Aspects of trapped surfaces and spacetime singularities

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As the candidate's supervisors, we have approved this dissertation for submission.

Signed:	Professor Sunil D. Maharaj	September 2019.
Signed:	Professor Rituparno Goswami	September 2019.

Abstract

This thesis studies various aspects of black hole and singular spacetimes, as well as the geometry of expansion-free stars. In particular, a theorem on the topological equivalence of marginally trapped 2-surfaces in locally rotationally symmetric (LRS) class II spacetimes and null normal foliation (NNF) spacetimes has been established. It is shown that, for isolated horizons, this equivalence leaves the surface gravity invariant, while an explicit proof has been provided which shows that the surface gravity cannot be zero for LRS II and NNF spacetimes. A detailed study is also carried out on the evolution of black hole horizons, and all of the necessary conditions satisfied by null, spacelike and timelike horizons are established, extending earlier results. We show that for there to exist trapping in an expansion-free dynamical star, the star must accelerate and radiate simultaneously. We show in particular that, under the condition of non-zero acceleration and radiation, these stars are necessarily conformally flat, which severely constrains their existence. We have studied minimal horizons and it is shown that the energy density completely determines the geometry of the leaves of minimal horizons in LRS II spacetimes. We show that *all minimal horizons in LRS II have spherical symmetry.* This result naturally extends to NNF spacetimes and is shown to hold for any 4-dimensional spacetime. Also, we have constructed a T_2-separable g boundary, which we denote by $\tilde{g}$, and established an explicit embedding of $\tilde{g}$ into the a boundary. We provide a counterexample to an earlier conjecture that all boundary constructions of a certain broad class will necessarily be topologically pathological.

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Declaration 2 - Publications

- *Marginally trapped surfaces in gravitational collapse: A semitetrad covariant study,* **arXiv Preprint, arXiV:1805.05684v1 (2018)** (with R. Goswami & S. D. Maharaj) – To be submitted (Physical Review D).

- *Some results on cosmological and astrophysical horizons and trapped Surfaces,* Class. Quantum Grav., **36**:215001 (2019) (with R. Goswami & S. D. Maharaj).

- *Properties of expansion-free dynamical stars,* Phys. Rev. D, **100**:044039 (2019) (with R. Goswami & S. D. Maharaj).

- *Geometric results for minimal hypersurfaces in generalized LRS II spacetimes,* (with R. Goswami & S. D. Maharaj) – Work in Progress.

- *A T_2-separable g-boundary and its relation to the a-boundary,* **arXiv Preprint, arXiV:1810.09766v2 (2019)** (with G. Amery) – Submitted (International Journal of Geometric Methods in Modern Physics).

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Dedication

To

“All the sad people of the world”

Contents

List of Figures

List of Abbreviations

GR	General Relativity
MTS	Marginally Trapped Surface
EFEs	Einstein's Field Equations
DH	Dynamical Horizon
LRS	Locally Rotationally Symmetric
NNF	Null Normal Foliation
MOTS	Marginally Outer Trapped Surface
MOTT	Marginally Outer Trapped Tube
TS	Trapped Surface
MTT	NMarginally Trapped Tube
NEH	Nonexpanding Horizon
IH	Isolated Horizon
TLM	Timelike Membrane
MMTS	Minimal Marginally Trapped Surface
MMTT	Minimal Marginally Trapped Tube
FOTH	Future Outer Trapping Horizon
SFOTH	Spacelike Future Outer Trapping Horizon
WEC	Weak Energy Condition
DEC	Dominant Energy Condition
NEC	Null Energy Condition
OS	Oppenheimer-Snyder
RW	Robertson-Walker
LTB	Lemaitre-Tolman-Bondi
BPP	Bounded Parameter Property

Chapter 1

Introduction

Einstein's theory of general relativity (GR) is a geometric theory encoded in the so-called Einstein field equations (EFEs)

$$G_{\mu\nu} \propto T_{\mu\nu}, \tag{1.1}$$

which can best be explained as relating the effect matter has on the geometry of spacetime, and how the matter evolves with said geometry. The quantity $G_{\mu\nu} = R_{\mu\nu} - (1/2)\, R g_{\mu\nu}$ ($g_{\mu\nu}$ is the symmetric metric 2-tensor on the spacetime) is called the *Einstein tensor*, and codifies the geometry (the curvature), while $T_{\mu\nu}$, the *energy momentum tensor*, carries the matter content. GR, as one of the two main pillars (the other being quantum mechanics) of theoretical physics that have shaped our understanding of the universe, has been unpacked over the past one hundred years to understand some of the profound implications of the theory. Among these implications are the formation of trapped surfaces in spacetimes and the existence of spacetime singularities, see Hawking & Penrose (1970) and Hawking & Ellis (1973).

1.1 Some background

1.1.1 Trapped regions and black hole horizons

Gravity determines the causal structure of spacetime via the deflection of light. As an obvious consequence, if sufficiently large amount of matter is concentrated in a small enough region in space, then this may deflect light coming out of this region to such an extent that it is actually dragged back inwards. This phenomenon can be well explained by the concept of closed trapped surfaces. Let us consider a spacetime manifold $(\mathscr{M}, g)$, with the metric 'g', and a Cauchy hypersurface 'σ' with induced metric 'h' and exterior curvature 'χ'. The formation of a closed trapped surface in (σ, h, χ) signals gravitational collapse and generally indicates geodesic incompleteness of $\mathscr{M}$ (and thus existence of spacetime singularities). Trapped surfaces and their properties have been extensively studied by various authors. Roger Penrose first defined a trapped surface in $\mathscr{M}$ as a closed compact spacelike 2-surface S such that the null expansions orthogonal to S are both converging (Hawking & Penrose, 1970; Hawking & Ellis, 1973; Joshi & Narlikar, 1982). Stephen Hawking then introduced the concept of a trapped region (Hawking & Penrose, 1970; Hawking & Ellis, 1973) and apparent horizons. This became the basis for formally defining a black hole to include globally hyperbolic spacetimes, defined as a region in $\mathscr{M}$ foliated by these surfaces (Joshi & Narlikar, 1982).

Marginally trapped surfaces (MTS), on which one of the null expansion scalars vanishes, have been extensively studied as these are used to describe trapping horizons in a spacetime. These 2-surfaces foliate a 3-dimensional submanifold of M. It has been shown in Booth & Fairhurst (2005), under the assumption that the intrinsic geometry and the normal bundle connection on the marginally trapped 2-surfaces are fixed, that any such surface serves as a suitable boundary in a quasi-local action formulation of general relativity. As such, these 3-submanifolds are suitable black hole boundaries and are used to study the local dynamics and evolution of black holes (Ashtekar & Krishnan, 2002; Ashtekar & Krishnan, 2003; Hayward, 2004). For example, dynamical horizons (DH), spacelike hypersurfaces in a spacetime foliated by MTS have been used in the local description of boundaries of black holes, see Ashtekar & Krishnan (2002), Ashtekar & Krishnan (2003) and Hayward (2004). Weakly trapped surfaces and trapped surfaces have been used to investigate uniqueness and geometric properties of dynamical horizons (Ashtekar & Galloway, 2005).

The topological properties of these 2-surfaces have also been investigated by various authors, with some authors extending these topological results to higher dimensional black holes (Hawking & Ellis, 1973; Newman, 1987; Friedman *et al.,* 1993; Galloway, 1995; Galloway *et al.*, 1999; Jacobson & Venkataramani, 2005; Bengtsson & Senovilla, 2011; Anderson *et al.*, 2015) producing some very interesting results. For example, Stephen Hawking theorized (see Hawking & Ellis (1973)) that cross sections of the event horizon for asymptotically flat and stationary spacetimes, satisfying the dominant energy condition, are topological 2-spheres. This is the well known Hawking black hole topology theorem. Newmann, (1987) was the first to constrain the result of Hawking by showing that the cross sections have to satisfy certain stability conditions.

A notion of stability of the MTS, analogous to minimal surfaces in Riemannian geometry, has also been established (Schoen & Yau, 1981; Andersson *et al.*, 2005; Andersson *et al.*, 2008). Scheon & Yau, (1981) used a blowup of Jang's equation (Jang, 1978; Andersson *et al.*, 2008; Galloway, 2011) to derive an evolution equation of the null expansion scalar on the MTS. This gave rise to a functional equation of some smooth function on the MTS. Conditions on the function and the principal eigenvalue associated with the function determine the stability of the MTS.

Ellis *et al.* (2014) studied the evolution of MTTs, hypersurfaces foliated by MTS in LRS II spacetimes. They introduced conditions on the slope of the tangent to the MTT curves which determine the nature of the MTT. This led the authors to describe black hole horizons in a real astrophysical setting and found that an initial 2-surface bifurcates into an outer and inner MTT, and that the inner MTT was timelike while the outer MTT was spacelike.

The interest in these hypersurfaces (called horizons) as a local description of black holes gave rise to the formulation of the laws of black hole mechanics and even flux laws locally. Trapping horizons, introduced by Hayward were first used to formulate dynamical laws of black hole mechanics (Hayward, 1994). Ashtekar and Krishnan (2002) and Ashtekar & Krishnan (2003) introduced the notion of an isolated horizon (IH), to characterize the equilibrium states of black holes. Truly dynamical black holes are characterized by dynamical horizon (DH) (Ashtekar & Krishnan, 2002; Ashtekar & Krishnan, 2003; Hayward, 2004; Bengtsson & Senovilla, 2011). Furthermore, it has been shown that flux laws can be formulated on DH (Ashtekar & Krishnan, 2002; Ashtekar & Krishnan, 2003).

The laws of black hole thermodynamics are formulated in the context of the properties of quantities such as the surface gravity and angular momentum on the horizon (Bardeen *et al.*, 1973; Brown & York, 1993; Ashtekar *et al.*, 2000; Ashtekar *et al.*, 2001). These are formulated so that they are intrinsic to the horizon geometry. For example, the notion of surface gravity, which is closely related to the temperature of a black hole, also plays an important role in the study of black hole dynamics (Hayward, 1994; Ashtekar *et al.*, 2000; Ashtekar *et al.*, 2000; Ashtekar *et al.*, 2001).

In this thesis we approach these various aspects of trapped surfaces by interpreting formalisms in terms of quantities from the $1+1+2$ covariant formalism.

The works in chapters 5 and 6 looks at different but related aspects of GR. Chapter 5 deals with expansion-free stars in an isolated neighborhood. This falls within the scope of stability analysis of self-gravitating systems (see May & White, 1966, Wilson, 1971, Santos, 1985, Goswami, 2007 and Joshi & Goswami, 2011).

Models of radiating stars in general relativity are important to describe astrophysical processes and to study gravitational collapse. Some recent examples of exact models which are physically reasonable were obtained by Tewari and Charan (2015), Tewari (2013), and Ivanov (2016a, 2016b, 2019). Anisotropy and dissipative effects have been shown to influence the collapse rate and temperature profiles in radiating stars by Reddy *et al.* (2015). It has been demonstrated that classes of exact solutions exist in general relativity, referred to as Euclidean stars, which regain Newtonian stars in the appropriate limit (Abebe *et al.*, 2014; Govender *et al.*, 2010; Herrera *et al.*, 2010). The Lie analysis of differential equations using symmetry invariance has proved to be a systematic method to produce general categories of exact solutions to the boundary condition of radiating objects (Abebe *et al.*, 2015; Mohanlal *et al.*, 2016; Mohanlal *et al.*, 2017). An important class of radiating stars which are expansion-free was introduced by Herrera *et al.* (2009). Expansion-free dynamical models imply the existence of a cavity or void. Matter distributions with a vanishing expansion scalar have to be inhomogeneous. These physical features have important astrophysical consequences for spherically symmetric distributions. Studies containing the description of physical properties of expansion-free dynamical radiating stars are contained in several treatments (Kumar & Srivastava, 2018a; Kumar & Srivastava, 2018b; Prisco *et al.,* 2011). Therefore, it is important to study the geometrical properties of expansion-free dynamical stars and find general conditions for their existence.

Various authors have explored expansion-free models with different considerations. The central theme of interest in these expansion-free models is the possibility that they could help explain voids on cosmological scales. Herrera and coauthors (2009) studied such models with non-zero shear and showed that the appearance of cavity (see reference Peebles, (2001) for more discussion), with matter which is anisotropic and dissipative, undergoing collapse is inevitable. The same authors followed this result (2010) in which they ruled out the Skripkin expansion-free model (see Skripkin, 1960) - with constant energy density and isotropic pressure - when considering the expansion-case along with the junction conditions. Herrera *et al.*, (2012), studied models collapsing adiabatically and showed that the instability was independent of the star stiffness. In particular, it was shown that the instability is entirely governed by the pressures and the radial profile of the energy density.

Chapter 6 deals with the geometry of hypersurfaces foliated by minimal 2-surfaces in certain classes of spacetimes in GR. Minimal surface theory in Riemannian geometry is a well studied and is a well understood topic (Fialkow, 1938; Takahashi, 1966; Calabi, 1967; Hsiang, 1967; Simons, 1968; Chern *et al.,* 1970; Dwivedi, 2019). Standard results in the literature can thus be evoked, given certain properties of spacetimes, to study the geometry of these surfaces in spacetimes.

