ON HUPPERT CONJECTURE FOR SOME QUASI-SIMPLE GROUPS

By

Philani Rodney Majozi

UNIVERSITY OF KWAZULU-NATAL

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE
AT
UNIVERSITY OF KWAZULU NATAL
PIETERMARITZBURG, SOUTH AFRICA
28 NOVEMBER 2014

UNIVERSITY OF KWAZULU NATAL
SCHOOL OF
MATHEMATICS, STATISTICS AND COMPUTING SCIENCE

The undersigned hereby certify that they have read and recommend to the Faculty of Science for acceptance a thesis entitled "**On Huppert Conjecture for some Quasi-simple Groups**" by **Philani Rodney Majozi** in partial fulfillment of the requirements for the degree of **Master of Science**.

Dated: 28 November 2014

Supervisor: ______________________
Dr. Hung P. Tong-Viet

Co-Supervisor: ______________________
Prof. Fortuné Massamba

Readers: ______________________
firstreader

____
secondreader

UNIVERSITY OF KWAZULU NATAL

Date: **28 November 2014**

Author: **Philani Rodney Majozi**

Title: **On Huppert Conjecture for some Quasi-simple Groups**

Department: **Mathematics, Statistics and Computing Science**

Degree: **M.Sc.** Convocation: **April** Year: **2015**

Signature of Author

Table of Contents

Abstract

We extend Huppert conjecture from finite non-abelian simple groups to finite quasi-simple groups. Let $\mathrm{cd}(G)$ denote the set of degrees of ordinary irreducible characters of G. Let H be a finite quasi-simple group and G a finite group such that $\mathrm{cd}(H) = \mathrm{cd}(G)$. We conjecture that $G \cong H \circ A$, a central product of H and A, where A is an abelian group.

To give some evidence, we have established the conjecture for all quasi-simple linear groups of dimension 2 in [14]. In this dissertation, we verify the conjecture for the family of special linear groups $\mathrm{SL}_3(q)$ with $q \geq 7$.

DEDICATION

To my dearest mother Makhosazana Majozi, the soul of my late father Mr Bongani Majozi, the soul of his brother Zibonele Majozi, to the soul of my late aunt Sangoma Khanyisile Majozi and to my whole family Thembeka, Philokuhle, Khumbulani, Vukani, Nonkululeko, Nomfundo MaGama Majozi, Mfundo Zenze, Zanele Sibeko, Amahle Ngubane, Mosiwa Ngubane, Mpilo Ngubane, Bongumusa Zondi, Sanele Ngema, Zama Majozi with his family, Lungi Majozi with all her siblings and Bheki Majozi with his family, aunt Nomusa Majozi with her family and aunt Thembisile Majozi with her family, I dedicate this work.

Acknowledgements

I would like to express deep feelings of gratitude to my supervisor Dr Hung P Tong-Viet for his invaluable guidance at different stages of the project. It would have been impossible to carry on with the quest of research work to its latter part of a written thesis without his able guidance, support and sympathetic words of encouragement. Through his open-minded and skeptical guidance, I have learnt so much from him. I am also greatly indebted to him for his understanding and believing in me. In particular, by granting me permission to tackle one of the open problems in character theory as a Master of Science project.

I am also grateful to my co-supervisor Professor Fortuné Massamba for his support in the last three months of the project. I have benefited greatly from his suggestions and comments on the layout and structure of the thesis. I appreciate his continuous encouragement.

My sincere thanks goes to Mr Muntukabongwa Mahlaba for his constant encouragement and support during the whole period of this project. Further, I am also grateful to Miss Christel Barnard, Bev Bonhomme and Faith Nzimande for their support and encouragement.

My sincere thanks are also due to my former colleagues at Edendale Technical High School, five old wise men Mr Ndabezinhle Hlongwane, Mr Siyabonga Zulu, Mr Mbongeni Lembethe, Mr Fana Luthuli and Mr Fano Ngubane (Principal), who always preach to me about perseverance, hope and excellence in education.

I am also grateful to Phelelani Hadebe, Nonjabulo Cele, Sthandiwe Goge, Thokozani

Zuma, Ayanda Dlamini, Mbali Ndlovu, Lukhona Yenzela, Philisiwe Ngcobo, Andile Ngcobo, Bongumusa Ngcobo, Dimpho Phello, Nompilo Nguse, Xolisa Nguse, Xolani Zondi, Thamsanqa Zondi, Nhlakanipho Ndlovu, Nduduzo Zuma, Mhlengi Nyathi, Sandile Ngubane, Mpilwenhle Shabalala, Nomfundo Dubazane, Sanele Dlamini, Sisanda Mkhwanazi, and all my learners at Edendale Technical High School for their encouragement during the whole period of this project.

Finally, I am grateful to my family for their love, support, encouragement and creating pleasant working environment for me to learn. This include My mother Makhosazana , Thembeka, Philokuhle, Khumbulani, Vukani, Nonkululeko, Nomfundo MaGama all represented by last name Majozi and My extended family members are Mfundo Zenze, Mosiwa Ngubane, Amahle Ngubane and Sanele Ngema.

List of Notation

$\mathbb{N}$	natural numbers
$\mathbb{Z}$	integer numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
tr	trace of a matrix
$\mathbb{F}$	field
G	finite group G
$\lvert G\rvert$	order of G
χ	character of finite group
St_S	Steinberg character of group S
$\mathrm{Irr}(G)$	set of all ordinary irreducible characters of G
$\mathrm{cd}(G)$	set of all complex irreducible character degrees of G
1	identity element of a finite group
S_n	symmetric group of n objects
A_n	alternating group of n objects
$\cong$	isomorphism of groups
$H \leq G$	H is a subgroup of G
$H \trianglelefteq G$	H is a normal subgroup of G
G/H	quotient group
$Z(G)$	center of G
C_i	conjugacy class of G
$\mathrm{Aut}(G)$	automorphism group of G
$[h,k]$	commutator of $h, k \in G$
$C_G(g)$	centralizer of $g \in G$
G'	commutator or derived subgroup of G

G''	commutator subgroup of G'
$H \times A$	direct product of groups
$H \circ A$	central product of groups
$[G : H]$	index of H in G
(h, k)	cartesian coordinate of a product of two sets
$\mathrm{lcm}(\alpha, \beta)$	lowest common multiple of $\alpha, \beta \in \mathbb{Z}$
(α, β) or $\gcd(\alpha, \beta)$	greatest common divisor of $\alpha, \beta \in \mathbb{Z}$
$\alpha \mid \beta$	α is a factor or divisor of β
$\mathrm{SL}_n(q)$	special linear group of dimension n
$\mathrm{PSL}_n(q)$	projective special linear group of dimension n over the field of q elements
$I_G(\vartheta)$	inertia group

Introduction

A group is abelian if every pair of its elements commutes, otherwise it is nonabelian. A group with only two normal subgroups, namely the trivial subgroup and the group itself, is called a *simple group*. A finite group H is said to be quasisimple if H is perfect and $H/\mathbf{Z}(H)$ is nonabelian simple. Other important algebraic structures of interest in our study are *rings*, *fields* and *modules*.

We shall work with a finite group G. Let $\mathbb{F}$ be a field. The elements of $\mathbb{F}[G]$ are formal linear combinations of finitely many elements of G with co-efficient in $\mathbb{F}$. $\mathbb{F}[G]$ is an algebra over $\mathbb{F}$ with respect to multiplication defined by a rule

$$\left(\sum_{g\in G} a_g g\right)\left(\sum_{g\in G} b_g g\right) = \sum_{g\in G, g'\in G} (a_g b_g) g * g'$$

A group P is a central product of two normal subgroups R and S if $P = RS$, and $[R, S] = 1$. If $\varphi : G \longrightarrow \mathrm{GL}(n, \mathbb{F})$ is a group homomorphism, then φ is a Matrix Representation of G of degree n, over the field $\mathbb{F}$. A representation $\varphi : G \longrightarrow \mathrm{GL}(n, \mathbb{F})$ is said to be irreducible if has only trivial representations. The function $\chi : G \longrightarrow \mathbb{F}$ defined by $\chi(g) = \mathrm{tr}(\varphi(g))$ is called the *character* of φ. The value $\chi(1)$ is called the *character degree*. A map $\chi : G \to \mathbb{C}$ is a class function if it is constant on each conjugacy class of G. Thus, the character is also a class function.

Lemma 0.0.1. (*Isaacs, [10]*) *Let G be a finite group.*

1. *A character is irreducible if it is afforded by some irreducible representation.*

2. *The irreducible characters of G form a basis of the vector space of class functions.*
3. *Two representations of G are equivalent if and only if the corresponding characters are equal.*

Let $\mathrm{cd}(G)$ be the set of all complex irreducible character degrees of G. The set of character degrees of group G can be used to determine a certain information regarding the structure of G. For example, the result of Isaacs and Passman in [8] asserts that if $\mathrm{cd}(G)$ contains only 1 and primes then G is solvable.

It is understood that the structure of a finite group is not always determined entirely by the set of its irreducible character degrees. For example, quaternion group (Q_8) is a solvable group with the same character degree set with S_3. However $Q_8 \ncong S_3 \times A$ for any abelian group.

In the late 1990's, Bertram Huppert posed a conjecture which, if it true, then a simple group can be almost identified by its character degrees. Formally, Huppert stated his conjecture as follows:

Huppert's Conjecture: Let G be a finite group and H a finite nonabelian simple group such that the sets of character degrees of G and H are the same. Then $G \cong H \times A$, where A is an abelian group.

Huppert himself in [9] has proposed five-step method to prove the conjecture a given simple group.

Let G be a finite group and H a finite nonabelian simple group.

Step 1: Show $G' = G''$. Then if G'/M is a chief factor of G, $G'/M \cong S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$, a nonabelian simple group.

Step 2: Show $G'/M \cong H$, where H is a simple group.

Step 3: If $\theta \in \mathrm{Irr}(M)$, $\theta(1) = 1$, then $I_{G'}(\theta) = G'$, hence $[M, G'] = M'$.

Step 4: Show $M = 1$.

Step 5: Show $G = G' \times \mathbf{C}_G(G')$. As $G/G' \cong \mathbf{C}_G(G')$ is abelian and $G' \cong H$

Then the proof will be complete.

Huppert verified the conjecture by a case-by-case basis for many nonabelian simple groups, including the Suzuki groups, many of the sporadic simple groups, and a few of the simple groups of Lie type [9]. Except for the Suzuki groups and the family of simple groups $\mathrm{PSL}_2(q)$, for $q \geq 4$ prime or a power of a prime, Huppert proves the conjecture for specific simple groups of Lie type of small, fixed rank. Huppert verified his conjecture only for low degrees of alternating groups, specifically $5 \leq n \leq 11$.

In recent years, a substantial amount of research has been done on verifying Huppert conjecture. Wakefield verified Huppert conjecture for the simple groups of lie type of rank two [24]. Tong-Viet, Wakefield, Nguyen and others verified Huppert conjecture for many other groups. In particular, they verified the conjecture for alternating groups A_n, $12 \leq n \leq 13$ and the family of projective simple groups $\mathrm{PSL}_4(q)$, for $q \geq 13$ prime or a power of a prime.

