FILTER CHARACTERISATIONS
OF THE
EXTENDIBILITY OF CONTINUOUS FUNCTIONS

BY

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PREFACE

The work culminating in the present dissertation was carried out in the Department of Mathematics, University of Natal, Pietermaritzburg, from January 1990 to January 1991, under the supervision of Professor J.Swart.

Except where specifically indicated to the contrary, these studies represent the original work of the author. Due acknowledgement has been given whenever use has been made of the work of others. No part of this work has been submitted in any form to another University.

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ABSTRACT

The concepts of z-, C-, and C^*-embedding are "global" in the sense that they are concerned with *every* zero-set/continuous (and bounded) real-valued function on a subspace. Each of these embeddings can be "localised": z-embedding of a subspace to z-embedding of a particular continuous real-valued function on the subspace ; C-embedding to the extendibility of a particular continuous real-valued function on the subspace ; and C^*-embedding to the extendibility of a particular bounded continuous real-valued function on the subspace. The aim of this dissertation is to obtain characterisations of these global embeddings, and to localise them to obtain characterisations of the corresponding local embeddings.

The results fall into two streams: the first uses classical concepts to characterise global embeddings, and these are localised in classical terms; the second uses various types of filters to characterise the global embeddings, and the localisations are cast in filter-theoretic terms. In both streams any characterisation of a localised version of a global embedding immediately yields the original global characterisation, and furthermore increases our understanding of the global characterisation.

Chapter 1 introduces most of the terminology that will be needed in subsequent chapters, and sketches necessary background.

Chapter 2 is dedicated to completely regular filters , which are used in the characterisations obtained in chapters 3 and 4 . Of particular importance are the maximal completely regular filters, and their relationship to z-ultrafilters. It is shown that there is a one-one correspondence between the maximal completely regular filters and the z-ultrafilters on a space. A further correspondence, central to later theory, shows that to each maximal completely regular filter $\mathcal{F}$ on a subspace S of X there corresponds a unique maximal completely regular filter $\mathcal{F}^*$ on the parent space that is coarser than $\mathcal{F}$.

Chapter 3 is concerned with characterising z-, C^*-, and C- embedding, with particular emphasis on filter characterisations. In the classical stream it is shown that $S \subseteq X$ is z-embedded in X *iff* every $f \in C(S)$ can be uniformly approximated on S by continuous real-valued functions on cozero-supersets of S ; and that S is C- (C^*-) embedded in X *iff* S is z-embedded in X and the collection of (bounded) continuous real-valued functions on S that extend over X is closed under the formation of (bounded) quotients. The Gillman and Jerison characterisations of C^*- and C- embedding are deduced from these. In the filter-theoretic stream it is shown that $S \subseteq X$ is z-embedded in X *iff* the trace on S of every z-ultrafilter on X that meets S is a z-ultrafilter on S *iff* every completely regular filter on S is z-embedded in X ; that S is C^*-embedded in X *iff* the trace on S of every maximal completely regular filter on X that meets S is maximal completely regular on S ; and that S is C-embedded in X *iff* every z-ultrafilter on S is the trace on S of some z-ultrafilter on X that meets S .

Chapter 4 studies localisations of the results of chapter 3. In the classical stream , the localisation of the z-embedding result shows that if $S \subseteq X$ and $f \in C(S)$ then f is z-embedded in X *iff* f can be uniformly approximated on S by continuous real-valued functions on cozero-supersets of X that contain S ; the localisation of the C^*- (C-) embedding results are very elegant localisations of the Gillman and Jerison characterisations, showing that $f \in C^*(S)$ ($f \in C(S)$) extends over X *iff* disjoint Lebesgue-sets of f are completely separated in X (and S is completely separated from every zero-set of X that is completely separated from f) . In the filter-theoretic stream, the localisation of the z-embedding result of chapter 3 shows that $f \in C(S)$ is z-embedded in X *iff* each member of a particular family of completely regular filters on X, associated with f , is z-embedded in X ; localisation of the C^*- and C- embedding results has to date been only partially successful, and only a necessary filter-theoretic condition for the extendihility of a given $f \in C^*(S)$ is derived.

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CHAPTER 1
Introduction

1.1 SYNOPSIS

We begin with an informal overview of the aims and content of the dissertation. Concepts introduced here will be discussed in greater detail in subsequent sections.

Let us establish just enough notation and make just enough definitions in order to briefly describe the subject matter. Throughout this dissertation S will denote a subspace of a topological space X , and we shall refer to X as the *parent space (of S)* . The reals $\mathbb{R}$ will always have their usual topology. By a *zero-set* of a space X we shall mean a set of the form $f^{-1}(0)$ where f is a continuous real-valued function on X . If f is a continuous real-valued function on the subspace S , then by an *extension of f to X* we mean a continuous real-valued function g on X whose restriction to S , $g \restriction S$, is just f . We shall denote by $C(X)$ (resp. $C^*(X)$) the set of all continuous (resp. continuous and bounded) real-valued functions on a space X .

We say that S is *z-embedded in X* if every zero-set of S is the intersection with S of a zero-set of X . If every continuous (resp. bounded continuous) real-valued function on S extends over the parent space X , then we say that S is *C-embedded* (resp. *C^*-embedded*) in X . These three embeddings are *global* in the sense that they concern themselves with *every* zero-set/continuous (and bounded) real-valued function on a subspace. It is easily seen that z-embedding is necessary for both C- and C^*- embedding.

One of the main aims of the dissertation is to obtain filter-theoretic characterisations of z-, C-, and C^*- embedding of a subspace. In the case of z-embedding this is achieved directly. For C-, and C^*- embedding it is achieved by first obtaining more standard characterisations of extendibility and then characterising these conditions in filter-theoretic terms. This results in the establishment of interesting relationships, in the presence of one of these embeddings, between various types of filter on the subspace with those on the parent space.

The other main objective is to obtain *localisations* of z-, C^*-, and C- embedding, dealing with a single continuous function[1] at a time. Whereas z-embedding is concerned with every zero-set of the subspace (and the collection of all zero-sets is determined by the collection of all continuous real-valued function on the subspace), the localised version of z-embedding is concerned with a particular class of zero-sets associated with any given continuous real-valued function on the subspace. The localised version of z-embedding of S in X is termed the *z-embedding in* X of a single given $f \in C(S)$ (defined later). What makes this a true localisation is that the global embedding is recoverable from the proposed localisation: S is z-embedded in X *iff* every $f \in C(S)$ is z-embedded in X . Hence by finding conditions for the z-embedding of any given function on a subspace we may proceed to conditions under which all functions will be z-embedded, i.e., conditions under which the subspace will be z-embedded. In this sense z-embedding of functions is at a level one deeper than z-embedding of subspaces. The localised version of C-embedding (resp. C^*-embedding) is extendibility of a single given continuous function (resp. bounded continuous function) on the subspace. Again the global condition is recoverable from the local version: S is C-embedded (resp. C^*-embedded) in X *iff* every continuous (resp. bounded continuous) function on the subspace extends over X . Of course these are just the definitions of C-embedding and C^*-embedding.

Localisations of classical results have been obtained by R.L.Blair ([Bl_2]) . We shall seek to localise the filter characterisations of the global embeddings in order to obtain filter characterisations of z-embedding of a function and of extendibility of a single given (possibly bounded) function. For instance, if some global condition holds *iff* all filters on the subspace satisfy a condition $\mathfrak{P}$ then we shall try to identify a class of filters on the subspace, associated with a single given function, such that the localisation of the global condition holds for the given function *iff* the class of filters identified satisfies $\mathfrak{P}$.

Note that an index of terminology is provided at the end of the dissertation.

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[1] Unless otherwise stated, all functions on a space are real-valued

1.2 DEFINITIONS AND NOTATION

We define all the general concepts that will be used in later chapters, and record those properties that will be of use to us. This section is written so as to make the dissertation practically self-contained.

(a) *General notation*

The reals will be denoted by $\mathbb{R}$, and will always have their usual topology. The set of all integers will be denoted by $\mathbb{Z}$, the set of positive integers by $\mathbb{N}$ and the set of non-negative integers by ω . The unit interval $[0,1]$ will always have the usual subspace topology. Throughout we shall consider S to be a subspace of a topological space X , and we shall term X the *parent space* of S . Unless specifically indicated to the contrary, no space shall ever be assumed to satisfy any separation axiom.

Unless otherwise stated, functions are considered to be real-valued. If $f : X \to \mathbb{R}$ and $A \subseteq \mathbb{R}$ then we write $f^{-1}(A)$ for $\{x \in X : f(x) \in A\}$ — we don't use $f^{\leftarrow}(A)$. We define $C(X)$ to be the set of all continuous real-valued functions on a space X , and denote by $C^*(X)$ the subcollection consisting of all bounded continuous real-valued functions on X . If $f \in C(X)$ then we denote by $f \mid S$ the restriction of f to S . The function on X with constant value $c \in \mathbb{R}$ will be written as $\mathbf{c}$, irrespective of the space X . If $f, g \in C(X)$ then we define $f \wedge g$ and $f \vee g$ by $(f \wedge g)(x) = \min\{f(x), g(x)\}$ and $(f \vee g)(x) = \max\{f(x), g(x)\}$; note that $f \wedge g, f \vee g \in C(X)$.

(b) *Zero- and cozero- sets of a topological space*

1.2.1 *Definition :* A *zero-set* of a space X is a set of the form $f^{-1}(0)$ where $f \in C(X)$. We will write $Z(f)$ for the zero-set of f , and $\mathfrak{Z}(X) = \{Z(f) : f \in C(X) \}$. A *cozero-set* of X is the complement of a zero-set; we denote the cozero-set of f by $\operatorname{coz} f$. Note that zero-sets are closed, that cozero-sets are open, and that $\emptyset$ and X are both zero- and cozero- sets.

First let us note that $C(X)$ and $C^*(X)$ determine the same zero-sets, i.e., that $\{Z(f) : f \in C(X)\} = \{Z(f) : f \in C^*(X)\}$: if $f \in C(X)$ then $Z(f) = Z((f \vee -1) \wedge 1)$. As such we may always assume that the function determining a given zero-set is bounded with range $[0,1]$, so $\mathfrak{Z}(X) = \{f^{-1}(0) : f \in C(X) \text{ and } 0 \leq f \leq 1\}$ and the family of cozero-sets is given by $\{f^{-1}(0,1] : f \in C(X) \text{ and } 0 \leq f \leq 1\}$.

Next we note that if $F \subseteq \mathbf{R}$ is closed, then $f^{-1}(F)$ is a zero-set for $f \in C(X)$, as is easily seen by considering the zero-set of the map $x \mapsto \mathrm{d}(f(x), F)$ (the distance of $f(x)$ from F). Dually, if $G \subseteq \mathbf{R}$ is open then $f^{-1}(G)$ is cozero for $f \in C(X)$.

The identity $Z(f) = \bigcap_{n \in \mathbf{N}} \{x \in X : |f(x)| < \frac{1}{n}\}$ shows that every zero-set can be written as a countable intersection of cozero-sets. Dually, every cozero-set can be written as a countable union of zero-sets.

Suppose that $f, g \in C(X)$. Then $Z(f) \cup Z(g) = Z(fg)$ and $Z(f) \cap Z(g) = Z(f^2 + g^2)$, showing that $(\mathfrak{Z}(X), \cap, \cup)$ is a lattice. This lattice has a least element $\emptyset$ and a greatest element X , and we show that it is also closed under countable intersections: Suppose $f_n \in C(X)$ for each $n \in \mathbf{N}$. Define $g_n = |f_n| \wedge 2^{-n}$ and $g = \sum_{n \in \mathbf{N}} g_n$ — since $|g_n| < 2^{-n}$ the series converges uniformly by the Weierstrass M-test. Now $Z(g) = \bigcap_{n \in \mathbf{N}} Z(g_n) = \bigcap_{n \in \mathbf{N}} Z(f_n)$. Dually the collection of cozero-sets is closed under finite intersections and countable unions.

(c) *Completely separated sets*

1.2.2 *Definition* : Subsets A and B of a space X are said to be ***completely separated in*** X if there is an $f \in C(X)$ with $0 \leq f \leq 1$, $f(A) = 0$ and $f(B) = 1$.

In completely separating A and B it clearly suffices to find an $f \in C(X)$ with $f(A) \leq r$ and $f(B) \geq s$ for some numbers r and s with $r < s$. It also suffices to contain A and B (respectively) in completely separated sets. Complete separation of sets is also most conveniently expressed in terms of zero-sets of the space, this being particularly useful in our work:

1.2.3 *Theorem :* ([GJ 1.15]) : *Two sets are completely separated in X iff they are contained in disjoint zero-sets of X .* □

(d) *$\mathfrak{P}$-filters and $\mathfrak{P}$-filterbases*

Recall that if ($\mathfrak{P}$, $\cap$, $\cup$) is a lattice of subsets of a set X , then a *$\mathfrak{P}$-filter* on X (or a *filter in the lattice* $\mathfrak{P}$) is a nonempty collection $\mathfrak{F}$ of nonempty members of $\mathfrak{P}$ such that i) if $P_1, P_2 \in \mathfrak{F}$ then $P_1 \cap P_2 \in \mathfrak{F}$, and ii) if $P_1 \in \mathfrak{F}$ and $P_1 \subseteq P_2 \in \mathfrak{P}$ then $P_2 \in \mathfrak{F}$. A *$\mathfrak{P}$-filterbase* on X is a nonempty collection $\mathfrak{B}$ of members of $\mathfrak{P}$ such that if $C_1, C_2 \in \mathfrak{B}$ then $C_1 \cap C_2 \supseteq C_3$ for some $C_3 \in \mathfrak{B}$, and the $\mathfrak{P}$-filter generated by (based on) $\mathfrak{B}$ is the collection of all elements of $\mathfrak{P}$ that are supersets of elements of $\mathfrak{B}$. If $\mathfrak{P}$ is the power set of X then the $\mathfrak{P}$-filters on X are just the filters on X ; if $\mathfrak{P}$ is the lattice of zero-sets of a space X , then the $\mathfrak{P}$-filters are just the z-filters.

It is clear that subset inclusion will partially order the set of all $\mathfrak{P}$-filters on a set X . Many of our conditions will involve the "coarser than" relation which allows us to compare filters in different lattices (e.g., filters with z-filters) :

1.2.4 *Definition* : If $\mathfrak{F}$ is a $\mathfrak{P}_1$-filterbase and $\mathfrak{G}$ is a $\mathfrak{P}_2$-filterbase then we say that $\mathfrak{G}$ is *finer* than $\mathfrak{F}$ (or that $\mathfrak{F}$ is *coarser* than $\mathfrak{G}$) , and write $\mathfrak{F} \leq \mathfrak{G}$, if every member of $\mathfrak{F}$ contains a member of $\mathfrak{G}$. In the case that $\mathfrak{P}_1 = \mathfrak{P}_2$ we have $\mathfrak{F} \leq \mathfrak{G}$ *iff* $\mathfrak{F} \subseteq \mathfrak{G}$. We say that $\mathfrak{F}$ and $\mathfrak{G}$ are *equivalent* if $\mathfrak{F} \leq \mathfrak{G}$ and $\mathfrak{G} \leq \mathfrak{F}$. Note that $\leq$ will not be used as a partial order of any set, except when it is restricted to a particular class of filters (in which case $\leq$ coincides with $\subseteq$) .

1.2.5 *Definition* : A set A is said to *meet* a $\mathfrak{P}$-filterbase $\mathfrak{B}$ if $A \cap B \neq \emptyset$ for each $B \in \mathfrak{B}$. $\mathfrak{P}$-filterbases $\mathfrak{F}$ and $\mathfrak{G}$ are said to *meet* if each member of $\mathfrak{F}$ meets $\mathfrak{G}$.

Regarding z-filters, we need to recall the following properties:

(i) every z-filter on X is contained in some z-ultrafilter on X ,

(ii) z-ultrafilters are prime ,

(iii) a prime z-filter on X is contained in a unique z-ultrafilter on X , and

(iv) if $\mathcal{U}$ is a z-ultrafilter on X and $U \in \mathfrak{Z}(X)$ then $U \in \mathcal{U}$ *iff* U meets $\mathcal{U}$.

1.2.6 *Definition* : If $\mathcal{F}$ and $\mathcal{G}$ are filterbases on X which meet, then we write $\sup\{\mathcal{F}, \mathcal{G}\} = \{U \subseteq X : U \supseteq F \cap G \text{ for some } F \in \mathcal{F}, G \in \mathcal{G}\}$. This is a filter on X containing (finer than) both $\mathcal{F}$ and $\mathcal{G}$. If $\mathcal{U}$ and $\mathcal{W}$ are z-filters on X which meet then we write $\sup_{\mathfrak{Z}(X)}\{\mathcal{U}, \mathcal{W}\} = \{Z \in \mathfrak{Z}(X) : Z \supseteq U \cap W \text{ for some } U \in \mathcal{U} , W \in \mathcal{W}\}$, a z-filter on X containing (finer than) both $\mathcal{U}$ and $\mathcal{W}$.

(e) *Definition of z-, C^*-, and C- embedding*

We introduce the three concepts that this dissertation is primarily concerned with. Note that the definitions are given in classical terms, i.e., in terms of zero-sets, complete separation etc. In later chapters we shall discover what consequences these embeddings have for various types of filters on the spaces involved.

1.2.7 *Definition* : Let $S \subseteq X$. If $f \in C(S)$ then we say that f *extends* over X if there exists a $g \in C(X)$ with $g \mid S = f$. Note that we do not insist that an extension of a bounded function be bounded, although it can be assumed bounded: if $|f| \leq \boldsymbol{m}$ and $g \in C(X)$ with $g \mid S = f$, then $(g \vee -\boldsymbol{m}) \wedge \boldsymbol{m}$ is a bounded extension of f .

1.2.8 *Definition* : Let $S \subseteq X$. We say that S is *C^*-embedded in* X if every bounded continuous function on S extends continuously over X , i.e., $C^*(S) = \{g \mid S : g \in C^*(X)\}$

The classical characterisation of C^*-embedding is the Urysohn Extension Theorem. We shall not use this in the passage to a filter characterisation of C^*-embedding — indeed we shall recover it from one of our characterisations and so provide an alternative proof.

1.2.9 <u>*Theorem*</u> (Urysohn Extension Theorem [GJ 1.17]) : *$S \subseteq X$ is C^*-embedded in X iff every pair of completely separated sets in S is completely separated in X .* □

Note that we can restate this theorem as follows : *$S \subseteq X$ is C^*-embedded in X iff disjoint zero-sets of S are contained in disjoint zero-sets of X .*

1.2.10 <u>*Definition :*</u> Let $S \subseteq X$. We say that S is *C-embedded in X* if every continuous function on S extends continuous over X , i.e., $C(S) = \{g \mid S : g \in C(X)\}$.

Note that C^*-embedding is obviously necessary for C-embedding. The Gillman and Jerison characterisation of C-embedding is as follows:

1.2.11 <u>*Theorem*</u> ([GJ 1.18]) : *A C^*-embedded subset S of X is C-embedded iff it is completely separated from every zero-set of X disjoint from S .* □

One of the best known applications of the Urysohn Extension Theorem is to couple it with the Urysohn Lemma (that in a normal space disjoint closed sets are completely separated) in order to prove the well known Tietze-Urysohn Theorem characterising C-embedding. This too will be reproved using our characterisations.

1.2.12 <u>*Theorem*</u> (Tietze-Urysohn Extension Theorem) : *If S is a closed subset of a normal space X , then S is C-embedded in X .*

We shall arrive at further characterisations of C^*-, and C- embeddings via characterisations of a particular condition that is necessary for both of these embeddings, and discovering what supplementary conditions will yield C^*- or C- embedding. The necessary condition is that of *z-embedding*:

1.2.13 *Definition* : If $S \subseteq X$ and $A \in \mathcal{Z}(S)$ then we shall say that A *extends* to a zero-set of X if it is the case that $A = Z \cap S$ for some $Z \in \mathcal{Z}(X)$. We shall apply this terminology to cozero-sets too, in the obvious way.

1.2.14 *Definition* (R.L.Blair [Bl_3]) : A subspace S of a space X is said to be *z-embedded in* X if every zero-set of S extends to a zero set of X — i.e., for each $Z \in \mathcal{Z}(S)$ there is a $Z' \in \mathcal{Z}(X)$ with $Z' \cap S = Z$. Equivalently, S is z-embedded in X *iff* every cozero-set of S extends to a cozero-set of X . Since the intersection with S of a zero set of X is always a zero set of S , z-embedding of S in X means that $\mathcal{Z}(S) = \{Z \cap S : Z \in \mathcal{Z}(X)\}$.

It is easily seen that z-embedding of a subspace S of X is a necessary condition for both C^*- and C- embedding of S in X : if $f \in C(S)$ and $f = g \mid S$ with $g \in C(X)$ then $Z(f) = Z(g) \cap S$. For this reason alone it is worth studying z-embedding and investigating what conditions must hold in addition to z-embedding to ensure C^*-, and C- embedding. The concept has found applications in other areas, such as in the investigation of function algebras, z-filters, lattices, and measures.

The following proposition will be very useful, and will often be used without reference.

1.2.15 *Proposition* (R.L.Blair and A.W.Hager [BH_1 1.1]) : *Every cozero-set is z-embedded.*

Proof. Let $g \in C(X)$ and $f \in C^*(\operatorname{coz} g)$, so that $Z(f)$ is a typical element of $\mathcal{Z}(\operatorname{coz} g)$ (recall that $C(Y)$ and $C^*(Y)$ generate the same zero-sets). Define h on X by:

$$h(x) = \begin{cases} f(x)\,g(x) & \text{if } x \in \operatorname{coz} g \\ 0 & \text{if } x \in Z(g) \end{cases} .$$

Let $x \in X$. We claim that h is continuous at x . There are three cases to consider: (i) If $x \in \operatorname{coz} g$ then, since $\operatorname{coz} g$ is open, there is a neighbourhood U of x in X with $U \subseteq \operatorname{coz} g$. Now $h \mid U = (f \mid U)(g \mid U)$, which is continuous on U . So h is continuous at x ; (ii) If $x \in Z(g) - Fr_X(Z(g))$ (where Fr_X denotes the frontier operator on X) , then there is a neighbourhood of x on which g , and hence h , is constant valued at 0 . Hence h is continuous at x ; (iii) If $x \in Z(g) \cap Fr_X(Z(g))$ then, given $\epsilon > 0$, then if $|f(y)| \leq M$ for all $y \in \operatorname{coz} g$, there is a neighbourhood U of x such that $g(U) \subseteq (-\frac{\epsilon}{M}, \frac{\epsilon}{M})$ since $g(x) = 0$. But $h(x) = 0$ and $h(u) \subseteq (-\epsilon, \epsilon)$, showing that h is continuous at x .

We have shown that $h \in C(X)$. It is clear that $Z(h) \cap \operatorname{coz} g = Z(f)$. □

There are a number of instances in later chapters where functions are defined in a manner very similar to h in the preceding proposition. Showing that these functions are continuous is tedious, and we shall simply say "as in proposition 1.2.15 , the function is continuous".

We have established the necessary notation for our work, but before we can begin our study of z-, C^*-, and C- embedding we must introduce completely regular filters. This is the content of the next chapter.

CHAPTER 2

Completely Regular Filters

2.1 INTRODUCTION

Our characterisations of the various embeddings and of extendibility will be mainly given in terms of the relative behaviour of z-filters and of *completely regular* filters on the subspace and on the parent space. The important properties of z-filters have been given in the introductory chapter. In this chapter we introduce tbe concept of completely regular filters and establish their most important properties in the context of extending continuous functions and characterising the various embeddings.

2.2 DEFINITIONS

The concept of a completely regular filter was introduced by P.S.Aleksandrov ([Al]), who used the term "completely regular system" to refer to a particular kind of filtersubbase. The term used here, as well as the application to filters, is due to Bourbaki ([Bou ; chap IV, §1 Ex. 8]). In [Al] completely regular systems are used to establish a characterisation of the Stone-Čech compactification of a Tychonoff space. Completely regular filters have found a number of uses in distinct, though related, fields. In the sequel we shall see their use in characterising the various types of embedding.

2.2.1 *Definition* : A filterbase $\mathcal{F}$ on X is said to be *completely regular* if for each $F \in \mathcal{F}$ there is an $F' \in \mathcal{F}$ such that F' and $X - F$ are completely separated in X . We also define the trivial filter $\{X\}$ on X to be completely regular. A completely regular filter on X is said to be *maximal completely regular* if there exists no strictly larger completely regular filter on X .

It is clear that a filter $\mathcal{F}$ is completely regular if, and only if, some filterbase for $\mathcal{F}$ is completely regular.

2.2.2 *Examples* : (i) The coarsest completely regular filterbase on a space X is $\{X\}$.

(ii) If $f: X \to \mathbb{R}$ is continuous, with $Z(f) \neq \emptyset$ and $f \geq 0$ then the family $\mathfrak{B} = \{f^{-1}[0,r] : r > 0\}$ is a completely regular filterbase on X . This sort of construction will be used in a number of proofs.

(iii) Later in this chapter we shall see that there is a one-one correspondence between the maximal completely regular filters and the z-ultrafilters on a space.

(iv) If τ is a Tychonoff topology on X , and if for each $U \in \tau$ we define $U^* = U \cup \{\mathfrak{F} : \mathfrak{F} \text{ is a free maximal completely regular filter with } U \in \mathfrak{F}\}$, then the family $\mathfrak{B} = \{U^* : U \in \tau\}$ is a base for a topology on X^* with respect to which X^* is homeomorphic to βX , the Stone-Čech compactification of X . This is the characterisation of βX due to P.S.Aleksandrov ([Al]) .

2.2.3 *Remark* : Note that a completely regular filterbase on X is a filterbase in the lattice of *all* subsets of X — there is no "lattice of completely regular sets" in which we are working. Note also that the "coarser than" relation $\leq$ coincides with subset inclusion $\subseteq$ when restricted to the family of all completely regular filters on a space, and hence partially orders this family.

2.2.4 *Definition* : If a subspace S of X meets a $\mathfrak{P}$-filterbase $\mathfrak{F}$ on X then the *trace* of $\mathfrak{F}$ on S, denoted $\mathfrak{F} \mid S$, is the family $\{F \cap S : F \in \mathfrak{F}\}$. It is easily verified that the trace on S of a completely regular filter(base) on X that meets S is a completely regular filter(base) on S , and also that the trace on S of a z-filter on X that meets S is a z-filterbase on X (the trace can fail to be a z-filter on S — closure under zero-supersets is not guaranteed; however, the trace will be a filter if S is z-embedded in X) .

2.2.5 <u>*Proposition*</u> : *If* $\mathcal{F}$ *and* $\mathcal{G}$ *are completely regular filterbases on* X *which meet, then* $\sup\{\mathcal{F},\mathcal{G}\}$ *is completely regular on* X .

