Applications of Lévy Processes in Finance

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by

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Abstract

The option pricing theory set forth by Black and Scholes assumes that the underlying asset can be modeled by Geometric Brownian motion, with the Brownian motion being the driving force of uncertainty. Recent empirical studies, Dotsis, Psychoyios & Skiadopolous (2007) [17], suggest that the use of Brownian motion alone is insufficient in accurately describing the evolution of the underlying asset. A more realistic description of the underlying asset's dynamics would be to include random jumps in addition to that of the Brownian motion.

The concept of including jumps in the asset price model leads us naturally to the concept of a Lévy process. Lévy processes serve as a building block for stochastic processes that include jumps in addition to Brownian motion. In this dissertation we first examine the structure and nature of an arbitrary Lévy process. We then introduce the stochastic integral for Lévy processes as well as the extended version of Itô's lemma, we then identify exponential Lévy processes that can serve as Radon-Nikodým derivatives in defining new probability measures.

Equipped with our knowledge of Lévy processes we then implement this process in a financial context with the Lévy process serving as driving source of uncertainty in some stock price model. In particular we look at jump-diffusion models such as Merton's(1976) [37] jump-diffusion model and the jump-diffusion model proposed by Kou and Wang (2004) [30]. As the Lévy processes we consider have more than one source of randomness we are faced with the difficulty of pricing options in an incomplete market.

The options that we shall consider shall be mainly European in nature, where exercise can only occur at maturity. In addition to the vanilla calls and puts we independently derive a closed form solution for an exchange option under Merton's jump-diffusion model making use of conditioning arguments and stochastic integral representations. We also examine some exotic options under the Kou and Wang model such as barrier options and lookback options where the solution to the option price is derived in terms of Laplace transforms. We then develop the Kou and Wang model to include only positive jumps, under this revised model we compute the value of a perpetual put option along with the optimal exercise point.

Keywords

Derivative pricing, Lévy processes, exchange options, stochastic integration

Preface

The work described in this dissertation was carried out in the School of Statistics and Actuarial Science at the University of KwaZulu-Natal, under the supervision of Prof. J.G. O'Hara.

Theses studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

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Chapter 1

Introduction

The purpose of this chapter is to introduce definitions, concepts as well as fix notation that we shall frequently encounter throughout this dissertation.

Definition 1.1. : Let S be a non-empty set and $\mathcal{F}$ be a collection of subsets of S. We call $\mathcal{F}$ a σ- algebra if the following conditions hold :

- $S \in \mathcal{F}$.
- If $A \subset S$ and $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$.
- If (A_n) is a sequence of subsets in $\mathcal{F}$ then $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(S, \mathcal{F})$ is called a measurable space. A measure on $(S, \mathcal{F})$ is a mapping $\mu : \mathcal{F} \to [0, \infty]$ that satisfies the following conditions:

- $\mu(\emptyset) = 0$.
- If (A_n) is a sequence of mutually disjoint sets in $\mathcal{F}$ then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n). \tag{1.1}$$

The triplet $(S, \mathcal{F}, \mu)$ is called a measure space. If $\mu(S) < \infty$ then the measure μ is said to be finite and if it is possible to find a sequence of disjoint events $\{A_n, n \in \mathbb{N}\}$ in $\mathcal{F}$ such that $S = \bigcup\limits_{n=1}^{\infty} A_n$ and each $\mu(A_n) < \infty$, then we say that the measure is σ-finite. For a *probability measure*, we usually write $S = \Omega$ and take Ω to represent the set of all possible outcomes. Elements in $\mathcal{F}$ are called event sets and any measure $\mathbf{P}$ on the measure space $(\Omega, \mathcal{F})$ must satisfy $\mathbf{P}(\Omega) = 1$. Here the triple $(\Omega, \mathcal{F}, \mathbf{P})$ is called a probability space.

An effective tool for modeling the uncertainty that can evolve over time is the concept of a filtration $(\mathcal{F}_t, t \geqslant 0)$ which represents a family of σ-algebras that satisfies $\mathcal{F}_s \subseteq \mathcal{F}_t$ for all $s \leqslant t$. A filtration can be thought of as capturing information that is evolving over time and as we have $\mathcal{F}_s \subseteq \mathcal{F}_t$, information is not forgotten. A probability space equipped with a filtration is known as a *filtered probability space*. With this idea of a probability space we shall now consider the notion of a stochastic process. A stochastic process is a sequence of time indexed random variables $\{X_t, t \geqslant 0\}$ that is defined on a probability space, note that we suppress the dependence of X_t on ω as we take it to be implicitly assumed. For our purposes we shall be interested in stochastic processes where the time index is assumed to be continuous, as such we shall be dealing with continuous time parameter stochastic processes. When dealing with a stochastic process there are certain desirable properties that are of interest to us as they lead to elegant results without the hazard computational difficulty. Among the many properties of stochastic processes, for our purposes there probably none more important than that of the martingale property. Martingale processes play an important role in the pricing of derivatives and financial instruments, needless to say it will be a recurrent theme throughout this dissertation.

Definition 1.2. : A continuous time parameter stochastic process $\{X_t, t \geqslant 0\}$ is said to have a martingale property with respect to a sequence of information sets $(\mathcal{F}_t, t \geqslant 0)$ under a given measure $\mathbf{P}$ if

$$\mathbf{E}^{\mathbf{P}}|X_t| < \infty \qquad \forall t \geqslant 0 \qquad \text{and} \tag{1.2}$$

$$\mathbf{E}^{\mathbf{P}}\big(X_t \big| \mathcal{F}_s\big) = X_s \qquad \mathbf{P}\text{−a.s.} \quad \text{for any } 0 \leqslant s < t < \infty. \tag{1.3}$$

When indicating that a stochastic process has the martingale property we will often say that X_t has a $(\mathbf{P}, \mathcal{F}_t)$-martingale property to indicate the important fact that the martingale property only exists with respect to a particular measure and filtration. Whenever Eq.(1.2) holds true we say that X is *integrable* with respect to the measure $\mathbf{P}$ and denote this by $X \in L^1(\Omega, \mathcal{F}, \mathbf{P})$, which we abbreviate sometimes to just $L^1(\Omega, \mathbf{P})$.

Remark 1.1. Note that we have defined the martingale property for process where the time index runs from $[0, \infty)$, for most of our applications however our focus shall be restricted to the the time interval $[0, T]$ where $T < \infty$. When verifying the martingale property over the time interval $[0, T]$ we still use Definition 1.2 with the only minor change being Eq.(1.3) must hold for all $t \leqslant T$.

For martingale processes that are defined on the time interval $[0, \infty)$ the following theorem allows us to associate a limiting random variable with the stochastic process.

Theorem 1.1. *Let $\{X_t, t \geqslant 0\}$ be a process with a $(\mathbf{P}, \mathcal{F}_t)$-martingale property that is also $L^1(\Omega, \mathcal{F}, \mathbf{P})$ bounded, i.e.*

$$\sup_{t \geqslant 0} \mathbf{E}^{\mathbf{P}}|X_t| < \infty, \tag{1.4}$$

then we have that

$$\lim_{t \to \infty} X_t = X_\infty \qquad \mathbf{P}\text{−}a.s. \tag{1.5}$$

Proof: See Elliott [14] ■

If we fix a particular $\omega \in \Omega$ and allow only the time parameter t to vary for the stochastic process $\{X_t, t \geqslant 0\}$ then we obtain what is known as a sample path trajectory of a stochastic process. Properties such as continuity and differentiability now become of interest to us; as far a continuity goes we have the following:

Definition 1.3. : A continuous time parameter process $\{X_t, t \geqslant 0\}$ is said to have a *càdlàg* property under a given measure $\mathbf{P}$ if its sample path trajectories are all right-continuous in a $\mathbf{P}$-a.s. sense and have a finite left limit.

In a similar manner process that have left-continuous sample path trajectories with finite right limits are said to be *càglàd*.

For a given set of time points $\{t_1, t_2, \ldots, t_n\}$ the following theorem provides an easy method of verification of independence between a sequence if random variables:

Theorem 1.2. *The random variables $X_1, \ldots, X_n$ are independent if, and only if*

$$\mathbf{E}\left(\exp\left[i\sum_{j=1}^{n} u_j X_j\right]\right) = \psi_{X_1}(u_1)\cdots\psi_{X_n}(u_n) \qquad \textit{for all } u_1, \ldots, u_n \in \mathbb{R},$$

where $\psi_{X_k}(u_k)$ is the characteristic function of X_k, i.e. $\psi_{X_k}(u_k) = \mathbf{E}[e^{iu_k X_k}]$.

Proof: See Applebaum [4] ∎

Two important examples of stochastic processes are standard Brownian motion and the Poisson process. Brownian motion serves as a fundamental building block for the modeling of continuous path processes, ever since Louis Bachelier [6] first postulated its potential use as a model for describing the evolution of stock prices.

Definition 1.4. : A continuous time parameter process $W = \{W_t, 0 \leqslant t \leqslant T\}$ is said to have a standard Brownian motion property under a given measure $\mathbf{P}$ if

- $W_0 = 0 \quad \mathbf{P}-$a.s.
- W has independent increments, i.e. for $0 \leqslant s < t,\ W_t - W_s$ is independent of $\mathcal{F}_s = \sigma(W_u,\ u \leqslant s)$.
- W has increments that are normally distributed for non-overlapping intervals of time, i.e. $W_t - W_s \sim N(0, t-s)$.

The Poisson process is a counting process that serves as the basic building block for modeling jump processes. It counts the number of times a certain random event occurs in some time interval.

Definition 1.5. : A continuous time parameter process $N = \{N_t, 0 \leqslant t \leqslant T\}$ is said to be a Poisson process with intensity parameter λ under a given measure $\mathbf{P}$ if

- $N_0 = 0 \quad \mathbf{P}-$a.s.
- N has stationary increments, i.e. $N_t - N_s$ and N_{t-s} both follow a Poisson distribution with mean parameter $\lambda(t-s)$.
- N has independent increments, i.e. for $0 \leqslant s < t, \quad N_t - N_s$ is independent of $\mathcal{F}_s = \sigma(N_u,\ u \leqslant s)$.

Both the standard Brownian motion process and the Poisson process have càdlàg sample paths and while they are different stochastic processes we shall see that they share certain characteristics.

We conclude this chapter by defining what is meant by an adapted process. Consider a stochastic process $\{X_t, t \geqslant 0\}$ and a filtration $(\mathcal{F}_t, t \geqslant 0)$, the process $\{X_t, t \geqslant 0\}$ is said to be adapted to the filtration $(\mathcal{F}_t, t \geqslant 0)$ if for any t X_t is $\mathcal{F}_t$-measurable. To avoid technical measurability problems we shall assume that when working with any filtration $(\mathcal{F}_t, t \geqslant 0)$, that the the filtration is complete. By this we meant that the filtration contains all the $\mathbf{P}$-null sets. For more thorough introduction to concepts in stochastic calculus we refer the interested reader to Protter [40].

Chapter 2

Stochastic calculus for Lévy processes

2.1 Lévy processes

The aim of this section is to introduce the reader to the concept of a Lévy process and the properties associated with this process as it will serve as our source of randomness when building a stock price model. To fully understand the nature of any arbitrary Lévy process we will examine the Lévy-Khintchine formula. However, before we can gain more insight into this formula we will require some understanding of jump processes, counting measures and integration with respect to jump process. After gaining an understanding of these aspects we shall be able to decompose the Lévy process into three independent processes, namely Brownain motion, a compound Poisson process and a final compensated jump process. Throughout this section we shall assume that we are under the real world measure or canonical measure $\mathbf{P}$. We shall now start by giving a precise definition of the Lévy process

Definition 2.1. : A stochastic process $X = \{X_t \ , t \geqslant 0\}$ is said to be a Lévy process if

- $X_0 = 0 \quad \mathbf{P}-$a.s.
- X has independent increments, i.e. $X_t - X_s$ is independent of $\mathcal{F}_u = \sigma\{X_u \ , u \leqslant s\}$.
- X has stationary increments, i.e. $X_t - X_s$ has the same distribution as X_{t-s}.
- X is stochastically continuous, i.e. $\forall \, \epsilon > 0$ and $s \geqslant 0$

$$\lim_{t \to s} \mathbf{P}(\mid X_t - X_s \mid \geqslant \epsilon) = 0.$$

If we are given any arbitrary process we can then use Definition 2.1 to determine whether or not such a process is indeed a Lévy process. However this definition of the Lévy process does not give us a real insight to the nature of the process, ideally we would like to have some representation of a general Lévy process. Such a representation is made possible with the aid of the Lévy-Khintchine formula. This formula reveals a great deal about the nature of any general Lévy process and is also connected with the Lévy-Itô decomposition. Such a decomposition will allow us to write X_t as linear combination of Brownian motion, compound Poisson process and a compensated jump process. The Lévy-Khintchine formula states that if X_t is a Lévy process then its characteristic function must be of the form

$$\mathbf{E}^{\mathbf{P}}\left[\mathrm{e}^{-i\theta X_t}\right] = \exp(-t\psi(\theta))$$

where $\psi(\theta)$ is called the Lévy exponent and is given by

$$\psi(\theta) = \frac{\theta^2 a^2}{2} + i\mu\theta + \int_{|x|<1} (1 - e^{-i\theta x} - i\theta x)\,\nu(dx) + \int_{|x|\geqslant 1} (1 - e^{-i\theta x})\,\nu(dx) \quad (2.1)$$

for $a,\ \theta \in \mathbb{R}$ and for some σ-finite measure ν on $\mathbb{R} - \{0\}$ satisfying

$$\nu\{0\} = 0 \qquad \text{and} \qquad \int \min\{x^2, 1\}\,\nu(dx) < \infty.$$

Eq.(2.1) may at first seem intimidating, however it contains all the necessary information to completely specify the Lévy process X_t. To fully appreciate the Lévy-Khintchine formula and to aid our understanding of Lévy processes we shall first introduce the *jump process* $\Delta X = \{\Delta X_t\ ,\ t \geqslant 0\}$ defined by

$$\Delta X_t = X_t - X_{t-} \quad (2.2)$$

where X_{t-} is the left limit of X at time point t (or using a more mathematical definition $X_{t-} = \lim_{s\uparrow t} X_s$). One particular problem that we are faced with when dealing with the jumps of a Lévy process is that the sum of all possible jumps may become unbounded, i.e. we may have

$$\sum_{0\leqslant s\leqslant t} \left|\Delta X_s\right| = \infty \qquad \mathbf{P}\text{−a.s.} \quad (2.3)$$

We overcome this problem by exploiting the fact that we will always have

$$\sum_{0\leqslant s\leqslant t} \left|\Delta X_s\right|^2 < \infty \qquad \mathbf{P}\text{−a.s.} \quad (2.4)$$

This property of the jump process is similar to that of Brownian motion, which has unbounded first order variation but finite quadratic variation. The justification of Eq.(2.4) will be shown later on. Furthermore the jump process is right continuous with finite left limits. Such a property implies that the sample path trajectories of the jump process are càdlàg. Rather than working directly with the jump process ΔX we will find it more convenient to work with a process that counts the number of jumps that have occurred in some time interval. In this regard we shall consider the following process, for $0 \leqslant t < \infty$ and $A \in \mathcal{B}(\mathbb{R}-\{0\})$ define

$$N(t, A) = \sum_{0\leqslant s\leqslant t} \mathbf{1}_{(\Delta X_s \in A)} = \#\{0 \leqslant s \leqslant t;\ \Delta X_s \in A\} \quad (2.5)$$

where $\mathbf{1}$ is the indicator function. $N(t, A)$ counts the number of jumps up to a time point t and the Borel subset A represents the range of possible jump sizes. Note that this process does depend on ω and is thus a stochastic process. So if we fix t then $N(t, A)$ is a random variable. Similarly if we fix a particular sample path $\omega \in \Omega$ then the set function $A \to N(t, A)(\omega)$ is a counting measure on $\mathcal{B}(\mathbb{R} - \{0\})$. Thus if we take the expected value of $N(t, A)$

$$\mathbf{E}^{\mathbf{P}}(N(t, A)) = \int_{\Omega} N(t, A)(\omega)\, dP(\omega)$$

we obtain a Borel measure on $\mathcal{B}(\mathbb{R} - \{0\})$. We shall define $\mathbf{E}^{\mathbf{P}}(N(1, A)) = \nu(A)$ for $A \in \mathcal{B}(\mathbb{R} - \{0\})$, calling $\nu(A)$ the intensity measure associated with X. The intensity measure represents the *expected number of jumps* of a certain height in a time interval of unit length 1. When dealing with Lévy processes the intensity measure is more commonly known as the *Lévy measure*. As we have seen in Eq.(2.3), the sum of all possible jumps can explode to infinity. We shall give a brief explanation why with the use of the following lemma.

Lemma 2.1. $N(t, A) < \infty$ $\mathbf{P}$-a.s *for all* $t \geqslant 0$ *if the Borel subset* A *is bounded away from zero.*

Proof: See [4] Chapter 2 Lemma 2.3.4 ■

We say that A is bounded away from zero if $0 \notin \bar{A}$ where $\bar{A}$ represents the closure of the subset A. The proof of this Lemma makes use of stopping times and the fact the sample paths of the jump process are càdlàg, which guarantees that there can be at most a countable number of jumps in any finite time interval. When A is not bounded away from zero Lemma 2.1 may no longer hold true, in which case we may have an infinite number of jumps occurring. The reason for the number of jumps becoming unbounded is due to an infinite number of *small jumps* around the point zero. We can thus conclude that it is only when the Borel subset A is not bounded away from zero that the sum of all possible jumps in Eq.(2.3) may explode towards infinity due to an accumulation of an infinite number of small jumps. Therefore zero is an accumulation point for the process $\{N(t, A), t \geqslant 0\}$. When dealing with a Lévy process X we consider the Lévy measure $\nu(\cdot)$ to determine whether or not the process we are dealing with has infinite number of jumps. The reason for this is that if we have $\nu(\cdot) < \infty$[1] we can conclude that there are finite amount of jumps in any finite time interval. Thus by examining the Lévy measure we shall be able to determine if the Lévy process admits an infinite number of jumps.

Theorem 2.1. *Let* $N(t, A)$ *be defined as in Eq.(2.5), then we have the following*

- *If* A *is bounded away from zero, then the stochastic process* $\{N(t, A), t \geqslant 0\}$ *is a Poisson process with intensity parameter* $\nu(A)$.
- *If* $A_1, \cdots, A_m$ *is a sequence of disjoint Borel subsets in* $\mathcal{B}(\mathbb{R} - \{0\})$*, then the random variables* $N(t, A_1), \ldots, N(t, A_m)$ *are independent.*

[1]Since $\nu(\cdot) = \mathbf{E}^{\mathbf{P}}(N(1, \cdot))$, if we have $\nu(\cdot) < \infty \Rightarrow N(t, \cdot) < \infty$ $\mathbf{P}$−a.s. for any finite t.

Proof: See Applebaum [4] chapter 2 ∎

From Theorem 2.1 we can now identify $N(t, A)$ as a Poisson process that counts the number of jumps up to time t and the Borel subset A indicates the magnitude of the jumps that are being counted by $N(t, A)$. However when A fails to be bounded away from zero we can no longer claim that $N(t, A)$ is a Poisson process. The reason for this is because a Poisson process can only experience a finite number of jumps in any finite time interval and when A fails to be bounded away from zero we may have an infinite number of jumps in any finite time interval.

Next we shall define a random measure, in particular we shall see that the Poisson process $N(t, A)$ satisfies the requirements of a random measure.

Remark 2.1. Since $N(t, A)$ is a Poisson process we will always have $\nu(A) < \infty$ for all A bounded away from zero.

Definition 2.2. : Let $(S, \mathcal{A})$ be a measurable space and $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. A random measure M on $(S, \mathcal{A})$ is a collection of random variables $\{M(B),\ B \in \mathcal{A}\}$ such that

- $M(\emptyset) = 0$.
- Given any sequence of mutually disjoint sets $\{A_n,\ n \in \mathbb{N}\}$ in $\mathcal{A}$,

$$M\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} M(A_n) \qquad \mathbf{P}-\text{a.s.}$$

- For each disjoint family $(B_1, \ldots, B_n)$ in $\mathcal{A}$, the random variables $M(B_1), \ldots, M(B_n)$ are independent.

Since $N(t, A)$ is a counting measure is satisfies the first two properties of in Definition 2.2. The final property of a random measure follows from Theorem 2.1. So $N(t, A)$ is indeed a random measure and is in fact a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$.

Having defined the process $\{N(t, A), t \geqslant 0\}$ and examined the properties that such a process possesses we can now proceed towards defining the Poisson integral. Let f be a Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$ and let the Borel subset A be bounded away from zero, then for each $t \geqslant 0$ and $\omega \in \Omega$ we may define the Poisson integral of f as a random finite sum as follows

$$\int_A f(x)N(t, dx)(\omega) = \sum_{x\in A} f(x)\ N(t, \{x\})(\omega). \tag{2.6}$$

Now if we make use of the fact that the integral in Eq.(2.6) is zero unless a jump occurs we can also define the Poisson integral as

$$\int_A f(x)\ N(t, dx)(\omega) = \sum_{0\leqslant u\leqslant t} f(\Delta X_u)\mathbf{1}_{(\Delta X_u \in A)}. \tag{2.7}$$

Eq.(2.6) and Eq.(2.7) both have the same meaning but are just different representations of the Poisson integral. Note that for each t, $\int_A f(x)N(t,dx)$ is a random variable. Hence if we vary t we will obtain a stochastic process with càdlàg sample paths since ΔX is càdlàg. By making use of the fact that the sample paths of $N(t,A)$ are bounded almost surely for A bounded away from zero we will always have for any Borel measurable function f that

$$\int_A |f(x)|\, N(t,dx) < \infty \qquad \mathbf{P}\text{−a.s.} \tag{2.8}$$

Having stated that for each fixed t that the Poisson integral is indeed a random variable we would now like to determine the distribution of such a random variable. Again we assume that the Borel subset A is bounded away from zero in the following theorem.

Theorem 2.2. *For each $t \geqslant 0$, $\int_A f(x)\ N(t,dx)$ is a compound Poisson process with characteristic function given by*

$$\mathbf{E}^{\mathbf{P}}\left(\exp\left[iu\int_A f(x)\ N(t,dx)\right]\right) = \exp\left(t\int_A (e^{iuf(x)}-1)\ \nu(dx)\right) \textit{ for each } u \in \mathbb{R}. \tag{2.9}$$

Proof: See Sato [42], pg 124 ■

For the above compound Poisson process it is the Borel measurable function f that governs the actual jump sizes. Furthermore we can obtain the moments of the compound Poisson process, provided that they exist, by differentiating Eq.(2.9) with respect to u. So the first moment the compound Poisson process is given by

$$\mathbf{E}^{\mathbf{P}}\left(\int_A f(x)\ N(t,dx)\right) = t\int_A f(x)\ \nu(dx). \tag{2.10}$$

Remark 2.2. Note that Eq.(2.10) will only be defined if $f \in L^1(A,\nu)$, i.e.

$$\int_A |f(x)|\ \nu(dx) < \infty.$$

Similarly higher moments of the compound Poisson process will only exist if $f \in L^n(A,\nu)$ for $n \geqslant 1$.

We shall require one more final result on the Poisson integral. Consider two stochastic processes defined by $\{\int_{A_1} f(x)\ N(t,dx), t \geqslant 0\}$ and $\{\int_{A_2} g(x)\ N(t,dx), t \geqslant 0\}$ where f and g are Borel measurable and the Borel subsets A_1 and A_2 are disjoint and bounded away from zero. Then it can be shown that since the the range of jumps that that are common to both processes is the $\emptyset$, both processes are *independent* (see [4] Chapter 2, Theorem 2.4.6).

In order to deal with the fact that the sum of all possible jumps may become unbounded we have to consider the following process $\{\tilde{N}(t,A), t \geqslant 0\}$ defined by

$$\tilde{N}(t,A) = N(t,A) - t\nu(A).$$

Such a process is known as a *compensated Poisson random measure*. This process has the desirable property of being a martingale-valued measure since $E(|\tilde{N}(t,A)|) < \infty$ and

$$\begin{aligned}\mathbf{E}^{\mathbf{P}}(\tilde{N}(t,A)|\mathcal{F}_s) &= \mathbf{E}^{\mathbf{P}}(N(t,A)|\mathcal{F}_s) - t\nu(A) \\ &= N(s,A) - s\nu(A) = \tilde{N}(s,A), \qquad (2.11)\end{aligned}$$

where the last line in Eq.(2.11) makes use of the fact that the Poisson process $N(t,A)$ has independent increments. We define the compensated Poisson integral in a similar manner to that in Eq.(2.6) (or Eq.(2.7)), however now we have to impose that condition that the Borel measurable function f be integrable with respect to the measure ν, i.e. we must have $f \in L^1(A,\nu)$ in order to give proper meaning to the following integral

$$\int_A f(x)\ \tilde{N}(t,dx) = \int_A f(x)\big[N(t,dx) - t\nu(dx)\big]. \qquad (2.12)$$

By making use of the independent increments property of the Poisson process, a straightforward exercise reveals that $\{\int_A f(x)\ \tilde{N}(t,dx),\ t \geqslant 0\}$ is a martingale process. Now if we make use of Theorem 2.9 and the fact that $\int_A f(x)\ \nu(dx)$ is deterministic, then we can conclude that the characteristic function of the compensated Poisson process must be of the following form

$$\mathbf{E}^{\mathbf{P}}\left(\exp\left[iu\int_A f(x)\ \tilde{N}(t,dx)\right]\right) = \exp\left(t\int_A (e^{iuf(x)} - 1 - iuf(x))\ \nu(dx)\right), \qquad (2.13)$$

where $u \in \mathbb{R}$. We are now in a position to identify part of the Lévy-Khintchine formula. Consider Eq.(2.13) and Eq.(2.9). If we assume that the Borel measurable function $f(x) = x$ and that the Borel subsets A_1 and A_2, which represent the range of jumps, are disjoint then the joint characteristic function of the random variables $\int_{A_1} x\ N(t,dx)$ and $\int_{A_2} x\ \tilde{N}(t,dx)$ must be of the following form

$$\begin{aligned}&\mathbf{E}\left(\exp\left[i\theta\int_{A_1} x\ N(t,dx) + i\theta\int_{A_2} x\ \tilde{N}(t,dx)\right]\right) \\ &= \mathbf{E}^{\mathbf{P}}\left(\exp\left[i\theta\int_{A_1} x\ N(t,dx)\right]\right)\cdot\mathbf{E}^{\mathbf{P}}\left(\exp\left[i\theta\int_{A_2} x\ \tilde{N}(t,dx)\right]\right) \qquad (2.14) \\ &= \exp\left(t\int_{A_1}(e^{i\theta x} - 1)\ \nu(dx) + t\int_{A_2}(e^{i\theta x} - 1 - i\theta x)\ \nu(dx)\right),\ \theta \in \mathbb{R}. \qquad (2.15)\end{aligned}$$

The justification for Eq.(2.14) follows from the fact that the Borel subsets A_1 and A_2 are assumed to be disjoint, hence both the random variables are independent and we can thus make use of Theorem 1.2. Now if we examine Eq.(2.15) carefully we see that it almost identical to part of the Lévy-Khintchine formula in Eq.(2.1), the only difference being that the Borel subsets are defined to be $A_1 = \{|x| \geqslant 1\}$ and $A_2 = \{|x| < 1\}$. We can now identify part of the Lévy process X as a linear combination of a *compound Poisson process and a compensated jump process*. The process is given by

$$\int_{|x|\geqslant 1} x\ N(t,dx) = \sum_{0\leqslant s\leqslant t} \Delta X_s \mathbf{1}_{(\Delta X_s \geqslant 1)} \qquad (2.16)$$

is the compound Poisson process and is responsible for all the jumps whose size is greater than or equal to one. Since the Borel subset defining the range of possible jumps is bounded away from zero we know that the Poisson integral defined in Eq.(2.16) is finite almost surely under the measure $\mathbf{P}$. The compensated jump process defined by

$$\int_{|x|<1} x \ \tilde{N}(t, dx) \tag{2.17}$$

deals with all the jumps whose size is less than one. The reason for working with a compensated sum of jumps lies in the fact that the uncompensated sum of all jumps less than one explodes towards infinity since the Borel subset is not bounded away from zero. A rigorous construction and proof that the compensated integral in Eq.(2.17) exists in a square integrable martingale space can be found in [4] or [42].

Remark 2.3. Note that the process defined by $\{\int_{|x|<1} x \ N(t, dx), t \geqslant 0\}$ is not a compensated compound Poisson process since the Borel subset which indicates the magnitude of all possible jumps is not bounded away from zero.

We are now in a position to decompose the Lévy process X. Firstly recall that the Lévy exponent in Eq.(2.1) was given by

$$\psi(\theta) = \frac{\theta^2 a^2}{2} + i\mu\theta + \int_{|x|<1} (1 - e^{-i\theta x} - i\theta x) \ \nu(dx) + \int_{|x|\geqslant 1} (1 - e^{-i\theta x}) \ \nu(dx).$$

We know that integrals in the above Lévy-Khintchine formula represent a compound Poisson process and a compensated jump process. To identify what remains of the Lévy-Khintchine formula we shall make use of the following result

Theorem 2.3. *Let $\{W_t, t \geqslant 0\}$ be standard Brownian motion under the measure $\mathbf{P}$ with filtration $\{\mathcal{F}_t, t \geqslant 0\}$ and let u be a constant. Then the process $\{Y_t, t \geqslant 0\}$ is a complex martingale where $Y_t = \exp(iuW_t - \frac{1}{2}u^2 t)$*

Proof:

Clearly we have that $\mathbf{E}^{\mathbf{P}}|Y_t| = 1 < \infty$. Now

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\left(Y_t \middle| \mathcal{F}_s\right) \\
&= \mathbf{E}^{\mathbf{P}}\left(\exp(iu(W_t + W_s - W_s) - \frac{1}{2}u^2 t) \middle| \mathcal{F}_s\right) \\
&= \exp(iuW_s) \ \mathbf{E}^{\mathbf{P}}\left(\exp\big(iu(W_t - W_s) - \frac{1}{2}u^2 t\big) \middle| \mathcal{F}_s\right) \\
&= \exp(iuW_s - \frac{1}{2}u^2 t) \ \mathbf{E}^{\mathbf{P}}(\exp(iu(W_t - W_s)) \quad \text{independent increments} \\
&= \exp(iuW_s - \frac{1}{2}u^2 t) \ \exp(u^2(t-s)) \qquad W_t - W_s \sim N(0, t-s) \\
&= \exp(iuW_s - \frac{1}{2}u^2 s) = Y_s \qquad \mathbf{P}\text{−a.s.} \qquad \blacksquare
\end{aligned} \tag{2.18}$$

By making use of Theorem 2.3 above and the fact that the martingale must have a constant mean we have

$$\begin{aligned} \mathbf{E}^{\mathbf{P}}(Y_t) &= Y_0 \\ \Rightarrow \mathbf{E}^{\mathbf{P}}\big(\exp\big(iuW_t - \frac{1}{2}u^2 t\big)\big) &= 1 \\ \Rightarrow \mathbf{E}^{\mathbf{P}}\big(\exp(iuW_t)\big) &= \exp\bigg(\frac{1}{2}u^2 t\bigg). \end{aligned} \tag{2.19}$$

We can now identify the rest of the Lévy-Khintchine formula as as Brownian motion with drift. Notice that we can factor the characteristic function of the Lévy process into three distinct components as follows (see Theorem 1.2)

$$\begin{aligned} \mathbf{E}^{\mathbf{P}}\left(\mathrm{e}^{-i\theta X_t}\right) &= \exp\bigg(-i\mu\theta t - t\frac{\theta^2 a^2}{2}\bigg)\cdot\exp\bigg(-t\int_{|x|\geqslant 1}(1-e^{-i\theta x})\,\nu(dx)\bigg) \\ &\quad \times\exp\bigg(-t\int_{|x|<1}(1-e^{-i\theta x}-i\theta x)\,\nu(dx)\bigg). \end{aligned} \tag{2.20}$$

Hence we can conclude that the Lévy process X_t must be of the form

$$X_t = \mu t + aW_t + \int_{|x|<1} x\,\tilde{N}(t,dx) + \int_{|x|\geqslant 1} x\,N(t,dx) \tag{2.21}$$

where W_t is standard Brownian motion and the drift μ is given by

$$\mu = \mathbf{E}^{\mathbf{P}}\bigg(X_1 - \int_{|x|\geqslant 1} x\,N(1,dx)\bigg). \tag{2.22}$$

Remark 2.4. Note that we have not imposed the integrability requirement that $\int_{|x|\geqslant 1} x\,\nu(dx) < \infty$. Even though $\int_{|x|\geqslant 1} x\,N(t,dx)$ is finite almost surely it does not imply that any of the moments of the random variable exist. However it can be shown that for any Lévy process of the form

$$X_t - \int_{|x|\geqslant c} x\,N(t,dx) \quad \forall c \geqslant 1, \tag{2.23}$$

will always have finite moments of all orders, see [4] chapter 2 pg 101. Hence Eq.(2.22) will be finite even if f is not integrable with respect to ν on the set $\{|x| \geqslant 1\}$.

Eq.(2.21) is often referred to as the Lévy-Itô Decomposition. We can now conclude that the Lévy process X must be a linear combination of a Brownian motion process, a compound Poisson process and a compensated jump process responsible for all jumps of size less than one. Furthermore the Brownian motion process and both the jump processes are independent of each other.

We need to mention a little more about the measure ν. As stated earlier ν must be a σ-finite measure satisfying

$$\nu\{0\} = 0 \quad \text{and} \quad \int \min\{x^2, 1\}\,\nu(dx) < \infty, \tag{2.24}$$

focusing on the interval where the jump sizes are less than one in magnitude we must have

$$\int_{|x|<1} x^2 \; \nu(dx) < \infty. \tag{2.25}$$

Since $\nu(\cdot) = \mathbf{E}^{\mathbf{P}}(N(1,\cdot))$ we can conclude that

$$\int_{|x|<1} x^2 \; \nu(dx) < \infty$$

$$\mathbf{E}^{\mathbf{P}}\left(\int_{|x|<1} x^2 \; N(t,dx)\right) < \infty$$

$$\text{and thus } \int_{|x|<1} x^2 \; N(t,dx) < \infty \qquad \mathbf{P}\text{−a.s.}, \tag{2.26}$$

but Eq.(2.26) is just

$$\sum_{0 \leqslant s \leqslant t} \Delta X_s^2 \mathbf{1}_{(\Delta X_s < 1)} < \infty \qquad \mathbf{P}\text{−a.s.}$$

Recall Eq.(2.8) which guarantees that for any Borel measurable function f and Borel set A bounded away from zero that the Poisson integral is finite almost surely. By taking $f(x) = x^2$ and $A = \{|x| \geqslant 1\}$ we have

$$\int_{|x|\geqslant 1} x^2 \; N(t,dx) = \sum_{0 \leqslant s \leqslant t} \Delta X_s^2 \mathbf{1}_{(\Delta X_s \geqslant 1)} < \infty \qquad \mathbf{P}\text{−a.s.}$$

This together with Eq.(2.26) guarantees that the sum of all jumps squared will remain finite, i.e.

$$\sum_{0 \leqslant s \leqslant t} \Delta X_s^2 < \infty \qquad \mathbf{P}\text{−a.s.}$$

as claimed in Eq.(2.4). Note that while we can claim that the sum of all jumps squared will remain finite we cannot guarantee that any of the moments exist for either of the jump processes. The *Lévy measure* ν contains all the information about the jump process. The drift μ, the coefficient of the Brownian motion which is represented by the constant a and the Lévy measure ν completely specify the Lévy process X_t and it is because of this that the following triplet (μ, a, ν) is called the *Lévy or characteristic triplet*. We shall now state some standard results concerning that of the Lévy measure and the coefficient of the Brownian motion

Proposition 2.1.

. *If* $\nu(\mathbb{R}) < \infty$ *then the sample paths of* X *have a finite number of jumps in any finite time interval. The Lévy process* X *is said to have* finite activity.

. *If* $\nu(\mathbb{R}) = \infty$ *then the sample paths* X *have an infinite number of jumps in any finite time interval. The Lévy process is said to have* infinite activity.

. *If* $a = 0$ *and* $\int_{|x|<1} |x| \; \nu(dx) < \infty$ *then the sample paths of the Lévy process* X *have* finite first order variation.

. *If* $a \neq 0$ *or* $\int_{|x|<1} |x| \; \nu(dx) = \infty$ *then the sample paths of the Lévy process* X *are said to have* infinite first order variation.

If we let $\nu(\cdot)$ and $\mu = 0$ in the Lévy triplet then the Lévy process is simply standard Brownian motion. Similarly if we let $a = 0$ and $\nu(\mathbb{R} - \{1\}) = 0$ with $\nu(1) = 1$ then we obtain a jumps process that can only take on jumps whose size is the value one, which is the Poisson process. Therefore we can conclude that the Brownian motion and Poisson process are special cases of the Lévy process. This should not be very surprising since both the Poisson process and Brownian motion satisfy the requirements of a Lévy process provided in Definition 2.1.

We shall conclude this section by stating some of the additional properties satisfied by the Lévy process X. An important property that is satisfied by the Lévy process X is that it belongs the class of processes known as *semimartingales*. More precisely a stochastic process $\{X_t, t \geqslant 0\}$ is said to be a semimartingale if it is an adapted process that admits representation as

$$X_t = X_0 + M_t + C_t, \tag{2.27}$$

where $M = \{M_t, t \geqslant 0\}$ is a *local martingale* and $C = (C_t, t \geqslant 0)$ is an adapted process which has finite first order variation. From the Lévy-Itô decomposition in Eq.(2.21) we know that standard Brownian motion is indeed a martingale and that the compensated jump process $\int_{|x|<1} x \; \tilde{N}(t, dx)$ is also a square integrable martingale. By grouping the drift term μt and the compound Poisson process $\int_{|x|\geqslant 1} x \; N(t, dx)$ it can be shown that the sum of the drift and compound Poisson process has finite first order variation. By defining

$$M_t = aW_t + \int_{|x|<1} x \; \tilde{N}(t, dx) \quad \text{and} \quad C_t = \mu t + \int_{|x|\geqslant 1} x \; N(t, dx),$$

we can see the Lévy process X does indeed admit representation as a semimartingale since it is of the form required by Eq.(2.27). For a formal definition of a local martingale see Definition 2.4.

2.2 Stochastic Integration

In this section we shall examine integrals of the form

$$\int_0^T h_t \; dX_t \tag{2.28}$$

for a selected class of integrand processes $\{h_t, 0 \leqslant t \leqslant T\}$ and where the integrator process $\{X_t, 0 \leqslant t \leqslant T\}$ is some Lévy process. Integrals of this form are commonly referred to as Lévy integrals. We shall start by reviewing the Brownian motion integral, as much of the theory concerning Lévy integrals can be closely linked to that of the Brownian motion or the *Itô integral*. While the main aim of this chapter is to define stochastic integrals with respect to Lévy processes there are other results that are of importance. In particular we shall be interested in Itô's Lemma for a function of a Lévy process and the product rule between any two Lévy process as these are the tools that we shall require later on when faced with problem of pricing financial instruments. Before proceeding we shall need to define some of the terminology that we shall use throughout this section.

Definition 2.3. : A discrete time parameter process $\{Y_k,\ k = 1, \ldots, n\}$ is said to be *predictable* with respect to a sequence of information sets $\{\mathcal{F}_k,\ k = 1, \ldots, n\}$ if Y_k is a $\mathcal{F}_{k-1}$ measurable function for each and every time point $k = 1, \ldots, n$.

Having defined X_{t-} as the left limit of a process X at time t, another process $\{Y_t,\ 0 \leqslant t \leqslant T\}$ is said to be predictable (with respect to X) if Y_t is progressively measurable with respect to the information sets $\mathcal{F}_t^{X-} = \sigma(X_{s-},\ 0 \leqslant s \leqslant t)$. The notion of predictability is important in defining the stochastic integral as we shall see. We begin now by recalling the Itô integral and our focus shall be on some finite time interval $[0, T]$ where $T \in [0, \infty)$. While integrals of the form

$$\int_0^T f(W_s)\, ds$$

can be given a Rieman interpretation the unbounded first order variation property of a Brownian motion sample path makes it impossible to develop a path-by-path Rieman-Stieltjies based interpretation for integral expressions of the form

$$I_T = \int_0^T h_s\, dW_s, \tag{2.29}$$

where $\{W_t, t \geqslant 0\}$ is standard Brownian motion. Instead we start by considering a special class of integrands $h = \{h_t, 0 \leqslant t \leqslant T\}$ that are predictable with respect to the filtration $(\mathcal{F}_t^W, 0 \leqslant t \leqslant T)$ and satisfy the following condition

$$\int_0^T \mathbf{E}^{\mathbf{P}}\left(h_s^2\right) ds < \infty. \tag{2.30}$$

Whenever Eq.(2.30) is satisfied we usually write $h \in \mathcal{H}_2(T)$. Note that Fubini's Theorem (see [45]) allows us to interchange expectation and integration operators so we can rewrite the condition in Eq.(2.30) as follows

$$\mathbf{E}^{\mathbf{P}}\left(\int_0^T h_s^2\, ds\right) < \infty. \tag{2.31}$$

Providing that $h \in \mathcal{H}_2(T)$ holds true we can then find a sequence $(\phi_t^{(n)},\ n = 1, 2, \ldots)$ of predictable step functions that converge to h_t in mean square, i.e.

$$\lim_{n\to\infty} \mathbf{E}^{\mathbf{P}}\left(\int_0^T (\phi_t^{(n)} - h_t)^2\, dt\right) = 0. \tag{2.32}$$

A proof of the above result can be found in Karatzas and Shreve (1988) [27] or Chung and Williams [25]. The reason for working with step functions is that the Itô integral is defined in a manner that "resembles" the Rieman-Sieltjies integral provided that the integrand is a simple predicable step function. On replacing the integrand h_t with the approximating sequence of predictable step functions $\phi_t^{(n)}$ we then partition the interval $[0,T]$ as follows, let $(\mathcal{P}_n,\ n \in \mathbb{N})$ be a sequence of partitions of the interval $[0,T]$ that satisfy $0 = t_0^n \leqslant t_1^n \leqslant \ldots \leqslant t_n^n = T$. The Itô integral is then defined in the following manner

$$\int_0^T h_s \, dWs = \lim_{n\to\infty} \sum_{i=1}^{n} \phi^{(i)} \big(W_{t_i} - W_{t_{i-1}}\big). \tag{2.33}$$

The limit in Eq.(2.33) can be shown to exist in $L^2(\Omega, \mathbf{P})$ and is invariant of the choice of predictable step functions $\phi_t^{(n)}$ so the limit is well defined. A full and detailed account of the Itô integral can be found in Kou [32] or Lamberton and Lapeyre [33] or Shreve [44].

Remark 2.5. Since the sample paths of Brownian motion are continuous $\mathbf{P}$-a.s. we will always have $W_{t-} = W_t$ and hence the σ-field being generated by W_{t-} will coincide with the σ-field being generated by W_t, i.e. $\mathcal{F}_t^{W_{t-}} = \mathcal{F}_t^{W_t}$. Therefore the requirement that the integrand process be predictable implies that $\{h_t, 0 \leqslant t \leqslant T\}$ is also adapted to the filtration $(\mathcal{F}_t^W, 0 \leqslant t \leqslant T)$. Such an implication only holds true due to the continuous nature of the Brownian motion sample paths. As we shall see when dealing with a Lévy process that does not have $\mathbf{P}$-a.s. continuous sample paths we can no longer claim that $\mathcal{F}_t^{X_{t-}} = \mathcal{F}_t^{X_t}$ and it is then that the notion of predicability becomes essential in defining the stochastic integral.