In this dissertation, we seek to extend the Huppert's conjecture from nonabelian simple groups to quasisimple groups.

Huppert's Conjecture for quasisimple groups: Let G be a finite group and H a finite nonabelian quasisimple group such that the sets of character degrees of G and H are the same. Then $G \cong H \circ A$, where A is an abelian group.

In our quest to achieve these result, we shall modify the Huppert's method to suit the quasisimple groups. The following pattern is proposed in [14] to approach the conjecture in setting of quasisimple groups.

Step 1: Show $G' = G''$.

Step 2: Suppose that G'/M is a chief factor of G, $G'/M \cong S_1 \times S_2 \times \cdots \times S_k$, where $S_i \cong S$, a nonabelian simple group. Show $G'/M \cong H/\mathbf{Z}(H)$, where H is a quasisimple group.

Step 3: Show that G' is isomorphic to a perfect central cover of $H/\mathbf{Z}(H)$.

Step 4: Show that $G = G' \circ \mathbf{C}_G(G')$ and $\mathbf{C}_G(G')$ is abelian.

Step 5: Show that the covers of $H/\mathbf{Z}(H)$ have distinct sets of character degrees. Combining Step 3 and 4, we have that $H \cong G'$ and it follows that G is isomorphic to the central product of H and the abelian group $\mathbf{C}_G(G')$.

Then the proof will be complete.

To give some evidence, we have established the conjecture for all quasi-simple linear groups of dimension 2 in [14]. This work was carried out as a joint research before planning Master of Science work.

In this dissertation, we verify the conjecture for the family of special linear groups $\mathrm{SL}_3(q)$ with $q \geq 7$. $\mathrm{SL}_3(q)$ is a proper quasisimple group. i.e. It has a nontrivial center if and only if $q \equiv 1$ modulo 3. The case when $\mathrm{SL}_3(q)$ has a trivial center was verified by Wakefield in [25].

Chapter 1

BACKGROUND RESULTS

In our quest to verify the Huppert conjecture, we shall invoke and use available results to enable us to prove our main theorem.

We first introduce the result that will enable us to verify that G is quasi-perfect.

Lemma 1.0.2. *[[24], Lemma 1.4] Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$. If ψ is a constituent of χ_N, then $\chi(1)/\psi(1)$ divides $|G : N|$. In particular,*

$$\frac{\chi(1)}{\gcd(\chi(1), |G:N|)} \text{ divides } \psi(1)$$

Definition 1.0.1. (See [[21], pp 359].) Let $\chi \in \mathrm{Irr}(G)$. χ is said to be *isolated* in G if $\chi(1)$ is divisible by no proper nontrivial character degree of G, and no proper multiple of $\chi(1)$ is a character degree of G.

The following result is useful in the proof of Step 1. Please refer to Huppert [9] and Tong-Viet [21].

Lemma 1.0.3. *Let G/N be a solvable factor group of G, minimal with respect to being nonabelian. Then two cases can occur.*

(*a*) *G/N is an r-group for some prime r. Hence there exists $\psi \in \mathrm{Irr}(G/N)$ such that $\psi(1) = r^b > 1$ for some prime r. If $\chi \in \mathrm{Irr}(G)$ and $r \nmid \chi(1)$, then $\chi\tau \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(G/N)$.*

(*b*) *G/N is a Frobenius group with elementary abelian Frobenius kernel F/N. Then $f = |G : F| \in \mathrm{cd}(G)$ and $|F : N| = r^a$, and F/N is an irreducible module for the cyclic group G/F, hence a is the smallest integer such that $r^a - 1 \equiv 0 \pmod f$. For every $\psi \in \mathrm{Irr}(F)$, either $|G : F|\,\psi(1) \in \mathrm{cd}(G)$ or $|F : N| \mid \psi(1)^2$. In the latter case, r divides $\psi(1)$. Furthermore:*

(*1*) *If no proper multiple of f is in $\mathrm{cd}(G)$, then $\chi(1) \mid f$ for all $\chi \in \mathrm{Irr}(G)$ such that $r \nmid \chi(1)$, and if $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) \nmid f$, then $r^a \mid \chi(1)^2$.*

(*2*) *If $\chi \in \mathrm{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\mathrm{cd}(G)$, then either f divides $\chi(1)$ or r^a divides $\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G, then $f = \chi(1)$ or $r^a \mid \chi(1)^2$.*

Proof. Statements (a) and (b) follow from [[10], Lemma 3.12] and [[10], Theorem 12.4]. Suppose G/N is a Frobenius group. Assume that no proper multiple of f is in $\mathrm{cd}(G)$, and let $\chi \in \mathrm{Irr}(G)$. Let ψ be an irreducible constituent of χ_F. By [[10], Lemma 6.8], we have that $\chi(1) = k\psi(1)$ and by [[10], Corollary 11.29] we obtain $k \mid f = |G : F|$. By (b), we have that either $f\psi(1) \in \mathrm{cd}(G)$ or $r^a \mid \psi(1)^2$. Suppose $r \nmid \chi(1)$. Then $r \nmid \psi(1)$ so that $f\chi(1)/k \in \mathrm{cd}(G)$. As no proper multiple of f is a character degree of G, we deduce that $f\chi(1)/k = f$ so that $\chi(1) \mid f$. Now assume $\chi(1) \nmid f$. Then $r \mid \chi(1)$. Since $r \nmid f$, we deduce that $r \nmid k$, hence $r \mid \psi(1)$ so that $f\psi(1) > f$. Thus $f\psi(1)$ is not a character degree of G and so $r^a \mid \psi(1)^2$. As $\psi(1) \mid \chi(1)$, thus (1) follows. Similarly, we can prove (2).

Suppose that χ is in $\mathrm{Irr}(G)$ such that no proper multiple of $\chi(1)$ is in $\mathrm{cd}(G)$. Let $\psi \in \mathrm{Irr}(F)$ be an irreducible constituent of χ_F. As above, we have that $\chi(1) = k\psi(1)$, $k \mid f$ and either $f\psi(1) \in \mathrm{cd}(G)$ or $r^a \mid \psi(1)^2$. If the latter case holds then we are done since $\psi(1) \mid \chi(1)$. Now assume $f\psi(1) \in \mathrm{cd}(G)$. Observe that $\psi(1) = \chi(1)/k$ so that $f\psi(1) = f\chi(1)/k \in \mathrm{cd}(G)$, where $f\psi(1)/k$ is a multiple of $\chi(1)$ since $k \mid f$. As no proper multiple of $\chi(1)$ belongs to $\mathrm{cd}(G)$, it follows that $f\chi(1)/k = \chi(1)$, which implies that $k = f$. Since k divides $\chi(1)$, we deduce that $f \mid \chi(1)$. The remaining statement is obvious. The proof is complete. $\square$

Definition 1.0.2. We say that θ is extendible to G if there exists $\chi \in \mathrm{Irr}(G)$ such that the restriction (χ_N) of χ to N is θ.

We shall need some results from Clifford theory. The next two lemmas are stated as Lemma 2 and Lemma 3 in [9].

Lemma 1.0.4. *Let $N \trianglelefteq G$ and $\chi \in \mathrm{Irr}(G)$.*

(*a*) *If $\chi_N = \theta_1 + \theta_2 + \theta_3 + \cdots + \theta_k$ with $\theta_j \in \mathrm{Irr}(N)$, then k divides $|G : N|$. In particular, if $\chi(1)$ is relatively prime to $|G : N|$, then $\chi_N \in \mathrm{Irr}(N)$.*

(*b*) (*Gallagher's Lemma*) *If $\chi_N \in \mathrm{Irr}(N)$ then $\chi\theta \in \mathrm{Irr}(G)$ for every $\theta \in \mathrm{Irr}(G/N)$.*

Definition 1.0.3. Let $H \triangleleft G$ and $\vartheta \in \mathrm{Irr}(H)$. Then

$$I_G(\vartheta) = \{g \in G | \vartheta^g = \vartheta\}$$

is the inertia group of ϑ in G.

Lemma 1.0.5. *Suppose $N \trianglelefteq G$ and $\vartheta \in \mathrm{Irr}(N)$. By $I = I_G(\vartheta)$ we denote the inertia subgroup of ϑ in G.*

(a) *If $\vartheta^I = \sum_{i=1}^{k} \varphi_i$ with $\varphi_i \in \mathrm{Irr}(I)$, then $\varphi_i^G \in \mathrm{Irr}(G)$. In particular $\varphi_i(1)\,|G : I| \in \mathrm{cd}(G)$.*

(b) *If ϑ allows an extension ϑ_0 to I, then $(\vartheta_0\tau)^G \in \mathrm{Irr}(G)$ for all $\tau \in \mathrm{Irr}(I/N)$. In particular $\vartheta(1)\tau(1)\,|G : I| \in \mathrm{cd}(G)$.*

(c) *If $\varrho \in \mathrm{Irr}(I)$ such that $\varrho_N = e\vartheta$, then $\varrho = \vartheta_0\tau_0$, where ϑ_0 is a character of an irreducible projective representation of I of degree $\vartheta(1)$ while τ_0 is the character of an irreducible projective representation I/N of degree e.*

Definition 1.0.4. The automorphism group of a finite group G, denoted by $\mathrm{Aut(G)}$, is the set all the automorphisms of G under the operation of composition.

The following lemma will be used to verify Step 2 and 3.

Lemma 1.0.6. *[[1], Lemma 5] Let N be a minimal normal subgroup of G so that $N \cong S_1 \times S_2 \times S_3 \times \cdots \times S_k$, where $S_i \cong S$, a nonabelian simple group. Let A be the automorphism group of S. If $\sigma \in \mathrm{Irr}(S)$ extends to A, then $\sigma \times \sigma \times \cdots \times \sigma \times \sigma \in \mathrm{Irr}(N)$ extends to G.*

This lemma assert that if an irreducible character χ of S extends to the automorphism group of S, then $\chi(1)^k$ is a degree of G.

We shall combine the results from Lemma 1.0.6 with the next lemma to prove Step 2.

Lemma 1.0.7. *[[19], Lemma 5.2] If S is a nonabelian simple group, then there exists a nontrivial irreducible character θ of S that extends to $\mathrm{Aut}(S)$. The following holds.*

(a) *If S is an alternating group* A_n *with* $n \geq 7$*, then* $\mathrm{Irr}(S)$ *contains at least five nonlinear irreducible characters of different degrees which extend to* $\mathrm{Aut}(A_n)$*. in particular, S has two consecutive characters of degrees* $\dfrac{n(n-3)}{2}$ *and* $\dfrac{(n-1)(n-2)}{2}$ *that both extend to* $\mathrm{Aut}(S)$.

(b) *If S is a sporadic simple group or the Tits group then there exist at least four distinct nonlinear irreducible characters of different degrees of S which extend to* $\mathrm{Aut}(S)$*. Moreover, if S is a sporadic simple group or the Tits group, then S has two nontrivial irreducible characters of relatively prime degrees which both extend to* $\mathrm{Aut}(S)$.

(c) (*Schmid, [20]*) *If S is a simple group of Lie type then the Steinberg character* St_S *of S of degree* $|S|_p$ *extends to* $\mathrm{Aut}(S)$.