Proof. We need only show that the base $\mathcal{B}=\{F\cap G : F\in\mathcal{F}\ ,\ G\in\mathcal{G}\ \}$ for $\sup\{\mathcal{F},\mathcal{G}\}$ is completely regular on X . Let $F\in\mathcal{F}, G\in\mathcal{G}$. There exist $F'\in\mathcal{F}$ and $G'\in\mathcal{G}$ together with functions $f,g\in C^*(X)$ with $0\le f\le 1$,$0\le g\le 1$, $f(F')=0$, $f(X-F)=1$, $g(G')=0$ and $g(X-G)=1$. Then we have $f\vee g\in C^*(X)$ with $0\le f\vee g\le 1$, $(f\vee g)(F'\cap G')=0$ and $(f\vee g)(X-(F\cap G))=(f\vee g)((X-F)\cup(X-G))=1$. Thus $F'\cap G'\in\mathcal{B}$ and $X-(F\cap G)$ are completely separated in X . □

Thus if $\mathcal{F}$ and $\mathcal{G}$ are completely regular filterbases on X which meet, then $\sup\{\mathcal{F},\mathcal{G}\}$ is a completely regular filter on X finer than both $\mathcal{F}$ and $\mathcal{G}$. Note that this means that distinct maximal completely regular filterbases on X cannot meet. Also, if a completely regular filterbase $\mathcal{G}$ on X meets a maximal completely regular filter $\mathcal{F}$ on X then we must have $\mathcal{G}\subseteq\mathcal{F}$.

2.3 PROPERTIES OF MAXIMAL COMPLETELY REGULAR FILTERS

It will be *maximal* completely regular filters that are of most use in our characterisations. It is the easy success of z-ultrafilters in characterising z-, and C- embedding (see theorems 3.2.13 and 3.4.9) and the correspondence that exists between z-ultrafilters and maximal completely regular filters on a space (see later in this section) that point to the potential use of maximal completely regular filters for characterising the various embeddings.

The theorems of this section contain the most important properties of maximal completely regular filters in this context. Following R.L.Blair ($[\mathrm{Bl}_1]$) , we obtain several very useful characterisations of maximality of completely regular filters and establish an all-important correspondence between the maximal completely regular filters and the z-ultrafilters on a space, as well as an equally important link between the maximal completely regular filters on a subspace with those on the parent space.

2.3.1 <u>*Theorem*</u> : *Every completely regular filter is contained in some maximal completely regular filter.*

Proof. Suppose that $\mathfrak{F}$ is a completely regular filter on X , and define $\mathcal{S} = \{\mathfrak{F}' : \mathfrak{F}' \text{ is a completely regular filter on } X \text{ with } \mathfrak{F} \subseteq \mathfrak{F}'\}$. Since $\mathfrak{F} \in \mathcal{S}$, $\mathcal{S} \neq \emptyset$. Partially order $\mathcal{S}$ by inclusion, and let $\mathcal{C}$ be a chain in $\mathcal{S}$. Then $\bigcup \mathcal{C}$ is a filter on X , and we claim it is completely regular : Let $U \in \bigcup \mathcal{C}$, say $U \in \mathfrak{F}' \in \mathcal{C}$. By complete regularity of $\mathfrak{F}'$, there is a $V \in \mathfrak{F}' \subseteq \bigcup \mathcal{C}$ which is completely separated from $X - U$. Hence $\bigcup \mathcal{C}$ is completely regular, and so is an upper bound for $\mathcal{C}$ in $\mathcal{S}$. By Zorn's lemma we conclude that $\mathcal{S}$ has a maximal element, say $\mathcal{G}$. It is clear that $\mathcal{G}$ is a maximal completely regular filter containing $\mathfrak{F}$. □

The following elementary lemma finds use in the next theorem as well as in later localisations of global results.

2.3.2 <u>*Lemma*</u> : *If a filter $\mathfrak{F}$ on a compact space X has exactly one cluster point, then $\mathfrak{F}$ converges (to that point).*

Proof. Suppose $\mathfrak{F}$ clusters at $c \in X$ and nowhere else. Let U be any open neighbourhood of c . For each $x \in X - U$, $\mathfrak{F}$ fails to cluster at x so we may choose an open neighbourhood U_x of x and an $F_x \in \mathfrak{F}$ such that $U_x \cap F_x = \emptyset$. Now $\{U\} \cup \{U_x : x \in X - U\}$ is an open cover of X , so by compactness there is a finite subset A of $X - U$ such that $X - U \subseteq \bigcup_{x \in A} U_x$. Now $(\bigcup_{x \in A} U_x) \cap \bigcap_{x \in A} F_x = \emptyset$, by construction, and so we have $(X - U) \cap \bigcap_{x \in A} F_x = \emptyset$, i.e., $\bigcap_{x \in A} F_x \subseteq U$. Since $\bigcap_{x \in A} F_x \in \mathfrak{F}$ it follows that $U \in \mathfrak{F}$. □

The following characterisations of maximal completely regular filters will be used often in our work, especially (ii) . J.W.Green proved (i) ⇒ (iii) ⇒ (iv) ⇒ (v) ⇒ (i) in his 1973 paper ([Gr_2 ; lemma 3]). The equivalence of (i) and (ii) is stated without proof by R.L.Blair in his 1976 paper ([Bl_1 ; proposition 2.1 (a)]).

2.3.3 <u>Theorem</u>: *If $\mathcal{F}$ is a completely regular filter on X, then the following are equivalent:*

(i) *$\mathcal{F}$ is maximal completely regular on X,*

(ii) *if $Z, Z' \in \mathcal{Z}(X)$ and if Z, Z' meet $\mathcal{F}$, then $Z \cap Z' \neq \emptyset$,*

(iii) *there is only one z-ultrafilter on X finer than $\mathcal{F}$,*

(iv) *there is only one maximal completely regular filter on X finer than $\mathcal{F}$,*

(v) *for every function $\varphi \in C^*(X)$, $\varphi(\mathcal{F})$ converges in the usual topology on $\mathbb{R}$.*

Proof. (i) $\Rightarrow$ (ii) : Suppose $\mathcal{F}$ is a maximal completely regular filter on X, and that $Z, Z' \in \mathcal{Z}(X)$ meet $\mathcal{F}$. Suppose $Z \cap Z' = \emptyset$. Then Z and Z' are completely separated in X, so there is a $\varphi \in C(X)$ with $0 \leq \varphi \leq 1$, $\varphi(Z) = 0$ and $\varphi(Z') = 1$. Define $\mathcal{G} = \sup\left\{ \mathcal{F}, \{\varphi^{-1}[0,e) : e \in (0,1)\} \right\}$. Since Z meets $\mathcal{F}$ and $Z \subseteq \varphi^{-1}[0,e)$ for all $e \in (0,1)$, it follows that $\mathcal{G}$ is a filter on X. Since $\{\varphi^{-1}[0,e) : e \in (0,1)\}$ is a completely regular filterbase, it follows from proposition 2.2.5 that $\mathcal{G}$ is completely regular. Now $\mathcal{G}$ is strictly finer than $\mathcal{F}$, for if $\mathcal{F} = \mathcal{G}$ then $\varphi^{-1}[0,e) \in \mathcal{F}$ for $e \in [0,1)$ but $\varphi^{-1}[0,e) \cap Z' = \emptyset$ and Z' should meet $\mathcal{F}$. We have contradicted the maximality of $\mathcal{F}$, so we conclude that $Z \cap Z' \neq \emptyset$.

(ii) $\Rightarrow$ (iii) : First we show that every completely regular filter on X is coarser than *some* z-ultrafilter on X. Let $\mathcal{G}$ be a completely regular filter on X. For each $G \in \mathcal{G}$ choose a zero set Z_G contained in G as follows : there is a $G' \in \mathcal{G}$ (with $G' \subseteq G$) such that G' and $X - G$ are completely separated in X. Hence G' and $X - G$ can be contained in two disjoint zero-sets of X, showing that there is a zero-set of X contained in G — let Z_G be such a zero set. Define $\mathcal{B} = \{Z_G : G \in \mathcal{G}\}$. Now every element of $\mathcal{B}$ is a superset of an element of $\mathcal{G}$, so $\mathcal{B} \subseteq \mathcal{G}$. Furthermore every member of $\mathcal{G}$ contains a member of $\mathcal{B}$, and so $\mathcal{B}$ is a filterbase for $\mathcal{G}$. Of course $\mathcal{B}$ is also a base for some z-filter on X, which will be contained in some (not necessarily unique) z-ultrafilter $\mathcal{U}$ on X. Each $G \in \mathcal{G}$ contains the element Z_G of $\mathcal{U}$, hence $\mathcal{G}$ is coarser than $\mathcal{U}$.

Now suppose that (ii) holds and let $\mathcal{U}$ and $\mathcal{U}'$ be z-ultrafilters on X that are finer than $\mathcal{F}$. If $\mathcal{U} \neq \mathcal{U}'$ then (by maximality) $\mathcal{U}$ and $\mathcal{U}'$ are incomparable, so we may choose a $Z \in \mathcal{U} - \mathcal{U}'$. Let $Z' \in \mathcal{U}'$. Now Z meets $\mathcal{F}$ (since $\mathcal{F} \leq \mathcal{U}$ and $Z \in \mathcal{U}$) and Z' meets $\mathcal{F}$ (since $\mathcal{F} \leq \mathcal{U}'$ and $Z' \in \mathcal{U}'$) , so by (ii) $Z \cap Z' \neq \emptyset$. So Z meets every member of the z-ultrafilter $\mathcal{U}'$, and therefore $Z \in \mathcal{U}'$. This is a contradiction, so we must have $\mathcal{U} = \mathcal{U}'$.

(iii) $\Rightarrow$ (iv) : By theorem 2.3.1 there is at least one maximal completely regular filter on X finer than $\mathcal{F}$. Suppose $\mathcal{F}$ is contained in two distinct maximal completely regular filters on X , say $\mathcal{G}$ and $\mathcal{G}'$. By maximality, $\mathcal{G}$ and $\mathcal{G}'$ cannot meet — otherwise $\sup\{\mathcal{G}, \mathcal{G}'\}$ is a completely regular filter on X finer than both $\mathcal{G}$ and $\mathcal{G}'$. From the argument of (ii) $\Rightarrow$ (iii) , there exist z-ultrafilters $\mathcal{U}$ and $\mathcal{U}'$ with $\mathcal{G} \leq \mathcal{U}$, $\mathcal{G}' \leq \mathcal{U}'$. We claim that $\mathcal{U} \neq \mathcal{U}'$, which contradicts (iii) : Choose $G \in \mathcal{G}$ and $G' \in \mathcal{G}'$ with $G \cap G' = \emptyset$ and choose $U \in \mathcal{U}$ and $U' \in \mathcal{U}'$ with $G \supseteq U$ and $G' \supseteq U'$; now $U \in \mathcal{U} - \mathcal{U}'$ since $U \cap U' = \emptyset$. We conclude that $\mathcal{G} = \mathcal{G}'$.

(iv) $\Rightarrow$ (v) : Suppose (iv) holds and that $\varphi \in C^*(X)$. Since $\varphi(X) \subseteq \mathbf{R}$ is bounded, $\varphi(\mathcal{F})$ has at least one cluster point in $\mathbf{R}$ ($\varphi(\mathcal{F})$ is a filter on the compact space $\overline{\varphi(X)}$) . Suppose a, b are cluster points of $\varphi(\mathcal{F})$ with $a < b$. For each $e > 0$, $\varphi^{-1}(-\infty, a+e)$ and $\varphi^{-1}(b-e, \infty)$ meet $\mathcal{F}$, thus $\mathcal{G}_1 = \sup\{\ \mathcal{F}, \{\varphi^{-1}(-\infty, a+e) : e > 0\}\ \}$ and $\mathcal{G}_2 = \sup\{\ \mathcal{F}, \{\varphi^{-1}(b-e, \infty) : e > 0\}\ \}$ are filters on X . If we take $e = \frac{b-a}{3}$ then we have $\varphi^{-1}(-\infty, a+e) \in \mathcal{G}_1$ and $\varphi^{-1}(b-e, \infty) \in \mathcal{G}_2$ showing that $\mathcal{G}_1$ and $\mathcal{G}_2$ are incomparable and do not meet. The filterbases $\{\varphi^{-1}(-\infty, a+e) : e > 0\}$ and $\{\varphi^{-1}(b-e, \infty) : e > 0\}$ are clearly completely regular, so that by proposition 2.2.5 $\mathcal{G}_1$ and $\mathcal{G}_2$ are completely regular.

So $\mathcal{G}_1$ and $\mathcal{G}_2$ are distinct completely regular filters strictly finer than $\mathcal{F}$ (neither can equal $\mathcal{F}$ for they are incomparable). Now $\mathcal{G}_1$ and $\mathcal{G}_2$ are contained in maximal completely regular filters on X , and these must be distinct since $\mathcal{G}_1$ and $\mathcal{G}_2$ do not meet. These maximal completely regular filters are also finer than $\mathcal{F}$ — contradicting (iv) .

Thus $\varphi(\mathcal{F})$ has exactly one cluster point. It follows by the preceding lemma that $\varphi(\mathcal{F})$ converges.

(v) $\Rightarrow$ (i) : Suppose (v) holds and $\mathfrak{F}$ is not maximal completely regular. Then there is a completely regular filter $\mathfrak{G}$ strictly finer than $\mathfrak{F}$. Choose $G_2 \in \mathfrak{G} - \mathfrak{F}$. There is a $G_1 \in \mathfrak{G}$ and a $\varphi \in C^*(X)$ with $0 \leq \varphi \leq 1$, $\varphi(G_1) = 0$ and $\varphi(X - G_2) = 1$. Let $e \in (0,1)$. Then $\varphi^{-1}[0,e)$ and $\varphi^{-1}(e,1]$ meet $\mathfrak{F}$, since $F \cap G_1 \neq \emptyset$ and $F \cap (X - G_2) \neq \emptyset$ for each $F \in \mathfrak{F} \subseteq \mathfrak{G}$. Now if $e \in (0,1)$ and $F \in \mathfrak{F}$, F meets both $\varphi^{-1}[0,e)$ and $\varphi^{-1}(1-e,1]$, so $\varphi(F) \cap [0,e) \neq \emptyset \neq \varphi(F) \cap (1-e,1]$. Hence $\varphi(\mathfrak{F})$ clusters at both 0 and 1, a contradiction. □

We note from (ii) $\Rightarrow$ (iii) of the preceding proof that for any completely regular filter on a space we can always choose a base for the filter consisting of zero-sets of the space. In particular, this means that any element of a completely regular filter contains a zero-set element of the filter.

Since every completely regular filter has a base consisting of zero-sets, every completely regular filter contains a base for a z-filter. This suggests the possibility of a relationship between the completely regular filters and the z-filters on a space. The following two theorems show that there is a one-one correspondence between z-ultrafilters on a space and maximal completely regular filters on that space. This correspondence plays a central role in later theory. In view of this correspondence, and of the Gillman and Jerison construction of βX for Tychonoff X [GJ 6.5], the characterisation of βX mentioned in example 2.4 (iv) is not too surprising. This correspondence is due to R.L.Blair. Theorems 2.3.4 and 2.3.5 are propositions 2.1 (b) and 2.1 (c) of [Bl_1], in which the proofs are outlined.

2.3.4 _Theorem_ : *If $\mathfrak{F}$ is a maximal completely regular filter on X, then there is a unique z-ultrafilter $\mathfrak{U}$ on X finer than $\mathfrak{F}$; furthermore, if $Z \in \mathfrak{Z}(X)$ then $Z \in \mathfrak{U}$ iff Z meets $\mathfrak{F}$.*

Proof. The existence and uniqueness of $\mathfrak{U}$ is given by theorem 2.3.3 (iii). Now let $Z \in \mathfrak{Z}(X)$. If $Z \in \mathfrak{U}$ and $F \in \mathfrak{F}$ then $F \supseteq U$ for some $U \in \mathfrak{U}$ and $Z \cap F \supseteq Z \cap U \neq \emptyset$, so Z meets $\mathfrak{F}$. On the other hand, suppose Z meets $\mathfrak{F}$ — we claim that Z must then be in $\mathfrak{U}$. Let $U \in \mathfrak{U}$

so that Z and U are zero-sets of X which meet $\mathcal{F}$ so by theorem 2.3.3 (ii) , $Z \cap U \neq \emptyset$. Since Z meets every member of $\mathcal{U}$, it follows that $Z \in \mathcal{U}$. □

2.3.5 <u>Theorem</u> : *If $\mathcal{U}$ is a z-ultrafilter on X , then there is a unique maximal completely regular filter on X coarser than $\mathcal{U}$.*

Proof. Let $\mathcal{U}$ be a z-ultrafilter on X . Define

$$\mathcal{B} = \{f^{-1}[0,r] : f \in C(X) \text{ , } f \geq 0 \text{ , } r > 0 \text{ , } Z \subseteq Z(f) \text{ for some } Z \in \mathcal{U}\} .$$

Certainly $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$. Let $f_1, f_2 \in C(X)$ with $f_1, f_2 \geq 0$, and suppose $Z_1, Z_2 \in \mathcal{U}$ with $Z_1 \subseteq Z(f_1)$ and $Z_2 \subseteq Z(f_2)$. Let $r_1, r_2 > 0$. Then $f_1 \vee f_2 \in C(X)$, $f_1 \vee f_2 \geq 0$, $Z_1 \cap Z_2 \in \mathcal{U}$ with $Z_1 \cap Z_2 \subseteq Z(f_1 \vee f_2) = Z(f_1) \cap Z(f_2)$ and if we take $r = \min\{r_1, r_2\} > 0$ then $(f_1 \vee f_2)^{-1}[0,r] \subseteq f_1^{-1}[0,r_1] \cap f_2^{-1}[0,r_2]$ and $(f_1 \vee f_2)^{-1}[0,r] \in \mathcal{B}$. So $\mathcal{B}$ is a filterbase on X .

If $f \in C(X)$ with $f \geq 0$, $Z \in \mathcal{U}$ with $Z \subseteq Z(f)$ and $r > 0$ then $f^{-1}[0,\frac{r}{2}] \in \mathcal{B}$ and is completely separated from $X - f^{-1}[0,r] = f^{-1}(r,\infty)$. Hence $\mathcal{B}$ is a completely regular filterbase on X .

Now $\mathcal{B}$ is contained in some maximal completely regular filter on X , say $\mathcal{G}$. We claim that $\mathcal{G}$ is coarser than $\mathcal{U}$. Let $\mathcal{H}$ be the unique z-ultrafilter on X finer than $\mathcal{G}$ (see theorem 2.3.4) . We show that $\mathcal{H} = \mathcal{U}$, for which it suffices to check that if $Z \in \mathcal{Z}(X)$ then Z meets $\mathcal{G}$ *iff* $Z \in \mathcal{U}$ (Z meets $\mathcal{G}$ *iff* $Z \in \mathcal{H}$, by theorem 2.3.4) . Let $Z \in \mathcal{Z}(X)$.

Suppose Z meets $\mathcal{G}$. If $Z \notin \mathcal{U}$ then, since $\mathcal{U}$ is a z-ultrafilter, Z does not meet $\mathcal{U}$. Hence there is $U \in \mathcal{U}$ with $Z \cap U = \emptyset$. Now U and Z are disjoint zero-sets, so there is an $f \in C(X)$ with $0 \leq f \leq 1$, $f(U) = 0$ and $f(Z) = 1$. Then $f^{-1}[0,\frac{1}{2}] \in \mathcal{B} \subseteq \mathcal{G}$, but $f^{-1}[0,\frac{1}{2}] \cap Z = \emptyset$ — a contradiction to Z meeting $\mathcal{G}$. So $Z \in \mathcal{U}$.

Now suppose $Z \in \mathcal{U}$. Let $G \in \mathcal{G}$, and suppose that $Z \cap G = \emptyset$ (so that Z does uot meet $\mathcal{G}$) . Let Z_G be a zero-set with $Z_G \subseteq G$ and $Z_G \in \mathcal{G}$. Choose $f \in C(X)$ with $0 \leq f \leq 1$, $f(Z) = 0$ and $f(Z_G) = 1$. Now, since $Z \in \mathcal{U}$, we have $f^{-1}[0,\frac{1}{2}] \in \mathcal{B} \subseteq \mathcal{G}$, but we have $f^{-1}[0,\frac{1}{2}] \cap Z_G = \emptyset$! Thus $Z \cap G \neq \emptyset$, so Z meets $\mathcal{G}$.

Hence $\mathcal{H} = \mathcal{U}$, and we have shown the existence of a maximal completely regular filter coarser than $\mathcal{U}$. It remains to prove uniqueness:

Suppose $\mathcal{G}_1, \mathcal{G}_2$ are maximal completely regular filters coarser than $\mathcal{U}$. If $Z \in \mathcal{Z}(X)$ then, from theorem 2.3.4, $Z \in \mathcal{U}$ *iff* Z meets $\mathcal{G}_1$, and $Z \in \mathcal{U}$ *iff* Z meets $\mathcal{G}_2$. Since each member of $\mathcal{G}_1$ and of $\mathcal{G}_2$ contains a zero-set element of that filter, it follows that $\mathcal{G}_1$ meets $\mathcal{G}_2$. By maximality of $\mathcal{G}_1$ and $\mathcal{G}_2$, we must have $\mathcal{G}_1 = \mathcal{G}_2$. □

The next theorem is also central to later theory on extendibility. It provides a link between the maximal completely regular filters on a subspace (whose continuous real-valued functions we wish to extend) and the maximal completely regular filters on the parent space (over which the extension must be defined). The result was proved by J.W.Green, first under the added assumption that the parent space is Tychonoff ([Gr_1 ; lemma]) , and then for arbitrary spaces in [Gr_2 ; lemma 4]

2.3.6 Theorem : *Let $S \subseteq X$. If $\mathcal{F}$ is a maximal completely regular filter on S , then there is a unique maximal completely regular filter on X coarser than $\mathcal{F}$.*

Proof. Suppose that $\mathcal{F}$ is a maximal completely regular filter on S. Define $\mathcal{S} = \{\mathcal{G} : \mathcal{G}$ is a completely regular filter on X coarser than $\mathcal{F}\}$. $\mathcal{S} \neq \emptyset$, since $\{X\} \in \mathcal{S}$. Partially order $\mathcal{S}$ by subset inclusion, and let $\mathcal{C}$ be a chain in $\mathcal{S}$. Now $\bigcup\mathcal{C}$ is a filter on X containing every member of $\mathcal{C}$; and if $U \in \bigcup\mathcal{C}$ then $U \in \mathcal{G}$ for some $\mathcal{G} \in \mathcal{C}$, so by complete regularity of $\mathcal{G}$ there is a $U' \in \mathcal{G} \subseteq \bigcup C$ with U' and $X - U$ completely separated. Hence $\bigcup\mathcal{C}$ is completely regular, and so $\bigcup\mathcal{C} \in \mathcal{S}$. By Zorn's lemma, $\mathcal{S}$ has a maximal element, say $\mathcal{G}$. Thus $\mathcal{G}$ is maximal among filters on X that are both completely regular on X and coarser than $\mathcal{F}$.

We claim that $\mathcal{G}$ is maximal completely regular on X . Suppose the contrary ; then there is a completely regular filter $\mathcal{H}$ on X strictly finer than $\mathcal{G}$ (i.e., $\mathcal{G} \subsetneq \mathcal{H}$) . Now $\mathcal{H}$ is not coarser than $\mathcal{F}$, by maximality of $\mathcal{G}$ in $\mathcal{S}$.

Claim : if $\sup\{\mathcal{H} \mid S, \mathcal{F}\}$ is a filter then it is a completely regular filter on S strictly finer than $\mathcal{F}$.

Proof. Suppose that $\sup\{\mathcal{H} \mid S, \mathcal{F}\}$ is a filter on S , i.e, that $\mathcal{H}$ meets S and $\mathcal{H} \mid S$ meets $\mathcal{F}$. Now $\mathcal{H} \mid S$ is completely regular on S , so $\sup\{\mathcal{H} \mid S, \mathcal{F}\}$ is completely regular on S . Of course $\mathcal{F} \subseteq \sup\{\mathcal{H} \mid S, \mathcal{F}\}$, and if we choose an $H \in \mathcal{H}$ such that $H \cap S \notin \mathcal{F}$ ($\mathcal{H} \subsetneq \mathcal{F}$ so $\mathcal{H} \mid S \subsetneq \mathcal{F}$) then for any $F \in \mathcal{F}$, $H \cap S \cap F \in \sup\{\mathcal{H} \mid S, \mathcal{F}\} - \mathcal{F}$ — thus $\mathcal{F} \subsetneq \sup\{\mathcal{H} \mid S, \mathcal{F}\}$. $\square_{\text{claim}}$

By maximality of $\mathcal{F}$, $\sup\{\mathcal{H} \mid S, \mathcal{F}\}$ cannot be a filter. Thus $\mathcal{H} \mid S$ cannot meet $\mathcal{F}$ (this includes the possibility of $\mathcal{H}$ not meeting S). Choose $H_1 \in \mathcal{H}$ and $F_1 \in \mathcal{F}$ with $H_1 \cap S \cap F_1 = \emptyset$ i.e., $H_1 \cap F_1 = \emptyset$ since $F_1 \subseteq S$. By complete regularity of $\mathcal{H}$ on X , there is an $H \in \mathcal{H}$ with H and $X - H_1$ completely separated in X ; and by complete regularity of $\mathcal{F}$ on S , there is an $F \in \mathcal{F}$ with F and $S - F_1$ completely separated in S . Note that $F \cap H = \emptyset$ (since $F \subseteq F_1$, $H \subseteq H_1$) , so that $F \subseteq X - H$.

By complete regularity of $\mathcal{H}$ on X , there is an $H' \in \mathcal{H}$ and a $\varphi \in C(X)$ with $0 \le \varphi \le 1$, $\varphi(H') = 1$ and $\varphi(X - H) = 0$. Define $\mathcal{G}' = \sup\left(\mathcal{G}, \left\{\varphi^{-1}[0,e) : 0 < e < 1\right\}\right)$.

Claim : $\mathcal{G}'$ is a completely regular filter on X .

Proof. It is clear that $\left\{\varphi^{-1}[0,e) : 0 < e < 1\right\}$ is a completely regular filterbase on X . So we need only show that $\mathcal{G}$ meets $\left\{\varphi^{-1}[0,e) : 0 < e < 1\right\}$. Let $G \in \mathcal{G}$, then since $\mathcal{G} \le \mathcal{F}$ we have $G \supseteq F'$ for some $F' \in \mathcal{F}$. Then $G \supseteq F' \cap F$, and $\emptyset \ne F' \cap F \subseteq X - H$. Now $\varphi(X - H) = 0$, so $\varphi(F' \cap F) = 0$ and we have shown that $\varphi^{-1}(0) \cap G \ne \emptyset$. Therefore $\varphi^{-1}[0,e) \cap G \ne \emptyset$ for $0 < e < 1$. $\square_{\text{claim}}$

Claim : $\mathcal{G}'$ is strictly finer than $\mathcal{G}$, and $\mathcal{G}'$ is coarser than $\mathcal{F}$.