1.1.2 Spacetime singularities

Another subject explored in this thesis is spacetime singularities. The study of spacetime singularities has always proved one of the most contentious subjects in all of GR. This is for the simple reason that there has always been disagreement on what the definition of a singularity should be. As these singularities are not interior manifold points, one would need to identify local sets containing them in order to assign them any properties, as is done in any physical theory. However, to do this would require the definition of what these missing manifold points are. One option to consider would be in terms of *geodesic completeness*, i.e. a spacetime being termed singular if it is geodesically incomplete (failure of extremal curves to terminate at points in the interior of the spacetime). While geodesic completeness and *metric completeness* (convergence of every Cauchy sequence to an interior manifold point with respect to some distance function) are equivalent notions in the Riemannian case (Kobayashi & Katsumi, 1963), this is not so in the Lorentzian case (Kundt, 1963) since it is possible to have the spacetime either null, timelike, or timelike geodesically complete and incomplete in the other two. Geroch's example (Geroch, 1968a) conformal to the 2-dimensional Minkowski spacetime illustrates this fact. As such, Hawking & Ellis (1973) adapted a minimum definition of geodesic completeness as being both timelike and null geodesically complete. Specifically, the choice of timelike completeness allows for the assignment of a physical interpretation to singularities as non-existent histories outside the bounds of the proper time parameter. There is however a caveat that is worth mentioning. Some of these singular points are only apparently singular and are actually due to the choice of coordinates. It is possible to choose a set of coordinates that *maximally extended* the spacetime, where the metric becomes analytic through these points; eg. the Kruskal embedding (Kruskal, 1960). Truly singular points are those which maximally analytic extensions cannot remove.

Over the past sixty years, schemes have been developed, (Geroch, 1968b; Schmidt, 1971; Geroch *et al.*, 1972; Scott & Szekeres, 1994), which "capture" these singular points in constructed boundaries. These boundaries are then attached to the spacetime manifold, a process known as manifold completion. The resulting manifold is called the manifold with boundary.

An early investigation of geodesic completeness was carried out by Szekeres (1960) using a power series expansion of the Schwarzschild solution around the coordinate singularity $r = 2m$ to obtain a transformation under which $r = 2m$ is a regular manifold point (see Ehresmann (1957)) was probably the first to suggest a definition of singular spacetimes). Geroch (1968b) provided the first boundary construction for singular spacetimes; the g boundary. This involves quotienting the set of incomplete geodesics in the spacetime under the equivalence that the geodesics limit to the same point.

Schmidt (1971) provided an alternative construction of singular spacetimes called the b boundary. This construction maps endpoints of incomplete curves in the bundle of frames of M, $\mathscr{L}(M)$ - generated via Cauchy completion - as additional points of M. It is best known for its application by Hawking and others to singularity theorems (Hawking & Penrose, 1970; Hawking & Ellis, 1973) and considers a broader class of curves than the g boundary. However, this boundary construction is fraught with undesirable properties. For instance, a 4-dimensional manifold has a 20-dimensional frame bundle. Though it was shown (Schmidt, 1971) that it is sufficient to consider only the bundle of orthonormal frames - which is 10-dimensional - intuition and computations are problematic. As a result, only situations with sufficient symmetries to reduce the dimensions can reasonably be approached by this scheme. Furthermore, there is the problem of the b boundary being non-Hausdorff which means that a b boundary point may be arbitrarily close to manifold points.

In (1972) Geroch and collaborators constructed the c boundary. This construction is made solely using the causal structure of the spacetime manifold. It attaches future (respectively, past) endpoints to every inextensible curve in the spacetime manifold. These endpoints are called ideal points, and their collection can be interpreted as the boundary at conformal infinity. The attachment is done in such a way that the ideal points only depend on the past (resp., future) of the curves. These ideal points are represented by *terminal indecomposable future* (resp., *past*) sets, (Flores, 2007), which are the maximal future (resp., past) sets that are not the chronological future (resp., past) of any manifold point and cannot be represented as the union of two proper subsets, both of which are open future (resp., past) sets.

As with the other constructions, the c boundary construction is problematic and carries a lot of complexities. The original topology constructed (Geroch *et al.*, 1972) has topological separation problems as well as non-intuitive results for certain solutions. As such there have been several modifications of the c boundary via the construction of different topologies (Budic & Sachs, 1974; Szabados, 1988; Harris, 1998; Harris, 2000; Ashley, 2002; Flores, 2007).

Geroch *et al.* (1982) constructed an example in which any singular spacetime boundary construction which falls in a certain class (including the g and b boundaries) will exhibit pathological topological properties. This led to their assertion that this may in general be true; one might always be able to construct an example for which a particular singular boundary construction fails.

Scott and Szekeres (1994) constructed the abstract boundary or simply the a boundary. Given the existence of open embeddings $\phi_i : M \longrightarrow \hat{M}_i$, the a boundary is constructed by defining an equivalence relation on the topological boundaries $\partial_{\phi_i} M$ with respect to the $\hat{M}_i$. This construction stands out as it considers a much broader class of curves relative to the other constructions, as well as by virtue of being T_2 separable, which is a crucial requirement for a physically reasonable spacetime (Barry & Scott, 2011). Its generality means that it is applicable in contexts in which one has only an affine connection, for example in Yang-Mills and Einstein-Cartan theories (Scott & Szekeres, 1994). This construction also turns out to be easier to implement in general compared to the other construction schemes. However, the a boundary is also plagued with its own shortcomings (see the discussions in Curiel (1999, 2009)). It would be nice to have a causal characterization of the a boundary, as well as being able to define differentiable and metric structures on the a boundary, and to establish what (if any) new information can be learned from these properties on the a boundary.

Curiel (1999, 2009) has argued that the use of curve incompleteness is an adequate definition in context of singular spacetimes and singular structures as far as existing spacetimes of physical relevance are concerned. This would seem to suggest that the g, b and a boundaries are all acceptable boundary constructions in a general relativistic context. This is not necessarily true outside of this immediate context, and this is where the a boundary - which is more broadly applicable - becomes the construction of choice. Nevertheless, the ubiquity of the application of the curve incompleteness approach motivates a better understanding of the relationship between such constructions and the a boundary.

To date, there have been no successful results relating the various boundary constructions, although there has been some work on relating b-completeness and g-completeness (Hawking & Ellis, 1973). Moreover, within different boundary constructions, there have been several modifications to deal with some of the problems faced by these constructions (Budic & Sachs, 1974; Szabados, 1988; Harris, 1998; Harris, 2000; Ashley, 2002; Flores, 2007). Finally, there does not seem to be a "natural" and general way to map between different boundary constructions (Ashley, 2002). This is one of the problem addressed in this thesis in chapter 7.

1.2 Structure of thesis

This thesis is structured in the following manner. In chapter 2 we introduce some fundamental concepts used through out the different chapters.

Chapter 3 is based on the reference Sherif *et al.*, (2018) which is joint work with Rituparno Goswami and Sunil D. Maharaj. The work considers various geometric properties of marginally trapped surfaces and topological relationships between the 3-surfaces they foliate in certain classes of spacetimes. This work extends the work carried out by Ellis *et al.* (2014) where a formalism was introduce to study the evolution of black hole horizons in LRS II spacetimes. We also consider stability of marginally trapped surfaces in LRS II spacetimes in the context of the $1+1+2$ semitetrad covariant formulation of GR.

Chapter 4 is based on the reference Sherif *et al.*, (2019a) which is joint work with Rituparno Goswami and Sunil D. Maharaj. In this work we consider both topological and causal relationships between certain classes of spacetimes and a general 4-dimensional spacetime. We also study the stability of marginally trapped surfaces and classify the spacelike future outer trapping horizons in the Robertson-Walker and the Lemaitre-Tolman-Bondi spacetimes. The surface gravity on a black hole horizon was calculated for certain classes of spacetimes (to be specified in the chapter) as well as a general 4-dimensional spacetime, using the formalism of Booth and collaborators (Booth & Fairhurst, 2004; Booth *et al.*, 2005) while adapting the calculations to the $1+1+2$ covariant formalism variables.

Chapter 5 is based on the reference Sherif *et al.*, (2019b), which is joint work with Rituparno Goswami and Sunil D. Maharaj. This work looks at the formation of a “cavity”, in a star with no expansion. We are interested in the conditions under which such cavity would form and what the geometry of the star should be for such mechanism to occur.

Chapter 6 is based on the reference Sherif *et al.*, (2019c), which is joint work with Rituparno Goswami and Sunil D. Maharaj. In this work we study the geometry of horizons foliated by minimal 2-surfaces. We evoke well known results from minimal surface theory to determine the geometry of these 2-surfaces.

Chapter 7 is based on the reference Sherif & Amery, (2019), which is joint work with Gareth Amery. In this work we formulate a method of constructing a Hausdorff separable g boundary, which we label as $\tilde{g}$, and establish a relationship between the g and a boundary constructions of spacetime.

In chapter 8 we conclude with the summary of the different results from each chapter.

Chapter 2

Preliminaries

Many different areas of mathematics have been employed to probe the various facets of Einstein's theory of general relativity. The foundational setup of the theory by Einstein utilized what was the then relatively well developed area of Riemannian and pseudo-Riemannian geometry. As the theory evolved over the decades, different areas of mathematics have been used to study varying aspects of the theory. Topology, differential geometry and differential topology have been used to study concepts like black holes and spacetime singularities. This chapter provides the necessary mathematical background for the concepts used in the subsequent chapters. The notes of section 2.1, follow the particular references Greenberg (1970), Tsamparlis & Mason (1983), Mason & Tsamparlis (1985), Tsamparlis (1992), Clarkson (2007) and Ellis *et al.* (2014), and the associated references therein. These notes provide some background for the $1+1+2$ semitetrad covariant formalism. We restrict the notes to the minimum requirements for chapters 3, 4, 5, and 6. We will mostly follow certain references (Hawking & Ellis, 1973; Munkres, 2000; Lee, 2001; Wasserman, 2004; Kostecki, 2009) for section 2.2 where we introduce definitions primarily used in Sherif & Amery (2019) of chapter 7. In some cases we will give provisional definitions that subsequently lead to those used in the chapter.

2.1 Some notes on the $1+1+2$ semitetrad covariant formulation of general relativity

Given a spacetime M, to covariantly describe M we first make a choice of a unit tangent vector field, u^a, to a defined timelike congruence (Ellis, 2009). Given any 4-vector U^a in the spacetime, the projection tensor $h_a^{\ b} \equiv g_a^{\ b} + u_a u^b$, projects U^a onto the 3-space as

$$U^a \quad = \quad U u^a + U^{\langle a \rangle},$$

where U is the scalar along u^a and $U^{\langle a \rangle}$ is the projected 3-vector (the angle bracket denotes the projected symmetric trace-free part of a vector or tensor). This naturally gives rise to two derivatives:

- The *covariant time derivative* (or simply the dot derivative) along the observers' congruence. For any tensor $S^{a..b}{}_{c..d}$,

$$\dot{S}^{a..b}{}_{c..d} \equiv u^e \nabla_e S^{a..b}{}_{c..d}.$$

- Fully orthogonally *projected covariant derivative* D with the tensor h_{ab}, where the total projection is on all the free indices. For any tensor $S^{a..b}{}_{c..d}$,

$$D_e S^{a..b}{}_{c..d} \equiv h^a{}_f h^p{}_c ... h^b{}_g h^q{}_d h^r{}_e \nabla_r S^{f..g}{}_{p..q}.$$

The kinematical and Weyl quantities associated with the $1+3$ splitting is given by the collection of the quantities $\rho, p, \Theta, H_{ab}, E_{ab}, \sigma_{ab}, \pi_{ab}, q_a, \omega_a$. We have $\rho = T_{ab} u^a u^b$ as the energy density, $p = \frac{1}{3} h^{ab} T_{ab}$ is the isotropic pressure, $\Theta = D_a u^a$ is expansion, H_{ab} is the gravito-magnetic tensor, E_{ab}

is the gravito-electric tensor, σ_{ab} is the shear tensor, $\pi_{ab} = \pi_{\langle ab \rangle}$ is the anisotropic stress tensor, $q_a = q_{\langle a \rangle} = -h_a^{\ b} T_{bc} u^c$ is the 3-vector defining the heat flux, and ω_a is the rotation vector. The quantity T_{ab} is the energy-momentum tensor. For LRS II spacetimes the Weyl is purely electric (Clarkson, 2007).

The 3-space can further be split by the choice of a vector field - which we denote by e^a - orthogonal to u^a, and the kinematical and Weyl variables described above can be split into an irreducible set of covariant scalar variables. The vector field e^a allows for the introduction of a projector tensor given by $N_a^{\ b} \equiv g_a^{\ b} + u_a u^b + e_a e^b$, which projects vectors orthogonal to u^a and e^a onto a 2-surface defined as the sheet $N_a^{\ a} = 2$. This further splitting introduces two new derivatives:

- The *hat derivative* is the spatial derivative along the vector e^a. For a 3-tensor $\psi_{a..b}^{\ \ \ \ c..d}$,

$$\hat{\psi}_{a..b}^{\ \ \ \ c..d} \equiv e^f D_f \psi_{a..b}^{\ \ \ \ c..d}.$$

- The *delta derivative* is the projected spatial derivative on the 2-sheet by $N_a^{\ b}$, projected on all the free indices. For any 3-tensor $\psi_{a..b}^{\ \ \ \ c..d}$,

$$\delta_e \psi_{a..b}^{\ \ \ \ c..d} \equiv N_a^{\ f} .. N_b^{\ g} N_h^{\ c} .. N_i^{\ d} N_e^{\ j} D_j \psi_{f..g}^{\ \ \ \ h..i}.$$

The resulting quantities fully describing a spacetime M are then given by the collection of the quantities

$$\left(\rho, p, A, \Theta, \phi, \Sigma, \mathcal{E}, \Pi, Q, \Omega, \Theta, \mathcal{H}_{ab}, \mathcal{E}_{ab}, \Sigma_{ab}, \Pi_{ab}, A^a, \mathcal{H}^a, \mathcal{E}^a, \Sigma^a, \Pi^a, q^a, \omega^a\right).$$

Some of these quantities, relevant to this thesis, are defined as follows:

$$
\begin{aligned}
\sigma_{ab} &= \Sigma\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Sigma_{(a}e_{b)} + \Sigma_{ab},\\
q_a &= Q e_a,\\
\dot{u}_a &= A e_a + A_a,\\
\phi &= \delta_a e^a,\\
\omega_a &= \Omega e_a + \Omega_a.
\end{aligned}
\tag{2.1}
$$

We also have

$$
\begin{aligned}
E_{ab} &= \mathcal{E}\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\mathcal{E}_{(a}e_{b)} + \mathcal{E}_{ab},\\
\pi_{ab} &= \Pi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Pi_{(a}e_{b)} + \Pi_{ab}.
\end{aligned}
\tag{2.2}
$$