Definition 1.0.5. If G is a group, then we denote by $\mathrm{Mult}(G)$ the Schur multiplier of G. A group H is called a *covering group* of G if there exists a subgroup A of H such that $A \leq H' \cap Z(H)$ and $H/A \cong G$. A covering group H of G is called a *universal covering group* if $|Z(H)| = |\mathrm{Mult}(G)|$.

In this study, the cases for small values of q where the Schur multiplier has many perfect covers, shall be called the exceptional cases. It follows that their Schur multiplier will be called the exceptional Schur multipliers.

The following table provides us with the list of exceptional Schur multipliers. (See Theorem 5.1.4 in [12])

Table 0.1. Exceptional Schur Multiplier	
L	Mult(L)
$PSL_2(4)$, $PSL_3(2)$, $PSL_4(2)$, $U_4(2)$, $Sp_6(2)$, $G_2(2)$, $F_4(2)$	$\mathbb{Z}_2$
$G_2(3)$	$\mathbb{Z}_3$
$PSL_2(9)$, $Sp_4(2)'$, $\Omega_7(3)$	$\mathbb{Z}_6$
$PSL_3(4)$	$\mathbb{Z}_4 \times \mathbb{Z}_{12}$
$U_4(3)$	$\mathbb{Z}_3 \times \mathbb{Z}_{12}$
$U_6(2)$, ${}^2E_6(2)$	$\mathbb{Z}_2 \times \mathbb{Z}_6$
$\Omega_8^+(2)$, ${}^2B_2(8)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$

The following lemma will be used to verify Step 3.

Lemma 1.0.8. *[[9], Lemma 6] Suppose $M \trianglelefteq G' = G''$ and $\theta^g = \theta$ for all $g \in G'$ and $\theta \in \mathrm{Irr}(M)$ such that $\theta(1) = 1$. Then $M' = [M, G']$ and $|M : M'|$ divides the order of the Schur multiplier of G'/M.*

Finally, we introduce the next two lemmas to extend the proof of Step 3 from simple groups to quasi-simple groups. This lemmas appear in [14] as Lemma 3 and Lemma 4.

Lemma 1.0.9. *If $S = \mathrm{PSL}_3(q)$ with $q \geq 7$ and let A be an abelian group of order less than or equal to $|\mathrm{Mult}(S)|$, then $|S| > |\mathrm{Aut}(A)|$.*

Proof. We consider the linear groups $S = \mathrm{PSL}_3(q)$ with $q \geq 7$. We have $|\mathrm{Mult}(S)| = (3, q-1) = 1$ or 3. Furthermore, $|A| \leq 1$ or 3 and $|S| \geq 1\ 876\ 896$. It follows that $|S| > |\mathrm{Aut}(A)|$, which imply the lemma. □

Lemma 1.0.10. *Let H be $\mathrm{SL}_3(q)$ with $q \geq 7$, G be a perfect group and M a normal subgroup of G such that $G/M \cong H$ and $|M : M'|$ divides $\left|\mathrm{Mult}(\mathrm{PSL}_3(q))\right|$. Then G/M' is isomorphic to the cover of $\mathrm{PSL}_3(q)$.*

Proof. As M/M' is abelian and normal in G/M', we have

$$\frac{M}{M'} \leq C_{G/M'}\left(\frac{M}{M'}\right) \trianglelefteq \frac{G}{M'}.$$

We first consider the case $C_{G/M'}(M/M') = G/M'$. Then M/M' is central in G/M' and we deduce that G/M' is a perfect central cover of $G/M \cong H$, which implies that G/M' is a cover of $H/Z(H)$, as wanted.

The lemma is proved if we can show that $C_{G/M'}(M/M')$ cannot be a *proper* normal subgroup of G/M'. Assume so, then

$$\frac{C_{G/M'}(M/M')}{M/M'} \triangleleft \frac{G/M'}{M/M'} \cong \frac{G}{M} = H.$$

Therefore,

$$\left|\frac{C_{G/M'}(M/M')}{M/M'}\right| \leq |Z(H)|$$

and hence

$$|C_{G/M'}(M/M')| \leq |M/M'||Z(H)|.$$

Thus

$$\left|\frac{G/M'}{C_{G/M'}(M/M')}\right| \geq \frac{|G/M|}{|Z(H)|} = |H/Z(H)|.$$

Since the quotient group on the left side can be embedded in $\mathrm{Aut}(M/M')$ and M/M' is abelian of order less than or equal to $|\mathrm{Mult}(H/Z(H))|$, this last inequality leads to a contradiction by Lemma 1.0.9. $\square$

Chapter 2

VERIFYING HUPPERT'S CONJECTURE FOR $\mathrm{SL}_3(q)$

2.1 Results concerning the character degrees for $\mathrm{SL}_3(q)$

Let $q = p^f$ for some prime p and integer f. We only need $q \equiv 1 \pmod 3$ and $q \geq 7$. We consider this case only as there are many perfect covers for the group $\mathrm{SL}_3(4)$. In particular, the multiplier of this group has a structure $\mathbb{Z}_4 \times \mathbb{Z}_{12}$. Moreover, the case when $\mathrm{SL}_3(q)$ has a trivial center was verified by Wakefield in [25]. We postpone the cases when $q \equiv 1 \pmod 3$ and $q < 7$ to another time.

The list of irreducible character degrees of $\mathrm{SL}_3(q)$ is obtained in [5]. $\mathrm{cd}(G) = \mathrm{cd}(\mathrm{SL}_3(q)) = \Big\{1, q^3, q(q+1), (q-1)^2(q+1), q(q^2+q+1), (q-1)(q^2+q+1), q^2+q+1, (q+1)(q^2+q+1), \frac{1}{3}(q-1)^2(q+1), \frac{1}{3}(q+1)(q^2+q+1)\Big\}$.

We shall examine the degrees of G and establish their properties. In particular, we shall identify which degrees of G are nontrivial powers, consecutive degrees and composite prime powers.

The following Lemmas are useful in the proof of step 1. All these statements but the last appear as Lemma 3.1 and Lemma 3.2 in [25].

Lemma 2.1.1. *The number $q^2 + q + 1$ cannot be written in the form y^n for $n > 1$.*

Proof. As Nagell showed in [17], $q^2 + q + 1 = y^n$ has no solutions unless n is 3. In that case, as proved in [13], the only solutions are $(q, y, n) = (18, 7, 3)$ and $(q, y, n) = (-19, 7, 3)$. As q is prime or a power of a prime, we see that neither of these cases is possible. Thus $q^2 + q + 1$ cannot be expressed as y^n for $n > 1$. □

Lemma 2.1.2. *For $q > 2$, the degrees q^3, $q(q + 1)$, $q(q^2 + q + 1)$ and $(q - 1)^2(q + 1)$ are maximal with respect to divisibility among the degrees of $\mathrm{SL}_3(q)$. Moreover, q^3 and $q(q + 1)$ are isolated degrees of $\mathrm{SL}_3(q)$.*

Proof. The degrees $q^3, (q-1)^2(q+1)$, $q(q+1)$, $q(q^2+q+1)$ are maximal with respect to divisibility among the degrees of $\mathrm{SL}_3(q)$, since $(q, (q-1)^2(q+1)) = 1$, $(q^3, (q-1)^2(q+1)) = 1$, $(q, q^2 + q + 1) = (q^3, q^2 + q + 1) = 1$, $(q + 1, q^2 + q + 1) = 1$, $(q, q + 1) = (q^3, q + 1) = 1$ and $(q^2 + q + 1, (q - 1)^2(q + 1)) = 1$. Examining the degrees of $\mathrm{SL}_3(q)$, we observe that q^3, $q(q + 1)$ have no nontrivial proper divisor in $\mathrm{cd}(\mathrm{SL}_3(q))$ as required. □

Lemma 2.1.3. *For $q \geq 7$, the only nontrivial powers among the degrees of G are q^3 and possibly $q^3 - 1, (q - 1)^2(q + 1)$ and $(q - 1)^2(q + 1)/3$. The only degree of the form p^b for some integer $b \geq 1$ is q^3.*

Proof. By Theorem 1 of [4], the product of consecutive integers $q(q + 1)$ is never a nontrivial power. As $q^2 + q + 1$ is not a power and $(q, q^2 + q + 1) = (q^2 + q + 1, q + 1) = 1$, we have that $q^2 + q + 1$, $q(q^2 + q + 1)$ and $(q + 1)(q^2 + q + 1)$ are not nontrivial powers.

In the result of Nagell [17], the equation $x^2 + x + 1 = 3y^n$ has only the trivial solution ($x = y = 1$) if $n \geq 3$. When $n = 2$, this equation has infinitely many solutions. Even when x is a prime power, we have for example $(x, y) = (313, 181)$ and $(2288805793, 1321442641)$.

We are left to show that $(q+1)(q^2+q+1)/3$ is not a nontrivial power. Suppose by contradiction that $(q+1)(q^2+q+1)/3 = y^n$ for some integers $n \geq 2$ and $y \geq 2$. By Nagell [21], we obtain that $n = 2$. Next, we have $3 \mid q^2+q+1$ since $q \equiv 1 \pmod 3$ and $q \geq 3$. Since $(q+1, (q^2+q+1)/3) = 1$, we deduce that $q^2+q+1 = 3a^2$ and $q+1 = b^2$ for some integer $a, b \geq 2$ with $y = ab$. Since b^2 is congruent to 0 or 1 modulo 3, we deduce that $q = b^2 - 1$ is congruent to -1 or 0 modulo 3, contradicting our assumption that $q \equiv 1 \pmod 3$.

□

Lemma 2.1.4. *The pairs of consecutive integers among character degrees of G, for $q \geq 2$, are*

$$q(q+1) \text{ and } q^2+q+1$$

as well as

$$q^3-1 \text{ and } q^3.$$

Proof. Consider $q > 2$. q^2+q and q^2+q+1 are coprime degrees of G, since for $(q, q^2+q+1) = 1$ and $(q+1, q^2+q+1) = 1$. Hence, it follows that $(q^3, q^2+q+1) = 1$. As $q > 2$, combining the fact that $(q, q^2+q+1) = 1$ and $(q, q-1) = 1$, we have $(q^3, (q^2+q+1)(q-1)) = 1$. □

Definition 2.1.1. A number x is an n^{th} *root of unity* if $x^n - 1 = 0$. x is said to be a *primitive* n^{th} *root of unity* if n is the least integer of $\alpha = 1, 2, 3, \cdots, n$ for which $x^\alpha - 1 = 0$. A polynomial given by

$$\Phi_n(x) = \prod_{k=1}^{n}(x - \xi_k) \text{ where } \xi_k \text{ are the primitive } n^{th} \text{ roots of unity in } \mathbb{C}$$

is called the nth cyclotomic polynomial.

In the verification of Step 2, we shall use the classification of finite simple groups. We shall eliminate other families of nonabelian simple groups to show that the only possible group isomorphic to G'/M is $\mathrm{PSL}_3(q)$. The simple groups of Lie type have character degrees which involve cyclotomic polynomials in q as factors. We need the following lemma to eliminate the simple groups of Lie type.

