The Coriolis effect and travelling waves in porous media convection subject to rotation

by

Mihail Radu Patrascoiu

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Per Asper Ad Astra....

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Nomenclature

Latin Symbols

Da	Darcy number
$\hat{\mathbf{e}}_x$	Unit vector in x direction
$\hat{\mathbf{e}}_y$	Unit vector in y direction
$\hat{\mathbf{e}}_z$	Unit vector in z direction
g_*	Gravitational Acceleration (9.81 ms^{-2})
H_*	Height of layer [m]
l_*	Length of layer [m]
l	Scaled length of layer
k_*	Permeability of the porous domain
k	Wave number
k_S	wave number containing streamlines
p	Reduced pressure (dimensionless)
Pr	Prandtl number
$\hat{\mathbf{q}}$	Dimensionless specific flow-rate vector
Ra	Rayleigh number
Ra_c^{st}	Characteristic Rayleigh number (stationary case)
$Ra_c^{(ov)}$	Characteristic Rayleigh number (over-stable case)
Ra_{cr}	Critical Rayleigh number

T	Dimensionless temperature
T_C	Coldest wall temperature
T_H	Hottest wall temperature
t	Time [s]
Ta	Taylor Number
u	horizontal x component of the specific flow rate
v	horizontal y component of the specific flow rate
w	vertical component of the specific flow rate
x	Horizontal length co-ordinate
y	Horizontal width co-ordinate
z	Vertical length co-ordinate
X	Slow space scale

Greek Symbols

α	Scaled wave number
β.	Thermal expansion coefficient
χ	$\phi \mathrm{Pr}/Da$
ϕ	Porosity
γ	χ/π^2
ε	Disturbance amplitude
λ	Wave length
ν.	Kinematic viscosity [m^2/s]

ρ	Fluid density [kg/m^3]
Ω, ω	Angular velocity of the layer
ψ	Stream function
ξ	Linear coefficient corresponding to stationary or over-stable convection, in the context.
∇^2	Laplacian
σ	Oscillatory frequency [s^{-1}]
σ_o	Critical frequency for stationary or over-stable convection
τ	Slow time scale at order ε^2
τ_o	Slow time scale at order ε

Superscripts

*	Dimensional quantities
$()'$	Refers to scaled terms
ov	Refers to over-stable conditions
c	Refers to characteristic values

Abstract

This study intends to recover and expand the analytical work of Vadasz (1998) for linear and weak non-linear stability of a rotating porous media heated form below and subject to gravity and Coriolis forces. It is shown that the viscosity has a destabilising effect at high rotation rate. It has been established that the critical wave number in a plane containing the streamlines is dependent on rotation. Finite amplitude calculations provide a set of differential equations for the amplitude and phase, corresponding to the stationary and over-stable convection, identifying the post-transient conditions that a fluid is subject to, i.e. a pitchfork bifurcation for the stationary case, or a Hopf bifurcation in the case of over-stable convection. The previous model (Vadasz [1998]) was extended with an additional time scale in order to represent amplitude fluctuations and a short space scale to include horizontal modes of oscillations. When the complete solution for the stream function or temperature is analysed, where left and right travelling waves are considered, we obtain a set of differential equations for the amplitude and phase. The solutions are discussed in this context

1. Introduction

1.1 Motivation

The study of flow in rotating porous media is motivated by its practical applications in geophysics and engineering. Flows in porous geological formations subject to earth rotation, the flow of magma in the earth mantle close to the earth crust (Fowler [2]) represent examples of geophysical applications. Among the applications of rotating flow in porous media to engineering disciplines, one can find the food process industry, chemical-processing industry, centrifugal filtration processes and rotating machinery.

More specifically, packed bed mechanically agitated vessels are used in the food processing and chemical engineering industries in batch processes. The packed bed consists of solid particles of fibres of material, which form the solid matrix while fluid flows through pores. As the solid matrix rotates, due to mechanical agitation, a rotating frame of reference is a necessity when investigating these flows. The role of the flow of fluid through these beds can vary from drying processes to extraction of soluble components from the solid particles. The molasses in centrifugal crystal separation processes in the sugar milling industry and the extraction of sodium alginate from kelp are just two examples of such processes.

Modelling of flow and heat transfer in porous media is also applied for the design of heat pipes using porous wicks and includes effects of boiling in unsaturated porous medium, surface tension driven flow with heat transfer and condensation in unsaturated porous media.

With the emerging utilisation of the porous medium approach in non traditional fields, including some applications in which the solid matrix is subjected to rotation (like physiological processes in human body subject to rotating trajectories, cooling of electronic equipment in rotating radar, cooling of turbomachinery blades, or cooling of rotors of electric machines) a thorough understanding of the flow in rotating porous medium becomes essential. Its results can be used in the more established industrial applications like food processes, chemical engineering or centrifugal processes, as well as to the aforementioned non-traditional applications of the porous medium approach.

Additional recent applications of the porous media approach are the flow of liquids in biological tissues like the human brain, the cardiovascular flow of blood in human heart or other physiological processes, pebble-bed nuclear reactors and cooling of turbine blades in the hot portion of the turbo-expander.

Regarding the last application, such a cooling process enables the expander inlet gas temperature to increase beyond the allowed metal temperature, bringing a significant contribution to the cost-effectiveness of the expander. The cooling process occurs by injecting air through channels in the internal part of the blade. As long as the geometry of the channels is not too complicated the traditional heat transfer approach can be applied to evaluate the cooling performance. However, for complicated channel geometry the porous medium approach will prove again the most effective way of simulating the phenomenon.

The macro-level porous media approach is gaining an increased level of interest in solving practical fluid flow problems, which are too difficult to solve by using a traditional micro-level approach. As such *Direct Chill* (DC) casting models apply the Darcy law to predict the heat transfer, fluid flow and ultimately the thermal stresses in the solidified metal. Another important application of rotating flows in porous media is in the design of a multi-pore distributor in a gas solid fluidised bed. A multi-pore distributor is a device, which is constructed from foraminous materials, wires compacts, filter cloth, compressed fibres, sintered metal or such like.

2. Literature survey

The main reason behind the apparent lack of interest of this type of flow is probably the fact that isothermal flow in homogeneous porous media following Darcy's law is irrotational (Bear [20]) hence the effect of rotation on this flow is not significant. However, for heterogeneous medium with spatial dependent permeability or for free convection in a non-isothermal homogeneous porous medium, the flow is not irrotational any more hence effects of rotation become significant. In some applications, these effects can be small, e.g. when the porous media Ekman number is high. Nevertheless, the effect of rotation is of interest as it may generate secondary flows in planes perpendicular to the main flow direction. Even when these secondary flows are weak, it is essential to understand their source, as they might be detectable in experiments. To support this claim, it is sufficient to look at the corresponding rotating flows in pure liquids (non-porous domains). There the Ekman number controls the Coriolis effect and secondary motion in planes perpendicular to the main flow direction. Experiments (Hart [21], Johnston , Haleen and Lezius [22] and Lezius and Johnston [23]) showed that this secondary motion is detectable , even for very low or very high Ekman numbers although the details of this motion may vary considerably according to pertaining conditions. It is therefore expected to obtain secondary motion when a solid porous matrix is present in a similar geometric configuration, although its details cannot be *a priori* predicted based on physical intuition only. This creates a strong motivation to investigate the effect of rotation in isothermal heterogeneous porous media. For high angular velocity, or extremely high permeability, conditions pertaining to some engineering applications, the Ekman number can become of unit order of magnitude or lower and then the effect of rotation becomes even more significant. The same motivation applies for investigating the effect of rotation on free convection in porous media.

Multi-pore distributor designs have been investigated in applications of rotating porous media (Whitehead [3]). Research results (Davidson and Harrison [4]) showed that the porous distributor allowed a more even expansion of the bed than the other distributors and its design affected the behaviour of the bed over most of its height. An even distribution of the gas is necessary to avoid instability in the fluidised bed, which can break down proper fluidisation. A commonly used solution to avoid maldistribution of gas and bed instability is cyclic interchange fluidisation (CIF) (Kvasha [5]), where the distributor is rotating at constant angular velocities which vary between 20 and 2500 rpm, depending on the size of the bed (the higher its diameter, the lower the angular velocity).

Some examples of applications of the cyclic interchange fluidisation are the highly exothermic synthesis of alkylchlorsilanes polymer filling the composites, treatment of finely dispersed solids, drying of paste-like polymers, permanganate of potash and iodine (Kvasa [5]). Therefore, evaluating the flow field through a porous rotating distributor becomes a design necessity.

Plumb [6] presented a comprehensive review of the heat transfer in unsaturated porous media flow with particular applications to the heat pipe technology. Again, when the heat pipe is used for cooling devices, which are subject to rotation the corresponding centrifugal, and Coriolis effects become relevant as well.

A Direct Cill model was applied by Katgerman [7] to analyse the heat transfer phenomena during continuous casting of aluminium alloys. When centrifugal casting processes are considered, rotation effects become relevant to the problem. The porous medium approach is also used in processing of composite materials. Güçeri [8] states that "most of the studies in resin transfer moulding (RTM) processes and structural reaction injection moulding (SRJM) treat the flow domain as an isotropic porous medium and perform a Darcy flow analysis utilising a continuum model".

Additional applications of the porous medium approach are discussed by Nield and Bejan [9] and Bejan [10] in comprehensive reviews of the fundamentals of heat convection in porous media. Bejan [10] mentions among the applications of heat transfer in porous media the process of cooling of winding structures in high power density electric machines. When this applies to a rotor of a an electric machine or generator (or motor), rotation effects become relevant as well. Mohanty [11] presented a study of natural and mixed convection in rod arrays motivated by safety related thermal-hydraulic modelling of nuclear reactors with particular attention to the rod-bundle geometry. The author concluded that "bundle average experimental friction factor values in forced convection are better explained through a porous medium model" and "the porous medium parameters so derived also yield quantitative corroboration of the flow through vertical bundles induced solely by buoyancy". The porous media approach was also successfully applied to stimulate complex transport phenomena in mass and heat exchangers (Roberson and Jacobs [12]) and in the cooling of electronic equipment (Vadasz [13]).

Chandrasekhar (1961) has shown a perfect agreement in the results of temperature dependence and stability in a porous layer subject to rotation and its corresponding problem in pure fluids.

Nevertheless, no reported research was found on *isothermal* flow in rotating porous media. Limited research results are available for natural convection in rotating porous media, e.g. Rudraiah, Shivakumara and Friedrich [14], Prabhanani and Vadyanathan [15], Jou and Liaw [16, 17] and Palm and Tyvand [18]. Nield [19] while presenting a comprehensive review of the stability of convective flows in porous media finds also that the effect of rotation on convection in a porous medium attracted limited interest and the lack of experimental results is particularly noticed. The problem of rotating porous layer subject to gravity and heated from below was originally investigated by Friedrich [25] and by Prabhamani and Vaidyanathan [15]. Both studies considered a non-Darcy model, which is probably subject to the limitations as shown by Nield [24]. Friedrich [25] focused on the effect of Prandtl number on the convective flow resulting from a linear stability analysis as well as a non-linear numerical solution, while Prabhamani and Vaidyanathan [15] dealt with the influence of variable viscosity on the stability condition. He latter concluded that variable viscosity has a destabilising effect. Although the non-Darcy model considered included the time derivative in the momentum equation the possibility of convection setting-in as an oscillatory instability was not explicitly investigated in ref. [15]. It should be pointed out for a pure fluid (non-porous domain) convection sets in a oscillatory instability. Jou and Liaw [16], [17] investigated a similar problem of gravity driven thermal convection in a rotating porous layer subject to transient heating from below. By using a non-Darcy model they established stability conditions for the marginal state without considering the possibility of oscillatory convection.

An important analogy was discovered by Palm and Tyvand [18] who showed by using a Darcy model that the onset of gravity driven convection in a rotating porous medium is equivalent to the case of an anisotropic porous medium. The critical Rayleigh number was developed in this study and matched theirs finding (equation 3.3 in the text).

The methodology adopted in this thesis consists of a presentation of dimensionless equations governing the flow and transport phenomena in a rotating frame of reference.

3. Objectives

This study is intended to compare results for convection in rotating porous media with the corresponding results in pure fluids (non-porous domain). The equations governing the flow and heat transfer in porous media can be obtained via an averaging procedure of the Navier-Stokes and energy equations over a *representative elementary volume*. As a result, the filtration velocity applicable at the macroscopic level will be considered and a set of new parameters are introduced such as porosity, defined as the ratio of the pore volume to the volume of the porous matrix, and permeability, which is a property describing the ability of the porous matrix to allow fluid flow.

The analysis will focus on the effect of the Coriolis force on the basic free convection and the travelling waves associated to the expansion around overstable solutions.

The study utilises the method and parameters used by Vadasz [1] and developed further to encompass the introduction of a large space scale and analysis of travelling waves on the expansion around over-stable solutions. The structure of the study is as follows: *Part 1*: Geometrical definition and Problem formulation. *Part 2*: Linear stability analysis for stationary and oscillatory cases, as in the case investigated by Vadasz (1998). *Part 3*:Weak non-linear solution for stationary convection; linear stability results. *Part 4*: Weak non-linear solution for stationary convection. *Part 5*: Weak non-linear solution of oscillatory convection for standing waves case and travelling waves case. *Part 6*: Discussion and conclusions.

2. Problem formulation and governing equations

2.1. Problem formulation

The objective of this study is to investigate the Coriolis effect in the analytical solution to the convection flow problem through a porous media in a rotating square channel.

Rotating flows in porous media can be dealt with by classifying them in three major categories.

(a) Isothermal flows in heterogeneous porous media subject to rotation.

(b) Convective flows in non-isothermal homogeneous porous media subject to rotation.

(c) Convective flows in non-isothermal heterogeneous porous media subject to rotation.

Case ***(b)*** is to be analysed in this study.

A non –isothermal flow allows, as a result of free convection, a non-vanishing vorticity field. Free convection is the phenomenon of **fluid flow** driven by **density variations** in a fluid **subject to body forces**. The relative orientation of the density gradient with respect to the body force is an important factor for providing a sufficient condition for convection to occur. Some of these forces are constant, like gravity for example, others can vary linearly with the distance from the axis of rotation.

The relative orientation of the density gradient with respect to the body force is also an important factor for convection to occur. This is shown graphically in Figure 1 for a particular case of thermal convection, where F represents the body force and $\nabla\rho = -\beta_T \nabla T$ is the direction of the density gradient.

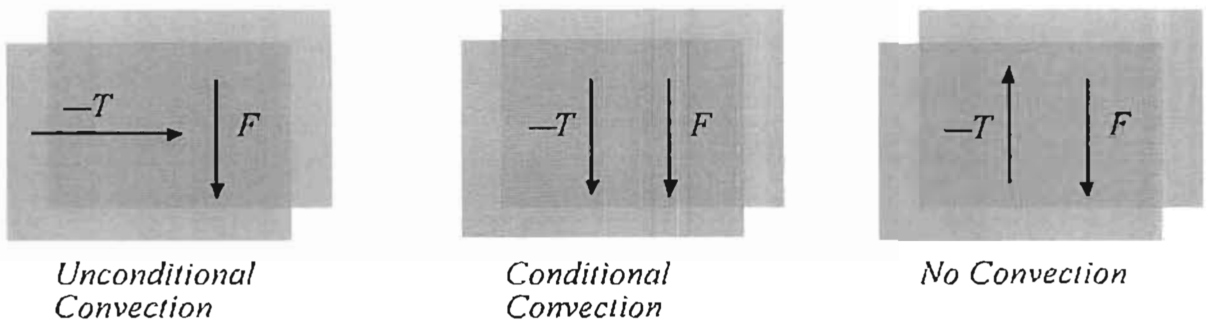

Figure 1: The effect of the relative orientation of the temperature gradient with respect to the body force on the set-up of convection

2.2. Governing equations

Let us consider a long horizontal square channel filled with fluid saturated porous material rotating with Ω angular velocity about an axis perpendicular to the horizontal walls, as shown in Figure 2. An axial horizontal flow parallel to the channel walls is imposed through an axial pressure gradient. The layer is heated from below, while the vertical distance between the top and bottom boundaries is H_*.

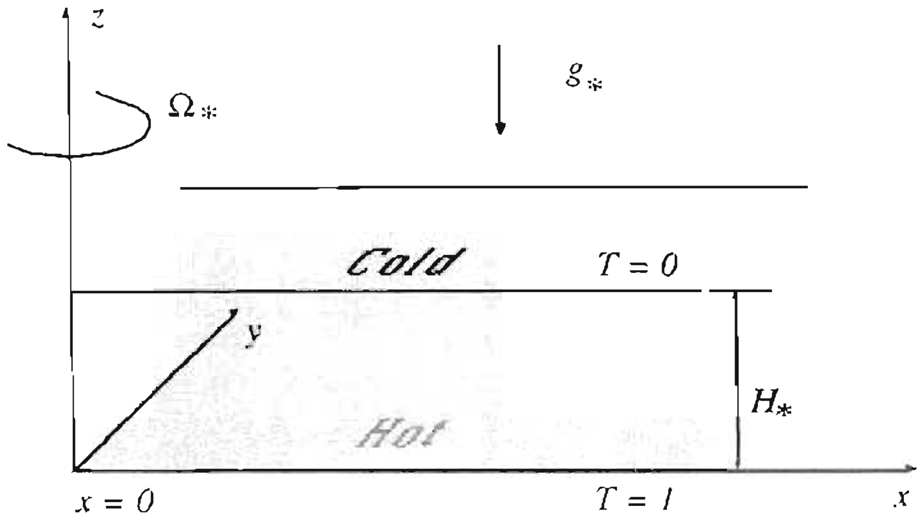

Figure 2: A rotating porous layer saturated fluid, heated form below

A negative temperature gradient along the vertical direction is expected due to the imposed thermal boundary conditions. At a distance $l_* << g_*/\Omega_*$ from the axis of rotation the gravity buoyancy can be assumed to be dominant and the centrifugal buoyancy can be neglected, hence limiting the effect of rotation to the Coriolis acceleration. Furthermore the centrifugal acceleration can be assumed as constant and absorbed in the reduced pressure term. Darcy law is extended only to include the time derivative and Coriolis terms; Boussinesq approximation is applied to account for the effects of density variations. Subject to these conditions the following dimensionless set of dimensionless set of governing equations for continuity, Darcy and energy, is obtained:

Continuity equation

$$\nabla \cdot \mathbf{q} = 0 \tag{2.1}$$

Darcy equation (including the rotation effects)

$$\frac{\partial \mathbf{q}}{\partial t} + Ta^{1/2} \cdot \hat{\mathbf{e}}_z \times \mathbf{q} + \mathbf{q} = -\nabla p + Ra \cdot T \cdot \hat{\mathbf{e}}_z \tag{2.2}$$

Energy equation

$$\chi \frac{\partial T}{\partial t'} + \mathbf{q} \cdot \nabla T = \nabla^2 T \tag{2.3}$$

Implicitly, in equations (1) – (3), the values of α_* / H_*, $\mu_* \alpha_* / k_*$ and $\Delta T_c = (T_H - T_C)$ are used to scale the dimensional filtration velocity components $(u_*, v_* w_*)$, reduced pressure (p_*) and temperature variations $(T_* - T_C)$ respectively. The height of the layer H_* was used to for scaling of the variables x_*, y_* and z_*. Accordingly, $x = x_* / H_*$, $y = y_* / H_*$ and $z = z_* / H_*$. The time variable was scaled initially by using the value H_*^2 / α_*, hence $t = t_* \alpha_* / H_*^2$, and thereafter re-scaled for convenience in the form $t' = \chi \cdot t$, where χ is a dimensionless number that includes the Prandtl and Darcy numbers as well as the porosity of the considered porous domain, defined as

$$\chi = \frac{\phi \cdot Pr}{Da} \tag{2.4}$$

In equation (4) $\Pr = \nu_* / \alpha_*$ and $Da = k_* / H_*^2$ and represent the Prandtl and Darcy numbers. The combined dimensionless group allows the Prandtl number to affect the flow in porous media. The interval Pr can take values expands from as little as 10^{-3} (for liquid metals) up to 10^{3} for oils and the corresponding value of χ will be multiplied by a factor of ϕ / Da which is usually a big number covering values from 10^{2} up to 10^{23}. The values χ can take in traditional porous media applications are large, fact that provides justification for neglecting the time derivative in Darcy equation. For modern porous media applications, however, its value may become of unit order of magnitude or even smaller, in which case the time derivative should not be neglected. In the present case, we consider the time derivative term in the equation in order to investigate the overstable convection and will analyse the behaviour of the overstable solution in respect to χ. A linear approximation was assumed between density and temperature in the form of $\rho = 1 - \beta T$, where $\beta = \beta_{*}(T_H - T_C)$. There are two dimensionless groups, which appear in equation (2), the porous media gravity related Rayleigh number, ***Ra*** and the porous media Taylor number, ***Ta***, defined as (***Appendix I***)

$$Ra = \frac{\beta_{*} \Delta T_{c} g_{*} H_{*} k_{*}}{\nu_{*} \alpha_{*}} \tag{2.5.1}$$

$$Ta = \left(\frac{2\omega_{*} k_{*}}{\phi \nu_{*}} \right) \tag{2.5.2}$$

As for the boundary conditions they have to comply with the fact that at the top and bottom of the considered porous media domain the solution must follow the impermeability conditions on the margins, i.e. $\mathbf{q} \cdot \hat{\mathbf{e}}_n = 0$. The temperature boundary conditions are $T = 1$ at $z = 0$ and $T = 0$ at $z = 1$. The lateral boundaries can be taken at the convection cell wavelength where $\mathbf{q} \cdot \hat{\mathbf{e}}_n = 0$ and $\nabla T \cdot \hat{\mathbf{e}}_n = 0$.

The system of equations (2.1), (2.2), and (2.3) form a three-dimensional non-linear coupled system, which together with the corresponding boundary conditions accepts a *basic* motionless solution. To determine a non-trivial solution to the system it is convenient to manipulate equation (2.2) by applying onto it the *curl* operator $(\nabla \times)$ in order to obtain an equation for vorticity, defined as $\omega = \nabla \times \mathbf{q}$

$$\nabla \times \left[\frac{\partial \mathbf{q}}{\partial t} + Ta^{1/2} \hat{\mathbf{e}}_z \times \mathbf{q} + \mathbf{q} \right] = \nabla \times \left[-\nabla p + Ra \cdot T \cdot \hat{\mathbf{e}}_z \right] \tag{2.6}$$

or (see *Appendix 2 for details*)

$$\frac{\partial \omega}{\partial t'} + \omega - Ta^{1/2} \frac{\partial \mathbf{q}}{\partial z} = Ra \left[\frac{\partial T}{\partial y} \hat{\mathbf{e}}_x - \frac{\partial T}{\partial x} \hat{\mathbf{e}}_y \right] \tag{2.7}$$

It is to be noted that the vertical component of equation (2.7) is independent of temperature. By manipulating further the equation (2.7) and using the fact that $\mathbf{q}$ is solenoidal, it can be written as

$$\left[\frac{\partial}{\partial t'} + 1 \right] \nabla^2 \mathbf{q} + Ta^{1/2} \frac{\partial \mathbf{q}}{\partial z} + Ra \left[\frac{\partial^2 T}{\partial x \partial z} \hat{\mathbf{e}}_x + \frac{\partial^2 T}{\partial y \partial z} \mathbf{e}_y - \nabla_H^2 T \hat{\mathbf{e}}_z \right] = 0 \tag{2.8}$$

3. Linear stability analysis

3.a. Basic flow solutions.

There is a set of basic steady-state solutions marked with the sub-script $(.)_b$ corresponding to the following conditions

(a) there is no flow in any preferential direction,

(b) the pressure (p) is not a function of x or y, and

(c) there is a two dimensional problem

The set satisfies the following system of equations:

$$\nabla \cdot \mathbf{q} = 0 \tag{3.1}$$

$$\mathbf{q} = -\nabla p + RaT\hat{\mathbf{e}}_z \tag{3.2}$$

$$\nabla^2 T = \mathbf{q} \cdot \nabla T \tag{3.3}$$

Subject to

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$
$$T = T(z) \tag{3.4}$$

and to the following boundary conditions

$$T = 1 \quad z = 0$$

$$T = 0 \quad z = 1 \qquad (3.5)$$

By solving the equations (2.1), (2.2), and (2.3) we obtain the basic solution for temperature (***Appendix 3; Section 1***)

$$T_b = 1 - z \qquad (3.6a)$$

$$\mathbf{q} = \omega_b = 0 \qquad (3.6b)$$

$$p_b = Ra\int(1 - z)dz + C, \qquad (3.6c)$$

where C stands for an integration constant.

3.b. Linear stability analysis

Assuming small perturbations around basic solutions in the form of $\mathbf{q} = \mathbf{q}_b + \mathbf{q}'$, $T = T_b + T'$ and $\omega = \omega_b + \omega'$ we investigate the growth and decay of infinitesimal disturbances around this solution.

Linearising the equations (2.8), (2.3) and (2.7) it will result the following linear system

$$\left[\frac{\partial}{\partial t'}+1\right]\nabla^2\mathbf{q}'+Ta^{1/2}\frac{\partial\omega'}{\partial z}+Ra\left[\frac{\partial^2T'}{\partial x\partial z}\hat{\mathbf{e}}_x+\frac{\partial^2T'}{\partial y\partial z}\hat{\mathbf{e}}_y-\nabla_H^2T'\hat{\mathbf{e}}_z\right]=0 \tag{3.7}$$

$$\left[\chi\frac{\partial}{\partial t'}-\nabla^2\right]T'-w'=0 \tag{3.8}$$

$$\left[\frac{\partial}{\partial t'}+1\right]\omega_z'=Ta^{1/2}\frac{\partial w'}{\partial z} \tag{3.9}$$

where ω' and w' are small perturbations of the vertical component of vorticity and filtration velocity, respectively (***Appendix 3; Section 2***)

The system of the equations (2.7), (2.8) and (2.9) can be de-coupled to provide one equation for the temperature perturbation

$$\left\{\left[\frac{\partial}{\partial t'}+1\right]^2\left[\chi\frac{\partial}{\partial t'}-\nabla^2\right]\nabla^2+Ta\left[\chi\frac{\partial}{\partial t'}-\nabla^2\right]-Ra\left[\frac{\partial}{\partial t'}+1\right]\nabla_H^2\right\}T'=0 \tag{3.10a}$$

or for the filtration velocity

$$\left\{\left[\frac{\partial}{\partial t'}+1\right]^2\left[\chi\frac{\partial}{\partial t'}-\nabla^2\right]\nabla^2+Ta\left[\chi\frac{\partial}{\partial t'}-\nabla^2\right]-Ra\left[\frac{\partial}{\partial t'}+1\right]\nabla_H^2\right\}w'=0 \tag{3.10b}$$

We assume a solution in the form of

$$T' = \theta(z) \cdot \exp\left[i(k_x x + k_y y) + \sigma t'\right] \tag{3.11}$$

that will provide an ordinary differential equation for $\theta(z)$ as shown below (***Appendix 3; Section 3***)

$$\left\{(\alpha+1)^2\left[D_z^2 - k^2 - \chi\sigma\right]\left(D_z^2 - k^2\right) + Ta\left[D_z^2 - k^2 - \chi\sigma\right]D_z^2 - Ra(\sigma+1)k^2\right\}\theta = 0 \tag{3.12}$$

where $k^2 = k_x^2 + k_y^2$, and $D_z^n = d^n/dz^n$ $(n = 2)$.

Equation (2.12) will accept a solution of the form $\theta(z) = A_{1n}\sin\lambda_n z + A_{2n}\cos\lambda_n z$, which for the boundary conditions of $z = 0 \Rightarrow T' = 0$ and $z = 1 \Rightarrow T' = 0$, will yield $A_{2n} = 0$ and

$$\theta(z) = b_n \sin(n\pi z) \tag{3.14}$$

$n = 1$ minimises the Rayleigh number in equation 3.12, indicating that $\theta(z) = b_n \sin(\pi z)$ is the eigenfunction for marginal stability.

It is convenient to re-scale the parameters k^2, Ra and χ in the form

$$\alpha = \frac{k^2}{\pi^2}; \quad R = \frac{Ra}{\pi^2}; \quad \gamma = \frac{\chi}{\pi^2} \tag{3.14.a}$$

Substituting these values in equation (3.12) yields the scaled parametric value for Raleigh number

$$R = \frac{\left[1 + a + \gamma\sigma\right]\left[(\sigma+1)^2(\alpha+1) + Ta\right]}{\alpha(\sigma+1)} \tag{3.15}$$

3.1. Linear stability analysis. Stationary convection

Analysing the solution (2.14) of equation (2.13) it is to be noted that for stationary convection σ in real; furthermore, if $\sigma = 0$ the stability is marginal. The corresponding characteristic values of Rayleigh number associated to this case are obtained by letting $\sigma = 0$ in the expression of the scaled equation

$$R = \left.\frac{[1 + a + \gamma\sigma]\left[(\sigma + 1)^2 (\alpha + 1) + Ta\right]}{\alpha(\sigma + 1)}\right|_{\sigma=0} = R_c^{(st)} \tag{3.1.0}$$

$$R_c^{(st)} = \frac{(1+\alpha)^2}{\alpha} + Ta\frac{(1+\alpha)}{\alpha} \tag{3.1.1}$$

At this point the analysis of equation (3.1.1) reveals that the first term alone represents the characteristic Rayleigh number for convection in the absence of rotation, while the second term is the contribution of rotation. The graphical representation of Rayleigh number as function of α is shown in Figure 3 for different values of Taylor number. From the graphical representation it can be seen that the critical Rayleigh number associated with stationary convection is strongly influenced by rotation.

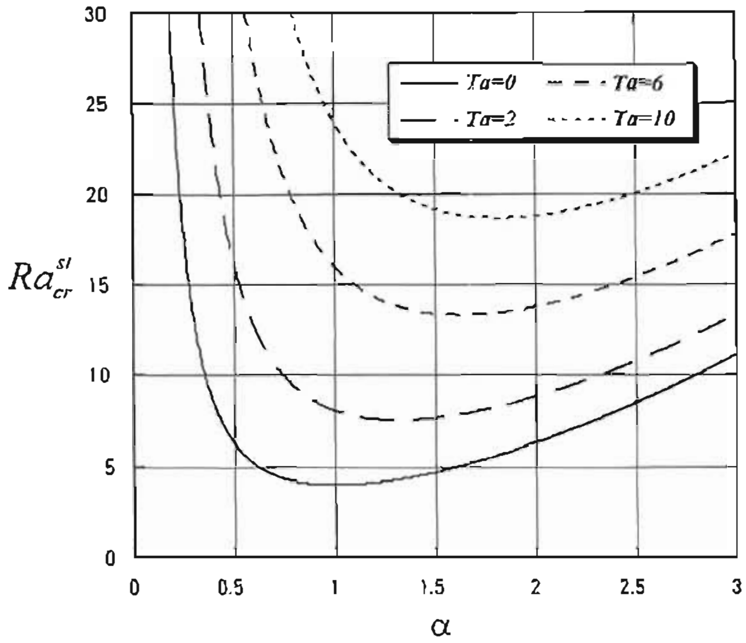

Figure 3: Dependence of Rayleigh number for stationary case with Taylor numbers

By minimising the expression (3.1) with respect to α, we obtain the critical value or Rayleigh and the critical wave number $k_{cr}^{(st)}/\pi$

$$\frac{\partial R_c^{(st)}}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[\frac{(1+\alpha)^2}{\alpha} + Ta\frac{(1+\alpha)}{\alpha}\right] = 0$$

from which results the critical wave number

$$\alpha_{cr}^{(st)} = \frac{k_{cr}^2}{\pi^2} = \sqrt{1+Ta} \qquad (3.1.2)$$

and critical value of Rayleigh

$$Ra_{cr}^{(st)} = \left[1+\sqrt{1+Ta}\right]^2 \qquad (3.1.3)$$

The dependence of the critical value of wave number on Taylor is illustrated in Figure 4

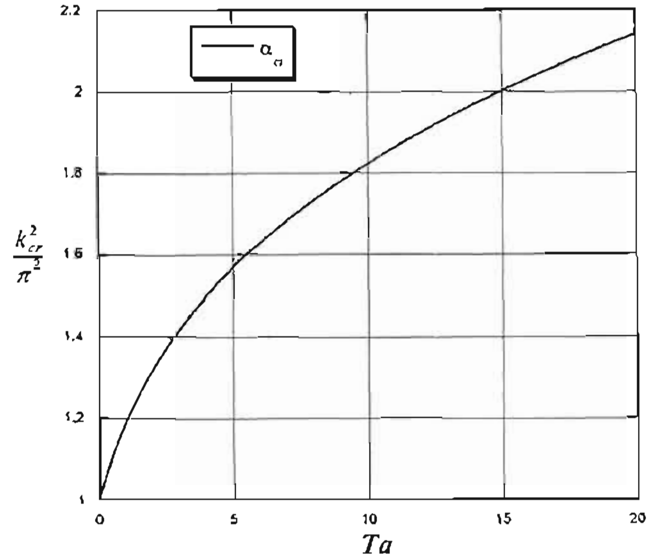

Figure 4: Variation of critical wave number as a function of Taylor number

The dependence of the critical Rayleigh number on Taylor is shown in Figure 5

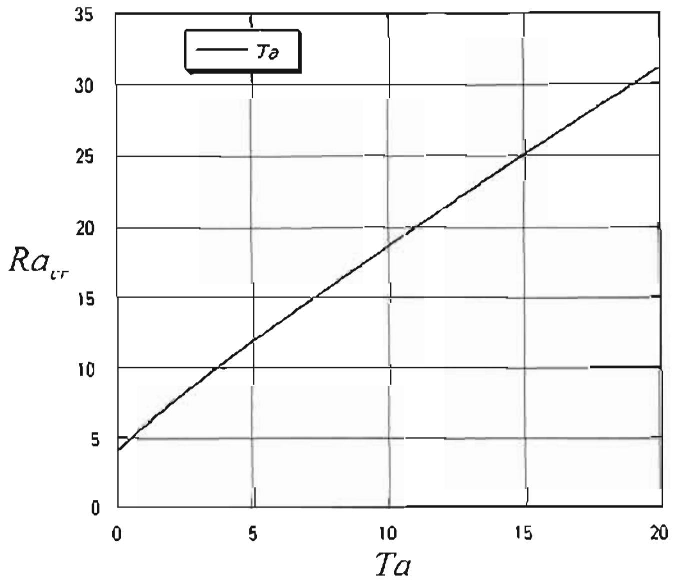

Figure 5: Variation of Rayleigh critical as a function of Taylor number

Palm and Tyvand, Friedrich (1984), Friedrich (1983) and Vadasz (1997) have presented a similar result for critical values of the wave and Rayleigh numbers. To investigate the effect of viscosity on stability we have to analyse the limiting conditions when $Ta \to \infty$ for $\omega_* \to \infty$ and $\nu_* \to 0$. For large values of Ta number, equations (3.2) and (3.3) become

$$\alpha_{cr}^{(st)}\Big|_{Ta\to\infty} \to Ta^{1/2} \qquad (3.1.4)$$

$$Ra_{cr}^{(st)}\Big|_{Ta\to\infty} \to Ta + O\left(Ta^{1/2}\right) \qquad (3.1.5)$$

From the definition of Rayleigh number

$$Ra = \frac{\beta_* \Delta T_c g_* H_* k_*}{\nu_* \alpha_*} = R\pi^2 \Rightarrow R = \frac{\beta_* \Delta T_c g_* H_* k_*}{\pi^2 \nu_* \alpha_*} \qquad (3.1.6)$$

$$\left[\frac{\beta_* \Delta T_c g_* H_* k_*}{\nu_* \alpha_*}\right]_{cr} = \pi^2 Ta \qquad (3.1.7)$$

According to 3.1.5 for large ***Ta*** we have

$$R = \frac{\beta_* \Delta T_c g_* H_* k_*}{\pi^2 \nu_* \alpha_*} = \left(\frac{2\omega_* k_*}{\phi \nu_*}\right)^2 + O\left(\frac{2\omega_* k_*}{\phi \nu_*}\right) \qquad (3.1.8)$$

Hence

$$\beta.\Delta T_c = \frac{\nu.\alpha.\pi^2}{\phi^2 \nu_\bullet^2} 4\omega_\bullet^2 k_\bullet^2 \frac{1}{g.H.k.} + O\left(\frac{2\pi k.}{g.H.k.} \frac{\nu.\alpha.\pi^2}{g.H.k.}\right) \qquad (3.1.9)$$

Using the expression for $\beta_{cr} = \beta.\Delta T_c$ we can write the critical temperature difference over the porous layer

$$\beta_{cr} = \frac{1}{\nu.} \frac{4\pi^2 k.\alpha.\omega^2.}{\phi^2 H.g.} + O\left(\frac{2\omega.k.}{\phi\nu.} \frac{\pi^2 \nu.\alpha.}{g.H.k.}\right) \rightarrow \frac{1}{\nu.} \frac{4\pi^2 k.\alpha.\omega^2.}{\phi^2 H.g.} \qquad (3.1.10)$$

Equation (3.1.10) shows that the critical temperature difference for rotating porous media is inversely proportional to viscosity and proportional to $\omega_\bullet^2$. As a result, very large values of ***Ta*** , or high angular velocity, have a destabilising effect.

Following to these results we shall investigate the complete solution. We shall consider the existence of a stream function ψ describing the pattern convection corresponding to longitudinal rolls. The variation along y direction of variables will vanish. As a result, the wave number $k_y = 0$ and therefore $k_x^2 = k^2$. The solution for T' becomes in this case $T' = \theta(z)e^{kx+\sigma t}$. For stationary case we have to take $\sigma = 0$ otherwise the system will become unstable. It can be seen that a positive real part of σ will increase the solution exponentially to infinity. A negative value it will bring the solution to zero. As a result, the expression of T' will be

$$T' = B\cos(kx)\sin(\pi z) \tag{3.1.11}$$

In *Appendix 2* the determination of w', ω_z', u' and v' is presented. However their values is as follows

For w':

$$w' = \left(k^2 + \pi^2\right)B\cos(kx)\sin(\pi z) \tag{3.1.12}$$

For vertical component of vorticity ω_z':

$$\omega' = \pi\left(k^2 + \pi^2\right)Ta^{1/2}B\cos(kx)\cos(\pi z) \tag{3.1.13}$$

For the horizontal components of filtration velocity

$$u' = \frac{\pi\left(k^2 + \pi^2\right)}{k}B\sin(kx)\cos(\pi z) \tag{3.1.14}$$

$$v' = \frac{\pi Ta^{1/2}\left(k^2 + \pi^2\right)}{k}B\sin(kx)\cos(\pi z) \tag{3.1.15}$$

From equations (3.10) and (3.11) the ration between horizontal and vertical components of filtration velocity, can be evaluated

$$\frac{v'}{u'} = -Ta^{1/2} \tag{3.1.16}$$

Let λ_s and k_s, a wavelength and its wave number of a roll containing streamlines represented in a plane as in Figure 7

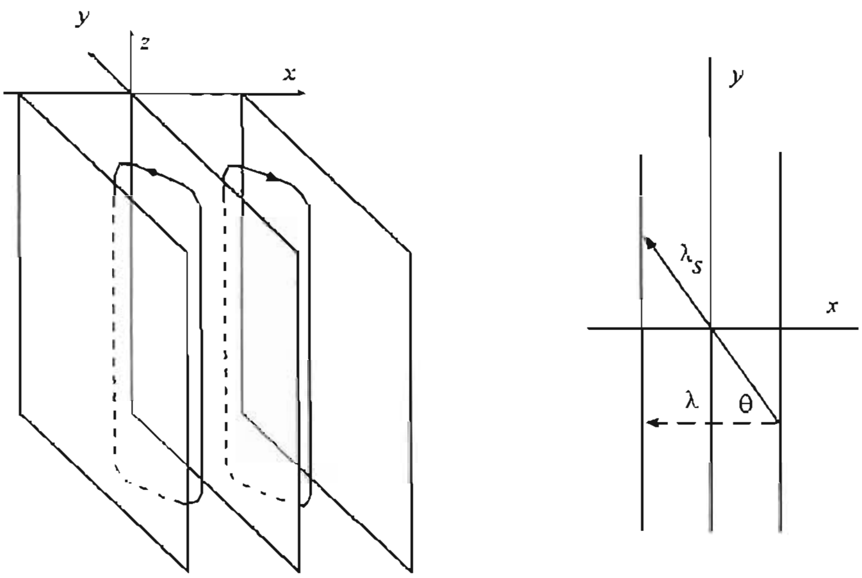

Figure 6: Oblique plane wavelength of a roll

$$\tan\theta = \frac{v'}{u'} \quad \Rightarrow \quad \theta = \arctan\left(\frac{v'}{u'}\right) \tag{3.1.17}$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\left(\frac{v'}{u'}\right)}} \tag{3.1.18}$$

From the previous two expressions and from Figure 7

$$k_s = k\cos\left(\arctan\left(\frac{v'}{u'}\right)\right) = \frac{k}{\sqrt{1+\left(\frac{v'}{u'}\right)^2}} = \frac{k}{\sqrt{1+Ta}} \qquad (3.1.19)$$

Knowing that the critical value of k is $k_c = \pi(1+Ta)^{1/4}$, yields that the wavelength in an oblique plane containing the streamlines is dependent of Taylor number and implicitly of rotation and it is given by

$$k_{s,cr}^{(st)} = \frac{\pi}{\sqrt[4]{1+Ta}} \qquad (3.1.20)$$

The stream function it is defined by its components as: $u' = \frac{\partial\psi}{\partial z}$ and $w' = -\frac{\partial\psi}{\partial x}$ (see ***Appendix 2***)

$$\psi' = A\sin(kx)\sin(\pi z) \qquad (3.1.21)$$

The relationship between A and B is $A = -B$; therefore

$$\psi' = -\frac{\left(k^2+\pi^2\right)}{k}\sin(kx)\sin(\pi z) \qquad (3.1.22)$$

3.2. Linear stability analysis. Oscillatory convection

Over-stable convection implies the possibility of an oscillatory motion and σ can be written as a complex number $\sigma = \sigma_r + i\sigma_i$. In the solution, the real part represents an exponential growth or decay and the imaginary part an oscillation. For $\sigma_r \neq 0$ we have an unstable situation where the solution can go to zero or infinity, depending if σ_r is less or bigger than zero. For $\sigma_r = 0$ however, the solution will oscillate about an equilibrium position. For small amplitude oscillations, the solution will be quasi-stable; for high amplitudes, the solution will "jump" out of equilibrium. The case when $\sigma_r \neq 0$ is called "marginal stability". Substituting $\sigma = i\sigma_{\mathrm{Im}}$ in the expression 2.11 and equation 2.12 and imposing that $\sigma_i^2 \geq 0$ in order to have over-stability result two equations, by calling that real and imaginary part to be equal to zero (***Appendix** 3.2; §1*) The two equations will provide an expression for the over-stable characteristic value of Rayleigh number and the corresponding oscillatory frequency σ_i

$$R_c^{(ov)} = \frac{2}{\alpha}\left[(1+\alpha)(1+\alpha+\gamma) + \frac{\gamma^2 Ta}{(1+\alpha+\gamma)}\right] \qquad (3.2.1)$$

$$\sigma_{\mathrm{Im}}^2 = \frac{(1+\alpha-\gamma)Ta}{(1+\alpha)(1+\alpha+\gamma)} - 1 \qquad (3.2.2)$$

Imposing the condition $\sigma_i^2 > 0$ it results an inequality

$$(1+\alpha)^2+(\gamma-Ta)(1+\alpha)+\gamma Ta<0 \tag{3.2.3}$$

from which we can obtain a further condition in order to allow positive values of α in order to have over-stable convection

$$\left|\frac{(Ta-\gamma)-\sqrt{\gamma^2-6\gamma Ta+Ta^2}}{2}\right|<(1+\alpha) \tag{3.2.4}$$

The quantity under the square root must also be positive

$$\gamma^2-6\gamma Ta+Ta^2>0 \tag{3.2.5}$$

resulting the domain of γ for which we have over-stable convection $\gamma \in \left[0,\left(3-2\sqrt{2}Ta\right)\right]$.