In (2.1), A_a is the acceleration vector, ϕ is the sheet expansion, and $\mathcal{E}$, Π and Σ are the scalars associated with the electric part of the Weyl tensor, the anisotropic stress and the shear tensor respectively. The evolution and propagation equations can be obtained from the Ricci identities of the vectors u^a and e^a as well as the doubly contracted Bianchi identities. Some of the evolution and propagation equations relevant to this paper are given below (see Clarkson (2007) which contains the complete set of equations):

- *Evolution:*

$$
\begin{aligned}
\frac{2}{3}\dot{\Theta} - \dot{\Sigma} \quad = \quad & A\phi - 2\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 - \frac{1}{3}\left(\rho + 3p - 2\Lambda\right) + \mathcal{E} - \frac{1}{2}\Pi - \Sigma_a\Sigma^a + \Omega_a\Omega^a\\
& - \left(2a_a - A_a - \delta_a\right)A^a + 2\Omega^2 + \varepsilon_{ab}\alpha^a\Omega^b - 2\alpha_a\Sigma^a - \Sigma_{ab}\Sigma^{ab},
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\dot{\phi} \quad = \quad & \left(\frac{2}{3}\Theta - \Sigma\right)\left(A - \frac{1}{2}\phi\right) + Q + 2\xi\Omega + \delta_a\alpha^a - \zeta^{ab}\Sigma_{ab} + A^a\left(\alpha_a - a_a\right)\\
& + \left(a^a - A^a\right)\left(\Sigma_a - \varepsilon_{ab}\Omega^b\right).
\end{aligned}
\tag{2.4}
$$

- *Propagation:*

$$\frac{2}{3}\hat{\Theta} - \hat{\Sigma} = \frac{3}{2}\phi\Sigma + Q + 2\xi\Omega + \delta_a\Sigma^a + \varepsilon_{ab}\delta^a\Omega^b - 2\Sigma_a a^a + 2\varepsilon_{ab}A^a\Omega^b - \Sigma_{ab}\zeta^{ab}, \quad (2.5)$$

$$\begin{aligned}\hat{\phi} &= \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{1}{2}\phi^2 - \frac{2}{3}\left(\rho + \Lambda\right) - \mathcal{E} - \frac{1}{2}\Pi + 2\xi^2 + \delta_a a^a \\ &\quad - a_a a^a - \zeta_{ab}\zeta^{ab} + 2\varepsilon_{ab}\alpha^a\Omega^b - \Sigma_a\Sigma^a + \Omega_a\Omega^a.\end{aligned} \quad (2.6)$$

The full covariant derivatives of the vectors u^a and e^a are given by (Clarkson, 2007)

$$\begin{aligned}\nabla_a u_b &= -Au_a e_b + e_a e_b\left(\frac{1}{3}\Theta + \Sigma\right) + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right) - u_a A_b + e_a\left(\Sigma_b + \varepsilon_{bm}\Omega^m\right) + \Omega\varepsilon_{ab} \\ &\quad + \left(\Sigma_a - \varepsilon_{am}\Omega^m\right)e_b + \Sigma_{ab},\end{aligned} \quad (2.7)$$

$$\begin{aligned}\nabla_a e_b = &- Au_a u_b + \left(\frac{1}{3}\Theta + \Sigma\right)e_a u_b + \frac{1}{2}\phi N_{ab} - u_a\alpha_b + \left(\Sigma_a - \varepsilon_{am}\Omega^m\right)u_b + e_a a_b \\ &+ \xi\varepsilon_{ab} + \zeta_{ab},\end{aligned} \quad (2.8)$$

where

$$\alpha_a \equiv \dot{e}_a = N_{ab}\dot{e}^b,$$

$$\varepsilon_{ab} \equiv \varepsilon_{abc}e^c = u^d\eta_{dabc}e^c,$$

$$\zeta_{ab} \equiv \delta_{\{a}e_{b\}},$$

$$\xi \equiv \frac{1}{2}\varepsilon^{ab}\delta_a e_b,$$

$$a_a \equiv e^c D_c e_a = \hat{e}_a.$$

The quantities $\Sigma, \Sigma_a, \Sigma_{ab}$ are related to the shear tensor and shear scalar via the relation

$$\sigma^2 \equiv \frac{1}{2}\sigma_{ab}\sigma^{ab} = \frac{3}{4}\Sigma^2 + \Sigma_a\Sigma^a + \frac{1}{2}\Sigma_{ab}\Sigma^{ab}. \quad (2.9)$$

We also have the relation $\hat{u}_a = \left(\frac{1}{3}\Theta + \Sigma\right)e_a + \Sigma_a + \varepsilon_{ab}\Omega^b$. The quantity ζ_{ab} is the shear of e^a (distortion of the sheet), a^a is its acceleration, and ξ is the twisting of the sheet (rotation of e^a).

2.2 Topological spaces and their properties

2.2.1 Topological spaces, continuous maps, homeomorphisms and embeddings

Definition 2.2.1 (**Topological space**). *Let* $\mathbf{S}$ *be a set and let* τ *be a collection of open subsets of* $\mathbf{S}$ *satisfying the following properties:*

a both $\mathbf{S}$ *and* $\varnothing$ *are in* τ*,*

b arbitrary union of elements of τ *are in* τ*,*

c finite intersections of elements of τ *are in* τ*.*

Then the pair $(\mathbf{S}, \tau)$ *is called a* ***topological space*** *and* τ *is called a topology on* $\mathbf{S}$*.*

The above definition of a topological space has obvious consequences. For example, the definition can be phrased in terms of closed sets by interchanging the requirement on unions and intersections: arbitrary intersections of elements of τ (elements of τ are now closed sets) are in τ, and finite unions of elements of τ are in τ. This complementarity is just a consequence of De Morgan's laws. A subset **A** of **S** is closed if and only if $\mathbf{A} = cl(A)$, where $cl(A)$ denotes the closure of **A** (the union of A and all of its limit points).

Every set **S** is endowed with two topologies, the *trivial topology* made up of the empty set and the **S** itself, and the *discrete topology* which is the collection of all subsets of **S**. Besides the trivial and discrete topologies, the set **S** may have multiple topologies defined on it. However these will not be discussed for the purpose of this thesis. Sometimes these topologies are subset related (the terminology usually used in the literature is τ_1 and τ_2 are *comparable*), i.e. one may be the subset of the other. Let **S** be a set and let τ_1 and τ_2 be two topologies defined on **S**. If $\tau_1 \subset \tau_2$, then we say that τ_1 is weaker (or coarser) than τ_2, or that τ_2 is finer than τ_1.

We will simply name a topological space by the underlying set with the understanding that we are talking about topological spaces, except otherwise stated. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two topological spaces. In general, the task of writing down all of the subsets of a topological space **S** may not be feasible. One could instead specify a "reasonable" reduced collection of subsets of **S** from which the topology can be recovered (or generated). This reduced collection is termed a *basis* of the topology. Formally,

Definition 2.2.2 (**Basis of a topology**). *A **basis** for a topology τ on a topological space S is a collection $\mathscr{B}$ of subsets of **S** such that*

a For any $s \in \mathbf{S}$, $\exists$ an element $B \in \mathscr{B}$ such that $s \in B$.

b If $s \in B_1 \cap B_2$, for $B_1, B_2 \in \mathscr{B}$, then $\exists$ an element $B_3 \subset B_1 \cap B_2$ of $\mathscr{B}$ such that $s \in B_3$.

Definition 2.2.3 (**Induced, subspace, or relative topology**). *Let **S** be a topological space with a topology τ, and let $A \subset \mathbf{S}$. Then the collection*

$$\tau_A = \{A \cap \mathscr{U} | \mathscr{U} \in \tau\}$$

*is a topology on A, the **induced, subspace or relative topology**.*

If $\mathscr{B}$ is a basis for τ, then the collection

$$\mathscr{B}_A = \{A \cap B | B \in \mathscr{B}\}$$

is a basis for τ_A.

Definition 2.2.4 (**Continuous map**). *A map $f : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ is called **continuous** if $\forall$ open $V \subseteq \mathbf{S}_2$, $f^{-1}(V)$ is open in $\mathbf{S}_1$.*

The map $f : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ is said to be *open* if $\forall$ open $U \subset \mathbf{S}_1$, the image $f(U)$ is open in $\mathbf{S}_1$.

Definition 2.2.5 (**Homeomorphism**)**.** *A map $f : \boldsymbol{S}_1 \longrightarrow \boldsymbol{S}_2$ is a **homeomorphism** if it is bijective, open and continuous.*

The definition 2.2.5 is equivalent to the statement that f is bijective and bi-continuous (both f and f^{-1} are continuous). By definition, it is clear that the bijectivity is carried over to the open sets of homeomorphic spaces. As such, any property of $\mathbf{S}_1$, defined entirely in terms of the open sets of $\mathbf{S}_1$ - which are called *topological properties* - will have a corresponding property in $\mathbf{S}_2$. It is easily checked that homeomorphism is an equivalence relation and as such homeomorphic spaces are treated as one and the same space.

It is possible to have a case where the map f is injective but only surjective onto a subset $f(\mathbf{S}_1) = A_2$ of $\mathbf{S}_2$. In this case we say that f is homeomorphic onto its image.

Definition 2.2.6 (**Embedding**)**.** *A map $f : \boldsymbol{S}_1 \longrightarrow \boldsymbol{S}_2$ is called an **embedding** (of $\boldsymbol{S}_1$ into $\boldsymbol{S}_2$) if f is injective and f is a homeomorphism onto its image $f(\boldsymbol{S}_1)$.*

2.2.2 Some properties of topological spaces

We now look at some of the topological properties of topological spaces mentioned earlier. We will in particular look at properties such as *connectedness, compactness, local compactness, para-compactness* and *density*.

Let $\mathbf{S}$ be a topological space, and let $\mathbf{S}'$ and $\mathbf{S}''$ be non-empty open subsets of $\mathbf{S}$ such that $\mathbf{S}' \cap \mathbf{S}'' = \varnothing$. If $\mathbf{S}' \cup \mathbf{S}'' = \mathbf{S}$, then the pair $(\mathbf{S}', \mathbf{S}'')$ is called a *separation* of $\mathbf{S}$.

Definition 2.2.7 (**Connectedness**)**.** *A topological space $\boldsymbol{S}$ for which there does not exist a separation is called a **connected** space.*

As the definition 2.2.7 is implicitly stated in terms of open subsets of $\mathbf{S}$, connectedness is clearly a topological property. If $\mathbf{S}$ contains a non-empty proper subset $\mathbf{S}'$, then a natural separation of $\mathbf{S}$ is the pair $\left(\mathbf{S}', \mathbf{S} \setminus \mathbf{S}'\right)$. An alternative but equivalent definition of connectnedness that is sometimes used is the following: *a topological space $\mathbf{S}$ is said to be connected if and only if $\varnothing$ and $\mathbf{S}$ are the only subsets of $\mathbf{S}$ that are both open and closed.*

Given a closed interval $I = [a, b]$ on the real line, every infinite subset of I has a limit point. This was a crucial ingredient in proving the *maximum value theorem* and the *uniform continuity theorem.* For an arbitrary topological space, a much stronger formulation was needed and this was done in terms of open coverings of a topological space which allows us to define the notion of compactness.

Definition 2.2.8 (**Covering (open)**)**.** *Let $\mathbf{S}$ be a topological space, and let $\mathscr{A}$ be a collection of open subsets of $\mathbf{S}$ such that $\bigcup\limits_{\mathscr{U} \in \mathscr{A}} \mathscr{U} = \mathbf{S}$. Then $\mathscr{A}$ is called an **open covering** of $\mathbf{S}$.*

Definition 2.2.9 (**Compactness**)**.** *A topological space $\mathbf{S}$ is said to be **compact** if for every open cover $\mathscr{A}$ of $\mathbf{S}$, $\exists$ a subcollection $\mathscr{A}'$ of $\mathscr{A}$ such that $\bigcup\limits_{\mathscr{U}' \in \mathscr{A}'} \mathscr{U}' = \mathbf{S}$.*

We state a theorem on compactness which is used in chapter7. The proof can be found in Munkres, (2000) p.167.

Theorem 2.2.10. *Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be two topological spaces, and let $f : \mathbf{S}_1 \longrightarrow \mathbf{S}_2$ be a bijective and continuous map between them. If $\mathbf{S}_1$ is compact and $\mathbf{S}_2$ is Hausdorff, then f is a homeomorphism.*

Definition 2.2.11 (**refinement**)**.** *Let $\mathbf{S}$ be a topological space and let $\mathscr{A}_1$ and $\mathscr{A}_2$ be two open coverings of $\mathbf{S}$. The covering $\mathscr{A}_2$ is called a **refinement** of $\mathscr{A}_1$ if, for every $\mathscr{U}_2 \in \mathscr{A}_2$, $\exists\, \mathscr{U}_1 \in \mathscr{A}_1$ such that $\mathscr{U}_1$ contains $\mathscr{U}_2$.*

Definition 2.2.12 (Locally finite open covering). *Let $\mathbf{S}$ be a topological space and let $\mathscr{A}$ be an open covering of $\mathbf{S}$ such that every $s \in \mathbf{S}$ has a neighborhood intersecting only a finte number of elements of $\mathscr{A}$. The $\mathscr{A}$ is a **locally finite open covering** of $\mathbf{S}$.*

Definition 2.2.13 (Paracompactness). *A topological space $\mathbf{S}$ such that every open covering of $\mathbf{S}$ has an open refinement that is locally finite is said to be **paracompact**.*

Another important notion of topological spaces is the closeness of points of the topological space, which is understood in terms of the separation axioms. There are many different separation axioms (Munkres, 2000). We will mention just one, *Hausdorff separability*, which is of physical relevance to the study of spacetimes. In fact, this is a property that is sometimes assumed by some authors in the definition of manifolds.