Next we shall explore the properties that are associated with the Itô integral with the help of the following theorem

Theorem 2.4. *An Itô interpretation applied to the stochastic integral* $I_t = \int_0^t h_s \, dW_s$ *will produce a stochastic process with* $\mathbf{P}$*-a.s. continuous sample paths and*

(i) $\mathbf{E}^{\mathbf{P}}[I_t] = 0$

(ii) $\mathbf{Var}^{\mathbf{P}}\left(\int_0^t h_s \, dW_s\right) = \mathbf{E}^{\mathbf{P}}\left(\int_0^t h_s^2 \, ds\right) = \int_0^t \mathbf{E}^{\mathbf{P}}\left(h_s^2\right) ds$ ***(Itô isometry)***

(iii) $\mathbf{E}^{\mathbf{P}}(I_t|\mathcal{F}_s^W) = I_s \quad \mathbf{P}-$*a.s.*

Proof: See Shreve [44] chapter 4 ■

The most important result in Theorem 2.4 which has significant implications in finance is the fact that the Itô integral is a square integrable martingale under the measure $\mathbf{P}$. This martingale property will prove essential when we wish to price some contingent claim. Note that the martingale property of the Itô integral requires that the integrand process satisfy $h \in \mathcal{H}_2(T)$. If we were to relax this condition and only require that the integrand process satisfy

$$\mathbf{P}\bigg(\int_0^T h_s^2 \, ds \Big\langle \infty) = 1, \tag{2.34}$$

then we would still be able to define the Itô integral in a manner similar to Eq.(2.33) but the limit is now taken in probability, a weaker mode of convergence as apposed to convergence in mean square. Whenever Eq.(2.34) holds we shall denote this by $h \in \mathcal{D}_2(T)$. However we can no longer claim that the stochastic process $\{I_t = \int_0^t h_s \, dW_s, 0 \leqslant t \leqslant T\}$ is a martingale under the measure $\mathbf{P}$ if $h \in \mathcal{D}_2(T)$. Instead we will only have the Itô integral being a *local martingale* and not necessarily a martingale under the measure $\mathbf{P}$. We now define what is meant by a local martingale.

Definition 2.4. : A process $\{X_t, t \geqslant 0\}$ is called a local martingale if there exists as sequence of stopping times $\{\tau_n,\ n = 1, 2 \ldots\}$ such that $\tau_n \to \infty \quad \mathbf{P}-$a.s. as $n \to \infty$ and for each fixed n the stopped process $\{X_{t\wedge\tau_n},\ t \geqslant 0,\ n \text{ fixed}\}$[2] is a uniformly integrable martingale.[3]

Clearly all martingales are local martingales but the converse need not be true. We therefore need to be careful when considering our integrand process as it has a direct effect on the martingale property of the Itô integral. One final result concerning the Itô integral that shall prove useful is that if the integrand h_t is a deterministic function of time then the distribution of $\int_0^t h_s \, dW_t$ can be shown to be that of a normal random variable, i.e. we have for each fixed t

$$\int_0^t h_s \, dW_s \sim N\left(0, \int_0^t h_s^2 \, ds\right). \tag{2.35}$$

For a proof of the above result see Kuo [32] chapter 2 Theorem 2.3.4 .

Let us now return to developing a proper meaning for stochastic integrals where the integrator is some arbitrary Lévy process and assume without loss of generality that the Lévy triplet is of the form $(1, 1, \nu)$. By making use of the Lévy-Itô decomposition we can write $\int_0^t h_s \, dX_s$ as

$$\begin{aligned}\int_0^t h_s \, dX_s \quad &= \quad \underbrace{\int_0^t h_s \, ds}_{I_1} \; + \; \underbrace{\int_0^t h_s \, dW_s}_{I_2} \; + \; \underbrace{\int_0^t \int_{|x|<1} x h_s \, \tilde{N}(ds, dx)}_{I_3} \\ &\quad + \; \underbrace{\int_0^t \int_{|x|\geqslant 1} x h_s \, N(ds, dx)}_{I_4}.\end{aligned}$$

The integral I_1 is the normal Riemann integral which is well defined for all suitably bounded integrands. As we have already discussed the construction of I_2, we focus our attention on I_3 and I_4. In particular we are interested in the construction of both integrals I_3 and I_4 and properties these integrals possess. We shall begin by considering I_4, since the integrand is a function of x and s it will be easier working with integrals of the form

$$\int_0^t \int_A F(s, x) \, N(ds, dx), \tag{2.36}$$

[2] $t \wedge \tau_n = min(t, \tau_n)$

[3] We say that X is uniformly integrable if $\lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} \mathbf{E}^{\mathbf{P}}(|X_t| \mathbf{1}_{(|X_t|>n)}) = 0$

where the Borel subset A is bounded away from zero and $N(\cdot,\cdot)$ is a random Poisson measure on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$ as defined Eq.(2.5). With this random measure we have an intensity measure $\nu(\cdot) = \mathbf{E}^{\mathbf{P}}[N(\cdot,1)]$ which is also the Lévy measure. Since the Borel subset is bounded away from zero we can define the integral I_4 in a manner similar to that of Eq.(2.7), as a random finite sum that may be expressed as

$$\int_0^t \int_A F(s,x)\; N(ds,dx) = \sum_{0 \leqslant u \leqslant t} F(u, \Delta X_u) \mathbf{1}_{(\Delta X_u \in A)}. \tag{2.37}$$

The only constraint that we shall place on the integrand process F is that it be predictable with respect to $(\mathcal{F}_t^X, 0 \leqslant t \leqslant T)$, the filtration being generated by the Lévy process X. We shall discuss briefly why we must work with a predictable integrand process. Suppose that in addition to the predictability requirement the integrand process F also satisfies

$$\mathbf{E}^{\mathbf{P}} \left(\int_0^T \int_A |F(s,x)|^2\; \nu(dx) ds \right) < \infty. \tag{2.38}$$

We can then define a compensated integral process by

$$\int_0^t \int_A F(s,x)\; \tilde{N}(ds,dx) = \int_0^t \int_A F(s,x)\; N(ds,dx) - \int_0^t \int_A F(s,x)\; \nu(dx) ds. \tag{2.39}$$

Clearly the integrator $\tilde{N}$ is a martingale as shown in Eq.(2.11). Within the theory of stochastic integration whenever we have a martingale process serving as the integrator it is highly desirable that the resulting stochastic integral should at the very least be a local martingale. So we would ideally like to have Eq.(2.39) being a local martingale if not a martingale. This can only be made possible if the integrand process is predictable. It is for this reason that we impose the constraint that F be predictable. The following simple but effective example illustrates the consequences of not working with predictable integrand processes. Consider the Poisson process $N = \{N_t, 0 \leqslant t \leqslant T\}$ with intensity parameter λ under the measure $\mathbf{P}$, define the compensated Poisson process which we know to be a martingale by $\tilde{N}_t = N_t - \lambda t$ and take ΔN_s to be the integrand

$$\Delta N_s = \begin{cases} 1 & \text{whenever a jump occurs at time } s \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\begin{aligned} \int_0^t \Delta N_s\; d\tilde{N}_s &= \int_0^t \Delta N_s\; dN_s - \lambda \int_0^t \Delta N_s\; ds \\ &= \int_0^t dN_s - \lambda \cdot 0 \\ &= N_t. \end{aligned} \tag{2.40}$$

In the above example, our integrator was a martingale process but the resulting stochastic integral was the Poisson process N which we know is *not* a martingale. The reason for this lies in the fact that our integrand process ΔN_s is not predictable since the value of ΔN_s is only determined by observing N_s up to the time point s and not before. The above example illustrates the importance of working with a predictable integrand process.

Finally let us now consider the integral represented by I_3. Again because the integrand is a function of both x and s we shall focus on integrals of the form

$$\int_0^t \int_{|x|<1} H(s,x)\ \tilde{N}(ds,dx). \tag{2.41}$$

Since the random Poisson measure $N(\cdot,\cdot)$ has an accumulation point at zero we know that the number of jumps, whose magnitude is less than one, may become unbounded. It is for this reason that we must work with a compensated integral process but even this is not enough to guarantee that the stochastic integral in Eq.(2.41) exists. Once again we start by considering a specific class of integrand processes $H = \{H(t,x),\ 0 \leqslant t \leqslant T\}$ that are predictable with respect to the filtration $(\mathcal{F}_t^X,\ 0 \leqslant t \leqslant T)$ and satisfies

$$\int_0^T \int_{|x|<1} \mathbf{E}^{\mathbf{P}}\left(|H(s,x)|^2\right)\ \nu(dx)ds < \infty. \tag{2.42}$$

Whenever Eq.(2.42) holds we shall denote this by $H \in \mathcal{H}_2(T,E)$, where $E = \{x \in \mathbb{R} - \{0\}, |x| < 1\}$. Provided that $H \in \mathcal{H}_2(T,E)$ we can then find a sequence of predictable step functions that converge to H in mean square. The construction of the integral in Eq.(2.41) then resembles that of the Brownian motion integral, where we replace the integrand by the predictable step functions and then take a limit which can be shown to exist in $L^2(\Omega,\mathbf{P})$. The limit is also independent of the choice of predictable step functions used and hence the limit is well defined. Therefore in order to define the integral in Eq.(2.41) we require that H be predictable and that $H \in \mathcal{H}_2(T,E)$ as this will ensure that the stochastic process defined by $\{\int_0^t \int_{|x|<1} H(s,x)\ \tilde{N}(ds,dx),\ 0 \leqslant t \leqslant T\}$ will be a square integrable martingale. We shall now list some of the properties of this stochastic integral with the aid of the following theorem.

Theorem 2.5. *If $H \in \mathcal{H}_2(T,E)$ and is predictable with respect to the filtration $(\mathcal{F}_t^X,\ 0 \leqslant t \leqslant T)$ then we have the following*

(i) $\mathbf{E}^{\mathbf{P}}\big(\int_0^t \int_{|x|<1} H(s,x)\ \tilde{N}(ds,dx)\big) = 0,$

(ii) $\mathbf{E}^{\mathbf{P}}\big(\big|\int_0^t \int_{|x|<1} H(s,x)\ \tilde{N}(ds,dx)\big|^2\big) = \int_0^t \int_{|x|<1} \mathbf{E}^{\mathbf{P}}\big|H(s,x)\big|^2\ \nu(dx)ds,$

(iii) $\mathbf{E}^{\mathbf{P}}\big(\int_0^t \int_{|x|<1} H(u,x)\ \tilde{N}(du,dx)\big|\mathcal{F}_s^X\big) = \int_0^s \int_{|x|<1} H(u,x)\ \tilde{N}(du,dx).$

Proof: See Applebaum [4], chapter 4 ■

The proof is carried out by considering the integrator to be any square integrable martingale valued measure on $\mathbb{R}^+ \times (\mathbb{R} - \{0\})$ with independent increments that has zero measure at time $t = 0$, for which the compensated Poisson measure can be shown to be a special case. Similar to that of the Itô integral if we wish to extend the class of permissible integrands to a much broader class of processes we could impose a less restrictive constraint on the integrand process by requiring that

$$\mathbf{P}\left(\int_0^T \int_{|x|<1} |H(s,x)|^2 \, \nu(dx)ds < \infty\right) = 1, \tag{2.43}$$

whenever Eq.(2.43) holds we say $H \in \mathcal{D}_2(T,E)$. With this constraint on the integrand process the stochastic integral defined in Eq.(2.41) arises as a limit in probability and no longer a limit in mean square. A consequence of extending the class of permissible integrands is that we can no longer guarantee that the process $\{\int_0^t \int_{|x|<1} H(s,x)\ \tilde{N}(ds,dx), 0 \leqslant t \leqslant T\}$ is a square integrable martingale under the measure $\mathbf{P}$ as we had in Theorem 2.5 but rather is now a local martingale.

So to give a brief summary, we are interested in defining integrals of the form $\int_0^t h_s\, dX_s$ where the integrator is some Lévy process. In order to define the Lévy integral we must have the integrand process satisfying some constraints, the first constraint that we shall impose on the integrand is that it be predictable with respect to the filtration $(\mathcal{F}_t^X, 0 \leqslant t \leqslant T)$. The second constraint takes the form of

$$\int_0^T \mathbf{E}^{\mathbf{P}}\left(h_s^2\right) ds < \infty, \tag{2.44}$$

for the Brownian motion integral, while for stochastic integrals of the form $\int_0^t \int_{|x|<1} H(s,x)\ \tilde{N}(ds,dx)$ the constraint is given by

$$\int_0^T \int_{|x|<1} \mathbf{E}^{\mathbf{P}}\left(|H(s,x)|^2\right) \nu(dx)ds < \infty. \tag{2.45}$$

We can easily extend the class of permissible integrands to a larger class of processes by imposing less restrictive constraints given by $h \in \mathcal{D}_2(T)$ or $H \in \mathcal{D}_2(T,E)$, however the consequences of doing so means that we may no longer have the resulting stochastic integral processes being square integrable martingales but rather local martingales. Integrals of the form $\int_0^t \int_A F(s,x)\ N(ds,dx)$ can be defined as a random finite sum as in Eq.(2.39), for A bounded away from zero and F a predictable integrand. It is also possible to give the integral process $\{\int_0^t \int_A F(s,x)\ \tilde{N}(ds,dx), 0 \leqslant t \leqslant T\}$ a martingale property under the measure $\mathbf{P}$ but we will have to place a constraint on the integrand similar to that of Eq.(2.45), i.e. we require

$$\int_0^T \int_A \mathbf{E}^{\mathbf{P}}\left(|F(s,x)|^2\right) \nu(dx)ds < \infty, \tag{2.46}$$

to ensure that we have a square integrable martingale process, see Elliott [14]. As we have seen the predictability constraint that we place on the integrand process is vital in ensuring that the resulting stochastic integral has the martingale property. Predictability with respect to the filtration being generated by the Lévy process X means that the integrand must be measurable with respect to the filtration $(\mathcal{F}_t^{X_{t-}},\ 0 \leqslant t \leqslant T)$. Since the Lévy process admits possible jumps we can no longer assume that $X_t = X_{t-}$ as was the case with Brownian motion, therefore $\mathcal{F}_t^{X_t} \neq \mathcal{F}_t^{X_{t-}}$ for all $t \in [0, T]$ since the sample paths of the Lévy process are not continuous in $\mathbf{P}$-a.s. manner whenever we have jumps occurring. Concluding, in this section we have introduced the stochastic integral where the integrator process was a Lévy process. In order to give an interpretation to this stochastic integral we required that the integrand be predicable and satisfy an integrability constraint.

2.2.1 Quadratic variation and Itô's lemma

Before proceeding towards Itô's Lemma for a Lévy process we need to discuss the concept of quadratic variation. It is this notion of quadratic variation that distinguishes stochastic calculus from ordinary calculus and as we shall see plays a significant role in Itô's Lemma. We start by formally defining what is meant by quadratic variation.

Definition 2.5. Let $(\mathcal{P}_n, n \in \mathbb{N})$ be a partition of the interval $[0, T]$ such that $\mathcal{P}_n = \{0 = t_0^n < t_1^n < \ldots < t_n^n = T\}$ and let $||\pi_n|| = \max\limits_{1 \leqslant i \leqslant n} |t_i - t_{i-1}|$. Then the quadratic variation for a stochastic process $\{X_t, t \geqslant 0\}$ up to time T, denoted by $[X, X]_T$, is defined by

$$[X, X]_T = \lim_{||\pi_n|| \to 0} \sum_{i=1}^{n} |X_{T \wedge t_i^n} - X_{T \wedge t_{i-1}^n}|^2,$$

provided that the limit can be shown to exist in some sense, i.e. in either a $L^2(\Omega, \mathbf{P})$ or in a $\mathbf{P}$-a.s. sense. As $[X, X]_T$ is a random variable, if we vary the time index t we then obtain a *quadratic variation process* $\{[X, X]_t, 0 \leqslant t \leqslant T\}$ where for each t, $[X, X]_t$ is defined by

$$[X, X]_t = \lim_{||\pi_n|| \to 0} \sum_{i=1}^{n} |X_{t \wedge t_i^n} - X_{t \wedge t_{i-1}^n}|^2. \tag{2.47}$$

For the sake of completeness we shall define first order variation, which we shall denote by $|X|_t$, for the same stochastic process $\{X_t, t \geqslant 0\}$, by

$$|X|_t = \lim_{||\pi_n|| \to 0} \sum_{i=1}^{n} |X_{t \wedge t_i} - X_{t \wedge t_{i-1}}|.$$

When we deal with ordinary calculus functions with continuous derivatives can be shown to have finite first order variation and quadratic variation equal to zero, it is for this reason that quadratic variation is not considered within ordinary calculus. However, within stochastic calculus processes like the Brownian motion have infinite first order variation and finite quadratic variation and it is this nonzero quadratic variation that becomes very important when we wish to form a chain rule for stochastic functions as it means the appearance of second order terms that would normally have been absent within the realms of ordinary calculus. For our purposes we shall mainly be focused on computing the quadratic variation for Lévy type stochastic integrals of the form $L_t = \int_0^t h_s \, dX_s$. Let us start again by considering the Brownian motion integral. Having developed an interpretation for $\int_0^t h_s \, dW_s$ where the integrator is standard Brownian motion under the measure $\mathbf{P}$ and the integrand process $h \in \mathcal{H}_2(T)$ we are now interested in calculating

$$\left[\int_0^\cdot h_s \, dW_s, \int_0^\cdot h_s \, dW_s \right]_t . \tag{2.48}$$

By making use of the well known fact that Brownian motion accumulates t units of quadratic variation in the interval $[0, t]$, i.e. $[W, W]_t = t$. It can be shown that the quadratic variation for the Brownian motion integral reduces to

$$\begin{aligned} \left[\int_0^\cdot h_s \, dW_s, \int_0^\cdot h_s \, dW_s \right]_t &= \int_0^t h_s^2 \, d[W, W]_s \\ &= \int_0^t h_s^2 \, ds, \end{aligned} \tag{2.49}$$

and since $h \in \mathcal{H}_2(T)$ Eq.(2.49) is finite $\mathbf{P}$-a.s. The above result can be found in Shreve chapter 4 [44] or Protter [40]. Before proceeding towards determining the quadratic variation where the stochastic integral in question has a random Poisson measure as the integrator we first need to determine $[N(t, A), N(t, A)]_t$. By definition we have for any Borel subset A bounded away from zero

$$[N(\cdot, A), N(\cdot, A)]_t = \lim_{||\pi_n|| \to 0} \sum_{i=1}^{n} |N(t_i \wedge t, A) - N(t_{i-1} \wedge t, A)|^2, \tag{2.50}$$

by making use of the fact that the increments of the Poisson process must satisfy

$$N(t_i, A) - N(t_{i-1}, A) = \begin{cases} 1 & \text{whenever a jump occurs at time } t_i \\ 0 & \text{otherwise} \end{cases}$$

we can compute the quadratic variation for $N(t, A)$ as follows

$$\begin{aligned} [N(\cdot, A), N(\cdot, A)]_t &= \lim_{||\pi_n|| \to 0} \sum_{i=1}^{n} |N(t_i \wedge t, A) - N(t_{i-1} \wedge t, A)|^2 \\ &= \lim_{||\pi_n|| \to 0} \sum_{i=1}^{N(t,A)} 1^2 \\ &= N(t, A), \end{aligned} \tag{2.51}$$

where the last line in Eq.(2.51) follows from the fact that $N(t, A)$ is independent of the partition and thus independent of $||\pi_n||$. We can then make use of this result when we wish to compute the quadratic variation for the compensated Poisson process $\tilde{N}(t, A) = N(t, A) - t\nu(A)$. By making use of the fact that

$$\Delta\tilde{N}(t_i, A)^2 = \Delta N(t_i, A)^2 - 2\nu(A)\Delta N(t_i, A)\Delta t_i + \nu(A)^2\Delta t_i^2, \tag{2.52}$$

where $\Delta t_i = t_i - t_{i-1}$ and $\Delta N(t_i, A) = N(t_i, A) - N(t_{i-1}, A)$, we then have

$$\begin{aligned}[\tilde{N}(\cdot, A), \tilde{N}(\cdot, A)]_t &= \lim_{||\pi_n||\to 0} \sum_{i=1}^{n} \Delta N(t_i, A)^2 \\ &\quad -2\nu(A) \lim_{||\pi_n||\to 0} \sum_{i=1}^{n} \Delta N(t_i, A)\Delta t_i \qquad (2.53) \\ &\quad +\nu(A)^2 \lim_{||\pi_n||\to 0} \sum_{i=1}^{n} \Delta t_i^2. \qquad (2.54)\end{aligned}$$

Now both summations in expressions (2.53) and (2.54) have limits equal to zero. To see this we shall consider expression (2.53)

$$\begin{aligned}\sum_{i=1}^{n} |\Delta N(t_i, A)||\Delta t_i| &\leqslant \max_{1\leqslant i\leqslant n} |\Delta t_i| \cdot \sum_{i=1}^{n} |\Delta N(t_i, A)| \\ &= ||\pi_n|| \cdot \sum_{i=1}^{n} |\Delta N(t_i, A)| \\ &= ||\pi_n|| \cdot N(t, A),\end{aligned} \tag{2.55}$$

now

$$||\pi_n|| \cdot N(t, A) \to 0 \quad \text{as } ||\pi_n|| \to 0, \tag{2.56}$$

since $N(t, A) < \infty$ **P**-a.s. In a similar manner we can show that the summation in expression (2.54) converges to zero and the proof mimics what we have done above, we can thus conclude that the quadratic variation for the compensated Poisson process is given by

$$[\tilde{N}(\cdot, A), \tilde{N}(\cdot, A)]_t = [N(\cdot, A), N(\cdot, A)]_t = N(t, A). \tag{2.57}$$

Returning to the problem of determining the quadratic variation for the stochastic integral where the integrator is a random Poisson measure, let us first consider the stochastic integrals of the form $\int_0^t \int_A F(s, x)\ N(ds, dx)$ where the integrand is predictable and the Borel set A is bounded away from zero. By making use of the fact that the value of the stochastic integral is zero whenever there are no jumps, the quadratic variation is then given by

$$\begin{aligned}&\left[\int_0^{\cdot} \int_A F(s, x)\ N(ds, dx), \int_0^{\cdot} \int_A F(s, x)\ N(ds, dx)\right]_t \\ &= \int_0^t \int_A F(s, x)^2\ [N(ds, dx), N(ds, dx)]_s \\ &= \int_0^t \int_A F(s, x)^2\ N(ds, dx),\end{aligned} \tag{2.58}$$

and by using the same arguments that precede Eq.(2.8), we will always have Eq.(2.58) being finite in a **P**-a.s. sense. Finally let us proceed towards the determining the quadratic variation for integrals of the form $\int_0^t \int_{|x|<1} H(s,x)\, \tilde{N}(ds,dx)$. We consider the case where $H \in \mathcal{H}_2(T,E)$ as this will ensure that

$$\begin{aligned} & \int_0^t \int_{|x|<1} \mathbf{E}^{\mathbf{P}} |H(s,x)|^2\, \nu(dx)dx < \infty \\ \Rightarrow \quad & \mathbf{E}^{\mathbf{P}} \left(\int_0^t \int_{|x|<1} H(s,x)^2\, N(ds,dx) \right) < \infty \\ \Rightarrow \quad & \int_0^t \int_{|x|<1} H(s,x)^2\, N(ds,dx) < \infty \qquad \mathbf{P}-\text{a.s.} \end{aligned}$$

The quadratic variation for $\int_0^t \int_{|x|<1} H(s,x)\, \tilde{N}(ds,dx)$ is then computed in a manner similar to that in Eq.(2.58)

$$\begin{aligned} & \left[\int_0^{cdot} \int_{|x|<1} H(s,x)\, \tilde{N}(ds,dx), \int_0^{\cdot} \int_{|x|<1} H(s,x)\, \tilde{N}(ds,dx) \right]_t \\ = \; & \int_0^t \int_{|x|<1} H(s,x)^2\, [\tilde{N}(ds,dx), \tilde{N}(ds,dx)]_s \\ = \; & \int_0^t \int_{|x|<1} H(s,x)^2\, N(ds,dx), \end{aligned} \tag{2.59}$$

where the last line in Eq.(2.59) follows from the fact that that the quadratic variation for the compensated Poisson measure reduces to that of $N(\cdot,\cdot)$ as shown in Eq.(2.57).

Remark 2.6. We have only considered quadratic variation for stochastic integrals where the integrand process either lies in $\mathcal{H}_2(T)$ or $\mathcal{H}_2(T,E)$. If we were to consider the case where the integrand process only satisfied the less restrictive conditions of $h \in \mathcal{D}_2(T)$ or $H \in \mathcal{D}_2(T,E)$ then the quadratic variation for the stochastic integrals would still exist and correspond to the results shown above but convergence is now taken in a probability sense.

One final result that we shall require concerns the quadratic covariation between two stochastic processes

Theorem 2.6. *Given a process $\{X_t, t \geqslant 0\}$ with* **P***-a.s. continuous sample paths and a process $\{Y_t, t \geqslant 0\}$ with finite first order variation, then*

$$[X,Y]_t = 0.$$

Proof:

$$\begin{aligned} [X,Y]_t &= \lim_{||\pi_n|| \to 0} \sum_{i=1}^n |X_{t\wedge t_i} - X_{t\wedge t_{i-1}}||Y_{t\wedge t_i} - Y_{t\wedge t_{i-1}}| \\ &\leqslant \lim_{||\pi_n|| \to 0} \max_{1 \leqslant i \leqslant n} |X_{t\wedge t_i} - X_{t\wedge t_{i-1}}| \cdot \lim_{||\pi_n|| \to 0} \sum_{i=1}^n |Y_{t\wedge t_i} - Y_{t\wedge t_{i-1}}| \\ &= \lim_{||\pi_n|| \to 0} \max_{1 \leqslant i \leqslant n} |X_{t\wedge t_i} - X_{t\wedge t_{i-1}}| \cdot |Y|_t. \end{aligned}$$

It follows from the **P**-a.s. sample path continuity of X_t that

$$\lim_{||\pi_n||\to 0} \max_{1\leqslant i\leqslant n} |X_{t\wedge t_i} - X_{t\wedge t_{i-1}}| = 0$$

and since Y_t has finite first order variation, it follows that $|Y|_t < \infty$ and hence we have

$$[X,Y]_t = 0 \cdot |Y|_t = 0 \qquad \blacksquare$$

The following useful technique is used when we wish to determine the quadratic variation for any process. In particular consider the Lévy process, if we break up the Lévy process X_t into continuous and discontinuous parts as such

$$X_t = X_t^c + X_t^d,$$

where X_t^c represents the continuous part of the the Lévy process and X_t^d represents the discontinuous part, then the quadratic variation can also be split up in a similar manner as follows

$$[X,X]_t = [X^c, X^c]_t + \sum_{0\leqslant s\leqslant t} (\Delta X_s^d)^2, \tag{2.60}$$

see Jacob and Shirayev [24]. By making use of our knowledge on Lévy processes we know that the Brownian motion together with the drift account for the continuous part of the Lévy process while the compound Poisson process together with the compensated sum of small jumps represent the discontinuous part of the Lévy process. Therefore we have, with the help of Theorem 2.6,

$$[X,X]_t = t + \int_{|x|<1} x^2 \, N(t,dx) + \int_{|x|\geqslant 1} x^2 \, N(t,dx). \tag{2.61}$$

With this expression for the quadratic variation of a Lévy process we can now compute the quadratic variation for the Lévy stochastic integral $L_t = \int_0^t h_s \, dX_s$ as follows

$$\begin{aligned}
[L,L]_t &= [L^c, L^c]_t + \sum_{0\leqslant s\leqslant t} (\Delta L_s^d)^2 \\
&= \int_0^t h_s^2 \, d[W,W]_s + \int_0^t \int_{|x|\geqslant 1} F(s,x)^2 \, [N(ds,dx), N(ds,dx)]_s \\
&\quad + \int_0^t \int_{|x|<1} H(s,x)^2 \, [\tilde{N}(ds,dx), \tilde{N}(ds,dx)]_s \\
&= \int_0^t h_s^2 \, ds + \int_0^t \int_{|x|\geqslant 1} F(s,x)^2 \, N(ds,dx) \\
&\quad + \int_0^t \int_{|x|<1} H(s,x)^2 \, N(ds,dx).
\end{aligned} \tag{2.62}$$

In view of previously used notation it should be noted that both $F(s,x)$ and $H(s,x)$ are equal to xh_s in Eq.(2.62).

We are now finally in a position to state Itô's Lemma which is one of the most important results within stochastic calculus.

Theorem 2.7. ***Itô's Lemma*** *Let $\{X_t, t \geqslant 0\}$ be a càdlàg semimartingale stochastic process, then for each $f \in \mathcal{C}^{1,2}(\mathbb{R}^+, \mathbb{R})$, we have with probability one that*

$$f(t, X_t) - f(0, X_0) = \int_0^t \frac{\partial f}{\partial s}(s, X_{s-})\, ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-})\, dX_s$$

$$+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-})\, d[X^c, X^c]_s$$

$$+ \sum_{0 \leqslant s \leqslant t} \big[f(s, X_s) - f(s, X_{s-}) - \frac{\partial f}{\partial x}(s, X_{s-})\Delta X_s\big]. \tag{2.63}$$

Proof: See Applebaum [4], Theorem 4.4.10 ■

Note that all the partial derivatives are evaluated at the left time point of the process X, the reason for this is that it will ensure that the integrands are predictable with respect to the filtration $(\mathcal{F}_t^X, t \geqslant 0)$. We end this section by considering the product rule between any two Lévy processes.

Theorem 2.8. *If Y_t and Z_t are any two Lévy processes(or any semimartingales), then the product $Y_t Z_t$ is given by*

$$Y_t Z_t = Y_0 Z_0 + \int_0^t Y_{s-}\, dZ_s + \int_0^t Z_{s-}\, dY_s + [Y, Z]_t \tag{2.64}$$

Proof: See Elliott [14], Corollary 12.22. ■

As always it will be easier to remember this product rule if we express it in shorthand differential form as

$$d\big(Y_t Z_t\big) = Y_{t-}\, dZ_t + Z_{t-}\, dY_t + d[Y, Z]_t. \tag{2.65}$$

For our purposes we shall mainly encounter stochastic integrals in differential form as seen in Eq.(2.65) which is used just as an abbreviation and has no other separate meaning. From now on we assume that when we are dealing with stochastic integrals of the form

$$\int_0^t Y_{s-}\, dZ_s,$$

provided that the integrator process $\{Z_t, 0 \leqslant t \leqslant T\}$ can be shown to have a martingale property, we will then assume without the the loss of generality that the resulting stochastic integral process $\{\int_0^t Y_{s-}\, dZ_s, 0 \leqslant t \leqslant T\}$ is a local martingale at the very least. If we wish that the stochastic integral process should be a martingale and not just a local martingale then it will have to satisfy the integrability constraints specified in the previous section.

2.3 Change of measure and the Stochastic Exponential

The purpose of this section is to introduce to idea of an equivalent probability measure. Of particular interest to us shall be those equivalent probability measures that result in Lévy process having a martingale property. Such measures are known as *equivalent martingale measures*, usually written as EMM for short. The justification for considering equivalent martingale measures will be made clear in the next chapter when we state the Fundamental Theorem for Asset pricing. The idea is to create an artificial probability measure $\mathbf{Q}$, which is equivalent to the real world measure $\mathbf{P}$, in which all contingent claims can be valued and priced in an arbitrage free manner. In view of this change of measure we shall focus on the stochastic exponential process and provided this process has certain properties it will then be used to facilitate the change of measure and create a new equivalent measure $\mathbf{Q}$. Once we have established the change of measure our primary interest will be the how the dynamics of the Lévy process change under the new measure $\mathbf{Q}$. In particular we shall be interested in how the Lévy process can be made into a local $\mathbf{Q}$-martingale and what further constraints need be imposed that will ensure that the Lévy process becomes a $\mathbf{Q}$-martingale. Let us begin by defining what is meant by an equivalent measure.

Definition 2.6. Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Another probability measure $\mathbf{Q}$ is said to be absolutely continuous with respect to $\mathbf{P}$, denoted $\mathbf{Q} \ll \mathbf{P}$ if the following condition holds true:

$$\mathbf{P}(A) = 0 \Rightarrow \mathbf{Q}(A) = 0,$$

for $A \in \mathcal{F}$. If the reverse implication holds true, i.e. $\mathbf{Q}(A) = 0 \Rightarrow \mathbf{P}(A) = 0$, then both measures share the same null sets and are said to be equivalent measures which we shall denote by $\mathbf{Q} \sim \mathbf{P}$.

Let us now assume that we are already under some measure $\mathbf{P}$. We wish to change this measure $\mathbf{P}$ to some other equivalent measure $\mathbf{Q}$. The question now arises: how do we go about changing the measure? The following theorem will aid us in this regard and show us how we can employ a change of measure formula.

Theorem 2.9. *Given a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and a* $\mathbf{P}$*-a.s. strictly positive* $\mathcal{F}$*-measurable function* $Z(\omega)$ *satisfying* $\mathbf{E}^{\mathbf{P}}(Z) = 1$*. Then for* $A \in \mathcal{F}$*, define*

$$\mathbf{Q}(A) = \int_A Z(\omega)\, dP(\omega) = \mathbf{E}^{\mathbf{P}}(Z\mathbf{1}_A). \tag{2.66}$$

$\mathbf{Q}$ *then defines another probability space on* $(\Omega, \mathcal{F})$ *that is equivalent to* $\mathbf{P}$.

Proof: To prove that $\mathbf{Q}$ is a probability measure on $(\Omega, \mathcal{F})$ we need to establish the three axioms of a probability space. The first two axioms are easily verified, since the function Z is strictly positive we have

$$\int_\Omega 0\, dP(\omega) \leqslant \int_\Omega Z(\omega)\mathbf{1}_A\, dP(\omega) \leqslant \int_\Omega Z(\omega)\, dP(\omega).$$

This in turn implies that

$$0 \leqslant \mathbf{Q}(A) \leqslant 1 \quad \text{for all } A \in \mathcal{F}.$$

Furthermore

$$\mathbf{Q}(\Omega) = \int_\Omega Z(\omega)\, dP(\omega) = \mathbf{E}^{\mathbf{P}}(Z) = 1.$$

Finally, let $\{B_n\}_{n\geqslant 1}$ be a sequence of disjoint $\mathcal{F}$ measurable sets in Ω and $Z_m = \sum_{n=1}^{m} Z\mathbf{1}_{B_n}$. We can then use the Monotone convergence Theorem (See Weir [45]) to show that

$$\mathbf{Q}\Big(\bigcup_{n=1}^{\infty} B_n\Big) = \sum_{n=1}^{\infty} \mathbf{Q}(B_n),$$

and this completes the proof. ■

The Radon-Nikodým Theorem (see Royden [41]) can then be used to ensure that the $\mathcal{F}$-measurable random variable Z is defined in a $\mathbf{P}$-a.s. unique manner.

Remark 2.7. Note that we must have the $\mathcal{F}$-measurable random variable Z being strictly positive to ensure the resulting measure $\mathbf{Q}$ is equivalent. To see this let Z be a nonnegative random variable that takes on the value of zero for some $\omega \in \Omega$. Then if we use Theorem 2.9 to implement a change of measure then the resulting measure $\mathbf{Q}$ will not be an equivalent measure since $\mathbf{Q}(A) = 0 \not\Rightarrow \mathbf{P}(A) = 0$ for $A \in \mathcal{F}$. Such a measure would be absolutely continuous but *not* equivalent.

Because the Radon-Nikodým Theorem implies that the $\mathcal{F}$-measurable random variable Z linking $\mathbf{Q}$ to $\mathbf{P}$ is in fact $\mathbf{P}$-a.s. unique we can, without further loss of generality, define

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}}(\omega) = Z(\omega)$$

calling this random variable the Radon-Nikodým derivative linking $\mathbf{Q}$ to $\mathbf{P}$ on the measure space $(\Omega, \mathcal{F})$. Henceforth we shall write R-N derivative when making reference to the Radon-Nikodým derivative. From a finance point of view Theorem 2.9 is of incredible importance, if we consider the problem of pricing some random payoff we can then employ a change of measure that will allow us to can change the probability weights that we associate with this random payoff without changing its actual values. It is this technique of changing the probability measure that is essential to asset pricing as we shall see when we discuss the Fundamental Theorem of Asset pricing.

Having briefly described how we would go about constructing an equivalent probability measure we now wish to extend our definition of the R-N derivative to a family of measure spaces $\{(\Omega, \mathcal{F}_t), 0 \leqslant t \leqslant T\}$ thereby constructing a R-N process. The reason for wanting to create such a process lies in the fact that modeling continuous time phenomena generally requires that we work with a filtration process $(\mathcal{F}_t, 0 \leqslant t \leqslant T)$ that records the flow of information over some time horizon. To create such a R-N derivative process we need consider a strictly positive valued process $\{Z_t, 0 \leqslant t \leqslant T\}$ that has the martingale property with $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ for all $t \in [0, T]$. Such a process will then enable us to change the measure for the measure space $(\Omega, \mathcal{F}_t)$ by prescribing

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t} = Z_t.$$

Let us concentrate on the time interval $[0, T]$ for some fixed T. We wish to implement a change of measure over this interval, once we have established that the process $\{Z_t, 0 \leqslant t \leqslant T\}$ is indeed a strictly positive martingale under the measure $\mathbf{P}$ and has $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ for all $t \in [0, T]$. We then define the change of measure formula by

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = Z_T. \tag{2.67}$$

Note that we used the $\mathcal{F}_T$-measurable random variable Z_T to define the change in measure, however we can still compute probabilities based on event sets that are $\mathcal{F}_t$-measurable rather than $\mathcal{F}_T$-measurable, where $\mathcal{F}_t \subset \mathcal{F}_T$, as follows. Let $A \in \mathcal{F}_t$ then

$$\begin{aligned} \mathbf{Q}(A) &= \mathbf{E}^{\mathbf{P}}(\mathbf{1}_A Z_T) \\ &= \mathbf{E}^{\mathbf{P}}(\mathbf{E}^{\mathbf{P}}(\mathbf{1}_A Z_T|\mathcal{F}_t)) \\ &= \mathbf{E}^{\mathbf{P}}(\mathbf{1}_A \mathbf{E}^{\mathbf{P}}(Z_T|\mathcal{F}_t)) \\ &= \mathbf{E}^{\mathbf{P}}(\mathbf{1}_A Z_t). \end{aligned} \tag{2.68}$$

Eq.(2.68) follows from the fact that if $A \in \mathcal{F}_t$ it must be measurable with respect to $\mathcal{F}_t$. In essence the random variable Z_T *closes* the martingale process. This means that in addition to being $L^1(\Omega, \mathbf{P})$ integrable we can obtain any of the random variables Z_t by simply setting

$$Z_t = \mathbf{E}^{\mathbf{P}}(Z_T|\mathcal{F}_t). \tag{2.69}$$

It is for this reason that we concentrate on the random variable Z_T. Therefore when we wish to implement a change of measure we shall define

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = Z_T \tag{2.70}$$

calling this $\mathcal{F}_T$-measurable random variable the R-N derivative linking $\mathbf{Q}$ to $\mathbf{P}$ on the measure space $(\Omega, \mathcal{F}_T)$. Because the martingale process is closed by the random variable Z_T, if we are interested in changing the measure for any other time $t < T$ we can do this by simply defining

$$\mathbf{E}^{\mathbf{P}}(Z_T|\mathcal{F}_t) = Z_t = \left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_t}. \tag{2.71}$$

The analysis for the time interval $[0, \infty)$ is not so straightforward, even if we have a martingale process $\{Z_t, t \geqslant 0\}$ which is a strictly positive and has a constant mean of one it is not enough to ensure that a closing random variable can be found. We briefly describe how one would go about closing a martingale process that is defined on the time interval $[0, \infty)$. We shall define $\mathcal{F}_\infty = \bigcup_{t \geqslant 0} \mathcal{F}_t$. Now it can be easily shown that

$$\sup_{t \geqslant 0} \mathbf{E}^{\mathbf{P}} |Z_t| = \sup_{t \geqslant 0} \mathbf{E}^{\mathbf{P}} (Z_t) = 1 < \infty,$$

since the process is strictly positive. Hence the process $\{Z_t,\ t \geqslant 0\}$ is bounded in $L^1(\Omega, \mathbf{P})$. We can therefore invoke Doob's Martingale Convergence Theorem (Theorem 1.1) to justify that Z_t will converge to some random variable Z_∞ $\mathbf{P}$-a.s. Without going into too much detail, provided it can be shown that the martingale process is uniformly integrable we can then conclude that the random variable Z_∞ *closes* the martingale process, i.e. the $L^1(\Omega, \mathbf{P})$ random variable Z_∞ has the property

$$Z_t = \mathbf{E}^{\mathbf{P}} (Z_\infty | \mathcal{F}_t). \tag{2.72}$$

The change of measure is the defined by

$$\left. \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \right|_{\mathcal{F}_\infty} = Z_\infty.$$

Remark 2.8. For our purposes we shall be mainly interested in implementing a change of measure over the time interval $[0, T]$ for some fixed finite T as most financial contracts are traded over some finite time horizon. However when trading in contracts that are perpetual in nature one needs to take this into consideration because any process used to employ a change of measure will need to satisfy additional requirements, i.e. the process will have to be uniformly integrable as well as a strictly positive martingale.

The following result is often useful if we wish to compute the conditional expectation of some random variable under the measure $\mathbf{Q}$:

Lemma 2.2. *Given any $\mathcal{F}_T$-measurable random variable X that is also $L^1(\Omega, \mathcal{F}_T, \mathbf{P})$ integrable and sub σ-field $\mathcal{F}_t \subset \mathcal{F}_T$. If $\mathbf{P}$ and $\mathbf{Q}$ are equivalent measures defined on $(\Omega, \mathcal{F}_T)$, then we have*

$$\mathbf{E}^{\mathbf{Q}} \big(X \big| \mathcal{F}_t \big) = \frac{\mathbf{E}^{\mathbf{P}} \big(X Z_T \big| \mathcal{F}_t \big)}{\mathbf{E}^{\mathbf{P}} \big(Z_T \big| \mathcal{F}_t \big)} \tag{2.73}$$

where

$$Z_T = \left. \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \right|_{\mathcal{F}_T} \tag{2.74}$$

is the R-N derivative linking $\mathbf{Q}$ to $\mathbf{P}$ on the measure space $(\Omega, \mathcal{F}_T)$.

Proof: See Kuo [32], Lemma 8.9.2 ■

We now turn our attention to the identification of the process $\{Z_t, 0 \leqslant t \leqslant T\}$ that we shall use to effect a change in measure. We are therefore interested in strictly positive stochastic processes that have the martingale property as well as $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ for all $t \in [0, T]$. This leads us to the stochastic exponential process. Consider the following stochastic differential equation

$$dZ_t = Z_{t-}\big[h_t\, dW_t + \int_{|x|<1} (H(t,x) - 1)\ \tilde{N}(dt, dx) + \int_{|x|\geqslant 1} (F(t,x) - 1)\ \tilde{N}(dt, dx)\big]. \tag{2.75}$$

If we let

$$Y_t = \int_0^t h_s\, dW_s + \int_0^t \int_{|x|<1} (H(s,x) - 1)\ \tilde{N}(ds, dx) + \int_0^t \int_{|x|\geqslant 1} (F(s,x) - 1) \tilde{N}(ds, dx). \tag{2.76}$$

Then we can identify Y_t as a Lévy type stochastic integral and thus the process $\{Y_t, 0 \leqslant t \leqslant T\}$ as being a Lévy process. Let us assume that all integrand processes in Eq.(2.76) are predicable and satisfy the integrability constraints listed in the previous section so as to ensure the Lévy integral Y_t is indeed a martingale. It is also understood that the martingale property exists under the measure $\mathbf{P}$. Bearing this in mind we rewrite Eq.(2.75) as

$$dZ_t = Z_{t-}\ dY_t. \tag{2.77}$$

The solution to the above SDE is given by the *stochastic exponential process* (also known as the *Doléans-Dade exponential* [5] after its discoverer) and is defined by

$$Z_t = \exp\bigg(Y_t - \frac{1}{2}[Y^c, Y^c]_t\bigg) \cdot \prod_{0 \leqslant s \leqslant t} \big[1 + \Delta Y_s\big] e^{-\Delta Y_s}. \tag{2.78}$$

Remark 2.9. The infinite product in Eq.(2.78) can be shown to be finite $\mathbf{P}$-a.s. This together with the proof that the stochastic exponential is indeed a solution Eq.(2.77) can be found in Elliott [14].

