begin{aligned} F_1 &= \left(\left(\cos\left(\tau_2\sigma\right) - i\sin\left(\tau_2\sigma\right)\right)(\epsilon - 1) - 1\right)\left(\cos\left(\sigma\tau_1\right) - i\sin\left(\sigma\tau_1\right)\right), \\ F_2 &= \left(\left(\cos\left(\tau_2\sigma\right) - i\sin\left(\tau_2\sigma\right)\right)\epsilon - \cos\left(\tau_2\sigma\right) + i\sin\left(\tau_2\sigma\right) - 1\right)\left(\cos\left(\sigma\tau_1\right) - i\sin\left(\sigma\tau_1\right)\right), \end{aligned}$$

Separating the real and the imaginary parts of equation (22) and squaring both parts and adding the two equations, it follows that

$$\sigma^4 + F_3\,\sigma^2 + F_4 = 0. \tag{23}$$

where

$$\begin{aligned} F_3 &= \frac{1}{{P_1}^2}\left(2\,{C_7}^2(\epsilon-1)\left(\cos\left(\tau_2\sigma\right)+1\right) + \left(1-\mathcal{R}_c\right)^2{C_5}^2 + \left(2\,P_1\,C_8 + 2\,C_5\ \left(C_4+C_6\right)\right)\left(1-\mathcal{R}_c\right)\right) \\ &\quad + \frac{1}{{P_1}^2}\left(-\epsilon^2{C_7}^2 + \left(2\,C_9 + 2\,D_1\right)P_1 + \left(C_4+C_6\right)^2\right), \\ F_4 &= \frac{1}{{P_1}^2}\left(2\,{D_2}^2(\epsilon-1)\left(\cos\left(\tau_2\sigma\right)+1\right) + {C_8}^2\left(1-R_c\right)^2 + 2\,C_8\ \left(C_9+D_1\right)\left(1-\mathcal{R}_c\right)\right) \\ &\quad - \frac{1}{{P_1}^2}\left({D_2}^2\epsilon^2 + \left(C_9+D_1\right)^2\right), \end{aligned}$$

and let assume that $\sigma^2 = Q$ and substitute into the polynomial function (23), we obtain,

$$Q^2 + F_3 Q + F_4 = 0. \tag{24}$$

Clearly $\cos\left(\sigma\,\tau_2\right) + 1 \geq 0$ for $\mathcal{R}_c < 1$

$$\begin{aligned} F_3 &\geq \frac{1}{P_1}\left(\left(2\,C_9 + 2\,D_1\right)P_1 + C_4 + C_6 + \left(2\,P_1\,C_8 + C_5 + 2\,C_5\ \left(C_4+C_6\right)\right)\left(1-\mathcal{R}_c\right)\right) \geq 0, \\ F_4 &\geq \frac{1}{P_1}\left(C_9 + D_1 + \left(C_8 + 2\,C_8\ \left(C_9+D_1\right)\right)\left(1-\mathcal{R}_c\right)\right) \geq 0, \end{aligned}$$

By RouthHurwitz criterion equation (24) has negative real roots. Hence, our assumption $\sigma > 0$ is contradicted and (22) has no positive roots. Hence, equation (17) has negative real roots when $\mathcal{R}_c < 1$, E_0 is locally asymptotically stable for all $\tau > 0$. This proves the theorem. □

Theorem 3. *The positive equilibrium* $(S^*, I_c^*, R^*, V_v^*, V_{ca}^*)$ *is globally asymptotically stable when* $\mathcal{R}_c > 1$.

Proof. We use Lyapunov functions to prove the endemic equilibrium is globally asymptotically stable. Let (S, I_c, R, V_v, V_{ca}) be a positive solution of system (2) - (6) with initial conditions (7). To find the Lyapunov function we used logarithmic functions [24, 29, 30].

$$
\begin{aligned}
U_S(t) &= \frac{S(t)}{S^*} - 1 - \ln\left(\frac{S(t)}{S^*}\right), \quad U_{T_v}(t) = \frac{T_v(t)}{T_v^*} - 1 - \ln\left(\frac{T_v(t)}{T_v^*}\right), \\
U_{I_c}(t) &= \frac{I_c(t)}{I_c^*} - 1 - \ln\left(\frac{I_c(t)}{I_c^*}\right), \quad U_R(t) = \frac{R(t)}{R^*} - 1 - \ln\left(\frac{R(t)}{R^*}\right), \\
U_{V_v(t)} &= \frac{V_v(t)}{V_v^*} - 1 - \ln\left(\frac{V_v(t)}{V_v^*}\right), \quad U_{V_{ca}(t)} = \frac{V_{ca}(t)}{V_{ca}^*} - 1 - \ln\left(\frac{V_{ca}(t)}{V_{ca}^*}\right), \\
U_{S+}(t) &= \int_{\tau=0}^{h} \left(\frac{S(t-\tau_1)}{S^*} - 1 - \ln\left(\frac{S(t-\tau_1)}{S^*}\right)\right) d\tau, \\
U_{I_c+}(t) &= \int_{\tau=0}^{h} \left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right) d\tau,
\end{aligned}
$$

Hence, we consider the following,

$$
U(t) = U_S(t) + U_{I_c}(t) + U_R(t) + U_{V_v(t)} + U_{V_{ca}(t)} + U_+(t),
$$

We calculate the derivatives of $U_S(t)$, $U_{I_c}(t)$, $U_R(t)$, $U_{V_v(t)}$, $U_{V_{ca}(t)}$ and $U_+(t)$ separately and combine to get the derivative of the desired Lyapunov function

$$
\begin{aligned}
\frac{dU_S(t)}{dt} &= \left(1 - \frac{S^*}{S}\right) \frac{dS(t)}{dt} \\
&= \left((1-\rho)\, b\, N - \frac{\beta\, (I_c\,(t-\tau_1) + \eta\, V_{ca})\, S}{N} - \frac{(1-\varepsilon)\, \beta\, (I_c\,(t-\tau_1) + \eta\, V_{ca})\, S\,(t-\tau_2)}{N}\right)\left(1 - \frac{S^*}{S}\right) \\
&- \left((\gamma\, N + \mu + \rho_1 + \rho_2)\, S + \omega\, R\right)\left(1 - \frac{S^*}{S}\right),
\end{aligned}
$$

Using the endemic equilibrium (14) we get the,

$$
\begin{aligned}
\frac{dU_S(t)}{dt} &= \left(1 - \frac{S^*}{S}\right) \frac{dS(t)}{dt} = \left(1 - \frac{S^*}{S}\right)\left(\frac{\beta\, (\eta\, V_{ca}^* + I_c^*)\, S^*}{N^{*2}} + \frac{(1-\varepsilon)\, \beta\, (\eta\, v_{ca} + I_c^*)\, S^*}{N^{*2}}\right) \\
&+ \left(1 - \frac{S^*}{S}\right)\left(\frac{(\gamma\, N^* + \mu + \rho_1 + \rho_2)\, S^*}{N^*} - \frac{\omega\, R^*}{N^*}\right) - \left(1 - \frac{S^*}{S}\right)\left(\frac{\beta\, (I_c\,(t-\tau_1) + \eta\, V_{ca})\, S}{N}\right. \\
&\left. + \frac{(1-\varepsilon)\, \beta\, (I_c\,(t-\tau_1) + \eta\, V_{ca})\, S\,(t-\tau_2)}{N}\right) - \left(1 - \frac{S^*}{S}\right)\left((\gamma\, N + \mu + \rho_1 + \rho_2)\, S + \omega\, R\right),
\end{aligned}
$$

After some calculations we obtain

$$\begin{aligned}\frac{dU_S(t)}{dt} &= h_2 S^* N\left(2-\frac{h_1 S}{h_2 S^* N}-\frac{h_2 S^* N}{h_1 S}\right)\left(1-\frac{S^*}{S}\right)\\
&+(1-\varepsilon)\,h_2 S^* N\left(2-\frac{h_1 S\,(t-\tau_2)}{h_2 S^* N}-\frac{h_2 S^* N}{h_1 S\,(t-\tau_2)}\right)\left(1-\frac{S^*}{S}\right)\\
&+\left(\frac{(\rho_1+\rho_2+\mu)\,S^* N}{n}\left(2-\frac{S N^*}{S^* N}-\frac{S^* N}{S N^*}\right)+\gamma\,S^* N\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right)\right)\left(1-\frac{S^*}{S}\right)\\
&+\omega\,R\left(2-\frac{rN}{N^* R}-\frac{N^* R}{R^* N}\right)\left(1-\frac{S^*}{S}\right)-\frac{{h_2}^2 S^{*2} N^2}{h_1 S}\left(\frac{h_1 S}{h_2 S^* N}-1-\ln\left(\frac{h_1 S}{h_2 S^* N}\right)\right)\left(1-\frac{S^*}{S}\right)\\
&-\frac{(1-\varepsilon)\,{h_2}^2 S^{*2} N^2}{h_1 S\,(t-\tau_2)}\left(\frac{h_1 S\,(t-\tau_2)}{h_2 S^* N}-1-\ln\left(\frac{h_1 S\,(t-\tau_2)}{h_2 S^* N}\right)\right)\left(1-\frac{S^*}{S}\right)\\
&-\frac{(\rho_1+\rho_2+\mu)\,S^{*2} N^2}{{N^*}^2 S}\left(\frac{S N^*}{S^* N}-1-\ln\left(\frac{S N^*}{S^* N}\right)\right)\left(1-\frac{S^*}{S}\right)\\
&-\left(\frac{\gamma\,S^{*2} N}{S}\left(\frac{S}{S^*}-1-\ln\left(\frac{S}{S^*}\right)\right)+\frac{\omega\,R^2 N^*}{R^* N}\left(\frac{R^* N}{N^* R}-1-\ln\left(\frac{R^* N}{N^* R}\right)\right)\right)\left(1-\frac{S^*}{S}\right)\\
&-\left(\frac{{h_2}^2 S^{*2} N^2}{h_1 S}\ln\left(\frac{h_1 S}{h_2 S^* N}\right)+\frac{(1-\varepsilon)\,{h_2}^2 S^{*2} N^2}{h_1 S\,(t-\tau_2)}\ln\left(\frac{h_1 S\,(t-\tau_2)}{h_2 S^* N}\right)\right)\left(1-\frac{S^*}{S}\right)\\
&-\left(\frac{(\rho_1+\rho_2+\mu)\,S^{*2} N^2}{{N^*}^2 S}\ln\left(\frac{S N^*}{S^* N}\right)+\frac{\gamma\,S^{*2} N}{S}\ln\left(\frac{S}{S^*}\right)+\frac{\omega\,R^2 N^*}{rN}\ln\left(\frac{rN}{N^* R}\right)\right)\left(1-\frac{S^*}{S}\right)\le 0,\end{aligned}$$

where

$$h_1=\frac{\beta\,\left(I_c\,(t-\tau_2)+\eta\,V_{ca}\right)}{N},\quad h_2=\frac{\beta\,\left(I_c^*+\eta\,V_{ca}^*\right)}{{N^*}^2},\ m_1=\frac{h_2\,S^*\,N}{h_1\,S},$$

and similarly

$$\begin{aligned}\frac{dU_{T_v(t)}}{dt} &= \left(\rho_1 S\left(2-\frac{S^* T_v}{S T_v^*}-\frac{S T_v^*}{S^* T_v}\right)+\varepsilon\,V_v\left(2-\frac{V_v^* T_v}{V_v T_v^*}-\frac{V_v T_v^*}{V_v^* T_v}\right)\right)\left(1-\frac{T_v^*}{T_v}\right)\\
&+\gamma\,n\left(2-\frac{N}{N^*}-\frac{N^*}{N}\right)T_v\left(1-\frac{T_v^*}{T_v}\right)-\frac{\rho_1 S^2 T_v^*}{S^* T_v}\left(\frac{S^* T_v}{S T_v^*}-1-\ln\left(\frac{S^* T_v}{S T_v^*}\right)\right)\left(1-\frac{T_v^*}{T_v}\right)\\
&-\frac{\rho_1 S^2 T_v^*}{S^* T_v}\ln\left(\frac{S^* T_v}{S T_v^*}\right)\left(1-\frac{T_v^*}{T_v}\right)-\frac{\varepsilon\,{V_v}^2 T_v^*}{V_v^* T_v}\left(\frac{V_v^* T_v}{V_v T_v^*}-1\right)\left(1-\frac{T_v^*}{T_v}\right)\\
&-\frac{\gamma\,{N^*}^2 T_v}{N}\left(\frac{N}{N^*}-1-\ln\left(\frac{N}{N^*}\right)\right)\left(1-\frac{T_v^*}{T_v}\right)-\frac{\gamma\,{N^*}^2 T_v}{N}\ln\left(\frac{N}{N^*}\right)\left(1-\frac{T_v^*}{T_v}\right),\end{aligned}$$

$$\frac{dU_{I_c}(t)}{dt} = \left(h_1 S\left(2 - \frac{h_2 S^* N^* I_c}{I_c^* h_1 S} - \frac{I_c^* h_1 S}{h_2 S^* N^* I_c}\right) + \gamma N^* I_c \left(2 - \frac{N}{N^*} - \frac{N^*}{N}\right)\right)\left(1 - \frac{I_c^*}{I_c}\right)$$
$$+ (1-\varepsilon) h_1 S(t-\tau_2)\left(2 - \frac{h_2 S^* N^* I_c}{I_c^* h_1 S(t-\tau_2)} - \frac{I_c^* h_1 S(t-\tau_2)}{h_2 S^* N^* I_c}\right)\left(1 - \frac{I_c^*}{I_c}\right)$$
$$- \frac{{h_1}^2 S^2 I_c^*}{h_2 S^* N^* I_c}\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S} - 1 - \ln\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S}\right)\right)\left(1 - \frac{I_c^*}{Ic}\right)$$
$$- \frac{(1-\varepsilon){h_1}^2 (S(t-\tau_2))^2 I_c^*}{h_2 S^* N^* I_c}\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S(t-\tau_2)} - 1 - \ln\left(\frac{h_2 S^* N^* Ic}{I_c^* h_1 S(t-\tau_2)}\right)\right)\left(1 - \frac{I_c^*}{I_c}\right)$$
$$- \left(\frac{\gamma N^{*2} I_c}{N}\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right) + \frac{{h_1}^2 S^2 I_c^*}{h_2 S^* N^* I_c}\ln\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S}\right)\right)\left(1 - \frac{I_c^*}{I_c}\right)$$
$$- \frac{(1-\varepsilon){h_1}^2 (S(t-\tau_2))^2 I_c^*}{h_2 s N^* I_c}\ln\left(\frac{h_2 s N^* I_c}{I_c^* h_1 S(t-\tau_2)}\right)\left(1 - \frac{I_c^*}{I_c}\right) - \frac{\gamma N^{*2} I_c}{N}\ln\left(\frac{N}{N^*}\right)\left(1 - \frac{I_c^*}{I_c}\right) \le 0,$$

$$\frac{dU_R(t)}{dt} = -\left(\alpha_2 I_c\left(\frac{I_c^* R}{I_c R^*} - 1 - \ln\left(\frac{I_c^* R}{I_c R^*}\right)\right) + \gamma N^* R\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right)\right)\left(1 - \frac{R^*}{R}\right)$$
$$- \left(\alpha_3 V_{ca}\left(\frac{V_{ca}^* R}{V_{ca} R^*} - 1 - \ln\left(\frac{V_{ca}^* R}{V_{ca} R^*}\right)\right) + T_v \alpha_4\left(\frac{T_v^* R}{T_v R^*} - 1 - \ln\left(\frac{T_v^* R}{T_v R^*}\right)\right)\right)\left(1 - \frac{r}{R}\right)$$
$$- \left(\alpha_2 I_c \ln\left(\frac{I_c^* R}{I_c R^*}\right) + \gamma N^* R \ln\left(\frac{N}{N^*}\right) + \alpha_3 V_{ca}\ln\left(\frac{V_{ca}^* R}{V_{ca} R^*}\right) + T_v \alpha_4 \ln\left(\frac{T_v^* R}{T_v R^*}\right)\right)\left(1 - \frac{R^*}{R}\right),$$

$$\frac{dU_{V_v}(t)}{dt} = \left(h_2 \phi V_v^* N\left(2 - \frac{h_1 V_v^*}{h_2 V_v^* N} - \frac{h_2 V_v^* N}{h_1 V_v}\right) + \frac{(\varepsilon+\mu) V_v^* N}{N^*}\left(2 - \frac{V_v N^*}{V_v^* N} - \frac{V_v^* N}{V_v N^*}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right)$$
$$- \frac{{h_2}^2 \phi {V_v^*}^2 N^2}{h_1 V_v}\left(\frac{h_1 V_v}{h_2 v_v N} - 1 - \ln\left(\frac{h_1 V_v}{h_2 V_v^* N}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right) - \frac{\gamma N (V_v^* - V_v)^2}{V_v}$$
$$- \frac{(\varepsilon+\mu){V_v^*}^2 N^2}{{N^*}^2 V_v}\left(\frac{V_v N^*}{V_v^* N} - 1 - \ln\left(\frac{V_v N^*}{V_v^* N}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right) - \rho_2 S\left(\frac{N}{SN^*} - 1 - \ln\left(\frac{S^* N}{SN^*}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right)$$
$$- \left(\frac{{h_2}^2 \phi {V_v^*}^2 N^2}{h_1 V_v}\ln\left(\frac{h_1 V_v}{h_2 V_v^* N}\right) + \frac{(\varepsilon+\mu){V_v^*}^2 N^2}{{N^*}^2 V_v}\ln\left(\frac{V_v N^*}{V_v^* N}\right) + \rho_2 S \ln\left(\frac{S^* N}{SN^*}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right),$$

$$\frac{dU_{V_{ca}}(t)}{dt} = h_1 \phi V_v\left(2 - \frac{h_2 V_v^* V_{ca} N^*}{V_{ca}^* h_1 V_v} - \frac{V_{ca}^* h_1 V_v}{h_2 V_v^* V_{ca} N^*}\right)\left(1 - \frac{V_{ca}^*}{V_{ca}}\right) - \gamma V_{ca} N^* \ln\left(\frac{N}{N^*}\right)\left(1 - \frac{V_{ca}^*}{V_{ca}}\right)$$
$$- \frac{{h_1}^2 \phi {V_v}^2 V_{ca}^*}{h_2 V_v^* V_{ca} N^*}\left(\frac{h_2 V_v^* V_{ca} N^*}{V_{ca}^* h_1 V_v} - 1 - \ln\left(\frac{h_2 V_v^* V_{ca} N^*}{V_{ca}^* h_1 V_v}\right)\right)\left(1 - \frac{V_{ca}^*}{V_{ca}}\right)$$
$$- \frac{{h_1}^2 \phi {V_v}^2 V_{ca}^*}{h_2 V_v^* V_{ca} N^*}\ln\left(\frac{h_2 V_v^* V_{ca} N^*}{V_{ca}^* h_1 V_v}\right)\left(1 - \frac{v_{ca}}{V_{ca}}\right) - \gamma V_{ca} N^*\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right)\left(1 - \frac{V_{ca}^*}{V_{ca}}\right),$$

$$\frac{dU_{I_c+}(t)}{dt} = \frac{d}{dt}\left(\int_{\tau=0}^{h}\left(\frac{I_c(t-\tau_1)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_1)}{I_c^*}\right)\right) d\tau\right)$$
$$= \int_{\tau_1=0}^{h} \frac{d}{dt}\left(\left(\frac{I_c(t-\tau_1)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_1)}{I_c^*}\right)\right) d\tau\right)$$
$$= -\int_{\tau_1=0}^{\infty} \frac{d}{d\,\tau_1}\left(\left(\frac{I_c(t-\tau_1)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_1)}{I_c^*}\right)\right) d\tau_1\right),$$

$$\frac{dU_{S+}(t)}{dt} = \frac{d}{dt}\left(\int_{\tau_2=0}^{h}\left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right)d\tau\right)$$
$$= \int_{\tau=0}^{h}\frac{d}{dt}\left(\left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right)d\tau\right)$$
$$= -\int_{\tau_2=0}^{\infty}\frac{d}{d\,\tau_2}\left(\left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right)d\tau_2\right)$$

The Lyapunov derivative of the function is

$$\frac{dU(t)}{dt} = \frac{dU_S(t)}{dt} + \frac{dU_{T_v}(t)}{dt} + \frac{dU_{I_c}(t)}{dt} + \frac{dU_R(t)}{dt} + \frac{dU_{V_v}(t)}{dt} + \frac{dU_{V_{ca}}(t)}{dt}$$
$$+ \frac{dU_{s+}(t)}{dt} + \frac{dU_{I_c+}(t)}{dt} \leq 0,$$

Using arithmetic and geometric principles we establish that $\frac{dU(t)}{dt}$ is negative or $\frac{dU(t)}{dt} = 0$ when $S = S^*$, $T_v = T_v^*$, $I_c = I_c^*$, $R = R^*$, $V_v = V_v^*$ and $V_{ca} = V_{ca}^*$. Thus the endemic equilibrium is globally asymptotically stable by LaSalle's invariant principle [31] □

4 Parameter estimation and numerical simulation

In this section, we present the parameter values for the model (2)-(6) from the relevant literature. We use the parameter values for numerical simulations that will assist understanding the model predictions. We give simulations to show the effects of time delay on the dynamics of the foot and mouth disease. The initial number of susceptible animals is 200 animals per km^2 with one infected animal [7].

4.1 Parameter estimation

The per-capita death rate γ is estimated using the carrying capacity, natural birth rate and the natural death rate, $\gamma = (b - \mu)/200$. The amplification rate η and vaccination rate ρ are estimated by $0 \leq \eta \leq 1$ and $0 \leq \rho \leq 1$ respectively. The minimum and maximum values of vaccination rate are 40% and 75% [32]. All other parameter values used in the numerical simulations are given in Table 1 with their sources. Some parameter values are taken as they appear in literature while others are determined based on estimating the given parameters using in literature.

4.2 Numerical simulations

In this section, we present the numerical simulations to further enhance our understanding of the model (2)-(6) and to explore the effects of prophylactic and reactive vaccination,

Table 1: Dimensional parameter values for the model

Parameter description	symbol	Units	Value	source
Transmission rate	β	Day^{-1}	1.4	[11, 33]
birth rate	b	Day^{-1}	0.3	[34]
Transforming rate from I_c to R	α_2	Day^{-1}	0.1	[5, 7, 34]
The vaccinated carrier rate constant	α_3	Day^{-1}	0.2	[6]
The recovery rate constant	ω	Day^{-1}	[0.01, 0.2]	[31, 35, 36]
The rate of protection loss	ϕ	Day^{-1}	[0, 1]	[35]
Delay one	τ_1	Day^{-1}	[1,6]	[35]
Delay two	τ_2	Day^{-1}	[1, 14]	[7, 35]
Natural death rate	μ	$year^{-1}$	0.05	[34]
Treating and vaccinating of susceptible animals	ρ_1	Day^{-1}	0.5	[5, 7, 34]
Culling of clinical infective and vaccinated carrier animals	δ	Day^{-1}	[0.01, 0.5]	[11, 35]
Treating of vaccinated animals	ϵ	Day^{-1}	0.1	[31, 37]
The rate of Vaccinating susceptible animals	ρ_2	Day^{-1}	0.1	Estimate
The rate recovery of treating animals	α_4	Day^{-1}	0.1	Estimate

prophylactic treatment and reactive culling of infected animals. We first examine the effects of different control strategies which are the rate of vaccination (ρ), the rate of treating and vaccinating susceptible animals (ρ_1), the rate of treating of susceptible animals (ρ_2), the rate of treating vaccinated animals (ϵ) and the rate of culling infected and vaccinated carrier animals (δ). Using the least and high rates of control strategies, we investigate the effects of prophylactic vaccination and treatment and culling of infected animals using a two-time delay FMD model.

The effect of the rate of vaccination (ρ) on the disease dynamics is shown in Figure 3. Increasing the rate of vaccination leads most animals entering to the treated and vaccinated of susceptible, and to vaccinated animal classes. The implication of the rate of treating and vaccinating of susceptible animals on the dynamics of the disease using a system of ordinary differential equations is shown in Figure 4. Increasing the rate of treating and vaccinating of susceptible animals results in most animals entering the vaccinated class which is a protected class. The effect of the rate of vaccinating susceptible animals on a system of ordinary differential equations is shown in Figure 5. The effect of the rate of treating vaccinated animals on a system of ordinary differential equations is shown in Figure 6. Increasing the rate of treating vaccinated animals increases the susceptible, and treated and vaccinated classes and decreases the other classes. Figure 7 shows the increasing of the rate of culling infected and vaccinated carrier animals decreases the infected class essential vaccinated carrier and theoretically increase the flow of animals into the susceptible animals, the treated and vaccinated, and vaccinated classes.

Therefore, increasing the rate of vaccination, the rate of treating and vaccination of susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and

vaccinated carrier animals results decrease the foot and mouth disease burden.

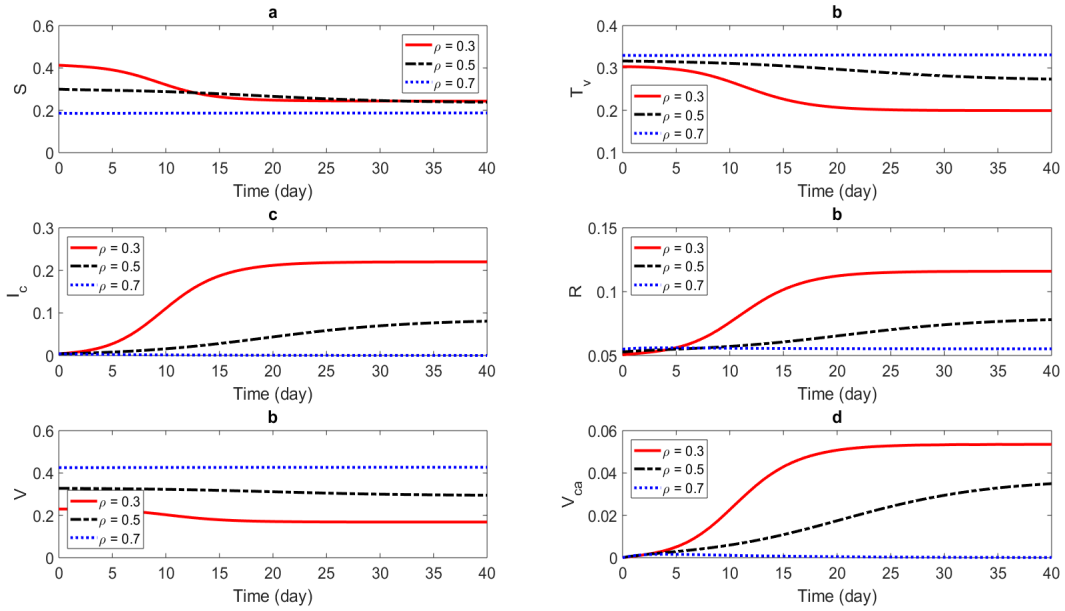

Figure 3: The effects of the rate of vaccination (ρ) on the dynamics of the disease.

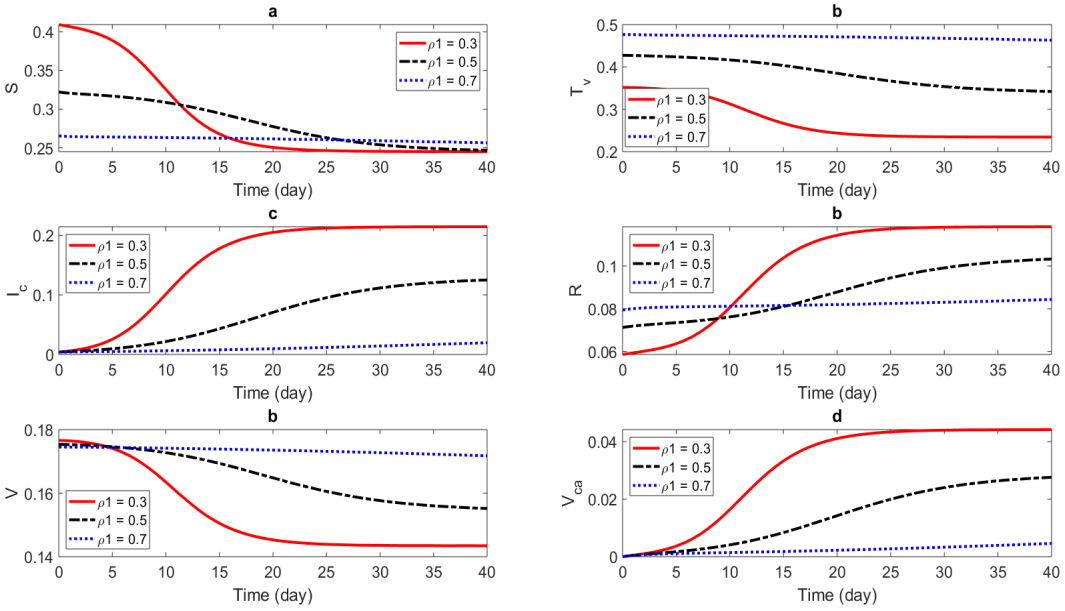

Figure 4: The effects of treating and vaccinating susceptible animals (ρ_1) on the dynamics of the disease.

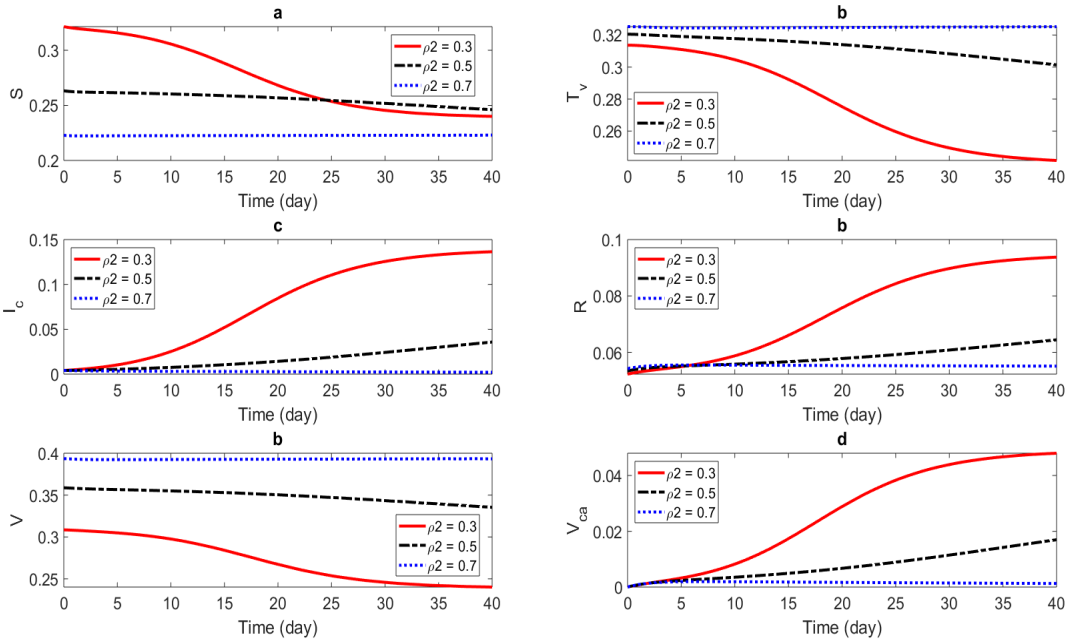

Figure 5: The effects of vaccinating susceptible animals (ρ_2) on the dynamics of the disease.

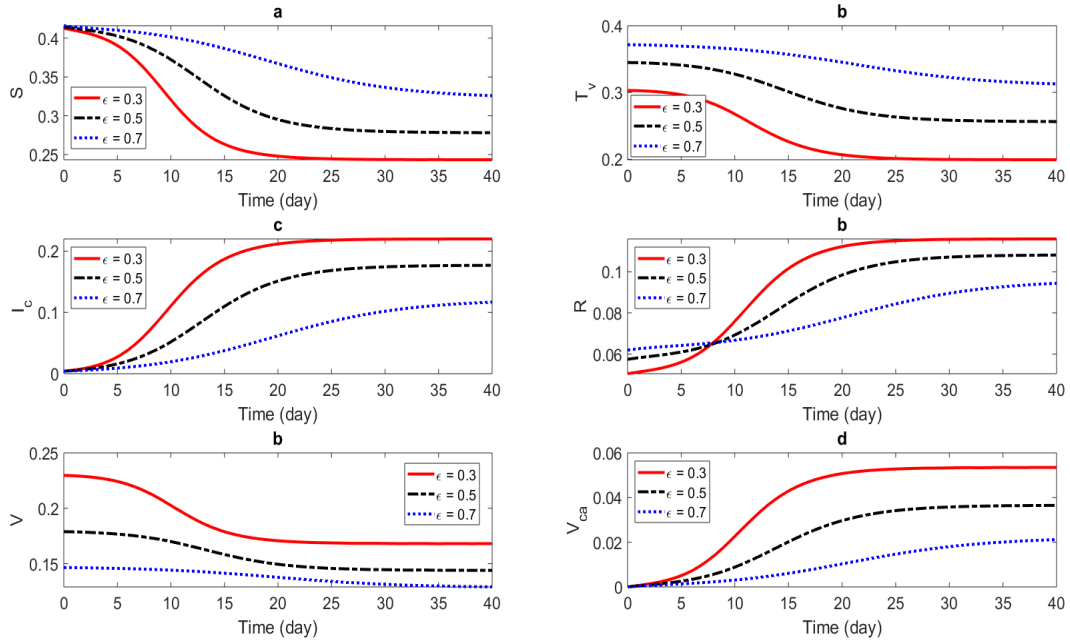

Figure 6: The effects of treating vaccinated animals (ϵ).

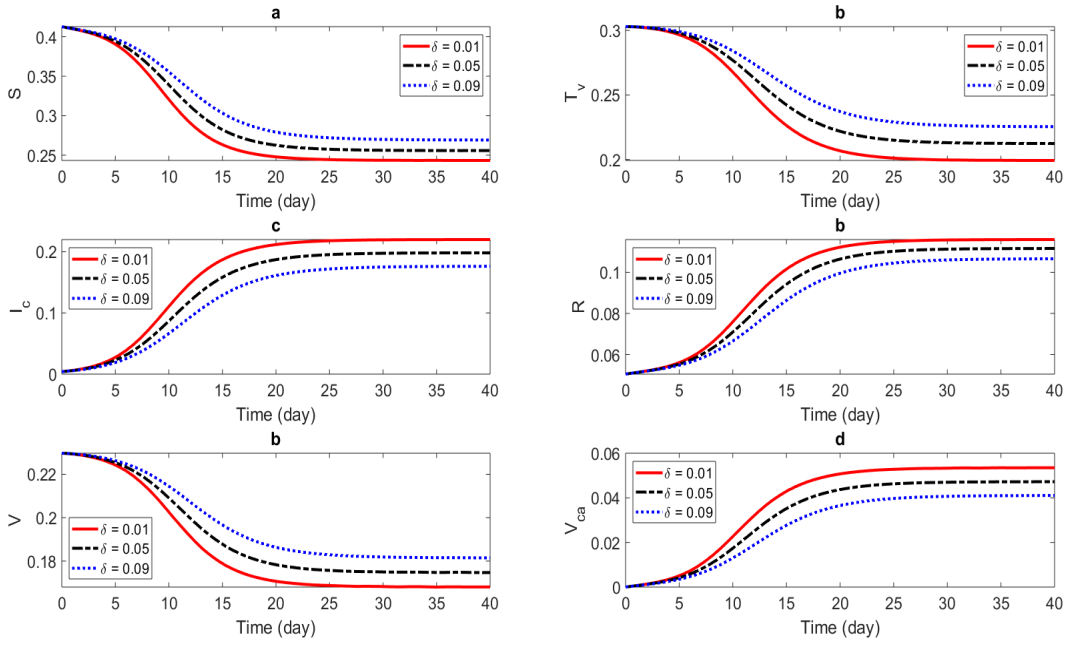

Figure 7: The effects of culling infected animals (δ) on system of solutions.

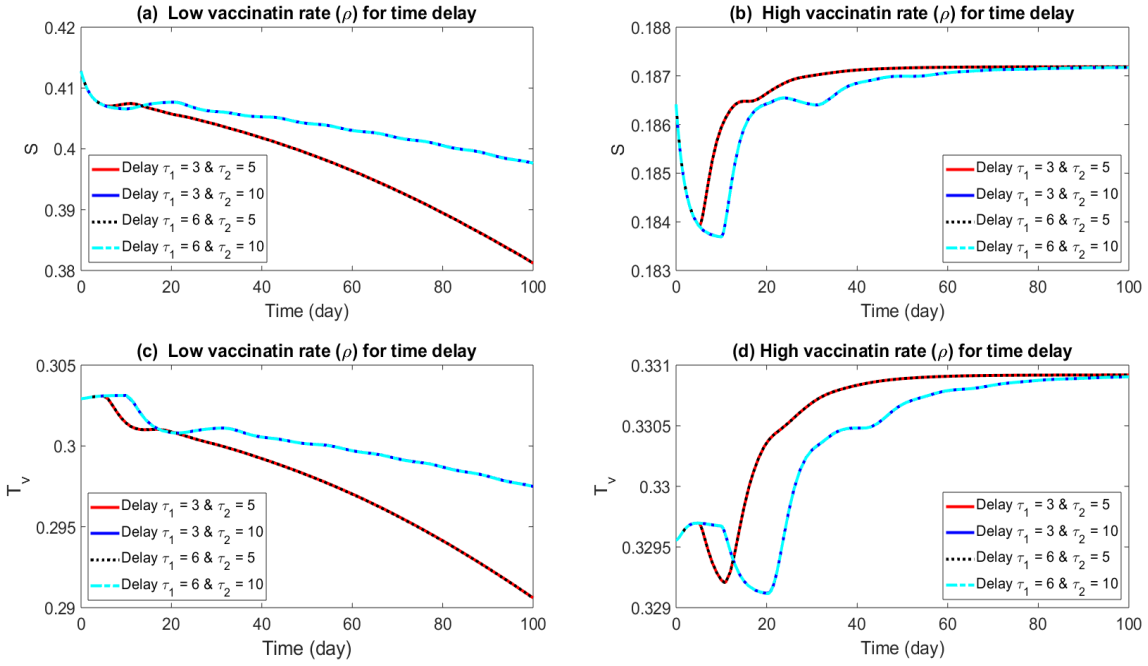

Figure 8: The effects of low ($\rho = 0.3$) and maximum ($\rho = 0.7$) rate of vaccination (ρ) on susceptible and treated animal classes with time delays.

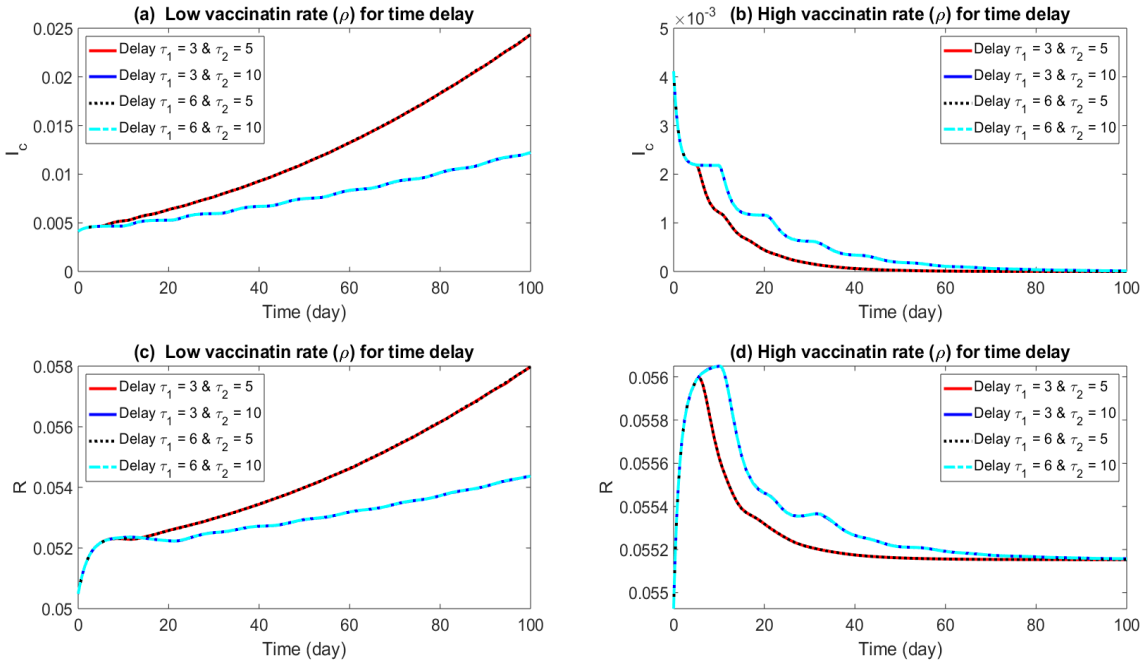

Figure 9: The effects of low ($\rho = 0.3$) and maximum ($\rho = 0.7$) rate vaccination (ρ) on clinically infected and recovery animal classes with time delays.

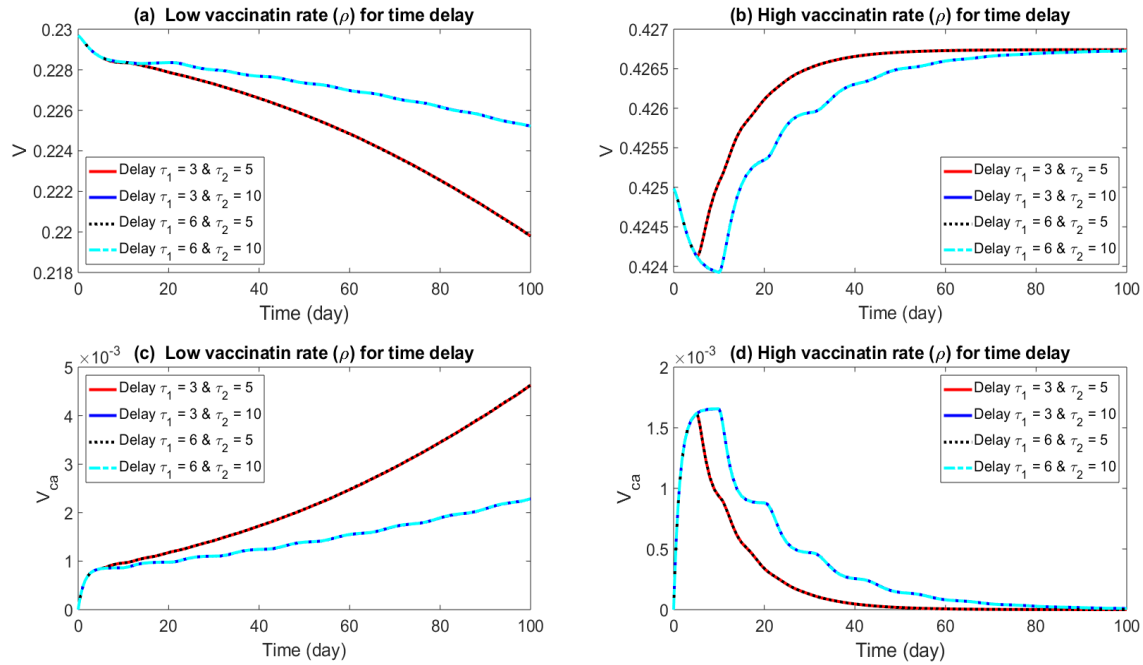

Figure 10: The effects of low ($\rho = 0.3$) and maximum ($\rho = 0.7$) rate of vaccination (ρ) on vaccinated and vaccinated carrier animal classes with time delays.

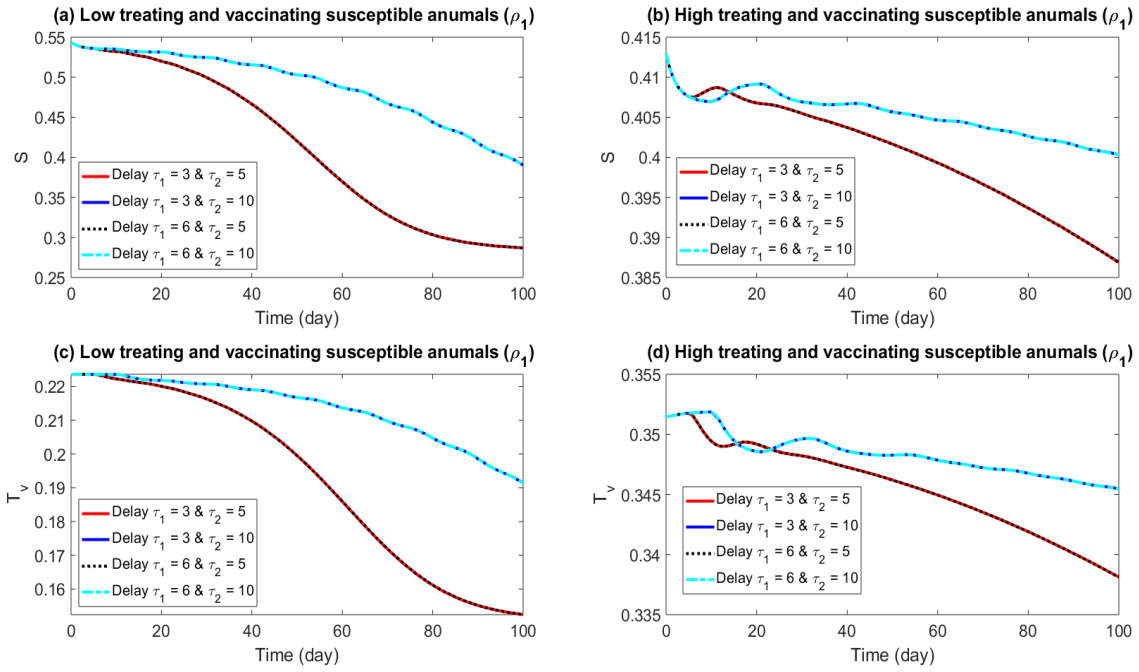

Figure 11: The effects of low ($\rho_1 = 0.1$)and maximum ($\rho_1 = 0.3$) rate of treating susceptible animals (ρ_1) on susceptible and treated animal classes with time delays.

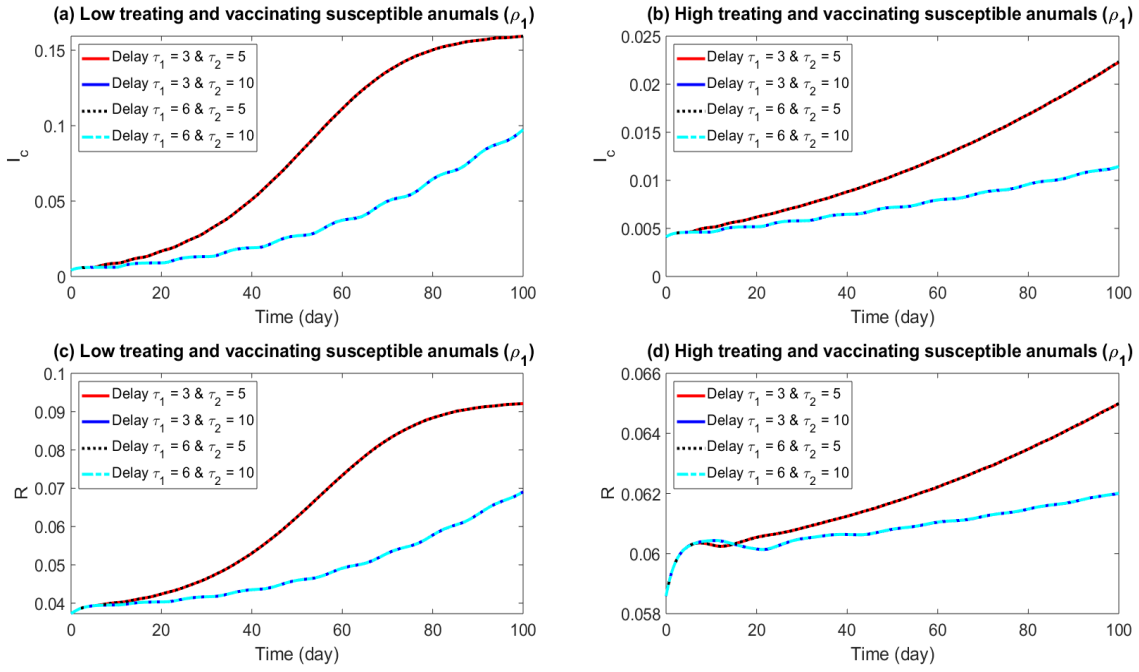

Figure 12: The effects of low ($\rho_1 = 0.1$)and maximum ($\rho_1 = 0.3$) rate of treating susceptible animals (ρ_1) on susceptible and treated animal classes with time delays.

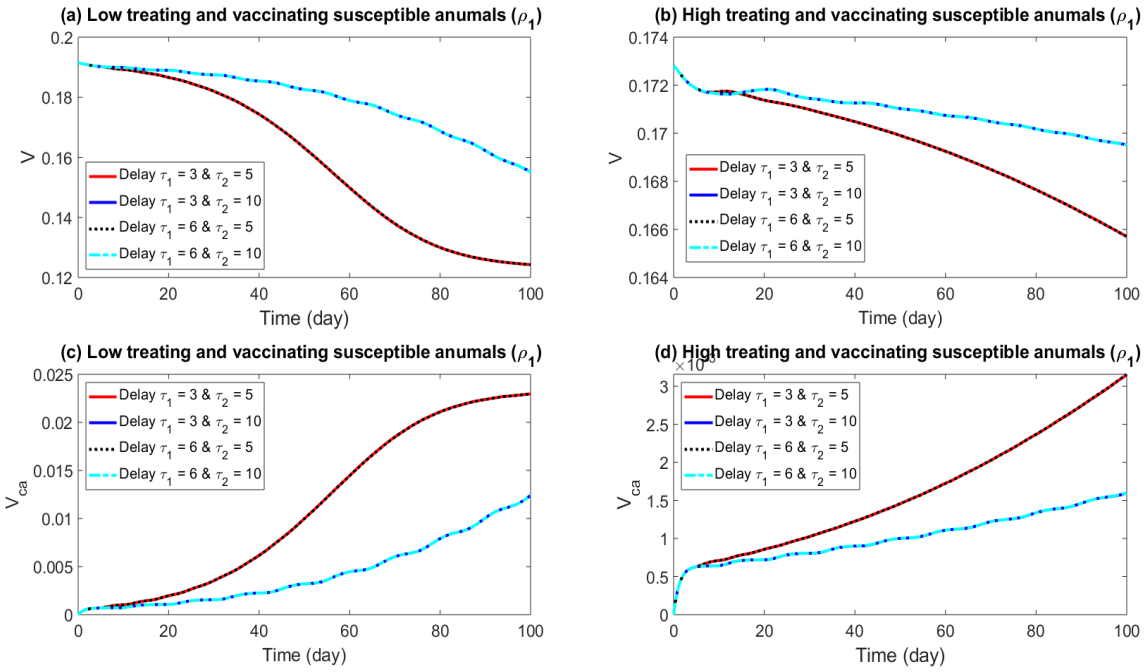

Figure 13: The effects of low ($\rho_1 = 0.1$)and maximum ($\rho_1 = 0.3$) rate of treating susceptible animals (ρ_1) on susceptible and treated animal classes with time delays.

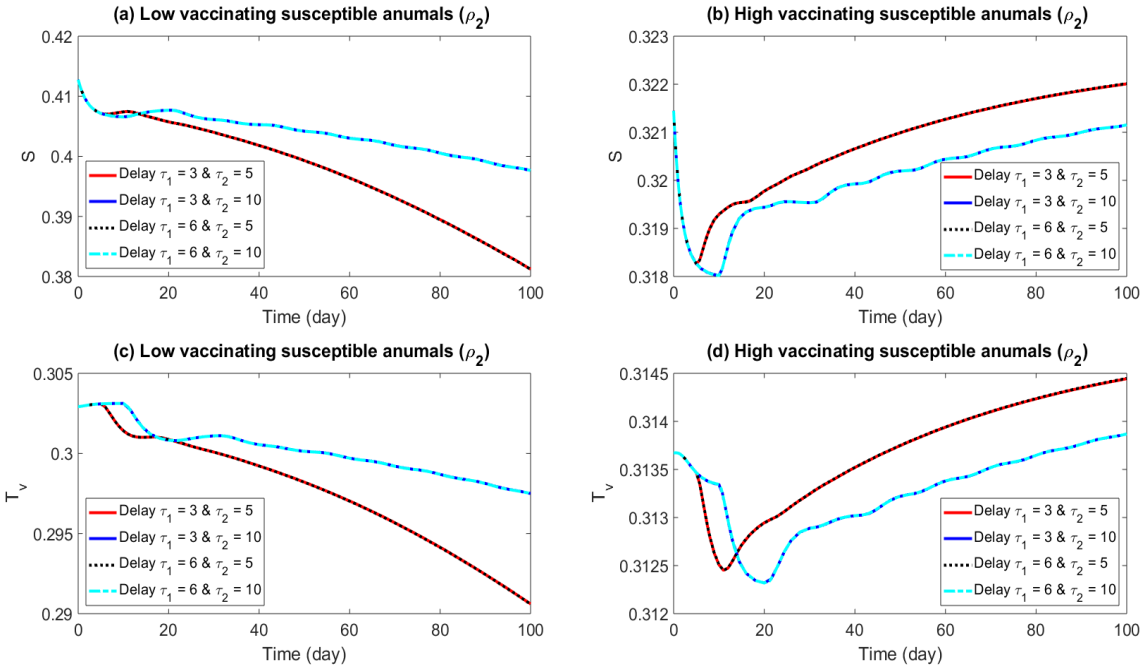

Figure 14: The effects of low ($\rho_2 = 0.1$)and maximum ($\rho_2 = 0.3$) rate of vaccinating susceptible animals (ρ_2) on susceptible and treated animal classes with time delays.