Definition 2.2.14 (Hausdorff (or T_2) separability). *A topological space $\mathbf{S}$ is said to be **Hausdorff** (or T_2 **separable**) if given any two distinct points $s_1, s_2 \in \mathbf{S}$, $\exists$ open neighborhoods U_{s_1} and V_{s_2} of s_1 and s_2 respectively such that $U_{s_1} \cap V_{s_2} = \varnothing$.*

With all these definitions out of the way we can now proceed to introduce manifolds, spacetimes, extensions and completions.

2.2.3 Manifolds, differentiable and smooth structures, spacetimes

We have briefly introduced topological spaces and some properties relevant to this thesis. We now introduce manifolds and differentiable manifolds, which are just topological spaces with additional structure.

Definition 2.2.15 (Chart). *Let U be an open subset of a topological space M, and let $\varphi : U \longrightarrow V$ be a homeomorphism from U to an open subset V of $\mathbb{R}^n$. The pair (U, φ) is called a **chart**.*

Definition 2.2.16 (**Atlas**). *A collection of charts $\left\{\left(U_a,\varphi_a\right)\right\}_{a\in I}$, where $\{U_a\}_{a\in I}$ covers M and the transition maps - $\varphi_2\circ\varphi_1^{-1}\Big|_{\varphi_1(U_1\cap U_2)}, \varphi_1\circ\varphi_2^{-1}\Big|_{\varphi_2(U_1\cap U_2)}$ for any pair of charts $\left(U_1,\varphi_1\right), \left(U_2,\varphi_2\right)$ in $\{\left(U_a,\varphi_a\right)\}_{a\in I}$, where $U_1\cap U_2\neq\varnothing$ - are homeomorphisms is called an **atlas**.*

Definition 2.2.17 (**Manifold**). *An n-dimensional **manifold** M is a Hausdorff and locally compact topological space, with second countable basis, that has an atlas placed on it. By definition, it follows that a manifold is paracompact (Wasserman, 2004).*

Definition 2.2.18 (c^k-**compatibility**). *Let $\mathscr{A}=\{\left(U_a,\varphi_a\right)\}_{a\in I}$ be an atlas. Two charts in $\mathscr{A}$ are said to be c^k-**compatible** if the transition maps are c^k (k-times differentiable). If all the charts in $\mathscr{A}$ are pairwise c^k-compatible then we say that $\mathscr{A}$ is a c^k-atlas. (We say the atlas is smooth if $k=\infty$.)*

Definition 2.2.19 (**Maximal atlas**). *Let $\mathscr{A}$ be an atlas on a manifold M. Let $\mathscr{A}'$ be any other atlas on M. If $\mathscr{A}$ is such that $\mathscr{A}'$ contains $\mathscr{A}$ implies $\mathscr{A}'=\mathscr{A}$, then the atlas $\mathscr{A}$ is said to be **maximal**.*

A maximal c^k-atlas (c^∞-atlas) is called a *differentiable structure* (*smooth structure*) on M. A *differentiable manifold* (*smooth manifold*) is a manifold endowed with a differentiable (smooth) structure.

Definition 2.2.20 (**Smooth function on a manifold**). *A function $f: M\longrightarrow\mathbb{R}$ is said to be smooth at a point $p\in M$ if $\exists$ a chart $\left(U,\varphi\right)$ in M, where U is an open neighborhood of p, such that $f\circ\varphi^{-1}:\varphi(U)\longrightarrow\mathbb{R}$ is smooth at $\varphi\left(p\right)$. The function f is called smooth on M if f is smooth at all points of M.*

Definition 2.2.21 (**Smooth maps between manifolds**). *Given two manifolds M and N, we say a map $Q: M\longrightarrow N$ is said to be smooth if for every $p\in M$, $\exists$ charts $\left(U,\varphi\right)$ at p and $\left(V,\psi\right)$ at $Q\left(p\right)$ such that $Q(U)\subset V$ and $\psi\circ Q\circ\varphi^{-1}$ is smooth.*

Definition 2.2.22 (**Spacetime**)**.** *A **spacetime** M is a smooth, connected, and orientable Lorentzian (has a Lorentzian signature) manifold, where by orientable we mean the determinant of the Jacobian of the transition maps are consistently positive for any pair of charts for some atlas on M,*

.

Definition 2.2.23 (**Extension**)**.** *Let M be a manifold and let $\mathscr{A}$ be a c^k-atlas on M. A manifold M' is an **extension** of M if $\exists$ a $c^{l\leq k}$-atlas $\mathscr{A}'$ on M' containing $\mathscr{A}$. If $\mathscr{A}'$ is maximal then M' is the maximal extension of M. Since spacetimes are smooth we simply drop the mention of the differentiability of the atlases.*

The containment relation between $\mathscr{A}$ and $\mathscr{A}'$ is simply the notion of embedding, which is a very important notion for some singular boundary constructions (the abstract boundary of Scott & Szekeres (1994), and more recently the $\tilde{g}$ boundary of Sherif & Amery (2019) - a modification of the g boundary of Geroch (1968b) - which is covered in the work of chapter 7). Given a spacetime M, a primary goal in the study of spacetime singularities is to construct a boundary of M on which incomplete curves terminate. This boundary is then adjoined to the M in a process known as spacetime completion. More formally,

Definition 2.2.24 (**Spacetime completion**)**.** *Let M be a spacetime. A manifold $\overline{M}$ is called a **manifold completion** if M is dense in $\overline{M}$, where $\overline{M} \setminus M = \partial M$ is the boundary of M.*

Chapter 3

Marginally trapped surfaces in gravitational collapse: A semitetrad covariant study

3.1 Introduction

In this chapter we extend the results in Ellis *et al.* (2014) to a broader class of spacetimes, notated NNF spacetimes for *null normal foliation* (to be defined in section 3.2). The NNF spacetime is a generalization of LRS II spacetimes which in general do not possess all the symmetries of LRS II spacetimes. The adaptation and interpretation of results in terms of the $1+1+2$ splitting provides a geometric (and therefore intuitive) picture of marginally trapped surfaces as well as the hypersurfaces they foliate. We then look at the stability of MTS (Andersson *et al.*, 2005; Andersson *et al.*, 2008; Galloway & Schoen, 2006; Cai & Galloway, 2001).

In section 3.2 we consider marginally trapped surfaces using this formalism in LRS II spacetime and a certain generalization of LRS II spacetimes, which we notate as NNF spacetimes. Evolution and stability analysis of marginally trapped surfaces are carried out. We then conclude in section 3.3.

3.2 Marginally trapped surfaces and their evolution

We start by providing some standard definitions. We recall that a codimension 2 surface S in a 4-dimensional spacetime $(\mathscr{M}, g)$ (with a Lorentzian signature $(-,+,+,+)$) can be expressed as a smooth embedding $\varphi : S \longrightarrow M$ of S into M, given by the parametric equation (Bengtsson & Senovilla, 2011)

$$x^\mu \quad = \quad \varphi^\mu \left(\lambda^A\right), \tag{3.1}$$

where $\{x^\mu\}$ (with $\mu \in \{0,1,2,3\}$) are local coordinates in $\mathscr{M}$, and $\{\lambda^A\}$ (with $A \in \{2,3\}$) are local coordinates in S.

The tangent vectors on S are given by the push forward (the differential of the map φ),

$$\varphi' \left(\partial_{\lambda^A}\right) = \frac{\partial \varphi^\mu}{\partial \lambda^A} = e^\mu_A, \tag{3.2}$$

while the first fundamental form of S in $\mathscr{M}$ is given as the pull back of g by φ:

$$\gamma = \varphi^* g = g \circ \varphi, \tag{3.3}$$

whose components are given by

$$\gamma_{AB}(\lambda) = g|_S (e_A, e_B) = g_{\mu\nu}(\varphi)\, e^\mu_A e^\nu_B.$$

Throughout we assume γ_{AB} is a positive definitive Riemannian metric. We will also simply write A, B as a, b where a, b takes the coordinates on S, since there is no ambiguity. We introduce a quantity called the *shape tensor* on S, χ which is given as the map

$$\chi : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)^\perp .$$

The sets $\mathfrak{X}(S), \mathfrak{X}(S)^{\perp}$ are the sets of smooth vector fields tangent to and perpendicular (normal) to S respectively. Let n be a normal vector field in $\mathfrak{X}(S)^{\perp}$. Then the second fundamental form on S relative to n is given by

$$\chi_{ab}|_n \equiv n_c \chi^c_{ab} = \gamma_a^{\ c} \gamma_b^{\ d} \nabla_d n_c.$$

We note that $\gamma_a^{\ b} \equiv N_a^{\ b}$, where $N_a^{\ b}$ is the projection tensor that projects vectors orthogonal to n_c onto S. From now on we will write

$$\chi_{ab}|_n \quad = \quad N_a^{\ c} N_b^{\ d} \nabla_d n_c. \tag{3.4}$$

which, for any normal vector n, is a 2-covariant symmetric tensor field on S. Let k_c, l_c be two future pointing (we assume time orientability on S) null vectors that are everywhere normal to S, given by the relations

$$k_c l^c = -1, \quad k_c k^c = 0, \quad l_c l^c = 0. \tag{3.5}$$

The expansion scalars are given as

$$\Theta_k = N^{ab} \chi_{ab}|_{k_c} = N^{cd} \nabla_d k_c, \tag{3.6}$$

$$\Theta_l = N^{ab} \chi_{ab}|_{l_c} = N^{cd} \nabla_d l_c. \tag{3.7}$$

Definition 3.2.1 (**Trapped Surface**). *A (future) trapped surface is a smooth, connected, closed, spacelike co-dimension* 2 *submanifold S of M such that the divergences Θ_k and Θ_l of the congruences generated by the null normal vector fields k^a and l^a (k^a is the outgoing null normal vector field and l^a is the ingoing null normal vector field) respectively are everywhere negative on S.*

Definition 3.2.2 (**Marginally Trapped Surface**). *A marginally trapped surface (MTS) is a smooth, connected, closed, spacelike co-dimension* 2 *submanifold S of M such that Θ_k is everywhere vanishing on S, and Θ_l is everywhere negative on S.*

Definition 3.2.3 (**Marginally Trapped Tube**). *A marginally trapped tube (MTT) is a co-dimension 1 submanifold H of M which is foliated by MTS.*

For more on the above definitions see the following references Ashtekar & Krishnan (2002), Ashtekar & Krishnan (2003), Hayward (2004), Bengtsson & Senovilla (2011) and Senovilla (1998; 2003; 2003).

3.2.1 Marginally trapped surfaces in LRS II spacetimes

We first start by briefly discussing null normal vectors and null geodesics in LRS II spacetimes. Given a spacetime $\mathscr{M}$, null geodesics are given as curves γ parametrized by an affine parameter λ. Tangent vectors to these curves are given by k^a. The null vector k^a obeys $k^a k_a = 0$. Since tangent vectors to null geodesics are parallelly transported along itself we write

$$k^b \nabla_b k^a = 0, \tag{3.8}$$

where the derivative $k^b \nabla_b$ is a derivative along the ray with respect to the affine parameter.

In LRS II spacetimes there is a preferred spatial direction. If the null geodesics move along this spatial direction, the sheet components of these null curves are zero. We can also define the notion of (locally) *incoming* and *outgoing* null geodesics with respect to the spatial direction. Let S be an open subset of $\mathscr{M}$ and γ be a null geodesic. Let k^a be the tangent to γ. Then γ is considered to be outgoing with respect to the spatial direction if $e^a k_a > 0$ and ingoing if $e^a k_a < 0$.

This allows us to write the equation of the tangent to the outgoing null geodesics as

$$k^a = \frac{E}{\sqrt{2}}\left(u^a + e^a\right), \tag{3.9}$$

where E is the energy of the light ray. We can similarly define the equation of the tangent to the incoming null geodesics as

$$l^a = \frac{1}{E\sqrt{2}}\left(u^a - e^a\right), \tag{3.10}$$

Without loss of generality we will set the energy E to unity (Ellis *et al.,* 2014; Stewart & Ellis, 1968).

In LRS II spacetimes we calculate the second fundamental forms along the outgoing and ingoing null normal directions as

$$\chi_{ab}|_{k_c} = \frac{1}{2\sqrt{2}}N_{ab}\left(\frac{2}{3}\Theta - \Sigma + \phi\right) \tag{3.11}$$

and

$$\chi_{ab}|_{l_c} = \frac{1}{2\sqrt{2}}N_{ab}\left(\frac{2}{3}\Theta - \Sigma - \phi\right), \tag{3.12}$$

Upon applying the projection tensor N^{ab} respectively on (3.11) and (3.12) we obtain the outgoing and ingoing null expansion scalars

$$\Theta_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma + \phi\right) \tag{3.13}$$

and

$$\Theta_l = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma - \phi\right). \tag{3.14}$$

Therefore, to locate a black hole it is sufficient to specify the quantities Θ, Σ and ϕ. A 2-surface in LRS II is thus said to be marginally trapped if

$$\frac{2}{3}\Theta - \Sigma + \phi = 0 \text{ and } \frac{2}{3}\Theta - \Sigma - \phi < 0. \tag{3.15}$$

Such simplification to locate black holes will have implications in numerical relativity, where these MTS are used as models for black holes (see for example Smarr *et al.* (Smarr *et al.,* 1976). The condition in (3.15) certainly satisfies the standard requirement for a surface to be marginally (outer) trapped, i.e. $\Theta_k = 0, \Theta_l < 0$. However (3.15) gives the additional constraint that the 2-sheet expansion must be positive, i.e $\phi > 0$. This then restricts the class of spacetimes in LRS II spacetimes admitting MTS and MTTs as defined in Definitions 4.2.2 and 4.2.3 respectively. The result implies that spacetimes in LRS II class (and by extension NNF class) with negative or zero sheet expansion cannot admit marginally trapped surfaces.