Lemma 2.1.5. *The only possible divisors of the yth cyclotomic polynomial Φ_y are the largest prime divisor of y (but not its square if $y > 2$) and numbers of the form $1 + ky$.*

2.2 Establishing that G is quasi-perfect when $H \cong \mathrm{SL}_3(q)$

A finite group is called quasi-perfect if $G' = G''$. By way of contradiction, suppose $G' \neq G''$. Then there exists a normal subgroup N of G such that G/N is solvable factor group minimal with respect to being nonabelian. By Lemma 1.0.3, G/N is an r-group or a Frobenius group.

Case 1: G/N is an r-group for some prime r. By Lemma 1.0.3 (a), G/N has a character degree $r^b > 1$. Therefore, r^b could be equal to r. Hence, we need to consider the case $r = p$ and $r \neq p$ separately.

Subcase 1(a): $r = p$. By Lemma 2.1.3, q^3 is the only nontrivial prime power degree of G, hence we deduce that $r^b = q^3 \in \mathrm{cd}(G/N)$. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = q^2 + q + 1$. Then $r \nmid \chi(1)$. By Lemma 1.0.4 (a), $\chi_N \in \mathrm{Irr}(N)$. Take $\eta \in \mathrm{Irr}(G/N)$ with $\eta(1) = r^b = q^3$. By lemma 1.0.4 (b), we obtain $\eta\chi \in \mathrm{Irr}(G)$, hence $\eta(1)\chi(1) = q^3(q^2 + q + 1) \in \mathrm{cd}(G)$, a contradiction.

Subcase 1(b): $r \neq p$. Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = q^3$. Then $(r, \chi(1)) = 1$ so $(|G/N|, \chi(1)) = 1$ and thus $\chi_N \in \mathrm{Irr}(N)$. Take $\varphi \in \mathrm{Irr}(G/N)$ with $\varphi(1) = r^b$.

By Lemma 1.0.6 (b), we have $\varphi\chi \in \mathrm{Irr}(G)$ hence $r^b q^3 \in \mathrm{cd}(G)$ which is impossible.

Case 2: G/N is a Frobenius group with elementary abelian Frobenius kernel F/N, where $|F : N| = r^c$ for some prime r. In addition, $f = |G : F| \in \mathrm{cd}(G)$.

Subcase 2(a): $r \neq p$. Let $\chi \in \mathrm{Irr}(G)$, $\chi(1) = q^3$. As $r \nmid \chi(1)$ and no proper multiples and no proper divisors of this degree are in $\mathrm{cd}(G)$, we deduce from Lemma 1.0.3 that $f = q^3$. Hence $r \mid \zeta(1)$ for any $\zeta \in \mathrm{Irr}(G)$ with $p \nmid \zeta(1)$. Let $\varphi \in \mathrm{Irr}(G)$ with $\varphi(1) = q(q+1)$ and let $\psi \in \mathrm{Irr}(F)$ be an irreducible constituent of φ_F. Since $f = q^3$, by Lemma 1.0.2, we have $\varphi(1)/\psi(1) \mid q^3$, which implies that $q + 1 \mid \psi(1)$. Moreover, we have $f\psi(1) \notin \mathrm{cd}(G)$. Since $r \neq p$ by Lemma 1.0.3, we have $r^c \mid \psi(1)^2_{p'}$. As $q \geq 7$ and $|G : F| \mid r^c - 1$, we have

$$f = q^3 \leq r^c - 1 \leq \psi(1)^2_{p'} - 1 < (q+1)^2 = q^2 + 2q + 1 < q^3,$$

a contradiction.

Subcase 2(b): $r = p$. We have $|G : F| = f \in \mathrm{cd}(G)$ and $f \neq q(q+1)$. By Lemma 2.1.2, $q(q+1)$ is an isolated degree of G and it follows by Lemma 1.0.3 that $r^c \mid q^2(q+1)^2$. Hence $r^c \mid q^2$. Furthermore by Lemma 1.0.3, $f \mid (r^c - 1)$ and we deduce that $f \leq q^2 - 1$. However $f \leq q^2 - 1 < q^2 < q(q+1)$, the smallest nontrivial character degree of G. Thus $f \notin \mathrm{cd}(G)$, leading to a contradiction.

Therefore we have $G' = G''$.

2.3 Establishing $G'/M \cong \mathrm{PSL}_3(q)$ when $H \cong \mathrm{SL}_3(q)$

Step 2 asserts that if G'/M is a chief factor of G, then $G'/M \cong H/\mathbf{Z}(H)$.

Suppose G'/M is a chief factor of G. By Step 1,

$$G'/M \cong S_1 \times S_2 \times S_3 \times \cdots \times S_k,$$

where $S_i \cong S$, a nonabelian simple group. We seek to prove that $G'/M \cong H/\mathbf{Z}(H) = \mathrm{PSL}_3(q)$.

By Lemma 6.8 in [10] and Clifford theory, if $N \trianglelefteq K$, then every degree of N must divide some degree of K. Hence, the degrees of S must divide the degrees of G. By the classification of the finite nonabelian simple groups, the possibilities for S include one of 26 sporadic simple groups, the Tits group, the alternating groups A_n for $n \geq 5$, the ten families of simple groups of exceptional Lie type and six families of simple groups of classical Lie type.

We must show that $k = 1$ and eliminate all possibilities except $\mathrm{PSL}_3(q)$.

2.3.1 Eliminating the alternating, sporadic, the Tits groups when $k \geq 1$

Proposition 2.3.1. *If S is isomorphic to an alternating group A_n with $n \geq 7$, a sporadic simple group, or the Tits group then $k = 1$.*

Proof. By the isomorphisms $A_5 \cong \mathrm{PSL}_2(5) \cong \mathrm{PSL}_2(4)$ and $A_6 \cong \mathrm{PSL}_2(9)$, the alternating groups A_5, A_6 can be considered as simple groups of Lie type. Thus, we consider alternating groups for $n \geq 7$. By way of contradiction, suppose that $k > 1$ and S is isomorphic to an alternating group for some $n \geq 7$. By Lemma 1.0.7, S has nonlinear irreducible characters of distinct degrees ψ_1,ψ_2,ψ_3, ψ_4 and ψ_5 which extend to $\mathrm{Aut}(S)$. By Lemma 1.0.6, we have ψ_1^k,ψ_2^k,ψ_3^k, ψ_4^k and ψ_5^k extend to G. However by Lemma 2.1.3, there are at most four nontrivial powers among the degrees of G. Hence, if $S \cong A_n, n \geq 7$ then $k = 1$.

Moreover, if S is a sporadic simple group or the Tits group, then by Lemma 1.0.7, S has at least five distinct nonlinear irreducible characters of different degrees which extend to $\mathrm{Aut}(S)$. However, G has at most four possible nontrivial power degrees and we obtain a contradiction as in the previous cases. □

Proposition 2.3.2. *The simple group S is not an alternating group A_n with $n \geq 7$ for $k = 1$.*

Proof. As $n \geq 7$, $S \cong A_n$ has irreducible characters $\theta_i, 1 \leq i \leq 3$, which extend to $\mathrm{Aut}(S) \cong \mathrm{S}_n$ of degree $n-1, n(n-3)/2$ and $n(n-3)/2+1 = (n-1)(n-2)/2$, respectively. So, $n-1, n(n-3)/2$ and $(n-1)(n-2)/2$ are degrees of G, where the last two are consecutive degrees. Since $n(n-3)/2 > n-1 > 1$ and $q(q+1)$ is a non-trivial minimal degree of G, we deduce from Lemma 2.1.4 that $n(n-3)/2 = q^3 - 1$ and $q^3 = (n-1)(n-2)/2$. Since $n \geq 7$ and $\gcd(n-1, n-2) = 1$, we can see that the latter equation cannot hold. □

Proposition 2.3.3. *The simple group S is not one of the 26 sporadic simple groups or Tits group for k=1.*

Proof. $\mathrm{SL}_3(q)$ has at most 10 irreducible character degrees. For this reason, we shall consider the following cases of sporadic simple groups with 15 or less extendible characters of distinct degrees to argue that S cannot be one of them.

Case 1: $S \cong M_{11}$, $S \cong M_{12}$, $S \cong M_{23}$ or $S \cong J_1$

For each of these sporadic simple groups, irreducible characters of consecutive degrees extend to its automorphism group. The higher degrees of these consecutive degrees are not a prime power. For $q \geq 4$, the higher degrees of G are q^3 and $q^3 - 1$. Hence this is impossible.

Case 2: $S \cong M_{22}$

The simple group M_{22} has irreducible characters of relatively prime degrees 45 and 154 which extend to $\mathrm{Aut}(M_{22})$. Examining the list of pair of relatively prime degrees of G, we observe that this is impossible.

Case 3: $S \cong^2 F_4(2)'$, the Tits group

This group has eight irreducible characters with nontrivial degrees which extend to its automorphism group. 3^3 is the only power of a prime power which extends. However, as $q \geq 7$, q^3 is the only character degree of G that is a power of a prime. Hence, we have a contradiction.

□

2.3.2 Eliminating the groups of Lie type when $k \geq 1$

To eliminate the groups of Lie type when $k \geq 1$, we will require the Steinberg character of these groups.

Definition 2.3.1. A *mixed degree* of S is a degree of S which is divisible by q but is not a power of q.

If S is a simple group of Lie type and χ is the Steinberg character of S, then $\chi(1)$ is a power of the prime p, where p is the defining characteristic of the group. By

Lemma 1.0.7, χ extends to the automorphism group of S. By Lemma 1.0.6, we have that $\chi(1)^k$ is a degree of G. As the only composite power of a prime among degrees of G is q^3, we must have that $\chi(1)^k = q^3$. Hence, the defining characteristic of the simple group S must be the same as the prime divisor of q^3, which is p. Now we will show that $k = 1$ for the simple groups of Lie type. Let $S = S(q_1)$ be defined over a field of q_1 elements.

We rely upon the following Lemma [Lemma 5.6 in [22]].

Lemma 2.3.4. *If S is a simple group of Lie type and $S \ncong \mathrm{PSL}_2(q_1)$, then S possesses an irreducible character of mixed degree.*

Proposition 2.3.5. *If S is a simple group of Lie type and $S \ncong \mathrm{PSL}_2(q_1)$, then k=1.*

Proof. Suppose that $k \geq 2$. The Steinberg character of S extends to $\mathrm{Aut}(S)$. Thus, we have $\mathrm{St}_S(1)^k = q^3$. Take $\mathrm{St}_S(1) = q_1^j$. Since $S \ncong \mathrm{PSL}_2(q_1)$, by Lemma 2.3.4, S possesses an irreducible character of mixed degree, say ψ. Since by step 1,

$$G'/M \cong S_1 \times S_2 \times S_3 \times \cdots \times S_k,$$

there is an irreducible character of G'/M found by multiplying $k-1$ copies of St_S with ψ. Then $(\mathrm{St}_S^{k-1}\psi)(1)$ is a mixed degree of G'/M. As the degrees of G'/M divides the degrees of G, we must have that the degree of this irreducible character divide one of the mixed degrees of G. The highest power of q on any mixed degree of G is 1. We have $q_1^{jk} = q^3$ then $q = q_1^{jk/3}$. The power of q_1 in $(\mathrm{St}_S^{k-1}\psi)(1)$ is at least $j(k-1)$. We must have $j(k-1) \leq jk/3$, which reduces to $2k \leq 3$. Thus $k < 2$, a contradiction. Hence $k = 1$ if $S \ncong \mathrm{PSL}_2(q_1)$ $\square$

Since an alternating, the Tits, and sporadic groups have been eliminated as possibilities for S, we have that S is a simple group of Lie type. We must eliminate the case when $k > 1$ and $S \cong \mathrm{PSL}_2(q_1)$.