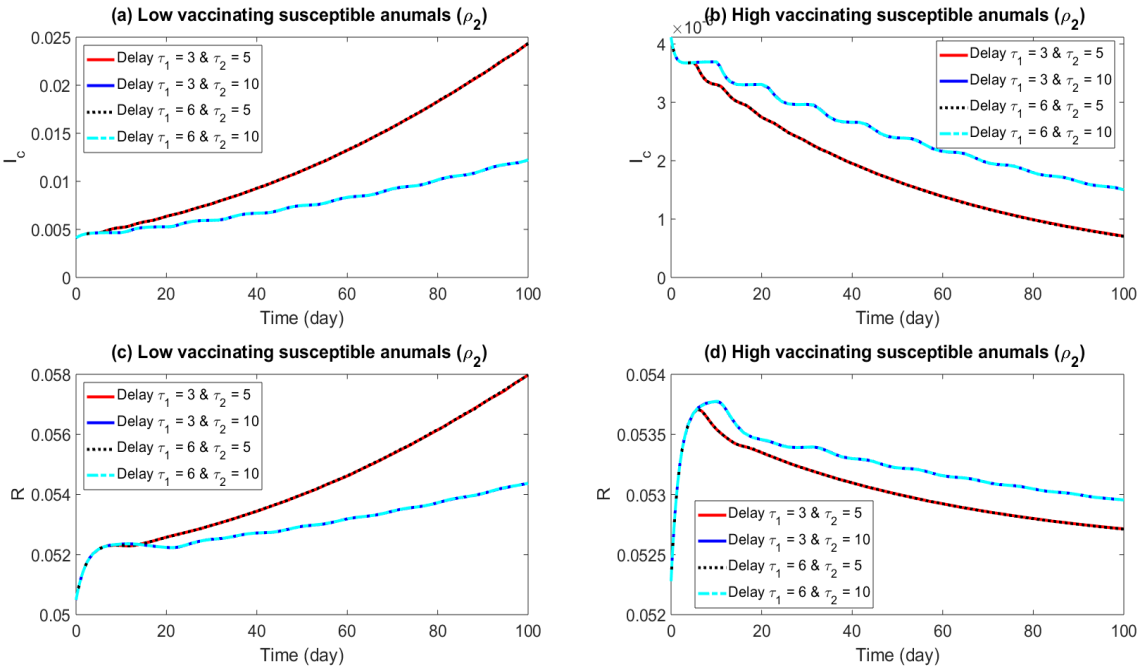

Figure 15: The effects of low ($\rho_2 = 0.1$)and maximum ($\rho_2 = 0.3$) rate of vaccinating susceptible animals (ρ_2) on susceptible and treated animal classes with time delays.

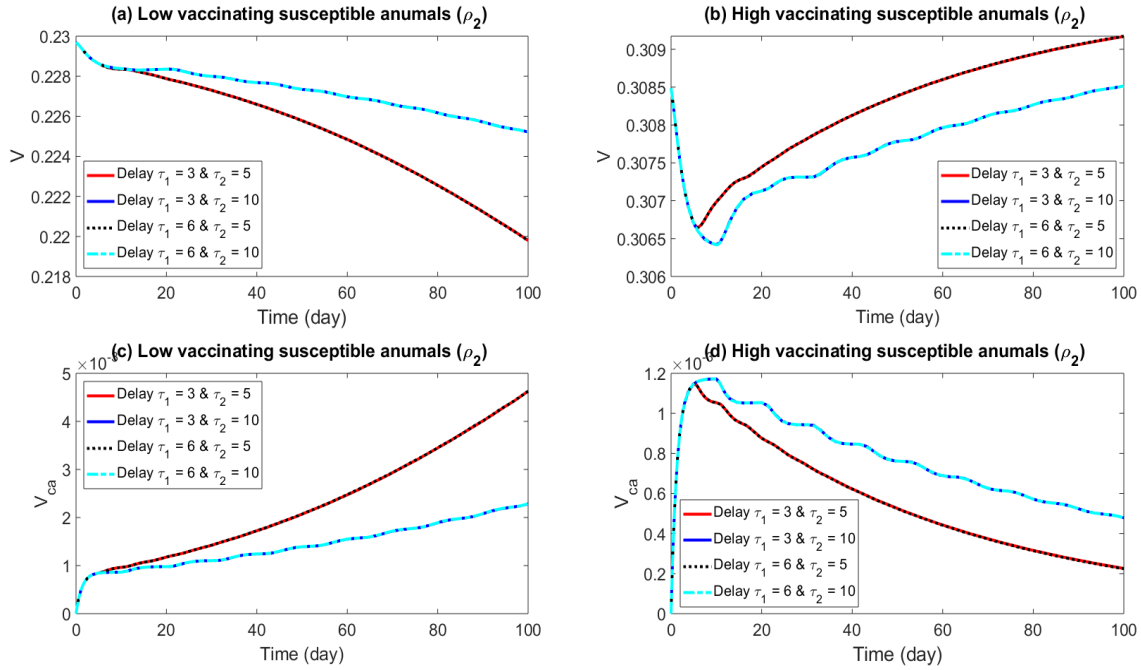

Figure 16: The effects of low ($\rho_2 = 0.1$)and maximum ($\rho_2 = 0.3$) rate of vaccinating susceptible animals (ρ_2) on susceptible and treated animal classes with time delays.

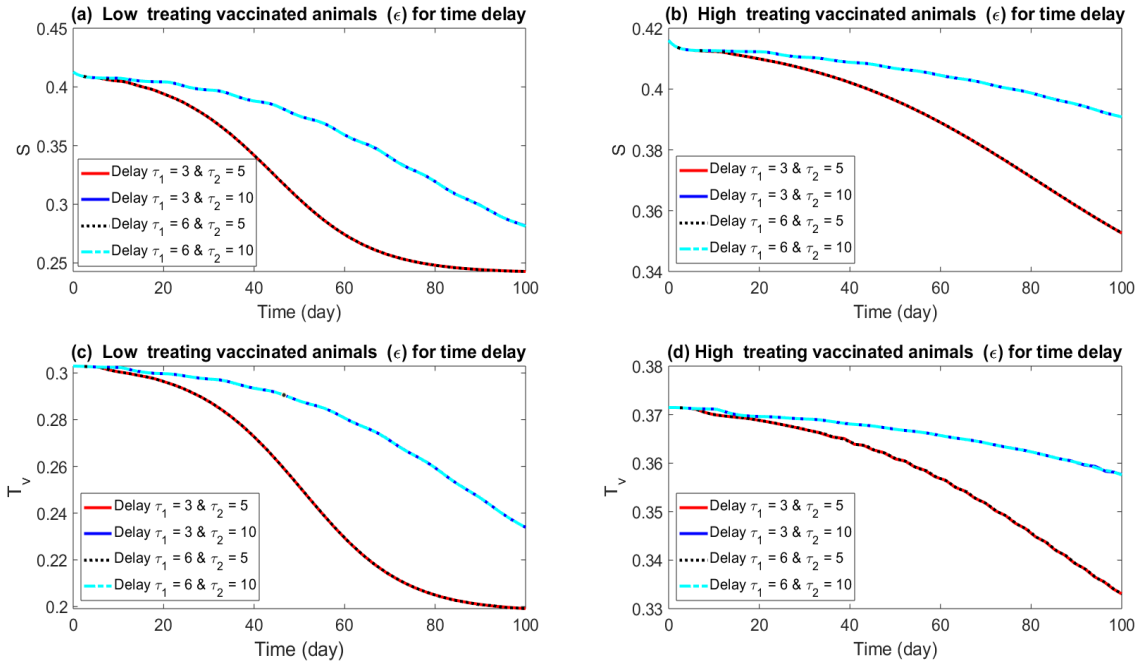

Figure 17: The effects of low ($\epsilon = 0.3$) and maximum ($\epsilon = 0.7$) rate of treating vaccinated animals (ϵ) on susceptible and treated animal classes with time delays.

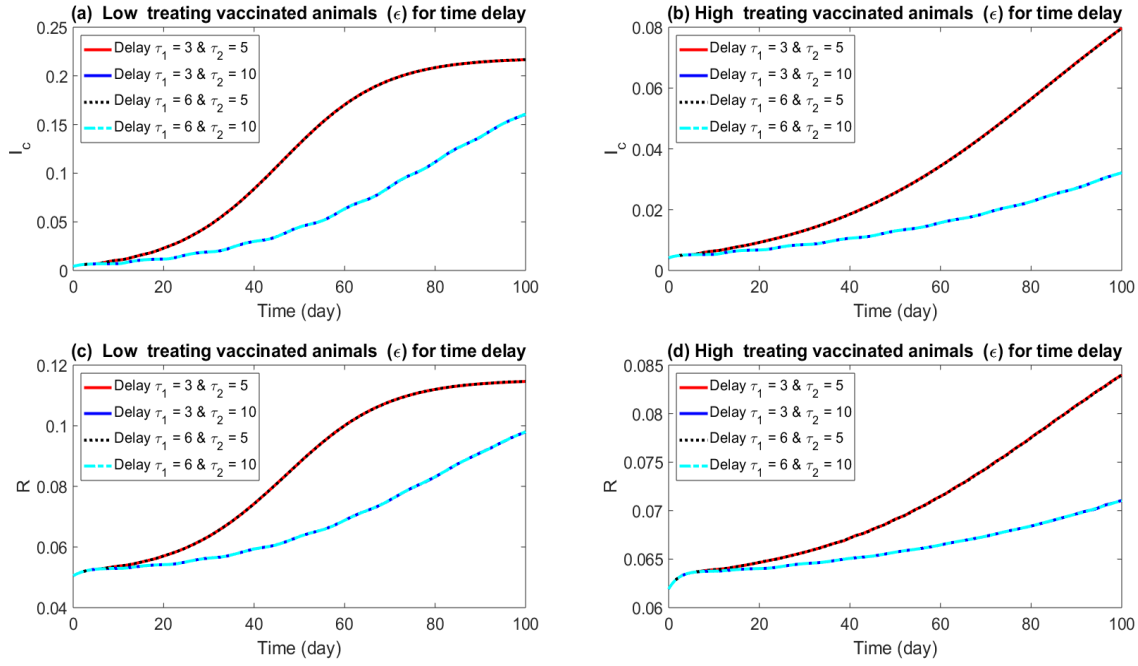

Figure 18: The effects of low ($\epsilon = 0.3$) and maximum ($\epsilon = 0.7$) rate of treating vaccinated animals (ϵ) on clinically infected and recovery animal classes with time delays.

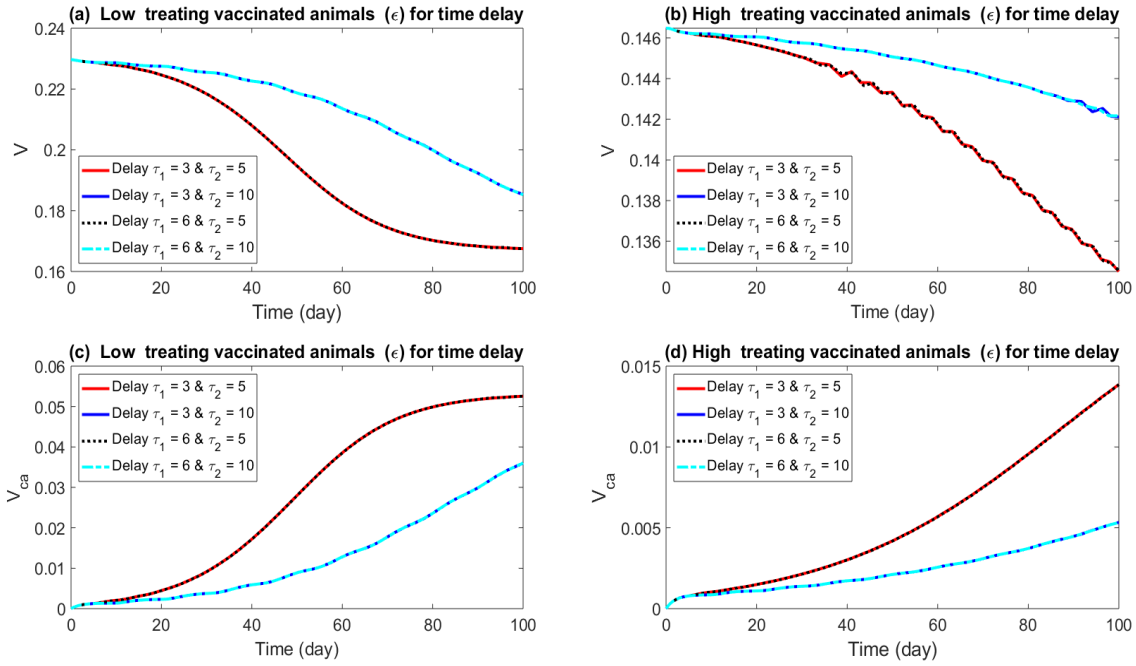

Figure 19: The effects of low ($\epsilon = 0.3$) and maximum ($\epsilon = 0.7$) of treating vaccinated animals (ϵ) on vaccinated and vaccinated carrier animal classes with time delays.

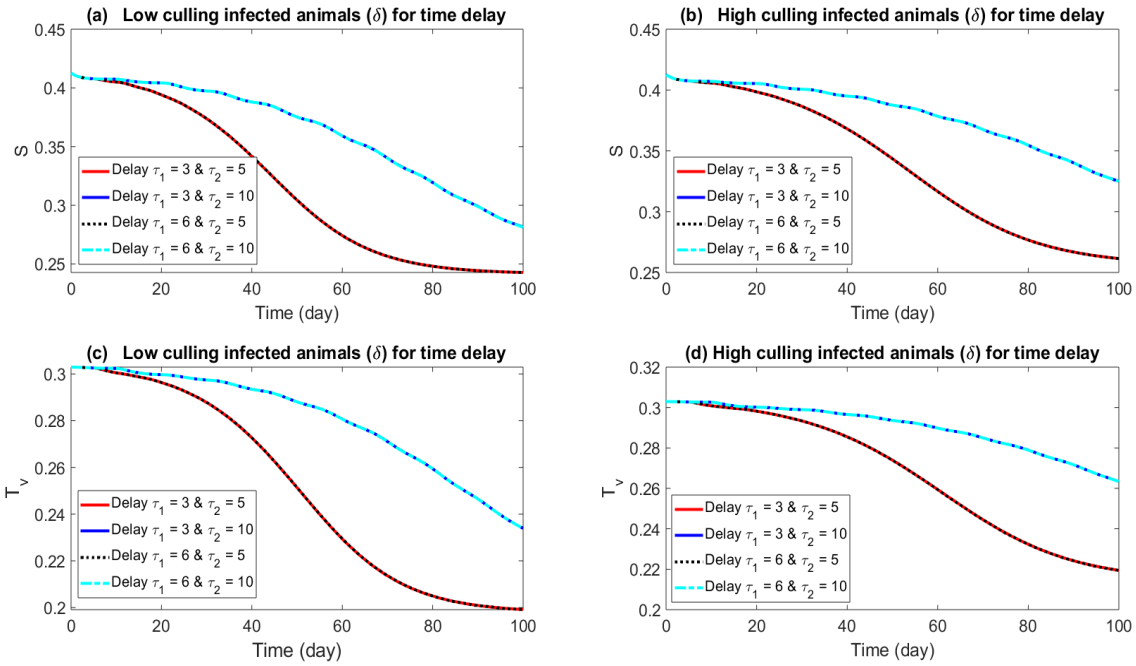

Figure 20: The effects of low ($\delta = 0.01$) and maximum ($\delta = 0.09$) rate of culling infected animals (δ) on susceptible and treated animal classes with time delays.

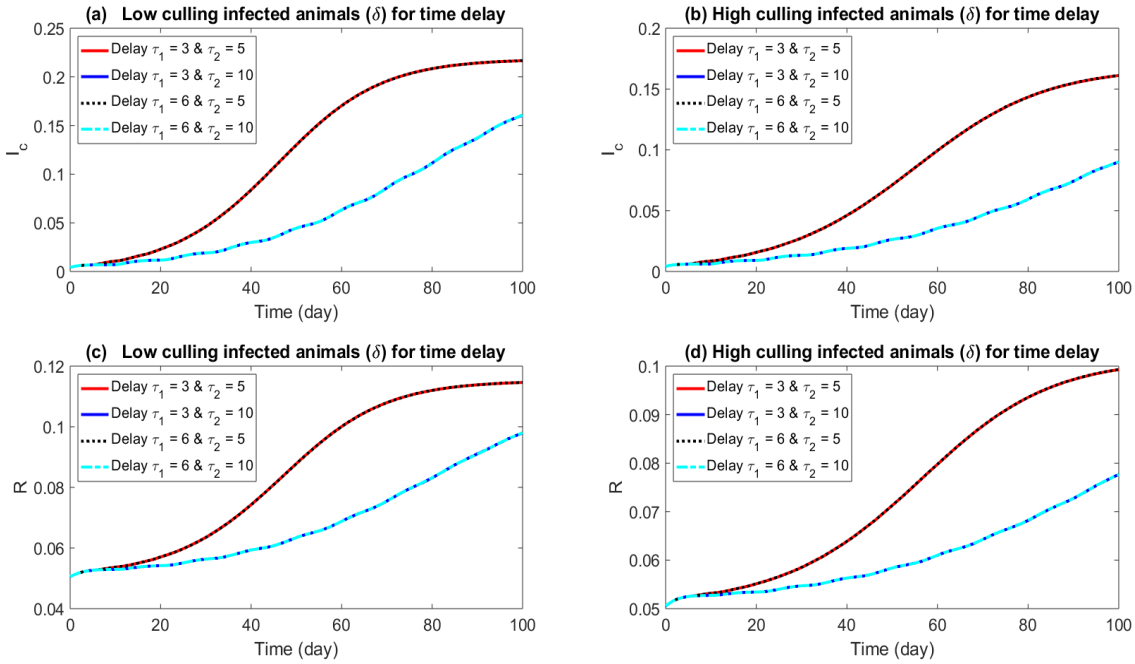

Figure 21: The effects of low ($\delta = 0.01$) and maximum ($\delta = 0.09$) rate of culling infected animals (δ) on clinically infected and recovery animal classes with time delays.

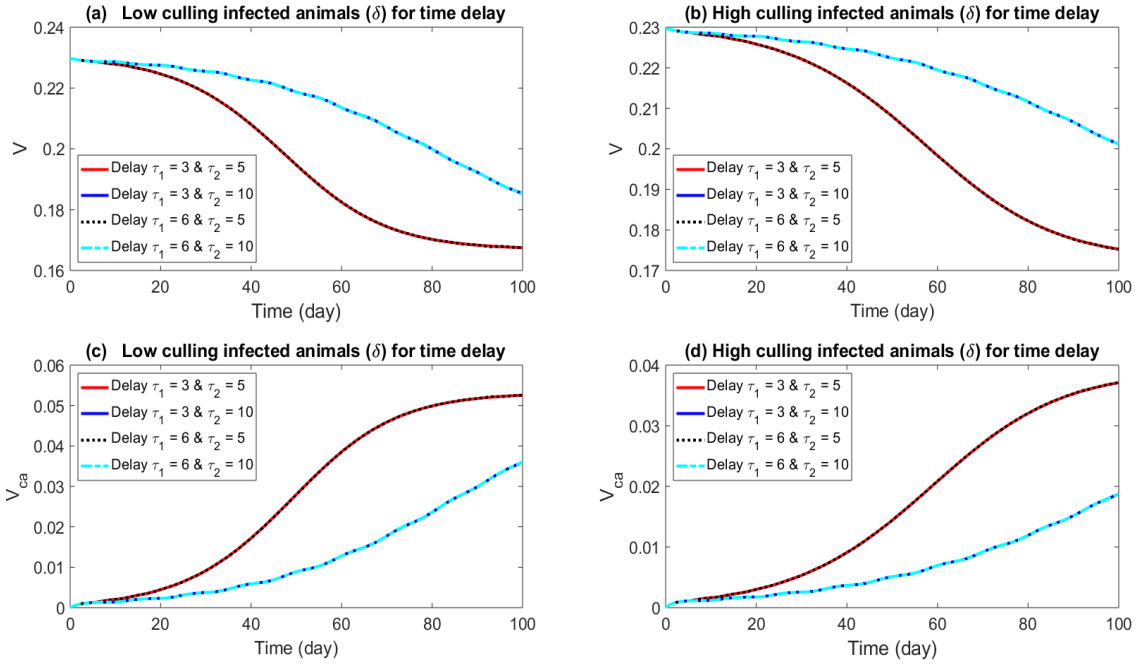

Figure 22: The effects of low ($\delta = 0.01$) and maximum ($\delta = 0.09$) rate of culling infected animals (δ) on vaccinated and vaccinated carrier animal classes with time delays.

The scenarios of varying the two-time delays by fixing the control parameters as least or high rates are explored in Figure 7 to Figure 22 by varying time delays we seek to investigate the effects of timing the development of clinical symptom after the animals have contracted the infection.

Sub-Figure (a) and (c) of Figure 8 to 19, shows that, fixing of $\rho = 0.3$ or $\rho_1 = 0.1$ or $\rho_2 = 0.1$ or $\epsilon = 0.3$ and increasing τ_2 increases susceptible, treated and vaccinated classes whilst decreases the clinically infected, recovered and vaccinated carrier classes. The decrease in τ_2 leads to the decrease in susceptible, treated and vaccinated, and vaccinated and increase in clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 seems to have no significant effect on the dynamic of the disease.

Sub-Figure (b) and (d) of Figure 8 to 19, shows that, fixing of $\rho = 0.7$ or $\rho_1 = 0.3$ or $\rho_2 = 0.3$ or $\epsilon = 0.7$ and increasing τ_2 increases susceptible, treated and vaccinated classes and decreases clinically infected, recovered and vaccinated carrier classes. Decreasing of τ_2 leads to decrease the susceptible, treated and vaccinated classes and increases clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 does not have any significant effect on the dynamic of the disease.

Sub-Figure (a) and (c) of Figure 20 to 22, shows that, fixing of $\delta = 0.01$ and increasing τ_2 increases susceptible, treated and vaccinated classes whilst decreases the clinically infected, recovered and vaccinated carrier classes. The decrease in τ_2 leads to the decrease in susceptible, treated and vaccinated, and vaccinated and increase in clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 seems to have no significant effect on the dynamic of the disease.

Sub-Figure (b) and (d) of Figure 20 to 22, show that, fixing of $\delta = 0.09$, and increasing τ_2 increases susceptible, treated and vaccinated classes and decreases clinically infected, recovered and vaccinated carrier classes. Decreasing of τ_2 leads to decrease the susceptible, treated and vaccinated classes and increases clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 does not have any significant effect on the dynamic of the disease.

Our numerical simulation results suggest that increasing τ_2 and increasing each of the control parameters minimizes the burden of foot and mouth disease and followed by increasing either the control or τ_2 parameters. However, decreasing both the control and τ_2 parameters increases the burden of foot and mouth disease. Increasing or decreasing of τ_1 in combination with increasing or decreasing of control parameters does not show a significant effect the dynamic of infection.

5 Discussion and results

The delay ordinary differential equation model for foot and mouth disease of cattle was presented in this paper to capture the effects of prophylactic vaccination, reactive vaccination,

prophylactic treatment, reactive culling and the effects of time delay. Mathematical analysis and numerical simulations were carried out to reveal the effects of the aforementioned control strategies and time delay on the burden of foot and mouth disease.

Mathematical analysis revealed the effects of the control reproduction number, $\mathcal{R}_c$ and the existence of two equilibria, namely the disease-free equilibrium and an endemic equilibrium. The disease-free equilibrium was locally asymptotically stable when $\mathcal{R}_c$ is less than unity. This means the FMD burden can be kept in check if the control strategies used suppress consistently the control reproduction number below unity. In fact, there is a possibility of eradicating the infection with such controls. However, if the control cannot reduce $\mathcal{R}_c$ below unity, then there is a possibility that the FMD can spread to endemic levels. This scenario may be characteristic of a control strategy that is either inadequately administered or less efficient. Evidence of controls such as the rate of vaccination, the rate of treating and vaccinating susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and vaccinated carrier animals is available where the burden of foot and mouth disease continued to increase [6, 13, 15, 38]. In particular, there are vaccines that do not induce rapid control [6]. The sensitivity analysis of $\mathcal{R}_c$ with respect to prophylactic vaccination showed that $\mathcal{R}_c$ decreases only when the rate of loss of vaccination is below a critical loss of vaccination otherwise the benefits of prophylactic vaccination alone may not be realized. This means that prophylactic vaccination as a single strategy may not successfully eradicate the foot and mouth disease. Simulations on $\mathcal{R}_c$ showed that the control reproduction number ($\mathcal{R}_c$) is less than one when the rate of treating and vaccinating of susceptible animals and rate of culling of clinically infected and vaccinated carrier animals are high.

Numerical simulations allowed us to observe the effects of time delays, prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling parameters on foot and mouth disease transmission in cattle. The numerical simulations suggested that increasing of both time delay two and control parameters or increasing of either of the time-delay two or control parameters decrease the burden of foot and mouth disease. But increasing of both time-delay one and control parameters does not show a significant effect on the dynamics of foot and mouth disease. Hence, time delay two has a significant effect on foot and mouth disease. Increasing time delay two means that the newly infected animals delay maximally to show clinical symptoms leading to less shedding of foot and mouth disease virus to other animals and subsequently the reduction of foot and mouth burden. Similarly, increasing of control parameters such as prophylactic vaccination, reactive vaccination, prophylactic treatment, and reactive culling parameters have a substantial significant effect on the decreasing of foot and mouth disease burden. Prophylactic and reactive vaccinations and treatment have been found to maintain immunity to FMD [17], but the high cost of vaccines and drugs limit the use vaccination and treatment control strategies for foot and mouth disease [15, 38, 39]. The results suggest that the strategy of decreasing time delay two whilst increasing the degree of control parameters contributes a significant effect on the reduction of foot and mouth disease but decreasing of both time delay two and control parameters increases the foot and mouth disease burden. Decreasing time delay two means that the newly infected animals fast to show clinical symptoms and leading to high shedding of foot and mouth disease virus to other animals and subsequently the increase of foot and mouth burden. Findings also suggest that the strategy of increasing of time delay two whilst decreasing the control parameters may not significantly reduce the foot and mouth disease burden. This outcome has consequences on systems that would want to reduce the costs of

control strategies but in the end, the disease burden will continue to hamper their efforts. Therefore, the strategy which has a significant effect on the protection of foot and mouse disease burden is increasing both the time delay two and the control parameters.

6 Conclusion

Findings suggested that the foot and mouth disease burden is decreased significantly when the unprotected animals delay maximally their time to show clinical symptoms and at the same time when the effectiveness of the control strategies are increased. It is imperative that control strategies play a significant role in moving the animals into the protected routes of infection than leaving more animals in the unprotected route of infection. By implication, strategies that directly protect and reduce the number of susceptible animals should be prioritized and effectively enhanced as these directly divert the animal flow into the protected route of the foot and mouth disease dynamics.

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CHAPTER 4

𝔸NALYSIS OF FOOT AND MOUTH DISEASE CONTROL USING A REACTION-DIFFUSION MODEL INCORPORATING ENVIRONMENTAL EFFECTS

In this Chapter we present a mathematical model of foot and mouth disease using a reaction-diffusion model incorporating control strategies for foot and mouth disease spread in both space and time.

Analysis of foot and mouth disease control using a reaction-diffusion model incorporating environmental effects

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Abstract

Foot and mouth disease spreads rapidly from animal to animal and hence, it is vital to employ effective strategies to control the disease spread. The disease may be contracted through contact between animals and through viruses shed into the environment. We develop a reaction-diffusion model incorporating control strategies for foot and mouth disease spread in both space and time. In this study, we investigate the effects of vaccination, quarantining of clinically infected animals, shedding of foot and mouth disease virus into the environment and rates of movement of animals and virus. Results from our study suggest that the foot and mouth disease burden is decreased significantly by increasing vaccination of newborn and the susceptible animals, increasing the quarantining of clinically infected animals, restricting the movement of infected animals and as well as decreasing the shedding and diffusion of virus particles in the environment.

Keywords: Foot and Mouth; Vaccination; Carriers; Stability; Diffusion; Travelling Waves.

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1 Introduction

Foot-and-mouth disease (FMD) is a cloven-hoofed animal infectious disease which is spreads by viral particles through direct and indirect contact with infected animals or contaminated environments [1]. The disease causes high morbidity rates in cattle when introduced into disease-free herds or areas. The symptoms of the disease include fever, blisters in the mouth and feet and loss of appetite [2]. The disease progresses rapidly through the latent and subclinical infection periods to clinical infections period before recovery. Clinically infected animals may be quarantined before they recover through natural immunity. Quarantined animals may also be removed through reactive culling [3]. In this study, we focus on natural recovery in quarantine, a strategy used by many subsistence farmers [4, 5]. Clinically infected animals are isolated into the quarantined animal class and progress to the recovered animal class through developing natural immunity [6, 7, 8, 9]. We assume that all clinically infected animals are quarantined. This is a simplifying assumption since some clinically infected animals may also recover directly before being quarantined [3]. The recovered animals' immunity wanes over time and the animals become susceptible again [10, 11]. Another control for FMD is when animals are protected from the disease through vaccination. However, the vaccine may not induce rapid protection leading to vaccinated carrier animals transmitting the infection to healthy animals [6, 8, 9, 12]. The infected animals shed off foot and mouth disease virus (FMDV) into the environment through aerosol spraying. The aerosol in the atmosphere may be inhaled air that leads to the transmission of FMDV [6].

A number of studies use ordinary differential equation (ODE) models and individual-based models [13, 14, 15]. A disadvantage of the ODEs models is that they are not able to capture the spatial spread of most infectious diseases. The spatial spread of the infection is a confounding factor in the dynamics of the infection. Maidana et al.[16] incorporated the spatial spread of FMD without any control strategies. The control strategies for foot and mouth disease are of paramount importance to animal health and for this reason, it is imperative to consider the effects of control strategies available when modelling the dynamics of FMD. In addition, the previous studies did not capture the effects of FMDV infection spread through aerosols. Aerosol transmission of FMDV has been found to be an important route for FMD transmission [6] and its inclusion in the models has a potential to alter the prediction of the infection progress. In areas where there is an outbreak of FMD, quarantining of infected animals is also an effective control measure to reduce contact between infected and healthy animals [13]. However, because quarantine animals can still shed the virus into the atmosphere through aerosols, this group of animals, together with the subclinically infected animals, clinically infected animals and vaccinated carrier animals can contribute towards the transmission of FMDV to healthy animals [6]. This, therefore, suggests that studies modelling FMD should capture the effects of aerosol transmission by including the concentration of FMDV in the atmospheric environment.

Using the studies [6, 7, 8, 9, 16] as building blocks, we develop a model with spatial spread of FMD incorporating vaccination that does not induce rapid protection, quarantine and shedding of the FMDV into the atmospheric environment through aerosols. Our goal is to assess the effects of diffusion, vaccination, quarantining and viral shedding rates on the prognosis of FMDV using a system of reaction-diffusion equations. We formulate the mathematical model and give an analysis of the model in Sections 2 and 3. Parameter estimation and numerical simulations shall be presented in Section 4 and a discussion of the results and conclusion are given in Section 5.

2 Model incorporating spatial transmission and control

In this section, we propose a model consisting of reaction-diffusion equations for the dynamics of the foot and mouth disease on a cattle population. The total population is subdivided into susceptible animals $S(x,t)$, exposed animals $E(x,t)$, subclinically infectious animals $I_s(x,t)$, clinically infectious animals $I_c(x,t)$, quarantined animals $Q(x,t)$, recovered animals $R(x,t)$, vaccinated animals $V_v(x,t)$

and vaccinated carrier animals $V_{ca}(x,t)$. We add another compartment which captures the concentration of FMDVs in the atmospheric environment through aerosols denoted by $F_v(x,t)$. Susceptible animals are animals which are free of FMDV, exposed animals are animals which are not yet infectious but have the virus, subclinical animals are infectious animals that are nearly or completely asymptomatic with no signs or symptoms of infection [17] and clinically infections animals are infectious animals with clinically diagnosed signs or symptoms. Quarantined animals are clinically infected animals which are isolated from the rest of animals with restricted movement [13]. The removed animals are either recovered or immune to the infection, and the immunity may wane with time and the recovered animals become susceptible again [18, 19, 20]. The vaccinated animals are animals which are protected from the disease by a vaccine and vaccinated carrier animals are animals which are vaccinated but get infected because the vaccine does not induce complete protection [6].

We assume a constant recruitment rate due to birth, Π, and that the per-capita death rate of the total population $N(x,t)$ is μ. A proportion of the newborns is given a vaccine at a constant rate ρ, where $0 \leq \rho \leq 1$, and the recruitment rate of susceptible animals is given by $(1-\rho)\Pi$. Susceptible animals are given a vaccine at a constant rate ρ_1, where $0 \leq \rho_1 \leq 1$ and move to the vaccinated class $V_v(x,t)$. The remaining susceptible animals go to the exposed class, $E(x,t)$ through the force of infection, $\beta(I_c(x,t) + \eta_1 I_s(x,t) + \eta_2 V_{ca}(x,t) + \eta_3 F_v(x,t))S(x,t)/N(x,t)$, where β is the rate new infections arise by successive contacts between susceptible and infected animals, $\eta_1 < 1$ is an amplification to show that $I_c(x,t)$ is more infectious than I_s, $\eta_2 < 1$ is an amplification to show that $V_{ca}(x,t)$ is less infectious compared to clinical infective animal, $I_c(x,t)$ and $\eta_3 < 1$ is an amplification to show that $F_v(x,t)$ is less infectious compared to $I_s(x,t)$, $I_c(x,t)$ and $V_{ca}(x,t)$. Thus, $\eta_3 < \eta_2 < \eta_1 < 1$. Exposed animals progress to subclinical infection class, $I_s(x,t)$ at a rate ϵ. Subclinical animals are capable of transmitting the infection and later show signs and symptoms of FMD and progress to clinically infected animals at a rate α_1. Clinically infected animals are isolated to the quarantined class $Q(x,t)$ at a rate α_2. Quarantined animals develop temporary immunity and move to the recovered class at a rate α_3 [13]. As stated earlier, we assume that all clinically infected animals are quarantined for simplicity. The recovered animal becomes susceptible again by losing immunity [21] at a rate ω. The rate of recruitment of vaccinated animals, $V_v(x,t)$ is $\rho\Pi$. The vaccinated animals progress to vaccinated carrier $V_{ca}(x,t)$ by the force of infection, $\beta\,\phi\,(I_c(x,t) + \eta_1 I_s(x,t) + \eta_2 V_{ca}(x,t) + \eta_3 F_v(x,t))V_v(x,t)/N(x,t)$, where ϕ is the rate of protection loss by the vaccine and $0 \leq \phi \leq 1$. $V_{ca}(x,t)$ animals progress to the recovered class, $R(x,t)$ at a rate of α_4. The vaccinated animals progression to the vaccinated carrier class is due to the fact that vaccine that does not induce rapid protection [6]. The subclinically infected, clinically infected, quarantined and vaccinated carrier animals shed off FMDV into the atmospheric environment at rates τ_{is}, τ_{ic}, τ_q, and τ_{vca} respectively [6]. The death rate of the FMDV in the atmospheric environment is denoted by μ_f. The average number of FMD viruses shedded into the atmospheric environment by subclinically infected, clinically infected, quarantined and vaccinated carrier animals are denoted by N_{is}, N_{ic}, N_q, andN_{vca} respectively. The spatial diffusion parameters of susceptible, exposed, subclinically and clinically infected classes are denoted by d_1, d_2, d_3 and d_4 respectively and the spatial diffusion parameters d_5, d_6, d_7, d_8 and d_9 are designed for the diffusion parameter of quarantined, recovered, vaccinated and vaccinated carrier classes respectively. The flow diagram for the model is presented in Figure 1.

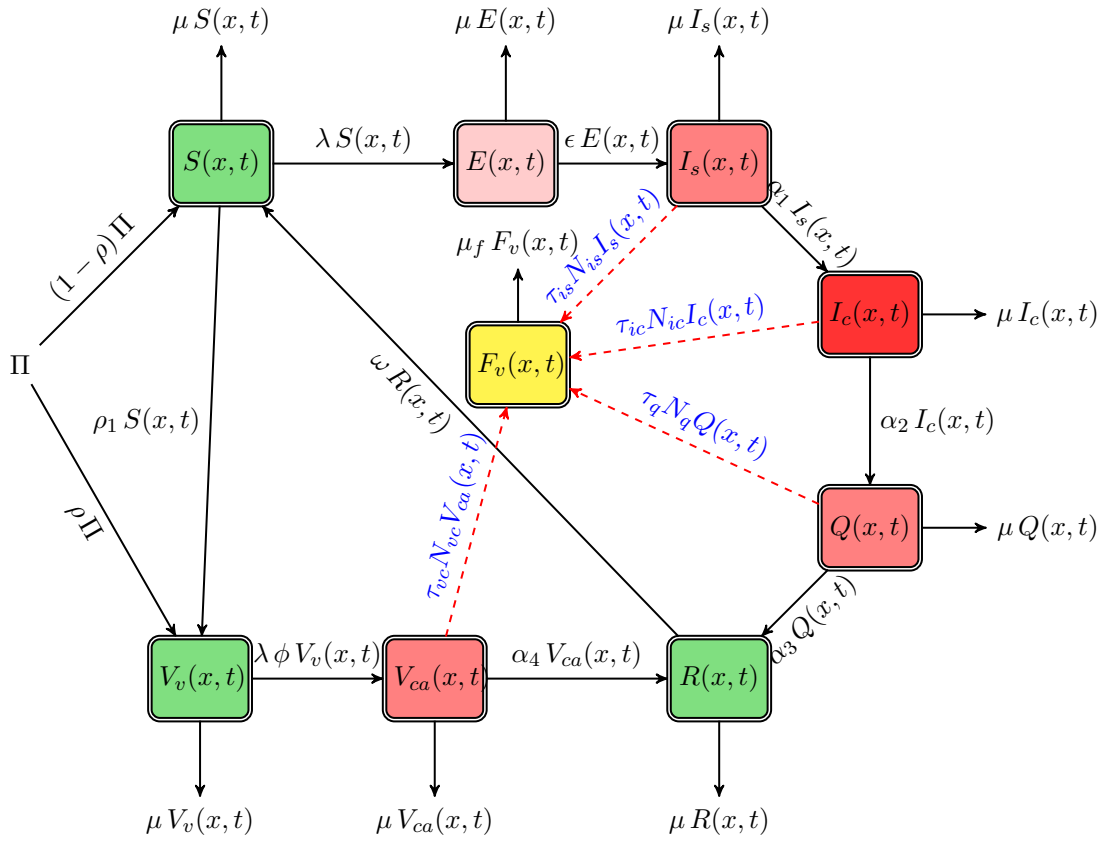

Figure 1: Flow diagram for the transmission dynamics of foot and mouth disease of cattle population model. The dashed arrows represent the shedding off of foot and mouth disease virus in the environment.

where $\lambda = \beta\,(I_c(x,t) + \eta_1\, I_s(x,t) + \eta_2\, V_{ca}(x,t) + \eta_3\, F_v(x,t))/N(x,t)$.

The model described by the following reaction-diffusion system of foot and mouth disease:

$$
\begin{aligned}
\frac{\partial S(x,t)}{\partial t} &= d_1 \frac{\partial^2 S(x,t)}{\partial x^2} + (1-\rho)\,\Pi - (\lambda+\mu)\, S(x,t) - \rho_1\, S(x,t) + \omega\, R(x,t),\\
\frac{\partial E(x,t)}{\partial t} &= d_2 \frac{\partial^2 E(x,t)}{\partial x^2} + \lambda\, S(x,t) - (\epsilon+\mu)\, E(x,t),\\
\frac{\partial I_s(x,t)}{\partial t} &= d_3 \frac{\partial^2 I_s(x,t)}{\partial x^2} + \epsilon\, E(x,t) - (\alpha_1+\mu)\, I_s(x,t),\\
\frac{\partial I_c(x,t)}{\partial t} &= d_4 \frac{\partial^2 I_c(x,t)}{\partial x^2} + \alpha_1\, I_s(x,t) - (\alpha_2+\mu)\, I_c(x,t),\\
\frac{\partial Q(x,t)}{\partial t} &= d_5 \frac{\partial^2 Q(x,t)}{\partial x^2} + \alpha_2\, I_c(x,t) - (\alpha_3+\mu)\, Q(x,t),\\
\frac{\partial R(x,t)}{\partial t} &= d_6 \frac{\partial^2 R(x,t)}{\partial x^2} + \alpha_3\, Q(x,t) - (\omega+\mu)\, R(x,t) + \alpha_4\, V_{ca}(x,t),\\
\frac{\partial V_v(x,t)}{\partial t} &= d_7 \frac{\partial^2 V_v(x,t)}{\partial x^2} + \rho\Pi - (\lambda\,\phi+\mu)\, V_v(x,t) + \rho_1\, S(x,t),\\
\frac{\partial V_{ca}(x,t)}{\partial t} &= d_8 \frac{\partial^2 V_{ca}(x,t)}{\partial x^2} + \lambda\,\phi V_v(x,t) - (\alpha_4+\mu)\, V_{ca}(x,t),\\
\frac{\partial F_v(x,t)}{\partial t} &= d_9 \frac{\partial^2 F_v(x,t)}{\partial x^2} + \tau_{is}\, N_{is}\, I_s(x,t) + \tau_{ic}\, N_{ic}\, I_c(x,t) + \tau_q\, N_q\, Q(x,t) + \tau_{vc}\, N_{vc}\, V_{ca}(x,t) - \mu_f\, F_v(x,t),
\end{aligned}
\tag{1}
$$

subject to the initial conditions of

$$\begin{aligned} &S(x,0)=\rho_1(x),\ E(x,0)=\rho_3(x),\ I_s(x,0)=\rho_2(x),\ I_c(x,0)=\rho_4(x),\ Q(x,0)=\rho_5(x),\\ &R(x,0)=\rho_6(x),\ V_v(x,0)=\rho_7(x),\ V_{ca}(x,0)=\rho_8(x),\ F_v(x,0)=\rho_9(x),\ \ t\geq 0,\ \ x\in\Omega. \end{aligned} \tag{2}$$

Ω in equation (2) is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$. The functions $\rho_i(x,t)\in\mathfrak{L}=C([0,t],Y)(i=1,2,3,\ldots,9)$ and Y is defined by,

$$\begin{aligned} Y=\Bigg\{&S,E,I_s,I_c,Q,R,V_v,V_{ca},F_v\in W^{2,2}(\Omega):\frac{\partial S(x,t)}{\partial n}=\frac{\partial E(x,t)}{\partial n}=\frac{\partial I_s(x,t)}{\partial n}=\frac{\partial I_c(x,t)}{\partial n}=\frac{\partial Q(x,t)}{\partial n}\\ &=\frac{\partial R(x,t)}{\partial n}=\frac{\partial V_v(x,t)}{\partial n}=\frac{\partial V_{ca}(x,t)}{\partial n}=\frac{\partial F_v(x,t)}{\partial n}=0,x\in\partial\Omega\Bigg\}. \end{aligned} \tag{3}$$

with the inner product $\langle\cdot,\cdot\rangle$, where $\frac{\partial}{\partial n}$ denotes the outward normal derivative on $\partial\Omega$ [22], $N(x,t)=S(x,t)+E(x,t)+I_s(x,t)+I_c(x,t)+Q(x,t)+R(x,t)+V_v(x,t)+V_{ca}(x,t)$, summing the equations of (1), by taking $D=d_i$ where $i=1,2,3,\ldots,9$ and leads to

$$\begin{aligned} &\frac{\partial N(x,t)}{\partial t}=D\frac{\partial^2 N(x,t)}{\partial x^2}+\Pi-\mu\,N(x,t),\\ &\frac{\partial F_v(x,t)}{\partial t}=D\frac{\partial^2 F_v(x,t)}{\partial x^2}+\tau_{is}\,N_{is}\,I_s(x,t)+\tau_{ic}\,N_{ic}\,I_c(x,t)+\tau_q\,N_q\,Q(x,t)+\tau_{vc}\,N_{vc}\,V_{ca}(x,t)-\mu_f\,F_v(x,t), \end{aligned}$$

Introducing the non-dimensional parameters to system (1) with the following dimensionless variables and parameters:

$$\begin{aligned} &x_1=\frac{S}{N},\ x_2=\frac{E}{N},\ x_3=\frac{I_S}{N},\ x_4=\frac{I_c}{N},\ x_5=\frac{Q}{N},\ x_6=\frac{R}{N},\ x_7=\frac{V_v}{N},\ x_8=\frac{V_{ca}}{N},\ x_9=\frac{F_v}{N},\ \tau=\mu\,t,\\ &\pi_1=\frac{(1-\rho)\,\Pi}{\mu\,N},\ b_1=\frac{1}{\mu},\ b_2=\frac{\rho_1}{\mu},\ c_1=\frac{\eta_1}{\mu},\ c_2=\frac{\eta_2}{\mu},\ c_3=\frac{\eta_3}{\mu},\ c_4=\frac{\omega}{\mu},\ c_5=\frac{\epsilon}{\mu},\ m_1=\frac{\alpha_1}{\mu},\ m_2=\frac{\alpha_2}{\mu},\\ &m_3=\frac{\alpha_3}{\mu},\ m_4=\frac{\alpha_4}{\mu},\ \pi_2=\frac{\rho\,\Pi}{\mu\,N},\ x=(\mu\,N)^{\frac{1}{2}}\,X,\ e_1=\frac{N_{is}\tau_{is}}{\mu},\ e_2=\frac{N_{ic}\tau_{ic}}{\mu},\ e_3=\frac{N_q\tau_q}{\mu},\ e_4=\frac{N_{vc}\tau_{vc}}{\mu},\\ &e_5=\frac{\mu_f}{\mu}, \end{aligned} \tag{4}$$

Then the corresponding non-dimensional model is becomes

$$\begin{aligned} &\frac{\partial}{\partial\tau}x_1\,(X,\tau)=d_1\,\frac{\partial^2}{\partial X^2}x_1\,(X,\tau)+\pi_1-(\Lambda+1+b_2)\,x_1(X,\tau)+c_4\,x_6(X,\tau),\\ &\frac{\partial}{\partial\tau}x_2\,(X,\tau)=d_2\,\frac{\partial^2}{\partial X^2}x_2\,(X,\tau)+\Lambda\,x_1(X,\tau)-(c_5+1)\,x_2(X,\tau),\\ &\frac{\partial}{\partial\tau}x_3\,(X,\tau)=d_3\,\frac{\partial^2}{\partial X^2}x_3\,(X,\tau)+c_5\,x_2(X,\tau)-(m_1+1)\,x_3(X,\tau),\\ &\frac{\partial}{\partial\tau}x_4\,(X,\tau)=d_4\,\frac{\partial^2}{\partial X^2}x_4\,(X,\tau)+m_1\,x_3(X,\tau)\ -(m_2+1)\,x_4(X,\tau),\\ &\frac{\partial}{\partial\tau}x_5\,(X,\tau)=d_5\,\frac{\partial^2}{\partial X^2}x_5\,(X,\tau)+m_2\,x_4(X,\tau)-(m_3+1)\,x_5(X,\tau),\\ &\frac{\partial}{\partial\tau}x_6\,(X,\tau)=d_6\,\frac{\partial^2}{\partial X^2}x_6\,(X,\tau)+m_3\,x_5(X,\tau)-(c_4+1)\,x_6(X,\tau)+m_4\,x_8(X,\tau),\\ &\frac{\partial}{\partial\tau}x_7\,(X,\tau)=d_7\,\frac{\partial^2}{\partial X^2}x_7\,(X,\tau)+\pi_2-(\phi\,\Lambda+1)\,x_7(X,\tau)+b_2\,x_1(X,\tau),\\ &\frac{\partial}{\partial\tau}x_8\,(X,\tau)=d_8\,\frac{\partial^2}{\partial X^2}x_8\,(X,\tau)+\phi\,\Lambda\,x_7(X,\tau)-(m_4+1)\,x_8(X,\tau),\\ &\frac{\partial}{\partial\tau}x_9\,(X,\tau)=d_9\,\frac{\partial^2}{\partial X^2}x_9\,(X,\tau)+e_1\,x_3(X,\tau)+e_2\,x_4(X,\tau)+e_3\,x_5(X,\tau)+e_4\,x_8(X,\tau)-e_5\,x_9(X,\tau), \end{aligned} \tag{5}$$

Subject to the initial conditions of

$$\begin{aligned} &x_1(X,0)=\rho_{11}(X),\ x_2(X,0)=\rho_{12}(X),\ x_3(X,0)=\rho_{13}(X),\ x_4(X,0)=\rho_{14}(X),\\ &x_5(X,0)=\rho_{15}(X),\ x_6(X,0)=\rho_{16}(X),\ x_7(X,0)=\rho_{17}(X),\ x_8(X,0)=\rho_{18}(X),\\ &x_9(X,0)=\rho_{19}(X),\quad \tau\geq 0,\quad X\in\partial\tilde{\Omega}. \end{aligned} \tag{6}$$

and with Neumann boundary conditions:

$$\begin{aligned} &\frac{\partial x_1(X,\tau)}{\partial n}=\frac{\partial x_2(X,\tau)}{\partial n}=\frac{\partial x_3(X,\tau)}{\partial n}=\frac{\partial x_4(X,\tau)}{\partial n}=\frac{\partial x_5(X,\tau)}{\partial n}=\frac{\partial x_6(X,\tau)}{\partial n}=\frac{\partial x_7(X,\tau)}{\partial n}\\ &=\frac{\partial x_8(X,\tau)}{\partial n}=\frac{\partial x_9(X,\tau)}{\partial n}=0,\quad \tau\geq 0,\quad X\in\partial\tilde{\Omega}. \end{aligned} \tag{7}$$

where

$$\Lambda=\beta\ \left(b_1\,x_4(X,\tau)+c_1\,x_3(X,\tau)+c_2\,x_8(X,\tau)+c_3\,x_9(X,\tau)\right),$$

2.1 The homogeneous non-dimensional model for foot and mouth disease

In this section, we analyze the homogeneous non-dimensional model. We considered the system of ordinary differential equations of the governing equations (8).

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}x_1(\tau)&=\pi_1-(\Lambda+1+b_2)\,x_1(\tau)+c_4\,x_6(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_2(\tau)&=\Lambda x_1(\tau)-(c_5+1)\,x_2(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_3(\tau)&=c_5\,x_2(\tau)-(m_1+1)\,x_3(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_4(\tau)&=m_1\,x_3(\tau)-(m_2+1)\,x_4(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_5(\tau)&=m_2\,x_4(\tau)-(m_3+1)\,x_5(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_6(\tau)&=m_3\,x_5(\tau)-(c_4+1)\,x_6(\tau)+m_4\,x_8(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_7(\tau)&=\pi_2-(\phi\,\Lambda+1)\,x_7(\tau)+b_2\,x_1(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_8(\tau)&=\phi\,\Lambda\,x_7(\tau)-(m_4+1)\,x_8(\tau),\\ \frac{\mathrm{d}}{\mathrm{d}\tau}x_9(\tau)&=e_1\,x_4(\tau)+e_2\,x_3(\tau)+e_3\,x_5(\tau)+e_4\,x_8(\tau)-e_5\,x_9(\tau), \end{aligned} \tag{8}$$

Subject to the initial conditions

$$\begin{aligned} &x_1(\tau)\geq x_1(0)\ \ x_2(\tau)\geq x_2(0),\quad x_3(\tau)\geq x_3(0),\ \ x_4(\tau)\geq x_4(0),\ \ x_5(\tau)\geq x_5(0),\\ &x_6(\tau)\geq x_6(0),\ \ x_7(\tau)\geq x_7(0),\ \ x_8(\tau)\geq x_8(0)\ \ x_9(\tau)\geq x_9(0), \end{aligned} \tag{9}$$

2.1.1 Feasible region.

We assume that all parameters of model (8) to be non-negative for the model to be biologically meaningful for time $\tau>0$.

Theorem 1. *The cattle foot and mouth disease transmission model (8) with initial condition (9), will then be analyzed in a suitable region given by*

$$\Gamma=\{G\in\Re_+^9, N(\tau)\leq\pi_1+\pi_2,\ x_9(\tau)\leq\frac{e_1+e_2+e_3+e_4}{e_5}\}, \tag{10}$$

where, $G=(x_1(\tau),x_2(\tau),x_3(\tau),x_4(\tau),x_5(\tau),x_6(\tau),x_7(\tau),x_8(\tau),x_9(\tau))$

Proof. For model (8) to be epidemiologically useful, it is important to show that all the state variables of the region Γ are non-negative for all time and it is positively invariant.

Let assume $\tau > 0$, $x_i(0) \geq 0$, for $1 \leq i \leq 9$. From the first equation of model (8), we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau} x_1(\tau) = \pi_1 - (\Lambda(\tau) + 1 + b_2)\, x_1(\tau) + c_4\, x_6(\tau), \tag{11}$$

Integrating equation (11) and using a differential inequality, we get

$$\begin{aligned} x_1(\tau) &= \left(\int_0^\tau \mathrm{e}^{-\int_0^\tau -\Lambda(\tau)-1-b_2\,\mathrm{d}\tau} \left(c_4\, x_6(\tau) + \pi_1\right) \mathrm{d}\tau + x_1(0) \right) \mathrm{e}^{\int_0^\tau (-\Lambda(\tau)-1-b_2)\,\mathrm{d}\tau} \\ &\geq x_1(0) \mathrm{e}^{\int_0^\tau (-\Lambda(\tau)-1-b_2)\,\mathrm{d}\tau} \geq 0, \end{aligned}$$

Hence, $x_1(\tau) \geq 0$, as $\tau \to 0$ and this implies that at any finite time, x_1 is non-negative. A similar analysis holds for the other equations of (8) where,

$$\begin{aligned} &x_3(\tau) \geq x_3(0)\, \mathrm{e}^{-(m_1+1)\tau} \geq 0, \qquad x_4(\tau) \geq x_4(0)\, \mathrm{e}^{-(m_2+1)\tau} \geq 0, \\ &x_5(\tau) = x_5(0)\, \mathrm{e}^{-(m_3+1)\tau} \geq 0, \qquad x_6(\tau) \geq x_6(0)\, \mathrm{e}^{-(c_4+1)\tau} \geq 0, \\ &x_9(\tau) \geq x_9(0)\, \mathrm{e}^{-e\,\tau} \geq 0, \qquad x_2(\tau) \geq x_2(0)\, \mathrm{e}^{-(c_5+1)\tau} \geq 0, \\ &x_7(\tau) \geq x_7(0)\, \mathrm{e}^{\int_0^\tau -\Lambda(\tau)\phi - 1\,\mathrm{d}\tau} \geq 0. \end{aligned}$$

Adding the above eight equations in model (8) gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau} N(\tau) = \pi_1 + \pi_2 - N(\tau), \tag{12}$$

where $N(\tau) = x_1(\tau) + x_2(\tau) + x_3(\tau) + x_4(\tau) + x_5(\tau) + x_6(\tau) + x_7(\tau) + x_8(\tau)$.