In the cases where there is no restriction on Θ_l, in which case ϕ sign is unrestricted, the co-dimension 1 submanifold is called a *marginally outer trapped tube* (MOTT), and the co-dimension 2 surfaces foliating the MOTT are called *marginally outer trapped surfaces* (MOTS). For example, for the time symmetric case with $\Theta = \Sigma = 0$, we have $\phi = 0$, which implies that in spacetimes with time symmetries the horizons are MOTTs but not MTTs. This is one of the key advantages of this approach to studying black holes: it allow us to determine permissible classes of spacetimes for certain results just by specifying certain geometric quantities. Such distinctions are not usually possible with other approaches. While subsequent results in the subsequent sections are specified to MTTs, we stress that these results equally apply to MOTTs.

3.2.2 Marginally trapped surfaces in NNF spacetimes

We start by defining a certain generalization of LRS II spacetimes.

Definition 3.2.4. *We will denote by NNF (for null normal foliation) the class of spacetimes for which there exists horizons foliated by MTS, and where null geodesics (both outgoing and ingoing) are normal to these MTS. Locally rotationally symmetric (LRS) spacetimes is a subclass of NNF.*

By definition, for NNF spacetimes the outgoing and ingoing null normal vector fields are given respectively as in (3.9) and (3.10). While generally for NNF spacetimes $\Omega \neq 0, \xi \neq 0$, and the tensor and vector quantities are nonvanishing, calculating the second fundamental forms along the outgoing and ingoing null normal directions we obtain exactly the expressions in (3.11) and (3.12) respectively. Furthermore, calculating the outgoing and ingoing null expansion scalars we again obtain the exact expressions in (3.13) and (3.14). This leads to the following results.

Lemma 3.2.5. *A marginally trapped surface S in an NNF spacetime is topologically LRS II, i.e. topologically equivalent to a marginally trapped surface S' in some spacetime $\mathscr{M}'$ of LRS II class.*

Theorem 3.2.6. *Let T be an MTT in an NNF spacetime foliated by MTS. Then T is topologically LRS II.*

The statement of theorem 3.2.6 follows from standard topological results: Suppose T is foliated by a collection $\{S_\alpha\}_{\alpha\in A}$ of MTS. Since S_α are closed, these surfaces are embedded and thus inherit the induced topology from T (see theorem 5 on page $51-52$, (Camaco, 1943)). Let T' be an MTT in LRS II foliated by a collection $\{S'_\beta\}_{\beta\in B}$, and for a fixed α and β let $f: S_\alpha \longrightarrow S'_\beta$ be a diffeomorphism. Then the topology of S_α and S_β coincide, and since the surfaces S'_β inherit the induced topology from T', T and T' must be topologically equivalent.

To differentiate black holes from white holes, or inner from outer black hole horizons, the definition of MTT here would not suffice (Booth & Fairhurst, 2005) and for that one requires the Lie derivative of Θ_k along the ingoing null normal direction, $\mathscr{L}_l\Theta_k$, to be strictly negative. For now, however, this is immaterial to the result in theorem 3.2.6. This will be briefly discussed upon in the next section when we discuss the evolution of trapped surfaces.

While in general an MTT may not be associated with the boundary of a black hole, for LRS II spacetime (and by extension NNF spacetimes), if there is indeed a trapped region and the signature of the induced metric is fixed all over the MTT, then the MTT is associated with the boundary of the black hole. It has been shown in Bengtsson & Senovilla (2011) that in an arbitrary spherical spacetime, there are closed trapped surfaces penetrating both sides of the MTT. In other words, there exists a vector n^a, penetrating the MTT, along which $\Theta_k < 0$ on both sides of the MTT. For the nonspacelike case of the vector n^a, a region that is not trapped *now* may get trapped later *later*, and so there is no problem here. In the spacelike case, the result means that the MTT is a local maximum for Θ_k, at the epoch, defined by the spacelike vector n^a. We show that

Proposition 3.2.7. *For an NNF spacetime (and therefore LRS II spacetimes), there can be no spacelike vector n^a, penetrating the MTT, along which $\Theta_k < 0$ on both sides of the MTT.*

Proof. **:** Owing to the coordinate freedom, we can always rotate the $[u, e]$ plane in such a way that e^a lies in the direction of n^a. Suppose there exists a spacelike vector n^a, that penetrates the MTT, and along this vector $\Theta_k < 0$ on both sides of the MTT. If the MTT is the local maximum for Θ_k, at an epoch, then we must have $\hat{\Theta}_k = 0$ and $\hat{\hat{\Theta}}_k < 0$, on the MTT at the given epoch.

Now, for NNF spacetimes, we have

$$\begin{aligned}
\hat{\Theta}_k &= \frac{3}{2}\phi\Sigma + Q - \frac{1}{2}\phi^2 + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) \\
&\quad - \frac{2}{3}\left(\rho + \Lambda\right) - \mathcal{E} - \frac{1}{2}\Pi + 2\xi\left(\Omega + 1\right) + \varepsilon_{ab}M^{ab} \\
&\quad - \left(\Sigma_{ab} + \zeta_{ab}\right)\zeta^{ab} + \tilde{M},
\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}
M^{ab} &= \left(\delta^a + 2A^a + 2\alpha^a\right)\Omega^b, \\
\tilde{M} &= \left(\delta_a - \Sigma_a\right)\left(a^a + \Sigma^a\right) - a^a\left(a_a + \Sigma_a\right).
\end{aligned}$$

On the MTT, where $\Theta_k = 0$, (3.16) becomes

$$\begin{aligned}
\hat{\Theta}_k|_{\Theta_k=0} &= -\frac{2}{3}\left(\rho+\Lambda\right) - \mathscr{E} - \frac{1}{2}\Pi + Q + 2\xi\left(\Omega+1\right) \\
&\quad +\varepsilon_{ab}M^{ab} - \left(\Sigma_{ab}+\zeta_{ab}\right)\zeta^{ab} \\
&\quad +\tilde{M}|_{\Theta_k=0}. \qquad (3.17)
\end{aligned}$$

Setting (3.17) to zero gives

$$\begin{aligned}
\mathscr{E}|_{\Theta_k=0} &= -\frac{2}{3}\left(\rho+\Lambda\right) - \frac{1}{2}\Pi + Q + 2\xi\left(\Omega+1\right) + \varepsilon_{ab}M^{ab} \\
&\quad - \left(\Sigma_{ab}+\zeta_{ab}\right)\zeta^{ab} + \tilde{M}|_{\Theta_k=0}, \qquad (3.18)
\end{aligned}$$

so that

$$\begin{aligned}
\hat{\mathscr{E}}|_{\Theta_k=0} &= -\frac{2}{3}\widehat{\left(\rho+\Lambda\right)} - \frac{1}{2}\hat{\Pi} + \hat{Q} + 2\widehat{\xi\left(\Omega+1\right)} + \widehat{\varepsilon_{ab}M^{ab}} \\
&\quad -\widehat{\left(\Sigma_{ab}+\zeta_{ab}\right)\zeta^{ab}} + \widehat{\tilde{M}}|_{\Theta_k=0}. \qquad (3.19)
\end{aligned}$$

Computing $\hat{\hat{\Theta}}_k$ and evaluating on the 2-surface gives

$$\begin{aligned}
\hat{\hat{\Theta}}_k|_{\Theta_k=0} &= -\frac{3}{2}\left(\hat{\phi}\Sigma+\phi\hat{\Sigma}\right) + -\frac{1}{3}\left(\hat{\Theta}\Sigma+\Theta\hat{\Sigma}\right) + \frac{4}{9}\Theta\hat{\Theta} \\
&\quad -\phi\hat{\phi} - 2\sigma\hat{\Sigma} - \frac{2}{3}\hat{\rho} - \hat{\mathscr{E}} - \frac{1}{2}\hat{\Pi} + \hat{Q} + 2\widehat{\xi\left(\Omega+1\right)} \\
&\quad +\widehat{\varepsilon_{ab}M^{ab}} - \widehat{\left(\Sigma_{ab}+\zeta_{ab}\right)\zeta^{ab}} + \widehat{\tilde{M}}|_{\Theta_k=0} \\
&= \frac{3}{2}\left(\frac{1}{3}\Sigma + \frac{4}{9}\Theta\right)\hat{\Theta}_k|_{\Theta_k=0} \\
&= 0. \qquad (3.20)
\end{aligned}$$

Thus in NNF spacetimes at a given epoch, we cannot have trapped surfaces penetrating both sides of the MTT. □

3.2.3 Evolution of MTTs

We will now consider evolution of leaves of the MTT, a major aim of this chapter. As the MTS evolves, it *traces* out the MTT, and so we can study the evolution to determine the causal character of the MTT. We recall the approach in Ellis *et al.* (2014). In LRS II spacetimes, the MTS can be viewed as a curve in the $[u, e]$ plane, defined by $\Theta_k = 0$. If Θ_k^a is the tangent vector to the MTS curve, then the slope of Θ_k^a, α/β, can be used to study the evolution of the MTS. To determine such slope, note that the tangent to $\Theta_k = 0$ can be written as

$$\Theta_k^a = \alpha u^a + \beta e^a, \tag{3.21}$$

where α, β are smooth functions on the MTS. Since $\nabla_a \Theta_k = -\dot{\Theta}_k u_a + \hat{\Theta}_k e_a$ is normal to the curve $\Theta_k = 0$ (and thus normal to the MTT), we must have $\Theta_k^a \nabla_a \Theta_k = 0$, from which we obtain the slope as

$$\frac{\alpha}{\beta} = -\frac{\hat{\Theta}_k}{\dot{\Theta}_k} = \frac{\frac{2}{3}\left(\rho + \Lambda\right) + \mathcal{E} + \frac{1}{2}\Pi - Q}{-\frac{1}{3}\left(\rho + 3p - 2\Lambda\right) + \mathcal{E} - \frac{1}{2}\Pi + Q}, \tag{3.22}$$

where $\dot{\Theta}_k$ and $\hat{\Theta}_k$ are obtained by adding up (2.3) and (2.4), and (2.5) and (2.6) respectively. To determine the causal character of the MTT, one then takes the square of the tangent vector Θ_k^a, i.e.

$$|\Theta_k^a|^2 = \beta^2 \left(1 - \frac{\alpha^2}{\beta^2}\right). \tag{3.23}$$

Therefore, at each point on the MTT, it is timelike if $\alpha^2/\beta^2 > 1$, spacelike if $\alpha^2/\beta^2 < 1$ and null if $\alpha^2/\beta^2 = 1$.

To generalize the above discussed result to NNF spacetimes, we will first generalize to any 4-dimensional spacetime. To do this, it becomes immediately clear that the approach used in Ellis *et al.* (2014) would not suffice. This is because the splitting of the covariant derivative of Θ_k has a sheet component, and a product of this covariant derivative with the tangent to an MTS gives an expression, which when set to zero, there is no way to explicitly write out the slope α/β. However, we note that $\nabla_a\Theta_k$ is normal to the MTT, and as such the square can be used to determine the causal character of the MTT. We will do this explicitly and we will show the limiting cases for NNF and LRS II.

We start by calculating

$$\dot{\Theta}_k = \dot{\Theta}_k|_{LRS} + T_1 + \varepsilon_{ab}R^a\Omega^b, \tag{3.24}$$

and

$$\hat{\Theta}_k = \hat{\Theta}_k|_{LRS} + T_2 + \varepsilon_{ab}\overline{R}^a\Omega^b, \tag{3.25}$$

where we have set

$$\begin{aligned}
T_1 &= 3\Omega^2 - \Sigma^2 + 2\Omega\xi + \delta_a\left(\mathscr{A}^a + \alpha^a\right) + \left(R_a - 2a_a\right)\mathscr{A}^a \\
&\quad + \left(a^a - \mathscr{A}^a\right)\Sigma_a - 2\alpha_a\Sigma^a - \Sigma_{ab}\left(\Sigma^{ab} + \zeta^{ab}\right), \\
T_2 &= 2\xi^2 + \Omega^2 + 2\xi\Omega - \left(\zeta_{ab} + \Sigma_{ab}\right)\zeta^{ab} + \delta_a\left(a^a + \Sigma^a\right) \\
&\quad - \left(2a^a + \Sigma^a\right)\Sigma_a, \\
R^a &= \mathscr{A}^a + \alpha^a - a^a, \\
\overline{R}^a &= \delta^a + 2\left(\alpha^a + \mathscr{A}^a\right).
\end{aligned}$$

The notation $(*)_{LRS\ II}$ denotes the part of a quantity $*$ restricted to those covariant scalars completely describing LRS II spacetimes (the covariant scalars completely describing LRS II spacetimes are given by the set $\{\rho, p, A, \Theta, \phi, \Sigma, \mathscr{E}, \Pi, Q\}$). The covariant derivative of Θ_k decomposes as

$$\nabla_a\Theta_k = -\dot{\Theta}_k u_a + \hat{\Theta}_k e_a + \delta_a\Theta_k,$$

where

$$\delta_a\Theta_k = P_a + \varepsilon_{ab}Z^b.$$

Here we have set

$$\begin{aligned}
Z^b &= 2\delta^b(\xi+\Omega)+\Theta_l\Omega^b-2\zeta^{bc}\Omega_c+4Y^b+2\Sigma^b(\xi-\Omega),\\
Y^b &= \Omega\alpha^b-2\xi\Sigma^b+\Omega\mathscr{A}^b+\frac{1}{2}\mathscr{H}^b-\xi a^b,
\end{aligned}$$

and P_a is given by

$$\begin{aligned}
P_a &= 2\delta^b(\Sigma_{ab}+\zeta_{ab})-\Theta_l\Sigma_a-\Pi_a-Q_a-2\mathscr{E}_a\\
&\quad -2\Sigma_{ab}\left(\Sigma^b-\varepsilon^{bc}\Omega_c+a^b\right)+2\Omega_a(\xi-\Omega)-2\zeta_{ab}\Sigma^b.
\end{aligned}$$