Proposition 2.3.6. *The simple group S is not $\mathrm{PSL}_2(q_1)$ for any $k \geq 1$.*

Proof. Suppose that $k = 1$. The degree of the Steinberg character of $\mathrm{PSL}_2(q_1)$ is q_1. If q_1 is prime, it is impossible for $q_1 = q^3$. If q_1 is composite, then $q_1 + 1 = q^3 + 1$ divides a degree of G, which is impossible.

Now consider the possibility that $k = 2$. Here $q_1^2 = q^3$. Again, if q_1 is prime, it is not possible that $q_1^2 = q^3$. If q_1 is composite, then $(\sqrt{q^3} + 1)^2$ is a degree of G'/M, hence must divide a degree of G. This is a contradiction.

Finally, consider the possibility that $k > 2$. Then $q_1^k = q^3$. Hence $q = q_1^{k/3}$. Consider the irreducible character of G'/M found by multiplying $k - 1$ copies of the Steinberg character with a character of S of degree $q_1 - 1$. As the degree of this character must divide a degree of G, we find that $k - 1 \leq k/3$, which implies $k \leq 2$, a contradiction. □

Proposition 2.3.7. *The group S is not a simple group of exceptional Lie type.*

Proof. We will examine each of the families of simple groups of exceptional Lie type. We will rely upon the results in Table 1 of [22]. We shall argue as in [22], [24] and [25], to verify that S is not a simple group of exceptional Lie type.

Case 1: $S \cong {}^2B_2(q_1^2)$, $q_1^2 = 2^{2m+1}$, $m \geq 1$. Now

$$\mathrm{cd}({}^2B_2(q_1^2)) = \{1, q_1^4, q_1^4 + 1, (q_1^2 - 1)a, (q_1^2 - 1)b, (q_1^2 - 1)u\},$$

for $q_1^2 = 2^{2m+1} \geq 8$, $u = \frac{1}{\sqrt{2}}q_1$, $a = q_1^2 + \frac{1}{\sqrt{2}}q_1 + 1$, and $b = q_1^2 - \frac{1}{\sqrt{2}}q_1 + 1$. The largest degree of ${}^2B_2(q_1^2)$ is $(q_1^2 - 1)a = (q_1^2 - 1)(q_1^2 + \frac{1}{\sqrt{2}}q_1 + 1)$. As the Steinberg character of S has a degree q_1^4, we have $q_1^4 = q^3$, so $q_1^{4/3} = q$. The largest degree of G is $(q + 1)(q^2 + q + 1) = q^3 + 2q^2 + 2q + 1 = q_1^4 + 2q_1^{8/3} + 2q_1^{4/3} + 1$.

Using section 13.9 in [2] and results of Malle in [15], $S =^2 B_2(q_1^2)$ has a unipotent character of degree $(q_1^2-1)u$ which is extendible to $\mathrm{Aut}(S)$. Hence G has a degree $u(q_1^2-1)$. Since $q = q_1^{4/3}$, q is even and so $u(q_1^2-1)$ is an even degree of G but not a 2-power, so $u(q_1^2-1) = q(q+1)$ or $q(q^2+q+1)$. Comparing the even parts, we deduce that $u = 2^m = q = 2^{(4m+2)/3}$ which is absurd.

Case 2: $S \cong G_2(q_1)$

Assume that $q_1 > 2$, otherwise $G_2(2)$ is not simple. As $\phi_{2,2}$ is a unipotent character of S, it extends to $\mathrm{Aut}(S)$ and thus G has a degree $\phi_{2,2}(1) = q_1(q_1+1)(q_1^3+1)/2$. By our assumption on q_1, $\phi_{2,2}(1)$ is divisible by p but not a power of p since $q_1^2 = q$ and q_1^3+1 is prime to p. So, $\phi_{2,2}(1) = q(q+1)$ or $q(q^2+q+1)$. Now if p is odd, then by comparing the p-part, we have $q_1 = q$ which is impossible as $q = q_1^2$. If p is even, then $q_1/2 = q = q_1^2$, a contradiction again.

Case 3: $S \cong^2 G_2(q_1^2)$, $q_1^2 = 3^{2m+1}$, $m \geq 1$.

Consider the character of degree

$$\frac{1}{\sqrt{3}}q_1(q_1-1)(q_1+1)(q_1^2+1) = \frac{1}{\sqrt{3}}q_1(q_1^2-1)(q_1^2+1).$$

This degree is a mixed degree, so it must divide either $q(q+1)$ or $q(q^2+q+1)$. This character degree is even as q_1 is a power of 3. Thus it must divide $q(q+1)$ since $q(q^2+q+1)$ is odd.

The Steinberg character of S has a degree q_1^6, so $q = q_1^2$. Hence we have

$$(q_1^2-1)(q_1^2+1) \mid q_1^2+1,$$

a contradiction.

Case 4: S is isomorphic to one of the remaining simple groups of exceptional Lie type.

For the remaining simple groups of exceptional Lie type, we will utilise the same argument. $S = S(q_1)$ is a simple group of exceptional Lie type defined over a field of q_1 elements. Now, suppose that the Steinberg character of S has degree q_1^j. By Lemma 1.0.6, $q^3 = q_1^j$, we have $q = q_1^{j/3}$. For each of the remaining possibilities for S, there is a mixed degree of S whose power on q_1 is greater than $j/3$. As the mixed degrees of G have power of at most $j/3$, we have a contradiction. Table 1 exhibits the degree of the Steinberg character of S and a character degree which will result in a contradiction.

□

Table 1. Eliminating simple groups of Lie type (See [22])

$S = S(q_1)$	St(1)	Char of S	$\chi(1)_p$
$L_{l+1}(q_1)$	$q_1^{l(l+1)/2}$	$(1,1,1,...,1,2)$	$q_1^{l(l-1)/2}$
$U_{l+1}(q_1^2)$	$q_1^{l(l+1)/2}$	$(1,1,1,...,1,2)$	$q_1^{l(l-1)/2}$
$O_{2l+1}(q_1)$	$q_1^{l^2}$	$\begin{pmatrix} 0 & 1 & 2 & \cdots & l-2 & l-1 & l \\ 1 & 2 & \cdots & l-2 & & & \end{pmatrix}$	$q_1^{l^2-2l+1}$
$S_{2l}(q_1), p = 2$	$q_1^{l^2}$	$\begin{pmatrix} 0 & 1 & 2 & \cdots & l-2 & l-1 & l \\ 1 & 2 & \cdots & l-2 & & & \end{pmatrix}$	$\frac{1}{2}q_1^{l^2-2l+1}$
$S_{2l}(q_1), p > 2$	$q_1^{l^2}$	$\begin{pmatrix} 0 & 1 & 2 & \cdots & l-2 & l-1 & l \\ 1 & 2 & \cdots & l-2 & & & \end{pmatrix}$	$q_1^{l^2-2l+1}$
$O_{2l}{}^{+}(q_1)$	$q_1^{l^2-l}$	$\begin{pmatrix} 0 & 1 & 2 & \cdots & l-3 & l-1 \\ 1 & 2 & 3 & \cdots & l-2 & l-1 \end{pmatrix}$	$q_1^{l^2-3l+3}$
$O_{2l}{}^{-}(q_1)$	$q_1^{l^2-l}$	$\begin{pmatrix} 0 & 1 & 2 & \cdots & l-2 & l \\ & 1 & 2 & \cdots & l-2 & \end{pmatrix}$	$q_1^{l^2-3l+2}$

${}^3D_4(q_1^3)$	q_1^{12}	$\phi''_{1,3}$	q_1^7
$F_4(q_1)$	q_1^{24}	$\phi_{9,10}$	q_1^{10}
${}^2F_4(q_1^2)$	q_1^{24}	ϵ''	q_1^{10}
$E_6(q_1)$	q_1^{36}	$\phi_{6,25}$	q_1^{25}
${}^2E_6(q_1^2)$	q_1^{36}	$\phi''_{2,16}$	q_1^{25}
$E_7(q_1)$	q_1^{63}	$\phi''_{7,46}$	q_1^{46}
$E_8(q_1)$	q_1^{120}	$\phi_{8,91}$	q_1^{91}

Remark 2.3.1. Char of S denotes the character of S and $\chi(1)_p$ denotes q_1-part of the degree of S.

2.3.3 Eliminating the Classical groups of Lie type when k=1

Proposition 2.3.8. *The simple group* $S \cong \mathrm{PSL}_3(q)$

Proof. $S = S(q_1)$ is a simple group of Classical Lie Type defined over a field of q_1 elements. The Steinberg character of S has a degree q_1^j for some j. To eliminate most of simple groups of classical Lie type, we rely upon the same reasoning as Case 4 of proposition 2.3.7. Now, suppose that the Steinberg character of S has degree q_1^j. By Lemma 1.0.6, $q^3 = q^j$, so $q = q^{j/3}$.

For most of the possibilities for S, we can find a mixed degree of S whose power on q_1 is greater than $j/3$. As the mixed degrees of G have power at most $j/3$ on q_1, we have a contradiction.

Table 1 exhibits the degree of the Steinberg character of S and the symbol corresponding to an irreducible character of S of appropriate degree which will result in a contradiction.

We shall use the same arguments as in [22], [23], [24], [25] and use the same notation as adopted in section 13.8 of [2]. We will proceed by examining each family of the simple groups of classical Lie type separately. For $S \cong \mathrm{PSU}_{l+1}(q_1^2)$ or $S \cong \mathrm{PSL}_{l+1}(q_1)$, so if $k = 1$, we must have $l(l-1)/2 \leq l(l+1)/6$, which is not satisfied for $l > 2$. Thus $l \leq 2$ if $S \cong \mathrm{PSU}_{l+1}(q_1^2)$ or $S \cong \mathrm{PSL}_{l+1}(q_1)$. Now, we will examine each of these possibilities for l separately.

Case 1: $S \cong \mathrm{PSU}_2(q_1^2)$ or $S \cong \mathrm{PSL}_2(q_1)$

This case was handled in Proposition 2.3.6.

Case 2(a): $S \cong \mathrm{PSL}_3(q_1)$

In this case, we have $q_1^3 = q^3$, such that we have $q = q_1$ and the divisibility condition is satisfied as the degrees of G are the degrees of $\mathrm{SL}_3(q)$ which are different from the degrees of $S = \mathrm{PSL}_3(q)$.

Case 2(b): $S \cong \mathrm{PSU}_3(q_1^2)$

Suppose the Steinberg character of S has a degree q_1^j. It follows from Lemma 1.0.6 that $q_1^j = q^3$, such that $q = q_1^{j/3}$. For $l \leq 2$, there is a mixed degree of $S \cong \mathrm{PSU}_{l+1}(q_1^2)$ which does not divide a degree of G.

Table 2 exhibits the degree of the Steinberg character of S and the symbol corresponding to an irreducible character of S of appropriate degree which will result in a contradiction.