Let us examine this stochastic exponential, for our purposes we shall require that Z_t be strictly positive for all t. This therefore implies that

$$\inf\{\Delta Y_t,\ 0 \leqslant t \leqslant T\} > -1\ \ \mathbf{P}\text{−a.s.}$$

in order for Z_t to be strictly positive. We are interested in under what conditions the process Z can be made into a $\mathbf{P}$-martingale. Writing Eq.(2.77) in integral form and assuming that $Z_0 = 1$ we have

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_s\, dY_s \\ &= 1 + \int_0^t Z_{s-} h_s\, dW_s + \int_0^t \int_{|x|<1} Z_{s-}(H(s,x) - 1)\ \tilde{N}(ds, dx) \qquad (2.79) \\ &\quad + \int_0^t \int_{|x|\geqslant 1} Z_{s-}(F(s,x) - 1)\ \tilde{N}(ds, dx). \qquad (2.80) \end{aligned}$$

As it stands $\{Z_t, 0 \leqslant t \leqslant T\}$ is a local martingale for each integrand suitably bounded. By this we mean that each integrand satisfies

$$\mathbf{P}\big(Z_t h_t < M,\ \ \forall t \in [0, T]\big) = 1, \tag{2.81}$$

where $M < \infty$. Having studied stochastic integration in the last section we know the necessary constraints that must be placed on each of the integrands in Eq.(2.79) and Eq.(2.80) that will ensure that the process $\{Z_t, 0 \leqslant t \leqslant T\}$ will be a martingale under the measure $\mathbf{P}$, namely we require that

$$\mathbf{E}^{\mathbf{P}}\left(\int_0^T Z_{s-}^2 h_s^2 ds\right) < \infty, \tag{2.82}$$

$$\mathbf{E}^{\mathbf{P}}\left(\int_0^T \int_{|x|<1} Z_{s-}^2 (H(s,x)-1)^2 \, \nu(dx)ds\right) < \infty, \quad \text{and} \tag{2.83}$$

$$\mathbf{E}^{\mathbf{P}}\left(\int_0^T \int_{|x|\geqslant 1} Z_{s-}^2 (F(s,x)-1)^2 \, \nu(dx)ds\right) < \infty. \tag{2.84}$$

Providing that each of these requirements are met we will have $\{Z_t, 0 \leqslant t \leqslant T\}$ being a martingale under the measure $\mathbf{P}$. As before $\nu(\cdot)$ is the Lévy measure. However to verify Eq.(2.82) - Eq.(2.84) is rather difficult in practice, what we would ideally like is to verify whether the SDE defined by Eq.(2.77) is indeed a martingale by using some method that posses less computational difficulty. Such a method of verification is made possible as follows:

Theorem 2.10. *Given a Lévy type stochastic integral of the form*

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_{s-} h_s \, dW_s + \int_0^t \int_{|x|<1} Z_{s-}(H(s,x)-1) \, \tilde{N}(ds,dx) \\ &\quad + \int_0^t \int_{|x|\geqslant 1} Z_{s-}(F(s,x)-1) \, \tilde{N}(ds,dx) \end{aligned}$$

then this local martingale is a martingale under the measure $\mathbf{P}$ *if and only if* $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ *for all* $t \geqslant 0$

Proof: See Elliott [14] ■

In actual fact Theorem 2.10 can be just as difficult to verify, however if we choose the functions h_t, $H(t,x)$ and $F(t,x)$ appropriately we will be able to verify if the stochastic exponential process is a martingale. For our purposes we shall mainly focus on the case where h_t, $H(t,x)$ and $F(t,x)$ are assumed to be constant or deterministic, in such a case Theorem 2.10 is not that difficult to establish. There are various other methods that can be employed to determine whether local martingales of the form in Theorem 2.10 are martingales. For instance if we take $Y_t = \int_0^t h_s \, dWs$ then the resulting stochastic exponential of the form

$$Z_t = \exp\left(Y_t - \frac{1}{2}[Y^c, Y^c]_t\right) \tag{2.85}$$

will be a martingale if the Novikov criterion is met. More precisely, if we have a process that has continuous sample paths such as $Y_t = \int_0^t h_s \, dW_s$ and the Novikov condition

$$\mathbf{E}^{\mathbf{P}}\left(\exp\left(\frac{1}{2}[Y,Y]_T\right)\right) = \mathbf{E}^{\mathbf{P}}\left(\exp\left(\frac{1}{2}\int_0^T h_s^2 \, ds\right)\right) < \infty, \tag{2.86}$$

holds true, then we will have the exponential process defined in Eq.(2.85) being a martingale under the measure $\mathbf{P}$. More general results concerning local martingales and semimartingales can be found in Chung and Williams [25] and Durrett [13].

So when we wish to implement a change of measure we use the stochastic exponential defined in Eq.(2.78) and then verify that it is a martingale for which $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$. Let us assume that we have now changed the measure $\mathbf{P}$ to some equivalent measure $\mathbf{Q}$, we now turn our attention to examining the dynamics of the Lévy process under the new equivalent measure $\mathbf{Q}$. Of particular interest to us is how we can express the Lévy process as $\mathbf{Q}$-martingale. We have already seen that the Lévy process is a semimartingale under the measure $\mathbf{P}$, we are therefore concerned with how we can go about altering the Lévy process so as to make it a martingale under the measure $\mathbf{Q}$. In such instances we shall call the resulting equivalent measure an *equivalent martingale measure*. We shall consider Girsanov's Theorem which allows us to change the standard Borwnian motion under $\mathbf{P}$ to a standard Brownian motion under $\mathbf{Q}$ as well as how to change a martingale jump process under $\mathbf{P}$ to a martingale jump process under $\mathbf{Q}$.

Theorem 2.11. ***Girsanov's Theorem*** *Let $\{W_t,\ t \geqslant 0\}$ be standard Brownian motion under the measure $\mathbf{P}$. Then if the stochastic exponential defined by*

$$Z_t = \exp\left(Y_t - \frac{1}{2}[Y^c, Y^c]_t\right) \cdot \prod_{0 \leqslant s \leqslant t} \left[1 + \Delta Y_s\right] e^{-\Delta Y_s} \tag{2.87}$$

is a martingale under the measure $\mathbf{P}$ for which $\mathbf{E}^{\mathbf{P}}(Z_T) = 1$ where the process $\{Y_t,\ t \geqslant 0\}$ is a Lévy process containing the Brownian motion, then a new process $\{\tilde{W}_t,\ 0 \leqslant t \leqslant T\}$ defined by

$$\tilde{W}_t = W_t - \int_0^t h_s \, ds \tag{2.88}$$

has standard Brownian motion under the measure $\mathbf{Q}$ defined by

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = Z_T. \tag{2.89}$$

Proof: See Elliott [14], chapter 13 ∎

Just as Girsanov's Theorem allows us to change the Brownian motion under $\mathbf{P}$ to a Brownian motion under $\mathbf{Q}$ we now wish to consider how the jump process, which is a martingale under the measure $\mathbf{P}$, can be represented as a martingale under the measure $\mathbf{Q}$. Let us consider the the process $\{M_t, 0 \leqslant t \leqslant T\}$ under the measure $\mathbf{P}$ that is defined by

$$M_t = \int_0^t \int_{|x|<1} L(s,x) \ \tilde{N}(ds, dx). \tag{2.90}$$

As it stands we have Eq.(2.90) as a local $\mathbf{P}$-martingale, if we impose the integrability condition

$$\mathbf{E}^{\mathbf{P}}\left(\int_0^T\int_{|x|<1} L(s,x)^2\ \nu(dx)ds\right) < \infty, \tag{2.91}$$

then we shall have the process $\{M_t, 0 \leqslant t \leqslant T\}$ being a martingale under the measure $\mathbf{P}$. Let us assume now that we have changed the measure to some equivalent measure $\mathbf{Q}$ defined by

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = Z_T,$$

where Z_t is the stochastic exponential defined in Eq.(2.87) that is driven by the Lévy type stochastic integral of the form in Eq.(2.76). Now if we define a new process $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ by

$$\begin{aligned}\tilde{M}_t &= M_t - \int_0^t\int_{|x|<1} L(s,x)(H(s,x)-1)\ \nu(dx)ds \\ &= \int_0^t L(s,x)\big[N(ds,dx) - H(s,x)\ \nu(dx)ds\big] \\ &= \int_0^t L(s,x)\ \tilde{N}_{\mathbf{Q}}(ds,dx), \end{aligned} \tag{2.92}$$

where $\tilde{N}_{\mathbf{Q}}(ds,dx) = N(ds,dx) - H(s,x)\nu(dx)ds$. Then it can be shown that $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ is a $\mathbf{Q}$-martingale, see Jacod and Shriyaev [24]. In a similar manner martingales of the form

$$J_t = \int_0^t\int_{|x|\geqslant 1} K(s,x)\ \tilde{N}(ds,dx), \tag{2.93}$$

under the measure $\mathbf{P}$ have representation as martingales under $\mathbf{Q}$ as

$$\tilde{J}_t = J_t - \int_0^t\int_{|x|\geqslant 1} K(s,x)(F(s,x)-1)\ \nu(dx)ds. \tag{2.94}$$

In general we have the following lemma which is of considerable assistance when we wish to verify whether a stochastic process is indeed a martingale under some $\mathbf{P}$-equivalent measure $\mathbf{Q}$.

Lemma 2.3. *A process $\{X_t, 0 \leqslant t \leqslant T\}$ is a local martingale under the measure $\mathbf{Q}$ if and only if the process $\{X_tZ_t, 0 \leqslant t \leqslant T\}$ is a local martingale under the measure $\mathbf{P}$, where Z_t is the R-N derivative defined by*

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T} = Z_T.$$

Proof: See Elliott [14] Lemma 13.10. ■

Equipped with the knowledge of how to construct an equivalent probability measure as well as how to change martingales under the some already existing measure $\mathbf{P}$ to martingales under an equivalent measure $\mathbf{Q}$ we are now in a position to apply these results in a financial setting.

In this chapter we have defined what is meant by a Lévy process and examined the properties that are associated with such a process. As the Lévy process allows for jumps it will serve as our building block when we wish to model asset price dynamics that include jumps. We have then considered stochastic integration with respect to Lévy processes and identified those stochastic integrals that possess the martingale property. Finally we have discussed how to change a probability measure $\mathbf{P}$ to another equivalent probability measure $\mathbf{Q}$. Of particular interest to us is how the Lévy process can be made into a $\mathbf{Q}$-martingale and which processes can serve as possible change of measure formulas. We shall now apply these results from stochastic calculus in a financial setting. More precisely, given a random payoff at some future date in the form of an option the questions we are faced with is: how do we model the dynamics of the asset in question and what is the fair price for the option? This shall be our focus and the implementation of results from stochastic calculus in answering financial based problems.

Chapter 3

Financial concepts and the Fundamental Theorems of Asset pricing

Up until now we have primarily focused on the stochastic calculus regarding Lévy processes, we now wish to apply our knowledge of Lévy processes in a financial context. Given that it is our desire to price some contingent claim that is driven by a Lévy process we shall begin this chapter by reviewing some financial concepts such as arbitrage opportunities and constructing arbitrage portfolios. We then examine the Fundamental Theorems of Asset pricing which will then enable us to develop a price process for a contingent claim in a manner that will preclude any arbitrage opportunity. The idea is as follows, suppose we have an economy containing two tradable assets (R_t, S_t) where S_t represents some stock price also known as the risky asset and R_t represents the price of a risk-free bond. We wish to add another tradable asset to the economy where the new tradable asset is some contingent claim whose value depends on the underlying stock price, the question we are faced with is this: what is the fair value of such a claim? In other words how can we develop a price process for the claim in such a manner that it becomes impossible to create an arbitrage opportunity. Clearly the fair valuation of the claim will depend on the pricing dynamics that we shall specify for the stock price process. At first we will assume that the stock price process dynamics contain only one source of randomness, namely Brownian motion. This gives rise to the Black-Scholes-Merton model where the stock price assumes the dynamics of a geometric Brownian model. We will then replace the Brownian motion with some general Lévy process that allows for possible jumps in the stock price model and examine the consequences of this action and how the valuation process of contingent claims is thus affected.

3.1 Arbitrage Free Economies

Let us begin by considering an economy made up of the following tradable assets (R_t, S_t), where R_t represents the price of a risk-free bond and S_t the price of some stock. We assume that trading is done on some finite time horizon

$[0, T]$ where $T \in [0, \infty)$. The first question that we are interested in is this: is the economy arbitrage free? The justification for wanting the economy to be arbitrage free lies in the fact that if the economy were to permit an arbitrage opportunity it would allow investors to make a profit without exposing themselves to the risk of incurring a loss and in finance we always assume that in a properly functioning market such opportunities do not exist as it means that an investor can make a "free lunch" at the expense of the market. Therefore within the theory of mathematical finance we must restrict ourselves to markets that prevent any arbitrage opportunities and if such an arbitrage opportunity were to exist the properly functioning market forces would act quickly to dispel these occurrences. Before addressing the question of how one would go about determining whether an economy is arbitrage free or not we define what is meant by trading strategies and portfolio processes with respect to the current economy under consideration.

Definition 3.1. A trading strategy is a process (φ_t, ϕ_t) where φ_t represents the number of units invested in the risk-free bond at time t and ϕ_t the number of units invested in the stock price at time t. The value of a portfolio based on such a trading strategy is given by

$$V_t = \varphi_t R_t + \phi_t S_t. \tag{3.1}$$

For our purposes we shall only be interested in *self-financing* portfolios and trading strategies. A self-financing portfolio once it is created receives no injection or withdrawal of funds but is only allowed to be re-balanced in the positions held in stock and bond. In terms of a mathematical description a self-financing portfolio has the following property

$$dV_t = \varphi_t dR_t + \phi_t dS_t. \tag{3.2}$$

Remark 3.1. There are other trading strategies that one might employ other than a self-financing strategy but for our purposes we shall not consider such strategies as most of the definitions and theorems that follow implicitly assume the use self-financing strategies. In addition there are certain constraints that need be imposed on the values of (φ_t, ϕ_t), such constraints prevent the portfolio value from becoming unbounded and leading to arbitrage opportunities. We shall not go into great detail here about the constraints placed on (φ_t, ϕ_t) but implicitly assume these constraints are satisfied, for more details on the restrictions placed on (φ_t, ϕ_t) we refer the reader to Karatzas and Shreve [26] and Lamberton and Lapeyre [33]

In light of the above definition of self-financing trading strategies and portfolios we can now define an arbitrage opportunity in terms of a self-financing portfolio. We assume that we are working under the real world measure or canonical measure $\mathbf{P}$.

Definition 3.2. An arbitrage opportunity in an economy (R_t, S_t) over the time interval $[0, T]$ is any self-financing trading strategy that has $V_0 = 0$ and has

$$\mathbf{P}(V_T \geqslant 0) = 1 \quad \text{and} \quad \mathbf{P}(V_T > 0) > 0. \tag{3.3}$$

Therefore in order to establish whether or not the economy is arbitrage free we must verify whether it is possible to construct some self-financing portfolio that has an initial setup cost of 0 and a terminal value at time T that is nonnegative, i.e. $V_T \geqslant 0$ $\mathbf{P}$-a.s. and the portfolio must also have a strictly positive payoff that has positive $\mathbf{P}$ measure for certain outcomes. Clearly the construction of such a portfolio depends on the type of trading strategies that are allowed within the economy and the underlying dynamics that we assume are governing the the tradable assets. Let us assume that we have created some portfolio that has an initial setup cost of 0, if it is then true that no allocation of (φ_t, ϕ_t) to (R_t, S_t) produces an arbitrage opportunity then we can conclude that the economy is arbitrage free. Such a verification may be easy to establish for an economy that has only two tradable assets. When faced with an economy that has multiple tradable assets the verification whether such an economy is arbitrage free using portfolio arguments may become difficult and tedious. We would like an alternative method of determining whether or not an economy is arbitrage free that does not require the use of portfolio arguments. Such a method of verification is indeed possible and has in recent times been called the Fundamental Theorem of Asset Pricing and was originally coined by Dybvig and Ross, see Kiesel and Bingham [7].

Theorem 3.1. ***First Fundamental Theorem of Asset Pricing*** *An economy or pricing system (R_t, S_t) does not admit an arbitrage opportunity if and only if there exists a probability measure $\mathbf{Q}$ equivalent to the real world probability measure $\mathbf{P}$ such that the deflated stock price process*

$$S_t^R = \frac{S_t}{R_t} \tag{3.4}$$

has a $(\mathbf{Q}, \mathcal{F}_t)$-martingale property.

Proof: See Karatzas and Shreve [26] ■

Remark 3.2. The proof of the above theorem considers an economy that has multiple tradable assets which is represented as a vector of tradable assets. The existence of an equivalent martingale measure implies that there are no arbitrage opportunities, however, in continuous time the converse need not be true, this is due mainly to the type of trading strategies that are allowed in continuous time. To address this problem of equivalence we can either impose constraints on the type of trading strategies that are allowed or redefine our notion of arbitrage. A stronger condition called the "no free lunch without vanishing risk" was proposed by Delabaen and Schachermayer [11],[12]. We will not dwell too much on this topic but assume that an arbitrage free economy implies and is implied by an equivalent martingale measure.

Hence the problem of determining whether an economy is arbitrage free or not can be reduced to establishing whether or not an equivalent martingale measure exists. This is done by selecting an asset in the economy, say R_t the risk-free bond, we then deflate all other assets in the economy with respect to

R_t and establish if there exists a measure $\mathbf{Q}$ which is equivalent to $\mathbf{P}$ such that

$$\mathbf{E}^{\mathbf{Q}}\left(\frac{S_T}{R_T}\middle|\mathcal{F}_t\right) = \frac{S_t}{R_t}. \tag{3.5}$$

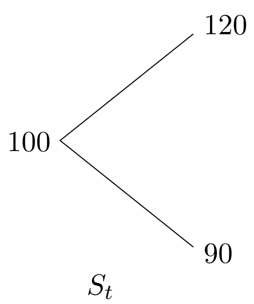
Note that depending on the dynamics of the assets in the economy there may exist more than one measure $\mathbf{Q}$ such that Eq.(3.5) holds true and if no equivalent martingale measure can be found then the economy is not arbitrage free. In Theorem 3.1 we chose R_t as the asset to which all other tradable assets are deflated; we call R_t the *numeraire*.

Definition 3.3. A numeraire is a price process that is strictly positive for all times.

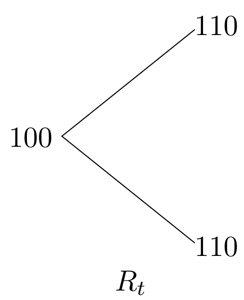
So any price process, provided that it is strictly positive, can serve as a numeraire, so we could have easily chosen S_t to serve as our numeraire in Theorem 3.1 as common convention assumes that all tradable assets in the economy are strictly positive. The pair $(R_t, \mathbf{Q})$ which represents the numeraire and EMM based on selecting R_t as the numeraire is called a numeraire pair, we can therefore reword Theorem 3.1 and say that *an economy is arbitrage free if and only if it admits at least one numeraire pair*. Note that all assets in the economy when deflated with respect to the numeraire must have a $\mathbf{Q}$-martingale property. In our economy it is easy to see this since we have

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}\left(\frac{S_T}{R_T}\middle|\mathcal{F}_t\right) &= \frac{S_t}{R_t} \\
\mathbf{E}^{\mathbf{Q}}\left(\frac{R_T}{R_T}\middle|\mathcal{F}_t\right) &= \frac{R_t}{R_t} = 1.
\end{aligned}$$

The following simple example illustrates how we can use the Theorem 3.1 to ascertain if an economy is arbitrage free or not. Consider the the following economy where trading is only allowed at the beginning and end of the time interval $[0, T]$



S_t



R_t

Clearly there are only two possible outcomes for the risky asset S_t at time T, i.e. we can only have $S_T = 120$ if the stock price goes up or $S_T = 90$ if the stock price goes down. The risk-free asset R_t has the same payoff at time T of 110 for both possible outcomes. Using portfolio arguments to establish if the economy admits an arbitrage opportunity, we must create a portfolio that has $V_0 = 0$ and $V_T > 0$ for some outcome while keeping the process $\{V_t, 0 \leqslant t \leqslant T\}$ strictly nonnegative.

$$\begin{aligned} \Rightarrow 100\varphi_0 + 100\phi_0 &= 0 \\ \Rightarrow \varphi_0 &= -\phi_0. \end{aligned}$$

Since trading can only take place at the beginning and at the end of the time interval $[0, T]$ the value of the portfolio at time T given by

$$V_T = \begin{cases} -110\phi_0 + 120\phi_0 = 10\phi_0 & \text{if } S_T = 120 \\ -110\phi_0 + 90\phi_0 = -10\phi_0 & \text{if } S_T = 90. \end{cases}$$

Clearly there does not exist any $\phi_0 \in \mathbb{R}$ that ensures $V_T > 0$ for at least one of the possible outcomes. Hence the economy is arbitrage free. Now making use of Theorem 3.1, selecting R_t as our pricing numeraire, if we can find at least one equivalent martingale measure $\mathbf{Q}$ such that

$$\mathbf{E}^{\mathbf{Q}}\left(\frac{S_T}{R_T}\middle|\mathcal{F}_0\right) = \frac{S_0}{R_0} \tag{3.6}$$

$$\mathbf{E}^{\mathbf{Q}}\left(\frac{R_T}{R_T}\middle|\mathcal{F}_0\right) = \frac{R_0}{R_0} \tag{3.7}$$

then we can conclude that the economy is arbitrage free. Since there are only two possible outcomes we can solve Eq.(3.6) and Eq.(3.7) as follows

$$\begin{aligned} \frac{120}{110}q + \frac{90}{110}(1-q) = \frac{100}{100} \\ \frac{110}{110}q + \frac{110}{110}(1-q) = \frac{100}{100}. \end{aligned}$$

Solving this system of equations we get $q = \frac{2}{3}$ and $(1 - q) = \frac{1}{3}$. So the economy does indeed admit a numeraire pair and hence we can conclude that the economy is arbitrage free.

Remark 3.3. Note that in the above example we did not require the probabilities being generated by the real world measure $\mathbf{P}$. All that we need to know is that both the possible outcomes for the risky asset have positive $\mathbf{P}$ measure which we make use of when constructing an equivalent martingale measure $\mathbf{Q}$.

The above example illustrates how we can verify that an economy is arbitrage free using both portfolio arguments and equivalent martingale methods. Due to the simple nature of the above economy the portfolio argument was easily established, for a more complex economy with multiple outcomes such portfolio arguments can become rather difficult to use in establishing if an economy is arbitrage free. For such economies the martingale technique is a much more efficient method of verifying whether the economy is arbitrage free or not.

3.2 Complete Economies and the Pricing of Attainable Claims

Once we have established that the economy under consideration is indeed arbitrage free we can now turn our attention to the problem of adding another tradable asset to the economy and developing an arbitrage free price process for this asset. More precisely we wish to add a contingent claim to our economy or pricing system (R_t, S_t).

Definition 3.4. A contingent claim can be characterized by a nonnegative valued random variable, say Y_T, that is measurable with respect to the measure space $(\Omega, \mathcal{F}_T)$.

As our interests have thus far been concerned with trading over a finite time interval $[0, T]$ we shall assume that the contingent claim is payable at time T. Given some contingent claim Y_T, suppose that we have developed a price process $(\pi_t^Y, 0 \leqslant t \leqslant T)$ for the claim. This price process must have the property that when added to the economy or pricing system (R_t, S_t) the augmented pricing system (R_t, S_t, π_t^Y), which represents the original economy plus the new security, should not generate any arbitrage opportunities. A price process for the claim that satisfies this property is said to be an *admissible price process*.

A possible starting point for the pricing of a $\mathcal{F}_T$-measurable claim is the concept of attainability. More specifically a claim Y_T is said to be attainable if there exists a self-financing trading strategy (φ_t, ϕ_t) that ensures that

$$V_T = \varphi_T R_T + \phi_T S_T = Y_T \qquad \mathbf{P}\text{−a.s.} \tag{3.8}$$

Any trading strategy that has this property is said to be a *replicating or hedging portfolio*. If we can construct a replicating portfolio for the claim Y_T then we can define the value of the claim at any time t to be the value of the replicating portfolio, i.e. define $\pi_t^Y = V_t$ for all $t \in [0, T]$. The reason behind this lies in the fact that if the replicating portfolio were at any time not equal to the value of the contingent claim then it would be possible to create an arbitrage opportunity, we can therefore define the price process of the claim to be the value of the replicating portfolio. Such an approach to the pricing of contingent claims raises the following questions:

(i) Can any $\mathcal{F}_T$ measurable claim be replicated?

(ii) Given a trading strategy (φ_t, ϕ_t) that replicates Y_T, does there possibly exist a different trading strategy (φ_t*, ϕ_t*) that generates a portfolio V_t* and has $V_t* = V_t$ for all $t \in [0, T]$?

The second question can be addressed with the aid of the following theorem:

Theorem 3.2. *Suppose that a pricing system (R_t, S_t) does not admit an arbitrage opportunity and that Y_T is an attainable claim with payment at time T.*

Then the arbitrage free price π_t^Y for this claim at any time $t \leqslant T$ is given by V_t the value of the portfolio for any (i.e. every) replicating strategy of Y_T. Moreover,

$$\frac{\pi_t^Y}{R_t} = \frac{V_t}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_t\right). \tag{3.9}$$

Proof: Since Y_T is attainable we can find some self-financing trading strategy (φ_t, ϕ_t) with portfolio process $\{V_t,\ 0 \leqslant t \leqslant T\}$ that replicates the contingent claim and yields $V_T = Y_T$ **P**-a.s. To avoid arbitrage the price of this claim at any time $t \leqslant T$ must be given by the time t value of the replicating portfolio, namely V_t. Since we are assuming that the economy is arbitrage free, at least one equivalent martingale measure exits or rather the economy admits at least one numeraire pair (by Theorem 3.1). Deflating the replicating portfolio with respect to the numeraire, which we shall take to be R_t we have

$$\tilde{V}_t = \varphi_t + \phi_t \tilde{S}_t. \tag{3.10}$$

The self-financing constraint on this replicating portfolio then implies that

$$d\tilde{V}_t = \phi_t d\tilde{S}_t. \tag{3.11}$$

Rewriting this in integral form we have

$$\tilde{V}_t = \tilde{V}_0 + \int_0^t \phi_u \, d\tilde{S}_u. \tag{3.12}$$

We know that $\tilde{S}_t$ is a martingale under the measure **Q**, hence $\tilde{V}_t$ is a local **Q**-martingale. Depending the dynamics of $\tilde{S}_t$ if we impose the constraint that

$$\mathbf{E}^{\mathbf{Q}}(\tilde{V}_T^2) < \infty, \tag{3.13}$$

then we will have $\tilde{V}_t$ being a **Q**-martingale. Therefore by the martingale property we have

$$\begin{aligned} \tilde{V}_t &= \mathbf{E}^{\mathbf{Q}}(\tilde{V}_T | \mathcal{F}_t) \\ \Rightarrow \frac{V_t}{R_t} &= \mathbf{E}^{\mathbf{Q}}\left(\frac{V_T}{R_T}\middle|\mathcal{F}_t\right) \end{aligned} \tag{3.14}$$

but $V_T = Y_T$ so we have

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_t\right). \qquad \blacksquare \tag{3.15}$$

Note that the left hand side of Eq.(3.14) is determined by the replicating portfolio while the right hand side is determined only by the choice of the measure **Q** and probabilistic methods. Therefore providing that the claim is attainable the price of the claim is invariant to ones choice of replicating portfolio. This result implies that for *any* equivalent martingale measure **Q**, and for any t, all self-financing portfolios replicating Y_T will have the same $\mathcal{F}_t$-measurable value providing that the original pricing system (R_t, S_t) is arbitrage free. We also see that from Eq.(3.14) that the problem of pricing an attainable claim using any replicating portfolio is equivalent to computing the deflated value of the $\mathcal{F}_T$-measurable claim under an equivalent martingale measure $\mathbf{Q}$ and that the value of the claim for any time $t \leqslant T$ follows a $(\mathbf{Q}, \mathcal{F}_t)$-martingale property.

Theorem 3.2 only shows how the pricing of a contingent claim can be reduced to computing its value under some equivalent martingale measure using probabilistic methods. This is often of limited value as it stands since we first need to establish that the economy is arbitrage free and that the claim we are interested in pricing is indeed attainable. The question of the economy being arbitrage free has already been established - providing that the pricing system (R_t, S_t) admits at least one EMM then the pricing system is arbitrage free and vice versa. The question of attainability can be addressed by either explicitly constructing the replicating portfolio beforehand or proving that *all claims* (or at least those within some suitable class) are attainable. We shall not take the route of attempting to construct a replicating portfolios to determine if the claim is attainable but rather make use of a result that deals with the existence of an equivalent martingale measure.

Definition 3.5. If every random variable Y is attainable, then we say that the economy or market is complete. Otherwise we have an incomplete economy.

The following theorem characterizes a complete economy in terms of a martingale measure:

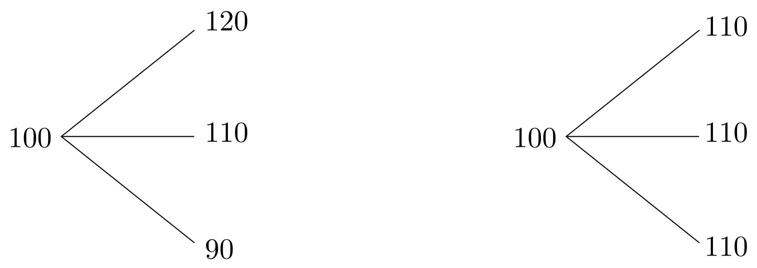
Theorem 3.3. ***Second fundamental theorem*** *Given a pricing system (R_t, S_t) that is arbitrage free. Then it will be complete if and only if there exists a unique equivalent martingale measure* $\mathbf{Q}$

Proof: See Harrison and Kreps [23]. ■

Therefore when we have established that the economy under consideration is indeed arbitrage free we know that there exists at least one EMM, provided that this EMM is unique we will then have all contingents claims being attainable and the valuation of these claims can be computed under the unique EMM $\mathbf{Q}$ as shown in Theorem 3.2. If the economy admits more than one EMM then the economy is still arbitrage free but since there exists more than one numeraire pair the economy is incomplete and not all claims are attainable. For claims that are not attainable no replicating portfolio exists that can be used to derive a fair price process. Given a unattainable claim we can still use the pricing formula from Theorem 3.2

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_t\right) \tag{3.16}$$

to derive an arbitrage free pricing formula for claim, however since there now exist more than one EMM we have more than one arbitrage free price process for such a claim. Therefore for each possible EMM we have an associated arbitrage free price process for the claim and the challenge we are now faced with is to select one EMM that will serve as our pricing measure under which all claims will be evaluated. The following example illustrates the difficulty of pricing in an incomplete economy.



S_t R_t

Again we assume that trading can only take place at the beginning and at the end of the time interval $[0, T]$. In this economy there are three different possible outcomes for the risky asset S_t at time T. We begin by establishing if the above economy is arbitrage free, selecting R_t as our pricing numeraire we seek an equivalent measure $\mathbf{Q}$ such that

$$\mathbf{E}^{\mathbf{Q}}\left(\frac{S_T}{R_T}\bigg|\mathcal{F}_0\right) = \frac{S_0}{R_0} \quad \text{and} \quad \mathbf{E}^{\mathbf{Q}}\left(\frac{R_T}{R_T}\bigg|\mathcal{F}_0\right) = \frac{R_0}{R_0}.$$

As before we have no interest in the real world probabilities, all that we require is that each of the three outcomes has positive measure under the real world measure $\mathbf{P}$. Since there are three different possible outcomes let us define

$$q_1 = \mathbf{Q}(S_T = 120) \quad q_2 = \mathbf{Q}(S_T = 110) \quad q_3 = \mathbf{Q}(S_T = 90).$$

Therefore an equivalent martingale measure, if it exists, must have the property that

$$\frac{120}{110}q_1 + \frac{110}{110}q_2 + \frac{90}{110}q_3 = \frac{100}{100} \tag{3.17}$$

$$\frac{110}{110}q_1 + \frac{110}{110}q_2 + \frac{110}{110}q_3 = \frac{110}{110}. \tag{3.18}$$

From Eq.(3.17) and Eq.(3.18) it can be seen that we have three unknowns, namely q_1, q_2 and q_3, but only two equations in the unknowns. Let $q_1 = \lambda$ then we have for all $0 < \lambda < \frac{2}{3}$ we have that

$$q_1 = \lambda \quad q_2 = 1 - \frac{3}{2}\lambda \quad q_3 = \frac{\lambda}{2}$$

and so there does exist at least one EMM and hence the economy is arbitrage free however since there is no unique EMM, there are in fact an infinite number of equivalent martingale measures for all $0 < \lambda < \frac{2}{3}$, the economy is therefore incomplete and thus not all claims are attainable. We now examine the problem of pricing in an incomplete economy. Consider the following $\mathcal{F}_T$-measurable claim

$$Y_T = y_1\mathbf{1}_{(S_T=120)} + y_2\mathbf{1}_{(S_T=110)} + y_3\mathbf{1}_{(S_T=90)}$$

then such a claim will be attainable if and only if

$$2y_1 - 3y_2 + y_3 = 0. \tag{3.19}$$

To establish this fact note that for any claim to be attainable there must exist some self-financing trading strategy (φ_t, ϕ_t) that ensures that

$$110\varphi_T + 120\phi_T = y_1 \tag{3.20}$$
$$110\varphi_T + 110\phi_T = y_2 \tag{3.21}$$
$$110\varphi_T + 90\phi_T = y_3. \tag{3.22}$$

Eq.(3.20) and Eq.(3.21) imply that

$$\phi_T = \frac{y_1 - y_2}{10} \tag{3.23}$$

while Eq.(3.21) and Eq.(3.22) implies that

$$\phi_T = \frac{y_2 - y_3}{20}. \tag{3.24}$$

A unique solution will therefore only be possible if

$$\frac{y_2 - y_3}{20} = \frac{y_1 - y_2}{10} \Rightarrow 2y_1 - 3y_2 + y_3 = 0.$$

Provided that the claim is indeed attainable, in our example attainability amounts to the claim Y_T satisfying Eq.(3.19), then the price process for the claim will be invariant to ones choice of EMM since the value of the claim at time $t = 0$ is given by

$$\begin{aligned}
\frac{Y_0}{R_0} &= \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_0\right) \\
\frac{Y_0}{100} &= \frac{y_1}{110}\lambda + \frac{y_2}{110}(1 - \frac{3}{2}\lambda) + \frac{y_3}{110}\frac{\lambda}{2} \\
Y_0 &= \frac{10}{22}\lambda(2y_1 - 3y_2 + y_3) + \frac{10}{11}y_2.
\end{aligned} \tag{3.25}$$

Such a price for the claim at time $t = 0$ will be independent of λ (our equivalent martingale pricing measure) only if the attainability condition of Eq.(3.19) is satisfied. If for example we were to price the claim $Y_T = \mathbf{1}_{(S_T=120)} + \mathbf{1}_{(S_T=110)} + \mathbf{1}_{(S_T=90)}$ then such a claim being attainable would be independent of our choice λ that shall serve as our pricing measure. However the claim $Y_T = \mathbf{1}_{(S_T=120)}$ is not attainable as it does not satisfy Eq.(3.19), such a claim will then depend on our choice of λ. So if we choose our pricing measure $\lambda = 0.1$ then the price of this claim will be given by

$$Y_0 = \frac{10}{22}\lambda(2) = \frac{10}{11}0.1 = \frac{1}{11}.$$

Although such a price process is not supported for the claim $Y_T = \mathbf{1}_{(S_T=120)}$ by a replicating portfolio the augmented pricing system (R_t, S_t, Y_t) that results by selecting the numeraire pair $(R_t, \mathbf{Q}^{(\lambda=0.1)})$ will be arbitrage free. Similarly we could have chosen the numeraire pair $(R_t, \mathbf{Q}^{(\lambda=0.15)})$ and obtained a price for the claim at time $t = 0$ as $Y_0 = 2/33$ and the augmented pricing system (R_t, S_t, Y_T) that results by selecting $\lambda = 0.15$ will once again be arbitrage free. Given non attainable claims we are therefore faced with an identification problem. For different choices of the pricing measure we obtain different augmented pricing systems each of which is arbitrage free. We would therefore like to select one of the many possible pricing measures, based on some criteria, to serve as our pricing measure which will then be used in the valuation of contingent claims.

The approach we have taken in developing an arbitrage free price process for a contingent claim can now be summarized as follows:

(i) Given an economy of tradable assets we first need to establish that the economy is arbitrage free. This is done by selecting an asset within the economy which is called the numeraire and determining it there exists at least one $\mathbf{P}$-equivalent measure $\mathbf{Q}$ such that when all the tradable assets are deflated with respect to the numeraire they have a $(\mathbf{Q}, \mathcal{F}_t)$-martingale property. Hence an arbitrage free economy implies and is implied by the existence of an equivalent martingale measure.

(ii) Next we must establish if the EMM $\mathbf{Q}$ is unique or if there exist multiple EMM's. A unique EMM implies that the market is complete and thus any contingent claim is attainable. We can therefore construct a replicating portfolio for any contingent claim and the price process for such a claim is given by

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\bigg|\mathcal{F}_t\right). \tag{3.26}$$

Such a pricing formula implies that the deflated value of the claim must have a martingale property under the measure $\mathbf{Q}$. If the EMM is not unique then we are faced with the problem of pricing in an incomplete economy where not all claims are attainable. For non attainable claims we can still use the pricing formula in Eq.(3.26) however for each different measure $\mathbf{Q}$ we obtain a host of different arbitrage free price processes. A key challenge is therefore to select one EMM from the available pool of equivalent martingale measures that will serve as our pricing measure. Such a selection must be based on the EMM $\mathbf{Q}$ satisfying some criteria which we shall explore in detail in the chapters to come.

Chapter 4

Pricing Contingent Claims in Continuous Time

In this chapter we will consider the pricing of contingent claims that makes use of the martingale concept. We shall specify the dynamics for the tradable assets in the economy which is composed of a risky asset S_t and a risk-free asset R_t. Together these assets form our pricing system or economy (R_t, S_t). At first we shall consider the case where the risky asset is driven by geometric Brownian motion and this leads to the Black-Scholes [8] model where the Brownian motion is the only source of randomness in the model. Then we shall replace the Brownian motion with a much more general Lévy process that allows for possible jumps to be included in the model. Since the pricing of contingent claims depends on the underlying dynamics that we specify we shall examine how the price process for the contingent claims are affected by the different dynamics that we assume for the model.

4.1 Model Assumptions

We restrict economic activity to the finite time horizon $[0, T]$ where $T \in [0, \infty)$. All uncertainty is modeled with the aid of a complete probability space $(\Omega, \mathcal{F}, \mathbf{P})$ where Ω is the set of all possible outcomes in the model, the sigma field $\mathcal{F}$ is a collection of subsets of Ω and $\mathbf{P}$ is the real world or canonical probability measure. To model the flow of information over time we use a filtration process $(\mathcal{F}_t, 0 \leqslant t \leqslant T)$. When dealing with the Black-Scholes model the uncertainty is being driven by the Brownian motion and we denote the the filtration by $(\mathcal{F}_t^W, 0 \leqslant t \leqslant T)$. When the Lévy process is the driving source of the uncertainty we shall denote the filtration process as $(\mathcal{F}_t^X, 0 \leqslant t \leqslant T)$. The economy is also assumed to have the following properties:

(i) The economy is frictionless, i.e. there are no taxes or transaction costs and assets are infinitely divisible with no restrictions on short sales.

(ii) Information is freely available to all and there is no insider trading. All traders prefer more wealth to less.

The risk-free asset R_t, also known as a bond or cash roll-over account which we assume is devoid of any systematic risk, has the following dynamics

$$dR_t = r_t R_t dt \qquad R_0 = 1 \tag{4.1}$$

where the process $\{r_t, 0 \leqslant t \leqslant T\}$ is interpreted as the risk-free interest rate process or the instantaneous short rate process. Note that the risk-free interest rate process can be interpreted as being a stochastic process but for our purposes we shall assume it to be a deterministic process that evolves over the time interval $[0, T]$. Solving Eq.(4.1) we have

$$R_t = R_0 \mathrm{exp}\bigg(\int_0^t r_s \, ds \bigg).$$

We also require that the risk-free rate of interest satisfy the following integrability constraint over the interval $[0, T]$

$$\int_0^T |r_t| \, dt < \infty \quad \mathbf{P} - \text{a.s.} \tag{4.2}$$

The risky asset which is some stock price that is subjected to systematic risk has the following dynamics under the real world measure $\mathbf{P}$

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \tag{4.3}$$

where the process $\{W_t, 0 \leqslant t \leqslant T\}$ is standard Brownian motion under the measure $\mathbf{P}$. The process $\{\mu_t, 0 \leqslant t \leqslant T\}$ is the appreciation rate of the stock while the process $\{\sigma_t, 0 \leqslant t \leqslant T\}$ is the volatility of the risky asset which models the intensity with which the source of uncertainty influences the stock price, the source of uncertainty in this case is the Brownian motion. There are also integrability restrictions placed on both μ_t and σ_t which are given as follows:

$$\int_0^T (|\mu_t| + \sigma_t^2) \, dt < \infty \quad \mathbf{P}\text{−a.s.} \tag{4.4}$$

We refer to r_t, μ_t and σ_t as the coefficients of the stock price model and assume that all coefficients of the model are progressively measurable with respect to the filtration $(\mathcal{F}_t^W, 0 \leqslant t \leqslant T)$. For our purposes we shall assume that the coefficients of the model are deterministic as this aids in reducing computational difficulty.