Since $\nabla_a\Theta_k$ is normal to the MTT, the surface will be timelike, spacelike or null if

$$\begin{aligned}
\nabla^a\Theta_k\nabla_a\Theta_k &> 0,\\
\nabla^a\Theta_k\nabla_a\Theta_k &< 0,\\
\nabla^a\Theta_k\nabla_a\Theta_k &= 0,
\end{aligned} \tag{3.26}$$

respectively, where

$$\nabla^a\Theta_k\nabla_a\Theta_k = -\dot{\Theta}_k^2+\hat{\Theta}_k^2+\delta^a\Theta_k\delta_a\Theta_k, \tag{3.27}$$

with

$$\delta_a\Theta_k\delta^a\Theta_k = \left(P^2+Z^2\right)+2\varepsilon_{ab}P^aZ^b, \tag{3.28}$$

where $P^2 = P_aP^a$ and $Z^2 = Z_bZ^b$. We also have

$$\begin{aligned}
\dot{\Theta}_k^2 &= \left(\dot{\Theta}_k|^2_{LRS}+T_1^2+\Omega^2R^2+2T_1\right)\\
&\quad +2\varepsilon_{ab}R^a\Omega^b(1+T_1)
\end{aligned} \tag{3.29}$$

and

$$
\begin{aligned}
\hat{\Theta}_k^2 &= \left(\hat{\Theta}_k|_{LRS}^2 + T_2^2 + \Omega^2\overline{R}^2 + 2T_2\right) \\
&\quad +2\varepsilon_{ab}\overline{R}^a\Omega^b\left(1+T_2\right). \qquad (3.30)
\end{aligned}
$$

So for example, the condition for the MTT to be null is given by

$$
\begin{aligned}
\dot{\Theta}_k|_{LRS}^2 - \hat{\Theta}_k|_{LRS}^2 &= -\left(T_1^2 - T_2^2\right) - \Omega^2\left(R^2 - \overline{R}^2\right) \\
&\quad -2\varepsilon_{ab}\{R^a\left(1+T_1\right) + 2\varepsilon_{ab}P^aZ^b \\
&\quad +\overline{R}^a\left(1+T_2\right)\}\Omega^b + P^2 \\
&\quad +Z^2. \qquad (3.31)
\end{aligned}
$$

For NNF spacetimes, (3.28) vanishes and the condition for the MTT to be null reduces to

$$
\begin{aligned}
\dot{\Theta}_k|_{LRS}^2 - \hat{\Theta}_k|_{LRS}^2 &= -\left(T_1^2 - T_2^2\right) - \Omega^2\left(R^2 - \overline{R}^2\right) \\
&\quad -2\varepsilon_{ab}\{R^a\left(1+T_1\right) \\
&\quad +\overline{R}^a\left(1+T_2\right)\}\Omega^b. \qquad (3.32)
\end{aligned}
$$

The MTT is spacelike and timelike if

$$
\begin{aligned}
\dot{\Theta}_k|_{LRS}^2 - \hat{\Theta}_k|_{LRS}^2 &> -\left(T_1^2 - T_2^2\right) - \Omega^2\left(R^2 - \overline{R}^2\right) \\
&\quad -2\varepsilon_{ab}\{R^a\left(1+T_1\right) \\
&\quad +\overline{R}^a\left(1+T_2\right)\}\Omega^b \qquad (3.33)
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\Theta}_k|_{LRS}^2 - \hat{\Theta}_k|_{LRS}^2 &< -\left(T_1^2 - T_2^2\right) - \Omega^2\left(R^2 - \overline{R}^2\right) \\
&\quad -2\varepsilon_{ab}\{R^a\left(1+T_1\right) \\
&\quad +\overline{R}^a\left(1+T_2\right)\}\Omega^b \qquad (3.34)
\end{aligned}
$$

respectively.

For LRS II spacetimes, all vector and tensor quantities vanish (Clarkson, 2007), i.e. $\overline{P} = T_1 = T_2 = R = \overline{R} = P = Z = 0$. We also have $\xi = \Omega = 0$. Thus

$$\begin{aligned}
\dot{\Theta}_k^2 &= \dot{\Theta}_k|^2_{LRS}, \\
\hat{\Theta}_k^2 &= \hat{\Theta}_k|^2_{LRS}.
\end{aligned}$$

The conditions in (3.26) then reduce to

$$\hat{\Theta}_k|^2_{LRS} > \dot{\Theta}_k|^2_{LRS}, \tag{3.35}$$

$$\hat{\Theta}_k|^2_{LRS} < \dot{\Theta}_k|^2_{LRS}, \tag{3.36}$$

and

$$\hat{\Theta}_k|^2_{LRS} = \dot{\Theta}_k|^2_{LRS}, \tag{3.37}$$

respectively, which recovers the results for the LRS II spacetime in Ellis *et al.* (2014).

We can now state precisely which MTT are markers of the existence of a trapped region in a dynamical situation, i.e. a black hole. For NNF spacetimes they are precisely those satisfying the condition

$$\begin{aligned}
\mathscr{L}_l\Theta_k &= \left(\dot{\Theta}_k - \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 - T_2) \\
&\quad + \varepsilon_{ab}\left(R^a - \bar{R}^a\right)\Omega^b < 0,
\end{aligned} \tag{3.38}$$

in addition to the condition (3.33) and for LRS II, they are those satisfying

$$\mathscr{L}_l\Theta_k = \frac{1}{3}\left(\rho + 3p\right) + 2\mathscr{E} < 0, \tag{3.39}$$

in addition to the condition (3.36) (we have set $\Lambda = 0$).

Our emphasis on LRS II and NNF spacetimes is due to the fact that we expect most physically interesting spacetimes should fall in either class, and therefore have implications for numerical studies of black hole mergers. NNF spacetimes encompass those with rotation as well as spatial twist and as such, should include rotating black holes, even those without the symmetries of LRS II.

3.2.4 Stability of marginally trapped surfaces

In this section, we examine the stability of leaves of the MTT as the leaves evolve, following the convention in Cai & Galloway (2001). As we have done through out this work, considering and interpreting results on marginally trapped surfaces in terms of the geometric and matter variables from the $1+1+2$ splitting, we will see how to interpret stability in terms of these variables. We will focus on stability of MTS (which holds for MOTS) in LRS II spacetimes.

Stability analysis of MTS, analogous to stability analysis in minimal surface theory has been well formulated (Andersson *et al.*, 2005; Andersson *et al.*, 2008; Cai & Galloway, 2001; Galloway & Schoen, 2006). An MTS S is said to be stable if, given a deformation S_t of S, the associated outgoing null expansion scalar is somewhere positive on S_t. We start by giving an overview.

Let S be an MTS in an initial data set $\left(\Sigma, h, \chi\right)$ with outward normal vector e^a. Consider variation $t \mapsto S_t$ of $S = S_0$ with the variation vector field

$$\begin{aligned} \mathcal{V}^a &= \frac{\partial}{\partial t}|_{t=0} \\ &= \Phi e^a,\ \Phi \in C^{\infty}(S). \end{aligned} \tag{3.40}$$

Let $\Theta(t)$ be the null expansion of S_t with respect to $k^a_t = u^a + e^a_t$ (where e^a_t is the unit normal vector field to S_t in Σ). Then one has

$$\frac{\partial\Theta}{\partial t}|_{t=0} \quad = \quad L(\Phi)\,, \tag{3.41}$$

where L is the operator

$$L : C^\infty(S) \longrightarrow C^\infty(S)\,,$$

given by

$$\begin{aligned} L(\Phi) \quad &= \quad \{-\Delta + 2X^a\nabla_a + F + \nabla_a X^a \\ &\qquad -X_a X^a\}\Phi, \end{aligned} \tag{3.42}$$

(Andersson *et al.*, 2005; Andersson *et al.*, 2008; Cai & Galloway, 2001; Galloway, 2011; Galloway & Schoen, 2006) where

$$\begin{aligned} F &= \frac{1}{2}R_S - \left(\mu + J^a e_a\right) - \frac{1}{2}\left(\chi_S\right)^{ab}\left(\chi_S\right)_{ab}, \\ \mu &= \frac{1}{2}\left(R_\Sigma + \left(\left(\chi_\Sigma\right)^a_{\ a}\right)^2 - \left(\chi_\Sigma\right)^{ab}\left(\chi_\Sigma\right)_{ab}\right), \\ J^a &= \nabla_b\left(\chi_\Sigma\right)^{ba} - d^a\left(\chi_\Sigma\right)^b_{\ b}. \end{aligned} \tag{3.43}$$

The quantity R_S denotes the scalar curvature on S and R_Σ is the scalar curvature on Σ. The quantities J^a and μ are the local momentum and local energy density respectively.

For LRS II spacetimes the vector field X^a becomes

$$X_a \quad = \quad -l_c N^b_{\ a}\nabla_b k^c = 0. \tag{3.44}$$

Therefore (3.42) reduces to the classical stability operator in minimal surface theory

$$L(\Phi) \quad = \quad -(\Delta - F)\,\Phi. \tag{3.45}$$

The MOTS is said to be stable if there exists a real number λ (called the principal eigenvalue of L) such that $L(\Phi) = \lambda\Phi$ (Φ is the associated eigenfunction), $\lambda \geq 0$ and Φ is a positive function. Strict stability requires $\lambda > 0$.

Given an initial data set, the dominant energy condition (DEC) is given by

$$\mu \geq \sqrt{J^a J_a}. \tag{3.46}$$

In addition to the stability condition, the DEC is usually required to be satisfied.

Explicitly we calculate the following quantities for LRS II spacetimes.

$$\begin{aligned}
\left(\chi_S\right)^{ab}\left(\chi_S\right)_{ab} &= -\frac{1}{2}\left(\frac{2}{3}\Theta-\Sigma\right)^2+\frac{1}{2}\phi^2, \\
\left(\chi_\Sigma\right)^{ab}\left(\chi_\Sigma\right)_{ab} &= \frac{1}{3}\Theta^2+\frac{3}{2}\Sigma^2, \\
\left(\chi_\Sigma\right)^{a}{}_{a} &= \Theta, \\
J^a &= \left(\frac{3}{2}\Sigma^2-\dot{\Theta}\right)u^a+\left(\frac{1}{2}A\Sigma-Q\right)e^a, \\
\mu &= \rho+\Lambda,
\end{aligned} \tag{3.47}$$

The function F then becomes

$$F = \frac{1}{2}A\Sigma-\frac{2}{3}\left(\rho+\Lambda\right)-\mathcal{E}-\frac{1}{2}\Pi-Q.$$

The scalar curvature on R_S on S given by

$$R_S = 2K, \tag{3.48}$$

and the scalar curvature R_Σ on Σ is given by

$$R_\Sigma = -2\left(\hat{\phi}+\frac{3}{4}\phi^2-K\right), \tag{3.49}$$

where K is the Gaussian curvature. The DEC in (3.46) reduces to

$$\rho+\Lambda \geq \sqrt{-\left(\frac{3}{2}\Sigma^2-\dot{\Theta}\right)^2+\left(\frac{1}{2}A\Sigma-Q\right)^2}. \tag{3.50}$$

The square root on the RHS of (3.50) provides the extra condition that

$$\left(\frac{1}{2}A\Sigma - Q\right)^2 \quad \geq \quad \left(\frac{3}{2}\Sigma^2 - \dot{\Theta}\right)^2. \tag{3.51}$$

Whenever the spacetime is such that, on the 2-surfaces we have $\Delta = 0$, then one obtains λ from $\lambda = F$. Let us now present some simple examples. For spherically symmetric spacetimes, the Laplacian is zero and ρ, Λ, Π, Q are all vanishing. Showing stability of the MOTS then amounts to solving

$$(\lambda + \mathscr{E})\,\Phi \quad = \quad 0. \tag{3.52}$$

Since Φ is required to be strictly positive, the solution to (3.52) is

$$\lambda \quad = \quad -\mathscr{E}. \tag{3.53}$$

The MTS are then stable if $\mathscr{E} \leq 0$ and strictly stable if $\mathscr{E} < 0$.

As an example, take the Schwarzschild spacetime with

$$\mathscr{E} \quad = \quad -\frac{2m}{r^3}.$$

Since $r, m > 0$, $\mathscr{E} < 0$, and we have $\lambda > 0$. We thus have strict stability. The combination of strict stability, and noting that the DEC in (3.50) is satisfied for a spherically symmetric case, implies that the surfaces S are topological 2-spheres (Newman, 1987). In fact we see that the solution in (3.53) implies that for a spherically symmetric vacuum spacetime $\mathscr{E}$ is necessarily less than zero, since the MTS must be topological spheres.

As another example, consider the case of the Oppenheimer-Snyder collapse. On the inner horizon Φ is time-dependent only, and the Laplacian is zero. We also have the vanishing of

$$\Sigma, A, Q, \Pi, \Lambda, \mathscr{E},$$

as well as all of the hat derivatives. Then the stability equation becomes

$$-\frac{2}{3}\rho\Phi \quad = \quad \lambda\Phi, \tag{3.54}$$

which implies $\lambda = -(2/3)\,\rho$. Since $\rho > 0$, λ is strictly negative and the MTS are unstable.

In the numerical simulation of black holes merger, from the inspiral to the ringdown atate, one expects λ to initialy have negative values for the indivitual MTS (or MOTS), pass through $\lambda = 0$, and then assume positive values. For simplification, consider the case of non-rotating black holes with $\Delta = 0$ on S. then S smoothly evolves so that F goes from negative to positive, passing through the marginally stable case $F = 0$.

For a specific example, consider the case where the black holes models are shear-free perfect fluid spacetimes ($\Pi = Q = \Sigma = 0$). During the inspiral, one has $\mathcal{E} > -2/3\rho$ on the individual MTSs (or MOTSs). We see that this condition holds through out the merger and immediately before the ringdown state we must have $\mathcal{E} = -2/3\rho$. At the ring down state , $\mathcal{E} < -2/3\rho$.

If the final state is the event horizon of the Schwarzschild vacuum solution, the $\mathcal{E}$ must be less than zero on the MOTS (we specifically use MOTS here since $\phi = 0$ on the horizon for the Schwarzschild solution) since $\rho = 0$, which is expected.