The values of α corresponding to the boundaries of the domain are those of the characteristic value of Rayleigh number associated with stationary convection. Graphical representation of the characteristic curves, for various values of Ta and γ are presented

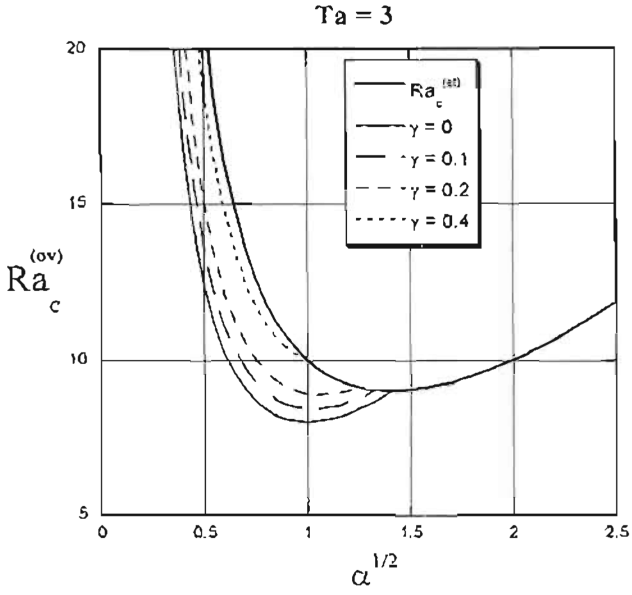

Figure 7: Marginal stability for over-stable convection at *Ta* = 3

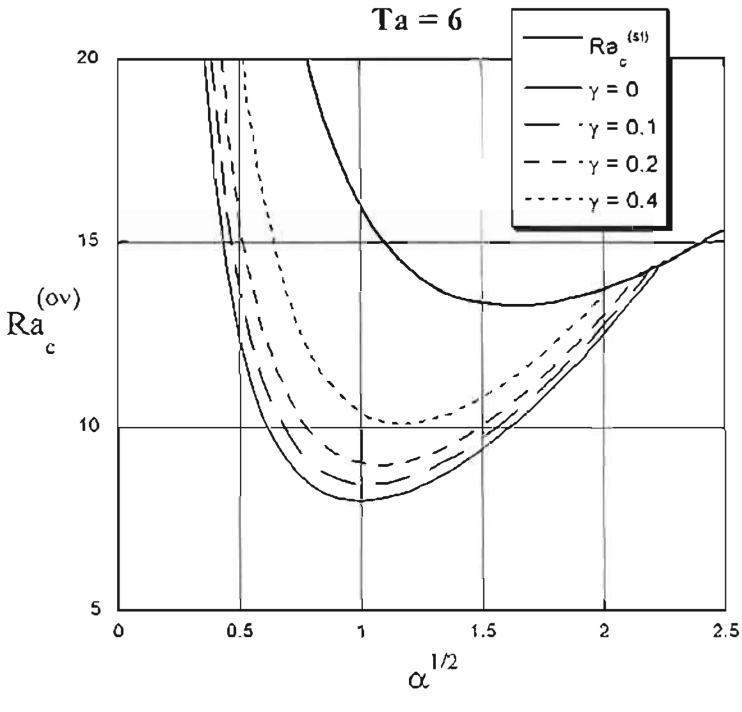

Figure 8: Marginal stability for over-stable convection at *Ta* = 6

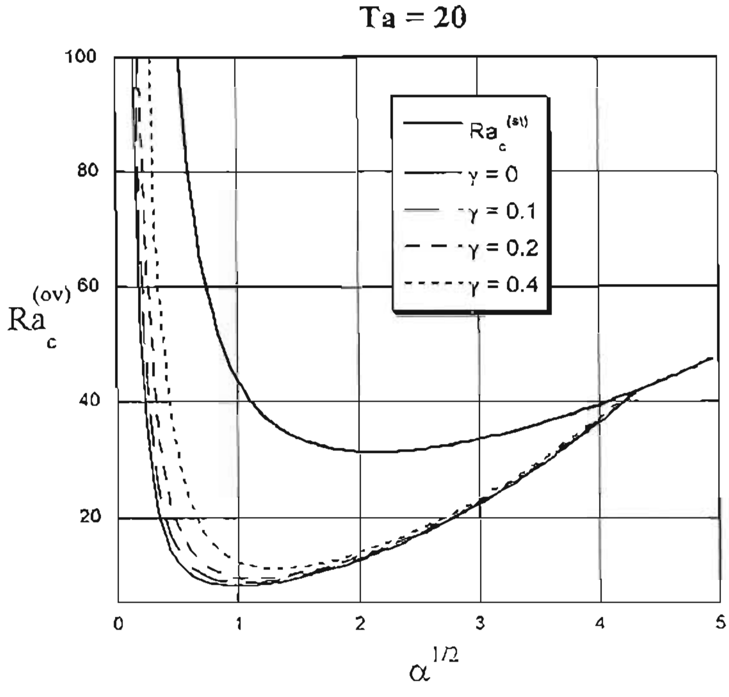

Figure 9: Marginal stability for over-stable convection at $Ta = 20$

The continuous line represents the upper limit of stationary convection.

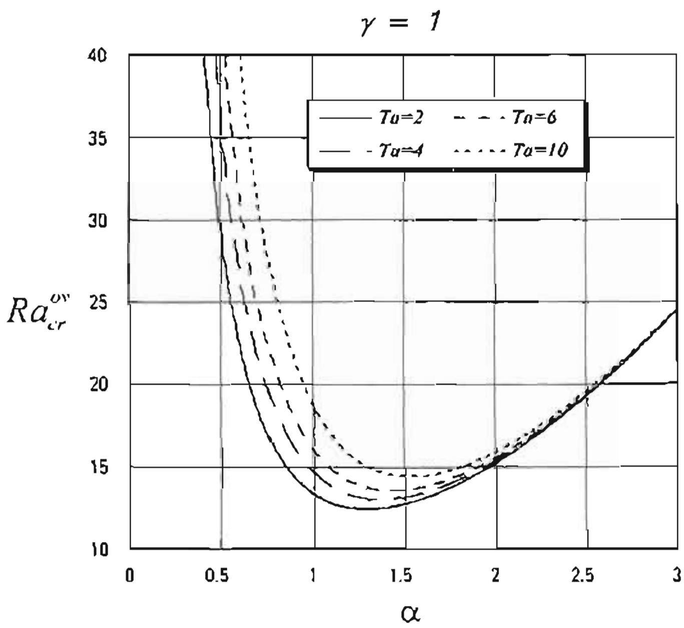

Figure 10: Variation of Rayleigh number for various values of Taylor number

It can be seen that as Taylor number increases, the branching-off points shift to the right. Furthermore, no limitations of the Prandtl number (γ) appears as a necessary condition for over-stability to set at the convection threshold.

The characteristic values of Ra associated with $Ta = 6,\ 20$, and 80 for higher values of γ are shown in the following graph. For high values of γ the curves of over-stability branch-in the stationary zone. The reason lies behind the fact that γ and Ta are both functions of porosity, but inversely proportional to each other.

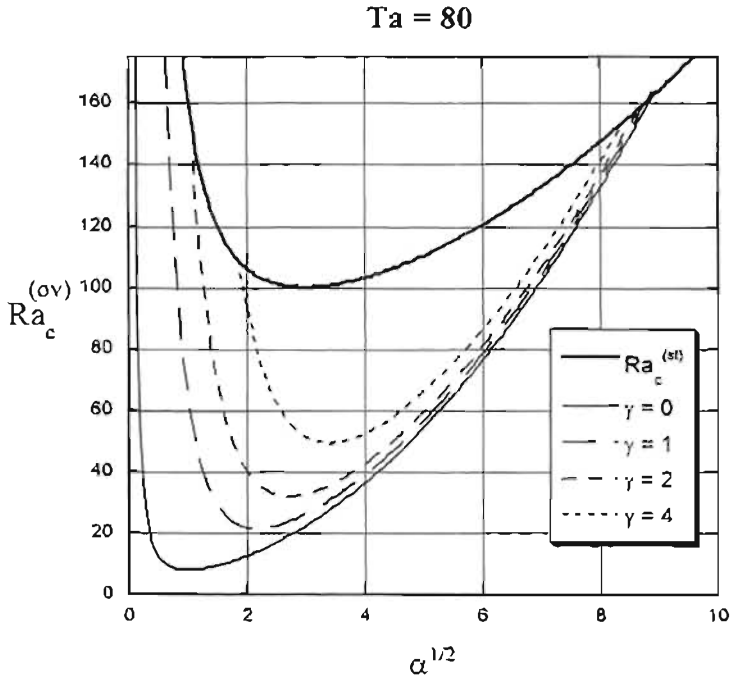

Figure 11: Marginal stability for over-stable convection at *Ta* = 80

The corresponding values of frequency variation is presented in the Figure 12 for the same parameters of γ and Ta

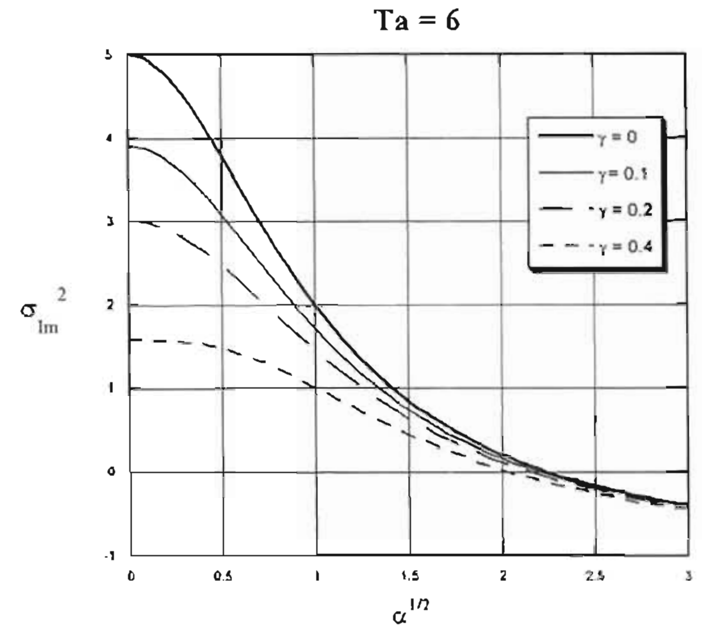

Figure 12: Variation of frequency at $Ta = 6$

Negative values of σ_i^2 show wave number domain where over-stable convection is impossible.

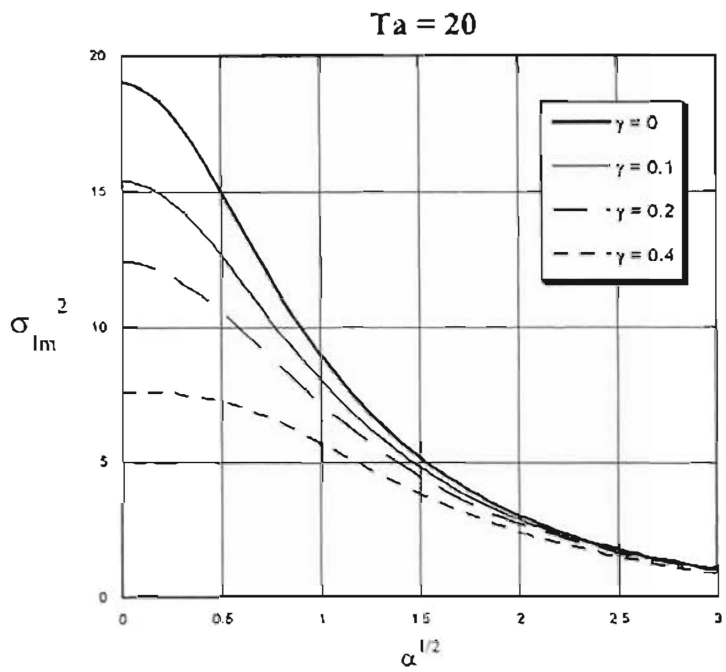

Figure 13: Variation of frequency for $Ta = 20$

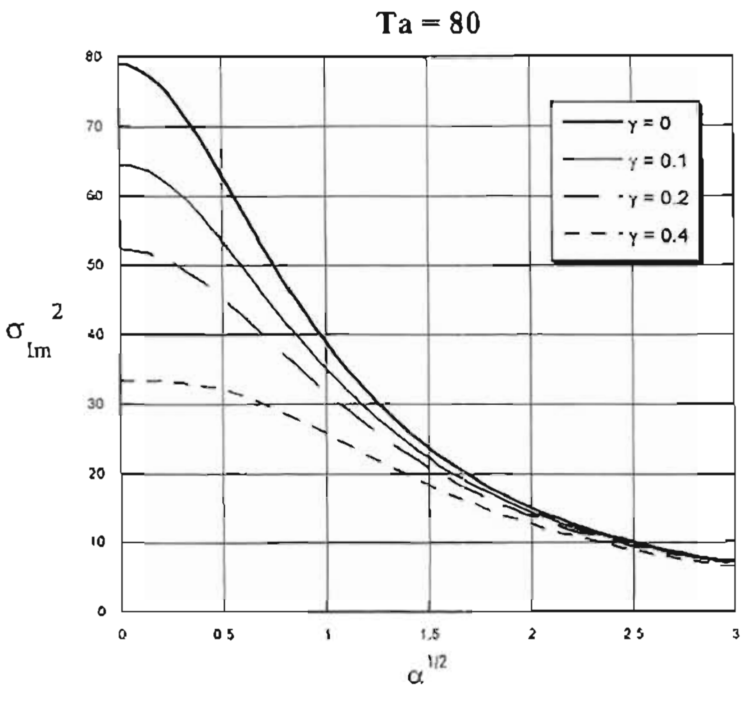

Figure 14: Variation of frequency for *Ta* = 80

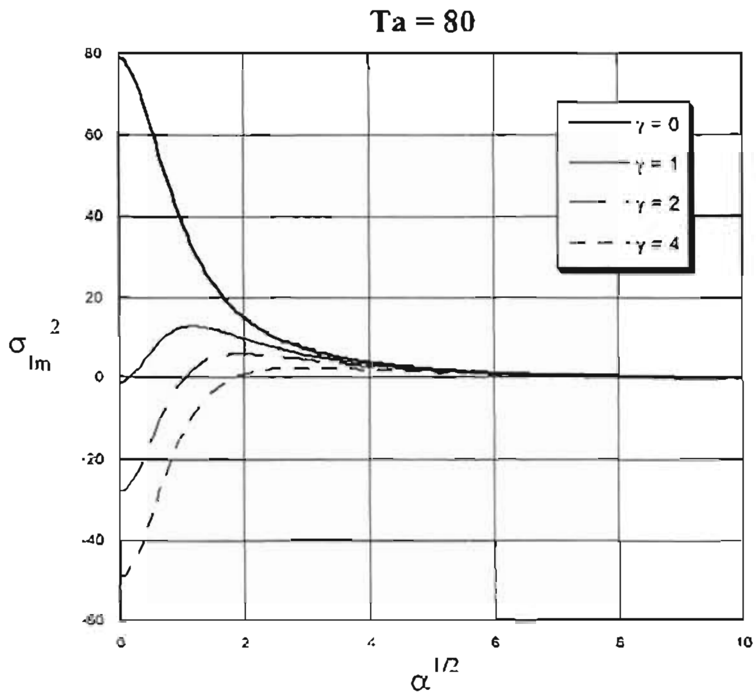

Figure 15: Variation of frequency for *Ta* = 80 and larger values of γ

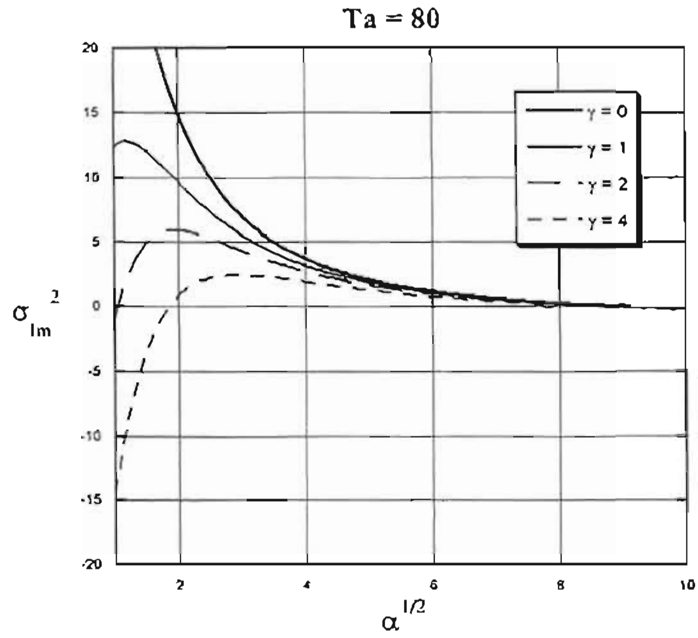

Figure 16: Detail of the frequency variation for *Ta* = 80 and larger values of γ

It is interesting to note the characteristic curves of Rayleigh numbers for a particular case when $\gamma = 0$. This corresponds to a limiting case for the over-stable curves. For such value of γ we have the equations 2.2.1 and 2.2.2 in the following form

$$R_c^{(ov)} = \frac{2}{\alpha}\left(1+\alpha^2\right) \tag{3.2.5}$$

$$\sigma_{Im}^2 = \frac{Ta}{1+\alpha} - 1 \tag{3.2.6}$$

It is obvious that the characteristic curves for $\gamma = 0$ are independent of the Taylor number, and theirs position is fixed in the plane determined by $R_c^{(ov)}$ and $\alpha^{1/2}$, and represent a lower limit for all characteristic curves.

By minimising the expression 2.2.5 with respect to α, results

$$\alpha_{cr}^{(ov)} = 1 \tag{3.2.7}$$

which in turn yields for the critical value of Rayleygh in the over-stable zone

$$R_{cr}^{(ov)} = 8 \tag{3.2.8}$$

The condition that expresses over-stability is derived from equation 2.2.2 by imposing $\sigma_{1m}^2 > o$, which leads to

$$(1+\alpha)^2 - (\gamma - Ta)(1+\alpha) + \gamma Ta < 0 \tag{3.2.9}$$

Equation 2.2.7 yields the condition for a positive range of α to have over-stable convection

$$\gamma \in \left(0,\ \left(3 - 2\sqrt{2}\right)Ta\right) \quad \forall Ta > 1 \tag{3.2.10}$$

The corresponding values of α consistent with over-stable convection is

$$\frac{(Ta-\gamma) - \sqrt{Ta^2 - 6\gamma Ta + \gamma^2}}{2} - 1 < \alpha < \frac{(Ta-\gamma) + \sqrt{Ta^2 - 6\gamma Ta + \gamma^2}}{2} - 1 \tag{3.2.11}$$

On these boundary values $\sigma_{1m}^2 = 0$ and stationary convection occurs. This was illustrated in Figures 9, 10, 11 and 12.

If $\alpha_{cr}^{(ov)} = 1 \Rightarrow Ta \geq 2$ in order to allow a real value for the frequency. The sufficiency of this condition is that $Ra_{cr}^{(ov)} \leq Ra_c^{(st)}$, which implies that at $\gamma = 0$ we have the following

$$8 \leq \left[1 + \sqrt{Ta+1}\right]^2 \tag{3.2.12}$$

From which

$$Ta \geq 4\left(2 - \sqrt{2}\right) \tag{3.2.13}$$

All the other characteristic curves for different values of γ will be situated between the $\gamma = 0$ curve and the characteristic stationary convection curves related to a certain Ta. To obtain the critical value of Rayleigh, wave numbers and corresponding frequency we have to minimise equation (2.2.1) with respect to α, yielding the quadratic algebraic equation

$$\alpha^4 + 2(\gamma+1)\alpha^3 + \gamma(\gamma+1)\alpha^2 - 2\left[(\gamma+1)^2 + \gamma^2 Ta\right]\alpha - \gamma^2(\gamma+1)Ta - (\gamma+1)^3 = 0 \quad (3.2.14)$$

The solution of equation 2.2.14 was obtained numerically. It has one positive and real solution within over-stability bound that is associated to γ and Ta. The other three roots are a pair of complex conjugate and one real, but negative. The over-stable critical wave number is presented in *Figure 17* as a function of γ for various values of Ta.

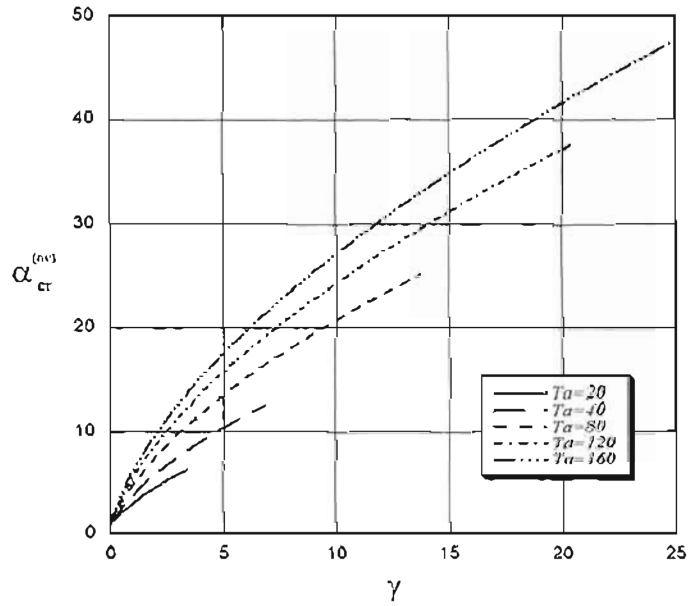

Figure 17: Variation of α associated with over-stable convection for various values of Taylor number

The end of each curve corresponds to a point where no more critical values are consistent with the condition $\sigma_{\shortparallel}^{2} > 0$. Furthermore each end-point of the curves corresponding to a maximum $\alpha_{cr,\max}^{(ov)}$, determine a straight line $\alpha_{cr,\max}^{(ov)} \cong 2$. Equation 2.2.1 will give us, upon substitution of the values of $\alpha_{cr}^{(ov)}$ the critical values of Rayleygh number for over-stability. The dependency of $Ra_{cr}^{(ov)}$ as a function of γ with various Ta numbers taken as parameters, is shown in *Figure 18*.

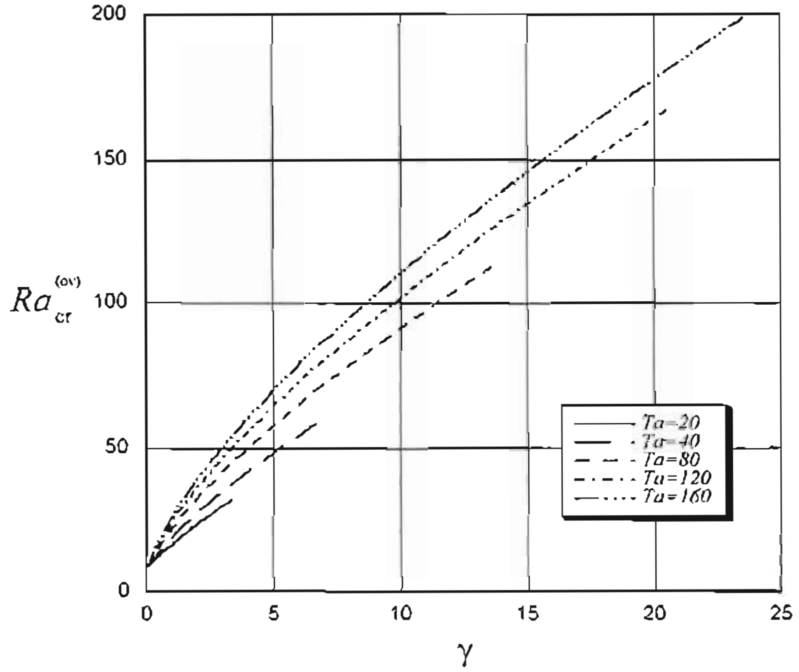

Figure 18: Variation of $Ra_{cr}^{(ov)}$ associated with over-stable convection for various values of Taylor number

It can be noted that the end-point curves corresponding to $Ra_{cr,\max}^{(ov)}$ are lined along a straight line $Ra_{cr,\max}^{(ov)} \cong 8$, which is consistent with the condition $\sigma_{\shortparallel}^{2} > 0$.

By substituting the critical wave number obtained from 2.2.14 into 2.2.2 we obtain the critical value for frequency, for various Ta, presented in *Figure 19*, where we take the abscissa as $\log(\gamma)$ and obtaining a similar graphical representation as in stationary case.

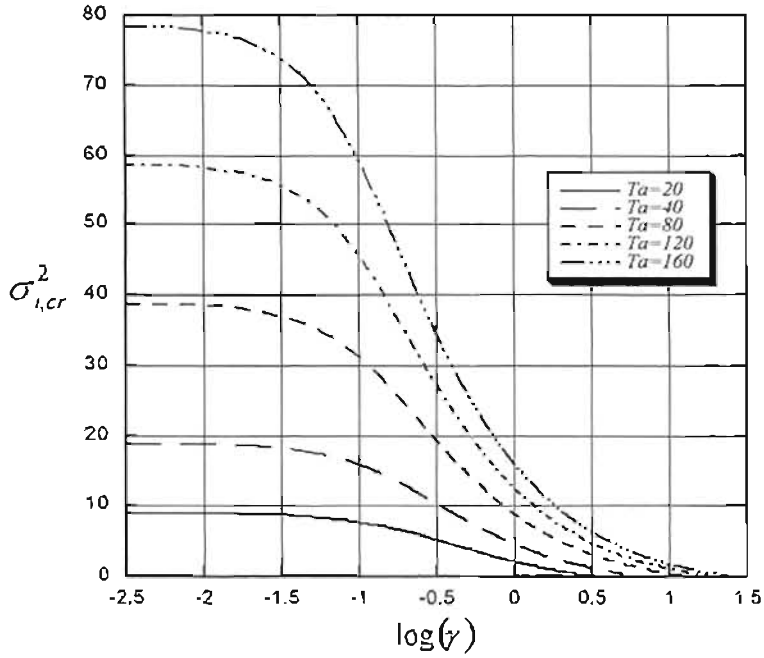

Figure 18: Variation of $Ra_{cr}^{(ov)}$ associated with over-stable convection as a function of $\log\gamma$ for various values of Taylor number

High values for frequency correspond to small values of γ. As in the case analysed before, when $\gamma \to 0$, the critical curves represent only a condition of necessity. For sufficiency to be fulfilled we have to have a further condition when $R_{cr}^{(ov)} \leq R_{cr}^{(st)}$. This is illustrated accurately in *Figure 20*, where (Ta, γ) plane is divided in two by a continuous line, corresponding to $Ra_{cr}^{(ov)} = Ra_{cr}^{(st)}$, where the zone below the line is consistent with over-stable convection and the zone above, for stationary convection where instability occurs. The dotted line corresponding to $\sigma_{i,cr}^2 = 0$, represents a separation between a zone below, where over-stable convection is possible but cannot occur because $Ra_{cr}^{(ov)} > Ra_{cr}^{(st)}$. The limit where over-stable convection occur for the same values of critical Rayleigh numbers, define the CTP (Co-dimension-2 Point). The dotted curve envelops the end-points corresponding to the wave numbers for high values of γ, approximated with a straight line $Ta = 6\gamma + 2$.

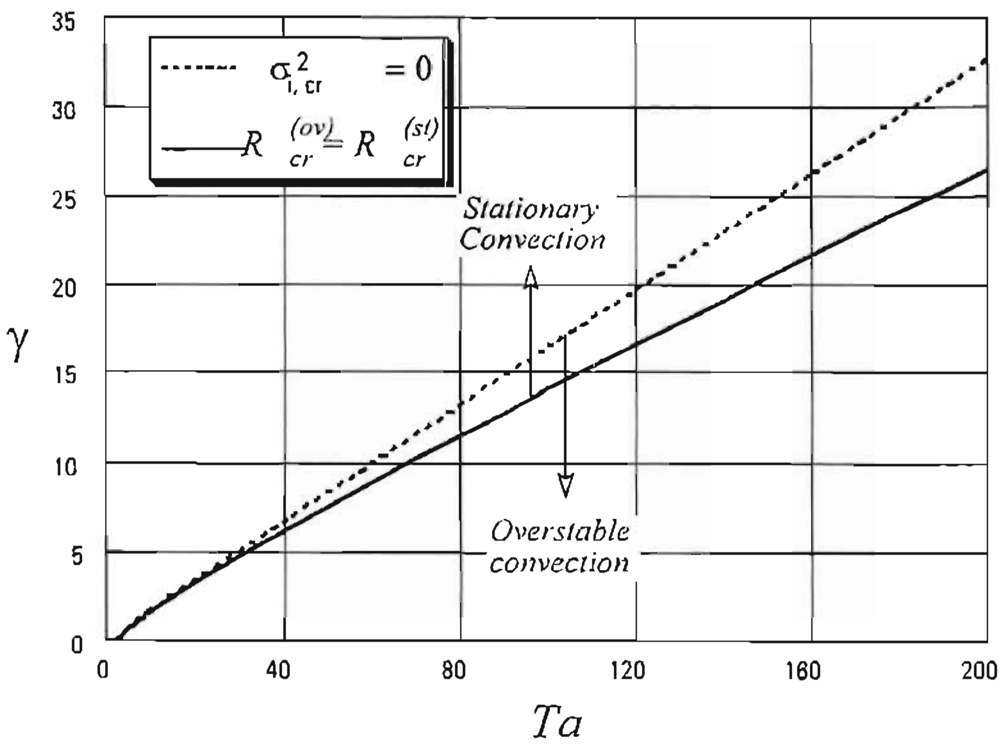

Figure 20: The stability map of the division of the plane in two zones corresponding to stationary and over-stable convection

4. Weak non-linear analysis

In this chapter we shall investigate the solutions of stream function and temperature in a porous layer subject to rotation where small non-linearity is considered. It is convenient to introduce the stream function as discussed in previous section, as $u' = \partial\psi'/\partial z$ and $w' = -\partial\psi'/\partial x$, to express them in the equations 2.3 and 2.7. By de-coupling the two equations and bearing in mind that we deal with a two-dimensional problem, therefore all the derivatives to y, we obtain

$$\left[\chi\frac{\partial}{\partial t'} - \nabla^2\right]T + \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = 0 \qquad (4.0.1)$$

$$\left[\frac{\partial}{\partial t'} + 1\right]^2\nabla^2\psi + Ta\frac{\partial^2\psi}{\partial z^2} + Ra\left[\frac{\partial}{\partial t'} + 1\right]\frac{\partial T}{\partial x} = 0 \qquad (4.0.2)$$

Where the Laplacian in this case it is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \qquad (4.0.3)$$

The derivation of 4.0.1 and 4.0.2 is presented in ***Appendix A4.0***.

4.1. Expansion around stationary solutions

The objective of the weak non-linear analysis is to provide quantitative and qualitative results of the amplitude of convection. The possibility of a co-dimension 2 bifurcation is anticipated at the intersection of the stationary and over-stable solutions (Brand, Hohenberg & Steinberg 1984; Cross & Kim 1988; Schöpf & Zimmermann 1993).

We know that the basic motionless solutions are $\psi_0 = 0$ and $T_0 = 1 - z$ and we can write the stream function and temperature expanded in a series of

$$[\psi] = [\psi_0] + \varepsilon[\psi_1] + \varepsilon^2[\psi_2] + \varepsilon^3[\psi_3] + \dots \qquad (4.1.1a)$$

$$[T] = [T_0] + \varepsilon[T_1] + \varepsilon^2[T_2] + \varepsilon^3[T_3] + \dots \qquad (4.1.1b)$$

where ε is a perturbation defined as $\varepsilon = [1 - Ra_{cr}/Ra]^{1/2}$.

The Rayleigh number can also be written as

$$Ra = Ra_{cr} + Ra_{cr}^{(2n)}\left[\varepsilon^2 + \varepsilon^4 + \dots + \varepsilon^{2n}\right] \qquad (4.1.2)$$

In order to reach finite values for amplitude at the steady state we have to choose a *slow time scale* $\tau = \varepsilon^2 t'$ (by allowing minimal time variations only, preventing exponential growth) and a slow space scale $X = \varepsilon x$. The new space scale was introduced by Newell & Whitehead (1969) and Segel (1969) in order to allow continuous horizontal band of modes of oscillation. Upon these transformations we have to consider a re-scaling of variables in the form

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon \frac{\partial}{\partial X} \tag{4.1.3}$$

$$\frac{\partial}{\partial t'} \to \frac{\partial}{\partial t'} + \varepsilon^2 \frac{\partial}{\partial \tau} \tag{4.3.4}$$

From the expression 4.1.2 can be written as

$$Ra = Ra_{cr}\left(1 + \varepsilon^2\right) \tag{4.1.2b}$$

Substituting the expansions 4.1.1, 4.1.2, and the slow time and large space scales into equations 4.0.1 and 4.0.2 and identifying the terms of equal powers of ε, produces a set of partial differential equations at each order.

For the leading order we have a set of equations identical to those solved for the linear stability case

$$\nabla^2 \psi_1 + Ta \frac{\partial^2 \psi_1}{\partial z^2} + Ra_{cr} \frac{\partial T_1}{\partial x} = 0 \tag{4.1.5a}$$

$$\nabla^2 T_1 - \frac{\partial \psi_1}{\partial x} = 0 \tag{4.1.5b}$$

For the second order we have the equations presented as

$$\nabla^2\psi_2 + Ta\frac{\partial^2\psi_2}{\partial z^2} + Ra_{cr}\frac{\partial T_2}{\partial x} = -2\frac{\partial}{\partial X}\frac{\partial\psi_1}{\partial x} \tag{4.1.6a}$$

$$\nabla^2 T_2 - \frac{\partial\psi_2}{\partial x} = -2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial\psi_1}{\partial X} + \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} \tag{4.1.6b}$$

The RHS of equations 4.1.6 (a and b) consist of non-homogeneous terms including the solutions already determined at order ε. The non-homogeneous terms fore a particular solution in addition to the solution of the homogeneous operator.

The third order equations are presented in the form

$$\nabla^2\psi_3 + Ta\frac{\partial^2\psi_3}{\partial z^2} + Ra_{cr}\frac{\partial T_3}{\partial x} = -2\frac{\partial}{\partial \tau}\nabla^2\psi_1 - Ra_{cr}\frac{\partial}{\partial \tau}\frac{\partial T_1}{\partial x} -$$
$$\frac{\partial^2\psi_1}{\partial X^2} - 2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} - Ra_{cr}\frac{\partial T_2}{\partial X} \tag{4.1.7a}$$

$$\nabla^2 T_3 - \frac{\partial\psi_3}{\partial x} = -2\frac{\partial}{\partial X}\frac{\partial T_2}{\partial x} + \frac{\partial\psi_2}{\partial X} +$$
$$\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z} + \frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z} + \tag{4.1.7b}$$
$$\chi\frac{\partial T_1}{\partial \tau} - \frac{\partial^2 T_1}{\partial X^2} + \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X} - \frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z}$$

We shall analyse the equations for each order apart.

For order ε, the solution is given by the eigenvalues of the stationary convection

$$\psi_1 = \left[A_1(\tau, X)e^{ikx} + A_1^*(\tau, X)e^{-ikx}\right]\sin(\pi z) \tag{4.1.8a}$$

$$T_1 = \left[B_1(\tau, X)e^{ikx} + B_1^*(\tau, X)e^{-ikx}\right]\sin(\pi z) \tag{4.1.8b}$$

The relationships between the amplitudes $A_1(\tau, X)$ and $B_1(\tau, X)$, respectively $A_1^*(\tau, X)$ and $B_1^*(\tau, X)$, are obtained by substituting the solutions 3.1.8 (a and b) into equations 3.1.5 (a and b).

$$B_1 = -\frac{i\sqrt{\alpha}}{\pi(1+\alpha)}A_1 \tag{4.1.9a}$$

$$B_1^* = \frac{i\sqrt{\alpha}}{\pi(1+\alpha)}A_1^* \tag{4.1.9b}$$

Solvability condition at order ε^3 will determine the amplitudes A_1 and A_1^*

The order ε^2 solutions result by de-coupling the equations 3.1.6 (a and b). The non-linear part of the RHS will generate a particular temperature solution as follows

$$\psi_2 = \left[A_2 e^{ikx} + A_2^* e^{-ikx}\right]\sin(\pi z) \tag{4.1.10a}$$

$$T_2 = \left[B_1(\tau, X)e^{ikx} + B_1^*(\tau, X)e^{-ikx}\right]\sin(\pi z) - \frac{\alpha}{2\pi(\alpha+1)}A_1 A_1^* \sin(2\pi z) \tag{4.1.10b}$$

The particular solution for the stream function converges to zero.

The relationship between A_2 and A_2^* at order ε^2 is identical as for the order ε.

The order ε^3, solutions consist of known solutions calculated from the previous orders, ε and ε^2. The equations at order ε^3 are non-homogeneous versions of those at order ε. From here it can be drawn a solvability condition that will impose constraints on the amplitudes at order ε enables their determination. The solvability condition results from the process of de-coupling the equations and evaluating the RHS forcing terms, which is represented in the form of $\left[A_1(\tau,X)e^{ikx} + A_1^*(\tau,X)e^{-ikx}\right]\sin(\pi z)$. All the other terms that containing higher harmonics of z, will be forced to zero. This condition will lead to determination of the solvability condition in the form of a partial differential equation, where the original time and space scales are restored:

$$\eta\frac{\partial A}{\partial t'} - (1+\alpha)\frac{\partial^2 A}{\partial x^2} = \frac{\pi^2\alpha^2}{2}\left(\xi_0^{st} - AA^*\right)A \qquad (4.1.11)$$

where A, A^*, ξ_0^{st} and η are as follows

$$A = \varepsilon A_1 \qquad (4.1.12a)$$

$$A^* = \varepsilon A_1^* \qquad (4.3.12b)$$

$$\xi_0^{st} = \frac{2(1+\alpha)}{\alpha}\left(\frac{R}{R_{cr}^{st}} - 1\right) \qquad (4.3.13)$$

$$\eta = \frac{(1+\alpha)(2-\alpha)+\alpha\gamma}{\gamma} \tag{4.1.14}$$

It is noted the appearance of a diffusion term corresponding to the slow space scale. By imposing a symmetry condition at the axis of rotation ($x=0$), implies $A_1 = -A_1^*$. This changes the solution in

$$\psi^{(1)} = C_1 \sin(kx)\sin(\pi z) \tag{4.1.15}$$

where $C_1 = 2iA_1$. In this case we do not have a phase angle and the result satisfies the equations and the boundary conditions without slow space scales. As a result, the diffusion term vanishes out from equation 3.1.11, which subsequently transforms into an ordinary differential equation of real amplitude $C = \varepsilon C_1$.

$$\eta\frac{dC}{dt} = \frac{\alpha^2\pi^2}{8}\left(\xi^{s1} - C^2\right)C \tag{4.1.16}$$

where $\xi^{s1} = 4\xi_0^{s1}$. The equation 3.1.16 yields a solution at the steady state in the form

$$C = \begin{cases} 0 & \forall \quad R < R_{cr}^{s1} \\ \pm\sqrt{\xi^{s1}} & \forall \quad R \geq R_{cr}^{s1} \end{cases} \tag{4.1.17}$$

The solution 3.1.17 shows that a pitchfork bifurcation occurs at the critical value of Rayleigh number for stationary convection (Figure 17). The relaxation time is positive as long as $\gamma > \gamma_t^{st}$ (γ_t^{st} is the transition value of γ).

$$\gamma_t^{st} = \sqrt{(1+Ta)} - \frac{2}{\sqrt{(1+Ta)}} - 1 \qquad (4.1.18)$$

Below this value the relaxation time becomes negative and the solution decays to $C = 0$. The values of Taylor number consistent with a positive relaxation time are represented by the condition $Ta \geq 3$. Determination of the amplitude coefficients provides a complete solution for the stationary convection at order ε. The complete calculation of such solution is provided in ***Appendix 4.1***.

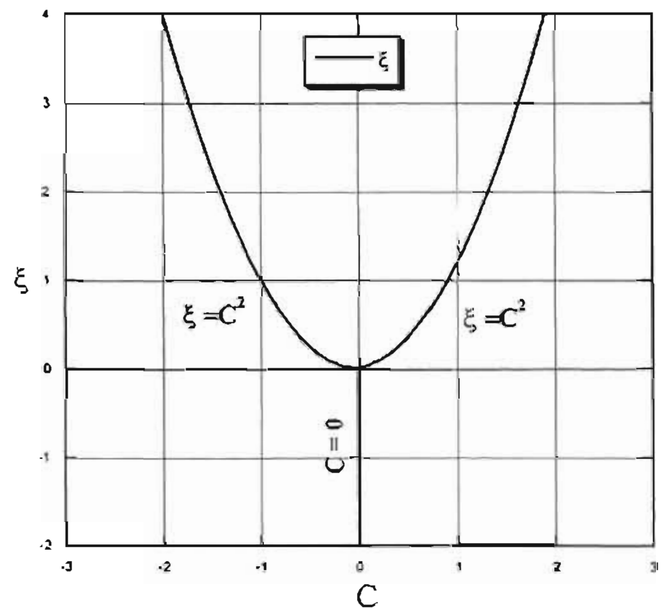

Figure 17: Graphical representation of the solution of the ψ_1 amplitude in $\xi - C$ plane

4.2. Expansion around over-stable solutions

We have identified in the previous section the solutions for both the stream function ψ_1 the and temperature T_1 satisfying their boundary conditions, in the form

$$\psi_1 = 2i\left(A_1 e^{ikx} - A_1^{*} e^{-ikx}\right)\sin(kx)\sin\pi z \tag{4.2.1}$$

$$T_1 = 2\left(C_1 e^{ikx} + C_1^{*} e^{-ikx}\right)\cos(kx)\sin(\pi z) \tag{4.2.2}$$

In the case of weak non-linear analysis of the over-stable convection, equations 3.0.1 and 3.0.2 will apply with the only requirement that we have to refer to the corresponding critical values of the over-stable convection. The expansions 3.1.1 a and b apply as well, but we have to introduce two slow time scales $\tau = \varepsilon^2 t'$, $\tau_o = \varepsilon t'$ and allow a short time scale to be present into equations in order to describe amplitude fluctuations. A further re-scaling of the short time scale is convenient in the form $\tilde{t} = \sigma_o t'$, where $\sigma_o = \sigma_i^{cr}$. By substituting these new scales into equations 3.0.1 and 3.0.2 we obtain, for the leading order, the following equations

$$\left[\sigma_o \frac{\partial}{\partial \tilde{t}} + 1\right]^2 \nabla^2 \psi_1 + Ta \frac{\partial^2 \psi_1}{\partial z^2} + Ra_{cr}\left[\sigma_o \frac{\partial}{\partial \tilde{t}} + 1\right]\frac{\partial T_1}{\partial x} = 0 \tag{4.2.3a}$$

$$\left[\chi \sigma_o \frac{\partial}{\partial \tilde{t}} - \nabla^2\right] T_1 + \frac{\partial \psi_1}{\partial x} = 0 \tag{4.3.3b}$$

The general solution for ψ_1 has the form

$$\psi_1 = \left[A_1 e^{i(kx+\tilde{t})} + B_1 e^{i(kx-\tilde{t})} + A_1^* e^{-i(kx+\tilde{t})} + B_1^* e^{-i(kx-\tilde{t})}\right]\sin(\pi z)$$

$$T_1 = \left[C_1 e^{i(kx+\tilde{t})} + D_1 e^{i(kx-\tilde{t})} + C_1^* e^{-i(kx+\tilde{t})} + D_1^* e^{-i(kx-\tilde{t})}\right]\sin(\pi z)$$

This case will be analysed in detail in Section 4. By imposing symmetry conditions at the axis of rotation one obtains upon substitution $A_1 = -B_1^*$ and $B_1 = -A_1^*$. This is a special case of standing waves while travelling waves are excluded.

At first order a relationship between coefficients is recovered.

$$C_1 = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}A_1 \qquad (4.2.4a)$$

$$C_1^* = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}A_1^* \qquad (4.2.4b)$$

The equations at order $O(\varepsilon^2)$ are as follows

$$\left[\sigma_o\frac{\partial}{\partial\tilde{t}} + 1\right]^2\nabla^2\psi_2 + Ta\frac{\partial^2\psi_2}{\partial z^2} + Ra_{cr}\left[\sigma_o\frac{\partial}{\partial\tilde{t}} + 1\right]\frac{\partial T_2}{\partial x} =$$

$$-2\frac{\partial}{\partial\tau_o}\left[\sigma_o\frac{\partial}{\partial\tilde{t}} + 1\right]\nabla^2\psi_1 - Ra_{cr}\frac{\partial}{\partial\tau_o}\frac{\partial T_1}{\partial x} \qquad (4.2.5a)$$

$$\left[\chi\sigma_o\frac{\partial}{\partial\tilde{t}}-\nabla^2\right]T_2+\frac{\partial\psi_2}{\partial x}=-\chi\frac{\partial T_1}{\partial\tau_o}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}+\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} \qquad (4.2.5b)$$

The solution to the equations 3.2.5 a and b represent a superposition of the homogeneous part and particular solutions due to the non-homogeneous terms. The homogeneous solution has the same form as in the case of order $O(\varepsilon)$

$$\psi_2=2i\left(A_2e^{ikx}+A_2^{*}e^{-ikx}\right)\sin(kx)\sin\pi z \qquad (4.2.6a)$$

$$T_2=2\left(C_2e^{ikx}+C_2^{*}e^{-ikx}\right)\cos(kx)\sin(\pi z) \qquad (4.2.6b)$$

The relationships that exist between the coefficients preserve.