Solving (12) gives

$$N(\tau) = \pi_1 + \pi_2 + N(0)\mathrm{e}^{-\tau},$$

where $N(0)$ represents the initial value of $N(\tau)$. Thus, as $\tau \to \infty$, $0 \leq N(\tau) \leq \pi_1 + \pi_2$. Therefore, the solutions of the model with non-negative initial conditions remain non-negative for all $0 \leq \tau < \infty$. Since $\frac{\mathrm{d}}{\mathrm{d}\tau} x_9(\tau) = e_1\, x_4(\tau) + e_2\, x_3(\tau) + e_3\, x_5(\tau) + e_4\, x_8(\tau) - e_5\, x_9(\tau) \leq e_1 + e_2 + e_3 + e_4 - e_5\, x_9(\tau)$. Thus, we can easily obtain that, $0 \leq x_9(\tau) \leq \frac{e_1+e_2+e_3+e_4}{e_5}$. Therefore, all variables are bounded. This shows that for initial conditions (9), the region Γ is positively invariant and attracting and therefore the region Γ is a feasible region for the model (8). □

2.1.2 The control reproduction ratio for the model

The control reproduction ratio is calculated using the next generation matrix method [23, 24]. We take only the infected classes of the model to calculate the control reproduction ratio. Using the notation in [23], the non-negative matrix of new infections F into compartments and the M-matrix, V, of the transition terms associated with the model (8), in and out of compartments at the disease free equilibrium point, $x_2 = x_3 = x_4 = x_5 = x_6 = x_8 = x_{10} = 0$, x_1 and x_7 are given by:

$$F = \begin{bmatrix} 0 & \frac{\beta\, c_1 \pi_1}{1+b_2} & \frac{\beta\, b_1 \pi_1}{1+b_2} & \frac{\beta\, c_2 \pi_1}{1+b_2} & \frac{\beta\, c_3 \pi_1}{1+b_2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \phi\,\beta\, c_1 \left(\frac{b_2 \pi_1}{1+b_2} + \pi_2\right) & \phi\,\beta\, b_1 \left(\frac{b_2 \pi_1}{1+b_2} + \pi_2\right) & \phi\,\beta\, c_2 \left(\frac{b_2 \pi_1}{1+b_2} + \pi_2\right) & \phi\,\beta\, c_3 \left(\frac{b_2 \pi_1}{1+b_2} + \pi_2\right) \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} c_5+1 & 0 & 0 & 0 & 0 \\ -c_5 & m_1+1 & 0 & 0 & 0 \\ 0 & -m_1 & m_2+1 & 0 & 0 \\ 0 & 0 & 0 & m_4+1 & 0 \\ 0 & -e_1 & -e_2 & -e_4 & e_5 \end{bmatrix},$$

so that

$$V^{-1} = \begin{bmatrix} (c_5+1)^{-1} & 0 & 0 & 0 & 0 \\ \frac{c_5}{(m_1+1)(c_5+1)} & (m_1+1)^{-1} & 0 & 0 & 0 \\ \frac{c_5 m_1}{(m_2+1)(m_1+1)(c_5+1)} & \frac{m_1}{(m_2+1)(m_1+1)} & (m_2+1)^{-1} & 0 & 0 \\ 0 & 0 & 0 & (m_4+1)^{-1} & 0 \\ \frac{c_5(e_2 m_1+e_1(m_2+1))}{(m_2+1)(m_1+1)(c_5+1)e_5} & \frac{e_2 m_1+e_1(m_2+1)}{(m_2+1)(m_1+1)e_5} & \frac{e_2}{(m_2+1)e_5} & \frac{e_4}{(m_4+1)e_5} & {e_5}^{-1} \end{bmatrix},$$

Following [23], the control reproduction number of model 8 is

$$\mathcal{R}_c = \rho(FV^{-1})$$

where ρ represents the spectral radius of the matrix FV^{-1}. $\mathcal{R}_c$ is FMD infection control reproduction number and it is given by

$$\begin{aligned} \mathcal{R}_c = & \frac{\beta\,\phi\,(c_2 e_5 + c_3 e_4)\,(\pi_1 b_2 + \pi_2 b_2 + \pi_2)}{(m_4+1)\,(1+b_2)\,e_5} + \frac{(b_1 m_1 + c_1 m_2 + c_1)\,c_5 \pi_1 \beta}{(m_2+1)\,(m_1+1)\,(c_5+1)\,(1+b_2)} \\ & + \frac{c_3\,(e_1 m_2 + e_2 m_1 + e_1)\,c_5 \pi_1 \beta}{(m_2+1)\,(m_1+1)\,(c_5+1)\,(1+b_2)\,e_5}. \end{aligned} \tag{13}$$

2.1.3 Equilibrium points of the system

The system of equations (8) has two steady states.The first one is the disease-free equilibrium point given by

$$\hat{E}_0 = (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{x}_4, \hat{x}_5, \hat{x}_6, \hat{x}_7, \hat{x}_8, \hat{x}_9) = \left(\frac{\pi_1}{1+b_2}, 0, 0, 0, 0, 0, \frac{\pi_1 b_2}{b_2+1} + \pi_2, 0, 0 \right).$$

The second one is the endemic state E_1 of the model is given in terms of λ^* and R^* with

$$E^* = (x_1^*, x_2^*, x_3^*, x_4^*, x_5^*, x_6^*, x_7^*, x_8^*, x_9^*),$$

where the force of infection at the equilibrium point is

$$\Lambda^* = \beta\,(b_1\,x_4^* + c_1\,x_3^* + c_2\,x_8^* + c_3\,x_9^*)\,, \tag{14}$$

and

$$x_1^* = \frac{c_4\, x_6^* + \pi_1}{\Lambda^* + 1 + b_2}, \quad x_2^* = \frac{\Lambda^*\ (c_4\, x_6^* + \pi_1)}{(\Lambda^* + 1 + b_2)\,(c_5 + 1)}, \quad x_3^* = \frac{c_5\, \Lambda^*\ (c_4\, x_6^* + \pi_1)}{(\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)},$$
$$x_4^* = \frac{c_5\, \Lambda^*\ (c_4\, x_6^* + \pi_1)\, m_1}{(\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)},$$
$$x_5^* = \frac{m_2\, c_5\, \Lambda^*\ (c_4\, x_6^* + \pi_1)\, m_1}{(\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)\,(m_3 + 1)},$$
$$x_6^* = \frac{m_3\, m_2\, m_1\, c_5\, \Lambda^*\, \pi_1\, H_1}{H_2\, H_3} + \frac{m_4\, \Lambda^*\, \phi\, b_2\, \pi_1\, H_1}{(\Lambda^* + 1 + b_2)\,(\Lambda^*\, \phi + 1)\,(m_4 + 1)\,(c_4 + 1)\, H_3}$$
$$+ \frac{m_4\, \Lambda^*\, \phi\, \pi_2\, H_1}{(\Lambda^*\, \phi + 1)\,(m_4 + 1)\,(c_4 + 1)\, H_3}, \quad x_7^* = \frac{\pi_2}{\Lambda^*\, \phi + 1} + \frac{b_2\ (c_4\, x_6^* + \pi_1)}{(\Lambda^* + 1 + b_2)\,(\Lambda^*\, \phi + 1)},$$
$$x_8^* = \frac{\Lambda^*\, \phi\, \pi_2}{(\Lambda^*\, \phi + 1)\,(m_4 + 1)} + \frac{\Lambda^*\, \phi\, b_2\ (c_4\, x_6^* + \pi_1)}{(\Lambda^* + 1 + b_2)\,(\Lambda^*\, \phi + 1)\,(m_4 + 1)},$$
$$x_9^* = \frac{e_1\, m_1\, c_5\, \Lambda^*\ (c_4\, x_6^* + \pi_1)}{e_5\ (\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)} + \frac{e_2\, c_5\, \Lambda^*\ (c_4\, x_6^* + \pi_1)}{e_5\ (\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)}$$
$$+ \frac{e_3\, m_2\, m_1\, c_5\, \Lambda^*\ (c_4\, x_6^* + \pi_1)}{e_5\ (\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)\,(m_3 + 1)} + \frac{e_4\, \Lambda^*\, \phi\, \pi_2}{(\Lambda^*\, \phi + 1)\,(m_4 + 1)\, e_5}$$
$$+ \frac{e_4\, \Lambda^*\, \phi\, b_2\ (c_4\, x_6^* + \pi_1)}{(\Lambda^* + 1 + b_2)\,(\Lambda^*\, \phi + 1)\,(m_4 + 1)\, e_5},$$

where

$$H_1 = (\Lambda^* + 1 + b_2)^2\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)\,(m_3 + 1)\,(c_4 + 1)^2\,(\Lambda^*\, \phi + 1)\,(m_4 + 1)\,,$$
$$H_2 = (\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)\,(m_3 + 1)\,(c_4 + 1)\,,$$
$$H_3 = H_1 - \Lambda^*\, c_5\, m_1\, m_2\, m_3\, c_4\ (\Lambda^* + 1 + b_2)\,(\Lambda^*\, \phi + 1)\,(m_4 + 1)\,(c_4 + 1)$$
$$- m_4\, \Lambda^*\, \phi\, b_2\, c_4\ (\Lambda^* + 1 + b_2)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)\,(m_3 + 1)\,(c_4 + 1)\,,$$

If x_3^* , x_4^*, x_8^* and x_9^* are substituted into (14), we obtain the equation in terms of Λ^*,

$$\Lambda^*(K_1 {\Lambda^*}^2 + K_2 \Lambda^* + K_3) = 0,$$

where

$$K_1 = e_5\, \phi\ (m_4 + 1)\,(m_3 + 1)\,(m_2 + 1)\,(m_1 + 1)\,(c_5 + 1)\,,$$
$$K_2 = -G_1\, \phi\, c_5\, \beta\ (c_4\, x_6^* + \pi_1)\,(m_4 + 1) - \frac{K_1\, \beta\, \pi_2\ (c_2\, e_5 + c_3\, e_4)}{e_5\ (m_4 + 1)} + \frac{K_1\ (b_2\, \phi + \phi + 1)}{\phi},$$
$$K_3 = \frac{K_1\ (b_2 + 1)\,(1 - \mathcal{R}_c + G_2 - G_4 - G_5)}{\phi},$$
$$G_1 = (m_1\, m_3 + m_1)\, c_3\, e_1 + c_3\ ((m_3 + 1)\,(m_2 + 1)\, e_2 + e_3\, m_1\, m_2)$$
$$+ e_5\ (m_3 + 1)\,(b_1\, m_1 + c_1\, m_2 + c_1)\,,$$
$$G_2 = \frac{(b_1 m_1 + c_1 m_2 + c_1)\, c_5 \pi_1 \beta}{(m_2 + 1)\,(m_1 + 1)\,(c_5 + 1)\,(b_2 + 1)} + \frac{c_3\,(e_1 m_2 + e_2 m_1 + e_1)\, c_5 \pi_1 \beta}{(m_2 + 1)\,(m_1 + 1)\,(c_5 + 1)\,(b_2 + 1)\, e_5},$$
$$G_3 = \frac{b_1\, \beta\, m_1\, c_5\ (c_4\, x_6^* + \pi_1)}{(b_2 + 1)\,(c_5 + 1)\,(m_1 + 1)\,(m_2 + 1)} + \frac{(c_4\, x_6^* + \pi_1)\, c_5\, c_1\, \beta}{(b_2 + 1)\,(c_5 + 1)\,(m_1 + 1)},$$
$$G_4 = \frac{(c_2\, e_5 + c_3\, e_4)\, b_2\, c_4\, x_6^*\, \beta\, \phi}{e_5\ (m_4 + 1)\,(b_2 + 1)} + G_3,$$
$$G_5 = \frac{(c_4\, x_6^* + \pi_1)\, c_5\, c_3\, \beta\, e_2}{(b_2 + 1)\, e_5\ (m_1 + 1)\,(c_5 + 1)} + \frac{\beta\, c_3\, c_5\ (c_4\, x_6^* + \pi_1)\, m_1\ (e_1\, m_3 + e_3\, m_2 + e_1)}{e_5\ (m_3 + 1)\,(m_2 + 1)\,(m_1 + 1)\,(c_5 + 1)\,(b_2 + 1)}.$$

The roots of equation (15) are $\Lambda^* = 0$ which corresponds to the disease free equilibrium point and

$$\Lambda^* = \frac{-K_2 \pm \sqrt{{K_2}^2 - 4\,K_1\,K_3}}{2\,K_1}.$$

The condition $K_1 > 0$ and $K_3 < 0$ for any values of K_2 when $\mathcal{R}_c > 1$ is satisfied to ensures positivity of Λ^* and subsequently the positivity of E_1. If $K_1 > 0$, $K_2 < 0$, $K_3 > 0$ and $K_2^2 - 4K_1K_3 > 0$ when $\mathcal{R}_c < 1$, then there is a possibility of existence of two real positive solutions. Since these two positive solutions exist when $\mathcal{R}_c < 1$, then there is a possibility of existence of a backward bifurcation.

2.1.4 Stability of disease free equilibrium point

Let

$$F = \begin{bmatrix} 0 & \beta\,c_1 x_1^* & \beta\,b_1 x_1^* & \beta\,c_2 x_1^* & \beta\,c_3 x_1^* \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \phi\,\beta\,c_1 x_7^* & \phi\,\beta\,b_1 x_7^* & \phi\,\beta\,c_2 x_7^* & \phi\,\beta\,c_3 x_7^* \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$V = \begin{bmatrix} c_5 + 1 & 0 & 0 & 0 & 0 \\ -c_5 & m_1 + 1 & 0 & 0 & 0 \\ 0 & -m_1 & m_2 + 1 & 0 & 0 \\ 0 & 0 & 0 & m_4 + 1 & 0 \\ 0 & -e_1 & -e_2 & -e_4 & e_5 \end{bmatrix},$$

$$M = F - V = \begin{bmatrix} -1 - c_5 & \beta\,c_1 x_1^* & \beta\,b_1 x_1^* & \beta\,c_2 x_1^* & \beta\,c_3 x_1^* \\ c_5 & -1 - m_1 & 0 & 0 & 0 \\ 0 & m_1 & -1 - m_2 & 0 & 0 \\ 0 & \phi\,\beta\,c_1 x_7^* & \phi\,\beta\,b_1 x_7^* & \phi\,\beta\,c_2 x_7^* - m_4 - 1 & \phi\,\beta\,c_3 x_7^* \\ 0 & e_1 & e_2 & e_4 & -e_5 \end{bmatrix},$$

Define the stability modulus, $s(M) = \max\{Re\lambda : \lambda \text{ is an eigenvale of } M\}$, hence, the simple eigenvalue of M is $s(M)$ with positive eigenvector [25, 26]. Using the van den Diressche and Watmough [24] of **Theorem 2**, these two equivalences hold.

$$\mathcal{R}_c > 0 \Longleftrightarrow s(M) > 0, \quad \mathcal{R}_c < 0 \Longleftrightarrow s(M) < 0,$$

Theorem 2. *The foot and mouth disease free equilibrium point of system (8), $\hat{E}_0$, is locally asymptotically stable for $\mathcal{R}_c < 1$ and unstable otherwise.*

Proof. To prove the local stability of disease free equilibrium of system (8), we verify the hypothesis $(A_1) - (A_5)$ in van den Diressche and Watmough [24]. F and V^{-1} are non-negatives, then by **Lemma** (9.1) of [27], V is nonsingular M-matrix therefore, all eigenvalues of $F - V$ have negative real parts. Hypotheses $(A_1) - (A_5)$ are easily verified, while (A_5) is satisfied if all eigenvalues of the 9×9 matrix.

$$J|_{\hat{E}_0} = \begin{bmatrix} M & 0 \\ J_3 & J_4 \end{bmatrix},$$

are negative real parts, where $J_3 = -J_4$. The matrix J_4 is given by

$$J_4 = \begin{bmatrix} -1-b_2 & c_4 & 0 \\ 0 & -c_4-1 & 0 \\ b_2 & 0 & -1 \end{bmatrix},$$

The eigenvalues of J_4 are given by

$$s(J_4) = \max\{-1, -(c_4+1), -(1+b_2)\},$$

If $\mathcal{R}_c < 1$, then $s(M) < 0$ and $s(J|_{\hat{E}_0}) < 0$, then the disease-free equilibrium $\hat{E}_0$ of system (8) is locally asymptotically stable. □

Theorem 3. *The foot and mouth disease-free equilibrium point of system (8), E^*, is globally asymptotically stable in the region for $\mathcal{R}_c < 1$ and unstable otherwise.*

Proof. We consider for system (8) on the space of the first eight variables only $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$. It is clear that if the disease-free equilibrium for the first eight equations is globally stable, then $x_9(t) \to 0$, and the disease-free equilibrium for the full system (8) is globally stable. Consider the following Lyapunov function of,

$$U = x_1\left(\frac{\hat{x}_1}{x_1} - 1 - \ln\frac{\hat{x}_1}{x_1}\right) + x_7\left(\frac{\hat{x}_7}{x_7} - 1 - \ln\frac{\hat{x}_7}{x_7}\right) + x_2 + \frac{(c_5+1)\,x_3}{c_5} + \frac{(c_5+1)\,(m_1+1)\,x_4}{c_5\,m_1} + \frac{(c_5+1)\,(m_1+1)\,(m_2+1)\,x_5}{c_5\,m_1\,m_2} + \frac{(m_3+1)\,(m_2+1)\,(m_1+1)\,(c_5+1)\,x_6}{c_5\,m_1\,m_2\,m_3} + x_8,$$

where $\hat{x}_1 = \frac{\pi_1}{1+b_2}$ and $\hat{x}_7 = \frac{\pi_1 b_2}{b_2+1} + \pi_2$. Notice that the function $f(x) = x - 1 - \ln x > 0$, for $x > 0$, and $x \neq 1$. $U = 0$ at the disease-free equilibrium and otherwise $U > 0$ for other than the disease-free equilibrium points. The derivative of U with respect to t is negative, when we differentiate with respect to t and replace $(x_1', x_2', x_3', \ldots, x_8')$ with their equals from (8),

$$U' = (\pi_1 - (\Lambda + 1 + b_2)\,x_1 + c_4\,x_6)\left(1 - \frac{X_1}{x_1}\right) + (\pi_2 - (\Lambda\,\phi + 1)\,x_7 + b_2\,x_1)\left(1 - \frac{X_7}{x_7}\right) + \Lambda\,x_1 - \frac{(m_3+1)\,(m_2+1)\,(c_4\,x_6 - m_4\,x_8 + x_6)\,(m_1+1)\,(c_5+1)}{c_5\,m_1\,m_2\,m_3} + \Lambda\,\phi\,x_7 - (m_4+1)\,x_8,$$

Rearranging the above equation, we get the following equation,

$$\begin{aligned} U' = {} & -\pi_1\left(\frac{x_1}{\pi_1} + \frac{\pi_1}{(1+b_2)\,x_1} - 2\right) - c_4\,x_6\left(\frac{\pi_1}{(1+b_2)\,x_1} + \frac{(1+b_2)\,x_1}{\pi_1} - 2\right) \\ & - \frac{(\pi_1+\pi_2)\,b_2 + \pi_2}{1+b_2}\left(\frac{x_7}{b_2\,x_1+\pi_2} + \frac{b_2\,x_1+\pi_2}{x_7} - 2\right) - \frac{1}{x_8}\left(\frac{1}{x_8} + x_8 - 2\right) - (1-\mathcal{R}_c)\,c_4\,x_6 - Y_1\,c_4\,x_6 \\ & - \frac{b_2\,\pi_1\,\Lambda\,\phi}{1+b_2}\left(\frac{1}{\Lambda\,\phi} - 1 - \ln\left(\frac{1}{\Lambda\,\phi}\right)\right) - \frac{\Lambda\,\pi_1}{1+b_2}\left(\frac{1+b_2}{\Lambda\,\pi_1} - 1 - \ln\left(\frac{1+b_2}{\Lambda\,\pi_1}\right)\right) - \frac{\Lambda\,\pi_1}{1+b_2}\ln\left(\frac{1+b_2}{\Lambda\,\pi_1}\right) \\ & - (\Lambda\,\phi\,\pi_2 + 1)\left(\frac{1}{\Lambda\,\phi\,\pi_2+1} - 1 - \ln\left(\frac{1}{\Lambda\,\phi\,\pi_2+1}\right)\right) - \frac{m_4\,x_8}{Y_2}\,(Y_2 - 1 - \ln(Y_2)) - Y_3 \\ & - Y_5\left(\frac{b_2\,x_1+x_1}{\pi_1} - 1 - \ln\left(\frac{b_2\,x_1+x_1}{\pi_1}\right)\right) - \frac{c_4\,x_6\,(1+b_2)\,x_1}{\pi_1}\left(\frac{1}{Y_4} - 1 - \ln\left(\frac{1}{Y_4}\right)\right) \\ & - \frac{c_4\,x_6\,(1+b_2)\,x_1}{\pi_1}\ln\left(\frac{1}{Y_4}\right) - \frac{m_4\,x_8}{Y_2}\ln(Y_2) - (\Lambda\,\phi\,\pi_2+1)\ln\left(\frac{1}{\Lambda\,\phi\,\pi_2+1}\right) - \frac{b_2\,\pi_1\,\Lambda\,\phi}{1+b_2}\ln\left(\frac{1}{\Lambda\,\phi}\right) \\ & - Y_5\ln\left(\frac{b_2\,x_1+x_1}{\pi_1}\right) - \frac{x_6}{Y_2} - \frac{x_8 - 1 + (x_8-1)^2\,x_8}{{x_8}^2}, \end{aligned}$$

where

$$Y_1 = \frac{\pi_1 b_2 \left(c_2 e_5 + c_3 e_4\right) \beta \, \phi}{e_5 \left(1 + b_2\right) \left(m_4 + 1\right)} + \frac{\beta \, \pi_1 c_5 \left(b_1 m_1 + c_1 m_2 + c_1\right)}{\left(m_2 + 1\right) \left(m_1 + 1\right) \left(c_5 + 1\right) \left(1 + b_2\right)} + \frac{\beta \, \pi_1 c_3 c_5 \left(e_1 m_2 + e_2 m_1 + e_1\right)}{\left(m_2 + 1\right) \left(m_1 + 1\right) \left(c_5 + 1\right) \left(1 + b_2\right) e_5},$$

$$Y_2 = \frac{c_5 \, m_1 \, m_2 \, m_3}{\left(m_3 + 1\right) \left(m_2 + 1\right) \left(m_1 + 1\right) \left(c_5 + 1\right)}, \qquad Y_3 = \frac{\left(\pi_2 b_2 + \pi_2\right) \left(c_2 e_5 + c_3 e_4\right) \beta \, \phi \, c_4 \, x_6}{e_5 \left(1 + b_2\right) \left(m_4 + 1\right)},$$

$$Y_4 = \frac{c_5 \, m_1 \, m_2 \, m_3 \, \left(1 + b_2\right) x_1}{\left(m_3 + 1\right) \left(m_2 + 1\right) \left(m_1 + 1\right) \left(c_5 + 1\right) \pi_1}, \qquad Y_5 = \frac{x_7 \, \pi_1 \, b_2}{\left(1 + b_2\right) \left(b_2 \, x_1 + \pi_2\right)},$$

For $\mathcal{R}_c < 1$ and using arithmetic and geometric principles the derivative of the Lyapunov function is non-positive ($U' < 0$) for all values of x_i, for $i = 1, 2, 3, \ldots, 9$ other than the disease-free equilibrium points. Therefore, by Lyapunovs theorem, the disease-free equilibrium is globally asymptotically stable. □

2.1.5 Global stability of the endemic equilibrium point

In this section, we show the global stability of the endemic equilibrium of E^* of system (8).

Theorem 4. *The foot and mouth disease endemic equilibrium point of system (8), E^*, is globally asymptotically stable in the region for $\mathcal{R}_c < 1$ and unstable for $\mathcal{R}_c > 1$.*

Proof. Let $V_1 = x_1 - x_1^* - x_1^* \, \ln \frac{x_1}{x_1^*} + x_2 - x_2^* - x_2^* \, \ln \frac{x_2}{x_2^*} + x_3 - x_3^* - x_3^* \, \ln \frac{x_3}{x_3^*}$,
$V_2 = x_4 - x_4^* - x_4^* \, \ln \frac{x_4}{x_4^*} + x_5 - x_5^* - x_5^* \, \ln \frac{x_5}{x_5^*} + x_6 - x_6^* - x_6^* \, \ln \frac{x_6}{x_6^*}$ and
$V_3 = x_7 - x_7^* - x_7^* \, \ln \frac{x_7}{x_7^*} + x_8 - x_8^* - x_8^* \, \ln \frac{x_8}{x_8^*} + x_9 - x_9^* - x_9^* \, \ln \frac{x_9}{x_9^*}$,
differentiating the given values using the equilibrium equations give

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau} V_1 \left(\tau\right) &= x_1' \left(1 - \frac{x_1^*}{x_1}\right) + x_2' \left(1 - \frac{x_2^*}{x_2}\right) + x_3' \left(1 - \frac{x_3^*}{x_3}\right) \\
&= \left(\pi_1 - \left(\Lambda + 1 + b_2\right) x_1 + c_4 \, x_6\right) \left(1 - \frac{x_1^*}{x_1}\right) + \left(x_1 \, \Lambda - \left(c_5 + 1\right) x_2\right) \left(1 - \frac{x_2^*}{x_2}\right) \\
&\quad + \left(c_5 \, x_2 - \left(m_1 + 1\right) x_3\right) \left(1 - \frac{x_3^*}{x_3}\right) \\
&= x_1^* \, \left(b_2 + 1 + \Lambda^*\right) \left(2 - \frac{x_1}{x_1^*} - \frac{x_1^*}{x_1}\right) + \Lambda \, x_1^* \left(2 - \frac{x_2}{x_2^*} - \frac{x_2^*}{x_2}\right) + c_5 \, x_2 \left(2 - \frac{x_3^*}{x_3} - \frac{x_3}{x_3^*}\right) \\
&\quad + \Lambda^* \, x_1^* \left(\frac{\Lambda}{\Lambda^*} - 1\right) \left(1 - \frac{x_1}{x_1^*}\right) + c_4 \, x_6^* \left(1 - \frac{x_1^*}{x_1}\right) \left(\frac{x_6}{x_6^*} - 1\right) + \Lambda \, x_1^* \left(1 - \frac{x_2^*}{x_2}\right) \left(\frac{x_1}{x_1^*} - 1\right) \\
&\quad - \frac{c_5 \, x_2 \, x_3}{x_3^*} \left(\frac{x_3^*}{x_3} - 1\right) - x_3^* \, \left(m_1 + 1\right) \left(\frac{x_3}{x_3^*} - 1\right).
\end{aligned} \tag{15}$$

Finally, equation (15) can be further simplify to give

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}V_1(\tau) &= x_1^*\,(b_2+1+\Lambda^*)\left(2-\frac{x_1}{x_1^*}-\frac{x_1^*}{x_1}\right)+\Lambda\,x_1^*\left(2-\frac{x_2}{x_2^*}-\frac{x_2^*}{x_2}\right)+c_5\,x_2\left(2-\frac{x_3^*}{x_3}-\frac{x_3}{x_3^*}\right)\\
&-x_1^*\,\Lambda\left(\frac{\Lambda^*}{\Lambda}-1-\ln\left(\frac{\Lambda^*}{\Lambda}\right)\right)-\Lambda^*\,x_1\left(\frac{\Lambda}{\Lambda^*}-1-\ln\left(\frac{\Lambda}{\Lambda^*}\right)\right)-c_4\,x_6\left(\frac{x_6^*}{x_6}-1-\ln\left(\frac{x_6^*}{x_6}\right)\right)\\
&-\frac{c_4\,x_6^*\,x_1^*}{x_1}\left(\frac{x_6}{x_6^*}-1-\ln\left(\frac{x_6}{x_6^*}\right)\right)-\Lambda\,x_1\left(\frac{x_1^*}{x_1}-1-\ln\left(\frac{x_1^*}{x_1}\right)\right)-\frac{\Lambda\,x_1^*\,x_2^*}{x_2}\left(\frac{x_1}{x_1^*}-1-\ln\left(\frac{x_1}{x_1^*}\right)\right)\\
&-\frac{c_5\,x_2\,x_3}{x_3^*}\left(\frac{x_3^*}{x_3}-1-\ln\left(\frac{x_3^*}{x_3}\right)\right)-x_3^*\,(m_1+1)\left(\frac{x_3}{x_3^*}-1-\ln\left(\frac{x_3}{x_3^*}\right)\right)-x_1^*\,\Lambda\,\ln\left(\frac{\Lambda^*}{\Lambda}\right)\\
&-x_1\,\Lambda^*\,\ln\left(\frac{\Lambda}{\Lambda^*}\right)-c_4\,x_6\,\ln\left(\frac{x_6^*}{x_6}\right)-\frac{c_4\,x_6^*\,x_1^*}{x_1}\ln\left(\frac{x_6}{x_6^*}\right)-\Lambda\,x_1\,\ln\left(\frac{x_1^*}{x_1}\right)-\frac{\Lambda\,x_1^*\,x_2^*}{x_2}\ln\left(\frac{x_1}{x_1^*}\right)\\
&-\frac{c_5\,x_2\,x_3}{x_3^*}\ln\left(\frac{x_3^*}{x_3}\right)-x_3^*\,(m_1+1)\ln\left(\frac{x_3}{x_3^*}\right).\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}V_2(\tau) &= x_3\,m_1\left(2-\frac{x_4^*}{x_4}-\frac{x_4}{x_4^*}\right)+m_2\,x_4\left(2-\frac{x_5^*}{x_5}-\frac{x_5}{x_5^*}\right)+(m_3\,x_5+m_4\,x_8)\left(2-\frac{x_6^*}{x_6}-\frac{x_6}{x_6^*}\right)\\
&-\frac{x_3\,m_1\,x_4}{x_4^*}\left(\frac{x_4^*}{x_4}-1-\ln\left(\frac{x_4^*}{x_4}\right)\right)-x_3^*\,m_1\left(\frac{x_4}{x_4^*}-1-\ln\left(\frac{x_4}{x_4^*}\right)\right)-\frac{m_2\,x_4\,x_5}{x_5^*}\left(\frac{x_5^*}{x_5}-1-\ln\left(\frac{x_5^*}{x_5}\right)\right)\\
&-m_2\,x_4^*\left(\frac{x_5}{x_5^*}-1-\ln\left(\frac{x_5}{x_5^*}\right)\right)-\frac{(m_3\,x_5+m_4\,x_8)\,x_6}{x_6^*}\left(\frac{x_6^*}{x_6}-1-\ln\left(\frac{x_6^*}{x_6}\right)\right)\\
&-\frac{(x_5^*\,m_3+x_8^*\,m_4)\,x_6}{x_6^*}\left(1-\frac{x_6^*}{x_6}-\ln\left(\frac{x_6^*}{x_6}\right)\right)-\frac{x_3\,m_1\,x_4}{x_4^*}\ln\left(\frac{x_4^*}{x_4}\right)-x_3^*\,m_1\,\ln\left(\frac{x_4}{x_4^*}\right)\\
&-\frac{m_2\,x_4\,x_5}{X_5}\ln\left(\frac{x_5^*}{x_5}\right)-m_2\,x_4^*\,\ln\left(\frac{x_5}{x_5^*}\right)-\frac{(m_3\,x_5+m_4\,x_8)\,x_6}{x_6^*}\ln\left(\frac{x_6^*}{x_6}\right)-\frac{(x_5^*\,m_3+x_8^*\,m_4)\,x_6}{x_6^*}\ln\left(\frac{x_6^*}{x_6}\right),\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}V_3(\tau) &= \Lambda^*\,x_7^*\,\phi\left(2-\frac{\Lambda\,x_7}{\Lambda^*\,x_7^*}-\frac{\Lambda^*\,x_7^*}{\Lambda\,x_7}\right)+x_7^*\left(2-\frac{x_7}{x_7^*}-\frac{x_7^*}{x_7}\right)+\frac{b_2\,x_1^*\,x_7^*}{x_7}\left(2-\frac{x_7}{x_7^*}-\frac{x_7^*}{x_7}\right)\\
&+\Lambda\,x_7\left(2-\frac{x_8\,x_7^*}{x_8^*\,x_7}-\frac{x_8^*\,x_7}{x_8\,x_7^*}\right)+(e_1\,x_3+e_2\,x_4+e_3\,x_5+e_4\,x_8)\left(2-\frac{x_9^*}{x_9}-\frac{x_9}{x_9^*}\right)\\
&-\frac{{\Lambda^*}^2{x_7^*}^2\phi}{\Lambda\,x_7}\left(\frac{\Lambda}{\Lambda^*}-1-\ln\left(\frac{\Lambda}{\Lambda^*}\right)\right)-x_7^*\,\phi\,\Lambda\left(\frac{\Lambda^*}{\Lambda}-1-\ln\left(\frac{\Lambda^*}{\Lambda}\right)\right)\\
&-\frac{b_2\,x_1^*\,{x_7^*}^2}{{x_7}^2}\left(\frac{x_7}{x_7^*}-1-\ln\left(\frac{x_7}{x_7^*}\right)\right)-b_2\,x_1\left(\frac{x_7^*}{x_7}-1-\ln\left(\frac{x_7^*}{x_7}\right)\right)\\
&-\frac{\Lambda\,{x_7}^2X_8}{x_8\,x_7^*}\left(\frac{x_8\,x_7^*}{x_8^*\,x_7}-1-\ln\left(\frac{x_8\,x_7^*}{x_8^*\,x_7}\right)\right)-\Lambda\,x_7^*\left(\frac{x_8^*\,x_7}{x_8\,x_7^*}-1-\ln\left(\frac{x_8^*\,x_7}{x_8\,x_7^*}\right)\right)\\
&-\frac{(e_1\,x_3+e_2\,x_4+e_3\,x_5+e_4\,x_8)\,x_9}{x_9^*}\left(\frac{x_9^*}{x_9}-1-\ln\left(\frac{x_9^*}{x_9}\right)\right)-e_5\,x_9^*\left(\frac{x_9}{x_9^*}-1-\ln\left(\frac{x_9}{x_9^*}\right)\right)\\
&-\frac{{\Lambda_1}^2{x_7^*}^2\phi}{\Lambda\,x_7}\ln\left(\frac{\Lambda}{\Lambda^*}\right)-x_7^*\,\phi\,\Lambda\,\ln\left(\frac{\Lambda^*}{\Lambda}\right)-\frac{b_2\,x_1^*\,{x_7^*}^2}{{x_7}^2}\ln\left(\frac{x_7}{x_7^*}\right)-b_2\,x_1\,\ln\left(\frac{x_7^*}{x_7}\right)\\
&-\frac{\Lambda\,{x_7}^2x_8^*}{x_8\,x_7^*}\ln\left(\frac{x_8\,x_7^*}{x_8^*\,x_7}\right)-\Lambda\,x_7^*\,\ln\left(\frac{x_8^*\,x_7}{x_8\,x_7^*}\right)-\frac{(e_1\,x_3+e_2\,x_4+e_3\,x_5+e_4\,x_8)\,x_9}{x_9^*}\ln\left(\frac{x_9^*}{x_9}\right)\\
&-e_5\,x_9^*\,\ln\left(\frac{x_9}{x_9^*}\right).\end{aligned}$$

Define the Lyapunov function

$$V(x_1,x_2...x_9)=V_1(x_1,x_2...x_9)+V_2(x_1,x_2...x_9)+V_3(x_1,x_2...x_9),$$

It follows that the Lyapunov function

$$\frac{\mathrm{d}}{\mathrm{d}\tau}V(\tau)=\frac{\mathrm{d}}{\mathrm{d}\tau}V_1(\tau)+\frac{\mathrm{d}}{\mathrm{d}\tau}V_2(\tau)+\frac{\mathrm{d}}{\mathrm{d}\tau}V_3(\tau),$$

Notice that the function $f(x)=x-1-\ln x\geq 0$, for $x>0$. Since the arithmetic mean exceeds the geometric mean, it follows that

$$2-\frac{x_1}{x_1^*}-\frac{x_1^*}{x_1}\leq 0,\quad 2-\frac{x_2}{x_2^*}-\frac{x_2^*}{x_2}\leq 0,\ 2-\frac{x_3}{x_3^*}-\frac{x_3^*}{x_3}\leq 0,\ 2-\frac{x_4}{x_4^*}-\frac{x_4^*}{x_4}\leq 0,\quad 2-\frac{x_5}{x_5^*}-\frac{x_5^*}{x_5}\leq 0,$$
$$2-\frac{x_6}{x_6^*}-\frac{x_6^*}{x_6}\leq 0,\quad 2-\frac{x_7}{x_7^*}-\frac{x_7^*}{x_{76}}\leq 0,\quad 2-\frac{x_8}{x_8^*}-\frac{x_8^*}{x_8}\leq 0,\quad 2-\frac{x_9}{x_9^*}-\frac{x_9^*}{x_9}\leq 0,\quad 2-\frac{x_8\,x_7^*}{x_8^*\,x_7}-\frac{x_8^*\,x_7}{x_8\,x_7^*},$$
$$2-\frac{\Lambda\,x_7}{\Lambda^*\,x_7^*}-\frac{\Lambda^*\,x_7^*}{\Lambda\,x_7}.$$

Using arithmetic and geometric principles we establish that $\frac{\mathrm{d}}{\mathrm{d}\tau}V(\tau)$ is negative or $\frac{\mathrm{d}}{\mathrm{d}\tau}V(\tau)=0$ when $x_1=x_1^*$, $x_2=x_2^*$, $x_3=x_3^*$, $x_4=x_4^*$, $x_5=x_5^*$, $x_6=x_6^*$, $x_7=x_7^*$, $x_7=x_7^*$, $x_8=x_8^*$ and $x_9=x_9^*$. Thus the endemic equilibrium is globally asymptotically stable by LaSalle's invariant principle [28]. □

3 Existence of Travelling Waves

In this section, we investigated the existence of travelling wave solutions for the system of equations (5). The method of the proofs is to use the Schauders fixed point theorem by applying method of upper–lower solutions and its associated cross-iteration scheme [29].

Let

$$\tilde{x_1}(X,\tau)=x_{10}-x_1(X,\tau),\ \tilde{x_2}(X,\tau)=x_2(X,\tau),\ \tilde{x_3}(X,\tau)=x_3(X,\tau),$$
$$\tilde{x_4}(X,\tau)=x_4(X,\tau),\ \tilde{x_6}(X,\tau)=x_6(X,\tau),\ \tilde{x_7}(X,\tau)=x_{70}-x_7(X,\tau),$$
$$\tilde{x_8}(X,\tau)=x_8(X,\tau),\ \tilde{x_9}(X,\tau)=x_9(X,\tau),$$

then system (5) is transformed into system of equations (16) by omitting tilde for simplicity

$$\begin{aligned}
\frac{\partial}{\partial\tau}x_1(X,\tau)&=d_1\frac{\partial^2}{\partial X^2}x_1(X,\tau)-\pi_1+(\Lambda+1+b_2)\,(x_{10}-x_1(X,\tau))-c_4\,x_6(X,\tau),\\
\frac{\partial}{\partial\tau}x_2(X,\tau)&=d_2\frac{\partial^2}{\partial X^2}x_2(X,\tau)+\Lambda\,x_1(X,\tau)-(c_5+1)\,x_2(X,\tau),\\
\frac{\partial}{\partial\tau}x_3(X,\tau)&=d_3\frac{\partial^2}{\partial X^2}x_3(X,\tau)+c_5\,x_2(X,\tau)-(m_1+1)\,x_3(X,\tau),\\
\frac{\partial}{\partial\tau}x_4(X,\tau)&=d_4\frac{\partial^2}{\partial X^2}x_4(X,\tau)+m_1\,x_3(X,\tau)-(m_2+1)\,x_4(X,\tau),\\
\frac{\partial}{\partial\tau}x_5(X,\tau)&=d_5\frac{\partial^2}{\partial X^2}x_5(X,\tau)+m_2\,x_4(X,\tau)-(m_3+1)\,x_5(X,\tau),\\
\frac{\partial}{\partial\tau}x_6(X,\tau)&=d_6\frac{\partial^2}{\partial X^2}x_6(X,\tau)+m_3\,x_5(X,\tau)-(c_4+1)\,x_6(X,\tau)+m_4\,x_8(X,\tau),\\
\frac{\partial}{\partial\tau}x_7(X,\tau)&=d_7\frac{\partial^2}{\partial X^2}x_7(X,\tau)-\pi_2+(\Lambda\,\phi+1)\,(x_{70}-x_7(X,\tau))-b_2\,(x_{10}-x_1(X,\tau)),\\
\frac{\partial}{\partial\tau}x_8(X,\tau)&=d_8\frac{\partial^2}{\partial X^2}x_8(X,\tau)+\phi\,\Lambda\,x_7(X,\tau)-(m_4+1)\,x_8(X,\tau),\\
\frac{\partial}{\partial\tau}x_9(X,\tau)&=d_9\frac{\partial^2}{\partial X^2}x_9(X,\tau)+e_1\,x_3(X,\tau)+e_2\,x_4(X,\tau)+e_3\,x_5(X,\tau)+e_4\,x_8(X,\tau)-e_5\,x_9(X,\tau).
\end{aligned}\tag{16}$$

We now look for travelling wave solutions to analyze the local stability of our transformed system of equations (16),

The usual form of travelling wave is the following.

$$\begin{aligned} &x_1(X,\tau)=u_1(z),\ x_2(X,\tau)=u_2(z),\ x_3(X,\tau)=u_3(z),\ x_4(X,\tau)=u_4(z),\ x_5(X,\tau)=u_5(z),\\ &x_6(X,\tau)=u_6(z),\ x_7(X,\tau)=u_7(z),\ x_8(X,\tau)=u_8(z),\ x_9(X,\tau)=u_9(z), \end{aligned} \tag{17}$$

where $u_1, u_2, u_3, \ldots, u_9 \in C^2(\mathbb{R},\mathbb{R})$, $z = X - c\tau$ is a new variable of the moving coordinate and c is a positive constant which corresponding to the speed of the wave to the right direction in the X-plane [30, 31]. Then we have $\frac{\partial u}{\partial \tau} = -c\,\frac{du}{dz}$ and $\frac{\partial u}{\partial X} = \frac{du}{dz}$ and substituting into to the system of equations (5). The partial differential equations in X and τ are transformed into second degree ordinary differential equations in z.

To analyze the system of equations (16), we assume that by taking $D = d_i$, where $i = 1, 2, 3, \ldots, 9$ and denoting the traveling wave $X - c\tau$ by t, we derive the system of equations, (18)

$$\begin{aligned} &Du_1''(t) + c\,u_1'(t) + f_{c1}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_2''(t) + c\,u_2'(t) + f_{c2}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_3''(t) + c\,u_3'(t) + f_{c3}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_4''(t) + c\,u_4'(t) + f_{c4}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_5''(t) + c\,u_5'(t) + f_{c5}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_6''(t) + c\,u_6'(t) + f_{c6}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_7''(t) + c\,u_7'(t) + f_{c7}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_8''(t) + c\,u_8'(t) + f_{c8}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0,\\ &Du_9''(t) + c\,u_9'(t) + f_{c9}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) = 0, \end{aligned} \tag{18}$$

where,

$$\begin{aligned} f_{c1}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= -\pi_1 + \left(\Lambda_2 + 1 + b_2\right)\left(x_{10} - u_1(t)\right) - c_4\,u_6(t),\\ f_{c2}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= \Lambda_2\,u_1(t) - \left(c_5 + 1\right)u_2(t),\\ f_{c3}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= c_5\,u_2(t) - \left(m_1 + 1\right)u_3(t),\\ f_{c4}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= m_1\,u_3(t) - \left(m_2 + 1\right)u_4(t),\\ f_{c5}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= m_2\,u_4(t) - \left(m_3 + 1\right)u_5(t),\\ f_{c6}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= m_3\,u_5(t) - \left(c_4 + 1\right)u_6(t) + m_4\,u_8(t),\\ f_{c7}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= -\pi_2 + \left(\Lambda_2\,\phi + 1\right)\left(x_{70} - x_7(t)\right) - b_2\,\left(x_{10} - x_1(t)\right),\\ f_{c8}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= \phi\,\Lambda_2\,\left(x_{70} - u_7\left(t\right)\right) - \left(m_4 + 1\right)u_8\left(t\right),\\ f_{c9}\left(u_{1t}, u_{2t}, u_{3t}, \ldots, u_{9t}\right) &= e_1\,u_4(t) + e_2\,u_3(t) + e_3\,u_5(t) + e_4\,u_8\left(t\right),\\ \Lambda_2 &= \beta\,\left(b_1\,u_4\left(t\right) + c_1\,u_3\left(t\right) + c_2\,u_8\left(t\right) + c_3\,u_9\left(t\right)\right), \end{aligned} \tag{19}$$

Equation (18) will be solved subject to the following boundary value condition.

$$\begin{aligned} &\lim_{t\to-\infty}\left(u_1(t), u_2(t), u_3(t), \ldots, u_9(t)\right) = (0,0,0,0,0,0,0,0,0),\\ &\lim_{t\to\infty}\left(u_1(t), u_2(t), u_3(t), \ldots, u_9(t)\right) = (k_1, k_2, k_3, \ldots, k_9) \triangleq \left(x_1^*, x_2^*, x_3^*, \ldots, x_9^*\right). \end{aligned} \tag{20}$$

We define the upper and lower solutions of system (3.1)

Definition 3.1. *A pair of continuous function $\overline{\Phi} = (\overline{u}_1, \overline{u}_2, \overline{u}_3, \ldots, \overline{u}_9)$ and $\underline{\Phi} = (\underline{u}_1, \underline{u}_2, \underline{u}_3, \ldots, \underline{u}_9)$ are called the upper–lower solutions of the systems of of system (18) if $\overline{\Phi}$ and $\underline{\Phi}$ are twice differentiable almost*

everywhere in $\mathbb{R}$ *and they are essentially bounded on* $\mathbb{R}$*,there hold the following properties,*

$$
\begin{aligned}
&D\underline{u}''_1(t) + c\,\underline{u}'_1(t) + f_{c1}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_2(t) + c\,\underline{u}'_2(t) + f_{c2}\left(\overline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_3(t) + c\,\underline{u}'_3(t) + f_{c3}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_4(t) + c\,\underline{u}'_4(t) + f_{c4}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_5(t) + c\,\underline{u}'_5(t) + f_{c5}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_6(t) + c\,\underline{u}'_6(t) + f_{c6}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_7(t) + c\,\underline{u}'_7(t) + f_{c7}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_8(t) + c\,\underline{u}'_8(t) + f_{c8}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},\\
&D\underline{u}''_9(t) + c\,\underline{u}'_9(t) + f_{c9}\left(\underline{u}_{1t}, \underline{u}_{2t}, \underline{u}_{3t}, \ldots, \underline{u}_{9t}\right) \geq 0, a.\ e.\ in\ \mathbb{R},
\end{aligned}
$$

and

$$
\begin{aligned}
&D\overline{u}''_1(t) + c\,\overline{u}'_1(t) + f_{c1}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_2(t) + c\,\overline{u}'_2(t) + f_{c2}\left(\underline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_3(t) + c\,\overline{u}'_3(t) + f_{c3}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_4(t) + c\,\overline{u}'_4(t) + f_{c4}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_5(t) + c\,\overline{u}'_5(t) + f_{c5}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_6(t) + c\,\overline{u}'_6(t) + f_{c6}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_7(t) + c\,\overline{u}'_7(t) + f_{c7}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_8(t) + c\,\overline{u}'_8(t) + f_{c8}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},\\
&D\overline{u}''_9(t) + c\,\overline{u}'_9(t) + f_{c9}\left(\overline{u}_{1t}, \overline{u}_{2t}, \overline{u}_{3t}, \ldots, \overline{u}_{9t}\right) \leq 0, a.\ e.\ in\ \mathbb{R},
\end{aligned}
$$

We use the values of f_{c1}*,* f_{c2}*,* f_{c3}*,* $\ldots$*,* f_{c8} *and* f_{c9} *as (19).*

Corresponding to (20) by using [32], [22] we assume that

(A1) $f(\tilde{\mathbf{0}}) = f(\tilde{\mathbf{K}}) = 0$ with $0 \leq K = (k_1, k_2, k_3, \ldots, k_9)$ and $\tilde{u}$ is a constant function of $[0, t) \to \mathbb{R}$ taking the value of u for all $t \in [0, t)$.

(PQMC) Partial quasi-monotonicity conditions (PQMC): There exist nine non-negative constants $\beta_i > 0,\ j = 1, 2, 3, \ldots, 9$ such that

$$
\begin{aligned}
&f_{c1}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c1}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_1[u_{12}\left(0\right) - u_{11}\left(0\right)] \geq 0,\\
&f_{c2}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c2}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_2[u_{22}\left(0\right) - u_{21}\left(0\right)] \geq 0,\\
&f_{c2}\left(u_{12}, u_{21}, u_{31}, u_{41}, u_{51}, u_{61}, u_{71}, u_{91}\right) - f_{c2}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) \leq 0,\\
&f_{c3}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c3}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_3[u_{32}\left(0\right) - u_{31}\left(0\right)] \geq 0,\\
&f_{c4}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c4}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_4[u_{42}\left(0\right) - u_{41}\left(0\right)] \geq 0,\\
&f_{c5}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c5}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_5[u_{52}\left(0\right) - u_{51}\left(0\right)] \geq 0,\\
&f_{c6}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c6}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_6[u_{62}\left(0\right) - u_{61}\left(0\right)] \geq 0,\\
&f_{c7}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c7}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_7[u_{72}\left(0\right) - u_{71}\left(0\right)] \geq 0,\\
&f_{c8}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c8}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_8[u_{82}\left(0\right) - u_{81}\left(0\right)] \geq 0,\\
&f_{c9}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c9}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) + \beta_9[u_{82}\left(0\right) - u_{81}\left(0\right)] \geq 0,
\end{aligned}
$$

where

$u_{1i}, u_{2i}, u_{3i}, \ldots, u_{9i}, \in C([0, t], \mathbb{R}), i = 1, 2$ with, $(0, 0, 0, 0, 0, 0, 0, 0, 0) \leq (u_{11}, u_{21}, u_{31}, \ldots, u_{91})(s)$ $\leq (u_{12}, u_{22}, u_{32}, \ldots, u_{92})(s) \leq (M_1, M_2, M_3, \ldots, M_9) > (k_1, k_2, k_3, \ldots, k_9),\ where\ \ s \in [0, t],\ M_j,$ and $k_j, j = 1, 2, \ldots, 9$ are positive constants.

Corresponding to the boundary condition (20), we can make the following lemma,

Lemma 1. *f_{cj}, $j = 1, 2, \ldots, 9$ of system (18) satisfy the partial quasi-monotonicity conditions (PQMC).*

Proof.

$$
\begin{aligned}
&f_{c1}\left(u_{12}, u_{22}, \ldots, u_{92}\right) - f_{c1}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = -\left(u_{12}\left(0\right)\Lambda_{22} - u_{11}\left(0\right)\Lambda_{21}\right) + x_{10}\ \left(\Lambda_{22} - \Lambda_{21}\right)\\
&\quad - \left(1 + b_2\right)\left(u_{12}\left(0\right) - u_{11}\left(0\right)\right) - c_4\ \left(u_{62}\left(0\right) - u_{61}\left(0\right)\right) \geq -\left(1 + b_2\right)\left(u_{12}\left(0\right) - u_{11}\left(0\right)\right),\\
&f_{c2}\left(u_{12}, u_{22}, \ldots, u_{92}\right) - f_{c2}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = \Lambda_{22}\, u_{12}\left(0\right) - \Lambda_{21}\, u_{11}\left(0\right) - \left(c_5 + 1\right) u_{22}\left(0\right)\\
&\quad + \left(c_5 + 1\right) u_{21}\left(0\right) \geq -\left(c_5 + 1\right)\left(u_{22}\left(0\right) - u_{21}\left(0\right)\right),\\
&f_{c2}\left(u_{12}, u_{21}, u_{31}, \ldots, u_{91}\right) - f_{c2}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = -\Lambda_2\ \left(u_{12}\left(0\right) - u_{11}\left(0\right)\right) \leq 0,\\
&f_{c3}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c3}\left(u_{11}, u_{21}, \ldots, u_{91}\right) = u_{22}\left(0\right) c_5 - \left(m_1 + 1\right) u_{32}\left(0\right) - u_{21}\left(0\right) c_5\\
&\quad + \left(m_1 + 1\right) u_{31}\left(0\right) \geq -\left(m_1 + 1\right)\left(u_{32}\left(0\right) - u_{31}\left(0\right)\right),\\
&f_{c4}\left(u_{12}, u_{22}, \ldots, u_{92}\right) - f_{c4}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = u_{32}\left(0\right) m_1 - \left(m_2 + 1\right) u_{42}\left(0\right) - u_{31}\left(0\right) m_1\\
&\quad + \left(m_2 + 1\right) u_{41}\left(0\right) \geq -\left(m_2 + 1\right)\left(u_{42}\left(0\right) - u_{41}\left(0\right)\right),\\
&f_{c5}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c5}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = u_{42}\left(0\right) m_2 - \left(m_3 + 1\right) u_{52}\left(0\right) - u_{41}\left(0\right) m_2\\
&\quad + \left(m_3 + 1\right) u_{51}\left(0\right) \geq -\left(m_3 + 1\right)\left(u_{52}\left(0\right) - u_{51}\left(0\right)\right),\\
&f_{c6}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c6}\left(u_{11}, u_{21}, \ldots, u_{91}\right) = u_{52}\left(0\right) m_3 - \left(c_4 + 1\right) u_{62}\left(0\right) + m_4\, u_{82}\left(0\right) - u_{51}\left(0\right) m_3\\
&\quad + \left(c_4 + 1\right) u_{61}\left(0\right) - m_4\, u_{81}\left(0\right) \geq -\left(c_4 + 1\right)\left(u_{62}\left(0\right) - u_{61}\left(0\right)\right),\\
&f_{c7}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c7}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = \left(\Lambda_{22}\,\phi + 1\right)\left(x_{70} - u_{72}\left(0\right)\right) - b_2\ \left(x_{10} - u_{12}\left(0\right)\right)\\
&\quad - \left(\Lambda_{21}\,\phi + 1\right)\left(x_{70} - u_{71}\left(0\right)\right) + b_2\ \left(x_{10} - u_{11}\left(0\right)\right) \geq -(u_{72}(0) - u_{71}(0)),\\
&f_{c8}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c8}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) = \Lambda_{22}\,\phi\ \left(x_{70} - u_{72}\left(0\right)\right) - b_2\ \left(x_{10} - u_{12}\left(0\right)\right)\\
&\quad - \Lambda_{21}\,\phi\left(x_{70} - u_{71}\left(0\right)\right) + b_2\ \left(x_{10} - u_{11}\left(0\right)\right) \geq -\left(m_4 + 1\right)\left(u_{82}\left(0\right) - u_{81}\left(0\right)\right),\\
&f_{c9}\left(u_{12}, u_{22}, u_{32}, \ldots, u_{92}\right) - f_{c9}\left(u_{11}, u_{21}, u_{31}, \ldots, u_{91}\right) \geq -e_5\ \left(u_{92}\left(0\right) - u_{91}\left(0\right)\right),
\end{aligned}
$$

where $\Lambda_{21} = \beta\ \left(b_1\, u_{41} + c_1\, u_{31} + c_2\, u_{81} + c_3\, u_{91}\right)$ and $\Lambda_{22} = \beta\ \left(b_1\, u_{42} + c_1\, u_{32} + c_2\, u_{82} + c_3\, u_{92}\right)$.