We therefore have an idea, in the case considered here, the correlation between the electric Weyl scalar $\mathcal{E}$ and the energy density ρ one is to expect during the three stages of black holes merger. In effect, one may specify the geometric quantities $\mathcal{E}$ and ρ when studying such mergers, which nicely correlates the three regimes of the merger.

3.3 Conclusion

We have established a topological relationship between marginally trapped surfaces (and the 3-surfaces they foliate) in LRS II spacetimes and those of a certain generalization of LRS II spacetimes, which we call NNF spacetime, utilizing the $1+1+2$ semitetrad covariant formalism. We then extended our earlier work on the evolution of MTTs, which was restricted to LRS II spacetimes, to NNF spacetimes as well as a general 4-dimensional spacetime. Our approach allows one to express quantities in terms of well defined curvature variables, which has the added advantage of providing relative ease in assigning physical interpretation once certain geometric and matter variables are specified. We have also interpreted the stability analysis of marginally trapped surfaces in LRS II in terms of the matter and curvature variables using this formalism, and indeed established the stability of marginally trapped surfaces in Schwarzschild spacetime and the instability of those in the Oppenheimer-Snyder collapse. A further study of implications of our approach is currently being considered to extend stability analysis to NNF spacetimes as these spacetimes should rotating black hole spacetimes with spatial twist as well as other sheet related quantities, but not with all the symmetries of LRS II.

Chapter 4

Some results on cosmological and astrophysical horizons and trapped surfaces

4.1 Introduction

In this chapter we investigate the evolution of black hole horizons in context of the $1+1+2$ covariant formalism. The formulation in this chapter allows us to make some very important observations about black hole horizons. We determine the forms of the horizon function determining the sign of the intrinsic metric on the horizon for certain classes of spacetimes as well as for the general case. We use these results to study horizons and trapped surfaces in some well known spacetimes. We then consider some results concerning surface gravity on the horizon.

The chapter is structured as follow: Section 4.3 looks at the nature of horizons in various classes of spacetimes and use the results to study properties of horizons and marginally trapped surfaces in the Robertson-Walker and the Lemaitre-Tolman-Bondi spacetimes. In section 4.4 we calculate the surface gravity for various classes of spacetimes as well as the general case, and in section 4.5 we conclude with discussion of the results.

4.2 Some definitions

Let us now recall some useful notions from the previous chapter (Senovilla, (1998); Ashtekar & Krishnan, (2002); Ashtekar & Krishnan, (2003); Senovilla, (2003); Senovilla, (2003); Hayward, (2004); Bengtsson & Senovilla, (2011)). As was mentioned in the Introduction, the authors in (Booth & Fairhurst, 2005) showed that a 3-surface foliated by marginally trapped 2-surfaces is a suitable boundary of a black hole under certain conditions. In the definitions and discussions that are to follow, k^a and l^a are respectively the outward and inward null normal vector fields to a leaf of such foliation, while Θ_k and Θ_l are the expansions of the congruences generated by k^a and l^a respectively.

The definitions that are to follow were introduced in the previous chapter. However, we will recall them in order to make this chapter self contained.

Definition 4.2.1 (**Trapped Surface**). *A (future) trapped surface (TS) is a smooth, connected, closed, spacelike co-dimension* 2 *submanifold S of M such that the divergences, Θ_k and Θ_l, of the congruences generated by the null normal vector fields k^a and l^a respectively are everywhere negative on S (k^a is the outgoing null normal vector field and l^a is the ingoing null normal vector field).*

Definition 4.2.2 (**Marginally Trapped Surface**). *A marginally trapped surface (MTS) is a smooth, connected, closed, spacelike co-dimension* 2 *submanifold S of M such that Θ_k is everywhere vanishing on S and Θ_l is everywhere negative on S.*

Definition 4.2.3 (**Marginally Trapped Tube**). *A marginally trapped tube (MTT) is a co-dimension 1 submanifold H of M which is foliated by MTS.*

In the case that there is no restriction on the sign of Θ_l, then the co-dimension 1 surface is called a *marginally outer trapped tube* (MOTT), and the co-dimension 2 surfaces foliating the MOTT are called *marginally outer trapped surfaces* (MOTS). For more on the above definitions see the following references (Ashtekar & Krishnan, 2002; Ashtekar & Krishnan, 2003; Bengtsson & Senovilla, 2011; Booth & Fairhurst, 2005; Hayward, 2004; Senovilla, 1998; Senovilla, 2003a; Senovilla, 2003b, as well as Ellis *et al.,* 2014). In general, the signature of the induced metric on H will vary over H. There are however cases where the signature is fixed all over H (C is a fixed real number everywhere on H). In such cases a spacelike MTT is called a dynamical horizon (DH), a timelike MTT is called a timelike membrane (TLM), and a null and non-expanding MTT is called an isolated horizon (IH). An MTT is a *future outer trapping* if and only if $\mathscr{L}_l\Theta_k$ is everywhere negative on H and a *future inner trapping horizon* if and only if $\mathscr{L}_l\Theta_k$ is everywhere positive on H, where $\mathscr{L}_l$ is the Lie derivative operator in the direction of the null normal vector field l^a. Timelike implies inner trapped and spacelike implies outer trapped.

We now introduce certain classes of spacetimes to be considered throughout this work. An LRS spacetime M is a spacetime in which at each point $p \in M$, there exists a continuous isotropy group generating a multiply transitive isometry group on M (Stewart & Ellis, 1968; Elst & Ellis, 1996; Clarkson & Barrett, 2003; Betschart & Clarkson, 2004; Singh *et al.,* 2016; Singh *et al.,* 2017). The general metric of LRS spacetimes is given by

$$\begin{aligned} ds^2 = &- A^2dt^2 + B^2d\chi^2 + \bar{F}^2dy^2 + \left[\left(\bar{F}\bar{D}\right)^2 + (Bh)^2 - \left(Ag\right)^2\right]dz^2 \\ &+ \left(A^2g\,dt - B^2h\,d\chi\right)dz, \end{aligned} \tag{4.1}$$

where $A^2, B^2, \bar{F}^2$ are functions of t and χ, $\bar{D}^2$ is a function of y and k (k fixes the geometry of the 2-surfaces), and g, h are functions of y. LRS II spacetimes are a subclass of LRS spacetimes with $g = 0 = h$.

The field equations and full covariant derivatives of u^a and e^a for LRS spacetimes are given by the vanishing of all tensor and vector quantities in (2.3) to (2.8). For LRS II spacetimes, in addition to the vanishing of the tensor and vector quantities in (2.3) to (2.8), we also have the vanishing of Ω and ξ.

LRS II class of spacetimes generalizes spherically symmetric solutions to Einstein field equations (EFEs). Examples of physically interesting spherically symmetric solutions that fall into the class of LRS II spacetimes include Schwarzschild, Friedman-Lemaitre-Robertson-Walker (FLRW), Lemaitre-Tolman-Bondi, Vaidya and the Oppeinheimer-Snyder dust solutions. LRS spacetimes, a generalization of LRS II spacetimes, include solutions with nonzero vorticity and nonzero spatial twist. Some of these solutions include the Gödel's world model, the Kantowski-Sachs models and the Bianchi models, invariant under the G_3 groups of types I, II, VIII and IX.

For LRS II spacetimes, the outward and inward pointing null normal vectors to the MTS - given as co-dimension 2 smooth embeddings of M - are expressed by

$$k^a = \frac{1}{\sqrt{2}}\left(u^a + e^a\right) \quad \text{and} \quad l^a = \frac{1}{\sqrt{2}}\left(u^a - e^a\right), \tag{4.2}$$

respectively (Ellis *et al.,* 2014; Goswami & Ellis, 2011).

The class of NNF spacetimes provides a certain generalization of LRS spacetimes in the sense that, a spacetime may possess the property that the preferred scalings of the null vector fields in the spacetime coincide with those of LRS spacetimes, but without all of the symmetries of LRS spacetimes. In the case of NNF spacetimes, there are nonscalar background quantities, i.e. under linear perturbations, vector and tensor quantities are not necessarily gauge invariant.

For LRS II spacetimes, the expansions Θ_k and Θ_l are given as

$$\Theta_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma + \phi\right) \quad \text{and} \quad \Theta_l = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma - \phi\right), \tag{4.3}$$

respectively (Sherif *et al.*, 2018).

Definition 4.2.4 (Hayward). *A spacelike future outer trapping horizon (SFOTH) is a spacelike MTT H of spacetime, such that $\mathscr{L}_l \Theta_k < 0$.*

Definition 4.2.5 (Regularity). *A DH $\mathscr{H}$ is regular if*

(1). H is achronal (for $p, q \in H$, $\nexists$ a timelike curve $\gamma : (a, b) \longrightarrow H$ such that $\gamma(a) = p$ and $\gamma(b) = q$ or $\gamma(a) = q$ and $\gamma(b) = p$).

(2). $2\sigma^2 + T_{ab} k^a k^b \neq 0$ on H $\forall$ null vectors k^a, where T_{ab} is the energy momentum tensor.

We next look at evolution of MTTs in LRS II and NNF classes using the prescription in Booth *et al.* 2005) in context of the $1+1+2$ splitting (also see Booth & Fairhurst (2004) and Booth & Fairhurst (2005)).

4.3 On Properties of MTTs

For NNF spacetimes, it transpires that the null expansion scalars take exactly the same form as (4.3) (Sherif *et al.*, 2018). This result implies topological equivalence of horizon types in these spacetimes to LRS II. In this section we consider the evolution of MTTs. Specifically, we study the evolution of MTTs in the LRS II class of spacetimes, the NNF class, then a general 4-dimensional spacetime. The approach in this work for determining the causal character of an MTT utilizes the formulation in Booth *et al.* (2005). We compute a certain smooth function on the MTT determining its causal character, denoted by C, in terms of quantities from the $1+1+2$ decomposition of spacetime admitting the MTTs (Clarkson, 2007; Ellis *et al.*, 2014). This allows us to phrase the definition of the causal nature of an MTTs in terms of constraints on the geometric and thermodynamic variables. We will see that the causal character of the MTT is determined by some energy condition on the MTT, which is expected as stated in Booth *et al.* (2005). The comparison of the three cases considered also provides us with a useful way to determine a certain equivalence under causal properties. Obtaining expressions in terms of these geometric and thermodynamic variables provides us with useful insights into the nature and properties of MTTs of certain well studied spacetimes. Interesting statements can be made about MTT and marginally trapped 2-surfaces in certain cosmological examples provided. We recover some existing results.

4.3.1 Evolution of MTTs

We now describe the procedure for determining the signature of the induced metric on an MTT H, following Booth *et al.* (2005) where the null expansions now are expressed in terms of quantities from the $1+1+2$ splitting. It is worth mentioning that in Booth *et al.* (2005), the analyses were restricted to spherically symmetric spacetimes. Consequently, by adapting the calculation of C to the variables from the $1+1+2$ splitting, we can extend to LRS II spacetimes. While it is true that in general this approach may not be suitable for extensive analysis of the evolution of MTTs, we will see that the form of C for more general spacetimes provides insight into causal relationships of MTTs in LRS II spacetimes and more general spacetimes.

Let us first briefly discuss the formalism in Booth *et al.* (2005). One begins by introducing a vector field $\bar{X}^a$ which is normal to an MTT H. A tangent vector field X^a to H (in the sense that $X^a\bar{X}_a = 0$) is also introduced which is everywhere orthogonal to the foliation. The vector field X^a generates a foliation preserving flow ($\mathscr{L}_X v = f(v)$, for some function $f(v)$ (v labels the foliation)). Both X^a and $\bar{X}^a$ are assumed to be future pointing in the sense that

$$X^a l_a < 0,\ \bar{X}^a l_a < 0.$$

There is a further requirement that

$$\bar{X}^a \bar{X}_a \quad = \quad -X^a X_a.$$

Of course if H is spacelike then $X^a X_a > 0$ and similarly $\bar{X}^a \bar{X}_a < 0$. The vector fields X^a and $\bar{X}^a$ can be written as

$$X^a = \alpha \left(k^a - C l^a\right) \text{ and } \bar{X}^a = \alpha \left(k^a + C l^a\right), \tag{4.4}$$

respectively, where C is some scalar field on H ($C \in C^\infty(H)$, where$C^\infty(H)$ is the set of smooth functions on H) and $\alpha \in \mathbb{R}^+$. Without loss of generality, we set $\alpha = 1$, and from the definition of X^a,

$$\mathscr{L}_X \Theta_k = \mathscr{L}_{[k-Cl]} \Theta_k = 0, \tag{4.5}$$

from which we write the explicit expression for the field C:

$$C = \frac{\mathscr{L}_k \Theta_k}{\mathscr{L}_l \Theta_k}. \tag{4.6}$$

The proportionality of $X^a X_a$ and $\bar{X}^a \bar{X}_a$ to C means that the sign of C, at a point of H, can be used to determine the causal nature of the MTT at the point: If $C < 0$ the MTT is timelike, if $C > 0$ the MTT is spacelike and if $C = 0$ or $C = \infty$ ($\mathscr{L}_k \Theta_k \neq 0$ and $\mathscr{L}_l \Theta_k = 0$) the MTT is null. The sign of C also determines whether the MTT is expanding ($C > 0$), contracting ($C < 0$) or unchanging in area ($C = 0$ or $C = \infty$). Thus the MTT is timelike if and only if it is contracting and spacelike if and only if it is expanding.