4.2 Black-Scholes and Merton Model

Among the most renowned models in mathematical finance is the Black-Scholes and Merton model, in the papers of Black & Scholes [8] and Merton [36] a closed form pricing formula for the value of a contingent claim known as a European option was derived. The Black & Scholes paper was based on constructing a partial differential equation along with an appropriate boundary condition for the valuation process of the claim and the solving this partial differential equation to obtain the value of the contingent claim. We will not take this path but rather make use of our knowledge that deals with stochastic integration and equivalent martingale measures. Under the Black-Scholes and Merton model we have the following dynamics for the evolution of the assets in our economy

$$\begin{aligned} dR_t &= rR_t dt & R_0 = 1 \quad (4.5)\\ dS_t &= \mu S_t dt + \sigma S_t dW_t & S_0 \in \mathbb{R}^+, \quad (4.6) \end{aligned}$$

with constant coefficients $\mu \in \mathbb{R}$ and $r, \sigma \in \mathbb{R}^+$. The solution to deterministic differential equation is easily seen as $R_t = R_0 e^{rt} = e^{rt}$ and an application of Itô's lemma to the linear stochastic differential equation yields the solution of the risky asset S_t at time t by

$$S_t = S_0 \exp\bigg((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\bigg). \tag{4.7}$$

The first question is of course to ask whether the model specified by the above dynamics for the the assets S_t and R_t is free of arbitrage. From Theorem 3.1 we know that an economy or pricing system is arbitrage free if and only if it admits at least one equivalent martingale measure (or numeraire pair). To verify if such an EMM exists we deflate the risky asset S_t with respect to R_t, which we take to be our pricing numeraire, and apply the product rule (Theorem 2.8) for stochastic processes as follows:

$$d\bigg(\frac{S_t}{R_t}\bigg) = \frac{1}{R_t} dS_t + S_t d\bigg(\frac{1}{R_t}\bigg) + d\bigg[S, \frac{1}{R}\bigg]_t, \tag{4.8}$$

now since the sample paths of the Brownian motion are continuous in a $\mathbf{P}$-a.s. sense we can conclude that the sample paths of S_t are continuous almost surely under the measure $\mathbf{P}$. In addition R_t has finite first order variation, hence by Theorem 2.6 we must have

$$\bigg[S, \frac{1}{R}\bigg]_t = 0.$$

We can therefore write Eq.(4.8) as

$$\begin{aligned} d\bigg(\frac{S_t}{R_t}\bigg) &= \frac{1}{R_t}\, dS_t + S_t\, d\bigg(\frac{1}{R_t}\bigg) \\ &= e^{-rt} dS_t - re^{-rt} S_t dt \\ &= e^{-rt}\big(\mu S_t dt + \sigma S_t dW_t\big) - re^{-rt} S_t dt \\ &= e^{-rt}\bigg((\mu - r) S_t dt + \sigma S_t dW_t\bigg) \\ \Rightarrow \frac{S_t}{R_t} &= S_0 + (\mu - r)\int_0^t e^{-ru} S_u\, du + \sigma \int_0^t e^{-ru} S_u\, dW_u. \end{aligned} \tag{4.9}$$

While $\int_0^t e^{-ru} S_u\, dW_u$ can be interpreted as a local $\mathbf{P}$-martingale the drift represented by $\int_0^t e^{-ru} S_u\, du$ prevents the delated stock price from having a local martingale property under the measure $\mathbf{P}$. In order to express the deflated stock price as a martingale under some equivalent measure $\mathbf{Q}$ we must remove the drift term form Eq.(4.9). In order to effect a change of measure let us consider the following stochastic exponential

$$Z_t = \exp\bigg(\theta W_t - \frac{1}{2}\theta^2 t\bigg). \tag{4.10}$$

It can be easily seen that the process $\{Z_t, 0 \leqslant t \leqslant T\}$ is strictly positive and satisfies the martingale property under the measure $\mathbf{P}$ since

$$\begin{aligned}
\mathbf{E}^{\mathbf{P}}\big(Z_t \big| \mathcal{F}_s^W\big) &= \mathbf{E}^{\mathbf{P}}\bigg(e^{\theta W_t - \frac{1}{2}\theta^2 t}\bigg|\mathcal{F}_s^W\bigg) \\
&= e^{-\frac{1}{2}\theta^2 t}\mathbf{E}^{\mathbf{P}}\bigg(e^{\theta W_t}\bigg|\mathcal{F}_s^W\bigg) \\
&= e^{-\frac{1}{2}\theta^2 t} \cdot e^{\frac{1}{2}\theta^2(t-s)+\theta W_s} \\
&= \exp\bigg(\theta W_s - \frac{1}{2}\theta^2 s\bigg) = Z_s.
\end{aligned}$$

Thus we have that the process $\{Z_t, 0 \leqslant t \leqslant T\}$ is a martingale under the measure $\mathbf{P}$. Note we could have just as well used the Novikov condition to verify that the stochastic exponential is a martingale under the measure $\mathbf{P}$ since

$$\mathbf{E}^{\mathbf{P}}\bigg(\exp\bigg(\frac{1}{2}[\theta W_t, \theta W_t]_t\bigg)\bigg) = \exp\bigg(\frac{1}{2}\theta^2 t\bigg) < \infty.$$

We also have that

$$\mathbf{E}^{\mathbf{P}}\big(Z_t\big) = \mathbf{E}^{\mathbf{P}}\big(Z_t \big| \mathcal{F}_0^W\big) = Z_0 = e^0 = 1.$$

Since the stochastic exponential defined in Eq.(4.10) is a strictly positive martingale under the measure $\mathbf{P}$ and has $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ we can make use of this process implement a change of measure. Under a new $\mathbf{P}$-equivalent measure $\mathbf{Q}$ defined by the R-N derivative

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}_T^W} = Z_T, \tag{4.11}$$

we know that $\tilde{W}_t = W_t - \theta t$ is standard Brownian motion under the measure $\mathbf{Q}$ by Girsanov's theorem (Theorem 2.11). By making use of this fact we can then rewrite Eq.(4.9) as

$$\begin{aligned}
\frac{S_t}{R_t} &= S_0 + (\mu - r)\int_0^t e^{-ru}S_u\, du + \sigma\int_0^t e^{-ru}S_u\, d(\tilde{W}_u + \theta u) \\
&= (\mu - r + \theta\sigma)\int_0^t e^{-ru}S_u\, du + \sigma\int_0^t e^{-ru}S_u\, d\tilde{W}_u. \qquad (4.12)
\end{aligned}$$

Now notice that by setting $\theta = \frac{r-\mu}{\sigma}$, the drift term will then be removed from Eq.(4.12) and so we have

$$\frac{S_t}{R_t} = S_0 + \sigma\int_0^t e^{-ru}S_u\, d\tilde{W}_u$$

which is a local **Q**-martingale. For the deflated risky asset to be a martingale under the measure **Q** we require that

$$\mathbf{E}^{\mathbf{Q}}\left(\sigma^2 \int_0^T e^{-2rt} S_t^2 \, dt\right) < \infty. \tag{4.13}$$

However we can again make use of the Novikov condition which is an easier method of verification that the deflated risky asset is a martingale under the measure **Q**. As the solution to the stochastic integral equation

$$\frac{S_t}{R_t} = S_0 + \sigma \int_0^t e^{-ru} S_u \, d\tilde{W}_u \tag{4.14}$$

is given by

$$\begin{aligned} \frac{S_t}{R_t} &= S_0 \exp\left(\int_0^t \sigma \, d\tilde{W}_u - \frac{1}{2}\int_0^t \sigma^2 \, du\right) \\ &= S_0 \exp\left(\sigma \tilde{W}_t - \frac{1}{2}\sigma^2 t\right). \end{aligned} \tag{4.15}$$

The Novikov condition is then verified since

$$\mathbf{E}^{\mathbf{Q}}\left(\exp\left(\frac{1}{2}[\sigma W_t, \sigma W_t]_t\right)\right) = \mathbf{E}^{\mathbf{Q}}\left(\exp\left(\frac{1}{2}\sigma^2 t\right)\right) = \exp\left(\frac{1}{2}\sigma^2 t\right) < \infty. \tag{4.16}$$

Hence the deflated risky asset is indeed a martingale under the measure **Q** defined by

$$\left.\frac{d\mathbf{Q}}{d\mathbf{P}}\right|_{\mathcal{F}_T^W} = Z_T. \tag{4.17}$$

We have established that there is at least one EMM for the economy (R_t, S_t) and hence we can conclude that the economy based on such dynamics of the Black-Scholes and Merton model is arbitrage free. Notice that the EMM is unique since the deflated risky asset only admits representation as **Q**-martingale if and only if we set $\theta = \frac{r-\mu}{\sigma}$ and so by Theorem 3.3 the economy is *complete* which implies that any contingent claim is attainable. Therefore faced with any $\mathcal{F}_T$ measurable contingent claim Y_T an arbitrage free price process for the claim $\{\pi_t^Y, 0 \leqslant t \leqslant T\}$ is given by

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_t^W\right).$$

The quantity $\theta = \frac{r-\mu}{\sigma}$ is called the *market price for risk* and is seen as a measure for the tradeoff between risk and return.

4.2.1 Options

Contingent claims or derivatives can be grouped under three general headings, namely: *Options, Forwards and Futures and Swaps*. For our purposes we shall mainly be interested in contingent claims that fall under the category of options. An option is a financial instrument that gives the holder the *right, but* *not the obligation*, to buy or sell the underlying asset at or before a specified date known as the maturity or expiry date for a specific price known as the exercise or strike price. Call options afford the holder the right to buy the specified underlying while put options afford the holder the right to sell the predetermined underlying. European style options have the property that only permits the holder to buy/sell the underlying at expiry when the option reaches maturity. American style options allow the holder to buy/sell the underlying at any time prior to or at the maturity date. The simple or standard call and put options are referred to as plain-vanilla type options while other path-dependent options fall under the category known as exotic options.

We now consider the pricing of a European call option under the Black-Scholes and Merton model; having established that the economy (R_t, S_t) is arbitrage free and that equivalent martingale measure is unique we now proceed with the pricing of another tradable asset that we wish to add to the economy in a manner that will prevent any arbitrage opportunity. Since a European call option gives the holder the right but not the obligation to buy the underlying asset, which we take to be the risky asset S_t, the payoff of such a claim at maturity date T is given by the $\mathcal{F}_T^W$-measurable random variable $Y_T = \max(S_T - K, 0) = (S_T - K)^+$ where S_T is the value of the risky asset at time T and K is the strike price agreed at creation of the option. By Theorem 3.2 the deflated price process of the option must follow a $(\mathbf{Q}, \mathcal{F}_t^W)$ martingale property. Therefore we must have that

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{\max(S_T - K, 0)}{R_T}\bigg|\mathcal{F}_t^W\right) \tag{4.18}$$

where the measure $\mathbf{Q}$ is defined by the R-N derivative

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T^W} = Z_T = \exp\left(\left(\frac{r-\mu}{\sigma}\right)W_T - \frac{1}{2}\left(\frac{r-\mu}{\sigma}\right)^2 T\right). \tag{4.19}$$

In order to compute the value of the conditional expectation in Eq.(4.18) we essentially need the distribution of $\max\left(S_T - K, 0\right)$ under the measure $\mathbf{Q}$. Now recall that from Eq.(4.15) that we have

$$\begin{aligned}
\frac{S_T}{R_T} &= S_t\exp\left(\sigma(\tilde{W}_T - \tilde{W}_t) - \frac{1}{2}\sigma^2(T-t)\right)\\
\Rightarrow S_T &= S_t\exp\left(\left(r - \frac{1}{2}\sigma^2\right)(T-t) + \sigma(\tilde{W}_T - \tilde{W}_t)\right).
\end{aligned}$$

We can compute Eq.(4.18) as follows

$$\begin{aligned}
\frac{\pi_t^Y}{R_t} &= \mathbf{E}^{\mathbf{Q}}\left(\frac{\max(S_T - K, 0)}{R_T}\bigg|\mathcal{F}_t^W\right)\\
\pi_t^Y &= \mathbf{E}^{\mathbf{Q}}\left(e^{-r(T-t)}\max(S_T - K, 0)\bigg|\mathcal{F}_t^W\right).
\end{aligned} \tag{4.20}$$

The $\max(S_T - K, 0)$ can be written as $S_T\mathbf{1}_{(S_T>K)} - K\mathbf{1}_{(S_T>K)}$, and so Eq.(4.20) becomes

$$\begin{aligned}\pi_t^Y &= \mathbf{E}^{\mathbf{Q}}\big(e^{-r(T-t)}S_T\mathbf{1}_{(S_T>K)}|\mathcal{F}_t^W\big) - \mathbf{E}^{\mathbf{Q}}\big(e^{-r(T-t)}K\mathbf{1}_{(S_T>K)}|\mathcal{F}_t^W\big)\\ &= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big(S_T\mathbf{1}_{(S_T>K)}|\mathcal{F}_t^W\big) - Ke^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big(\mathbf{1}_{(S_T>K)}|\mathcal{F}_t^W\big). \quad (4.21)\end{aligned}$$

By making use of the fact that the increment $\tilde{W}_T - \tilde{W}_t \sim N(0, T-t)$ under the measure $\mathbf{Q}$ and independent of $\mathcal{F}_t^W$ we can compute the first term in Eq.(4.21) as follows

$$e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big(S_T\mathbf{1}_{(S_T>K)}|\mathcal{F}_t^W\big) = S_t\mathbf{Q}\bigg(Z \leqslant \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\bigg),$$

while the second term in Eq.(4.21) turns out to be

$$Ke^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big(\mathbf{1}_{(S_T>K)}|\mathcal{F}_t^W\big) = Ke^{-r(T-t)}\mathbf{Q}\bigg(Z \leqslant \frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\bigg),$$

where $Z \sim N(0,1)$ random variable. Hence the price process $\{\pi_t^Y, 0 \leqslant t \leqslant T\}$ for the $\mathcal{F}_T^W$ measurable claim known as a European call option is given by

$$\pi_t^Y = S_tN(d_1) - Ke^{-r(T-t)}N(d_2), \quad (4.22)$$

where d_1 and d_2 are given by

$$d_1 = \frac{\ln\frac{S_t}{K} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = \frac{\ln\frac{S_t}{K} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

and $N(x)$ is the standard normal cumulative distribution $\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\,dy$.

The formula in Eq.(4.22) is the famous Black-Scholes formula for the price of a European call option. If we were interested in computing the corresponding European put option with strike price K and payoff function given by $\max(K - S_T, 0)$ then we can make use of the relation that exists between the European call and put option, known as the *put-call-parity*. If we denote that value of the European call option with strike price K at time t by c_t and the corresponding European put by p_t. Then using the following payoff identity

$$\max(K - S_T, 0) = \max(S_T - K, 0) + K - S_T, \quad (4.23)$$

the value of the European put can be determined by the value the European call in the following manner by deflating all terms in Eq.(4.23) with respect to the numeriare and taking conditional expectations under the measure $\mathbf{Q}$ we

$$\begin{aligned}&\mathbf{E}^{\mathbf{Q}}\bigg(e^{-r(T-t)}\max(K - S_T, 0)\bigg|\mathcal{F}_t^W\bigg) = \mathbf{E}^{\mathbf{Q}}\bigg(e^{-r(T-t)}\max(S_T - K, 0)\bigg|\mathcal{F}_t^W\bigg)\\ &+\mathbf{E}^{\mathbf{Q}}\big(e^{-r(T-t)}K\big|\mathcal{F}_t^W\big) - \mathbf{E}^{\mathbf{Q}}\big(e^{-r(T-t)}S_T\big|\mathcal{F}_t^W\big)\\ &\Rightarrow p_t = c_t + Ke^{-r(T-t)} - S_t. \quad (4.24)\end{aligned}$$

Eq.(4.24) follows from the fact that the deflated stock price has a martingale property under the measure $\mathbf{Q}$. The relationship between the European call and put option as expressed in Eq.(4.24) is known as the *put-call parity (PCP)* and as can be seen offers a less computational method for determining the value of a European put option provided that the value of the corresponding European call option is known. The beauty of the Black-Scholes and Merton model lies in its computational efficiency and elegant solutions to the option pricing problem, however the are several drawbacks associated with this model as well most notably the models inability to incorporate the effects of sudden price jumps in the risky asset. It is this reason that we will now consider using a Lévy process as our source of randomness as such a process allows jumps to be included in the dynamics of the risky asset. As a result of using the Lévy process as the driving force of uncertainty we are equipped with a model that is more flexible than that of the Black-Scholes and Merton model.

4.3 Geometric Lévy model

Having studied the dynamics for the risky asset under the Black-Scholes and Merton framework we have established that the model proposed for the economy (R_t, S_t) is indeed arbitrage free and that the market is complete. Hence given any $\mathcal{F}_T^W$-measurable claim Y_T we can define a unique arbitrage free price process $\{\pi_t^Y, 0 \leqslant t \leqslant T\}$ for the claim by the prescription

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_t^W\right),$$

where $\mathbf{Q}$ is the unique EMM that gives all tradable assets in the economy a $(\mathbf{Q}, \mathcal{F}_t)$ martingale property. We now wish to replace the driving force of uncertainty within the model which is represented by the Brownian motion with a much more general Lévy process that will allow for jumps to be included in the risky assets dynamics. By using a Lévy process to model the uncertainty we obtain a model that is more realistic in depicting the behavior of a stock price process since it is well known that stock prices exhibit sudden jumps in relation to internal and external factors. The model that we will now consider for the economy (R_t, S_t) is given by the following dynamics under the real world measure $\mathbf{P}$

$$\begin{aligned} dR_t &= r_t R_t dt & R_0 = 1 \qquad (4.25)\\ dS_t &= \mu_t S_{t-} dt + \sigma_t S_{t-} dX_t & S_0 \in \mathbb{R}^+. \qquad (4.26) \end{aligned}$$

The coefficients of the stock price model, namely μ_t, σ_t and r_t are all assumed to be deterministic functions of time and the constraints imposed in Eq.(4.2) and Eq.(4.4) still hold. The process $\{X_t, 0 \leqslant t \leqslant T\}$ is a Lévy process satisfying some additional requirements.

Remark 4.1. While the dynamics for the risk-free asset R_t remains the same there is a change in the SDE for the risky asset S_t. The reason for this is more easily seen if we write the dynamics for S_t in integral form as

$$S_t = S_0 + \int_0^t \mu_u S_{u-}\ du + \int_0^t \sigma_u S_{u-}\ dX_u.$$

In order to give the integral $\int_0^t \sigma_u S_{u-}\ dX_u$ a stochastic interpretation the integrand must be predictable, and since the sample paths of a Lévy process are càdlàg (right continuous), this can only be done by working with the left version of S_t as seen in the SDE of Eq.(4.26).

We shall require that the Lévy process $\{X_t, \leqslant t \leqslant T\}$ satisfy

$$\mathbf{E}^{\mathbf{P}}\left[e^{-\theta X_t}\right] < \infty \qquad \text{for all } \theta \in (-\theta_1, \theta_2), \tag{4.27}$$

where $0 < \theta_1, \theta_2 \leqslant \infty$. Such a requirement will then imply that X_t has finite moments of all orders and in particular $\mathbf{E}^{\mathbf{P}}(X_1) < \infty$, see Chan [9]. We also have the Lévy measure ν satisfying the following conditions as a result of Eq.(4.27)

$$\int_{|x|\geqslant 1} e^{-\theta x}\ \nu(dx) < \infty \qquad \text{and} \qquad \int_{|x|\geqslant 1} x\ \nu(dx) < \infty.$$

With this additional requirement imposed on the Lévy process $\{X_t, 0 \leqslant t \leqslant T\}$ we can now write the Lévy-Itô decomposition as

$$X_t = ct + W_t + \int_{\mathbb{R}} x\ \tilde{N}(t, dx), \tag{4.28}$$

where $c = \mathbf{E}^{\mathbf{P}}(X_1)$, $\{W_t, 0 \leqslant t \leqslant T\}$ is standard Brownian motion and $\{M_t = \int_{\mathbb{R}} x\ \tilde{N}(t, dx), 0 \leqslant t \leqslant T\}$ is the process responsible for all the jumps which is independent of the Brownian motion. We assume without the loss of generality that the Lévy characteristic triplet is of the form $(c, 1, \nu)$. Returning to the SDE in Eq.(4.26), in order to find a solution to this equation we apply Itô's lemma to the function $\ln S_t$, noting that $\Delta S_t = \sigma_t S_{t-} \Delta X_t = \sigma_t S_{t-} \Delta M_t$ and $[X^c, X^c]_t = [W, W]_t = t$ we have

$$\begin{aligned}
\ln S_t &= \ln S_0 + \int_0^t \frac{1}{S_{u-}}\ dS_u + \frac{1}{2}\int_0^t \left(\frac{-1}{S_{u-}^2}\right) d[S^c, S^c]_u \\
&\quad + \sum_{0 \leqslant u \leqslant t} \left[\ln S_u - \ln S_{u-} - \frac{1}{S_{u-}}\Delta S_u\right] \\
&= \ln S_0 + \int_0^t \left(\mu_u\ du + \sigma_u\ dX_u\right) - \frac{1}{2}\int_0^t \sigma_u^2\ du \\
&\quad + \sum_{0 \leqslant u \leqslant t} \left[\ln(\sigma_u \Delta M_u + 1) - \sigma_u \Delta M_u\right] \\
&= \ln S_0 + \int_0^t (\mu_u - \frac{1}{2}\sigma_u^2)\ du + \int_0^t \sigma_u\ dX_u + \sum_{0 \leqslant u \leqslant t} \left[\ln(\sigma_u \Delta M_u + 1) - \sigma_u \Delta M_u\right].
\end{aligned}$$

Finally we get

$$S_t = S_0 \mathrm{exp}\left(\int_0^t \sigma_u\ dX_u + \int_0^t (\mu_u - \frac{1}{2}\sigma_u^2)\ du\right) \prod_{0 \leqslant u \leqslant t} (\sigma_u \Delta M_u + 1) e^{-\sigma_u \Delta M_u}. \tag{4.29}$$

In order to insure that the stock price S_t remains nonnegative we need $\sigma_t \Delta M_t > -1$ for all t, one possible way to ensure this is to restrict the Lévy measure ν to some interval $[-c_1, c_2]$ where $c_1, c_2 \geqslant 0$. By restricting the Lévy measure in this way we will always have $\Delta M_t \in [-c_1, c_2]$ for all t (i.e. the stock price can only experience jumps of a certain magnitude) and the stock price will remain nonnegative as long as $-\frac{1}{c_2} \leqslant \sigma_t \leqslant \frac{1}{c_1}$.

Having specified the dynamics for the risky asset S_t in a manner that allows for jumps to be included in the stock price model we must now establish whether the economy (R_t, S_t), that is being based on the new dynamics for the risky asset, is arbitrage free. We proceed as before and begin by selecting R_t as our pricing numeraire, by Theorem 3.1 we need to verify that at least one EMM exists for the economy (R_t, S_t) under the new dynamics. An application of Itô's lemma to the deflated stock price $\tilde{S}_t = \frac{S_t}{R_t}$ yields

$$\begin{aligned}
\tilde{S}_t &= \tilde{S}_0 + \int_0^t (-r_u \tilde{S}_{u-})\, du + \int_0^t \frac{1}{R_u}\, dS_u + \frac{1}{2}\int_0^t \frac{0}{R_t}\, d[S^c, S^c]_u \\
&\quad + \sum_{0 \leqslant u \leqslant t} \big[\tilde{S}_u - \tilde{S}_{u-} - \frac{1}{R_u}\Delta S_u\big] \\
&= \tilde{S}_0 + \int_0^t \tilde{S}_{u-}(\mu_u - r_u)\, du + \int_0^t \sigma_u \tilde{S}_{u-}\, dX_u \\
&\quad + \sum_{0 \leqslant u \leqslant t} \big[\tilde{S}_{u-}\sigma_u \Delta M_u - \tilde{S}_{u-}\sigma_u \Delta M_u\big] \\
&= \tilde{S}_0 + \int_0^t \tilde{S}_{u-}(\mu_u - r_u)\, du + \int_0^t \sigma_u \tilde{S}_{u-}\, \big(c\, du + dW_u + dM_u\big) \\
&= \tilde{S}_0 + \int_0^t \tilde{S}_{u-}(\mu_u - r_u + c\sigma_u)\, du + \int_0^t \sigma_u \tilde{S}_{u-}\, dW_u \\
&\quad + \int_0^t \sigma_u \tilde{S}_{u-}\, dM_u.
\end{aligned} \tag{4.30}$$

Now in order to effect a change in measure from the real world measure $\mathbf{P}$ to some other equivalent measure $\mathbf{Q}$ we must implement a change of measure formula such as the stochastic exponential defined in Eq.(2.75) by the SDE

$$dZ_t = Z_{t-}\big[h_t\, dW_t + \int_{|x|<1} (H(t,x) - 1)\, \tilde{N}(dt, dx) + \int_{|x|\geqslant 1} (F(t,x) - 1)\, \tilde{N}(dt, dx)\big]. \tag{4.31}$$

Since we have imposed the restriction that the Lévy process $\{X_t, 0 \leqslant t \leqslant T\}$ must have finite moments of all orders we can without loss of generality rewrite the SDE in Eq.(4.31) as

$$\begin{aligned}
dZ_t &= Z_{t-}\big[h_t\, dW_t + \int_{\mathbb{R}} (H(t,x) - 1)\tilde{N}(dt, dx)\big] && (4.32)\\
&= Z_{t-}\, dY_t, && (4.33)
\end{aligned}$$

where $\{Y_t, 0 \leqslant t \leqslant T\}$ is a Lévy integral process given by

$$Y_t = \int_0^t h_s \, dW_s + \int_0^t \int_{\mathbb{R}} (H(s,x) - 1) \, \tilde{N}(ds, dx). \tag{4.34}$$

The solution to the SDE in Eq.(4.33) is the Doléans-Dade exponential defined in Eq.(2.78) as

$$\begin{aligned} Z_t = \exp\bigg(\int_0^t h_s \, dW_s + \int_0^t \int_{\mathbb{R}} (H(s,x) - 1) \, \tilde{N}(ds, dx) - \frac{1}{2} \int_0^t h_s^2 \, ds \bigg) \\ \times \prod_{0 \leqslant s \leqslant t} H(s, \Delta X_s) e^{-(H(s, \Delta X_s) - 1)}, \end{aligned} \tag{4.35}$$

in order for Z_t to remain strictly positive we shall impose the constraint that $H(t, \Delta X_t) > 0$ **P**-a.s. As can be seen from Eq.(4.32) Z_t is a local martingale under the measure **P**, if we select the functions/processes $\{h_t, 0 \leqslant t \leqslant T\}$ and $\{H(t,x), 0 \leqslant t \leqslant T\}$ carefully so as to guarantee that $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ for all t then by Theorem 2.10 we will have the process $\{Z_t, 0 \leqslant t \leqslant T\}$ being a strictly positive martingale under the measure **P** and hence we can use this stochastic exponential to implement a change in measure by defining

$$\left. \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \right|_{\mathcal{F}_T^X} = Z_T. \tag{4.36}$$

Calling Z_T the R-N derivative linking **Q** to **P** on the measure space $(\Omega, \mathcal{F}_T^X)$.Under this new **P**-equivalent measure **Q** Girsanov's theorem tells us that there exists a process $\{\tilde{W}_t, 0 \leqslant t \leqslant T\}$ defined by

$$\tilde{W}_t = W_t - \int_0^t h_s \, ds, \tag{4.37}$$

that is standard Brownian motion under the measure **Q**. Further more the jump process $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ defined by

$$\begin{aligned} \tilde{M}_t &= M_t - \int_0^t \int_{\mathbb{R}} x(H(s,x) - 1) \, \nu(dx) ds \\ &= \int_0^t \int_{\mathbb{R}} x \big[\tilde{N}(ds, dx) - H(s,x) \, \nu(dx) ds \big] \\ &= \int_0^t \int_{\mathbb{R}} x \tilde{N}_{\mathbf{Q}}(ds, dx) \end{aligned} \tag{4.38}$$

is a **Q**-martingale and also independent of the Brownian motion process $\{\tilde{W}_t, 0 \leqslant t \leqslant T\}$. Here

$$\tilde{N}_{\mathbf{Q}}(ds, dx) = \tilde{N}(ds, dx) - H(s,x) \, \nu(dx) ds.$$

Making use of Eq.(4.37) and Eq.(4.38) we can now rewrite the deflated stock price in Eq.(4.30) as

$$
\begin{aligned}
\tilde{S}_t &= \tilde{S}_0 + \int_0^t \tilde{S}_{u-}(\mu_u - r_u + c\sigma_u)\,du + \int_0^t \sigma_u \tilde{S}_{u-}\,\left(d\tilde{W}_u + h_u\,du\right) \\
&\quad + \int_0^t \sigma_u \tilde{S}_{u-}\,\left(d\tilde{M}_u + \int_{\mathbb{R}} x(H(u,x)-1)\,\nu(dx)du\right) \\
&= \int_0^t \tilde{S}_u \left(\mu_u - r_u + c\sigma_u + \sigma_u h_u + \int_{\mathbb{R}} \sigma_u x(H(u,x)-1)\nu(dx)\right) du \quad (4.39) \\
&\quad + \tilde{S}_0 + \int_0^t \sigma_u \tilde{S}_{u-}\,d\tilde{W}_u + \int_0^t \sigma_u \tilde{S}_{u-}\,d\tilde{M}_u. \quad (4.40)
\end{aligned}
$$

Since an arbitrage free economy implies and is implied by the existence of at least one EMM, in order to give the deflated stock price a local martingale property under the measure $\mathbf{Q}$ we must remove the drift component from the deflated stock price which is represented expression (4.39). Hence if we set

$$
\mu_u - r_u + c\sigma_u + \sigma_u h_u + \int_{\mathbb{R}} \sigma_u x(H(u,x)-1)\nu(dx) = 0, \tag{4.41}
$$

then we obtain the following dynamics for the deflated stock price under the measure $\mathbf{Q}$

$$
\tilde{S}_t = \tilde{S}_0 + \int_0^t \sigma_u \tilde{S}_u\,d\tilde{W}_u + \int_0^t \sigma_u \tilde{S}_u\,d\tilde{M}_u. \tag{4.42}
$$

Such a representation implies that the deflated stock price process is a local $\mathbf{Q}$-martingale. In order to prove that $\tilde{S}_t$ is a $(\mathbf{Q}, \mathcal{F}_t^X)$ martingale we need to impose the following integrability constraints on the stochastic integrals

$$
\mathbf{E}^{\mathbf{Q}}\left(\left|\int_0^T \sigma_u \tilde{S}_{u-}\,d\tilde{W}_u\right|^2\right) = \mathbf{E}^{\mathbf{Q}}\left(\int_0^T \sigma_u^2 \tilde{S}_u^2\,du\right) < \infty, \tag{4.43}
$$

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left(\left|\int_0^T \sigma_u \tilde{S}_{u-}\,d\tilde{M}_u\right|^2\right) \\
&= \mathbf{E}^{\mathbf{Q}}\left(\int_0^T \int_{\mathbb{R}} \sigma_u^2 \tilde{S}_{u-}^2 x^2 H(s,x)^2\,\nu(dx)ds\right) < \infty. \quad (4.44)
\end{aligned}
$$

Only once we have verified that Eq.(4.43) and Eq.(4.44) hold can we then conclude that the deflated stock price process has a $(\mathbf{Q}, \mathcal{F}_t^X)$ martingale property. Provided that this martingale property can be shown to exist we will then be able to deduce that the economy (R_t, S_t) is arbitrage free being based on the dynamics of the Geometric Lévy process. In actual practise it can become very complicated to verify Eq.(4.44), it is for this reason that we shall consider the cases where $H(t,x)$ is a deterministic function of time and bounded on all finite intervals $[0,t] \subseteq [0,T]$, such an assumption will also facilitate the computation in proving that the stochastic exponential has $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$. While Eq.(4.41) is a necessary requirement for the deflated stock price to have a $\mathbf{Q}$-martingale property it does not uniquely determine the functions h_t and $H(t,x)$ since we only have a single equation involving two unknown functions, all that we know is that $H(t,x) > -1$ **P**-a.s. Since these functions are not uniquely specified by Eq.(4.41) the resulting EMM defined by the R-N derivative

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T^X} = Z_T, \tag{4.45}$$

will not be a unique equivalent martingale measure. In fact there are infinitely many EMM's, one for each choice of h_t and $H(t,x)$. Hence while the economy (R_t, S_t) is indeed arbitrage free there exists no unique EMM and thus we are faced with the problem of pricing claims in an incomplete market. Suppose that we now wish to add some $\mathcal{F}_T$-measurable claim Y_T to the economy (R_t, S_t). Market incompleteness implies that such a claim may no longer be attainable, and while we can still develop an arbitrage free price process for the claim $\{\pi_t^Y, 0 \leqslant t \leqslant T\}$ by defining

$$\frac{\pi_t^Y}{R_t} = \mathbf{E}^{\mathbf{Q}}\left(\frac{Y_T}{R_T}\middle|\mathcal{F}_t^X\right),$$

such a price process for the claim will no longer be unique as it depends on the measure **Q**. Thus there exist infinitely many possible price process for the claim Y_T, one for each possible EMM. The question we are now faced with is this: given that we have an incomplete economy (R_t, S_t), how do we select one particular EMM under which all claims can be priced? We shall examine this problem in the following chapter.

Chapter 5

Selecting an Equivalent Martingale Measure in an Incomplete Market

Having verified the economy (R_t, S_t) being based on the dynamics of the the Geometric Lévy model of Eq.(4.25) and Eq.(4.26) is arbitrage free we are now faced with the problem of market *incompleteness*. The purpose of this chapter is to develop criteria that will aid us in pricing contingent claims in an incomplete market. Since market incompleteness implies that there no longer exists a replicating portfolio that can be used to determine the arbitrage free price process $\{\pi_t^Y, 0 \leqslant t \leqslant T\}$ for the $\mathcal{F}_T$-measurable contingent claim Y_T, in light of Theorem 3.3 market incompleteness translates into the fact that the equivalent martingale measure **Q** is no longer unique, in fact there exist infinitely many possible EMM's. The approach that we shall take deals with selecting a specific EMM from this pool of possible measures and the justification for selecting this measure will be based on the fact that this measure satisfies certain requirements. This measure will then be used in the pricing of contingent claims and will be "unique" in the sense that the price process associated with a contingent claim corresponds only to that particular EMM selected. We shall now look at these criteria.

5.1 Föllmer-Schweizer minimal martingale measure

Recall that Eq.(4.41) was a necessary condition for the existence of an EMM

$$\mu_u - r_u + c\sigma_u + \sigma_u h_u + \int_{\mathbb{R}} \sigma_u x(H(u,x) - 1)\nu(dx) = 0, \tag{5.1}$$

however as this equation did not specify both $H(u,x)$ and h_u the resulting martingale measure was not unique implying market incompleteness. The first approach that we shall take in order to determine both $H(u,x)$ and h_u is referred to as the *Föllmer-Schweizer minimal martingale measure*, see [15]. This minimal measure is constructed with the aid of a replicating or hedging portfolio.

We shall assume that all processes that we encounter hence forth are square integrable, i.e. any process Y_t belongs to the space $L^2(\Omega, \mathbf{P})$. The need to work with square integrable processes will enable us to make use of representation theorems which will prove extremely useful in this section. In an incomplete market not all claims are attainable since all self-financing trading strategies can no longer guarantee that the terminal value of the portfolio replicates the random payoff, i.e. we no longer have $V_T = Y_T$ $\mathbf{P}$-a.s. In order to deal with this difficulty Föllmer and Schweizer suggest that a cumulative cost process $\{C_t, 0 \leqslant t \leqslant T\}$ be included in the hedging portfolio that is defined by

$$C_t = \tilde{V}_t - \int_0^t \phi_u \, d\tilde{S}_u, \tag{5.2}$$

where $\tilde{V}_t$ is the deflated value of the portfolio at time t with respect to the numeraire R_t and $\int_0^t \phi_u \, d\tilde{S}_u$ represents any gains in the portfolio with $\tilde{S}_t$ being the deflated value of the risky asset; for detailed motivation behind this definition see Föllmer and Sondermann [16]. From Eq.(5.2) we see that the hedging portfolio is no longer self-financing since the portfolio requires external injections of funds represented by the cost process. In light of the fact that funds can now be injected into the portfolio we can now define an admissible strategy as one which guarantees $V_T = Y_T$ $\mathbf{P}$-a.s. This does not mean that the claim is attainable since we have defined attainability in terms of a *self-financing portfolio* and the portfolio process that we now deal with is of the form

$$\tilde{V}_t = \int_0^t \phi_u \, d\tilde{S}_u + C_t. \tag{5.3}$$

The cost process represents the external funds required to ensure that the portfolio yields $V_T = Y_T$ $\mathbf{P}$-a.s.

Remark 5.1. Note that we still invest in both risky and risk-free assets and so the process $\{V_t, 0 \leqslant t \leqslant T\}$ defined by

$$V_t = \phi_t S_t + \varphi_t R_t,$$

is still the value of the trading strategy however as this trading strategy can no longer fully hedge the claim an additional cost must be incurred. It is for this reason why we focus on the process defined in Eq.(5.3) as it represents gains from trading along with the cumulative cost component that is required to hedge the claim.

We can interpret this cost process $\{C_t, 0 \leqslant t \leqslant T\}$ as the additional or *intrinsic* risk that associated with the hedging strategy. If we recall the Black-Scholes and Merton framework, since the model that was specified for the asset dynamics was complete we could replicate any contingent claim without incurring any additional risk. In a market where claims are non attainable we are exposed to the intrinsic risk of not being able to perfectly replicate the claim, and naturally we would like this intrinsic risk to be as minimal as possible. So as a measure of the additional risk Föllmer and Schweizer introduced the conditional mean square error process defined by

$$\mathbf{E}^{\mathbf{P}}\big[\big(C_T - C_t\big)^2 \big| \mathcal{F}_t^X\big]. \tag{5.4}$$

Remark 5.2. Note that we have used the real world measure $\mathbf{P}$ in the definition of the conditional mean square error process. The justification is given as follows. If we consider the case where the contingent claim is a call option, then the call writer would want to minimize his additional risk based on his own *subjective beliefs*.

In the Black-Scholes setup Eq.(5.4) is always zero. Since we wish for the additional risk to be minimal we must therefore identify trading strategies that minimize that conditional mean square error process defined in Eq.(5.4). Before we can proceed in identifying such trading strategies we need define what is meant by *orthogonal martingales* since these martingales play a crucial role in identifying strategies that minimize the conditional mean square error process.

Definition 5.1. Let $\{X_t, 0 \leqslant t \leqslant T\}$ and $\{Y_t, 0 \leqslant t \leqslant T\}$ be two square integrable martingales under the measure $\mathbf{P}$, i.e. X_t and $Y_t \in L^2(\Omega, \mathbf{P})$. Then X_t and Y_t are said to be orthogonal if and only if $[X, Y]_t = 0$ $\mathbf{P}$-a.s. We denote orthogonality by $X_t \perp Y_t$.

In order to understand Definition 5.1 we use the idea of quadratic covariation between two L^2-integrable martingales, see Meyer [38]. Under Meyer's definition the quadratic covariation is defined to be an increasing process $[X, Y]_t$ satisfying

$$\mathbf{E}^{\mathbf{P}}\big((X_t - X_s)(Y_t - Y_s)\big|\mathcal{F}_s\big) = \mathbf{E}^{\mathbf{P}}\big([X, Y]_t - [X, Y]_s\big|\mathcal{F}_s\big). \tag{5.5}$$

To show that orthogonality occurs if and only if $[X, Y]_t = 0$, let us assume that $[X, Y]_t = 0$ $\mathbf{P}$-a.s. first. According to Meyer's definition the quadratic covariation is an increasing process, hence $[X, Y]_t = 0$ implies that $[X, Y]_s = 0$ for all $s \leqslant t$. So we must have

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\big((X_t - X_s)(Y_t - Y_s)\big|\mathcal{F}_s\big) = 0 \\
&\mathbf{E}^{\mathbf{P}}(X_t Y_t|\mathcal{F}_s) - X_s \mathbf{E}^{\mathbf{P}}(Y_t|\mathcal{F}_s) - Y_s \mathbf{E}^{\mathbf{P}}(X_t|\mathcal{F}_s) + X_s Y_s = 0 \\
&\mathbf{E}^{\mathbf{P}}(X_t Y_t|\mathcal{F}_s) - X_s Y_s = 0 \\
&\Rightarrow \mathbf{E}^{\mathbf{P}}(X_t Y_t|\mathcal{F}_s) = X_s Y_s \qquad\qquad \mathbf{P} - \text{a.s.}
\end{aligned} \tag{5.6}$$

Therefore we have $[X, Y]_t = 0$ $\mathbf{P}$-a.s. implying that the process $\{X_t Y_t, 0 \leqslant t \leqslant T\}$ is a martingale under the measure $\mathbf{P}$. Conversely let us now assume that the process $\{X_t Y_t, 0 \leqslant t \leqslant T\}$ is a martingale under the measure $\mathbf{P}$, then by the product rule (Theorem 2.8) we have

$$X_t Y_t = \int_0^t X_{s-}\, dY_s + \int_0^t Y_{s-}\, dX_s + [X, Y]_t, \tag{5.7}$$

noting that $X_0 Y_0 = 0$. Now since both X_t and Y_t are square integrable martingales the stochastic integrals are in Eq.(5.7) are well defined and are both local martingales under the measure $\mathbf{P}$, but we have assumed that the process $\{X_t Y_t, 0 \leqslant t \leqslant T\}$ is a martingale under the measure $\mathbf{P}$ therefore both the stochastic integrals in Eq.(5.7) must be $\mathbf{P}$-martingales. Now since the quadratic covariation is an increasing function in order to preserve the martingale property which we now know exists we must have $[X, Y]_t = 0$ $\mathbf{P}$-a.s. Hence the martingale property of the process $\{X_tY_t, 0 \leqslant t \leqslant T\}$ implies that the quadratic covariation between X_t and Y_t must be equal to zero.

Returning now to the Föllmer Schweizer minimal martingale measure, we seek to minimize the conditional mean square error process represented by Eq.(5.4). To be more precise the minimization we shall discuss is in fact local minimization in that the additional risk should be minimal under all "infinitesimal perturbations", see Schweizer [43] for a more detailed account of this concept. To fix ideas let us consider the special case where the measure $\mathbf{P}$ is already a martingale measure for the deflated stock price process. This then implies that the claim Y_T must be a square integrable martingale under the measure $\mathbf{P}$. Furthermore it can be shown that the space generated by the all square integrable random variables of the form $\int_0^t \phi_u \, d\tilde{S}_u$ forms a closed subspace of the space $L^2(\Omega, \mathbf{P})$. The Kunita-Watanabe decomposition, see [31], then states that any claim $Y_T \in L^2(\Omega, \mathbf{P})$ has a unique decomposition given by

$$Y_T = Y_0 + \int_0^T \hat{\phi}_u \, d\tilde{S}_u + L_T, \tag{5.8}$$

where $\{L_t, 0 \leqslant t \leqslant T\}$ is a square integrable martingale orthogonal to $\tilde{S}_t$ under the measure $\mathbf{P}$. Then setting

$$\phi_t = \hat{\phi}_t \qquad\qquad \varphi_t = \tilde{V}_t - \phi_t \tilde{S}_t,$$

along with

$$\tilde{V}_t = Y_0 + \int_0^T \hat{\phi}_u \, d\tilde{S}_u + L_t, \tag{5.9}$$

we obtain a trading strategy where the accumulated cost process is a square integrable martingale orthogonal to $\tilde{S}_t$ under the measure $\mathbf{P}$, i.e. $C_t = L_t$. It can further be shown that such a trading strategy minimizes the conditional mean square error process, see Schweizer [43]. Thus the problem of minimizing the conditional mean square error process can be solved using the Kunita-Watanabe projection theorem provided the the measure $\mathbf{P}$ is a martingale measure. We refer to any trading strategy satisfying

$$\mathbf{E}^{\mathbf{P}}(C_T - C_t | \mathcal{F}_t) = 0,$$

as *mean-self-financing*, i.e. the cost process $\{C_t, 0 \leqslant t \leqslant T\}$ is $\mathbf{P}$-martingale. Based on our previous results with the geometric Lévy model we know that the deflated stock price is not a martingale under the measure $\mathbf{P}$ and since there is no analogous Kunita-Watanabe decomposition for an arbitrary semimartingale we cannot make use of the above results in our current situation. However we can make use of the above ideas, under the geometric Lévy model the deflated stock price under the measure $\mathbf{P}$ was given by the semimartingale

$$\begin{aligned} \tilde{S}_t &= \tilde{S}_0 + \int_0^t \tilde{S}_{u-}(\mu_u - r_u + c\sigma_u) \, du + \int_0^t \sigma_u \tilde{S}_{u-} \, dW_u \\ &\quad + \int_0^t \sigma_u \tilde{S}_{u-} \, dM_u, \end{aligned} \tag{5.10}$$

where the martingale part of $\tilde{S}_t$ under the measure **P** is

$$\int_0^t \sigma_u \tilde{S}_{u-}\, dW_u + \int_0^t \sigma_u \tilde{S}_{u-}\, dM_u.$$

The following proposition is essential in finding a trading strategy that locally minimizes the conditional mean square error process.

Proposition 5.1. *A trading strategy which "locally" minimizes the conditional mean square error process is equivalent to a mean-self-financing strategy that has the cumulative cost process $\{C_t, 0 \leqslant t \leqslant T\}$ being orthogonal to the martingale part of the deflated stock price under the measure* **P**.

Proof: See Schweizer [43] ■

So we need to identify a trading strategy that has the cost process as a **P**-martingale which is orthogonal to the martingale part of the deflated stock price under the measure **P**, such a strategy is called an optimal trading strategy. The existence of such optimal strategies is given in Föllmer and Schweizer [15] which states that an optimal strategy is equivalent to the deflated claim $\tilde{Y}_T$ having the following decomposition under the measure **P**

$$\tilde{Y}_T = Y_0 + \int_0^T \phi_u\, d\tilde{S}_u + L_T, \tag{5.11}$$

where $\tilde{Y}_T$ is the deflated claim and the process $\{L_t, 0 \leqslant t \leqslant T\}$ is a square integrable martingale orthogonal to the martingale part of $\tilde{S}_t$ under **P**. To see that such a decomposition is indeed equivalent to an optimal trading strategy let us suppose that the claim Y_T admits representation as in Eq.(5.11), then by setting

$$\begin{aligned} \tilde{V}_t &= Y_0 + \int_0^t \phi_u\, d\tilde{S}_u + L_t \\ \varphi_t &= \tilde{V}_t - \phi_t \tilde{S}_t, \end{aligned}$$

will produce a trading strategy that is optimal. Conversely any optimal trading strategy will yield a decomposition of the form in Eq.(5.11) with $L_t = C_t$ as a square integrable martingale orthogonal to the martingale part of $\tilde{S}_t$ under the measure **P**. From our knowledge of equivalent martingales measure we know that $\tilde{V}_t$ must be a martingale under some **P**-equivalent measure, as this negates any possibility of arbitrage. If we accept that the deflated value of the claim must be of the form as in Eq.(5.11) and set $\tilde{V}_T = \tilde{Y}_T$ we can then compute the value of the claim for any $t \leqslant T$ by setting

$$\tilde{V}_t = \mathbf{E}^{\mathbf{Q}}\big(\tilde{V}_T | \mathcal{F}_t\big),$$

for some equivalent martingale measure **Q**. In order to determine the measure **Q** we shall make use of the decomposition of $\tilde{V}_t$. More specifically since the stochastic integral process $\{\int_0^t \phi_u\, d\tilde{S}_u, 0 \leqslant t \leqslant T\}$ is a **Q**-martingale, in order to ensure that $\tilde{V}_t$ be a **Q**-martingale we require that the process $\{L_t, 0 \leqslant t \leqslant T\}$

defined in Eq.(5.11) also be a martingale under the measure $\mathbf{Q}$. We therefore seek a measure $\mathbf{Q}$ that gives the deflated stock price a martingale property but at the same time retains the martingale property of any square integrable $\mathbf{P}$-martingale orthogonal to the martingale part of $\tilde{S}_t$ under $\mathbf{P}$. This brings us to the idea of a *minimal martingale measure*.