Proof. Certainly $\mathcal{G} \subseteq \mathcal{G}'$. Suppose that $\mathcal{G} = \mathcal{G}'$. Then $\varphi^{-1}[0,e) \in \mathcal{G} \subseteq \mathcal{H}$ for each $0 < e < 1$. But $H' \in \mathcal{H}$ and $H' \cap \varphi^{-1}[0,e) = \emptyset$ for each $0 < e < 1$. Therefore $\mathcal{G} \subsetneq \mathcal{G}'$.

Let $G' \in \mathcal{G}'$. Then $G' \supseteq G \cap \varphi^{-1}[0,e)$ for some $G \in \mathcal{G}$, $e \in (0,1)$. Now $\varphi^{-1}[0,e) \supseteq \varphi^{-1}(0) \supseteq X - H \supseteq F \in \mathcal{F}$, so $\varphi^{-1}[0,e) \cap S \in \mathcal{F}$. Since $\mathcal{G} \leq \mathcal{F}$, $G \supseteq F'$ for some $F' \in \mathcal{F}$. Now we have $G' \supseteq G \cap \varphi^{-1}[0,e) \supseteq F' \cap \varphi^{-1}[0,e) = (F' \cap S) \cap \varphi^{-1}[0,e) = F' \cap (\varphi^{-1}[0,e) \cap S) \in \mathcal{F}$. Thus $\mathcal{G}' \leq \mathcal{F}$. $\square_{\text{claim}}$

Thus $\mathcal{G}'$ is a completely regular filter on X coarser than $\mathcal{F}$ and strictly finer than $\mathcal{G}$. This contradicts the maximality of $\mathcal{G}$ in $\mathcal{F}$. Therefore $\mathcal{G}$ is maximal completely regular.

It remains to show that $\mathcal{G}$ is unique: Suppose $\mathcal{U}$ is also a maximal completely regular filter on X with $\mathcal{U} \leq \mathcal{F}$. If $U \in \mathcal{U}$ and $G \in \mathcal{G}$ then $U \supseteq F_1$, $G \supseteq F_2$ for some $F_1, F_2 \in \mathcal{F}$. Now $U \cap G \supseteq F_1 \cap F_2 \neq \emptyset$. So $\mathcal{U}$ meets $\mathcal{G}$, and it follows by maximality that $\mathcal{U} = \mathcal{G}$. $\square$

2.3.7 *Notation* : If $\mathcal{F}$ is a maximal completely regular filter on $S \subseteq X$ then we shall denote by $\mathcal{F}^*$ the unique maximal completely regular filter on X coarser than $\mathcal{F}$.

2.3.8 *Remark* : It is easily verified that the results of this section continue to hold with "maximal completely regular filterbase" substituted for "maximal completely regular filter".

We have developed all the infrastructure necessary for our characterisations of z-, C^*-, and C- embedding as well as the localisations of these embeddings. Chapter 3 will deal with characterisations of the global embeddings, and chapter 4 with their localisations.

CHAPTER 3

Filter characterisations of z-, C*-, and C- embedding

3.1 INTRODUCTION

In this chapter we establish characterisations of z-embedding, C^*-embedding, and C-embedding of a subspace in terms of z-ultrafilters and maximal completely regular filters. Typically we shall be concerned with correspondences between maximal completely regular filters on the parent space and maximal completely regular filters on the subspace; or between z-ultrafilters on the parent space and z-ultrafilters on the subspace. We shall see that natural, and desirable, correspondences exist when the subspace is z-, C^*-, or C- embedded in the parent space, and that these correspondences are also sufficient conditions for the respective embeddings.

The properties of z-, C^*-, and C- embedding are global in the sense that they are concerned with the simultaneous extendibility of every zero-set, continuous function, or bounded continuous function of a subspace. In the next chapter we will investigate localisations of the characterisations in the present chapter, i.e., investigate conditions, arising from characterisations of the present chapter, under which a particular class of zero-sets (the Lebesgue-sets) associated with a continuous function will extend, and conditions under which a particular (possibly bounded) continuous function will extend.

Theorem 3.2.13 characterises z-embedding of a subspace in terms of the relationship between z-ultrafilters, maximal completely regular filters and real z-ultrafilters on the parent space with those on the subspace. The proofs of the seven equivalences listed in this theorem depend only on well known properties of z-ultrafilters and on the properties of maximal completely regular filters developed in chapter 2 . In particular, no non-filter-theoretic characterisation of z-embedding of a subspace is used in the proofs. Besides being a characterisation of z-embedding of a subspace and relating filters on the parent space to those on the subspace, this theorem is also important in that it is used in establishing filter-theoretic characterisations of C^*-, and C- embedding (z-embedding is necessary for these embeddings).

Theorem 3.4.3 characterises C^*-embedding of a subspace in terms of the behaviour of maximal completely regular filters on the parent space with respect to those on the subspace. The essential part of the proof of this theorem uses a non-filter-theoretic characterisation of C^*-embedding : that $S \subseteq X$ is C^*-embedded in X *iff* S is z-embedded in X and a certain condition (β) holds. This characterisation is transformed into a filter-theoretic one by the use of the earlier filter-theoretic characterisation of z-embedding and through finding a filter-theoretic characterisation of (β) . In addition to being a characterisation of C^*-embedding and establishing a relationship between maximal completely regular filters on the parent space with those on the subspace, this result is also used in establishing a filter characterisation of C-embedding.

Theorem 3.4.9 uses z-ultrafilters and maximal completely regular filters to characterise C-embedding of a subspace. The proof of this result is in much the same spirit as that of the characterisation of C^*-embedding. It uses a non-filter-theoretic characterisation of C-embedding (that $S \subseteq X$ is C-embedded in X *iff* S is z-embedded in X and a condition (γ) holds) in the essential part of the proof. As before the filter-theoretic result is formulated by using the earlier characterisations of z-embedding and by finding a filter characterisation of (γ). The proof also makes use of the filter-theoretic characterisation of C^*-embedding.

The major part of the present chapter is derived from four papers, [Gr_1], [Gr_2], [BH_1] and [Bl_1] . In [Gr_1] filter characterisations of C^*-, and C- embedding are obtained for Tychonoff spaces, with intricate proofs. In [Gr_2] the essential part of a lemma in [Gr_1] is proved without the Tychonoff requirement, as is the characterisation of C^*-embedding. Since the other results of [Gr_1] require the Tychonoff axiom only in that they rely on these two results, they then carry across to arbitrary spaces. In [BH_1] non-filter-theoretic characterisations of z-, C^*-, and C- embeddings (for arbitrary spaces) are obtained, and these are used in [Bl_1] to derive filter characterisations of the three types of embedding which include, and improve upon, those of [Gr_1] and [Gr_2] . The theorem statements are, for the most part, those of [BH_1] and [Bl_1] — there being very little room for improvement of the wording in these papers. Our notation too has been adopted from these two papers.

3.2 CHARACTERISATIONS OF z-EMBEDDING

We begin our study of z-embedding by characterising it in terms of the uniform approximation of continuous functions on the subspace by continuous functions on cozero-supersets from the parent space (theorem 3.2.4) . This leads very quickly to a characterisation of both C^*- and C- embedding (3.3.3) , from which we shall see that theorem 3.2.4 has depth comparable to that of the Urysohn Extension Theorem.

3.2.1 *Definition* : A ***partition of unity*** on a space X is a collection Φ of continuous non-negative real-valued functions on X such that, at each $x \in X$, $\varphi(x) \neq 0$ for only finitely many $\varphi \in \Phi$, and $\sum_{\varphi \in \Phi} \varphi(x) = 1$. Φ is said to be *locally-finite* if each $x \in X$ has a neighbourhood on which all but finitely many $\varphi \in \Phi$ vanish .

3.2.2 *Definition* : A family $\{A_\alpha\}_{\alpha \in I}$ of subsets of a space X is *locally finite* if every point of X has a neighbourhood which meets only finitely many of the A_α's . The family is said to be ***star-finite*** if every member meets only finitely many other members.

The following lemma is due to R.L.Blair and A.W.Hager [BH_1 2.1] . It is used only in proving the above-mentioned characterisation of z-embedding. The first step of the proof (constructing a countable locally-finite refinement of a cover of a space by cozero-sets) follows R.Engelking in [Eng_1 p 221 ; alternatively, Eng_2 p 394] .

3.2.3 Lemma : *If $\{C_n\}_{n=1}^{\infty}$ is a countable cover of X by cozero-sets, then there is a countable locally-finite partition of unity $\{g_n\}$ on X, with $coz\, g_n \subseteq C_n$ for each n.*

Proof. Let $C_n = f_n^{-1}(0,1]$ where $f_n \in C(X)$ and $f_n : X \to [0,1]$. Define $f = \sum_n \frac{1}{2^n} f_n$. By the Weierstrass M-test, the series for f is uniformly convergent, and so $f \in C(X)$. Also $0 \le f \le \sum_n 2^{-n} = 1$, and since $\bigcup_n C_n = X$ we see that $f > 0$; so $0 < f \le 1$.

Define $V_k = f^{-1}(\frac{1}{k},1]$ and $F_k = f^{-1}[\frac{1}{k},1]$. The collections $\{V_k\}_{k=1}^{\infty}$ and $\{F_k\}_{k=1}^{\infty}$ cover X. Note that $V_{k+1} - F_{k-1} = f^{-1}(\frac{1}{k+1},\frac{1}{k-1})$ is cozero. Define $F_0 = \emptyset$.

We claim that the collection $\mathcal{U} = \{C_n \cap (V_{k+1} - F_{k-1}) : 1 \le n \le k \text{ and } k = 2,3,\ldots\}$ forms a countable locally-finite cozero-set refinement of $\{C_n\}$: It is clear that each member of $\mathcal{U}$ is cozero, and that if $\mathcal{U}$ is a cover of X then it will certainly refine $\{C_n\}$. Let $x \in X$, and denote by K the smallest integer such that $x \in F_K$. Suppose that $x \notin \bigcup_{n \le K} C_n$, i.e., that $f_1(x) = f_2(x) = \ldots = f_K(x) = 0$. Then we have $f(x) \le 1 - \left(\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{2^K}\right) = \frac{1}{2^K} < \frac{1}{K}$ — contradicting the definition of K. Hence there is an $n \le K$ with $x \in C_n$. By minimality of K we have $x \notin F_{K-1}$, so that $x \in C_n \cap (F_K - F_{K-1}) \subseteq C_n \cap (V_{K+1} - F_{K-1})$. Thus $\mathcal{U}$ covers X.

It remains to show that $\mathcal{U}$ is locally-finite: Define $C_{k,n} = C_n \cap (V_{k+1} - F_{k-1})$, and note that for every $n \le k$ we have $C_{k,n} \subseteq V_{k+1} \subseteq F_{k+1}$. If $m \ge k+2$ then $F_{m-1} \supseteq F_{k+1}$, so if $i \le m$ then $C_{k,n} \cap C_{m,i} = \emptyset$ since $C_{m,i} = C_i \cap (V_{m+1} - F_{m-1}) \subseteq X - F_{m-1} \subseteq X - F_{k+1}$. Thus $C_{k,n}$ meets only finitely many other members of $\mathcal{U}$, showing that $\mathcal{U}$ is star-finite. Now a star-finite open cover of a topological space is locally-finite, so we conclude that $\mathcal{U}$ is locally-finite.

Rename the members of the countable family $\mathcal{U}$, so that we may write $\mathcal{U} = \{C_j'\}_{j=1}^{\infty}$. For each $j \in \mathbb{N}$ choose an $n(j)$ such that $C_j' \subseteq C_{n(j)}$, possible since $\mathcal{U}$ is a refinement of $\{C_n\}$. For each $n \in \mathbb{N}$, set $D_n = \bigcup\{C_j' : n(j) = n\}$. Notice that $D_n \subseteq C_n$ for each n, and that each D_n is a countable union of cozero-sets so is cozero. If $x \in X$ then by local-finiteness of $\mathcal{U}$ there is some neighbourhood N of x which meets only finitely many members of $\mathcal{U}$; since D_n meets N *iff* some $C'_j \in \mathcal{U}$ with $n(j) = n$ meets N , it follows that $\{D_n\}$ is locally-finite. Also $\{D_n\}$ is a cover of X , for if $x \in X$ then there is some j such that $x \in C'_j \subseteq D_{n(j)}$.

For each $n \in \mathbf{N}$ choose an $h_n \in C(X)$ with $h_n \geq 0$ and $\operatorname{coz} h_n = D_n$. By local finiteness $\sum h_i(x)$ is a finite sum for each $x \in X$; and since $\{D_n\}$ covers X it follows that $\sum h_i > 0$. Define $g_n = h_n \Big/ \sum h_i$. If $x \in X$ then there is a neighbourhood of x on which all but finitely many of the h_n vanish ; on this neighbourhood $\sum h_i$ is a finite linear combination of continuous functions, making $\sum h_i$ and hence g_n continuous on X . By construction we have $\operatorname{coz} g_n = \operatorname{coz} h_n = D_n \subseteq C_n$. Clearly $g_n \geq 0$ and each $x \in X$ has a neighbonrhood on which all but finitely many g_n vanish; and if $x \in X$, then $\sum g_n(x) = \left(\sum \frac{h_n}{\sum h_i}\right)(x) = 1$. Thus $\{g_n\}$ is a locally-finite partition of unity on X with $\operatorname{coz} g_n \subseteq C_n$ for each n . $\square$

3.2.4 <u>*Theorem*</u> (R.L.Blair and A.W.Hager [BH$_1$ 2.2]) : *Let* $S \subseteq X$. *Then* S *is* *z-embedded in* X *iff each (bounded)* $f \in C(S)$ *can be approximated uniformly on* S *by continuous functions on cozero-sets of* X *which contain* S .

Proof. $\Leftarrow$: Suppose each $f \in C(S)$ can be approximated uniformly on S by continuous functions on cozero-sets of X which contain S . Let $Z(f) \in \mathfrak{Z}(S)$ with $f \in C^*(S)$. For each $n \in \mathbf{N}$ we can choose a cozero-set S_n of X and an $f_n \in C(S_n)$ with $S_n \supseteq S$ and $|f_n(s) - f(s)| < \frac{1}{n}$ for each $s \in S$. For each $n \in \mathbf{N}$, define $Z'_n = \{x \in S_n : |f_n(x)| \leq \frac{1}{n}\}$ — then Z'_n is a zero-set of S_n (being the preimage under a continuous map on S_n of the closed subset $[-\frac{1}{n}, \frac{1}{n}]$ of $\mathbf{R}$). By proposition 1.2.15 we know that each S_n is z-embedded in X , hence for each $n \in \mathbf{N}$ there is a $Z_n \in \mathfrak{Z}(X)$ with $Z_n \cap S_n = Z'_n$. We shall show that $(\bigcap Z_n) \cap S = Z(f)$.

Let $x \in Z(f)$; then $x \in S$, so for each $n \in \mathbf{N}$ we have $|f_n(x)| = |f_n(x) - f(x)| < \frac{1}{n}$. So $x \in Z'_n = Z_n \cap S_n \subseteq Z_n$ for each $n \in \mathbf{N}$. Thus $x \in (\bigcap Z_n) \cap S$.

Now let $x \in (\bigcap Z_n) \cap S$. Then $x \in (\bigcap Z_i) \cap S_n$ for each $n \in \mathbb{N}$, so $x \in Z_n \cap S_n = Z'_n$ for each $n \in \mathbb{N}$. Thus $|f_n(x)| \le \frac{1}{n}$ for all $n \in \mathbb{N}$; and $|f(x)| \le |f_n(x)| + |f(x) - f_n(x)| \le \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$ for each $n \in \mathbb{N}$. So $f(x) = 0$, i.e., $x \in Z(f)$.

$\Rightarrow$: Now suppose that S is z-embedded in X. Let $f \in C(S)$ and $\epsilon > 0$. For each integer $n \in \mathbb{Z}$, define $A_n = \{ s \in S : n - 1 < \frac{1}{\epsilon} \cdot f(s) < n + 1 \}$ and choose a cozero-set C_n of X such that $C_n \cap S = A_n$.

Define $P = \bigcup\limits_n C_n$, so that $\{C_n\}$ is a countable cover of P by cozero-sets of P and lemma 3.2.3 applies (the intersection with P of a cozero-set of X is a cozero-set of P). Note that P is cozero. Let $\{g_n\}_{n \in \mathbb{Z}}$ be a countable locally-finite partition of unity on P with $\operatorname{coz} g_n \subseteq C_n$ for each $n \in \mathbb{Z}$. Let $g = \sum \epsilon n g_n$ — if $p \in P$ then $g_n(p) = 0$ for all but finitely many $n \in \mathbb{Z}$, so this is a finite sum. For each $p \in P$ there is a neighbourhood of p on which all but finitely many of the g_n vanish, and so on this neighbourhood g is a finite linear combination of continuous functions; so g is continuous on P. We will show that $|f(s) - g(s)| < \epsilon$ for every $s \in S$.

Let $s \in S$. Then s belongs to at most two (consecutively indexed) C_n, hence at most two (consecutively indexed) g_n's are nonzero at s. Let k be the largest integer with $k \le \frac{1}{\epsilon} \cdot f(s)$. Then $g_k(s) + g_{k+1}(s) = 1$, and $g(s) = \epsilon k\, g_k(s) + \epsilon (k+1)\, g_{k+1}(s)$ and thus we have $g(s) = \epsilon k \cdot 1 + \epsilon g_{k+1}(s)$, so $\epsilon k \le g(s) \le \epsilon k + \epsilon$. Also we have chosen k so that $k \le \frac{1}{\epsilon} \cdot f(s) < k + 1$, so we have $\epsilon k \le f(s) < \epsilon k + \epsilon$. Thus $|g(s) - f(s)| < \epsilon$.

Now g is a continuous function on the cozero-set $P \supseteq S$ of X with $|g - f| < \epsilon$ on S. □

The theorem was first proved for dense subspaces by A.W.Hager in [**Ha**$_1$ 3.6]. We shall see that this theorem leads to a version of the Urysohn Extension Theorem (3.3.8), and that it can also be used to prove the Tietze-Urysohn Extension Theorem (something for which the Urysohn Extension Theorem is usually employed). It is theorem 3.3.8 that indicates the direction in which we ought to look for filter characterisations of C^*-, and C- embedding, emphasising the importance of the above characterisation of z-embedding. The technique of proof of theorem 3.2.4 will be adapted in order to prove theorem 3.3.5 .

The following lemma provides a sufficient condition for extendibility of a given $f \in C(S)$ over $X \supseteq S$. It is often the case in the conditions we will investigate that a sequence of approximating functions to the function we wish to extend will arise in a natural manner. It is in this way that the lemma will prove useful in later proofs.

3.2.5 *Lemma* (R.L.Blair and A.W.Hager [BH$_1$ 2.3]) : *Let $S \subseteq X$. If $f_n \in C(X)$ for each n , $f \in C(S)$, and if $f_n \to f$ uniformly on S , then f extends continuously over X .*

Proof. Extract a subsequence of $\{f_n\}$ in order to arrange it that for each n , $|f_n(s) - f_{n-1}(s)| < 2^{-n}$ for all $s \in S$. Define $g_n = ((f_n - f_{n-1}) \wedge 2^{-n}) \vee -2^{-n}$. Then each g_n is continuous on X and by the M-test we see that $\sum g_n$ converges uniformly on X , say $\sum g_n = g \in C(X)$.

Let $s \in S$, then $(g_i | S)(s) = g_i(s) = f_i(s) - f_{i-1}(s)$ since on S we have $|f_n - f_{n-1}| < 2^{-n}$. So $(\sum_{i=1}^{n} g_i | S)(s) = ((f_n - f_0) | S)(s)$. Now on S we have : $g + f_0 = \sum_{n=1}^{\infty} g_n + f_0 = \lim_{n \to \infty} \sum_{i=1}^{n} g_i + f_0 = \lim_{n \to \infty} (f_n - f_0) + f_0 = f$. Thus $g + f_0$ is the desired extension of f . □

Note that we may restate this lemma as follows: *If $S \subseteq X$, $g_n \in C(S)$ for each n , $f \in C(S)$, and if $g_n \to f$ uniformly , then if each of the g_n's extends continuously over X so too does f .*

The lemma and theorem 3.2.4 yield the following result concerning partial extendibility of a given function on a z-embedded subspace.

3.2.6 <u>*Corollary*</u> (R.L.Blair and A.W.Hager [BH_1 2.4]) : *If S is z-embedded in X then for each $f \in C(S)$ there are cozero-sets $R_1, R_2, \ldots$ of X, each containing S, such that f extends continuously over $\bigcap R_n$.*

Proof. By theorem 3.2.4 , for each $n \in \mathbb{N}$ we may choose a cozero-set R_n of X and a $g_n \in C(R_n)$ with $|f(s) - g_n(s))| < \frac{1}{n}$ for each $s \in S$. Since $S \subseteq \bigcap R_n$, $\{g_i | \bigcap R_n\} \subseteq C(\bigcap R_n)$ and $g_n \to f$ uniformly on S, it follows from the lemma that f extends continuously over $\bigcap R_n$. □

The converse of this corollary is not true ([BH_1 2.5 (a)]) : It suffices to find a subset S of X which is a countable intersection of cozero-sets of X, yet is not z-embedded in X. Let X be the Moore plane Γ, and let S be the x-axis (see, for instance, [GJ 3K.1]) . Then S is a zero-set of X and hence a countable intersection of cozero-sets. Since S is discrete, $\mathfrak{Z}(S) = \mathfrak{P}(S)$ (the power set of S) , and so $|\mathfrak{Z}(S)| = 2^c$. But $|\mathfrak{Z}(X)| \leq |C(X)| = c$. Thus S is not z-embedded.

We now turn our attention to filter characterisations of z-embedding. These establish interesting relationships, in the presence of z-embedding, between z-filters on the parent space with those on the subspace. Before we can state the theorem, we need to make a number of definitions.

3.2.7 <u>*Definition*</u> : Let S be a topological space and $f \in C(S)$. For each $a \in \mathbb{R}$ we define the *lower Lebesgue-set of f at a*, denoted $L_a(f)$, and the *upper Lebesgue-set of f at a*, denoted $L^a(f)$, by $L_a(f) = \{s \in S : f(s) \leq a\}$ and $L^a(f) = \{s \in S : f(s) \geq a\}$.

Note that both $L_a(f)$ and $L^a(f)$ are zero-sets of S. In this section we shall use Lebesgue-sets in order to define z-embedding of functions, and of filters, on a subspace — concepts that will be needed in our characterisations of z-embedding. In the next chapter, Lebesgue-sets will be used in necessary and sufficient conditions for z-embedding of, and extendibility of, a single given function on a subspace. Lebesgue-sets were first used by H.Lebesgue to characterise various classes of functions ([Le]) .

3.2.8 *Definition* (R.L.Blair [Bl_1 p 286]) : Let $S \subseteq X$ and $f \in C(S)$. We say that *f is z-embedded in X* if every Lebesgue-set of f extends to a zero-set of X .

It is clear that if S is z-embedded in X , then every member of $C(S)$ will be z-embedded in X . Conversely, if all of $C(S)$ is z-embedded in X and $Z \in \mathfrak{Z}(S)$, say $Z = Z(f)$ with $f \in C(S)$, then $Z = L_0(f) \cap L^0(f) = (Z_1 \cap S) \cap (Z_2 \cap S) = (Z_1 \cap Z_2) \cap S$ for some $Z_1, Z_2 \in \mathfrak{Z}(X)$ — hence S is z-embedded in X . Since $C(S)$ and $C^*(S)$ determine the same zero-sets, we have that S is z-embedded in X *iff* all of $C(S)$ is z-embedded in X *iff* all of $C^*(S)$ is z-embedded in X .

Suppose $S \subseteq X$ and $f \in C(S)$ extends over X , say $f = g \mid S$ with $g \in C(X)$. Clearly the upper and lower Lebesgue-sets of f at any $a \in \mathbb{R}$ are just the intersection with S of the corresponding Lebesgue-sets of g . Hence f is z-embedded in X , so we see that z-embedding of a function is necessary for extendibility. It is easily checked that if f is z-embedded in X then so too are $|f|$, $-f$, $c+f$, cf, $f \vee c$ and $f \wedge c$ ($c \in \mathbb{R}$) ; in particular, if $f \in C(S)$ is z-embedded in X , then so too is $(r \vee f) \wedge s$ for $r < s$ in $\mathbb{R}$ (see next definition).

3.2.9 *Definition* (R.L.Blair [Bl_1 p 286]) : Let $S \subseteq X$ and $\mathfrak{F}$ be a filter on S . We shall say that $\mathfrak{F}$ is *z-embedded in X* if it is the case that for every $F \in \mathfrak{F}$ there exists an $F' \in \mathfrak{F}$ such that F' and $S - F$ can be completely separated in S by some continuous function on S that is z-embedded in X (in view of the remark at the end of the last paragraph, it does not matter exactly how this function completely separates F' and $S - F$) . It is clear that $\mathfrak{F}$ will then be completely regular on S .

3.2.10 Definition ([GJ 4.12]) : Let $f: X \to Y$ be continuous, and let $\mathcal{F}$ be a z-filter on X. We define the *sharp mapping* $f^{\#}$ by $f^{\#}(\mathcal{F}) = \{Z \in \mathcal{Z}(Y) : f^{-1}(Z) \in \mathcal{F}\}$, a z-filter on Y. It is easily seen that if $\mathcal{F}$ is a prime z-filter then $f^{\#}(\mathcal{F})$ is also prime; but if $\mathcal{F}$ is a z-ultrafilter then $f^{\#}(\mathcal{F})$ need not necessarily be a z-ultrafilter. See [GJ 4.12]. Note that if $\mathcal{F}$ is a z-filter on $S \subseteq X$ and if $\varphi: S \to X$ is the inclusion map of S into X, then $\varphi^{\#}(\mathcal{F}) = \{Z \in \mathcal{Z}(X) : Z \cap S \in \mathcal{F}\}$ — we call this the *z-filter on X determined by* $\mathcal{F}$. If $\mathcal{F}$ is a z-ultrafilter on S then, since $\varphi^{\#}(\mathcal{F})$ is a prime z-filter on X, $\varphi^{\#}(\mathcal{F})$ is contained in a unique z-ultrafilter on X — we call this the *z-ultrafilter on X determined by* $\mathcal{F}$.