Let $\beta_1 = b_2 + 1$, $\beta_2 = c_5 + 1$, $\beta_3 = m_1 + 1$, $\beta_4 = m_2 + 1$, $\beta_5 = m_3 + 1$, $\beta_6 = c_4 + 1$, $\beta_7 = 1$, $\beta_8 = m_4 + 1$ and $\beta_9 = e_5$ then the proof is completed. □

For the constants β_1, β_2, β_3, β_4 β_5, β_6, β_7, β_8 and β_9; $H : C(\mathbb{R}, \mathbb{R}^9) \to C(\mathbb{R}, \mathbb{R}^9)$ and then

$$
\begin{aligned}
H_1(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c1}(u_1, u_2, u_3, \ldots, u_9) + \beta_1\, u_1(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_2(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c2}(u_1, u_2, u_3, \ldots, u_9) + \beta_2\, u_2(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_3(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c3}(u_1, u_2, u_3, \ldots, u_9) + \beta_3\, u_3(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_4(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c4}(u_1, u_2, u_3, \ldots, u_9) + \beta_4\, u_4(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_5(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c5}(u_1, u_2, u_3, \ldots, u_9) + \beta_5\, u_5(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_6(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c6}(u_1, u_2, u_3, \ldots, u_9) + \beta_6\, u_2(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_7(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c7}(u_1, u_2, u_3, \ldots, u_9) + \beta_7\, u_7(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_8(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c8}(u_1, u_2, u_3, \ldots, u_9) + \beta_8\, u_8(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}),\\
H_9(u_1, u_2, u_3, \ldots, u_9)(t) &= f_{c9}(u_1, u_2, u_3, \ldots, u_9) + \beta_9\, u_9(t), \quad \text{for } u_1, u_2, u_3, \ldots, u_9 \in C(\mathbb{R}, \mathbb{R}).
\end{aligned}
\tag{21}
$$

Equation (18) can be rewritten by substituting equation (21) as

$$\begin{aligned}
&Du_1''(t) + c\,u_1'(t) - \beta_1\,u_1(t) + H_1(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_2''(t) + c\,u_2'(t) - \beta_2\,u_2(t) + H_3(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_3''(t) + c\,u_3'(t) - \beta_3\,u_3(t) + H_3(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_4''(t) + c\,u_4'(t) - \beta_4\,u_4(t) + H_4(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_5''(t) + c\,u_5'(t) - \beta_5\,u_5(t) + H_5(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_6''(t) + c\,u_6'(t) - \beta_6\,u_6(t) + H_6(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_7''(t) + c\,u_7'(t) - \beta_7\,u_7(t) + H_7(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_8''(t) + c\,u_8'(t) - \beta_8\,u_8(t) + H_8(u_1,u_2,u_3,\ldots,u_9)(t) = 0,\\
&Du_9''(t) + c\,u_9'(t) - \beta_9\,u_9(t) + H_9(u_1,u_2,u_3,\ldots,u_9)(t) = 0,
\end{aligned} \tag{22}$$

and define

$$\begin{aligned}
C_K(\mathbb{R},\mathbb{R}^9) = \big\{&(u_1,u_2,u_3,\ldots,u_9) \in C_K(\mathbb{R},\mathbb{R}^9) : (0,0,0,0,0,0,0,0,0)\\
&\leq (u_1,u_2,u_3,\ldots,u_9) \leq (k_1,k_2,k_3,\ldots,k_9)\big\},
\end{aligned}$$

and define $F_i = (F_1,F_2,F_3,\ldots,F_9) : C_K(\mathbb{R},\mathbb{R}^9) \to C(\mathbb{R},\mathbb{R}^9)$ by

$$\begin{aligned}
F_i(u_1,u_2,u_3,\ldots,u_9)(t) = \frac{1}{D\,(\lambda_{2i}-\lambda_{1i})}\Bigg[&\int_{-\infty}^{t} e^{\lambda_{1i}(t-s)}\,H_i(u_1,u_2,u_3,\ldots,u_9)(s)\,ds\\
&+ \int_{t}^{\infty} e^{\lambda_{2i}(t-s)} H_i(u_1,u_2,u_3,\ldots,u_9)(s)\,ds\Bigg],
\end{aligned}$$

where,

$$\begin{aligned}
&(0,0,0,0,0,0,0,0,0) \leq (u_1,u_2,u_3,\ldots,u_9) \leq (k_1,k_2,k_3,\ldots,k_9) \text{ and}\\
&\lambda_{1i} = \frac{c-\sqrt{c^2+4\,\beta_i\,D}}{2\,D} < 0,\ \ \lambda_{2i} = \frac{c+\sqrt{c^2+4\,\beta_i\,D}}{2\,D} > 0,\ \ i = 1,2,\ldots,9,
\end{aligned}$$

for $(u_1,u_2,u_3,\ldots,u_9) \in C_K(\mathbb{R},\mathbb{R}^9)$. we can see that $F : C_K(\mathbb{R},\mathbb{R}^9) \to C(\mathbb{R},\mathbb{R}^9)$ is well defined, and for any $(u_1,u_2,u_3,\ldots,u_9) \in C_K(\mathbb{R},\mathbb{R}^9)(u_1,u_2,u_3,\ldots,u_9)$, then, $F_1(u_1,u_2,u_3,\ldots,u_9)$, $F_2(u_1,u_2,u_3,\ldots,u_9),\ldots,F_9(u_1,u_2,u_3,\ldots,u_9),\ \ i = 1,2,\ldots,9$ satisfy

$$\begin{aligned}
&DF_i''(u_1,u_2,u_3,\ldots,u_9) + cF_i'(u_1,u_2,u_3,\ldots,u_9) - \beta_i F_i(u_1,u_2,u_3,\ldots,u_9)\\
&+ H_i(u_1,u_2,u_3,\ldots,u_9) = 0,\ \ i = 1,2,\ldots,9,
\end{aligned} \tag{23}$$

Thus, if $F(u_1,u_2,u_3,\ldots,u_9) = (F_1(u_1,u_2,u_3,\ldots,u_9),F_2(u_1,u_2,u_3,\ldots,u_9),\ldots,F_9(u_1,u_2,u_3,\ldots,u_9)) = (u_1,u_2,u_3,\ldots,u_9),\ \ i = 1,2,\ldots,9$, i.e., $(u_1,u_2,u_3,\ldots,u_9)$ is a fixed point of F, (23) reduce to (22), it mean that (22) has a solution $(u_1,u_2,u_3,\ldots,u_9)$. Then if this the solution further satisfies the boundary condition (20), then it gives a traveling wave solutions.

Therefore, there exists a traveling wave solutions of system (18) satisfying (PQMC) and there is a pair of upper and lower solutions of $\big(\underline{u_1}(t),\underline{u_2}(t),\ldots,\underline{u_9}(t)\big)$ and $(\overline{u}_1(t),\overline{u}_2(t),\overline{u}_3(t),\ldots,\overline{u}_9(t))$, which satisfying the following conditions:

$$\begin{aligned}
&(C1)\ \ (0,0,0,0,0,0,0,0,0) \leq \big(\underline{u_1}(t),\underline{u_2}(t),\ldots,\underline{u_9}(t)\big) \leq (\overline{u}_1(t),\overline{u}_2(t),\overline{u}_3(t),\ldots,\overline{u}_9(t))\\
&\quad \leq (M_1,M_2,M_3,\ldots,M_9),\ \ t \in \mathbb{R}\\
&(C2)\ \ \lim_{t\to-\infty}\big(\underline{u_1}(t),\underline{u_2}(t),\ldots,\underline{u_9}(t)\big) = (0,0,0,0,0,0,0,0,0),\\
&\quad \lim_{t\to\infty}(\overline{u}_1(t),\overline{u}_2(t),\overline{u}_3(t),\ldots,\overline{u}_9(t)) = (k_1,k_2,k_3,\ldots,k_9),
\end{aligned} \tag{24}$$

$\Phi = (u_1,u_2,u_3,\ldots,u_9)$ is a the function which has upper and lower value of $\underline{\Phi} = (\underline{u_1},\underline{u_2},\underline{u_3},\ldots,\underline{u_9})$ and $\overline{\Phi} = (\overline{u}_1,\overline{u}_2,\overline{u}_3,\ldots,\overline{u}_9)$ then we define the profile for

$$\Gamma\left(\underline{\Phi},\overline{\Phi}\right) = \{\Phi \in C(\mathbb{R},\mathbb{R}^9) \mid \underline{\Phi} \leq \Phi \leq \overline{\Phi}\}.$$

From ([22], [32]) we have the following Lemmas (2) and (3)

Lemma 2. *Assume that (PQM) hold then* $F(\Gamma(\underline{\Phi}, \overline{\Phi})) \subset \Gamma(\underline{\Phi}, \overline{\Phi})$*, where* $F = F_i$*,* $i = 1, 2, 3, ..., 9$.
Lemma 3. $F : (\Gamma(\underline{\Phi}, \overline{\Phi})) \to \Gamma(\underline{\Phi}, \overline{\Phi})$*, is compact.*

Now, to construct and prove the upperlower solutions, note that $\mathcal{R}_c > 1$ can be written in the form

$$
\begin{aligned}
&\Delta_1 = -\pi_1 + (\beta\ (c_2\, M_8 + W_1) + 1 + b_2)\,(x_{10} - k_1) - c_4\, M_6 > 0,\\
&\Delta_2 = \beta\ (c_2\, M_8 + W_1)\ (x_{10} - M_1) - (c_5 + 1)\, k_2 > 0, \quad \Delta_3 = c_5\, M_2 - (m_1 + 1)\, k_3 > 0,\\
&\Delta_4 = M_3\, m_1 - (m_2 + 1)\, k_4 > 0, \quad \Delta 5 = m_2\, M_4 - (m_3 + 1)\, k_5 > 0,\\
&\Delta_6 = m_3\, M_5 + m_4\, M_8 - (c_4 + 1)\, k_6 > 0,\\
&\Delta_7 = -\pi_2 + (\beta\, \phi\ (c_2\, M_8 + W_1) + 1)\,(x_{70} - k_7) - b_2\ (x_{10} - M_1) > 0,\\
&\Delta_8 = \beta\, \phi\ (c_2\, k_8 + W_1)\ (x_{70} - M_7) - (m_4 + 1)\, k_8 > 0,\\
&\Delta_9 = M_3\, e_2 + M_4\, e_1 + M_5\, e_3 + M_8\, e_4 - e_5\, k_9 > 0,
\end{aligned}
$$

where $W_1 = M_3\, c_1 + M_4\, b_1 + M_9\, c_3$. Using Wang and Xu [22, 33] assumption,

$$
c > c^* \triangleq \max\left\{2\sqrt{D\Delta_1}, 2\sqrt{D\Delta_2}, 2\sqrt{D\Delta_3}, 2\sqrt{D\Delta_4}, 2\sqrt{D\Delta_6}, 2\sqrt{D\Delta_7}, 2\sqrt{D\Delta_8}, 2\sqrt{D\Delta_9},\right\}
$$

then there exists $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5\ \lambda_6,\ \lambda_7,\ \lambda_8$, and λ_9, such that

$$
\begin{aligned}
&D{\lambda_1}^2 + c\lambda_1 - \frac{\pi_1}{k_1} + \frac{(\beta\ (M_8\, c_2 + W_1) + 1 + b_2)\,(x_{10} - k_1)}{k_1} - \frac{c_4\, M_6}{k_1} = 0,\\
&D{\lambda_2}^2 + c\lambda_2 + \frac{\beta\ (M_8\, c_2 + W_1)\,(x_{10} - M_1)}{k_2} - c_5 - 1 = 0,\\
&D{\lambda_3}^2 + c\lambda_3 + \frac{c_5\, M_2}{k_3} - m_1 - 1 = 0,\\
&D{\lambda_4}^2 + c\lambda_4 + \frac{M_3\, m_1}{k_4} - m_2 - 1 = 0,\\
&D{\lambda_5}^2 + c\lambda_5 + \frac{m_2\, M_4}{k_5} - m_3 - 1 = 0,\\
&D{\lambda_6}^2 + c\lambda_6 + \frac{m_3\, M_5}{k_6} + \frac{m_4\, M_8}{k_6} - c_4 - 1 = 0,\\
&D{\lambda_7}^2 + c\lambda_7 - \frac{\pi_2}{k_7} + \frac{(\beta\, \phi\ (M_8\, c_2 + W_1) + 1)\,(x_{70} - k_7)}{k_7} - \frac{b_2\ (x_{10} - M_1)}{k_7} = 0,\\
&D{\lambda_8}^2 + c\lambda_8 + \frac{\phi\, \beta\ (c_2\, k_8 + W_1)\,(x_{70} - M_7)}{k_8} - m_4 - 1 = 0,\\
&D{\lambda_9}^2 + c\lambda_9 + \frac{M_3\, e_2 + M_4\, e_1 + M_5\, e_3 + M_8\, e_4}{k_9} - e_5 = 0,
\end{aligned}
$$

There exists a pair of upper and lower solutions $(\overline{u_1}, \overline{u_2}, \overline{u_3}, \ldots, \overline{u_9})$, and $(\underline{u_1}, \underline{u_2}, \underline{u_3}, \ldots, \underline{u_9})$ of (18) respectively, satisfying the partial quasi-monotonicity conditions $(C1)$. Assuming that $\mathcal{R}_c > 1$, and $m_4 > m_2, m_3$, there exist $\epsilon_i > 0,\ (i = 1, 2, 3, \ldots, 18)$ where $\epsilon_1, \epsilon_2 \in (0, k_1)$, $\epsilon_3, \epsilon_4 \in (0, k_2)$, $\epsilon_5, \epsilon_6 \in (0, k_3)$, $\epsilon_7, \epsilon_8 \in (0, k_4)$, $\epsilon_9, \epsilon_{10} \in (0, k_5)$ $\epsilon_{11}, \epsilon_{12} \in (0, k_6)$, $\epsilon_{13}, \epsilon_{14} \in (0, k_7)$, $\epsilon_{15}, \epsilon_{16} \in (0, k_8)$, $\epsilon_{17}, \epsilon_{18} \in$

$(0, k_9)$ such that

$$\begin{aligned}
&\pi_1 - (\beta\, R_1 + b_2 + 1)\,(x_{10} - k_1 + \epsilon_1) + c_4\ (k_6 - \epsilon_{11}) < 0, \quad -\beta\, R_1\ (x_{10} - k_1 - \epsilon_2) + (c_5 + 1)\,(k_2 - \epsilon_3) < 0,\\
&-c_5\ (k_2 - \epsilon_3) + (m_1 + 1)\,(k_3 - \epsilon_5) < 0, \quad -(k_3 - \epsilon_5)\, m_1 + (m_2 + 1)\,(k_4 - \epsilon_7) < 0,\\
&m_2\ (k_4 - \epsilon_7) - (m_3 + 1)\,(k_5 - \epsilon_9) < 0, \quad -m_3\ (k_5 - \epsilon_9) + (c_4 + 1)\,(k_6 - \epsilon_{11}) - m_4\ (k_8 - \epsilon_{15}) < 0,\\
&\pi_2 - (\beta\, \phi\, R_1 + 1)\,(x_{70} - k_7 + \epsilon_{13}) + b_2\ (x_{10} - k_1 + \epsilon_1) < 0,\\
&-\beta\, \phi\, R_1\ (x_{70} - k_7 + \epsilon_{13}) + (m_4 + 1)\,(k_8 - \epsilon_{15}) < 0,\\
&(k_9 - \epsilon_{17})\, e_5 + (\epsilon_7 - k_4)\, e_1 + (\epsilon_5 - k_3)\, e_2 + e_3\ (\epsilon_9 - k_5) + e_4\ (\epsilon_{15} - k_8) < 0,\\
&-\pi_1 + (\beta\, R_2 + b_2 + 1)\,(x_{10} - k_1 - \epsilon_2) - c_4\ (k_6 + \epsilon_{12}) < 0,\\
&\beta\, R_2\ (x_{10} - k_1 + \epsilon_1) - (c_5 + 1)\,(k_2 + \epsilon_4) < 0, \quad c_5\ (k_2 + \epsilon_4) - (m_1 + 1)\,(k_3 + \epsilon_6) < 0,\\
&(k_3 + \epsilon_6)\, m_1 - (m_2 + 1)\,(k_4 + \epsilon_8) < 0, \quad m_2\ (k_4 + \epsilon_8) - (m_3 + 1)\,(k_5 + \epsilon_{10}) < 0,\\
&m_3\ (k_5 + \epsilon_{10}) - (c_4 + 1)\,(k_6 + \epsilon_{12}) + m_4\ (k_8 + \epsilon_{16}) < 0,\\
&-\pi_2 + (\beta\, R_2\, \phi + 1)\,(x_{70} - k_7 - \epsilon_{14}) - b_2\ (x_{10} - k_1 - \epsilon_2) < 0,\\
&\beta\, \phi\, R_2\ (x_{70} - k_7 - \epsilon_{14}) - (m_4 + 1)\,(k_8 + \epsilon_{16}) < 0 < 0,\\
&e_1\ (k_4 + \epsilon_8) + e_2\ (k_3 + \epsilon_6) + e_3\ (k_5 + \epsilon_{10}) + e_4\ (k_8 + \epsilon_{16}) - e_5\ (k_9 + \epsilon_{18}) < 0,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
R_1 &= b_1\ (k_4 - \epsilon_7) + c_1\ (k_3 - \epsilon_5) + c_2\ (k_8 - \epsilon_{15}) + c_3\ (k_9 - \epsilon_{17}),\\
R_2 &= b_1\ (k_4 + \epsilon_8) + c_1\ (k_3 + \epsilon_6) + c_2\ (k_8 + \epsilon_{16}) + c_3\ (k_9 + \epsilon_{18}),\\
F_1 &= e_1\ (k_4 - \epsilon_7) + e_2\ (k_3 - \epsilon_5) + e_3\ (k_5 - \epsilon_9) + e_4\ (k_8 - \epsilon_{15}),\\
F_2 &= e_1\ (k_4 + \epsilon_8) + e_2\ (k_3 + \epsilon_6) + e_3\ (k_5 + \epsilon_{10}) + e_4\ (k_8 + \epsilon_{16}).
\end{aligned}$$

Then we can find that $\epsilon_i > 0,\ (i = 1, 2, 3, \ldots, 18)$ which satisfying the above 18 equations. Assume that

$$x_{10} - \frac{c_4\, k_6 + \pi_1}{R_1\, \beta + b_2 + 1} = k_1, \quad x_{10} - \frac{c_4\, k_6 + \pi_1}{R_2\, \beta + b_2 + 1} = k_1,$$

For $\epsilon_3, \epsilon_4 \in (0, k_2)$, $\epsilon_5, \epsilon_6 \in (0, k_3)$, $\epsilon_9, \epsilon_{10} \in (0, k_5)$ we can find $\epsilon_1, \epsilon_2 \in (0, k_1)$ such that

$$\begin{aligned}
k_1 > \epsilon_1 > \frac{c_4\, \epsilon_{11}}{R_1\, \beta + b_2 + 1} > -\frac{c_4\, \epsilon_{11}}{R_1\, \beta + b_2 + 1} &= -\frac{c_4\, (\epsilon_{11} + k_6)}{R_1\, \beta + b_2 + 1} - \left(x_{10} - \frac{\pi_1}{R_1\, \beta + b_2 + 1} - k_1\right),\\
k_1 > \epsilon_2 > \frac{c_4\, \epsilon_{12}}{R_2\, \beta + b_2 + 1} > -\frac{c_4\, \epsilon_{12}}{R_2\, \beta + b_2 + 1} &= -\frac{c_4\, (\epsilon_{12} + k_6)}{R_2\, \beta + b_2 + 1} + \left(x_{10} - \frac{\pi_1}{R_2\, \beta + b_2 + 1} - k_1\right),
\end{aligned}$$

which give

$$\begin{aligned}
&\pi_1 - (\beta\, R_1 + b_2 + 1)\,(x_{10} - k_1 + \epsilon_1) + c_4\ (k_6 - \epsilon_{11}) < 0,\\
&-\pi_1 + (\beta\, R_2 + b_2 + 1)\,(x_{10} - k_1 - \epsilon_2) - c_4\ (k_6 + \epsilon_{12}) < 0.
\end{aligned}$$

Assume that $\dfrac{\beta\, R_1\ (x_{10} - k_1)}{c_5 + 1} = k_2$ and $\dfrac{\beta\, R_2\ (x_{10} - k_1)}{c_5 + 1} = k_2$. For $\epsilon_1, \epsilon_2 \in (0, k_1)$, $\epsilon_3, \epsilon_4 \in (0, k_2)$, $\epsilon_9, \epsilon_{10} \in (0, k_5)$ there exist $\epsilon_3, \epsilon_4 \in (0, k_2)$ such that

$$\begin{aligned}
&\epsilon_3 > \frac{\beta\, R_1\, \epsilon_2}{c_5 + 1}, \quad \epsilon_3 > \frac{\beta\, \epsilon_2}{c_5 + 1}, \quad \epsilon_4 > \frac{\beta\, R_2\, \epsilon_1}{c_5 + 1}, \quad \epsilon_4 > \frac{\beta\, \epsilon_1}{c_5 + 1},\\
&k_2 > \epsilon_3 > \frac{\beta\, R_1\, \epsilon_2}{c_5 + 1} = \frac{\beta\, R_1\, (\epsilon_2 + k_1)}{c_5 + 1} - \left(\frac{\beta\, R_1\, x_{10}}{c_5 + 1} - k_2\right),\\
&k_2 > \epsilon_4 > \frac{\beta\, R_2\, \epsilon_1}{c_5 + 1} = \frac{\beta\, R_2\, (\epsilon_1 - k_1)}{c_5 + 1} + \left(\frac{\beta\, R_2\, x_{10}}{c_5 + 1} - k_2\right),
\end{aligned}$$

which give

$$-\beta\,(x_{10}-k_1-\epsilon_2)\,R_1+(c_5+1)\,(k_2-\epsilon_3)<0,,\quad \beta\,(x_{10}+\epsilon_1-k_1)\,R_2-(c_5+1)\,(k_2+\epsilon_4)<0.$$

For $k_3=c_5\,k_2/(m_1+1)$, $\epsilon_1,\epsilon_2\in(0,k_1)$, $\epsilon_3,\epsilon_4\in(0,k_2)$, $\epsilon_9,\epsilon_{10}\in(0,k_5)$ there exist $\epsilon_5,\epsilon_6\in(0,k_3)$ such that

$$k_3>\epsilon_5>\frac{c_5\,\epsilon_3}{m_1+1}=\frac{c_5\,(\epsilon_3-k_2)}{m_1+1}+k_3,\quad k_3>\epsilon_6>\frac{c_5\,\epsilon_4}{m_1+1}=\frac{c_5\,(\epsilon_4+k_2)}{m_1+1}-k_3,$$

which give

$$-c_5\,(k_2-\epsilon_3)+(m_1+1)\,(k_3-\epsilon_5)<0,\quad c_5\,(k_2+\epsilon_4)-(m_1+1)\,(k_3+\epsilon_6)<0.$$

For $k_4=k_3m_1/(m_2+1)$, $\epsilon_1,\epsilon_2\in(0,k_1)$, $\epsilon_3,\epsilon_4\in(0,k_2)$, $\epsilon_5,\epsilon_6\in(0,k_3)$,
$\epsilon_5,\epsilon_6\in(0,k_3)$, $\epsilon_9,\epsilon_{10}\in(0,k_5)$ there exist $\epsilon_7,\epsilon_8\in(0,k_4)$ such that

$$k_4>\epsilon_7>\frac{\epsilon_5m_1}{m_2+1}=\frac{m_1\,(\epsilon_5-k_3)}{m_2+1}+k_4,\quad k_4>\epsilon_8>\frac{\epsilon_6m_1}{m_2+1}=\frac{m_1\,(\epsilon_6+k_3)}{m_2+1}-k_4,$$

which give

$$-(k_3-\epsilon_5)\,m_1+(m_2+1)\,(k_4-\epsilon_7)<0,\qquad (k_3+\epsilon_6)\,m_1-(m_2+1)\,(k_4+\epsilon_8)<0.$$

For $m_2\,k_4/(m_3+1)=k_5$, $\epsilon_1,\epsilon_2\in(0,k_1)$, $\epsilon_3,\epsilon_4\in(0,k_2)$, $\epsilon_5,\epsilon_6\in(0,k_3)$ there exist $\epsilon_9,\epsilon_{10}\in(0,k_5)$ such that

$$k_5>\epsilon_9<\frac{m_2\,\epsilon_7}{m_3+1}=\frac{m_2\,(\epsilon_7-k_4)}{m_3+1}+k_5),\quad k_5>\epsilon_{10}<\frac{m_2\,\epsilon_8}{m_3+1}=\frac{m_2\,(\epsilon_8+k_4)}{m_3+1}-k_5,$$

which give

$$m_2\,(k_4-\epsilon_7)-(m_3+1)\,(k_5-\epsilon_9)<0,\qquad m_2\,(k_4+\epsilon_8)-(m_3+1)\,(k_5+\epsilon_{10})<0.$$

For $(m_3\,k_5+m_4\,k_8)/(c_4+1)=k_6$, $\epsilon_1,\epsilon_2\in(0,k_1)$, $\epsilon_3,\epsilon_4\in(0,k_2)$, $\epsilon_5,\epsilon_6\in(0,k_3)$ there exist $\epsilon_{11},\epsilon_{12}\in(0,k_6)$ such that

$$\begin{aligned}
k_6>\epsilon_{11}>\frac{2\,m_3\,\epsilon_9}{c_4+1}&=\frac{m_3\,(2\,\epsilon_9-k_5)}{c_4+1}-\left(\frac{k_8m_4}{c_4+1}-k_6\right),\\
k_6>\epsilon_{11}>\frac{2\,m_4\,\epsilon_{15}}{c_4+1}&=\frac{m_4\,(2\,\epsilon_{15}-k_8)}{c_4+1}-\left(\frac{m_3\,k_5}{c_4+1}-k_6\right),\\
k_6>\epsilon_{12}>\frac{2\,m_3\,\epsilon_{10}}{c_4+1}&=\frac{m_3\,(2\,\epsilon_{10}-k_5)}{c_4+1}-\left(\frac{k_8m_4}{c_4+1}-k_6\right),\\
k_6>\epsilon_{12}>\frac{2\,m_4\,\epsilon_{16}}{c_4+1}&=\frac{m_4\,(2\,\epsilon_{16}-k_8)}{c_4+1}-\left(\frac{m_3\,k_5}{c_4+1}-k_6\right),
\end{aligned}$$

which give

$$\begin{aligned}
&-m_3\,(k_5-\epsilon_9)+(c_4+1)\,(k_6-\epsilon_{11})-m_4\,(k_8-\epsilon_{15})<0,\\
&m_3\,(k_5+\epsilon_{10})-(c_4+1)\,(k_6+\epsilon_{12})+m_4\,(k_8+\epsilon_{16})<0.
\end{aligned}$$

For $x_{70}-\dfrac{\pi_2+b_2\,(x_{10}-k_1)}{R_1\,\beta\,\phi+1}=k_7$ and $x_{70}-\dfrac{\pi_2+b_2\,(x_{10}-k_1)}{R_2\,\beta\,\phi+1}=k_7$, $\epsilon_1,\epsilon_2\in(0,k_1)$, $\epsilon_3,\epsilon_4\in(0,k_2)$, $\epsilon_5,\epsilon_6\in(0,k_3)$ there exist $\epsilon_{13},\epsilon_{14}\in(0,k_7)$ such that

$$\begin{aligned}
k_7>\epsilon_{13}>\frac{b_2\,\epsilon_1}{R_1\,\beta\,\phi+1}&=\frac{b_2\,(\epsilon_1-k_1)}{R_1\,\beta\,\phi+1}-\left(x_{70}-\frac{\pi_2+b_2\,x_{10}}{R_1\,\beta\,\phi+1}-k_7\right),\\
k_7>\epsilon_{14}>\frac{b_2\,\epsilon_2}{R_2\,\beta\,\phi+1}&=\frac{b_2\,(\epsilon_2+k_1)}{R_2\,\beta\,\phi+1}+\left(x_{70}-\frac{\pi_2+b_2\,x_{10}}{R_2\,\beta\,\phi+1}-k_7\right),
\end{aligned}$$

which give

$$\pi_2 - (R_1\,\beta\,\phi + 1)\,(x_{70} - k_7 + \epsilon_{13}) + b_2\,(x_{10} - k_1 + \epsilon_1) < 0,$$
$$-\pi_2 + (R_2\,\beta\,\phi + 1)\,(x_{70} - k_7 - \epsilon_{14}) - b_2\,(x_{10} - k_1 - \epsilon_2) < 0.$$

For $\dfrac{\beta\,\phi\,R_1\,(x_{70} - k_7)}{m_4 + 1} - k_8$ and $\dfrac{\beta\,\phi\,R_2\,(x_{70} - k_7)}{m_4 + 1} - k_8$, $\epsilon_1, \epsilon_2 \in (0, k_1)$, $\epsilon_3, \epsilon_4 \in (0, k_2)$, $\epsilon_5, \epsilon_6 \in (0, k_3)$ there exist $\epsilon_{15}, \epsilon_{16} \in (0, k_8)$ such that

$$k_8 > \epsilon_{15} > \frac{\beta\,\phi\,\epsilon_{13}}{m_4 + 1} > -\frac{\beta\,\phi\,R_1\,\epsilon_{13}}{m_4 + 1} = -\frac{\beta\,\phi\,R_1\,(\epsilon_{13} + k_7)}{m_4 + 1} - \left(\frac{\beta\,\phi\,R_1\,x_{70}}{m_4 + 1} - k_8\right),$$
$$k_8 > \epsilon_{16} > \frac{\beta\,\phi\,\epsilon_{14}}{m_4 + 1} > -\frac{\beta\,\phi\,R_2\,\epsilon_{14}}{m_4 + 1} = -\frac{\beta\,\phi\,R_2\,(\epsilon_{14} - k_7)}{m_4 + 1} + \left(\frac{\beta\,\phi\,R_2\,x_{70}}{m_4 + 1} - k_8\right),$$

which give

$$-\beta\,\phi\,R_1\,(x_{70} - k_7 + \epsilon_{13}) + (m_4 + 1)\,(k_8 - \epsilon_{15}) < 0, \quad \beta\,\phi\,R_2\,(x_{70} - k_7 - \epsilon_{14}) - (m_4 + 1)\,(k_8 + \epsilon_{16}) < 0.$$

For $\dfrac{e_1\,k_4 + e_2\,k_3 + e_3\,k_5 + e_4\,k_8}{e_5} = k_9$, $\epsilon_1, \epsilon_2 \in (0, k_1)$, $\epsilon_3, \epsilon_4 \in (0, k_2)$, $\epsilon_5, \epsilon_6 \in (0, k_3)$ there exist $\epsilon_{17}, \epsilon_{18} \in (0, k_9)$ such that

$$k_9 > \epsilon_{17} > \frac{4\,e_2\,\epsilon_5}{e_5} = \frac{e_2\,(4\,\epsilon_5 - k_3)}{e_5} - \left(\frac{e_1\,k_4 + e_3\,k_5 + e_4\,k_8}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{17} > \frac{4\,e_1\,\epsilon_7}{e_5} = \frac{e_1\,(4\,\epsilon_7 - k_4)}{e_5} - \left(\frac{e_2\,k_3 + e_3\,k_5 + e_4\,k_8}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{17} > \frac{4\,e_3\,\epsilon_9}{e_5} = \frac{e_3\,(4\,\epsilon_9 - k_5)}{e_5} - \left(\frac{e_1\,k_4 + e_2\,k_3 + e_4\,k_8}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{17} > \frac{4\,e_4\,\epsilon_{15}}{e_5} = \frac{e_4\,(4\,\epsilon_{15} - k_8)}{e_5} - \left(\frac{e_1\,k_4 + e_2\,k_3 + e_3\,k_5}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{18} > \frac{4\,e_2\,\epsilon_6}{e_5} = \frac{e_2\,(4\,\epsilon_6 + k_3)}{e_5} + \left(\frac{e_1\,k_4 + e_3\,k_5 + e_4\,k_8}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{18} > \frac{4\,e_1\,\epsilon_8}{e_5} = \frac{e_1\,(4\,\epsilon_8 + k_4)}{e_5} + \left(\frac{e_2\,k_3 + e_3\,k_5 + e_4\,k_8}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{18} > \frac{4\,e_3\,\epsilon_{10}}{e_5} = \frac{e_3\,(4\,\epsilon_{10} + k_5)}{e_5} + \left(\frac{e_1\,k_4 + e_2\,k_3 + e_4\,k_8}{e_5} - k_9\right),$$
$$k_9 > \epsilon_{18} > \frac{4\,e_4\,\epsilon_{16}}{e_5} = \frac{e_4\,(4\,\epsilon_{16} + k_8)}{e_5} + \left(\frac{e_1\,k_4 + e_2\,k_3 + e_3\,k_5}{e_5} - k_9\right),$$

which give

$$(k_9 - \epsilon_{17})\,e_5 + (\epsilon_7 - k_4)\,e_1 + (\epsilon_5 - k_3)\,e_2 + e_3\,(\epsilon_9 - k_5) + e_4\,(\epsilon_{15} - k_8) < 0,$$
$$e_1\,(k_4 + \epsilon_8) + e_2\,(k_3 + \epsilon_6) + e_3\,(k_5 + \epsilon_{10}) + e_4\,(k_8 + \epsilon_{16}) - e_5\,(k_9 + \epsilon_{18}) < 0.$$

Furthermore, we can choose $\epsilon_1, \epsilon_2 \in (0, k_1)$, $\epsilon_3, \epsilon_4 \in (0, k_2)$, $\epsilon_5, \epsilon_6 \in (0, k_3)$, $\epsilon_7, \epsilon_8 \in (0, k_4)$, $\epsilon_9, \epsilon_{10} \in (0, k_5)$, $\epsilon_{11}, \epsilon_{12} \in (0, k_6)$, $\epsilon_{13}, \epsilon_{14} \in (0, k_7)$, $\epsilon_{15}, \epsilon_{16} \in (0, k_8)$ and $\epsilon_{17}, \epsilon_{18} \in (0, k_9)$ satisfying 28 (see from the Appendix)

Let $\epsilon_i > 0$ be defined as in (25) and λ_i for $i = 1, 2, \ldots, 18$, we can define the continuous functions $\underline{\Phi} = (\underline{u_1}, \underline{u_2}, \underline{u_3}, \ldots, \underline{u_9})$ and $\overline{\Phi} = (\overline{u_1}, \overline{u_2}, \overline{u_3}, \ldots, \overline{u_9})$ as follows:

$$\underline{u}_1(t) = \begin{cases} 0 & t \le t_1 \\ k_1 - \epsilon_1\,e^{-\lambda t} & t > t_1 \end{cases}, \quad \overline{u}_1(t) = \begin{cases} k_1\,e^{\lambda_1 t} & t \le t_{10} \\ k_1 + \epsilon_2\,e^{-\lambda t} & t > t_{10} \end{cases},$$

$$\underline{u}_2(t)=\begin{cases}0 & t\le t_2\\ k_2-\epsilon_3\,e^{-\lambda t} & t>t_2\end{cases},\quad \overline{u}_2(t)=\begin{cases}k_2\,e^{\lambda_2 t} & t\le t_{11}\\ k_2+\epsilon_4\,e^{-\lambda t} & t>t_{11}\end{cases},$$

$$\underline{u}_3(t)=\begin{cases}0 & t\le t_3\\ k_3-\epsilon_5\,e^{-\lambda t} & t>t_3\end{cases},\quad \overline{u}_3(t)=\begin{cases}k_3\,e^{\lambda_3 t} & t\le t_{12}\\ k_3+\epsilon_6\,e^{-\lambda t} & t>t_{12}\end{cases},$$

$$\underline{u}_4(t)=\begin{cases}0 & t\le t_4\\ k_4-\epsilon_7\,e^{-\lambda t} & t>t_4\end{cases},\quad \overline{u}_4(t)=\begin{cases}k_4\,e^{\lambda_4 t} & t\le t_{13}\\ k_4+\epsilon_8\,e^{-\lambda t} & t>t_{13}\end{cases},$$

$$\underline{u}_5(t)=\begin{cases}0 & t\le t_5\\ k_5-\epsilon_9\,e^{-\lambda t} & t>t_5\end{cases},\quad \overline{u}_5(t)=\begin{cases}k_5\,e^{\lambda_5 t} & t\le t_{14}\\ k_5+\epsilon_{10}\,e^{-\lambda t} & t>t_{14}\end{cases},$$

$$\underline{u}_6(t)=\begin{cases}0 & t\le t_6\\ k_2-\epsilon_{11}\,e^{-\lambda t} & t>t_6\end{cases},\quad \overline{u}_6(t)=\begin{cases}k_2\,e^{\lambda_6 t} & t\le t_{15}\\ k_2+\epsilon_{12}\,e^{-\lambda t} & t>t_{15}\end{cases},$$

$$\underline{u}_7(t)=\begin{cases}0 & t\le t_7\\ k_3-\epsilon_{13}\,e^{-\lambda t} & t>t_7\end{cases},\quad \overline{u}_7(t)=\begin{cases}k_3\,e^{\lambda_7 t} & t\le t_{16}\\ k_3+\epsilon_{14}\,e^{-\lambda t} & t>t_{16}\end{cases},$$

$$\underline{u}_8(t)=\begin{cases}0 & t\le t_8\\ k_4-\epsilon_{15}\,e^{-\lambda t} & t>t_8\end{cases},\quad \overline{u}_8(t)=\begin{cases}k_4\,e^{\lambda_8 t} & t\le t_{17}\\ k_4+\epsilon_{16}\,e^{-\lambda t} & t>t_{17}\end{cases},$$

$$\underline{u}_9(t)=\begin{cases}0 & t\le t_9\\ k_5-\epsilon_{17}\,e^{-\lambda t} & t>t_9\end{cases},\quad \overline{u}_9(t)=\begin{cases}k_5\,e^{\lambda_9 t} & t\le t_{18}\\ k_5+\epsilon_{18}\,e^{-\lambda t} & t>t_{18}\end{cases}.$$

We can see that $k_i<\sup_{t\in\mathbb{R}}\overline{\Phi}=M_i$, $\overline{\Phi}$ and $\underline{\Phi}$ satisfy (C_1) and (C_2), $(i=1,2,\ldots,9)$.

Lemma 4. *Let $\mathcal{R}_c>1$, then the function $\underline{\Phi}(t)=(\underline{u_1}(t),\underline{u_2}(t),\ldots,\underline{u_9}(t))$ defines the lower solution of system (18).*

Proof.

$$\begin{aligned}
Q_1(t)&=D\,\underline{u}_1''(t)+c\,\underline{u}_1'(t)+f_{c1}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_2(t)&=D\,\underline{u}_2''(t)+c\,\underline{u}_2'(t)+f_{c2}\left(\overline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_3(t)&=D\,\underline{u}_3''(t)+c\,\underline{u}_3'(t)+f_{c3}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_4(t)&=D\underline{u}_4''(t)+c\,\underline{u}_4'(t)+f_{c4}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_5(t)&=D\underline{u}_5''(t)+c\,\underline{u}_5'(t)+f_{c5}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_6(t)&=D\,\underline{u}_6''(t)+c\,\underline{u}_6'(t)+f_{c6}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_7(t)&=D\,\underline{u}_7''(t)+c\,\underline{u}_7'(t)+f_{c7}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_8(t)&=D\underline{u}_8''(t)+c\,\underline{u}_8'(t)+f_{c8}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right),\\
Q_9(t)&=D\underline{u}_9''(t)+c\,\underline{u}_9'(t)+f_{c9}\left(\underline{u_1}(t),\underline{u_2}(t),\underline{u_3}(t),\ldots,\underline{u_9}(t)\right).
\end{aligned}\tag{26}$$

If $t\le t_1$, $\underline{u}_1(t)=0$ and $\underline{u}_i(t)\ge 0$, where $i=2...9$, then $Q_1(t)=-\pi_1+(\Lambda_2+b_2+1)\,x_{10}+c_4\,u_6(t)\ge 0$.

If $t>t_1$, $\underline{u}_1(t)=k_1-\epsilon_2\mathrm{e}^{-\lambda t}$, $\underline{u}_i(t)\ge k_2-\epsilon_j\mathrm{e}^{-\lambda(t)}$, where $i=2,3,...,9$ and $j=4,6,...18$,

$$\begin{aligned}
Q_1(t)&\ge-\epsilon_1\mathrm{e}^{-\lambda t}\left(D\lambda^2+c\lambda+\frac{\pi_1}{\epsilon_1\mathrm{e}^{-\lambda t}}-\frac{(\beta R_2+1+b_2)\left(x_{10}-k_1+\epsilon_1\mathrm{e}^{-\lambda t}\right)}{\epsilon_1\mathrm{e}^{-\lambda t}}+\frac{c_4\left(k_6-\epsilon_{11}\mathrm{e}^{-\lambda t}\right)}{\epsilon_1\mathrm{e}^{-\lambda t}}\right)\\
&\ge-\epsilon_1\mathrm{e}^{-\lambda t}\left(D\lambda^2+c\lambda+\pi_1-\frac{(\beta R_1+1+b_2)\,(x_{10}-k_1+\epsilon_1)}{\epsilon_1}+\frac{c_4\,(k_6-\epsilon_{11})}{\epsilon_1}\right),
\end{aligned}$$

where $R_2 = b_1\ \left(k_4 - \epsilon_7 \mathrm{e}^{-\lambda t}\right) + c_1\ \left(k_3 - \epsilon_5 \mathrm{e}^{-\lambda t}\right) + c_2\ \left(k_8 - \epsilon_{15} \mathrm{e}^{-\lambda t}\right) + c_3\ \left(k_9 - \epsilon_{17} \mathrm{e}^{-\lambda t}\right)$,
Note that $\pi_1 - (R_1 + 1 + b_2)\,(x_{10} - k_1 + \epsilon_1) + c_4\ (k_6 - \epsilon_{11}) < 0$ such that $Q_1(\lambda) > 0$ for all $\lambda \in (0, \lambda_{10}^*)$.

If $t \leq t_2$, $\underline{u}_2(t) = 0$, $\underline{u}_1(t) = k_1 - \epsilon_2 \mathrm{e}^{-\lambda t}$, and $\underline{u}_i(t) \geq 0$, where $i = 1, 3, ..., 9$, then $Q_2(t) = \Lambda_2\ (x_{10} - u_1(t)) \geq 0$.

If $t > t_2$, $\underline{u}_2(t) = k_2 - \epsilon_3 \mathrm{e}^{-\lambda t}$, $\underline{u}_i(t) \geq k_2 - \epsilon_j \mathrm{e}^{-\lambda(t)}$, where $i = 1, 3, ..., 9$ and $j = 2, 6, ..., 18$.

$$
\begin{aligned}
Q_2(t) &\geq -\mathrm{e}^{-\lambda t}\epsilon_3 \left(D\,\lambda^2 + c\lambda - \frac{\beta\, R_3\ \left(x_{10} - k_1 - \epsilon_2 \mathrm{e}^{-\lambda t}\right)}{\epsilon_3 \mathrm{e}^{-\lambda t}} + \frac{(c_5 + 1)\left(k_2 - \epsilon_3 \mathrm{e}^{-\lambda t}\right)}{\epsilon_3 \mathrm{e}^{-\lambda t}} \right) \\
&\geq -\mathrm{e}^{-\lambda t}\epsilon_3 \left(D\,\lambda^2 + c\lambda - \frac{\beta\, R_1\ (x_{10} - k_1 - \epsilon_2)}{\epsilon_3} + \frac{(c_5 + 1)\,(k_2 - \epsilon_3)}{\epsilon_3} \right) = Q_2(\lambda),
\end{aligned}
$$

Since $Q_2(0) = -\beta\, R_3\ (x_{10} - k_1 - \epsilon_2) + (c_5 + 1)\,(k_2 - \epsilon_3) < 0$, then there exists a $\lambda_2^* > 0$ such that $Q_2 > 0$ for all $\lambda \in (0, \lambda_2^*)$.

If $t \leq t_3$, $\underline{u}_3(t) = 0$ and $\underline{u}_i(t) \geq 0$, where $i = 1, 2, 4, ..., 9$, then $Q_3(t) = c_5\, u_2(t) \geq 0$.

If $t > t_3$, $\underline{u}_3(t) = k_3 - \epsilon_5 \mathrm{e}^{-\lambda t}$, $\underline{u}_i(t) \geq k_i - \epsilon_j \mathrm{e}^{-\lambda(t)}$, where $i = 1, 2, 4, ..., 9$ and $j = 2, 6, ..., 18$,

$$
\begin{aligned}
Q_3(t) &\geq -\mathrm{e}^{-\lambda t}\epsilon_5 \left(D\,\lambda^2 + c\lambda - \frac{c_5\ \left(k_2 - \epsilon_3 \mathrm{e}^{-\lambda t}\right)}{\epsilon_5 \mathrm{e}^{-\lambda t}} + \frac{(m_1 + 1)\left(k_3 - \epsilon_5 \mathrm{e}^{-\lambda t}\right)}{\epsilon_5 \mathrm{e}^{-\lambda t}} \right) \\
&\geq -\mathrm{e}^{-\lambda t}\epsilon_5 \left(D\,\lambda^2 + c\lambda - \frac{c_5\ (k_2 - \epsilon_3)}{\epsilon_5} + \frac{(m_1 + 1)\,(k_3 - \epsilon_5)}{\epsilon_5} \right),
\end{aligned}
$$

Note that $-c_5\ (k_2 - \epsilon_4) + (m_1 + 1)\,(k_3 - \epsilon_5) < 0$, then there exists a $\lambda_3^* > 0$ such that $Q_1(\lambda) > 0$ for all $\lambda \in (0, \lambda_3^*)$.

If $t \leq t_4$, $\underline{u}_4(t) = 0$ and $\underline{u}_i(t) \geq 0$, where $i = 1, 2, 3, 5, ..., 9$, then $Q_4(t) = u_3(t) m_1 \geq 0$.

If $t > t_4$, $\underline{u}_4(t) = k_4 - \epsilon_7 \mathrm{e}^{-\lambda t}$, $\underline{u}_i(t) \geq k_i - \epsilon_j \mathrm{e}^{-\lambda(t)}$, where $i = 1, 3, ...9$ and $j = 2, 6, ..., 18$,

$$
\begin{aligned}
Q_4(t) &\geq -\mathrm{e}^{-\lambda t}\epsilon_7 \left(D\,\lambda^2 + c\lambda - \frac{\left(k_3 - \epsilon_5 \mathrm{e}^{-\lambda t}\right) m_1}{\epsilon_7 \mathrm{e}^{-\lambda t}} + \frac{(m_2 + 1)\left(k_4 - \epsilon_7 \mathrm{e}^{-\lambda t}\right)}{\epsilon_7 \mathrm{e}^{-\lambda t}} \right) \\
&\geq -\mathrm{e}^{-\lambda t}\epsilon_7 \left(D\,\lambda^2 + c\lambda - \frac{(k_3 - \epsilon_5)\, m_1}{\epsilon_7} + \frac{(m_2 + 1)\,(k_4 - \epsilon_7)}{\epsilon_7} \right),
\end{aligned}
$$

Note that $-(k_3 - \epsilon_5)\, m_1 + (m_2 + 1)\,(k_4 - \epsilon_7) < 0$, then there exists a $\lambda_4^* > 0$ such that $Q_4(\lambda) > 0$ for all $\lambda \in (0, \lambda_4^*)$.

If $t \leq t_5$, $\underline{u}_5(t) = 0$ and $\underline{u}_i(t) \geq 0$, where $i = 1, 2, 3, 5, ..., 9$, then $Q_5(t) = m_2\, u_4(t) \geq 0$.

If $t > t_5$, $\underline{u}_5(t) = k_5 - \epsilon_9 \mathrm{e}^{-\lambda t}$, $\underline{u}_i(t) \geq k_i - \epsilon_j \mathrm{e}^{-\lambda(t)}$, where $i = 1, 3, ...9$ and $j = 2, 6, ..., 18$,

$$
\begin{aligned}
Q_5(t) &\geq \epsilon_9 \mathrm{e}^{-\lambda t} \left(D\,\lambda^2 + c\lambda + \frac{m_2\ \left(k_4 - \epsilon_7 \mathrm{e}^{-\lambda t}\right)}{\epsilon_9 \mathrm{e}^{-\lambda t}} - \frac{(m_3 + 1)\left(k_5 - \epsilon_9 \mathrm{e}^{-\lambda t}\right)}{\epsilon_9 \mathrm{e}^{-\lambda t}} \right) \\
&\geq \epsilon_9 \mathrm{e}^{-\lambda t} \left(D\,\lambda^2 + c\lambda + \frac{m_2\ (k_4 - \epsilon_7)}{\epsilon_9} - \frac{(m_3 + 1)\,(k_5 - \epsilon_9)}{\epsilon_9} \right),
\end{aligned}
$$

Note that $m_2\ (k_4 - \epsilon_7) - (m_3 + 1)\,(k_5 - \epsilon_9) < 0$, then there exists a $\lambda_5^* > 0$ such that $Q_5(\lambda) > 0$ for all $\lambda \in (0, \lambda_5^*)$.

If $t \leq t_6$, $\underline{u}_6(t) = 0$ and $\underline{u}_i(t) \geq 0$, where $i = 1, 2, 3, 5, ..., 9$, then $Q_6(t) = m_3\, u_5(t) + m_4\, u_8(t) \geq 0$.

If $t > t_6$, $\underline{u}_6(t) = k_6 - \epsilon_{11} \mathrm{e}^{-\lambda t}$, $\underline{u}_i(t) \geq k_i - \epsilon_j \mathrm{e}^{-\lambda(t)}$, where $i = 1, 3, ..., 9$ and $j = 2, 6, ..., 18$,

$$
\begin{aligned}
Q_6(t) &\geq -\mathrm{e}^{-\lambda t}\epsilon_{11} \left(D\,\lambda^2 + c\lambda - \frac{m_3\ \left(k_5 - \epsilon_9 \mathrm{e}^{-\lambda t}\right)}{\epsilon_{11} \mathrm{e}^{-\lambda t}} + \frac{(c_4 + 1)\left(k_6 - \epsilon_{11} \mathrm{e}^{-\lambda t}\right)}{\epsilon_{11} \mathrm{e}^{-\lambda t}} - \frac{m_4\ \left(k_8 - \epsilon_{15} \mathrm{e}^{-\lambda t}\right)}{\epsilon_{11} \mathrm{e}^{-\lambda t}} \right) \\
&\geq -\mathrm{e}^{-\lambda t}\epsilon_{11} \left(D\,\lambda^2 + c\lambda - \frac{m_3\ (k_5 - \epsilon_9)}{\epsilon_{11}} + \frac{(c_4 + 1)\,(k_6 - \epsilon_{11})}{\epsilon_{11}} - \frac{m_4\ (k_8 - \epsilon_{15})}{\epsilon_{11}} \right),
\end{aligned}
$$

Note that $-m_3\ (k_5-\epsilon_9)+(c_4+1)\,(k_6-\epsilon_{11})-m_4\ (k_8-\epsilon_{15})<0$, then there exists a $\lambda_6^*>0$ such that $Q_6(\lambda)>0$ for all $\lambda\in(0,\lambda_6^*)$.

If $t\le t_7$, $\underline{u}_7(t)=0$ and $\underline{u}_i\,(t)\ge 0$, where $i=1,2,3,5...9$, then $Q_7(t)=-\pi_2+(\Lambda_2\,\phi+1)\,x_{70}-b_2\ (x_{10}-u_1\,(t))\ge 0$.

If $t>t_7$, $\underline{u}_7(t)=k_7-\epsilon_{14}\mathrm{e}^{-\lambda\,t}$, $\underline{u}_i(t)\ge k_i-\epsilon_j\mathrm{e}^{-\lambda\,(t)}$, where $i=1,3,...,9$ and $j=2,6,...,18$,

$$\begin{aligned}Q_7(t)&\ge -\mathrm{e}^{-\lambda\,t}\epsilon_{13}\left(D\,\lambda^2+c\lambda+\frac{\pi_2}{\epsilon_{13}\mathrm{e}^{-\lambda\,t}}-\frac{(\beta\,\phi\,R_2+1)\left(x_{70}-k_7+\epsilon_{13}\mathrm{e}^{-\lambda\,t}\right)}{\epsilon_{13}\mathrm{e}^{-\lambda\,t}}\right.\\&\quad\left.+\frac{b_2\ \left(x_{10}-k_1+\epsilon_1\mathrm{e}^{-\lambda\,t}\right)}{\epsilon_{13}\mathrm{e}^{-\lambda\,t}}\right)\\&\ge -\mathrm{e}^{-\lambda\,t}\epsilon_{13}\left(D\,\lambda^2+c\lambda+\frac{\pi_2}{\epsilon_{13}}-\frac{(\beta\,\phi\,R_1+1)\,(x_{70}-k_7+\epsilon_{13})}{\epsilon_{13}}+\frac{b_2\ (x_{10}-k_1+\epsilon_1)}{\epsilon_{13}}\right),\end{aligned}$$

Note that $\pi_2-(\beta\,\phi\,R_1+1)\,(x_{70}-k_7+\epsilon_{13})+b_2\ (x_{10}-k_1+\epsilon_1)<0$, then there exists a $\lambda_7^*>0$ such that $Q_7(\lambda)>0$ for all $\lambda\in(0,\lambda_7^*)$.

If $t\le t_8$, $\underline{u}_8(t)=0$ and $\underline{u}_i\,(t)\ge 0$, where $i=1,2,3,5,...,9$, then $Q_8(t)=\phi\,\Lambda_2\,(x_{70}-u_7\,(t))\ge 0$.

If $t>t_8$, $\underline{u}_8(t)=k_8-\epsilon_{16}\mathrm{e}^{-\lambda\,t}$, $\underline{u}_i(t)\ge k_i-\epsilon_j\mathrm{e}^{-\lambda\,(t)}$, where $i=1,3,...,9$ and $j=2,6,...,18$,

$$\begin{aligned}Q_8(t)&\ge -\mathrm{e}^{-\lambda\,t}\epsilon_{15}\left(D\lambda^2+c\lambda-\frac{\beta\,\phi\,R_2\ \left(x_{70}-k_7+\epsilon_{13}\mathrm{e}^{-\lambda\,t}\right)}{\epsilon_{15}\mathrm{e}^{-\lambda\,t}}+\frac{(m_4+1)\left(k_8-\epsilon_{15}\mathrm{e}^{-\lambda\,t}\right)}{\epsilon_{15}\mathrm{e}^{-\lambda\,t}}\right)\\&\ge -\mathrm{e}^{-\lambda\,t}\epsilon_{15}\left(D\,\lambda^2+c\lambda-\frac{\beta\,\phi\,R_1\ (x_{70}-k_7+\epsilon_{13})}{\epsilon_{15}}+\frac{(m_4+1)\,(k_8-\epsilon_{15})}{\epsilon_{15}}\right),\end{aligned}$$

Note that $-\beta\,\phi\,R_1\ (x_{70}-k_7+\epsilon_{13})+(m_4+1)\,(k_8-\epsilon_{15})<0$, then there exists a $\lambda_8^*>0$ such that $Q_8(\lambda)>0$ for all $\lambda\in(0,\lambda_8^*)$.

If $t\le t_9$, $\underline{u}_9(t)=0$ and $\underline{u}_i\,(t)\ge 0$, where $i=1,2,3,5...9$, then $Q_9(t)=e_1\,u_4\,(t)+e_2\,u_3\,(t)+e_3\,u_5\,(t)+e_4\,u_8\,(t)\ge 0$.