By evaluating the right hand side of equations 4.2.5 a and b, it is clear that a particular solution will emerge in the form of

$$\psi_2^p=\tilde{t}\sin(\tilde{t})\sin(kx)\sin(\pi z) \qquad (4.2.7a)$$

$$T_2^p=\tilde{t}\cos(\tilde{t})\sin(kx)\sin(\pi z) \qquad (4.2.7b)$$

Which are secular terms in the solution. In this case we have a condition of resonance, unless we set $\partial A_1/\partial\tau_o=0$ in order to avoid it. The particular solutions for the stream function and temperature are

$$\psi_2^p=0 \qquad (4.2.8a)$$

$$T_2^p=\left(b_2+a_1e^{2i\tilde{t}}+a_1^{*}e^{-ik\tilde{t}}\right)\sin(2\pi z) \qquad (4.2.8b)$$

where the coefficients a_1, a_1^* and b_2 are as follows

$$b_2 = -\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1A_1^* \tag{4.2.9a}$$

$$a_1 = \frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_0^2-i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+(\gamma\sigma_o)^2\right](4+\gamma^2\sigma_o^2)}A_1^2 \tag{4.2.9b}$$

$$a_1^* = \frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_0^2+i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+(\gamma\sigma_o)^2\right](4+\gamma^2\sigma_o^2)}(A_1^*)^2 \tag{4.2.9c}$$

The de-coupled equation at $O(\varepsilon^3)$ for ψ is shown below

$$\left\{\left[\chi\sigma\frac{\partial}{\partial\tilde{t}}-\nabla^2\right]\left[\left(\sigma_o\frac{\partial}{\partial\tilde{t}}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\sigma_o\frac{\partial}{\partial\tilde{t}}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3=$$

$$-\left(\chi\frac{\partial}{\partial\tilde{t}}-\nabla^2\right)\left[\left(\frac{\partial}{\partial\tilde{t}}+1\right)^2\frac{\partial^2}{\partial X^2}+4\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial\tilde{t}}+1\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\left(\frac{\partial^2}{\partial\tau_o^2}+2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial\tilde{t}}+1\right)\right)\nabla^2\right]\psi_1-$$

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial\tilde{t}}+1\right)\nabla^2+2\left(\frac{\partial}{\partial\tilde{t}}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_1-$$

$$\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial\tilde{t}}+1\right)^2\nabla^2\psi_1+Ra\left[\left(\frac{\partial}{\partial\tilde{t}}+1\right)\frac{\partial^2}{\partial x^2}\psi_1+\left(\frac{\partial}{\partial\tilde{t}}+1\right)\frac{\partial^2}{\partial X^2}\psi_1\right]+ \qquad (4.2.10)$$

$$Ra_{cr}\left[2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1+\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\psi_1\right]-Ra_{cr}\left(\chi\sigma_o^2\frac{\partial}{\partial x}\frac{\partial^2}{\partial\tilde{t}^2}+\chi\sigma_o\frac{\partial}{\partial x}\frac{\partial}{\partial\tilde{t}}\right)T_1-$$

$$Ra_{cr}\left(\frac{\partial}{\partial x}\frac{\partial}{\partial\tau}\nabla^2+\frac{\partial}{\partial x}\frac{\partial}{\partial\tilde{t}}\nabla^2+\frac{\partial}{\partial x}\nabla^2+\chi\frac{\partial}{\partial x}\frac{\partial}{\partial\tau}\right)T_1+\Theta(\psi_2,\ T_2)-$$

$$Ra_{cr}\left[\left(\frac{\partial}{\partial\tilde{t}}+1\right)\frac{\partial}{\partial X}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]-$$

$$Ra_{cr}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial\tilde{t}}+1\right)\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}-\frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z}+\frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial X}-\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x}\right]$$

The right hand side of equation 3.2.10 consists of terms evaluated from the order $O(\varepsilon)$ and $O(\varepsilon^2)$ written under a general term $\Theta(\psi_2,\ T_2)$. To avoid resonance due to secular terms appearing in the equation, we have to set all the forcing terms coefficients to zero. In order to determine the amplitudes at order $O(\varepsilon)$ we shall consider only the terms containing ψ_1 and T_1. They are terms of form $e^{it}\sin(kx)\sin(\pi z)$ and $e^{-it}\sin(kx)\sin(\pi z)$. The others are the non-resonant harmonics or various convection terms associated to the homogeneous operator. This operation yields a differential equation for the unknown complex amplitude at order $O(\varepsilon)$ in the form

$$M_{12}\frac{\partial^2 A}{\partial \tau_o^2} + M_3\frac{\partial^2 A}{\partial X^2} + M_4\frac{\partial A}{\partial \tau} + M_5 A + M_6 M_7 A^2 A^* = 0 \tag{4.2.11}$$

where M_{ij} are coefficients analysed in detail in the Appendix 4.2 and $A = \varepsilon A_1$ and $A^* = \varepsilon A_1^*$. By setting $\partial^2 A/\partial \tau_0^2$ and $\partial^2 A/\partial X^2$ equal to zero we obtain an ordinary differential equation

$$\frac{\partial A}{\partial t} = Z_1\left[\xi^{ov} - Z_2 AA^*\right]A \tag{4.2.12}$$

where $Z_1 = M_5/M_4$ and $Z_2 = M_6 M_7/M_5$

The following notations were used

$$Z_1 = z_{1R} + iz_{1I} = \frac{\pi^2 \alpha\gamma s R_{cr}\left(\sigma_o P^\circ + Q^\circ\right)}{q} + i\frac{\pi^2 \alpha\gamma s R_{cr}\left(\sigma_o Q^\circ - P^\circ\right)}{q} \tag{4.2.13}$$

where

$$P^\circ = 2\sigma_o p(p+\gamma)s + \alpha\gamma\sigma_o R_{cr}(p-\gamma) \tag{4.2.14}$$

$$Q^\circ = 2p\left(p - \gamma\sigma_o^2\right)s - \alpha p R_{cr}(p-\gamma) \tag{4.2.15}$$

$$q = \sigma_o^2\left[p(p+\gamma)s + \alpha\gamma R_{cr}(p-\gamma)\right]^2 + p^2\left[2s\left(p-\gamma\sigma_o^2\right) - \alpha R_{cr}(p-\gamma)\right]^2 \tag{4.2.16}$$

$$p = \alpha + 1 \tag{4.2.17}$$

$$s = (\alpha + 1)^2 + \gamma^2 \sigma_o^2 \tag{4.2.18}$$

$$Z_2 = z_{2r} + iz_{2i} = \frac{\alpha\left[6p + \alpha\gamma^2\sigma_o^2\right]}{s\left(4 + \gamma^2\sigma_o^2\right)} + i\left(-\frac{\alpha\gamma\sigma_o(\alpha + 3)}{s\left(4 + \gamma^2\sigma_o^2\right)}\right) \tag{4.2.19}$$

$$\xi^{ov} = \varepsilon^2 = \left(\frac{R}{R_{cr}^{ov}} - 1\right) \tag{4.2.20}$$

It is useful to express the equation 3.2.11 as two equations, one for amplitude $r = |A|$, one for phase θ

$$A = re^{i\theta} \qquad A = re^{-i\theta} \qquad AA^* = r^2 \tag{4.2.21}$$

$$J_1 \frac{dr}{dt} = \left[\xi^{ov} - J_2 r^2\right] r \tag{4.2.22}$$

$$\frac{d\theta}{dt} = J_3 \xi^{ov} - J_4 r^2 \tag{4.2.23}$$

Where

$$J_1 = \frac{1}{z_{1R}} \qquad J_2 = \frac{z_{1R} z_{2R} - z_{1I} z_{2I}}{z_{1R}} \tag{4.2.24}$$

$J_3 = z_{1i}$ $\qquad J_4 = z_{1i}z_{2r} + z_{1r}z_{2i}$ (4.2.25)

The sign of the coefficient of the non-linear term indicates the direction of the bifurcation, i.e. forward (supercritical) or inverse (subcritical). If $J_2 > 0$ the bifurcation is forward. If $J_2 < 0$, the bifurcation is inverse.

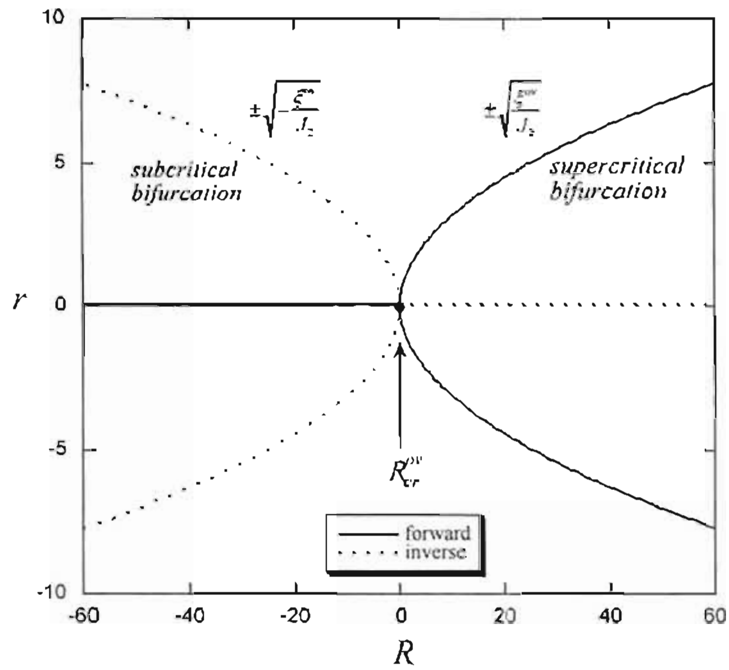

Figure 18: Graphical representation of the bifurcation as a function of the signature of the coefficient of the non-linear term in the amplitude equation

The change of sign of the non-linear term implies the transition from equilibrium to non-equilibrium and the specific point where this occurs is called *non-equilibrium* or *tri-critical* point. By representing the coefficient J_2 as a function of γ, and implicitly as a function of α_{cr} we obtain a series of representations for various values of Taylor number, as a parameter.

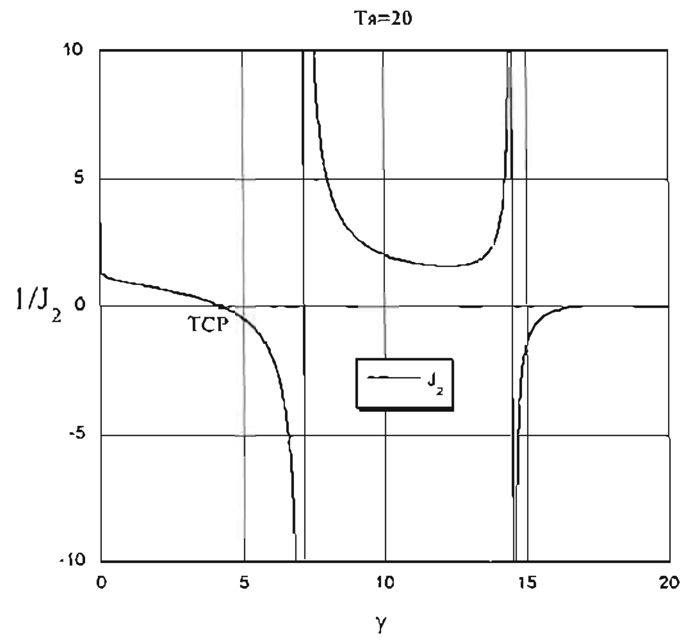

Figure 19: Variation of the coefficient J_2 **as function of** γ **for** $Ta = 20$

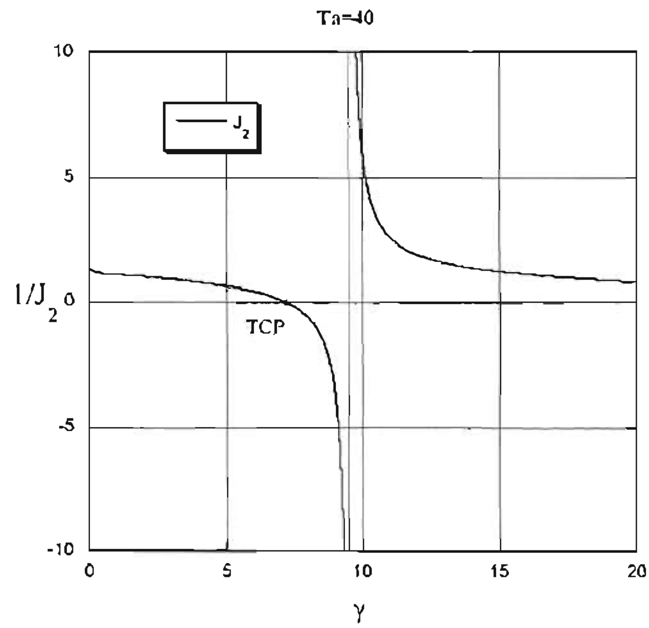

Figure 20: Variation of the coefficient J_2 **as function of** γ **for** $Ta = 40$

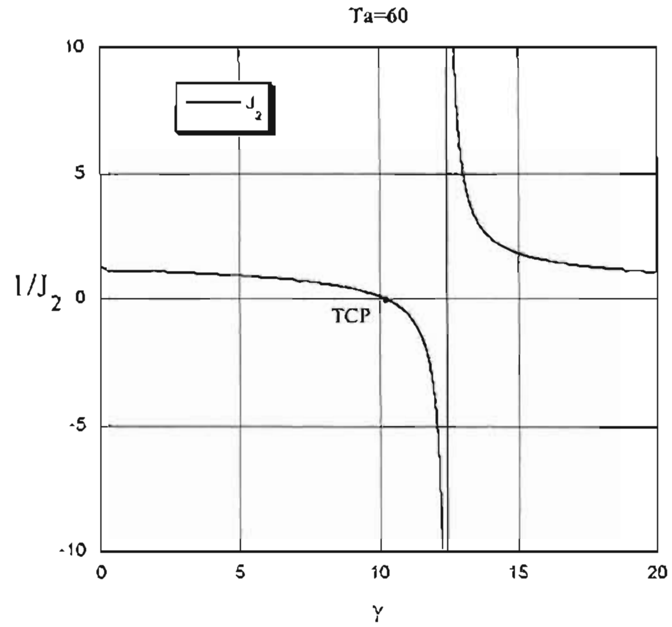

Figure 21: Variation of the coefficient J_2 as function of γ for $Ta = 60$

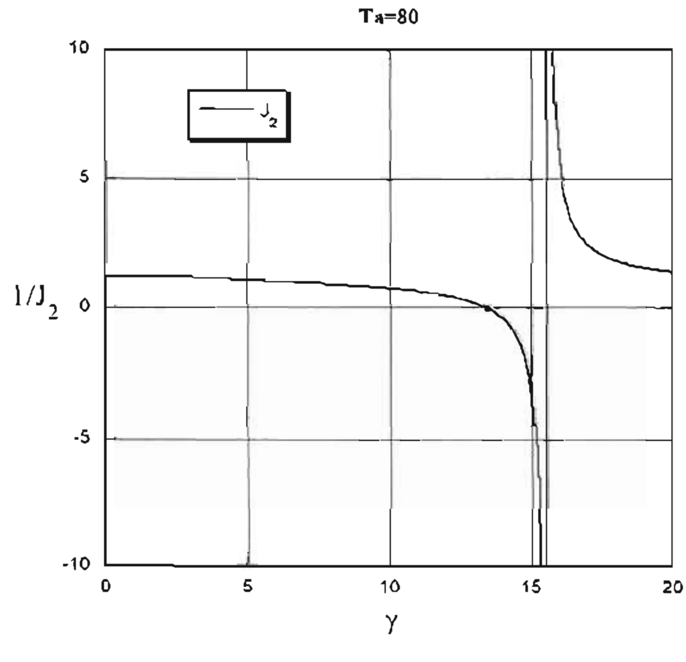

Figure 22: Variation of the coefficient J_2 as function of γ for $Ta = 80$

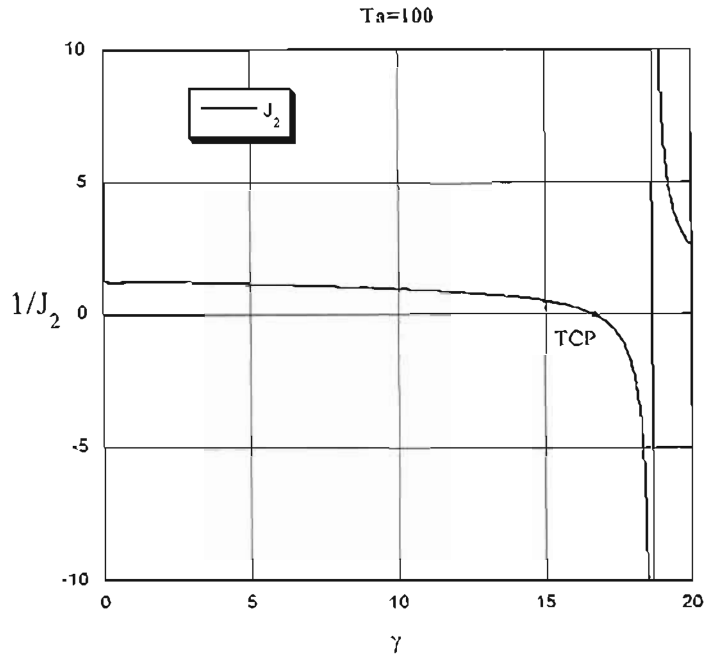

Figure 23: Variation of the coefficient J_2 as function of γ for $Ta = 100$

The behaviour of the coefficient J_2 is in general similar for all values of Ta before the tri-critical point (TCP). For each value of Taylor number there is a maximum allowed value for γ, which is associated with the over-stability of the convection process. For values of $\gamma < \gamma_{tc}$ the bifurcation is forward, while for $\gamma > \gamma_{tc}$ the bifurcation is inverse. The point of singularity that makes J_2 to diverge is always situated at $\gamma > \gamma_{\max}$. A diagram representing the variation of γ_{tc} as a function of Taylor number is presented in Figure 24.

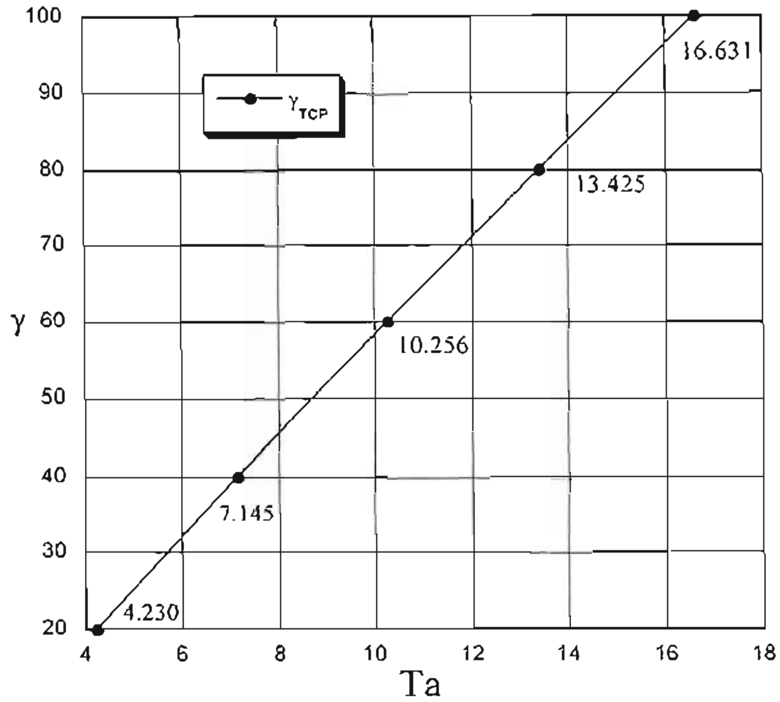

Figure 24: Variation of γ_{tc} with Ta

It can be noted that the variation of γ_{tc} follows the approximate path of a straight line $\gamma_{tc} = aTa + b$, where $a = 6.456$ and $b = -7.23$. The values of γ_{tc} corresponding to various Ta are shown on the graph above.

Each curve corresponding to the non-linear term coefficient contains another zero and a singularity, as illustrated in Figure 19 for $Ta = 20$, but this is located in all the cases beyond the over-stable zone. As Ta increases the next singularity and change of sign shifts out of the graph range.

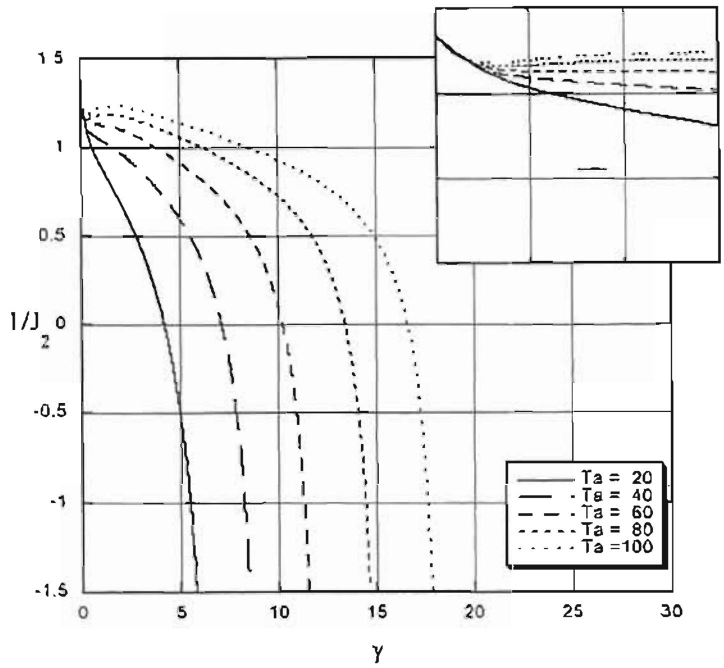

Figure 25: Variation of J_2 for various Taylor numbers

It can be seen from the graph above that for small values of γ the behaviour of the non-linear term coefficient is the same. In general the location of γ_{tc} is different form γ_{max} and dependent of Ta, as shown in Figure 26.

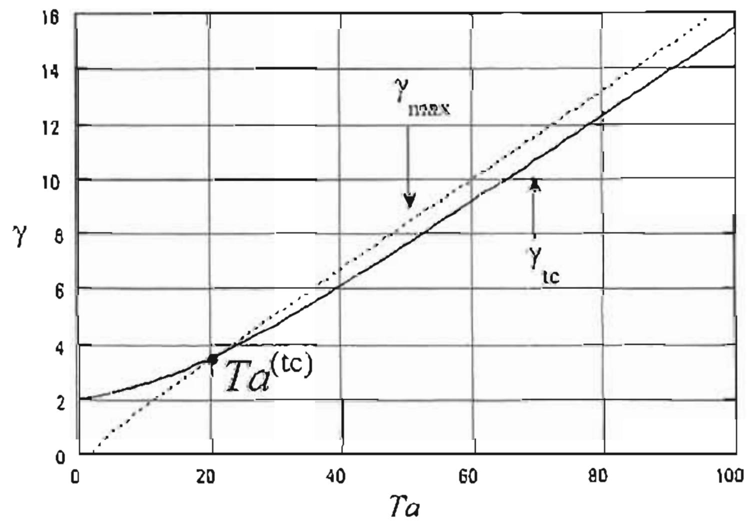

Figure 26: Curves representing the maximum and tri-critical values for γ

It can be noted the point corresponding to the Taylor number $Ta^{(ic)}$ from which $\gamma_{max} > \gamma_{ic}$. The meaning of this transition is that for values of γ below $Ta^{(ic)}$ we have a forward bifurcation and for values of γ above $Ta^{(ic)}$ we have an inverse bifurcation over entire over-stable zone.

As seen from 3.2.22, J_1 is the relaxation time. If the relaxation time is positive, the forward bifurcation is stable. Otherwise, the inverse bifurcation becomes stable (See Figure 18). Figure 27 shows the relaxation time J_1 as a function of γ for different values of Taylor numbers.

From the figure it is evident that the relaxation time seems to be independent of Ta and linearly dependent of γ. Also we can notice that it is positive over all the range of parameters considered.

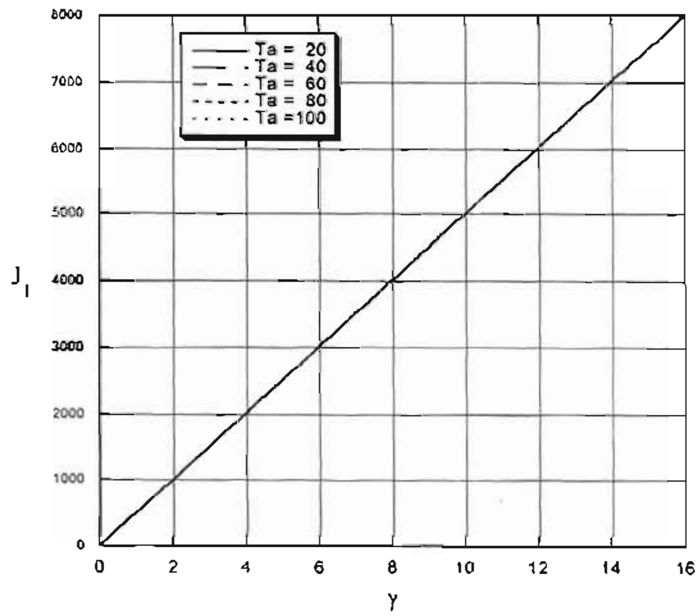

Figure 27: Variation of the relaxation time coefficient J_1 as a function of γ for various Taylor numbers

For a steady state situation, the expression 3.2.22 can be written as

$$J_2^* \xi^{ov} = r^2 \tag{4.2.25}$$

which represent the post transient state for supercritical values of R, where $J_2^* = 1/J_2$ and yielding a solution in the form

$$r = \begin{cases} 0 & \forall R < R_{cr}^{(ov)} \\ \pm\sqrt{\xi^{ov} J_2^*} & \forall R \geq R_{cr}^{(ov)} \end{cases} \tag{4.2.26}$$

Therefore the solution for the amplitude can be expressed as

$$A = r\exp[i\theta] = r\exp\left[i\dot{\theta}t\right] = \pm\sqrt{\xi^{ov} J_2^*}\exp\left[i\dot{\theta}t\right] + c.c. \tag{4.2.27}$$

where c.c. stands cor the complex conjugate part.

The non-linear corection for frequency can be obtained from equation 4.2.23 in which we substituted the solution for r^2

$$\frac{d\theta}{dt} = J_3\xi^{ov} - J_4 J_2^* \xi^{ov} = \xi^{ov}\left(J_3 - J_4 J_2^*\right) \tag{4.2.28}$$

Again we have, according to 4.2.26 a Hopf bifurcation occurring at critical values of Rayleigh number consistent with over-stable convection

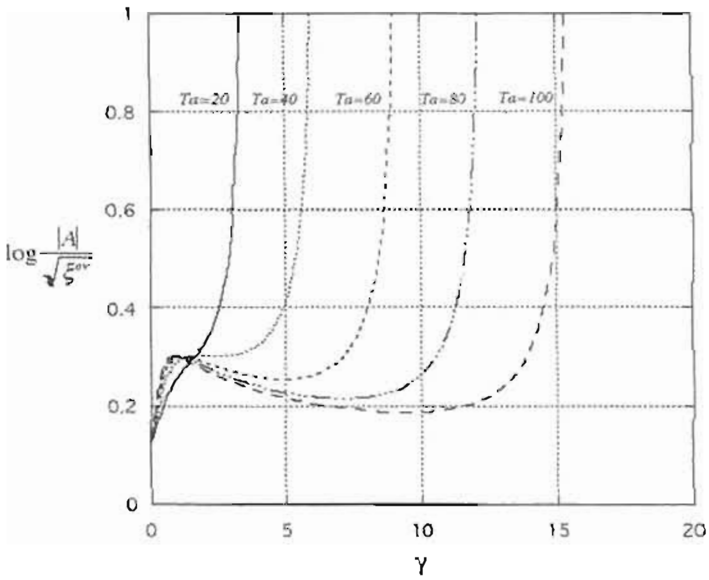

Figure 28: Post-transient amplitude as a function of γ and Ta as a parameter

It can be noted that the solutions diverge as they approach the tri-critical point which are 4.23, 7,14, 10.25, 13.42 and 16.63.

Similarly, the post transient solutions for the non-linear frequency solutions are analysed by ploting them in terms of the $\log\left(\dot{\theta}/\xi^{ov}\right)$ from 4.2.28

$$\log\left(\frac{\dot{\theta}}{\xi^{ov}}\right)=\log\left(J_3-J_4J_2^*\right) \tag{4.2.29}$$

From the Figure 29 we can see that (a) the frequency corrections diverge as they approach the tri-critical value of γ and (b) the frequency correction is inversely proportional with Ta.

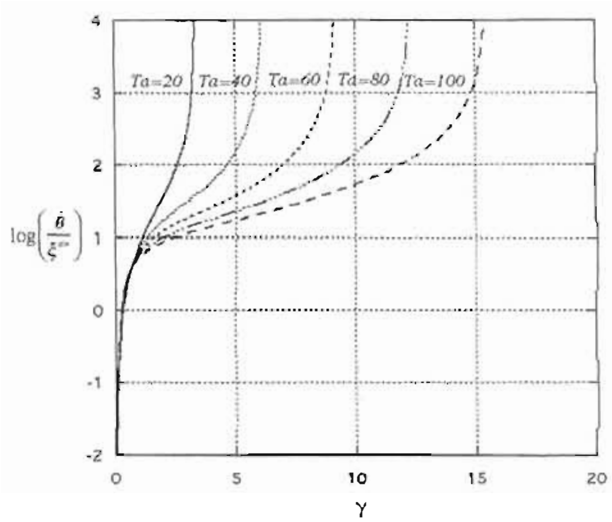

Figure 29: Post transient frequency correction as function of γ and Ta as parameter

As in the linear stability case, by setting $\gamma = 0$, as per 3.2.6 from *Stationary convection; over-stable case*, the corresponding value for α is $\alpha_{cr}^{ov} = 1$. As a result,

$$\lim_{\gamma \to 0} \frac{1}{J_2} = \frac{6\alpha}{4(\alpha+1)} = \frac{3}{4}$$

$$\lim_{\gamma \to 0} \frac{1}{J_1} \sim \frac{1}{\gamma} \to \infty$$

The amplitude and frequency correction

$$|A| \to 2\sqrt{\frac{\xi^{ov}}{3}}$$

$$\dot{\theta} \to 0$$

In conclusion, for small values of γ ($\gamma \cong 0$) the over-stable solution will oscillate with a post transient amplitude of $|A| = 2\varepsilon/\sqrt{3}$ at a frequency of $\sigma_o^{cr} = (Ta/2 - 1)^{1/2}$.

5. Expansion around over-stable solutions. Travelling waves

Equations 3.0.1 and 3.0.2 can be considered for this case, with the same scaling for time and space. The leading order equations are as in the previous section, is

$$\left[\sigma_o \frac{\partial}{\partial \tilde{t}} + 1\right]^2 \nabla^2 \psi_1 + Ta\frac{\partial^2 \psi_1}{\partial z^2} + Ra_{cr}\left[\sigma_o \frac{\partial}{\partial \tilde{t}} + 1\right]\frac{\partial T_1}{\partial x} = 0 \tag{5.1}$$

$$\left[\chi\sigma_o \frac{\partial}{\partial \tilde{t}} - \nabla^2\right]T_1 + \frac{\partial \psi_1}{\partial x} = 0 \tag{5.2}$$

and the general solutions can be expressed as

$$\psi_1 = \left[A_1 e^{i(kx+\tilde{t})} + B_1 e^{i(kx-\tilde{t})} + A_1^* e^{-i(kx+\tilde{t})} + B_1^* e^{-i(kx-\tilde{t})}\right]\sin(\pi z) \tag{5.3}$$

$$T_1 = \left[C_1 e^{i(kx+\tilde{t})} + D_1 e^{i(kx-\tilde{t})} + C_1^* e^{-i(kx+\tilde{t})} + D_1^* e^{-i(kx-\tilde{t})}\right]\sin(\pi z) \tag{5.4}$$

where the amplitudes $A_1 = A_1(\tau_o, \tau, X)$, $A_1^* = A_1^*(\tau_o, \tau, X)$, $B_1 = B_1(\tau_o, \tau, X)$, $B_1^* = B_1^*(\tau_o, \tau, X)$, $C_1 = C_1(\tau_o, \tau, X)$, $C_1^* = C_1^*(\tau_o, \tau, X)$, $D_1 = D_1(\tau_o, \tau, X)$ and $D_1^* = D_1^*(\tau_o, \tau, X)$ describe modulations of the wave on the slow time ($\tau_o = \varepsilon t'$, $\tau = \varepsilon^2 t'$) and space scales ($X = \varepsilon x$) for a Hopf bifurcation. These solutions represent travelling waves. Unlike in the previous case we do not neglect the slow space scale, therefore the resulting equation is expected to contain a diffusion term. As in the previous case we can determine a relationship between coefficients at order one that will prove useful in a later calculation In this case we have four relationships as following

$$C_1 = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}A_1 \tag{5.5}$$

$$C_1^* = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}A_1^* \tag{5.6}$$

$$D_1 = -\frac{\sqrt{\alpha}\left[-\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}B_1 \tag{5.7}$$

$$D_1^* = -\frac{\sqrt{\alpha}\left[-\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}B_1^* \tag{5.8}$$

The de-coupled equation for the stream function, at order $O(\varepsilon^2)$ is expressed by

$$\begin{aligned}
&\left\{\left(\chi\frac{\partial}{\partial t} - \nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right] - Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_2 = \\
&-2\frac{\partial}{\partial\tau_o}\left(\chi\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t}+1\right)\nabla^2\psi_1 - 2\frac{\partial}{\partial x}\frac{\partial}{X}\left(\chi\frac{\partial}{\partial t} - \nabla^2\right)\left(\frac{\partial}{\partial t}+1\right)^2\psi_1 - \\
&\left(\chi\frac{\partial}{\partial\tau_o} - 2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1 - \left(\chi\frac{\partial}{\partial\tau_o} - 2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1 + \\
&Ra_{cr}2\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{X}\psi_1 + Ra_{cr}\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial x^2}\psi_1 - Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]
\end{aligned} \tag{5.9}$$

The above expression can be analyse as two entities, a first one expressing a linear relationship between slow time and space scale which would play a role in establishing the diffusion term and a second one expressing a non-linear dependency of the amplitudes. The solution to the equation above will be a superposition of a homogeneous part consistent with the homogeneous part of the equation and a particular part consistent with the right hand side. Because the homogeneous part of the equation is similar with that at $O(\varepsilon)$, the solution at the current order will be:

$$\psi_2 = \left[A_2 e^{i(kx+\bar{i})} + B_2 e^{i(kx-\bar{i})} + A_2^{*} e^{-i(kx+\bar{i})} + B_2^{*} e^{-i(kx-\bar{i})}\right]\sin(\pi z) + \psi_2^{p} \qquad (5.10)$$

$$T_2 = \left[C_2 e^{i(kx+\bar{i})} + D_2 e^{i(kx-\bar{i})} + C_2^{*} e^{-i(kx+\bar{i})} + D_2^{*} e^{-i(kx-\bar{i})}\right]\sin(\pi z) + T_2^{p} \qquad (5.11)$$

where ψ_2^p and T_2^p are the particular solutions.

The relationships between coefficients will be similar to those at $O(\varepsilon)$. The particular solution for the stream function $\psi_2^p = 0$. The particular solution which is identical to the homogeneous solution must be forced to be zero in order to avoid resonance. That will provide a relationship between the coefficients of the derivatives to the slow scales introduced.

$$\frac{\partial A_1}{\partial X} = f\left(\frac{\partial A_1}{\partial \tau_0}\right) \qquad (5.12)$$

The particular solution of the temperature can be expressed as

$$T_p^{ov,t} = \left[b_2 + a_1 e^{2i\tau} + a_1^* e^{-2i\tau}\right]\sin(2\pi z) \tag{5.12}$$

where

$$a_1^* = -\frac{\alpha}{\pi}\frac{2(\alpha+1) - \gamma^2\sigma_o^2 + i\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)} A_1^* B_1 \tag{5.13}$$

$$a_1 = -\frac{\alpha}{\pi}\frac{2(\alpha+1) - \gamma^2\sigma_o^2 - i\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)} A_1 B_1^* \tag{5.14}$$

$$b_2 = -\frac{1}{2}\frac{\alpha(\alpha+1)}{\pi\left[(1+\alpha)^2 + \gamma^2\sigma_o^2\right]}\left[A_1 A_1^* + B_1 B_1^*\right] \tag{5.15}$$

The complete solution can be written as

$$\psi_2 = \psi_2^p \tag{5.16}$$

$$T_2 = T_2^h + T_2^p \tag{5.17}$$

The de-coupled equation for the stream function at order $O(\varepsilon^3)$ is as follows

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3=$$

$$\Theta(\psi_1,\ T_1)+\Theta(\psi_2,\ T_2)-Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial X}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]J_2-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}J_3 \qquad (5.18)$$

Only the terms containing ψ_1 and T_1 on the right hand side, or combinations of both are relevant for this study. Consequently equation (5.18) can be written as

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3=$$

$$\Theta(\psi_1,\ T_1)-Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial X}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]J_2-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}J_3 \qquad (5.19)$$

In order to avoid resonance we have to set all the coefficients of the forcing terms to zero. The terms that carry convection modes other than the natural modes are not relevant and will not be considered. This yields for the $\exp[i(kx+\tilde{t})]$ and $\exp[i(kx-\tilde{t})]$ parts of the solution a set of two non-linear equations

$$\left[h_{11}\frac{\partial^2 A}{\partial t^2}+h_{12}\frac{\partial^2 A}{\partial x^2}\right]+h_{13}\left\{\frac{\partial A}{\partial t}-h_{14}\left[\xi^{ov}-h_{15}AA^*-h_{16}BB^*\right]A\right\}=0 \qquad (5.20)$$

$$\left[h_{21}\frac{\partial^2 B}{\partial t^2}+h_{22}\frac{\partial^2 B}{\partial x^2}\right]+h_{23}\left\{\frac{\partial B}{\partial t}-h_{24}\left[\xi^{ov}-h_{15}BB^*-h_{26}AA^*\right]B\right\}=0 \qquad (5.21)$$

where the coefficients h_{ij} are determined in the *Appendix 5*.

$$h_{11} = 2\pi^4\gamma(1+\alpha)(1+i\sigma_0) + 2\pi^2 k\left[2\sigma_o(\alpha+\gamma+1) - i\left(\gamma\left(1-\sigma_0^2\right) + 2(\alpha+1) - R_{cr}\right)\right]\times$$

$$\frac{\left\{\left[(1+3\alpha)(2-\gamma\sigma_0) + \gamma(\alpha+Ta+1) - R_{cr}\right] + i2\sigma_o(\alpha+1)(1+\alpha+2\gamma)\right\}}{\left\{\left(4\sigma_0 + 4\sigma_0\alpha + 2\sigma_0\gamma - \sigma_0 R_{cr} - \sigma_0^3\gamma\right) - i\left[2(1+\alpha)\left(1-\sigma_0^2\right) - 2\sigma_0\gamma + Ta - R_{cr}\right]\right\}} \tag{5.22}$$

$$h_{12} = \pi^2\left[(5\alpha+1)\left(\sigma_0^2 - 1\right) - Ta + R_{cr} - i\left(2\sigma_0(5\alpha+1) - \sigma_0 R_{cr}\right)\right] \tag{5.23}$$

$$h_{13} = \pi^4\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} + i2\gamma\sigma_0(\alpha+1)\right] \tag{5.24}$$

$$h_{14} = \frac{\alpha R_{cr}(1+i\sigma_0)\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} - i2\gamma\sigma_0(\alpha+1)\right]}{\left[\left(\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr}\right)^2 + \left(2\gamma\sigma_0(\alpha+1)\right)^2\right]} \tag{5.25}$$

$$h_{15} = \frac{1}{2}\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]} \tag{5.26}$$

$$h_{16} = \frac{\alpha\left[8(\alpha+1) + \sigma_0^2\gamma^2(\alpha-1)\right]}{2\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2 + 4\right)} - i\frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2+4\right)} \tag{5.27}$$

$$h_{21} = 2\pi^4\gamma(1+\alpha)(1-i\sigma_0) - 2\pi^2 k\left[2\sigma_o(\alpha+\gamma+1) + i\left(\gamma\left(1-\sigma_0^2\right) + 2(\alpha+1) - R_{cr}\right)\right]\times$$

$$\frac{\left\{\left[(1+3\alpha)(2-\gamma\sigma_0) + \gamma(\alpha+Ta+1) - R_{cr}\right] - i2\sigma(\alpha+1)(1+\alpha+2\gamma)\right\}}{\left\{\left(4\sigma_0 + 4\sigma_0\alpha + 2\sigma_0\gamma - \sigma_0 R_{cr} - \sigma_0^3\gamma\right) + i\left[2(1+\alpha)\left(1-\sigma_0^2\right) - 2\sigma_0\gamma + Ta - R_{cr}\right]\right\}}$$

(5.28)

$$h_{22} = \pi^2\left[(5\alpha+1)\left(\sigma_0^2 - 1\right) - Ta + R_{cr} + i(2\sigma_0(5\alpha+1) - \sigma_0 R_{cr})\right] \tag{5.29}$$

$$h_{23} = \pi^4\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} - i2\gamma\sigma_0(\alpha+1)\right] \tag{5.30}$$

$$h_{24} = \frac{\alpha R_{cr}(1-i\sigma_0)\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} + i2\gamma\sigma_0(\alpha+1)\right]}{\left[\left(\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr}\right)^2 + (2\gamma\sigma_0(\alpha+1))^2\right]} \tag{5.31}$$

$$h_{25} = \frac{1}{2}\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]} = h_{15} \tag{5.32}$$

$$h_{26} = \frac{\alpha\left[8(\alpha+1) + \sigma_0^2\gamma^2(\alpha-1)\right]}{2\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2 + 4\right)} + i\frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2 + 4\right)} \tag{5.33}$$

In equations 5.20 and 5.21, $\xi^{ov} = \varepsilon^2 = \left(\frac{R}{R_{cr}^{ov}} - 1\right)$ and we used the original time and space scale, $\tau = \varepsilon^2 t$, $\tau_o = \varepsilon\, t$, $X = \varepsilon\, t$, $\bar{t} = \tau_o t'$ and recalling that $A = \varepsilon A_1$, $B = \varepsilon B_1$, $A^* = \varepsilon A_1^*$, $B^* = \varepsilon B_1^*$.

An interesting case is when we set the first brackets of the equations to zero. In this case we have

$$\frac{\partial A}{\partial t} - h_{14}\left[\xi^{ov} - h_{15}AA^* - h_{16}BB^*\right]A = 0 \tag{5.34}$$

$$\frac{\partial B}{\partial t} - h_{24}\left[\xi^{ov} - h_{15}BB^* - h_{26}AA^*\right]B = 0 \qquad (5.35)$$

By presenting the equations 5.22 and 23 in terms of a complex amplitude we obtain a set of four equations, two for the absolute value of amplitude $r_A = |A|$, $r_B = |B|$ and two for the corresponding phases θ_A and θ_B.

$$A = r_A \exp(i\theta_A) \qquad A^* = r_A \exp(-i\theta_A) \qquad B = r_B \exp(i\theta_B) \qquad B^* = r_B \exp(-i\theta_B) \qquad (5.36)$$

The new set of equations are listed below

$$\begin{cases} \dfrac{dr_A}{dt} = \left(h_1^r \xi_{ov} - h_2^r r_A^2 - s_1 r_B^2\right) r_A \\ \\ \dfrac{dr_B}{dt} = \left(h_1^r \xi_{ov} - s_1 r_A^2 - h_2^r r_B^2\right) r_B \end{cases} \qquad (5.37)$$

$$\begin{cases} \dfrac{d\theta_A}{dt} = h_{14}^i \xi_{ov} - h_{15} h_{14}^i r_A^2 - s_2 r_B^2 \\ \\ \dfrac{d\theta_B}{dt} = h_{14}^i \xi_{ov} - s_2 r_A^2 - h_{15} h_{14}^i r_B^2 \end{cases} \qquad (5.38)$$

where

$$h_1^r = h_{14}^r + i h_{14}^i \qquad (5.39)$$

$$h_{14}^r = \frac{\alpha R_{cr}\left[\gamma(\alpha+1)(1+\sigma_o^2) + \gamma Ta - R_{cr}\right]}{\left[\gamma(\alpha+1)(1-\sigma_o^2) + \gamma Ta - R_{cr}\right]^2 + 4\gamma\sigma_o^2(1+\alpha)^2} \qquad (5.40)$$

$$h_{14}^i = -\frac{\alpha\sigma_o R_{cr}\left[\gamma(\alpha+1)(1+\sigma_o^2) - \gamma Ta + R_{cr}\right]}{\left[\gamma(\alpha+1)(1-\sigma_o^2) + \gamma Ta - R_{cr}\right]^2 + 4\gamma\sigma_o^2(1+\alpha)^2} \qquad (5.41)$$

$$h_2^r = h_{15} h_{14}^r \tag{5.42}$$

$$h_{15} = \frac{1}{2} \frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \sigma_0^2 \gamma^2\right]} \tag{5.43}$$

The expressions for $h_{i,j}^m$ can be found in the *Appendix 5*. At steady state we derive the following set of equations, able to express a relationship between amplitudes r_A and r_B.

$$r_A^2 = r_B^2 \tag{5.44}$$

In this case the Reduced amplitude equation can be written as

$$\frac{dr_A}{dt} = \left[h_1^r \cdot \xi_{ov} + h_3 \cdot r_A^2\right] \cdot r_A \tag{5.45}$$

where

$$h_3 = -\frac{1}{2} \frac{\alpha R_{cr}\left[\gamma(\alpha+1)\left(1+\sigma_o^2\right) + \gamma Ta - R_{cr}\right]}{\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr}\right]^2 + \left[2\gamma\sigma_o(\alpha+1)\right]^2} \times \left[\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]} + \frac{\alpha\left[8(\alpha+1) + \sigma_o^2\gamma^2(\alpha-1)\right]}{\left[(\alpha+1)^2 + \gamma^2\sigma^2\right]\left(4 + \sigma_o^2\gamma^2\right)}\right] +$$

$$+ \frac{\alpha R_{cr}\left[\gamma(\alpha+1)\left(1+\sigma_o^2\right) - \gamma Ta + R_{cr}\right]}{\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr}\right]^2 + \left[2\gamma\sigma_o(\alpha+1)\right]^2} \times \frac{\alpha\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]\left(4 + \sigma_o^2\gamma^2\right)} \tag{5.46}$$

The *Reduced Phase Equation* in the case when $r_A^2 = r_B^2$ can be written as

$$\frac{d\theta}{dt} = h_1^i \cdot \xi_{ov} + \tilde{h}_3 \cdot r_A^2 \tag{5.47}$$

where

$$\tilde{h}_3 = \frac{1}{2}\frac{\alpha R_{cr}\left[\gamma(\alpha+1)(1+\sigma_o^2)-\gamma Ta+R_{cr}\right]}{\left[\gamma(\alpha+1)(1-\sigma_o^2)+\gamma Ta-R_{cr}\right]^2+\left[2\gamma\sigma_o(\alpha+1)\right]^2}\times\left[\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}+\frac{\alpha\left[8(\alpha+1)+\sigma_o^2\gamma^2(\alpha-1)\right]}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right](4+\sigma_o^2\gamma^2)}\right]+$$

$$-\frac{\alpha R_{cr}\left[\gamma(\alpha+1)(1+\sigma_o^2)-\gamma Ta+R_{cr}\gamma\right]}{\left[\gamma(\alpha+1)(1-\sigma_o^2)+\gamma Ta-R_{cr}\right]^2+\left[2\gamma\sigma_o(\alpha+1)\right]^2}\times\frac{\alpha\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right](4+\sigma_o^2\gamma^2)}$$

(5.48)

Returning to the full set of equations, we can write them as

$$\left[h_{11}^r\frac{\partial^2 r_A}{\partial t^2}+h_{13}^r\frac{\partial r_A}{\partial t}+h_{12}^r\frac{\partial^2 r_A}{\partial x^2}\right]-\left[h_{11}^i\frac{\partial^2\theta_A}{\partial t^2}-h_{13}^i\frac{\partial\theta_A}{\partial t}+h_{12}^r\frac{\partial^2\theta_A}{\partial x^2}\right]r_A-\left[N_1\xi_{ov}+N_2r_A^2+N_3r_B^2\right]r_A=0$$

(5.49)

$$\left[h_{11}^r\frac{\partial^2 r_B}{\partial t^2}+h_{13}^r\frac{\partial r_B}{\partial t}+h_{12}^r\frac{\partial^2 r_B}{\partial x^2}\right]-\left[h_{11}^r\frac{\partial^2\theta_B}{\partial t^2}-h_{13}^i\frac{\partial\theta_B}{\partial t}+h_{12}^r\frac{\partial^2\theta_B}{\partial x^2}\right]r_A-\left[N_1\xi_{ov}+N_2r_A^2+N_3r_B^2\right]r_B=0$$

(5.50)

$$\left[h_{11}^i\frac{\partial^2 r_A}{\partial t^2}+h_{13}^i\frac{\partial r_A}{\partial t}+h_{12}^i\frac{\partial^2 r_A}{\partial x^2}\right]-\left[h_{11}^i\frac{\partial^2\theta_A}{\partial t^2}-h_{13}^r\frac{\partial\theta_A}{\partial t}+h_{12}^i\frac{\partial^2\theta_A}{\partial x^2}\right]r_A-\left[\tilde{N}_1\xi_{ov}+\tilde{N}_2r_A^2+\tilde{N}_3r_B^2\right]r_A=0$$

(5.51)

$$\left[h_{11}^i\frac{\partial^2 r_B}{\partial t^2}+h_{13}^i\frac{\partial r_B}{\partial t}+h_{12}^i\frac{\partial^2 r_B}{\partial x^2}\right]-\left[h_{11}^i\frac{\partial^2\theta_B}{\partial t^2}-h_{13}^r\frac{\partial\theta_B}{\partial t}+h_{12}^i\frac{\partial^2\theta_B}{\partial x^2}\right]r_A-\left[\tilde{N}_1\xi_{ov}+\tilde{N}_2r_A^2+\tilde{N}_3r_B^2\right]r_B=0$$

(5.52)

These are the final set of equations and it can be noted that there is a strong coupling between them.