Case 3: $S \cong \mathrm{P}\Omega_{2l}{}^{\pm}(q_1)$, where $l \geq 4$. The degree of the steinberg character of $S \cong \mathrm{P}\Omega_{2l}{}^{\pm}(q_1)$ is q^{l^2-l}. As shown in Table 1, there is a mixed degree of S whose power on q_1 is $l^2 - 3l + 2$. Using the same argument as in Case 4 of proposition 2.3.7, we must have that

$$l^2 - 3l + 2 \leq \frac{l(l-1)}{3},$$

which is not satisfied for $l \geq 1$.

Case 4: $S \cong O_{2l+1}(q_1)$ or $S \cong \mathrm{PSp}_{2l}(q_1)$.

The degree of the steinberg character of $S \cong O_{2l+1}(q_1)$ or $S \cong \mathrm{PSp}_{2l}(q_1)$ is q^{l^2}. As shown in Table 1, there is a mixed degree of S whose power on q_1 is $l^2 - 2l + 1$. Using the same argument as in Case 4 of proposition 2.3.7, we must have

$$l^2 - 2l + 1 \leq \frac{l^2}{3},$$

which is not satisfied for $l \geq 3$. If q_1 is even, then the exponent on q_1 in the degree is at least

$$l^2 - 2l + 1 - (l - 1).$$

But

$$l^2 - 2l + 1 - (l - 1) \leq \frac{l^2}{3}$$

is not satisfied for $l \geq 4$. Suppose that $l = 2$. For $q_1 \neq 3$, the degree of the Steinberg character is 3^4, which cannot possibly extend to q^3. S possesses a unipotent character of degree $q_1(q_1^2 + 1)/2$ and $q_1(q_1 - 1)^2/2$. The degree of the Steinberg character of S is q_1^4. Thus we must have that $q = q_1^{4/3}$. But these degrees cannot divide $q_1^{4/3}(q_1^{4/3} + 1)$ and so they must divide $q(q^2 + q + 1) = q_1^{4/3}(q_1^{8/3} + q_1^{4/3} + 1)$. Both these degrees must divide $q(q^2 + q + 1)$. Hence $q^2 + q + 1$ has factors $q_1 - 1$ and $q_1^2 + 1$. This implies $q^2 + q + 1$ has a divisor congruent -1 modulo 3, in contradiction to Lemma 2.1.5.

From Section 13.8 of [2], we have that $S \cong O_{2l+1}(q_1)$ or $S \cong \mathrm{PSp}_{2l}(q_1)$ possess a unipotent character corresponding to the symbol

$$\alpha = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \dots & \lambda_a \\ & \mu_1 & \mu_2 & \dots & \mu_b \end{pmatrix}.$$

where $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a$ and $0 \leq \mu_1 < \mu_2 < \cdots < \mu_b$, $a - b$ is odd and positive, λ_1, μ_1 are not both 0.

Now suppose that $l = 3$. In particular, from Section 13.8 of [2], we see that S has a unipotent character χ^{α} of degree

$$\chi^{\alpha}(1) = \frac{1}{2}q_1^4(q_1 + 1)(q_1^2 - q_1 + 1)$$

corresponding to the symbol

$$\alpha = \begin{pmatrix} 1 & & 2 & & 3 \\ & 0 & & 1 & \end{pmatrix}.$$

When $l = 3$, the degree of the Steinberg character of S is q_1^9, so we must have that $q = q_1^3$. Hence $\chi^{\alpha}(1)$ divides no degree of G.

Thus, this complete the proof, we have eliminated all possibilities of S except when S is $\mathrm{PSL}_3(q)$ as required.

$\square$

Table 2. Eliminating $PSL_{l+1}(q_1)$ and $PSU_{l+1}(q_1^2)$, $2 \leq l \leq 3$ (See [23])

$S = S(q_1)$	St(1)	Char of S	$\chi(1)_p$
$PSL_3(q_1)$	q_1^3	$\chi_{q_1(q_1+1)}$	$q_1\Phi_2$
$PSU_3(q_1^2)$	q_1^3	$\chi_{q_1(q_1-1)}$	$q_1\Phi_1$
$PSL_4(q_1)$	q_1^6	$\chi^{(2,2)}$	$q_1^2\Phi_4$
$PSU_4(q_1^2)$	q_1^6	$\chi^{(1,1,2)}$	$q_1^3\Phi_6$

Remark 2.3.2. Char of S denotes the character of S and $\chi(1)_p$ denotes q_1-part of the degree of S.

2.4 Establishing that G' is isomorphic to a cover of $\mathrm{PSL}_3(q)$

We shall introduce lemmas that we are required to establish that G' is isomorphic to a cover of $H/Z(H) = \mathrm{PSL}_3(q)$ when $H \cong \mathrm{SL}_3(q)$.

The two Lemmas below appear in [23] as Lemma 5 and Lemma 6 respectively.

Lemma 2.4.1. *If q is a prime power with $q \geq 5$ and $q \neq 9$, then the Schur multiplier of $\mathrm{SL}_2(q)$ is trivial. The Schur multiplier of $\mathrm{SL}_2(4)$ and $\mathrm{SL}_2(9)$ are cyclic of order 2 and 3, respectively.*

Lemma 2.4.2. *Assume that q is a prime power with $q \geq 3$. If K is a maximal subgroup of $\mathrm{SL}_2(q)$ whose index divides $q \pm 1$ or q, then one of the following cases holds.*

1. *If $q \geq 13$, q odd or $q \geq 4$, q even then K is the Borel subgroup of index $q+1$. Moreover $q+1$ is the smallest index of a maximal subgroup of $\mathrm{SL}_2(q)$.*

2. *If $q \in \{3, 5, 7, 11\}$ then either K is the Borel subgroup of index $q+1$ or K is the nonabelian factor group of index q.*

3. *If $q = 9$, then K is the Borel subgroup of index 10 and 6 is the smallest index of a maximal subgroup of $\mathrm{SL}_2(9)$.*

Definition 2.4.1. Suppose $N \lhd G$ and $\vartheta \in \mathrm{Irr}(N)$. Then the induction of ϑ from N to G is denoted by ϑ^G and the set of irreducible constituents of ϑ^G is denoted by $\mathrm{Irr}(G|\vartheta)$.

Lemma 2.4.3. *Suppose $Y \trianglelefteq L$ is such that $L/Y \cong \mathrm{SL}_2(q)$ and let $\delta \in \mathrm{Irr}(Y)$. If $\chi(1)$ divides q or $q+1$ for any $\chi \in \mathrm{Irr}(L \mid \delta)$, then δ is L-invariant.*

Proof. Let δ be not L-invariant and $V = I_L(\delta)$. Express

$$\delta^V = \sum_i \eta_i, \text{ where } \eta_i \in \mathrm{Irr}(V \mid \delta).$$

It follows that for each i, $\eta_i^L \in \mathrm{Irr}(L \mid \delta)$ and $\eta_i^L = |L : V|\,\eta(1)$ divides q and $q+1$. Let M be a subgroup of L such that $V \leq M$ and M/Y is maximal in L/Y. We have the index $|L : M|$ divides $q+1$ or q, thus we have the following cases.

Case $q = 7, 11$ and $|L : M| = q$. We have that M/Y is a nonabelian factor group of index q. It follows that $\eta_i^L(1) = |L : M| \cdot |M : V|\, \eta_i(1) = q \cdot |M : V|\, \eta_i(1)$ divides q which forces $M = V$ and all $\eta_i(1) = 1$. Hence $V/Y = M/Y$ is abelian as δ extends to V and all irreducible characters of V/Y are linear. However, this leads to a contradiction as M/Y is nonabelian.

Case $q \geq 7$, $q \neq 7$ and $q \neq 11$. By Lemma 2.4.2, we have M/Y isomorphic to a Borel Subgroup and $|L : M| = q + 1$ such that $|M : V|\, \eta_i(1)$ divides q or $q + 1$.

Hence, we must have

$$\eta_i^L(1) = |L : M| \cdot |M : V|\eta_i(1) = (q + 1) \cdot |M : V|\eta_i(1)$$

divides $q + 1$ and we deduce that $|M : V|\eta_i(1) = 1$. This implies that $M = V$ and all $\eta_i(1) = 1$. $M/Y = V/Y$ is isomorphic to the Borel subgroup of $\mathrm{SL}_2(q)$ of index $q + 1$ and that all constituents of δ^V are linear. By Lemma 1.0.4, we deduce that the Borel subgroup M/Y is abelian, leading to a contradiction as $q \geq 7$. It follows that δ is L-invariant as required.

$\square$

Lemma 2.4.4. *Let $q \equiv 1 \pmod 3$. Let X be a perfect group and $M \lhd X$ such that X/M is a cover of $\mathrm{PSL}_3(q)$ and every character degree of X divides a degree of $\mathrm{SL}_3(q)$. Then every linear character of M is stable under X.*

Proof. Suppose $\vartheta \in \mathrm{Irr}(M)$ with $\vartheta(1) = 1$. If ϑ is not stable under X, we have $I_X(\vartheta) \lneq X$. Hence $I_X(\vartheta)$ is contained in a maximal subgroup of X and the index of $I_X(\vartheta)$ in X must divide a degree of G.

We will prove that there are no such maximal subgroups containing $I_X(\vartheta)$. Let

$I_X(\vartheta) = I \lneq X$ for some $\vartheta \in \mathrm{Irr}(M)$. Let P be maximal such that $I \leq P \lneq X$. If

$$\vartheta^I = \sum_i \varphi_i, \text{ for } \varphi_i \in \mathrm{Irr}(I),$$

then by Lemma 1.0.5, $\varphi_i(1)\,|X : I|$ is a degree of X and divide some degree of $\mathrm{SL}_3(q)$.

Thus, we will need to find indices of maximal subgroups of $\mathrm{PSL}_3(q)$ which divide some character degrees of $\mathrm{SL}_3(q)$. From a list of maximal subgroups of $\mathrm{PSL}_2(q)$ and $\mathrm{PSL}_3(q)$ in [11] and [19], we have Tables 3 and 4 respectively. Let $q = p^f$.