3.2.11 Definition : A z-ultrafilter is said to be a *real z-ultrafilter* if it has the countable intersection property (i.e., every countable intersection of members of $\mathcal{U}$ is nonempty). See [GJ 5.15].

We shall need the following elementary proposition in our filter-theoretic characterisations of z-embedding.

3.2.12 Proposition (R.L.Blair [Bl_4 2.2]) : *A z-filter $\mathcal{F}$ on X is a real z-ultrafilter on X iff $\mathcal{F}$ is prime and closed under countable intersections.*

Proof. $\Rightarrow$: Assume $\mathcal{F}$ is a real z-ultrafilter on X. Of course, $\mathcal{F}$ is prime. Suppose that $\{Z_n\} \subseteq \mathcal{F}$ with $\bigcap Z_n \notin \mathcal{F}$. Since a real z-ultrafilter has the countable intersection property, $\bigcap Z_n \neq \emptyset$. Now a zero-set of X is in $\mathcal{F}$ exactly when it meets $\mathcal{F}$. Hence there is a $Z \in \mathcal{F}$ such that $(\bigcap Z_n) \cap Z = \emptyset$ — but this contradicts the countable intersection property enjoyed by $\mathcal{F}$.

$\Leftarrow$: Now assume that $\mathcal{F}$ is a prime z-filter on X closed under countable intersections. Let $Z \in \mathcal{Z}(X)$, say $Z = Z(f)$ with $f \in C(X)$. Suppose that Z meets $\mathcal{F}$ — we must show that $Z \in \mathcal{F}$. For $n \in \mathbb{N}$, set $Z_n = \{x \in X : |f(x)| \leq \frac{1}{n}\}$ and $Z'_n = \{x \in X : |f(x)| \geq \frac{1}{n}\}$. Now Z meets no Z'_n, so that $Z'_n \notin \mathcal{F}$ for each $n \in \mathbb{N}$. But for each $n \in \mathbb{N}$ we have $X = Z_n \cup Z'_n \in \mathcal{F}$, so by primeness we conclude that $Z_n \in \mathcal{F}$. Since $\mathcal{F}$ is closed under countable intersections,

$Z = \bigcap Z_n \in \mathfrak{F}$. Hence $\mathfrak{F}$ is a z-ultrafilter. Closure under countable intersections certainly implies the countable intersection property, and so we conclude that $\mathfrak{F}$ is a real z-ultrafilter. □

We have developed the necessary terminology and notation to proceed with the filter-theoretic characterisation of z-embedding. Other than the preceding elementary proposition, this theorem makes no use of earlier results in this chapter. The equivalence of (a) – (f) was first proved in [$\mathbf{Bl}_3$] . In [$\mathbf{Gr}_1$ Theorem 3] the implication (a) ⇒ (c) is proved for Tychonoff spaces, but under the stronger assumption of C^*-embedding (and with considerably more effort). The statement and proof of the theorem presented here is essentially that of R.L.Blair ([$\mathbf{Bl}_1$ 3.1]) , though he omitted (b) in favour of (b)′ . The addition of (g) is due to R.L.Blair.

We have noted that if $S \subseteq X$ then $\mathfrak{Z}(S) \supseteq \{Z \cap S : Z \in \mathfrak{Z}(X)\}$, and that equality holds if S is z-embedded in X . Thus z-embedding of S in X means that the lattice $\mathfrak{Z}(S)$ of zero-sets of S is entirely determined by those zero-sets of X that meet S , and this leads us to enquire if this implies any relationship between z-filters on S (filters in the lattice $\mathfrak{Z}(S)$) and z-filters on X . It is clear that we will probably have to limit our attention to those z-filters on X that meet S .

Suppose that $\mathfrak{F}$ is a z-filter on X that meets S , with S z-embedded in X . It is clear that $\mathfrak{F} \mid S$ is then a z-filter on S , and we ask: what properties enjoyed by $\mathfrak{F}$ will continue to be enjoyed by $\mathfrak{F} \mid S$?

Suppose $\mathfrak{F}$ is a z-ultrafilter on X that meets S , and suppose that $Z \in \mathfrak{Z}(S)$ meets $\mathfrak{F} \mid S$. By z-embedding, there is a $Z' \in \mathfrak{Z}(X)$ with $Z' \cap S = Z$. Now $\emptyset \neq Z \cap F \cap S$ $= Z' \cap S \cap F \cap S \subseteq Z' \cap F$ for all $F \in \mathfrak{F}$, hence Z' meets $\mathfrak{F}$ and so since $\mathfrak{F}$ is a z-ultrafilter we conclude that $Z' \in \mathfrak{F}$. Now $Z = Z' \cap S \in \mathfrak{F} \mid S$, showing that $\mathfrak{F} \mid S$ is a z-ultrafilter on S .

Suppose $\mathfrak{F}$ is a real z-ultrafilter on X that meets S , then (by the above) $\mathfrak{F} \mid S$ is a z-ultrafilter on S (and therefore prime) ; and if $Z_1, Z_2, \ldots \in \mathfrak{F} \mid S$ then there exist $Z'_1, Z'_2, \ldots \in \mathfrak{F}$

with $Z'_n \cap S = Z_n$ and so $\bigcap Z_n = (\bigcap Z'_n) \cap S \in \mathfrak{F} \mid S$ since $\mathfrak{F}$ is closed under countable intersections. So by proposition 3.2.12 $\mathfrak{F} \mid S$ is a real z-ultrafilter on S.

Thus z-embedding of S in X ensures that the trace on S of every (real) z-ultrafilter on X that meets S will be a (real) z-ultrafilter on S. The theorem will show that these conditions are sufficient too.

Let $\mathfrak{F}$ be a z-filter on $S \subseteq X$, and consider the z-filter $\varphi^{\#}(\mathfrak{F})$ on X determined by $\mathfrak{F}$ (where $\varphi \colon S \rightarrow X$ is the inclusion map). It is clear that $\varphi^{\#}(\mathfrak{F})$ meets S, and it is natural to enquire as to the relation between $\varphi^{\#}(\mathfrak{F}) \mid S$ and $\mathfrak{F}$. It is clear that we will always have $\varphi^{\#}(\mathfrak{F}) \mid S \subseteq \mathfrak{F}$, and it is easy to see ((a) $\Rightarrow$ (b) below) that the reverse inequality will hold if S is z-embedded in X. Conversely, the theorem shows that if this reverse inequality holds for all z-filters on S then S is z-embedded in X. Hence S is z-embedded in X *iff* distinct z-filters on S determine distinct z-filters on X.

3.2.13 <u>*Theorem*</u> (R.L.Blair [Bl_1 3.1]): *If $S \subseteq X$ then the following are equivalent:*

(a) *S is z-embedded in X,*

(b) *if $\mathfrak{G}$ is any z-filter on S, and if $\varphi : S \rightarrow X$ is the inclusion map of S into X, then $\mathfrak{G} \subseteq \varphi^{\#}(\mathfrak{G}) \mid S$ (in fact $\mathfrak{G} = \varphi^{\#}(\mathfrak{G}) \mid S$),*

(b)′ *if $\mathfrak{G}$ is any z-ultrafilter on S, and if $\varphi : S \rightarrow X$ is the inclusion map of S into X, then $\mathfrak{G} \subseteq \varphi^{\#}(\mathfrak{G}) \mid S$ (in fact $\mathfrak{G} = \varphi^{\#}(\mathfrak{G}) \mid S$),*

(c) *if $\mathfrak{F}$ is any z-ultrafilter on X which meets S, then $\mathfrak{F} \mid S$ is a z-ultrafilter on S,*

(d) *if $\mathfrak{F}$ is any z-ultrafilter on X which meets S, then $\mathfrak{F} \mid S$ is a prime z-filter on S,*

(e) *if $\mathfrak{F}$ is any real z-ultrafilter on X which meets S, then $\mathfrak{F} \mid S$ is a real z-ultrafilter on S,*

(f) *if A_1 and A_2 are completely separated subsets of S, then there exist zero-sets $Z_1, Z_2 \in \mathfrak{Z}(X)$ such that $A_1 \subseteq Z_1$, $A_2 \subseteq Z_2$, and $Z_1 \cap Z_2 \cap S = \emptyset$,*

(g) *every (maximal) completely regular filter on S is z-embedded in X.*

Proof. (a) $\Rightarrow$ (b) : Let $\mathcal{G}$ be a z-filter on S and let $G \in \mathcal{G}$. By z-embedding there is some $Z \in \mathcal{Z}(X)$ such that $G = Z \cap S$. Now $\varphi^{\#}(\mathcal{G}) \mid S = \{Z' \cap S : Z' \in \mathcal{Z}(X) \text{ and } Z' \cap S \in \mathcal{G}\}$, so it is clear that $G \in \varphi^{\#}(\mathcal{G}) \mid S$. It is always true that $\varphi^{\#}(\mathcal{G}) \mid S \subseteq \mathcal{G}$.

(b) $\Rightarrow$ (b)$'$: this is trivial.

(b)$'$ $\Rightarrow$ (c) : Let $\mathcal{F}$ be a z-ultrafilter on X which meets S . Then, clearly, $\mathcal{F} \mid S$ is a base for a z-filter on S , and hence $\mathcal{F} \mid S \subseteq \mathcal{G}$ for some z-ultrafilter $\mathcal{G}$ on S . Certainly $\mathcal{F} \subseteq \varphi^{\#}(\mathcal{G}) = \{Z \in \mathcal{Z}(X) : Z \cap S \in \mathcal{G}\}$, and so (since $\varphi^{\#}(\mathcal{G})$ is a z-filter on X) by maximality of $\mathcal{F}$ we have $\mathcal{F} = \varphi^{\#}(\mathcal{G})$. Now by (b)$'$ we have $\mathcal{G} \subseteq \varphi^{\#}(\mathcal{G}) \mid S = \mathcal{F} \mid S$, and we already have $\mathcal{F} \mid S \subseteq \mathcal{G}$. So $\mathcal{F} \mid S = \mathcal{G}$, a z-ultrafilter on S .

(c) $\Rightarrow$ (d) : Any z-ultrafilter is prime, so this is immediate.

(d) $\Rightarrow$ (e) : Let $\mathcal{F}$ be a real z-ultrafilter on X which meets S . Then $\mathcal{F}$ is closed under countable intersection, and hence so too is $\mathcal{F} \mid S$. By (d) , $\mathcal{F} \mid S$ is a prime z-filter. By proposition 3.2.12 , $\mathcal{F} \mid S$ is a real z-ultrafilter on S .

(e) $\Rightarrow$ (f) : Let A_1 and A_2 be completely separated subsets of S . We may assume that $A_1 \neq \emptyset \neq A_2$. Since A_1 and A_2 are completely separated in S , we may choose zero-sets $Z_1', Z_2' \in \mathcal{Z}(S)$ such that $A_1 \subseteq Z_1'$, $A_2 \subseteq Z_2'$ and $Z_1' \cap Z_2' = \emptyset$. Choose $x \in A_1$, and let $\mathcal{F} = \{Z \in \mathcal{Z}(X) : x \in Z\}$. Then $\mathcal{F}$ is a z-filter on X and has the countable intersection property; and if $Y_1, Y_2 \in \mathcal{Z}(X)$ with $Y_1 \cup Y_2 \in \mathcal{F}$ then $x \in Y_1 \cup Y_2$ so $x \in Y_1$ or $x \in Y_2$, so one of Y_1, Y_2 is in $\mathcal{F}$ — showing that $\mathcal{F}$ is prime. By proposition 3.2.12 , $\mathcal{F}$ is a real z-ultrafilter. Thus, by (e), $\mathcal{F} \mid S$ is a real z-ultrafilter on S . Now since Z_1' meets $\mathcal{F} \mid S$ ($x \in A_1 \subseteq Z_1'$ and $x \in \bigcap(\mathcal{F} \mid S)$) and $\mathcal{F} \mid S$ is a z-ultrafilter, we must have $Z_1' \in \mathcal{F} \mid S$. Thus $Z_1' = Z_1 \cap S$ for some $Z_1 \in \mathcal{F} \subseteq \mathcal{Z}(X)$. Similarly $Z_2' = Z_2 \cap S$ for some $Z_2 \in \mathcal{Z}(X)$, and we have $Z_1 \cap Z_2 \cap S = Z_1' \cap Z_2' = \emptyset$.

(f) $\Rightarrow$ (g) : It clearly suffices to show that every $f \in C(S)$ is z-embedded in X . Let $f \in C(S)$ and $a \in \mathbb{R}$. If $n \in \mathbb{N}$ then the Lebesgue-sets $L_a(f)$ and $L^{a+\frac{1}{n}}(f)$ are disjoint zero-sets of

S , so are completely separated in S . By (f) , there is a $Z_n \in \mathfrak{Z}(X)$ with $L_a(f) \subseteq Z_n$ and $Z_n \cap L^{a+\frac{1}{n}}(f) = \emptyset$. Then $\bigcap_n Z_n \in \mathfrak{Z}(X)$ and $L_a(f) = \bigcap_n Z_n \cap S$. Similarly for $L^a(f)$.

(g) $\Rightarrow$ (a) : Let $f \in C(S)$. Set $A_0 = Z(f)$ and for each $n \in \mathbb{N}$, set

$$A_n = \{x \in S : \ |f(x)| \geq \tfrac{1}{n}\} = L_{-\frac{1}{n}}(f) \cup L^{\frac{1}{n}}(f) \quad .$$

We may assume that $f \neq 0$ and that $Z(f) \neq \emptyset$, so that $A_0 \neq \emptyset$ and after some stage all the A_n's will be nonempty, say for $n \geq N$.

For $n = 0$ and for $n \geq N$, define

$$\mathfrak{B}_n = \left\{ g^{-1}[0,r] : r > 0 \ , \ g \in C(S) \ , \ g \geq 0 \ , \ g \text{ is } z\text{-embedded in } X \ , \ A_n \subseteq Z(g) \right\}.$$

For $n = 0$ and for $n \geq N$, $A_n \neq \emptyset$ so that $\emptyset \notin \mathfrak{B}_n$. Also, for $n = 0$ and for $n \geq N$, $\mathfrak{B}_n \neq \emptyset$ as is shown by considering the function $g = 0$ on S .

We will show that each $\mathfrak{B}_n$, for $n = 0$ and for $n \geq N$, is a completely regular filterbase on S . Let $n = 0$ or $n \geq N$ and let $r_1, r_2 > 0$, g_1, g_2 be non-negative functions in $C(S)$ that are z-embedded in X with $A_n \subseteq Z(g_1)$ and $A_n \subseteq Z(g_2)$. Let $r = \min\{r_1, r_2\} > 0$. Certainly $g_1 \vee g_2 \geq 0$, and we also have $Z(g_1 \vee g_2) = Z(g_1) \cap Z(g_2) \supseteq A_n$. Now $(g_1 \vee g_2)^{-1}[0,r] \subseteq g_1^{-1}[0,r_1] \cap g_2^{-1}[0,r_2]$, so it remains to show that $g_1 \vee g_2$ is z-embedded in X . Let $a \in \mathbb{R}$. There exist $Z_1, Z_1', Z_2, Z_2' \in \mathfrak{Z}(X)$ such that $L_a(g_1) = Z_1 \cap S$, $L^a(g_1) = Z_1' \cap S$, $L_a(g_2) = Z_2 \cap S$ and $L^a(g_2) = Z_2' \cap S$. Now we have $L_a(g_1 \vee g_2) = L_a(g_1) \cap L_a(g_2)$ $= (Z_1 \cap Z_2) \cap S$ and $L^a(g_1 \vee g_2) = L^a(g_1) \cup L^a(g_2) = (Z_1' \cup Z_2') \cap S$. Hence $g_1 \vee g_2$ is z-embedded in X . We have shown that $\mathfrak{B}$ is a filterbase on S ; it is obvious that this filterbase is completely regular.

Suppose that $\mathfrak{B}_0$ meets some $\mathfrak{B}_n$ with $n \geq N$. Then $\sup\{\mathfrak{B}_0, \mathfrak{B}_n\}$ is a completely regular filter on S containing both $\mathfrak{B}_0$ and $\mathfrak{B}_n$. Let $\mathfrak{F}$ be a maximal completely regular filter on S containing $\sup\{\mathfrak{B}_0, \mathfrak{B}_n\}$, and hence so too $\mathfrak{B}_0$ and $\mathfrak{B}_n$. By (g) , $\mathfrak{F}$ is z-embedded in X . We claim that A_0 meets $\mathfrak{F}$: Suppose the contrary, say $F \in \mathfrak{F}$ is such that $F \cap A_0 = \emptyset$, i.e., such that $A_0 \subseteq S - F$. By z-embedding of $\mathfrak{F}$ in X , there is an $F' \in \mathfrak{F}$ and a non-negative z-embedded in X $g \in C(S)$ such that $g(F') = 1$ and $g(S - F) = 0$. But then, since

$A_0 \subseteq S - F \subseteq Z(g)$, $g^{-1}[0,\frac{1}{2}] \in \mathfrak{B}_0 \subseteq \mathfrak{F}$. Now $g^{-1}[0,\frac{1}{2}] \cap F' = \emptyset$ — a contradiction. So A_0 must meet $\mathfrak{F}$. Similarly, A_n meets $\mathfrak{F}$. But now A_0 and A_n are zero-sets of S that meet the maximal completely regular filter $\mathfrak{F}$, so their intersection ought to be nonempty — but $A_0 \cap A_n = \emptyset$. We conclude that, for $n \geq N$, $\mathfrak{B}_0$ does not meet $\mathfrak{B}_n$.

Thus for each $n \geq N$ there is a z-embedded $g_n \in C(S)$ with $Z(f) = A_0 \subseteq Z(g_n)$ such that $Z(g_n) \cap A_n = \emptyset$ (A_n is contained in each member of $\mathfrak{B}_n$) . By z-embedding of g_n in X , there is a $Z_n \in \mathfrak{Z}(X)$ such that $Z(g_n) = Z_n \cap S$ (simply note that $Z(g_n) = L_0(g_n) \cap L^0(g_n)$) . Now $\bigcap_{n \geq N} Z_n \in \mathfrak{Z}(X)$ and $Z(f) = \bigcap_{n \geq N} Z_n \cap S$ — that $Z(f) \subseteq \bigcap_{n \geq N} Z_n \cap S$ is clear and the reverse inclusion follows from the fact that $Z(g_n) \cap A_n = \emptyset$ for $n \geq N$. □

The equivalence of (a), (c), and (e) highlights the filter-theoretic importance of z-embedding of a subspace. This is particularly pertinent in the study of Tychonoff spaces, where z-ultrafilters and real z-ultrafilters can be used to characterise a large number of properties. Thus, if the parent space X has a property $\mathfrak{P}$ that can be characterised in terms of z-ultrafilters on X then (c) will help in determining whether a particular z-embedded subspace has property $\mathfrak{P}$. An obviously interesting case is that where the parent space is βS and the subspace is S .

The essential part of the proof of 3.2.13 is the implication (g) ⇒ (a). The proof of this implication will be greatly simplified in the next chapter, where we shall establish new filter-theoretic characterisations for the z-embedding of a single given function on a subspace (i.e., a localisation of the results of theorem 3.2.13) , recalling that S is z-embedded in X *iff* all of $C(S)$ is z-embedded in X . This new proof will show more clearly why (a) and (g) should be expected to be equivalent.

Theorem 3.2.13 will be used in the remaining two major theorems of this chapter, characterising C^*-, and C- embedding of a subspace in filter-theoretic terms. In the presence of z-embedding of S in X we have found that every z-ultrafilter on X that meets S yields a z-ultrafilter on S . Asking when *all* z-ultrafilters on S will arise in this way leads us to a characterisation of C-embedding. Considering maximal completely regular filters, instead of z-ultrafilters, in the questions we asked regarding z-embedding leads to a characterisation of C^*-embedding.

The Urysohn Extension Theorem may be phrased: *$S \subseteq X$ is C^*-embedded in X iff whenever A_1 and A_2 are completely separated subsets of S, there exist zero-sets $Z_1, Z_2 \in \mathcal{Z}(X)$ such that $A_1 \subseteq Z_1$, $A_2 \subseteq Z_2$ and $Z_1 \cap Z_2 = \emptyset$*, i.e., *completely separated subsets of S are completely separated in X*. Note how condition (f) in theorem 3.2.13 is a weakening of this, and that we could rephrase (f) as (f)$'$: *the complete separation of completely separated subsets of S can be effected by two disjoint zero-sets of S (containing the subsets) both of which extend to zero-sets of X.*

If a z-filter $\mathcal{U}$ on X meets $S \subseteq X$, then $\mathcal{U} \mid S$ is a z-filterbase on S. It is clear that if S is z-embedded in X then $\mathcal{U} \mid S$ will be a z-filter on S. The following proposition shows that the converse is true if X is Tychonoff. The result was pointed out by the referee of the paper $[\mathrm{Bl}_1]$.

3.2.14 <u>*Proposition*</u> ($[\mathrm{Bl}_1$ 3.2 (d) (i) $]$): *Let $S \subseteq X$. If X is a Tychonoff space, and if the trace on S of every z-filter on X which meets S is a z-filter on S (i.e., not just a z-filterbase), then S is z-embedded in X.*

Proof. We show that if there is a function $f \in C(X)$ which is non-constant on S then S is z-embedded in X (of course we need not worry about the case $|S| = 1$). Suppose that f is such a function, and pick $x_1, x_2 \in S$ with $f(x_1) \neq f(x_2)$ — if X is Tychonoff then we could choose f to be a function which completely separates $\{x_1\}$ and $\{x_2\}$. Let $\mathcal{F}_1 = \{Z \in \mathcal{Z}(X) : x_1 \in Z\}$ and $\mathcal{F}_2 = \{Z \in \mathcal{Z}(X) : x_2 \in Z\}$ — these are clearly z-filters on X. Choose $Z_1 \in \mathcal{F}_1$ and $Z_2 \in \mathcal{F}_2$ such that $Z_1 \cap Z_2 = \emptyset$ $\Big($e.g., $Z_1 = f^{-1}\big(\overline{N_r(f(x_1))}\big)$ and $Z_2 = f^{-1}\big(\overline{N_r(f(x_2))}\big)$ with $r = \frac{1}{3}|f(x_1) - f(x_2)|$, and where $N_r(a) = (a - r, a + r)$ $\Big)$. By hypothesis, $\mathcal{F}_1 \mid S$ and $\mathcal{F}_2 \mid S$ are z-filters on S. Let $A \in \mathcal{Z}(S)$. For $i = 1,2$ we have

$(A \cup Z_i) \cap S \supseteq Z_i \cap S$ and so $(A \cup Z_i) \cap S \in \mathfrak{F}_i \mid S$. Thus $(A \cup Z_i) \cap S = Z_i' \cap S$ for some $Z_i' \in \mathfrak{F}_i \subseteq \mathfrak{Z}(X)$. Now $(Z_1' \cap Z_2') \cap S = (Z_1' \cap S) \cap (Z_2' \cap S) = (A \cup Z_1) \cap S \cap (A \cup Z_2) \cap S = S \cap (A \cup (Z_1 \cap Z_2)) = S \cap A = A$. Hence S is z-embedded in X . □

This result need not hold if X is not Tychonoff ([Bl$_1$ 3.2 (d) (ii)]) : Let X be a regular T_1-space on which every real-valued continuous function is constant (see [He]). Choose $a, b \in X$ with $a \neq b$, and define $S = \{a, b\}$. Then obviously the trace on S of any z-filter on X that meets S is a z-filter on S , but S is not z-embedded in X for the only zero-set of X is X itself.

3.3 CHARACTERISATIONS OF C^*-, AND C- EMBEDDING

In this section we develop non-filter-theoretic characterisations of C^*-, and C-embedding. These will be used in obtaining filter-theoretic characterisations in the next section. We start by proving a characterisation (theorem 3.3.3) that is almost immediate from theorem 3.2.4 . We then isolate three conditions ((α), (β) and (γ)) in order to study theorem 3.3.3 more closely, and we show that this theorem contains the Gillman and Jerison characterisations of both C^*- and C- embedding. We also show how the results can be coupled with Urysohn's Lemma in order to prove the Tietze-Urysohn Extension Theorem, and argue that theorem 3.2.4 has depth comparable to that of the Urysohn Extension Theorem.

3.3.1 *Definition* : If $S \subseteq X$ we define $C(X) \mid S = \{f \mid S : f \in C(X)\}$ — the collection of continuous real-valued functions on S that extend over X . If $h \in C(S)$ then we say that h is a *quotient from* $C(X) \mid S$ if $h = \frac{f}{g}$ for some $f, g \in C(X) \mid S$ with $Z(g) = \emptyset$. If it is the case that each (bounded) quotient from $C(X) \mid S$ is again in $C(X) \mid S$, we say that $C(X) \mid S$ is *closed under (bounded) quotients.*

This definition is due to A.W.Hager ([Ha$_2$]) who spoke of *closure under (bounded) inversion.*

3.3.2 *Proposition* (R.L.Blair and A.W.Hager [$\mathbf{BH_1}$ 3.1]) : *Let $S \subseteq X$ and let $h \in C(S)$. Then h is a quotient from $C(X) \mid S$ iff h extends continuously over some cozero-set of X which contains S .*

Proof. $\Rightarrow$: Let $h = \dfrac{f_1}{f_2}$ with $f_1 = g_1 \mid S$ and $f_2 = g_2 \mid S$ for $g_1, g_2 \in C(X)$ with $Z(f_2) = \emptyset$. Then $\operatorname{coz} g_2 \supseteq S$, $u = \dfrac{g_1 \mid \operatorname{coz} g_2}{g_2 \mid \operatorname{coz} g_2} \in C(\operatorname{coz} g_2)$ and $u \mid S = \dfrac{f_1}{f_2} = h$.

$\Leftarrow$: Choose $f \in C^*(X)$ so that $\operatorname{coz} f \supseteq S$ and h extends continuously over $\operatorname{coz} f$, say $g \in C(\operatorname{coz} f)$ with $g \mid S = h$. Define $g_1 = \dfrac{fg}{1+g^2}$ on $\operatorname{coz} f$ and $g_1 = 0$ on $Z(f)$. Define $g_2 = \dfrac{f}{1+g^2}$ on $\operatorname{coz} f$ and $g_2 = 0$ on $Z(f)$. As in the proof of proposition 1.2.15 , we have $g_1, g_2 \in C(X)$. Now $h = (g_1 \mid S) \big/ (g_2 \mid S)$, so h is a quotient from $C(X) \mid S$. □

We have noted that z-embedding is necessary for both C^*- and C- embedding of a subspace. The following theorem isolates what it is that must be added to z-embedding in order to produce C^*- or C- embedding.