Definition 5.2. An equivalent martingale measure $\mathbf{Q}$ is called minimal if every square integrable $\mathbf{P}$-martingale which is orthogonal to the martingale part of $\tilde{S}_t$ under $\mathbf{P}$ remains a martingale under the measure $\mathbf{Q}$.

The minimal measure $\mathbf{Q}$ ensures that any optimal strategy with decomposition as in Eq.(5.11) is also a $\mathbf{Q}$-martingale which justifies us taking $\tilde{V}_t = \mathbf{E}^{\mathbf{Q}}(\tilde{V}_T|\mathcal{F}_t)$ to be the deflated value of the claim at time t. The minimal measure has the property that apart from changing the deflated stock price into a $\mathbf{Q}$-martingale, it leaves intact the remaining structure of the model. In particular it preserves any orthogonality relations. The next theorem furnishes us with a result that will then enable us to compute the minimal martingale measure for the geometric Lévy model.

Theorem 5.1. *The minimal martingale measure* $\mathbf{Q}$ *exists if and only if the R-N derivative* $Z_T = \frac{d\mathbf{Q}}{d\mathbf{P}}|_{\mathcal{F}_T}$ *satisfies*

$$Z_t = 1 + \int_0^t \gamma_s Z_{s-}\, dG_t, \tag{5.12}$$

where $\{G_t, 0 \leqslant t \leqslant T\}$ *is the martingale part of the Lévy process* X_t *under the measure* $\mathbf{P}$. *Furthermore the measure is unique.*

Proof: See Föllmer and Schweizer [15] ■

In the geometric Lévy model the martingale part of the Lévy process X_t under the measure $\mathbf{P}$ is given by

$$\begin{aligned} G_t &= W_t + M_t \\ &= W_t + \int_{\mathbb{R}} x\, \tilde{N}(ds, dx). \end{aligned} \tag{5.13}$$

The minimal martingale measure is therefore of the form

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_{s-}\gamma_s\, d\big(W_s + M_s\big) \\ &= 1 + \int_0^t Z_{s-}\gamma_s\, dW_s + \int_0^t \int_{\mathbb{R}} Z_{s-}\gamma_s x\, \tilde{N}(ds, dx). \end{aligned} \tag{5.14}$$

Since our change of measure formula was given by the stochastic integral equation

$$Z_t = 1 + \int_0^t Z_{s-} h_s\, dW_s + \int_0^t \int_{\mathbb{R}} Z_{s-}(H(s,x) - 1)\, \tilde{N}(t, dx). \tag{5.15}$$

In order for this change of measure formula to be a minimal martingale measure it must be of the form in Eq.(5.14). So we require that

$$h_s = \gamma_s \qquad \text{and} \qquad H(s,x) - 1 = x\gamma_s = x h_s. \tag{5.16}$$

We can now write Eq.(4.41) as

$$\begin{aligned}
\mu_u - r_u + c\sigma_u + \sigma_u h_u + \int_{\mathbb{R}} \sigma_u x(H(u,x)-1)\nu(dx) &= 0 \\
\mu_u - r_u + c\sigma_u + \sigma_u h_u + \int_{\mathbb{R}} \sigma_u x(xh_u)\,\nu(dx) &= 0,
\end{aligned} \tag{5.17}$$

setting $v = \int_{\mathbb{R}} x^2\,\nu(dx)$ we finally get

$$\begin{aligned}
h_u &= \frac{r_u - c\sigma_u - \mu_u}{\sigma_u(1+v)} \\
H(u,x) - 1 &= \left(\frac{r_u - c\sigma_u - \mu_u}{\sigma_u(1+v)}\right)x.
\end{aligned} \tag{5.18}$$

We have thus specified both functions h_u and $H(u,x)$ and in doing so have specified the equivalent martingale measure. However in order for this stochastic exponential to remain strictly positive, and thus serve as a probability measure, we must impose some constraints on the function $H(u,x)$. From Eq.(4.35) we know that $H(u,x) > 0$ **P**-a.s. This is then equivalent to the condition that

$$-\frac{1}{c_2} \leqslant \left(\frac{r_u - c\sigma_u - \mu_u}{\sigma_u(c+v)}\right) \leqslant \frac{1}{c_1},$$

since $\Delta X_t \in [-c_1, c_2]$. To see that the orthogonality is indeed preserved, let $\{B_t, 0 \leqslant t \leqslant T\}$ be any square integrable **P**-martingale orthogonal to the martingale part of the Lévy process G_t under the measure **P**, i.e.

$$\begin{aligned}
&[B,G]_t = 0 \\
&\left[G, W + \int_{\mathbb{R}} x\,\tilde{N}(\cdot,dx)\right]_t = 0 \\
&[G,W]_t + \left[G, \int_{\mathbb{R}} x\,\tilde{N}(\cdot,dx)\right]_t = 0.
\end{aligned} \tag{5.19}$$

Then we will have that B_t is orthogonal to the martingale part of $\tilde{S}_t$ under the measure **P** since

$$\begin{aligned}
&\left[B, \int_0^{\cdot} \sigma_u \tilde{S}_{u-}\,dW_u + \int_0^{\cdot} \sigma_u \tilde{S}_{u-}\,dM_u\right]_t \\
&= \left[\int_0^{\cdot} dB_u, \int_0^{\cdot} \sigma_u \tilde{S}_{u-}\,dW_u\right]_t + \left[\int_0^{\cdot} dB_u, \int_0^{\cdot} \sigma_u \tilde{S}_{u-}\,dM_u\right]_t \\
&= \int_0^t \sigma_u \tilde{S}_{u-}\,d[B,W]_u + \int_0^t \sigma_u \tilde{S}_u\,d[B,M]_u \\
&= \int_0^t \sigma_u \tilde{S}_{u-}\,d[B,W]_u + \int_0^t \sigma_u \tilde{S}_u\,d\left[B, \int_{\mathbb{R}} x\,\tilde{N}(\cdot,dx)\right]_u,
\end{aligned} \tag{5.20}$$

from Eq.(5.19) we conclude that Eq.(5.20) must be equal to zero and hence the square integrable martingale B_t is orthogonal to the martingale part of the deflated stock price under the measure **P**. Similar arguments show that B_t is orthogonal to the **P** martingale Z_t in Eq.(5.15), we therefore have that $[B_t, Z_t]_t = 0$. From the product rule we then have

$$\begin{aligned}
B_t Z_t &= \int_0^t B_{s-}\, dZ_s + \int_0^t Z_{s-}\, dB_s + [B,Z]_t \\
B_t Z_t &= \int_0^t B_{s-}\, dZ_s + \int_0^t Z_{s-}\, dB_s,
\end{aligned} \tag{5.21}$$

since $\{B_t, 0 \leqslant t \leqslant T\}$ and $\{Z_t, 0 \leqslant t \leqslant T\}$ are **P**-martingales which are orthogonal, $\{B_t Z_t, 0 \leqslant t \leqslant T\}$ is a local **P**-martingale. From Lemma 2.3 we can now conclude that $\{B_t, 0 \leqslant t \leqslant T\}$ must also be a local **Q**-martingale. The Cauchy-Schwartz inequality (see Weir [45]) can then be used along with Lemma 2.2 to deduce that

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}|B_t| \\
&= \mathbf{E}^{\mathbf{P}}(|B_t Z_t|) \\
&\leqslant \mathbf{E}^{\mathbf{P}}(|B_t|^2) \cdot \mathbf{E}^{\mathbf{P}}(|Z_t|^2)
\end{aligned} \tag{5.22}$$

and since both Z_t and $B_t \in L^2(\Omega, \mathbf{P})$, we conclude that Eq.(5.22) is finite and hence $\mathbf{E}^{\mathbf{Q}}|B_t| < \infty$, therefore $B_t \in L^1(\Omega, \mathbf{Q})$ and thus a **Q**-martingale.

5.2 Minimal relative entropy measure

The next criteria that we shall consider involves choosing an equivalent martingale measure that has *minimum relative entropy*. More precisely, for a fixed measure **P** we define the relative entropy $I_{\mathbf{P}}(\mathbf{Q})$ of any measure **Q** with respect to **P** as

$$I_{\mathbf{P}}(\mathbf{Q}) = \begin{cases} \int \ln \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big|_{\mathcal{F}_T}\, \mathrm{d}\mathbf{Q} & \text{if } \mathbf{Q} \text{ is absolutely continuous with respect to } \mathbb{P} \\ \infty & \text{otherwise.} \end{cases}$$

We therefore are interested in the measure **Q** that minimizes $I_{\mathbf{P}}(\mathbf{Q})$ over the set of all possible equivalent martingale measures. If we think of the real world measure **P** as encapsulating information about how the economy behaves then measure **Q** with minimum relative entropy is closest to the measure **P** in terms of its information content. Recall that the R-N derivative used to implement a change of measure was given by $\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\big|_{\mathcal{F}_T} = Z_T$, where Z_t was given by the stochastic exponential

$$Z_t = \exp\left(\int_0^t h_s\, dW_s + \int_0^t \int_{\mathbb{R}} (H(s,x) - 1)\, \tilde{N}(ds, dx) - \frac{1}{2}\int_0^t h_s^2\, ds \right) \times \prod_{0 \leqslant s \leqslant t} H(s, \Delta X_s) e^{-(H(s,\Delta X_s) - 1)},$$

therefore the relative entropy $I_{\mathbf{P}}(\mathbf{Q})$ of $\mathbf{Q}$ with respect to $\mathbf{P}$ is

$$\begin{aligned} I_{\mathbf{P}}(\mathbf{Q}) &= \int_{\Omega} \ln \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}_T} \mathrm{d}\mathbf{Q} \\ &= \mathbf{E}^{\mathbf{Q}}\bigg(\ln \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} \bigg) \\ &= \mathbf{E}^{\mathbf{Q}}\bigg(\int_0^T h_s\, dW_s - \frac{1}{2}\int_0^T h_s^2\, ds + \int_0^T \int_{\mathbb{R}} (H(s,x)-1)\, \tilde{N}(ds,dx) \bigg) \\ &\quad + \mathbf{E}^{\mathbf{Q}}\bigg(\ln \bigg[\prod_{0\leqslant s\leqslant T} H(s,\Delta X_s) e^{-(H(s,\Delta X_s)-1)} \bigg] \bigg). \end{aligned} \tag{5.23}$$

However we can write

$$\begin{aligned} &\ln \bigg(\prod_{0\leqslant s\leqslant T} H(s,\Delta X_s) e^{-(H(s,\Delta X_s)-1)} \bigg) \\ &= \sum_{0\leqslant s\leqslant T} \Big(\ln H(s,\Delta X_s) - (H(s,\Delta X_s)-1) \Big) \\ &= \int_0^T \int_{\mathbb{R}} \big[\ln H(s,x) - (H(s,x)-1) \big]\, N(ds,dx), \end{aligned} \tag{5.24}$$

where the last follows from Eq.(2.37). We can now rewrite Eq.(5.23) as

$$\begin{aligned} I_{\mathbf{P}}(\mathbf{Q}) &= \mathbf{E}^{\mathbf{Q}}\bigg(\int_0^T h_s\, dW_s - \frac{1}{2}\int_0^T h_s^2\, ds + \int_0^T \int_{\mathbb{R}} (H(s,x)-1)\, \tilde{N}(ds,dx) \bigg) \\ &\quad + \mathbf{E}^{\mathbf{Q}}\bigg(\int_0^T \int_{\mathbb{R}} \big[\ln H(s,x) - (H(s,x)-1) \big]\, N(ds,dx) \bigg). \end{aligned} \tag{5.25}$$

If we express W_t as $\tilde{W}_t$ and $\tilde{N}(ds,dx)$ as $\tilde{N}_{\mathbf{Q}}(ds,dx)$ so as to make the stochastic integrals martingales under the measure $\mathbf{Q}$, then by the martingale property of the stochastic integrals, the expected value of these particular stochastic integrals is equal to zero. Finally we have the relative entropy of the measure $\mathbf{Q}$ with respect to the measure $\mathbf{P}$ as

$$\begin{aligned} I_{\mathbf{P}}(\mathbf{Q}) &= \mathbf{E}^{\mathbf{Q}}\bigg(\int_0^T h_s\, d\tilde{W}_s + \frac{1}{2}\int_0^T h_s^2\, ds + \int_0^T \int_{\mathbb{R}} \ln(H(s,x))\, \tilde{N}_{\mathbf{Q}}(ds,dx) \bigg) \\ &\quad + \mathbf{E}^{\mathbf{Q}}\bigg(\int_0^T \int_{\mathbb{R}} \big[H(s,x)(\ln H(s,x)-1)+1 \big]\, \nu(dx)ds \bigg) \\ &= \mathbf{E}^{\mathbf{Q}}\bigg(\frac{1}{2}\int_0^T h_s^2\, ds + \int_0^T \int_{\mathbb{R}} \big[H(s,x)(\ln H(s,x)-1)+1 \big]\, \nu(dx)ds \bigg). \end{aligned}$$

The problem of then finding an equivalent martingale measure $\mathbf{Q}$ that has minimum relative entropy can then be reduced to minimizing

$$\mathbf{E}^{\mathbf{Q}}\bigg(\frac{1}{2}\int_0^T h_s^2\, ds + \int_0^T \int_{\mathbb{R}} \big[H(s,x)(\ln H(s,x)-1)+1 \big]\, \nu(dx)ds \bigg), \tag{5.26}$$

subject to the martingale constraint of

$$\mu_s - r_s + c\sigma_s + \sigma_s h_s + \int_{\mathbb{R}} \sigma_s x(H(s,x)-1)\nu(dx) = 0. \tag{5.27}$$

Since we are assuming that the processes $\{H(t,x), 0 \leqslant t \leqslant T\}$ and $\{h_t, 0 \leqslant t \leqslant T\}$ are deterministic this optimization problem can then be further reduced to that of minimizing

$$\frac{1}{2}h_s^2 + \int_{\mathbb{R}} [H(s,x)(\ln H(s,x)-1)+1]\nu(dx), \tag{5.28}$$

subject to the constraint of Eq.(5.27), see Chan [9]. The optimization problem can then be solved using the method of Lagrange multipliers. More precisely, by forming the Lagrangian

$$\begin{aligned} L(h_s, H(s,x), \lambda_s) \;&=\; \lambda_s\Big(\mu_s - r_s + c\sigma_s + \sigma_s h_s + \int_{\mathbb{R}} \sigma_s x(H(s,x)-1)\nu(dx)\Big) \\ &\quad + \frac{1}{2}h_s^2 + \int_{\mathbb{R}} H(s,x)(\ln H(s,x)-1)+1]\nu(dx) \end{aligned} \tag{5.29}$$

where λ_s is the Lagrangian multiplier and is assumed to be continuous. We then compute the partial derivatives and equate them to zero, i.e.

$$\frac{\partial L}{\partial h} = 0 \qquad \frac{\partial L}{\partial H} = 0 \qquad \frac{\partial L}{\partial \lambda} = 0. \tag{5.30}$$

Solving these equations we then get

$$H(s,x) = \exp(-\lambda_s \sigma_s x) \qquad \text{and} \qquad h_u = -\sigma_s \lambda_s. \tag{5.31}$$

The second derivative test then shows that the functions $H(s,x) = e^{-\lambda_s \sigma_s x}$ and $h_s = -\lambda_s \sigma_s$ do indeed produce a minimum value for Eq.(5.29), see Chan [9] for a complete proof of this result. Substituting the functions for $H(s,x)$ and h_s back into the stochastic exponential and letting $\theta_s = -\lambda_s \sigma_s$ we have

$$\begin{aligned} Z_t \;&=\; \exp\Big(\int_0^t \theta_s\, dW_s + \int_0^t \int_{\mathbb{R}} (e^{\theta_s x}-1)\, \tilde{N}(ds,dx) - \frac{1}{2}\int_0^t \theta_s^2\, ds\Big) \\ &\quad \times \prod_{0 \leqslant s \leqslant t} e^{\theta_s \Delta X_s} \exp\big(-(e^{\theta_s \Delta X_s - 1})\big). \end{aligned}$$

Again we make use of the fact that the infinite product can be written as follows

$$\begin{aligned} &\prod_{0 \leqslant s \leqslant t} e^{\theta_s \Delta X_s} \exp\big(-(e^{\theta_s \Delta X_s - 1})\big) \\ &= \exp\Big(\sum_{0 \leqslant s \leqslant t} (\theta_s \Delta X_s - (e^{\theta_s \Delta X_s} - 1))\Big) \\ &= \exp\Big(\int_0^t \int_{\mathbb{R}} \big(\theta_s x - (e^{\theta_s x} - 1)\big)\Big). \end{aligned} \tag{5.32}$$

We therefore have the stochastic exponential as

$$\begin{aligned} Z_t &= \exp\Big(\int_0^t \theta_s \, dW_s + \int_0^t \int_{\mathbb{R}} (e^{\theta_s x} - 1) \, \tilde{N}(ds, dx) - \frac{1}{2} \int_0^t \theta_s^2 \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \big(\theta_s x - (e^{\theta_s x} - 1)\big) \, N(ds, dx) \Big) \\ &= \exp\Big(\int_0^t \theta_s \, dW_s + \int_0^t \int_{\mathbb{R}} \theta_s x \, \tilde{N}(ds, dx) - \frac{1}{2} \int_0^t \theta_s^2 \, ds \\ &\quad - \int_0^t \int_{\mathbb{R}} (e^{\theta_s x} - 1 - \theta_s x) \, \nu(dx) ds \Big). \end{aligned} \tag{5.33}$$

We can still simplify the stochastic exponential by noting that

$$\begin{aligned} &\int_0^t \theta_s \, dW_s + \int_0^t \int_{\mathbb{R}} \theta_s x \, \tilde{N}(ds, dx) \\ &= \int_0^t \theta_s \Big(dW_s + \int_{\mathbb{R}} x \, \tilde{N}(ds, dx) \Big) \\ &= \int_0^t \theta_s \big(dX_s - c \, ds \big). \end{aligned} \tag{5.34}$$

Recall that the Lévy process $X_t = ct + W_t + M_t$ was the source of randomness in our stock price model. Since θ_s is deterministic and c is a constant we can compute the moment generating function of the following random variable as

$$\begin{aligned} &\mathbf{E}^{\mathbf{P}} \Big[\exp\Big(\int_0^t \theta_s \big(dX_s - c \, ds \big) \Big) \Big] \\ &= \exp\Big(\int_0^t \Big(\frac{\theta_s^2}{2} - c\theta_s \Big) \, ds + \int_0^t \int_{\mathbb{R}} (e^{\theta_s x} - 1 - \theta_s x) \nu(dx) ds \Big) \\ &= \exp\Big(\int_0^t \big(\upsilon(\theta_s) - c\theta_s \big) \, ds \Big), \end{aligned} \tag{5.35}$$

where $\upsilon(\theta_s) = \frac{\theta_s^2}{2} + \int_{\mathbb{R}} (e^{\theta_s x} - 1 - \theta_s x) \nu(dx)$. We can now finally write the stochastic exponential as

$$\begin{aligned} Z_t &= \exp\Big(\int_0^t \theta_s \big(dX_s - c \, ds \big) - \int_0^t \big(\upsilon(\theta_s) - c\theta_s \big) \, ds \Big) \\ &= \exp\Big(\int_0^t \theta_s \, dX_s - \int_0^t \upsilon(\theta_s) \, ds \Big). \end{aligned} \tag{5.36}$$

Eq.(5.36) is more commonly known as the *generalized Esscher transform*. So after much manipulation we can conclude that the measure $\mathbf{Q}$ which has minimum relative entropy is given by the generalized Esscher transform. The martingale property of the generalized Esscher transform is easily verified since θ_s is deterministic and the Lévy process has independent increments. To verify that $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ we simply note that the generalized Esscher transform can be written as

$$Z_t = \frac{\exp\left(\int_0^t \theta_s \, dX_s\right)}{\mathbf{E}^{\mathbf{P}}\left[\exp\left(\int_0^t \theta_s \, dX_s\right)\right]}, \tag{5.37}$$

and since we have assumed that the Lévy process has finite moments of all orders we will always have $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$. Originally proposed by Gerber and Shui [21] for the pricing of contingent claims where the stock price process has independent and stationary increments, the Esscher transform offers an attractive way of identifying at least one EMM in an incomplete market. Another justification for selecting the Esscher transform as change of measure formula is that it leads to an expected utility maximization for an investor but for our purposes it is the property of minimal relative entropy that makes the Esscher transform so appealing. In most cases that follow we will not require the generalized form Esscher transform but a more simplistic form given by

$$Z_t = \exp\big(\theta X_t - tv(\theta)\big), \tag{5.38}$$

where $\theta \in \mathbb{R}$ and $\mathbf{E}^{\mathbf{P}}(e^{\theta X_t}) = e^{tv(\theta)}$. Eq.(5.38) is often called the Esscher transform of parameter θ. We have one final result regarding the Esscher transform which is quite useful when computing expectations. Suppose that once we have solved the stochastic differential equation for the risky asset we can write the stock price as

$$S_t = S_0 e^{Y_t},$$

where $\{Y_t, 0 \leqslant t \leqslant T\}$ is a process with stationary and independent increments. The Esscher transform is then defined as

$$Z_t = \frac{e^{\theta Y_t}}{\mathbf{E}^{\mathbf{P}}\left[e^{\theta Y_t}\right]} = \frac{S_t^{\theta}}{\mathbf{E}^{\mathbf{P}}\left[S_t^{\theta}\right]}, \tag{5.39}$$

we then determine the value of θ so as to give the deflated stock price a $\mathbf{Q}$ martingale property. For notational convenience we shall write $\mathbf{E}^{\mathbf{Q}}(S_t)$ as $\mathbf{E}(S_t;\theta)$ indicating that the EMM is represented by the choice of θ. We can now state the following result which enables us to factorize an expectation provided that the Esscher transform is being used as the change of measure formula.

Lemma 5.1. *For measurable function g and θ, k and t real numbers with $t \geqslant 0$*

$$\mathbf{E}\left[S_t^k g(S_t);\theta\right] = \mathbf{E}[S_t^k;\theta]\ \mathbf{E}[g(S_t);k+\theta]. \tag{5.40}$$

Proof: See Bingham and Kiesel [7] ■

Thus far we have looked at two possible criteria that one could employ when faced with the problem of pricing contingent claims in an incomplete market, namely the Föllmer-Schweizer martingale measure and the Esscher transform measure which has minimum relative entropy. There are numerous other methods that could also have been employed to select a specific martingale measure such as subreplication, superreplication, Utility indifference principles, mean variance hedging etc. Each one of these approaches would have then produced an equivalent martingale measure. Given a non attainable contingent claim, the price process for this contingent claim would consequently depend on the martingale measure chosen. So in essence when pricing contingent claims in an incomplete market there exist a range of arbitrage free prices to choose from, each price dependent on the martingale measure that has been selected using the various criteria. For an introduction to other possible criteria used in selecting an EMM we refer the interested reader to Bingham and Kiesel [7].

Chapter 6

The Merton model

6.1 Pricing a European call option under the Merton model

We shall now concentrate on those models that contain Lévy processes as the driving source on uncertainty and remain tractable. The models that we shall explore fall under the category of jump-diffusion models which are a subclass of the Lévy process. A prime example of a jump-diffusion model is the model proposed by Merton [37] which is the focus of this chapter. The Merton jump-diffusion model behaves like that of the the classical geometric Brownian motion model except that now the model is interlaced with random jumps. The model is specified by the following dynamics for the risk-free and risky asset under the measure $\mathbf{P}$

$$dR_t = rR_t\,dt \qquad R_0 = 1 \tag{6.1}$$

$$dS_t = S_{t-}\Big(\mu dt + \sigma dW_t + \int_{\mathbb{R}} (e^{y_i} - 1)\, N(dt, dy_i)\Big) \qquad S_0 \in \mathbb{R}. \tag{6.2}$$

W_t is the standard Brownian motion, $N(dt, dy_i)$ is a random Poisson measure on $[0, T] \times \mathbb{R} - \{0\}$ and the Y_i are a sequence of independently and identically distributed (i.i.d.) normal random variables with mean δ and variance β, i.e. $Y_i \sim N(\delta, \beta)$. The instantaneous rate of return is represented by μ and the volatility is given by σ while the risk-free rate of interest is given by r. Note that the coefficients of the stock price model are all constant and hence deterministic. The Lévy measure is given by $\nu(dx) = \lambda f_Y(y)dy$, where λ is the intensity rate of the Poisson process $\{N_t, 0 \leqslant t \leqslant T\}$ and $f_Y(y)$ is the distribution of the jumps sizes. All sources of randomness are assumed to be independent, i.e. W_t, $N(dt, dy_i)$ and Y_i are all independent of each other. As can be seen from the SDE for the risky asset, the stock price behaves like that of the geometric Brownian motion model until some random time when there is a jump in the stock price. The Poisson process N_t governs the arrival of the jumps while the actual sizes of the jumps are determined by the i.i.d. random variables Y_i. From the Lévy measure we can see that the jumps in the model are governed by a compound Poisson process while the Brownian motion represents the continuous randomness experienced in the model, the Brownian motion represents the “diffusion” part of the model while the compound Poisson process accounts for the "jump" of the model. Since the Lévy measure is of the form $\nu(dx) = \lambda f_Y(y)dy$ we will have

$$\int_{\mathbb{R}} x \, \nu(dx) < \infty, \tag{6.3}$$

since the Y_i are normally distributed and have a finite mean. Hence the jumps of the compound Poisson process have finite variation, see Proposition 2.1. Since the Poisson process is responsible for the arrival of the jumps in the model the stochastic integral representing the jumps can be written as follows

$$\int_0^t \int_{\mathbb{R}} (e^{y_i} - 1) \; N(dt, dy_i) = \sum_{i=1}^{N_t} (e^{Y_i} - 1). \tag{6.4}$$

From Eq.(6.4) we can readily see that the number of jumps experienced arrive according to a Poisson process while the actual size of the jumps is determined by the random variables Y_i. Applying Itô's lemma to the stochastic function $\ln S_t$ will yield the following solution for the stock price SDE in Eq.(6.2)

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right) \prod_{i=1}^{N_t} e^{Y_i}. \tag{6.5}$$

The contingent claims that are of interest to us shall be the European call option with strike price K and the European exchange option. We assume that all trading activity occurs over the finite time interval $[0, T]$ and that all model assumptions stated in section 4.1 hold. Let us start with the European call option, we seek to price the following $\mathcal{F}_T$-measurable claim $\Gamma_T = \max(S_T - K, 0)$. The theory that we have developed in pricing contingent claims requires that we first verify that the model represented by the dynamics in Eq.(6.1) and Eq.(6.2) is arbitrage free. We therefore start by establishing the existence of at least one EMM, selecting the risk-free asset R_t as our pricing numeraire we apply the product rule to the deflated stock price process $\tilde{S}_t = S_t/R_t$

$$\tilde{S}_t = \int_0^t \frac{1}{R_{u-}} \, dS_u + \int_0^t S_{u-} \; d\left(\frac{1}{R_u}\right) + \left[S, \frac{1}{R}\right]_t, \tag{6.6}$$

since the Brownian motion has continuous sample paths **P**-a.s. and R_t has finite first order variation, by Theorem 2.6 the quadratic covariation is given as

$$\left[\int_0^{\cdot} \sigma S_{u-} \; dW_u, \frac{1}{R}\right]_t = 0.$$

Hence the quadratic covariation between S_t and $\frac{1}{R_t}$ is reduced to

$$\left[\int_0^{\cdot} \int_{\mathbb{R}} S_{u-}(e^{y_i} - 1) \; N(du, dy_i), \frac{1}{R}\right]_t,$$

but from Eq.(6.3) we know that the jumps of the compound Poisson process have finite variation and since the risk-free asset has continuous sample paths we therefore conclude that the quadratic covariation between S_t and $\frac{1}{R_t}$ must be equal to zero, i.e. $\left[S, \frac{1}{R}\right]_t = 0$. Eq.(6.6) then becomes

$$\begin{aligned}\tilde{S}_t &= \int_0^t \frac{1}{R_{u-}}\, dS_u + \int_0^t S_{u-}\, d\left(\frac{1}{R_u}\right)\\ &= \int_0^t \tilde{S}_{u-}(\mu - r)\, du + \int_0^t \tilde{S}_{u-}\sigma\, dW_u + \int_0^t \int_{\mathbb{R}} \tilde{S}_{u-}(e^{y_i} - 1)\, N(du, dy_i).\end{aligned}$$

We have already seen that the use of the Lévy process as the source of randomness in the stock price model results in market incompleteness, i.e. the EMM measure is not unique. In the previous chapter we have come across various criteria that is used to select a particular EMM from the pool of available EMM's, for the pricing of contingent claims. In this chapter we shall focus on the equivalent martingale measure that has minimum relative entropy with respect to the real world measure $\mathbf{P}$. We shall therefore use the Esscher transform as our change of measure formula. Since the risky asset can be written in the form

$$\begin{aligned}S_t &= S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i\right)\\ &= S_0 e^{X_t},\end{aligned} \tag{6.7}$$

where the process $\{X_t, 0 \leqslant t \leqslant T\}$ is given by

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t + \sum_{i=1}^{N_t} Y_i.$$

We therefore define the Esscher transform of parameter θ as

$$\frac{e^{\theta \tilde{X}_t}}{\mathbf{E}^{\mathbf{P}}\left[e^{\theta \tilde{X}_t}\right]} \qquad \theta \in \mathbb{R}, \tag{6.8}$$

where $\tilde{X}_t$ is given by

$$\tilde{X}_t = \mu t + W_t + \sum_{i=1}^{N_t} Y_i. \tag{6.9}$$

In order to compute $\mathbf{E}^{\mathbf{P}}\left[e^{\theta \tilde{X}_t}\right]$ we make use of the independence between the Poisson process, Brownian motion process and the i.i.d. random variables Y_i.

This yields

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\big[e^{\theta \tilde{X}_t}\big] \\
&= e^{\theta\mu t}\ \mathbf{E}^{\mathbf{P}}\big[\exp(\theta W_t)\big]\ \mathbf{E}^{\mathbf{P}}\Bigg[\prod_{i=1}^{N_t} e^{\theta Y_i}\Bigg] \\
&= e^{\theta\mu t} e^{\frac{1}{2}\theta^2 t}\ \mathbf{E}^{\mathbf{P}}\Bigg[\mathbf{E}^{\mathbf{P}}\Bigg(\prod_{i=1}^{N_t} e^{\theta Y_i}\Bigg| N_t\Bigg)\Bigg] \\
&= e^{\theta\mu t} e^{\frac{1}{2}\theta^2 t}\mathbf{E}^{\mathbf{P}}\Bigg[\prod_{i=1}^{N_t} e^{\theta\delta + \frac{1}{2}\beta\theta^2}\Bigg] \\
&= \exp\big(\theta\mu t + \frac{1}{2}\theta^2 t + \lambda t(e^{\theta\delta+\frac{1}{2}\beta\theta^2} - 1)\big).
\end{aligned} \tag{6.10}
$$

Combining this with Eq.(6.8) we finally have that the Esscher transform of parameter θ is given by

$$
Z_t = \exp\Bigg(\theta W_t + \theta \sum_{i=1}^{N_t} Y_i - \frac{1}{2}\theta^2 t - \lambda t(e^{\theta\delta - \frac{1}{2}\theta^2\beta} - 1)\Bigg). \tag{6.11}
$$

It is easily verified that $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ for all $t \in [0,T]$ and that $\{Z_t, 0 \leqslant t \leqslant T\}$ has a $\mathbf{P}$-martingale property, therefore the Esscher transform is a suitable process to facilitate a change of measure. This change in measure is implemented with the use of the the following R-N derivative

$$
\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\bigg|_{\mathcal{F}_T^X} = Z_T, \tag{6.12}
$$

where Z_T is given by Eq.(6.11). Under this measure $\mathbf{Q}$, Girsanov's theorem (Theorem 2.11) states that there exists a process $\{\tilde{W}_t, 0 \leqslant t \leqslant T\}$ defined by

$$
\tilde{W}_t = W_t - \theta t, \tag{6.13}
$$

that is standard Brownian motion. In order to work with a compound Poisson process that is a $\mathbf{Q}$-martingale we make use of the result given in Eq.(2.94). The process $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ is therefore a $\mathbf{Q}$-martingale where $\tilde{M}_t$ is defined by

$$
\begin{aligned}
\tilde{M}_t &= \int_0^t \int_{\mathbb{R}} (e^{y_i} - 1)\ \tilde{N}_{\mathbf{Q}}(ds, dy_i) \\
&= \int_0^t \int_{\mathbb{R}} (e^{y_i} - 1)\big[N(ds, dy_i) - e^{\theta y}\ \nu(dy)ds\big] \\
&= \int_0^t \int_{\mathbb{R}} (e^{y_i} - 1)\ N(ds, dy_i) - \int_0^t \int_{\mathbb{R}} (e^{y} - 1)e^{\theta y}\ \nu(dy)ds \\
&= \int_0^t \int_{\mathbb{R}} (e^{y_i} - 1)\ N(ds, dy_i) - \lambda t \int_{\mathbb{R}} (e^{y} - 1)e^{\theta y} f_Y(y)\ dy,
\end{aligned} \tag{6.14}
$$

where $f_Y(y)$ the distribution of the jumps. If we let

$$\mathbf{E}^{\mathbf{P}}[e^{\theta Y_i}] = \phi(\theta),$$

then since the Y_i are i.i.d. normally distributed random variables with mean δ and variance β we have $\phi(\theta) = e^{\theta\delta + \frac{1}{2}\theta^2\beta}$. Therefore the **Q**-martingale $\tilde{M}_t$ is given by

$$\tilde{M}_t = \int_0^t \int_{\mathbb{R}} (e^{y_i} - 1)\ N(ds, dy_i) - \lambda t(\phi(\theta+1) - \phi(\theta)) \tag{6.15}$$

Remark 6.1. Note that the process $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ is indeed a martingale under the measure **Q** and not just a local **Q**-martingale. Since $\tilde{N}_{\mathbf{Q}}(ds, dy_i) = N(ds, dy_i) - e^{\theta y}\nu(dy)ds$, using Theorem 2.5 it is then easily verified that

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left(\left|\int_0^t \int_{\mathbb{R}} (e^{y_i} - 1)\ \tilde{N}_{\mathbf{Q}}(ds, dy_i)\right|^2\right) \\
&= \mathbf{E}^{\mathbf{Q}}\left(\lambda t \int_{\mathbb{R}} |e^y - 1|^2 e^{\theta y} f_Y(y)\ dy\right) \\
&= \lambda t \int_{\mathbb{R}} |e^y - 1|^2 e^{\theta y} f_Y(y)\ dy < \infty.
\end{aligned} \tag{6.16}$$

Eq.(6.16) follows from the fact that $\lambda t \int_{\mathbb{R}} |e^y - 1|^2 f_Y(y)\ dy$ is non random and hence remains unaffected by the expectation operator. It is this integrability property that ensures the stochastic integral $\tilde{M}_t$ is indeed a martingale under the measure **Q**.

Returning to the deflated stock price process $\tilde{S}_t$, making use of the two processes $\{\tilde{W}_t, 0 \leqslant t \leqslant T\}$ and $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ we can express the deflated stock price as

$$\begin{aligned}
\tilde{S}_t &= \int_0^t \tilde{S}_{u-}(\mu - r)\ du + \int_0^t \tilde{S}_{u-}\sigma\ d\big(\tilde{W}_u + \theta u\big) \\
&\quad + \int_0^t \int_{\mathbb{R}} \tilde{S}_{u-}\ d\big(\lambda u(\phi(\theta+1) - \phi(\theta)) + \tilde{M}_u\big), \\
\tilde{S}_t &= \big(\mu - r + \lambda(\phi(\theta+1) - \phi(\theta)) + \theta\sigma\big) \int_0^t \tilde{S}_{u-}\ du + \sigma \int_0^t \tilde{S}_{u-}\ d\tilde{W}_u \\
&\quad + \int_0^t \tilde{S}_{u-}\ d\tilde{M}_u.
\end{aligned} \tag{6.17}$$

Since an arbitrage free economy implies and is implied by the existence of at least one numeraire pair, we are therefore interested in how the deflated stock price process can be made into a **Q**-martingale. As it stands the deflated stock price does not have a **Q**-martingale property because the drift term, represented by the Rieman integral, does not allow for a constant mean which is a necessary condition for the martingale property. We therefore require that the drift term be removed from the deflated stock price process, to do this we set

$$\mu - r + \lambda(\phi(\theta+1) - \phi(\theta)) + \theta\sigma = 0, \tag{6.18}$$

the dynamics for the delated stock price process then becomes

$$\tilde{S}_t = \sigma \int_0^t \tilde{S}_{u-}\ d\tilde{W}_u + \int_0^t \tilde{S}_{u-}\ d\tilde{M}_u, \tag{6.19}$$

which is a local $\mathbf{Q}$-martingale. Adding the integrability requirements that

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}\bigg(\Big|\sigma \int_0^T \tilde{S}_{u-}\ d\tilde{W}_u\Big|^2\bigg) &= \sigma^2 \int_0^T \mathbf{E}^{\mathbf{Q}}\big(S_{u-}^2\big)\ du < \infty \\
\mathbf{E}^{\mathbf{Q}}\bigg(\Big|\int_0^T \tilde{S}_{u-}\ d\tilde{M}_u\Big|^2\bigg) &= \int_0^T \int_{\mathbb{R}} \mathbf{E}^{\mathbf{Q}}\big(S_{u-}|e^y - 1|\big)^2 e^{\theta y}\ \nu(dy)du < \infty,
\end{aligned}$$

will then ensure that the deflated stock price process has a $\mathbf{Q}$-martingale property. In light of Eq.(6.16) these integrability conditions can be replaced with the sufficient condition that

$$\int_0^T \mathbf{E}^{\mathbf{Q}}\big(S_{u-}\big)^2\ du < \infty. \tag{6.20}$$

Hence we have established that the model based on the Merton dynamics is indeed arbitrage free since it admits a numeraire pair. It should be noted that the EMM $\mathbf{Q}$ is determined by the value of θ such that Eq.(6.18) holds true, this value of θ is also interpreted as the *market price for risk* which measures the excess rate of return that is required of the risky asset over the risk-free asset R_t. In the Black-Scholes-Merton model the market price for risk was given by $\frac{r-\mu}{\sigma}$, in the current setting of the Merton model the risk associated with the stock is greater since we have included jumps in the dynamics of the stock price process. We therefore expect that the market price for risk under the Merton model be greater than that of the Black-Scholes-Merton model.