Table 3. Maximal subgroups of $\mathrm{PSL}_2(q)$ (see [11] and [19])

Subgroup	Condition	Index
$\mathrm{D}_{(q-1)}$	$q \geq 13$, odd	$\frac{1}{2}q\Phi_2$
$\mathrm{D}_{2(q-1)}$	q even	$\frac{1}{2}q\Phi_2$
$\mathrm{D}_{(q+1)}$	$q \neq 7, 9$, odd	$\frac{1}{2}q\Phi_1$
$\mathrm{D}_{2(q+1)}$	q even	$\frac{1}{2}q\Phi_1$
Borel subgroup		Φ_2
$\mathrm{PSL}_2(q_0) \cdot (2, \alpha)$	$q = q_0^\alpha$	
S_4	$q = p \equiv \pm 1 \bmod 8$	
	$q = p^2, 3 < p \equiv \pm 3 \bmod 10$	
A_4	$q = p \equiv \pm 3 \bmod 8, q > 3$	
A_5	$q = p \equiv \pm 1 \bmod 10$	
	$q = p^2, p \equiv \pm 3 \bmod 10$	

Table 4. Maximal subgroups of $\mathrm{PSL}_3(q)$ (see [5], [11] and [19])

Subgroup	Condition
$^\wedge[q^2] : GL_2(q)$	
$^\wedge(\mathbb{Z}_{q-1})^2 \cdot S_3$	$q \geq 5$
$^\wedge\mathbb{Z}_{q^2+q+1} \cdot 3$	$q \neq 4$
$\mathrm{PSL}(q_0) \cdot ((q-1, 3), b)$	$q = q_0^b$, b prime
$3^2 \cdot \mathrm{SL}_2(3)$	$q = p \equiv 1 \bmod 9$
$3^2 \cdot \mathrm{Q}_8$	$q = p \equiv 4, 7 \bmod 9$
$\mathrm{SO}_3(q)$	q odd
$\mathrm{PSU}_3(q_0)$	$q = q_0^2$
A_6	$p \equiv 1, 2, 4, 7, 8, 13 \bmod 15$
$\mathrm{PSL}_2(7)$	$2 < q = p \equiv 1, 2, 4 \bmod 7$

Remark 2.4.1. The symbol $^\wedge$ means we are giving the structure of the preimage in special linear or symplectic groups. $^\wedge[q^2]$ denotes an unspecified group of order q^2, $A : B$ denotes a split extension, $A \circ B$ denotes a central product, and $A \cdot B$ denotes a non-split extension.

We now eliminate the possible cases, by examining Tables 3 and 4.

Case P/M is isomorphic to the image in $\mathrm{PSL}_3(q)$ of $[q^2] : \mathrm{GL}_2(q)$. i.e. $P/M \cong [q^2] : \mathrm{SL}_2(q) \cdot (q-1)/3$. Then $|X : P| = q^2 + q + 1$ and for every i, $|P : I|\varphi_i(1)$ divides q and $q + 1$.

Let S, R and T be subgroups of P such that $S/M \cong [q^2]$, $R/M \cong \mathrm{SL}_2(q)$ and $T = SR \trianglelefteq P$. If $\chi \in \mathrm{Irr}(T \mid \vartheta)$, then $\chi(1)$ divides q and $q + 1$.

Subcase 1(a): $S \leq I \leq P$. We have $T/S \cong \mathrm{SL}_2(q)$. Express

$$\vartheta^S = \sum_{i=1}^{k} \delta_i, \text{ where } \delta_i \in \mathrm{Irr}(S \mid \vartheta).$$

We have to show that $S/\mathrm{Ker}\ \vartheta$ is abelian. This is equivalent to show that $S' \leq \mathrm{Ker}\ \delta_i$ for all i, and

$$S' \leq \cap_{i=1}^{k} \mathrm{Ker}\ \delta_i = \mathrm{Ker}\ \vartheta^S$$

Suppose $S/\mathrm{Ker}\ \vartheta$ is nonabelian. This is equivalent to showing that $S' \nleq \mathrm{Ker}\ \delta_j$, for some j, hence $1 = \vartheta(1) < \delta_j(1)$ and $p \mid \delta_j(1)$. As $\mathrm{Irr}(T \mid \delta_j) \subseteq \mathrm{Irr}(T \mid \vartheta)$, by Lemma 2.4.3, we obtain that δ_j is T-invariant.

As $q \equiv 1 \pmod 3$, we observe that $q \neq 9$ so the Schur multiplier of $T/S \cong \mathrm{SL}_2(q)$ is trivial. Then we have by Theorem 11.7 (Isaacs, [10]), δ_j extends to $\delta_0 \in \mathrm{Irr}(T)$ and hence by Lemma 1.0.4, $\delta_0\eta$ are all the irreducible constituents of δ^T where $\eta \in \mathrm{Irr}(T/S)$. Take $\eta \in \mathrm{Irr}(T/S)$ with $\eta(1) = q^2$. We have $\delta_0(1)\eta(1) = q^2\delta_j(1)$ must divide q and $q+1$, which is impossible. Hence $S/\mathrm{Ker}\ \vartheta$ is abelian. By Lemma 2.4.3, each linear δ_i is T-invariant. Therefore

$$[T, S] \leq \cap_{i=1}^{k} \mathrm{Ker}\ \delta_i \leq \mathrm{Ker}\ \vartheta^S \leq M$$

which is a contradiction as T acts nontrivially on $S/M \cong [q^2]$. Hence $S \nleq I \leq P$.

Subcase 1(b): $S \nleq I \leq P$. Since $I \cap S \lneq S$, one can find $\delta \in \mathrm{Irr}(S \mid \vartheta)$ with $p \mid \delta(1)$. Since $\mathrm{Irr}(T|\delta) \subseteq \mathrm{Irr}(T|\vartheta)$, for every $\chi \in \mathrm{Irr}(T|\delta)$, we have $\chi(1)$ divides q or $q+1$ and hence by Lemma 2.4.3, δ is T-invariant. Now since $q \geq 7, q \neq 9$, the Schur multiplier of $T/S \cong \mathrm{SL}_2(q)$ is trivial and by Theorem 11.7 in [10] δ extends to $\delta_0 \in \mathrm{Irr}(T|\vartheta)$. Hence by Lemma 1.0.4, $\delta_0\eta$ are all the irreducible constituents of δ^T where $\eta \in \mathrm{Irr}(T/S)$. Take $\eta \in \mathrm{Irr}(T/S)$ with $\eta(1) = q^2$. We have $\delta_0(1)\eta(1) = q^2\delta_j(1)$ must divide q and $q+1$, which is impossible. □

By combining Steps 1 and 2, we have established that $G'/M \cong \mathrm{PSL}_3(q)$. As every degree of G' divides a degree of G, Lemma 2.4.4 implies that every linear character of M is stable under G'. It follows by Lemma 1.0.8 that $|M : M'|$ divides $|\mathrm{Mult}(G'/M)| = |\mathrm{Mult}(\mathrm{PSL}_3(q))|$. Using Lemma 1.0.10, we deduce that G'/M' is isomorphic to a cover of $\mathrm{PSL}_3(q)$.

Repeating the above arguments by using Lemmas 2.4.4 and 1.0.10 and [9, Lemma 6], we have $G'/M^{(i)}$ is isomorphic to a cover of $\mathrm{PSL}_3(q)$ for every $i \geq 0$. Therefore, if M is solvable then G' is isomorphic to a cover of $\mathrm{PSL}_3(q)$, as wanted.

We will eliminate the case when M is nonsolvable. Assume so, then there is an integer i such that

$$M^{(i)} = M^{(i+1)} > 1.$$

Let $N \leq M^{(i)}$ be a normal subgroup of G' so that $M^{(i)}/N \cong T^k$ for some non-abelian simple group T. By [16, Lemma 4.2], T has a non-principal irreducible character φ that extends to $\mathrm{Aut}(T)$. Now [1, Lemma 5] implies that φ^k extends to $\Phi \in \mathrm{Irr}(G/N)$. Therefore, by Gallagher's Theorem, $\Phi\chi \in \mathrm{Irr}(G'/N)$ for every $\chi \in \mathrm{Irr}(G'/M^{(i)})$. In particular,

$$\varphi(1)^k\chi(1) \in \mathrm{cd}(G'/N) \subseteq \mathrm{cd}(G').$$

However, we have that $G'/M^{(i)}$ is isomorphic to a cover of $\mathrm{PSL}_3(q)$. Taking χ to be the Steinberg character of degree q^3 of $G'/M^{(i)}$, we deduce that $q^3\varphi^k(1)$ is a degree of G' and therefore it divides a degree of G. As $\mathrm{cd}(G) = \mathrm{cd}(H)$, $q^3\varphi^k(1)$ divides a degree of H. This is impossible by inspecting the character degree set of $\mathrm{SL}_3(q)$.

Therefore, we have that $G'/M \cong \mathrm{PSL}_3(q)$ with $q \geq 7$ and deduce that $|M| = 1$ or $M \cong \mathbb{Z}_3$ according to whether $G' = \mathrm{PSL}_3(q)$ or $\mathrm{SL}_3(q)$, the covers of $\mathrm{PSL}_3(q)$.

2.5 Establishing that $G \cong G' \circ C_G(G')$ and $C_G(G')$ is abelian

Since G' is quasisimple, we have that $G/C_G(G')$ is an almost simple group with $\mathrm{PSL}_3(q) \trianglelefteq G/C_G(G') \leq \mathrm{Out}(\mathrm{PSL}_3(q))$. As $q \geq 7$ and $q \equiv 1 \pmod 3$, by Theorem 2.5.12 in [7], we have $\mathrm{Out}(\mathrm{PSL}_3(q)) = \langle d \rangle : (\langle \sigma \rangle \times \langle \tau \rangle)$, where τ is a graph automorphism of order 2, σ is a field automorphism of order f with $q = p^f$ and d is a diagonal automorphism of order 3.

We claim that $G'C_G(G') = G' \circ C_G(G') = G' \circ C$, where $C := C_G(G')$. Let $Z := Z(G')$. As G' is quasisimple, $G'/Z \cong \mathrm{PSL}_3(q)$ with $q \geq 7$, we know that $|Z| = 1$ or 3 according to whether $G' = \mathrm{PSL}_3(q)$ or $\mathrm{SL}_3(q)$. We first have that $C \cap G' = Z = Z(G')$ and $[G', C] = [C, G'] \leq Z$. It follows that $[G', C, G'] \leq [Z, G'] = 1$ and $[C, G', G'] \leq [Z, G'] = 1$ so by Three Subgroup Lemma, (see [6]), we have $1 = [G', G', C] = [G', C]$ as $[G', G'] = G'$. Thus $G'C = G' \circ C$ as required.

Next, we have to show that $C := C_G(G')$ is abelian. Since $Z = Z(G')$, and $G' \trianglelefteq G$, we see that Z is normal in G, so $G'/Z \times C/Z \trianglelefteq G/Z$. Firstly, C/Z is abelian as otherwise, we can choose $\lambda \in \mathrm{Irr}(C/Z)$ with $\lambda(1) > 1$ and let $\mu \in \mathrm{Irr}(G'/Z)$ with $\mu(1) = q^3$, we know that $\mu \times \lambda \in \mathrm{Irr}(G'/Z \times C/Z)$ and so $(\mu \times \lambda)(1) = q^3\lambda(1) > q^3$ must divide some degree of $\mathrm{SL}_3(q)$ which is impossible.

If $|Z| = 1$ then $G' \cong \mathrm{PSL}_3(q)$, $C/Z = C$ is abelian and we are done. Now assume that $|Z| = 3$. Then $G' \cong \mathrm{SL}_3(q)$. Let $1 \neq \nu \in \mathrm{Irr}(Z)$. It follows from Frame and Simpson [5] that $\mathrm{SL}_3(q)$ has an irreducible faithful character of degree $q(q^2 + q + 1)$. So, there exists $\chi \in \mathrm{Irr}(G'|\nu)$ with $\chi(1) = q(q^2 + q + 1)$. Now if C is nonabelian, then we can choose $\lambda \in \mathrm{Irr}(C|\nu)$ such that $\lambda(1) > 1$. However, if $C' = Z$ then C is nonabelian. It follows from Theorem 3.7.1 and 3.7.2 in [6] that $\mu := \chi \cdot \lambda \in \mathrm{Irr}(G' \circ C)$ by the theory of representations of central products. So $G' \circ C$ has a degree $\lambda(1)q(q^2 + q + 1)$ and thus this degree must divide some degree of G, which is impossible as $q(q^2+q+1)$ is maximal with respect to divisibility among the degrees of $\mathrm{SL}_3(q)$. Therefore, C is always abelian. In particular, by Theorem 3.7.1 and 3.7.2 in [6], we know that $\mathrm{cd}(G' \circ C) = \mathrm{cd}(G')$.