If $t>t_9$, $\underline{u}_9(t)=k_9-\epsilon_{18}\mathrm{e}^{-\lambda\,t}$, $\underline{u}_i(t)\ge k_i-\epsilon_j\mathrm{e}^{-\lambda\,(t)}$, where $i=1,3,...,9$ and $j=2,6,...,18$,

$$\begin{aligned}Q_9(t)&\ge -\mathrm{e}^{-\lambda\,t}\epsilon_{17}\left(D\,\lambda^2+c\lambda-\frac{F_2}{\epsilon_{17}\mathrm{e}^{-\lambda\,t}}+\frac{e_5\ \left(k_9-\epsilon_{17}\mathrm{e}^{-\lambda\,t}\right)}{\epsilon_{17}\mathrm{e}^{-\lambda\,t}}\right)\\&\ge -\mathrm{e}^{-\lambda\,t}\epsilon_{17}\left(D\,\lambda^2+c\lambda-\frac{F_1}{\epsilon_{17}}+\frac{e_5\ (k_9-\epsilon_{17})}{\epsilon_{17}}\right),\end{aligned}$$

where $F_2=e_1\ \left(k_4-\epsilon_7\mathrm{e}^{-\lambda\,t}\right)+e_2\ \left(k_3-\epsilon_5\mathrm{e}^{-\lambda\,t}\right)+e_3\ \left(k_5-\epsilon_9\mathrm{e}^{-\lambda\,t}\right)+e_4\ \left(k_8-\epsilon_{15}\mathrm{e}^{-\lambda\,t}\right)$.

Note that $-F_1+e_5\ (k_9-\epsilon_{17})<0$, then there exists a $\lambda_{18}^*>0$ such that $Q_9(\lambda)>0$ for all $\lambda\in(0,\lambda_9^*)$.

Hence, by taking $\lambda\in(0,\min\lambda_i^*,i=1,2,\ldots,9)$, we can see that $\underline{\Phi}(t)$ is a lower solutions of system (18). □

Lemma 5. *Let $\mathcal{R}_c>1$, then $\overline{\Phi}(t)=(\overline{u_1}(t),\overline{u_2}(t),\ldots,\overline{u_9}(t))$ defines the upper solution of system (18).*

Proof.

$$
\begin{aligned}
P_1(t) &= D\overline{u''}_1(t) + c\,\overline{u'}_1(t) + f_{c1}\left(\overline{u}_1, \overline{u}_2(t), \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_2(t) &= D\overline{u''}_2(t) + c\,\overline{u'}_2(t) + f_{c2}\left(\underline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_3(t) &= D\overline{u''}_3(t) + c\,\overline{u'}_3(t) + f_{c3}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_4(t) &= D\overline{u''}_4(t) + c\,\overline{u'}_4(t) + f_{c4}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_5(t) &= D\overline{u''}_5(t) + c\,\overline{u'}_5(t) + f_{c5}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_6(t) &= D\overline{u''}_6(t) + c\,\overline{u'}_6(t) + f_{c6}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_7(t) &= D\overline{u''}_7(t) + c\,\overline{u'}_7(t) + f_{c7}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_8(t) &= D\overline{u''}_8(t) + c\,\overline{u'}_8(t) + f_{c8}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right),\\
P_9(t) &= D\overline{u''}_9(t) + c\,\overline{u'}_9(t) + f_{c9}\left(\overline{u}_1(t), \overline{u}_2, \overline{u}_3(t), \ldots, \overline{u}_9(t)\right).
\end{aligned}
\tag{27}
$$

If $t \le t_{10}$, $\overline{u}_1(tt) = k_1\,\mathrm{e}^{\lambda_1 t}$, $\overline{u}_i(t) \le k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 2, 3, ..., 9$,

$$
\begin{aligned}
P_1(t) &\le \mathrm{e}^{\lambda_1 t} k_1 \left(D\,{\lambda_1}^2 + c\lambda_1 - \frac{\pi_1}{k_1 \mathrm{e}^{\lambda_1 t}} + \frac{\left(\beta\,\left(W_2 + c_2\,k_8 \mathrm{e}^{\lambda_8 t}\right) + 1 + b_2\right)\left(x_{10} - k_1 \mathrm{e}^{\lambda_1 t}\right)}{k_1 \mathrm{e}^{\lambda_1 t}} - \frac{c_4\,k_6 \mathrm{e}^{\lambda_6 t}}{k_1 \mathrm{e}^{\lambda_1 t}} \right)\\
&\le \mathrm{e}^{\lambda_1 t} k_1 \left(D\,{\lambda_1}^2 + c\lambda_1 - \frac{\pi_1}{k_1} + \frac{\left(\beta\,\left(M_8\,c_2 + W_1\right) + 1 + b_2\right)\left(x_{10} - k_1\right)}{k_1} - \frac{c_4\,M_6}{k_1} \right) = 0,
\end{aligned}
$$

If $t \ge t_{10}$, $\overline{u}_1(t) \le k_1 - \epsilon_1\,\mathrm{e}^{\lambda t}$, $\overline{u}_1(t) = k_1 + \epsilon_2\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \le k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 2, 3, ..., 9$ and $j = 1, 3, 5, 7, ..., 17$,

$$
\begin{aligned}
P_1(t) &\le \epsilon_2 \mathrm{e}^{-\lambda t} \left(D\lambda^2 + c\lambda - \frac{\pi_1}{\epsilon_2 \mathrm{e}^{-\lambda t}} + \frac{\left(\beta\,R_4 + b_2 + 1\right)\left(x_{10} - k_1 - \epsilon_2 \mathrm{e}^{-\lambda t}\right)}{\epsilon_2 \mathrm{e}^{-\lambda t}} - \frac{c_4\,\left(k_6 - \epsilon_{11} \mathrm{e}^{-\lambda t}\right)}{\epsilon_2 \mathrm{e}^{-\lambda t}} \right)\\
&\le \epsilon_2 \mathrm{e}^{-\lambda t} \left(D\lambda^2 + c\lambda - \frac{\pi_1}{\epsilon_2} + \frac{\left(\beta\,R_2 + b_2 + 1\right)\left(x_{10} - k_1 - \epsilon_2\right)}{\epsilon_2} - \frac{c_4\,\left(k_6 + \epsilon_{12}\right)}{\epsilon_2} \right),
\end{aligned}
$$

where where $R_4 = b_1\,\left(k_4 + \epsilon_8 \mathrm{e}^{-\lambda t}\right) + c_1\,\left(k_3 + \epsilon_6 \mathrm{e}^{-\lambda t}\right) + c_2\,\left(k_8 + \epsilon_{16} \mathrm{e}^{-\lambda t}\right) + c_3\,\left(k_9 + \epsilon_{18} \mathrm{e}^{-\lambda t}\right)$.

Since $-\pi_1 + \left(\beta\,R_2 + b_2 + 1\right)\left(x_{10} - k_1 - \epsilon_2\right) - c_4\,\left(k_6 + \epsilon_{12}\right) < 0$, then there exists a $\lambda^*_{10} > 0$ such that $P_1(t) < 0$ for all $\lambda \in (0, \lambda^*_{10})$.

If $t \le t_{11}$, $\overline{u}_2(t) = k_2\,\mathrm{e}^{\lambda_2 t}$, $\overline{u}_i(t) \le k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 1, 3, 4, ..., 9$,

$$
\begin{aligned}
P_2(t) &\le \mathrm{e}^{\lambda_2 t} k_2 \left(D\,{\lambda_2}^2 + c\lambda_2 + \frac{\beta\,\left(W_2 + c_2\,k_8 \mathrm{e}^{\lambda_8 t}\right)\left(x_{10} - k_1 \mathrm{e}^{\lambda_1 t}\right)}{k_2 \mathrm{e}^{\lambda_2 t}} - c_5 - 1 \right)\\
&\le \mathrm{e}^{\lambda_2 t} k_2 \left(D\,{\lambda_2}^2 + c\lambda_2 + \frac{\beta\,\left(M_8\,c_2 + W_1\right)\left(x_{10} - M_1\right)}{k_2} - c_5 - 1 \right) = 0,
\end{aligned}
$$

If $t > t_{11}$, $\overline{u}_2(t) = k_2 + \epsilon_4\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \le k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 3, ..., 9$ and $j = 1, 5, 7, ..., 17$,

$$
\begin{aligned}
P_2(t) &\le \mathrm{e}^{-\lambda t} \epsilon_4 \left(D\lambda^2 + c\lambda + \frac{\beta\,R_4\,\left(x_{10} - k_1 - \epsilon_2 \mathrm{e}^{-\lambda t}\right)}{\epsilon_4 \mathrm{e}^{-\lambda t}} - \frac{\left(c_5 + 1\right)\left(k_2 + \epsilon_4 \mathrm{e}^{-\lambda t}\right)}{\epsilon_4 \mathrm{e}^{-\lambda t}} \right)\\
&\le \mathrm{e}^{-\lambda t} \epsilon_4 \left(D\lambda^2 + c\lambda + \frac{\beta\,R_2\,\left(x_{10} - k_1 + \epsilon_1\right)}{\epsilon_4} - \frac{\left(c_5 + 1\right)\left(k_2 + \epsilon_4\right)}{\epsilon_4} \right) = P_2(\lambda),
\end{aligned}
$$

Since $P_2\left(0\right) \le \beta\,R_2\,\left(k_{10} - k_1 + \epsilon_1\right) - \left(c_5 + 1\right)\left(k_2 + \epsilon_4\right) < 0$, then there exists a $\lambda^*_{11} > 0$ such that $P_2(t) < 0$ for all $\lambda \in (0, \lambda^*_{11})$.

If $t \le t_{12}$, $\overline{u}_3(t) = k_3\,\mathrm{e}^{\lambda_3 t}$, $\overline{u}_i(t) \le k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 1, 3, 4, ..., 9$,

$$\begin{aligned} P_3(t) &\le \mathrm{e}^{\lambda_3 t} k_3 \left(D\,{\lambda_3}^2 + c\lambda_3 + \frac{c_5\,k_2 \mathrm{e}^{\lambda_2 t}}{k_3 \mathrm{e}^{\lambda_3 t}} - m_1 - 1 \right) \\ &\le \mathrm{e}^{\lambda_3 t} k_3 \left(D\,{\lambda_3}^2 + c\lambda_3 + \frac{c_5\,M_2}{k_3} - m_1 - 1 \right) = 0, \end{aligned}$$

If $t > t_{12}$, $\overline{u}_3(t) = k_3 + \epsilon_6\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \le k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 2, 4, ..., 9$ and $j = 1, 3, 7, 9, ..., 17$,

$$\begin{aligned} P_3(t)\,\epsilon_6 \mathrm{e}^{-\lambda t} &\left(D\lambda^2 + c\lambda + \frac{c_5\,\left(k_2 + \epsilon_4 \mathrm{e}^{-\lambda t}\right)}{\epsilon_6 \mathrm{e}^{-\lambda t}} - \frac{(m_1+1)\left(k_3 + \epsilon_6 \mathrm{e}^{-\lambda t}\right)}{\epsilon_6 \mathrm{e}^{-\lambda t}} \right) \\ &\le \epsilon_6 \mathrm{e}^{-\lambda t} \left(D\lambda^2 + c\lambda + \frac{c_5\,(k_2+\epsilon_4)}{\epsilon_6} - \frac{(m_1+1)\,(k_3+\epsilon_6)}{\epsilon_6} \right), \end{aligned}$$

Note that $c_5\,(k_2+\epsilon_4) - (m_1+1)\,(k_3+\epsilon_6) < 0$, then there exists a $\lambda_{12}^* > 0$ such that $P_3(t) < 0$ for all $\lambda \in (0, \lambda_{12}^*)$.

If $t \le t_{13}$, $\overline{u}_4(t) = k_4\,\mathrm{e}^{\lambda_4 t}$, $\overline{u}_i(t) \le k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 1, 2, 3, 5, ..., 9$,

$$\begin{aligned} P_4(t) &\le \mathrm{e}^{\lambda_4 t} k_4 \left(D\,{\lambda_4}^2 + c\lambda_4 + \frac{k_3 \mathrm{e}^{\lambda_3 t} m_1}{k_4 \mathrm{e}^{\lambda_4 t}} - m_2 - 1 \right) \\ &\le \mathrm{e}^{\lambda_4 t} k_4 \left(D\,{\lambda_4}^2 + c\lambda_4 + \frac{M_3\,m_1}{k_4} - m_2 - 1 \right) = 0, \end{aligned}$$

If $t > t_{13}$, $\overline{u}_4(t) = k_4 + \epsilon_8\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \le k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 2, 3, 5, ..., 9$ and $j = 1, 3, 5, 9, 11, ..., 17$,

$$\begin{aligned} P_4(t) &\le \mathrm{e}^{-\lambda t} \epsilon_8 \left(D\lambda^2 + c\lambda + \frac{\left(k_3 + \epsilon_6 \mathrm{e}^{-\lambda t}\right) m_1}{\epsilon_8 \mathrm{e}^{-\lambda t}} - \frac{(m_2+1)\left(k_4 + \epsilon_8 \mathrm{e}^{-\lambda t}\right)}{\epsilon_8 \mathrm{e}^{-\lambda t}} \right) \\ &\le \mathrm{e}^{-\lambda t} \epsilon_8 \left(D\lambda^2 + c\lambda + \frac{(k_3+\epsilon_6)\,m_1}{\epsilon_8} - \frac{(m_2+1)\,(k_4+\epsilon_8)}{\epsilon_8} \right), \end{aligned}$$

Note that $(k_3+\epsilon_6)\,m_1 - (m_2+1)\,(k_4+\epsilon_8) < 0$ for all $\lambda \in (0, \lambda_4^*)$.

If $t \le t_{14}$, $\overline{\vartheta}(t) = k_5 \mathrm{e}^{-\lambda_5 t}$, $\overline{\varphi}(t) = k_5 \mathrm{e}^{-\lambda_5 t}$, $\overline{\theta}(t) \le k_4 \mathrm{e}^{-\lambda_5 t}$ and $\overline{\varphi}(t) \le k_2 \mathrm{e}^{-\lambda_2 t}$,

$$P_5(t) \le k_5 \mathrm{e}^{\lambda_5 z} \left(D\lambda_5^2 + c\lambda_5 + \frac{m_2\,k_4 \mathrm{e}^{\lambda_4 z}}{k_5 \mathrm{e}^{\lambda_5 z}} - m_3 - 1 \right) \le k_5 \mathrm{e}^{\lambda_5 z} \left(D\lambda_5^2 + c\lambda_5 + \frac{m_2\,M_4}{k_5} - m_3 - 1 \right) = 0,$$

If $t > t_{14}$, $\overline{u}_5(t) = k_5 + \epsilon_{10}\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \le k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 2, 3, 4, 6, ..., 9$ and $j = 1, 3, 5, 7, 11, ..., 17$,

$$\begin{aligned} P_5(t) &\le \mathrm{e}^{-\lambda t} \epsilon_{10} \left(D\lambda^2 + c\lambda + \frac{m_2\,\left(k_4 + \epsilon_8 \mathrm{e}^{-\lambda t}\right)}{\epsilon_{10} \mathrm{e}^{-\lambda t}} - \frac{(m_3+1)\left(k_5 + \epsilon_{10} \mathrm{e}^{-\lambda t}\right)}{\epsilon_{10} \mathrm{e}^{-\lambda t}} \right) \\ &\le \mathrm{e}^{-\lambda t} \epsilon_{10} \left(D\lambda^2 + c\lambda + \frac{m_2\,(k_4+\epsilon_8)}{\epsilon_{10}} - \frac{(m_3+1)\,(k_5+\epsilon_{10})}{\epsilon_{10}} \right), \end{aligned}$$

Note that $m_2\,(k_4+\epsilon_8) - (m_3+1)\,(k_5+\epsilon_{10}) < 0$, then there exists a $\lambda_{14}^* > 0$ such that $P_5(t) < 0$ for all $\lambda \in (0, \lambda_{14}^*)$.

If $t \le t_{15}$, $\overline{u}_6(t) = k_6\,\mathrm{e}^{\lambda_6 t}$, $\overline{u}_i(t) \le k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 1, 2, ...5, 7, ..., 9$,

$$\begin{aligned} P_6(t) &\le \mathrm{e}^{\lambda_6 t} k_6 \left(D\,{\lambda_6}^2 + c\lambda_6 + \frac{m_3\,k_5 \mathrm{e}^{\lambda_5 t}}{k_6 \mathrm{e}^{\lambda_6 t}} + \frac{m_4\,k_8 \mathrm{e}^{\lambda_8 t}}{k_6 \mathrm{e}^{\lambda_6 t}} - c_4 - 1 \right) \\ &\le \mathrm{e}^{\lambda_6 t} k_6 \left(D\,{\lambda_6}^2 + c\lambda_6 + \frac{m_3\,M_5}{k_6} + \frac{m_4\,M_8}{k_6} - c_4 - 1 \right) = 0, \end{aligned}$$

If $t > t_{15}$, $\overline{u}_6(t) = k_6 + \epsilon_{12}\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \leq k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 2, ..., 5, 7, ..., 9$ and $j = 1, 3, 5, 7, 9, 13, ..., 17$,

$$\begin{aligned} P_6\left(t\right) &\leq \epsilon_{12}\mathrm{e}^{-\lambda t}\left(D\lambda^2 + c\lambda + \frac{m_3\,\left(k_5 + \epsilon_{10}\mathrm{e}^{-\lambda t}\right)}{\epsilon_{12}\mathrm{e}^{-\lambda t}} - \frac{\left(c_4+1\right)\left(k_6 + \epsilon_{12}\mathrm{e}^{-\lambda t}\right)}{\epsilon_{12}\mathrm{e}^{-\lambda t}}\right. \\ &\quad \left. + \frac{m_4\,\left(k_8 + \epsilon_{16}\mathrm{e}^{-\lambda t}\right)}{\epsilon_{12}\mathrm{e}^{-\lambda t}}\right) \\ &\leq \epsilon_{12}\mathrm{e}^{-\lambda t}\left(D\lambda^2 + c\lambda + \frac{m_3\,\left(k_5+\epsilon_{10}\right)}{\epsilon_{12}} - \frac{\left(c_4+1\right)\left(k_6+\epsilon_{12}\right)}{\epsilon_{12}} + \frac{m_4\,\left(k_8+\epsilon_{16}\right)}{\epsilon_{12}}\right), \end{aligned}$$

Note that $m_3\,\left(k_5+\epsilon_{10}\right) - \left(c_4+1\right)\left(k_6+\epsilon_{12}\right) + m_4\,\left(k_8+\epsilon_{16}\right) < 0$, then there exists a $\lambda_{15}^* > 0$ such that $P_6(t) < 0$ for all $\lambda \in (0, \lambda_{15}^*)$.

If $t \leq t_{16}$, $\overline{u}_7(t) = k_7\,\mathrm{e}^{\lambda_7 t}$, $\overline{u}_i(t) \leq k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 1, 2, ...6, 8, 9$,

$$\begin{aligned} P_7(t) &\leq \mathrm{e}^{\lambda_7 t}k_7\left(D\,{\lambda_7}^2 + c\lambda_7 - \frac{\pi_2}{k_7\mathrm{e}^{\lambda_7 t}} + \frac{\left(\beta\,\left(W_2 + c_2\,k_8\mathrm{e}^{\lambda_8 t}\right)\phi + 1\right)\left(x_{70} - k_7\mathrm{e}^{\lambda_7 t}\right)}{k_7\mathrm{e}^{\lambda_7 t}}\right. \\ &\quad \left. - \frac{b_2\,\left(x_{10} - k_1\mathrm{e}^{\lambda_1 t}\right)}{k_7\mathrm{e}^{\lambda_7 t}}\right) \\ &\leq \mathrm{e}^{\lambda_7 t}k_7\left(D\,{\lambda_7}^2 + c\lambda_7 - \frac{\pi_2}{k_7} + \frac{\left(\beta\,\phi\,\left(M_8\,c_2 + W_1\right) + 1\right)\left(x_{70} - k_7\right)}{k_7}\right. \\ &\quad \left. - \frac{b_2\,\left(x_{10} - M_1\right)}{k_7}\right) = 0, \end{aligned}$$

where $W_2 = b_1\,k_4\mathrm{e}^{\lambda_4 t} + c_1\,k_3\mathrm{e}^{\lambda_3 t} + c_3\,k_9\mathrm{e}^{\lambda_9 t}$.

If $t > t_{16}$, $\overline{u}_7(t) = k_7 + \epsilon_{14}\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \leq k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 2, ..., 6, 8, 9$ and $j = 1, 3, 5...11, 15, 17$,

$$\begin{aligned} P_7\left(t\right) &\leq \mathrm{e}^{-\lambda t}\epsilon_{14}\left(D\lambda^2 + c\lambda - \frac{\pi_2}{\epsilon_{14}\mathrm{e}^{-\lambda t}} + \frac{\left(\beta\,R_4\,\phi + 1\right)\left(x_{70} - k_7 - \epsilon_{14}\mathrm{e}^{-\lambda t}\right)}{\epsilon_{14}\mathrm{e}^{-\lambda t}}\right. \\ &\quad \left. - \frac{b_2\,\left(x_{10} - k_1 - \epsilon_2\mathrm{e}^{-\lambda t}\right)}{\epsilon_{14}\mathrm{e}^{-\lambda t}}\right) \\ &\leq \mathrm{e}^{-\lambda t}\epsilon_{14}\left(D\lambda^2 + c\lambda - \frac{\pi_2}{\epsilon_{14}} + \frac{\left(R_3\,\beta\,\phi + 1\right)\left(x_{70} - k_7 - \epsilon_{14}\right)}{\epsilon_{14}} - \frac{b_2\,\left(x_{10} - k_1 - \epsilon_2\right)}{\epsilon_{14}}\right), \end{aligned}$$

Note that $-\pi_2 + \left(R_2\,\beta\,\phi + 1\right)\left(x_{70} - k_7 - \epsilon_{14}\right) - b_2\,\left(x_{10} - k_1 - \epsilon_2\right) < 0$, then there exists a $\lambda_{16}^* > 0$ such that $P_7(t) < 0$ for all $\lambda \in (0, \lambda_{16}^*)$.

If $t \leq t_{17}$, $\overline{u}_8(t) = k_8\,\mathrm{e}^{\lambda_8 t}$, $\overline{u}_i(t) \leq k_i\,\mathrm{e}^{\lambda_i t}$ where $i = 1, 2, ..., 7, 9$,

$$\begin{aligned} P_8(t) &\leq \mathrm{e}^{\lambda_8 t}k_8\left(D\,{\lambda_8}^2 + c\lambda_8 + \frac{\phi\,\beta\,\left(W_2 + c_2\,k_8\mathrm{e}^{\lambda_8 t}\right)\left(x_{70} - k_7\mathrm{e}^{\lambda_7 t}\right)}{k_8\mathrm{e}^{\lambda_8 t}} - m_4 - 1\right) \\ &\leq \mathrm{e}^{\lambda_8 t}k_8\left(D\,{\lambda_8}^2 + c\lambda_8 + \frac{\phi\,\beta\,\left(c_2\,k_8 + W_1\right)\left(x_{70} - M_7\right)}{k_8} - m_4 - 1\right) = 0, \end{aligned}$$

If $t > t_{17}$, $\overline{u}_8(t) = k_8 + \epsilon_{16}\,\mathrm{e}^{\lambda t}$, $\overline{u}_i(t) \leq k_i + \epsilon_j\,\mathrm{e}^{\lambda t}$, where $i = 1, 2, .., 7, 9$ and $j = 1, 3, ..., 13, 17$,

$$\begin{aligned} P_8\left(t\right) &\leq \epsilon_{16}\mathrm{e}^{-\lambda t}\left(D\lambda^2 + c\lambda + \frac{\phi\,\beta\,R_4\,\left(x_{70} - k_7 - \epsilon_{14}\mathrm{e}^{-\lambda t}\right)}{\epsilon_{16}\mathrm{e}^{-\lambda t}} - \frac{\left(m_4+1\right)\left(k_8 + \epsilon_{16}\mathrm{e}^{-\lambda t}\right)}{\epsilon_{16}\mathrm{e}^{-\lambda t}}\right) \\ &\leq \epsilon_{16}\mathrm{e}^{-\lambda t}\left(D\,\lambda^2 + c\lambda + \frac{\phi\,\beta\,R_2\,\left(x_{70} - k_7 - \epsilon_{14}\right)}{\epsilon_{16}} - \frac{\left(m_4+1\right)\left(k_8+\epsilon_{16}\right)}{\epsilon_{16}}\right), \end{aligned}$$

Note that $\phi\,\beta\,R_2\,(x_{70}-k_7-\epsilon_{14})-(m_4+1)\,(k_8+\epsilon_{16})<0$, then there exists a $\lambda_{17}^*>0$ such that $P_8(t)<0$ for all $\lambda\in(0,\lambda_{17}^*)$.

If $t\le t_{18}$, $\overline{u}_9(t)=k_9\,\mathrm{e}^{\lambda_9\,t}$, $\overline{u}_i(t)\le k_i\,\mathrm{e}^{\lambda_i\,t}$ where $i=1,2,...,8$,

$$\begin{aligned}P_9(t)&\le \mathrm{e}^{\lambda_9 z}k_9\left(D\,{\lambda_9}^2+c\lambda_9+\frac{F_4}{k_9\mathrm{e}^{\lambda_9 z}}-e_5\right)\\&\le \mathrm{e}^{\lambda_9 z}k_9\left(D\,{\lambda_9}^2+c\lambda_9+\frac{M_3\,e_2+M_4\,e_1+M_5\,e_3+M_8\,e_4}{k_9}-e_5\right)=0,\end{aligned}$$

where $F_4=e_1\,k_4\mathrm{e}^{\lambda_4 t}+e_2\,k_3\mathrm{e}^{\lambda_3 t}+e_3\,k_5\mathrm{e}^{\lambda_5 t}+e_4\,k_8\mathrm{e}^{\lambda_8 t}$,

If $t>t_{18}$, $\overline{u}_9(t)=k_9+\epsilon_{18}\,\mathrm{e}^{\lambda\,t}$, $\overline{u}_i(t)\le k_i+\epsilon_j\,\mathrm{e}^{\lambda\,t}$, where $i=1,2,...,8$ and $j=1,3,...,15$,

$$\begin{aligned}P_9\,(t)&\le \epsilon_{18}\mathrm{e}^{-\lambda\,t}\left(D\,\lambda^2+c\lambda+\frac{F_5}{\epsilon_{18}\mathrm{e}^{-\lambda\,t}}-\frac{e_5\,\left(k_9+\epsilon_{18}\mathrm{e}^{-\lambda\,t}\right)}{\epsilon_{18}\mathrm{e}^{-\lambda\,t}}\right)\\&\le \epsilon_{18}\mathrm{e}^{-\lambda\,t}\left(D\,\lambda^2+c\lambda+\frac{F_2}{\epsilon_{18}}-\frac{e_5\,(k_9+\epsilon_{18})}{\epsilon_{18}}\right),\end{aligned}$$

where $F_5=e_1\,\left(k_4+\epsilon_8\mathrm{e}^{-\lambda\,t}\right)+e_2\,\left(k_3+\epsilon_6\mathrm{e}^{-\lambda\,t}\right)+e_3\,\left(k_5+\epsilon_{10}\mathrm{e}^{-\lambda\,t}\right)+e_4\,\left(k_8+\epsilon_{16}\mathrm{e}^{-\lambda\,t}\right)$,

Note that $F_2-e_5\,(k_9+\epsilon_{18})<0$, then there exists a $\lambda_{19}^*>0$ such that $P_9(t)<0$ for all $\lambda\in(0,\lambda_{19}^*)$.

Hence, for all $\lambda\in(0,\min\{\lambda_i,i=1,2,\ldots,9\})$, $P_i(t)<0(i=1,2,\ldots,9)$. This completes the proof. □

By combining the Schauders fixed point theorem and lemmas (1- 5), we know that there exists a fixed point $(u_1^*(t),u_2^*(t),\ldots,u_9^*(t))$ for F in $\Gamma(\overline{u}_1(t),\overline{u}_2(t),\overline{u}_3(t),\ldots,\overline{u}_9(t))$, which gives the solutions of system (18). Moreover, from $C2$ (24), we can find that

$$\lim_{t\to-\infty}(u_1^*(t),u_2^*(t),\ldots,u_9^*(t))=(0,0,0,0,0,0,0,0,0),\quad \lim_{t\to+\infty}(u_1^*(t),u_2^*(t),\ldots,u_9^*(t))=(k_1,k_2,\ldots,k_9)$$

This shows that the fixed point satisfies the asymptotic boundary conditions (7). Consequently, there exists a traveling wave solution for system (18) connecting the steady state $(0,0,0,0,0,0,0,0,0)$ and $(k_1,k_2,k_3,\ldots,k_9)$, hence, we have the following result:

Theorem 5. *Assume that $d_1=d_2=d_3=d_4=d_5=d_6=d_8=d_9=D$, $\mathcal{R}_c>1$ then for every $c>c^*$, system (1) has a traveling wave solution with speed c connecting the disease-free steady state $\hat{E}_0$ and the endemic steady state E^*.*

4 Parameter estimation and numerical simulation

In this section, we present the parameter values for the model (1) from the relevant literature to carry out numerical simulations that enhance further understanding of the model predictions. The simulations show the effects of vaccination, diffusion and shedding of foot and mouth disease virus on the dynamics of the foot and mouth disease.

4.1 Parameter estimation

The amplification rates η_i, where $i=1,2,3$ are estimated to be between 0 and 1 satisfying $\eta_3<\eta_2<\eta_1<1$. The vaccination rates ρ and ρ_1 are also estimated to be between 0 and 1. The average number of FMD viruses shedded into the atmospheric environment are estimated as $N_{is}=800$, $N_{ic}=3000$, $N_q=3000$, and $N_{vca}=500$. We use $d_1=d_6=d_7$, these are diffusion constants associated with healthy animals namely, susceptible, recovered and vaccinated animals. The movement of these animals is expected to more than that of infected animals. We assume that the diffusion of clinically infected and the vaccinated carrier animals are equal ($d_2=d_8$), since the movement of uninfected

animals are active and cover a wide area as compared to subclinically and clinically infected animals. We also use $d_3 > d_4$ since the subclinically infected animals are active and cover a wide area compared to clinically infected animals class. The movement of quarantining clinical animals is restricted and the diffusion rate of quarantining the clinical animals estimated as $d_5 = 0$. We use Figure 2 as baseline simulations of FMD dynamics and compare it to the rest of the Figures (see Figure 3, 4,..., 14) as we vary the parameters. All parameter values used in the numerical simulations are given in Table 1 with their sources. Some parameter values are taken as they appear in literature while others are determined based on estimating the given parameters.

Table 1: Dimensional parameter values for the model

symbol	Units	Value	source	symbol	Units	Value	source
Π	Day^{-1}	0.13	Calculated	α_4	Day^{-1}	0.3	[6, 34, 38]
β	Day^{-1}	0.3	[8, 34, 35]	ρ	Day^{-1}	[0, 1]	[34]
μ	$year^{-1}$	0.05	[36]	ρ_1	Day^{-1}	[0, 1]	[34]
η_1	Day^{-1}	[0, 1]	estimated	ϕ	Day^{-1}	0.2	[13]
η_2	Day^{-1}	[0, 1]	estimated	$\tau_{is}, \tau_{ic}, \tau_q, \tau_{vc}$	Day^{-1}	[0, 1]	estimated
η_3	Day^{-1}	[0, 1]	estimated	$d_1, d_2, d_3, d_4, d_5,$			
ϵ	Day^{-1}	0.6	[16, 36, 37]	d_6, d_7, d_8, d_9	Day^{-1}	[0, 1]	[16, 34, 39]
$\alpha 1$	Day^{-1}	0.3	[16, 36, 37]	N_{is}	Day^{-1}	800	estimated
α_2	Day^{-1}	0.3	[16, 36, 37]	N_{ic}	Day^{-1}	3000	estimated
α_3	Day^{-1}	0.4	[36, 37]	N_q	Day^{-1}	3000	estimated
ω	Day^{-1}	[0.01, 0.2]	[13, 21, 28, 38]	N_{vc}	Day^{-1}	500	estimated

4.2 Numerical simulations

In this section, we present numerical simulations to enrich our understanding of the model (1) and to explore the effects of spatial diffusion, vaccination and shedding off of foot and mouth disease virus into the environment. In particular, we investigate the effects of control parameters for vaccination (ρ, ρ_1), quarantining clinical infected animals (α_2), the viral shedding rates (τ_{is}, τ_{ic}, τ_q, τ_{vca}) and diffusion rates ($d_1, d_2, d_3, d_4, d_6, d_7, d_8, d_9$).

Even though simulations of all the state variables were performed, the effects of each of the aforementioned parameters are alternatively interesting on how they affect infected animal classes in the dynamics of infection. We therefore deliberately show only the simulation outputs for the infectious animal classes I_s, I_c, Q, V_{ca} and F_v as they are the drivers of infection.

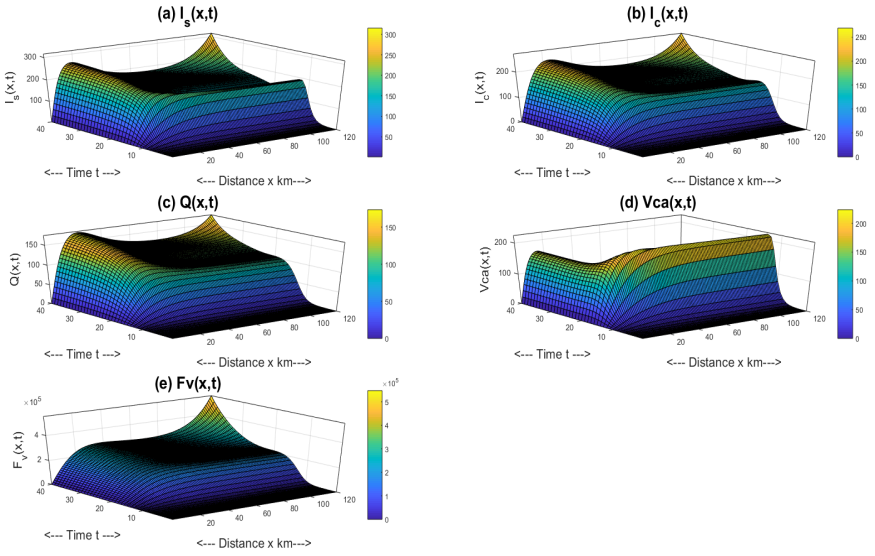

Figure 2: The graph on the dynamics of the disease with parameters: $\Pi = 0.13,\ \beta = 0.3,\ \eta_1 = 0.4,\ \eta_2 = 0.3,\ \eta_3 = 0.08, \mu = 0.0013,\ \epsilon = 0.6,\ \alpha 1 = 0.3,\ \alpha_2 = 0.3,\ \alpha_3 = 0.4,\ \omega = 0.1,\ \alpha_4 = 0.3,\ \rho = 0.2,\ \rho_1 = 0.0001,\ \phi = 0.2,\ \tau_{is} = \tau_{is} = \tau_{ic} = \tau_q = \tau_{vc} = 0.1,\ d_5 = 0,\ d_1 = d_6 = d_7 = 0.8,\ d_1 = d_6 = d_7 = 0.5,\ d_2 = d_8 = 0.3,\ d_9 = 0.6,\ d_3 = 0.2,\ d_4 = 0.1,\ N_{is} = 800,\ N_{ic} = N_q = 3000,\ N_{vc} = 500.$

4.2.1 Effects of vaccination parameters (ρ and ρ_1)

In this section, we investigate the effects of the rate of vaccination parameters (ρ and ρ_1). The effects of the rate of vaccination new born animals on FMD are shown in Figure 3 by varying $\rho = 0.2$ to $\rho = 0.7$ keeping the other parameters as in Figure 2. Figure 3 shows increasing the rate of new born animals vaccination (ρ) is associated with the increase in vaccinated carrier animals and decrease on the other infected animals, as well as a decrease in the quantity of the FMD virus in the environment. This suggests that more animals are protected and move to the protected routs but the increase in vaccinated carrier animals may pass as the threat to the control efforts applied. The effects of the rate of vaccination of susceptible animals (ρ_1) on the dynamics of the disease are shown in Figure 4 where ρ_1 is vaccine from 0.0001 to 0.0008 whilst keeping the other parameters as in Figure 2. Increasing the rate of vaccination of susceptible animals increases the vaccinated carrier animal classes and decrease the other infected animal classes and the quantity of the FMD virus in the environment.

4.2.2 Effects of quarantining the clinically infected animals parameter (α_2)

In this section, we investigate the effects of the rate of quarantining clinical infected animal (α_2) by varying (α_2) from 0.2 to 0.7 whilst keeping the other parameters as in Figure 2. An increase in the rate of quarantining clinically infected animals (α_2) is associated with the decrease in all infected animals, as well as a decrease in the quantity of the FMD virus in the environment.

4.2.3 Effects of FMDV shedding rates into the environment (τ_{is}, τ_{ic}, τ_q, τ_{vc})

In this section, we investigate the effects of parameters varying the parameter τ_j, where $j = is, ic, q, vc$ from 0.1 to 0.5 in Figures 6 to 9 while keeping the rest of the parameters as in Figure 2. Figure 6 shows the effects of the rate of FMDV shedding from subclinically infected animals (τ_{is}). Increasing the rate of FMDV shedding from subclinically infected animals increases all the infected animal classes as well

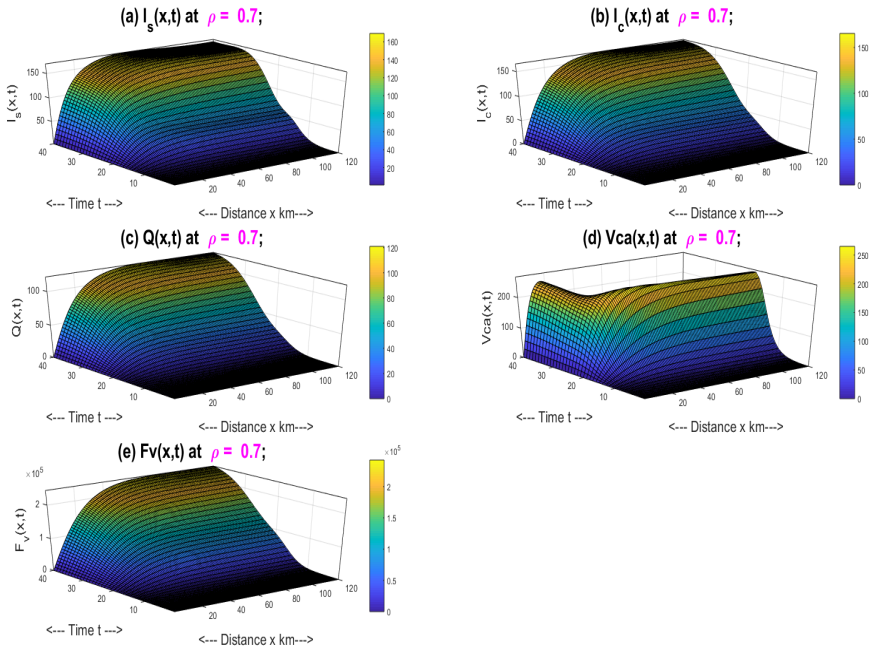

Figure 3: The effects of the rate of vaccination on the dynamics of the disease by increasing the rate of vaccination from $\rho = 0.2$ to $\rho = 0.7$ and keeping the other parameters as in Figure 2.

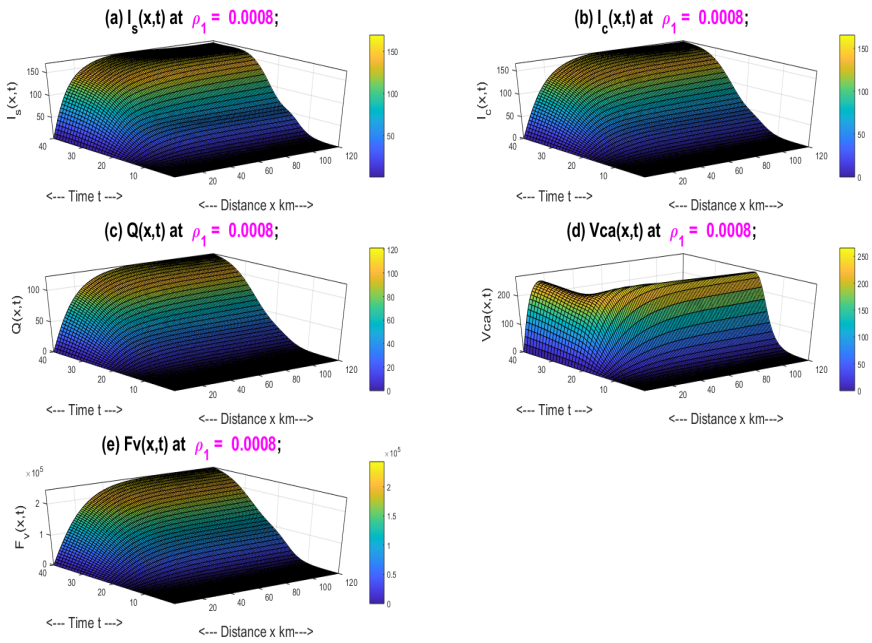

Figure 4: The effects of the rate of vaccination of susceptible animals on the dynamics of the disease by increasing the rate of vaccination from $\rho_1 = 0.0001$ to $\rho_2 = 0.0008$ and keeping the other parameters as in Figure 2.

as FMDV in the environment. Figures 7 shows the effects of the rate of FMDV shedding from the clinically infected animals (τ_{ic}). Increasing the rate of FMDV shedding from clinically infected animals increase all the infected animal classes and the FMD in the environment. Figures 8 shows the effects of the rate of FMDV shedding from the quarantined clinically infected animals (τ_q). Increasing the rate of FMDV shedding from quarantined clinically infected animals increase all the infected animal classes and the FMD in the environment. Figures 9 shows the effects of the rate of FMDV shedding from the vaccinated carrier animals (τ_{vc}). Increasing the rate of FMDV shedding from vaccinated carrier animals increase all the infected animal classes and the FMD in the environment.

Overall, our simulations suggest that increasing any shedding rate increases the FMD burden. In particular, using the parameters estimates in the current simulations, more damage seen to come from shedding from, followed in that order. Thus, strategies should target environmental shedding rates of FMDV.

4.2.4 Effects of diffusion parameters (d_1, d_2, d_3, d_4, d_6, d_7, d_8, d_9)

In this section, we investigate the effects of diffusion parameters (d_1, d_2, d_3, d_4, d_6, d_7, d_8, d_9). The parameters $d_1 = d_6 = d_7$ shall be varied from 0.5 to 0.8, $d_2 = d_8$ varied from 0.3 to 0.4, d_3 from 0.3 to 0.4, d_3 from 0.3 to 0.4 and d_9 from 0.6 to 0.9 in Figure 10 to 14. We shall vary these parameters and compare the outcomes from the ones on Figure 2. Figure 10 shows the impact of the rate of diffusion on the dynamics of the disease by increasing the rates of diffusion $d_1 = d_6 = d_7$ by varying from 0.5 to 0.8 Increasing the rates of diffusion $d_1 = d_6 = d_7$ increase all the infected animal classes as well as FMDV in the environment. Figure 11 shows the impact of the rate of diffusion parameter $d_2 = d_8$ on the dynamics of the disease by varying $d_2 = d_8$ from 0.3 to 0.4. Increasing the rates of diffusion $d_2 = d_8$ decreases all the infected animal classes as well as FMDV in the environment. Figure 12 shows the impact of the rate of diffusion parameter d_3 on the dynamics of the disease by varying d_3 from 0.2 to 0.4. Increasing the rates of diffusion d_3 decrease all the infected animal classes as well as FMDV in the environment. Figure 13 shows the impact of the rate of diffusion parameter d_4 on the dynamics of the disease by varying d_4 from 0.1 to 0.2. Increasing the rates of diffusion d_4 decrease all the infected animal classes as well as FMDV in the environment. Figure 14 shows the impact of the rate of diffusion parameter d_9 on the dynamics of the disease by varying d_9 from 0.6 to 0.9. Increasing the rates of diffusion d_9 decrease all the infected animal classes as well as FMDV in the environment. Our simulations suggest that when the movement of health animals is low and increasing the movement of unhealthy animals may increase the burden of foot and mouth disease and may not eradicate the infection either.

5 Discussion and results

The reaction-diffusion equations model for foot and mouth disease of cattle was presented in this paper to capture the effects of the rate of vaccination, spatial diffusion, quarantining of clinically infected animals and shedding of foot and mouth disease virus into the environment. Mathematical analysis and numerical simulations were carried out to reveal the effects of the above-mentioned control strategies on the foot and mouth disease burden.

Our mathematical analysis revealed the existence of the disease-free and endemic equilibrium points. When the control reproduction number, $\mathcal{R}_c$ is less than unity, the system has a unique disease-free steady state which is locally asymptotically stable. This suggests that foot and mouth disease burden can be kept in check when control reproduction number is below unity. However, when the control can not reduce $\mathcal{R}_c$ below unity, there is a possibility that the foot and mouth disease can spread to endemic levels. Evidence of controls such as the rate of vaccination and the rate of quarantining of clinically infected animals parameters on foot and mouth disease transmission in cattle is available

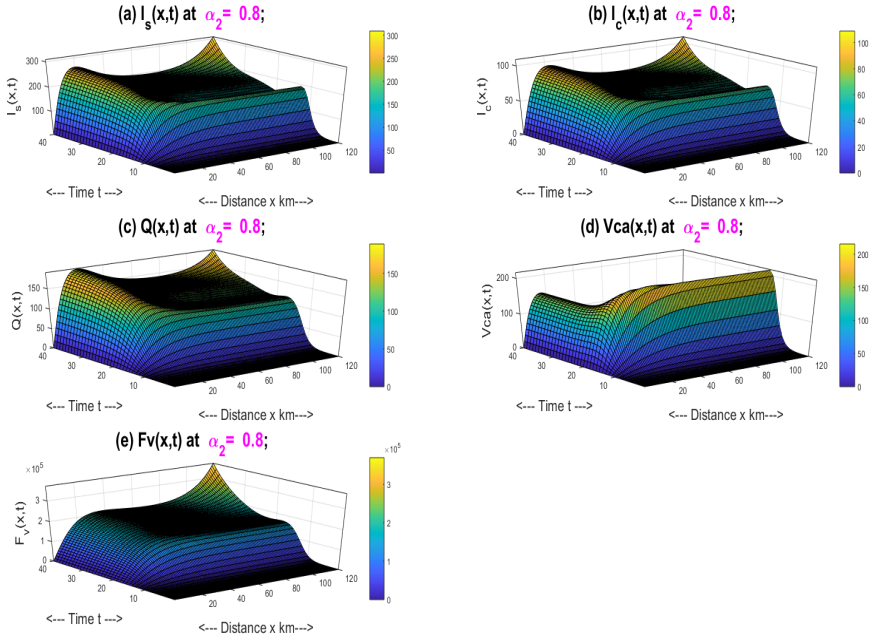

Figure 5: The effects of the rate of quarantining clinical infected class on the dynamics of the disease by increasing the rate of quarantining from $\alpha_2 = 0.3$ to $\alpha_2 = 0.8$ and keeping the other parameters as in Figure 2.

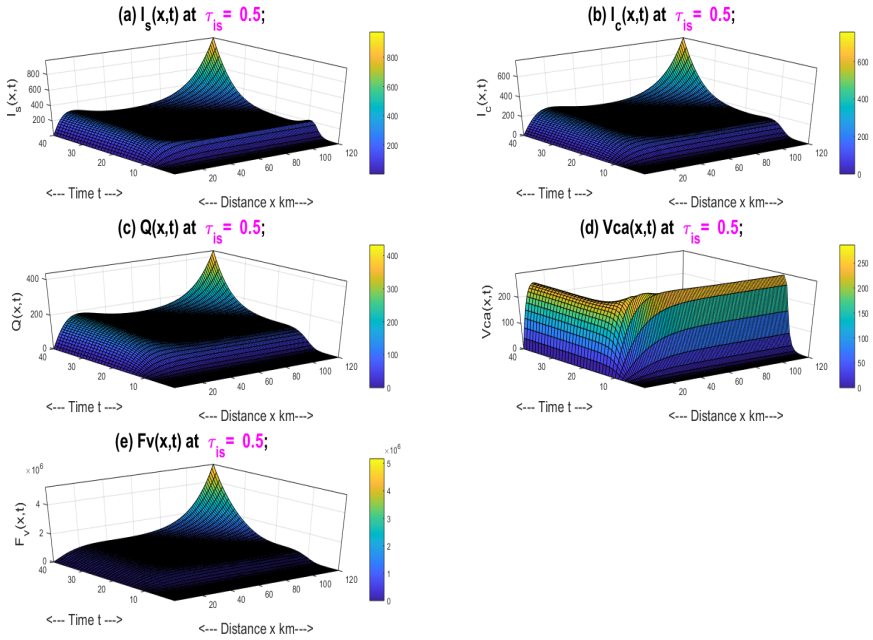

Figure 6: The effects of the rate of FMDV on the dynamics of the disease by increasing the rate of subclinically infected animals shed off FMDV from$\tau_{is} = 0.1$ to $\tau_{is} = 0.5$ and keeping the other parameters as in Figure 2.

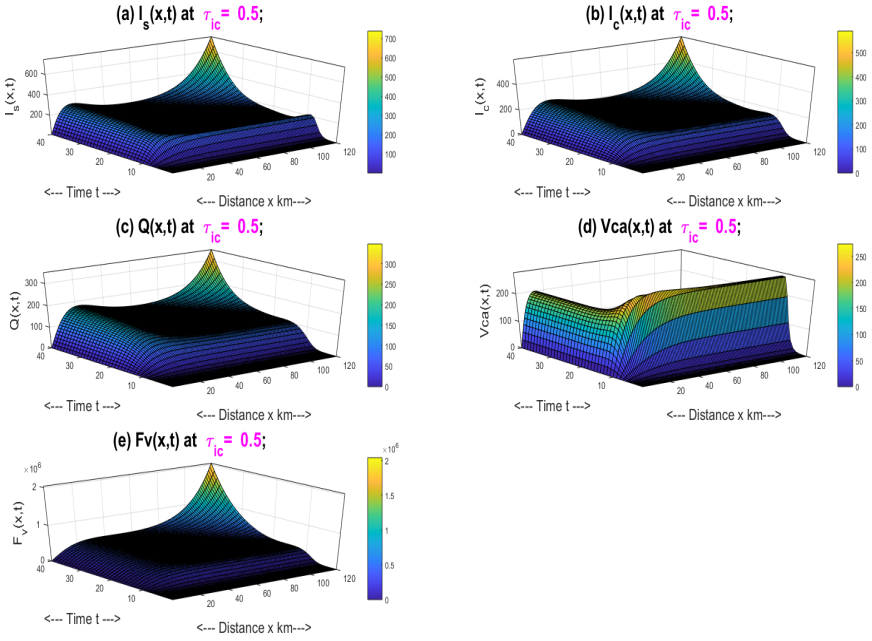

Figure 7: The effects of the rate of FMDV on the dynamics of the disease by increasing the rate of clinically infected animals shed off FMDV from $\tau_{ic} = 0.1$ to $\tau_{ic} = 0.5$ and keeping the other parameters as in Figure 2.

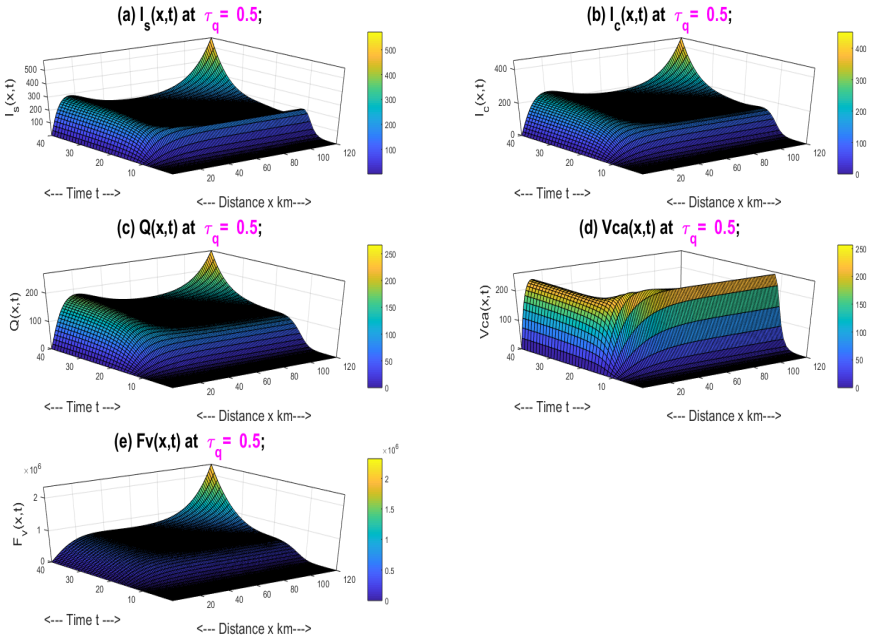

Figure 8: The effects of the rate of FMDV on the dynamics of the disease by increasing the rate of quarantining animals shed off FMDV from $\tau_q = 0.1$ to $\tau_q = 0.5$ and keeping the other parameters as in Figure 2.

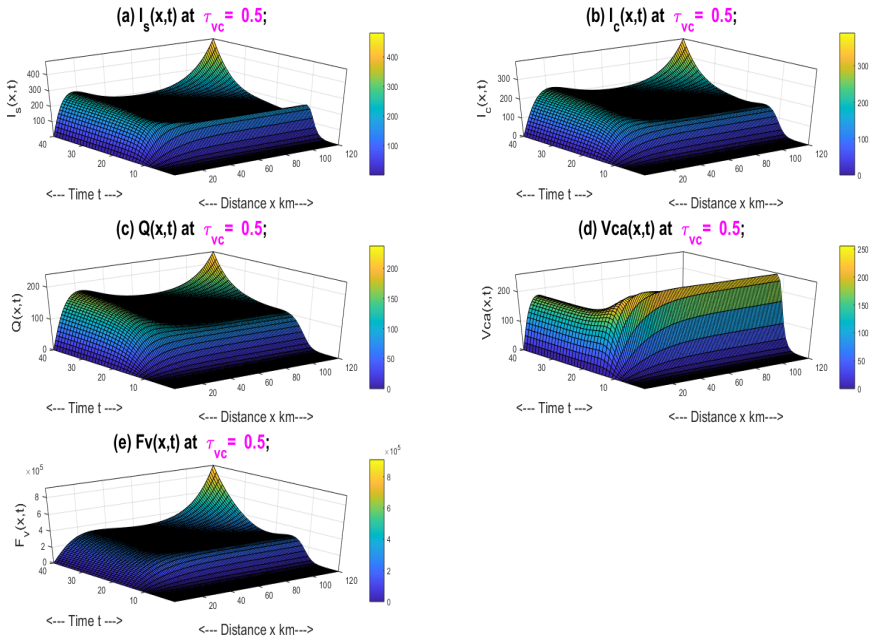

Figure 9: The effects of the rate of FMDV on the dynamics of the disease by increasing the rate of vaccinated carrier animals shed off FMDV from $\tau_{vc} = 0.1$ to $\tau_{vc} = 0.5$ and keeping the other parameters as in Figure 2.