We now proceed with the pricing of contingent claims based on the dynamics of the Merton model, starting with the European call option we have the following $\mathcal{F}_T$-measurable claim $\Gamma_T = \max(S_T - K, 0)$. A martingale based method to pricing contingent claims gives the price for the claim Γ_t as

$$\begin{aligned}
\Gamma_t &= \mathbf{E}^{\mathbf{Q}}\bigg(\frac{R_t \max(S_T - K, 0)}{R_T}\bigg|\mathcal{F}_t^X\bigg) \\
&= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big(\max(S_T - K, 0)\big|\mathcal{F}_t^X\big).
\end{aligned} \tag{6.21}$$

As we have used the Esscher transform as the change of measure formula we can make use of the factorization formula given in Lemma 5.1. This then allows us to compute the price of the European call as follows

$$\begin{aligned}
\Gamma_t &= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[\big(S_T - K\big)\mathbf{1}_{(S_T>K)};\theta\big|\mathcal{F}_T^X\big] \\
&= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[S_T\mathbf{1}_{(S_T>K)};\theta\big|\mathcal{F}_T^X\big] - Ke^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[\mathbf{1}_{(S_T>K)};\theta\big|\mathcal{F}_T^X\big] \\
&= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[S_T;\theta\big|\mathcal{F}_t^X\big]\ \mathbf{E}^{\mathbf{Q}}\big[\mathbf{1}_{(S_t>K)};\theta+1\big|\mathcal{F}_t^X\big] \qquad (6.22) \\
&\quad -Ke^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[\mathbf{1}_{(S_T>K)} : \theta\big|\mathcal{F}_T^X\big]. \qquad (6.23)
\end{aligned}$$

Focusing on Eq.(6.22) first, as the deflated stock price must follow a $(\mathbf{Q}, \mathcal{F}_t^X)$ martingale property it follows that

$$\mathbf{E}^{\mathbf{Q}}\left(\frac{S_T}{e^{rT}}; \theta \middle| \mathcal{F}_t^X\right) = \frac{S_t}{e^{rt}}, \tag{6.24}$$

which leaves us with

$$\begin{aligned} & \mathbf{E}^{\mathbf{Q}}\big[\mathbf{1}_{(S_t>K)}; \theta + 1 | \mathcal{F}_t^X\big] \\ = \;& \mathbf{Q}\big(S_T > K; \theta + 1 \big| \mathcal{F}_t^X\big). \end{aligned} \tag{6.25}$$

In order to compute the probability in Eq.(6.25) we first need to determine the conditional distribution of the deflated stock price under the measure $\mathbf{Q}$. The solution to the deflated stock price in Eq.(6.19) is given by the Doléans-Dade exponential process in Eq.(2.78) as

$$\tilde{S}_t = S_0 \exp\left(\sigma \tilde{W}_t + \int_0^t \int_{\mathbb{R}} y_i \, N(ds, dy_i) - (\lambda(\phi(\theta+1) - \phi(\theta)) + \frac{1}{2}\sigma^2)t\right). \tag{6.26}$$

We can make use of Eq.(6.26) to determine $\ln S_T$ as this will aid in reducing computational difficulty, we therefore have

$$\ln S_T = \ln S_t + \sigma(\tilde{W}_T - \tilde{W}_t) + \int_t^T \int_{\mathbb{R}} y_i \, N(ds, dy_i) + (r - \lambda(\phi(\theta+1) - \phi(\theta)) - \frac{1}{2}\sigma^2)(T-t), \tag{6.27}$$

as the stochastic integral $\int_t^T \int_{\mathbb{R}} y_i \, N(ds, dy_i)$ has its jumps governed by the Poisson process $\{N_t, 0 \leqslant t \leqslant T\}$, it can be interpreted as the random finite sum $\sum_{i=1}^{N_T - N_t} Y_i$, hence we have that

$$\ln S_T = \ln S_t + \sigma(\tilde{W}_T - \tilde{W}_t) + \sum_{i=1}^{N_T - N_t} Y_i + (r - \lambda(\phi(\theta+1) - \phi(\theta)) - \frac{1}{2}\sigma^2)(T-t). \tag{6.28}$$

Under the measure $\mathbf{Q}$ we know that the increments of the Brownian motion process $\{\tilde{W}_t, 0 \leqslant t \leqslant T\}$ are normally distributed, what remains to determine is the distribution of the process $\{N_t, 0 \leqslant t \leqslant T\}$ and the Y_i under the measure $\mathbf{Q}$. From Lemma 2.2 we can compute the conditional distribution of the increment $N_T - N_t$ under the measure $\mathbf{Q}$ as

$$\mathbf{E}^{\mathbf{Q}}\big[e^{iu(N_T - N_t)} \big| \mathcal{F}_t\big] = \frac{\mathbf{E}^{\mathbf{P}}\big[e^{iu(N_T - N_t)} Z_T \big| \mathcal{F}_t\big]}{\mathbf{E}^{\mathbf{P}}\big[Z_T \big| \mathcal{F}_t\big]} \qquad \forall u \in \mathbb{R}, \tag{6.29}$$

where Z_T is the R-N derivative (Esscher transform) linking $\mathbf{Q}$ to $\mathbf{P}$ on the measure space $(\Omega, \mathcal{F}_T^X)$ defined in Eq.(6.11). By exploiting the the fact that both the Brownian motion and Poisson process have independent increments under the measure $\mathbf{P}$ and $\{Z_t, 0 \leqslant t \leqslant T\}$ has a $(\mathbf{P}, \mathcal{F}_t^X)$ martingale property, Eq.(6.29) reduces to

$$\exp\big(-\lambda(T-t)(\phi(\theta) - 1)\big)\, \mathbf{E}^{\mathbf{P}}\left[\exp\left(iu(N_T - N_t) + \theta \sum_{i=1}^{N_T - N_t} Y_i\right)\middle| \mathcal{F}_t\right]. \tag{6.30}$$

Since the increments of the Poisson process are independent of the filtration process $(\mathcal{F}_t^X, 0 \leqslant t \leqslant T)$ and the random variables Y_i are i.i.d. the conditional expectation in Eq.(6.30) then amounts to

$$\begin{aligned}
& \mathbf{E}^{\mathbf{P}}\Big[\exp\Big(iu(N_T - N_t) + \theta \sum_{i=1}^{N_T - N_t} Y_i\Big)\Big] \\
= \;& \mathbf{E}^{\mathbf{P}}\Big[\mathbf{E}^{\mathbf{P}}\Big[\exp\Big(iu(N_T - N_t) + \theta \sum_{i=1}^{N_T - N_t} Y_i\Big)\Big| N_T - N_t\Big]\Big] \\
= \;& \mathbf{E}^{\mathbf{P}}\Big[\exp\Big((iu + \theta\delta + \frac{1}{2}\theta^2\beta)(N_T - N_t)\Big)\Big] \\
= \;& \exp\big(\lambda(T-t)(e^{iu+\theta\delta+\frac{1}{2}\theta^2\beta} - 1)\big) \\
= \;& \exp\big(\lambda(T-t)(e^{iu}\phi(\theta) - 1)\big).
\end{aligned} \tag{6.31}$$

Hence we have that

$$\begin{aligned}
& \mathbf{E}^{\mathbf{Q}}\big[e^{iu(N_T - N_t)}\big|\mathcal{F}_t\big] \\
= \;& \exp\big(\lambda(T-t)(e^{iu}\phi(\theta) - 1)\big)\; \exp\big(-\lambda(T-t)(\phi(\theta) - 1)\big) \\
= \;& \exp\big(\lambda\phi(\theta)(T-t)(e^{iu} - 1)\big),
\end{aligned} \tag{6.32}$$

which we recognize as the characteristic function of a Poisson random variable with intensity parameter $\tilde{\lambda} = \lambda\phi(\theta)$. It is now easily verified that the process $\{N_t, 0 \leqslant t \leqslant T\}$ satisfies all the requirements of Definition 1.5 and is thus a Poisson process with intensity parameter $\tilde{\lambda}$ under the measure $\mathbf{Q}$. In a similar way we can determine the distribution of the i.i.d random variables Y_i under the measure $\mathbf{Q}$. However since the Y_i are not time indexed random variables, i.e. not a stochastic process, the following formula is used to determine the distribution under the measure $\mathbf{Q}$

$$\frac{e^{\theta y} f_Y(y)}{\mathbf{E}^{\mathbf{P}}[e^{\theta y}]}, \tag{6.33}$$

known as the Esscher transform of a random variable, this formula specifies a new probability density for the Y_i. Evaluating Eq.(6.33) we see that only the mean of the Y_i change while the variance remains unaffected by the change in measure. The mean of the Y_i is now $\delta + \theta\beta$ and the variance is still β under the measure $\mathbf{Q}$. Having determined the distribution for the process $\{N_t, 0 \leqslant t \leqslant T\}$ and the Y_i under the new measure $\mathbf{Q}$ we can now compute the value of the European call option. Recall that the value of the European call was given by

$$\Gamma_t = S_t\mathbf{Q}(S_T > K; \theta + 1|\mathcal{F}_t^X) - Ke^{-r(T-t)}\mathbf{Q}(S_T > K; \theta|\mathcal{F}_t^X). \tag{6.34}$$

Focusing on $\mathbf{Q}(S_T > K; \theta|\mathcal{F}_t^X)$ first, this probability can also be expressed as

$$\mathbf{Q}(\ln S_T > \ln K; \theta|\mathcal{F}_t^X) = \mathbf{Q}(\ln S_T > \ln K; \theta), \tag{6.35}$$

as the stock price has independent increments. From Eq.(6.28) we see that the $\mathcal{F}_T$-measurable function $\ln S_T$ contains both Brownian motion and Poisson increments and since both the Brownian motion and Poisson process have independent increments under the measure $\mathbf{Q}$ we have that $\ln S_T$ is independent of the filtration $(\mathcal{F}_t^X, 0 \leqslant t \leqslant T)$. Note that all sources of randomness are still independent of each other under the measure new $\mathbf{Q}$. If we condition on the event set $\{N_T - N_t = n\}$ then the random function $\ln S_t$ can be expressed as follows

$$\ln S_T\big|_{N_T - N_t = n} = \sigma(\tilde{W}_T - \tilde{W}_t) + \sum_{i=1}^{n} Y_i + (r - \lambda(\phi(\theta+1) - \phi(\theta)) - \frac{1}{2}\sigma^2)(T-t). \quad \textbf{(6.36)}$$

As the Brownian motion has normally distributed increments and the Y_i are still i.i.d normal random variables under the measure $\mathbf{Q}$, it follows that conditional on the event set $\{N_T - N_t = n\}$ the $\mathcal{F}_T$-measurable random variable is normally distributed with mean

$$\mathbf{E}^{\mathbf{Q}}\big(\ln S_T \big| N_T - N_t = n\big) = \ln S_t + n(\delta + \theta\beta) + (r - \lambda(\phi(\theta+1) - \phi(\theta)) - \frac{1}{2}\sigma^2)(T-t),$$

and variance given by

$$\mathbf{Var}^{\mathbf{Q}}\big(\ln S_T \big| N_T - N_t = n\big) = \sigma^2(T-t) + n\beta.$$

Hence the conditional probability is computed making use of the normal distribution as follows

$$\begin{aligned}
&\mathbf{Q}(\ln S_T > \ln K | N_T - N_t = n) \\
&= \mathbf{Q}\left(Z > \frac{\ln \frac{K}{S_t} - n(\delta + \theta\beta) - (r - \lambda(\phi(\theta+1) - \phi(\theta)) - \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t) + n\beta}}\right),
\end{aligned}$$

where Z is a normal random variable with mean equal to zero and variance equal to one. The unconditional probability is then computed as follows

$$\begin{aligned}
&\mathbf{Q}(\ln S_T > \ln K) \\
&= \sum_{n=0}^{\infty} \mathbf{Q}(\ln S_T > \ln K | N_T - N_t = n)\mathbf{Q}(N_T - N_t = n) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}(T-t)}(\tilde{\lambda}(T-t))^n}{n!}\mathbf{Q}(\ln S_T > \ln K | N_T - N_t = n). \qquad \textbf{(6.37)}
\end{aligned}$$

Eq.(6.37) follows from the fact that the process $\{N_t, 0 \leqslant t \leqslant T\}$ is a Poisson process with intensity parameter $\tilde{\lambda}$ under the measure $\mathbf{Q}$. In a similar manner we can compute the probability $\mathbf{Q}(\ln S_T > \ln K; \theta + 1 | \mathcal{F}_t^X)$ using the same arguments as we have done above, the only difference being that the market price for risk θ must be adjusted to $\theta + 1$ when determining the respective distributions. Finally we have that the price of the European call option under the dynamics of the Merton model is given by

$$\begin{aligned}
\Gamma_t &= S_t\mathbf{Q}(S_T > K; \theta+1|\mathcal{F}_t^X) - Ke^{-r(T-t)}\mathbf{Q}(S_T > K; \theta|\mathcal{F}_t^X) \\
&= \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}(T-t)}(\tilde{\lambda}(T-t))^n}{n!}\big[S_t\mathbf{Q}(\ln S_T > \ln K; \theta+1|N_T - N_t = n) \\
&\quad -Ke^{-r(T-t)}\mathbf{Q}(\ln S_T > \ln K; \theta|\mathcal{F}_t^X)\big] \\
&= \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}(T-t)}(\tilde{\lambda}(T-t))^n}{n!}\big[S_tN(d_{1,n}) - Ke^{-r(T-t)}N(d_{2,n})\big], \qquad (6.38)
\end{aligned}$$

where $d_{1,n}$ and $d_{2,n}$ are given by

$$d_{1,n} = \frac{\ln\frac{S_t}{K} + (r - \lambda(\phi(\theta+1) - \phi(\theta)) + \frac{1}{2}\sigma^2)(T-t) + n(\beta(\theta+1)+\delta)}{\sqrt{\sigma^2(T-t)+n\beta}} \qquad (6.39)$$

$$d_{2,n} = d_{1,n} - \sqrt{\sigma^2(T-t)+n\beta}, \qquad (6.40)$$

and the function $N(x)$ is the standard normal cumulative distribution given by

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2}\, dy.$$

If we set $\tilde{\lambda}$ and $n = 0$ in Eq.(6.38) then we obtain the Black-Scholes Merton pricing formula for the European call option derived in Eq.(4.22). We can interpret the pricing formula in Eq.(6.38) as a weighted average of European call options under the the Black-Scholes framework, in a sense we evaluate the call option under the Black-Scholes setup conditional on the underlying stock price experiencing n jumps and to this we multiply by the corresponding weight/probability that the asset experiences n jumps. In Merton's [37] paper, it was assumed that the jumps experienced by the asset are uncorrelated to the market as a whole, hence the risk associated with the jumps in the model is considered as non-systematic risk and should therefore earn no risk premium. As our derivation of the market price for risk included the possible jumps experienced by the asset, our approach differs from the reasoning of Merton's in that the jumps are treated as systematic risk which must be incorporated into the the risk premium associated with the asset. Since the underlying asset contains more risk than in the Black-Scholes model, the price for the call option in Eq.(6.38) is expected to be greater than the corresponding Black-Scholes call option of Eq.(4.22). We could have easily calculated the value of the call option using the Föllmer-Schweizer EMM, the techniques remain the same for deriving the value of the call option. The only adjustment is that the market price for risk will change since we are now using a different equivalent martingale measure. Each possible EMM has its own unique market price for risk. From a practical point of view in terms of implementing Eq.(6.38) a common truncation of $n = 20$, see Kou [30], is used when evaluating the call option with specific values.

The value of the European put with strike price K under the Merton jump diffusion model can computed with the aid of the put-call-parity. The value of the European put $\{p_t, 0 \leqslant t \leqslant T\}$ is given by

$$p_t = \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}(T-t)}(\tilde{\lambda}(T-t))^n}{n!} \big[Ke^{-r(T-t)}N(-d_{2,n}) - S_t N(-d_{1,n})\big], \tag{6.41}$$

where $d_{1,n}$ and $d_{2,n}$ are given by Eq.(6.39) and Eq.(6.40).

6.2 Pricing a European Exchange option under the Merton model

We now look at the pricing of a European exchange option where the underlying dynamics of the risky assets follow that of the Merton model. An exchange option is gives the holder the right but not the obligation to exchange one asset for another, since the option we are considering is European in nature exercise can only occur at maturity. The payoff for this contingent is claim is given by the $\mathcal{F}_t^X$-measurable random variable $\Gamma_T = \max\{S_{1,T} - S_{2,T}, 0\}$, where $S_{1,t}$ and $S_{2,t}$ are two different tradable assets in the economy. The approach that we shall take in the pricing of the exchange option makes extensive use of stochastic integration along with the Esscher transform and conditional arguments; an alternative method may be found in Cheang and Chiarella [22] or Cheang, Chiarella and Ziogas [18] where the derivation for the value of the exchange option is based on change of numeraire techniques and Margrabe's [35] formula. To the best of our knowledge the techniques that we shall use in determining a closed form solution for the value of an exchange option, under the Merton jump-diffusion setup, have not been published before. We assume that the dynamics for the two tradable assets are given by the following linear stochastic differential equations

$$\begin{aligned} dS_{1,t} &= S_{1,t-}\left[\mu_1 dt + \sigma_1 dW_{1,t} + \int_{\mathbb{R}} (e^{x_i} - 1)\, N(dt, dx_i)\right] \\ dS_{2,t} &= S_{2,t-}\left[\mu_2 dt + \sigma_2 dW_{2,t} + \int_{\mathbb{R}} (e^{y_i} - 1)\, N(dt, dy_i)\right], \end{aligned} \tag{6.42}$$

where $W_{i,t}$ are two correlated Brownian motions with correlation $[W_{1,t}, W_{2,t}]_t = \rho t$. $N(dt, \cdot)$ is a random Poisson measure on $[0,T] \times (\mathbf{R} - \{0\})$, the Lévy measure associated with the this random Poisson measure is if the form $\nu(\cdot) = \lambda f(\cdot)$. We assume that the process governing jumps for both the assets in the economy is the Poisson process $\{N_t, 0 \leqslant t \leqslant T\}$ with intensity parameter λ. The actual size of the jumps is determined by the i.i.d. normal random variables X_i for asset 1 while for asset 2 it is the i.i.d. normal random variables Y_i that determine the size of the jumps experienced by the risky asset. The mean of the X_i is δ_1 and variance β_1 while the mean for the Y_i is δ_2 and variance β_2. The appreciation rate for the i^{th} asset is given by μ_i while the volatility is given by σ_i, where $i = 1, 2$. We still assume that a risk-free asset is trading in the economy, i.e. $R_t = R_0 e^{rt}$ is still a tradable asset however for our purposes we shall not be interested in this asset when it comes to deriving an arbitrage-free pricing formula for the value of the option exchange option. Since the Poisson process $\{N_t, 0 \leqslant t \leqslant T\}$ is responsible for the arrival of jumps for both tradable assets, the economic implication of this assumption is that both assets experience jumps at the *same time* but the actual size of the jumps need not be the same for both the assets. If we think of the Poisson process as representing some source of systematic risk in the economy then both the tradable assets are exposed to this systematic risk but in varying degrees. Again we assume that all sources of randomness are independent of each other, i.e. the Brownian motions, the Poisson process as well as the i.i.d. normal random variables are all assumed independent.

We now wish to price the $\mathcal{F}_T$-measurable claim $\Gamma_T = \max\{S_{1,T} - S_{2,T}, 0\}$. Our martingale based approach to pricing contingent claims requires that we select a numeraire and establish the existence of at least one equivalent martingale measure. For this particular claim we will select the asset $S_{2,t}$ as our pricing numeraire, arbitrage free price process for the claim $\{\Gamma_t, 0 \leqslant t \leqslant T\}$ may then be given by

$$\begin{aligned} \frac{\Gamma_t}{S_{2,t}} &= \mathbf{E}^{\mathbf{Q}}\left[\frac{\max\{S_{1,T} - S_{2,T}, 0\}}{S_{2,T}}\bigg|\mathcal{F}_t^X\right] \\ \Gamma_t &= S_{2,t}\mathbf{E}^{\mathbf{Q}}\left[\max\left\{\frac{S_{1,T}}{S_{2,T}} - 1, 0\right\}\bigg|\mathcal{F}_t^X\right], \end{aligned} \tag{6.43}$$

where $\mathbf{Q}$ is the EMM measure based on selecting $S_{2,t}$ as our pricing numeraire. We now deflate $S_{1,t}$ with respect to $S_{2,t}$ and apply the product rule from Theorem 5.7 which yields

$$\frac{S_{1,t}}{S_{2,t}} = \int_0^t \frac{1}{S_{2,u-}}\, dS_{1,u} + \int_0^t S_{1,u-}\, d\left(\frac{1}{S_{2,u}}\right) + \left[S_1, \frac{1}{S_2}\right]_t. \tag{6.44}$$

In order to evaluate Eq.(6.44) we need first to compute $1/S_{2,t}$ and then determine the quadratic covariation between $S_{1,t}$ and $1/S_{2,t}$. We apply Itô's lemma (Theorem 2.7) to the stochastic function $1/S_{2,t}$, making use of the fact that

$\Delta S_{2,t} = S_{2,t-}[e^{Y_i} - 1]$ we have

$$\begin{aligned}
\frac{1}{S_{2,t}} &= \frac{1}{S_{2,0}} - \int_0^t \frac{1}{S_{2,u-}^2}\, dS_{2,u} + \frac{1}{2}\int_0^t \frac{2}{S_{2,u-}^3}\, d[S_2^c\ ,\ S_2^c]_{(u)} \\
&\quad + \sum_{0\leqslant u\leqslant t}\Big(\frac{1}{S_{2,u}} - \frac{1}{S_{2,u-}} + \frac{1}{S_{2,u-}^2}\Delta S_{2,u}\Big) \\
&= \frac{1}{S_{2,0}} - \int_0^t \frac{1}{S_{2,u-}}\Big(\sigma_2 dW_{2,u} + (\mu_2 - \sigma_2^2)du + \int_{\mathbb{R}}(e^{y_i} - 1)\ N(du, dx_i)\Big) \\
&\quad + \sum_{i=1}^{N(t)}\Big(\frac{1}{S_{2,u-}e^{Y_i}} - \frac{1}{S_{2,u-}} + \frac{1}{S_{2,u-}}(e^{Y_i} - 1)\Big) \\
&= \frac{1}{S_{2,0}} - \int_0^t \frac{1}{S_{2,u-}}\big(\sigma_2 dW_{2,u} + (\mu_2 - \sigma_2^2)du\big) \\
&\quad + \int_0^t\int_{\mathbb{R}} \frac{1}{S_{2,u-}}(e^{-y_i} - 1)\ N(du, dx_i).
\end{aligned} \tag{6.45}$$

Next we examine the quadratic covariation between $S_{1,t}$ and $1/S_{2,t}$. Making use of the stochastic integral representation of $1/S_{2,t}$ we have

$$\begin{aligned}
\left[S_1\ ,\ \frac{1}{S_2}\right]_t &= \Bigg[\int_0^\cdot \sigma_1 S_{1,u-}\ dW_{1,u} + \int_0^\cdot\int_{\mathbb{R}} S_{1,u-}(e^{x_i} - 1)\ N(du, dx_i) \\
&\qquad , -\int_0^\cdot \sigma_2 \frac{1}{S_{2,u-}}\ dW_{2,u} + \int_0^\cdot\int_{\mathbb{R}} \frac{1}{S_{2,u-}}(e^{-y_i} - 1)\ N(du, dy_i)\Bigg]_t,
\end{aligned}$$

since the Brownian motions has **P**-a.s. continuous sample paths and the random Poisson measure has finite first order variation under the Merton model, any quadratic covariation between the between the Brownian motions and the random Poisson measure must be zero, i.e. $[W_1, N(\cdot,\cdot)]_t = 0$ and $[W_2, N(\cdot,\cdot)]_t = 0$, by Theorem 2.6. Making use of this fact, we can factor the quadratic covariation between $S_{1,t}$ and $1/S_{2,t}$ as follows

$$\begin{aligned}
\left[S_1, \frac{1}{S_2}\right]_t &= \left[\int_0^\cdot \sigma_1 S_{1,u-}\ dW_{1,u}, -\int_0^\cdot \sigma_2 \frac{1}{S_{2,u-}}\ dW_{2,u}\right]_t \\
&+ \left[\int_0^\cdot\int_{\mathbb{R}} S_{1,u-}(e^{x_i} - 1)\ N(du, dx_i), \int_0^\cdot\int_{\mathbb{R}} \frac{1}{S_{2,u-}}(e^{-y_i} - 1)\ N(du, dy_i)\right]_t \\
&= -\int_0^t \sigma_1\sigma_2 \frac{S_{1,u-}}{S_{2,u-}}\ d\big[W_1\ ,\ W_2\big]_u \\
&\quad + \int_0^t \frac{S_{1,u-}}{S_{2,u-}} d\left[\int_{\mathbb{R}}(e^{x_i} - 1)\ N(u, dx_i), \int_{\mathbb{R}}(e^{-y_i} - 1)\ N(u, dy_i)\right]_t.
\end{aligned} \tag{6.46}$$

Now in order to calculate the quadratic covariation in Eq.(6.46) which represents the jumps experienced by the assets, it will be easier if we write the stochastic integrals as random finite sums. We can therefore write $\int_{\mathbb{R}}(e^{x_i}-1)\ N(t,dx_i)$ as

$$\sum_{i=1}^{N_t}(e^{X_i}-1), \tag{6.47}$$

hence the quadratic covariation in Eq.(6.46) can be expressed as

$$\left[\sum_{i=1}^{N.}(e^{X_i}-1),\sum_{i=1}^{N.}(e^{-Y_i}-1)\right]_t. \tag{6.48}$$

Since the Poisson process $\{N_t, 0\leqslant t\leqslant T\}$ is responsible for the arrival of the jumps in both stock prices Eq.(6.48) reduces to

$$\left[\sum_{i=1}^{N.}(e^{X_i}-1),\sum_{i=1}^{N.}(e^{-Y_i}-1)\right]_t=\sum_{i=1}^{N_t}(e^{X_i}-1)(e^{-Y_i}-1). \tag{6.49}$$

Thus we have the quadratic covariation between $S_{1,t}$ and $1/S_{2,t}$ as

$$\left[S_1,\frac{1}{S_2}\right]_t=-\int_0^t\rho\sigma_1\sigma_2\frac{S_{1,u-}}{S_{2,u-}}\,du+\int_0^t\frac{S_{1,u-}}{S_{2,u-}}\,d\left[\sum_{i=1}^{N.}(e^{X_i}-1)(e^{-Y_i}-1)\right]_u, \tag{6.50}$$

where we have made use of the fact that $[W_1,W_2]_t=\rho t$. Having determined both $1/S_{2,t}$ and the quadratic covariation between $S_{1,t}$ and $1/S_{2,t}$ we can now return to finding a stochastic integral representation for $S_{1,t}/S_{2,t}$. Hence Substituting Eq.(6.45) and Eq.(6.50) into Eq.(6.44) we have the following

$$\begin{aligned}\frac{S_{1,t}}{S_{2,t}} &= \int_0^t\frac{S_{1,u-}}{S_{2,u-}}\Bigg[(\mu_1-\mu_2+\sigma_2^2-\sigma_1\sigma_2\rho)du+\sigma_1dW_{1,u}-\sigma_2dW_{2,u}\\ &\quad+\int_{\mathbb{R}}(e^{x_i}-1)\ N(du,dx_i)+\int_{\mathbb{R}}(e^{-y_i}-1)\ N(du,dy_i)\\ &\quad+d\left(\sum_{i=1}^{N_u}(e^{Y_i}-1)(e^{-X_i}-1)\right)\Bigg],\end{aligned}$$

making use of the fact that the jump stochastic integrals can be written as random finite sums we note that

$$\begin{aligned}&\int_{\mathbb{R}}(e^{x_i}-1)\ N(du,dx_i)+\int_{\mathbb{R}}(e^{-y_i}-1)\ N(du,dy_i)+\sum_{i=1}^{N_t}(e^{X_i}-1)(e^{-Y_i}-1)\\ =&\sum_{i=1}^{N_t}\left[(e^{X_i}-1)+(e^{-Y_i}-1)+(e^{X_i}-1)(e^{-Y_i}-1)\right]\\ =&\sum_{i=1}^{N_t}(e^{X_i-Y_i}-1).\end{aligned} \tag{6.51}$$

Thus we have the following expression for the deflated stock price with respect to $S_{2,t}$ as

$$\begin{aligned}\frac{S_{1,t}}{S_{2,t}} &= \int_0^t \frac{S_{1,u-}}{S_{2,u-}}\Bigg[(\mu_1-\mu_2+\sigma_2^2-\sigma_1\sigma_2\rho)du+\sigma_1 dW_{1,t}-\sigma_2 dW_{2,u} \\ &\quad +d\Bigg(\sum_{i=1}^{N_u}(e^{Y_i-X_i}-1)\Bigg)\Bigg]. \end{aligned} \tag{6.52}$$

Since both the X_i and Y_i are i.i.d. normal random variables, it follows immediately that X_i-Y_i still has a normal distribution, i.e. $X_i-Y_i\sim N(\delta_1-\delta_2,\beta_1+\beta_2)$. To ease notation we shall set $K_i=X_i-Y_i$, where K_i are i.i.d. normal random variables with mean equal to $\delta_1-\delta_2$ and variance $\beta_1+\beta_2$. Rather than working with correlated Browninan motions if we define a new process $\{B_t, 0\leqslant t\leqslant T\}$ by

$$B_t=\frac{1}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\big(\sigma_1 W_{1,t}-\sigma_2 W_{2,t}\big). \tag{6.53}$$

It can be shown that the process $\{B_t, 0\leqslant t\leqslant T\}$ satisfies all the conditions of Definition 1.4 and is thus a standard Brownian motion process under the measure $\mathbf{P}$. See Appendix A.2 for a proof of this result. Setting $\sigma_{12}=\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$ we can now write the deflated stock price of $S_{1,t}$ with respect to $S_{2,t}$ as

$$\begin{aligned}\frac{S_{1,t}}{S_{2,t}} &= \int_0^t \frac{S_{1,u-}}{S_{2,u-}}\Bigg[(\mu_1-\mu_2+\sigma_2^2-\sigma_1\sigma_2\rho)du+\sigma_{12} dB_t \\ &\quad +\, d\Bigg(\sum_{i=1}^{N_u}(e^{K_i}-1)\Bigg)\Bigg]. \end{aligned} \tag{6.54}$$

We now need to establish if the economy represented by the two tradable assets is arbitrage free, we therefore need to verify if there exists at least one equivalent measure that gives the deflated stock price of $S_{1,t}$ a $\mathbf{Q}$-martingale property. In order to facilitate a change in measure we must identify a stochastic process $\{Z_t, 0\leqslant t\leqslant T\}$ that is strictly positive and has the property $\mathbf{E}^{\mathbf{P}}(Z_t)=1$ for all $t\in[0,T]$. Due to additional sources of randomness in the model we are faced with the difficulty of market incompleteness, as such we must select a specific EMM, provided that at least one exists. We shall again make use of the Esscher transform as our change of measure formula since it is the measure with minimum relative entropy as well as for its simplicity and elegance. Based on the dynamics of the deflated stock price process of $S_{1,t}$ with respect to $S_{2,t}$, the Esscher transform for the exchange option is given by

$$Z_t=\frac{\exp\Big(\theta B_t+\theta\int_0^t\int_{\mathbb{R}}(e^{k_i}-1)\;N(ds,dk_i)\Big)}{\mathbf{E}^{\mathbf{P}}\Big[\exp\Big(\theta B_t+\theta\int_0^t\int_{\mathbb{R}}(e^{k_i}-1)\;N(ds,dk_i)\Big)\Big]}, \tag{6.55}$$

where we have written the finite random sum of jumps in integral form. Making use of the independence between $\{B_t, 0\leqslant t\leqslant T\}$ and $\{N_t, 0\leqslant t\leqslant T\}$ the expectation in Eq.(6.55) can be easily computed, the Esscher transform is then defined by the process $\{Z_t, 0 \leqslant t \leqslant T\}$ where Z_t is given by

$$Z_t = \exp\left(\theta B_t + \theta \int_0^t \int_{\mathbb{R}} (e^{k_i} - 1)\; N(du, dk_i) - \frac{1}{2}\theta^2 t - \lambda t (e^{\theta(\delta_1 - \delta_2) + \frac{1}{2}\theta^2(\beta_1 + \beta_2)} - 1)\right).$$

It is easily verified that the process $\{Z_t, 0 \leqslant t \leqslant T\}$ is strictly positive and satisfies the condition $\mathbf{E}^{\mathbf{P}}(Z_t) = 1$ and so a new $\mathbf{P}$-equivalent measure $\mathbf{Q}$ can be defined by the prescription

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T^X} = Z_T. \tag{6.56}$$

Under this measure $\mathbf{Q}$ Girsanov's Theorem asserts the existence of the process $\{\tilde{B}_t, 0 \leqslant t \leqslant T\}$ that has a standard Brownian motion property, where $\tilde{B}_t$ is defined by

$$\tilde{B}_t = B_t - \theta t. \tag{6.57}$$

In a similar manner we must identify a jump process that has a $\mathbf{Q}$-martingale property, making use of the result in Eq.(2.94) the process $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ has a $\mathbf{Q}$-martingale property and $\tilde{M}_t$ is given by

$$\begin{aligned} \tilde{M}_t &= \int_0^t \int_{\mathbb{R}} (e^{k_i} - 1)\; N(du, dk_i) - \int_0^t \int_{\mathbb{R}} e^{\theta k}(e^k - 1)\; \nu(dk) du \\ &= \int_0^t \int_{\mathbb{R}} (e^{k_i} - 1)\; N(du, dk_i) - \lambda t(\phi_k(\theta + 1) - \phi_k(\theta)), \end{aligned} \tag{6.58}$$

where we have made use of the fact that $\nu(dk) = \lambda f_k(k) dk$ and the K_i are i.i.d. normal random variables, we also define $\phi_k(\theta) = \mathbf{E}^{\mathbf{P}}[e^{\theta k}]$. Writing the deflated stock price in Eq.(6.54) in terms of the new processes $\{\tilde{B}_t, 0 \leqslant t \leqslant T\}$ and $\{\tilde{M}_t, 0 \leqslant t \leqslant T\}$ we have

$$\begin{aligned} \frac{S_{1,t}}{S_{2,t}} &= \int_0^t \frac{S_{1,u-}}{S_{2,u-}} \Bigg[(\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2 + \lambda(\phi_k(\theta+1) - \phi_k(\theta)) + \theta\sigma_{12})\; du \\ &\qquad + \sigma_{12} d\tilde{B}_t + d\tilde{M}_u \Bigg]. \end{aligned} \tag{6.59}$$

Since we require that the deflated stock price have a $(\mathbf{Q}, \mathcal{F}_t)$ martingale property, as this is then equivalent to an arbitrage free economy, we must remove the drift term from the deflated stock which is represented by the Riemann integral. This requires that we set

$$\mu_1 - \mu_2 + \sigma_2^2 - \rho\sigma_1\sigma_2 + \lambda(\phi_k(\theta+1) - \phi_k(\theta)) + \theta\sigma_{12} = 0, \tag{6.60}$$

setting $F(\theta) = \lambda(\phi_k(\theta+1) - \phi_k(\theta)) + \theta\sigma_{12}$ we can write Eq.(6.60) as

$$F(\theta) = \mu_2 - \mu_1 - \sigma_2^2 + \rho\sigma_1\sigma_2. \tag{6.61}$$

Solving Eq.(6.61) for θ will then ensure that the deflated stock is a local $\mathbf{Q}$-martingale as well as determine the market price for risk for this particular exchange option, adding the integrability requirements

$$\int_0^T \mathbf{E}^{\mathbf{Q}}\left[\frac{S_{1,u-}}{S_{2,u-}}\right]^2 du < \infty$$

$$\int_0^T \int_{\mathbb{R}} \mathbf{E}^{\mathbf{Q}}\left[\frac{S_{1,u-}}{S_{2,u-}}\right]^2 e^{\theta k}|e^k - 1|^2 \ \nu(dk)du < \infty, \tag{6.62}$$

will then ensure that the deflated stock price has a $(\mathbf{Q}, \mathcal{F}_t)$ martingale property. Note that the value of θ ultimately determines the equivalent martingale measure $\mathbf{Q}$ defined by

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T^X} = Z_T. \tag{6.63}$$

Once the value of θ is determined that satisfies Eq.(6.61) the dynamics of the deflated stock price is then given by

$$\frac{S_{1,t}}{S_{2,t}} = \sigma_{12}\int_0^t \frac{S_{1,u-}}{S_{2,u-}} d\tilde{B}_u + \int_0^t \frac{S_{1,u-}}{S_{2,u-}} d\tilde{M}_u. \tag{6.64}$$

The solution to this integral equation is given by the Doléans-Dade exponential process

$$\frac{S_{1,T}}{S_{2,T}} = \frac{S_{1,t}}{S_{2,t}} \exp\left(\sigma_{12}(\tilde{B}_T - \tilde{B}_t) + \int_t^T \int_{\mathbb{R}} K_i \ N(du, dk_i) - (\lambda(\phi_k(\theta+1) - \phi_k(\theta)) + \sigma_{12}^2)(T-t)\right). \tag{6.65}$$

Taking natural logarithms on both sides of Eq.(6.65) and writing the stochastic jump integral as a random finite sum we have

$$\ln\frac{S_{1,T}}{S_{2,T}} = \ln\frac{S_{1,t}}{S_{2,t}} + \sigma_{12}(\tilde{B}_T - \tilde{B}_t) + \sum_{i=1}^{N_T - N_t} K_i - (\lambda(\phi_k(\theta+1) - \phi_k(\theta)) + \sigma_{12}^2)(T-t). \tag{6.66}$$

Now if we condition on the event set $\{N_T - N_t = n\}$ then we will have the $\mathcal{F}_T$-measurable random variable $\ln\frac{S_{1,T}}{S_{2,T}}$ being a function of the Brownian motion increment and a finite sum of i.i.d random variables. Provided that the K_i are still normal random variables under $\mathbf{Q}$, then we can make use of the independence between the K_i and the Brownian motion to conclude that the $\mathcal{F}_T$-measurable random variable $\ln\frac{S_{1,T}}{S_{2,T}}$ will also be a normal random conditional on the event set $\{N_T - N_t = n\}$. So we need to determine the distribution of the K_i under $\mathbf{Q}$, this is done by computing

$$\frac{e^{\theta k} f_k(k)}{\mathbf{E}^{\mathbf{P}}\left[e^{\theta k}\right]}, \tag{6.67}$$

by making use of the fact that $K_i \sim N(\delta_1 - \delta_2, \beta_1 + \beta_2)$ under the measure $\mathbf{P}$ we find that the K_i are still normal random variables under $\mathbf{Q}$ but the distribution is now $K_i \sim N(\delta_1 - \delta_2 + \theta(\beta_1 + \beta_2), \beta_1 + \beta_2)$. Essentially we make use of Lemma 2.2, hence only the mean of the K_i is altered under the new measure $\mathbf{Q}$. In a similar way we can determine the distribution of the process $\{N_t, 0 \leqslant t \leqslant T\}$ under $\mathbf{Q}$. From Lemma 2.2 it follows that

$$\mathbf{E}^{\mathbf{Q}}\left[e^{iu(N_T - N_t)}\middle|\mathcal{F}_t\right] = \frac{\mathbf{E}^{\mathbf{P}}\left[e^{iu(N_T - N_t)} Z_T\middle|\mathcal{F}_t\right]}{\mathbf{E}^{\mathbf{P}}\left[Z_T\middle|\mathcal{F}_t\right]} \qquad \forall u \in \mathbb{R}, \tag{6.68}$$

where Z_T is the R-N derivative linking $\mathbf{Q}$ to $\mathbf{P}$ of the measure space $(\Omega, \mathcal{F}_T)$ defined in Eq.(6.56). Eq.(6.68) is then solved in precisely the same manner as in the previous section yielding

$$\mathbf{E}^{\mathbf{Q}}\big[e^{iu(N_T-N_t)}\big|\mathcal{F}_t\big] = \exp\big(\lambda\phi(\theta)(T-t)(e^{iu}-1)\big). \tag{6.69}$$

Hence $\{N_t, 0 \leqslant t \leqslant T\}$ is still a Poisson process under the measure $\mathbf{Q}$ but now the intensity parameter is given by $\tilde{\lambda} = \lambda\phi(\theta)$. Having shown that the K_i are still i.i.d. normal random variables we can conclude that, conditional on the event set $\{N_T - N_t = n\}$, the $\mathcal{F}_T$ measurable random variable $\ln\frac{S_{1,T}}{S_{2,T}}$ is a normal random variable with mean and variance given by

$$\begin{aligned}
\mathbf{E}^{\mathbf{Q}}\left(\ln\frac{S_{1,T}}{S_{2,T}}\middle|N_T - N_t = n\right) &= \ln\frac{S_{1,t}}{S_{2,t}} + n(\delta_1 - \delta_2 + \theta(\beta_1+\beta_2)) \\
&\quad -(\lambda(\phi_k(\theta+1) - \phi_k(\theta) + \sigma_{12}^2)(T-t) \\
\mathbf{Var}^{\mathbf{Q}}\left(\ln\frac{S_{1,T}}{S_{2,T}}\middle|N_T - N_t = n\right) &= \sigma_{12}^2(T-t) + n(\beta_1+\beta_2).
\end{aligned}$$

We are now in a position to compute the value of the exchange option under the Merton dynamics

$$\begin{aligned}
\Gamma_t &= S_{2,t}\mathbf{E}^{\mathbf{Q}}\left[\max\left\{\frac{S_{1,T}}{S_{2,T}} - 1\right\}\middle|\mathcal{F}_t\right] \\
&= S_{2,t}\sum_{n=0}^{\infty}\mathbf{E}^{\mathbf{Q}}\left[\max\left\{\frac{S_{1,T}}{S_{2,T}} - 1\right\}\middle|N_T - N_T = n\right]\mathbf{Q}(N_T - N_t = n). \quad (6.70)
\end{aligned}$$

Now since $\ln\frac{S_{1,T}}{S_{2,T}}|_{N_T-N_t=n}$ is normally distributed with variance $\sigma_{12}^2(T-t) + n(\beta_1+\beta_2)$ we can make use of the Black's formula, see Appendix A.1, to compute the conditional expectation as follows

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left[\max\left\{\frac{S_{1,T}}{S_{2,T}} - 1\right\}\middle|N_T - N_T = n\right] \\
&\mathbf{E}^{\mathbf{Q}}\left[\frac{S_{1,T}}{S_{2,T}}\middle|N_T - N_t = n\right]N(d_{1,n}) - N(d_{2,n}), \quad (6.71)
\end{aligned}$$

where $d_{1,n}$ and $d_{2,n}$ is given by

$$\begin{aligned}
d_{1,n} &= \frac{\ln\mathbf{E}^{\mathbf{Q}}\left[\frac{S_{1,T}}{S_{2,T}}\middle|N_T - N_t = n\right] + \frac{1}{2}(\sigma_{12}^2(T-t) + n(\beta_1+\beta_2))}{\sqrt{\sigma_{12}^2(T-t) + n(\beta_1+\beta_2)}} \\
d_{2,n} &= d_{1,n} - \sqrt{\sigma_{12}^2(T-t) + n(\beta_1+\beta_2)}. \quad (6.72)
\end{aligned}$$

We are not quite done yet, we still need to evaluate the conditional expectation of $\mathbf{E}^{\mathbf{Q}}\left[\frac{S_{1,T}}{S_{2,T}}\middle|N_T - N_t = n\right]$. Making use of Eq.(6.65) and the independence between

$\{B_t, 0 \leqslant t \leqslant T\}, \{N_t, 0 \leqslant t \leqslant T\}$ and the K_i we have the

$$\begin{aligned}
& \mathbf{E}^{\mathbf{Q}}\left[\frac{S_{1,T}}{S_{2,T}}\bigg| N_T - N_t = n\right] \\
= & \frac{S_{1,t}}{S_{2,t}} \exp\bigg(-(\lambda(\phi_k(\theta+1) - \phi_k(\theta)) + \sigma_{12}^2)(T-t)\bigg) \\
& \times \mathbf{E}^{\mathbf{Q}}\left[\exp\left(\sigma_{12}(\tilde{B}_T - \tilde{B}_t) + \sum_{i=1}^{n} K_i \big| N_T - N_t = n\right)\right] \\
= & \frac{S_{1,t}}{S_{2,t}} \exp\bigg(-\lambda(\phi_k(\theta+1) - \phi_k(\theta))(T-t) + n(\delta_1 - \delta_2) + n(\beta_1+\beta_2)(\theta + \frac{1}{2})\bigg) \\
= & \frac{S_{1,t}}{S_{2,t}} e^{\zeta_{n,t}}, \qquad (6.73)
\end{aligned}$$

where $\zeta_{n,t}$ is given by

$$\zeta_{n,t} = -\lambda(\phi_k(\theta+1) - \phi_k(\theta))(T-t) + n(\delta_1 - \delta_2) + n(\beta_1+\beta_2)(\theta + \frac{1}{2}).$$

Hence we can finally compute the value of the European exchange option as

$$\begin{aligned}
\Gamma_t &= S_{2,t} \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}(T-t)}(\tilde{\lambda}(T-t))^n}{n!} \left[\frac{S_{1,t}}{S_{2,t}} e^{\zeta_{n,t}} N(d_{1,n}) - N(d_{2,n})\right] \\
&= \sum_{n=0}^{\infty} \frac{e^{-\tilde{\lambda}(T-t)}(\tilde{\lambda}(T-t))^n}{n!} \left[S_{1,t} e^{\zeta_{n,t}} N(d_{1,n}) - S_{2,t} N(d_{2,n})\right], \qquad (6.74)
\end{aligned}$$

where $d_{1,n}$ and $d_{2,n}$ are given by

$$\begin{aligned}
d_{1,n} &= \frac{\ln\frac{S_{1,t}}{S_{2,t}} - \lambda(\phi_k(\theta+1) - \phi_k(\theta)) + n(\delta_1 - \delta_2 + \theta(\beta_1+\beta_2)) + \frac{1}{2}(\sigma_{12}^2(T-t) + 2n(\beta_1+\beta_2))}{\sqrt{\sigma_{12}^2(T-t) + n(\beta_1+\beta_2)}} \\
d_{2,n} &= d_{1,n} - \sqrt{\sigma_{12}^2(T-t) + n(\beta_1+\beta_2)}.
\end{aligned}$$

As before we have $N(x)$ as the cumulative normal distribution defined by

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy.$$

If we were to set both n and λ equal to zero we would obtain the pricing formula for an exchange option under the Black-Scholes and Merton model. Again, from a practical point of view we need only focus on the first 15 to 20 terms in the infinite series in Eq.(6.74) in order to obtain a numerical value for the the exchange option.

The exchange option pricing formula can also be easily extended to include jumps that are correlated provided that the jump sizes for the assets are still normally distributed. In addition we could have just as well incorporated two autonomous Poisson processes for governing the jumps for the respective stocks, i.e. we could have the Poisson process $\{N_{1,t}, 0 \leqslant t \leqslant T\}$ governing the jumps for asset $S_{1,t}$ and the another Poisson process $\{N_{2,t}, 0 \leqslant t \leqslant T\}$ different from $\{N_{1,t}, 0 \leqslant t \leqslant T\}$ governing the jumps for the asset $S_{2,t}$. The consequence of having two autonomous Poisson processes for the respective assets would mean that the pricing formula for the value of the exchange option would have to be adjusted to having two infinite summations, one for each Poisson distribution.

Chapter 7

Kou and Wang Model

We now turn our attention to the jump-diffusion model developed by Kou [28] and Wang [30]. This jump-diffusion model is similar to that of the Merton jump-diffusion model [37] explored in chapter 6, however where as the jump sizes followed a normal distribution under the Merton model under the Kou and Wang model the jump sizes are assumed to exponentially distributed. This may seem as a minor change in the assumptions of the risky assets dynamics but the reward for doing so is quite dramatic. More precisely, while we were able to derive closed form solutions for the standard European call and put options under the Merton dynamics no mention was made of any exotic options such as perpetual options, lookback options, barrier options and other path-dependent options. The reason being that a martingale approach to pricing these exotic options invariably requires knowledge of the *first passage of time* of a jump-diffusion process to a flat boundary. While the first passage of time for Brownian motion to a flat boundary is a well known result, see Shreve [44], for jump-diffusion processes and Lévy processes in general the first passage of time can prove difficult if not impossible to determine. The reason for this is that it is possible for the jump-diffusion process to "overshoot" flat boundaries, we shall explore this concept in greater detail as we proceed.

The most appealing aspect of the Kou-Wang model lies in the fact that there are closed form solutions to a number of exotic options. The existence of closed form solutions to these exotic options is largely due to jump sizes being exponentially distributed as this facilitates the computation of the first passage of time as shown in Kou and Wang [29]. Another important property that the Kou-Wang model possesses is the *leptokurtic* feature which means that the return distribution is skewed to the left with higher peaks. In addition the return distribution has heavier tails than that of the normal distribution and based on empirical evidence this property is highly desirable. While the Merton jump-diffusion model exhibits the leptokurtic feature, it is more pronounced in the Kou-Wang model as the return distribution has significantly heavier tails in comparison.

In this chapter we shall start off by examining the European call option.

We then consider exotic options such as the perpetual put, barrier options and lookback options. In particular when deriving a pricing formula for the perpetual put option we develop the Kou-Wang model to include only positive jumps as it leads to an elegant closed form solution for the value of the perpetual put option as well as highlighting the overshoot problem faced when pricing path-dependent options based on Lévy processes.