We claim that if $\lambda \in \mathrm{Irr}(C)$ then λ is G−invariant. Suppose that $\lambda \in \mathrm{Irr}(C)$ is not G−invariant. If Z lies in the kernel of λ then $\chi = \mathrm{St}_{G'/Z} \times \lambda$ is an irreducible character of $G'C/Z$ and is not G−invariant, so G has a degree which is a proper multiple of q^3, a contradiction. Now suppose that Z does not lie in the kernel of λ. Suppose that $\lambda \in \mathrm{Irr}(C|\nu)$ where $1 \neq \nu \in \mathrm{Irr}(Z)$. We know that $G' \cong \mathrm{SL}_3(q)$ has a faithful character μ of degree $(q-1)^2(q+1)$ and is G-invariant. So, $\chi = \mu\lambda$ is an irreducible character of $G'C$ which is not G-invariant and thus G has a degree which is a proper multiple of $(q-1)^2(q+1)$ which is impossible.

If $G/C_G(G')$ induces non-trivial automorphism, say α, we shall apply the same argument as in the proof of Theorem 6.1 in [19] to show that $\mathrm{cd}(G/C_G(G')) \nsubseteq \mathrm{cd}(\mathrm{SL}_3(q))$.

(i) Assume $\alpha = d^a\tau$. From Frame and Simpson [5], $\mathrm{SL}_3(q)$ has a semi-simple non-real valued irreducible character λ of degree $(q-1)^2(q+1)$. So, λ is invariant under all diagonal automorphisms but not τ−invariant. So G possesses an irreducible character whose degree is a multiple of $2(q-1)^2(q+1)$, which is impossible.

(ii) Assume $\alpha = d^a\sigma^b\tau^c$, where $0 < b < f$, $0 \leq a \leq (q-1, 3)$ and $0 \leq c \leq 1$. By a classical result of Zsigmondy in [26], we can choose $\omega \in \mathbb{F}$ of order which is a primitive prime divisor of $p^{3f} - 1 = q^3 - 1$, that is a prime divisor of $p^{3f} - 1$ that does not divide $\prod_{i=1}^{3f-1}(p^i - 1)$. We then choose a semisimple element $s \in \mathrm{SL}_3(q)$ with eigenvalues ω, ω^q and ω^{q^2}. The image of s under the canonical projection $\mathrm{GL}_3(q) \to \mathrm{PGL}_3(q)$ is a semisimple element of $\mathrm{PGL}_3(q)$. Abusing the notation, we denote it by s. Since s, s^{-1}, and $\tau(s)$ are all conjugates in $\mathrm{PGL}_3(q)$, the semisimple character $\chi_s \in \mathrm{SL}_3(q)$ of degree $(q-1)^2(q+1)$ is real by [4, Lemma 2.5] and hence

$$(\chi_s)^\tau = \chi_{\tau(s)} = \chi_{s^{-1}} = \overline{\chi_s} = \chi_s$$

by Corollary 2.5 of [18]. In other words, χ_s is invariant under τ. Since $\mathrm{PGL}_3(q)$ has no degree which is a proper multiple of $(q-1)^2(q+1)$, we obtain that χ_s is also d-invariant. By checking the multiplicities of character degrees of $\mathrm{SL}_3(q)$ and $\mathrm{PGL}_3(q)$, we see that there exists an s as above so that $\chi_s \in \mathrm{Irr}(\mathrm{PSL}_3(q))$. Using Lemma 2.5 of [3] again, we have χ_s is not σ^b-invariant, since $|s| = |\omega|$ does not divide $|\mathrm{PSL}_3(p^b)|$. We have shown that χ_s is not $\alpha-$invariant. Therefore G has a degree which is a proper multiple of $\chi_s(1) = (q-1)^2(q+1)$, a contradiction.

(iii) Finally, assume $\alpha = d^a$ with $0 < a < 3$. Then $G/C \cong \mathrm{PGL}_3(q)$. From [5] again, $\mathrm{SL}_3(q)$ has a faithful irreducible character χ of degree $(q-1)^2(q+1)/3$ which is not α-invariant. Let $\phi \in \mathrm{Irr}(G' \circ C)$ be an irreducible constituent of the restriction χ to $G' \circ C$. As $|G : G'C| = 3$, and $\chi(1) > 3, 1 < \phi(1) \leq \chi(1)$. Since $\mathrm{cd}(G' \circ C) = \mathrm{cd}(G') \subseteq \mathrm{cd}(\mathrm{SL}_3(q))$, $G' \circ C$ has no irreducible character whose degree is nontrivial proper divisor of $\chi(1)$, this forces $\phi(1) = \chi(1)$ and hence ϕ extends to G. However, we know that $\mu = \phi_{G'} \in \mathrm{Irr}(G')$ so μ extends to $\chi \in \mathrm{Irr}(G)$. This is a contradiction as μ is not α-invariant but $\phi_C = \phi(1)\lambda$ with $\lambda \in \mathrm{Irr}(C)$ is G-invariant so that $\phi = \phi_{G'} \cdot \lambda$ is not G-invariant. We conclude that $G = G' \circ C$ where C is abelian and $G' \cong \mathrm{PSL}_3(q)$ or $\mathrm{SL}_3(q)$.

2.6 Establishing that covers of $H/Z(H)$ have distinct sets of character degrees

We have seen from Section 2.5 that $C_G(G')$ is abelian and $G = G' \circ C$ where $C := C_G(G')$ is abelian and $G' \cong \mathrm{SL}_3(q)$ or $\mathrm{PSL}_3(q)$. We need to show that $G' \cong \mathrm{SL}_3(q)$.

It suffices to show that G' cannot be $\mathrm{PSL}_3(q)$. Suppose by contradiction that $G' \cong \mathrm{PSL}_3(q)$. Then $G = G' \times C$ since $Z = Z(G') = 1$. So $\mathrm{cd}(G) = \mathrm{cd}(G')$ which means that $\mathrm{cd}(\mathrm{SL}_3(q)) = \mathrm{cd}(\mathrm{PSL}_3(q))$ which is impossible, since $(q-1)^2(q+1)/3 \in$

$\mathrm{cd}(\mathrm{SL}_3(q)) \setminus \mathrm{cd}(\mathrm{PSL}_3(q))$.

This completes the proof.

Bibliography

[1] M. Bianchi, D. Chillag, M.L. Lewis, and E. Pacifici, *Character degree graphs that are complete graphs*, Proc. Amer. Math. Soc. **135** (2007), 671–676.

[2] R.W. Carter, *Finite Groups of Lie Type: Conjugacy Classes and Complex Characters*, Wiley, New York, 1985.

[3] S Dolfi, G. Navarro, and P.H. Tiep, *Primes dividing the degrees of the real characters*, Math. Z. **259** (2008), 755–774.

[4] P Erdös and J.L. Selfridge, *The product of consecutive integers is never a power*, Illinois J. Math **19** (1975), 292–301.

[5] J.S Frame and W.A Simpson, *The character tables for* $\mathrm{SL}(3,q)$, $\mathrm{SU}(3,q^2)$, $\mathrm{PSL}(3,q)$, $\mathrm{PSU}(3,q^2)$, Canadian J. Math. **25** (1973), 486–494.

[6] D. Gorenstein, *Finite groups*, Second ed., Chelsea Publishing Co., New York, 1980.

[7] D. Gorenstein, R. Lyons, and R. Solomon, *The classification of finite simple groups*, Math. Surveys. Monogr, No. 3, Amer. Math. Soc., Providence (1998), 671–676.

[8] B. Huppert, *Character theory of finite groups*, Walter de Gruyter, Berlin, 1998.

[9] ______, *Some simple groups which are determined by their character degrees* I, Illinois J. Math. **44** (2000).

[10] I.M. Isaacs, *Character theory of finite groups*, Dover Publications, New York, 1994.

[11] O.H. King, *The subgroup structure of finite classical groups in terms of geometric configurations*, London. Math. Soc. Lecture Notes Series **327** (2005), 29–56.

[12] P. Kleidman and M. Liebeck, *The Subgroup Structure of the Finite Classical Groups*, Cambridge University Press, Cambridge, 1990.

[13] W. Ljunggren, *Einige Bemerkungen über die Darstellung ganzer zahlen durch binäre kubische Formen mit positiver Diskriminante*, Acta Math **75** (1943), 1–21.

[14] P.R. Majozi, H.N. Nguyen, H.P. Tong-Viet, and T.P. Wakefield, *Extending Huppert conjecture from non-abelian simple groups to quasisimple groups*, in preparation.

[15] G. Malle, *Extensions of unipotent characters and the inductive McKay condition*, J. Algebra **320** (7), 2963–2980.

[16] A. Moretó, *Complex group algebras of finite groups: Brauer's problem 1*, Adv. Math **208** (2007), 236–248.

[17] T. Nagell, *Des équations indéterminées $x^2 + x + 1 = y^n$ et $x^2 + x + 1 = 3y^n$*, Norsk. mat. Foren. Skr. Serie I **2** (1921), 14 pp.

[18] G. Navarro, P.H. Tiep, and A. Turull, *Brauer characters with cyclotomic field of values*, J. Pure Appl. Algebra **212** (2008), 628–635.

[19] H.N Nguyen, H.P. Tong-Viet, and T.P. Wakefield, *Projective special linear groups $\mathrm{PSL}_4(q)$ are determined by the set of their character degrees*, Journal of Algebra and Its Applications **11 (6)** (2012), 26 pp.

[20] P. Schmid, *Extending the steinberg representation*, J. Algebra **150** (1992), 254–256.

[21] H.P. Tong-Viet, *The simple Ree groups ${}^2\mathrm{F}_4(q^2)$ are determined by the set of the character degrees*, J. Algebra **339** (2011), 357–369.

[22] H.P. Tong-Viet and T.P. Wakefield, *On Huppert's conjecture for* $G_2(q), q \geq 7$, Journal of Pure and Applied Algebra, **216 (12)** (2012), 2720–2729.

[23] ______, *On Huppert's conjecture for* ${}^3D_4(q), q \geq 3$, Algebr. Represent. Theory **16 (2)** (2013), 470–490.

[24] T.P. Wakefield, *Verifying Huppert's conjecture for the simple groups of Lie type of rank two*, Ph.D. thesis, Kent State University, 2008.

[25] ______, *Verifying Huppert's conjecture for* $\mathrm{PSL}_3(q)$ *and* $\mathrm{PSU}_3(q^2)$, Comm. Algebra. **37** (2009), 2887–2906.

[26] K. Zsigomondy, *Zur theorie der Potenzreste*, Monath. Math. Phys. **3** (1892), 265–284.