6. Conclusions

The present study can be divided into two main streams. In one stream we have recovered and expanded the analytical work of Vadasz (1998) for weak non-linear analysis for a porous media layer subject to Coriolis forces. Our results proved to be in concordance with the previous outcomes. Further more we have enhanced the previous model by considering a long space scale $X = \varepsilon x$ and an additional slow time scale $\tau_o = \varepsilon t'$ in order to obtain a complete equation for the amplitude and frequency of oscillation for the convection in porous media.

The tri-critical value of gamma for various values of Taylor number appears to follow a straight-line.

The complete equation consists of two distinct terms grouped as a non-linear part $A - Z_1\left(\xi^{ov} - Z_2 AA^*\right)A$ that we are familiar with, and a second part where we introduced the above mentioned scales $\partial^2 A/\partial t^2 - Z_3 \partial^2 A/\partial x^2$. Z_1, Z_2 and Z_3 are constants to be found in the text [p.52-53]. Interestingly, the way the scales came out into the final equation indicates the wave characteristics of the convection. It also indicates the existence of a group velocity term associated to the normal modes of oscillation. The finite amplitude results indicate that a pitchfork bifurcation occurs for the stationary case and a Hopf bifurcation for over-stable convection at critical values of Rayleigh numbers.

The results also suggest the possibility of controlling a more general case the codimensional-2 point. However, this analysis requires further work.

We further analysed the case where travelling waves are considered and determined a set of eight coupled equations for amplitude and phase. If we impose the conditions for the stream function that $A_1^* = \pm B_1$ and $A_1 = \pm B_1^*$, and for temperature $C_1^* = \pm D_1$ and $C_1 = \pm D_1^*$, we obtain the equations for the standing waves case.

By neglecting the time and space slow scales, for a steady state case we obtained the following relationship for the amplitudes $r_A^2 = r_B^2$, reducing the equation to a set of equations for amplitude and phase that can be de-coupled.

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Table of Contents (Appendices)

Appendix 1: The non-dimensional analysis of the governing equations corresponding to flow and heat transfer in rotating porous media

The quantities marked $(.)_*$ represent dimensional quantities; items marked $(.)_C$ represent the characteristic values: items marked with no subscript index represent a-dimensional quantities and those marked $(.)_0$ represent their reference values.

The continuity, Darcy and energy equations are

$$\nabla_* \hat{\mathbf{q}}_* = 0 \tag{A1.1}$$

$$\hat{\mathbf{q}}_* = -\frac{k_*}{\mu_*}\left[\nabla_* p_* + \rho_* g_* \hat{\mathbf{e}}_z\right] \tag{A1.2}$$

$$\alpha_* \nabla_*^2 T_* = \mathbf{q}_* \nabla_* T_* \tag{A1.3}$$

Additional expressions for density, length, gradient and temperature are

$$\rho_* = \rho_0\left[1 - \beta_*(T_* - T_0)\right] = \rho_0 \rho \tag{A1.4}$$

$$l_* = l_c l \tag{A1.5}$$

$$\nabla_* = \frac{1}{l_c}\frac{\partial}{\partial l_{i,*}} = \frac{1}{l_c}\nabla \tag{A1.6}$$

$$T = \frac{T_* - T_0}{\Delta T_C} \tag{A1.7}$$

By conveniently choosing all the quantities we obtain for continuity equation

$$\nabla.\mathbf{q}_* = \frac{q_c}{l_c}\nabla\mathbf{q} = 0 \Rightarrow \nabla\mathbf{q} = 0 \tag{A.8}$$

It is to be noted that the continuity equation is considered as time independent.

For the equation (A1.2) we consider taking

$$\rho = 1 - \beta.\Delta T_c T = 1 - CT \tag{A1.9}$$

where C stands for a constant.

$$\mathbf{q}_c\mathbf{q} = -\frac{k_*}{\mu_*}\left[\frac{1}{l_c}\nabla p_* + \rho_0\rho g.\hat{\mathbf{e}}_z\right] = -\frac{k_*}{\mu_*}\left[\frac{1}{l_c}\nabla p_* + \rho g.\hat{\mathbf{e}}_z - \rho CTg.\hat{\mathbf{e}}_z\right] =$$

$$-\frac{k_*}{\mu_*}\left[\frac{1}{l_c}\nabla(p_* + \rho_0\rho g.zl_c) - \rho CTg.\hat{\mathbf{e}}_z\right] = -\frac{k_*}{\mu_*}\frac{\Delta p_c}{l_c}\nabla p + \frac{k_*}{\mu_*}\frac{g.\rho C}{l_c}T\hat{\mathbf{e}}_z \tag{A1.10}$$

Therefore

$$\mathbf{q} = -\frac{k_*\Delta p_c}{q_c\mu_* l_c}\nabla p + \frac{k_* g.\rho C}{\mu_* q_c l_c}T\hat{\mathbf{e}}_z \tag{A1.11}$$

From the equation (A1.3) we have

$$\frac{\alpha_*}{l_c^2}\Delta T_c^2\nabla^2 T = q_c\mathbf{q}\frac{1}{l_c}\Delta T_c\nabla T \tag{A1.12}$$

Simplifying, we obtain

$$\frac{\alpha_*}{l_c q_c}\Delta T_c\nabla^2 T = \mathbf{q}\nabla T \tag{A1.13}$$

By imposing $\alpha_*\Delta T_c / q_c l_c = 1$ we obtain the energy equation in the form

$$\mathbf{q}\nabla T = \nabla^2 T \tag{A1.14}$$

From $\alpha.\Delta T_c / q_c l_c = 1$ we can further express $q_c = \alpha.\Delta T_c / l_c$ and also, by imposing $k.\Delta p_c / \mu.\alpha. = 1$ we can express $\Delta p_c = \mu.\alpha./k.$ which introduced into A1.11 we obtain the Darcy equation

$$\mathbf{q} = -\nabla p_o + R_c T\mathbf{e}_z \tag{A1.15}$$

where

$$Ra = \frac{\beta.g.k.\Delta T_c \rho_c}{l_f q_c \mu.} = \frac{\beta.g.k.\Delta T_c H.}{\alpha.\mu.} \tag{A1.16}$$

Appendix 2: Derivation of the equation 2.6

Let the equations below be the continuity, Darcy and energy equations and $\omega = \nabla \times \mathbf{q}$ the vorticity.

$$\nabla \cdot \mathbf{q} = 0 \tag{A2.1}$$

$$\frac{\partial \mathbf{q}}{\partial t} + Ta^{1/2} \cdot \hat{\mathbf{e}}_z \times \mathbf{q} + \mathbf{q} = -\nabla p + Ra \cdot T \cdot \hat{\mathbf{e}}_z \tag{A2.2}$$

$$\chi \frac{\partial T}{\partial t'} + \mathbf{q} \cdot \nabla T = \nabla^2 T \tag{A2.3}$$

Where:

$$\mathbf{q} = u\hat{\mathbf{e}}_x + v\hat{\mathbf{e}}_y + w\hat{\mathbf{e}}_z$$

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{e}}_x + \frac{\partial}{\partial y}\hat{\mathbf{e}}_y + \frac{\partial}{\partial z}\hat{\mathbf{e}}_z$$

Hence

$$\nabla \times \left[\frac{\partial \mathbf{q}}{\partial t} + Ta^{1/2}\hat{\mathbf{e}}_z \times \mathbf{q} + \mathbf{q}\right] = \nabla \times \left[-\nabla p + Ra \cdot T \cdot \hat{\mathbf{e}}_z\right] \tag{A2.4}$$

The operator ∇ will transform each term in the LHS of equation (4)

$$\nabla \times \frac{\partial \mathbf{q}}{\partial t} = \frac{\partial \omega}{\partial t} \tag{A2.5}$$

$$\nabla \times \left[Ta^{1/2} \hat{\mathbf{e}}_z \times \mathbf{q} \right] = Ta^{1/2} \left\{ \nabla \times \left[\hat{\mathbf{e}}_z \times \mathbf{q} \right] \right\}$$

The triple product $\nabla \times \left[\hat{\mathbf{e}}_z \times \mathbf{q} \right]$, which is known as a "triple vectorial product", will yield

$$\nabla \times \left[\hat{\mathbf{e}}_z \times \mathbf{q} \right] = \left[\frac{\partial}{\partial x} \hat{\mathbf{e}}_x + \frac{\partial}{\partial y} \hat{\mathbf{e}}_y + \frac{\partial}{\partial z} \hat{\mathbf{e}}_z \right] \times \left[-v \hat{\mathbf{e}}_x + u \hat{\mathbf{e}}_y \right] = \begin{bmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \partial_x & \partial_y & \partial_z \\ -v & u & 0 \end{bmatrix} \tag{A2.6}$$

It its to be noted that the vertical component of the flow is assumed zero since we have no flow on that direction.

Expanding the determinant from equation (6) we obtain

$$\nabla \times \left[\hat{\mathbf{e}}_z + \mathbf{q} \right] = -\hat{\mathbf{e}}_x \frac{\partial u}{\partial z} - \hat{\mathbf{e}}_y \frac{\partial v}{\partial z} + \hat{\mathbf{e}}_z \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right] = -\hat{\mathbf{e}}_x \frac{\partial u}{\partial z} - \hat{\mathbf{e}}_y \frac{\partial v}{\partial z} + \hat{\mathbf{e}}_z \left[\nabla \mathbf{q} - \frac{\partial w}{\partial z} \right]$$

As $\nabla \mathbf{q} = 0$ it results that

$$\nabla \times Ta^{1/2} \hat{\mathbf{e}}_z \times \mathbf{q} = -Ta^{1/2} \frac{\partial q}{\partial z} \tag{A2.7}$$

On the right hand side of the equation (4) we have

$$\nabla \times (\nabla \cdot p) = 0$$

as a rotor operator applied to a divergence, and

$$\nabla \times Ra \cdot T\hat{\mathbf{e}}_z = Ra \nabla \times T\hat{\mathbf{e}}_z = Ra\left[(\nabla T) \times \hat{\mathbf{e}}_z + T(\nabla \times \hat{\mathbf{e}}_z)\right] =$$

$$= Ra\left[\frac{\partial T}{\partial y}\hat{\mathbf{e}}_x - \frac{\partial T}{\partial x}\hat{\mathbf{e}}_y + T\frac{\partial \hat{e}_z}{\partial z} \cdot \hat{\mathbf{e}}_z\right] = Ra\left(\frac{\partial T}{\partial y}\hat{\mathbf{e}}_x - \frac{\partial T}{\partial x}\hat{\mathbf{e}}_y\right) \tag{A2.8}$$

The resulting equation is

$$\frac{\partial \omega}{\partial t} + \omega - Ta^{1/2}\frac{\partial q}{\partial z} = Ra\left(\frac{\partial T}{\partial y}\hat{e}_x - \frac{\partial T}{\partial x}\hat{e}_y\right) \tag{A2.9}$$

To introduce the vorticity we have to apply the operator ∇ once again to equation (A1.9)

$$\nabla \times \left[\frac{\partial \omega}{\partial t} + \omega - Ta^{1/2}\frac{\partial q}{\partial z}\right] = \nabla \times \left[Ra\left(\frac{\partial T}{\partial y}\hat{e}_x - \frac{\partial T}{\partial x}\hat{e}_y\right)\right]$$

On the LHS the operator applied to each term will give

$$\nabla \times \frac{\partial \omega}{\partial t} = \frac{\partial}{\partial t}[\nabla \times \omega] = \frac{\partial}{\partial t}[\nabla \times \nabla \times \mathbf{q}] = \frac{\partial}{\partial t}\left[\nabla(\nabla \mathbf{q}) - \nabla^2 \mathbf{q}\right] = -\frac{\partial}{\partial t}\left[\nabla^2 \mathbf{q}\right] \tag{A2.10}$$

$$\nabla \times \omega = -\nabla^2 \mathbf{q} \tag{A2.11}$$

$$\nabla \times Ta^{1/2}\frac{\partial \mathbf{q}}{\partial z} = Ta^{1/2}\nabla \times \frac{\partial \mathbf{q}}{\partial z} = Ta^{1/2}\frac{\partial}{\partial z}\nabla \times \mathbf{q} = Ta^{1/2}\frac{\partial}{\partial z}\omega \tag{A2.12}$$

On the RHS, the operator applied to each term will give

$$\nabla \times Ra\left[\frac{\partial T}{\partial y}\hat{\mathbf{e}}_x - \frac{\partial T}{\partial x}\hat{\mathbf{e}}_y\right] = Ra\begin{bmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial T}{\partial y} & \dfrac{\partial T}{\partial x} & 0 \end{bmatrix} = Ra\left[\frac{\partial^2 T}{\partial x \partial z}\hat{\mathbf{e}}_x + \frac{\partial^2 T}{\partial y \partial z}\hat{\mathbf{e}}_y - \nabla_H^2 T\hat{\mathbf{e}}_z\right] \quad (A2.13)$$

Adding the resulting terms from LHS and RHS we obtain

$$\left[\frac{\partial}{\partial t'} + 1\right]\nabla^2 \mathbf{q} + Ta^{1/2}\frac{\partial \mathbf{q}}{\partial z} + Ra\left[\frac{\partial^2 T}{\partial x \partial z}\hat{\mathbf{e}}_x + \frac{\partial^2 T}{\partial y \partial z}\mathbf{e}_y - \nabla_H^2 T\hat{\mathbf{e}}_z\right] = 0 \quad (A2.14)$$

Appendix 3

Section 1: Basic flow solutions

The basic solution in a two dimensional problem assumes no flow, the temperature a function of z only and the pressure not a function of x and y. As a result, the equations (2.1), (2.2) and (2.3) will be written as follows

$$\nabla_* \cdot \hat{\mathbf{q}}_* = 0 \tag{A3.0.1}$$

$$0 = -\frac{k_*}{\mu_*}\left[\nabla_* p_* + \rho . g_* \hat{\mathbf{e}}_z\right] \tag{A3.0.2}$$

$$\nabla^2 T = 0 \tag{A3.0.3}$$

Equation (A3.2) will yield

$$\frac{\partial p}{\partial x}\hat{e}_x + \frac{\partial p}{\partial y}\hat{e}_y + \frac{\partial p}{\partial z}\hat{e}_z = Ra \cdot T\hat{e}_z \tag{A3.0.4}$$

as pressure is not a function of x or y and T is a function of z only, it results

$$\frac{\partial p}{\partial z} = Ra \cdot T \tag{A3.0.5}$$

$$\frac{\partial^2 T}{\partial z^2} = 0 \tag{A3.0.6}$$

From (A3.5) results that $T = Az + B$, where A and B are two constants to be determined according to the boundary conditions

$$\begin{matrix} T = 1 & z = 0 & & A = -1 \\ & & \Rightarrow & \\ T = 0 & z = 1 & & B = \ \ 1 \end{matrix} \tag{A3.0.7}$$

The pressure can be obtained from (A3.4) yielding $p_b = Ra\int(1-z)dz + C$

Section 2: Small perturbations around stationary solutions

Let $q = q_b + \varepsilon q'$ $\quad T = T_b + \varepsilon T'$ $\quad \omega = \omega_b + \varepsilon\omega'$ the small perturbations around basic solutions. We know that $\hat{q}_b = \omega_b = 0$ and $T_b = 1 - z$; therefore equation (A1.7) can be written

$$\frac{\partial(\omega_b + \omega')}{\partial t'} + (\omega_b + \omega') - Ta^{1/2}\frac{\partial(\mathbf{q}_b + \mathbf{q}')}{\partial z} = Ra\left[\frac{\partial(T_b + T')}{\partial y}\hat{\mathbf{e}}_x - \frac{\partial(T_b + T')}{\partial x}\hat{\mathbf{e}}_y\right] \tag{A3.0.8}$$

Opening the brackets and separating the terms in basic on the left-hand side and small perturbations on the right hand side, we obtain

$$\frac{\partial \omega_b}{\partial t'} + \omega_b - Ta^{1/2}\frac{\partial \mathbf{q}_b}{\partial z} - Ra\left[\frac{\partial T_b}{\partial y}\hat{\mathbf{e}}_x - \frac{\partial T_b}{\partial x}\hat{\mathbf{e}}_y\right] = 0 = \left[\frac{\partial}{\partial t} + 1\right]\omega' - Ta^{1/2}\frac{\partial w}{\partial z} \qquad \text{(A3.0.9)}$$

I considered the value of $\mathbf{q}'$ along z-axis as w and the products $\frac{\partial T'}{\partial x_i}\hat{\mathbf{e}}_j \equiv 0$. Therefore

$$\left[\frac{\partial}{\partial t} + 1\right]\omega' = Ta^{1/2}\frac{\partial w}{\partial z} \qquad \text{(A3.0.10)}$$

In the energy equation we insert the small perturbation solution

$$\chi\frac{\partial(T_b + T')}{\partial t'} + (\mathbf{q}_b + \mathbf{q}')\nabla(T_b + T') = \nabla^2(T_b + T') \qquad \text{(A3.0.11)}$$

Developing the terms and separating the equation in two parts we obtain

$$\chi\frac{\partial}{\partial t'}T_b + \mathbf{q}_b\nabla T_b - \nabla^2 T_b = -\chi\frac{\partial}{\partial t}T' - \mathbf{q}_b\nabla T' - \mathbf{q}'\nabla T_b - \mathbf{q}'\nabla T' + \nabla^2 T' \qquad \text{(A3.0.12)}$$

The LHS of the equation is zero since it represents the energy equation in its basic form; also $\mathbf{q}_b\nabla T' = \mathbf{q}'\nabla T' = 0$, because $\mathbf{q}_b$ is zero (basic no-flow situation) and the product between two perturbations will yield a second order perturbation that can be neglected. As $T_b = 1 - z$ and $\mathbf{q}'$ along z-axis is w' the final form of the energy equation is

$$\left(\chi\frac{\partial}{\partial t'} - \nabla^2\right)T' - w' = 0 \qquad \text{(A3.0.13)}$$

From equation (A3.8) we can express w' as

$$w' = \left(\chi \frac{\partial}{\partial t'} - \nabla^2\right) T' \tag{A3.0.14}$$

and from equation (A3.6), by multiplying to the left with the operator $\left[\frac{\partial}{\partial t'} + 1\right]^{-1}$ we get

$$\omega_z' = \left[\frac{\partial}{\partial t'} + 1\right]^{-1} Ta^{1/2} \frac{\partial \hat{\mathbf{q}}'}{\partial z} \tag{A3.0.15}$$

introduced together with (A3.9) in (1.8) yields

$$\left[\frac{\partial}{\partial t'} + 1\right]^2 \nabla^2 \hat{\mathbf{q}}' + Ta \frac{\partial^2 \hat{\mathbf{q}}'}{\partial z^2} + \left[\frac{\partial}{\partial t'} + 1\right] Ra \left[\frac{\partial^2}{\partial x \partial z} \hat{\mathbf{e}}_x + \frac{\partial^2}{\partial y \partial z} \hat{\mathbf{e}}_y - \nabla_H^2 \hat{\mathbf{e}}_z\right] T' = 0 \tag{A3.0.16}$$

Taking $\frac{\partial^2 T'}{\partial x \partial z} \hat{\mathbf{e}}_x = \frac{\partial^2 T'}{\partial y \partial z} \hat{\mathbf{e}}_y \equiv 0$. Finally, the temperature perturbation equation is

$$\left\{\left[\frac{\partial}{\partial t'} + 1\right]^2 \left[\chi \frac{\partial}{\partial t'} - \nabla^2\right] \nabla^2 + Ta \left[\chi \frac{\partial}{\partial t'} - \nabla^2\right] - Ra \left[\frac{\partial}{\partial t'} + 1\right] \nabla_H^2\right\} T' = 0 \tag{A3.0.17}$$

Section 3: Amplitude differential equation of T'

The assumed solution (2.11) can be written as $T' = \theta(z) \cdot F(x,y,t)$, which introduced into equation (2.10), opening the brackets and taking $k_x^2 + k_y^2 = k$ yields an algebraic equation

$$\chi\sigma^3 k^2 + \chi\sigma^3 D^2 + 2\chi\sigma^2 k^2 + 2\chi\sigma^2 D^2 + \chi\sigma k^2 + \chi\sigma D^2 - \sigma^2\left(k^4 - 2k^2 D^2 + D^4\right) -$$
$$-2\sigma\left(k^4 - 2k^2 D^2 + D^4\right) - \left(k^4 - 2k^2 D^2 + D^4\right) + Ta\left(\chi\sigma D^2 + D^2 k^2 - D^4\right) - Ra(\sigma + 1)k^2 = 0$$

(A3.0.18)

Simplifying

$$\left\{(\alpha + 1)^2\left[D_z^2 - k^2 - \chi\sigma\right]\left(D_z^2 - k^2\right) + Ta\left[D_z^2 - k^2 - \chi\sigma\right]D_z^2 - Ra(\sigma + 1)k^2\right\}\theta = 0$$

(A3.0.19)

For $\theta = b_n \sin(n\pi z)$ as a solution for (2.12) the equation above will yield

$$R = \frac{\left[1 + \alpha + \gamma\sigma\right] \cdot \left[(\sigma + 1)^2(\alpha + 1) + Ta\right]}{\alpha(\sigma + 1)} \qquad \text{(A3.0.20)}$$

Appendix 3.1: ***Determination of*** $\mathbf{w}'$, ω', $\mathbf{u}'$ ***and*** $\mathbf{v}'$

By substituting $T' = B\cos(kx)\sin(\pi z)$ in equations (2.8), (2.9) and (2.7) along the respective directions we obtain

$$\left[\chi\frac{\partial}{\partial t'} - \nabla^2\right]B\cos(kx)\sin(\pi z) - w' = 0 \qquad \text{(A3.1.1)}$$

There is no time dependency in the expression above and the value of w' is

$$w' = -\nabla^2 T' = -\nabla^2 B\cos(kx)\sin(\pi z) = \left(k^2 + \pi^2\right)B\cos(kx)\sin(\pi z) \qquad \text{(A3.1.2)}$$

$$w' = \left(k^2 + \pi^2\right)T' \qquad \text{(A3.1.4)}$$

For the vertical component of vorticity,

$$\left[\frac{\partial}{\partial t'} + 1\right]\omega'_z = Ta^{1/2}\frac{\partial w'}{\partial z} \qquad \text{(A3.1.5)}$$

$$\omega'_z = Ta^{1/2}\frac{\partial}{\partial z}w' = Ta^{1/2}\frac{\partial}{\partial z}\left(k^2 + \pi^2\right)B\cos(kx)\sin(\pi z) \qquad \text{(A3.1.6)}$$

$$\omega'_z = \pi\left(k^2 + \pi^2\right)Ta^{1/2}B\cos(kx)\cos(\pi z) \qquad \text{(A3.1.7)}$$

To determine u' and v' we consider equation 3.7 taken by components, knowing that

$$\text{(i)}\quad \nabla^2\hat{\mathbf{q}}' = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\cdot\left[u'\hat{\mathbf{e}}_x + v'\hat{\mathbf{e}}'_y + w'\hat{\mathbf{e}}'_z\right] = \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial y^2} + \frac{\partial^2 w'}{\partial z^2} \qquad \text{(A3.1.9)}$$

$$\text{(ii)}\quad \frac{\partial}{\partial z}\omega' = \frac{\partial}{\partial z}\left[\nabla\times\hat{\mathbf{q}}'\right] \qquad \text{(A3.1.10)}$$

(iii)

$$\frac{\partial}{\partial z}\omega' = \frac{\partial}{\partial z}\left[\hat{\mathbf{e}}_x\left(\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}\right) + \hat{\mathbf{e}}_y\left(\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right) + \hat{\mathbf{e}}_z\left(\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right)\right] = \qquad \text{(A3.1.11)}$$

$$= -\frac{\partial^2 v'}{\partial z^2}\hat{\mathbf{e}}_z + \left(\frac{\partial^2 u'}{\partial z^2} - \frac{\partial^2 w'}{\partial x \partial z}\right)\hat{\mathbf{e}}_y + \frac{\partial^2 v}{\partial x \partial z}\hat{\mathbf{e}}_z$$

(iv) $\frac{\partial(\cdot)}{\partial y} \equiv 0$ and (v) $\frac{\partial(\cdot)}{\partial t} \equiv 0$

we can write the following equations

$$\nabla^2 u' - Ta^{1/2}\frac{\partial^2 v'}{\partial z^2} = -Ra\frac{\partial^2 T'}{\partial x \partial z} \qquad \text{(A3.1.12)}$$

$$\nabla^2 v' + Ta^{1/2}\frac{\partial^2 u'}{\partial z^2} = Ta^{1/2}\frac{\partial^2 w'}{\partial x \partial z} \qquad \text{(A3.1.13)}$$

Since w' and T' have the same representation in sine and cosine, we can write

$$u' = C\cos(kx)\sin(\pi z) \qquad\qquad v' = D\cos(kx)\sin(\pi z)$$

$$\begin{cases} \dfrac{\partial^2 T'}{\partial x \partial z} = -\pi k B\sin(kx)\cos(\pi z) \\[2ex] \dfrac{\partial^2 w'}{\partial x \partial z} = -\pi k\left(k^2 + \pi^2\right)B\sin(kx)\cos(\pi z) \end{cases} \qquad \text{(A3.1.14)}$$

$$\begin{cases} \nabla^2 u' = -\left(k^2 + \pi^2\right)u' \\[2ex] \nabla^2 v' = -\left(k^2 + \pi^2\right)v' \end{cases} \qquad \text{(A3.1.15)}$$

The de-coupling of equations A2.2.12 and A2.2.13

$$\begin{bmatrix} -\left(k^2+\pi^2\right) & Ta^{1/2}\pi^2 \\ -Ta^{1/2}\pi^2 & -\left(k^2+\pi^2\right) \end{bmatrix} \cdot \begin{pmatrix} C \\ D \end{pmatrix} = \begin{bmatrix} Ra\pi k B \\ -Ta^{1/2}\pi k\left(k^2+\pi^2\right)B \end{bmatrix} \tag{A3.1.16}$$

$$\Delta = \left(k^2+\pi^2\right)^2 + Ta\pi^2$$

$$\Delta_C = \begin{vmatrix} Ra\pi k B & Ta^{1/2}\pi^2 \\ -Ta^{1/2}\pi k B & -\left(k^2+\pi^2\right) \end{vmatrix} = k\pi B\left(k^2+\pi^2\right)\left(\pi^2 Ta - Ra\right) \tag{A3.1.17}$$

$$\Delta_D = \begin{vmatrix} -\left(k^2+\pi^2\right) & \pi k Ra B \\ -Ta^{1/2}\pi^2 & -Ta^{1/2}k\pi\left(k^2+\pi^2\right)B \end{vmatrix} = Ta^{1/2}\pi k B\left(\pi^2 Ra + \left(k^2+\pi^2\right)\right) \tag{A3.1.18}$$

$$C = \frac{k\pi\left(k^2+\pi^2\right)\left(Ta\pi^2 - Ra\right)}{\left(k^2+\pi^2\right) + Ta\pi^4} B \tag{A.3.1.19}$$

$$D = \frac{k\pi Ta^{1/2}\left(k^2+\pi^2\right)\left[\left(k^2+\pi^2\right)^2 + Ra\pi^2\right]}{\left(k^2+\pi^2\right) + Ta\pi^4} B \tag{A3.1.20}$$

We remind here that the critical value of Rayleigh Ra_c at $\sigma = 0$ is

$$Ra_c = \frac{\left(k^2+\pi^2\right)^2}{k^2} + Ta\frac{\pi^2\left(k^2+\pi^2\right)}{k^2}$$, and therefore

$$\left[Ta\pi^2 - Ra_c\right] = -\frac{\left[Ta\pi^4 + \left(k^2+\pi^2\right)^2\right]}{k^2} \tag{A3.1.21}$$

Results the value for C

$$C = -\frac{\pi\left(k^2+\pi^2\right)}{k} B \tag{A3.1.22}$$

For D, a similar calculation will reveal

$$D = \frac{\pi Ta^{1/2}\left(k^2 + \pi^2\right)}{k} B \qquad \text{(A3.1.23)}$$

Finally, we can write the expressions for

$$u' = -\frac{\pi\left(k^2 + \pi^2\right)}{k} B\sin(kx)\cos(\pi z) \qquad \text{(A3.1.24)}$$

$$v' = \frac{\pi Ta^{1/2}\left(k^2 + \pi^2\right)}{k} B\sin(kx)\cos(\pi z) \qquad \text{(A3.1.25)}$$

Appendix 3.2: Determination of $R_c^{(ov)}$ and σ_i^2 [§1]

The equation 2.12 is listed below

$$\left\{(\alpha+1)^2\left[D_z^2-k^2-\chi\sigma\right]\left(D_z^2-k^2\right)+Ta\left[D_z^2-k^2-\chi\sigma\right]D_z^2-Ra(\sigma+1)k^2\right\}\theta=0 \qquad \text{(A.3.2.1)}$$

By substituting $\sigma=i\sigma_i$ in the solution $T'=\theta(z)\cdot e^{i(k_x x+k_y y)+\sigma t'}$ and knowing that $D_z^2T'=-\pi^2T'$ and $D_z^4T'=\pi^4T'$

$$(i\sigma_i+1)^2\left[D_z^4-2D_z^2k^2+k^4-D_z^2\chi i\sigma_i+\chi i\sigma_i k^2\right]+Ta\left[D_z^4-D_z^2k^2-\chi D_z^2 i\sigma_i\right]-Ra(i\sigma_i+1)k^2=0$$

(A.3.2.2)

Recalling that $\alpha=\dfrac{k^2}{\pi^2}$, $R=\dfrac{Ra}{\pi^2}$ and $\gamma=\dfrac{\chi}{\pi^2}$ we can write the expanded equation

$$-\pi^2\sigma_i^2-2\pi^2\sigma_i^2\alpha-\pi^2\sigma_i^2\alpha^2-2\pi^2\sigma_i^2\gamma-2\pi^2\sigma_i^2\gamma\alpha+\pi^2+2\pi^2\alpha+$$

$$\pi^2\alpha^2+Ta\pi^2+Ta\pi^2\alpha-Ra\alpha-i\pi^2\sigma_i^3\gamma-i\pi^2\sigma_i^3\gamma\alpha+2i\pi^2\sigma_i+ \qquad \text{(A3.2.3)}$$

$$4i\pi^2\sigma_i\alpha+2i\pi^2\sigma_i\alpha^2+i\pi^2\gamma\sigma_i+i\pi^2\gamma\alpha\sigma_i+iTa\pi^2\gamma\sigma_i-iRa\alpha\sigma_i=0$$

Which can be further separated in two parts, real and imaginary, both equals to zero

$$-\pi^2\sigma_i^2-2\pi^2\sigma_i^2\alpha-\pi^2\sigma_i^2\alpha^2-2\pi^2\sigma_i^2\gamma-2\pi^2\sigma_i^2\gamma\alpha+\pi^2+2\pi^2\alpha+$$

(A3.2.4)

$$\pi^2\alpha^2+Ta\pi^2+Ta\pi^2\alpha-Ra\alpha=0$$

$$-i\pi^2\sigma_i^3\gamma-i\pi^2\sigma_i^3\gamma\alpha+2i\pi^2\sigma_i+4i\pi^2\sigma_i\alpha+2i\pi^2\sigma_i\alpha^2+$$

(A3.2.5)

$$i\pi^2\gamma\sigma_i+i\pi^2\gamma\alpha\sigma_i+iTa\pi^2\gamma\sigma_i-iRa\alpha\sigma_i=0$$

From equation A2.2.4 we can express

$$Ra = \frac{\left(1-\sigma_i^2\right)\pi^2(1+\alpha)^2 - 2\sigma_i^2\pi^2\gamma(1+\alpha) + Ta(1+\alpha)\pi^2}{\alpha} \tag{A3.2.6}$$

which introduced into A2.2.5 yields

$$(1+\alpha)\gamma - (1+\alpha)^2\gamma\sigma_i^2 + 2(1+\alpha)^2 + Ta\gamma - (1+\alpha)^2 + (1+\alpha)^2\sigma_i^2 + 2\gamma(1+\alpha)\sigma_i^2 - Ta(1+\alpha) = 0 \tag{A3.2.7}$$

it can be noted that the power of σ_i is constant and equal to 2 all along the expression, hence

$$\sigma_i^2 = \frac{(1+\alpha-\gamma)Ta}{(1+\alpha)(1+\alpha+\gamma)} - 1 \tag{A3.2.8}$$

The above expression is introduced next into the equation A2.2.6 by replacing all σ_i^2's.

$$R_c^{(ov)} = \frac{Ra_c^{(ov)}}{\pi^2} = \frac{1}{\alpha}\left[\left(1 - \frac{(1+\alpha-\gamma)Ta}{(1+\alpha)(1+\alpha+\gamma)} + 1\right)(1+\alpha)^2 - 2\left(\frac{(1+\alpha-\gamma)Ta}{(1+\alpha)(1+\alpha+\gamma)}\right)(1+\alpha)\gamma + Ta(1+\alpha)\right] \tag{A3.2.9}$$

Resulting

$$R_c^{(ov)} = \frac{2}{\alpha}\left[(1+\alpha)(1+\alpha+\gamma) + \frac{\gamma^2 Ta}{(1+\alpha+\gamma)}\right] \tag{A3.2.10}$$

§2. By minimising the above expression with respect to α we have

$$\frac{dR_c^{(ov)}}{d\alpha} = \frac{d}{d\alpha}\left\{\frac{2}{\alpha}\left[(1+\alpha)(1+\alpha+\gamma) + \frac{\gamma^2 Ta}{(1+\alpha+\gamma)}\right]\right\} = 0 \tag{A3.2.11}$$

$$\frac{dR_c^{(or)}}{d\alpha} = 2\frac{d}{d\alpha}\left\{\frac{(1+\alpha)(1+\alpha+\gamma)}{\alpha}\right\} + 2\gamma^2 Ta\frac{d}{d\alpha}\left\{\frac{1}{\alpha(1+\alpha+\gamma)}\right\} =$$

(A3.2.12)

$$-\frac{(1+\alpha)(1+\alpha+\gamma)+\dfrac{\gamma^2 Ta}{1+\alpha+\gamma}}{\alpha^2} + 2\frac{2+2\alpha+\gamma-\dfrac{\gamma^2 Ta}{1+\alpha+\gamma}}{\alpha} = 0$$

$$-(1+\alpha)(1+\alpha+\gamma) - \frac{\gamma^2 Ta}{1+\alpha+\gamma} + (2+2\alpha+\gamma)\alpha - \frac{\alpha\gamma^2 Ta}{(1+\alpha+\gamma)^2} = 0 \qquad \text{(A3.2.13)}$$

By simplifying the expression A2.2.13 we obtain

$$\alpha^4 + 2(\gamma+1)\alpha^3 + \gamma(\gamma+1)\alpha^2 - 2\left[(\gamma+1)^2 + \gamma^2 Ta\right]\alpha - \gamma^2(\gamma+1)Ta - (\gamma+1)^3 = 0$$

(A3.2.14)

Appendix 4.0: Weak non-linear analysis

Let us consider the continuity equation in the form $\nabla \cdot \hat{\mathbf{q}} = 0$. Since all derivatives with respect to y are zero, we can write

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \qquad \text{(A4.0.1)}$$

From the energy equation we have

$$\left[\chi \frac{\partial}{\partial t'} - \nabla^2\right] T' + \mathbf{q}' \nabla T' = 0 \qquad \text{(A4.0.2)}$$

Knowing that $u' = \partial \psi' / \partial z$ and $w' = -\partial \psi' / \partial x$

$$\hat{\mathbf{q}}' \nabla T' = \left(u' \hat{\mathbf{e}}_x + v' \hat{\mathbf{e}}_y + w' \hat{\mathbf{e}}_z\right) \cdot \left(\frac{\partial T'}{\partial x} \hat{\mathbf{e}}_x + \frac{\partial T'}{\partial z} \hat{\mathbf{e}}_z\right) =$$

$$\text{(A4.0.3)}$$

$$u' \frac{\partial T}{\partial x} + w' \frac{\partial T'}{\partial z} = \frac{\partial \psi}{\partial z} \frac{\partial T'}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial T'}{\partial z}$$

All the mixed vector scalar products are $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$, where δ stands for Kroneker function

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \qquad \text{(A4.0.4)}$$

To determine the equation 3.0.2 we have to consider the following

$$\hat{\omega}' = \nabla \times \hat{\mathbf{q}}' = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u' & v' & w' \end{vmatrix} =$$

$$\left[\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}\right]\hat{\mathbf{e}}_x + \left[\frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x}\right]\hat{\mathbf{e}}_y + \left[\frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}\right]\hat{\mathbf{e}}_z = \qquad \text{(A4.0.5)}$$

$$\omega'_x\hat{\mathbf{e}}_x + \omega'_y\hat{\mathbf{e}}_y + \omega'_z\hat{\mathbf{e}}_z$$

The components above can be written

$$\omega'_x = -\frac{\partial v'}{\partial z} \qquad \text{(A4.0.6)}$$

$$\omega'_y = \frac{\partial u'}{\partial z} - \frac{\partial w'}{\partial x} = \frac{\partial}{\partial z}\frac{\partial \psi}{\partial z} - \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = \nabla^2 \psi \qquad \text{(A4.0.6)}$$

$$\omega'_z = \frac{\partial v'}{\partial x} \qquad \text{(A4.0.7)}$$

Re-writing the equation 1.7 by components

$$\hat{\mathbf{e}}_x: \quad \left[\frac{\partial}{\partial t'} + 1\right]\omega'_x - Ta^{1/2}\frac{\partial^2 \psi}{\partial z^2} = 0 \qquad \text{(A4.0.8)}$$

$$\hat{\mathbf{e}}_y: \quad \left[\frac{\partial}{\partial t'} + 1\right]\omega'_y - Ta^{1/2}\frac{\partial v'}{\partial z} = -Ra\frac{\partial T'}{\partial x} \qquad \text{(A4.0.9)}$$

$$\hat{\mathbf{e}}_z: \quad \left[\frac{\partial}{\partial t'} + 1\right]\omega'_z + Ta^{1/2}\frac{\partial^2 \psi}{\partial z \partial x} = 0 \qquad \text{(A4.0.10)}$$

Taking equation A3.0.9 we proceed by multiplying the first equation with $Ta^{1/2}$ and the second with $[\partial/\partial t + 1]$

$$\begin{cases} \left[\frac{\partial}{\partial t'}+1\right]\left(-\frac{\partial v'}{\partial z}\right) - Ta^{1/2}\frac{\partial^2\psi}{\partial z^2}=0 & \qquad Ta^{1/2} \\ \left[\frac{\partial}{\partial t'}+1\right]\nabla^2\psi - Ta^{1/2}\frac{\partial v'}{\partial z} + Ra\frac{\partial T'}{\partial x}=0 & \qquad \left[\frac{\partial}{\partial t'}+1\right] \end{cases} \tag{A4.0.11}$$

$$\begin{cases} -Ta^{1/2}\left[\frac{\partial}{\partial t'}+1\right]\left(-\frac{\partial v'}{\partial z}\right) - Ta\frac{\partial^2\psi}{\partial z^2}=0 \\ \left[\frac{\partial}{\partial t'}+1\right]^2\nabla^2\psi - Ta^{1/2}\frac{\partial v'}{\partial z} - Ta^{1/2}\left[\frac{\partial}{\partial t'}+1\right]\left(-\frac{\partial v'}{\partial z}\right) + Ra\frac{\partial T'}{\partial x}=0 \end{cases} \tag{A4.0.12}$$

Subtracting the first from the second equation in A3.0.12 we obtain

$$\left[\frac{\partial}{\partial t'}+1\right]^2\nabla^2\psi + Ta\frac{\partial^2\psi}{\partial z^2} + Ra\left[\frac{\partial}{\partial t'}+1\right]\frac{\partial T'}{\partial x}=0 \tag{A4.0.13}$$

Appendix 4.1: Expansion around stationary solutions

The equations 4.0.1 and 4.0.2 are shown below

$$\left[\chi\frac{\partial}{\partial t'} - \nabla^2\right] + \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = 0 \qquad \text{(A4.1.1)}$$

$$\left[\frac{\partial}{\partial t'} + 1\right]^2 \nabla^2\psi + Ta\frac{\partial^2\psi}{\partial z^2} + Ra\left[\frac{\partial}{\partial t'} + 1\right]\frac{\partial T}{\partial x} = 0 \qquad \text{(A4.1.2)}$$

In order to obtain a solution we need to re-scale the variable as follows

$$X = \varepsilon x \qquad \text{(A4.1.3)}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X} \qquad \text{(A4.1.4)}$$

$$\tau = \varepsilon^2 t' \qquad \text{(A4.1.5)}$$

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t'} + \varepsilon^2\frac{\partial}{\partial \tau} \qquad \text{(A4.1.6)}$$

Because of the stationary character of the problem the expression A4.1.5 will be written

$$\frac{\partial}{\partial t'} = \varepsilon^2\frac{\partial}{\partial \tau} \qquad \text{(A4.1.7)}$$

$$Ra = Ra\left(1 + \varepsilon^2\right) \qquad \text{(A4.1.8)}$$

We shall proceed to expand the expression A4.1.1 by splitting it into more convenient components

Part 1: $\left(\chi\frac{\partial}{\partial t'}-\nabla^2\right)T$

$$\left(\chi\frac{\partial}{\partial t'}-\nabla^2\right)T=\left(\chi\varepsilon^2\frac{\partial}{\partial\tau}-\nabla^2\right)\left(T_0+\varepsilon T_1+\varepsilon^2T_2+\varepsilon^3T_3\right)=$$

$$\left(\chi\varepsilon^2\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial z^2}\right)\left(T_0+\varepsilon T_1+\varepsilon^2T_2+\varepsilon^3T_3\right)=$$

(A4.1.9)

$$\left(\chi\varepsilon^2\frac{\partial}{\partial\tau}-\left(\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X}\right)^2-\frac{\partial^2}{\partial z}\right)\left(T_0+\varepsilon T_1+\varepsilon^2T_2+\varepsilon^3T_3\right)=$$

$$\left[\varepsilon^2\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial x^2}-2\varepsilon\frac{\partial}{\partial x}\frac{\partial}{\partial X}-\varepsilon^2\frac{\partial^2}{\partial X^2}-\frac{\partial}{\partial z}\right]\left(T_0+\varepsilon T_1+\varepsilon^2T_2+\varepsilon^3T_3\right)=$$

$$\varepsilon^3\chi\frac{\partial}{\partial\tau}T_1-\frac{\partial^2}{\partial x^2}T_0-\varepsilon\frac{\partial^2}{\partial x^2}T_1-\varepsilon^2\frac{\partial^2}{\partial x^2}T_2-\varepsilon^3\frac{\partial^2}{\partial x^2}T_3-2\varepsilon^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}T_1$$

$$-2\varepsilon^3\frac{\partial}{\partial x}\frac{\partial}{\partial X}T_2-\varepsilon^3\frac{\partial^2}{\partial X^2}T_1-\varepsilon\frac{\partial^2}{\partial z^2}T_1-\varepsilon^2\frac{\partial^2}{\partial z^2}T_2-\varepsilon^3\frac{\partial^2}{\partial z^2}T_3$$

We neglect all the terms that are zero i.e. all the derivatives of T_0 with respect to any variable except z, and all the terms that contain powers of ε higher than 3. We can write the partial result as:

$$\left(\chi\frac{\partial}{\partial t'}-\nabla^2\right)T=\varepsilon\left(-\nabla^2T_1\right)+\varepsilon^2\left(-\nabla^2T_2-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}T_1\right)+$$

(A4.1.10)

$$\varepsilon^3\left(-\nabla^2T_3-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}T_2+\chi\frac{\partial}{\partial\tau}T_1-\frac{\partial^2}{\partial X^2}T_1\right)$$

Part 2: $\frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x}$

$$\frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x}=\frac{\partial}{\partial z}\left(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\varepsilon^3\psi_3\right)\left(\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X}\right)\left(T_0+\varepsilon T_1+\varepsilon^2 T_2+\varepsilon^3 T_3\right)=$$

$$\varepsilon^2\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}+\varepsilon^3\frac{\partial \psi_1}{\partial z}\frac{\partial T_2}{\partial x}+\varepsilon^3\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial X}+\varepsilon^3\frac{\partial \psi_2}{\partial z}\frac{\partial T_1}{\partial x}$$

$$\frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x}=\varepsilon^2\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}+\varepsilon^3\left(\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial X}+\frac{\partial \psi_1}{\partial z}\frac{\partial T_2}{\partial x}+\frac{\partial \psi_2}{\partial z}\frac{\partial T_1}{\partial x}\right) \qquad \text{(A4.1.11)}$$

Part 3: $\frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z}$

$$\frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z}=\left(\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X}\right)\left(\psi_0+\varepsilon\psi_1+\varepsilon^2\psi_2+\varepsilon^3\psi_3\right)\frac{\partial}{\partial z}\left(T_0+\varepsilon T_1+\varepsilon^2 T_2+\varepsilon^3 T_3\right)=$$

$$-\varepsilon\frac{\partial}{\partial x}\psi_1+\varepsilon^2\frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}+\varepsilon^3\frac{\partial \psi_1}{\partial x}\frac{T_2}{\partial z}-\varepsilon^2\frac{\partial \psi_2}{\partial x}+\varepsilon^3\frac{\partial \psi_2}{\partial x}\frac{\partial T_1}{\partial z}-\varepsilon^3\frac{\partial \psi_3}{\partial x}-\varepsilon^2\frac{\partial \psi_1}{\partial X}-\varepsilon^3\frac{\partial \psi_2}{\partial X}$$

$$\frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z}=\varepsilon\left(-\frac{\partial \psi_1}{\partial x}\right)+\varepsilon^2\left(\frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial \psi_1}{\partial X}-\frac{\partial \psi_2}{\partial x}\right)+\varepsilon^3\left(\frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z}+\frac{\partial \psi_2}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial \psi_2}{\partial X}-\frac{\partial \psi_3}{\partial x}\right)$$

(A4.1.12)

By adding A4.1.9, A4.1.10 and A4.1.11 and separating according to the power of ε we obtain for order $O(\varepsilon)$

$$\nabla^2 T_1 - \frac{\partial \psi_1}{\partial x} = 0$$

Which is the equation 4.1.5b

For order $O(\varepsilon^2)$

$$\nabla^2 T_2 - \frac{\partial \psi_2}{\partial x} = -2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial X} +$$

Which is the equation 4.1.6b

$$\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}$$

For order $O(\varepsilon^3)$

$$\nabla^2 T_3 - \frac{\partial \psi_3}{\partial x} = -2\frac{\partial}{\partial X}\frac{\partial T_2}{\partial x} + \frac{\partial \psi_2}{\partial X} +$$

$$\frac{\partial \psi_2}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_2}{\partial x}\frac{\partial T_1}{\partial z} + \frac{\partial \psi_1}{\partial z}\frac{\partial T_2}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z} +$$

Which is the equation 4.1.7b

$$\chi\frac{\partial T_1}{\partial \tau} - \frac{\partial^2 T_1}{\partial X^2} + \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial X} - \frac{\partial \psi_1}{\partial X}\frac{\partial T_1}{\partial z}$$

We shall proceed to expand the expression A4.1.2 by splitting it into more manageable components and neglecting all the higher than 3 powers of ε and all the derivatives that equal to zero.