We now give the dynamics of the risky and risk-free asset in the Kou-Wang model. Under the real world measure $\mathbf{P}$ we have

$$\begin{aligned} dR_t &= rR_t\,dt & R_0 = 1 \qquad (7.1)\\ dS_t &= S_{t-}\Big(\mu dt + \sigma dW_t + \int_{\mathbb{R}} (e^{y_i}-1)\; N(dt, dy_i)\Big) & S_0 \in \mathbb{R}^+, \qquad (7.2)\end{aligned}$$

where the process $\{W_t, 0 \leqslant t \leqslant T\}$ is standard Brownian motion, $N(dt, dy_i)$ is a Poisson measure on $[0,T] \times \big(\mathbb{R} - \{0\}\big)$ and the Y_i are a sequence of i.i.d. asymmetrical double exponentially distributed random variables with density function given by

$$f_Y(y) = p \cdot \alpha_1 e^{-\alpha_1 y}\mathbf{1}_{(y \geqslant 0)} + q \cdot \alpha_2 e^{\alpha_2 y}\mathbf{1}_{(y \leqslant 0)}. \tag{7.3}$$

Here $p, q \geqslant 0$ and $p + q = 1$. In order for the risky asset S_t to have a finite expectation we shall require that $\alpha_1 > 1$ and $\alpha_2 > 0$. The instantaneous rate of return of the risky asset is given by $\mu \in \mathbb{R}$ and the volatility $\sigma \in \mathbb{R}^+$. The Lévy measure is of the form $\nu(dy) = \lambda f_Y(y)dy$, where λ is the intensity rate of the the Poisson process $\{N_t, 0 \leqslant t \leqslant T\}$ and $f_Y(y)$ is as in Eq.(7.3). Again we assume that all sources of randomness, W_t, N_t and Y_i, are independent. We shall examine the properties of the jump sizes before proceeding. Since the jumps are now assumed to exponentially distributed they will possess what is known as the memoryless property. More precisely if Y is exponentially distributed then the memoryless property states that

$$\mathbf{P}\big(Y > s + t \big| Y > t\big) = \mathbf{P}\big(Y > s\big), \tag{7.4}$$

for all $s, t \geqslant 0$, see Shreve [44] chapter 11 for a discussion of the memoryless property. It is this property that aids in the computation of the first passage of time to a flat boundary and allows us to express exotic options in elegant closed form solutions. At present is seems impossible to derive closed form solutions for exotic options under any other jump-diffusion model including the Merton model. Finally we look at the moment generating function of the Y_i,

$$\mathbf{E}^{\mathbf{P}}\big[e^{Y_i}\big] = p\frac{\alpha_1}{\alpha_1 - 1} + q\frac{\alpha_2}{\alpha_2 + 1}. \tag{7.5}$$

From here it can be seen that in order to have $\mathbf{E}^{\mathbf{P}}[e^{Y_i}] < \infty$ we must have $\alpha_1 > 1$, this essentially prevents the upward jumps from exceeding 100%. Note that in Eq.(7.3) the upward jumps are governed by the exponential random variable defined on $[0, \infty)$ while downward jumps are governed by the exponential random variables defined on $(-\infty, 0]$.

7.1 Pricing a European call option

We shall start by examining a standard European call option where the dynamics for the economy (R_t, S_t) is given by Eq.(7.1) and Eq.(7.2). We do this to introduce the relevant notion that will encounter throughout the rest of the chapter. We are therefore interested in pricing the $\mathcal{F}_T$-measurable claim Γ_T, where $\Gamma_T = \max\{S_T - K, 0\}$ and K is the strike price of the option. Our arbitrage free pricing approach to contingent claims requires that we select a numeraire and establish if there exist at least one EMM. Since the dynamics proposed for the risky asset incorporates jumps we know that economy or market will be incomplete and as such there may well exist infinitely many EMM's. We shall select the risk-free asset R_t as our numeraire and deflate the risky asset S_t with respect to this chosen numeraire. We then apply the product rule (Theorem 2.8) to the deflated stock price $\tilde{S}_t = S_t/R_t$. We shall not go into too much detail here as the application of the product rule to the deflated stock price is identical to what we have already done in chapter 6 when pricing a European call option under the Merton dynamics. The dynamics of the deflated stock price are given by

$$\tilde{S}_t = \int_0^t \tilde{S}_{u-}\big(\mu - r\big)\ du + \int_0^t \tilde{S}_{u-}\sigma\ dW_u + \int_0^t \int_{\mathbb{R}} \tilde{S}_{u-}(e^{y_i} - 1)\ N(ds, dy_i).$$

In order to verify that the economy is arbitrage free we require that $\tilde{S}_t$ be a martingale under some equivalent measure $\mathbf{Q}$. As market incompleteness implies that the measure $\mathbf{Q}$ is not necessarily unique we select the Esscher transform as our change of measure formula, the motivation behind this being that the Esscher transform has minimum relative entropy with respect to the measure $\mathbf{P}$. There are of course many other EMM's that one could have chosen, Kou [28] suggests the use of utility functions in determining the fair value of the option. With the help of the Esscher transform we define the measure $\mathbf{Q}$ by the following R-N derivative

$$\left.\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}}\right|_{\mathcal{F}_T^X} = Z_T, \tag{7.6}$$

where the Esscher transform process $\{Z_t, 0 \leqslant t \leqslant T\}$ is defined by

$$Z_t = \frac{e^{\theta X_t}}{\mathbf{E}^{\mathbf{P}}\big[e^{\theta X_t}\big]} \qquad \theta \in (-\alpha_2, \alpha_1 - 1). \tag{7.7}$$

The process $\{X_t, 0 \leqslant t \leqslant T\}$ is given by

$$X_t = \mu t + W_t + \sum_{i=1}^{N_t} Y_i. \tag{7.8}$$

Remark 7.1. Note that the Esscher transform is only defined for $\theta \in (-\alpha_2, \alpha_1 - 1)$, the reason for this is once we apply our change of measure formula the distribution of the jump sizes will change. In order to ensure that the jumps have a finite expectation under the new measure $\mathbf{Q}$ we shall require that the following integral be finite

$$\int_{\mathbb{R}} \left(e^{y(\theta+1)} - e^{\theta y}\right) f_Y(y)\, dy$$
$$= q\alpha_2 \int_{-\infty}^{0} \left(e^{y(\theta+\alpha_2+1)} - e^{y(\theta+\alpha_2)}\right) dy \tag{7.9}$$
$$+ p\alpha_1 \int_{0}^{\infty} \left(e^{-y(\alpha_1-\theta-1)} - e^{-y(\alpha_1-\theta)}\right) dy. \tag{7.10}$$

From Eq.(7.9) and Eq.(7.10) we can see that we must have $\theta \in (-\alpha_2, \alpha_1 - 1)$ in order for the above integral to be finite.

Next we apply Girsanov's theorem (Theorem 2.11) and the results from Eq.(2.94) so as to express the deflated stock as a local $\mathbf{Q}$-martingale, we therefore have

$$\begin{aligned}\tilde{S}_t \;&=\; \left(r - \mu + \sigma\theta + \lambda\left[\frac{p\alpha_1}{(\alpha_1-\theta-1)(\alpha_1-1)} + \frac{q\alpha_2}{(\alpha_2+\theta+1)(\alpha_2+\theta)}\right]\right) \\ &\quad \times \int_0^t \tilde{S}_{u-}\, du + \sigma \int_0^t \tilde{S}_{u-}\, d\tilde{W}_u + \int_0^t \int_{\mathbb{R}} \tilde{S}_{u-}(e_i^y - 1)\, \tilde{N}_{\mathbf{Q}}(du, dy_i),\end{aligned}$$

where $\{\tilde{W}_t, 0 \leqslant t \leqslant T\}$ is standard Brownian motion under the measure $\mathbf{Q}$ and the process $\{\tilde{N}_{\mathbf{Q}}(t,\cdot), 0 \leqslant t \leqslant T\}$ is a $\mathbf{Q}$-martingale with $\tilde{N}_{\mathbf{Q}}(du, dy_i) = N(du, dy_i) - \lambda e^{\theta y} f_Y(y) dy du$. By setting the drift term, represented by the Riemann integral, equal to zero we have

$$\tilde{S}_t = \sigma \int_0^t \tilde{S}_{u-}\, d\tilde{W}_u + \int_0^t \int_{\mathbb{R}} \tilde{S}_{u-}(e_i^y - 1)\, \tilde{N}_{\mathbf{Q}}(du, dy_i), \tag{7.11}$$

which is a local $\mathbf{Q}$-martingale. Adding the integrability constraint that

$$\mathbf{E}^{\mathbf{Q}}\left(\left|\sigma \int_0^T \tilde{S}_{u-}\, d\tilde{W}\right|^2\right) = \sigma^2 \int_0^T \mathbf{E}^{\mathbf{Q}}\left(\tilde{S}_{u-}\right)^2 du < \infty,$$

along with

$$\begin{aligned}&\mathbf{E}^{\mathbf{Q}}\left(\left|\int_0^T \int_{\mathbb{R}} \tilde{S}_{u-}(e^{y_i} - 1)\, \tilde{N}_{\mathbf{Q}}(du, dy_i)\right|^2\right) \\ &= \int_0^T \mathbf{E}^{\mathbf{Q}}\left(\tilde{S}_{u-}|e^y - 1|\right)^2 e^{\theta y} \nu(dy) du < \infty,\end{aligned}$$

will then ensure that the deflated stock price has a $(\mathbf{Q}, \mathcal{F}_t^X)$-martingale property which in turn verifies that the economy is indeed arbitrage free. Note that the equation

$$r - \mu + \sigma\theta + \lambda\left[\frac{p\alpha_1}{(\alpha_1-\theta-1)(\alpha_1-1)} + \frac{q\alpha_2}{(\alpha_2+\theta+1)(\alpha_2+\theta)}\right] = 0, \tag{7.12}$$

determines the market price for risk and hence the measure. Having verified that the model (R_t, S_t) is arbitrage free when the dynamics for the tradable assets are given by Eq.(7.1) and Eq.(7.2) we now focus on the pricing of the $\mathcal{F}_T$-measurable contingent claim $\Gamma_T = \max\{S_T - K, 0\}$.

Let us assume that we already working under some EMM $\mathbf{Q}$, that need not necessarily be by the use of the Esscher transform, and that the dynamics of the deflated stock price under the measure $\mathbf{Q}$ are as in Eq.(7.11) which can be written as the following SDE

$$d\tilde{S}_t = \tilde{S}_t\bigg(\sigma\tilde{W}_t + \int_{\mathbb{R}} (e^{y_i} - 1)\ \tilde{N}_{\mathbf{Q}}(dt, dy_i)\bigg). \tag{7.13}$$

The solution to the above SDE can be obtained by applying Itô's Lemma (Theorem 2.7) to the stochastic function $\ln\tilde{S}_t$, which yields

$$\begin{aligned}
\tilde{S}_t &= \tilde{S}_0 \exp\left(\sigma\tilde{W}_t - \frac{1}{2}\sigma^2 t - \lambda\delta t\right)\prod_{i=1}^{N_t} e^{Y_i} \\
\Rightarrow S_t &= S_0 \exp\left(\sigma\tilde{W}_t + (r - \frac{1}{2}\sigma^2 - \lambda\delta)t\right)\prod_{i=1}^{N_t} e^{Y_i},
\end{aligned} \tag{7.14}$$

where

$$\delta = \mathbf{E}^{\mathbf{Q}}\big[e^{Y_i}\big] - 1. \tag{7.15}$$

In order to compute Eq.(7.15) we will need the distribution of the jump sizes under the measure $\mathbf{Q}$, the measure $\mathbf{Q}$ in turn depends on our choice of R-N derivative. Hence, depending on our choice of measure we will have different distributions for the jump sizes. If we had for instance chosen the Esscher transform as our R-N derivative then the jumps will still be exponentially distributed but with the original parameters now adjusted, so we would have the probability density function $f_Y(y)$ under the measure $\mathbf{Q}$ as being

$$f_Y(y) = \tilde{p}\cdot\tilde{\alpha_1}e^{-\tilde{\alpha_1}y}\mathbf{1}_{(y\geqslant 0)} + \tilde{q}\cdot\tilde{\alpha_2}e^{\tilde{\alpha_2}y}\mathbf{1}_{(y\leqslant 0)}, \tag{7.16}$$

where $\tilde{p}, \tilde{q} \geqslant 0, \tilde{p} + \tilde{q} = 1$ and $\tilde{\alpha_1} > 1, \tilde{\alpha_2} > 0$. A price process $\{\Gamma_t, 0 \leqslant t \leqslant T\}$ for the European call option is given by

$$\begin{aligned}
\Gamma_t &= \mathbf{E}^{\mathbf{Q}}\left[\frac{\max\{S_T - K, 0\}}{e^{r(T-t)}}\middle|\mathcal{F}_t\right] \\
&= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[S_T\mathbf{1}_{(S_T>K)}\big|\mathcal{F}_t^X\big] - Ke^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[\mathbf{1}_{(S_T>K)}\big|\mathcal{F}_t^X\big] \\
&= e^{-r(T-t)}\mathbf{E}^{\mathbf{Q}}\big[S_T\mathbf{1}_{(S_T>K)}\big|\mathcal{F}_t^X\big] - Ke^{-r(T-t)}\mathbf{Q}\big(S_T > K\big|\mathcal{F}_t^X\big).
\end{aligned} \tag{7.17}$$

Focusing on the first term in Eq.(7.17) and taking $t = 0$ we have

$$e^{-rT}\mathbf{E}^{\mathbf{Q}}\big[S_T\mathbf{1}_{(S_T>K)}\big], \tag{7.18}$$

we now shall implement a change of numeraire technique to ease computation. More precisely, since the deflated stock price must be a martingale under the measure $\mathbf{Q}$ we know that

$$\mathbf{E}^{\mathbf{Q}}\left[\frac{S_T}{e^{rT}}\right] = S_0. \tag{7.19}$$

A new measure $\tilde{\mathbf{Q}}$ can now be defined with the aid the following R-N derivative

$$\left.\frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}\mathbf{Q}}\right|_{\mathcal{F}_T^X} = \frac{S_T}{e^{rT}S_0}. \tag{7.20}$$

Note that $\tilde{\mathbf{Q}}$ is a well defined probability measure since the process $\{e^{-rt}\frac{S_t}{S_0}, 0 \leqslant t \leqslant T\}$ is strictly positive martingale under the measure $\mathbf{Q}$ and satisfies the requirement

$$\mathbf{E}^{\mathbf{Q}}\left[\frac{S_T}{e^{rT}S_0}\right] = 1. \tag{7.21}$$

With this new measure $\tilde{\mathbf{Q}}$ we can evaluate the expectation in Eq.(7.18) with the aid of Theorem 2.2 as follows

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left[\frac{S_T}{e^{rT}}\mathbf{1}_{(S_T>K)}\right] \\
&= \mathbf{E}^{\tilde{\mathbf{Q}}}\left[\frac{S_T}{e^{rT}}\mathbf{1}_{(S_T>K)}\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\tilde{\mathbf{Q}}}\right] \\
&= \mathbf{E}^{\tilde{\mathbf{Q}}}\left[\frac{S_T}{e^{rT}}\mathbf{1}_{(S_T>K)}\frac{S_0 e^{rT}}{S_T}\right] \\
&= S_0\mathbf{E}^{\tilde{\mathbf{Q}}}\big[\mathbf{1}_{(S_T>K)}\big] \\
&= S_0\tilde{\mathbf{Q}}\big(S_T > K\big).
\end{aligned} \tag{7.22}$$

Thus we have that the price for the call option at time $t = 0$ is given by

$$\begin{aligned}
\Gamma_0 &= S_0\tilde{\mathbf{Q}}\big(S_T > K\big) - Ke^{-rT}\mathbf{Q}\big(S_T > K\big) \\
&= S_0\tilde{\mathbf{Q}}\big(\ln S_T > \ln K\big) - Ke^{-rT}\mathbf{Q}\big(\ln S_T > \ln K\big),
\end{aligned} \tag{7.23}$$

we are therefore left with the task of determining the probabilities, under the respective measures, in Eq.(7.23) in order to determine the value for the European call option. From Eq.(7.14) we have that

$$\ln S_T = \ln S_0 + \big(r - \frac{1}{2}\sigma^2 - \lambda\delta\big)T + \sigma\tilde{W}_T + \sum_{i=1}^{N_T} Y_i, \tag{7.24}$$

under the measure $\mathbf{Q}$. We therefore need the distribution of $\ln S_T$ under the measure $\mathbf{Q}$ which is a combination of a normal random variable along with a sequence of exponential random variables. The following result, due to Kou, aids us in this regard.

Proposition 7.1. *Let Y_i be a sequence of i.i.d. random variables with density given by Eq.(7.3) then we have the following decomposition for $n \geqslant 1$*

$$\sum_{i=1}^{n} Y_i \stackrel{d}{=} \begin{cases} \sum_{i=1}^{k} Y_i^+, & \text{with probability } P_{n,k}, k = 1,2,\ldots,n \\ -\sum_{i=1}^{k} Y_i^-, & \text{with probability } Q_{n,k}, k = 1,2,\ldots,n \end{cases}$$

where Y_i^+ is an exponential random variable with density function $f_{Y^+}(y) = \alpha_1 e^{-\alpha_1 y}\mathbf{1}_{(y\geqslant 0)}$ and Y_i^- is also an exponentially distributed random variable with density $f_{Y^-}(y) = \alpha_2 e^{\alpha_2 y}\mathbf{1}_{(y\leqslant 0)}$. Furthermore $P_{n,k}$ and $Q_{n,k}$ are given by

$$\begin{aligned} P_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^{i-k}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^{n-i} p^i q^{n-i} \\ Q_{n,k} &= \sum_{i=k}^{n-1} \binom{n-k-1}{i-k}\binom{n}{i}\left(\frac{\alpha_1}{\alpha_1+\alpha_2}\right)^{n-i}\left(\frac{\alpha_2}{\alpha_1+\alpha_2}\right)^{i-k} p^{n-i} q^i. \end{aligned} \tag{7.25}$$

Proof: See Kou [28], Appendix B ■

Remark 7.2. Note that since we are working under the EMM $\mathbf{Q}$ we can no longer assume that the parameters $p, q, \alpha_1, \alpha_2, \lambda, \delta$ remain invariant from the change in measure. These parameters will be adjusted as a result from the change in measure from $\mathbf{P}$ to $\mathbf{Q}$. However for the sake of notational convenience we shall continue with the use of the parameters $p, q, \alpha_1, \alpha_2, \lambda, \beta$ with the understanding that these parameters have already been adjusted subsequent to the change in measure.

We shall concentrate on computing the probability $\mathbf{Q}(\ln S_T > \ln K)$ first as it simplifies computation when evaluating the probability $\tilde{\mathbf{Q}}(\ln S_T > \ln K)$. From Proposition 7.1 we can compute the probability $\mathbf{Q}\big(\ln S_T > \ln K\big)$ as follows

$$\begin{aligned} &\mathbf{Q}\big(\ln S_T > \ln K\big) \\ &= \mathbf{Q}\Big((r - \frac{1}{2}\sigma^2 - \lambda\beta)T + \sigma\tilde{W}_T + \sum_{i=1}^{N_T} Y_i > \ln K\Big) \\ &= \sum_{n=0}^{\infty} \mathbf{Q}\big(N_T = n\big) \cdot \mathbf{Q}\Big((r - \frac{1}{2}\sigma^2 - \lambda\beta)T + \sigma\tilde{W}_T + \sum_{i=1}^{n} Y_i > \ln K\Big) \qquad (7.26) \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{n} \mathbf{Q}\big(N_T = n\big)\, P_{n,k}\, \mathbf{Q}\Big((r - \frac{1}{2}\sigma^2 - \lambda\beta)T + \sigma\tilde{W}_T + \sum_{i=1}^{n} Y_i^+ > \ln K\Big) \\ &+ \sum_{n=0}^{\infty}\sum_{k=1}^{n} \mathbf{Q}\big(N_T = n\big)\, Q_{n,k}\, \mathbf{Q}\Big((r - \frac{1}{2}\sigma^2 - \lambda\beta)T + \sigma\tilde{W}_T - \sum_{i=1}^{n} Y_i^- > \ln K\Big). \end{aligned}$$

Eq.(7.26) follows from the independence between W_t, N_t and the Y_i. Now in order to compute probabilities of the form

$$\mathbf{Q}\Big((r - \frac{1}{2}\sigma^2 - \lambda\beta)T + \sigma\tilde{W}_T + \sum_{i=1}^{n} Y_i^+ > \ln K\Big), \tag{7.27}$$

we make use of Hh functions. The Hh function is a non-increasing function defined by

$$Hh_n(x) = \int_x^\infty Hh_{n-1}(y)\, dy = \frac{1}{n!}\int_x^\infty (y-x)^n e^{-\frac{y^2}{2}}\, dy \qquad n = 0,1,2\ldots \tag{7.28}$$

Note that for $n = -1, 0$ we have $Hh_{-1}(x) = e^{-\frac{x^2}{2}}$ and $Hh_0(x) = \sqrt{2\pi}Hh_{-1}(x)$. The Hh can also be expressed in terms of confluent hypergeometric functions, see Abramowitz and Stegun [3]. It is the Hh function that allows us to compute probabilities of the form in Eq.(7.27). To see this, note that the random variable

$$(r - \frac{1}{2}\sigma^2 - \lambda\beta)T + \sigma\tilde{W}_T + \sum_{i=1}^{n} Y_i^+, \tag{7.29}$$

is a sum of a normal random variable along with exponentially distributed random variables. If we let Z be a normal random variable with mean equal to zero and variance equal to $\sigma^2 T$ then we can rewrite Eq.(7.29) as

$$(r - \frac{1}{2}\sigma^2 - \lambda\beta)T + Z + \sum_{i=1}^{n} Y_i^+, \tag{7.30}$$

where the Y_i^+ are i.i.d. exponentially distributed with probability density function $\alpha_1 e^{-\alpha_1 y}\mathbf{1}_{(y\geqslant 0)}$. By making use of the independence property that exists between the normal random variable and the exponentially distributed random variables the probability density function of the random variable $Z + \sum\limits_{i=1}^{n} Y_i^+$ is computed as follows

$$\begin{aligned} f_{Z+\sum\limits_{i=1}^{n} Y_i^+}(t) &= \int_{-\infty}^{\infty} f_{\sum\limits_{i=1}^{n} Y_i^+}(t-x) f_Z(x)\, dx \\ &= e^{-t\alpha_1}(\alpha_1)^n \int_{-\infty}^{t} \frac{e^{\alpha_1 x}(t-x)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{x^2}{2\sigma^2 T}}\, dx. \end{aligned} \tag{7.31}$$

In Eq.(7.31) we have made use of the fact that a sum of i.i.d. exponential random variables has a gamma distribution, see Shreve [44]. By completing the square in the natural exponent we have

$$f_{Z+\sum\limits_{i=1}^{n} Y_i^+}(t) = e^{-t\alpha_1}(\alpha_1)^n e^{\frac{T(\sigma\alpha_1)^2}{2}} \int_{-\infty}^{t} \frac{(t-x)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi\sigma^2 T}} e^{-\frac{(x-\sigma^2 T\alpha_1)^2}{2\sigma^2 T}}\, dx, \tag{7.32}$$

now letting $a = \frac{x - \sigma^2 T\alpha_1}{\sigma\sqrt{T}}$ from which we get with $da = \frac{dx}{\sigma\sqrt{T}}$, yields

$$\begin{aligned} f_{Z+\sum\limits_{i=1}^{n} Y_i^+}(t) &= e^{-t\alpha_1} e^{\frac{T(\sigma\alpha_1)^2}{2}} (\sqrt{T}\sigma)^{n-1}(\alpha_1)^n \\ &\quad \times \int_{-\infty}^{\frac{t-\sigma^2 T\alpha_1}{\sigma\sqrt{T}}} \frac{\left(\frac{t}{\sigma\sqrt{T}} - y - \sigma\sqrt{T}\alpha_1\right)^{n-1}}{(n-1)!} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \\ &= \frac{e^{\frac{T(\sigma\alpha_1)^2}{2}}}{\sqrt{2\pi}} (\sqrt{T}\sigma)^{n-1}(\alpha_1)^n e^{-t\alpha_1}\, Hh_{n-1}\Big(-\frac{t}{\sigma\sqrt{T}} + \sigma\sqrt{T}\alpha_1\Big), \end{aligned}$$

since

$$\frac{1}{(n-1)!}\int_{-\infty}^{x}(x-y)^{n-1}e^{-\frac{y^2}{2}}\,dy = Hh_{n-1}(-x). \tag{7.33}$$

With this density function for the random variable $Z+\sum_{i=1}^{n}Y_i^{+}$ we can compute probabilities by integrating the density function across the relevant range. The probability density function of the random variable $Z-\sum_{i=1}^{n}Y_i^{-}$ is derived in a similar manner as above and is given by

$$f_{Z-\sum\limits_{i=1}^{n}Y_i^{-}}(t) = \frac{e^{T\frac{(\sigma\alpha_2)^2}{2}}}{\sqrt{2\pi}}(\sigma\sqrt{T})^{n-1}(\alpha_2)^n e^{t\alpha_2}\,Hh_{n-1}\Big(\frac{t}{\sigma\sqrt{T}}+\sigma\sqrt{T}\alpha_2\Big). \tag{7.34}$$

Having derived the probability density functions for the random variables $Z+\sum_{i=1}^{n}Y_i^{+}$ and $Z-\sum_{i=1}^{n}Y_i^{-}$ we can compute tail probabilities as such

$$\begin{aligned}
&\mathbf{Q}\Big(Z+\sum_{i=1}^{n}Y_i^{+}\geqslant x\Big)\\
&= \frac{(\sigma\sqrt{T}\alpha_1)^n e^{\frac{T(\sigma\alpha_1)^2}{2}}}{\sigma\sqrt{2\pi}}\int_x^{\infty}e^{-t\alpha_1}Hh_{n-1}\Big(-\frac{t}{\sigma\sqrt{T}}+\sigma\sqrt{T}\alpha_1\Big)\,dt, &(7.35)\\
&\mathbf{Q}\Big(Z-\sum_{i=1}^{n}Y_i^{-}\geqslant x\Big)\\
&= \frac{(\sigma\sqrt{T}\alpha_2)^n e^{\frac{T(\sigma\alpha_2)^2}{2}}}{\sigma\sqrt{2\pi}}\int_x^{\infty}e^{t\alpha_2}Hh_{n-1}\Big(\frac{t}{\sigma\sqrt{T}}+\sigma\sqrt{T}\alpha_1\Big)\,dt. &(7.36)
\end{aligned}$$

This the enables us to compute the the probability $\mathbf{Q}\big(\ln S_T > K\big)$. While no explicit antiderivative exists for the integrals in Eq.(7.35) and Eq.(7.35) Kou derives a recursive relationship for these integrals by using integration by parts formula which is useful for computer programming, we shall not go into to much detail here and refer the interested reader to Appendix $B.2.$ of Kou [28].

We can now compute the probability $\tilde{\mathbf{Q}}\big(\ln S_T > \ln K\big)$ in much a similar manner as done above. Recall that $\tilde{\mathbf{Q}}$ is the measure defined by

$$\frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}\mathbf{Q}}\bigg|_{\mathcal{F}_T^X} = \frac{S_T}{e^{rT}S_0}. \tag{7.37}$$

In order to compute the probability we must first examine how the measure $\tilde{\mathbf{Q}}$ has affected the dynamics of the risky-asset S_t once the change of measure is implemented. Starting with the Brownian motion, we have by the Girsanov theorem (Theorem 2.11) the process $\{\hat{W}_t, 0\leqslant t\leqslant T\}$ defined by $\hat{W}_t = \tilde{W}_t - \sigma t$ that is standard Brownian motion under the measure $\tilde{\mathbf{Q}}$. Hence the risky asset has the following representation under the measure $\tilde{\mathbf{Q}}$

$$S_t = S_0 \exp\left((r + \frac{1}{2}\sigma^2 - \lambda\delta)t + \sigma \hat{W}_t + \sum_{i=1}^{N_t} Y_i\right). \tag{7.38}$$

The process $\{N_t, 0 \leqslant t \leqslant T\}$ is still a Poisson process but the with intensity parameter $\hat{\lambda} = \lambda \mathbf{E}^{\mathbf{Q}}[e^{Y_i}] = \lambda(\delta + 1)$. To see this we make use of Lemma 2.2 to determine the distribution of N_t under the measure $\tilde{\mathbf{Q}}$ as such

$$\mathbf{E}^{\tilde{\mathbf{Q}}}\left[e^{N_t}\middle|\mathcal{F}_t\right] = \frac{\mathbf{E}^{\mathbf{Q}}\left[e^{N_t}\frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}\mathbf{Q}}\middle|\mathcal{F}_t\right]}{\mathbf{E}^{\mathbf{Q}}\left[\frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}\mathbf{Q}}\middle|\mathcal{F}_t\right]}. \tag{7.39}$$

By making use of the independence between $\tilde{W}_t, N_t$ and the Y_i under the measure $\mathbf{Q}$, Eq.(7.39) then reduces to

$$\begin{aligned}
\mathbf{E}^{\tilde{\mathbf{Q}}}\left[e^{N_t}\middle|\mathcal{F}_t\right] &= \exp\left(\lambda \mathbf{E}^{\mathbf{Q}}[e^{Y_i}](T-t)(e-1)\right) \\
&= \exp\left(\hat{\lambda}(T-t)(e-1)\right).
\end{aligned} \tag{7.40}$$

To determine the distribution of the jumps under the measure $\tilde{\mathbf{Q}}$, a similar application of Lemma 2.2 to the Y_i yields that the jump sizes are still exponentially distributed but the with the following parameters

$$\hat{p} = \frac{p}{1+\delta}\cdot\frac{\alpha_1}{\alpha_1 - 1}, \qquad \hat{q} = \frac{q}{1+\delta}\cdot\frac{\alpha_2}{\alpha_2+1}, \qquad \hat{\alpha_1} = \alpha_1 - 1, \qquad \hat{\alpha_2} = \alpha_2 + 1. \tag{7.41}$$

The computation of the probability $\tilde{\mathbf{Q}}\big(\ln S_T > \ln K\big)$ then proceeds in exactly the same manner as our computation of $\mathbf{Q}\big(\ln S_T > \ln K\big)$ involving the Hh function, the only difference being that the parameters are adjusted under the measure $\tilde{\mathbf{Q}}$. If we set

$$\mathbf{Q}\big(\ln S_T > \ln K\big) = \Upsilon(r, \sigma, \lambda, p, q, \alpha_1, \alpha_2, T), \tag{7.42}$$

then

$$\tilde{\mathbf{Q}}\big(\ln S_T > \ln K\big) = \Upsilon(r, \sigma, \hat{\lambda}, \hat{p}, \hat{q}, \hat{\alpha_1}, \hat{\alpha_2}, T). \tag{7.43}$$

Thus the value of the European call option at time $t = 0$ is given by

$$\Gamma_0 = S_0 \Upsilon(r, \sigma, \hat{\lambda}, \hat{p}, \hat{q}, \hat{\alpha_1}, \hat{\alpha_2}, T) - K e^{-rT} \Upsilon(r, \sigma, \lambda, p, q, \alpha_1, \alpha_2, T), \tag{7.44}$$

such a pricing formula is similar in form to the famous Black-Scholes and Merton call option price. As usual the the European put option can be deduced with the aid of the put-call-parity.

7.2 Perpetual options

We shall now consider the simplest American type option known as the *perpetual American put*. Being American in nature the holder of such an option can choose at which point in time to exercise his right to sell the underlying asset at a predetermined strike price. The perpetual nature of this option then affords the holder the additional benefit of choosing exactly when the option is exercised regardless of any time constraint. As such the holder of the perpetual put option is not subjected to trading in a confined time interval $[0, T]$ where $T < \infty$ but can elect to exercise the option at any time with no expiration date on the option. Since time is no longer a factor in deciding when the option should be exercised we expect that the exercising option will depend only upon the price of the risky asset. The holder of the perpetual put must then decide when exactly to exercise the option based on the price of the risky asset immediately dropping below some level. The idea of "immediately" exercising the option at some future date leads us naturally to the concept of stopping times.

Definition 7.1. A stopping time τ is a random variable taking values in $[0, \infty]$ and satisfying

$$\{\tau \leqslant t\} \in \mathcal{F}_t \qquad \forall t \geqslant 0. \tag{7.45}$$

An interpretation of this definition is that the random variable τ can only depend on all information the is available at the current time but may not depend on any future information. With this definition of a stopping time we can define the price of the perpetual American put as follows

Definition 7.2. Let $\mathcal{T}$ be the set of all stopping times. The price of the perpetual American put is defined to be

$$v(S_0) = \sup_{\tau \in \mathcal{T}} \mathbf{E}^{\mathbf{Q}}\big[e^{-r\tau}\big(K - S_\tau\big)\big], \tag{7.46}$$

where S_0 is the initial stock price. In the event that $\tau = \infty$, i.e. the option is never exercised, we interpret $e^{-r\tau}(K - S_\tau)$ to be zero.

Note that the above definition already assumes that the economy of tradable assets is arbitrage free as we take the measure $\mathbf{Q}$ to be some equivalent martingale measure. From Definition 7.2 it is clear that the holder can only exercise the option based on current information and any future information is considered to be irrelevant with regards to the decision of exercising. As stated in the beginning of this chapter, the difficulty with pricing exotic and path dependent options based on jump-diffusion models resides in the "overshoot" problem. In attempt to highlight this problem we shall first consider the a jump-diffusion model that has only positive jumps. The dynamics for the tradable assets in the economy remain exactly the same as given by Eq.(7.1) and Eq.(7.2), however the restriction that the jumps be only positive now implies that the Lévy measure is of the from $\nu(dy) = \lambda f_{Y^+}(y)dy$ and the probability distribution of the jump sizes is given by

$$f_{Y^+}(y) = \begin{cases} \alpha_1 e^{-\alpha_1 y} & y \geqslant 0 \\ 0 & \text{elsewhere,} \end{cases} \tag{7.47}$$

where $\alpha_1 > 1$ as before. We now wish to price an American perpetual put option based on the dynamics proposed by Kou with the modification that the jumps can only be positive, in other words we assume that the risky asset can only jump upwards. Since we are considering a put option the holder will only exercise his right to sell the risky asset when the stock price drops below a certain level, say L. Therefore our stopping time rule is given by

$$\tau_L = \inf\{t \geqslant 0; S_t = L\} \qquad L < K, \tag{7.48}$$

note that at present we do not know the value of L and take it to be an arbitrary number less than the strike price K. Eq.(7.48) is commonly referred to as the first passage of time of a jump-diffusion process to a flat boundary L. In order to price the American perpetual put it is essential that we compute the following expectation

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-r\tau_L}\right] \qquad r \geqslant 0. \tag{7.49}$$

Eq.(7.49) is known as the Laplace transform of the first passage of time for the jump-diffusion process. Before we proceed with computing the Laplace transform of the first passage of time we require a few preliminary results. Consider the process $\{H_t, 0 \leqslant t \leqslant T\}$ defined by

$$H_t = \beta t + \sigma \tilde{W}_t + \sum_{i=1}^{N_t} Y_i^+, \tag{7.50}$$

where $\beta, \sigma > 0$. The moment generating function of H_t, for $x < \alpha_1$, is obtained as

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left[e^{xH_t}\right] \\
&= \exp\left(x\beta t + \frac{1}{2}x^2\sigma^2 t + \lambda t\left[\frac{\alpha_1}{\alpha_1 - x} - 1\right]\right) \\
&= \exp\big(G(x)t\big),
\end{aligned}$$

where $G(x)$ is defined as

$$G(x) = \beta x + \frac{1}{2}x^2\sigma^2 + \lambda\left[\frac{\alpha_1}{\alpha_1 - x} - 1\right]. \tag{7.51}$$

Lemma 7.1. *The equation*

$$G(x) = z \qquad \textit{for all } z > 0$$

has exactly three roots: $\rho_{1,z},\ \rho_{2,z},\ -\rho_{3,z}$*, where*

$$0 < \rho_{1,z} < \alpha_1 < \rho_{2,z} < \infty \qquad 0 < \rho_{3,z} < \infty.$$

Proof: Consider the equation $G(x) - z = 0$, where we know $z > 0$ and $G(x)$ is continuous for all $x \neq \alpha_1$. Now both $G'(x) > 0$ and $G''(x) > 0$ for all $x \in (0, \alpha_1)$, hence $G(x)$ is convex on $(0, \alpha_1)$. Since $G(0) = 0$ and $G(\alpha_1^-) = +\infty$, there is one solution to the equation $G(x) - z = 0$ on the interval $(0, \alpha_1)$ which we denote by $\rho_{1,z}$. On the interval (α_1, ∞) we have $G(\alpha_1^+) = -\infty$ and $G(+\infty) = +\infty$, therefore there exists at least one solution to the equation $G(x) - z = 0$ on (α_1, ∞). Similarly on the interval $(-\infty, 0)$ we have $G(-\infty) = +\infty$ and $G(0) = 0$, therefore the exists at least one solution to the equation $G(x) - z = 0$ on $(-\infty, 0)$. But $G(x)$ is a polynomial of degree three and hence can have at most three real roots. We therefore conclude that there is only one root on the interval (α_1, ∞), denoted by $\rho_{2,z}$, and one root on the interval $(-\infty, 0)$, denoted by $-\rho_{3,z}$ ■

We need one final result before determining the Laplace transform of the first passage of time for the jump-diffusion process, namely the optional stopping theorem for martingales.

Theorem 7.1. *If the process $\{Z_t, t \geqslant 0\}$ is a martingale under some measure* $\mathbf{Q}$ *and τ is a stopping time, then the stopped process $\{Z_{t\wedge\tau}, t \geqslant 0\}$ is also a martingale under the measure* $\mathbf{Q}$. *Moreover,*

$$\mathbf{E}^{\mathbf{Q}}\big[Z_{t\wedge\tau}\big] = \mathbf{E}^{\mathbf{Q}}\big[Z_0\big] \tag{7.52}$$

Proof: See Elliott [14] ■

We are now in a position to determine the Laplace transform of the first passage of time for a jump-diffusion process. Consider the process $\{\mathcal{Z}_t, t \geqslant 0\}$ defined by

$$\mathcal{Z}_t = \exp\big(uH_t - tG(u)\big) \qquad u < -\rho_{3,0}, \tag{7.53}$$

where H_t and $G(\cdot)$ are defined by Eq.(7.50) and Eq.(7.51) respectively. The restriction that $u < -\rho_{3,0}$ will then ensure that $G(u)$ is strictly positive with $-\rho_{3,0}$ being a solution to the equation $G(u) = 0$ as defined in Lemma 7.1. Let us assume that all processes are defined with respect to the measure $\mathbf{Q}$, which is taken to be some EMM. A straight forward calculation reveals that the process $\{\mathcal{Z}_t, t \geqslant 0\}$ is a martingale under the measure $\mathbf{Q}$. Define the stopping time

$$\tau_m = \inf\{t \geqslant 0; H_t = m\} \qquad m < 0, \tag{7.54}$$

by the optional stopping theorem (Theorem 7.1), the stopped process $\{\mathcal{Z}_{t\wedge\tau_m}, t \geqslant 0\}$ is still a martingale under the measure $\mathbf{Q}$. We therefore have that

$$\mathbf{E}^{\mathbf{Q}}\Big[e^{uH_{t\wedge\tau_m} - (t\wedge\tau_m)G(u)}\Big] = \mathbf{E}^{\mathbf{Q}}\big[\mathcal{Z}_0\big] = e^{uH_0 - 0G(u)} = 1. \tag{7.55}$$

The process $\{\mathcal{Z}_{t\wedge\tau_m}, t \geqslant 0\}$ can be written as

$$\mathcal{Z}_{t\wedge\tau_m} = e^{uH_{\tau_m} - \tau_m G(u)}\mathbf{1}_{(\tau_m < t)} + e^{uH_t - tG(u)}\mathbf{1}_{(\tau_m > t)},$$

with this expression for $\mathcal{Z}_{t\wedge\tau_m}$ and making use of the fact that $H_{\tau_m} = m$, Eq.(7.55) can be written as

$$\mathbf{E}^{\mathbf{Q}}\Big[e^{uH_{\tau_m} - \tau_m G(u)}\mathbf{1}_{(\tau_m < t)}\Big] + \mathbf{E}^{\mathbf{Q}}\Big[e^{uH_t - tG(u)}\mathbf{1}_{(\tau_m > t)}\Big] = 1. \tag{7.56}$$

Consider the random variables $e^{um - \tau_m G(u)}\mathbf{1}_{(\tau_m < t)}$. These nonnegative random variables increase with t and the limit is given by $e^{um - \tau_m G(u)}\mathbf{1}_{(\tau_m < \infty)}$, so we have that

$$0 \leqslant e^{um - \tau_m G(u)}\mathbf{1}_{(\tau_m < 1)} \leqslant e^{um - \tau_m G(u)}\mathbf{1}_{(\tau_m < 2)} \leqslant \dots \quad \mathbf{Q}\text{-a.s.},$$

and

$$\lim_{t\to\infty} e^{um-\tau_m G(u)}\mathbf{1}_{(\tau_m<t)} = e^{um-\tau_m G(u)}\mathbf{1}_{(\tau_m<\infty)} \qquad \mathbf{Q}\text{-a.s.}$$

We can now apply the Monotone convergence theorem (See Weir [45]), which allows us to interchange limit and expectation operator to obtain

$$\lim_{t\to\infty} \mathbf{E}^{\mathbf{Q}}\Big[e^{um-\tau_m G(u)}\mathbf{1}_{(\tau_m<t)}\Big] = \mathbf{E}^{\mathbf{Q}}\big[e^{um-\tau_m G(u)}\mathbf{1}_{(\tau_m<\infty)}\big]. \tag{7.57}$$

Now consider the random variable $e^{uH_t-tG(u)}\mathbf{1}_{(\tau_m>t)}$. Since the event set $(\tau_m > t)$ implies that $H_t > m$, we will therefore have $uH_t < um$ as we have assumed that $u < -\rho_{3,0}$. Hence the random variable satisfies

$$0 \leqslant e^{uH_t-tG(u)}\mathbf{1}_{(\tau_m>t)} \leqslant e^{um-tG(u)},$$

taking the limit as t tends to infinity we have

$$\lim_{t\to\infty} e^{uH_t-tG(u)}\mathbf{1}_{(\tau_m>t)} \leqslant \lim_{t\to\infty} e^{um-tG(u)} = 0. \tag{7.58}$$

Eq.(7.58) follows from the fact that $G(u)$ is strictly positive on the interval $(-\infty, -\rho_{3,0})$ and we have assumed that $u < -\rho_{3,0}$. We can now invoke the Lebesgue Dominated Theorem of Convergence to conclude that

$$\lim_{t\to\infty} \mathbf{E}^{\mathbf{Q}}\Big[e^{uH_t-tG(u)}\mathbf{1}_{(\tau_m>t)}\Big] = \mathbf{E}^{\mathbf{Q}}\Big[\lim_{t\to\infty} e^{uH_t-tG(u)}\mathbf{1}_{(\tau_m>t)}\Big] = 0. \tag{7.59}$$

Hence we can conclude that Eq.(7.56) reduces to

$$\mathbf{E}^{\mathbf{Q}}\big[e^{um-\tau_m G(u)}\mathbf{1}_{(\tau_m<\infty)}\big] = 1, \tag{7.60}$$

or equivalently

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-\tau_m G(u)}\mathbf{1}_{(\tau_m<\infty)}\big] = e^{-um}.$$

We are almost done, if we set $G(u) = z$ where $z > 0$ and recall that $u < -\rho_{3,0}$ we will then have

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-z\tau_m}\mathbf{1}_{(\tau_m<\infty)}\big] = e^{m\rho_{3,z}}, \tag{7.61}$$

where $-\rho_{3,z}$ is the only negative solution to the equation $G(u) = z$. We can now drop the indicator function in Eq.(7.61) as we have assumed that $e^{-z\tau_m} = 0$ if $\tau_m = \infty$ **Q**-a.s. This then completes the derivation of the Laplace transform for the first passage of time of a jump-diffusion process with positive jumps.