We shall use theorem 3.2.4 , lemma 3.2.5 and proposition 3.3.2 . Suppose that S is z-embedded in X and that $f \in C(S)$. We shall construct a sequence of continuous functions on S which converges uniformly to f , with each member of the sequence extending continuously over X . By lemma 3.2.5 this implies that f extends over X . Theorem 3.2.4 provides functions on cozero-supersets of S that approximate f arbitrarily closely on S . Since these functions are defined on cozero-sets of X , we may apply proposition 3.3.2 to conclude that they are quotients from $C(X) \mid S$, and so (under the added assumption that $C(X) \mid S$ is closed under quotients) extend them to continuous functions on X that approximate f arbitrarily closely on S .

3.3.3 *Theorem* : *Let $S \subseteq X$. Then S is C^*-embedded (resp. C-embedded) in X iff S is z-embedded in X and $C(X) \mid S$ is closed under bounded quotients (resp. quotients)*

Proof. $\Rightarrow$: is obvious.

$\Leftarrow$: Let S be z-embedded in X and let $f \in C(S)$. Let $n \in \mathbb{N}$. By theorem 3.2.4 we may choose a cozero-set P_n of X containing S and a $g_n \in C(P_n)$ such that $|f(s) - g_n(s)| < \frac{1}{n}$ for each $s \in S$. By proposition 3.3.2 $g_n \mid S$ is a quotient from $C(X) \mid S$, and $g_n \mid S$ is bounded if f is. Assuming $C(X) \mid S$ is closed under bounded quotients or quotients, according as f is bounded or not, $g_n \mid S$ extends to $f_n \in C(X)$. Now $f_n \to f$ uniformly on S, and so by lemma 3.2.5 we conclude that f extends over X. $\square$

This theorem is due to A.W.Hager, in [$\mathbf{Ha_2}$], and S.Mrówka, in [$\mathbf{Mr_1}$]. Both proofs rely on the Urysohn Extension Theorem or on its usual technique of proof. We shall show that we can recover the Urysohn Extension Theorem as well as the Gillman and Jerison characterisation of C-embedding ([GJ 1.18]) from theorem 3.3.3.

Consider the following three conditions on a subspace S of X (R.L.Blair and A.W.Hager [$\mathbf{BH_1}$ p 45]):

(α) disjoint zero-sets of S are completely separated in X,

(β) if $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \cap Z_2 \cap S = \emptyset$, then $Z_1 \cap S$ and $Z_2 \cap S$ are completely separated in X,

(γ) S is completely separated from every disjoint zero-set of X.

The Gillman and Jerison version of the Urysohn Extension Theorem asserts that (α) is equivalent to C^*-embedding of S in X ([GJ 1.17]). They also show that C-embedding of S in X is equivalent to C^*-embedding (i.e., (α)) together with (γ) ([GJ 1.18]). We will reprove these equivalences in what follows. It is clear that (α) implies (β), and we prove:

3.3.4 <u>*Proposition*</u> (R.L.Blair and A.W.Hager [$\mathbf{BH_1}$ 3.3]): (γ) $\Rightarrow$ (β)

Proof. Let $f_1, f_2 \in C(X)$ with $Z(f_1) \cap Z(f_2) \cap S = \emptyset$. By ($\gamma$) we may choose an $f \in C(X)$ with $f(S) = 0$ and $f(Z(f_1) \cap Z(f_2)) = 1$. Define $g = f^2 + f_1^2 + f_2^2 \in C(X)$. Now f is nonzero when both f_1 and f_2 are zero, so $Z(g) = \emptyset$. Now $f_1^2/g \in C(X)$ and $(f_1^2/g)(Z(f_1) \cap S) = 0$ and $(f_1^2/g)(Z(f_2) \cap S) = 1$. $\square$

Condition (γ) is sometimes referred to as *well-embedding* of S in X, the terminology being due to W.Moran in [Mo 6.1].

Together with theorem 3.3.3, the next two theorems will yield the Gillman and Jerison characterisations of C^*-, and C- embedding.

3.3.5 <u>*Theorem*</u> (R.L.Blair and A.W.Hager [BB_1 3.4A]) : *Let* $S \subseteq X$. *Then the following are equivalent:*

(i) (β) *holds*,

(ii) *if* $f \in C^*(S)$ *extends over a z-embedded set, then* f *extends over* X,

(iii) $C(X) \mid S$ *is closed under bounded quotients*.

Proof. (i) $\Rightarrow$ (ii) : Let $f \in C^*(S)$ extend over z-embedded $P \subseteq X$ with $P \supseteq S$, say $h \in C^*(P)$ with $h \mid S = f$ (as always, we may assume that an extension of a bounded function is bounded).

Choose $m \in \mathbb{N}$ with $|f| \leq m-1$, and let $k \in \mathbb{N}$. For each $n \in \mathbb{Z}$ with $|n| \leq mk+1$, define $A'_n = \{p \in P : k\,h(p) \leq n\}$ and $B'_n = \{p \in P : k\,h(p) \geq n\}$. The A'_n and B'_n are zero-sets of P (they are Lebesgue-sets of h), so by z-embedding of P in X there exist $A_n, B_n \in \mathcal{Z}(X)$ with $A_n \cap P = A'_n$ and $B_n \cap P = B'_n$. Now, for each integer n with $|n| \leq mk+1$, we have $(A'_{n+1} \cap B'_{n-1}) \cap (A'_{n-2} \cup B'_{n+2}) = \emptyset$, so $(A_{n+1} \cap B_{n-1}) \cap (A_{n-2} \cup B_{n+2}) \cap S = \emptyset$ so by (β) there is a $u_n \in C(X)$ with $0 \leq u_n \leq 1$, $u_n((A_{n-2} \cup B_{n+2}) \cap S) = 0$ and $u_n((A_{n+1} \cap B_{n-1}) \cap S) = 1$.

Define $u = \sum_{|n| \leq mk+1} u_n$. If $s \in S$ then $s \in A_{n+1} \cap B_{n-1}$ for some integer n with $|n| \leq mk+1$, so $u(s) \geq 1$ for each $s \in S$. Hence $\operatorname{coz} u \supseteq S$. Define g_n on $\operatorname{coz} u$ by $g_n = \dfrac{u_n \mid \operatorname{coz} u}{u \mid \operatorname{coz} u}$, and let $g = \sum_{|n| \leq mk+1} \frac{n}{k} g_n$. Now define $f_k = (u \wedge 1)\, g$ on $\operatorname{coz} u$ and $f_k = 0$ on $Z(u)$. Thus $f_k = g$ on S. As in proposition 1.2.15 we can easily check that f_k is continuous on X.

For $s \in S$, let j be the largest integer with $j \leq k f(s)$. Then $\sum_{n=j-1}^{j+2} g_n(s) = \frac{1}{u(s)} \sum_{n=j-1}^{j+2} u_n(s) = 1$ since only u_{j-1}, u_j, u_{j+1} and u_{j+2} can be nonzero at s. Also $g(s) = \sum_{|n| \leq mk+1} \frac{n}{k} g_n(s) = \sum_{n=j-1}^{j+2} \frac{n}{k} g_n(s)$, so $\frac{j-1}{k} \leq g(s) \leq \frac{j+2}{k}$, and we already have $\frac{j}{k} \leq f(s) < \frac{j+1}{k}$ by choice of j. Hence $|f(s) - g(s)| \leq \frac{2}{k}$. Thus, since $f_k = g$ on S, $f_k \to f$ uniformly on S. Now, by lemma 3.2.5, f extends over X.

(ii) $\Rightarrow$ (iii) : This follows from proposition 3.3.2 .

(iii) $\Rightarrow$ (i) : Let $f_1, f_2 \in C(X)$ with $Z(f_1) \cap Z(f_2) \cap S = \emptyset$. Define $g_1 = f_1^2 | S$ and $g_2 = f_2^2 | S$. Then $\frac{g_1}{g_1 + g_2}$ is a bounded quotient from $C(X) | S$, so by (iii) there is an $h \in C(X)$ such that $h | S = \frac{g_1}{g_1 + g_2}$. Now $h(Z(f_1) \cap S) = 0$ and $h(Z(f_2) \cap S) = 1$. □

3.3.6 Theorem (R.L.Blair and A.W.Hager [BH$_1$ 3.4B]) : *Let $S \subseteq X$. Then the following are equivalent:*

(i) *(γ) holds ,*

(ii) *if $f \in C(S)$ extends over a z-embedded set, then f extends over X ,*

(iii) *$C(X) | S$ is closed under quotients .*

Proof. (i) $\Rightarrow$ (ii) : Suppose $f \in C(S)$ extends over z-embedded $P \supseteq S$, say $h \in C(P)$ with $h | S = f$. Let $\varphi : \mathbf{R} \cong (-1,1)$ be a homeomorphism. Now $\varphi \circ h \in C^*(P)$, so by (i) $\Rightarrow$ (ii) of the preceding theorem (and since $(\gamma) \Rightarrow (\beta)$ — proposition 3.3.4) there is an extension $g \in C(X)$ of $\varphi \circ h$ (see note below) . By (γ) we may choose a $u \in C(X)$ with $0 \leq u \leq 1$, $u(S) = 1$ and $u\left(\{x \in X : |g(x)| \geq 1\}\right) = 0$. Now we have $|u(x)g(x)| < 1$ for all $x \in X$, so that $\varphi^{-1} \circ (ug)$ is properly defined. Since $(ug) | S = (\varphi \circ h) | S$, $(\varphi^{-1} \circ (ug)) | S = h | S = f$ — so $\varphi^{-1} \circ (ug)$ extends f.

Note : It is possible to show that $\varphi \circ h$ extends over X without appealing to theorem 3.3.5 ; we use theorem 3.2.4 instead. By theorem 3.2.4 , for each $n \in \mathbf{N}$ there is a cozero-set P_n of X containing P and a $g_n \in C(P_n)$ such that $|(\varphi \circ h)(x) - g_n(x)| < \frac{1}{n}$ for each $x \in P$. Since

$\varphi \circ h$ is bounded, we may assume that g_n is bounded. By (γ) , there is an $h_n \in C(X)$ with $h_n(S) = 1$ and $h_n(X - P_n) = 0$. Define f_n on X by $f_n = g_n h_n$ on $\mathrm{coz}\, h_n$ and $f_n = 0$ on $Z(h_n)$. Then, as in proposition 1.2.15 , $f_n \in C(X)$. Since $f_n = g_n$ on S , we see that $f_n \to \varphi \circ h$ uniformly on S . By lemma 3.2.5 , $\varphi \circ h$ extends over X .

(ii) $\Rightarrow$ (iii) : This is immediate from proposition 3.3.2 and since cozero-sets are z-embedded .

(iii) $\Rightarrow$ (i) : Let $f \in C(X)$ with $Z(f) \cap S = \emptyset$. Then $1/(f \mid S)$ is a quotient from $C(X) \mid S$, so by (iii) there is an $h \in C(X)$ such that $h \mid S = 1/(f \mid S)$. Now $(hf)(S) = 1$ and $(hf)(Z(f)) = 0$. □

3.3.7 __*Proposition*__ (R.L.Blair and A.W.Hager [BH$_1$ 3.5]) : *For $S \subseteq X$, (α) is equivalent to the conjunction of (β) and z-embedding .*

Proof. $\Rightarrow$: We have already noted that (α) implies (β) . Let $f \in C(S)$, and for each $n \in \mathbb{N}$ define $A_n = \{s \in S : |f(s)| \geq \frac{1}{n}\}$. By (α) , A_n and $Z(f)$ are completely separated in X , so there is a $Z_n \in \mathfrak{Z}(X)$ with $Z(f) \subseteq Z_n$ and $Z_n \cap A_n = \emptyset$. Now $\bigcap_n Z_n \cap \bigcup_n A_n = \emptyset$, and $\bigcup_n A_n = \mathrm{coz}\, f$. Hence $\bigcap_n Z_n \cap S = Z(f)$, and so S is z-embedded in X .

$\Leftarrow$: Let $Z_1', Z_2' \in \mathfrak{Z}(S)$ with $Z_1' \cap Z_2' = \emptyset$. By z-embedding there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ such that $Z_1 \cap S = Z_1'$ and $Z_2' \cap S = Z_2'$. Now $Z_1 \cap Z_2 \cap S = \emptyset$, so by (β) we have $Z_1 \cap S = Z_1'$ and $Z_2' \cap S = Z_2'$ are completely separated in X . □

3.3.8 __*Theorem*__ (R.L.Blair and A.W.Hager [BH$_1$ 3.6]) : *Let $S \subseteq X$.*

A. The following are equivalent:

(i) *S is z-embedded in X and (β) holds ,*

(ii) *S is C^*-embedded in X ,*

(iii) *(α) holds .*

B. The following are equivalent:

(i) *S is z-embedded and (γ) holds ,*

(ii) *S is C-embedded in X .*

Proof. A (i) $\Rightarrow$ (ii) : this is clear from theorem 3.3.5 (i) $\Rightarrow$ (ii) ,

(ii) $\Rightarrow$ (iii) : is obvious ,

(iii) $\Rightarrow$ (i) : follows from proposition 3.3.7 .

B (i) $\Rightarrow$ (ii) : this is clear from theorem 3.3.6 (i) $\Rightarrow$ (ii) ,

(ii) $\Rightarrow$ (i) : Let $f \in C(X)$ with $Z(f) \cap S = \emptyset$. Define $g(s) = \frac{1}{f(s)}$ for $s \in S$, so that $g \in C(S)$. Let $h \in C(X)$ with $h \mid S = g$. Now $fh \in C(X)$, $(fh)(S) = 1$ and $(fh)(\,Z(f)\,) = 0$. □

In the next section we will find filter characterisations of (β) and (γ) and couple these with the earlier filter characterisations of z-embedding in order to transform theorem 3.3.8 into a characterisation of both C^*- and C- embedding in filter-theoretic terms.

Note that, in the last theorem, A (ii) $\Leftrightarrow$ (iii) is just the Gillman and Jerison version of the Urysohn Extension Theorem which now has an alternative proof. In addition we have a new characterisation of both C^*-embedding and C-embedding.

In view of theorems 3.3.5 and 3.3.6 , we may restate theorem 3.3.3 as follows: *Let $S \subseteq X$. Then S is C^*-embedded (resp. C-embedded) in X iff S is z-embedded in X and (β) (resp. (γ)) holds.* This is just theorem 3.3.8 , showing that theorem 3.3.3 contains a version of the Urysohn Extension Theorem. Now theorem 3.2.4 is used in the proof of theorem 3.3.3 and we have seen that it can be used in the proof of theorem 3.3.6 (see (†) above) , showing theorem 3.2.4's role in leading to a version of the Urysohn Extension Theorem. Note also that theorem 3.3.3 (as restated above) and proposition 3.3.7 yield the following: *$S \subseteq X$ is C-embedded in X iff both (α) and (γ) hold.* This is just the Gillman and Jerison result [GJ 1.18] . Since theorem 3.3.3 is an almost immediate consequence of theorem 3.2.4 , this is further testimony to the depth of theorem 3.2.4 .

3.3.9 *Remark:* One of the important applications of the Urysohn Extension Theorem is to couple it with the Urysohn Lemma (that a space is normal *iff* disjoint closed sets of the space are completely separated in the space) to prove the Tietze-Urysohn Extension Theorem. We show that theorem 3.2.4 also succeeds in this application ([BH$_1$ 3.7 (d)]) : Let X be normal, and let F be closed in X. Then

(i) F is z-embedded in X: Let $Z \in \mathfrak{Z}(F)$. Any zero-set may be written as a countable intersection of cozero-sets, so we may choose a family $\{Z_n\}_{n \in \mathbb{N}} \subseteq \mathfrak{Z}(F)$ such that $F - Z = \bigcup_{n \in \mathbb{N}} Z_n$. By the Urysohn Lemma, Z is completely separated from each of the Z_n (Z and Z_n are disjoint closed sets in F, and hence so too in X) so we may choose $Z'_n \in \mathfrak{Z}(X)$ with $Z'_n \supseteq Z$ and $Z'_n \cap Z_n = \emptyset$. Now $\bigcap_{n \in \mathbb{N}} Z'_n \in \mathfrak{Z}(X)$ and $\bigcap_{n \in \mathbb{N}} Z'_n \cap F = Z$.

(ii) (γ) holds : If $Z \in \mathfrak{Z}(X)$ with $Z \cap F = \emptyset$, then Z and F are disjoint closed sets in X, and so are completely separated in X.

(iii) F is C-embedded : This follows from (i), (ii) and theorem 3.3.8 B. We have seen how theorem 3.2.4 can be used in the proof of theorem 3.3.6 (which yields 3.3.8 B).

3.4 FILTER-THEORETIC CHARACTERISATIONS OF C^*-, AND C- EMBEDDING

The characterisations in section 3.3 of C^*-, and C- embeddings in terms of z-embedding, (β), and (γ) will be used to establish filter-theoretic characterisations of these embeddings. They will also be our starting point in the next chapter for localisations of the global results. We start by characterising (β) and (γ) in filter-theoretic terms.

Considering the proof A (i) $\Rightarrow$ (ii) below makes the characterisation 3.4.1 A (ii) of (β) seem quite plausible. The equivalence of (a) and (e) in theorem 3.4.3 (characterising C^*-embedding) together with the relatively easy deduction of (e) from (a) in theorem 3.4.9

(characterising C-embedding) and the knowledge that C^*-embedding and (γ) yield C-embedding makes the initial consideration of the condition 3.4.1 B (ii) as a characterisation of (γ) more plausible.

3.4.1 Theorem (R.L.Blair [Bl$_1$ 4.2]) : *Let* $S \subseteq X$.

A. *The following are equivalent:*

(i) *(β) holds ,*

(ii) *If* $\mathfrak{F}$ *is any maximal completely regular filter on* X , *and if* $\mathfrak{F}\,|\,S$ *meets* $Z_1, Z_2 \in \mathfrak{Z}(X)$, *then* $Z_1 \cap Z_2 \cap S \neq \emptyset$.

B. *The following are equivalent:*

(i) *(γ) holds ,*

(ii) *If* $\mathfrak{F}$ *is a maximal completely regular filter on* S, *and if* $\mathfrak{U}$ *is the unique z-ultrafilter on* X *finer than the unique maximal completely regular filter on* X *coarser than* $\mathfrak{F}$, *then* $\mathfrak{U}$ *meets* S ,

(iii) *If* $\mathfrak{G}$ *is any z-filter on* X *which meets* S , *then there exists a z-ultrafilter* $\mathfrak{U}$ *on* X *which meets* $\mathfrak{G}\,|\,S$.

Proof. A. (i) $\Rightarrow$ (ii) : Suppose that (ii) fails. Then there is a maximal completely regular filter $\mathfrak{F}$ on X meeting S and there are zero-sets $Z_1, Z_2 \in \mathfrak{Z}(X)$ such that $\mathfrak{F}\,|\,S$ meets both Z_1 and Z_2 , but $Z_1 \cap Z_2 \cap S = \emptyset$. Let $Z_1', Z_2' \in \mathfrak{Z}(X)$ with $Z_1 \cap S \subseteq Z_1'$, $Z_2 \cap S \subseteq Z_2'$. Now $\mathfrak{F}$ meets both $Z_1 \cap S$ and $Z_2 \cap S$, so $\mathfrak{F}$ meets both Z_1' and Z_2' . It follows from theorem 2.3.3 (ii) that $Z_1' \cap Z_2' \neq \emptyset$. Thus we are unable to completely separate $Z_1 \cap S$ and $Z_2 \cap S$ in X — i.e., (β) fails.

(ii) $\Rightarrow$ (i) : Let $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $Z_1 \cap Z_2 \cap S = \emptyset$. For $i = 1,2$ let

$$\mathfrak{B}_i = \{ f^{-1}[0,r] : r > 0 \, , \, f \in C(X) \, , \, f \geq 0 \, , \, Z_i \cap S \subseteq Z(f) \ \} \ .$$

Claim : $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are completely regular filterbases on X

Proof. We may assume that $Z_i \cap S \neq \emptyset$ for $i = 1,2$, and this means that $\emptyset \notin \mathfrak{B}_i$. Choose $g_i \in C(X)$ with $Z_i = Z(g_i)$, then $(g_i \vee 0)^{-1}[0,1] \in \mathfrak{B}_i$ so that $\mathfrak{B}_i \neq \emptyset$. Suppose $r, s > 0$; $f, h \in C(X)$; $f, h \geq 0$; $Z_i \cap S \subseteq Z(f) \cap Z(h)$. Now $C(X) \ni f \vee h \geq 0$ and $Z_i \cap S \subseteq Z(f \vee h) = Z(f) \cap Z(h)$ and so $f^{-1}[0,r] \cap h^{-1}[0,s] \supseteq (f \vee h)^{-1}[0, r \wedge s] \in \mathfrak{B}_i$. We have shown that $\mathfrak{B}_i$ is a filterbase. Now let $r > 0$, $f \in C(X)$ with $f \geq 0$ and $Z_i \cap S \subseteq Z(f)$ so that $f^{-1}[0,r] \in \mathfrak{B}_i$. Then $f^{-1}[0,\frac{r}{2}] \in \mathfrak{B}_i$ and is completely separated from $X - f^{-1}[0,r]$. $\square_{\text{claim}}$

Suppose that $\mathfrak{B}_1$ meets $\mathfrak{B}_2$, so that there is a maximal completely regular filter $\mathfrak{F}$ on X finer than both $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Note that $\mathfrak{F}$ meets S .

Claim : $\mathfrak{F}$ meets both $Z_1 \cap S$ and $Z_2 \cap S$.

Proof. Suppose that $\mathfrak{F}$ does not meet $Z_1 \cap S$. Then there is an $F \in \mathfrak{F}$ such that $F \cap Z_1 \cap S = \emptyset$. By complete regularity of $\mathfrak{F}$ there is an $f \in C(X)$ and an $F' \in \mathfrak{F}$ with $0 \leq f \leq 1$, $f(F') = 1$ and $f(X - F) = 0$. Now we have $Z(f) \supseteq X - F \supseteq Z_1 \cap S$, and so $f^{-1}[0,\frac{1}{2}] \in \mathfrak{B}_1 \subseteq \mathfrak{F}$. But $f^{-1}[0,\frac{1}{2}] \cap F' = \emptyset$, a contradiction. So $\mathfrak{F}$ meets $Z_1 \cap S$, and similarly $\mathfrak{F}$ meets $Z_2 \cap S$. $\square_{\text{claim}}$

It follows that $\mathfrak{F} | S$ meets both Z_1 and Z_2 , so by (ii) it is the case that $Z_1 \cap Z_2 \cap S \neq \emptyset$. This is a contradiction, and we conclude that $\mathfrak{B}_1$ and $\mathfrak{B}_2$ cannot meet. Consequently we may choose, for $i = 1,2$, a $Z'_i \in \mathfrak{B}_i$ such that $Z'_1 \cap Z'_2 = \emptyset$. Now Z'_1 and Z'_2 are pre-images of closed sets under a continuous real-valued function and so are zero-sets of X . We have, for $i = 1,2$, $Z_i \cap S \subseteq Z'_i$ so that $Z_1 \cap S$ and $Z_2 \cap S$ are contained in disjoint zero-sets of X and are therefore completely separated in X .

B. (i) $\Rightarrow$ (ii) : Assume (γ) , and let $\mathfrak{F}$ be a maximal completely regular filter on S . Let $\mathfrak{F}^*$ be the unique maximal completely regular filter on X coarser than $\mathfrak{F}$, and let $\mathfrak{U}$ be the unique z-ultrafilter on X finer than $\mathfrak{F}^*$. Suppose that (ii) fails so that there exists a $Z \in \mathfrak{U}$ with $Z \cap S = \emptyset$. Now, by (γ) , there is an $f \in C(X)$ with $f(Z) = 0$, $f(S) = 1$ and $f \geq 0$. Define

$\mathfrak{B} = \{f^{-1}[0,r] : r > 0\}$. Obviously $\mathfrak{B}$ is a completely regular filterbase on X , and $\mathfrak{B} \subseteq \mathfrak{U}$ since $\mathfrak{Z}(X) \ni f^{-1}[0,r] \supseteq f^{-1}(0) \supseteq Z \in \mathfrak{U}$ for $r > 0$. Since both $\mathfrak{B}$ and $\mathfrak{F}^*$ are coarser than $\mathfrak{U}$ it follows that $\mathfrak{B}$ and $\mathfrak{F}^*$ meet, and so by maximality of $\mathfrak{F}^*$ we must have $\mathfrak{B} \subseteq \mathfrak{F}^*$. But we then have $f^{-1}[0,\frac{1}{2}] \in \mathfrak{F}^*$ and so $f^{-1}[0,\frac{1}{2}] \cap S \neq \emptyset$ since $\mathfrak{F}^*$ meets S , contradicting $f(S) = 1$.

(ii) $\Rightarrow$ (iii) : Let $\mathcal{G}$ be a z-filter on X which meets S . Then $\mathcal{G} \mid S$ is a z-filterbase on S , so $\mathcal{G} \mid S \subseteq \mathfrak{U}'$ for some z-ultrafilter $\mathfrak{U}'$ on S . Let $\mathfrak{F}$ be the unique maximal completely regular filter on S such that $\mathfrak{F} \leq \mathfrak{U}'$ and let $\mathfrak{F}^*$ be as usual. Now let $\mathfrak{U}$ be the unique z-ultrafilter on X finer than $\mathfrak{F}^*$. Now $\mathcal{G}$ meets $\mathfrak{F}^*$, since $\mathfrak{F}^* \leq \mathfrak{U}'$ and $\mathcal{G} \leq \mathcal{G} \mid S \leq \mathfrak{U}'$. By proposition 2.3.4 we know that if $Z \in \mathfrak{Z}(X)$ then it is the case that $Z \in \mathfrak{U}$ *iff* Z meets $\mathfrak{F}^*$. Since $\mathcal{G}$ consists of zero-sets of X that meet $\mathfrak{F}^*$, we have $\mathcal{G} \subseteq \mathfrak{U}$. By (ii) $\mathfrak{U}$ meets S , and it follows that $\mathfrak{U}$ meets $\mathcal{G} \mid S$.