Part 1 $\quad \nabla^2 \psi$

$$\nabla^2\psi = \left[\frac{\partial^2}{\partial x^2} + 2\varepsilon\frac{\partial}{\partial x}\frac{\partial}{\partial X} + \varepsilon^2\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial z^2}\right]\left(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3\right) =$$

$$\varepsilon\left(\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial z^2}\right) + \varepsilon^2\left(\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial z^2} + 2\frac{\partial}{\partial X}\frac{\partial\psi_1}{\partial x}\right) + \tag{A4.1.13}$$

$$\varepsilon^3\left(\frac{\partial^2\psi_3}{\partial x^2} + \frac{\partial^2\psi_3}{\partial z^2} + 2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} + \frac{\partial^2\psi_1}{\partial X}\right)$$

Part 2 $\left[\frac{\partial}{\partial t'} + 1\right]^2 \times$ Part 1

$$\left[\frac{\partial}{\partial t'} + 1\right]^2 \nabla^2\psi = \left[2\varepsilon^2\frac{\partial}{\partial\tau} + 1\right]\nabla^2\psi =$$

$$\left[2\varepsilon^2\frac{\partial}{\partial\tau} + 1\right]\left\{\varepsilon\left(\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial z^2}\right) + \varepsilon^2\left(\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial z^2} + 2\frac{\partial}{\partial X}\frac{\partial\psi_1}{\partial x}\right) +\right.$$

$$\left.\varepsilon^3\left(\frac{\partial^2\psi_3}{\partial x^2} + \frac{\partial^2\psi_3}{\partial z^2} + 2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} + \frac{\partial^2\psi_1}{\partial X}\right)\right\}$$

$$\left[\frac{\partial}{\partial t'} + 1\right]^2 \nabla^2\psi = \varepsilon\left(\nabla^2\psi_1\right) + \varepsilon^2\left(\nabla^2\psi_2 + 2\frac{\partial}{\partial X}\frac{\partial\psi_1}{\partial x}\right) +$$

$$\varepsilon^3\left(\nabla^2\psi_3 + 2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} + 2\frac{\partial}{\partial\tau}\nabla^2\psi_1 + \frac{\partial^2\psi_1}{\partial X^2}\right) \tag{A4.1.14}$$

Part 3: $Ta\frac{\partial^2\psi}{\partial z^2}$

$$Ta\frac{\partial^2\psi}{\partial z^2} = Ta\frac{\partial^2}{\partial z^2}\left(\psi_0 + \varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3\right) =$$

$$\varepsilon\left(Ta\frac{\partial^2\psi_1}{\partial z^2}\right) + \varepsilon^2\left(Ta\frac{\partial^2\psi_2}{\partial z^2}\right) + \varepsilon^3\left(Ta\frac{\partial^2\psi_3}{\partial z^2}\right) \tag{A4.1.15}$$

Part 4 $\quad Ra\left[\frac{\partial}{\partial t'} + 1\right]\frac{\partial T}{\partial x}$

$$Ra\left[\frac{\partial}{\partial t'} + 1\right]\frac{\partial T}{\partial x} = Ra_{cr}\left(1 + \varepsilon^2\right)\left(\varepsilon^2\frac{\partial}{\partial \tau} + 1\right)\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X}\right)\left(T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3\right) =$$

$$\varepsilon\left(Ra_{cr}\frac{\partial T_1}{\partial x}\right) + \varepsilon^2\left[Ra_{cr}\left(\frac{\partial T_2}{\partial x} + \frac{\partial T_1}{\partial X}\right)\right] + \varepsilon^3\left[Ra_{cr}\left(\frac{\partial T_3}{\partial x} + \frac{\partial T_2}{\partial X} + \frac{\partial T_1}{\partial x} + \frac{\partial}{\partial \tau}\frac{\partial T_1}{\partial x}\right)\right]$$

(A4.1.16)

Summing all the parts according to each power of ε we obtain

For order $O(\varepsilon)$

$$\nabla^2\psi_1 + Ta\frac{\partial^2\psi_1}{\partial z^2} + Ra_{cr}\frac{\partial T_1}{\partial x} = 0$$ Which is the equation 4.1.5a

$$\nabla^2\psi_2 + Ta\frac{\partial^2\psi_2}{\partial z^2} + Ra_{cr}\frac{\partial T_2}{\partial x} = -2\frac{\partial}{\partial X}\frac{\partial\psi_1}{\partial x} - Ra\frac{\partial T_1}{\partial x}$$ Which is the equation 4.1.6a

$$\nabla^2\psi_3 + Ta\frac{\partial^2\psi_3}{\partial z^2} + Ra_{cr}\frac{\partial T_3}{\partial x} = -2\frac{\partial}{\partial \tau}\nabla^2\psi_1 - Ra\frac{\partial}{\partial \tau}\frac{\partial T_1}{\partial x} -$$

Which is the equation 4.1.7a

$$\frac{\partial^2\psi_1}{\partial X^2} - 2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} - Ra_{cr}\frac{\partial T_2}{\partial X}$$

To establish the correlation between amplitudes A_1 and B_1 we need to consider the pair of equations 4.1.5 a and b into which we replace the eigenfunction corresponding to the amplitudes with the undetermined solution for the respective order. We remind that $\psi^{(1)} = \left[A_1(\tau, X)e^{ikx} + A_1^*(\tau, X)e^{-ikx}\right]\sin(\pi z)$ and $T^{(1)} = \left[B_1(\tau, X)e^{ikx} + B_1^*(\tau, X)e^{-ikx}\right]\sin(\pi z)$.

Like before we shall proceed analysing the equations by parts

$$\begin{cases} \nabla^2\psi_1 + Ta\dfrac{\partial^2\psi_1}{\partial z^2} + Ra_{cr}\dfrac{\partial T_1}{\partial x} = 0 \\ \\ \nabla^2 T^{(1)} - \dfrac{\partial \psi^{(1)}}{\partial x} = 0 \end{cases}$$

Part 1

$$\nabla^2\psi_1 = \nabla^2\left(A_1e^{ikx} + A_1^*e^{-ikx}\right)\sin(\pi z) = -\left(k^2 + \pi^2\right)\left(A_1e^{ikx} + A_1^*e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.17)}$$

Part 2

$$\frac{\partial^2}{\partial z^2}\psi_1 = \frac{\partial^2}{\partial z^2}\left(A_1e^{ikx} + A_1^*e^{-ikx}\right)\sin(\pi z) = -\pi^2\left(A_1e^{ikx} + A_1^*e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.18)}$$

Part 3

$$\nabla^2 T_1 = \nabla^2\left(B_1e^{ikx} + B_1^*e^{-ikx}\right)\sin(\pi z) = -\left(k^2 + \pi^2\right)\left(B_1e^{ikx} + B_1^*e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.18)}$$

Part 4

$$\frac{\partial}{\partial x}T_1 = \frac{\partial}{\partial x}\left(B_1 e^{ikx} + B_1^{*} e^{-ikx}\right)\sin(\pi z) = ik\left(B_1 e^{ikx} - B_1^{*} e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.19)}$$

By summing the Parts corresponding to each expression, separating according to the exponential argument and recalling that $\alpha = k^2/\pi^2$ we obtain the following system of equations

$$(\alpha + 1)A_1 + Ta A_1 + iRa_{cr}\frac{1}{\pi}\sqrt{\alpha}B_1 = 0 \qquad \text{(A4.1.20)}$$

$$(\alpha + 1)A_1^{*} + Ta A_{1*} - iRa_{cr}\frac{1}{\pi}\sqrt{\alpha}B_1^{*} = 0 \qquad \text{(A4.1.21)}$$

$$(\alpha + 1)B_1 + i\frac{1}{\pi}\sqrt{\alpha}A_1 = 0 \qquad \text{(A4.1.22)}$$

$$(\alpha + 1)B_1^{*} - i\frac{1}{\pi}\sqrt{\alpha}A_1^{*} = 0 \qquad \text{(A4.1.23)}$$

From the last two equations we can draw the relationships between coefficients of the solutions

$$B_1 = -\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}A_1 \qquad B_1^{*} = -\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}A_1^{*} \qquad \text{(A4.1.24)}$$

To determine the solutions at order $O(\varepsilon^2)$ we first consider the set of equations 4.1.6 a and b. The de-coupling process for temperature is shown below

$$\begin{cases} \nabla^2\psi_2 + Ta\dfrac{\partial^2\psi_2}{\partial z^2} + Ra_{cr}\dfrac{\partial T_2}{\partial x} = RHS_1(\psi_1,T_1) \\ \\ \nabla^2 T_2 - \dfrac{\partial\psi_2}{\partial x} = RHS_2(\psi_1,T_1)\end{cases} \qquad \left| \begin{array}{l} \dfrac{\partial}{\partial x} \\ \\ \left(\nabla^2 + Ta\dfrac{\partial^2}{\partial z^2}\right)\end{array}\right. \tag{A4.1.25}$$

and for stream function

$$\begin{cases} \nabla^2\psi_2 + Ta\dfrac{\partial^2\psi_2}{\partial z^2} + Ra_{cr}\dfrac{\partial T_2}{\partial x} = RHS_1(\psi_1,T_1) \\ \\ \nabla^2 T_2 - \dfrac{\partial\psi_2}{\partial x} = RHS_2(\psi_1,T_1)\end{cases} \qquad \left| \begin{array}{l} \nabla^2 \\ \\ -Ra_{cr}\dfrac{\partial}{\partial x}\end{array}\right. \tag{A4.1.26}$$

Where

$$RHS_1(\psi_1,T_1) = -2\frac{\partial}{\partial X}\frac{\partial\psi_1}{\partial x} - Ra\frac{\partial T_1}{\partial x} \tag{A4.1.27}$$

$$RHS_2(\psi_1,T_1) = -2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial\psi_1}{\partial X} + \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} \tag{A4.1.28}$$

The de-coupled functions are shown below, first the stream-function equation and second the temperature equation.

$$\nabla^2\left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]\psi_2 + Ra_{cr}\frac{\partial^2}{\partial x^2}\psi_2 = \nabla^2 RHS_1(\psi_1,T_1) - Ra_{cr}\frac{\partial}{\partial x}RHS_2(\psi_1,T_1)$$

(A4.1.28)

$$\left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]\nabla^2 T_2 + Ra_{cr}\frac{\partial^2}{\partial x^2}T_2 = \frac{\partial}{\partial x}RHS_1(\psi_1, T_1) + \left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]RHS_2(\psi_1, T_1)$$

(A4.1.29)

Since the left hand side of the homogeneous equations are similar to those at the first order the solutions will have to look similar up to a particular function as shown below

$$\psi_1 = \left(A_2 e^{ikx} + A_2^* e^{-ikx}\right)\sin(\pi z) + f_1(RHS) \qquad \text{(A4.1.30)}$$

$$T_1 = \left(B_2 e^{ikx} + B_2^* e^{-ikx}\right)\sin(\pi z) + f_2(RHS) \qquad \text{(A4.1.31)}$$

We shall proceed as before, calculating parts of the expressions starting with stream-function equation A4.1.27

$$\nabla^2\left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]\psi_2 + Ra_{cr}\frac{\partial^2}{\partial x^2}\psi_2 = \nabla^2\left\{-2\frac{\partial}{\partial X}\frac{\partial \psi_1}{\partial x} - Ra_{cr}\frac{\partial T_1}{\partial X}\right\} -$$

(A4.1.32)

$$Ra_{cr}\frac{\partial}{\partial x}\left\{-2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}\right\}$$

Part 1

$$2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1 = 2ik\left(\frac{\partial A_1}{\partial X}e^{ikx} - \frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.33)}$$

Part 2

$$Ra_{cr}\frac{\partial T_1}{\partial X} = Ra_{cr}\frac{\partial}{\partial X}\left(B_1e^{ikx} + B_1^{*}e^{-ikx}\right)\sin(\pi z) =$$

$$Ra_{cr}\frac{\partial}{\partial X}\left(-\frac{i\sqrt{\alpha}}{\pi(1+\alpha)}A_1e^{ikx} + \frac{i\sqrt{\alpha}}{\pi(1+\alpha)}A_1^{*}e^{-ikx}\right)\sin(\pi z) =$$

(A4.1.34)

$$(1+\alpha)^2\frac{\partial}{\partial X}\left(-\frac{i\sqrt{\alpha}}{\pi(1+\alpha)}A_1e^{ikx} + \frac{i\sqrt{\alpha}}{\pi(1+\alpha)}A_1^{*}e^{-ikx}\right)\sin(\pi z) =$$

$$-i\frac{k}{\pi^2}(1+\alpha)\left(\frac{\partial A_1}{\partial X}e^{ikx} - \frac{\partial A_1^{*}}{\partial X}e^{-ikx}\right)$$

$Part\ 3 = \nabla^2(Part\ \ 1 + \ \ Part\ \ 2)$

$$\nabla^2\left(-2\frac{\partial}{\partial X}\frac{\partial \psi_1}{\partial x} - Ra_{cr}\frac{\partial T_1}{\partial X}\right) = ik(2-(1+\alpha))(1+\alpha)\left(\frac{\partial A_1}{\partial X}e^{ikx} - \frac{\partial A_1^{*}}{\partial X}e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.35)}$$

Part 4

$$2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} = 2\frac{\alpha}{(\alpha+1)}\left(\frac{\partial A_1}{\partial X}e^{ikx} + \frac{\partial A_1^{*}}{\partial X}\right)\sin(\pi z) \qquad \text{(A4.1.36)}$$

Part 5

$$\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} = \frac{\partial}{\partial z}\left[\left(A_1e^{ikx} + A_1^{*}e^{-ikx}\right)\sin(\pi z)\right]\frac{\partial}{\partial x}\left[\left(B_1e^{ikx} + B_1^{*}e^{-ikx}\right)\sin(\pi z)\right] =$$

(A4.1.37)

$$\frac{\pi\alpha}{2(\alpha+1)}\left(A_1e^{ikx} + A_1^{*}e^{-ikx}\right)^2\sin(2\pi z)$$

Part 6

$$\frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} = \frac{\partial}{\partial x}\left[\left(A_1 e^{ikx} + A_1^* e^{-ikx}\right)\sin(\pi z)\right]\frac{\partial}{\partial z}\left[\left(B_1 e^{ikx} + B_1^* e^{-ikx}\right)\sin(\pi z)\right] =$$

(A4.1.38)

$$\frac{\pi\alpha}{2(\alpha+1)}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)^2 \sin(2\pi z)$$

Part 7 = *Part* 5 − *Part* 6

$$\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} = \frac{2\pi\alpha}{(\alpha+1)} A_1 A_1^* \sin(2\pi z) \tag{A4.1.39}$$

Part 8

$$\frac{\partial \psi_1}{\partial X} = \left(\frac{\partial A_1}{\partial X} e^{ikx} + \frac{\partial A_1^*}{\partial X} e^{-ikx}\right)\sin(\pi z) \tag{A4.1.40}$$

$$Part\ 8 = Ra_{cr}\frac{\partial}{\partial x}\left[-Part\ 4 + Part\ 8 + Part\ 7\right]$$

$$Ra_{cr}\frac{\partial}{\partial x}\left\{-2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}\right\} =$$

(A4.1.41)

$$ik\left[(\alpha+1)^2 - 2(\alpha+1)\alpha\right]\left(\frac{\partial A_1}{\partial X} e^{ikx} - \frac{A_1^*}{\partial X} e^{-ikx}\right)\sin(\pi z)$$

Finally, by adding A4.1.34 to A4.1.41 we obtain the non-homogeneous part of stream equation

$$f_1(RHS) = ik\left[2(\alpha+1) - (\alpha+1)^2 - (\alpha+1)^2 + 2(\alpha+1)\alpha\right]\left(\frac{\partial A_1}{\partial X} e^{ikx} - \frac{\partial A_1^*}{\partial X} e^{-ikx}\right)\sin(\pi z) \equiv 0$$

(A4.1.42)

That means that the solution of stream function at order $O(\varepsilon^2)$ will be similar to that at $O(\varepsilon)$. We shall proceed with the calculation of $f_2(RHS)$ differently from $f_1(RHS)$. The equation for temperature is explicitly shown below.

$$\left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]\nabla^2 T_2 + Ra_{cr}\frac{\partial^2}{\partial x^2}T_2 = \frac{\partial}{\partial x}\left[-2\frac{\partial}{\partial X}\frac{\partial \psi_1}{\partial x} - Ra_{cr}\frac{\partial T_1}{\partial X}\right] +$$

$$\left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]\left[-2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}\right] \tag{A4.1.43}$$

$$\frac{\partial}{\partial x}\left[-2\frac{\partial}{\partial X}\frac{\partial \psi_1}{\partial x} - Ra_{cr}\frac{\partial T_1}{\partial X}\right] + \left[\nabla^2 + Ta\frac{\partial}{\partial x}\right]\times$$

$$\left[-2\frac{\partial}{\partial X}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial X} + \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z}\right] =$$

$$\frac{\partial}{\partial x}\left[-2ik\left(\frac{\partial A_1}{\partial X}e^{ikx} - \frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z) + \pi^2(\alpha+1)^2\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}\left(\frac{\partial A_1}{\partial X}e^{ikx} - \frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z)\right] +$$

$$\left[\frac{\partial^2}{\partial x^2} + \alpha^2\frac{\partial^2}{\partial z^2}\right]\left[2ik\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}\left(\frac{\partial A_1}{\partial X}e^{ikx} + \frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z) + \left(\frac{\partial A_1}{\partial X}e^{ikx} + \frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z)\right] +$$

$$\left[\frac{\partial^2}{\partial x^2} + \alpha^2\frac{\partial^2}{\partial z^2}\right]\left[\pi\left(A_1e^{ikx} - A_1^*e^{-ikx}\right)^2\cos(\pi z)\left(-\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}\right)ik\sin(\pi z)\right] -$$

$$\left[\frac{\partial^2}{\partial x^2} + \alpha^2\frac{\partial^2}{\partial z^2}\right]\left[\pi\left(A_1e^{ikx} + A_1^*e^{-ikx}\right)^2\sin(\pi z)\left(-\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}\right)ik\cos(\pi z)\right] =$$

$$\left[-2(ik)^2\left(\frac{\partial A_1}{\partial X}e^{ikx}+\frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z)+\pi^2(1+\alpha)^2\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}(ik)\left(\frac{\partial A_1}{\partial X}e^{ikx}+\frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z)\right]+$$

$$\left[\frac{\partial^2}{\partial x^2}+\alpha^2\frac{\partial^2}{\partial z^2}\right]\left\{\left(1-2\frac{\alpha}{\alpha+1}\right)\left(\frac{\partial A_1}{\partial X}e^{ikx}+\frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z)+\right.$$

$$\left.\pi\frac{\alpha}{2(\alpha+1)}\left(A_1e^{ikx}+A_1^*e^{-ikx}\right)^2\sin(2\pi z)-\pi\frac{\alpha}{2(\alpha+1)}\left(A_1e^{ikx}-A_1^*e^{-ikx}\right)^2\sin(2\pi z)\right\}=$$

$$\left[2k^2-(1+\alpha)\pi^2\alpha-k^2-\pi^2\alpha^2+2\frac{\alpha}{\alpha+1}\left(k^2+\pi^2\alpha^2\right)\right]\left(\frac{\partial A_1}{\partial X}e^{ikx}+\frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(\pi z)-$$

$$-4\pi^2\frac{2\sqrt{\alpha}k}{\alpha+1}\alpha^2A_1A_1^*\sin(2\pi z)$$

Finally,

$$f_2(RHS)=-8\pi^2\alpha^2\frac{k\sqrt{\alpha}}{\alpha+1}A_1A_1^*\sin(2\pi z) \qquad \text{(A4.1.44)}$$

The temperature equation can be written now

$$\left[\nabla^2+Ta\frac{\partial}{\partial x}\right]\nabla^2T_2+Ra_{cr}\frac{\partial^2}{\partial x^2}T_2=8\pi^2k\frac{\alpha^2\sqrt{\alpha}}{\alpha+1}A_1A_1^*\sin(2\pi z) \qquad \text{(A4.1.45)}$$

Let $T_2^p=Y\sin(2\pi z)$ be a solution that satisfies equation A3.1.36, where Y is a polynomial expression of constants. The second term of the equation will yield zero, since the assumed solution it is not a function of x. Some algebraic work to the equation will transform it

$$16Y\pi^4\alpha^2\sin(2\pi z) = -8\pi^2\frac{k^2}{\pi(\alpha+1)}\alpha^2 A_1 A_1^* \sin(2\pi z) \qquad \text{(A4.1.46)}$$

$$Y = -\frac{\alpha}{2\pi(\alpha+1)} A_1 A_1^* \sin(2\pi z) \qquad \text{(A4.1.47)}$$

The solution for temperature at order $O(\varepsilon^2)$ is

$$T_2 = \left(B_2 e^{ikx} + B_2^* e^{-ikx}\right)\sin(\pi z) - \frac{\alpha}{2\pi(\alpha+1)} A_1 A_1^* \sin(2\pi z) \qquad \text{(A4.1.48)}$$

For the $O(\varepsilon^3)$ solutions we can write the equation 4.1.7

$$\nabla^2\psi_3 + Ta\frac{\partial^2\psi_3}{\partial z^2} + Ra_{cr}\frac{\partial T_3}{\partial x} = RHS_1(\psi_1, T_1) \qquad \text{(A4.1.49)}$$

$$\nabla^2 T_3 - \frac{\partial\psi_3}{\partial x} = RHS_2(\psi_1, T_1) \qquad \text{(A4.1.50)}$$

In order to determine ψ_3 and T_3 we have to de-couple the equations above. For ψ_3 we have

$$\begin{cases} \nabla^2\psi_3 + Ta\dfrac{\partial^2\psi_3}{\partial z^2} + Ra_{cr}\dfrac{\partial T_3}{\partial x} = RHS_1(\psi_1, T_1) \\ \\ \nabla^2 T_3 - \dfrac{\partial\psi_3}{\partial x} = RHS_2(\psi_1, T_1) \end{cases} \quad \left| \begin{array}{l} \nabla^2 \\ \\ -Ra\dfrac{\partial}{\partial x} \end{array} \right. \tag{A4.1.51}$$

By adding the two expressions we get the de-coupled equation for ψ_3

$$\left\{\nabla^2\left(\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right) + Ra_{cr}\frac{\partial^2}{\partial x^2}\right\}\psi_3 = \nabla^2 RHS_1 - Ra_{cr}\frac{\partial}{\partial x}RHS_2 \tag{A4.1.52}$$

$$RHS_1 = -2\frac{\partial}{\partial\tau}\nabla^2\psi_1 - Ra_{cr}\frac{\partial}{\partial\tau}\frac{\partial T_1}{\partial x} - \frac{\partial^2\psi_1}{\partial X^2} - 2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} - Ra_{cr}\frac{\partial T_2}{\partial X} \tag{A3.1.53}$$

$$\begin{aligned} RHS_2 = &-2\frac{\partial}{\partial X}\frac{\partial T_2}{\partial x} + \frac{\partial\psi_2}{\partial X} + \frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x} - \frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z} + \\ &\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z} + \chi\frac{\partial T_1}{\partial\tau} - \frac{\partial^2 T_1}{\partial X^2} + \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X} - \frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z} \end{aligned} \tag{A4.1.54}$$

We shall work the expression 4.1.53 by parts, in the end compiling them into a final expression that will yield the solvability condition for the amplitude.

Part 1

$$\nabla^2\psi_1 = -\left(k^2 + \pi^2\right)\left(A_1 e^{ikx} + A_1^* e^{-ikx}\right)\sin(\pi z) \tag{A4.1.55}$$

Part 2

$$2\frac{\partial}{\partial\tau}\nabla^2\psi_1 = -2\left(k^2+\pi^2\right)\left(\frac{\partial A_1}{\partial\tau}e^{ikx}+\frac{\partial A_1^*}{\partial\tau}e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.56)}$$

Part 3

$$Ra_{cr}\frac{\partial}{\partial\tau}\frac{\partial T_1}{\partial x} = \pi^2\alpha(\alpha+1)\left(\frac{\partial A_1}{\partial\tau}e^{ikx}+\frac{\partial A_1^*}{\partial\tau}e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.57)}$$

Part 4

$$Ra_{cr}\frac{\partial T_1}{\partial x} = k^2(\alpha+1)\left(A_1e^{ikx}+A_1^*e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.58)}$$

Part 5

$$\frac{\partial^2\psi_1}{\partial X^2} = \left(\frac{\partial^2 A_1}{\partial X^2}e^{ikx}+\frac{\partial A_1^*}{\partial X^2}e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.59)}$$

Part 6

$$2\frac{\partial}{\partial X}\frac{\partial\psi_2}{\partial x} = 2ik\left(\frac{\partial A_2}{\partial X}e^{ikx}-\frac{\partial A_2^*}{\partial X}e^{-ikx}\right)\sin(\pi z) \qquad \text{(A4.1.60)}$$

Part 7

$$\chi\frac{\partial T_1}{\partial\tau} = -\chi\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}\left(\frac{\partial A_1}{\partial\tau}e^{ikx} - \frac{\partial A_1^*}{\partial\tau}e^{-ikx}\right)\sin(\pi z) \tag{A4.1.61}$$

Part 8

$$Ra_{cr}\frac{\partial T_2}{\partial X} = \pi^2(\alpha+1)^2\frac{\partial}{\partial X}\left[\left(B_2e^{ikx}+B_2^*e^{-ikx}\right)\sin(\pi z) - \frac{\alpha}{2\pi(\alpha+1)}A_1A_1^*\sin(2\pi z)\right] =$$

(A4.1.62)

$$-(\alpha+1)\left[ik\left(\frac{\partial A_2}{\partial X}e^{ikx} - \frac{\partial A_2^*}{\partial X}e^{-ikx}\right)\sin(\pi z) + \frac{\alpha\pi}{2}\left(\frac{\partial A_1}{\partial X}A_1^* + \frac{\partial A_1^*}{\partial X}A_1\right)\sin(2\pi z)\right]$$

Part 9

$$\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x} = \frac{\alpha\pi}{2(\alpha+1)}\left(A_1e^{ikx}+A_1^*e^{-ikx}\right)\left(A_2e^{ikx}+A_2^*e^{-ikx}\right)\sin(2\pi z) \tag{A4.1.63}$$

Part 10

$$\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x} = \frac{\partial}{\partial z}\left[\left(A_1e^{ikx}+A_1^*e^{-ikx}\right)\sin(\pi z)\right]\times$$

$$\frac{\partial}{\partial x}\left[\left(B_2e^{ikx}+B_2^*e^{-ikx}\right)\sin(\pi z) - \frac{\alpha}{2\pi(\alpha+1)}A_1A_1^*\sin(2\pi z)\right] = \tag{A4.1.64}$$

$$\frac{\pi\alpha}{2(\alpha+1)}\left(A_1e^{ikx}+A_1^*e^{-ikx}\right)\left(A_2e^{ikx}+A_2^*e^{-ikx}\right)\sin(2\pi z)$$

Part 11

$$\frac{\partial \psi_2}{\partial x}\frac{\partial T_1}{\partial z} = \frac{\alpha\pi}{2(\alpha+1)}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)\left(A_2 e^{ikx} - A_2^* e^{-ikx}\right)\sin(2\pi z) \qquad \text{(A4.1.65)}$$

Part 12

$$\frac{\partial \psi_2}{\partial x}\frac{\partial T_2}{\partial z} = \frac{\partial}{\partial x}\left[\left(A_1 e^{ikx} + A_1^* e^{-ikx}\right)\sin(\pi z)\right]\times$$

$$\frac{\partial}{\partial z}\left[\left(B_2 e^{ikx} + B_2^* e^{-ikx}\right)\sin(\pi z) - \frac{\alpha}{2\pi(\alpha+1)} A_1 A_1^* \sin(2\pi z)\right] =$$

$$\text{(A4.66)}$$

$$\frac{\alpha\pi}{2(\alpha+1)}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)\left(A_2 e^{ikx} - A_2^* e^{-ikx}\right)\sin(2\pi z) -$$

$$\frac{ik\alpha}{\alpha+1}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)A_1 A_1^* \cos(2\pi z)\sin(\pi z)$$

This expression needs a little attention. We have a term $\cos(2\pi x)\sin(\pi z)$, which can be expressed with the help of the following trigonometric formulas

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right] \qquad \text{(A4.1.67)}$$

$$\sin(\pi z)\cos(2\pi z) = \frac{1}{2}\left[\sin(3\pi z) - \sin(\pi z)\right] \qquad \text{(A4.1.68)}$$

$$\frac{\partial \psi_2}{\partial x}\frac{\partial T_2}{\partial z} = \frac{\alpha\pi}{2(\alpha+1)}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)\left(A_2 e^{ikx} - A_2^* e^{-ikx}\right)\sin(2\pi z) -$$

$$\text{(A4.1.69)}$$

$$\frac{ik\alpha}{2(\alpha+1)}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)A_1 A_1^* \sin(3\pi z) + \frac{ik\alpha}{2(\alpha+1)}\left(A_1 e^{ikx} - A_1^* e^{-ikx}\right)A_1 A_1^* \sin(\pi z)$$

Part 13

$$2\frac{\partial}{\partial X}\frac{\partial T_2}{\partial x}=\frac{2\alpha}{\alpha+1}\left(\frac{\partial A_2}{\partial X}e^{ikx}+\frac{\partial A_2^*}{\partial X}e^{-ikx}\right)\sin(\pi z) \tag{A4.1.70}$$

Part 14

$$\frac{\partial \psi_2}{\partial X}=\left(\frac{\partial A_2}{\partial X}e^{ikx}+\frac{\partial A_2^*}{\partial X}e^{-ikx}\right)\sin(\pi z) \tag{A4.1.71}$$

Part 15

$$\frac{\partial^2 T_1}{\partial X^2}=-\frac{i\sqrt{\alpha}}{\pi(\alpha+1)}\left(\frac{\partial^2 A_1}{\partial X^2}e^{ikx}-\frac{\partial^2 A_1^*}{\partial X^2}e^{-ikx}\right)\sin(\pi z) \tag{A4.1.72}$$

Part 16

$$\frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial X}=-\frac{i\sqrt{\alpha}}{2(\alpha+1)}\left(A_1e^{ikx}+A_1^*e^{-ikx}\right)\left(\frac{\partial A_1}{\partial X}e^{ikx}-\frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\sin(2\pi z) \tag{A4.1.73}$$

Part 17

$$\frac{\partial \psi_1}{\partial X}\frac{\partial T_1}{\partial z}=-\frac{i\sqrt{\alpha}}{2(\alpha+1)}\left(\frac{\partial A_1}{\partial X}e^{ikx}+\frac{\partial A_1^*}{\partial X}e^{-ikx}\right)\left(A_1e^{ikx}-A_1^*e^{-ikx}\right)\sin(2\pi z) \tag{A4.1.74}$$

The right hand side of equation A4.1.53 can be expanded in terms of all the parts expressed so far, less all the terms that carry higher harmonics of the solution, which would be relevant for an order $O(\varepsilon^4)$ analysis.

$$\nabla^2 RHS_1 - Ra_{cr}\frac{\partial}{\partial x}RHS_2 = \nabla^2 RHS_1 - \pi^2(\alpha+1)^2\frac{\partial}{\partial x}RHS_2 =$$

$$\Bigg\{\Bigg[-2\left(k^2+\pi^2\right)^2\frac{\partial A_1}{\partial \tau} + \pi^2\alpha(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_1}{\partial \tau} + k^2(\alpha+1)\left(k^2+\alpha^2\right)A_1 +$$

$$\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial X^2} + 2ik\left(k^2+\pi^2\right)\frac{\partial A_2}{\partial X} - ik(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_2}{\partial X}\Bigg]e^{ikx} +$$

$$\Bigg[-2\left(k^2+\pi^2\right)^2\frac{\partial A_1^*}{\partial \tau} + \pi^2\alpha(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_1^*}{\partial \tau} + k^2(\alpha+1)\left(k^2+\alpha^2\right)A_1^* +$$

$$\left(k^2+\pi^2\right)\frac{\partial^2 A_1^*}{\partial X^2} - 2ik\left(k^2+\pi^2\right)\frac{\partial A_2^*}{\partial X} + ik(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_2^*}{\partial X}\Bigg]e^{-ikx} \tag{A4.1.75}$$

$$\Bigg[\pi^2(\alpha+1)^2\chi\frac{k\sqrt{\alpha}}{\pi(\alpha+1)}\frac{\partial A_1}{\partial \tau} - \pi^2(\alpha+1)^2\frac{\alpha k^2}{2(\alpha+1)}A_1^2A_1^* - \pi^2(\alpha+1)^2\frac{2ik\alpha}{\alpha+1}\frac{\partial A_2}{\partial X} +$$

$$ik\pi^2(\alpha+1)^2\frac{\partial A_2}{\partial X} - \pi^2(\alpha+1)^2\frac{k\sqrt{\alpha}}{\pi(\alpha+1)}\frac{\partial^2 A_1}{\partial X^2}\Bigg]e^{ikx} -$$

$$\Bigg[\pi^2(\alpha+1)^2\chi\frac{k\sqrt{\alpha}}{\pi(\alpha+1)}\frac{\partial A_1}{\partial \tau} - \pi^2(\alpha+1)^2\frac{\alpha k^2}{2(\alpha+1)}A_1^2A_1^* + \pi^2(\alpha+1)^2\frac{2ik\alpha}{\alpha+1}\frac{\partial A_2}{\partial X} -$$

$$ik\pi^2(\alpha+1)^2\frac{\partial A_2}{\partial X} - \pi^2(\alpha+1)^2\frac{k\sqrt{\alpha}}{\pi(\alpha+1)}\frac{\partial^2 A_1}{\partial X^2}\Bigg]e^{-ikx}\Bigg\}\sin(\pi z)$$

Separating A4.1.76 according to the power of the exponent, and equating to zero the resulting terms in order to obtain the solvability condition at $O\left(\varepsilon^3\right)$, we have for e^{ikx}:

$$(\bullet)e^{ikx} = -2\left(k^2+\pi^2\right)^2\frac{\partial A_1}{\partial\tau} + \pi^2\alpha(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_1}{\partial\tau} + k^2(\alpha+1)\left(k^2+\pi^2\right)A_1 +$$

$$\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial X^2} + 2ik\left(k^2+\pi^2\right)\frac{\partial A_2}{\partial X} - ik(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_2}{\partial X} -$$

(A4.1.76)

$$\pi^2(\alpha+1)^2\chi\frac{\alpha}{(\alpha+1)}\frac{\partial A_1}{\partial\tau} - \pi^2(\alpha+1)^2\frac{k^2\alpha}{2(\alpha+1)}A_1^2A_1^* + \pi^2(\alpha+1)^2\frac{2ik\alpha}{\alpha+1}\frac{\partial A_2}{\partial X} -$$

$$ik\pi^2(\alpha+1)^2\frac{\partial A_2}{\partial X} + \pi^2(\alpha+1)^2\frac{\alpha}{(\alpha+1)}\frac{\partial^2 A_1}{\partial X^2} = 0$$

Similarly for e^{-ikx}

$$(\bullet)e^{ikx} = -2\left(k^2+\pi^2\right)^2\frac{\partial A_1^*}{\partial\tau} + \pi^2\alpha(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_1^*}{\partial\tau} + k^2(\alpha+1)\left(k^2+\pi^2\right)A_1^* +$$

$$\left(k^2+\pi^2\right)\frac{\partial^2 A_1^*}{\partial X^2} - 2ik\left(k^2+\pi^2\right)\frac{\partial A_2^*}{\partial X} + ik(\alpha+1)\left(k^2+\pi^2\right)\frac{\partial A_2^*}{\partial X} -$$

(A4.1.77)

$$\pi^2(\alpha+1)^2\chi\frac{\alpha}{(\alpha+1)}\frac{\partial A_1^*}{\partial\tau} - \pi^2(\alpha+1)^2\frac{k^2\alpha}{2(\alpha+1)}A\left(A_1^*\right)^2 + \pi^2(\alpha+1)^2\frac{2ik\alpha}{\alpha+1}\frac{\partial A_2^*}{\partial X} +$$

$$ik\pi^2(\alpha+1)^2\frac{\partial A_2^*}{\partial X} + \pi^2(\alpha+1)^2\frac{\alpha}{(\alpha+1)}\frac{\partial^2 A_1^*}{\partial X^2} = 0$$

The two expressions A4.1.77 and 78 are identical up to the coefficients of A_2 and A_2^*. We expect that the partial algebraic sum of those coefficients to be zero.

$$2ik(k^2+\pi^2) - ik(\alpha+1)(k^2+\pi^2) + ik\pi^2(\alpha+1)^2\frac{2\alpha}{(\alpha+1)} - ik\pi^2(\alpha+1)^2 \equiv 0 \quad \text{(A3.1.78)}$$

The equation A3.3.77 can be written symbolically as

$$P \times \frac{\partial A_1}{\partial \tau} + Q \times A_1 + M \times A_1^2 A_1^* + N \times \frac{\partial^2 A_1}{\partial X^2} = 0 \qquad \text{(A4.1.79)}$$

Where P is the algebraic sum of the coefficients of $\partial A_1 / \partial \tau$

$$\left(k^2 + \pi^2\right)^2 (\alpha - 2) - k^2 \chi(\alpha + 1) \qquad \text{(A4.1.80)}$$

Q is the algebraic sum of the coefficients of A_1

$$\alpha \pi^2 (\alpha + 1)\left(k^2 + \pi^2\right) \qquad \text{(A4.1.81)}$$

M is the algebraic sum of the coefficients of $A_1^2 A_1^*$

$$\frac{-\alpha^2 \pi^2 \left(k^2 + \pi^2\right)}{2} \qquad \text{(A4.1.82)}$$

N is the algebraic sum of the coefficients of $\partial^2 A_1 / \partial X^2$

$$(1 + \alpha)\left(k^2 + \pi^2\right) \qquad \text{(A4.1.83)}$$

It follows that the equation for A_1

$$\pi^2 \left[(\alpha + 1)(\alpha - 2) - \alpha \gamma\right] \frac{\partial A_1}{\partial \tau} + \left[\alpha \pi^2 - \frac{\pi^2 \alpha^2}{2} A_1 A_1^*\right] A_1 + (\alpha + 1) \frac{\partial^2 A_1}{\partial X^2} = 0 \qquad \text{(A4.1.84)}$$

By replacing the variables in their original time and space scale

$$\eta\frac{\partial A}{\partial t}-(\alpha+1)\frac{\partial^2 A}{\partial x^2}=\frac{\pi^2\alpha^2}{2}\left(\xi_0^{st}-AA^*\right)A \qquad \text{(A4.1.90)}$$

Where η and ξ_0^{st} are

$$\eta=\frac{(\alpha+1)(2-\alpha)+\alpha\gamma}{\gamma}$$

$$\xi_0^{st}=\frac{2(\alpha+1)}{\alpha}\left(\frac{R}{R_{cr}}-1\right) \qquad \text{(A4.1.91)}$$

From the conditions that $\psi_1=0$ and $\partial\psi_1/\partial t=0$ results that $A_1=-A_1^*$

From trigonometry we have

$$e^{ikx}=\cos(kx)+i\sin(kx) \qquad \text{(A4.1.92)}$$

$$e^{-ikx}=\cos(kx)-i\sin(kx) \qquad \text{(A4.1.93)}$$

By subtracting the two expressions above

$$\sin(kx)=\frac{e^{ikx}-e^{-ikx}}{2i}$$

$$\psi_1=\left(A_1e^{ikx}-A_1e^{-ikx}\right)\sin(\pi z)=A_1\left(e^{ikx}-e^{-ikx}\right)\sin(\pi z)=2iA_1\sin(kx)\sin(\pi z) \qquad \text{(A3.1.94)}$$

$$\psi_1 = C_1 \sin(kx)\sin(\pi z) \tag{A4.1.95}$$

Therefore the solvability condition when the space scale has been removed (the diffusion term), can be expressed as

$$\eta \frac{dC}{dt} = \frac{\pi^2 \alpha^2}{8}\left(\xi_0^{st} - C^2\right)C \tag{A4.1.96}$$

$$\eta \frac{dC}{dt} - \frac{\pi^2 \alpha^2}{8}\xi_0^{st} C = -\frac{\pi^2 \alpha^2}{8} C^3 \tag{A4.1.97}$$

The equation above is Bernoulli type equation

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \tag{A4.1.98}$$

In this case

$$n = 3 \qquad P(x) = -\frac{\pi^2 \alpha^2}{8}\xi_0^{st} \qquad Q(x) = -\frac{\pi^2 \alpha^2}{8}$$

In equation A4.1.98 we call the variable $v = y^{1-n}$. It yields the integrating factor equation

$$v e^{(1-3)\int -\frac{\pi^2\alpha^2}{8}\xi_0^{st} dt} = (1-3)\int -\frac{\pi^2 \alpha^2}{8} e^{(1-3)\int -\frac{\pi^2\alpha^2}{8}\xi_0^{st} dt} dt + const \tag{A4.1.99}$$

The integrating constant is zero for $\forall t$

$$\nu = y^{-2} = \frac{1}{\xi_0^{st}} = C^2$$

For $Ra \geq Ra_{cr}^{st}$

$$C = \pm\sqrt{\xi_o^{st}} \qquad \text{(A4.1.100)}$$

For $Ra < Ra_{cr}$ we have $\xi_0^{st} < 0$ and equation A3.1.99 will yield

$$C^2 = -\xi_0^{st} \qquad \text{(A4.1.101)}$$

Implying that

$$C = \begin{cases} 0 & \forall \quad R < R_{cr} \\ \pm\sqrt{\xi_o^{st}} & \forall \quad R \geq R_{cr} \end{cases} \qquad \text{(A4.1.102)}$$

The condition $\eta > 0$ implies

$$(\alpha + 1)(2 - \alpha) + \alpha\gamma > 0 \qquad \text{(A4.1.103)}$$

Using for α, $\alpha_{cr}^{st} = \sqrt{1 + Ta}$ it yields

$$\left(\sqrt{1 + Ta} + 1\right)\left(2 - \sqrt{1 + Ta}\right) + \gamma\sqrt{1 + Ta} > 0 \qquad \text{(A4.1.104)}$$

In this case-limit, γ must be replaced with γ_t^{st} representing a transitional value relating to the relaxation time.

$$\gamma_t^{st}\sqrt{1+Ta} = -\left(\sqrt{1+Ta}+1\right)\left(2-\sqrt{1+Ta}\right) \tag{A4.1.105}$$

$$\gamma_t^{st} = \sqrt{1+Ta} - \frac{2}{\sqrt{1+Ta}} - 1 \tag{A4.1.106}$$

By imposing the condition for γ_t^{st} of being real and positive, $\gamma_t^{st} > 0$, we obtain an equation in Ta

$$\sqrt{Ta+1} > 2 \tag{A4.1.107}$$

From which

$$Ta > 3 \tag{A4.1.108}$$

Appendix 4.2: Expansion around over-stable solutions

The coupled equations at the leading order for stream function and temperature are (4.2.3a) and (4.2.3b)

$$\left[\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right]^2 \nabla^2 \psi_1 + Ta \frac{\partial^2 \psi_1}{\partial z^2} + Ra_{cr}\left[\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right]\frac{\partial T_1}{\partial x} = 0 \qquad \text{(A4.2.1)}$$

$$\left[\chi \sigma_o \frac{\partial}{\partial \bar{t}} - \nabla^2\right] T_1 + \frac{\partial \psi_1}{\partial x} = 0 \qquad \text{(A4.2.2)}$$

The corresponding solutions (4.2.1) and (4.2.2) are

$$\psi_1 = 2i\left(A_1 e^{i\bar{t}} - A_1^* e^{-i\bar{t}}\right)\sin(kx)\sin \pi z$$

$$T_1 = 2\left(C_1 e^{i\bar{t}} + C_1^* e^{-i\bar{t}}\right)\cos(kx)\sin(\pi z)$$

Working by parts the two equations above we have the following segments that can be added up in the end.