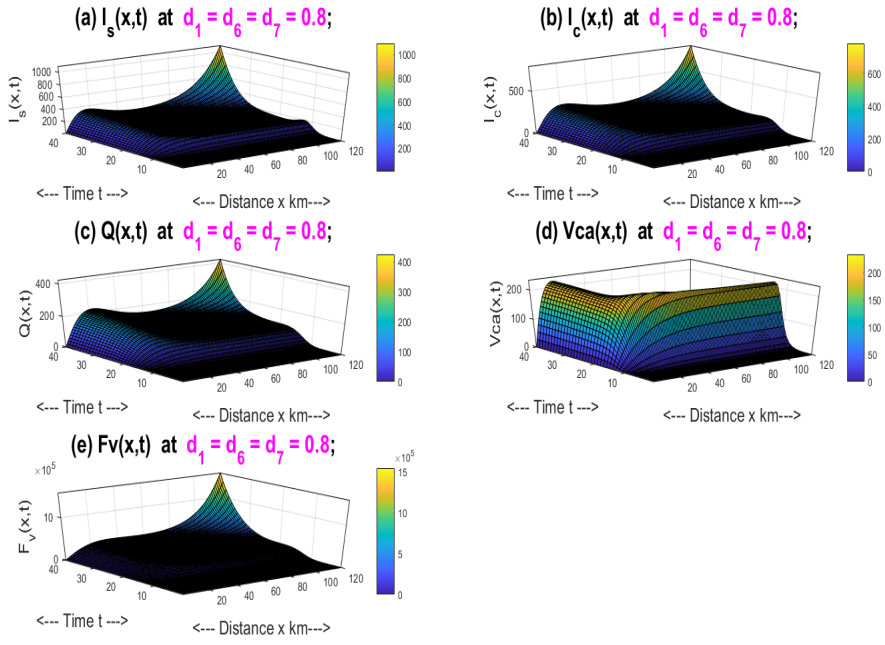

Figure 10: The effects of the rate of diffusion on the dynamics of the disease by increasing the rate of diffusion from $d_1 = d_6 = d_7 = 0.5$ to $d_1 = d_6 = d_7 = 0.8$ and keeping the other parameters as in Figure 2.

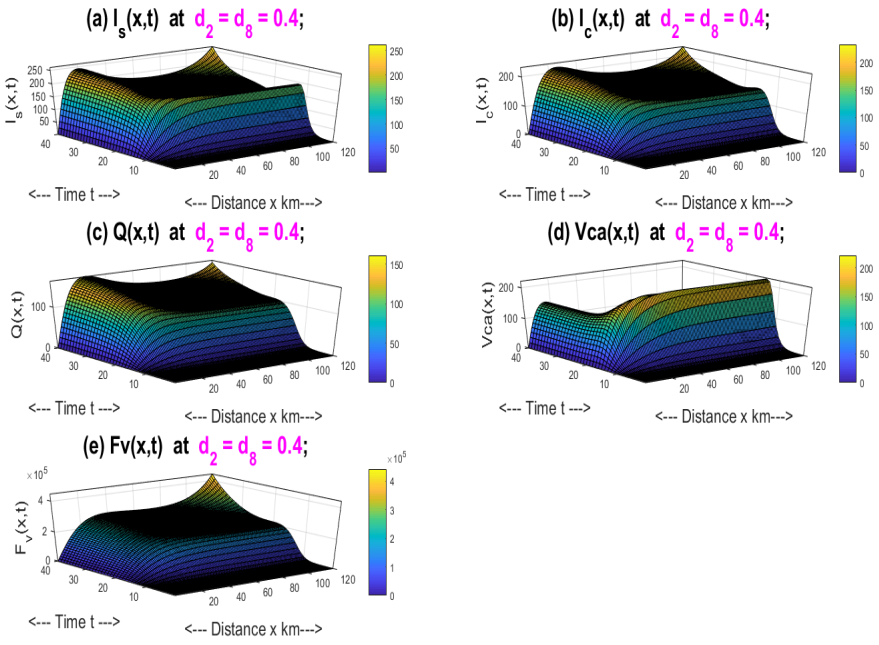

Figure 11: The effects of the rate of diffusion on the dynamics of the disease by increasing the rate of diffusion from $d_2 = d_8 = 0.4$ to $d_2 = d_8 = 0.5$ and keeping the other parameters as in Figure 2.

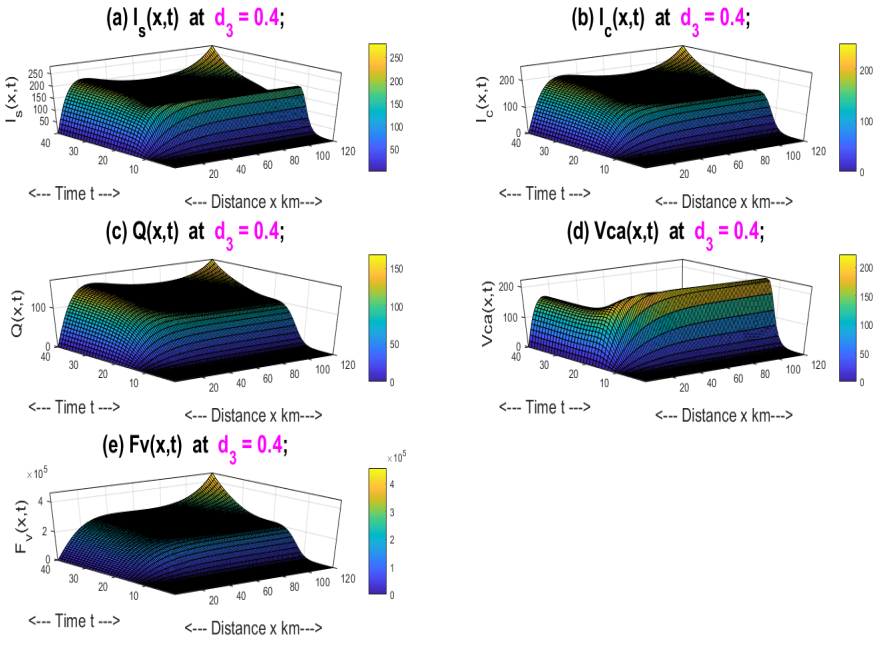

Figure 12: The effects of the rate of diffusion on the dynamics of the disease by increasing the rate of diffusion from $d_3 = 0.2$ to $d_3 = 0.4$ and keeping the other parameters as in Figure 2.

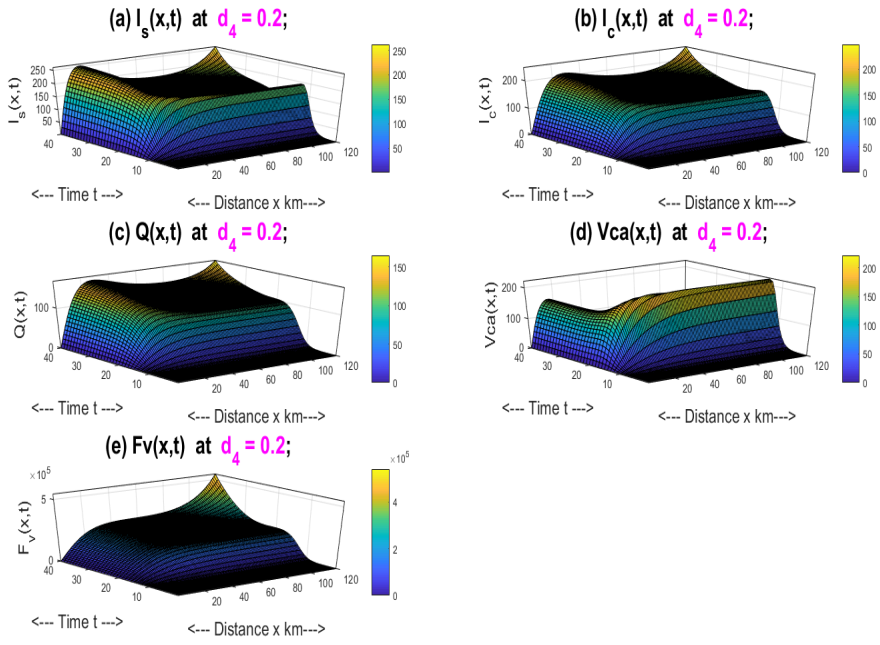

Figure 13: The effects of the rate of diffusion on the dynamics of the disease by increasing the rate of diffusion from $d_4 = 0.1$ to $d_4 = 0.2$ and keeping the other parameters as in Figure 2.

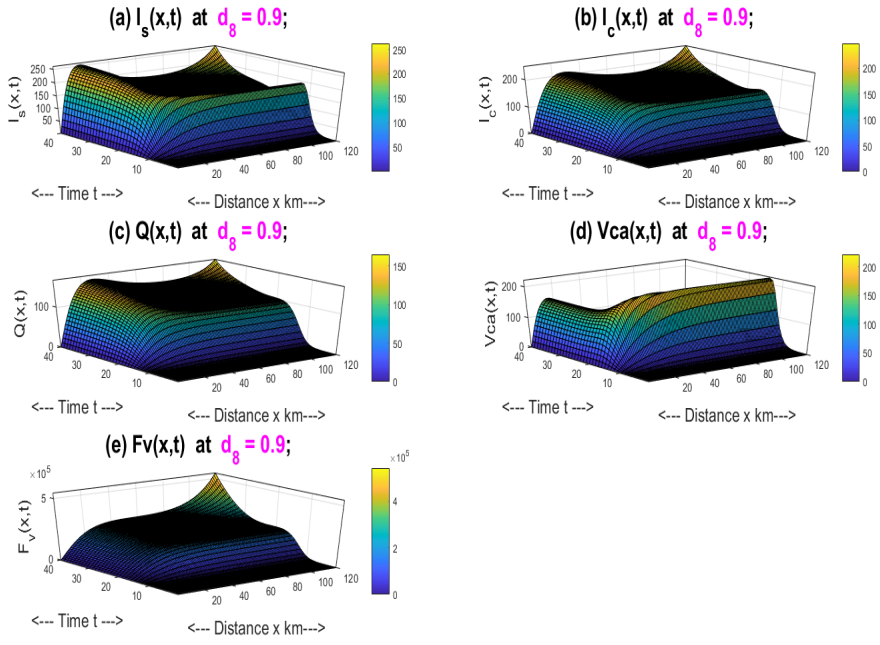

Figure 14: The effects of the rate of diffusion on the dynamics of the disease by increasing the rate of diffusion from $d_9 = 0.6$ to $d_9 = 0.9$ and keeping the other parameters as in Figure 2.

where the burden of foot and mouth disease continues to increase [6, 9, 15]. Such scenarios may be sources of development of resistance especially on controls associated with drugs and vaccines. The mathematical analysis also showed the existence of a travelling wave solution connecting the disease-free steady state and the endemic steady state by using the technique of upper and lower solutions and Schauders fixed point theorem. This shows that the spatial dynamics of the infection contributes towards the progression of infection in cattle [22, 29, 40]. Models that do not capture the spatial dynamics of foot and mouth disease are therefore limited in terms of explaining the factors that drive the foot and mouth disease infection [6, 9, 15].

Numerical simulations allowed us to observe the effects of the rate of vaccination, quarantining of clinically infected animals, foot and mouth disease virus shedding to the atmospheric environment and diffusion parameters on foot and mouth disease transmission in cattle. The results suggested that increasing either the rate of vaccination of newborn or susceptible animals would channel the movement of animals into the protected route of disease dynamics, that increase the vaccinated animal population and at the same time reduced the spread of foot and mouth disease virus into the environment. The benefits of vaccination through mass vaccination have been recorded in model M. J. Keeling *et al*.[15] which did not capture the spatial dynamics and environmental transmission. The inclusion of the spatial dynamics and environmental transmission may affect the estimates of the quantum of mass vaccinations. Increasing the rate of quarantining clinically infected animals was associated with the decrease in the infectious animals' classes as well as the virus in the environment. The quarantined animals have no contact with other healthy animals and so findings suggest that the environmental transmission alone may not be effective in as far as the increase in foot and mouth disease burden is concerned. Some studies did not capture the spatial spread of foot and mouth disease [41] showed that the quarantine of affected animals only reduced the spread of the disease in animals. However, since the subclinically infected and vaccinated carrier animals are still mixing with the healthy animals and the fact that there is shedding of the virus into the environment, the effects of quarantining may not effectively reduce the burden. Control strategies that also target the less infectious animals, as well as the virus in the environment, are needed in order to effectively control foot and mouth disease when one considers time and spatial spread of foot and mouth disease. Such controls should be optimized enough to consistently suppress if not eradicate foot and mouth disease. Our results also showed that increasing the rate of shedding from subclinically, clinically, quarantine clinically infected animals and the vaccinated carrier is associated with the increase in foot and mouth disease burden in cattle. Control strategies that target the reduction of shedding of the virus into the atmospheric environment need to be implemented effectively but this may not work alone if the other avenues of transmission are not taken of in the control strategy [6, 42, 43]. Spatial diffusion of animals was implemented in Maidana and Yang [16] without including the shedding and diffusion of the virus into the atmospheric environment. Their study only checked the existence of travelling wave solutions and determined the wave speed as a function of diffusion but fell short of testing the effects of the rate of diffusion and other parameters. Therefore, increasing the rate of movement of healthy animals when there is a low rate of movement of infectious animals decreases the burden of foot and mouth disease, but increasing the rate of movement of subclinically, clinically and vaccinated carrier animals and decreasing the diffusion rate of foot and mouth disease virus is associated with the increase in foot and mouth disease burden in cattle. Thus, reducing the movement of infected animals reduces the burden on foot and mouth disease.

A combination of control strategies that protects healthy animals and significantly reducing the infectious animals and as well as the shedding and diffusion rates of the virus in the environment is needed. Our study exposed the potential drivers of foot and mouth disease in cattle and the use of single strategies may not yield a desirable effect of eradicating the disease. As a consequence of this, it is important to implement an effective combination of rates of control, movement of animals, shedding rate and diffusion rate of the virus in the environment. This is consistent with the study by Mugabi *et al.* [44] and Bani-Yaghoub *et al.* [45], where it was found out that by applying environmental decontamination in combination with other control measures eliminate an endemic equilibrium of

the disease.

The models formulated in this study only focus on the cattle population, but we can extend this study by incorporating multi-transmission routes, such as using cattle and Sheep or other animal groups. We can improve this study in the future by incorporating quarantining all infected animal classes or study using optimizing the combination of control strategies using optimal control in partial differential equations. We can also study this partial differential equation using stochastic processes.

Appendices

$$
\begin{aligned}
\left(\frac{4\,e_2}{e_5}+\frac{m_1}{m_2+1}\right)\epsilon_5 &= \left(\frac{4\,e_2}{e_5}+\frac{m_1}{m_2+1}\right)\epsilon_5-\left(\frac{k_3 m_1}{m_2+1}-k_4\right)\\
&\quad-\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{e_2}{e_5}+\frac{m_1}{m_2+1}\right)\epsilon_6 &= \left(\frac{e_2}{e_5}+\frac{m_1}{m_2+1}\right)\epsilon_6+\left(\frac{k_3 m_1}{m_2+1}-k_4\right)\\
&\quad+\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{4\,e_1}{e_5}+\frac{m_2}{m_3+1}\right)\epsilon_7 &= \left(\frac{4\,e_1}{e_5}+\frac{m_2}{m_3+1}\right)\epsilon_7-\left(\frac{m_2\,k_4}{m_3+1}-k_5\right)\\
&\quad-\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{e_1}{e_5}+\frac{m_2}{m_3+1}\right)\epsilon_8 &= \left(\frac{e_1}{e_5}+\frac{m_2}{m_3+1}\right)\epsilon_8+\left(\frac{m_2\,k_4}{m_3+1}-k_5\right)\\
&\quad+\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{4\,e_3}{e_5}+\frac{2\,m_3}{c_4+1}\right)\epsilon_9 &= \left(\frac{4\,e_3}{e_5}+\frac{2\,m_3}{c_4+1}\right)\epsilon_9-\left(\frac{m_3\,k_5+k_8 m_4}{c_4+1}-k_6\right)\\
&\quad-\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{e_3}{e_5}+\frac{2\,m_3}{c_4+1}\right)\epsilon_{10} &= \left(\frac{e_3}{e_5}+\frac{2\,m_3}{c_4+1}\right)\epsilon_{10}+\left(\frac{m_3\,k_5+k_8 m_4}{c_4+1}-k_6\right)\\
&\quad+\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{4\,e_4}{e_5}+\frac{2\,m_4}{c_4+1}\right)\epsilon_{15} &= \left(\frac{4\,e_4}{e_5}+\frac{2\,m_4}{c_4+1}\right)\epsilon_{15}-\left(\frac{m_3\,k_5+k_8 m_4}{c_4+1}-k_6\right)\\
&\quad-\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),\\
\left(\frac{e_4}{e_5}+\frac{2\,m_4}{c_4+1}\right)\epsilon_{16} &= \left(\frac{2\,m_3\,\epsilon_{16}}{c_4+1}+\frac{e_4}{e_5}\right)\epsilon_{16}+\left(\frac{m_3\,k_5+k_8 m_4}{c_4+1}-k_6\right)\\
&\quad+\left(\frac{e_1\,k_4+e_2\,k_3+e_3\,k_5+e_4\,k_8}{e_5}-k_9\right),
\end{aligned}
\tag{28}
$$

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CHAPTER 5

ℂONCLUSION AND FUTURE DIRECTIONS

5.1 Conclusion

Foot and mouth disease is a livestock diseases that limits economic growth, increases poverty among subsistence farmers and reduces food security [1, 2]. The application of mathematical models is essential in studies of the dynamics of the foot and mouth disease [5, 10, 18]. However, most prior models are often included isolated control strategies without combining control strategies and a consideration of reaction-diffusion effects [17]. This thesis focused on developing epidemiological mathematical models of the spread of the foot and mouth disease that sought to improve understanding of the transmission dynamics in cattle populations. We developed and analyzed mathematical models for the control strategies of foot and mouth disease (FMD) by considering the environmental causes of infection, incorporating the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment, reactive culling, quarantining of clinically infected classes and the effects of time delay.

Chapter 1 gave a detailed background and motivation for research on the dynamics of the foot and mouth disease. Important aspects of the disease highlighted included the biology of foot and mouth virus and mathematical models previously used, such as ordinary differential equations, delay differential equations and partial differential equations.

In Chapter 2, we investigated the transmission dynamics of the foot and mouth disease using a system of differential equations. The purpose of this chapter is to show the effect of vaccination that does not induce rapid protection on disease dynamics. Livestock productivity and food security are improved through vaccination programs. However, there is a limitation to vaccination

as this does not induce rapid protection. The results suggested that even though vaccines may not induce rapid protection using a high rate of vaccination and low loss of vaccine protection rate may be successful in reducing the foot and mouth burden provided critical vaccination thresholds are taken into consideration. Stability analysis showed that the disease-free equilibrium is stable when $\mathcal{R}_c < 1$ and the endemic equilibrium is also locally stable when $\mathcal{R}_c > 1$. We investigated the effects of the vaccination coverage and loss of vaccination protection on both the vaccinated and unvaccinated animal populations. The findings show that a high vaccination rate and low loss of protection is the best strategy to reduce the foot and mouth disease burden, followed by high vaccination rate and high loss of protection. However, low vaccination rate and low loss of protection is the least strategy to protect the foot and mouth disease. On the other hand, a low vaccination and high loss of protection rates are the worst strategies for foot and mouth disease because in this strategy the flow of animals is high into the unprotected route of infection but flow into the vaccination route of infection is low.

The study further suggests that increasing the rate of recovery in vaccinated carrier class increases the recovered class. The recovered animals return to the susceptible class with a slight decrease in infectious classes and high decrease in vaccination carrier class. Hence, this strategy reduces the foot and mouth disease burden. Moreover, if infections occur in the population, the vaccination should be implemented as soon as possible. The study is not all encompassing, and can be improved by incorporating the spread of foot and mouth disease in both space and time.

In Chapter 3, we developed a delay ordinary differential equation model of the foot and mouth disease. The goals were to investigate the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling on the spread of foot and mouth disease with two-time delays. We determined the control reproduction number, $\mathcal{R}_c$ and the existence of a disease-free equilibrium and an endemic equilibrium using mathematical analysis. The disease-free equilibrium was locally asymptotically stable when $\mathcal{R}_c$ is less than unity. In this case the

FMD can be kept in check if the control strategies used consistently reduce the reproduction number below unity. In fact, there is a possibility of eradicating the infection with such controls. However, if controls cannot reduce $\mathcal{R}_c$ below unity, then there is a possibility that the FMD can spread to endemic levels. This scenario may be characteristic of a control strategy that is either inadequately administered or less efficient. Evidence of controls such as the rate of vaccination, the rate of treating and vaccinating susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and vaccinated carrier animals is available where the burden of foot and mouth disease continued to increase [1, 10, 12, 81]. In particular, there are vaccines that do not induce rapid control [12]. The sensitivity analysis of $\mathcal{R}_c$ with respect to prophylactic vaccination showed that $\mathcal{R}_c$ decreases only when the rate of loss of vaccination is below a critical level of vaccination otherwise the benefits of prophylactic vaccination alone may not be realized. This means that prophylactic vaccination as a single strategy may not successfully eradicate the foot and mouth disease. Simulations showed that the control reproduction number ($\mathcal{R}_c$) is less than one when the rate of treating and vaccinating susceptible animals and rate of culling of clinically infected and vaccinated carrier animals are high. The numerical simulations suggest that increasing of both time delay two and control parameters or increasing either of the time delay two or control parameters reduces the burden of foot and mouth disease. On the other hand, increasing of both time delay one and control parameters does not show a significant effect on the dynamics of foot and mouth disease. Therefore, time delay two has a significant effect on foot and mouth disease. The result also showed that newly infected animals are significantly delayed in showing clinical symptoms leading to lower shedding of foot and mouth disease virus in the environment and subsequently a reduction in the foot and mouth burden due to increasing time delay two. Similarly, increasing control parameters such as prophylactic vaccination, reactive vaccination, prophylactic treatment, and reactive culling parameters has a significant effect on the decreasing of foot and mouth disease burden. The results suggest that the strategy of decreasing time delay two whilst increasing the degree of control parameters contributes to the reduction of foot and mouth disease but reducing of both time delay two and control parameters increases the foot and mouth disease burden. Reducing time delay

two means that the newly infected animals fast to show clinical symptoms and leading to high shedding of foot and mouth disease virus to other animals and subsequently the increase of foot and mouth burden. The findings also suggest that the strategy of increasing of time delay two whilst reducing the control parameters may not significantly reduce the foot and mouth disease burden. This finding has consequences when the intention is to reduce the costs of control strategies but inadvertently, the disease burden continues to spread. The strategy with a significant effect on the protection from the foot and mouse disease involves both the time delay two and control parameters.

In Chapter 4, we extended the model in Chapter 2 to a reaction-diffusion model. The purpose was to investigate the effects of vaccination, quarantining clinically infected animals, shedding of foot and mouth disease virus into the environment and rates of movement of animals and virus. The mathematical analysis showed the existence of a travelling wave solution connecting the disease-free steady state and the endemic steady state. Here used the technique of upper and lower solutions and Schauders fixed point theorem.

The numerical results suggested that increasing either the rate of vaccination of newborn or susceptible animals would channel the movement of animals into the protected route of disease dynamics, that is, the vaccinated animal population and at the same time reduced the spread of foot and mouth disease virus into the environment. The results also showed that increasing the rate of quarantining clinically infected animals was associated with a decrease in the infectious animals' classes as well as the virus in the environment. The quarantined animals have no contact with other healthy animals and findings suggest that the environmental transmission alone may not be effective in increasing the foot and mouth disease virus. However, since the subclinically infected and vaccinated carrier animals are still able to mix with healthy animals and there is shedding of the virus into the environment, quarantining may not effectively reduce the burden. Control strategies that also target the less infectious animals, as well as the virus in the environment, are needed in order to effectively control foot and mouth disease when one

considers time and spatial spread of the virus. Such controls should be optimized to consistently suppress and eradicate foot and mouth disease. The findings show that increasing the rate of shedding from subclinically, clinically, quarantine clinically infected animals and the vaccinated carrier is associated with an increase in foot and mouth disease in cattle. Control strategies that target the reduction of shedding of the virus into the atmospheric environment need to be implemented effectively but this may not work alone if the other avenues of transmission are not taken of in the control strategy [12, 82, 83]. Increasing the rate of movement of healthy animals when there is a low rate of movement of infectious animals decreases the burden of foot and mouth disease, but increasing the rate of movement of subclinically, clinically and vaccinated carrier animals and decreasing the diffusion rate of foot and mouth disease virus is associated with the increase in foot and mouth disease burden in cattle. Thus, reducing the movement of infected animals reduces the burden of the foot and mouth disease. A combination of control strategies that protects healthy animals and significantly reduces the infectious animals and as well as the shedding and diffusion rates of the virus in the environment is needed. The study exposed the potential drivers of foot and mouth disease in cattle and suggests that the use of single control strategies may not yield the desirable effect of eradicating the disease. As a consequence, it is important to implement an effective combination of control, movement of animals, shedding rate and diffusion rates of the virus in the environment.

5.1.1 Summary of Recommendations

The findings in this study have important implications for foot and mouth disease control management. Hence, We summarize the recommendations in this study as follows:

(a) Vaccines may not induce rapid protection but taking a combination of high rates of vaccination and low loss of vaccine protection rate is the best strategy in reducing the foot and mouth burden but taking a low vaccination with a high loss of protection rates is the worst

strategy for foot and mouth disease protection.

(b) Reducing time delay in removing clinically infected animals and increasing the control parameters contributes significantly to the reduction of foot and mouth disease burden.

(c) Prophylactic vaccination as a single strategy may not successfully eradicate the foot and mouth disease but if combined with prophylactic and reactive vaccinations or other control strategies such as reactive culling is useful to eradicate foot and mouth disease burden.

(d) Increasing either the rate of vaccination of newborn or susceptible animals moves the animals into the protected route, this reduces the shedding of foot and mouth virus into the environment.

(e) Increasing the rate of shedding from subclinically, clinically, quarantine clinically infected animals and the vaccinated carrier is associated with an increase in foot and mouth disease burden in cattle.

(f) Restricting the movement of infected animals reduces the burden on foot and mouth disease.

5.2 Future Directions

This study has added some insights into FMD dynamics and highlighted specific areas that require attention if efforts to contain the disease are to be successful. Nevertheless, there are other aspects not considered in this study that could provide additional insights pertaining to FMDs.

(a) Considering multi-transmission routes such as cattle and sheep or wild animals.

(b) The model in Chapter 3 could be improved by incorporating time delay to the reaction-diffusion equations to provide more insights into the dynamics of FMD in cattle population.

(c) Incorporating age-structure using the partial differential equation to study the foot and mouth disease in animals.

(d) Combining stochastic and partial differential equations to study the foot and mouth disease of animals.

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APPENDICES

This article is about the Modelling control of foot and mouth disease with two time delays. It has been described in Chapter 3 in this thesis. This paper was published in International Journal of Biomathematics, https://doi.org/10.1142/S179352451930001X.

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Modeling control of foot and mouth disease with two time delays

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We develop a delay ordinary differential equation model that captures the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling on the spread of foot and mouth disease (FMD) with time delays. Simulation results from the study suggest that increasing time delay while increasing the control strategies decreases the burden of FMD. Further, the results reveal, that decreasing time delay while decreasing the control strategies increases the burden of FMD. The intermediate scenarios of either (i) increasing time delay while decreasing control or (ii) decreasing time delay while increasing control have intermediate effects on burden reduction. Thus, the implementation of effective control strategies combination can play an important role in mitigating against the FMD burden.

Keywords: Foot and mouth; vaccination; prophylactic vaccination; treatment; culling; time delay.

Mathematics Subject Classification 2010: 92B05, 92D30

1. Introduction:

One of the most common contagious animal viral diseases that can cause devastating economic, social, and environmental damages is foot and mouth disease (FMD) [24]. FMD affects cloven-hoofed animals [4, 25, 34] and is communicated through viral particles in the air transported by the wind and through direct and indirect contact. Susceptible animals that are exposed to FMD remain exposed for 2 to 4 days and subsequently proceed to a subclinical state. A clinically infected animal can recover by developing natural immunity [6, 9, 21] which can wane with time

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making the animal susceptible again [3, 16, 20, 25]. The vaccinated animals may become carriers and transmit the infection subclinically. The animal may recover by developing natural immunity [9, 32, 33, 36].

The spread of FMD may be controlled by vaccination and treatment of different animal groups. Vaccination can be administered as either a prophylactic vaccination or as a reactive vaccination strategy [2, 15, 33]. Prophylactic vaccination is carried out before the introduction of the disease whereas reactive vaccination is carried out after the outbreak [2, 15, 27, 28]. Vaccination alone is inadequate to completely protect the animals from FMD [27]. Evidence available shows the occurrence of several outbreaks of FMD in places where vaccination was administered [9, 15]. Prophylactic treatment has also been used in the control of FMD. The treatment uses antiviral drugs to protect animals from infection and to reduce the risk of carrier animals spreading the virus during emergency vaccination programs [1, 9]. Reactive culling is another measure that has been used in combination with other control measures such as reactive vaccination [2, 15]. It is administered after the outbreak of FMD. Prophylactic and reactive vaccination are to some extent, effective in preventing FMD. However, it remains to be determined how effective the combination of prophylactic and reactive vaccination is. It is also important to investigate the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling.

Differential equations have been used to model the dynamics of a number of diseases [10, 23, 35]. These equations include delay differential equations that can be used to describe epidemiological phenomena at a certain time in terms of the values of a given function at previous times [2]. A number of delay differential equations have been used to model other infections with one time delay [8, 18, 22].

In this study, we investigate the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling on the dynamics of FMD infection using a two-time delay model. We formulate the mathematical model in Sec. 2, provide some model analysis in Sec. 3, present parameter estimation and numerical simulations in Sec. 4 and give a discussion of the results in Sec. 5.

2. Model Incorporating Two Time Delays

We propose a model of time delay differential equations for the spread of FMD in animals by subdividing the total population into susceptible animals $S(t)$, "treated and vaccinated" animals $T_v(t)$, clinically infectious animals $I_c(t)$, recovered animals $R(t)$, vaccinated animals $V_v(t)$ and vaccinated carrier animals $V_{\text{ca}}(t)$. Susceptible animals are free of the foot and mouth disease virus (FMDV), treated and vaccinated animals are animals which are treated with prophylactic drugs and vaccinated with prophylactic reactive vaccines, the clinically infectious animals are those with clinically diagnosed symptoms. The vaccinated animals are protected from the disease by a prophylactic vaccine as well as prophylactic reactive vaccines, and vaccinated carrier animals may get infected because the prophylactic vaccine does not

induce complete protection. The removed animals are either recovered or immune to the infection. The immunity may wane with time and the recovered animals become susceptible again [3, 16, 20, 25].

The susceptible animals are recruited at the rate $bN(t)$ where b is the per-capita birth rate and $N(t)$ is the total population. A proportion of new birth is given a prophylactic vaccine at a constant rate of ρ where $0 \leq \rho \leq 1$, and hence, the net recruitment of the susceptible animal is given by $(1-\rho)bN(t)$. Susceptible animals are given either reactive vaccines at a constant rate ρ_2 where $0 \leq \rho_2 \leq 1$ or given prophylactic treatment [1]. ρ_1 is a rate of treating susceptible animals and subsequently vaccinating them so that they end up in the $T_v(t)$ class. Prophylactic drugs are preventive drugs that are administered to animals that are free of infection. Prophylactic drugs have been administered successfully for other viruses and diseases, for instance, HIV and malaria infections [5, 31]. Susceptible animals which are given a reactive vaccine move to the vaccinated class $V_v(t)$. Prophylactically treated susceptible animals will in principle move to the treated susceptible class and when subsequently vaccinated will move ultimately to the treated and vaccinated class $T_v(t)$. We introduce a time delay $\tau_1 > 0$ to replace the treated susceptible class and capture the time required to move the treated and subsequently vaccinated animals to the treated and vaccinated class $T_v(t)$. However, because the treated susceptible class is not immune to infection, they may be infected and move in principle to the sub-clinically infected animals which we replace by the time delay $\tau_2 > 0$ and allow movement of infected animals into the clinically infected animals $I_c(t)$. We capture the force of infection in the treated susceptible animals by $\beta(1-\epsilon)(I_c(t-\tau_2)+\eta V_{\mathrm{ca}}(t))S(t-\tau_1)/N(t)$ with ϵ the rate of treating susceptible animals where $0 \leq \epsilon \leq 1$ and $(1-\epsilon)$ capturing the treatment protection failure. The new infections through successive contacts between susceptible and infected animals occur at a rate β. Since V_{ca} animals are less infectious as compared to I_c we introduce an amplification factor $\eta < 1$.

Susceptible animals which are not treated or vaccinated may get infected and move in principle to the sub-clinically infected class and progress to the clinically infected class $I_c(t)$. As in the treated susceptible animals, we replace the sub-clinically infected animals by the time delay τ_2 and capture the force of infection by $\beta(I_c(t-\tau_2)+\eta V_{\mathrm{ca}}(t))S(t)/N(t)$. Susceptible animals are also subjected to natural death at a rate of μ. They also suffer from density-dependent death rate that comes due to crowding and we capture the combined death rate by the term $\mu+\gamma N$, where γ is the per-capita density-dependent death rate. The density-dependent death rate has the effect of inducing the logistic growth in the total population which is one of the realistic ways of capturing the growth of populations.

The clinically infected class are recruited from infection of susceptible animals and from infection of treated susceptible animals. They may recover naturally and move to the recovered class $R(t)$ at a rate α_2. They are also subjected to the combined death rate above. As a control measure, we introduce culling at a rate of δ and move animals out of $I_c(t)$ through this control measure. We assume that

$I_c(t)$ animals cannot move to $T_v(t)$, due to the fact that (i) there is no known post-infection treatment for FMD and also that (ii) even if treatment existed, they would need to be subsequently vaccinated to be able to join to $T_v(t)$ class. A vaccine cannot be administered to sick animals. Hence, the natural recovery and culling of sick animals are assumed to be sufficient to capture all the dynamics of $I_c(t)$ animals.

Vaccinated animals are recruited through prophylactic vaccination at a rate $\rho bN(t)$ and also from reactive vaccination of susceptible animals at a rate ρ_2. Since the vaccination do not induce rapid protection [9] and is not perfect, some vaccinated animals are infected and become vaccinated carriers class $V_{\text{ca}}(t)$ with a force of infection $\beta\phi(I_c(t-\tau_2)+\eta V_{\text{ca}}(t))S(t)/N(t)$, where ϕ is the rate of protection loss by the vaccination where $0 \leq \phi \leq 1$. Some vaccinated animals can be given prophylactic treatment at a rate ϵ and move to the treated and vaccinated class $T_v(t)$. They are also subjected to the combined death rate. The vaccinated carrier animals are recruited from infection of vaccinated animals, they recover naturally at a rate α_3, removed through culling at a rate δ and as well as through combined death rate.

The treated and vaccinated animals are recruited from the treating of vaccinated animals and from treated and subsequently vaccinated susceptible animals. They recover at a rate α_4 and die due to combined death. The recovered animals are recruited from the recovery of clinically infected animals, vaccinated carrier animals and, treated and vaccinated animals. The immunity wanes at a rate ω which moves the recovered animals back to the susceptible class otherwise the recovered animal are removed from their class through combined death. Figure 1 shows the flow diagram for the model proposed.

The model representing the dynamics of FMD infection is represented as a system of delay differential equations (DDEs) as follows:

$$\begin{aligned}\frac{dS(t)}{dt} &= (1-\rho)bN(t) - \lambda S(t) - (1-\epsilon)\lambda S(t-\tau_1) - \rho_1 S(t-\tau_1) \\ &\quad - (\rho_2 + \mu + \gamma N(t))S(t) + \omega R(t), \qquad (2.1)\\ \frac{dT_v(t)}{dt} &= \rho_1 S(t-\tau_1) + \epsilon V_v(t) - (\alpha_4 + \mu + \gamma N(t))T_v(t), \qquad (2.2)\\ \frac{dI_c(t)}{dt} &= \lambda S(t) + (1-\epsilon)\lambda S(t-\tau_1) - (\delta + \alpha_2 + \mu + \gamma N(t))Ic(t), \qquad (2.3)\\ \frac{dR(t)}{dt} &= \alpha_2 Ic(t) - (\omega + \mu + \gamma N(t))R(t) + \alpha_3 V_{\text{ca}}(t) + \alpha_4 T_v(t), \qquad (2.4)\\ \frac{dV_v(t)}{dt} &= \rho bN(t) + \rho_2 S(t) - \phi\lambda V_v(t) - (\epsilon + \mu + \gamma N(t))V_v(t), \qquad (2.5)\\ \frac{dV_{\text{ca}}(t)}{dt} &= \phi\lambda V_v(t) - (\delta + \alpha_3 + \mu + \gamma N(t))V_{\text{ca}}(t), \qquad (2.6)\end{aligned}$$

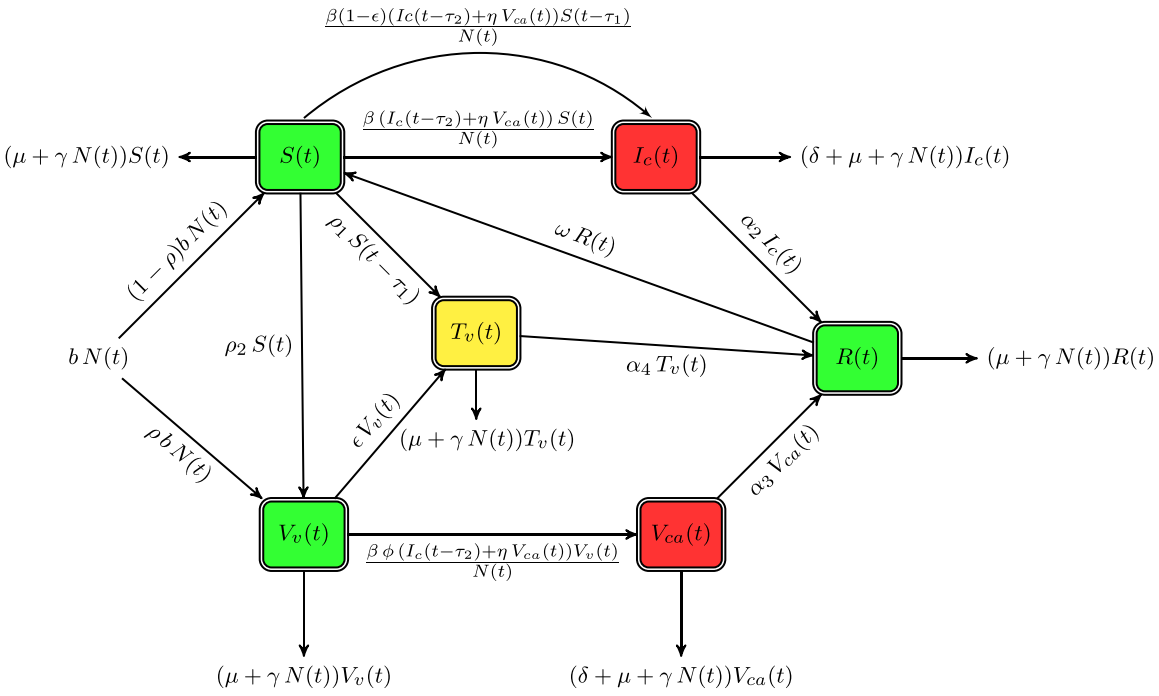

Fig. 1. Flow diagram for the FMD model with two time delay.

where $\lambda = \beta(I_c(t-\tau_2) + \eta V_{\rm ca}(t))/N(t)$ and subject to the initial conditions

$$\begin{aligned} &S(0) = S_0 \geq 0, \quad T_v(0) = T_{v0} \geq 0, \quad I_c(0) = I_{c0} \geq 0, \\ &R(0) = R_0 \geq 0, \quad V_v(0) = V_{v0} \geq 0, \quad V_c(0) = V_{\rm ca0} \geq 0, \\ &t \in [-\tau_i, 0], \quad \text{where } \tau_i \text{ is the maximum delay.} \end{aligned} \tag{2.7}$$

Adding equations of the model (2.1)–(2.6), leads to a logistic differential equation

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \frac{\gamma N(t)}{r}\right) - \delta(I_c(t) + V_{\rm ca}(t)). \tag{2.8}$$

From (2.8), we note that

$$\frac{dN(t)}{dt} \leq rN(t)\left(1 - \frac{N(t)}{K}\right), \quad \text{where } K = \frac{r}{\gamma} \text{ is the carrying capacity and } r = b - \mu > 0, \text{ is the growth rate.}$$

The solution satisfies

$$N(t) \leq \frac{K}{1 + \psi {\rm e}^{-rt}}, \quad \text{where } \psi = \left(\frac{K}{N(0)} - 1\right), \quad N(0) \leq K, \text{ and } \lim_{t\to\infty} N(t) \leq \lim_{t\to\infty} \frac{K}{1 + \psi {\rm e}^{-rt}} = K.$$

3. Model Analysis

In this section, we define a feasible region for the model (2.1)–(2.6), where Γ is positively invariant and attracting. We also calculate the reproduction ratio and determine the equilibrium points and their stability.

3.1. *Feasible region*

All the variables and parameters are assumed to be non-negative for the model to be biologically meaningful.

Theorem 3.1. *Let the system of equations* (2.1)–(2.6) *have initial conditions* (2.7). *Then, the region* Γ *defined by* (3.1) *is positively invariant and attracting where*

$$\Gamma = \{(S(t), T_v(t), I_c(s), R(t), V_v(t), V_{\text{ca}}(t)) \in \Re^6_+ \mid N(t) \leq K\}. \tag{3.1}$$

Proof. Assume for $t > 0$, $N(0) \geq 0$, $S(0) \geq 0$, $T_v(0) \geq 0$, $I_c(0) \geq 0$ $R(0) \geq 0$, $V_v(0) \geq 0$ and $V_{\text{ca}}(0) \geq 0$. From Eq. (2.6), we get

$$\frac{d}{dt}V_{\text{ca}}(t) = \frac{\beta\phi(I_c(t-\tau_2) + \eta V_{\text{ca}}(t))V_v(t)}{N(t)} \tag{3.2}$$

$$- (\delta + \alpha_3 + \mu + \gamma N(t))V_{\text{ca}}(t). \tag{3.3}$$

Integrating equation (3.2) and using a differential inequality, we get

$$V_{\text{ca}}(t) \geq V_{\text{ca}}(0) \exp\left(-(\delta + \alpha_3 + \mu)t - \gamma \int_0^t N(s)ds\right) \geq 0.$$

So $V_{\text{ca}}(t) \to V_{\text{ca}}(0)$ as $t \to 0$ hence $V_{\text{ca}} \geq 0$. This implies that at any finite time, V_{ca} is positive. A similar analysis holds for Eqs. (2.1)–(2.6), where

$$R(t) \geq R(0) \exp\left(-(\omega + \mu)t - \gamma \int_0^t N(s)ds\right) \geq 0,$$

$$I_c(t) \geq I_c(0) \exp\left(-(\delta + \alpha_2 + \mu)t - \gamma \int_0^t N(s)ds\right) \geq 0,$$

$$T_v(t) \geq T_v(0) \exp\left(-(\alpha_4 + \mu)t - \gamma \int_0^t N(s)ds\right) \geq 0,$$

$$V_v(t) \geq V_v(0) \exp\left(-(\epsilon + \mu)t - \int_0^t \left(\frac{\beta\phi(I_c(s-\tau_1) + \eta V_{\text{ca}}(s))}{N(s)} + \gamma N(s)\right) ds\right) \geq 0,$$

$$S(t) \geq S(0) \exp\Bigg(-(\rho_1 + \rho_2 + \mu)t$$

$$- \int_0^t \left(\frac{\beta(2-\epsilon)I_c(s-\tau_1) + \eta V_{\text{ca}}(s))}{N(s)} + \gamma N(s)\right) ds\Bigg) \geq 0.$$

Therefore, the solutions of the model with non-negative initial conditions remain non-negative for all $0 \leq t < \infty$. Since $0 \leq S(t), T_v, I_c(t), R(t), V_v(t), V_{\text{ca}}(t) \leq K$, all variables are bounded in $[0, K]$. This shows that for initial conditions (2.7), the region Γ is positively invariant and attracting and therefore the region Γ is a feasible region for the model (2.1)–(2.6). □

3.2. *The control reproduction ratio for the model*

The control reproduction ratio is calculated using the next generation matrix method [7, 37]. We take only the infected classes of the model to calculate the control reproduction ratio. At the disease free equilibrium point, $I_c = V_{\text{ca}} = 0$, S_0, R_0 and V_{v0}. The control reproduction number is given by

$$\mathcal{R}_c = \frac{\beta\phi\eta\rho b}{(\epsilon + b)(\alpha_3 + b + \delta)} + \frac{\beta(2-\epsilon)(1-\rho)b}{(b+\rho_1)(b+\delta+\alpha_2)},$$

where $b = (\mu + \gamma K)$.

To test the parameters that significantly affect the transmission dynamics of FMD in cattle, sensitivity analysis on $\mathcal{R}_c$ was carried out through differentiating $\mathcal{R}_c$ with respect to parameters of the model. The following results were obtained:

$$\frac{\partial \mathcal{R}_c}{\partial \epsilon} = -\frac{\beta\phi\eta b\rho}{(\epsilon+b)^2(\alpha_3+b+\delta)} - \frac{\beta(1-\rho)b}{(b+\rho_1)(b+\delta+\alpha_2)} < 0,$$

$$\frac{\partial \mathcal{R}_c}{\partial \delta} = -\frac{\beta\phi\eta b\rho}{(\epsilon+b)(\alpha_3+b+\delta)^2} - \frac{\beta(2-\epsilon)(1-\rho)b}{(b+\rho_1)(b+\delta+\alpha_2)^2} < 0,$$

$$\frac{\partial \mathcal{R}_c}{\partial \rho_1} = -\frac{\beta(2-\epsilon)(1-\rho)b}{(b+\rho_1)^2(b+\delta+\alpha_2)} < 0,$$

$$\frac{\partial \mathcal{R}_c}{\partial \rho} = \frac{\beta\phi\eta b}{(\epsilon+b)(\alpha_3+b+\delta)} - \frac{\beta(2-\epsilon)b}{(b+\rho_1)(b+\delta+\alpha_2)}$$

$$= \frac{\beta b\eta(\phi - \phi_{\text{crit}})}{(\epsilon+b)(\alpha_3+b+\delta)} < 0,$$

where $\phi_{\text{crit}} = ((2-\epsilon)(\epsilon+b)(\alpha_3+b+\delta))/(\eta(b+\rho_1)(b+\delta+\alpha_2))$. $\partial\mathcal{R}_c/\partial\rho > 0$, $\phi > \phi_{\text{crit}}$ and $\partial\mathcal{R}_c/\partial\rho < 0$ when $\phi < \phi_{\text{crit}}$ and $\partial\mathcal{R}_c/\partial\rho = 0$ when $\phi = \phi_{\text{crit}}$. For vaccination to be effective $\mathcal{R}_c < 1$, the loss of protection from vaccination should be less than the critical value of loss of protection from the vaccine, that is, $\phi < \phi_{\text{crit}}$.

The sensitivity analysis showed that the derivatives of $\mathcal{R}_c$ with respect to the rate of vaccination, the rate of treating susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and vaccinated carrier animals are all less than zero.

Figure 2 shows the effects of vaccination, treating and vaccinating susceptible animals, treating vaccinated animals and culling infected and vaccinated carrier animals on $\mathcal{R}_c$. Figures 2(a) and 2(b) show that $\mathcal{R}_c$ is reduced and less than one as the parameters, ρ, ρ_1 and δ increase. The graph of ϵ vs. ρ shows that increasing

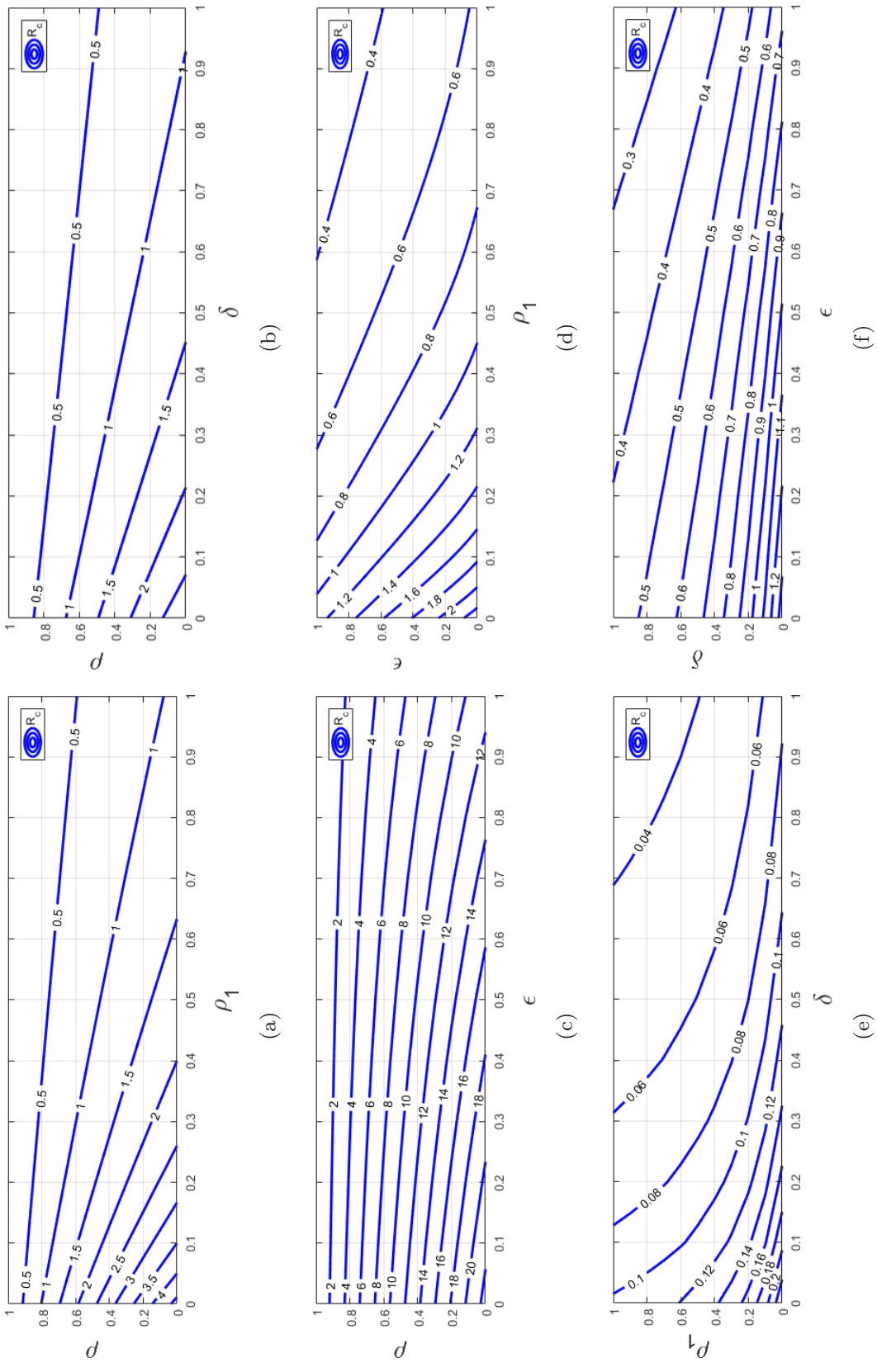

Fig. 2. The effects of the rate of vaccination (ρ), the rate of treating susceptible animals (ρ_1), the rate of treating vaccinated animals (ϵ) and the rate of culling infected and vaccinated carrier animals (δ) on the control reproduction number.

both ϵ and ρ reduces $\mathcal{R}_c$ to a value still above unity in Fig. 2(c). The graph in Fig. 2(d) illustrates that $\mathcal{R}_c$ reduces as both ρ_1 and ϵ increase. In addition, we note that $\mathcal{R}_c$ is reduced to a value less than one as the parameters δ, ρ_1 and ϵ are increased as shown in Figs. 2(e) and 2(f). Reducing the control reproduction number ($\mathcal{R}_c$) corresponds to decreasing the number of newly infected animals leading to low shedding of FMDV to other animals and subsequently decreasing the FMD burden.