(iii) $\Rightarrow$ (i) : Suppose that (i) fails. Then there is a zero-set Z of X with $Z \cap S = \emptyset$ but such that Z and S are not completely separated. Define

$$\mathfrak{B} = \{f^{-1}[0,r] : r > 0 , \ f \in C(X) , \ f \geq 0 , \ Z \subseteq Z(f)\} .$$

This is obviously a base for a z-filter on X , say $\mathcal{G}$. Now $\mathfrak{B}$ must meet S , for otherwise we could completely separate S and Z . Therefore $\mathcal{G}$ meets S and so, if (iii) holds , there is a z-ultrafilter $\mathfrak{U}$ on X which meets $\mathcal{G} \mid S$. Since $Z \cap S = \emptyset$ we have $Z \notin \mathfrak{U}$, and so there is a $Z' \in \mathfrak{U}$ such that $Z \cap Z' = \emptyset$. Let $f \in C(X)$ completely separate the disjoint zero-sets Z and Z' , say $f \geq 0$ with $f(Z') = 1$ and $f(Z) = 0$. But now $f^{-1}[0,\frac{1}{2}] \cap S \in \mathcal{G} \mid S$, so $f^{-1}[0,\frac{1}{2}] \cap S \cap Z' \neq \emptyset$, since $\mathcal{G} \mid S$ meets $\mathfrak{U}$. This is a contradiction. Thus (iii) must fail. □

We come now to the filter characterisation of C^*-embedding. The theorem has much the same flavour as theorem 3.2.13 (characterising z-embedding). We know that any completely regular filter $\mathfrak{F}$ on $X \supseteq S$ that meets S gives rise to a completely regular filter $\mathfrak{F} \mid S$ on S , and we ask: when is the trace of a maximal completely regular filter on X that meets S a maximal completely regular filter on S ? It is easily seen (3.4.3 (a) $\Rightarrow$ (b)) that C^*-embedding is enough to ensure this. In chapter 2 we showed that every maximal completely regular filter $\mathfrak{F}$ on S yields a maximal completely regular filter $\mathfrak{F}^*$ on X, with $\mathfrak{F}^* \leq \mathfrak{F}$. In the presence of C^*-embedding of S in X, $\mathfrak{F}^* \mid S$ is maximal completely regular on S, and it is natural to enquire as to the relation between $\mathfrak{F}$ and $\mathfrak{F}^* \mid S$. It will always be true that $\mathfrak{F}^* \mid S \leq \mathfrak{F}$ — will C^*-embedding ensure that $\mathfrak{F} \leq \mathfrak{F}^* \mid S$? This consideration leads from (b) to condition (e) in theorem 3.4.3, and (e) together with the characterisation of C^*-embedding of section 3.3 and the characterisation of (β) in this section easily imply (a).

The statement and proof presented here are those of R.L.Blair ([Bl_1 5.1]). The equivalence of (a), (b) and (c) is due to J.W.Green. He proved (a) $\Leftrightarrow$ (b) and (a) $\Leftrightarrow$ (c) for Tychonoff spaces in [Gr_1 theorems 1 and 2]. In [Gr_2 theorem 2] he strengthened (a) $\Leftrightarrow$ (b) to apply to arbitrary spaces. In [Gr_1] a characterisation of the Stone-Čech compactification due to P.S.Aleksandrov (see 2.2.2 (iv)) is used to establish the characterisations of C^*- and C-embedding. In [Gr_2] the notion of generalised boundary (due to A.D.Myškis [**My**]) is used to generalise to arbitrary spaces those results in [Gr_1] whose proof relied on the Tychonoff requirement.

3.4.2 <u>*Lemma*</u> (R.L. Blair [Bl_1 2.2]): *Let $S \subseteq X$. If $\mathfrak{F}$ is a maximal completely regular filter on S, and if $\mathfrak{F}$ is coarser than the trace on S of some completely regular filter $\mathfrak{G}$ on X which meets S, then $\mathfrak{F}$ is z-embedded in X.*

Proof. Let $\mathfrak{F}^*$ be the unique maximal completely regular filter on X with $\mathfrak{F}^* \leq \mathfrak{F}$. We claim that $\mathfrak{G}$ meets $\mathfrak{F}^*$: Let $G \in \mathfrak{G}$ and $F^* \in \mathfrak{F}^*$. Now $F^* \supseteq F$ for some $F \in \mathfrak{F}$, and since $\mathfrak{F} \leq \mathfrak{G} \mid S$ we have $F \supseteq G' \cap S$ for some $G' \in \mathfrak{G}$. Now $G \cap (G' \cap S) \neq \emptyset$ since $\mathfrak{G}$ meets S, and since $G' \cap S \subseteq F \subseteq F^*$ we conclude that $G \cap F^* \neq \emptyset$.

Since $\mathfrak{G}$ meets $\mathfrak{F}^*$ we must have, by maximality of $\mathfrak{F}^*$, $\mathfrak{G} \subseteq \mathfrak{F}^*$. Let $F \in \mathfrak{F}$. Then, since $\mathfrak{F} \leq \mathfrak{G} \mid S$, $F \supseteq G \cap S$ for some $G \in \mathfrak{G}$ and, by completely regularity of $\mathfrak{G}$, there is a $G' \in \mathfrak{G}$ such that G' and $X - G$ are completely separated in X. Now since $G' \in \mathfrak{G} \subseteq \mathfrak{F}^*$ and $\mathfrak{F}^* \leq \mathfrak{F}$, we have $G' \supseteq F'$ for some $F' \in \mathfrak{F}$. Choose an $f \in C(X)$ which completely separates G' and $X - G$. Then $f \mid S$ completely separates $F' \subseteq G' \cap S$ and $S - F \subseteq (X - G) \cap S$, and since $f \mid S$ extends over X, $f \mid S$ is z-embedded in X. □

3.4.3 <u>*Theorem*</u> (R.L.Blair [Bl_1 5.1]) : *If $S \subseteq X$ then the following are equivalent:*

(a) *S is C^*-embedded in X,*

(b) *if $\mathfrak{F}$ is a maximal completely regular filter on X which meets S, then $\mathfrak{F} \mid S$ is a maximal completely regular filter on S,*

(c) *every maximal completely regular filter on S is the trace on S of some maximal completely regular filter on X,*

(d) *every maximal completely regular filter on S is coarser than the trace on S of some (maximal) completely regular filter on X that meets S,*

(e) *if $\mathfrak{F}$ is a maximal completely regular filter on S, and if $\mathfrak{F}^*$ is the unique maximal completely regular filter on X coarser than $\mathfrak{F}$, then $\mathfrak{F} \leq \mathfrak{F}^* \mid S$.*

Proof. (a) $\Rightarrow$ (b) : Assume (a) and let $\mathfrak{F}$ be a maximal completely regular filter on X that meets S. That $\mathfrak{F} \mid S$ is a completely regular filter on S is clear. In order to show that $\mathfrak{F} \mid S$ is a maximal completely regular filter we will show that any two zero-sets of S that meet $\mathfrak{F} \mid S$ mnst meet each other. Let $A_1, A_2 \in \mathfrak{Z}(S)$ such that both A_1 and A_2 meet $\mathfrak{F} \mid S$. Suppose that $A_1 \cap A_2 = \emptyset$. Then A_1 and A_2 are disjoint zero-sets of S so by C^*-embedding (i.e., (α)) A_1 and A_2 are completely separated in X. Hence there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $A_1 \subseteq Z_1$, $A_2 \subseteq Z_2$ and $Z_1 \cap Z_2 = \emptyset$. But $\mathfrak{F}$ meets both Z_1 and Z_2, so by maximality of $\mathfrak{F}$ we should have $Z_1 \cap Z_2 \neq \emptyset$! We conclude that $\mathfrak{F} \mid S$ is a maximal completely regnlar filter.

(b) $\Rightarrow$ (c) : Let $\mathfrak{G}$ be a maximal completely regular filter on S. Let $\mathfrak{G}^*$ be the unique maximal completely regular filter on X coarser than $\mathfrak{G}$. Then $\mathfrak{G}^* \mid S \leq \mathfrak{G}$, and by (b) $\mathfrak{G}^* \mid S$ is a maximal completely regular filterbase on S. By maximality we have $\mathfrak{G} = \mathfrak{G}^* \mid S$.

(c) $\Rightarrow$ (d) : this is trivial.

(d) $\Rightarrow$ (e) : Let $\mathfrak{F}$ be a maximal completely regular filter on S , and let $\mathfrak{F}^*$ be the unique maximal completely regular filter on X coarser than $\mathfrak{F}$. By (d) there is a completely regular filter $\mathfrak{G}$ on X such that $\mathfrak{G}$ meets S and $\mathfrak{F} \leq \mathfrak{G} | S$. We have $\mathfrak{F}^* \leq \mathfrak{F} \leq \mathfrak{G} | S$, hence $\mathfrak{G}$ meets $\mathfrak{F}^*$. By maximality of $\mathfrak{F}^*$ we must have $\mathfrak{G} \subseteq \mathfrak{F}^*$. Thus $\mathfrak{G} | S \subseteq \mathfrak{F}^* | S$, so we have $\mathfrak{F} \leq \mathfrak{F}^* | S$.

(e) $\Rightarrow$ (a) : By theorem 3.3.8 we need only show that S is z-embedded in X and that (β) holds.

If $\mathfrak{F}$ is a maximal completely regular filter on S then by (e) $\mathfrak{F}$ is coarser than the trace on S of some (maximal) completely regular filter on X . By lemma 3.4.2 , $\mathfrak{F}$ is z-embedded in X . Now by theorem 3.2.13 it follows that S is z-embedded in X .

Let $\mathfrak{G}$ be a maximal completely regular filter on X , and suppose $\mathfrak{G} | S$ meets $Z_1, Z_2 \in \mathfrak{Z}(X)$. Then $\mathfrak{G}$ meets S so $\mathfrak{G} | S \subseteq \mathfrak{F}$ for some maximal completely regular filter $\mathfrak{F}$ on S ($\mathfrak{G} | S$ is a completely regular filter on S) . Let $\mathfrak{F}^*$ be the unique maximal completely regular filter on X coarser than $\mathfrak{F}$. By (e) we have $\mathfrak{F} \leq \mathfrak{F}^* | S$. Since $\mathfrak{G} | S \subseteq \mathfrak{F}$ and $\mathfrak{F}^* \leq \mathfrak{F}$, $\mathfrak{F}^*$ meets $\mathfrak{G}$. By maximality of $\mathfrak{F}^*$ and $\mathfrak{G}$ we must have $\mathfrak{F}^* = \mathfrak{G}$. Hence we have $\mathfrak{F}^* | S$ meeting Z_1 and Z_2 . Now since $\mathfrak{F} \leq \mathfrak{F}^* | S$ it follows that $\mathfrak{F}$ meets $Z_1 \cap S$ and $Z_2 \cap S$, and so by maximality of $\mathfrak{F}$ we have $Z_1 \cap Z_2 \cap S = (Z_1 \cap S) \cap (Z_2 \cap S) \neq \emptyset$. Now by theorem 3.4.1 A , (β) holds. □

Recall that theorem 3.2.13 showed that $S \subseteq X$ is z-embedded in X *iff* distinct z-filters on S determine distinct z-filters on X . We show now that S is C^*-embedded in X *iff* distinct z-ultrafilters on S determine distinct z-ultrafilters on X .

3.4.4 *Definition* (J.W.Green [Gr_2 p 104]) : Let $S \subseteq X$. We say that filterbases $\mathfrak{B}$ and $\mathfrak{B}'$ on S are ***completely separated in*** X if some member of $\mathfrak{B}$ is completely separated in X from some member of $\mathfrak{B}'$.

The equivalence of (a) and (b) in the following corollary is due to J.W.Green ([Gr_2 theorem 7]) . The equivalence of (a), (c) and (d) is given in [Bl_1 5.2] .

3.4.5 Corollary : *If $S \subseteq X$ then the following are equivalent:*

(a) S is C^*-embedded in X ,

(b) *distinct maximal completely regular filters on S are completely separated in X ,*

(c) *distinct z-ultrafilters on S determine distinct z-ultrafilters on X ,*

(d) *distinct z-ultrafilters on S are completely separated in X .*

Proof. (a) $\Rightarrow$ (b) : Let $\mathcal{G}_1, \mathcal{G}_2$ be distinct maximal completely regular filters on S . Let $\mathcal{G}_1^*, \mathcal{G}_2^*$ be the unique maximal completely regular filters on X with $\mathcal{G}_1^* \leq \mathcal{G}_1$, $\mathcal{G}_2^* \leq \mathcal{G}_2$.

We claim that $\mathcal{G}_1^* \neq \mathcal{G}_2^*$. Suppose not; then $\mathcal{G}_1^* \mid S = \mathcal{G}_2^* \mid S$ and by C^*-embedding (3.4.3 (b)) $\mathcal{G}_1^* \mid S$ and $\mathcal{G}_2^* \mid S$ are maximal completely regular filters on S such that (3.4.3 (e)) $\mathcal{G}_1 \leq \mathcal{G}_1^* \mid S$, $\mathcal{G}_2 \leq \mathcal{G}_2^* \mid S$. By maximality of $\mathcal{G}_1$ and $\mathcal{G}_2$ we must have $\mathcal{G}_1 = \mathcal{G}_2$, a contradiction.

So $\mathcal{G}_1^* \neq \mathcal{G}_2^*$ and so by maximality $\mathcal{G}_1^*$ and $\mathcal{G}_2^*$ cannot meet. Choose $G_1 \in \mathcal{G}_1^*$ and $G_2 \in \mathcal{G}_2^*$ such that $G_1 \cap G_2 = \emptyset$. By complete regularity of $\mathcal{G}_1^*$, G_1 contains a $G_1' \in \mathcal{G}_1^*$ such that G_1' is completely separated from $X - G_1 \supseteq G_2 \in \mathcal{G}_2^*$. Thus $\mathcal{G}_1^*$ and $\mathcal{G}_2^*$ are completely separated in X . Since $\mathcal{G}_1^* \leq \mathcal{G}_1$ and $\mathcal{G}_2^* \leq \mathcal{G}_2$ it follows that $\mathcal{G}_1$ and $\mathcal{G}_2$ are completely separated in X .

(b) $\Rightarrow$ (a) : We verify theorem 3.4.3 (b) . Let $\mathcal{F}$ be any maximal completely regular filter on X which meets S . That $\mathcal{F} \mid S$ is a completely regular filter is clear and to prove maximality we verify theorem 2.2.2 (iii) — that there is only one maximal completely regular filter on S finer than $\mathcal{F} \mid S$. Suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are maximal completely regular filters on S finer than $\mathcal{F} \mid S$, and suppose further that $\mathcal{G}_1 \neq \mathcal{G}_2$. Then by (b) $\mathcal{G}_1$ and $\mathcal{G}_2$ are completely separated in X , and so there exist $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ with G_1 and G_2 completely separated in X . Choose $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \supseteq G_1$, $Z_2 \supseteq G_2$ and $Z_1 \cap Z_2 = \emptyset$. But Z_1 and Z_2 meet $\mathcal{F}$ (they meet $\mathcal{G}_1$, $\mathcal{G}_2$ respectively and hence $\mathcal{F}$ since $\mathcal{F} \mid S$ is coarser than both $\mathcal{G}_1$ and $\mathcal{G}_2$) , so by maximality of $\mathcal{F}$ we must have $Z_1 \cap Z_2 \neq \emptyset$! Thus $\mathcal{G}_1 = \mathcal{G}_2$.

(a) $\Rightarrow$ (c) : Let $\varphi : S \to X$ be the inclusion map of S into X. Let $\mathcal{G}_1$, $\mathcal{G}_2$ be distinct z-ultrafilters on S. Now for $i = 1,2$ $\varphi^{\#}(\mathcal{G}_i) = \{ Z \in \mathcal{Z}(X) : Z \cap S \in \mathcal{G}_i \}$ is a prime z-filter on X (see 3.2.10) so there exist unique z-ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ on X with $\mathcal{U}_1 \supseteq \varphi^{\#}(\mathcal{G}_1)$ and $\mathcal{U}_2 \supseteq \varphi^{\#}(\mathcal{G}_2)$ ($\mathcal{U}_1$ and $\mathcal{U}_2$ are the z-ultrafilters on X determined by $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively).

By maximality of $\mathcal{G}_1$ and of $\mathcal{G}_2$ they cannot meet, so there exist $G_1 \in \mathcal{G}_1$, $G_2 \in \mathcal{G}_2$ with $G_1 \cap G_2 = \emptyset$. Now G_1 and G_2 are disjoint zero-sets of S so by (α) (which holds since by (a) , S is C^*-embedded in X) there exist $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \supseteq G_1$, $Z_2 \supseteq G_2$ and $Z_1 \cap Z_2 = \emptyset$. Now for $i = 1,2$ $\mathcal{Z}(S) \ni Z_i \cap S \supseteq G_i \cap S = G_i$, so $Z_i \cap S \in \mathcal{G}_i$ and thus $Z_i \in \varphi^{\#}(\mathcal{G}_i)$. If $\mathcal{U}_1 = \mathcal{U}_2$ then, since $\varphi^{\#}(\mathcal{G}_i) \subseteq \mathcal{U}_i$, we have $Z_1, Z_2 \in \mathcal{U}_1$ with $Z_1 \cap Z_2 = \emptyset$! Hence $\mathcal{U}_1$ and $\mathcal{U}_2$ are distinct.

(c) $\Rightarrow$ (d) : Let $\mathcal{G}_1, \mathcal{G}_2$ be distinct z-ultrafilters on S. Let $\varphi : S \to X$ be inclusion, and let $\mathcal{U}_1, \mathcal{U}_2$ be the unique z-ultrafilters on X with $\varphi^{\#}(\mathcal{G}_i) \subseteq \mathcal{U}_i$. By (c) $\mathcal{U}_1 \neq \mathcal{U}_2$ and it follows that $\varphi^{\#}(\mathcal{G}_1)$ cannot meet $\varphi^{\#}(\mathcal{G}_2)$ (if they meet then $\sup_{\mathcal{Z}(X)} \{ \varphi^{\#}(\mathcal{G}_1), \varphi^{\#}(\mathcal{G}_2) \}$ would be contained in a z-ultrafilter $\mathcal{U}$ on X strictly finer than both $\varphi^{\#}(\mathcal{G}_1)$ and $\varphi^{\#}(\mathcal{G}_2)$ — but now the prime z-filter $\varphi^{\#}(\mathcal{G}_1)$ is contained in distinct z-ultrafilters $\mathcal{U}$ and $\mathcal{U}_1$!) . Thus there are $Z_i \in \varphi^{\#}(\mathcal{G}_i)$ with $Z_1 \cap Z_2 = \emptyset$. Since Z_1 and Z_2 are disjoint zero-sets of X, they are completely separated. Hence the elements $Z_1 \cap S \in \mathcal{G}_1$ and $Z_2 \cap S \in \mathcal{G}_2$ are completely separated in X.

(d) $\Rightarrow$ (a) : We verify theorem 3.4.3 (b) . Let $\mathcal{F}$ be a maximal completely regular filter on X which meets S. To show $\mathcal{F} \mid S$ is a maximal completely regular filter on S we show that there is only one z-ultrafilter on S finer than $\mathcal{F} \mid S$. Suppose that $\mathcal{G}_1, \mathcal{G}_2$ are z-ultrafilters on S both finer than $\mathcal{F} \mid S$. If $\mathcal{G}_1 \neq \mathcal{G}_2$ then by (d) there exist $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \cap Z_2 = \emptyset$ and with $Z_i \cap S \in \mathcal{G}_i$. But Z_1, Z_2 meet $\mathcal{F}$, so by maximality of $\mathcal{F}$ we should have $Z_1 \cap Z_2 \neq \emptyset$. Thus $\mathcal{G}_1 = \mathcal{G}_2$. $\square$

3.4.6 <u>*Corollary*</u> (J.W.Green [Gr_2 p 104]) : *A zero-set S of X is C-embedded in X iff any two distinct z-ultrafilters on S are completely separated in X .*

Proof. $\Rightarrow$: This is clear from corollary 3.4.5 (d) .

$\Leftarrow$: By corollary 3.4.5 (d) , S is C^*-embedded in X . Since S is a zero-set of X it is completely separated from every disjoint zero-set — i.e., (γ) holds . Thus S is C-embedded in X . □

The following corollary is proved in [Gr_1 p 577] for Tychonoff spaces. The proof does not use the Tychonoff requirement explicitly — it is used in the proofs of results upon which the corollary relies. In [Gr_2] these results are proved for arbitrary spaces, and the remark is made that results like the following will then hold for arbitrary spaces.

3.4.7 <u>*Corollary :*</u> *If S is a discrete subspace of a space X , then S is C^*-embedded in X iff the trace on S of every maximal completely regular filter on X that meets S is an ultrafilter on S .*

Proof. $\Rightarrow$: Suppose S is C^*-embedded in X and that $\mathfrak{F}$ is a maximal completely regular filter on X which meets S . By theorem 3.4.3 (b) $\mathfrak{F} \mid S$ is a maximal completely regular filter on S . But, since S is discrete, all filters on S are completely regular. So $\mathfrak{F} \mid S$ is a maximal filter on S — an ultrafilter on S .

$\Leftarrow$: We verify theorem 3.4.3 (b) . If $\mathfrak{F}$ is a maximal completely regular filter on X which meets S , then by the hypothesis $\mathfrak{F} \mid S$ is an ultrafilter on S . By discreteness of S , $\mathfrak{F} \mid S$ is completely regular on S . Hence $\mathfrak{F} \mid S$ is a maximal completely regular filterbase.

□

We come now to the filter characterisation of C-embedding. We know that S is z-embedded in $X \supseteq S$ *iff* the trace on S of every z-ultrafilter on X that meets S is a z-ultrafilter on S . The theorem shows that *all* z-ultrafilters on S will arise in this way exactly when S is C-

embedded in X . The statement and proof presented here are those of R.L.Blair [$\mathbf{Bl}_1$ 5.3] . The equivalence of (a) and (b) is due to J.W.Green ([$\mathbf{Gr}_1$ theorem 4 for Tychonoff spaces; $\mathbf{Gr}_2$ theorem 5 for arbitrary spaces]) .

Asking when *all* z-ultrafilters on $S \subseteq X$ will arise as traces on S of z-ultrafilters on X that meet S is natural, given our characterisation 3.2.13 of z-embedding. That C-embedding of S in X guarantees this is easily shown (3.4.9 (a) $\Rightarrow$ (b)) , and we show that the converse is true (R.L.Blair [$\mathbf{Bl}_1$ 5.4]) : Consider the following condition on the embedding $S \subseteq X$:

(γ') each z-ultrafilter on S is finer than some z-ultrafilter on X ,

introduced by J.W.Green ([$\mathbf{Gr}_2$ corollary p 103]) and formalised by R.L.Blair in [$\mathbf{Bl}_1$] .

3.4.8 *Proposition* (R.L.Blair [Bl_1 4.4]) : *For any embedding $S \subseteq X$, (γ) holds if (γ') holds .*

Proof. Suppose that $S \subseteq X$ and that (γ') holds. We shall show that 3.4.1 B (iii) holds. Let $\mathcal{G}$ be a z-filter on X that meets S . Then $\mathcal{G} \mid S$ is a z-filterbase on S and so is contained in some z-ultrafilter $\mathcal{F}$ on S . By (γ') there is a z-ultrafilter $\mathcal{U}$ on X coarser than $\mathcal{F}$. Clearly $\mathcal{U}$ meets $\mathcal{G} \mid S$. □

The converse to this proposition is false ([$\mathbf{Bl}_1$ 4.5 (a)]) : The x-axis S of the Moore plane Γ is a zero-set in Γ , and so (γ) holds trivially. In [$\mathbf{Gr}_2$ lemma 8] it is shown that (γ') fails.

Suppose that every z-ultrafilter on S is the trace on S of some z-ultrafilter on X that meets S . Then the condition (γ') on S is satisfied, and so (γ) holds by the preceding proposition. To conclude that S is C-embedded in X , it remains to verify z-embedding of S in X , for which we verify 3.2.13 (b)$'$. Let $\mathcal{G}$ be a z-ultrafilter and let $\varphi\colon S \to X$ be the inclusion map. By hypothesis, there is a z-ultrafilter $\mathcal{U}$ on X meeting S such that $\mathcal{U} \mid S = \mathcal{G}$. Then $\mathcal{U} \subseteq \varphi^{\#}(\mathcal{G})$, and so $\mathcal{G} = \mathcal{U} \mid S \subseteq \varphi^{\#}(\mathcal{G}) \mid S$.

Knowing this equivalence and knowing the correspondence between z-ultrafilters on a space and maximal completely regular filters on the space of chapter 2 makes it not too difficult a task to recast the equivalence. This is the content of the following theorem.

3.4.9 <u>*Theorem*</u> (R.L.Blair [Bl_1 5.3]) : *If $S \subseteq X$ then the following are equivalent:*

(a) *S is C-embedded in X ,*

(b) *every z-ultrafilter on S is the trace on S of some z-ultrafilter on X ,*

(c) *every z-ultrafilter on S is coarser than the trace on S of some z-ultrafilter on X which meets S ,*

(d) *every maximal completely regular filter on S is coarser than the trace on S of some z-ultrafilter on X ,*

(e) *if $\mathfrak{F}$ is a maximal completely regular filter on S , if $\mathfrak{F}^*$ is the unique maximal completely regular filter on X coarser than $\mathfrak{F}$ and if $\mathfrak{U}$ is the unique z-ultrafilter on X finer than $\mathfrak{F}^*$, then $\mathfrak{F} \leq \mathfrak{F}^* \mid S$ and $\mathfrak{U}$ meets S .*

Proof. (a) $\Rightarrow$ (b) : Let $\mathfrak{F}$ be a z-ultrafilter on S . Let $\varphi\colon S \to X$ be the inclusion map of S iuto X . Then there is a (unique) z-ultrafilter $\mathfrak{U}$ on X with $\varphi^{\#}(\mathfrak{F}) \subseteq \mathfrak{U}$. Suppose there were a $Z \in \mathfrak{U}$ with $Z \cap S = \emptyset$. By C-embedding S is completely separated from the disjoint zero-set Z , hence there is a $Z' \in \mathfrak{Z}(X)$ with $Z' \supseteq S$ and $Z' \cap Z = \emptyset$. But now $Z' \in \varphi^{\#}(\mathfrak{F}) \subseteq \mathfrak{U}$, so $\emptyset = Z \cap Z' \in \mathfrak{U}$! Thus $\mathfrak{U}$ meets S , so $\mathfrak{U} \mid S$ is a z-ultrafilter on S (theorem 3.2.13 (c)). By (b) of theorem 3.2.13 , $\mathfrak{F} \subseteq \varphi^{\#}(\mathfrak{F}) \mid S \subseteq \mathfrak{U} \mid S$. Thus, by maximality of $\mathfrak{F}$, $\mathfrak{F} = \mathfrak{U} \mid S$.

(b) $\Rightarrow$ (c) : this is trivial.

(c) $\Rightarrow$ (d) : Let $\mathfrak{F}$ be a maximal completely regular filter on S . Then there is a (unique) z-ultrafilter $\mathfrak{G}$ on S finer than $\mathfrak{F}$. By (c) , $\mathfrak{G} \leq \mathfrak{U} \mid S$ for some z-ultrafilter $\mathfrak{U}$ on X which meets S . Now $\mathfrak{F} \leq \mathfrak{U} \mid S$.