Part 1

$$\left[\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right]^2 \psi_1 = 2i\left(\sigma_o^2 - 1\right)\left(k^2 + \pi^2\right)\left(A_1 e^{i\bar{t}} - A_1^* e^{-i\bar{t}}\right)\sin(kx)\sin(\pi z) +$$

$$4\sigma_o\left(k^2 + \pi^2\right)\left(A_1 e^{i\bar{t}} + A_1^* e^{-i\bar{t}}\right)\sin(kx)\sin(\pi z) \qquad \text{(A4.2.3)}$$

Part 2

$$Ta\frac{\partial^2\psi_1}{\partial z^2} = 2i(1-\alpha)(k^2+\pi^2)(A_1e^{i\bar{t}} - A_1^*e^{-i\bar{t}})\sin(kx)\sin(\pi z) \tag{A4.2.4}$$

Part 3

$$Ra_{cr}\sigma_o\frac{\partial}{\partial\bar{t}}\frac{\partial T_1}{\partial x} = -2ik\sigma_o\pi^2(1+\alpha)^2\left(C_1e^{i\bar{t}} - C_1^*e^{-i\bar{t}}\right)\sin(kx)\sin(\pi z) \tag{A4.2.5}$$

Part 5

$$Ra_{cr}\frac{\partial T_1}{\partial x} = -2k\pi^2(1+\alpha)^2\left(C_1e^{i\bar{t}} + C_1^*e^{-i\bar{t}}\right)\sin(kx)\sin(\pi z) \tag{A4.2.6}$$

Part 6

$$\chi\sigma_o\frac{\partial T_1}{\partial\bar{t}} = 2i\chi\sigma_o\left(C_1e^{i\bar{t}} - C_1^*e^{-i\bar{t}}\right)\cos(kx)\sin(\pi z) \tag{A4.2.7}$$

Part 7

$$\frac{\partial\psi_1}{\partial x} = 2ik\left(A_1e^{i\bar{t}} - A_1^*e^{-i\bar{t}}\right)\cos(kx)\sin(\pi z) \tag{A4.2.8}$$

Part 8

$$\nabla^2T_1 = -2(k^2+\pi^2)\left(C_1e^{i\bar{t}} + C_1^*e^{-i\bar{t}}\right)\cos(kx)\sin(\pi z) \tag{A4.2.9}$$

Assembling the parts in terms of equations A4.2.1 and 2, we obtain

$$2i(\sigma_o^2 - 1)(k^2 + \pi^2)(A_1 e^{it} - A_1^* e^{-it}) + 4\sigma_o(k^2 + \pi^2)(A_1 e^{it} + A_1^* e^{-it}) +$$

$$2i(1-\alpha)(k^2 + \pi^2)(A_1 e^{it} - A_1^* e^{-it}) - 2ik\sigma_o \pi^2 (1+\alpha)^2 (C_1 e^{it} - C_1^* e^{-it}) - \qquad (A4.2.10)$$

$$2k\pi^2(1+\alpha)^2 (C_1 e^{it} + C_1^* e^{-it}) = 0$$

$$2i\sigma_o \chi(C_1 e^{it} - C_1^* e^{-it}) + 2\pi^2(k^2 + \pi^2)(C_1 e^{it} + C_1^* e^{-it}) +$$

$$2ik(A_1 e^{it} - A_1^* e^{-it}) = 0 \qquad (A4.2.11)$$

We further separate the terms according to the exponential power

$$(k^2 + \pi^2)(\sigma_o^2 - 2i\sigma_o - \alpha)A_1 = k(\sigma_o - i)Ra_{cr}C_1 \qquad (A4.2.12a)$$

$$(k^2 + \pi^2)(\sigma_o^2 + 2i\sigma_o - \alpha)A_1 = k(\sigma_o + i)Ra_{cr}C_1 \qquad (A4.2.12b)$$

$$[2i\sigma_o \chi + 2(k^2 + \pi^2)]C_1 + 2ikA_1 = 0 \qquad (A4.2.13)$$

$$[-2i\sigma_o \chi + 2(k^2 + \pi^2)]C_1^* - 2ikA_1^* = 0 \qquad (A4.2.13b)$$

From equations A4.2.11 and 12 we can draw an expression for Ra_{cr}, while the equations A4.2.13 and 14 will give us the relationships between coefficients ant order one.

$$C_1 = -\frac{\sqrt{\alpha}[\gamma\sigma_o + i(\alpha+1)]}{\pi[(\alpha+1)^2 + \gamma^2\sigma_o^2]}A_1 \quad \text{and} \quad C_1^* = -\frac{\sqrt{\alpha}[\gamma\sigma_o - i(\alpha+1)]}{\pi[(\alpha+1)^2 + \gamma^2\sigma_o^2]}A_1^* \qquad (A4.2.14)$$

In order to determine the solutions corresponding to each order we have to de-couple the equations 4.0.1 and 4.0.2

$$\left[\frac{\partial}{\partial \tilde{t}}+1\right]^2 \nabla^4\psi + Ta\frac{\partial^2\psi}{\partial z^2} + Ra_{cr}\left[\frac{\partial}{\partial \tilde{t}}+1\right]\frac{\partial T}{\partial x} = 0 \qquad \text{(A4.2.15a)}$$

$$\left[\chi\frac{\partial}{\partial \tilde{t}} - \nabla^2\right]T + \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} = 0 \qquad \text{(A4.2.15b)}$$

To achieve that we make the following helping assumption

$$\tilde{T} = T_0 + T = (1 - z) + T$$

$$\frac{\partial\tilde{T}}{\partial x} = \frac{\partial T}{\partial x} \qquad \frac{\partial\tilde{T}}{\partial\tilde{t}} = \frac{\partial T}{\partial\tilde{t}} \qquad \nabla^2\tilde{T} = \nabla^2 T \qquad \frac{\partial\tilde{T}}{\partial z} = -1 + \frac{\partial T}{\partial z} \qquad \text{(A4.2.16a)}$$

We recall that

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X} \qquad \frac{\partial}{\partial\tilde{t}} \to \frac{\partial}{\partial\tilde{t}} + \varepsilon\frac{\partial}{\partial\tau_o} + \varepsilon^2\frac{\partial}{\partial\tau} \qquad \text{(A4.2.16b)}$$

$$A = A(\tau_o, \tau, X) \qquad B = B(\tau_o, \tau, X) \qquad \text{(A4.2.16c)}$$

After we replaced the new variables, equations A4.2.15 a and b will then appear in the form

$$\left[\frac{\partial}{\partial \tilde{t}}+1\right]^2 \nabla^4\psi + Ta\frac{\partial^2\psi}{\partial z^2} + Ra_{cr}\left[\frac{\partial}{\partial \tilde{t}}+1\right]\frac{\partial T}{\partial x} = 0 \qquad \text{(A4.2.17)}$$

$$\left[\chi\frac{\partial}{\partial \tilde{t}} - \nabla^2\right]T + \frac{\partial\psi}{\partial x} = \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial\psi}{\partial z}\frac{\partial T}{\partial x} \qquad \text{(A4.2.18)}$$

Let $\frac{\partial \psi}{\partial x}\frac{\partial T}{\partial z} - \frac{\partial \psi}{\partial z}\frac{\partial T}{\partial x} = J(\psi, T)$. We have

$$\begin{bmatrix} \left[\frac{\partial}{\partial \bar{t}} + 1\right]^2 \nabla^2 + Ta\frac{\partial^2}{\partial z^2} & Ra_{cr}\left[\frac{\partial}{\partial \bar{t}} + 1\right]\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \chi\frac{\partial}{\partial t} - \nabla^2 \end{bmatrix} \begin{Bmatrix} \psi \\ T \end{Bmatrix} = \begin{Bmatrix} 0 \\ J \end{Bmatrix} \tag{A4.2.19}$$

This vectorial equation can be solved using Cramer's method.

$$\Delta = \begin{vmatrix} \left[\frac{\partial}{\partial \bar{t}} + 1\right]^2 \nabla^2 + Ta\frac{\partial^2}{\partial z^2} & Ra_{cr}\left[\frac{\partial}{\partial \bar{t}} + 1\right]\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \chi\frac{\partial}{\partial t} - \nabla^2 \end{vmatrix} = \tag{A4.2.20}$$

$$\left\{\left[\frac{\partial}{\partial \bar{t}} + 1\right]^2 \nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right\}\left\{\chi\frac{\partial}{\partial t} - \nabla^2\right\} - \frac{\partial}{\partial x} Ra_{cr}\left[\frac{\partial}{\partial \bar{t}} + 1\right]\frac{\partial}{\partial x}$$

$$\Delta\psi = \Delta_{\psi}$$

$$\Delta_{\psi} = \begin{vmatrix} 0 & Ra\left(\frac{\partial}{\partial t} + 1\right)\frac{\partial}{\partial x} \\ J & \chi\frac{\partial}{\partial t} - \nabla^2 \end{vmatrix} = Ra\left(\frac{\partial}{\partial t} + 1\right)\frac{\partial}{\partial x} J \tag{A4.2.22}$$

$$\Delta T = \Delta_T$$

$$\Delta_T = \begin{vmatrix} \left(\frac{\partial}{\partial t} + 1\right)^2 \nabla^2 + Ta\frac{\partial^2}{\partial z^2} & 0 \\ \frac{\partial}{\partial x} & J \end{vmatrix} = \left\{\left(\frac{\partial}{\partial t} + 1\right)^2 \nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right\} J \tag{A4.2.23}$$

The analysis for J will give us a non-linear string of terms that will be responsible for the shape of the solution at order $O(\varepsilon^2)$ that in turn will affect the solution at order $O(\varepsilon^3)$
It will be seen that the weight of non-linearity at order $O(\varepsilon)$ is inexistent.

$$J = \frac{\partial \psi}{\partial x}\frac{T}{\partial z} - \frac{\partial \psi}{\partial z}\frac{T}{\partial x} =$$

$$\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{X}\right)\{\varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3\}\frac{\partial}{\partial z}\{\varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3\} -$$

$$\frac{\partial}{\partial z}\{\varepsilon\psi_1 + \varepsilon^2\psi_2 + \varepsilon^3\psi_3\}\left(\frac{\partial}{\partial x} + \varepsilon\frac{\partial}{X}\right)\{\varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3\} = \tag{A4.2.24}$$

$$\varepsilon^2 J_2 + \varepsilon^3 J_3$$

Where

$$J_2 = \frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x} \tag{A4.2.25}$$

$$J_3 = \frac{\partial \psi_1}{\partial x}\frac{\partial T_2}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_2}{\partial x} + \frac{\partial \psi_2}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_2}{\partial z}\frac{\partial T_1}{\partial x} + \frac{\partial \psi_1}{\partial X}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial X} \tag{A4.2.26}$$

The de-coupled equation for ψ will be

$$\left\{\left[\chi\frac{\partial}{\partial t} + \nabla^2\right]\left[\left(\frac{\partial}{\partial t} + 1\right)^2\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right] - Ra\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial t} + 1\right)\right\}\psi = Ra\left(\frac{\partial}{\partial t} + 1\right)\frac{\partial}{\partial x}J \tag{A4.2.27}$$

The de-coupled equation for T will be

$$\left\{\left[\chi\frac{\partial}{\partial t} + \nabla^2\right]\left[\left(\frac{\partial}{\partial t} + 1\right)^2\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right] - Ra\frac{\partial^2}{\partial x^2}\left(\frac{\partial}{\partial t} + 1\right)\right\}T = \left\{\left(\frac{\partial}{\partial t} + 1\right)^2\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right\}J \tag{A4.2.28}$$

The homogeneous parts of both equations are identical, showing that the associated homogeneous solutions will have similar forms. However, the particular solutions will differ according to each order.

Analysis of equation A4.2.27

By replacing the slow scales for time and space, we have

$$\left\{\left[\chi\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial \tau_o}+\varepsilon^2\frac{\partial}{\partial \tau}\right)-\nabla^2-2\varepsilon\frac{\partial}{\partial x}\frac{\partial}{\partial X}-\varepsilon^2\frac{\partial^2}{\partial X^2}\right]\left[\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial \tau_o}+\varepsilon^2\frac{\partial}{\partial \tau}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-\right.$$

$$\left.Ra_\sigma(1+\varepsilon^2)\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial \tau_o}+\varepsilon^2\frac{\partial}{\partial \tau}+1\right)\left(\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X}\right)^2\right\}(\varepsilon\psi_1+\varepsilon^2\psi_2+\varepsilon^3\psi_3)= \tag{A4.2.29}$$

$$-Ra_\sigma(1+\varepsilon^2)\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial \tau_o}+\varepsilon^2\frac{\partial}{\partial \tau}+1\right)\left(\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X}\right)(\varepsilon^2 J_2+\varepsilon^3 J_3)$$

This algebraic equation is very tedious to solve; however we shall proceed to solve it by parts.

Part 1

$$\chi\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial \tau_o}+\varepsilon^2\frac{\partial}{\partial \tau}+1\right)-\nabla^2-2\varepsilon\frac{\partial}{\partial X}\frac{\partial}{\partial x}-\varepsilon^2\frac{\partial^2}{\partial X^2}=$$

$$\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)+\varepsilon\left(\chi\frac{\partial}{\partial \tau_0}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)+\varepsilon^2\left(\chi\frac{\partial}{\partial \tau}-\frac{\partial^2}{\partial X^2}\right) \tag{A4.2.30}$$

Part 2

$$\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial\tau_o}+\varepsilon^2\frac{\partial}{\partial\tau}+1\right)^2\left(\nabla^2+2\varepsilon\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\varepsilon^2\frac{\partial^2}{\partial X^2}\right)+Ta\frac{\partial^2}{\partial z^2}=$$

$$\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}+$$

$$\varepsilon\left[2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}+2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2\right]+ \qquad \text{(A4.2.31)}$$

$$\varepsilon^2\left[\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial^2}{\partial X^2}+4\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\left(2\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial t}+1\right)+\frac{\partial^2}{\partial\tau_o^2}\right)\nabla^2\right]+$$

$$\varepsilon^3\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}+2\left(2\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial t}+1\right)+\frac{\partial^2}{\partial\tau_o^2}\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}+2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial\tau}\nabla^2\right]$$

$$Part\;\;3=Part\;\;1\times Part\;\;2\times\left(\varepsilon\psi_1+\varepsilon^2\psi_2+\varepsilon^3\psi_3\right)$$

$$\varepsilon\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\psi_1+$$

$$\varepsilon^2\left[\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}+2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2\right)\psi_1+\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\psi_1+\right. \qquad \text{(A4.2.32)}$$

$$\left.\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\psi_2\right]+$$

$$\varepsilon^3\left[\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial^2}{\partial X^2}+4\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\left(\frac{\partial^2}{\partial\tau_o^2}+2\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial t}+1\right)\right)\nabla^2\right)\psi_1+\right.$$

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}+2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2\right)\psi_1+\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X}\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\psi_1+$$

$$\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}+2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2\right)\psi_2+\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\psi_2+$$

$$\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\psi_3$$

Part 4

$$Ra_{\sigma}\left(\varepsilon^{2}+1\right)\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial\tau_{o}}+\varepsilon^{2}\frac{\partial}{\partial\tau}+1\right)\left(\frac{\partial^{2}}{\partial x^{2}}+2\varepsilon\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\varepsilon^{2}\frac{\partial^{2}}{\partial X^{2}}\right)\left(\varepsilon\psi_{1}+\varepsilon^{2}\psi_{2}+\varepsilon^{3}\psi_{3}\right)=$$

$$\varepsilon Ra_{\sigma}\frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial}{\partial t}+1\right)\psi_{1}+$$

$$\varepsilon^{2}Ra_{\sigma}\left[\frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial}{\partial t}+1\right)\psi_{2}+2\frac{\partial}{\partial X}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+1\right)\psi_{1}+\frac{\partial}{\partial\tau_{o}}\frac{\partial}{\partial x^{2}}\psi_{1}\right]+ \tag{A4.2.32}$$

$$\varepsilon^{3}\left[\frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial}{\partial t}+1\right)\psi_{3}+2\frac{\partial}{\partial X}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+1\right)\psi_{2}+\frac{\partial}{\partial\tau_{o}}\frac{\partial}{\partial x^{2}}\psi_{2}+\frac{\partial^{2}}{\partial x^{2}}\left(\frac{\partial}{\partial t}+1\right)\psi_{1}+\frac{\partial^{2}}{\partial X^{2}}\left(\frac{\partial}{\partial t}+1\right)\psi_{1}+\right.$$

$$\left.2\frac{\partial}{\partial\tau_{o}}\frac{\partial}{\partial X}\frac{\partial}{\partial x}\psi_{1}+\frac{\partial}{\partial\tau}\frac{\partial^{2}}{\partial x^{2}}\psi\right]$$

The right hand side of the equation A4.2.27

$$-Ra_{\sigma}\left(1+\varepsilon^{2}\right)\left(\frac{\partial}{\partial t}+\varepsilon\frac{\partial}{\partial\tau_{o}}+\varepsilon^{2}\frac{\partial}{\partial\tau}+1\right)\left(\frac{\partial}{\partial x}+\varepsilon\frac{\partial}{\partial X}\right)\left(\varepsilon^{2}J_{2}+\varepsilon^{3}J_{3}\right)=$$

$$-\varepsilon^{2}Ra_{\sigma}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+1\right)J_{2}- \tag{A4.2.330}$$

$$\varepsilon^{3}Ra\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial X}J_{2}+\frac{\partial}{\partial\tau_{o}}\frac{\partial}{\partial x}J_{2}+\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}J_{3}\right]$$

We can write the expressions of the de-coupled stream function equations for the corresponding orders

Order $O(\varepsilon)$

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^{2}\right)\left[\left(\frac{\partial}{\partial t}+1\right)^{2}\nabla^{2}+Ta\frac{\partial^{2}}{\partial z^{2}}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^{3}}{\partial x^{2}}\right\}\psi_{1}=0 \tag{A4.2.31}$$

We see that in the RHS expression of A4.2.27 there is no first order of ε

Order $O(\varepsilon^{2})$

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_\sigma\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_2=$$

$$-2\frac{\partial}{\partial\tau_o}\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}+1\right)\nabla^2\psi_1-2\frac{\partial}{\partial x}\frac{\partial}{X}\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}+1\right)^2\psi_1- \tag{A4.2.32}$$

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1-\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1+$$

$$Ra_\sigma 2\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{X}\psi_1+Ra_\sigma\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial x^2}\psi_1-Ra_\sigma\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]$$

The order $O(\varepsilon^2)$ would hold information about dependency of the still undetermined amplitudes and various variables

Order $O(\varepsilon^3)$

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{\sigma}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3=$$

$$-\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_2-$$

$$\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial^2}{\partial X^2}+4\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{X}+\left(\frac{\partial^2}{\partial\tau_o^2}+2\frac{\partial}{\partial\tau}\left(\frac{\partial}{\partial t}+1\right)\right)\nabla^2\right]\psi_2- \qquad (A4.2.33)$$

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_2-\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_1-$$

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)Ta\frac{\partial^2}{\partial z^2}\psi_2-\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1-\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1+$$

$$Ra_{\sigma}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\psi_1+2\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_2+\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial x^2}\psi_2+\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}\psi_1+2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1+\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\psi_1\right]-$$

$$Ra_{\sigma}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial X}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]-$$

$$Ra_{\sigma}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}+\frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z}+\frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X}-\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x}\right]$$

The homogeneous equation for T has the same form as for ψ, only the RHS will differ for order $O(\varepsilon^2)$ and $O(\varepsilon^3)$. We called the RHS as J

$$J=\left(\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)\left(\varepsilon^2 J_2+\varepsilon^3 J_3\right) \qquad (A4.2.34)$$

By introducing the slow scales for time and space we get an expression for J in terms of powers of ε.

$$J=\varepsilon^2\left\{\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2 J_2+Ta\frac{\partial^2}{\partial z^2}J_2\right\}+$$

$$\varepsilon^3\left\{\left(\frac{\partial}{\partial t}+1\right)^2\nabla J_3+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}J_2+2\frac{\partial}{\partial \tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2 J_2+Ta\frac{\partial^2}{\partial z^2}J_3\right\} \tag{A4.2.35}$$

The calculation process is identical to that for ψ and we shall resume to write only the results

Order $O(\varepsilon)$

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}T_1=0 \tag{A4.2.36}$$

Order $O(\varepsilon^2)$

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}T_2=$$

$$\left\{-2\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial t}+1\right)\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\nabla^2-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\left(\frac{\partial}{\partial t}+1\right)^2\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)-\right.$$

$$\left(\frac{\partial}{\partial t}+1\right)^2\left(\chi\frac{\partial}{\partial \tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\nabla^2-\left(\chi\frac{\partial}{\partial \tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)Ta\frac{\partial^2}{\partial z^2}+ \tag{A4.2.37}$$

$$\left.Ra_{cr}2\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}+Ra_{cr}\frac{\partial}{\partial \tau_o}\frac{\partial^2}{\partial x^2}\right\}T_1+\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2 J_2+Ta\frac{\partial^2}{\partial z^2}J_2$$

Order $O(\varepsilon^3)$

$$
\begin{aligned}
&\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3=\\
&-\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[2\frac{\partial}{\partial \tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_2-\\
&\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial^2}{\partial X^2}+4\frac{\partial}{\partial \tau_o}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{X}+\left(\frac{\partial^2}{\partial \tau_o^2}+2\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial t}+1\right)\right)\nabla^2\right]\psi_1-\\
&\left(\chi\frac{\partial}{\partial \tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_2-\left(\chi\frac{\partial}{\partial \tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial \tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_1-\\
&\left(\chi\frac{\partial}{\partial \tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)Ta\frac{\partial^2}{\partial z^2}\psi_2-\left(\chi\frac{\partial}{\partial \tau}-\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1-\left(\chi\frac{\partial}{\partial \tau}-\frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1+\\
&Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\psi_1+2\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_2+\frac{\partial}{\partial \tau_o}\frac{\partial^2}{\partial x^2}\psi_2+\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}\psi_1+2\frac{\partial}{\partial \tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1+\frac{\partial}{\partial \tau}\frac{\partial^2}{\partial x^2}\psi_1\right]-\\
&+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}J_2+2\frac{\partial}{\partial \tau_c}\left(\frac{\partial}{\partial t}+1\right)\nabla^2 J_2+\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2 J_3+Ta\frac{\partial^2}{\partial z^2}J_3
\end{aligned}
\tag{A4.2.38}
$$

The solutions at this order are

$$
\begin{aligned}
\psi_2 &= \psi_2^h+\psi_2^p\\
T_2 &= T_2^h+T_2^p
\end{aligned}
\tag{A4.2.39}
$$

Because the homogeneous part at order two is identical to that at order three, the homogeneous solutions will be

$$
\begin{aligned}
\psi_2^h &= 2i\left(A_2e^{it}-A_2^*e^{-it}\right)\sin(kx)\sin(\pi z)\\
T_2^h &= 2\left(C_2e^{it}+C_2e^{-it}\right)\cos(kx)\sin(\pi z)
\end{aligned}
\tag{A4.2.40}
$$

By introducing the solutions into the secular homogeneous equation we will obtain similar expressions between amplitudes and also a set of relationships between Ra_{cr} and the amplitudes. The relationships between coefficients at order two will preserve as well and they can be presented as

$$C_2 = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}A_2$$

$$C_2^* = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}A_2^* \tag{A4.2.41}$$

The analysis of the stream function at $O(\varepsilon^2)$ will give us information about the relationship established between time and space slow scales related to the amplitudes. For that we have to investigate the RHS of A4.2.32. The analysis of the non-linear terms follows by introducing the solutions from the previous order and performing some algebraic manipulations. The result will indicate that the solution for stream function at this order is not influenced by non-linearity induced. However this is not true for temperature which will appear to be strongly influenced by perturbations introduced at previous order.

$$Rc_\sigma \frac{\partial}{\partial x}\left(\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right)\left[\frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right] =$$

$$Rc_\sigma \frac{\partial}{\partial x}\left(\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right)\left(\frac{\partial}{\partial x}2i\left(A_1e^{t} - A_1^*e^{-it}\right)\sin(kx)\sin(\pi z)\frac{\partial}{\partial z}2\left(C_1e^{it} + C_1^*e^{-it}\right)\cos(kx)\sin(\pi z) -\right.$$

$$\left.\frac{\partial}{\partial z}2i\left(A_1e^{t} - A_1^*e^{-it}\right)\sin(kx)\sin(\pi z)\frac{\partial}{\partial x}2\left(C_1e^{it} + C_1^*e^{-it}\right)\cos(kx)\sin(\pi z)\right) = \tag{A4.2.42}$$

$$Rc_\sigma \frac{\partial}{\partial x}\left(\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right)\left[2ik\pi\left(A_1C_1e^{2it} + A_1C_1^* - A_1^*C_1 - A_1^*C_1^*e^{-2it}\right)\cos^2(kx)\sin(2\pi z) +\right.$$

$$\left.2ik\pi\left(A_1C_1e^{2it} + A_1C_1^* - A_1^*C_1 - A_1^*C_1^*e^{-2it}\right)\sin^2(kx)\sin(2\pi z)\right] =$$

$$Rc_\sigma \frac{\partial}{\partial x}\left(\sigma_o \frac{\partial}{\partial \bar{t}} + 1\right)\left[A_1C_1e^{2it} + A_1C_1^* - A_1^*C_1 - A_1^*C_1^*e^{-2it}\right]\sin(2\pi z) = 0$$

The remaining part of $RHS(\psi_1)$ of A4.2.32 will be regarded as a differential operator operating upon ψ_1. The result must be forced to zero in order to obtain the required relationship

$$RHS = -2\sigma_o^2\chi\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial t^2}\nabla^2 - 2\sigma_o\chi\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial t}\nabla^2 + 2\sigma_o\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial t}\nabla^4 + 2\frac{\partial}{\partial\tau_o}\nabla^4 -$$

$$2\chi\sigma_o^2\frac{\partial^3}{\partial t^3}\frac{\partial}{\partial x}\frac{\partial}{\partial X} - 4\chi\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x}\frac{\partial}{\partial X} - 2\chi\sigma_o\frac{\partial}{\partial x}\frac{\partial}{\partial t}\frac{\partial}{\partial X} + 2\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2 +$$

$$4\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2 + 2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2 - \chi\sigma_o^2\frac{\partial}{\partial\sigma_o}\frac{\partial^2}{\partial t^2}\nabla^2 - 2\chi\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial\tau_o}\nabla^2 - \qquad \text{(A4.2.43)}$$

$$\chi\frac{\partial}{\partial\tau_o}\nabla^2 + 2\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2 + 4\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2 + 2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2 - Ta\chi\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial z^2} +$$

$$2Ta\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial^2}{\partial z^2} + 2Ra_{cr}\frac{\partial}{\partial t}\frac{\partial}{\partial x}\frac{\partial}{\partial X} + 2Ra_{cr}\frac{\partial}{\partial x}\frac{\partial}{\partial X} + Ra_{cr}\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial x^2}$$

Applying the operator A4.2.43 to "$e^{\sigma t}$" part of the solution and equating it to zero, we obtain a relationship between $\partial A_1/\partial\tau_o$ and $\partial A_1/\partial X$

$$P\frac{\partial A_1}{\partial\tau_o}\sin(kx) = Q\frac{\partial A_1}{\partial X}\cos(kx) \qquad \text{(A4.2.44)}$$

Where

$$P = \pi^2\left[2\sigma_o(1+\alpha)^2 + 4\sigma_o\gamma(1+\alpha) + i\left(3\sigma_o^2\gamma(1+\alpha) - 2(1+\alpha)^2 - \gamma(1+\alpha) - \gamma Ta + \alpha Ra_{cr}\right)\right]$$

(A4.2.45)

$$Q = 2k\left[\gamma\sigma_o - \gamma\sigma_o^3 + 4\sigma_o(1+\alpha) - \sigma_o Ra_{cr} + i\left(2\gamma\sigma_o^2 + 2\sigma_o^2(1+\alpha) - 2(1+\alpha) - Ta + Ra_{cr}\right)\right]$$

(A4.2.46)

We can see that there is a correspondence between the wave number and α on one hand and the ratio between slow space scale and slow time scale.

$$f(\alpha)\tan(kx) = \frac{\partial X}{\partial \tau_o} \qquad \text{(A4.2.48)}$$

We can call the expression above the equivalent of a slow velocity scale. $f(\alpha)$ is a function not only of α, but also of Ra_{cr} and Ta, implicitly will be dependent of the rotation of the layer.

For the e^{-it} part we obtain a similar expression

$$\overline{P}\frac{\partial A_1}{\partial \tau_o}\sin(kx) = \overline{Q}\frac{\partial A_1}{\partial X}\cos(kx) \qquad \text{(A4.2.47)}$$

Where

$$\overline{P} = \pi^2\left[2\sigma_o(1+\alpha)^2 + 4\sigma_o\gamma(1+\alpha) - i\left(3\sigma_o^2\gamma(1+\alpha) - 2(1+\alpha)^2 - \gamma(1+\alpha) - \gamma Ta + \alpha Ra_{cr}\right)\right]$$

(A4.2.48)

$$\overline{Q} = 2k\left[\gamma\sigma_o - \gamma\sigma_o^3 + 4\sigma_o(1+\alpha) - \sigma_o Ra_{cr} - i\left(2\gamma\sigma_o^2 + 2\sigma_o^2(1+\alpha) - 2(1+\alpha) - Ta + Ra_{cr}\right)\right]$$

(A4.2.49)

As for the T equation, the linear terms are the same as in ψ equation and they can be forced to zero, remaining to analyse the non-linear part of T, which differs from the stream function equation

$$RHS = \left(\sigma_o\frac{\partial}{\partial t} + 1\right)^2 \nabla^2 J_2 + Ta\frac{\partial^2}{\partial z^2}J_2 \qquad \text{(A4.2.50)}$$

We found that J_2 is a function of t and z only

$$RHS = \left(\sigma_o^2 \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2} + 2\sigma_o \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2} + (1+Ta)\frac{\partial^2}{\partial z^2} \right) J_2 \qquad \text{(A4.2.51)}$$

$$RHS = 32 i \sigma_o^2 k \pi^3 \left(A_1 C_1 e^{2it} - A_1^* C_1^* e^{-2it} \right) \sin(2\pi z) +$$

$$32 \sigma_o k \pi^3 \left(A_1 C_1 e^{2it} + A_1^* C_1^* e^{-2it} \right) \sin(2\pi z) - \qquad \text{(A4.2.52)}$$

$$(1+Ta) 8 i k \pi^3 \left(A_1 C_1 e^{2it} + A_1 C_1^* - A_1^* C_1 - A_1^* C_1^* e^{-2it} \right) \sin(2\pi z)$$

In the end we can write

$$RHS = \left[b_2' + a_1' e^{2it} + a_1'^* e^{-2it} \right] \sin(2\pi z) \qquad \text{(A4.2.53)}$$

Where

$$b_2' = -8 i k \pi^3 (1+Ta)\left(A_1 C_1^* - A_1^* C_1 \right)$$

$$a_1' = \left(32 k \sigma_o \pi^3 + i\left(32 k \sigma_o^2 \pi^3 - 8 k \pi^3 (1+Ta) \right) \right) A_1 C_1 \qquad \text{(A4.2.54)}$$

$$a_1'^* = \left(32 k \sigma_o \pi^3 - i\left(32 k \sigma_o^2 \pi^3 - 8 k \pi^3 (1+Ta) \right) \right) A_1^* C_1^*$$

We ascertain that the particular solution of T_2 must be of the form

$$T_2^p = T_2^{p.1} + T_2^{p.2} + T_2^{p.3} = -b_2 \sin(2\pi z) + a_1 e^{2it} \sin(2\pi z) + a_1^* e^{-2it} \sin(2\pi z) \qquad \text{(A4.2.55)}$$

We proceed to analyse the structure of b_2

$$b_2' = -8ik\pi^3(1+Ta)\left(A_1C_1^* - A_1^*C_1\right) =$$

$$-8ik\pi^3\left(1+\alpha^2-1\right)\left(-\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1^*A_1 + -\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1A_1^*\right) = \quad (A4.2.56)$$

$$16k^2\pi^2\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1A_1^*$$

By introducing $T_2^{p.1} = b_2\sin(2\pi z)$ into A3.2.37

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}b_2\sin(2\pi z) = 16k^2\pi^2\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1A_1^*$$

(A4.2.57)

Knowing that b_2 is a constant we obtain the following equation

$$-\nabla^2\left(\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right)b_2\sin(2\pi z) = 16k^2\pi^2\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1A_1^* \quad (A4.2.58)$$

The expression for b_2 is

$$b_2 = -\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1A_1^* \quad (A4.2.59)$$

Introducing $T_2^{p.2} = a_1\sin(2\pi z)$ in equation A3.2.37

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}a_2e^{2it}\sin(2\pi z)$$

(A4.2.60)

$$-8k\pi^3\left[\sigma_o+4i\left(\sigma_o^2-\alpha^2\right)\right]\frac{\sqrt{\alpha}\left[\gamma\sigma_o+i(1+\alpha)\right]}{\pi\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]}A_1^2e^{2it}\sin(2\pi z)$$

It yields the following expression

$$\left[\left(64\sigma_o^2+32\gamma\sigma_o^2-16\alpha^2\right)-i\left(64\sigma_o-32\gamma\sigma_o^3+8\gamma\sigma_o\alpha^3\right)\right]\pi^4 a. -$$

(A4.2.61)

$$-\sqrt{\alpha}\frac{1}{\pi}\frac{\left[32\sigma_o+i\left(32\sigma_o^2-8\alpha^2\right)\right]k\pi^3\left[\gamma\sigma_o+i(1+\alpha)\right]}{\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}A_1^2$$

From which we can determine

$$a_1=\frac{\alpha}{\pi}\frac{2(\alpha+1)-\gamma^2\sigma_o^2-i\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}\left(A_1\right)^2 \qquad \text{(A4.2.62)}$$

Similarly we work out the expression for a_2^* by introducing $T_2^{p,3}=a_2^*e^{-2it}\sin(2\pi z)$ in the equation for T_2. It yields

$$a_2^*=\frac{\alpha}{\pi}\frac{2(\alpha+1)-\gamma^2\sigma_o^2+i\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}\left(A_1^*\right)^2 \qquad \text{(A4.2.63)}$$

The full equation ψ_3 at $O(\varepsilon)$ is presented below, where we retained from the right hand side of the equation only relevant terms linked to ψ_1

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_\sigma\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3=$$

$$-\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_1-$$

$$\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1-\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1+ \qquad (A4.2.64)$$

$$Ra_\sigma\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\psi_1+\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}\psi_1+2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1+\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\psi_1\right]-$$

$$Ra_\sigma\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial X}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]-$$

$$Ra_\sigma\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}+\frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z}+\frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X}-\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x}\right]$$

We notice that the right hand side consists of a string of linear terms and others non-linear in terms of ψ_1 and T_2. We shall proceed to analyse the linear terms of equation above by parts

Part 1

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\sigma_o\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\sigma_o\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{X}\right]=$$

$$2\sigma_o\chi\frac{\partial^2}{\partial\tau_o^2}\frac{\partial}{t}\nabla^2+2\chi\frac{\partial^2}{\partial\tau_o^2}\nabla^2+2\sigma_o^2\chi\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{X}+4\sigma_o\chi\frac{\partial}{\partial t}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{X}+ \qquad (A4.2.65)$$

$$2\chi\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{X}-4\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{X}\nabla^2-4\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{X}\nabla^2-4\sigma_o^2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial t^2}-$$

$$8\sigma_o\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}-4\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}$$

Part 2

$$\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)\left(\sigma_o\frac{\partial}{\partial t}+1\right)^2\nabla^2 =$$

$$\chi\sigma_o^2\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial t^2}\nabla^2+2\tau_o\chi\frac{\partial}{\partial\tau}\frac{\partial}{\partial t}\nabla^2+\chi\frac{\partial}{\partial\tau}\nabla^2-\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial X^2}\nabla^2- \tag{A4.2.66}$$

$$2\sigma_o\frac{\partial}{\partial t}\frac{\partial^2}{\partial X^2}\nabla^2-\frac{\partial^2}{\partial X^2}\nabla^2$$

Part 3

$$\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}=\chi Ta\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial z^2}-Ta\frac{\partial^2}{\partial X^2}\frac{\partial^2}{\partial z^2} \tag{A4.2.67}$$

Part 4

$$Ra\left[\left(\sigma_o\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}+2\frac{\partial}{\partial\sigma_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\frac{\partial^2}{\partial X^2}\left(\sigma_o\frac{\partial}{\partial t}+1\right)+\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\right]=$$

$$Ra_{cr}\sigma_o\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}+Ra_{cr}\frac{\partial^2}{\partial x^2}+2Ra_{cr}\frac{\partial}{\partial\sigma_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}+Ra_{cr}\sigma_o\frac{\partial}{\partial t}\frac{\partial^2}{\partial X^2}+ \tag{A4.2.68}$$

$$Ra_{cr}\frac{\partial^2}{\partial X^2}+Ra_{cr}\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}$$

As for the non-linear part, we have to split the terms in parts as well.

Parts 5, 6, 7 and 8 will deal with the non-linear terms containing mixed products of stream function and temperature.

Part 5

$$Ra_{cr}\left[\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial x}+\frac{\partial}{\partial x}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]=$$

$$2ik\pi Ra_{cr}\left[\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial x}+\frac{\partial}{\partial x}+\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]\times \tag{A4.2.69}$$

$$\left\{A_1C_1e^{2if}+A_1C_1-A_1^*C_1-A_1^*C_1^*e^{2if}\right\}\sin(2\pi z)=0$$

The result yields zero because the argument of the differential operator contains no x or t terms

Part 6

$$\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x}=\frac{\partial\psi_1}{\partial x}\frac{\partial}{\partial z}\left(T_2^h+T_2^p\right)-\frac{\partial\psi_1}{\partial z}\frac{\partial}{\partial x}\left(T_2^h+T_2^p\right)=$$

$$\frac{\partial\psi_1}{\partial x}\frac{\partial T_2^h}{\partial z}+\frac{\partial\psi_1}{\partial x}\frac{\partial T_2^p}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2^h}{\partial x}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2^p}{\partial x} \tag{A4.2.70}$$

We shall consider only the relevant terms in which appear ψ_1 and T_1. They are those where the particular solution for temperature appears explicitly.

$$\frac{\partial\psi_1}{\partial x}\frac{\partial T_2^p}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2^p}{\partial x}=\frac{\partial}{\partial x}\left[2i\left(A_1e^{it}-A_1^*e^{-it}\right)\sin(kx)\sin(\pi z)\right]\frac{\partial}{\partial z}\left[\left(b_2+a_1e^{2it}+a_1^*e^{-2it}\right)\sin(2\pi z)\right]-$$
$$\frac{\partial}{\partial z}\left[2i\left(A_1e^{it}-A_1^*e^{-it}\right)\sin(kx)\sin(\pi z)\right]\frac{\partial}{\partial x}\left[\left(b_2+a_1e^{2it}+a_1^*e^{-2it}\right)\sin(2\pi z)\right] \tag{A4.2.71}$$

We see that the second term where temperature appears is not a function of x, therefore it will vanish and the result is shown

$$\frac{\partial\psi_1}{\partial x}\frac{\partial T_2^p}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2^p}{\partial x}=4ik\pi\left(A_1e^{it}-A_1^*e^{-it}\right)\left(b_2+a_1e^{2it}+a_1^*e^{-2it}\right)\cos(kx)\sin(\pi z)\cos(2\pi z)=$$
$$4ik\pi\left(A_1e^{it}-A_1^*e^{-it}\right)\left(b_2+a_1e^{2it}+a_1^*e^{-2it}\right)\cos(kx)\frac{1}{2}(\sin(3\pi z)-\sin(\pi z))= \tag{A4.2.72}$$
$$2ik\pi\left(A_1e^{it}-A_1^*e^{-it}\right)\left(b_2+a_1e^{2it}+a_1^*e^{-2it}\right)\cos(kx)\sin(3\pi z)-$$
$$2ik\pi\left(A_1e^{it}-A_1^*e^{-it}\right)\left(b_2+a_1e^{2it}+a_1^*e^{-2it}\right)\cos(kx)\sin(\pi z)$$

The term containing $\sin(3\pi z)$ can be neglected at this stage since it does not contain the basic solution. By performing the multiplication between brackets we obtain a string of terms that can be further neglected because they are different that the resonant ones. These terms might become significant if we proceed analysing orders higher than $O(\varepsilon^3)$ or relationships between amplitude functions at order higher that $O(\varepsilon^2)$.

$$Ra_{cr}\frac{\partial}{\partial x}\left(\sigma_o\frac{\partial}{\partial t}+1\right)\left(\frac{\partial\psi_1}{\partial x}\frac{\partial T_2^p}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2^p}{\partial x}\right)=$$

$$Ra_{cr}\frac{\partial}{\partial x}\left(\sigma_o\frac{\partial}{\partial t}+1\right)\left\{-2ik\pi\left[\left(A_1b_2-A_1^*a_1\right)e^{it}-\left(A_1^*b_2-A_1a_1^*\right)e^{-it}\right]\cos(kx)\sin(\pi z)\right\}= \qquad (A4.2.73)$$

$$-2k^2\pi(\sigma_o-i)\left(A_1b_2-A_1^*a_1\right)e^{it}\sin(kx)\sin(\pi z)-$$

$$2k^2\pi(\sigma_o+i)\left(A_1^*b_2-A_1a_1^*\right)e^{-it}\sin(kx)\sin(\pi z)$$

Part 7

$$Ra_{cr}\frac{\partial}{\partial x}\left(\sigma_o\frac{\partial}{\partial t}+1\right)\left\{\frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X}\right\}=$$

$$Ra_{cr}\frac{\partial}{\partial x}\left(\sigma_o\frac{\partial}{\partial t}+1\right)\left\{i\pi\left[\frac{\partial A_1}{\partial X}e^{it}-\frac{\partial A_1^*}{\partial X}e^{-it}\right]\left[C_1e^{it}+C_1^*e^{-it}\right]\sin(2kx)\sin(2\pi z)-\right. \qquad (A4.2.74)$$

$$\left.i\pi\left[A_1e^{it}+A_1^*e^{-it}\right]\left[\frac{\partial C_1}{\partial X}e^{it}-\frac{\partial C_1^*}{\partial X}e^{-it}\right]\sin(2kx)\sin(2\pi z)\right\}$$

This is a non-resonant term and will not be considered for further calculations.

Part 8

$$Ra_{cr}\frac{\partial}{\partial x}\left(\sigma_o\frac{\partial}{\partial t}+1\right)\left\{\frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x}\right\} \qquad (A4.2.75)$$

It contains A_2 coefficients, therefore not to be calculated.