3.3. *Equilibrium points of the system*

The disease-free equilibrium point of the system of equation is given by

$$E_0 = (S_0, T_{v0}, 0, R_0, V_{v0}, 0), \tag{3.4}$$

where

$$S_0 = \frac{((1-\rho)(b^3 + (\omega+\epsilon+\alpha_4)b^2 + ((\omega+\alpha_4)\epsilon + \omega\alpha_4)b) + \epsilon\omega\alpha_4)K}{M_1},$$

$$T_{v0} = \frac{(b+\omega)K(\epsilon(b\rho+\rho_1+\rho_2) + b\rho_1(1-\rho))}{M_1},$$

$$R_0 = \frac{K\alpha_4(\epsilon(b\rho+\rho_1+\rho_2) + b\rho_1(1-\rho))}{M_1},$$

$$V_{v0} = \frac{Kb(b+\omega+\alpha_4)\rho\rho_1 + K(b+\alpha_4)(b+\omega)(b\rho+\rho_2)}{M_1},$$

$$M_1 = b^3 + (\omega+\epsilon+\rho_1+\rho_2+\alpha_4)b^2 + ((\omega+\rho_1+\rho_2+\alpha_4)\epsilon + (\omega+\rho_1+\rho_2)\alpha_4 + \omega(\rho_1+\rho_2))b + ((\omega+\rho_1+\rho_2)\alpha_4 + \omega(\rho_1+\rho_2))\epsilon + \omega\alpha_4\rho_2.$$

The force of infection at the equilibrium point is

$$\lambda^* = \frac{\beta(I_c^* + \eta V_{\text{ca}}^*)}{K}. \tag{3.5}$$

The endemic equilibrium E_1 of the model is given in terms of λ^* and R^* with

$$E_1 = (S^*, T_v^*, I_c^*, R^*, V_v^*, V_{\text{ca}}^*), \tag{3.6}$$

where

$$S^* = \frac{(1-\rho)bK + \omega R^*}{(2-\epsilon)\lambda^* + b + \rho_1 + \rho_2},$$

$$T_v^* = \frac{\rho_1((1-\rho)bK + \omega R)}{A_4} + \frac{\epsilon K b\rho}{(b+\alpha_4)(\lambda^*\phi + b + \epsilon)} + \frac{\epsilon\rho_2((1-\rho)bK + \omega R^*)}{(\lambda^*\phi + b + \epsilon)A_4},$$

$$I_c^* = \frac{(2-\epsilon)\lambda^*((1-\rho)bK + \omega R^*)}{(\delta + b + \alpha_2)((2-\epsilon)\lambda^* + b + \rho_1 + \rho_2)},$$

$$R^* = \frac{1}{1-A_2}\left(\frac{A_3}{(b+\omega)(\delta+b+\alpha_2)A_4} + A_5\left(\frac{\rho}{(b+\alpha_4)(\delta+b+\alpha_3)} + \frac{\rho_2(1-\rho)}{(\delta+b+\alpha_3)A_4}\right)\right),$$

$$V_v^* = \frac{Kb\rho}{\lambda^*\phi+b+\epsilon} + \frac{\rho_2((1-\rho)bK+\omega R^*)}{(\lambda^*\phi+b+\epsilon)((2-\epsilon)\lambda^*+b+\rho_1+\rho_2)},$$

$$V_{\text{ca}}^* = \frac{\lambda^*\phi}{\delta+b+\alpha_3}\left(\frac{Kb\rho}{\lambda^*\phi+b+\epsilon} + \frac{\rho_2((1-\rho)bK+\omega R^*)}{(\lambda^*\phi+b+\epsilon)((2-\epsilon)\lambda^*+b+\rho_1+\rho_2)}\right)$$

the parameters are given by

$$A_1 = (b+\omega)((2-\epsilon)\lambda^*+b+\rho_1+\rho_2),$$

$$A_2 = \frac{\omega}{A_1}\left(\frac{\alpha_2(2-\epsilon)\lambda^*}{\delta+b+\alpha_2} + \frac{\alpha_4(\lambda^*\phi\rho_1+(b+\epsilon)\rho_1+\epsilon\rho_2)}{(b+\alpha_4)(\lambda^*\phi+b+\epsilon)} + \frac{\alpha_3\phi\lambda^*\rho_2}{(\delta+b+\alpha_3)(\lambda^*\phi+b+\epsilon)}\right),$$

$$A_3 = (1-\rho)bK((2-\epsilon)\lambda^*(b+\alpha_4)\alpha_2 + \alpha_4\rho_1(\delta+b+\alpha_2)),$$

$$A_4 = ((2-\epsilon)\lambda^*+b+\rho_1+\rho_2)(b+\alpha_4),$$

$$A_5 = \frac{bK(\alpha_3\phi\lambda^*(b+\alpha_4)+\epsilon\alpha_4(\delta+b+\alpha_3))}{(b+\omega)(\lambda^*\phi+b+\epsilon)},$$

$$A_6 = (b+\epsilon)(b+\rho_1)(\alpha_3+b+\delta)(b+\delta+\alpha_2)K.$$

If I_c^* and V_{ca}^* are substituted into (3.5), we obtain the equation in terms of λ^*:

$$\lambda_e^*(B_1{\lambda_e^*}^2 + B_2\lambda_e^* + B_3) = 0, \tag{3.7}$$

where

$$B_1 = \phi(2-\epsilon)(\alpha_3+b+\delta)(b+\delta+\alpha_2)K,$$

$$B_2 = ((b+\epsilon)(2-\epsilon)+\phi(b+\rho_1+\rho_2))(\delta+b+\alpha_3)(\delta+b+\alpha_2)K - \beta(Kb(2-\epsilon)\phi(\rho(\delta+b+\alpha_2)\eta+(1-\rho)(\delta+b+\alpha_3)) + R^*(2-\epsilon)\omega\phi(\delta+b+\alpha_3)),$$

$$B_3 = (b+\epsilon)\rho_2(\alpha_3+b+\delta)(b+\delta+\alpha_2)K - (b+\delta+\alpha_2)K\beta b\eta\phi\rho_2 - \beta\omega R^*(\eta\phi\rho_2(b+\delta+\alpha_2)+(b+\epsilon)(\alpha_3+b+\delta)(2-\epsilon)) + A_6(1-\mathcal{R}_c).$$

The roots of equation (3.7) are $\lambda^* = 0$ which corresponds to the disease free equilibrium point and

$$\lambda_e^* = \frac{-B_2+\sqrt{{B_2}^2-4B_1B_3}}{2B_1} > 0, \quad \text{when } B_3 < 0. \tag{3.8}$$

The condition $B_3 < 0$ is satisfied when $\mathcal{R}_c > 1$ which ensures positivity of λ_e^* and subsequently the positivity of E_1.

3.4. *Stability analysis*

In this section, we study the stability of disease-free (E_0) and endemic (E_1) equilibria of the system of delay differential model (2.1)–(2.6) with the initial condition (2.7).

Theorem 3.2. *The disease-free equilibrium point E_0 of system* (2.1)–(2.7) *is locally asymptotically stable for $\mathcal{R}_c < 1$ and unstable for $\mathcal{R}_c > 1$.*

Proof. The characteristic equation of the Jacobian matrix at the disease-free equilibrium E_0 of system (2.1)–(2.7) takes the form

$$P(\lambda)\left(\lambda^2 + \frac{1}{P_1}(C_4 + C_5(1-\mathcal{R}_c) + C_6 + C_7(e^{-\lambda\tau_2}(\epsilon-1)-1)e^{-\lambda\tau_1})\lambda \right.$$
$$\left. + \frac{1}{P_1}(C_8(1-\mathcal{R}_c) + C_9 + D_1 + D_2(e^{-\lambda\tau_2}\epsilon - e^{-\lambda\tau_2} - 1)e^{-\lambda\tau_1})\right) = 0, \quad (3.9)$$

where

$$P(\lambda) = (K\gamma + \lambda)(\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3),$$
$$P_1 = (b+\omega+\rho_1)(b+\epsilon)\alpha_4 + \rho_2(b+\epsilon+\omega)\alpha_4 + (b+\rho_1+\rho_2)(b+\omega)(b+\epsilon),$$
$$C_1 = 3b + \epsilon + \omega + \alpha_4 + \rho_1 + \rho_2,$$
$$C_2 = (2b+\epsilon+\omega+\rho_1+\rho_2)\alpha_4 + (2b+\epsilon+\omega)\rho_2 + 3b^2 + 2b\epsilon + 2b\omega$$
$$+ 2b\rho_1 + \epsilon\omega + \epsilon\rho_1 + \omega\rho_1,$$
$$C_3 = (b+\epsilon+\omega)\alpha_4\rho_2 + (b+\epsilon)(b+\omega+\rho_1)\alpha_4 + (b+\omega)(b+\epsilon)\rho_2$$
$$+ (b+\rho_1)(b+\omega)(b+\epsilon),$$
$$C_4 = (\epsilon+b+\omega)(2b+2\delta+\alpha_2+\alpha_3)\alpha_4\rho_2 + (b+\omega+\rho_1)(\epsilon+b)(b+\delta+\alpha_2)\alpha_4$$
$$+ (b+\omega)(\epsilon+b)(b^2 + b\delta + b\alpha_2 + b\rho_1 + 2b\rho_2 + \delta\rho_1 + 2\delta\rho_2 + \alpha_2\rho_1$$
$$+ \alpha_2\rho_2 + \alpha_3\rho_2),$$
$$C_5 = (b+\alpha_3+\delta)(\epsilon+b)(b+\omega)(b+\rho_1) + \alpha_4(b+\omega+\rho_1)(b+\alpha_3+\delta)(\epsilon+b),$$
$$C_6 = \frac{(\epsilon+b)b\beta(\epsilon-2)(b+\alpha_3+\delta)(\rho-1)(b^2+b\omega+b\alpha_4+b\rho_1+\omega\alpha_4+\omega\rho_1+\alpha_4\rho_1)}{(b+\delta+\alpha_2)(b+\rho_1)}$$
$$- \eta\phi b(b+\omega)\rho_2\beta - \eta\phi(b+\omega)\beta\alpha_4\rho_2,$$
$$C_7 = \beta(b(1-\rho)(\epsilon+b+\omega)\alpha_4 + \epsilon\omega\alpha_4 + b(1-\rho)(b+\omega)(\epsilon+b)),$$

$$C_8 = (\epsilon + b + \omega)(b + \alpha_3 + \delta)\alpha_4\rho_2 + \alpha_4(b + \omega + \rho_1)(b + \alpha_3 + \delta)(\epsilon + b)$$
$$+ (b + \rho_1 + \rho_2)(b + \omega)(\epsilon + b)(b + \alpha_3 + \delta),$$
$$C_9 = (b + \delta + \alpha_2)(b + \alpha_3 + \delta)\rho_2(b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\omega + \epsilon\alpha_4 + \omega\alpha_4)$$
$$+ \frac{(b + \alpha_3 + \delta)(\epsilon + b)(b + \omega + \rho_1)\alpha_4 b(1 - \rho)(2 - \epsilon)\beta}{b + \rho_1},$$
$$D_1 = (b + \alpha_3 + \delta)(\epsilon + b)(b + \omega)b(1 - \rho)(2 - \epsilon)\beta$$
$$- (b + \delta + \alpha_2)\eta\phi(b + \omega)\beta\rho_2(\alpha_4 + b),$$
$$D_2 = (b + \alpha_3 + \delta)(\beta b(1 - \rho)(b + \omega + \alpha_4)(\epsilon + b) + \beta(-b\omega\rho\alpha_4 + b\omega\alpha_4 + \epsilon\omega\alpha_4)).$$

The characteristic equation (3.9) has clearly one negative real root ($\lambda_4 = -\gamma K$,) and since $C_1C_2 - C_3 > 0$ the other three negative real valued roots are granted by Routh–Hurwitz criterion.

The remaining roots are given by the roots of equation (3.10)

$$g(\lambda) \equiv \lambda^2 + \frac{1}{P_1}(C_4 + C_5(1 - \mathcal{R}_c) + C_6 + C_7(e^{-\lambda\tau_2}(\epsilon - 1) - 1)e^{-\lambda\tau_1})\lambda$$
$$+ \frac{1}{P_1}(C_8(1 - \mathcal{R}_c) + C_9 + D_1 + D_2(e^{-\lambda\tau_2}\epsilon - e^{-\lambda\tau_2} - 1)e^{-\lambda\tau_1}) = 0. \tag{3.10}$$

If $\mathcal{R}_c > 1$, we can get the real λ,

$$g(0) = E_1 + E_2 + E_3(1 - \mathcal{R}_c) < 0, \quad \lim_{\lambda\to\infty} g(\lambda) = +\infty,$$

where

$$E_1 = (b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\omega + \epsilon\alpha_4 + \omega\alpha_4)(2b + 2\delta + \alpha_2 + \alpha_3)\rho_2$$
$$+ \omega\alpha_4(\epsilon - \rho_1)(b + \delta + \alpha_2)$$
$$+ \frac{(b + \epsilon)b\beta(2 - \epsilon)(b + \alpha_3 + \delta)(1 - \rho)(b^2 + b\omega + b\alpha_4 + b\rho_1 + \omega\alpha_4 + \omega\rho_1 + \alpha_4\rho_1)}{(b + \delta + \alpha_2)(b + \rho_1)},$$
$$E_2 = \frac{(b + \delta + \alpha_2)(b + \rho_1)b\eta\phi\rho\beta(b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\omega + \epsilon\alpha_4 + \omega\alpha_4)}{(b + \alpha_3 + \delta)(b + \epsilon)},$$
$$E_3 = (b + \alpha_3 + \delta)(b + \epsilon)(b + \omega)(b + \rho_1) + \alpha_4(b + \omega + \rho_1)(b + \alpha_3 + \delta)(b + \epsilon)$$
$$+ (b + \delta + \alpha_2)(b + \rho_1)(b + \epsilon + \omega)\alpha_4 + (b + \delta + \alpha_2)(b + \rho_1)(b + \omega)(b + \epsilon).$$

Therefore, Eq. (3.10) has positive real roots for $\mathcal{R}_c > 1$, hence E_0 of (2.1)–(2.7) is unstable for $\mathcal{R}_c > 1$.

For $\mathcal{R}_c < 1$ and $\tau_1 = \tau_2 = 0$, Eq. (3.10) becomes

$$g(\lambda) = \lambda^2 + \frac{(E_1 + E_2 + E_3(1 - \mathcal{R}_c))\lambda}{P_1} + \frac{E_5 + E_6 + E_4(1 - \mathcal{R}_c)}{P_1} = 0, \quad (3.11)$$

where

$$\begin{aligned}
E_4 &= (b + \alpha_3 + \delta)(b + \epsilon)(b + \delta + \alpha_2)(b + \omega + \rho_1)\alpha_4 \\
&\quad + (b + \delta + \alpha_2)(b + \rho_1)(b + \alpha_3 + \delta)(b + \epsilon)(b + \omega) \\
&\quad + (b + \rho_1)(b + \delta + \alpha_2)(b + \alpha_3 + \delta)(b + \omega + \alpha_4)(b + \epsilon), \\
E_5 &= (b + \delta + \alpha_2)(b + \alpha_3 + \delta)\rho_2(b^2 + b\epsilon + b\omega + b\alpha_4 + \epsilon\omega + \epsilon\alpha_4 + \omega\alpha_4) \\
&\quad + (b + \alpha_3 + \delta)(b + \epsilon)(b + \omega)b(1 - \rho)(2 - \epsilon)\beta \\
&\quad + (b + \rho_1)(b + \delta + \alpha_2)(b + \omega + \alpha_4)b\eta\phi\rho\beta, \\
E_6 &= \frac{\beta(2 - \epsilon)(b + \alpha_3 + \delta)\alpha_4(b(1 - \rho)(b + \rho_1)(b + \epsilon) + b\omega\rho\rho_1)}{b + \rho_1}.
\end{aligned}$$

Equation (3.11) is quadratic and for $\mathcal{R}_c < 1$

$$E_1 + E_2 + E_3(1 - \mathcal{R}_c) > 0, \quad E_5 + E_6 + E_4(1 - \mathcal{R}_c) > 0.$$

Hence, by the Routh–Hurwitz criterion, the roots of equation (3.11) have negative real parts for $\mathcal{R}_c < 1$. Therefore, when $\tau_1 = \tau_2 = 0$, the disease-free equilibrium E_0 is locally asymptotically stable if $\mathcal{R}_c < 1$ and it is unstable if $\mathcal{R}_c > 1$.

For the general nonzero delay values ($\tau_1 \neq 0$, $\tau_2 \neq 0$), we first rearrange Eq. (3.10) in the following form:

$$\begin{aligned}
&\lambda^2 + \frac{1}{P_1}(C_4 + C_5(1 - \mathcal{R}_c) + C_6 + C_7(e^{-\lambda\tau_2}(\epsilon - 1) - 1)e^{-\lambda\tau_1})\lambda \\
&\quad = -\frac{1}{P_1}(C_8(1 - \mathcal{R}_c) + C_9 + D_1 + D_2(e^{-\lambda\tau_2}\epsilon - e^{-\lambda\tau_2} - 1)e^{-\lambda\tau_1}). \quad (3.12)
\end{aligned}$$

Suppose in Eq. (3.12) λ is a real and denote the left-hand side by $L(\lambda)$ and the right-hand side by $H(\lambda)$. We can see that $L(0) = 0$ and as $\lambda \to \infty$ the value $L(\lambda)$ approaches infinity, so the left-hand side of Eq. (3.12) is an increasing function. On the other hand, $H(\lambda)$ is a decreasing function as the value of λ increases and

$$H(0) = \frac{1}{P_1}(C_8(\mathcal{R}_c - 1) + C_9 + D_1 + D_2(e^{-\lambda\tau_2}\epsilon - e^{-\lambda\tau_2} - 1)e^{-\lambda\tau_1}) > 0. \quad (3.13)$$

The two functions should intersect at a positive value λ^* which is greater than zero. Hence, Eq. (3.13) has a positive real root. Therefore, the disease-free equilibrium point, E_0 is unstable for $\mathcal{R}_c > 1$. For $\mathcal{R}_c < 1$ $L(\lambda)$ is increasing and $H(\lambda)$ decreasing but $H(0) > 0$. Thus, Eq. (3.13) has negative real roots and therefore E_0 is unstable for $\mathcal{R}_c > 1$ and stable for $\mathcal{R}_c < 1$.

For positive delays ($\tau_1 \neq 0$, $\tau_2 \neq 0$), assume that $\lambda = i\sigma$ without loss of generality, where $\sigma > 0$ is a root of equation (3.10). Substituting into Eq. (3.10) shows that

$$-\sigma^2 + \frac{1}{P_1}(i(C_5(1-\mathcal{R}_c) + C_7F_1 + C_4 + C_6)\sigma) + \frac{1}{P_1}(C_8(1-\mathcal{R}_c) + D_2F_2 + C_9 + D_1) = 0, \tag{3.14}$$

where

$$F_1 = ((\cos(\tau_2\sigma) - i\sin(\tau_2\sigma))(\epsilon - 1) - 1)(\cos(\sigma\tau_1) - i\sin(\sigma\tau_1)),$$

$$F_2 = ((\cos(\tau_2\sigma) - i\sin(\tau_2\sigma))\epsilon - \cos(\tau_2\sigma) + i\sin(\tau_2\sigma) - 1)(\cos(\sigma\tau_1) - i\sin(\sigma\tau_1)).$$

Separating the real and the imaginary parts of equation (3.14) and squaring both parts and adding the two equations, it follows that

$$\sigma^4 + F_3\sigma^2 + F_4 = 0, \tag{3.15}$$

where

$$\begin{aligned} F_3 = {} & \frac{1}{{P_1}^2}(2{C_7}^2(\epsilon - 1)(\cos(\tau_2\sigma) + 1) + (1-\mathcal{R}_c)^2{C_5}^2 \\ & + (2P_1C_8 + 2C_5(C_4 + C_6))(1-\mathcal{R}_c)) \\ & + \frac{1}{{P_1}^2}(-\epsilon^2{C_7}^2 + (2C_9 + 2D_1)P_1 + (C_4 + C_6)^2), \\ F_4 = {} & \frac{1}{{P_1}^2}(2{D_2}^2(\epsilon - 1)(\cos(\tau_2\sigma) + 1) + {C_8}^2(1-R_c)^2 \\ & + 2C_8(C_9 + D_1)(1-\mathcal{R}_c)) - \frac{1}{{P_1}^2}({D_2}^2\epsilon^2 + (C_9 + D_1)^2), \end{aligned}$$

and let assume that $\sigma^2 = Q$ and substitute into the polynomial function (3.15), we obtain

$$Q^2 + F_3Q + F_4 = 0. \tag{3.16}$$

Clearly $\cos(\sigma\tau_2) + 1 \geq 0$ for $\mathcal{R}_c < 1$

$$\begin{aligned} F_3 \geq {} & \frac{1}{P_1}((2C_9 + 2D_1)P_1 + C_4 + C_6 + (2P_1C_8 + C_5 \\ & + 2C_5(C_4 + C_6))(1-\mathcal{R}_c)) \geq 0, \\ F_4 \geq {} & \frac{1}{P_1}(C_9 + D_1 + (C_8 + 2C_8(C_9 + D_1))(1-\mathcal{R}_c)) \geq 0. \end{aligned}$$

By Routh–Hurwitz criterion Eq. (3.16) has negative real roots. Hence, our assumption $\sigma > 0$ is contradicted and (3.14) has no positive roots. Hence, Eq. (3.9) has negative real roots when $\mathcal{R}_c < 1$, E_0 is locally asymptotically stable for all $\tau > 0$. This proves the theorem. □

Theorem 3.3. *The positive equilibrium* $(S^*, I_c^*, R^*, V_v^*, V_{\text{ca}}^*)$ *is globally asymptotically stable when* $\mathcal{R}_c > 1$.

Proof. We use Lyapunov functions to prove the endemic equilibrium is globally asymptotically stable. Let $(S, I_c, R, V_v, V_{\text{ca}})$ be a positive solution of system (2.1)–(2.6) with initial conditions (2.7). To find the Lyapunov function, we used logarithmic functions [22, 38, 39].

$$U_S(t) = \frac{S(t)}{S^*} - 1 - \ln\left(\frac{S(t)}{S^*}\right), \quad U_{T_v}(t) = \frac{T_v(t)}{T_v^*} - 1 - \ln\left(\frac{T_v(t)}{T_v^*}\right),$$

$$U_{I_c}(t) = \frac{I_c(t)}{I_c^*} - 1 - \ln\left(\frac{I_c(t)}{I_c^*}\right), \quad U_R(t) = \frac{R(t)}{R^*} - 1 - \ln\left(\frac{R(t)}{R^*}\right),$$

$$U_{V_v(t)} = \frac{V_v(t)}{V_v^*} - 1 - \ln\left(\frac{V_v(t)}{V_v^*}\right), \quad U_{V_{\text{ca}}(t)} = \frac{V_{\text{ca}}(t)}{V_{\text{ca}}^*} - 1 - \ln\left(\frac{V_{\text{ca}}(t)}{V_{\text{ca}}^*}\right),$$

$$U_{S+}(t) = \int_{\tau=0}^{h} \left(\frac{S(t-\tau_1)}{S^*} - 1 - \ln\left(\frac{S(t-\tau_1)}{S^*}\right)\right) d\tau,$$

$$U_{I_c+}(t) = \int_{\tau=0}^{h} \left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right) d\tau.$$

Hence, we consider the following:

$$U(t) = U_S(t) + U_{I_c}(t) + U_R(t) + U_{V_v(t)} + U_{V_{ca}(t)} + U_+(t).$$

We calculate the derivatives of $U_S(t), U_{I_c}(t), U_R(t), U_{V_v(t)}, U_{V_{\text{ca}}(t)}$ and $U_+(t)$ separately and combine to get the derivative of the desired Lyapunov function

$$\begin{aligned}
\frac{dU_S(t)}{dt} &= \left(1 - \frac{S^*}{S}\right) \frac{dS(t)}{dt} \\
&= \left((1-\rho)bN - \frac{\beta(I_c(t-\tau_1) + \eta V_{\text{ca}})S}{N}\right. \\
&\quad \left. - \frac{(1-\varepsilon)\beta(I_c(t-\tau_1) + \eta V_{\text{ca}})S(t-\tau_2)}{N}\right)\left(1 - \frac{S^*}{S}\right) \\
&\quad - ((\gamma N + \mu + \rho_1 + \rho_2)S + \omega R)\left(1 - \frac{S^*}{S}\right).
\end{aligned}$$

Using the endemic equilibrium (3.6), we get the

$$\begin{aligned}
\frac{dU_S(t)}{dt} &= \left(1 - \frac{S^*}{S}\right) \frac{dS(t)}{dt} \\
&= \left(1 - \frac{S^*}{S}\right)\left(\frac{\beta(\eta V_{\text{ca}}^* + I_c^* t)S^*}{N^{*2}} + \frac{(1-\varepsilon)\,\beta(\eta v_{\text{ca}} + I_c^*)S^*}{N^{*2}}\right) \\
&\quad + \left(1 - \frac{S^*}{S}\right)\left(\frac{(\gamma N^* + \mu + \rho_1 + \rho_2)S^*}{N^*} - \frac{\omega R^*}{N^*}\right) - \left(1 - \frac{S^*}{S}\right)
\end{aligned}$$

$$\times\left(\frac{\beta(I_c(t-\tau_1)+\eta V_{\mathrm{ca}})S}{N}+\frac{(1-\varepsilon)\beta(I_c(t-\tau_1)+\eta V_{\mathrm{ca}})S(t-\tau_2)}{N}\right)$$
$$-\left(1-\frac{S^*}{S}\right)((\gamma N+\mu+\rho_1+\rho_2)S+\omega R).$$

After some calculations, we obtain

$$\begin{aligned}\frac{dU_S(t)}{dt}&=h_2S^*N\left(2-\frac{h_1S}{h_2S^*N}-\frac{h_2S^*N}{h_1S}\right)\left(1-\frac{S^*}{S}\right)\\&+(1-\varepsilon)h_2S^*N\left(2-\frac{h_1S(t-\tau_2)}{h_2S^*N}-\frac{h_2S^*N}{h_1S(t-\tau_2)}\right)\left(1-\frac{S^*}{S}\right)\\&+\left(\frac{(\rho_1+\rho_2+\mu)S^*N}{n}\left(2-\frac{SN^*}{S^*N}-\frac{S^*N}{SN^*}\right)+\gamma S^*N\left(2-\frac{S}{S^*}-\frac{S^*}{S}\right)\right)\\&\times\left(1-\frac{S^*}{S}\right)+\omega R\left(2-\frac{rN}{N^*R}-\frac{N^*R}{R^*N}\right)\left(1-\frac{S^*}{S}\right)\\&-\frac{{h_2}^2S^{*2}N^2}{h_1S}\left(\frac{h_1S}{h_2S^*N}-1-\ln\left(\frac{h_1S}{h_2S^*N}\right)\right)\left(1-\frac{S^*}{S}\right)\\&-\frac{(1-\varepsilon)\,{h_2}^2S^{*2}N^2}{h_1S\,(t-\tau_2)}\left(\frac{h_1S(t-\tau_2)}{h_2S^*N}-1-\ln\left(\frac{h_1S(t-\tau_2)}{h_2S^*N}\right)\right)\left(1-\frac{S^*}{S}\right)\\&-\frac{(\rho_1+\rho_2+\mu)\,{S^*}^2N^2}{{N^*}^2S}\left(\frac{SN^*}{S^*N}-1-\ln\left(\frac{SN^*}{S^*N}\right)\right)\left(1-\frac{S^*}{S}\right)\\&-\left(\frac{\gamma{S^*}^2N}{S}\left(\frac{S}{S^*}-1-\ln\left(\frac{S}{S^*}\right)\right)+\frac{\omega R^2N^*}{R^*N}\left(\frac{R^*N}{N^*R}-1-\ln\left(\frac{R^*N}{N^*R}\right)\right)\right)\\&\times\left(1-\frac{S^*}{S}\right)-\left(\frac{{h_2}^2S^{*2}N^2}{h_1S}\ln\left(\frac{h_1S}{h_2S^*N}\right)+\frac{(1-\varepsilon)\,{h_2}^2S^{*2}N^2}{h_1S\,(t-\tau_2)}\right.\\&\left.\times\ln\left(\frac{h_1S(t-\tau_2)}{h_2S^*N}\right)\right)\left(1-\frac{S^*}{S}\right)-\left(\frac{(\rho_1+\rho_2+\mu)\,{S^*}^2N^2}{{N^*}^2S}\ln\left(\frac{SN^*}{S^*N}\right)\right.\\&\left.+\frac{\gamma{S^*}^2N}{S}\ln\left(\frac{S}{S^*}\right)+\frac{\omega R^2N^*}{rN}\ln\left(\frac{rN}{N^*R}\right)\right)\left(1-\frac{S^*}{S}\right)\le 0,\end{aligned}$$

where

$$h_1=\frac{\beta(I_c(t-\tau_2)+\eta V_{\mathrm{ca}})}{N},\quad h_2=\frac{\beta(I_c^*+\eta V_{\mathrm{ca}}^*)}{{N^*}^2},\quad m_1=\frac{h_2S^*N}{h_1S},$$

and similarly

$$\begin{aligned}\frac{dU_{T_v(t)}}{dt}&=\left(\rho_1S\left(2-\frac{S^*T_v}{ST_v^*}-\frac{ST_v^*}{S^*T_v}\right)+\varepsilon V_v\left(2-\frac{V_v^*T_v}{V_vT_v^*}-\frac{V_vT_v^*}{V_v^*T_v}\right)\right)\left(1-\frac{T_v^*}{T_v}\right)\\&+\gamma n\left(2-\frac{N}{N^*}-\frac{N^*}{N}\right)T_v\left(1-\frac{T_v^*}{T_v}\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\rho_1 S^2 T_v^*}{S^* T_v}\left(\frac{S^* T_v}{S T_v^*} - 1 - \ln\left(\frac{S^* T_v}{S T_v^*}\right)\right)\left(1 - \frac{T_v^*}{T_v}\right)\\
&- \frac{\rho_1 S^2 T_v^*}{S^* T_v}\ln\left(\frac{S^* T_v}{S T_v^*}\right)\left(1 - \frac{T_v^*}{T_v}\right) - \frac{\varepsilon {V_v}^2 T_v^*}{V_v^* T_v}\left(\frac{V_v^* T_v}{V_v T_v^*} - 1\right)\left(1 - \frac{T_v^*}{T_v}\right)\\
&- \frac{\gamma {N^*}^2 T_v}{N}\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right)\left(1 - \frac{T_v^*}{T_v}\right)\\
&- \frac{\gamma {N^*}^2 T_v}{N}\ln\left(\frac{N}{N^*}\right)\left(1 - \frac{T_v^*}{T_v}\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dU_{I_c}(t)}{dt} = &\left(h_1 S\left(2 - \frac{h_2 S^* N^* I_c}{I_c^* h_1 S} - \frac{I_c^* h_1 S}{h_2 S^* N^* I_c}\right)\right.\\
&\left. + \gamma N^* I_c\left(2 - \frac{N}{N^*} - \frac{N^*}{N}\right)\right)\left(1 - \frac{I_c^*}{I_c}\right) + (1-\varepsilon)h_1 S(t-\tau_2)\\
&\times\left(2 - \frac{h_2 S^* N^* I_c}{I_c^* h_1 S(t-\tau_2)} - \frac{I_c^* h_1 S(t-\tau_2)}{h_2 S^* N^* I_c}\right)\left(1 - \frac{I_c^*}{I_c}\right)\\
&- \frac{{h_1}^2 S^2 I_c^*}{h_2 S^* N^* I_c}\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S} - 1 - \ln\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S}\right)\right)\left(1 - \frac{I_c^*}{Ic}\right)\\
&- \frac{(1-\varepsilon){h_1}^2 (S(t-\tau_2))^2 I_c^*}{h_2 S^* N^* I_c}\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S(t-\tau_2)} - 1\right.\\
&\left. - \ln\left(\frac{h_2 S^* N^* Ic}{I_c^* h_1 S(t-\tau_2)}\right)\right)\left(1 - \frac{I_c^*}{I_c}\right) - \left(\frac{\gamma {N^*}^2 I_c}{N}\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right)\right.\\
&\left. + \frac{{h_1}^2 S^2 I_c^*}{h_2 S^* N^* I_c}\ln\left(\frac{h_2 S^* N^* I_c}{I_c^* h_1 S}\right)\right)\left(1 - \frac{I_c^*}{I_c}\right) - \frac{(1-\varepsilon){h_1}^2 (S(t-\tau_2))^2 I_c^*}{h_2 s N^* I_c}\\
&\times\ln\left(\frac{h_2 s N^* I_c}{I_c^* h_1 S(t-\tau_2)}\right)\left(1 - \frac{I_c^*}{I_c}\right) - \frac{\gamma {N^*}^2 I_c}{N}\ln\left(\frac{N}{N^*}\right)\left(1 - \frac{I_c^*}{I_c}\right) \le 0,
\end{aligned}
$$

$$
\begin{aligned}
\frac{dU_R(t)}{dt} = &-\left(\alpha_2 I_c\left(\frac{I_c^* R}{I_c R^*} - 1 - \ln\left(\frac{I_c^* R}{I_c R^*}\right)\right) + \gamma N^* R\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right)\right)\\
&\times\left(1 - \frac{R^*}{R}\right) - \left(\alpha_3 V_{ca}\left(\frac{V_{\mathrm{ca}}^* R}{V_{\mathrm{ca}} R^*} - 1 - \ln\left(\frac{V_{\mathrm{ca}}^* R}{V_{\mathrm{ca}} R^*}\right)\right)\right.\\
&\left. + T_v \alpha_4\left(\frac{T_v^* R}{T_v R^*} - 1 - \ln\left(\frac{T_v^* R}{T_v R^*}\right)\right)\right)\left(1 - \frac{r}{R}\right)\\
&- \left(\alpha_2 I_c \ln\left(\frac{I_c^* R}{I_c R^*}\right) + \gamma N^* R \ln\left(\frac{N}{N^*}\right) + \alpha_3 V_{\mathrm{ca}} \ln\left(\frac{V_{\mathrm{ca}}^* R}{V_{\mathrm{ca}} R^*}\right)\right.\\
&\left. + T_v \alpha_4 \ln\left(\frac{T_v^* R}{T_v R^*}\right)\right)\left(1 - \frac{R^*}{R}\right),
\end{aligned}
$$

$$\frac{dU_{V_v}(t)}{dt} = \left(h_2\phi V_v^* N\left(2 - \frac{h_1 V_v^*}{h_2 V_v^* N} - \frac{h_2 V_v^* N}{h_1 V_v}\right)\right.$$
$$\left. + \frac{(\varepsilon+\mu)\, V_v^* N}{N^*}\left(2 - \frac{V_v N^*}{V_v^* N} - \frac{V_v^* N}{V_v N^*}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right)$$
$$- \frac{{h_2}^2\phi {V_v^*}^2 N^2}{h_1 V_v}\left(\frac{h_1 V_v}{h_2 v_v N} - 1 - \ln\left(\frac{h_1 V_v}{h_2 V_v^* N}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right)$$
$$- \frac{\gamma N(V_v^* - V_v)^2}{V_v} - \frac{(\varepsilon+\mu){V_v^*}^2 N^2}{{N^*}^2 V_v}\left(\frac{V_v N^*}{V_v^* N} - 1 - \ln\left(\frac{V_v N^*}{V_v^* N}\right)\right)$$
$$\times\left(1 - \frac{V_v^*}{V_v}\right) - \rho_2 S\left(\frac{N}{SN^*} - 1 - \ln\left(\frac{S^* N}{SN^*}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right)$$
$$- \left(\frac{{h_2}^2\phi {V_v^*}^2 N^2}{h_1 V_v}\ln\left(\frac{h_1 V_v}{h_2 V_v^* N}\right) + \frac{(\varepsilon+\mu)\,{V_v^*}^2 N^2}{{N^*}^2 V_v}\ln\left(\frac{V_v N^*}{V_v^* N}\right)\right.$$
$$\left. + \rho_2 S\ln\left(\frac{S^* N}{SN^*}\right)\right)\left(1 - \frac{V_v^*}{V_v}\right),$$

$$\frac{dU_{V_{\mathrm{ca}}}(t)}{dt} = h_1\phi V_v\left(2 - \frac{h_2 V_v^* V_{\mathrm{ca}} N^*}{V_{\mathrm{ca}}^* h_1 V_v} - \frac{V_{\mathrm{ca}}^* h_1 V_v}{h_2 V_v^* V_{\mathrm{ca}} N^*}\right)\left(1 - \frac{V_{\mathrm{ca}}^*}{V_{\mathrm{ca}}}\right)$$
$$- \gamma V_{\mathrm{ca}} N^*\ln\left(\frac{N}{N^*}\right)\left(1 - \frac{V_{\mathrm{ca}}^*}{V_{\mathrm{ca}}}\right) - \frac{{h_1}^2\phi {V_v}^2 V_{\mathrm{ca}}^*}{h_2 V_v^* V_{\mathrm{ca}} N^*}\left(\frac{h_2 V_v^* V_{\mathrm{ca}} N^*}{V_{\mathrm{ca}}^* h_1 V_v} - 1\right.$$
$$\left. - \ln\left(\frac{h_2 V_v^* V_{\mathrm{ca}} N^*}{V_{\mathrm{ca}}^* h_1 V_v}\right)\right)\left(1 - \frac{V_{\mathrm{ca}}^*}{V_{\mathrm{ca}}}\right) - \frac{{h_1}^2\phi {V_v}^2 V_{\mathrm{ca}}^*}{h_2 V_v^* V_{\mathrm{ca}} N^*}\ln\left(\frac{h_2 V_v^* V_{\mathrm{ca}} N^*}{V_{\mathrm{ca}}^* h_1 V_v}\right)$$
$$\times\left(1 - \frac{v_{\mathrm{ca}}}{V_{\mathrm{ca}}}\right) - \gamma V_{\mathrm{ca}} N^*\left(\frac{N}{N^*} - 1 - \ln\left(\frac{N}{N^*}\right)\right)\left(1 - \frac{V_{\mathrm{ca}}^*}{V_{\mathrm{ca}}}\right),$$

$$\frac{dU_{I_c+}(t)}{dt} = \frac{d}{dt}\left(\int_{\tau=0}^{h}\left(\frac{I_c(t-\tau_1)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_1)}{I_c^*}\right)\right)d\tau\right)$$
$$= \int_{\tau_1=0}^{h}\frac{d}{dt}\left(\left(\frac{I_c(t-\tau_1)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_1)}{I_c^*}\right)\right)d\tau\right)$$
$$= -\int_{\tau_1=0}^{\infty}\frac{d}{d\tau_1}\left(\left(\frac{I_c(t-\tau_1)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_1)}{I_c^*}\right)\right)d\tau_1\right),$$

$$\frac{dU_{S+}(t)}{dt} = \frac{d}{dt}\left(\int_{\tau_2=0}^{h}\left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right)d\tau\right)$$
$$= \int_{\tau=0}^{h}\frac{d}{dt}\left(\left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right)d\tau\right)$$
$$= -\int_{\tau_2=0}^{\infty}\frac{d}{d\tau_2}\left(\left(\frac{I_c(t-\tau_2)}{I_c^*} - 1 - \ln\left(\frac{I_c(t-\tau_2)}{I_c^*}\right)\right)d\tau_2\right).$$

The Lyapunov derivative of the function is

$$\frac{dU(t)}{dt} = \frac{dU_S(t)}{dt} + \frac{dU_{T_v}(t)}{dt} + \frac{dU_{I_c}(t)}{dt} + \frac{dU_R(t)}{dt} + \frac{dU_{V_v}(t)}{dt} + \frac{dU_{V_{\text{ca}}}(t)}{dt} + \frac{dU_{s+}(t)}{dt} + \frac{dU_{I_c+}(t)}{dt} \leq 0.$$

Using arithmetic and geometric principles, we establish that $\frac{dU(t)}{dt}$ is negative or $\frac{dU(t)}{dt} = 0$ when $S = S^*$, $T_v = T_v^*$, $I_c = I_c^*$, $R = R^*$, $V_v = V_v^*$ and $V_{\text{ca}} = V_{\text{ca}}^*$. Thus, the endemic equilibrium is globally asymptotically stable by LaSalle's invariant principle [19]. □

4. Parameter Estimation and Numerical Simulation

In this section, we present the parameter values for the model (2.1)–(2.6) from the relevant literature. We use the parameter values for numerical simulations that will assist understanding the model predictions. We give simulations to show the effects of time delay on the dynamics of the FMD. The initial number of susceptible animals is 200 animals per km^2 with one infected animal [21].

4.1. *Parameter estimation*

The per-capita death rate γ is estimated using the carrying capacity, natural birth rate and the natural death rate, $\gamma = (b - \mu)/200$. The amplification rate η and vaccination rate ρ are estimated by $0 \leq \eta \leq 1$ to be $0 \leq \rho \leq 1$, respectively. The minimum and maximum values of vaccination rate are 40% and 75% [14]. All other parameter values used in the numerical simulations are given in Table 1 with their

Table 1. Dimensional parameter values for model.

Parameter description	Symbol	Units	Value	Source
Transmission rate	β	Day^{-1}	1.4	[26, 32]
Birth rate	b	Day^{-1}	0.3	[40]
Transforming rate from I_s to R	α_2	Day^{-1}	0.1	[6, 21, 40]
The vaccinated carrier rate constant	α_3	Day^{-1}	0.2	[9]
The recovery rate constant	ω	Day^{-1}	[0.01, 0.2]	[13, 19, 30]
The rate of protection loss	ϕ	Day^{-1}	[0, 1]	[13]
Delay one	τ_1	Day^{-1}	[1,6]	[13]
Delay two	τ_2	Day^{-1}	[1, 14]	[13, 21]
Natural death rate	μ	Day^{-1}	0.0324	[40]
Treating and vaccinating of susceptible animals	ρ_1	Day^{-1}	0.5	[6, 21, 40]
Culling of clinical infective and vaccinated carrier animals	δ	Day^{-1}	[0.01, 0.5]	[12, 13, 32]
Treating of vaccinated animals	ϵ	Day^{-1}	0.1	[19, 29]
The rate of Vaccinating susceptible animals	ρ_2	Day^{-1}	0.1	Estimate
The rate recovery of treating animals	α_4	Day^{-1}	0.1	Estimate

sources. Some parameter values are taken as they appear in literature while others are determined based on estimating the given parameters using in literature.

4.2. *Numerical simulations*

In this section, we present the numerical simulations to further enhance our understanding of the model (2.1)–(2.6) and to explore the effects of prophylactic and reactive vaccination, prophylactic treatment and reactive culling of infected animals. We first examine the effects of different control strategies which are the rate of vaccination (ρ), the rate of treating and vaccinating susceptible animals (ρ_1), the rate of treating of susceptible animals (ρ_2), the rate of treating vaccinated animals (ϵ) and the rate of culling infected and vaccinated carrier animals (δ). Using the least and high rates of control strategies, we investigate the effects of prophylactic vaccination and treatment and culling of infected animals using a two-time delay FMD model.

The effect of the rate of vaccination (ρ) on the disease dynamics is shown in Fig. 3. Increasing the rate of vaccination leads most animals entering to the treated and vaccinated of susceptible, and to vaccinated animal classes. The implication of the rate of treating and vaccinating of susceptible animals on the dynamics of the disease using a system of ordinary differential equations is shown in Fig. 4. Increasing the rate of treating and vaccinating of susceptible animals results in most animals entering the vaccinated class which is a protected class. The effect of the rate of vaccinating susceptible animals on a system of ordinary differential equations is shown in Fig. 5. The effect of the rate of treating vaccinated animals on a system of ordinary differential equations is shown in Fig. 6. Increasing the rate of treating vaccinated animals increases the susceptible, and treated and vaccinated classes and decreases the other classes. Figure 7 shows the increasing of the rate of culling infected and vaccinated carrier animals decreases the infected class essential vaccinated carrier and theoretically increase the flow of animals into the susceptible animals, the treated and vaccinated, and vaccinated classes.

Therefore, increasing the rate of vaccination, the rate of treating and vaccination of susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and vaccinated carrier animals results decrease the FMD burden.

The scenarios of varying the two-time delays by fixing the control parameters as least or high rates are explored in Figs. 7–22 by varying time delays we seek to investigate the effects of timing the development of clinical symptom after the animals have contracted the infection.

Subfigures (a) and (c) of Figs. 8–19, shows that, fixing of $\rho = 0.3$ or $\rho_1 = 0.1$ or $\rho_2 = 0.1$ or $\epsilon = 0.3$ and increasing τ_2 increases susceptible, treated and vaccinated classes while decreases the clinically infected, recovered and vaccinated carrier classes. The decrease in τ_2 leads to the decrease in susceptible, treated and vaccinated, and vaccinated and increase in clinically infected, recovered and

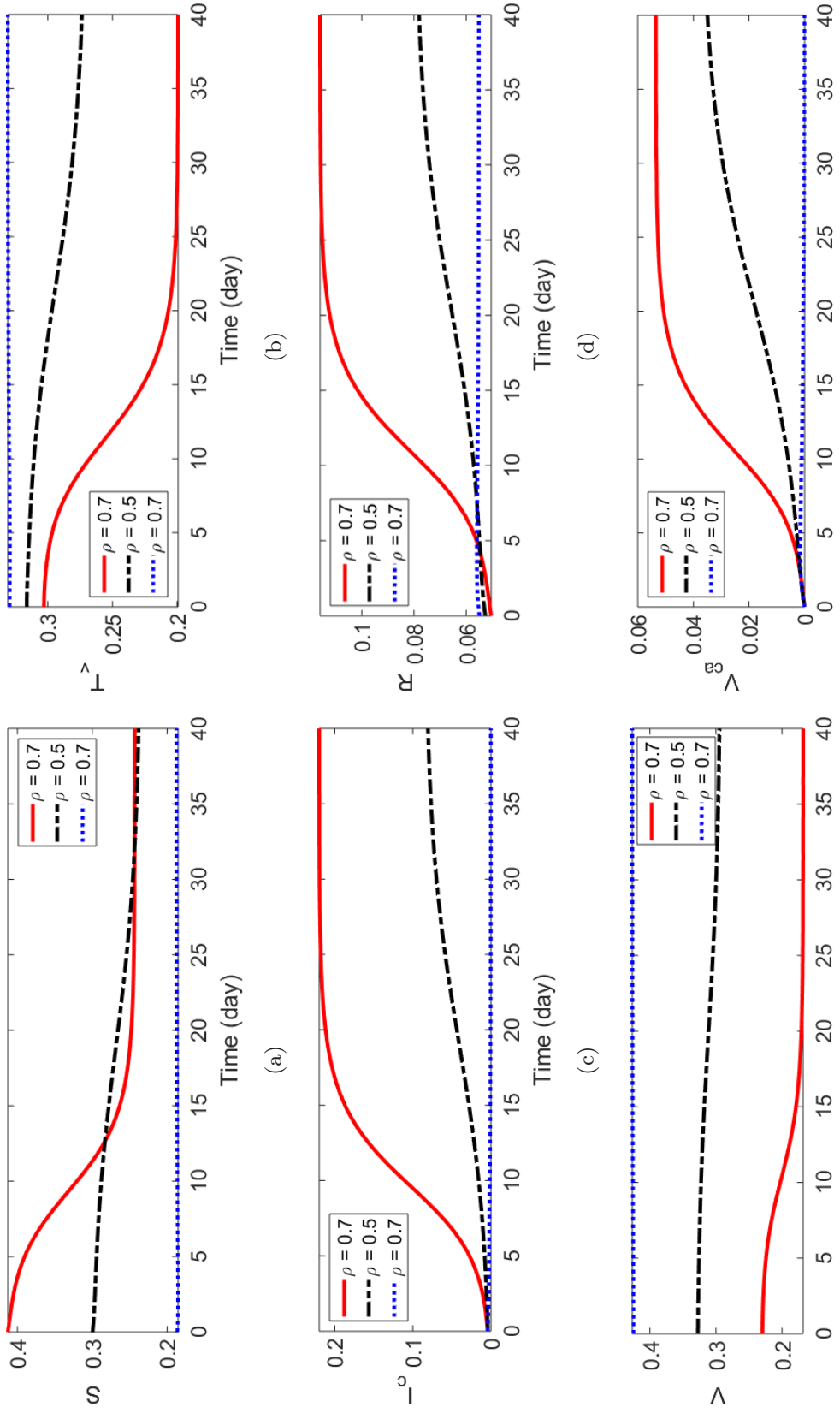

Fig. 3. The effects of the rate of vaccination (ρ) on the dynamics of the disease.

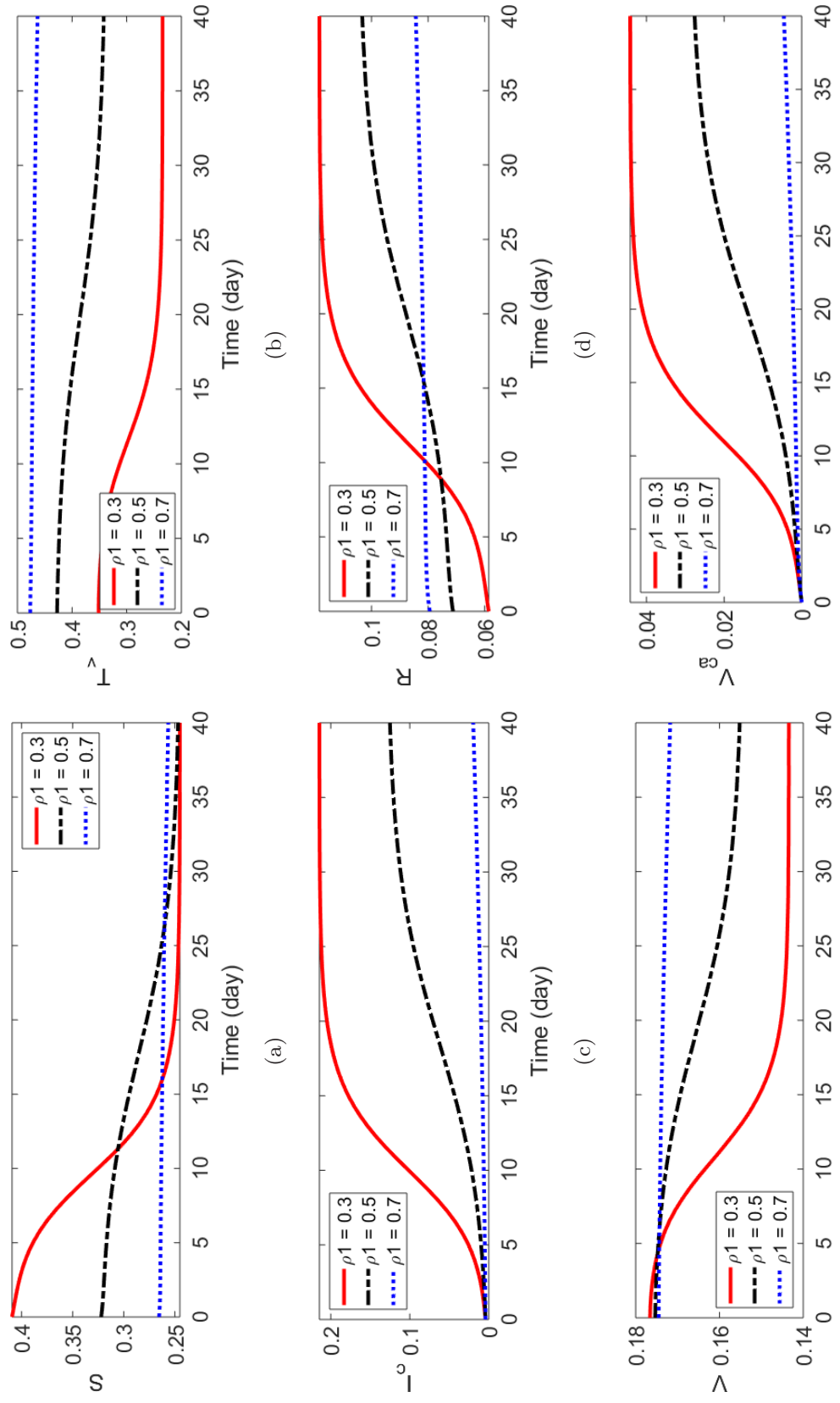

Fig. 4. The effects of treating and vaccinating susceptible animals (ρ_1) on the dynamics of the disease.

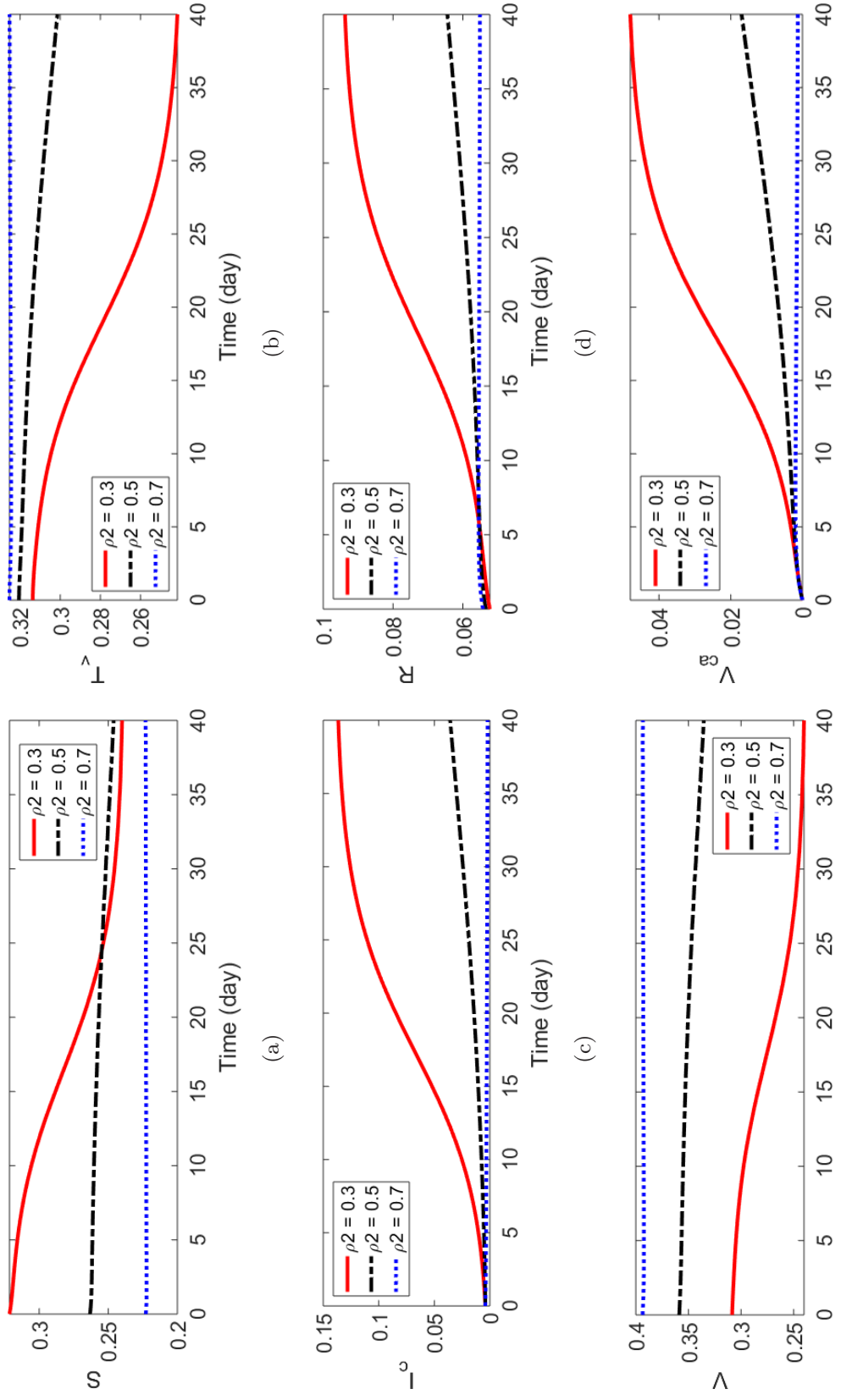

Fig. 5. The effects of vaccinating susceptible animals (ρ_2) on the dynamics of the disease.