(d) $\Rightarrow$ (e) : Let $\mathfrak{F}$, $\mathfrak{F}^*$ and $\mathfrak{U}$ be as in (e) . By (d) $\mathfrak{F} \leq \mathfrak{U}' \mid S$ for some z-ultrafilter $\mathfrak{U}'$ on X which meets S . Since $\mathfrak{F}^*$ is coarser than $\mathfrak{F}$ we have $\mathfrak{F}^* \leq \mathfrak{U}' \mid S$, hence $\mathfrak{U}'$ meets $\mathfrak{F}^*$.

We know that a zero-set of X will be in $\mathcal{U}$ exactly when it meets $\mathcal{F}^*$ (theorem 2.3.4) , thus $\mathcal{U}' \subseteq \mathcal{U}$. Since both $\mathcal{U}$ and $\mathcal{U}'$ are z-ultrafilters, we have $\mathcal{U} = \mathcal{U}'$. This means that $\mathcal{U}$ meets S .

Suppose it is not the case that $\mathcal{F} \leq \mathcal{F}^* \mid S$. Then there is an $F \in \mathcal{F}$ with $(A \cap S) - F \neq \emptyset$ for every $A \in \mathcal{F}^*$. By complete regularity of $\mathcal{F}$ we may choose an $F' \in \mathcal{F}$ and an $f \in C(S)$ with $f \geq 0$, $f(F') = 1$ and $f(S - F) = 0$. Define $\mathcal{B} = \{f^{-1}[0,r] : r > 0\}$. Easily, $\mathcal{B}$ is a completely regular filterbase on S which meets $\mathcal{F}^* \mid S$. Let $\mathcal{G}$ be a maximal completely regular filter on S containing both $\mathcal{B}$ and $\mathcal{F}^* \mid S$. By (d) there is a z-ultrafilter $\mathcal{U}''$ on X which meets S and satisfies $\mathcal{G} \leq \mathcal{U}'' \mid S$. So $\mathcal{B} \leq \mathcal{U}'' \mid S$ and $\mathcal{F}^* \mid S \leq \mathcal{U}'' \mid S$. Hence $\mathcal{U}''$ meets $\mathcal{F}^*$, so (as above) $\mathcal{U}'' = \mathcal{U}$. Thus $\mathcal{U}'' \mid S \geq \mathcal{F}$ (since $\mathcal{F} \leq \mathcal{F}^* \mid S \leq \mathcal{U}'' \mid S$) . But we already have $\mathcal{U}'' \geq \mathcal{B}$, so $F' \cap f^{-1}[0,\frac{1}{2}] \neq \emptyset$, a contradiction. So $\mathcal{F} \leq \mathcal{F}^* \mid S$.

(e) $\Rightarrow$ (a) : By theorem 3.4.3 (e) $\Rightarrow$ (a) , S is C^*-embedded in X . By theorem 3.4.1 B (ii) , (γ) holds . Now by theorem 3.3.8, S is C-embedded in X . □

The condition (γ') introduced above can also be used in a characterisation of C-embedding. First we find an alternative form for (γ') :

3.4.10 *Proposition* (R.L.Blair [Bl_1 4.3]) : *If $S \subseteq X$ then the following are equivalent:*

(i) *(γ') holds ,*

(ii) *if $\mathcal{F}$ is any z-ultrafilter on S , then the z-filter on X determined by $\mathcal{F}$ is a z-ultrafilter on X .*

Proof. (i) $\Rightarrow$ (ii) : Let $\varphi: S \to X$ be the inclusion map of S into X , and let $\mathcal{F}$ be a z-ultrafilter on S . By (γ') there is a z-ultrafilter $\mathcal{U}$ on X with $\mathcal{U} \leq \mathcal{F}$. Now $\mathcal{U} \subseteq \varphi^{\#}(\mathcal{F})$, for if $U \in \mathcal{U}$ then $U \supseteq F$ for some $F \in \mathcal{F}$ and so $\mathcal{Z}(S) \ni U \cap S \supseteq F \cap S = F \in \mathcal{F}$. Thus, by maximality of $\mathcal{U}$, $\varphi^{\#}(\mathcal{F}) = \mathcal{U}$, so $\varphi^{\#}(\mathcal{F})$ is a z-ultrafilter on X .

(ii) $\Rightarrow$ (i) : Let $\mathcal{F}$ be any z-ultrafilter on S . Let $\varphi: S \to X$ be inclusion. By (ii) $\varphi^{\#}(\mathcal{F})$ is a z-ultrafilter on X , and it is clear that $\varphi^{\#}(\mathcal{F}) \leq \mathcal{F}$. □

3.4.11 *Corollary* (J.W.Green [Gr_2 corollary p 103]) : *Let $S \subseteq X$. Then S is C-embedded in X iff S satisfies (γ') and $\mathcal{F} \mid S$ is a z-ultrafilter on S for all z-ultrafilters $\mathcal{F}$ on X that meet S (i.e., S is z-embedded in X) .*

Proof. $\Rightarrow$: By theorem 3.4.9 (a) $\Rightarrow$ (b) , (γ') holds . The second part holds by theorem 3.2.13 (a) $\Rightarrow$ (c) .

$\Leftarrow$: Condition (γ) holds by proposition 3.4.8 , and S is z-embedded in X . □

Thus $S \subseteq X$ is C-embedded in X *iff* S is z-embedded in X and (γ') holds *iff* S is z-embedded in X and (γ) holds , even though (γ') and (γ) are not equivalent.

3.4.12 *Theorem* (J.W.Green [Gr_1 theorem 6] ; [Gr_2] for arbitrary spaces) : *Let S be a zero-set of X . Then S is C-embedded in X iff the trace on S of every z-ultrafilter on X that meets S is a z-ultrafilter on S .*

Proof. $\Rightarrow$: This follows from theorem 3.2.13 (a) $\Rightarrow$ (c) .

$\Leftarrow$: By theorem 3.2.13 (c) $\Rightarrow$ (a) , S is z-embedded . Since S is a zero-set of X , (γ) holds. Hence S is C-embedded in X . □

Our filter-theoretic characterisation of z-, C^*-, and C- embedding is complete. In the next chapter we will present localisations of the characterisations of section 3.3 together with some localisations of section 3.4 , all of which shed more light on the results of this chapter.

CHAPTER 4

Localisation of global characterisations

4.1 INTRODUCTION

In this chapter we obtain localisations of some of the global results of the preceding chapter, in the sense explained on page 1.2 . Thus we obtain characterisations of z-embedding of a given function on a subspace, of extendibility of a given bounded function and of extendibility of a given unbounded function. In localising the classical results (like the Urysohn Extension Theorem) we follow R.L.Blair ([Bl_2]) . We then present some apparently new filter characterisations of z-embedding of a function (localising theorem 3.2.13) , and a partially complete localisation of theorem 3.4.3 (characterising C^*-embedding).

4.2 LOCALISATION OF CLASSICAL RESULTS

(a) *Localisation of classical results on z-embedding of a subspace*

One of the central results of chapter 3 is theorem 3.2.4 : *$S \subseteq X$ is z-embedded in X iff each (bounded) $f \in C(S)$ can be approximated uniformly on S by continuous functions on cozero-sets of X that contain S* . This led to characterisations of both C^*- and C- embedding, and these were the starting point for filter characterisations of both embeddings. We begin our localisations by obtaining a very pleasing localisation of theorem 3.2.4 . What makes this localisation particularly elegant is that much of its proof is extracted from that of theorem 3.2.4 , making the latter's proof more transparent.

4.2.1 Theorem (R.L.Blair [Bl_2 2.2]) : *Let $S \subseteq X$ and $f \in C(S)$. Then f is z-embedded in X iff f can be uniformly approximated on S by continuous functions on cozero-sets of X that contain S* .

Proof. $\Rightarrow$: Suppose that $f \in C(S)$ is z-embedded in X . Let $\epsilon > 0$ be given. For each $n \in \mathbb{Z}$ define $A_n = \{ s \in S : n - 1 < \frac{1}{\epsilon} f(s) < n + 1 \}$. Now A_n is the preimage of an open set in $\mathbb{R}$ under a continuous map, so A_n is a cozero-set of S . By z-embedding of f in X we may, for each $n \in \mathbb{Z}$, choose a cozero-set C_n of X such that $C_n \cap S = A_n$ (simply note that $A_n = S - \left(L_{(n-1)\epsilon}(f) \cup L^{(n+1)\epsilon}(f)\right)$ and that $L_{(n-1)\epsilon}(f)$ and $L^{(n+1)\epsilon}(f)$ both extend to zero-sets of X) . The proof now proceeds exactly as in the proof of theorem 3.2.4 (we have simply obtained the C_n's in a different manner).

$\Leftarrow$: Let $a \in \mathbb{R}$. For $n \in \mathbb{N}$ choose a cozero-set P_n of X and an $f \in C(P_n)$ with $P_n \supseteq S$ and $|f_n(x) - f(x)| \leq \frac{1}{n}$ for $x \in S \quad \cdots (\bigstar)$.

Define $Z'_n = \{x \in P_n : f_n(x) \leq a + \frac{1}{n}\}$. Now Z'_n is a zero-set of P_n , and P_n is z-embedded in X (being a cozero-set of X) . So for each n there is a $Z_n \in \mathcal{Z}(X)$ with $Z_n \cap P_n = Z'_n$. Now $\bigcap_n Z_n \in \mathcal{Z}(X)$ and we have $\left(\bigcap_n Z_n\right) \cap S = L_a(f)$: it is clear that $L_a(f) \subseteq \left(\bigcap_n Z_n\right) \cap S$; and if $x \in \left(\bigcap_n Z_n\right) \cap S$ then $x \in Z_n \cap S \subseteq Z_n \cap P_n$ for all n so, $x \in \bigcap_n Z'_n$ meaning $f_n(x) \leq a + \frac{1}{n}$ for all n ; by $(\bigstar)$ we now have $f(x) \leq a + \frac{2}{n}$ for all n , so $f(x) \leq a$ — i.e., $x \in L_a(f)$.

Since $L^a(f) = L_{-a}(-f)$ it follows that $L^a(f)$ also extends to a zero-set of X . $\square$

Note how the global characterisation 3.2.4 of z-embedding is easily recovered from this localisation, thus giving more insight into theorem 3.2.4 .

(b) *Localisation of classical results on C^*-embedding*

Recall from chapter 3 the following condition on an embedding $S \subseteq X$:

(α) disjoint zero-sets of S are completely separated in X .

We showed that S is C^*-embedded in X *iff* (α) holds (the Urysohn Extension Theorem). We shall localise condition (α) in order to obtain a necessary and sufficient condition for the extendibility of a given bounded continuous function on S . The result is an aesthetically pleasing localisation of the Urysohn Extension Theorem.

Let $f \in C(S)$ and consider the following condition on f , introduced in [$\mathbf{Bl}_2$ p66] :

(α_f) if $a < b$ in $\mathbf{R}$, then $L_a(f)$ and $L^b(f)$ are completely separated in X .

Note that whereas (α) demands that *all* pairs of disjoint zero-sets of S be completely separated in X , (α_f) demands this of only some smaller collection of disjoint zero-set pairs that is associated with f . It is in this sense that we view (α_f) as a localisation of its global counterpart. Furthermore, (α) can be recovered from its proposed localisation:

4.2.2 <u>*Proposition*</u> (R.L.Blair [$\mathbf{Bl}_2$ 3.1 (a)]) : *Let $S \subseteq X$. Then (α) holds iff (α_f) holds for every (bounded) $f \in C(S)$.*

Proof. $\Rightarrow$: this is obvious.

$\Leftarrow$: Let Z, Z' be disjoint zero-sets of S . Then Z and Z' are completely separated in S , so there is an $f \in C(S)$ with $0 \leq f \leq 1$, $f(Z) = 0$ and $f(Z') = 1$. By (α_f) , $L_0(f) \supseteq Z$ and $L^1(f) \supseteq Z'$ are completely separated in X . $\square$

The next theorem is our single function analogue of the Urysohn Extension Theorem. It shows that (α_f) is a successful localisation of (α) , and provides an alternative (and in some ways more transparent) proof of the Urysohn Extension Theorem. The theorem is due to S.Mrówka ([$\mathbf{Mr}_1$ 4.11]) who deduces it from a powerful general approximation theorem ([$\mathbf{Mr}_1$ 2.7]) . The proof presented here is that of R.L.Blair in [$\mathbf{Bl}_2$ 3.2] , based on a proof of [$\mathbf{Mr}_1$ 2.7] communicated to Blair by H.E.White, Jr.

4.2.3 <u>*Theorem :*</u> *Let $S \subseteq X$ and $f \in C^*(S)$. Then f has a continuous extension over X iff (α_f) holds.*

Proof. $\Rightarrow$: this is clear.

$\Leftarrow$: Suppose (α_f) holds. Choose a positive integer m with $|f| \leq m$. For $n \in \omega$, define $p(n) = m2^{n+2} - 1$. Now, for $n \in \omega$ and for integers j with $0 \leq j \leq p(n)$,

$A_{nj} = \{x \in S : f(x) \leq -m + j2^{-n-1}\}$ and $B_{nj} = \{x \in S : f(x) \geq -m + (j+1)2^{-n-1}\}$ are disjoint lower and upper Lebesgue-sets of f , so by (α_f) there is an $f_{nj} \in C^*(X)$ with $0 \leq f_{nj} \leq 1$, $f_{nj}(A_{nj}) = 0$ and $f_{nj}(B_{nj}) = 1$. Note that for a given n , $-m + j\,2^{-n-1}$ for $0 \leq j \leq p(n)$ ranges from $-m$ to $m - 2^{-n-1}$ in steps of 2^{-n-1} ; and $-m + (j+1)\,2^{-n-1}$ for $0 \leq j \leq p(n)$ ranges from $-m + 2^{-n-1}$ to m in steps of 2^{-n-1} .

Define $f_n = -m + 2^{-n-1}\sum_{j=0}^{p(n)} f_{nj}$. Note that $2^{-n-1}\sum_{j=0}^{p(n)} f_{nj} \leq 2^{-n-1}(1 + p(n))$ $= 2^{-n-1}(m2^{n+2}) = 2m$, so that $-m \leq f_n \leq m$. Note also that each f_n is continuous on X . We will show that $f_n \to f$ uniformly on S .

Let $n \in \omega$. We will show that $|f_{n+1} - f_n| \leq 2^{-n}$ on S . Let $x \in S$. Since $0 \leq \dfrac{(f+m)}{2m} \leq 1$, we can choose an integer k with $0 \leq k \leq p(n)$ and $\dfrac{k}{m2^{n+2}} \leq \dfrac{f(x)+m}{2m} \leq \dfrac{k+1}{m2^{n+2}}$, i.e., $-m + k2^{-n-1} \leq f(x) \leq -m + (k+1)2^{-n-1} \cdots (\bigstar)$.

Now if $j \leq k-1$ then $f_{nj}(x) = 1$ since by $(\bigstar)$ $x \in B_{nj}$; and if $j \geq k+1$ then $f_{nj}(x) = 0$ since by $(\bigstar)$ $x \in A_{nj}$. Substituting in the defining expression for f_n we find that $f_n(x) = -m + 2^{-n-1}(k + f_{nk}(x))$. Thus $f(x), f_n(x) \in [-m + k\,2^{-n-1}, -m + (k+1)\,2^{-n-1}]$ — by $(\bigstar)$ and since $0 \leq f_{nk}(x) \leq 1$. So we have that $|f(x) - f_n(x)| \leq 2^{-n-1}$ for $n \in \omega$. Now we have $|f_{n+1}(x) - f_n(x)| \leq |f_{n+1}(x) - f(x)| + |f_n(x) - f(x)| \leq 2^{-n-2} + 2^{-n-1}$ $< 2^{-n}$.

We have shown that $f_n \to f$ uniformly on S . It follows by lemma 3.2.5 that f extends contiuuously over X . □

Note that the Urysohn Extension Theorem follows immediately from 4.2.3 and 4.2.2 . In that the proof of 4.2.3 makes it clear why (α_f) succeeds in guaranteeing the extendibility of f , this new proof of the Urysohn Extension Theorem makes the success of (α) in characterising C^*-embedding more transparent.

(c) *Localisation of classical results on C-embedding*

We know that the conjunction of (α) and the following condition on $S \subseteq X$ characterises C-embedding:

(γ) S is completely separated from every disjoint zero-set of X .

We shall use (α_f) and a localisation of (γ) to obtain a necessary and sufficient condition for the extendibility of any given continuous function on a subspace. The result is an elegant localisation of the Gillman and Jerison characterisation of C-embedding.

4.2.4 *Definition* (R.L.Blair [Bl_2 p66]) : Let $S \subseteq X$, $f \in C(S)$ and $A \subseteq X$. We shall say that A is *completely separated from* f if A and $f^{-1}[a,b]$ are completely separated in X for each $a < b$ in $\mathbb{R}$.

Let $f \in C(S)$ and consider the following condition on f , introduced in [Bl_2 p66] :

(γ_f) S is completely separated from every zero-set of X that is completely separated from f .

Note that (γ_f) demands that some particular class of zero-sets of X that are disjoint from S be completely separated from S , whereas (γ) demands this of *all* zero-sets disjoint from S . As before, (γ) is recoverable from its global counterpart:

4.2.5 *Theorem* (R.L.Blair [Bl_2 3.1 (b)]) : *Let $S \subseteq X$. Then (γ) holds iff (γ_f) holds for every $f \in C(S)$.*

Proof. $\Rightarrow$: this is obvious.

$\Leftarrow$: Suppose (γ_f) holds for each $f \in C(S)$. Suppose $g \in C(X)$ with $Z(g) \cap S = \emptyset$. Define $f = \left|\frac{1}{g\,|\,S}\right|$. Then $f \in C(S)$, and if $0 \leq a < b$ in $\mathbb{R}$ then $|\,g\,| \geq \frac{1}{b}$ on $f^{-1}[a,b]$ so $Z(g)$ and $f^{-1}[a,b]$ are completely separated in X . By (γ_f) , S and $Z(g)$ are completely separated in X . $\square$

First let us note the following corollary to theorem 4.2.3 , which shows that (α_f) admits the extendibility of any truncation of $f \in C(S)$, no matter how large the cut-off level.

4.2.6 <u>*Definition*</u> : By a *truncation* of $f \in C(S)$ we mean a function of the form $(f \wedge a) \vee -a$, where $0 \leq a \in \mathbb{R}$.

4.2.7 <u>*Corollary*</u> (R.L.Blair [Bl_2 3.3]) : *Let $S \subseteq X$ and $f \in C(S)$. Then (α_f) holds iff every truncation of f has a continuous extension over X .*

Proof. $\Rightarrow$: Suppose (α_f) holds. Let $0 \leq c \in \mathbb{R}$; we verify $(\alpha_{(f \wedge c) \vee -c})$. Let $a < b$ in $\mathbb{R}$. If $a > c$ then $L^b((f \wedge c) \vee -c) = \emptyset$; and if $a < -c$ then $L_a((f \wedge c) \vee -c) = \emptyset$. Now suppose that $-c \leq a < b \leq c$. Then $L_a((f \wedge c) \vee -c) = L_a(f)$ and $L^b((f \wedge c) \vee -c) = L^b(f)$ and so are completely separated in X since by (α_f) $L_a(f)$ and $L^b(f)$ are completely separated in X .

$\Leftarrow$: Let $a < b$ in $\mathbb{R}$. Choose $c > \max\{\, |a|, |b| \,\}$. Then $L_a((f \wedge c) \vee -c) = L_a(f)$ and $L^b((f \wedge c) \vee -c) = L^b(f)$ so by $(\alpha_{(f \wedge c) \vee -c})$, which holds since $(f \wedge c) \vee -c \in C^*(S)$ extends over X , $L_a(f)$ and $L^b(f)$ are completely separated in X . $\square$

Of course, (α_f) is necessary for the extendibility of $f \in C(S)$. The following theorem shows that (α_f) actually guarantees the partial extendibility of f (over some cozero-superset of S) , and that if (γ_f) also holds then this leads to the extendibility over X of f .

4.2.8 <u>*Theorem*</u> : *Let $S \subseteq X$ and $f \in C(S)$. Then:*

(a) *if (α_f) holds then f extends continuously over some cozero-set $P \supseteq S$ with $X - P$ completely separated from f , and*

(b) *if (γ_f) holds then if $f = g \mid S$ for some $g \in C(T)$ where $T \supseteq S$ is a cozero-set of X with $X - T$ completely separated from f , then f extends continuously over X .*

Proof. (a) Suppose that (α_f) holds. Let $\varphi : \mathbb{R} \cong (-1,1)$ be an order-preserving homeomorphism. Now for $a \in \mathbb{R}$, $L_a(f) = L_{\varphi(a)}(\varphi \circ f)$ and $L^a(f) = L^{\varphi(a)}(\varphi \circ f)$. Since (α_f) holds, we see that $(\alpha_{\varphi \circ f})$ holds. Thus $\varphi \circ f$, being bounded and satisfyiug $(\alpha_{\varphi \circ f})$, extends over X — say $k \in C(X)$ with $k \mid S = \varphi \circ f$.

Define $P = X - \left(L_{-1}(k) \cup L^1(k)\right)$. Then P is a cozero-set of X, and $P \supseteq S$. Let $a < b$ in $\mathbf{R}$. Then $L_{-1}(k) \cup L^1(k)$ and $L^{\varphi(a)}(k) \cap L_{\varphi(b)}(k)$ are completely separated (by $|k|$) in X. Now $X - P = L_{-1}(k) \cup L^1(k)$ and $L^a(f) \cap L_b(f) = L^{\varphi(a)}(\varphi \circ f) \cap L_{\varphi(b)}(\varphi \circ f) = L^{\varphi(a)}(k \mid S) \cap L_{\varphi(b)}(k \mid S) \subseteq L^{\varphi(a)}(k) \cap L_{\varphi(b)}(k)$, so that $X - P$ and $L^a(f) \cap L_b(f) = f^{-1}[a,b]$ are also completely separated in X. Thus $X - P$ is completely separated from f.

Now $\varphi^{-1} \circ (k \mid P)$ is a continuous map on P (note that $|k \mid P| < 1$, so that $\varphi^{-1} \circ (k \mid P)$ is properly defined), and $(\varphi^{-1} \circ (k \mid P)) \mid S = \varphi^{-1} \circ (k \mid S) = \varphi^{-1} \circ (\varphi \circ f) = f$. Thus $\varphi^{-1} \circ (k \mid P)$ is a continuous extension of f over P.

(b) Suppose T is a cozero-set of X with $T \supseteq S$ and $X - T$ completely separated from f, and suppose $f = g \mid S$ with $g \in C(T)$. Suppose (γ_f) holds. Let $\varphi : \mathbf{R} \cong (-1,1)$ be an order preserving homeomorphism. As in (a) above, $L_a(g) = L_{\varphi(a)}(\varphi \circ g)$ and $L^a(g) = L^{\varphi(a)}(\varphi \circ g)$. Thus all the Lebesgue-sets of $\varphi \circ g$ are just Lebesgue-sets of g, and by z-embedding of g in X (which holds since T is z-embedded in X, being a cozero-set of X) these extend to zero-sets of X. So $\varphi \circ g$ is z-embedded in X. By theorem 4.2.1 there is, for each $n \in \mathbf{N}$, a cozero-set P_n of X and a $g_n \in C(P_n)$ with $P_n \supseteq T$ and $|(\varphi \circ g)(t) - g_n(t)| < \frac{1}{n}$ for all $t \in T$. Since $\varphi \circ g$ is bounded we may assume that each g_n is bounded.

By hypothesis we have $X - T$ completely separated from f, so by (γ_f) we conclude that $X - T$ is completely separated from S ($X - T$ is a zero-set of X since T is cozero). Choose $Z \in \mathfrak{Z}(X)$ with $S \subseteq Z$ and $Z \cap (X - T) = \emptyset$. Note that $S \subseteq Z \subseteq T$. Now Z and $X - P_n \subseteq X - T$ are disjoint zero-sets of X, so we may choose an $h_n \in C(X)$ with $h_n(Z) = 1$ and $h_n(X - P_n) = 0$.

Note that $\operatorname{coz} h_n \subseteq P_n = \operatorname{dom} g_n$, and define $f_n = \begin{cases} g_n h_n & \text{on } \operatorname{coz} h_n \\ 0 & \text{on } Z(h_n) \end{cases}$. As in proposition 1.2.15, $f_n \in C(X)$ (g_n is bounded). Also $f_n(z) = g_n(z)$ for each $z \in Z \subseteq T$, and thus have $|(\varphi \circ g)(z) - f_n(z)| < \frac{1}{n}$ for $z \in Z$. So $f_n \to (\varphi \circ g) \mid Z$ uniformly on Z, and by lemma 3.2.5 we deduce that $(\varphi \circ g) \mid Z$ extends continuously over X, say $h \in C(X)$ with $h \mid Z = (\varphi \circ g) \mid Z$.

Define $Q = \{x \in X : |h(x)| \geq 1\}$. Then $\mathfrak{Z}(X) \ni Q \subseteq X - Z$, i.e., $Q \cap Z = \emptyset$ and so Q and $S \subseteq Z$ are completely separated in X. So there is a $u \in C(X)$ with $0 \leq u \leq 1$, $u(S) = 1$ and $u(Q) = 0$. If $x \in X$ then $|h(x)| \geq 1 \Rightarrow x \in Q \Rightarrow u(x) = 0 \Rightarrow u(x)h(x) = 0$; and $|h(x)| < 1 \Rightarrow |u(x)h(x)| < 1$ since $0 \leq u(x) \leq 1$. Thus $|u(x)h(x)| < 1$ for all $x \in X$, and so $\varphi^{-1} \circ (uh)$ is properly defined. Now $(uh) | S = (u | S)((\varphi \circ g) | S) = (u | S)(\varphi \circ f) = \varphi \circ f$ since $u | S = 1$ and $g | S = f$. Thus we have $(\varphi^{-1} \circ uh) | S = f$, so f extends over X. □

This theorem is due to R.L.Blair in [Bl_2 3.6] , although the "proof" of (b) presented there is flawed (lemma 3.2.5 is applied incorrectly, making the second half of his proof entirely invalid) . The proof of (b) presented here is based on Blair's "proof", though some new constructs were found to be necessary in order to fill in the gaps left by his error. The statement of (b) in [Bl_2] is also not quite correct — it omits to require that T be cozero in X, even though this is in fact necessary in his proof ; instead he requires the weaker condition that g be z-embedded in X.

4.2.9 Corollary (R.L.Blair [Bl_2 3.7]) : *Let $S \subseteq X$ and $f \in C(S)$. If (α_f) holds then f is z-embedded in X.*

Proof. By the preceding theorem, f can be uniformly approximated on S by continuous functions on cozero-sets of X which contain S. By theorem 4.2.1 , f is z-embedded in X. □

We come now to the localisation of the Gillman and Jerison characterisation of C-embedding. Again, despite the complexity of its proof, this result (together with theorem 4.2.8 , of course) gives an intuitive feel as to why it succeeds and sheds new light on its global counterpart.