We are now in a position to price an American perpetual put option. Recall that we have used the dynamics of Kou [28] for the risky asset with the modification that the jumps experienced by the stock price are strictly positive and have an exponential distribution. The risky asset can be written in the form

$$\begin{aligned} S_t &= S_0 \exp\Bigg(\Big(r - \frac{1}{2}\sigma^2 - \lambda\Big[\frac{\alpha_1}{\alpha_1 - 1} - 1\Big]\Big)t + \sigma\tilde{W}_t + \sum_{i=1}^{N_t} Y_i^+\Bigg) \\ &= S_0 e^{\tilde{X}_t}, \end{aligned} \tag{7.62}$$

where the process $\{\tilde{X}_t, 0 \leqslant t \leqslant T\}$ is defined by

$$\tilde{X}_t = \Big(r - \frac{1}{2}\sigma^2 - \lambda\Big[\frac{\alpha_1}{\alpha_1 - 1} - 1\Big]\Big)t + \sigma\tilde{W}_t + \sum_{i=1}^{N_t} Y_i^+. \tag{7.63}$$

Note the similarity between H_t as defined in Eq.(7.50) and $\tilde{X}_t$ defined in Eq.(7.63). We can therefore apply results based on the jump-diffusion process relating to $\{H_t, 0 \leqslant t \leqslant T\}$ to the process $\{\tilde{X}_t, 0 \leqslant t \leqslant T\}$ with only minor modifications, one such constraint being that $r + \lambda > \frac{\sigma^2}{2} + \frac{\lambda\alpha_1}{\alpha_1 - 1}$. Returning to the pricing of the perpetual put option, the option in question is exercised according to a stopping time rule given by $\tau_L = \inf\{t \geqslant 0; S_t = L\}$. Let us assume that the current price of the asset S_0 is greater than the arbitrary level L. If the current stock price is at or below the level L then the holder will exercise the option immediately. In such a case we will have that the value of the perpetual American put is given by $v(S_0) = K - S_0$. If the current stock price S_0 is above the level L exercising the option will then take place at the stopping time τ_L. At the time of exercise the put pays $K - S_{\tau_L} = K - L$. The expression

$$\mathbf{E}^{\mathbf{Q}}\Big[e^{-r\tau_L}(K - S_{\tau_L})\Big] \qquad L < K,$$

therefore reduces to

$$\mathbf{E}^{\mathbf{Q}}\Big[e^{-r\tau_L}(K - L)\Big] = (K - L)\mathbf{E}^{\mathbf{Q}}\big[e^{-r\tau_L}\big] \qquad \forall S_0 \geqslant L. \tag{7.64}$$

Eq.(7.64) is then computed be determining the Laplace transform of the random variable τ_L. Note that we must have $r > 0$ and we interpret $e^{-r\tau_L}$ to be zero if $\tau_L = \infty$ **Q**-a.s. Since $S_t = S_0 e^{\tilde{X}_t}$, the stopping rule $S_t = L$ can be written as $\tilde{X}_t = \ln\frac{L}{S_0}$ and as we have assumed that $S_0 > L$ we will have $\ln\frac{L}{S_0} < 0$. We can now make use of the results concerning the Laplace transform for jump-diffusion processes. The Laplace transform of the random variable τ_L is given by

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-r\tau_L}\big] = \exp\left(\rho_{3,r}\ \ln\frac{L}{S_0}\right), \tag{7.65}$$

this follows from Eq.(7.61) with $z = r$ and $m = \ln\frac{L}{S_0}$. We can now conclude that the America perpetual put option is given by the following function

$$v(S_0) = \begin{cases} K - S, & 0 \leqslant S < L, \\ (K - L)\Big(\frac{L}{S}\Big)^{\rho_{3,r}}, & S \geqslant L. \end{cases} \tag{7.66}$$

We now need only to determine the level L at which the American put should be exercised. This done by simply differentiating the function $(K - L)\Big(\frac{L}{S}\Big)^{\rho_{3,r}}$ with respect to L and setting the result equal to zero. As such the value L maximizes the value of the put option. Let

$$g(L) = (K - L)\left(\frac{L}{S}\right)^{\rho_{3,r}},$$

we then have that

$$\frac{\partial g}{\partial L} = -\left(\frac{L}{S}\right)^{\rho_{3,r}} + \rho_{3,r}(K-L)\left(\frac{L}{S}\right)^{\rho_{3,r}-1}\frac{1}{S}. \tag{7.67}$$

Setting thus derivative equal to zero then yields the value for L as

$$L = K\left(\frac{\rho_{3,r}}{1+\rho_{3,r}}\right), \tag{7.68}$$

which is a number between 0 and K. A little more arithmetic reveals that the value of L given in Eq.(7.68) is indeed a maximum for the function $g(L)$, we shall denote this value of L as $L_\star$. We can now conclude that the value of the American put option is given by

$$v(S_0) = \begin{cases} K - S, & 0 \leqslant S < L_\star, \\ (K - L_\star)\left(\frac{L_\star}{S}\right)^{\rho_{3,r}}, & S \geqslant L_\star, \end{cases} \tag{7.69}$$

with $L_\star$ given by

$$L_\star = K\left(\frac{\rho_{3,r}}{1+\rho_{3,r}}\right).$$

Having gone through the computation of the American perpetual put option where the dynamics of the risky asset allows for only positive jumps, we are now in a better position to fully appreciate the complexity of pricing exotic and path-dependent options where the risky asset incorporates both positive and negative jumps. In our derivation of the perpetual put option pricing formula we modified the model presented by Kou [28] to only include positive exponentially distributed jumps. With these modified dynamics we considered the stopping time τ_m where $m < 0$, so our flat boundary was in the opposite direction to that of the jumps. As such we could always guarantee that when the jump-diffusion process touched the boundary m, the value of the jump-diffusion process was indeed equal to m. Using previously defined notation, we always had $H_{\tau_m} = m$ $\mathbf{Q}$-a.s. If we were to define the flat boundary m to be positive or allow for negative jumps we would no longer be able to claim that $H_{\tau_m} = m$ $\mathbf{Q}$-a.s. The reason for this if the jumps are pointed in the direction of the flat boundary then it is possible for the jump-diffusion process to *overshoot* to boundary in question. The overshoot problem then has the following consequence

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-r\tau_L}(K - S_{\tau_L})\right] \neq (K-L)\mathbf{E}^{\mathbf{Q}}\left[e^{-r\tau_L}\right]. \tag{7.70}$$

What we now have is that

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-r\tau_L}(K - S_{\tau_L})\right] = K\mathbf{E}^{\mathbf{Q}}\left[e^{-r\tau_m}\right] - \mathbf{E}^{\mathbf{Q}}\left[S_{\tau_L}e^{-r\tau_L}\right], \tag{7.71}$$

and it is the computation of the second term in Eq(7.71) that poses immense difficulty. Chesney and Jeanblanc [10] give a nice exposition of the overshoot problem as well as computational methods for Eq.(7.71). However the model the consider only allows for jumps with non random magnitudes, i.e. the size of the jumps are fixed. The idea of using martingale methods for the pricing of perpetual options was considered by Gerber and Shui [20] and provides an excellent insight into the use of optimal stopping theorem. Kou and Wang derive a soultion for the American perpetual put option making use of the result found in Mordecki [39]. The use of exponentially distributed random variables as a model for the jumps experienced by the risky asset is the key point in determining the first passage of time for the jump diffusion-process considered by Kou and Wang [29]. The problem of pricing perpetual options using ordinary differential equations is considered by Aase [1] and yields similar results to that of Chesney and Jeanblanc [10], see also Gapeev [19]. We refer the interested reader to the mentioned authors above for more detail on the subject.

7.3 Lookback options

We now return to the original model proposed by Kou and Wang where the risky asset has the following representation under some equivalent martingale measure $\mathbf{Q}$

$$S_t = S_0 \exp\left(\sigma \tilde{W}_t + (r - \frac{1}{2}\sigma^2 - \lambda\delta)t\right) \prod_{i=1}^{N_t} e^{Y_i}, \tag{7.72}$$

where $\delta = \mathbf{E}^{\mathbf{Q}}\left[e^{Y_i}\right] - 1$ and the Y_i are double asymmetrical exponential random variables as defined in Eq.(7.3). The exotic option that we shall examine in this section is the lookback put option. This contingent claim has a payoff function that is based on the underlying asset reaching either a maximum or minimum value over some time interval prior to expiration of the option. As such we are now restricted to trading in a finite time horizon $[0, T]$ where $T < \infty$. In addition the ability to exercise the option can only take place on the maturity date of the contract, i.e. at time T. A lookback call option has the payoff function given by

$$\max\left\{S_T - \min_{0\leqslant t\leqslant T} S_t, 0\right\} = S_T - \min_{0\leqslant t\leqslant T} S_t, \tag{7.73}$$

notice that since $S_T \geqslant \min_{0\leqslant t\leqslant T} S_t$ $\mathbf{Q}$-a.s. the option will always be exercised, we therefore expect that the lookback option to be more expensive than the European call option. The term “option” is misleading in the case of lookback options since such contingent claims are always exercised on maturity of the contract. The corresponding lookback put option is given by

$$\max\left\{\max_{0\leqslant t\leqslant T} S_t - S_T, 0\right\} = \max_{0\leqslant t\leqslant T} S_t - S_T. \tag{7.74}$$

There are slight variations on the payoff structure for lookback options, for example one could also define the payoff for lookback put option as follows

$$\max\left\{M, \max_{0\leqslant t\leqslant T} S_t\right\} - S_T, \tag{7.75}$$

where M is a fixed constant representing the prefixed maximum at the inception date of the contract, note that $M \geqslant S_0$. For our purposes it is the lookback put option that we shall focus on and we shall assume that the pay-off of such a contingent claim is given by Eq.(7.75). Having already verified that the dynamics for the tradable assets under the Kou-Wang model preclude any arbitrage opportunity we know that there exists at least one EMM, in fact since the economy is incomplete there exist infinitely many EMM's. Let us assume that we have already selected one such EMM **Q** from the pool of available EMM's. The justification of selecting one EMM from the infinitely many available equivalent martingale measures can be based on the use of the Föllmer-Schweizer techniques, minimum relative entropy arguments, utility functions or calibration methods. Once the EMM **Q** is fixed an arbitrage free price process $\{V_t, 0 \leqslant t \leqslant T\}$ for the claim $V_T = \max\left\{M, \max_{0\leqslant t\leqslant T} S_t\right\} - S_T$ may be given by

$$V_t = \mathbf{E}^{\mathbf{Q}}\big[e^{-r(T-t)}V_T \big| \mathcal{F}_t^X\big].$$

Focusing on the time point $t = 0$, the arbitrage free price for the lookback put option at inception dated is then given by

$$\begin{aligned} V_0 &= \mathbf{E}^{\mathbf{Q}}\Big[e^{-rT}\big(\max\{M, \max_{0\leqslant t\leqslant T} S_t\} - S_T\big)\Big] \\ &= \mathbf{E}^{\mathbf{Q}}\Big[e^{-rT}\ \max\Big\{M, \max_{0\leqslant t\leqslant T} S_t\Big\}\Big] - \mathbf{E}^{\mathbf{Q}}\big[e^{-rT}S_T\big] \\ &= \mathbf{E}^{\mathbf{Q}}\Big[e^{-rT}\ \max\Big\{M, \max_{0\leqslant t\leqslant T} S_t\Big\}\Big] - S_0, \end{aligned} \tag{7.76}$$

the last line in Eq.(7.76) follows from the fact that the deflated stock price has martingale property under the measure **Q**. To ease notation we shall write

$$\max_{0\leqslant t\leqslant T} S_t = S_0 \exp\big(M_X(T)\big), \tag{7.77}$$

where $M_X(T)$ is given by

$$M_X(T) = \max_{0\leqslant t\leqslant T}\left(\sigma\tilde{W}_t + (r - \frac{1}{2}\sigma^2 - \lambda\delta)t + \sum_{i=1}^{N_t} Y_i\right). \tag{7.78}$$

In order to determine the value of the lookback put option we need to evaluate the following expectation

$$\mathbf{E}^{\mathbf{Q}}\Big[e^{-rT}\ \max\Big\{M, S_0 e^{M_X(T)}\Big\}\Big]. \tag{7.79}$$

We can write Eq.(7.79) as

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}\left(\max\left\{S_0e^{M_X(T)}-M,0\right\}+M\right)\right]$$
$$=\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}\ \max\left\{S_0e^{M_X(T)}-M,0\right\}\right]+Me^{-rT}$$
$$=\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}\big(S_0e^{M_X(T)}-M\big)\mathbf{1}_{\left(S_0e^{M_X(T)}>M\right)}\right]+Me^{-rT}$$
$$=\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}S_0e^{M_X(T)}\mathbf{1}_{\left(M_X(T)>\ln\frac{M}{S_0}\right)}\right] \tag{7.80}$$
$$-Me^{-rT}\mathbf{Q}\left(M_X(T)>\ln\frac{M}{S_0}\right)+Me^{-rT}. \tag{7.81}$$

Consider Eq.(7.80), we have that

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}S_0e^{M_X(T)}\mathbf{1}_{\left(M_X(T)>\ln\frac{M}{S_0}\right)}\right]=e^{-rT}S_0\int_a^\infty e^z\ d\mathbf{Q}\big(M_X(T)\leqslant z\big), \tag{7.82}$$

where $a=\ln\frac{M}{S_0}$. By noting that $d\mathbf{Q}\big(M_X(T)\leqslant z\big)=-d\mathbf{Q}\big(M_X(T)\geqslant z\big)$ and using integration by parts the integral expression in Eq.(7.82) reduces to

$$-S_0e^{-rT}\left[e^z\mathbf{Q}\big(M_X(T)\geqslant z\big)\Big|_a^\infty-\int_a^\infty e^z\mathbf{Q}\big(M_X(T)\geqslant z\big)\ dz\right]. \tag{7.83}$$

Kou and Wang show that

$$\lim_{z\to\infty}e^z\mathbf{Q}\big(M_X(T)\geqslant z\big)=0,\qquad\forall\, T\geqslant 0,$$

see Lemma 4 [30]. Eq.(7.83) therefore becomes

$$S_0e^{-rT}\left[\frac{M}{S_0}\mathbf{Q}\left(M_X(T)\geqslant\ln\frac{M}{S_0}\right)+\int_a^\infty e^z\mathbf{Q}\big(M_X(T)\geqslant z\big)\ dz\right]. \tag{7.84}$$

Combining this with Eq.(7.81) we can now write the value of the lookback put option at time $t=0$ as

$$\begin{aligned}V_0 &= S_0e^{-rT}\int_a^\infty e^z\mathbf{Q}\big(M_X(T)\geqslant z\big)\ dz+Me^{-rT}\mathbf{Q}\left(M_X(T)\geqslant\ln\frac{M}{S_0}\right)-S_0\\ &\quad -Me^{-rT}\mathbf{Q}\left(M_X(T)\geqslant\ln\frac{M}{S_0}\right)+Me^{-rT}\\ &= S_0e^{-rT}\int_a^\infty e^z\mathbf{Q}\big(M_X(T)\geqslant z\big)\ dz+Me^{-rT}-S_0. \end{aligned} \tag{7.85}$$

The method proposed by Kou and Wang for the valuation of the lookback put option involves taking the Laplace transform of Eq.(7.85). Once we have the Laplace transform of the lookback put option we can the invert it by making use of numerical techniques such as the Gaver-Stehfest algorithm [2]. We shall not go into detail about the inversion process but focus primarily on finding the

Laplace transform for the lookback put option. Denoting the Laplace transform of the lookback put option by $\mathcal{L}[V_0]$, we have that for all $s > 0$

$$\begin{aligned}
\mathcal{L}[V_0] &= \int_0^\infty e^{-sT} S_0 e^{-rT} \int_a^\infty e^z \mathbf{Q}\big(M_X(T) \geqslant z\big)\, dz\, dT \\
&\quad + M \int_0^\infty e^{-(r+s)T}\, dT - S_0 \int_0^\infty e^{-sT}\, dT \\
&= S_0 \int_a^\infty e^z \int_0^\infty e^{-(s+r)T} \mathbf{Q}\big(M_X(T) \geqslant z\big)\, dT\, dz + \frac{M}{r+s} - \frac{S_0}{s}. \quad (7.86)
\end{aligned}$$

Turning our attention to the integral expression

$$\int_0^\infty e^{-(s+r)T} \mathbf{Q}\big(M_X(T) \geqslant z\big)\, dT,$$

since $\mathbf{Q}\big(M_X(T) \geqslant z\big) = \mathbf{Q}\big(\tau_z \leqslant T\big)$, where τ_z is the first passage of time for a jump-diffusion process to a flat barrier z. If we employ the integration by parts formula we will have that

$$\begin{aligned}
&\int_0^\infty e^{-(s+r)T} \mathbf{Q}\big(M_X(T) \geqslant z\big)\, dT \\
&= \int_0^\infty e^{-(s+r)T} \mathbf{Q}\big(\tau_z \leqslant T\big)\, dT \\
&= -\frac{e^{-(s+r)T} \mathbf{Q}\big(\tau_z \leqslant T\big)}{s+r}\bigg|_0^\infty + \frac{1}{s+r} \int_0^\infty e^{-(s+r)T}\, d\mathbf{Q}\big(\tau_z \leqslant T\big) \\
&= \frac{1}{s+r} \mathbf{E}^{\mathbf{Q}}\big[e^{-(s+r)\tau_z}\big].
\end{aligned}$$

Hence we have that the Laplace transform of the lookback put option at $t = 0$ is given by

$$\mathcal{L}[V_0] = \frac{S_0}{s+r} \int_a^\infty e^z \mathbf{E}^{\mathbf{Q}}\big[e^{-(s+r)\tau_z}\big]\, dz + \frac{M}{s+r} - \frac{S_0}{s}. \quad (7.87)$$

In order to compute the Laplace transform for the lookback put option we need to evaluate the quantity

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-(s+r)\tau_z}\big],$$

which is the Laplace transform of the the first passage of time of a jump-diffusion process. The Laplace transform for the first passage of time with respect to a jump-diffusion process is essential if we are to compute the Laplace transform for the lookback put option, hence we see the importance that the first passage of time plays in pricing exotic and path-dependent options. It is the ability to determine the Laplace transform for the first passage of time in the Kou and Wang jump-diffusion model that holds much appeal as it leads to elegant pricing formulas for exotic options, which at present, is not the case for other Lévy models. An application of the Laplace transform for the first passage of time under the Kou-Wang jump-diffusion model yields the following result

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-(r+s)\tau_z}\big] = \frac{\alpha_1 - \eta_{1,s+r}}{\alpha_1}\frac{\eta_{2,s+r}}{\eta_{2,s+r}-\eta_{1,s+r}}e^{-z\eta_{1,s+r}} + \frac{\eta_{2,s+r}-\alpha_1}{\alpha_1}\frac{\eta_{2,s+r}}{\eta_{1,s+r}-\eta_{1,s+r}}e^{-z\eta_{2,s+r}}, \tag{7.88}$$

where $\eta_{1,\cdot}$ and $\eta_{2,\cdot}$ are the two positive solutions to the equation

$$(r - \frac{1}{2}\sigma^2 - \lambda\delta)x + \frac{1}{2}\sigma^2 x^2 + \lambda\left(\frac{p\alpha_1}{\alpha_1 - 1} + \frac{q\alpha_2}{\alpha_2+1} - 1\right) = z \qquad z \geqslant 0.$$

Both α_1 and α_2 are the parameters of the double asymmetrical exponentially distributed random variables defined in Eq.(7.3) that govern the magnitude of the jumps experienced by the risky asset. For a detailed account of the Laplace transform of the first passage of time for the jump-diffusion model proposed by Kou and Wang, see [29], such a result depends heavily on the jump sizes being exponential random variables.

We can now compute the Laplace transform for the lookback put option by first evaluating the following integral

$$\begin{aligned}
&\int_a^{\infty} e^z \mathbf{E}^{\mathbf{Q}}\big[e^{-(s+r)\tau_z}\big]\,dz \\
&= \frac{\alpha_1 - \eta_{1,s+r}}{\alpha_1}\frac{\eta_{2,s+r}}{\eta_{2,s+r}-\eta_{1,s+r}}\int_a^{\infty} e^z e^{-z\eta_{1,s+r}}\,dz \\
&+ \frac{\eta_{2,s+r}-\alpha_1}{\alpha_1}\frac{\eta_{2,s+r}}{\eta_{2,s+r}-\eta_{1,s+r}}\int_a^{\infty} e^z e^{-z\eta_{2,s+r}}\,dz \\
&= \frac{\eta_{2,s+r}(\alpha_1 - \eta_{1,s+r})}{\alpha_1(\eta_{2,s+r}-\eta_{1,s+r})}\frac{e^{-a(\eta_{1,s+r}-1)}}{\eta_{1,s+r}-1} + \frac{\eta_{2,s+r}(\eta_{2,s+r}-\alpha_1)}{\alpha_1(\eta_{2,s+r}-\eta_{1,s+r})}\frac{e^{-a(\eta_{2,s+r}-1)}}{\eta_{2,s+r}-1}.
\end{aligned}$$

With this expression for the we can now give the Laplace transform of the lookback put option as

$$\begin{aligned}
\mathcal{L}[V_0] \;=\; & \frac{S_0}{s+r}\left(\frac{\eta_{2,s+r}(\alpha_1 - \eta_{1,s+r})}{\alpha_1(\eta_{2,s+r}-\eta_{1,s+r})}\frac{e^{-a(\eta_{1,s+r}-1)}}{\eta_{1,s+r}-1}\right. \\
&\left.+\frac{\eta_{2,s+r}(\eta_{2,s+r}-\alpha_1)}{\alpha_1(\eta_{2,s+r}-\eta_{1,s+r})}\frac{e^{-a(\eta_{2,s+r}-1)}}{\eta_{2,s+r}-1}\right) + \frac{M}{s+r} - \frac{S_0}{s}.
\end{aligned} \tag{7.89}$$

To obtain the value of the lookback option one can then make use of numerical inversion techniques which is can easily be done with the aid of many computer packages. From our perspective it is deriving the Laplace transform of the lookback option that serves as our focal point of interest as such a result would not have been possible if we were to consider another stock price model different from the exponential jump-diffusion model.

7.4 Barrier options

Barrier options have a similar payoff structure to that of the vanilla call and put options. Barrier options are a type of contingent claim where the payoff function is based on the underlying asset either passing above or below a certain level known as the barrier. The term *knock out* is used to indicate that option becomes worthless when the underlying asset crosses a certain level. An "up-and-out" option is therefore a contingent claim where the payoff is exactly the same as a vanilla option *provided* that the underlying stays below a certain level, the moment the underlying crosses the level the option payoff is zero. A "down-and-out" option has the barrier below the initial underlying, as soon as the the underlying falls below this barrier the option then becomes worthless. In a similar manner we also have *knock in* options, where the payoff function of the option is considered to be zero unless the underlying passes a certain level. The "up-and-in" option requires that the underlying cross above a certain level in order to activate the payoff function of the option while the "down-and-in" option requires that the underlying falls below some level to activate the payoff function. There are more complex barrier options that require the underlying to spend a certain amount of time either above or below a certain level in order to activate the payoff function. Our interests shall be in pricing a up-and-in call option where the barrier is at flat fixed level, say H. There are also variations as to the types of barriers used, in our case we have chosen to work with a flat barrier H but it is also possible to have the barrier as a function of time in which case the barrier would be changing throughout the life span of the option.

Let $H > S_0$ be a flat barrier, then the payoff function for the up-and-in call option is given by

$$U_T = \max\left\{S_T - K, 0\right\}\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H\right)}$$

As can be seen the $\mathcal{F}_T$-measurable random variable U_T has a similar payoff to that of the vanilla call option with strike price K. We shall assume that the option is European in nature, i.e. exercise can only occur at the expiry date T. As such all economic activity is restricted to the finite time interval $[0, T]$. An arbitrage free price process $\{U_t, 0 \leqslant t \leqslant T\}$ may be given by

$$U_t = \mathbf{E}^{\mathbf{Q}}\left[e^{-r(T-t)}\max\left\{S_T - K, 0\right\}\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H\right)}\middle|\mathcal{F}_t^X\right], \tag{7.90}$$

where $\mathbf{Q}$ is some equivalent martingale measure, we shall not dwell too much on how the measure $\mathbf{Q}$ was chosen as the addition of jumps to any stock price model results in market incompleteness. Rather we assume that some EMM $\mathbf{Q}$ is chosen on the basis that involves either using the results of Föllmer and Schweizer, minimal relative entropy arguments or other methods of justification. Having verified that the dynamics of the Kou and Wang model are indeed arbitrage free we know that there exits at least one EMM. Let us consider the price of the up-and-in call at $t = 0$, we therefore have that

$$\begin{aligned}
U_0 &= \mathbf{E}^{\mathbf{Q}}\left[e^{-rT}\max\left\{S_T - K, 0\right\}\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H\right)}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[e^{-rT}\left\{S_T - K, 0\right\}\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[e^{-rT}S_T\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right] - Ke^{-rT}\mathbf{E}^{\mathbf{Q}}\left[\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right] \\
&= \mathbf{E}^{\mathbf{Q}}\left[e^{-rT}S_T\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right] - Ke^{-rT}\mathbf{Q}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right).
\end{aligned} \tag{7.91}$$

As can be seen from Eq.(7.91) we shall require the joint distribution of the random variables $\max\limits_{0\leqslant t\leqslant T} S_t$ and S_T. Turning our attention to the expectation in Eq.(7.91), in order to reduce computational difficulty we shall employ a change of numeraire technique. In order to calculate the value of the up-and-in call option we must compute the following expectation

$$\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}S_T\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right]. \tag{7.92}$$

We now implement a change in measure using the following R-N derivative

$$\left.\frac{\mathrm{d}\tilde{\mathbf{Q}}}{\mathrm{d}\mathbf{Q}}\right|_{\mathcal{F}_T^X} = \frac{S_T}{e^{rT}S_0}. \tag{7.93}$$

Note that the new measure $\tilde{\mathbf{Q}}$ is a well defined probability measure since we have that

$$\mathbf{E}^{\mathbf{Q}}\left[\frac{S_T}{e^{rT}S_0}\right] = 1,$$

and the deflated risk asset is a strictly positive process under the measure $\mathbf{Q}$. As such, with the aid of Lemma 2.2 we can now write Eq.(7.92) as

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left[e^{-rT}S_T\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right] \\
&= \mathbf{E}^{\tilde{\mathbf{Q}}}\left[e^{-rT}S_T\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\tilde{\mathbf{Q}}}\right] \\
&= \mathbf{E}^{\tilde{\mathbf{Q}}}\left[e^{-rT}S_T\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\frac{e^{rT}S_0}{S_T}\right] \\
&= S_0\mathbf{E}^{\tilde{\mathbf{Q}}}\left[\mathbf{1}_{\left(\max\limits_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right)}\right] \\
&= S_0\tilde{\mathbf{Q}}\left(\max_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\right).
\end{aligned}$$

Hence we can now write the value of the up-and-in call option at $t = 0$ as

$$U_0 = S_0\tilde{\mathbf{Q}}\Big(\max_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\Big) - Ke^{-rT}\mathbf{Q}\Big(\max_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\Big). \quad (7.94)$$

Again note the similarity in the pricing formula between the up-and-in call option and that of the plain vanilla call option, the only difference been that in the case of the up-and-in call option the probability that we must now compute involves two random variables in the form of a joint distribution. Turning our attention firstly to the calculation of the probability

$$\mathbf{Q}\Big(\max_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\Big). \quad (7.95)$$

As before we shall write

$$S_0 \exp(M_X(T)) = \max_{0\leqslant t\leqslant T} S_t,$$

where $M_X(T)$ is defined as follows

$$M_X(T) = \max_{0\leqslant t\leqslant T}\Big(\sigma\tilde{W}_t + (r - \frac{1}{2}\sigma^2 - \lambda\delta)t + \sum_{i=1}^{N_t} Y_i\Big).$$

In order to work directly with the jump-diffusion process we shall also write S_T as follows

$$S_T = S_0 \exp(X_T),$$

where X_T is defined as

$$X_T = \sigma\tilde{W}_T + (r - \frac{1}{2}\sigma^2 - \lambda\delta)T + \sum_{i=1}^{N_T} Y_i.$$

With these slight alterations the probability in Eq.(7.95) can be rewritten as

$$\mathbf{Q}\big(M_X(T) \geqslant b, X_T \geqslant c\big),$$

where $b = \ln\frac{H}{S_0}$ and $c = \ln\frac{K}{S_0}$. Note that $b > 0$ as we have assumed from the outset that the barrier H was greater than the initial stock price S_0. Due the overshoot problem calculating this probability explicitly can prove difficult if not impossible to determine, the approach we shall take involves finding the Laplace transform of this probability. The justification behind this being that for this particular jump-diffusion model it was possible to derive a closed form solution for the first passage to a flat boundary. We now take the Laplace transform of this probability as follows

$$\int_0^\infty e^{-sT}\mathbf{Q}\big(M_X(T) \geqslant b, X_T \geqslant c\big)\, dT. \quad (7.96)$$

At present it seems easier to deal with finding the Laplace transform as in Eq.(7.96) as opposed to explicitly determining the joint distribution. Since the event set $\{M_X(T) \geqslant b\}$ is equivalent to the event set $\{\tau_b \leqslant T\}$, where $\tau_b = \inf\{t \geqslant 0; X_t \geqslant b\}$ the Laplace transform in Eq.(7.96) can be written as

$$\int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c\big)\; dT. \tag{7.97}$$

The advantage of working with the Laplace transform as in Eq.(7.97) is that we already have an explicit solution for the Laplace transform of the first passage of time, this will be of significant importance.

We now give a brief description of how the Laplace transform in Eq.(7.97) is evaluated,

$$\begin{aligned}
&\int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c\big)\; dT \\
&= \int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c, X_{\tau_b} = b\big)\; dT \qquad (7.98)\\
&+ \int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c, X_{\tau_b} \geqslant b\big)\; dT. \qquad (7.99)
\end{aligned}$$

Starting with Eq.(7.98)

$$\begin{aligned}
&\int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c, X_{\tau_b} = b\big)\; dT \\
&= \int_0^\infty e^{-sT}\int_0^T \mathbf{Q}\big(\tau_b \in dz, X_T \geqslant c, X_{\tau_b} = b\big)\; dT \\
&= \int_0^\infty \int_0^T e^{-sT}\mathbf{Q}\big(\tau_b \in dz, X_{\tau_b} = b\big)\mathbf{Q}\big(X_{T-z} \geqslant c-b\big)\; dT \\
&= \int_0^\infty e^{-sz}\mathbf{Q}\big(\tau_b \in dz, X_{\tau_b} = b\big)\int_0^\infty e^{-su}\mathbf{Q}\big(X_u \geqslant c-b\big)\; du \\
&= \mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}=b)}\big]\int_0^\infty e^{-su}\mathbf{Q}\big(X_u \geqslant c-b\big)\; du, \qquad (7.100)
\end{aligned}$$

we have used the strong Markov property in the third equality, the fourth equality follows from the fact that the Laplace transform of a convolution is the product of the individual Laplace transforms. The Laplace transform of expression (7.99) follows in an similar manner, by noting that

$$\mathbf{Q}\big(\tau_b \in dz, X_T \geqslant b, X_{\tau_b} \geqslant b\big) = \mathbf{Q}\big(\tau_b \in dz, X_{\tau_b} \geqslant b\big)\mathbf{Q}\big(X_T - z + Y_i^+ \geqslant c-b\big),$$

where Y^+ is a exponentially distributed random variable defined in Proposition 7.1. From an intuitive point of view if we have $X_{\tau_b} = b$ then we would have no overshoot and thus no jumps would have occurred when the barrier b was touched, however if $X_{\tau_b} \geqslant b$ then the jump-diffusion process may have overshot the barrier b and as we have assumed that $b \geqslant 0$ the only jumps that are positive valued are the Y_i^+. Expression (7.99) therefore reduces to

$$\int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c, X_{\tau_b} \geqslant b\big)\,dT$$
$$= \mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}\geqslant b)}\big]\int_0^\infty e^{-su}\mathbf{Q}\big(X_u + Y_i^+ \geqslant c-b\big)\,du.$$

We can thus conclude that

$$\int_0^\infty e^{-sT}\mathbf{Q}\big(\tau_b \leqslant T, X_T \geqslant c\big)\,dT$$
$$= \mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}=b)}\big]\int_0^\infty e^{-su}\mathbf{Q}\big(X_u \geqslant c-b\big)\,du$$
$$+ \mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}\geqslant b)}\big]\int_0^\infty e^{-su}\mathbf{Q}\big(X_u + Y_i^+ \geqslant c-b\big)\,du. \tag{7.101}$$

Hence we have an expression for the Laplace transform for the probability in Eq.(7.95). It is worth noting that the integral expressions

$$\int_0^\infty e^{-su}\mathbf{Q}\big(X_u \geqslant c-b\big)\,du, \qquad \int_0^\infty e^{-su}\mathbf{Q}\big(X_u + Y^+ \geqslant c-b\big)\,du,$$

are evaluated in terms of the Hh function by means of a recursive relationship. One can then employ various numerical methods for inverting the Laplace transform and finding the actual probability defined in Eq.(7.95). For our purposes we are more interested in finding the Laplace transform than the actual inversion which can be done with the aid of certain computer programmes.

Finally the computing of the probability

$$\tilde{\mathbf{Q}}\bigg(\max_{0\leqslant t\leqslant T} S_t \geqslant H, S_T > K\bigg),$$

is exactly the same as we have done above with only certain parameters being adjusted. As we have used the same change of measure formula when computing the value of the European call option, the adjusted parameters under the measure $\tilde{\mathbf{Q}}$ are given in Eq.(7.41). Once the Laplace transforms of the probabilities have been inverted we will then be able to determine the value U_0 of the up-and-in barrier option.

For the sake of completeness we now give the value of the following expectations that were involved in the computation of the barrier option.

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}=b)}\big] \qquad \mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}\geqslant b)}\big].$$

We have that

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}=b)}\big] = \frac{\alpha_1 - \eta_{1,s}}{\eta_{2,s} - \eta 1, s}e^{-b\eta_{1,s}} + \frac{\eta_{2,s} - \alpha_1}{\eta_{2,s} - \eta 1, s}e^{-b\eta_{2,s}}, \tag{7.102}$$

and that

$$\mathbf{E}^{\mathbf{Q}}\big[e^{-s\tau_b}\mathbf{1}_{(X_{\tau_b}\geqslant b)}\big] = \frac{(\alpha_1-\eta_{1,s})(\eta_{2,s}-\eta_{1,s})}{\alpha_1(\eta_{2,s}-\eta_{1,s})}\big[e^{-b\eta_{1,s}}-e^{-b\eta_{2,s}}\big], \tag{7.103}$$

where $\eta_{1,s}$ and $\eta_{2,s}$ are the positive solutions to the equation

$$(r-\frac{1}{2}\sigma^2-\lambda\delta)x+\frac{1}{2}\sigma^2x^2+\lambda\left(\frac{p\alpha_1}{\alpha_1-1}+\frac{q\alpha_2}{\alpha_2+1}-1\right)=s \qquad s\geqslant 0. \tag{7.104}$$

Chapter 8

Conclusions

Contingent claims or derivatives are used extensively in today's financial markets across the world. In particular derivatives such as options attract a significant amount of attention from academics and practitioners, both from a valuation and implementation point of view.

The option pricing theory introduced by Black & Scholes [8], and later expanded by Merton [36], still serves as a benchmark by which other option pricing models are judged. While the beauty of the Black & Scholes model lies in its tractability there are several drawbacks in the model that have been exposed over time, the inability of the model it incorporate sudden jumps into the dynamics of the underlying asset being an example. These sudden jumps experienced by tradable assets are known to exist in financial markets and are normally triggered by the release of information. This suggests that when modeling the underling asset a jump process should be included to account for random shocks in the market.

As a more realistic description of the underlying asset includes a jump process, we have considered the use of Lévy processes as the driving source of uncertainty when specifying the dynamics of the underlying asset as Lévy processes contain random jumps in addition to the Brownian motion. The geometric Lévy model we considered showed that once an additional source of randomness was introduced the market becomes incomplete and as a result we could no longer perfectly hedge a variety of contingent claims, options been one of them. This market incompleteness essentially meant the the equivalent martingale measure Q was no longer unique, as was the case in the Black & Scholes model. To combat the difficulty of pricing in an incomplete market the Esscher transform [21] was introduced as it has minimal relative entropy. As an alternative to the Esscher transform the Föllmer-Schweizer [15] minimal martingale measure was also considered. As yet there seems to be no definitive method to option pricing in an incomplete market.

Having introduced the Lévy process and examined its implications as far as the valuation process of contingent claims is concerned we then looked at jump-diffusion models, a subclass of Lévy processes. Jump-diffusion models retain an amount of tractability when compared with other models such as the Variance Gamma model of Madan and Seneta [34]. The Merton model extends the Black & Scholes model to include lognormal jumps. In addition to the vanilla calls and puts we independently derived a closed form solution to the value of a European exchange option with the aid of the Esscher transform.

As far as exotic and path dependent options are concerned, the introduction of a Lévy process makes it almost impossible to determine any closed form solutions for options such as lookback and barrier options. This is due to the overshoot problem when attempting to determine the first passage of time of a Lévy process to a flat boundary. The jump-diffusion model proposed by Kou & Wang [30] provides closed form solutions to a number of exotic options since the distribution of the first passage of time is known for this particular jump-diffusion model. The fact that the distribution of the first passage of time is known stems form the fact that the jumps are exponentially. At present there seems to be no other model for which the first passage of time can be computed, even the Merton model.

The Kou & Wang model also exhibits the leptokurtic feature evident in industry, where the tails of the asset distribution are more pronounced. Having looked at the lookback and barrier option, where the solutions were derived in terms of Laplace transforms we then modified the Kou & Wang model to experience only positive jumps and considered an American perpetual put option. As long as the jumps experienced by the asset are in the opposite direction to that the barrier we will have a elegant solution to the price of the perpetual put option.

Appendix A

Black's Formula

Theorem A.1. *If the conditional distribution of* V *is lognormal, that is* $\ln V|\mathcal{F}_t \sim N(m, s^2)$*, then*

$$\mathbf{E}\big[\max(V-K,0)\big|\mathcal{F}_t\big] = \mathbf{E}\big(V\big|\mathcal{F}_t\big)N(d_1) - KN(d_2), \tag{A.1}$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{\mathbf{E}\left(V\middle|\mathcal{F}_t\right)}{K} + \frac{1}{2}s^2}{s} \\ d_2 &= \frac{\ln \frac{\mathbf{E}\left(V\middle|\mathcal{F}_t\right)}{K} - \frac{1}{2}s^2}{s} \\ &= d_1 - s \end{aligned} \tag{A.2}$$

and $N(x)$ *is given by the standard normal cumulative distribution*

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2}\,dy. \tag{A.3}$$

Proof:

$$\begin{aligned} \mathbf{E}\big[\max(V-K,0)\big|\mathcal{F}_t\big] &= \frac{1}{\sqrt{2\pi s^2}} \int_0^{\infty} \max(v-K,0)\,\frac{1}{v}\,e^{-\frac{1}{2}\left(\frac{\ln v - m}{s}\right)^2}\,dv \\ &= \frac{1}{\sqrt{2\pi s^2}} \int_K^{\infty} (v-K)\frac{1}{v} e^{-\frac{1}{2}\left(\frac{\ln v - m}{s}\right)^2}\,dv, \end{aligned} \tag{A.4}$$

now set $y = \frac{\ln v - m}{s}$ in Eq.(A.4) this implies that $v = e^{m+ys}$ and that $dy = \frac{dv}{vs}$. The bounds of integration will also change to $y|_{\frac{\ln K - m}{s}}^{\infty}$, we therefore have

$$\begin{aligned} &\mathbf{E}\big[\max(V-K,0)\big|\mathcal{F}_t\big] \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln K - m}{s}}^{\infty} (e^{m+ys} - K) e^{-\frac{1}{2}y^2}\,dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln K - m}{s}}^{\infty} e^{m+ys} e^{-\frac{1}{2}y^2}\,dy - K\frac{1}{\sqrt{2\pi}} \int_{\frac{\ln K - m}{s}}^{\infty} e^{-\frac{1}{2}y^2}\,dy \end{aligned} \tag{A.5}$$

Now the first integral in Eq.(A.5) can be simplified by completing the square in the exponent as follows

$$\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K-m}{s}}^{\infty} e^{m+ys}e^{-\frac{1}{2}y^2}\,dy = \frac{1}{\sqrt{2\pi}}e^{m+\frac{1}{2}s^2}\int_{\frac{\ln K-m}{s}}^{\infty} e^{-\frac{1}{2}(y-s)^2}\,dy, \tag{A.6}$$

another change of variable of the form $x = y-s$, with integration bounds changing to $x|_{\frac{\ln K-m-s^2}{s}}^{\infty}$ allows us to write

$$\begin{aligned}
&\mathbf{E}\big[\max(V-K,0)\big|\mathcal{F}_t\big] \\
&= \frac{1}{\sqrt{2\pi}}e^{m+\frac{1}{2}s^2}\int_{\frac{\ln K-m-s^2}{s}}^{\infty} e^{-\frac{1}{2}x^2}\,dx - K\frac{1}{\sqrt{2\pi}}\int_{\frac{\ln K-m}{s}}^{\infty} e^{-\frac{1}{2}y^2}\,dy \\
&= e^{m+\frac{1}{2}s^2}\mathbf{P}\Big(Z > \frac{\ln K-m-s^2}{s}\Big) - K\mathbf{P}\Big(Z > \frac{\ln K-m}{s}\Big)
\end{aligned}$$

where $Z \sim N(0,1)$ is a standard normal random variable. Making use if the fact that if $V|\mathcal{F}_t$ is lognormally distributed then $\mathbf{E}(V|\mathcal{F}_t) = e^{m+\frac{1}{2}s^2}$ and the symmetric properties of the normal distribution we finally have

$$\begin{aligned}
&\mathbf{E}\big[\max(V-K,0)\big|\mathcal{F}_t\big] \\
=\ &\mathbf{E}(V|\mathcal{F}_t)\mathbf{P}\Big(Z < \frac{\ln\frac{\mathbf{E}(V|\mathcal{F}_t)}{K}+\frac{1}{2}s^2}{s}\Big) - K\mathbf{P}\Big(Z < \frac{\ln\frac{\mathbf{E}(V|\mathcal{F}_t)}{K}-\frac{1}{2}s^2}{s}\Big) \\
=\ &\mathbf{E}(V|\mathcal{F}_t)N(d_1) - KN(d_2).
\end{aligned} \tag{A.7}$$

A.1 Correlated Brownian motions

Theorem A.2. *If the process $\{W_{1,t}, t \geqslant 0\}$ and the process $\{W_{2,t}, t \geqslant 0\}$ are correlated Brownian motions under the measure $\mathbf{P}$ with correlation given by $\big[W_{1,t}, W_{2,t}\big]_t = \rho t$ the the process $\{B_t, t \geqslant 0\}$ defined by*

$$B_t = \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\big(\sigma_1 W_{1,t} - \sigma_2 W_{2,t}\big), \tag{A.8}$$

is standard Brownian motion under the measure $\mathbf{P}$ with σ_1 and σ_2 are any positive constants.

Proof: We need to verify the axioms of Definition 1.4. The first condition is simple enough as we have that

$$B_0 = \frac{1}{\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}}\big(\sigma_1 W_{1,0} - \sigma_2 W_{2,0}\big) = 0, \tag{A.9}$$

as the both $W_{1,0}$ and $W_{2,0}$ is equal to zero at time zero. In order to verify the second and third condition of Definition 1.4 we must show that $\{B_t, t \geqslant 0\}$ has independent and normally distributed increments. To do this we evaluate the following conditional expectation

$$\mathbf{E}^{\mathbf{P}}\big[e^{i(B_t-B_s)}\big|\mathcal{F}_t\big] \qquad s < t. \tag{A.10}$$

Now

$$B_t - B_s = \frac{1}{\sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}}\big[\sigma_1(W_{1,t} - W_{1,s}) - \sigma_2(W_{2,t} - W_{2,s})\big], \qquad \text{(A.11)}$$

as $\{B_t, t \geqslant 0\}$ is a function of the correlated Brownian motions, which have normally distributed increments, B_t will also be a normal random variable with mean and given by

$$\mathbf{E}^{\mathbf{P}}[B_t - B_s|\mathcal{F}_t] = 0$$

and variance given by

$$\begin{aligned}
\mathbf{Var}^{\mathbf{P}}(B_t - B_s|\mathcal{F}_t) &= \frac{1}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}\Big\{\sigma_1^2\mathbf{Var}^{\mathbf{P}}(W_{1,t} - W_{1,s}|\mathcal{F}_t) \\
&\quad + \sigma_2^2\mathbf{Var}^{\mathbf{P}}(W_{2,t} - W_{2,s}|\mathcal{F}_t) \\
&\quad - 2\rho\sigma_1\sigma_2\mathbf{Cov}^{\mathbf{P}}(W_{1,t} - W_{1,s}, W_{2,t} - W_{2,s}|\mathcal{F}_t)\Big\} \\
&= \frac{\big(\sigma_1^2(t-s) + \sigma_2^2(t-s) - 2\rho\sigma_1\sigma_2(t-s)\big)}{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \qquad \text{(A.12)} \\
&= (t-s).
\end{aligned}$$

Eq.(A.12) follows from the fact that both $\{W_{1,t}, t \geqslant 0\}$ and $\{W_{2,t}, t \geqslant 0\}$ have independent increments and are thus independent of the filtration $\mathcal{F}_t$. Hence we have that

$$\mathbf{E}^{\mathbf{P}}\big[e^{i(B_t - B_s)}\big|\mathcal{F}_t\big] = e^{-\frac{1}{2}(t-s)}, \qquad \text{(A.13)}$$

which we recognize as the characteristic function of a normal random variable with mean equal to zero and variance equal to $t - s$. So the process $\{B_t, t \geqslant 0\}$ is indeed standard Brownian motion under the measure $\mathbf{P}$.

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