The compilation of all the parts of the right hand side of equation A4.2.64 will be presented. We group the terms according to the power of the exponent $e^{i\tau}$ and to the argument that contains the amplitude. The expansion corresponding to $e^{-i\tau}$ is the complex conjugate of the first. For our analysis will be sufficient the $e^{i\tau}$ part

$$M_1\frac{\partial^2 A_1}{\partial \tau_o^2}\sin(kx)\sin(\pi z)+M_2\frac{\partial^2 A_1}{\partial \tau_o \partial X}\cos(kx)\sin(\pi z)+M_3\frac{\partial^2 A_1}{\partial X^2}\sin(kx)\sin(\pi z)+$$
(A4.2.76)
$$M_4\frac{\partial A_1}{\partial \tau}\sin(kx)\sin(\pi z)+M_5A_1\sin(kx)\sin(\pi z)+M_6\left(A_1b_2-A_1^*a_1\right)\sin(kx)\sin(\pi z)=0$$

Where

$$M_1=\left[4\pi^4\gamma(\alpha+1)+4i\pi^4\gamma\sigma_o(\alpha+1)\right] \tag{A4.2.77}$$

$$M_2=\left[8\pi^2\sigma_o k(\gamma+\alpha+1)+4ik\pi^2\left(\gamma\left(\sigma_o^2-1\right)-2(\alpha+1)+R_{cr}\right)\right] \tag{A4.2.78}$$

$$M_3=\left[\pi^2\left((10\alpha+2)\left(\sigma_o^2-1\right)-Ta+R_{cr}\right)-2i\sigma_o\pi^2\left(10\alpha+2-R_{cr}\right)\right] \tag{A4.2.79}$$

$$M_4=-4\pi^2\left[\frac{2\sigma_0 p(p+\gamma)+\alpha\gamma\sigma_0(p-\gamma)}{s\gamma}-i\frac{2p(p-\gamma\sigma_0^2)s-\alpha R_c p(p-\gamma)}{s\gamma}\right]$$

(A4.2.80)

where

$$p = \alpha + 1 \qquad (\alpha + 1)^2 + \gamma^2 \sigma_o^2 = s \tag{A4.2.81}$$

$$M_5 = 2\pi^4 \alpha R_{cr} (\sigma_o - i) \tag{A4.2.81}$$

$$M_6 = 2\pi^4 \alpha R_{cr} \pi (\sigma_o - i) \tag{A4.2.81}$$

Using A3.2.45 we can replace

$$M_2 \frac{\partial^2 A_1}{\partial \tau_o \partial X} \cos(kx)\sin(\pi z) = M_2 \frac{\partial A_1}{\partial \tau_o} \cos(kx) \frac{\partial A_1}{\partial X} \sin(\pi z) = M_2 \frac{P}{Q} \frac{\partial^2 A_1}{\partial \tau_o^2} \sin(kx)\sin(\pi z) \tag{A4.2.83}$$

EquationA4.2.76 can be written as

$$\begin{aligned} &M_1 \frac{\partial^2 A_1}{\partial \tau_o^2} \sin(kx)\sin(\pi z) + M_2 \frac{P}{Q} \frac{\partial^2 A_1}{\partial \tau_o^2} \sin(kx)\sin(\pi z) + M_3 \frac{\partial^2 A_1}{\partial X^2} \sin(kx)\sin(\pi z) + \\ &M_4 \frac{\partial A_1}{\partial \tau} \sin(kx)\sin(\pi z) + M_5 A_1 \sin(kx)\sin(\pi z) + M_6 \left(A_1 b_2 - A_1^* a_1\right)\sin(kx)\sin(\pi z) = 0 \end{aligned} \tag{A4.2.84}$$

Since we have all along the terms the mixed product $\sin(kx)\sin(\pi z)$ we can neglect it

$$M_1 \frac{\partial^2 A_1}{\partial \tau_o^2} + M_2 \frac{P}{Q} \frac{\partial^2 A_1}{\partial \tau_o^2} + M_3 \frac{\partial^2 A_1}{\partial X^2} + M_4 \frac{\partial A_1}{\partial \tau} + M_5 A_1 + M_6 \left(A_1 b_2 - A_1^* a_1\right) = 0 \tag{A4.2.85}$$

By setting to zero the diffusion part of the equation

$$\left(M_1 + M_2 \frac{P}{Q}\right)\frac{\partial^2 A_1}{\partial \tau_o^2} + M_3 \frac{\partial^2 A_1}{\partial X^2} = 0 \qquad \text{(A4.2.86)}$$

we remain with an equation of unknown amplitude of the convection at order $O(\varepsilon)$

$$-M_4 \frac{\partial A_1}{\partial \tau} + M_5 A_1 + M_6\left(A_1 b_2 - A_1^* a_1\right) \qquad \text{(A4.2.87)}$$

The quantity within the bracket can be analysed

$$A_1 b_2 - A_1^* a_1 = -\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]} A_1^2 A_1^* - \frac{\alpha\left[2(\alpha+1) - \gamma^2\sigma_o^2 - i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]\left(4 + \gamma^2\sigma_o^2\right)} A_1^2 A_1^*$$

$$-\frac{\alpha}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]}\left[(\alpha+1) + \frac{(2 - i\gamma\sigma_o)\left[(\alpha+1) - i\gamma\sigma_o\right]}{4 + \gamma^2\sigma_o^2}\right] A_1^2 A_1^* = -M_7 A_1^2 A_1^*$$

(A4.2.88)

Equation A4.2.87 can be written now

$$\frac{\partial A_1}{\partial \tau} + \frac{M_5}{M_4}\left(1 - \frac{M_6 M_7}{M_5} A_1 A_1^2\right) A_1 = 0 \qquad \text{(A4.2.89)}$$

We have to analyse A4.2.89.

$$\frac{M_6 M_7}{M_5} = \frac{\alpha\left[6(\alpha+1)+\gamma^2\sigma_o^2\alpha\right]}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)} - i\frac{\alpha\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)} = J_2 = z_{2R} + z_{2I}$$

(A4.2.90)

$$\frac{M_5}{M_4} = \frac{\pi^2\alpha R_{cr}(\sigma_o - i)s\gamma}{\left\{\left[2\sigma_o p(p+\gamma)s + \alpha\gamma\sigma_o R_{cr}(p-\gamma)\right] - i\left[2p\left(p-\gamma\sigma_o^2\right)s - \alpha p R_{cr}(p-\gamma)\right]\right\}}$$

(A4.2.91)

Let

$$\mathbf{P} = 2\sigma_o p(p+\gamma)s + \alpha\gamma\sigma_o R_{cr}(p-\gamma) \tag{A4.2.92}$$

$$\mathbf{Q} = 2p\left(p-\gamma\sigma_o^2\right)s - \alpha p R_{cr}(p-\gamma) \tag{A4.2.93}$$

$$q = \sigma_o^2\left[p(p+\gamma)s + \alpha\gamma\sigma_o R_{cr}(p-\gamma)\right]^2 + p^2\left[\alpha R_{cr}(p-\gamma) - 2\left(p-\gamma\sigma_o^2\right)s\right]^2 \tag{A4.2.94}$$

Then

$$\frac{M_5}{M_4} = \frac{\pi^2\alpha s\gamma R_{cr}(\sigma_o - i)(\mathbf{P}+i\mathbf{Q})}{q} \tag{A4.2.95}$$

$$\frac{M_5}{M_4} = \frac{\pi^2\alpha s\gamma R_{cr}(\sigma_o\mathbf{P}+\mathbf{Q})}{q} + i\frac{\pi^2\alpha s\gamma R_{cr}(\sigma_o\mathbf{Q}-\mathbf{P})}{q} = J_1 = z_{1r} + iz_{1i} \tag{A4.2.96}$$

$$\xi_{1r} = \frac{\pi^2 \alpha s \gamma R_{cr}}{q}\left[p(\sigma_o^2+1)(2sp+\alpha\gamma R_{cr}) - \alpha R_{cr} s\right] \tag{A4.2.97}$$

$$\frac{1}{\xi_{1r}} = \frac{q}{\pi^2 \alpha s \gamma R_{cr}\left[p(\sigma_o^2+1)(2sp+\alpha\gamma R_{cr}) - \alpha R_{cr} s\right]} \tag{A4.2.98}$$

$$\xi_{1i} = -\frac{\pi^2 \alpha s \gamma R_{cr} \sigma_o}{q}\left[2ps\gamma(\sigma_o^2+1) - \alpha R_{cr}(p^2+\gamma^2)\right]$$

Therefore equation A4.2.89 can be expressed as

$$\frac{\partial A_1}{\partial \tau} - J_1\left[1 - J_2 A_1 A_1^*\right]A_1 = 0 \tag{A4.2.99}$$

Multiplying the equation above with ε^3 and replacing the original scales, we have

$$\frac{1}{\varepsilon^2}\frac{\partial A_1}{\partial t} - J_1\left[1 - J_2 A_1 A_1^*\right]A_1 = 0 \qquad \Big| \, \varepsilon^3 \tag{A4.2.100}$$

$$\frac{\partial(\varepsilon A_1)}{\partial t} - J_1\left[\varepsilon^2 - J_2(\varepsilon A_1)(\varepsilon A_1^*)\right](\varepsilon A_1) = 0 \tag{A4.2.101}$$

By calling $A = \varepsilon A_1$ and $A^* = \varepsilon A_1^*$ we get a final form for the amplitude equation

$$\frac{\partial A}{\partial t} = J_1\left[\xi_{ov} - J_2 A A^*\right]A \tag{A4.2.102}$$

We need to separate the equation above into real and imaginary parts in order to extract the amplitude and the phase of the oscillatory motion. For that we have to write $A = re^{i\theta}$ and $A^* = re^{-i\theta}$, where r stands for the real amplitude and θ for the phase. By replacing these values into equation A4.2.95

$$\frac{\partial}{\partial t}\left[re^{i\theta}\right] = J_1\left[\xi_{ov} - J_2 r^2\right]re^{i\theta} \qquad (A4.2.103)$$

$$\frac{\partial r}{\partial t}e^{i\theta} + ir\frac{\partial \theta}{\partial t}e^{i\theta} = J_1\left[\xi_{ov} - J_2 r^2\right]re^{i\theta} \qquad (A4.2.104)$$

$$\frac{\partial r}{\partial t} + ir\frac{\partial \theta}{\partial t} = J_1\left[\xi_{ov} - J_2 r^2\right]r \qquad (A4.2.105)$$

In order to process to process this equation we need to express J_1 and J_2 as complex numbers $J_1 = z_{1R} + iz_{1I}$ and $J_2 = z_{2R} + iz_{2I}$.

$$\frac{\partial r}{\partial t} + ir\frac{\partial \theta}{\partial t} = \left(z_{1R} + iz_{1I}\right)\xi_{ov}r - \left(z_{1R} + iz_{1I}\right)\left(z_{2R} + iz_{2I}\right)r^3 \qquad (A4.2.106)$$

$$\frac{\partial r}{\partial t} + ir\frac{\partial \theta}{\partial t} = z_{1R}\xi_{ov}r + iz_{1I}\xi_{ov}r - \left(z_{1R}z_{2R} - z_{1I}z_{2I}\right)r^3 - i\left(z_{1I}z_{2R} + z_{1R}z_{2I}\right)r^3 \qquad (A4.2.107)$$

By separating the real from imaginary part we obtain two distinct equations, one for amplitude and one for the phase of the oscillatory motion

$$\begin{cases} \dfrac{\partial r}{\partial t} = \xi_{ov} z_{1R} r - (z_{1R} z_{2R} - z_{1I} z_{2I}) r^3 \\ \\ \dfrac{\partial \theta}{\partial t} = \xi_{ov} z_{1I} - (z_{1I} z_{2R} - z_{1R} z_{2I}) r^2 \end{cases} \qquad \text{(A4.2.108)}$$

It is convenient to re-arrange the terms above

$$\begin{cases} z_{1R}^{r} \dfrac{\partial r}{\partial t} = \left[\xi_{ov} - z_{12}^{R} r^2\right] r \\ \\ \dfrac{\partial \theta}{\partial t} = z_{1I} \xi_{ov} - z_{12}^{I} r^2 \end{cases} \qquad \text{(A4.2.109)}$$

Where

$$J_1 = \frac{1}{z_{1R}} \qquad J_2 = \frac{z_{1R} z_{2R} - z_{1I} z_{2I}}{z_{1R}} \qquad J_3 = z_{1I} z_{2R} + z_{1R} z_{2I} \qquad J_4 = z_{1I} \qquad \text{(A4.2.110)}$$

$$z_{1r} = \frac{\pi^2 \alpha s \gamma R_{cr}}{q} \left[p(\sigma_o^2 + 1)(2sp + \alpha \gamma R_{cr}) - \alpha R_{cr} s\right] \qquad \text{(A4.2.111)}$$

$$z_{1I} = -\frac{\pi^2 \alpha s \gamma R_{cr} \sigma_o}{q} \left[2ps\gamma(\sigma_o^2 + 1) - \alpha R_{cr}(p^2 + \gamma^2)\right] \qquad \text{(A4.2.112)}$$

$$z_{2R} = \frac{\alpha\left[6(\alpha + 1) + \gamma^2 \sigma_0^2 \alpha\right]}{\left[(\alpha + 1)^2 + \gamma^2 \sigma_0^2\right]\left(\gamma^2 \sigma_0^2 + 4\right)} \qquad \text{(A4.2.113)}$$

$$z_{2I} = -\frac{\alpha\gamma(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_0^2\right]\left(\gamma^2\sigma_0^2+4\right)} \tag{A4.2.114}$$

Appendix 5: Determination of the amplitude equation for the travelling waves case

5.1. Relationship between amplitude coefficients at order $O(\varepsilon)$

The stream function and temperature solutions can be written in the form

$$\psi_1 = \left[A_1 e^{i(kx+\bar{t})} + B_1 e^{i(kx-\bar{t})} + A_1^* e^{-i(kx+\bar{t})} + B_1^* e^{-i(kx-\bar{t})} \right] \sin(\pi z) \qquad \text{(A5.1)}$$

$$T_1 = \left[C_1 e^{i(kx+\bar{t})} + D_1 e^{i(kx-\bar{t})} + C_1^* e^{-i(kx+\bar{t})} + D_1^* e^{-i(kx-\bar{t})} \right] \sin(\pi z) \qquad \text{(A5.2)}$$

The equations at the leading order are

$$\left[\sigma_o \frac{\partial}{\partial \bar{t}} + 1 \right]^2 \nabla^2 \psi_1 + Ta \frac{\partial^2 \psi_1}{\partial z^2} + Ra_{cr} \left[\sigma_o \frac{\partial}{\partial \bar{t}} + 1 \right] \frac{\partial T_1}{\partial x} = 0 \qquad \text{(A5.3)}$$

$$\left[\chi \sigma_o \frac{\partial}{\partial \bar{t}} - \nabla^2 \right] T_1 + \frac{\partial \psi_1}{\partial x} = 0 \qquad \text{(A5.4)}$$

In order to determine the relationships between coefficients at the leading order, we introduce the solutions A4.1 and 2 into equations A4.3 and 4.

Working by parts each term of the first equation

Part 1

$$\sigma_o^2 \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\psi_1 = \sigma_o^2 k^2\left(A_1 e^{i(kx+t)} + B_1 e^{i(kx-t)} + A_1^* e^{-i(kx+t)} + B_1^* e^{-i(kx-t)}\right)\sin(\pi z)$$

(A5.5)

Part 2

$$2\sigma_o \frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2}\psi_1 = \sigma_o^2 \pi^2\left(A_1 e^{i(kx+t)} + B_1 e^{i(kx-t)} + A_1^* e^{-i(kx+t)} + B_1^* e^{-i(kx-t)}\right)\sin(\pi z)$$

(A5.6)

Part 3

$$2\sigma_o \frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\psi_1 = -2i\,\sigma_o k^2\left(A_1 e^{i(kx+t)} - B_1 e^{i(kx-t)} - A_1^* e^{-i(kx+t)} + B_1^* e^{-i(kx-t)}\right)\sin(\pi z)$$

(A5.7)

Part 4

$$2\sigma_o \frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}\psi_1 = -2i\sigma_o \pi^2\left(A_1 e^{i(kx+t)} - B_1 e^{i(kx-t)} - A_1^* e^{-i(kx+t)} + B_1^* e^{-i(kx-t)}\right)\sin(\pi z)$$

(A5.8)

Part 5

$$\left(\nabla^2 + \frac{\partial^2}{\partial z^2} Ta\right)\psi_1 = -k^2\left(A_1 e^{i(kx+t)} + B_1 e^{i(kx-t)} + A_1^* e^{-i(kx+t)} + B_1^* e^{-i(kx-t)}\right)\sin(\pi z) -$$

$$(1+Ta)\pi^2\left(A_1 e^{i(kx+t)} + B_1 e^{i(kx-t)} + A_1^* e^{-i(kx+t)} + B_1^* e^{-i(kx-t)}\right)\sin(\pi z) \tag{A5.9}$$

Part 6

$$Ra\sigma_o \frac{\partial}{\partial t}\frac{\partial}{\partial x} T_1 = -Ra\sigma_o k\left(C_1 e^{i(kx+t)} - D_1 e^{i(kx-t)} + C_1^* e^{-i(kx+t)} - D_1^* e^{-i(kx-t)}\right)\sin(\pi z) \tag{A5.10}$$

Part 7

$$Ra\frac{\partial}{\partial x} T_1 = -ikRa\left(C_1 e^{i(kx+t)} + D_1 e^{i(kx-t)} - C_1^* e^{-i(kx+t)} - D_1^* e^{-i(kx-t)}\right)\sin(\pi z) \tag{A5.11}$$

We shall group all the terms according to their exponent argument. For $e^{i(kx+t)}$ in the first equation:

$$\left[\sigma_o^2\left(k^2+\pi^2\right) - 2i\sigma_o\left(k^2+\pi^2\right) - \left(k^2+\pi^2\right) - \pi^2 Ta\right]A_1 - kRa\left(\sigma_o - i\right)C_1 = 0 \tag{A5.12}$$

For $e^{i(kx-t)}$ term

$$\left[\sigma_o^2\left(k^2+\pi^2\right)+2i\sigma_o\left(k^2+\pi^2\right)-\left(k^2+\pi^2\right)-\pi^2 Ta\right]B_1+kRa\left(\sigma_o+i\right)D_1=0 \qquad \text{(A5.13)}$$

For $e^{-i(kx+\bar{t})}$ term

$$\left[\sigma_o^2\left(k^2+\pi^2\right)+2i\sigma_o\left(k^2+\pi^2\right)-\left(k^2+\pi^2\right)-\pi^2 Ta\right]A^*-kRa\left(\sigma_o+i\right)C_1^*=0 \qquad \text{(A5.14)}$$

For $e^{-i(kx-\bar{t})}$ term

$$\left[\sigma_o^2\left(k^2+\pi^2\right)-2i\sigma_o\left(k^2+\pi^2\right)-\left(k^2+\pi^2\right)-\pi^2 Ta\right]B_1^*+kRa\left(\sigma_o-i\right)D_1^*=0 \qquad \text{(A5.15)}$$

Working for the second equation

Part 1

$$\chi\sigma_o\frac{\partial}{\partial t}T_1=i\chi\sigma_o\left[C_1e^{i(kx+\bar{t})}-D_1e^{i(kx-\bar{t})}-D_1^*e^{-i(kx+\bar{t})}+D_1^*e^{-i(kx-\bar{t})}\right]\sin(\pi z) \qquad \text{(A5.16)}$$

Part 2

$$\frac{\partial}{\partial x}\psi_1=ik\left[A_1e^{i(kx+\bar{t})}+B_1e^{i(kx-\bar{t})}-A_1^*e^{-i(kx+\bar{t})}-B_1^*e^{-i(kx-\bar{t})}\right]\sin(\pi z) \qquad \text{(A5.17)}$$

Part 3

$$-\nabla^2 T_1 = \left(k^2 + \pi^2\right)\left[C_1 e^{i(kx+t)} + D_1 e^{i(kx-t)} + D_1^{*} e^{-i(kx+t)} + D_1^{*} e^{-i(kx-t)}\right]\sin(\pi z) \qquad \text{(A5.18)}$$

Grouping the terms of the second equation according to the exponent argument

For $e^{i(kx+t)}$ term we have

$$\left[\left(k^2 + \pi^2\right) + i\chi\sigma_o\right]C_1 + ikA_1 = 0 \qquad \text{(A5.19)}$$

For $e^{i(kx-t)}$ term

$$\left[\left(k^2 + \pi^2\right) - i\chi\sigma_o\right]D_1 + ikB_1 = 0 \qquad \text{(A5.20)}$$

For $e^{-i(kx+t)}$ term

$$\left[\left(k^2 + \pi^2\right) - i\chi\sigma_o\right]C_1^{*} - ikA_1^{*} = 0 \qquad \text{(A5.21)}$$

For $e^{-i(kx-t)}$ term

$$\left[\left(k^2 + \pi^2\right) + i\chi\sigma_o\right]D_1^{*} - ikB_1^{*} = 0 \qquad \text{(A5.22)}$$

From the second equation we can draw the relationships between the coefficients, while from the first equation an expression for Ra can be recovered.

$$C_1 = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]} A_1 \tag{A5.23}$$

$$D_1 = -\frac{\sqrt{\alpha}\left[-\gamma\sigma_o + i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]} B_1 \tag{A5.24}$$

$$C_1^* = -\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]} A_1^* \tag{A5.25}$$

$$D_1^* = -\frac{\sqrt{\alpha}\left[-\gamma\sigma_o - i(\alpha+1)\right]}{\pi\left[(\alpha+1)^2 + \gamma^2\sigma_o^2\right]} B_1^* \tag{A5.26}$$

5.2. Order $O(\varepsilon^2)$ analysis

The de-coupled equation for ψ_2 can be written as:

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_2=$$

$$-2\frac{\partial}{\partial\tau_o}\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}+1\right)\nabla^2\psi_1-2\frac{\partial}{\partial x}\frac{\partial}{X}\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left(\frac{\partial}{\partial t}+1\right)^2\psi_1-$$

$$\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1-\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1+ \tag{A5.27}$$

$$Ra_{cr}2\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\frac{\partial}{X}\psi_1+Ra_{cr}\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial x^2}\psi_1-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]$$

We shall proceed to analyse first the non-linear term. We expect that the influence of this term at this order is zero, therefore its value for any ψ or T must be zero.

$$\left[\frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right]=$$

$$ik\pi\left(A_1e^{i(kx+t)}+B_1e^{i(kx-t)}-A_1^2e^{-i(kx+t)}-B_1^2e^{-i(kx-t)}\right)\sin(\pi z)\times$$

$$\left(C_1e^{i(kx+t)}+D_1e^{i(kx-t)}+C_1^2e^{-i(kx+t)}+D_1^2e^{-i(kx-t)}\right)\cos(\pi z)- \tag{A5.28}$$

$$ik\pi\left(A_1e^{i(kx+t)}+B_1e^{i(kx-t)}+A_1^2e^{-i(kx+t)}+B_1^2e^{-i(kx-t)}\right)\sin(\pi z)\times$$

$$\left(C_1e^{i(kx+t)}+D_1e^{i(kx-t)}-C_1^2e^{-i(kx+t)}-D_1^2e^{-i(kx-t)}\right)\cos(\pi z)$$

Simplifying, we obtain

$$\left[\frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right] =$$

$$ik\pi\left[\left(A_1 C_1^* - A_1^* C_1\right) + \left(A_1 D_1^* - B_1^* C_1\right)e^{2u} + \left(B_1 C_1^* - A_1^* D_1\right)e^{2u} + \left(B_1 D_1^* - B_1^* D_1\right)\right]\sin(2\pi z)$$

(A5.29)

Since the expression above is not a function of x

$$Ra\frac{\partial}{\partial x}\left(\sigma_o \frac{\partial}{\partial t} + 1\right)\left[\frac{\partial \psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial \psi_1}{\partial z}\frac{\partial T_1}{\partial x}\right] = 0 \qquad \text{(A5.30)}$$

The remaining right hand side of equation A4.27 can be expanded as one differential operator acting upon ψ_1. ψ_1, in turn is a function of amplitudes which are dependent of the slow time and space scales

$$RHS=\left\{-2\sigma_o^2\chi\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial t^2}\nabla^2-2\sigma_o\chi\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial t}\nabla^2+2\sigma_o\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial t}\nabla^4+2\frac{\partial}{\partial\tau_o}\nabla^4-\right.$$

$$2\sigma_o^3\chi\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial^3}{\partial t^3}-4\sigma_o^2\chi\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial^2}{\partial t^2}-2\sigma_o\chi\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial}{\partial t}+2\sigma_o^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial^2}{\partial t^2}\nabla^2+$$

$$4\sigma_o\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial}{\partial t}\nabla^2+2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2+2\sigma_o^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial^2}{\partial t^2}\nabla^2+4\sigma_o\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial}{\partial t}\nabla^2+$$

$$2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2-\chi\sigma_o^2\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial t^2}\nabla^2-2\chi\sigma_o\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial t}\nabla^2+2Ta\frac{\partial^2}{\partial z^2}\frac{\partial}{\partial x}\frac{\partial}{\partial X}-$$

$$\chi Ta\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial z^2}+2Ra_{cr}\sigma_o\frac{\partial}{\partial x}\frac{\partial}{\partial X}\frac{\partial}{\partial t}+2Ra_{cr}\frac{\partial}{\partial x}\frac{\partial}{\partial X}+$$

$$\left.Ra_{cr}\frac{\partial}{\partial\tau_o}\frac{\partial^2}{\partial t^2}\right\}\left[A_1e^{i(kx+t)}+B_1e^{i(kx-t)}+A_1^*e^{-i(kx+t)}+B_1^*e^{-i(kx-t)}\right]\sin(\pi z)$$

(A5.31)

The analysis of $e^{i(kx+t)}$ argument for the RHS of equation A4.27 will result in a split equation for A_1.

$$\frac{\partial A_1}{\partial X}\left[-4ik^3+4i\sigma_o^2k\pi^2+8\sigma_ok\pi^2+4i\sigma_o^2k^3+2\sigma_o\chi k+4i\sigma_o^2\chi k-\right.$$

$$\left.2Ra_{cr}\sigma_ok-4ik\pi^2-2\sigma_o^3\chi k-2iTak\pi^2+2ikRa_{cr}\right]\sin(\pi z)+$$

(A5.32)

$$\frac{\partial A_1}{\partial\tau_o}\left[2\pi^4-Ra_{cr}k^2+2i\sigma_o\pi^4+4i\sigma_o\chi k^2+4i\sigma_o\chi\pi^2+2i\sigma_ok^4+2k^4+\right.$$

$$\left.\pi^2\chi Ta+4i\sigma_ok^2\pi^2-\sigma_o^2\chi\pi^2-3\sigma_o^2\chi k^2+\chi k^2+4k^2\pi^2\right]\sin(\pi z)$$

The equation A4.32 can be written in short

$$P\frac{\partial A_1}{\partial X} = Q\frac{\partial A_1}{\partial \tau_o} \tag{A5.33}$$

where

$$P = 2k\left[\left(4\sigma_o + 4\sigma_o\alpha + 2\sigma_o\gamma - \sigma_o R_{cr} - \sigma_o^3\gamma\right) - \right.$$

$$\left. i\left(2\alpha - 2\sigma_o^2 - 2\sigma_o^2\alpha - 2\sigma_o\gamma + 2 + Ta - R_{cr}\right)\right] \tag{A5.34}$$

$$Q = \pi^2\left[\left(2 - R_{cr} + 2\alpha + \gamma Ta - \sigma_o^2\gamma - 3\sigma_o^2\gamma\alpha + \alpha\gamma + \gamma + 4\alpha\right) + \right.$$

$$\left. i\left(2\sigma_o + 4\sigma_o\alpha\gamma + 4\sigma_o\gamma + 2\sigma_o\alpha^2 + 4\sigma_o\alpha\right)\right] \tag{A5.35}$$

The analysis of $e^{i(kx-t)}$ argument for the RHS of equation A4.27 will result in a split equation for B_1.

$$P'\frac{\partial B_1}{\partial X} = Q'\frac{\partial B_1}{\partial \tau_o} \tag{A5.36}$$

where

$$P' = -2k\left[\left(4\sigma_o + 4\sigma_o\alpha + 2\sigma_o\gamma - \sigma_o R_{cr} - \sigma_o^3\gamma\right) + \right.$$

$$\left. i\left(2\alpha - 2\sigma_o^2 - 2\sigma_o^2\alpha - 2\sigma_o\gamma + 2 + Ta - R_{cr}\right)\right] \tag{A5.37}$$

$$Q' = -\pi^2\left[\left(2 - R_{cr} + 2\alpha + \gamma Ta - \sigma_o^2\gamma - 3\sigma_o^2\gamma\alpha + \alpha\gamma + \gamma + 4\alpha\right) - \right.$$

$$\left. i\left(2\sigma_o + 4\sigma_o\alpha\gamma + 4\sigma_o\gamma + 2\sigma_o\alpha^2 + 4\sigma_o\alpha\right)\right] \tag{A5.48}$$

The corresponding expressions for A_1^* and B_1^* will be the complex conjugates of expressions A4.33 and A4.36.

The relationships A4.33 and a4.36 will be used later to establish an expression for the amplitude equation at $O(\varepsilon)$ from the analysis of order $O(\varepsilon^3)$ equations for stream function and temperature.

We shall proceed now with the analysis of the non-linear term in T_2 equation. We recall from Appendix 3.2 that

$$J_2 = \frac{\partial\psi_1}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial x}$$

$$J_3 = \frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x} + \frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x} + \frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z} - \frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X}$$

For order $O(\varepsilon^2)$ we have no J_3 and the non-linear term can be written as

$$\left(\frac{\partial}{\partial t}+1\right)^2 \nabla^2 J_2 + Ta\frac{\partial^2}{\partial z^2} J_2$$

From A5.29 the expression above will transform to

$$\left(\frac{\partial}{\partial t}+1\right)^2 \nabla^2 J_2 + Ta\frac{\partial^2}{\partial z^2} J_2 =$$

$$ik\pi\left(\frac{\partial}{\partial t}+1\right)^2 \nabla^2\left[\left(A_1C_1^* - A_1^*C_1\right) + \left(A_1D_1^* - B_1^*C_1\right)e^{2it} + \left(B_1C_1^* - A_1^*D_1\right)e^{-2it} + \left(B_1D_1^* - B_1^*D_1\right)\right]\sin(2\pi z) +$$

$$ik\pi Ta\frac{\partial^2}{\partial z^2}\left[\left(A_1C_1^* - A_1^*C_1\right) + \left(A_1D_1^* - B_1^*C_1\right)e^{2it} + \left(B_1C_1^* - A_1^*D_1\right)e^{-2it} + \left(B_1D_1^* - B_1^*D_1\right)\right]\sin(2\pi z) =$$

$$-4i(1+Ta)k\pi^3\left[\left(A_1C_1^* - A_1^*C_1\right) + \left(B_1D_1^* - B_1^*D_1\right)\right]\sin(2\pi z) +$$

$$\left[16k\sigma_0\pi^3 + i\left(16k\sigma_0^2\pi^3 - 4(1+Ta)k\pi^3\right)\right]\left(A_1D_1^* - B_1^*C_1\right)e^{2it}\sin(2\pi z) -$$

$$\left[16k\sigma_0\pi^3 - i\left(16k\sigma_0^2\pi^3 - 4(1+Ta)k\pi^3\right)\right]\left(B_1C_1^* - A_1^*D_1\right)e^{-2it}\sin(2\pi z)$$

(A5.49)

Since the linear part of the right-hand side of the temperature equation is the same with that of stream function we shall confine to the determination of the particular solution for temperature, which contain non-resonant terms. From A5.49 we can write

$$\left(\frac{\partial}{\partial t}+1\right)^2 \nabla^2 J_2 + Ta\frac{\partial^2}{\partial z^2} J_2 =$$

(A5.50)

$$\left[\mathbf{b}_2 + \mathbf{a}_1 e^{2t} + \mathbf{a}_1^* e^{-2t}\right]\sin(2\pi z)$$

Let $T_{p,1}^{ov,t}$, $T_{p,2}^{ov,t}$ and $T_{p,3}^{ov,t}$ represent the first, second and third particular over-stable solutions of temperature in the travelling waves case. The quantities $\mathbf{b}_2$, $\mathbf{a}_1$ and $\mathbf{a}_1^*$ are given by

$$\mathbf{b}_2 = -4i(1+Ta)k\pi^3\left[\left(A_1C_1^* - A_1^*C_1\right) + \left(B_1D_1^* - B_1^*D_1\right)\right] \tag{A5.51}$$

$$\mathbf{a}_1 = \left[16k\sigma_o\pi^3 + i\left(16k\sigma_o^2\pi^3 - 4(1+Ta)k\pi^3\right)\right]\left(A_1D_1^* - B_1^*C_1\right)e^{2it} \tag{A5.52}$$

$$\mathbf{a}_1^* = \left[16k\sigma_o\pi^3 - i\left(16k\sigma_o^2\pi^3 - 4(1+Ta)k\pi^3\right)\right]\left(B_1C_1^* - A_1^*D_1\right)e^{-2it} \tag{A5.53}$$

The analysis of $\mathbf{b}_2$ in correlation with A5.23-26 reveals the following

$$\mathbf{b}_2 = -4i(1+Ta)k\pi^3\left[\left(A_1C_1 - A_1^*C_1\right) + \left(B_1D_1^* - B_1^*D_1\right)\right] =$$

$$-4ik\pi^3(1+Ta)\left[-\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}A_1A_1^* + \frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}A_1A_1^* - \right.$$

$$\left. \frac{\sqrt{\alpha}\left[-\gamma\sigma_o - i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}B_1B_1^* + \frac{\sqrt{\alpha}\left[-\gamma\sigma_o + i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}B_1B_1^*\right] = \tag{A5.54}$$

$$-\frac{8\alpha\pi^3(\alpha+1)(1+Ta)}{\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}\left[A_1A_1 + B_1B_1^*\right]$$

Let $T_{p,1}^{ov,l} = b_2 \sin(2\pi z)$ be the first solution that satisfies equation. Introduced into the temperature equation we obtain

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}b_2\sin(2\pi z)=$$

$$-\frac{8\alpha\pi^3(\alpha+1)(1+Ta)}{\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}\left[A_1A_1^*+B_1B_1^*\right]\sin(2\pi z) \tag{A5.55}$$

There is no time dependence and all time-derivative will vanish, remaining only with the following expression

$$-\nabla^2\left[\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]b_2\sin(2\pi z)=-\frac{8\alpha\pi^3(\alpha+1)(1+Ta)}{\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}\left[A_1A_1^*+B_1B_1^*\right]\sin(2\pi z) \tag{A5.56}$$

$$b_2\left[\frac{\partial^2}{\partial x^2}\nabla^2+(1+Ta)\frac{\partial^2}{\partial z^2}\nabla^2\right]\sin(2\pi z)=-\frac{8\alpha\pi^3(\alpha+1)(1+Ta)}{\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}\left[A_1A_1^*+B_1B_1^*\right]\sin(2\pi z) \tag{A5.57}$$

$$b_2=-\frac{1}{2}\frac{\alpha(\alpha+1)}{\pi\left[(1+\alpha)^2+\gamma^2\sigma^2\right]}\left[A_1A_1^*+B_1B_1^*\right] \tag{A5.58}$$

Similarly, based on A5.23-26, we compute

$$A_1 D_1^* - B_1^* C_1 = 2\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2 + \gamma^2\sigma_o^2\right]} B_1^* A_1 \qquad \text{(A5.59)}$$

$$B_1 C_1^* - A_1^* D_1 = -2\frac{\sqrt{\alpha}\left[\gamma\sigma_o - i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2 + \gamma^2\sigma_o^2\right]} A_1^* B_1 \qquad \text{(A4.60)}$$

Let $T_{p,2}^{ov,t} = a_1 e^{2it}\sin(2\pi z)$ be the second particular solution, which introduced into the temperature equation it yields

$$\left\{\left(\chi\frac{\partial}{\partial t} - \nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right] - Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}a_1 e^{2it}\sin(2\pi z) =$$
$$\left[16k\sigma_o\pi^3 + i\left(16k\sigma_o^2\pi^3 - 4(1+Ta)k\pi^3\right)\right]2\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2 + \gamma^2\sigma_o^2\right]} B_1^* A_1 \sin(2\pi z) \qquad \text{(A5.61)}$$

All derivatives with respect to x will disappear

$$\left\{\left(\chi\frac{\partial}{\partial t} - \frac{\partial^2}{\partial z^2}\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial^2}{\partial z^2} + Ta\frac{\partial^2}{\partial z^2}\right] - Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}a_1 e^{2it}\sin(2\pi z) =$$
$$\left[16k\sigma_o\pi^3 + i\left(16k\sigma_o^2\pi^3 - 4(1+Ta)k\pi^3\right)\right]2\frac{\sqrt{\alpha}\left[\gamma\sigma_o + i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2 + \gamma^2\sigma_o^2\right]} B_1^* A_1 \sin(2\pi z) \qquad \text{(A5.62)}$$

The calculation of the left-hand side of equation A5.62 will result

$$\{\sigma_o^3\chi\frac{\partial^3}{\partial t^3}\frac{\partial^2}{\partial z^2}+2\sigma_o^2\chi\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2}+\sigma_o\chi(1+Ta)\frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}-\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial^4}{\partial z^4}-$$

$$2\sigma_o\frac{\partial}{\partial t}\frac{\partial^4}{\partial z^4}-(1+Ta)\frac{\partial^2}{\partial z^2}\Big\}a_1e^{2it}\sin(2\pi z)=$$

$$\left[16k\sigma_o\pi^3+i\left(16k\sigma_o^2\pi^3-4(1+Ta)k\pi^3\right)\right]2\frac{\sqrt{\alpha}\left[\gamma\sigma_o+i(1+\alpha)\right]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}B_1^*A_1e^{2it}\sin(2\pi z)$$

(A5.63)

from which results the value of a_1

$$a_1=-\frac{\alpha}{\pi}\frac{2(\alpha+1)-\gamma^2\sigma_o^2-i\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}A_1B_1^* \qquad \text{(A5.64)}$$

In the same manner we calculate a_1^* by replacing the right-hand side of the temperature equation with $\mathbf{a}_1^*e^{-2it}\sin(2\pi z)$ and assume the particular solution $T_{p,3}^{ov,t}=a_1^*e^{-2it}\sin(2\pi z)$

$$\left\{\left(\chi\frac{\partial}{\partial t}-\nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2+Ta\frac{\partial^2}{\partial z^2}\right]-Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}T_{p,3}^{ov,t}=\mathbf{a}_1^*e^{-2it}\sin(2\pi z)$$

(A5.65)

Derivatives that contain x will vanish, and following the same process will result the following differential equation

$$\{\sigma_o^3\chi\frac{\partial^3}{\partial t^3}\frac{\partial^2}{\partial z^2}+2\sigma_o^2\chi\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial z^2}+\sigma_o\chi(1+Ta)\frac{\partial}{\partial t}\frac{\partial^2}{\partial z^2}-\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial^4}{\partial z^4}-$$

$$2\sigma_o\frac{\partial}{\partial t}\frac{\partial^4}{\partial z^4}-(1+Ta)\frac{\partial^2}{\partial z^2}\Big\}a_1^* e^{-2it}\sin(2\pi z)=$$

$$\left[16k\sigma_o\pi^3-i\left(16k\sigma_o^2\pi^3-4(1+Ta)k\pi^3\right)\right]2\frac{\sqrt{\alpha}[\gamma\sigma_o-i(1+\alpha)]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}B_1A_1^* e^{-2it}\sin(2\pi z)$$

(A5.66)

that will yield an algebraic equation in he form of

$$\{32\sigma_o^2\gamma\pi^4+64\sigma_o^2\pi^4-16(1+Ta)\pi^4-i(32\sigma_o^3\gamma\pi^4-64\sigma_o\pi^4-8\sigma_o\gamma\pi^4(1+Ta)\}\times$$

$$a_1^* e^{-2it}\sin(2\pi z)=$$

$$\left[16k\sigma_o\pi^3-i\left(16k\sigma_o^2\pi^3-4(1+Ta)k\pi^3\right)\right]\left[-2\frac{\sqrt{\alpha}[\gamma\sigma_o-i(1+\alpha)]}{\pi\left[(1+\alpha)^2+\gamma^2\sigma_o^2\right]}\right]B_1A_1^* e^{-2it}\sin(2\pi z)$$

(A5.67)

After simplifications the value for a_1^* results

$$a_1^*=-\frac{\alpha}{\pi}\frac{2(\alpha+1)-\gamma^2\sigma_o^2+i\gamma\sigma_o(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma_o^2\right](4+\gamma^2\sigma_o^2)}A_1B_1 \qquad \text{(A5.68)}$$

The particular solution for temperature at $O(\varepsilon^2)$ in the travelling waves case is

$$T_p^{ov,t} = \left[b_2 + a_1 e^{2i\bar{t}} + a_1^* e^{-2i\bar{t}}\right]\sin(2\pi z) \tag{A5.70}$$

with b_2, a_1 and a_1^* stated in A4.58, A4.64 and A4.68 respectively.

5.3. Order $O(\varepsilon^3)$ analysis

In this section we make use of the slow time-space scales according to the following notations for various variables as $\bar{t} = \sigma_o t'$, $\sigma = \varepsilon^2 t'$, $\sigma_o = \varepsilon t'$, $\sigma_o = \sigma_1^{cr}$ and $X = \varepsilon x$. The calculation will involve the steam function amplitudes A_1, A_1^*, B_1 and B_1^*.

The de-coupled equation for ψ at $O(\varepsilon^3)$ is exhibited below taking into consideration in the right-hand side only the terms related to ψ_1, forcing them to zero in order to obtain the relationship between amplitudes.

$$\left\{\left(\chi\frac{\partial}{\partial t} - \nabla^2\right)\left[\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2 + Ta\frac{\partial^2}{\partial z^2}\right] - Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\right\}\psi_3 =$$

$$-\left(\chi\frac{\partial}{\partial\tau_o} - 2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2 + 2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_1 -$$

$$\left(\chi\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1 - \left(\chi\frac{\partial}{\partial\tau} - \frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1 +$$

$$Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\psi_1 + \left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}\psi_1 + 2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1 + \frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\psi_1\right] -$$

$$Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial X} + \frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\right]J_2 - Ra_{cr}\left(\frac{\partial}{\partial t}+1\right)\frac{\partial}{\partial x}J_3$$

(A5.71)

The partial expression for J_2 has been calculated in A4.29. We shall proceed to calculate J_3: For that we split it into three groups

Group 1 $$\frac{\partial\psi_1}{\partial x}\frac{\partial T_2}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_2}{\partial x} \tag{A5.72}$$

Group 2 $$\frac{\partial\psi_2}{\partial x}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_2}{\partial z}\frac{\partial T_1}{\partial x} \tag{A5.73}$$

Group 3 $$\frac{\partial\psi_1}{\partial X}\frac{\partial T_1}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_1}{\partial X} \tag{A5.74}$$

The first group poses the most interesting features because it consists of mixed non-linear terms analysed in the previous paragraph. The second group contains the stream function solution at order $O(\varepsilon^2)$ and therefore we can ignore it in out calculation, since we deal only with order $O(\varepsilon)$. Group three contains non-resonant terms, which cannot be forced to zero, therefore will be neglected as irrelevant. Consequently, only Group 1 deserves attention and will be analysed later.

$$J_3=\frac{\partial\psi_1}{\partial x}\frac{\partial T_{p,2}^{ov,l}}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_{p,2}^{ov,l}}{\partial x} \tag{A5.75}$$

The operator associated to J_2 is calculated below

$$-Ra_{cr}\left[\left(\sigma_o\frac{\partial}{\partial \tilde{t}}+1\right)\frac{\partial}{\partial x}+\frac{\partial}{\partial \tau_o}\frac{\partial}{\partial x}\right](ik\pi\times$$

$$\left[\left(A_1C_1^*-A_1^*C_1\right)+\left(A_1D_1^*-B_1^*C_1\right)e^{2i\tilde{t}}+\left(B_1C_1^*-A_1^*D_1\right)e^{2i\tilde{t}}+\left(B_1D_1^*-B_1^*D_1\right)\right]\sin(2\pi z))=0$$

(A5.76)

Calculation of $-Ra\dfrac{\partial}{\partial x}\left(\sigma_o\dfrac{\partial}{\partial \tilde{t}}+1\right)J_3$ from equation A5.71 will follow

$$-Ra\frac{\partial}{\partial x}\left(\sigma_o\frac{\partial}{\partial \tilde{t}}+1\right)J_3=-\sigma_o Ra\frac{\partial}{\partial x}\frac{\partial}{\partial \tilde{t}}J_3-Ra_{cr}\frac{\partial}{\partial x}J_3=$$

$$-\sigma_o Ra\frac{\partial}{\partial x}\frac{\partial}{\partial \tilde{t}}\left(\frac{\partial \psi_1}{\partial x}\frac{\partial T_{p,2}^{ov,t}}{\partial z}-\frac{\partial \psi_1}{\partial z}\frac{\partial T_{p,2}^{ov,t}}{\partial x}\right)-Ra_{cr}\frac{\partial}{\partial x}\left(\frac{\partial \psi_1}{\partial x}\frac{\partial T_{p,2}^{ov,t}}{\partial z}-\frac{\partial \psi_1}{\partial z}\frac{\partial T_{p,2}^{ov,t}}{\partial x}\right)$$

(A5.77)

First we have to evaluate A5.75

$$\frac{\partial \psi_1}{\partial x}\frac{\partial T_{p,2}^{ov,t}}{\partial z}-\frac{\partial \psi_1}{\partial z}\frac{\partial T_{p,2}^{ov,t}}{\partial x}=$$

$$\left(ikA_1e^{i(kx+\tilde{t})}+ikB_1e^{i(kx-\tilde{t})}-ikA_1^*e^{-i(kx+\tilde{t})}-ikB_1^*e^{-i(kx-\tilde{t})}\right)\sin(\pi z)\times \qquad \text{(A5.78)}$$

$$2\pi\left(b_2+a_1e^{2i\tilde{t}}+a_1^*e^{-2i\tilde{t}}\right)\cos(2\pi z)$$

$$\frac{\partial\psi_1}{\partial x}\frac{\partial T_{p,2}^{ov,1}}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_{p,2}^{ov,1}}{\partial x}=$$

$$2ik\pi\left(ikA_1e^{i(kx+\tilde{t})}+ikB_1e^{i(kx-\tilde{t})}-ikA_1^*e^{-i(kx+\tilde{t})}-ikB_1^*e^{-i(kx-\tilde{t})}\right)\times \tag{A5.79}$$

$$\left(b_2+a_1e^{2i\tilde{t}}+a_1^*e^{-2i\tilde{t}}\right)\sin(\pi z)\cos(2\pi z)$$

The product $\sin(\pi z)\cos(2\pi z)$ can be transformed

$$\sin(\pi z)\cos(2\pi z)=\frac{1}{2}\left[\sin(3\pi z)-\sin(\pi z)\right]=\frac{1}{2}\sin(3\pi z)-\frac{1}{2}\sin(\pi z) \tag{A5.80}$$

However, we need to keep for further calculations only the $\left[-\frac{1}{2}\sin(\pi z)\right]$ term, as the resonant term, the other being neglected. Consequently the expression for J_3 changes to

$$J_3=\frac{\partial\psi_1}{\partial x}\frac{\partial T_{p,2}^{ov,1}}{\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial T_{p,2}^{ov,1}}{\partial x}=$$

$$ik\pi\left(ikA_1e^{i(kx+\tilde{t})}+ikB_1e^{i(kx-\tilde{t})}-ikA_1^*e^{-i(kx+\tilde{t})}-ikB_1^*e^{-i(kx-\tilde{t})}\right)\left(b_2+a_1e^{2i\tilde{t}}+a_1^*e^{-2i\tilde{t}}\right)\sin(\pi z) \tag{A5.81}$$

Expanding A5.81 we get

$$J_3 = -ik\pi\left[A_1 b_2 e^{i(kx+\tilde{t})} + A_1 a_1 e^{i(kx+\tilde{t})} e^{2i\tilde{t}} + A_1 a_1^* e^{i(kx-\tilde{t})} e^{-2i\tilde{t}} + B_1 b_2 e^{i(kx-\tilde{t})} + B_1 a_1 e^{i(kx+\tilde{t})} e^{2i\tilde{t}} + \right.$$

$$B_1 a_1^* e^{i(kx-\tilde{t})} e^{-2i\tilde{t}} - A_1^* b_2 e^{-i(kx+\tilde{t})} - A_1^* a_1 e^{-i(kx-\tilde{t})} e^{2i\tilde{t}} - A_1^* a_1^* e^{-i(kx+\tilde{t})} e^{-2i\tilde{t}} - B_1^* b_2 e^{-i(kx-\tilde{t})} -$$

$$\left. B_1^* a_1 e^{-i(kx-\tilde{t})} e^{2i\tilde{t}} - B_1^* a_1^* e^{-i(kx+\tilde{t})} e^{-2i\tilde{t}}\right]\sin(\pi z)$$

(A5.82)

$$-\sigma_o Ra \frac{\partial}{\partial x}\frac{\partial}{\partial \tilde{t}} J_3 - Ra_{cr}\frac{\partial}{\partial x} J_3 = -ik^2\sigma_o \pi Ra_{cr}\left[A_1 b_2 e^{i(kx+\tilde{t})} + 3A_1 a_1 e^{i(kx+3\tilde{t})} - A_1 a_1^* e^{i(kx-t)} - \right.$$

$$B_1 b_2 e^{i(kx-t)} + B_1 a_1 e^{i(kx+t)} - 3B_1 a_1^* e^{i(kx-3\tilde{t})} - A_1^* b_2 e^{-i(kx+\tilde{t})} + A_1^* a_1 e^{-i(kx-\tilde{t})} - 3A_1^* a_1^* e^{-i(kx+3\tilde{t})} +$$

$$\left. B_1^* b_2 e^{-i(kx-\tilde{t})} + 3B_1^* a_1 e^{-i(kx-3\tilde{t})} - B_1^* a_1^* e^{-i(kx+3\tilde{t})}\right]\sin(\pi z) - \pi k^2 Ra_{cr}\left[A_1 b_2 e^{i(kx+\tilde{t})} + A_1 a_1 e^{i(kx+3\tilde{t})} + \right.$$

$$A_1 a_1^* e^{i(kx-t)} + B_1 b_2 e^{i(kx-t)} + B_1 a_1 e^{i(kx+t)} + B_1 a_1^* e^{i(kx-3\tilde{t})} + A_1^* b_2 e^{-i(kx+\tilde{t})} + A_1^* a_1 e^{-i(kx-\tilde{t})} +$$

$$\left. A_1^* a_1^* e^{-i(kx+3\tilde{t})} + B_1^* b_2 e^{-i(kx-\tilde{t})} + B_1^* a_1 e^{-i(kx-3\tilde{t})} + B_1^* a_1^* e^{-i(kx+3\tilde{t})}\right]\sin(\pi z)$$