4.2.10 *Theorem* (R.L.Blair [BL_2 3.8]) : *Let* $S \subseteq X$ *and* $f \in C(S)$. *Then* f *extends continuously over* X *iff both* (α_f) *and* (γ_f) *hold.*

Proof. $\Rightarrow$: Suppose $f = g \mid S$ with $g \in C(X)$. It is obvious that (α_f) holds. Now suppose that $A \in \mathfrak{Z}(X)$ is completely separated from f . For $n \in \mathbb{Z}$, $L_{n-2}(g) \cup L^{n+2}(g)$ and $L^{n-1}(g) \cap L_{n+1}(g)$ are disjoint zero-sets of X so there is an $h_n \in C(X)$ with $h_n \geq 0$, $h_n\Big(L_{n-2}(g) \cup L^{n+2}(g)\Big) = 0$ and $h_n\Big(L^{n-1}(g) \cap L_{n+1}(g)\Big) = 1$.

Now A is completely separated from each $f^{-1}[a,b]$. In particular A is completely separated from $f^{-1}[n-1,n+1] = L^{n-1}(f) \cap L_{n+1}(f)$. So we may choose a $k_n \in C(X)$ with $k_n \geq 0$, $k_n(A) = 0$ and $k_n\Big(L^{n-1}(f) \cap L_{n+1}(f)\Big) = 1$.

Define $u_n = h_n k_n$. Now the family $\{\operatorname{coz} u_n : n \in \mathbb{Z}\}$ is locally finite in X , since it is the case that $\operatorname{coz} u_n \subseteq \operatorname{coz} h_n \subseteq X - \Big(L_{n-2}(g) \cup L^{n+2}(g)\Big) = \{x \in X : n-2 < g(x) < n+2\}$ (if $x \in X$ and we choose $m \in \mathbb{Z}$ with $m-2 < g(x) < m+2$, then $g^{-1}(m-2,m+2)$ is an open neighbourhood of x meeting only finitely many members of $\{\operatorname{coz} u_n : n \in \mathbb{Z}\}$). Hence $u = \sum_{n \in \mathbb{Z}} u_n \in C(X)$, since every point of X has a neighbourhood on which u is continuous.

If $x \in S$ then if we choose m to be the largest integer with $m \leq f(x)$ we have $x \in L^{m-1}(f) \cap L_{m+1}(f) \subseteq L^{m-1}(g) \cap L_{m+1}(g)$ so that $h_m(x) = 1$ and $k_m(x) = 1$, and thus $u_m(x) = 1$. Hence $u(x) \geq 1$ for every $x \in S$. Since each k_n , and hence each u_n , is zero on A we have $u(A) = 0$. Thus u completely separates S and A . So (γ_f) holds.

$\Leftarrow$: Suppose both (α_f) and (γ_f) hold. By theorem 4.2.8 (a) there exists a cozero-set T of X and a $g \in C(T)$ with $T \supseteq S$, $g \mid S = f$ and $X - T$ completely separated from f . All the conditions to theorem 4.2.8 (b) are satisfied, and we conclude that f extends continuously over X . $\square$

Again note that the global Gillman and Jerison characterisation of C-embedding follows immediately from 4.2.10 , 4.2.2 and 4.2.5 .

In chapter 3 we showed that C-embedding of S in X is equivalent to z-embedding of S in X in addition to condition (γ) holding. The obvious candidate for a localisation of this would be that $f \in C(S)$ extends over X *iff* f is z-embedded in X and (γ_f) holds. But in [$\mathbf{Bl_2}$ 3.12] it is shown that this is not the case, and Blair remarks that no condition *on* f has been isolated whose conjunction with z-embedding of f will yield extendibility of f. However, he shows that certain conditions *on the subspace* in addition to z-embedding of f yield extendibility:

4.2.11 Proposition: *Let $S \subseteq X$ and $f \in C^*(S)$. If f is z-embedded in X and if (β) holds, then f extends continuously over X.*

Proof. Let $a < b$ in $\mathbb{R}$, and choose $Z_1, Z_2 \in \mathcal{Z}(X)$ with $Z_1 \cap S = L_a(f)$, $Z_2 \cap S = L^b(f)$. Then $Z_1 \cap Z_2 \cap S = \emptyset$ so, by (β), $L_a(f)$ and $L^b(f)$ are completely separated in X. Thus (α_f) holds, and so by theorem 4.2.3 we conclude that f extends over X. □

4.2.12 Proposition: *Let $S \subseteq X$ and $f \in C(S)$. If f is z-embedded in X and if (γ) holds, then f extends continuously over X.*

Proof. If (γ) holds then by 3.3.4 (β) will hold. By proposition 4.2.5 (γ_f) will also hold. As in the preceding proposition, (α_f) holds. So by theorem 4.2.10 we conclude that f extends continuously over X. □

4.3 LOCALISATIONS OF FILTER-THEORETIC RESULTS

Just as the preceding results of this chapter are localisations of the classical characterisations of z-, C^*- and C- embedding, we wish to obtain localisations of the filter-theoretic characterisations of these embeddings (i.e., of theorems 3.2.13, 3.4.3 and 3.4.9). This is achieved in full for z-embedding of a function, but only in part for extendibility of a single given function.

We begin by localising the non-filter-theoretic condition 3.2.13 (f) , characterising z-embedding of a subspace. This is the key to a localisation of theorem 3.2.13 (a) $\Leftrightarrow$ (g) .

4.3.1 Theorem : *Let $S \subseteq X$ and $f \in C(S)$. Then f is z-embedded in X iff for each $a < b$ there exist zero-sets $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $L_a(f) \subseteq Z_1$, $L^b(f) \subseteq Z_2$ and $Z_1 \cap Z_2 \cap S = \emptyset$.*

Proof. $\Rightarrow$: Let $a < b$ in $\mathbb{R}$. By z-embedding of f , there exist $Z_1, Z_2 \in \mathfrak{Z}(X)$ with $Z_1 \cap S = L_a(f)$ and $Z_2 \cap S = L^b(f)$. Now $L_a(f) \subseteq Z_1$, $L^b(f) \subseteq Z_2$ and $Z_1 \cap Z_2 \cap S = L_a(f) \cap L^b(f) = \emptyset$.

$\Leftarrow$: Let $a \in \mathbb{R}$. For each $n \in \mathbb{N}$, there is a $Z_n \in \mathfrak{Z}(X)$ with $L_a(f) \subseteq Z_n$ and $Z_n \cap L^{a+\frac{1}{n}}(f) = \emptyset$. Now $(\bigcap Z_n) \cap S = L_a(f)$. Similarly for $L^a(f)$. ◻

Next we use the preceding theorem to localise condition 3.2.13 (g) .

4.3.2 *Definition* : If $S \subseteq X$ and $f \in C(S)$ then for each $a \in \mathbb{R}$ for which $L_a(f) \neq \emptyset$ we define $\mathfrak{B}_a(f) = \{ A \subseteq S : A \supseteq L_{a+r}(f) \text{ for some } r > 0 \}$. It is plain to see that $\mathfrak{B}_a(f)$ is a completely regular filter on S . Similarly, for each $b \in \mathbb{R}$ with $L^b(f) \neq \emptyset$ we define $\mathfrak{B}^b(f) = \{ A \subseteq S : A \supseteq L^{b-r}(f) \text{ for some } r > 0 \}$, and $\mathfrak{B}^b(f)$ is a completely regular filter on S .

It is surprising that just the filters $\mathfrak{B}_a(f)$ (or just the filters $\mathfrak{B}^b(f)$) associated with an $f \in C(S)$ can be used to characterise z-embedding of f . We start by using the filters $\mathfrak{B}_a(f)$:

4.3.3 Theorem : *Let $S \subseteq X$ and $f \in C(S)$. Then f is z-embedded in X iff $\mathfrak{B}_a(f)$ is z-embedded in X for each $a \in \mathbb{R}$ with $L_a(f) \neq \emptyset$.*

Proof. $\Rightarrow$: Let $r > 0$. Then $L_{a+\frac{r}{2}}(f) \in \mathfrak{B}_a(f)$ and is completely separated from $S - L_{a+r}(f)$ by the z-embedded function f . We have shown that a base for $\mathfrak{B}_a(f)$ is z-embedded, and it follows easily that $\mathfrak{B}_a(f)$ is z-embedded.

$\Leftarrow$: We apply theorem 4.3.1 . Let $a < b$ in $\mathbb{R}$. We may assume that $L_a(f) \neq \emptyset$ and $L^b(f) \neq \emptyset$. Let $m = \frac{a+b}{2} = a + \frac{b-a}{2}$. Now $L_m(f) \in \mathfrak{B}_a(f)$ which is z-embedded, so there is an $A \in \mathfrak{B}_a(f)$ and a z-embedded $g \in C(S)$ with $A \subseteq L_m(f)$, $g(A) \leq 0$ and $g(S - L_m(f)) \geq 1$. So we have $A \subseteq L_0(g)$ and $S - L_m(f) \subseteq L^1(g)$, and by z-embedding of g there exist $Z, Z' \in \mathfrak{Z}(X)$ with $L_0(g) = Z \cap S$ and $L^1(g) = Z' \cap S$. Now $L_a(f) \subseteq A \subseteq Z \cap S \subseteq Z$ and $L^b(f) \subseteq S - L_m(f)$ $\subseteq Z' \cap S \subseteq Z'$ and $Z \cap Z' \cap S = L_0(g) \cap L^1(g) = \emptyset$. Thus f is z-embedded in X . □

In view of the preceding theorem, we can offer an alternative proof of the equivalence of 3.2.13 (a) and 3.2.13 (g) :

4.3.4 <u>*Theorem :*</u> *Let $S \subseteq X$. Then S is z-embedded in X iff every completely regular filter on S is z-embedded in X .*

Proof. $\Rightarrow$: this is obvious.

$\Leftarrow$: Let $f \in C(S)$. Whenever $a \in \mathbb{R}$ with $L_a(f) \neq \emptyset$ the filter $\mathfrak{B}_a(f)$ is (by hypothesis) z-embedded in X , so by the preceding theorem f is z-embedded in X . So all of $\mathfrak{Z}(S)$ is z-embedded in X , and therefore S is z-embedded in X . □

Again we see the global result being recovered from its localisation, and this time making the proof of the global result considerably more simple.

As already mentioned, we are able to use just the filters $\mathfrak{B}^b(f)$ to characterise z-embedding of f :

4.3.5 <u>*Theorem :*</u> *Let $S \subseteq X$ and $f \in C(S)$. Then f is z-embedded in X iff $\mathfrak{B}^b(f)$ is z-embedded in X for each $b \in \mathbb{R}$ with $L^b(f) \neq \emptyset$.*

Proof. Similar to that of theorem 4.3.3 . □

4.3.6 <u>*Remark :*</u> The filters $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$ need not be maximal completely regular: If $c \in \mathbb{R}$ and $a > c$ then $L_a(c) = S$, so that $\mathfrak{B}_a(c) = \{S\}$. Maximality can still fail for non-constant functions: Suppose $f \in C(S)$ and $c_1 < d_1 < c_2 < d_2 < a$ with $L_a(f) \neq \emptyset$, $f^{-1}[c_1, d_1] \neq \emptyset$ and $f^{-1}[c_2, d_2] \neq \emptyset$. Then $f^{-1}[c_1, d_1]$ and $f^{-1}[c_2, d_2]$ are disjoint zero-sets of S that meet $\mathfrak{B}_a(f)$, so that $\mathfrak{B}_a(f)$ cannot be maximal.

The following theorem is a localisation of 3.2.13 (a) $\Leftrightarrow$ (b) . Whereas 3.2.13 (b) demands a condition of *all* z-filters on the subspace, the localisation demands this condition of only a collection of z-filters associated with a given function.

4.3.7 *Definition* : Note that, when defined, the bases $\{L_{a+r}(f) : r > 0\}$ and $\{L^{b-r}(f) : r > 0\}$ for $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$ are z-filterbases on S . We denote by $\mathfrak{U}_a(f)$ (resp. $\mathfrak{U}^b(f)$) the z-filter on S generated by $\{L_{a+r}(f) : r > 0\}$ (resp. $\{L^{b-r}(f) : r > 0\}$) .

4.3.8 *Theorem* : *Let $\varphi : S \to X$ be the inclusion map. The following are equivalent:*

(i) *$f \in C(S)$ is z-embedded in X ,*

(ii) *for each $\mathfrak{U}_a(f)$ that is defined and for each $\mathfrak{U}^b(f)$ that is defined, $\mathfrak{U}_a(f) \subseteq \varphi^{\#}(\mathfrak{U}_a(f)) \mid S$ and $\mathfrak{U}^b(f) \subseteq \varphi^{\#}(\mathfrak{U}^b(f)) \mid S$.*

Proof. $\Rightarrow$: Suppose that $U \in \mathfrak{U}_a(f)$. Then $U \supseteq L_{a+r}(f)$ for some $r > 0$. Now $\mathfrak{U}_a(f) \ni L_{a+r}(f) = Z \cap S$ for some $Z \in \mathfrak{Z}(X)$, by z-embedding of f . So $Z \in \varphi^{\#}(\mathfrak{U}_a(f))$, and $Z \cap S = L_{a+r}(f) \in \varphi^{\#}(\mathfrak{U}_a(f)) \mid S$. Since $\mathfrak{Z}(S) \ni U \supseteq L_{a+r}(f)$, $U \in \varphi^{\#}(\mathfrak{U}_a(f)) \mid S$. Similarly we have $\mathfrak{U}^b(f) \subseteq \varphi^{\#}(\mathfrak{U}^b(f)) \mid S$.

$\Leftarrow$: Let $a \in \mathbb{R}$. We may assume that $L_a(f) \neq \emptyset$. Then by hypothesis we have $\mathfrak{U}_a(f) = \varphi^{\#}(\mathfrak{U}_a(f)) \mid S$ (the reverse inclusion is always true). On expanding this we have $\mathfrak{U}_a(f) = \{Z \cap S : Z \in \mathfrak{Z}(X) \text{ and } Z \cap S \in \mathfrak{U}_a(f)\}$. Then for each $n \in \mathbb{N}$, $\mathfrak{U}_a(f) \ni L_{a+\frac{1}{n}}(f)$ $= Z_n \cap S$ for some $Z_n \in \mathfrak{Z}(X)$. Now $L_a(f) = (\bigcap Z_n) \cap S$. Similarly for $L^a(f)$. □

The next theorem is really a trivial restatement of theorem 4.2.3 , recasting (α_f) in terms of our filters $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$. Note that this theorem is a localisation of 3.4.5 (h) .

4.3.9 *Theorem* : *Let $S \subseteq X$ and $f \in C^*(S)$. The following are equivalent:*

(i) *f extends continuously over X ,*

(ii) *for each $a < b$ in $\mathbb{R}$ for which $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$ are defined, $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$ are completely separated in X .*

Proof. $\Rightarrow$: Let $a < b$ in $\mathbb{R}$ with $L_a(f) \neq \emptyset$ and $L^b(f) \neq \emptyset$. Choose r_1, r_2 with $a < r_1 < r_2 < b$. Now by (α_f) , $L_{r_1}(f) \in \mathfrak{B}_a(f)$ and $L^{r_2}(f) \in \mathfrak{B}^b(f)$ are completely separated in X . Hence $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$ are completely separated in X .

$\Leftarrow$: We verify (α_f) . Let $a < b$ in $\mathbb{R}$. Choose $A \in \mathfrak{B}_a(f)$ and $B \in \mathfrak{B}^b(f)$ with A and B completely separated in X . Since $A \supseteq L_a(f)$ and $B \supseteq L^b(f)$, $L_a(f)$ and $L^b(f)$ are also completely separated in X . $\square$

From theorems 4.3.3 , 4.3.4 and 4.3.5 we deduce that S *is z-embedded in* X *iff every completely regular filter on* S *is z-embedded in* X *iff for each* $f \in C^*(S)$ *and each* $a \in \mathbb{R}$ *for which* $\mathfrak{B}_a(f)$ *is defined,* $\mathfrak{B}_a(f)$ *is z-embedded in* X *iff for each* $f \in C^*(S)$ *and each* $b \in \mathbb{R}$ *for which* $\mathfrak{B}^b(f)$ *is defined,* $\mathfrak{B}^b(f)$ *is z-embedded in* X . Also, from theorems 4.3.9 and 3.4.5 (a) $\Leftrightarrow$ (h) we deduce that S *is* C^**-embedded in* X *iff distinct maximal completely regular filters on* S *are completely separated in* X *iff for each* $f \in C^*(S)$ *and for each* $a < b$ *in* $\mathbb{R}$ *for which* $\mathfrak{B}_a(f)$ *and* $\mathfrak{B}^b(f)$ *are defined,* $\mathfrak{B}_a(f)$ *and* $\mathfrak{B}^b(f)$ *are completely separated in* X . These suggest that the filters $\mathfrak{B}_a(f)$ and $\mathfrak{B}^b(f)$ for $f \in C^*(S)$ are able to fulfil much of the role of the collection of *all* completely regular filters on S , at least in the contexts of z-, and C^*- embedding. At this prompting, the condition that every $\mathfrak{B}_a(f)$ and every $\mathfrak{B}^b(f)$ be the trace on S of some maximal completely regular filter on X seems a likely localisation of 3.4.3 (c) . Investigation showed that this is indeed a sufficient condition for the extendibility of a given $f \in C^*(S)$ (this having a non-trivial proof) , but that the condition is excessively strong (e.g., on a compact S it holds only for constant f !) with the consequence that necessity fails.

Recall the following characterisation of maximality for completely regular filters (2.3.3 (v)) : A completely regular filter $\mathfrak{F}$ on S is maximal completely regular *iff* $\varphi(\mathfrak{F})$ converges for each $\varphi \in C^*(S)$. Using this characterisation we may restate 3.4.3 (a) $\Leftrightarrow$ (b) as follows: $S \subseteq X$ is C^*-embedded in X *iff* $\mathfrak{F} \mid S$ is a maximal completely regular filter on S for each maximal completely regular filter $\mathcal{F}$ on X that meets S *iff* for each maximal completely regular filter $\mathcal{F}$ on X that meets S, $\varphi(\mathcal{F} \mid S)$ converges for every $\varphi \in C^*(S)$. With this in mind we consider the following condition on $f \in C^*(S)$:

($\dagger_f$) for each maximal completely regular filter $\mathcal{F}$ on X that meets S, $f(\mathcal{F} \mid S)$ converges in $\mathbb{R}$.

($\dagger_f$) is thus a localisation of the condition 3.4.3 (b).

From the preceding discussion we have:

4.3.10 <u>*Proposition:*</u> *($\dagger_f$) holds for every $f \in C^*(S)$ iff S is C^*-embedded.* □

This leads us to wonder whether ($\dagger_f$) is necessary and/or sufficient for the extendibility of a single given bounded function. We are able to prove necessity:

4.3.11 <u>*Theorem:*</u> *Let $f \in C^*(S)$. Then ($\dagger_f$) is a necessary condition for the extendibility of f over X.*

Proof. Let $\mathcal{F}$ be a maximal completely regular filter on X that meets S. Suppose that $f(\mathcal{F} \mid S)$ does not converge.

Now since f is bounded, $f(\mathcal{F} \mid S)$ is a filter on the compact Hausdorff space $\overline{f(S)}$. Thus $f(\mathcal{F} \mid S)$ has at least one cluster point, and since $f(\mathcal{F} \mid S)$ does not converge we conclude from lemma 2.3.2 that $f(\mathcal{F} \mid S)$ has at least two distinct cluster points, say a and b.

Choose disjoint closed neighbourhoods A and B of a and b respectively. Now A and B meet $f(\mathcal{F} \mid S)$ since a and b are cluster points of $f(\mathcal{F} \mid S)$. If $F \in \mathcal{F}$ then $f(F \cap S) \in f(\mathcal{F} \mid S)$, and since A and B meet $f(\mathcal{F} \mid S)$ we have $A \cap f(F \cap S) \neq \emptyset \neq B \cap f(F \cap S)$. Thus we have $f^{-1}(A) \cap F \cap S \neq \emptyset \neq f^{-1}(B) \cap F \cap S$, and so $f^{-1}(A) \cap F \neq \emptyset \neq f^{-1}(B) \cap F$. We have shown that $f^{-1}(A)$ and $f^{-1}(B)$ meet $\mathcal{F}$.

Suppose $g \in C(X)$ with $g \mid S = f$. Consider $g^{-1}(A)$ and $g^{-1}(B)$. These are disjoint zero-sets of X with $g^{-1}(A) \supseteq f^{-1}(A)$ and $g^{-1}(B) \supseteq f^{-1}(B)$, and since $f^{-1}(A)$ and $f^{-1}(B)$ meet

$\mathfrak{F}$, we conclude that $g^{-1}(A)$ and $g^{-1}(B)$ meet $\mathfrak{F}$. We have produced *disjoint* zero-sets of X meeting the maximal completely regular filter $\mathfrak{F}$ on X — a contradiction. □

Note that in the last paragraph of the proof it does not suffice to extend $f^{-1}(A)$ and $f^{-1}(B)$ to zero-sets of X (if A and B were suitably chosen, this would be possible by z-embedding of f) . It seems that we do need something stronger than mere z-embedding of f for $(\dagger_f)$ to hold. The theorem shows that extendibility of f is sufficiently strong, but is it too strong? To answer this we must decide whether or not $(\dagger_f)$ is sufficient for the extendibility of $f \in C(S)$. To date this remains undecided, though indicators like 4.3.10 are encouraging.

4.4 CONCLUSION

The aim of this dissertation has been achieved as far as we are currently able. There are, to the author's knowledge, no results in the literature concerning localisation of our global filter-theoretic characterisation of z-, C^*-, and C- embedding — even the filter characterisations of z-embedding in this chapter do not appear anywhere.

Very little attempt has been made in this dissertation to study the various embeddings and their localisations when the subspace involved has additional properties (e.g., is dense in the parent space, is a zero-set of the parent space, satisfies separation axioms, etc.) . Results in this line do exist, but none have a filter-theoretic flavour to them. The fact that our characterisations hold without having to demand conditions of the subspace, and that they are elegant in spite of this, makes their appeal even greater. There are many instances (e.g., in functional analysis) where the subspace is known to have certain properties, and for this purpose it would be rewarding to find conditions which, in the presence of these subspace properties, are equivalent to the conditions of the various theorems of this dissertation.

The stream of results of a classical nature is, by the evidence of chapter 4, very well rounded. From each of the main classical characterisations of z-, C^*-, and C- embedding we have seen a very pleasing localisation unfold, and these localisations have given increased insight into the global characterisations. Furthermore, we now have classical conditions that allow us to decide the z-embedding status, and extendibility status, of a single given function on a subspace independent of the status of other continuous functions on the subspace. The localisation in the filter-theoretic stream is only complete with respect to z-embedding of a function. The necessary condition $(\dagger_f)$ appears to be a likely candidate for characterising extendibility of a bounded function on a subspace; in time we hope to decide the validity of this claim. As for extendibility of an unbounded function ... a few conditions arising from theorem 3.4.9 present themselves as potential characterisations, but no one has yet revealed itself as either necessary or sufficient. Since it usually the case that characterisations of the bounded case are used in the passage to a characterisation of the unbounded case, perhaps it is wise to settle the former first.

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REFERENCES

[Al] P.S.Aleksandrov, *Bikompakte Erweiterung topologischer Räume*, Mat. Sb. **5** (**47**) (1939) , pp 403–423 (Russian) , MR 1 , 318

[AIS] R.A.Alò, L.Imler and H.L.Shapiro, *P- and z- embedded subspaces*, Math. Ann. 188 (1970) , pp 13–22

[BH_1] R.L.Blair and A.W.Hager, *Extensions of Zero-sets and of Real-valued Functions*, Math. Z. **136** (1974), pp 41–57

[Bl_1] R.L.Blair, *Filter characterisations of z-, C^*-, and C-embeddings,* Fund. Math **XC** (1976) , pp 285–300

[Bl_2] R.L.Blair, *Extensions of Lebesgue Sets and of Real-Valued Functions*, Czechoslovak Math. J. **31** (**106**) (1981) , pp 63–74

[Bl_3] R.L.Blair, *z-embedding and preservation of realcompactness*, unpublished manuscript, 1963

[Bl_4] R.L.Blair, *On v-embedded sets in topological spaces*, TOPO 72 — General Topology and its Applications (Second Pittsburgh International Conference, December 18–22, 1972), Lecture Notes in Math. **378**, Berlin-Heidelberg-New York, 1974, pp 46–79

[Bou] N.Bourbaki, *Éléments de mathématique.* Part 1. *Les structures fondamentales de l'analyse.* Livre III: *Topologie générale*, Actualités Sci. Indust., no 1029, Hermann, Paris, 1947; English transl., Addison-Wesley, Reading, Mass., 1966.

[Eng_1] R.Engelking, *Outline of general topology*, PWN — Polish Scientific Publishers, Warsaw, 1965 ; English translation: Amsterdam: North-Holland, 1968 , New York: Interscience, 1968

[Eng_2] R.Engelking, *General Topology,* Translated from Polish by the author, Monografie Matematyczne TOM 60, PWN — Polish Scientific Publishers, Warsaw, 1977

[GJ] L.Gillman and M.Jerison, *Rings of Continuous Functions,* University Series in Higher Math., Princeton, N.J., 1960

[Gr_1] J.W.Green, *Filter Characterizations of C- and C^*-embeddings,* Proc. Amer. Math. Soc. **35** (1972) , pp 574–580

[Gr_2] J.W.Green, *Filters and C- and C^*-embeddings,* Topology Appl. **3** (1973) , pp 97–109

[Ha_1] A.W.Hager, *On Inverse-Closed Subalgebras of $C(X)$*, Proc. London Math. Soc (3) **19** (1969) , pp 233–257

[Ha_2] A.W.Hager, *C-, C^*-, and z-embedding* , Unpublished manuscript , 1967

[He] E.Hewitt, *On Two Problems of Urysohn*, Ann. of Math. (2) 47 (1946) , pp 503–509

[Le] H.Lebesgue, *Sur l'approximation des fonctions*, Bull. Sci. Math. 22 (1898) , pp 278–287

[Mo] W.Moran, *Measures on metacompact spaces*, Proc. London Math. Soc. (3) 20 (1970) , pp 507–524

[Mr_1] S.Mrówka, *On Some Approximation Theorems*, Nieuw Arch. Wisk. (3) **16** (1968) , pp 94–111

[My] A.D.Myškis, *The definition of boundary by means of continuous maps*, Mat. Sb. **26** (68) (1950), pp 225–227 (Russian) ; English translation: Am. Math. Soc. Transl. **51** (1951)