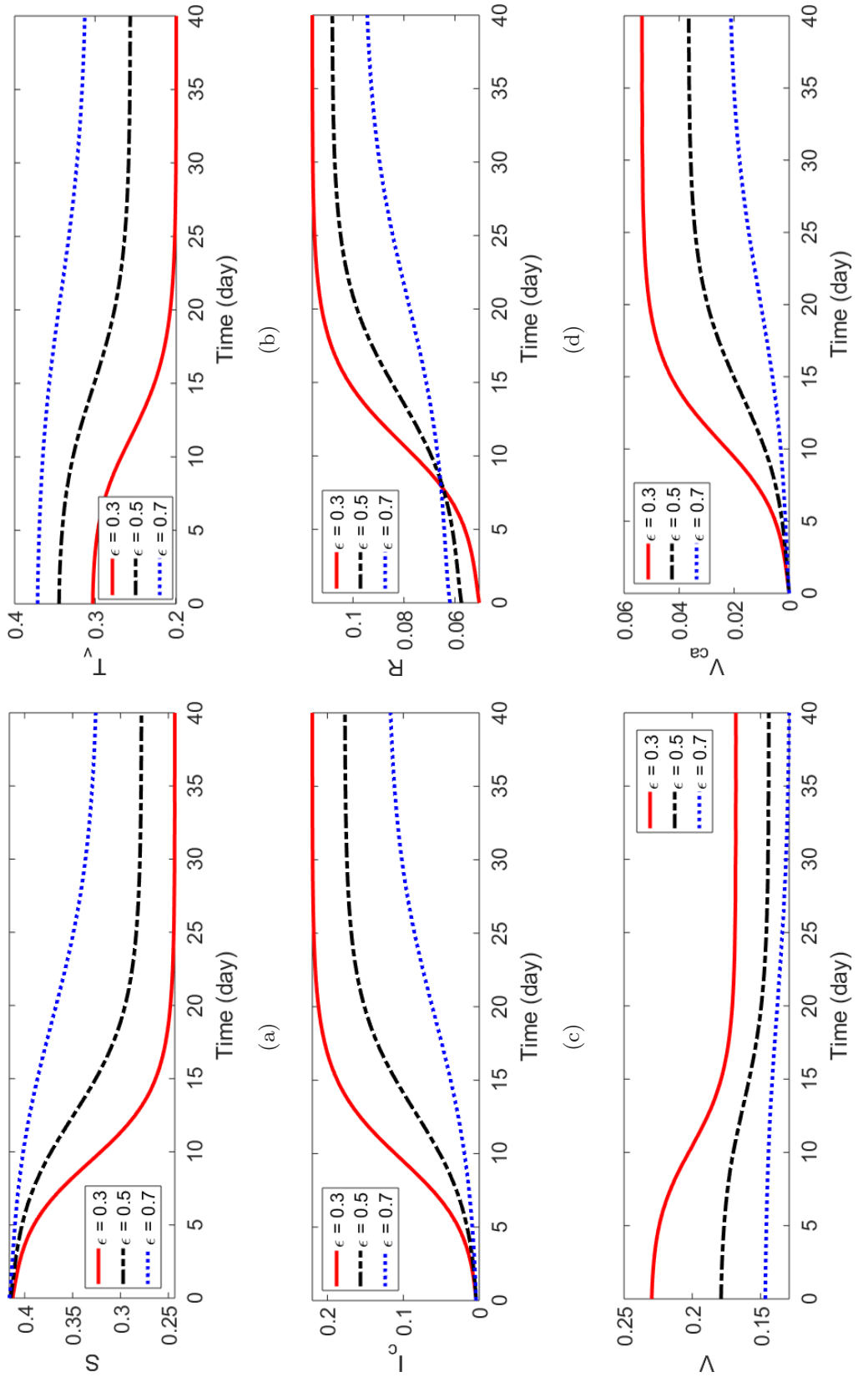

Fig. 6. The effects of treating vaccinated animals (ϵ).

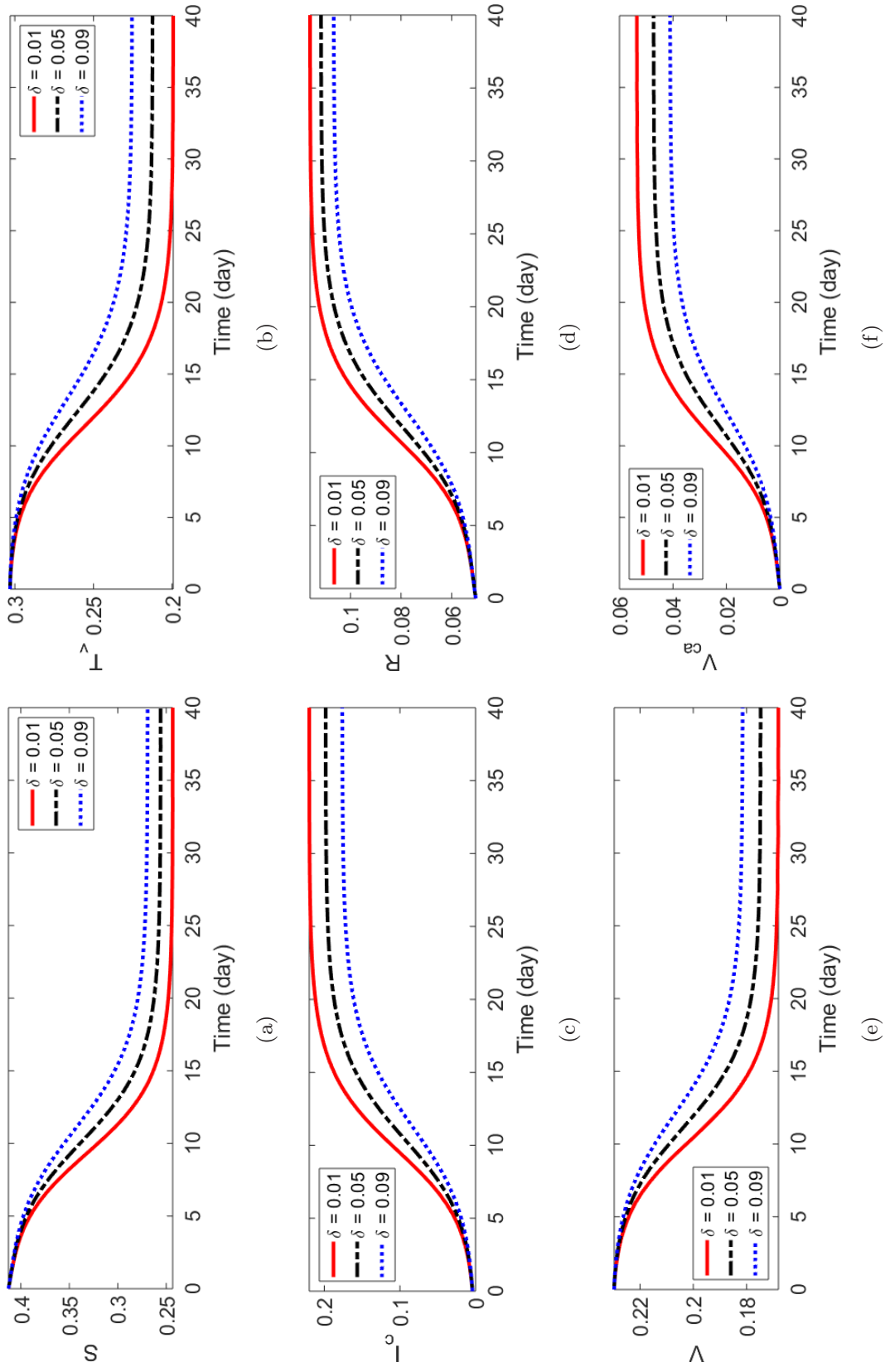

Fig. 7. The effects of culling infected animals (δ) on system of solutions.

vaccinated carrier classes. Varying of τ_1 seems to have no significant effect on the dynamic of the disease.

Subfigures (b) and (d) of Figs. 8–19, shows that, fixing of $\rho = 0.7$ or $\rho_1 = 0.3$ or $\rho_2 = 0.3$ or $\epsilon = 0.7$ and increasing τ_2 increases susceptible, treated and vaccinated classes and decreases clinically infected, recovered and vaccinated carrier classes. Decreasing of τ_2 leads to decrease the susceptible, treated and vaccinated classes and increases clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 does not have any significant effect on the dynamic of the disease.

Subfigures (a) and (c) of Figs. 20–22, shows that, fixing of $\delta = 0.01$ and increasing τ_2 increases susceptible, treated and vaccinated classes while decreases the clinically infected, recovered and vaccinated carrier classes. The decrease in τ_2 leads to the decrease in susceptible, treated and vaccinated, and vaccinated and increase in clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 seems to have no significant effect on the dynamic of the disease.

Subfigures (b) and (d) of Figs. 20–22, show that, fixing of $\delta = 0.09$, and increasing τ_2 increases susceptible, treated and vaccinated classes and decreases clinically infected, recovered and vaccinated carrier classes. Decreasing of τ_2 leads to decrease the susceptible, treated and vaccinated classes and increases clinically infected, recovered and vaccinated carrier classes. Varying of τ_1 does not have any significant effect on the dynamic of the disease.

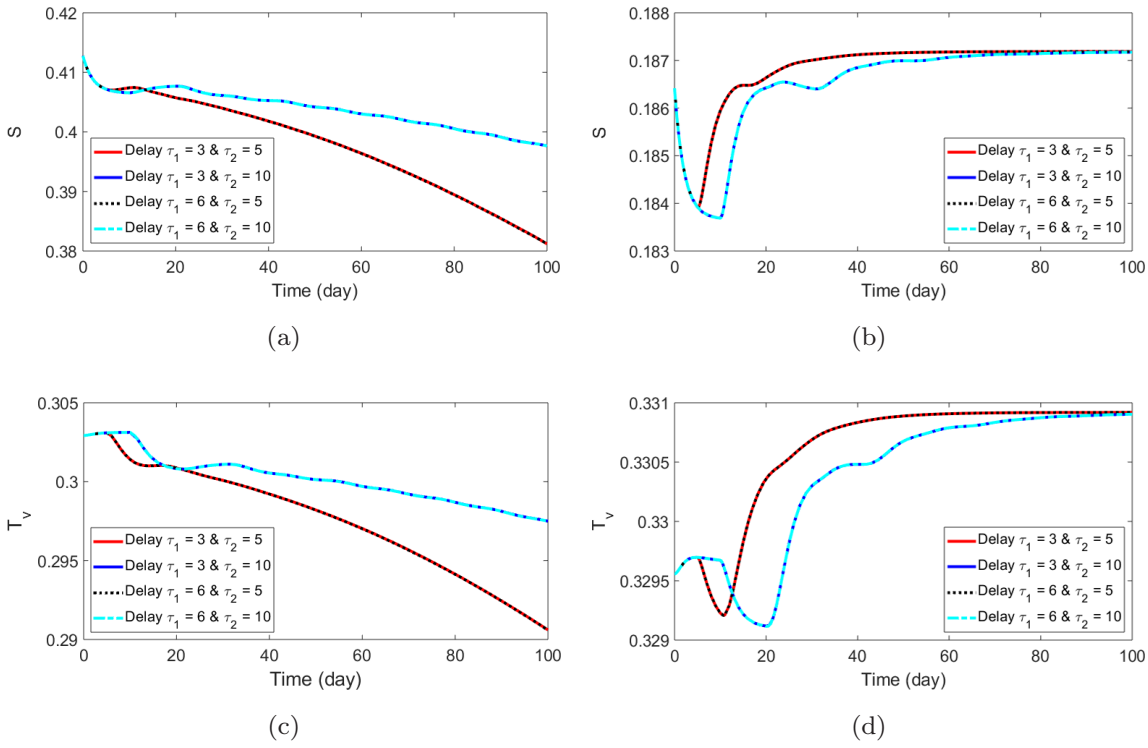

Fig. 8. The effects of low ($\rho = 0.3$) and maximum ($\rho = 0.7$) rate of vaccination (ρ) on susceptible and treated animal classes with time delays. (a) and (c) Low vaccinatin rate (ρ) for time delay. (b) and (d) High vaccinatin rate (ρ) for time delay.

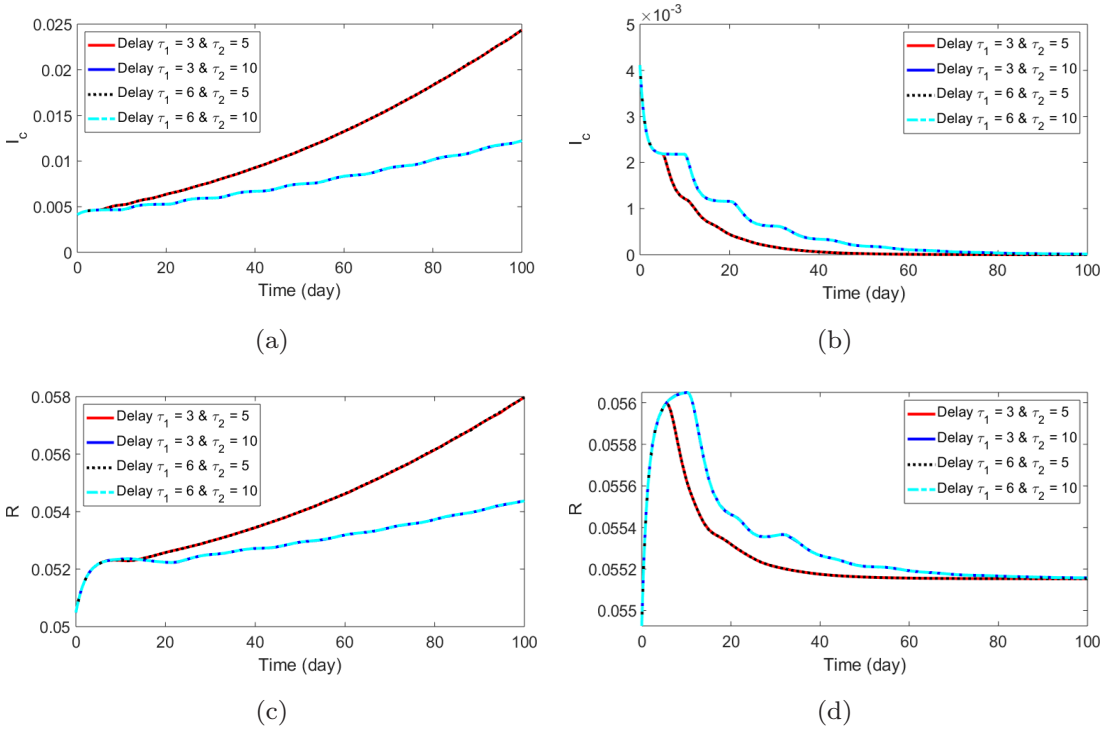

Fig. 9. The effects of low ($\rho = 0.3$) and maximum ($\rho = 0.7$) rate vaccination (ρ) on clinically infected and recovery animal classes with time delays. (a) and (c) Low vaccinatin rate (ρ) for time delay. (b) and (d) High vaccinatin rate (ρ) for time delay.

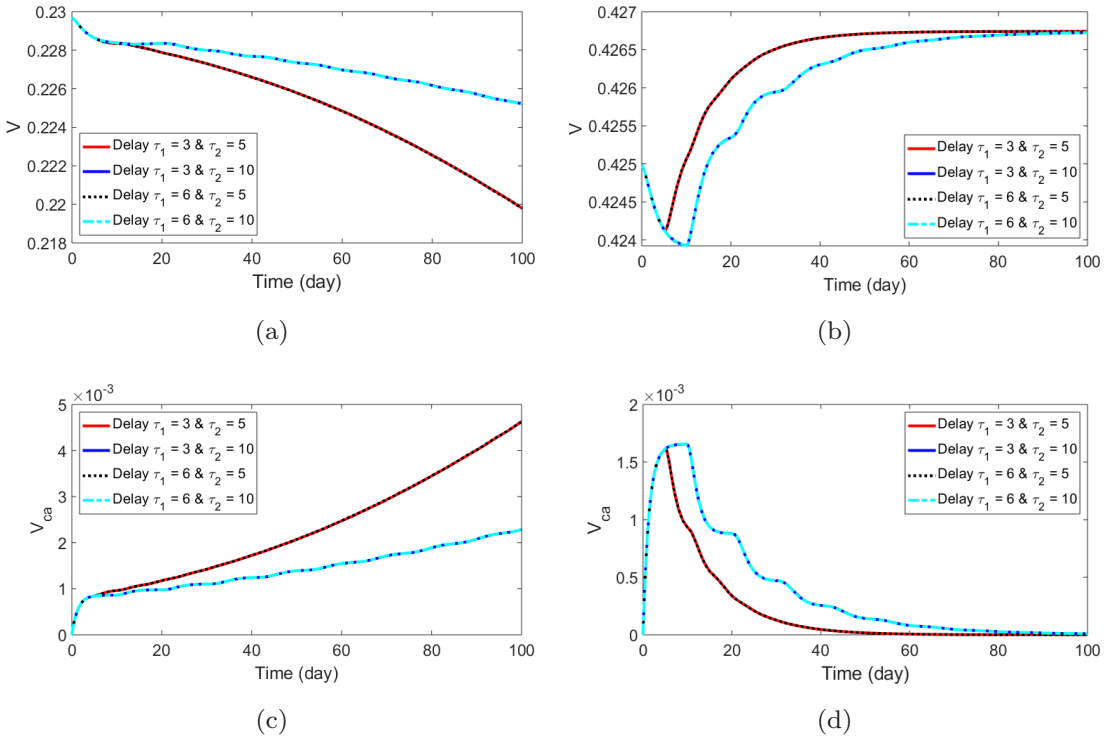

Fig. 10. The effects of low ($\rho = 0.3$) and maximum ($\rho = 0.7$) rate of vaccination (ρ) on vaccinated and vaccinated carrier animal classes with time delays. (a) and (c) Low vaccinatin rate (ρ) for time delay. (b) and (d) High vaccinatin rate (ρ) for time delay.

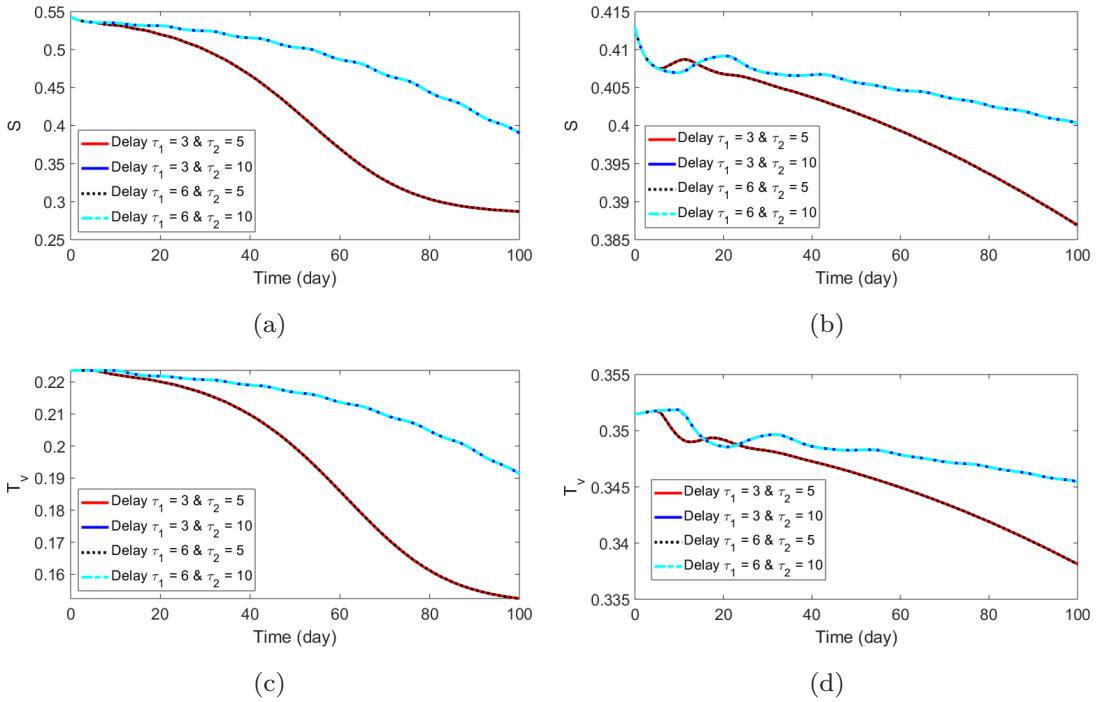

Fig. 11. The effects of low ($\rho_1 = 0.1$) and maximum ($\rho_1 = 0.3$) rate of treating susceptible animals (ρ_1) on susceptible and treated animal classes with time delays. (a) and (c) Low treating and vaccinating susceptible animals (ρ_1). (b) and (d) High treating and vaccinating susceptible animals (ρ_1).

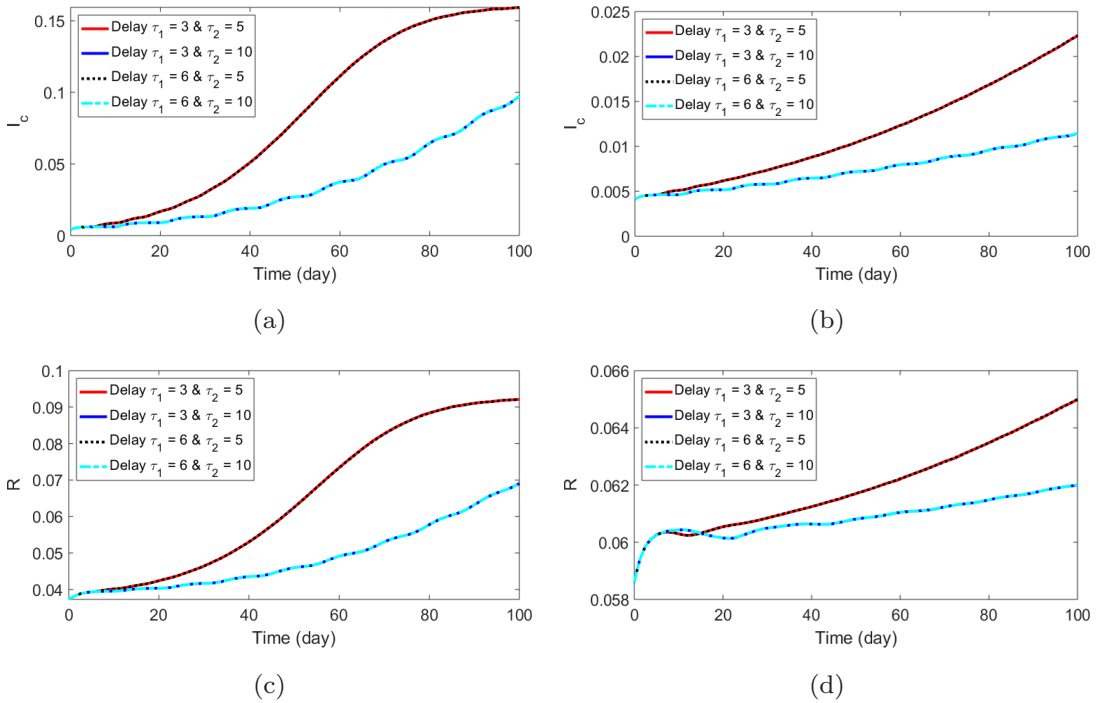

Fig. 12. The effects of low ($\rho_1 = 0.1$) and maximum ($\rho_1 = 0.3$) rate of treating susceptible animals (ρ_1) on susceptible and treated animal classes with time delays. (a) and (c) Low treating and vaccinating susceptible animals (ρ_1). (b) and (d) High treating and vaccinating susceptible animals (ρ_1).

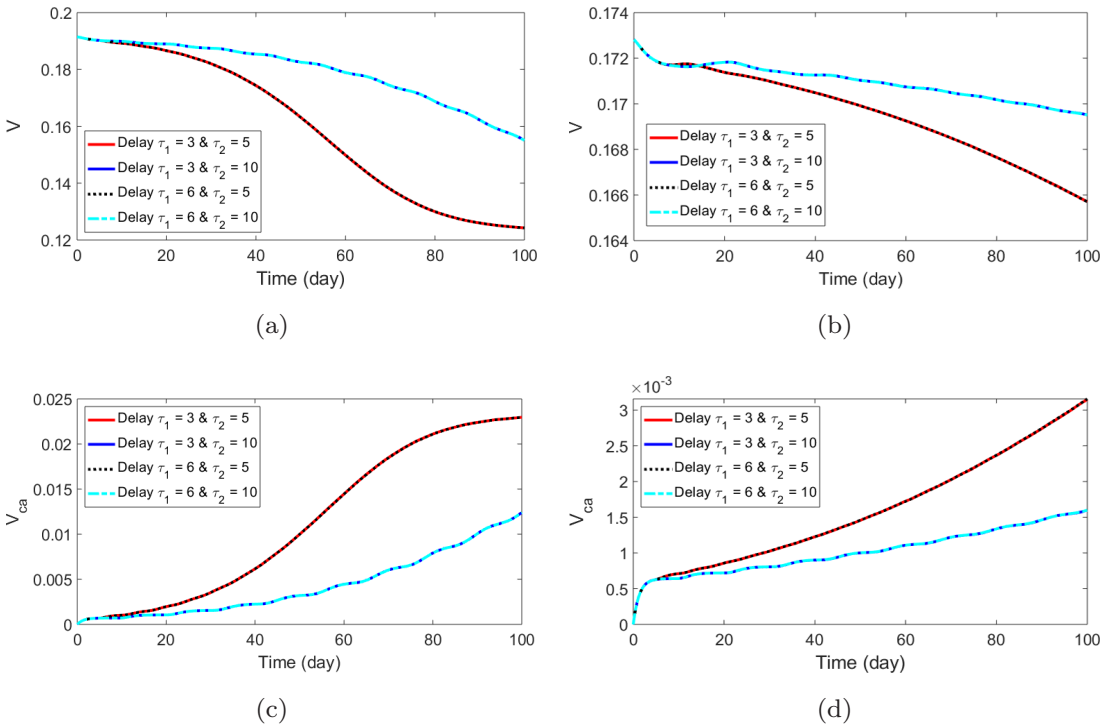

Fig. 13. The effects of low ($\rho_1 = 0.1$) and maximum ($\rho_1 = 0.3$) rate of treating susceptible animals (ρ_1) on susceptible and treated animal classes with time delays. (a) and (c) Low treating and vaccinating susceptible animals (ρ_1). (b) and (d) High treating and vaccinating susceptible animals (ρ_1).

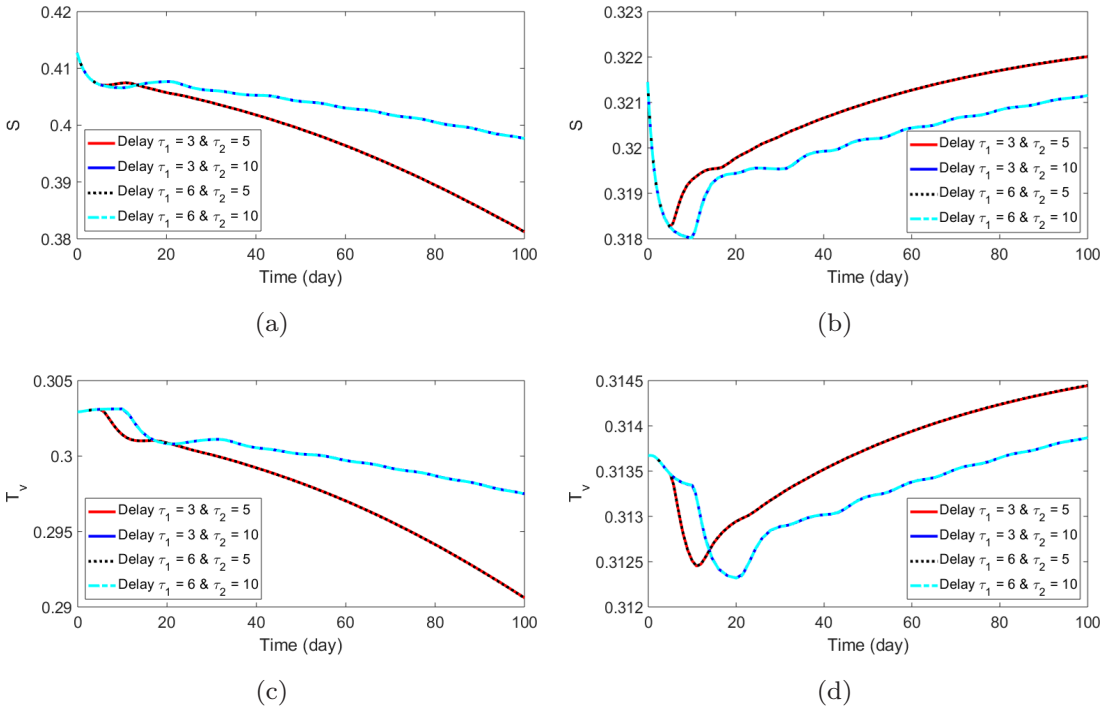

Fig. 14. The effects of low ($\rho_2 = 0.1$) and maximum ($\rho 2_2 = 0.3$) rate of vaccinating susceptible animals (ρ_2) on susceptible and treated animal classes with time delays. (a) and (c) Low vaccinating susceptible animals (ρ_2). (b) and (d) High vaccinating susceptible animals (ρ_2).

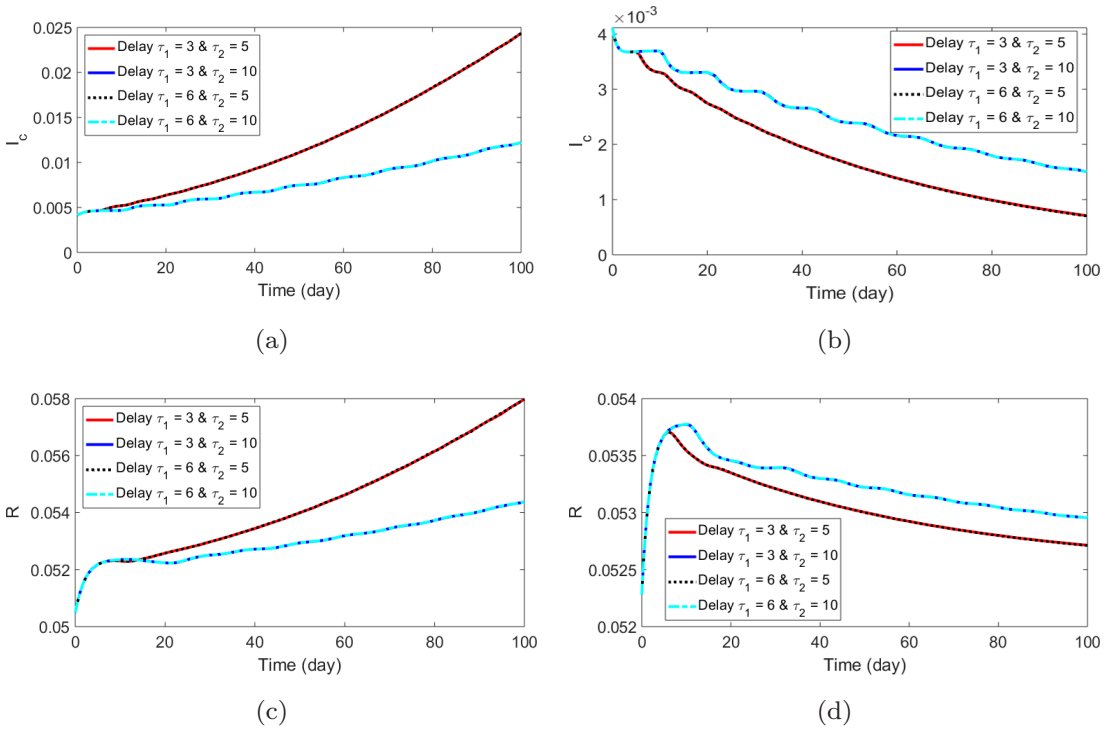

Fig. 15. The effects of low ($\rho_2 = 0.1$) and maximum ($\rho 2_2 = 0.3$) rate of vaccinating susceptible animals (ρ_2) on susceptible and treated animal classes with time delays. (a) and (c) Low vaccinating susceptible animals (ρ_2). (b) and (d) High vaccinating susceptible animals (ρ_2).

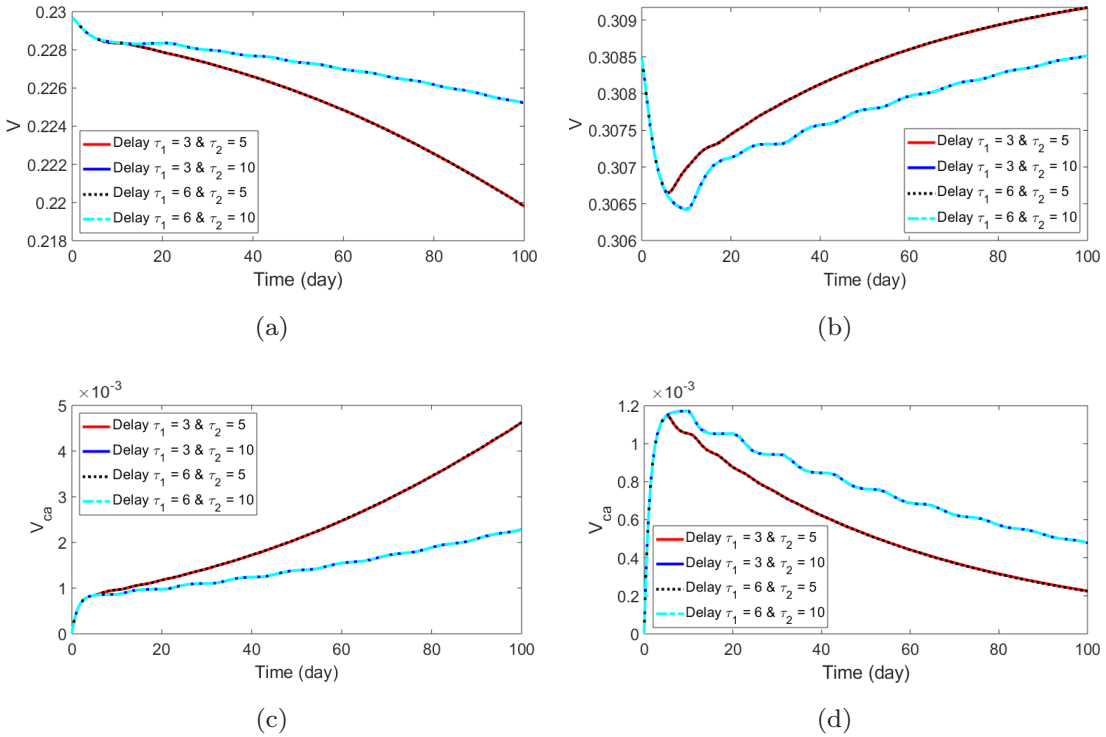

Fig. 16. The effects of low ($\rho_2 = 0.1$) and maximum ($\rho_2 = 0.3$) rate of vaccinating susceptible animals (ρ_2) on susceptible and treated animal classes with time delays. (a) and (c) Low vaccinating susceptible animals (ρ_2). (b) and (d) High vaccinating susceptible animals (ρ_2).

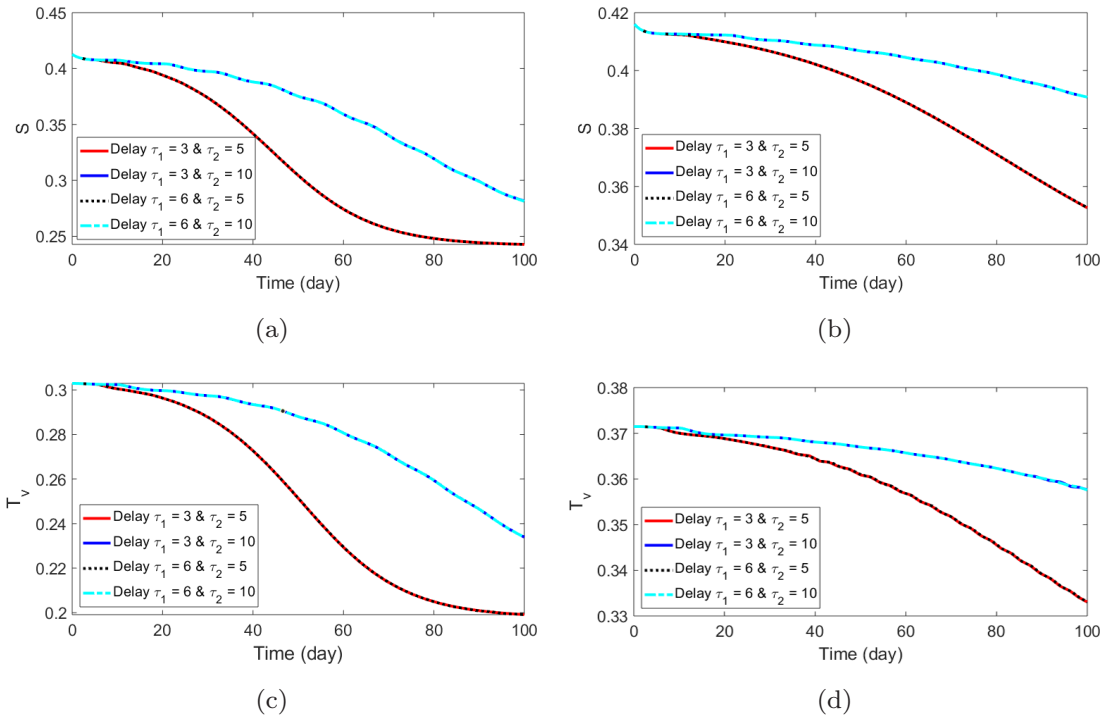

Fig. 17. The effects of low ($\epsilon = 0.3$) and maximum ($\epsilon = 0.7$) rate of treating vaccinated animals (ϵ) on susceptible and treated animal classes with time delays. (a) and (c) Low treating vaccinated animals (ϵ) for time delay. (b) and (d) High treating vaccinated animals (ϵ) for time delay.

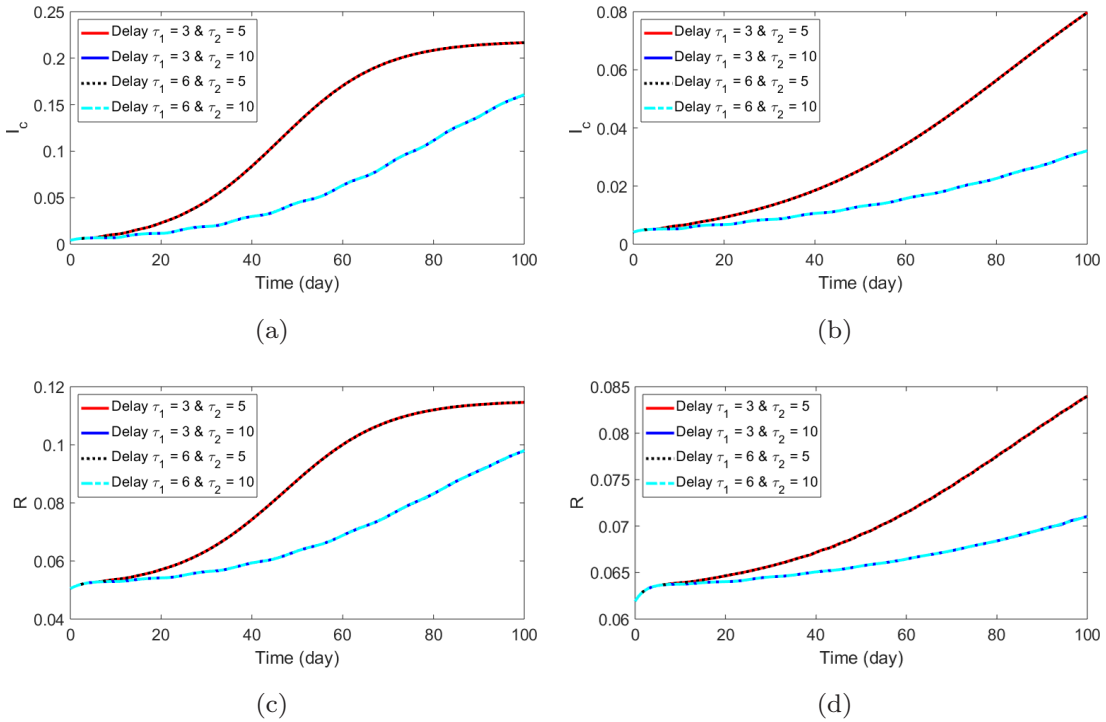

Fig. 18. The effects of low ($\epsilon = 0.3$) and maximum ($\epsilon = 0.7$) rate of treating vaccinated animals (ϵ) on clinically infected and recovery animal classes with time delays. (a) and (c) Low treating vaccinated animals (ϵ) for time delay. (b) and (d) High treating vaccinated animals (ϵ) for time delay.

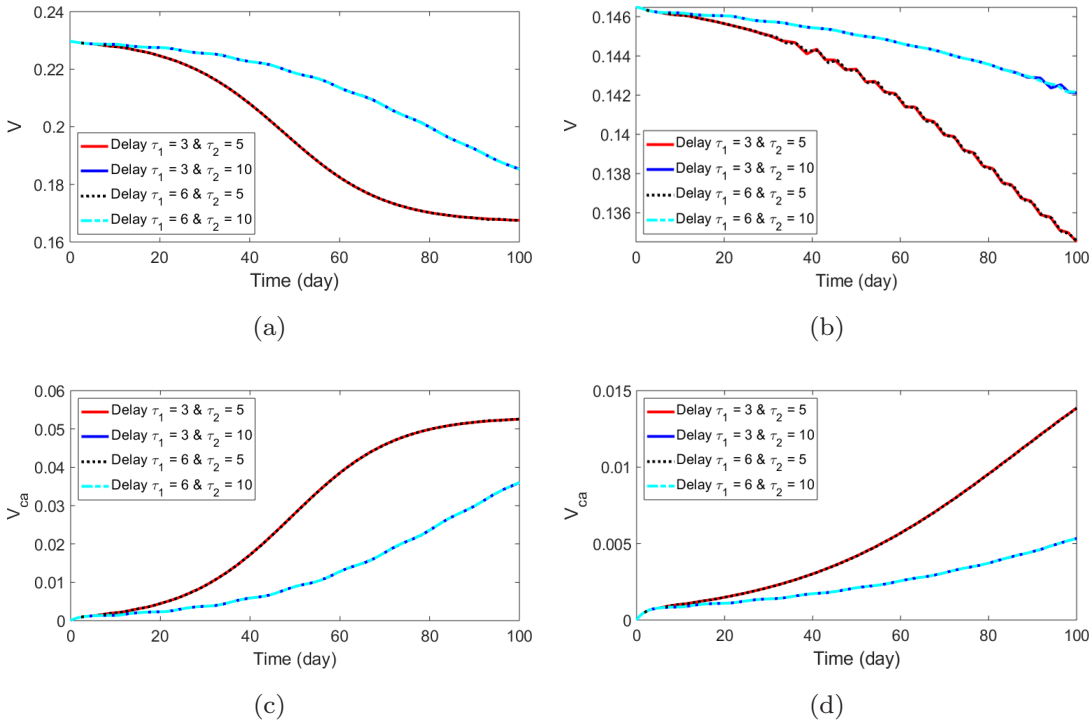

Fig. 19. The effects of low ($\epsilon = 0.3$) and maximum ($\epsilon = 0.7$) of treating vaccinated animals (ϵ) on vaccinated and vaccinated carrier animal classes with time delays. (a) and (c) Low treating vaccinated animals (ϵ) for time delay. (b) and (d) High treating vaccinated animals (ϵ) for time delay.

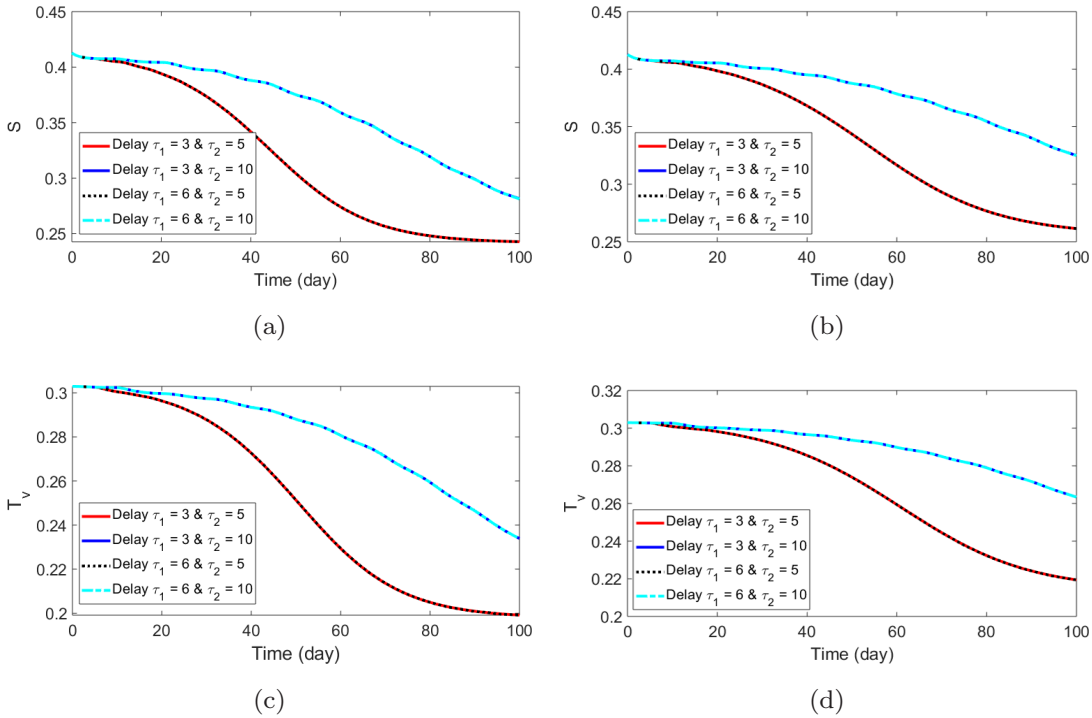

Fig. 20. The effects of low ($\delta = 0.01$) and maximum ($\delta = 0.09$) rate of culling infected animals (δ) on susceptible and treated animal classes with time delays. (a) and (c) Low culling infected animals (δ) for time delay. (b) and (d) High culling infected animals (δ) for time delay.

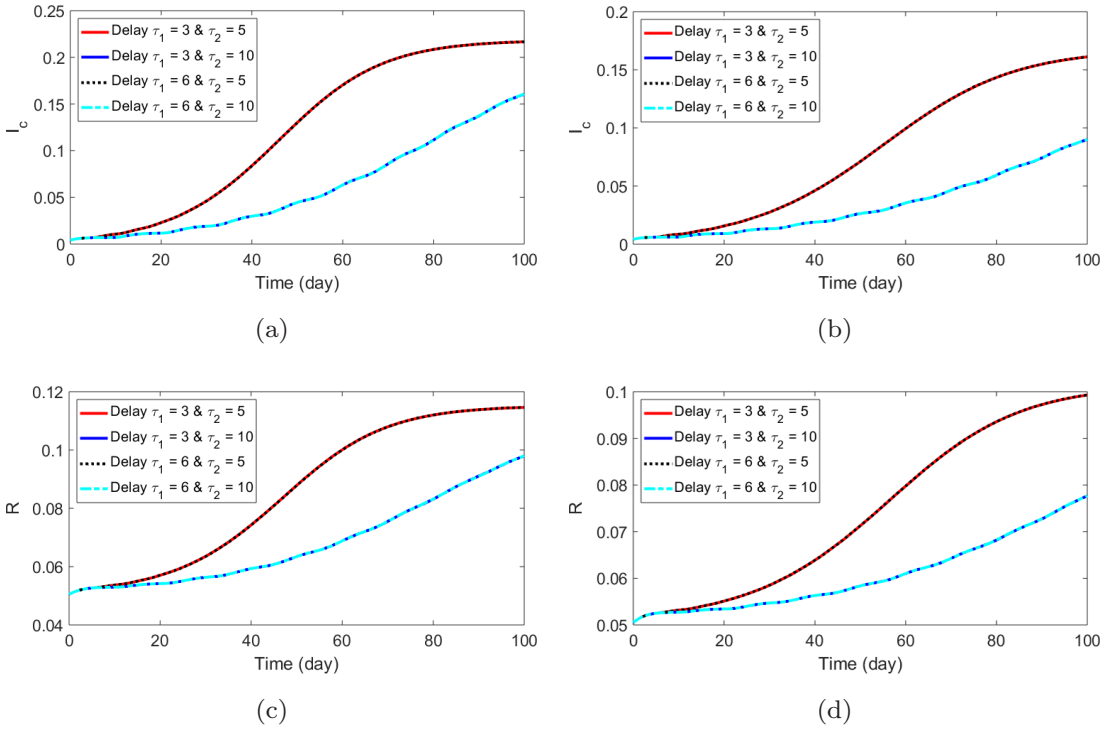

Fig. 21. The effects of low ($\delta = 0.01$) and maximum ($\delta = 0.09$) rate of culling infected animals (δ) on clinically infected and recovery animal classes with time delays. (a) and (c) Low culling infected animals (δ) for time delay. (b) and (d) High culling infected animals (δ) for time delay..

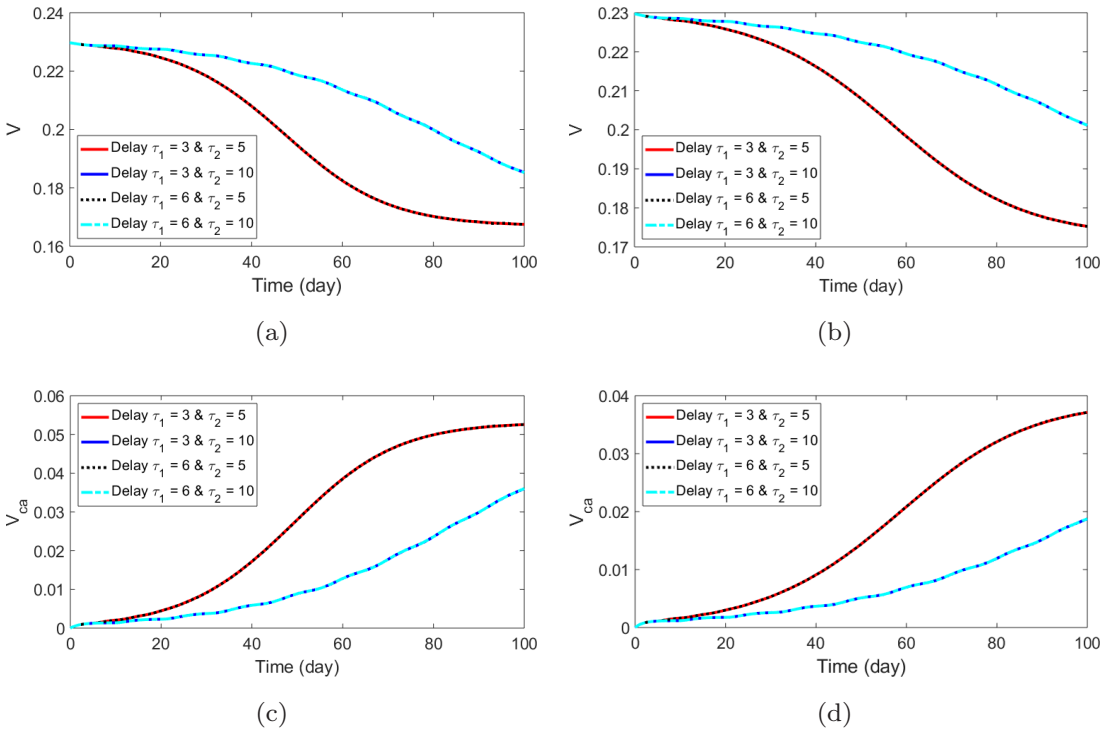

Fig. 22. The effects of low ($\delta = 0.01$) and maximum ($\delta = 0.09$) rate of culling infected animals (δ) on vaccinated and vaccinated carrier animal classes with time delays. (a) and (c) Low culling infected animals (δ) for time delay. (b) and (d) High culling infected animals (δ) for time delay.

Our numerical simulation results seem to suggest that increasing τ_2 and increasing each of the control parameters minimizes the burden of FMD and followed by increasing either the control or τ_2 parameters. However, decreasing both the control and τ_2 parameters increases the burden of FMD. Increasing or decreasing of τ_1 in combination with increasing or decreasing of control parameters does not show a significant effect the dynamic of infection.

5. Discussion and Results

The delay ordinary differential equation model for FMD of cattle was presented in this paper to capture the effects of prophylactic vaccination, reactive vaccination, prophylactic treatment, reactive culling and the effects of time delay. Mathematical analysis and numerical simulations were carried out to reveal the effects of the aforementioned control strategies and time delay on the burden of FMD.

Mathematical analysis revealed the effects of the control reproduction number, $\mathcal{R}_c$ and the existence of two equilibria, namely the disease-free equilibrium and an endemic equilibrium. The disease-free equilibrium was locally asymptotically stable when $\mathcal{R}_c$ is less than unity. This means the FMD burden can be kept in check if the control strategies used suppress consistently the control reproduction number below unity. In fact, there is a possibility of eradicating the infection with such controls. However, if the control cannot reduce $\mathcal{R}_c$ below unity, then there is a possibility that the FMD can spread to endemic levels. This scenario may be characteristic of a control strategy that is either inadequately administered or less efficient. Evidence of controls such as the rate of vaccination, the rate of treating and vaccinating susceptible animals, the rate of treating vaccinated animals and the rate of culling infected and vaccinated carrier animals is available where the burden of FMD continued to increase [9, 15, 17, 36]. In particular, there are vaccines that do not induce rapid control [9]. The sensitivity analysis of $\mathcal{R}_c$ with respect to prophylactic vaccination showed that $\mathcal{R}_c$ decreases only when the rate of loss of vaccination is below a critical loss of vaccination otherwise the benefits of prophylactic vaccination alone may not be realized. This means that prophylactic vaccination as a single strategy may not successfully eradicate the FMD. Simulations on $\mathcal{R}_c$ showed that the control reproduction number ($\mathcal{R}_c$) is less than one when the rate of treating and vaccinating of susceptible animals and rate of culling of clinically infected and vaccinated carrier animals are high.

Numerical simulations allowed us to observe the effects of time delays, prophylactic vaccination, reactive vaccination, prophylactic treatment and reactive culling parameters on FMD transmission in cattle. The numerical simulations suggested that increasing of both time delay two and control parameters or increasing of either of the time delay two or control parameters decrease the burden of FMD. But increasing of both time delay one and control parameters does not show a significant effect on the dynamics of FMD. Hence, time delay two has a significant effect on FMD. Increasing time delay two means that the newly infected animals

delay maximally to show clinical symptoms leading to less shedding of FMDV to other animals and subsequently the reduction of foot and mouth burden. Similarly increasing of control parameters such as prophylactic vaccination, reactive vaccination, prophylactic treatment, and reactive culling parameters have a substantial significant effect on the decreasing of FMD burden. Prophylactic and reactive vaccinations and treatment have been found to maintain immunity to FMD [28], but the high cost of vaccines and drugs limit the use vaccination and treatment control strategies for FMD [11, 15, 17]. The results suggest that the strategy of decreasing time delay two while increasing the degree of control parameters contributes a significant effect on the reduction of FMD but decreasing of both time delay two and control parameters increases the FMD burden. Decreasing time delay two means that the newly infected animals fast to show clinical symptoms and leading to high shedding of FMDV to other animals and subsequently the increase of FMB. Our results also suggest that the strategy of increasing of time delay two while decreasing the control parameters may not significantly reduce the FMD burden. This outcome has consequences on systems that would want to reduce the costs of control strategies but in the end, the disease burden will continue to hamper their efforts. Therefore, the strategy which has a significant effect on the protection of FMD burden is increasing both the time delay two and the control parameters.

6. Conclusion

Our result suggested that the FMD burden is decreased significantly when the unprotected animals delay maximally their time to show clinical symptoms and at the same time when the effectiveness of the control strategies are increased. It is imperative that control strategies play a significant role in moving the animals into the protected routes of infection than leaving more animals in the unprotected route of infection. By implication, strategies that directly protect and reduce the number of susceptible animals should be prioritized and effectively enhanced as these directly divert the animal flow into the protected route of the FMD dynamics.

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