(A5.83)

We note that the terms that have a power three at the exponent can be ignored since they are non-resonant terms, and for further analysis we will focus on the terms of exponent argument $\exp\left[i(kx+\tilde{t})\right]$ and $\exp\left[i(kx-\tilde{t})\right]$. The terms containing $\exp\left[-i(kx+\tilde{t})\right]$ and $\exp\left[-i(kx-\tilde{t})\right]$ will generate complex conjugate expressions of the amplitudes A and B. From A5.83 we select

$$-k^2\pi Ra_{cr}(1+i\sigma_o)\left[A_1 b_2 + B_1 a_1\right] \qquad \text{for} \qquad \exp\left[i(kx+\tilde{t})\right] \qquad \text{(A5.84)}$$

$$-k^2\pi Ra_{cr}(1-i\sigma_o)[A_1a_1 + B_1b_2] \qquad \text{for} \qquad \exp[i(kx-\tilde{t})] \tag{A5.85}$$

We now proceed to calculate by parts the linear part of A4.71

$$-\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]\psi_1 -$$

$$\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1-\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}\psi_1 + \tag{A5.86}$$

$$Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\psi_1+\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}\psi_1+2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1+\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\psi_1\right]$$

Part 1

$$-\left(\chi\frac{\partial}{\partial\tau_o}-2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right)\left[2\frac{\partial}{\partial\tau_o}\left(\frac{\partial}{\partial t}+1\right)\nabla^2+2\left(\frac{\partial}{\partial t}+1\right)^2\frac{\partial}{\partial x}\frac{\partial}{\partial X}\right]=$$

$$-2\chi\sigma_o\frac{\partial^2}{\partial\tau_o^2}\frac{\partial}{\partial t}\nabla^2-2\chi\frac{\partial^2}{\partial\tau_o^2}\nabla^2-2\chi\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}-4\chi\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}-$$

$$2\chi\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}+4\sigma_o\frac{\partial}{\partial t}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2+4\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\nabla^2+4\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}+$$

$$8\sigma_o\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}+4\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial X^2}$$

(A5.87)

Part 2

$$\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)\left(\frac{\partial}{\partial t}+1\right)^2\nabla^2\psi_1=-\chi\sigma_o^2\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial t^2}\nabla^2-2\chi\sigma_o\frac{\partial}{\partial\tau}\frac{\partial}{\partial t}\nabla^2-$$

(A5.88)

$$\chi\frac{\partial}{\partial\tau}\nabla^2+\sigma_o^2\frac{\partial^2}{\partial t^2}\frac{\partial^2}{\partial X^2}\nabla^2+2\sigma_o\frac{\partial}{\partial t}\frac{\partial^2}{\partial X^2}\nabla^2+\frac{\partial^2}{\partial X^2}\nabla^2$$

Part 3

$$-\left(\chi\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial X^2}\right)Ta\frac{\partial^2}{\partial z^2}=-\chi Ta\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial z^2}+Ta\frac{\partial^2}{\partial z^2}\frac{\partial^2}{\partial X^2} \qquad \text{(A5.89)}$$

Part 4

$$Ra_{cr}\left[\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial x^2}\psi_1+\left(\frac{\partial}{\partial t}+1\right)\frac{\partial^2}{\partial X^2}\psi_1+2\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}\psi_1+\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}\psi_1\right]=$$

$$\sigma_o Ra_{cr}\frac{\partial}{\partial t}\frac{\partial^2}{\partial x^2}+Ra_{cr}\frac{\partial^2}{\partial x^2}+2Ra_{cr}\frac{\partial}{\partial\tau_o}\frac{\partial}{\partial x}\frac{\partial}{\partial X}+\sigma_o Ra_{cr}\frac{\partial}{\partial t}\frac{\partial^2}{\partial X^2}+Ra_{cr}\frac{\partial^2}{\partial X^2}+Ra_{cr}\frac{\partial}{\partial\tau}\frac{\partial^2}{\partial x^2}$$

(A5.90)

By compiling all the calculated parts we reach for an expression of the right-hand side of the equation A5.71, associated to $e^{i(kx+\bar{t})}$ as follows

$$RHS = 2i\pi^2\gamma\sigma_o\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial\tau_o^2} + 2\pi^2\gamma\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial\tau_o^2} + 2i\pi^2\gamma k\sigma_o^2\frac{\partial^2 A_1}{\partial\tau_o\partial X} +$$

$$4\pi^2\gamma k\sigma_o\frac{\partial^2 A_1}{\partial\tau_o\partial X} - 2i\pi^2\gamma k\frac{\partial^2 A_1}{\partial\tau_o\partial X} + 4k\sigma_o\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial\tau_o\partial X} - 4ik\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial\tau_o\partial X} +$$

$$4k^2\sigma_o^2\frac{\partial^2 A_1}{\partial X^2} - 8ik^2\sigma_o\frac{\partial^2 A_1}{\partial X^2} - 4k^2\frac{\partial^2 A_1}{\partial X^2} - \pi^2\gamma\sigma_o^2\left(k^2+\pi^2\right)\frac{\partial A_1}{\partial\tau} + 2i\pi^2\gamma\sigma_o\left(k^2+\pi^2\right)\frac{\partial A_1}{\partial\tau} +$$

$$\pi^2\left(k^2+\pi^2\right)\gamma\frac{\partial A_1}{\partial\tau} + \sigma_o^2\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial X^2} - 2i\sigma_o\left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial X^2} - \left(k^2+\pi^2\right)\frac{\partial^2 A_1}{\partial X^2} +$$

$$\pi^4\gamma Ta\frac{\partial A_1}{\partial\tau} - \pi^2 Ta\frac{\partial^2 A_1}{\partial X^2} - iRa_{cr}k^2\sigma_o A_1 - k^2 Ra_{cr}A_1 + 2ikRa_{cr}\frac{\partial^2 A_1}{\partial\tau_o\partial X} + i\sigma_o Ra_{cr}\frac{\partial^2 A_1}{\partial X^2} +$$

$$Ra_{cr}\frac{\partial^2 A_1}{\partial X^2} - k^2 Ra_{cr}\frac{\partial A_1}{\partial\tau} - \pi k^2 Ra_{cr}\left(1+i\sigma_o\right)\left[A_1 b_2 + B_1 a_1\right]$$

(A5.91)

By compiling all the calculated parts we reach for an expression of the right-hand side of the equation A5.71, associated to $e^{i(kx-\bar{t})}$ as follows

$$RHS = -2i\pi^2\gamma\sigma_o\left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial\tau_o^2} + 2\pi^2\gamma\left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial\tau_o^2} + 2i\pi^2\gamma k\sigma_o^2\frac{\partial^2 B_1}{\partial\tau_o\partial X} -$$

$$4\pi^2\gamma k\sigma_o\frac{\partial^2 B_1}{\partial\tau_o\partial X} - 2i\pi^2\gamma k\frac{\partial^2 B_1}{\partial\tau_o\partial X} - 4k\sigma_o\left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial\tau_o\partial X} - 4ik\left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial\tau_o\partial X} +$$

$$4k^2\sigma_o^2\frac{\partial^2 B_1}{\partial X^2} + 8ik^2\sigma_o\frac{\partial^2 B_1}{\partial X^2} - 4k^2\frac{\partial^2 B_1}{\partial X^2} - \pi^2\gamma\sigma_o^2\left(k^2+\pi^2\right)\frac{\partial B_1}{\partial\tau} - 2i\pi^2\gamma\sigma_o\left(k^2+\pi^2\right)\frac{\partial B_1}{\partial\tau} +$$

$$\pi^2\left(k^2+\pi^2\right)\gamma\frac{\partial B_1}{\partial\tau} + \sigma_o^2\left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial X^2} + 2i\sigma_o\left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial X^2} - \left(k^2+\pi^2\right)\frac{\partial^2 B_1}{\partial X^2} +$$

$$\pi^4\gamma Ta\frac{\partial B_1}{\partial\tau} - \pi^2 Ta\frac{\partial^2 B_1}{\partial X^2} + iRa_{cr}k^2\sigma_o B_1 - k^2 Ra_{cr}B_1 + 2ikRa_{cr}\frac{\partial^2 B_1}{\partial\tau_o\partial X} - i\sigma_o Ra_{cr}\frac{\partial^2 B_1}{\partial X^2} +$$

$$Ra_{cr}\frac{\partial^2 B_1}{\partial X^2} - k^2 Ra_{cr}\frac{\partial B_1}{\partial\tau} - \pi k^2 Ra_{cr}(1-i\sigma_o)\left[A_1 a_1^* + B_1 b_2\right]$$

(A5.92)

We analyse the expressions A5.91 and A5.92 by forcing $RHS = 0$

$$M_1\frac{\partial^2 A_1}{\partial\tau_o^2} + M_2\frac{\partial^2 A_1}{\partial\tau_o\partial X} + M_3\frac{\partial^2 A_1}{\partial X^2} + M_4\frac{\partial A_1}{\partial\tau} - M_5 A_1 + M_6\left[A_1 b_2 + B_1 a_1\right] = 0 \qquad \text{(A5.93)}$$

$$M_1'\frac{\partial^2 B_1}{\partial\tau_o^2} + M_2'\frac{\partial^2 B_1}{\partial\tau_o\partial X} + M_3'\frac{\partial^2 B_1}{\partial X^2} + M_4'\frac{\partial B_1}{\partial\tau} - M_5' B_1 + M_6'\left[A_1 a_1^* + B_1 b_2\right] = 0 \qquad \text{(A5.94)}$$

where

$$M_1 = 2\gamma\pi^4(1+\alpha)\left[1+i\sigma_o\right] \qquad \text{(A5.95)}$$

$$M_1' = 2\gamma\pi^4(1+\alpha)[1 - i\sigma_o] \tag{A5.96}$$

$$M_2 = 2k\pi^2\left[2\gamma\sigma_o + 2\sigma_o(1+\alpha) + i\left(\gamma\sigma_o^2 - \gamma - 2(1+\alpha) + R_{cr}\right)\right] \tag{A5.97}$$

$$M_2 = -2k\pi^2\left[2\gamma\sigma_o + 2\sigma_o(1+\alpha) - i\left(\gamma\sigma_o^2 - \gamma - 2(1+\alpha) + R_{cr}\right)\right] \tag{A5.98}$$

$$M_3 = \pi^2\left[4\alpha\sigma_o^2 - 4\alpha + \sigma_o^2(1+\alpha) - (1+\alpha) - Ta + R_{cr} - i\left(8\sigma_o\alpha + 2\sigma_o(1+\alpha) - \sigma_o R_c\right)\right]$$

(A5.99)

$$M_3 = \pi^2\left[4\alpha\sigma_o^2 - 4\alpha + \sigma_o^2(1+\alpha) - (1+\alpha) - Ta + R_{cr} + i\left(8\sigma_o\alpha + 2\sigma_o(1+\alpha) - \sigma_o R_c\right)\right]$$

(A5.100)

$$M_4 = \pi^4\left[\gamma(1+\alpha) - \gamma\sigma_o^2(1+\alpha) + \gamma Ta - R_c + 2i\gamma\sigma_o(1+\alpha)\right] \tag{A5.101}$$

$$M_4' = \pi^4\left[\gamma(1+\alpha) - \gamma\sigma_o^2(1+\alpha) + \gamma Ta - R_c - 2i\gamma\sigma_o(1+\alpha)\right] \tag{A5.102}$$

$$M_5 = \alpha\pi^4 R_{cr}(1 + i\sigma_o) \tag{A5.103}$$

$$M_5' = \alpha\pi^4 R_{cr}(1 - i\sigma_o) \tag{A45104}$$

$$M_6 = -\alpha\pi R_{cr}\pi^4(1 + i\sigma_o) \tag{A5.105}$$

$$M_6' = -\alpha\pi R_{cr}\pi^4(1 - i\sigma_o) \tag{A5.106}$$

In the equations A5.93 and 94 we can replace the mixed derivative term with the argument M_2 and M_3 by using A5.33 and A5.36, as follows

$$M_2 \frac{\partial^2 A_1}{\partial \tau_o \partial X} = M_2 \frac{\partial A_1}{\partial \tau_o} \frac{\partial A_1}{\partial X} = M_2 \frac{Q}{P} \frac{\partial^2 A_1}{\partial \tau_o^2} = M_{PQ} \frac{\partial^2 A_1}{\partial \tau_o^2} \tag{A5.107}$$

$$M_2' \frac{\partial^2 B_1}{\partial \tau_o \partial X} = M_2' \frac{\partial B_1}{\partial \tau_o} \frac{\partial B_1}{\partial X} = M_2' \frac{Q'}{P'} \frac{\partial^2 B_1}{\partial \tau_o^2} = M_{PQ}' \frac{\partial^2 B_1}{\partial \tau_o^2} \tag{A5.108}$$

where

$$M_{PQ} = \pi^{-1}\left[2\sigma_o(\alpha+\gamma+1) + i\left(\gamma\left(\sigma_o^2+1\right) + Rc - 2(\alpha+\gamma+1)\right)\right] \times$$

$$\frac{\left\{\left[(1+3\alpha)(2-\gamma\sigma_o) + \gamma(\alpha+1+Ta) - R_c\right] + 2i\sigma_o(\alpha+1)(1+\alpha+2\gamma)\right\}}{\left\{\left(4\sigma_o + 4\sigma_o\alpha + 2\gamma\sigma_o - \sigma_o R_c - \sigma_o^3\gamma\right) - i\left[2(1+\alpha)\left(1-\sigma_o^2\right) - 2\sigma_o\gamma + Ta - R_{cr}\right]\right\}}$$

(A5.109)

$$M_{PQ}' = -\pi^{-1}\left[2\sigma_o(\alpha+\gamma+1) - i\left(\gamma\left(\sigma_o^2+1\right) + Rc - 2(\alpha+\gamma+1)\right)\right] \times$$

$$\frac{\left\{\left[(1+3\alpha)(2-\gamma\sigma_o) + \gamma(\alpha+1+Ta) - R_c\right] - 2i\sigma_o(\alpha+1)(1+\alpha+2\gamma)\right\}}{\left\{\left(4\sigma_o + 4\sigma_o\alpha + 2\gamma\sigma_o - \sigma_o R_c - \sigma_o^3\gamma\right) + i\left[2(1+\alpha)\left(1-\sigma_o^2\right) - 2\sigma_o\gamma + Ta - R_{cr}\right]\right\}}$$

(A5.110)

We have to evaluate at this stage the products $M_6\left[A_1 b_2 + B_1 a_1\right]$ and $M_6'\left[A_1 a_1^* + B_1 b_2\right]$ respectively.

$$M_6\left[A_1 b_2 + B_1 a_1\right] = -\alpha\pi R_{cr}\pi^{-1}\left(1 + i\sigma_o\right)\left[A_1 b_2 + B_1 a_1\right] =$$

$$-\alpha\pi R_{cr}\pi^{-1}\left(1+i\sigma_o\right)\left[-\frac{1}{2}\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]}\left(A_1A_1^* + B_1B_1^*\right)A_1 -\right.$$

$$\left.-\frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_o^2-i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}B_1B_1^*A_1\right] =$$

$$\alpha\pi R_{cr}\pi^{-1}\left(1+i\sigma_o\right)\frac{1}{2}\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]}A_1A_1^*A_1 +$$

$$\alpha\pi R_{cr}\pi^{-1}\left(1+i\sigma_o\right)\left[\frac{\alpha(\alpha+1)}{2\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]}+\frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_o^2-i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}\right]B_1B_1^*A_1$$

(A5.111)

$$M_6\left[A_1 a_1^* + B_1 b_2\right] = -\alpha\pi R_{cr}\pi^{-1}\left(1 - i\sigma_o\right)\left[A_1 a_1^* + B_1 b_2\right] =$$

$$-\alpha\pi R_{cr}\pi^{-1}\left(1-i\sigma_o\right)\left[-\frac{1}{2}\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]}\left(A_1A_1^* + B_1B_1^*\right)B_1 -\right.$$

$$\left.-\frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_o^2+i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}A_1^*B_1A_1\right] =$$

$$\alpha\pi R_{cr}\pi^{-1}\left(1-i\sigma_o\right)\frac{1}{2}\frac{\alpha(\alpha+1)}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]}B_1B_1^*B_1 +$$

$$\alpha\pi R_{cr}\pi^{-1}\left(1-i\sigma_o\right)\left[\frac{\alpha(\alpha+1)}{2\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]}+\frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_o^2+i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)}\right]A_1B_1A_1^*$$

(A5.112)

For simplification we call

$$M_{7,1} = -\frac{\alpha(\alpha+1)}{2\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]} \tag{A5.113}$$

$$M'_{7,1} = -\frac{\alpha(\alpha+1)}{2\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]} \tag{A5.114}$$

It is interesting to note that $M_{7,1} = M'_{7,1}$

$$M_{7,2} = \frac{\alpha(\alpha+1)}{2\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]} + \frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_o^2-i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)} \tag{A5.115}$$

$$M_{7,2} = \frac{\alpha(\alpha+1)}{2\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]} + \frac{\alpha\left[2(\alpha+1)-\gamma^2\sigma_o^2+i\gamma\sigma_o(\alpha+3)\right]}{\pi\left[(\alpha+1)^2+\gamma\sigma_o^2\right]\left(4+\gamma^2\sigma_o^2\right)} \tag{A5.116}$$

Also it is convenient to call

$$M_1 + M_{PQ} = L_1 \tag{A5.117}$$

$$M'_1 + M_{PQ} = L'_1 \tag{A5.118}$$

where M_1, M'_1, M_{PQ} and M'_{PQ} are defined by the relations A5.95 and 96 and A5.109 and 110 respectively. As a result, the expressions for the amplitude of the oscillatory convection in the travelling waves case, is given by the following formulae

$$L_1\frac{\partial^2 A_1}{\partial\tau_o^2}+M_3\frac{\partial^2 A_1}{\partial X^2}+M_4\frac{\partial A_1}{\partial\tau}-M_5A_1+M_6\left[M_{7,1}A_1A_1^*+M_{7,2}B_1B_1^*\right]A_1=0 \qquad \text{(A5.119)}$$

$$L_1'\frac{\partial^2 A_1}{\partial\tau_o^2}+M_3'\frac{\partial^2 A_1}{\partial X^2}+M_4'\frac{\partial A_1}{\partial\tau}-M_5'A_1+M_6'\left[M_{7,1}'B_1B_1^*+M_{7,2}'A_1A_1^*\right]B_1=0 \qquad \text{(A5.120)}$$

Rearranging the terms we obtain

$$L_1\frac{\partial^2 A_1}{\partial\tau_o^2}+M_3\frac{\partial^2 A_1}{\partial X^2}+M_4\left\{\frac{\partial A_1}{\partial\tau}-\frac{M_5}{M_4}\left[1-\frac{M_6}{M_5}\left[M_{7,1}A_1A_1^*+M_{7,2}B_1B_1^*\right]\right]A_1\right\}=0 \quad \text{(A5.121)}$$

$$L_1'\frac{\partial^2 B_1}{\partial\tau_o^2}+M_3'\frac{\partial^2 B_1}{\partial X^2}+M_4'\left\{\frac{\partial B_1}{\partial\tau}-\frac{M_5'}{M_4'}\left[1-\frac{M_6'}{M_5'}\left[M_{7,1}'B_1B_1^*+M_{7,2}'A_1A_1^*\right]\right]B_1\right\}=0 \quad \text{(A5.122)}$$

Further, we restore the original time and space scale, $\tau=\varepsilon^{2}t$, $\tau_o=\varepsilon t$, $X=\varepsilon x$; $\bar{t}=\tau_o t'$ and by multiplying the expressions with ε^3 and recalling that $A=\varepsilon A_1$, $B=\varepsilon B_1$, $A^*=\varepsilon A_1^*$, $B^*=\varepsilon B_1^*$, we obtain

$$L_1\frac{\partial^2 A}{\partial t^2}+M_3\frac{\partial^2 A}{\partial x^2}+M_4\left\{\frac{\partial A}{\partial t}-\frac{M_5}{M_4}\left[\varepsilon^2-\frac{M_6}{M_5}\left[M_{7,1}AA^*+M_{7,2}BB^*\right]\right]A\right\}=0 \qquad \text{(A5.123)}$$

$$L_1'\frac{\partial^2 B}{\partial t^2}+M_3'\frac{\partial^2 B}{\partial x^2}+M_4'\left\{\frac{\partial B}{\partial t}-\frac{M_5'}{M_4'}\left[\varepsilon^2-\frac{M_6'}{M_5'}\left[M_{7,1}'BB^*+M_{7,2}'AA^*\right]\right]B\right\}=0 \qquad \text{(A5.124)}$$

It is required to further simplify the expressions above to a simpler form, by calling the following parameters

$$L_1 = h_{11} \qquad\qquad L_1' = h_{21} \tag{A5.125}$$

$$M_3 = h_{12} \qquad\qquad M_3' = h_{22} \tag{A5.126}$$

$$M_4 = h_{13} \qquad\qquad M_4' = h_{23} \tag{A5.127}$$

$$\frac{M_5}{M_4} = h_{14} \qquad\qquad \frac{M_5'}{M_4'} = h_{24} \tag{A5.128}$$

$$\frac{M_6 M_{7.1}}{M_5} = h_{15} \qquad = \qquad \frac{M_6' M_{7.1}'}{M_5'} = h_{15} \tag{A5.129}$$

$$\frac{M_6 M_{7.2}}{M_5} = h_{16} \qquad\qquad \frac{M_6' M_{7.2}'}{M_5'} = h_{26} \tag{A5.130}$$

The equations A5.123 and 124 can be written as

$$\left[h_{11}\frac{\partial^2 A}{\partial t^2} + h_{12}\frac{\partial^2 A}{\partial x^2}\right] + h_{13}\left\{\frac{\partial A}{\partial t} - h_{14}\left[\xi^{ov} - h_{15}AA^* - h_{16}BB^*\right]A\right\} = 0 \tag{A5.131}$$

$$\left[h_{21}\frac{\partial^2 B}{\partial t^2} + h_{22}\frac{\partial^2 B}{\partial x^2}\right] + h_{23}\left\{\frac{\partial B}{\partial t} - h_{24}\left[\xi^{ov} - h_{15}BB^* - h_{26}AA^*\right]B\right\} = 0 \tag{A5.132}$$

where the coefficients for equation A5.131 are listed below

$$h_{11} = 2\pi^4\gamma(1+\alpha)(1+i\sigma_0) + 2\pi^2 k\left[2\sigma_o(\alpha+\gamma+1) - i\left(\gamma\left(1-\sigma_0^2\right) + 2(\alpha+1) - R_{cr}\right)\right]\times$$

$$\frac{\left\{\left[(1+3\alpha)(2-\gamma\sigma_0) + \gamma(\alpha+Ta+1) - R_{cr}\right] + i2\sigma_0(\alpha+1)(1+\alpha+2\gamma)\right\}}{\left\{\left(4\sigma_0 + 4\sigma_0\alpha + 2\sigma_0\gamma - \sigma_0 R_{cr} - \sigma_0^3\gamma\right) - i\left[2(1+\alpha)\left(1-\sigma_0^2\right) - 2\sigma_0\gamma + Ta - R_{cr}\right]\right\}}$$

(A5.133)

$$h_{12} = \pi^2\left[(5\alpha+1)\left(\sigma_0^2 - 1\right) - Ta + R_{cr} - i\left(2\sigma_0(5\alpha+1) - \sigma_0 R_{cr}\right)\right] \qquad \text{(A5.134)}$$

$$h_{13} = \pi^4\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} + i2\gamma\sigma_0(\alpha+1)\right] \qquad \text{(A5.135)}$$

$$h_{14} = \frac{\alpha R_{cr}(1+i\sigma_0)\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} - i2\gamma\sigma_0(\alpha+1)\right]}{\left[\left(\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr}\right)^2 + \left(2\gamma\sigma_0(\alpha+1)\right)^2\right]} \qquad \text{(A5.136)}$$

$$h_{15} = \frac{1}{2}\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]} \qquad \text{(A5.137)}$$

$$h_{16} = \frac{\alpha\left[8(\alpha+1) + \sigma_0^2\gamma^2(\alpha-1)\right]}{2\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2 + 4\right)} - i\frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2 + 4\right)} \qquad \text{(A5.138)}$$

The coefficients for the equation A5.132 are presented

$$h_{21} = 2\pi^4\gamma(1+\alpha)(1-i\sigma_0) - 2\pi^2 k\left[2\sigma_o(\alpha+\gamma+1) + i\left(\gamma\left(1-\sigma_0^2\right)+2(\alpha+1) - R_{cr}\right)\right]\times$$

$$\frac{\left\{\left[(1+3\alpha)(2-\gamma\sigma_0) + \gamma(\alpha+Ta+1) - R_{cr}\right] - i2\sigma_0(\alpha+1)(1+\alpha+2\gamma)\right\}}{\left\{\left(4\sigma_0 + 4\sigma_0\alpha + 2\sigma_0\gamma - \sigma_0 R_{cr} - \sigma_0^3\gamma\right) + i\left[2(1+\alpha)\left(1-\sigma_0^2\right) - 2\sigma_0\gamma + Ta - R_{cr}\right]\right\}}$$

(A5.139)

$$h_{22} = \pi^2\left[(5\alpha+1)\left(\sigma_0^2-1\right) - Ta + R_{cr} + i\left(2\sigma_0(5\alpha+1) - \sigma_0 R_{cr}\right)\right] \qquad \text{(A5.140)}$$

$$h_{23} = \pi^4\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} - i2\gamma\sigma_0(\alpha+1)\right] \qquad \text{(A5.141)}$$

$$h_{24} = \frac{\alpha R_{cr}(1-i\sigma_0)\left[\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr} + i2\gamma\sigma_0(\alpha+1)\right]}{\left[\left(\gamma(\alpha+1)\left(1-\sigma_0^2\right) + \gamma Ta - R_{cr}\right)^2 + \left(2\gamma\sigma_0(\alpha+1)\right)^2\right]} \qquad \text{(A5.142)}$$

$$h_{25} = \frac{1}{2}\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]} = h_{15} \qquad \text{(A5.143)}$$

$$h_{26} = \frac{\alpha\left[8(\alpha+1) + \sigma_0^2\gamma^2(\alpha-1)\right]}{2\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2+4\right)} + i\frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2 + \sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2+4\right)} \qquad \text{(A5.144)}$$

It should be noted that the two equations A5.131 and 132 for amplitude of the oscillatory convection in the travelling waves case are **not** complex conjugates to eachother. The difference consists in the coefficients associated to $\partial^2(\bullet)/\partial t^2$, respectively, h_{11} and h_{21}, which it is characteristic to a wave velocity group.

Equations A5.131 and 132 have indeed their complex conjugate counter-parts in the amplitude equations for A_1^* and B_1^* corresponding to the exponential argument in the original solution $\exp[-i(kx+t)]$ and $\exp[-i(kx-t)]$, equations which were not explicitly developed here. However, they too, will withstand the same relationship that exists between A5.131 and 132. By setting

$$\frac{\partial^2 A}{\partial t^2} = \frac{\partial^2 A^*}{\partial x^2} \equiv 0 \qquad \text{for A4.131} \tag{A5.145}$$

$$\frac{\partial^2 B}{\partial t^2} = \frac{\partial^2 B^*}{\partial x^2} \equiv 0 \qquad \text{for A5.132} \tag{A5.146}$$

We can investigate the following system of unknown amplitudes

$$\frac{\partial A}{\partial t} - h_{14}\left[\xi^{ov} - h_{15}AA^* - h_{16}BB^*\right]A = 0 \tag{A5.147}$$

$$\frac{\partial B}{\partial t} - h_{24}\left[\xi^{ov} - h_{15}BB^* - h_{26}AA^*\right]B = 0 \tag{A5.148}$$

It is convenient to express the coefficients of the two equations above as real and imaginary parts.

$$h_{14} = h_{14}^r + ih_{14}^i \tag{A5.149}$$

$$h_{24} = h_{14}^r - ih_{14}^i \tag{A5.150}$$

$$h^r_{14} = \frac{\alpha R_{cr}\left[\gamma(\alpha+1)\left(1+\sigma_o^2\right)+\gamma Ta - R_{cr}\right]}{\left[\gamma(\alpha+1)\left(1-\sigma_o^2\right)+\gamma Ta - R_{cr}\right]^2 + 4\gamma\sigma_o^2(1+\alpha)^2} \tag{A5.151}$$

$$h^i_{14} = -\frac{\alpha\sigma_o R_{cr}\left[\gamma(\alpha+1)\left(1+\sigma_o^2\right)-\gamma Ta + R_{cr}\right]}{\left[\gamma(\alpha+1)\left(1-\sigma_o^2\right)+\gamma Ta - R_{cr}\right]^2 + 4\gamma\sigma_o^2(1+\alpha)^2} \tag{A5.152}$$

$$h_{16} = h^r_{16} + i h^i_{16} \tag{A5.153}$$

$$h_{26} = h^r_{16} - i h^i_{16} \tag{A5.154}$$

$$h^r_{16} = \frac{\alpha\left[8(\alpha+1)+\sigma_0^2\gamma^2(\alpha-1)\right]}{2\left[(\alpha+1)^2+\sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2+4\right)} \tag{A5.155}$$

$$h^i_{16} = -\frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2+\sigma_0^2\gamma^2\right]\left(\sigma_0^2\gamma^2+4\right)} \tag{A5.156}$$

Let the following quantities be

$|A| = r_A$ $\qquad \theta_A$ - Phase angle of A $\qquad A = r_A e^{i\theta_A}$ $\qquad A^* = r_A e^{-i\theta_A}$ (A5.157)

$|B| = r_B$ $\qquad \theta_B$ - Phase angle of B $\qquad B = r_B e^{i\theta_B}$ $\qquad B^* = r_B e^{-i\theta_B}$ (A5.158)

$$\frac{d}{dt}\left[r_A e^{i\theta_A}\right] = \frac{dr_A}{dt} + i r_A \frac{d\theta_A}{dt} \tag{A5.159}$$

$$\frac{d}{dt}\left[r_B e^{i\theta_B}\right] = \frac{dr_B}{dt} + ir_B \frac{d\theta_B}{dt} \tag{A5.160}$$

In this instance equations A5.147 and 148 can be written as

$$\frac{dr_A}{dt} + ir_A \frac{d\theta_A}{dt} = \left(h_{14}^r + ih_{14}^i\right)\left[\xi^{ov} - h_{15}r_A^2 - \left(h_{16}^r + ih_{16}^i\right)r_B^2\right]r_A \tag{A5.161}$$

$$\frac{dr_B}{dt} + ir_B \frac{d\theta_B}{dt} = \left(h_{14}^r - ih_{14}^i\right)\left[\xi^{ov} - \left(h_{16}^r - ih_{16}^i\right)r_A^2 - h_{15}r_B^2\right]r_B \tag{A5.162}$$

Expanding the right hand side of the equations above we shall separate them into real and imaginary parts

$$\left(h_{14}^r + ih_{14}^i\right)\left[\xi^{ov} - h_{15}r_A^2 - \left(h_{16}^r + ih_{16}^i\right)r_B^2\right]r_A =$$

$$\left(h_{14}^r\xi^{ov} - h_{15}h_{14}^r r_A^2 - \left(h_{16}^r h_{14}^r - h_{16}^i h_{14}^i\right)r_B^2\right)r_A + i\left(h_{14}^i\xi^{ov} - h_{15}h_{14}^i r_A^2 - \left(h_{16}^r h_{14}^i + h_{16}^i h_{14}^r\right)r_B^2\right)r_A \tag{A5.163}$$

Let $s_1 = h_{16}^r h_{14}^r - h_{16}^i h_{14}^i$ and $s_2 = h_{16}^r h_{14}^i + h_{16}^i h_{14}^r$, equation A4.161 can be written now

$$\frac{dr_A}{dt} = \left(h_{14}^r\xi_{ov} - h_{15}h_{14}^r r_A^2 - s_1 r_B^2\right)r_A \tag{A5.164}$$

$$\frac{d\theta_A}{dt} = h_{14}^i\xi_{ov} - h_{15}h_{14}^i r_A^2 - s_2 r_B^2 \tag{A5.165}$$

Similarly, we work out the real and imaginary parts for equation A4.162

$$\left(h_{14}^r - ih_{14}^i\right)\left[\xi^{ov} - \left(h_{16}^r - ih_{16}^i\right)r_A^2 - h_{15}r_B^2\right]r_B =$$

$$\left(h_{14}^r\xi^{ov} - h_{15}h_{14}^r r_B^2 - \left(h_{16}^r h_{14}^r - h_{16}^i h_{14}^i\right)r_A^2\right)r_B - i\left(h_{14}^i\xi^{ov} - h_{15}h_{14}^i r_B^2 - \left(h_{16}^r h_{14}^i + h_{16}^i h_{14}^r\right)r_A^2\right)r_B$$

(A5.166)

With the same notations for s_1 and s_2 we have the equation A5.162 in the form

$$\frac{dr_B}{dt} = \left(h_{14}^r\xi_{ov} - s_1 r_A^2 - h_{15}h_{14}^r r_B^2\right)r_B \qquad \text{(A5.167)}$$

$$\frac{d\theta_B}{dt} = h_{14}^i\xi_{ov} - s_2 r_A^2 - h_{15}h_{14}^i r_B^2 \qquad \text{(A5.168)}$$

Finally, we have a system of four equations, two for absolute amplitudes of A and B and two for their corresponding phases

$$\begin{cases} \dfrac{dr_A}{dt} = \left(h_{14}^r\xi_{ov} - h_{15}h_{14}^r r_A^2 - s_1 r_B^2\right)r_A \\ \\ \dfrac{dr_B}{dt} = \left(h_{14}^r\xi_{ov} - s_1 r_A^2 - h_{15}h_{14}^r r_B^2\right)r_B \end{cases} \qquad \text{(A5.169)}$$

$$\begin{cases} \dfrac{d\theta_A}{dt} = h^i_{14}\xi_{ov} - h_{15}h^i_{14}r_A^2 - s_2 r_B^2 \\ \\ \dfrac{d\theta_B}{dt} = h^i_{14}\xi_{ov} - s_2 r_A^2 - h_{15}h^i_{14}r_B^2 \end{cases} \tag{A5.170}$$

In the system A5.169 we can further simplify the appearance by letting

$$\begin{matrix} h^r_{14} = h^r_1 & \qquad h_{15}h^r_{14} = h^r_2 \\ \\ h^i_{14} = h^i_1 & \qquad h_{15}h^i_{14} = h^i_2 \end{matrix} \tag{A5.171}$$

Hence, we can write the systems A5.169 and 170

$$\begin{cases} \dfrac{dr_A}{dt} = \left(h^r_1\xi_{ov} - h^r_2 r_A^2 - s_1 r_B^2\right) r_A \\ \\ \dfrac{dr_B}{dt} = \left(h^r_1\xi_{ov} - s_1 r_A^2 - h^r_2 r_B^2\right) r_B \end{cases} \tag{A5.172}$$

$$\begin{cases} \dfrac{d\theta_A}{dt} = h^i_{14}\xi_{ov} - h_{15}h^i_{14}r_A^2 - s_2 r_B^2 \\ \\ \dfrac{d\theta_B}{dt} = h^i_{14}\xi_{ov} - s_2 r_A^2 - h_{15}h^i_{14}r_B^2 \end{cases} \tag{A5.173}$$

For a steady state situation the system A5.172 becomes

$$\begin{cases} h_2^r r_A^2 + s_1 r_B^2 = h_1^r \xi_{ov} \\ \\ s_1 r_A^2 + h_2^r r_B^2 = h_1^r \xi_{ov} \end{cases} \tag{A5.174}$$

which for $r_A^2 \neq 0$ and $r_B^2 \neq 0$ we can solve it with Cramer's method

$$\begin{bmatrix} h_2^r & s_1 \\ s_1 & h_2^r \end{bmatrix} \begin{pmatrix} r_A^2 \\ r_B^2 \end{pmatrix} = \begin{pmatrix} h_1^r \xi_{ov} \\ h_1^r \xi_{ov} \end{pmatrix} \tag{A5.175}$$

To determine r_A^2 and r_B^2 we calculate Δ

$$\Delta = \begin{vmatrix} h_2^r & s_1 \\ s_1 & h_2^r \end{vmatrix} = \left(h_2^r + s_1\right)\left(h_2^r - s_1\right) \tag{A5.176}$$

$$\Delta r_A^2 = \begin{vmatrix} h_1^r \xi_{ov} & s_1 \\ h_1^r \xi_{ov} & h_2^r \end{vmatrix} = h_1^r \xi_{ov} \left(h_2^r - s_1\right) \tag{A5.177}$$

$$\Delta r_B^2 = \begin{vmatrix} h_2^r & h_1^r \xi_{ov} \\ s_1 & h_1^r \xi_{ov} \end{vmatrix} = h_1^r \xi_{ov} \left(h_2^r - s_1\right) \tag{A5.178}$$

Concluding that

$$r_A^2 = r_B^2 \tag{A5.179}$$

The *Reduced Amplitude Equation* in the case when $r_A^2 = r_B^2$ can be written as

$$\frac{dr_A}{dt} = \left[h_1^r \cdot \xi_{ov} + h_3 \cdot r_A^2\right] \cdot r_A \tag{A5.180}$$

where

$$h_3 = -\frac{1}{2}\frac{\alpha R_{\sigma}\left[\gamma(\alpha+1)(1+\sigma_0^2)+\gamma Ta - R_{\sigma}\right]}{\left[\gamma(\alpha+1)(1-\sigma_0^2)+\gamma Ta - R_{\sigma}\right]^2+\left[2\gamma\sigma(\alpha+1)\right]^2} \times \left[\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2+\gamma^2\sigma^2\right]}+\frac{\alpha\left[8(\alpha+1)+\sigma_0^2\gamma^2(\alpha-1)\right]}{\left[(\alpha+1)^2+\gamma^2\sigma^2\right](4+\sigma_0^2\gamma^2)}\right]+$$

$$+\frac{\alpha R_{\sigma}\left[\gamma(\alpha+1)(1+\sigma_0^2)-\gamma Ta + R_{\sigma}\right]}{\left[\gamma(\alpha+1)(1-\sigma_0^2)+\gamma Ta - R_{\sigma}\right]^2+\left[2\gamma\sigma(\alpha+1)\right]^2} \times \frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2+\gamma^2\sigma^2\right](4+\sigma_0^2\gamma^2)}$$

(A5.181)

$$\xi_{ov} = \left[\frac{R}{R_{cr}^{ov}} - 1\right] \qquad R_{cr}^{ov} = \frac{2}{\alpha}\left[(1+\alpha)(1+\alpha+\gamma)+\frac{\gamma^2 Ta}{(1+\alpha+\gamma)}\right]$$

$$\sigma_o^2 = \frac{(1+\alpha-\gamma)Ta}{(1+\alpha)(1+\alpha+\gamma)} - 1$$

$R_{cr} = \dfrac{Ra_{cr}}{\pi^2}$ and α results from

$$\alpha^4 + 2(\gamma+1)\alpha^3 + \gamma(\gamma+1)\alpha^2 - 2\left[(\gamma+1)^2+\gamma^2 Ta\right]\alpha - \gamma^2(\gamma+1)Ta - (\gamma+1)^3 = 0$$

The *Reduced Phase Equation* in the case when $r_A^2 = r_B^2$ can be written as

$$\frac{d\theta}{dt} = h_1^i \cdot \xi_{ov} + \tilde{h}_3 \cdot r_A^2 \tag{A5.182}$$

where

$$\tilde{h}_3 = \frac{1}{2}\frac{\alpha R_\sigma\left[\gamma(\alpha+1)(1+\sigma_0^2) - \gamma Ta + R_\sigma\right]}{\left[\gamma(\alpha+1)(1-\sigma_0^2) + \gamma Ta - R_\sigma\right]^2 + \left[2\gamma\sigma(\alpha+1)\right]^2} \times \left[\frac{\alpha(\alpha+1)}{\left[(\alpha+1)^2 + \gamma^2\sigma^2\right]} + \frac{\alpha\left[8(\alpha+1) + \sigma_0^2\gamma^2(\alpha-1)\right]}{\left[(\alpha+1)^2 + \gamma^2\sigma^2\right](4+\sigma_0^2\gamma^2)}\right] +$$

$$- \frac{\alpha R_\sigma\left[\gamma(\alpha+1)(1+\sigma_0^2) - \gamma Ta + R_\sigma r\right]}{\left[\gamma(\alpha+1)(1-\sigma_0^2) + \gamma Ta - R_\sigma\right]^2 + \left[2\gamma\sigma(\alpha+1)\right]^2} \times \frac{\alpha\gamma\sigma_0(\alpha+3)}{\left[(\alpha+1)^2 + \gamma^2\sigma^2\right](4+\sigma_0^2\gamma^2)}$$

(A5.183)

5.4. The full solvability condition

By considering that $r_{A/B} = r_{A/B}(x,t)$ and $\theta_{A/B} = \theta_{A/B}(x,t)$ we have in A4.131 first and then in A4.132

$$\frac{\partial^2 A}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[r_A e^{i\theta_A}\right] = e^{i\theta_A}\frac{\partial^2 r_A}{\partial t^2} - r_A e^{i\theta_A}\frac{\partial^2 \theta_A}{\partial t^2} \qquad \text{(A5.184)}$$

$$\frac{\partial^2 A}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[r_A e^{i\theta_A}\right] = e^{i\theta_A}\frac{\partial^2 r_A}{\partial x^2} - r_A e^{i\theta_A}\frac{\partial^2 \theta_A}{\partial x^2} \qquad \text{(A5.185)}$$

$$\frac{\partial^2 B}{\partial t^2} = \frac{\partial^2}{\partial t^2}\left[r_B e^{i\theta_B}\right] = e^{i\theta_B}\frac{\partial^2 r_B}{\partial t^2} - r_B e^{i\theta_B}\frac{\partial^2 \theta_B}{\partial t^2} \qquad \text{(A5.186)}$$

$$\frac{\partial^2 B}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left[r_B e^{i\theta_B}\right] = e^{i\theta_B}\frac{\partial^2 r_B}{\partial x^2} - r_B e^{i\theta_B}\frac{\partial^2 \theta_B}{\partial x^2} \qquad \text{(A5.187)}$$

As a result, both equations can be split into a amplitude and phase equations.

Amplitude equation for r_A

$$\left[h_{11}^{r}\frac{\partial^2 r_A}{\partial t^2}+h_{13}^{r}\frac{\partial r_A}{\partial t}+h_{12}^{r}\frac{\partial^2 r_A}{\partial x^2}\right]-\left[h_{11}^{r}\frac{\partial^2\theta_A}{\partial t^2}-h_{13}^{i}\frac{\partial\theta_A}{\partial t}+h_{12}^{r}\frac{\partial^2\theta_A}{\partial x^2}\right]r_A -$$

(A5188)

$$\left[N_1\xi_{ov}+N_2 r_A^2+N_3 r_B^2\right]r_A=0$$

Phase equation for r_A

$$\left[h_{11}^{i}\frac{\partial^2 r_A}{\partial t^2}+h_{13}^{i}\frac{\partial r_A}{\partial t}+h_{12}^{i}\frac{\partial^2 r_A}{\partial x^2}\right]-\left[h_{11}^{i}\frac{\partial^2\theta_A}{\partial t^2}-h_{13}^{r}\frac{\partial\theta_A}{\partial t}+h_{12}^{i}\frac{\partial^2\theta_A}{\partial x^2}\right]r_A -$$

(A5.189)

$$\left[\tilde{N}_1\xi_{ov}+\tilde{N}_2 r_A^2+\tilde{N}_3 r_B^2\right]r_A=0$$

Amplitude equation for r_B

$$\left[h_{11}^{r}\frac{\partial^2 r_B}{\partial t^2}+h_{13}^{r}\frac{\partial r_B}{\partial t}+h_{12}^{r}\frac{\partial^2 r_B}{\partial x^2}\right]-\left[h_{11}^{r}\frac{\partial^2\theta_B}{\partial t^2}-h_{13}^{i}\frac{\partial\theta_B}{\partial t}+h_{12}^{r}\frac{\partial^2\theta_B}{\partial x^2}\right]r_A$$

(A5.190)

$$-\left[N_1\xi_{ov}+N_2 r_A^2+N_3 r_B^2\right]r_B=0$$

Phase equation for r_B

$$\left[h_{11}^{i}\frac{\partial^2 r_B}{\partial t^2}+h_{13}^{i}\frac{\partial r_B}{\partial t}+h_{12}^{i}\frac{\partial^2 r_B}{\partial x^2}\right]-\left[h_{11}^{i}\frac{\partial^2\theta_B}{\partial t^2}-h_{13}^{r}\frac{\partial\theta_B}{\partial t}+h_{12}^{i}\frac{\partial^2\theta_B}{\partial x^2}\right]r_A$$

(A5.191)

$$-\left\{\tilde{N}_1\xi_{ov}+\tilde{N}_2 r_A^2+\tilde{N}_3 r_B^2\right]r_B=0$$

where all h_{ij} coefficients are expressed in terms of their real and imaginary parts.

Furthermore

$$N_1 = h_{13}^r h_{14}^r - h_{13}^i h_{14}^i \tag{A5.192}$$

$$N_2 = h_{13}^r h_{14}^r h_{15} - h_{13}^i h_{14}^i h_{15} \tag{A5.193}$$

$$N_3 = h_{13}^r h_{14}^r h_{16}^r - h_{13}^r h_{14}^i h_{16}^i - h_{13}^i h_{14}^r h_{16}^i - h_{13}^i h_{14}^i h_{16}^r \tag{A5.194}$$

$$\tilde{N}_1 = h_{13}^r h_{14}^i - h_{13}^i h_{14}^r \tag{A5.195}$$

$$\tilde{N}_2 = h_{13}^r h_{14}^i h_{15} - h_{13}^i h_{14}^r h_{15} \tag{A5.196}$$

$$\tilde{N}_3 = h_{13}^r h_{14}^r h_{16}^i - h_{13}^r h_{14} h_{16}^r - h_{13}^i h_{14}^r h_{16}^r - h_{13}^i h_{14}^i h_{16}^i \tag{A5.197}$$