As an example we can explicitly calculate C for the LRS II class of spacetimes in terms of the scalars of the $1+1+2$ formalism:

$$
\begin{aligned}
\mathscr{L}_k\Theta_k &= k^a\nabla_a\Theta_k \\
&= \dot{\Theta}_k + \hat{\Theta}_k \\
&= -\left(\rho + p\right) - \Pi + 2Q,
\end{aligned} \tag{4.7}
$$

and

$$
\begin{aligned}
\mathscr{L}_l\Theta_k &= l^a\nabla_a\Theta_k \\
&= \dot{\Theta}_k - \hat{\Theta}_k \\
&= \frac{1}{3}\left(\rho + 3p\right) + 2\mathscr{E},
\end{aligned} \tag{4.8}
$$

which gives

$$
C_{LRS\ II} = \frac{-\left(\rho + p\right) - \Pi + 2Q}{\frac{1}{3}\left(\rho - 3p\right) + 2\mathscr{E}}. \tag{4.9}
$$

The quantities $\dot{\Theta}_k$ and $\hat{\Theta}_k$ are computed using the appropriate linear combination of (2.3) to (2.6). We note that the Lie derivative is along the null normals and evaluated on the MTS (thus on the MTT). We now apply (4.9) to some simple cases. For example, consider the case for the spherically symmetric vacuum spacetimes. In this case, $\rho = p = \Pi = Q = 0$ and so the numerator of (4.9) is zero. Since $\mathscr{E} \neq 0$, H is null, i.e. H is an NEH. Considering Oppenheimer-Snyder collapse, $p = \Pi = Q = \mathscr{E} = 0$, and $C = -3$. Thus H is a TLM and contracting. Some authors argue that TLMs cannot be associated with the surface of a black hole during its evolution since timelike curves can traverse the membrane in both directions. However, this is a case where a TLM is clearly associated with the evolution of the black hole,and in such case can be considered as the boundary of the black hole.

Since an MTT H is timelike if and only if it is inner trapped (collapsing), i.e. $\mathscr{L}_l\Theta_k > 0$, for the LRS II class of spacetimes we can rephrase this in terms of the matter and scalar variables as follows:

Proposition 4.3.1. *An MTT in a spacetime in the LRS II class of spacetimes is timelike if and only if* $\rho > 3p - 6\mathscr{E}$.

But as the MTT is timelike, $C < 0$, which implies $\left(\rho + p\right) + \Pi - 2Q > 0$. Suppose we consider an ideal case like the perfect fluid, we have $\Pi = Q = 0$. If the weak energy condition (WEC) is satisfied then the condition for the MTT to be timelike is that $\rho > 3p - 6\mathscr{E}$ which agrees with the result in Ellis *et al.* (2014). Similarly, an MTT in a spacetime in the LRS II class is spacelike if and only if $\rho \leq 3p - 6\mathscr{E}$.

Analogously, we can compute C for NNF spacetimes. We shall denote by $(*)_{LRS\ II}$ the part of a quantity $*$ restricted to those covariant scalars completely describing LRS II spacetimes (the covariant scalars completely describing LRS II spacetimes are given by the set $\{\rho, p, A, \Theta, \phi, \Sigma, \mathscr{E}, \Pi, Q\}$).

$$
\begin{aligned}
\mathscr{L}_k\Theta_k &= k^a\nabla_a\Theta_k \\
&= \left(\dot{\Theta}_k + \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 + T_2) + \varepsilon_{ab}\left(R^a + \bar{R}^a\right)\Omega^b,
\end{aligned}
\tag{4.10}
$$

and

$$
\begin{aligned}
\mathscr{L}_l\Theta_k &= l^a\nabla_a\Theta_k \\
&= \left(\dot{\Theta}_k - \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 - T_2) + \varepsilon_{ab}\left(R^a - \bar{R}^a\right)\Omega^b,
\end{aligned}
\tag{4.11}
$$

where we set

$$
\begin{aligned}
T_1 &= 3\Omega^2 - \Sigma^2 + 2\Omega\xi + \delta_a\left(\mathscr{A}^a + \alpha^a\right) + (R_a - 2a_a)\mathscr{A}^a + \left(a^a - \mathscr{A}^a\right)\Sigma_a \\
&\quad -2\alpha_a\Sigma^a - \Sigma_{ab}\left(\Sigma^{ab} + \zeta^{ab}\right), \\
T_2 &= 2\xi^2 + \Omega^2 + 2\xi\Omega - (\zeta_{ab} + \Sigma_{ab})\zeta^{ab} - \left(2a^a + \Sigma^a\right)\Sigma_a + \delta_a\left(a^a + \Sigma^a\right), \\
R^a &= \mathscr{A}^a + \alpha^a - a^a, \\
\overline{R}^a &= \delta^a + 2\left(\alpha^a + \mathscr{A}^a\right).
\end{aligned}
$$

This gives

$$C_{NNF} = \frac{\left(\dot{\Theta}_k + \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 + T_2) + \varepsilon_{ab}\left(R^a + \bar{R}^a\right)\Omega^b}{\left(\dot{\Theta}_k - \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 - T_2) + \varepsilon_{ab}\left(R^a - \bar{R}^a\right)\Omega^b}. \tag{4.12}$$

Suppose we have a case where $R^a = \bar{R}^a$ and $T_1 = T_2$. Then (4.12) can be written as

$$C_{NNF} = C\big|_{LRS\ II} + \bar{C}, \tag{4.13}$$

where the function $\bar{C}$ is given by

$$\bar{C} = 6\frac{T_1 + \varepsilon_{ab}R^a\Omega^b}{\rho - 3p + 6\mathscr{E}}. \tag{4.14}$$

While not of any physical relevance and no physical motivation for the particular case given here, from (4.13) we can conclude that, in general, though MTT in NNF are topologically equivalent to LRS II, the MTTs will evolve differently. We note that the conditions on R^a and $\bar{R}^a$ are actually conditions on $\varepsilon_{ab}R^a\Omega^b$ and $\varepsilon_{ab}\bar{R}^a\Omega^b$.

We now provide some useful definitions and theorems. We mention the following theorem by Ashtekar & Galloway (2005), with proof making use of (4.9):

Theorem 4.3.2. *Let M be a 4-dimensional spacetime and let H be a DH in M satisfying the NEC. If H satisfies the second condition of regularity, then H is a (future) outer trapping horizon (FOTH).*

Proof. Condition (2) in definition 4.2.5 implies $\mathscr{L}_l\Theta_k$ is nowhere vanishing on H (Ashtekar & Galloway, 2005). If H satisfies the NEC, then by the Raychaudhuri equation, we know that $\mathscr{L}_k\Theta_k$ is nonpositive on H. Using (4.6), since H is spacelike, $\mathscr{L}_k\Theta_k$ is strictly negative ($\mathscr{L}_k\Theta_k = 0$ implies H is null). This implies that $\mathscr{L}_l\Theta_k$ has to be everywhere negative on H (for C to be positive). □

Such DH H is the spacelike (future) outer trapping horizon (SFOTH) put forward by Hayward, and whose existence is an indicator of the presence of a black hole (Ashtekar & Galloway, 2005).

Using (4.7) and (4.8), the signs of $\mathscr{L}_k\Theta_k$ and $\mathscr{L}_l\Theta_k$ on an SFOTH H imply the following result:

Theorem 4.3.3. *A DH H in a spacetime M of the LRS II class of spacetimes satisfying the energy condition*

$$2Q - \Pi < \rho + p < 4p - 6\mathscr{E}, \tag{4.15}$$

is an SFOTH.

Theorem 4.3.3 thus gives us a relatively easy way of detecting the possible existence of a black hole in LRS II spacetimes. Similar results can be stated for NNF class. For example, a DH in a spacetime in the NNF class satisfying the condition $2Q - \Pi - p + (T_1 + T_2) + \varepsilon_{ab}\left(R^a + \bar{R}^a\right)\Omega^b < \rho < 3p - 6\mathscr{E} - 3(T_1 - T_2) - 3\varepsilon_{ab}\left(R^a - \bar{R}^a\right)\Omega^b$ is an SFOTH.

Combining the result on topological equivalence between horizon types in the LRS II and NNF classes, and the case considered in (4.13), we state the following result:

Theorem 4.3.4. *Let H' be an MTT in the NNF class and let H' satisfy the case considered in* (4.13). *Suppose H is a null MTT in the LRS II class to which H' is topologically equivalent. Furthermore, suppose H' is regular and satisfies the NEC. Then H' is an SFOTH if $T_1 + \varepsilon_{ab}R^a\Omega^b < 0$ and a TLM if $T_1 + \varepsilon_{ab}R^a\Omega^b > 0$.*

This result also follows from the fact that H' being regular and satisfying the NEC implies that $\mathscr{L}_l\Theta_k$ is everywhere negative on H. From theorem 4.3.4 it is clear that it is entirely possible, in principle, to have topological equivalence between MTTs satisfying the same energy conditions, but exhibiting different causal character. We will state this as follows:

Proposition 4.3.5. *In general, topologically equivalent MTTs will exhibit different causal character.*

This is known and is expected in general as the causal character relies on a much *stronger* structure than the topology (see chapter 2 of Dribus (2017)).

For a general 4-dimensional spacetime, the generators of the outgoing and ingoing null geodesics to the two surfaces contain a sheet component, i.e. there is a component of k^a, l^a, denoted $\tau^a, -\tau^a$ respectively, projected onto the 2-sheet via N_{ab}. The generators for the outgoing and ingoing null geodesics are written as

$$k^a = \frac{1}{\sqrt{2}}\left(u^a + e^a + \tau^a\right),\ l^a = \frac{1}{\sqrt{2}}\left(u^a - e^a - \tau^a\right), \tag{4.16}$$

respectively. There is a coefficient, τ (the magnitude of k^a along e^a), of e^a for both null normals, but can be set to unity without loss of generality. We calculate the null expansions in the k^a direction and obtain

$$\Theta_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma + \phi + W\right), \tag{4.17}$$

where $W = \nabla_a \tau^a = (\delta_a - a_a)\,\tau^a$ (this decomposition, obtained from the decomposition of the fully orthogonally projected covariant derivative, D, of a vector orthogonal to u^a and e^a, can be found in (Clarkson, 2007)). It is clear that for a general 4-dimensional spacetime, if the sheet component of the null normal vector fields to the MTS is divergence free, then these MTS are topologically LRS II (see Sherif *et al.* (2018)). We therefore have

$$\begin{aligned}
\mathscr{L}_k\Theta_k &= \left(\dot{\Theta}_k + \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 + T_2) + \varepsilon_{ab}\left(R^a + \bar{R}^a\right)\Omega^b + \dot{W} + \hat{W} + \tau^a\delta_a\Theta_k,\\
\mathscr{L}_l\Theta_k &= \left(\dot{\Theta}_k - \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 - T_2) + \varepsilon_{ab}\left(R^a - \bar{R}^a\right)\Omega^b + \dot{W} - \hat{W} - \tau^a\delta_a\Theta_k,
\end{aligned} \tag{4.18}$$

so that

$$C = \frac{\left(\dot{\Theta}_k + \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 + T_2) + \varepsilon_{ab}\left(R^a + \bar{R}^a\right)\Omega^b + \dot{W} + \hat{W} + \tau^a\delta_a\Theta_k}{\left(\dot{\Theta}_k - \hat{\Theta}_k\right)\Big|_{LRS\ II} + (T_1 - T_2) + \varepsilon_{ab}\left(R^a - \bar{R}^a\right)\Omega^b + \dot{W} - \hat{W} - \tau^a\delta_a\Theta_k}. \tag{4.19}$$

where

$$\hat{\Theta}_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\hat{\Theta} - \hat{\Sigma} + \hat{\phi} + \hat{W}\right),\ \dot{\Theta}_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\dot{\Theta} - \dot{\Sigma} + \dot{\phi} + \dot{W}\right). \tag{4.20}$$

4.3.2 A causal classification

We have considered evolution of MTTs for general 4-dimensional spacetimes, NNF and LRS II spacetimes. The approach used here, where we have combined the formulation in Booth *et al.* (2005) coupled with the $1+1+2$ decomposition of spacetime allow us to give a particular classification of MTTs causally, i.e. classifying causally equivalent MTTs.

Given C, we will assume that the sign of denominator and numerator of C remains fixed by translation. This then leaves the sign of C and therefore the causal character of the MTT unchanged (notice that this ensures that causally equivalent MTTs satisfy the same energy conditions). It is also clear that if the support of certain functions is non-empty, C changes sign as well. This implies that the overall causal character of these MTTs will change at these support points. It is with these in mind we proceed with the following classification.

4.3.2.1 LRS II and NNF

Suppose we want to look at the particular case in (4.13). For a null MTT H in the LRS II class of spacetimes, i.e. $C|_{LRS\ II} = 0$, we may determine the causal character of the corresponding MTT, H', in the NNF class, which is topologically equivalent to H, by some restrictions on the function $\bar{C}$. For such case as that of (4.13), the evolution of the MTT will be precisely LRS II if $T_1 = -\varepsilon_{ab}R^a\Omega^b$. In fact, in general, MTT in the NNF class satisfying the condition that $\bar{R}^a = T_2 = 0$ will be both causally and diffeomorphically LRS II.

4.3.2.2 NNF and the general case

An MTT in a general 4-dimensional spacetime satisfying one of the conditions below is causally NNF:

a. $\hat{W} = -\tau^a \delta_a \Theta_k$.

b. $\dot{W} = 0$ and $\hat{W} = -\tau^a \delta_a \Theta_k$.

c. $W = 0$ and $\tau^a \delta_a \Theta_k = 0$.

In the case of the last item c., the MTT is also diffeomorphically NNF.

4.3.2.3 LRS II and the general case

An MTT in a general 4-dimensional spacetime satisfying one of the conditions below is causally LRS II:

a. $\hat{W} = -\tau^a \delta_a \Theta_k$ and $T_2 = \bar{R}^a = 0$.

b. $\dot{W} = 0$, $\hat{W} = -\tau^a \delta_a \Theta_k$ and $T_2 = \bar{R}^a = 0$.

c. $W = 0$, $\tau^a \delta_a \Theta_k = 0$ and $T_2 = \bar{R}^a = 0$.

d. The case considered in (4.13) and the additional condition that $T_1 = -\varepsilon_{ab} R^a \Omega^b$, with $\hat{W} = -\tau^a \delta_a \Theta_k$, $\dot{W} = 0$ and $\hat{W} = -\tau^a \delta_a \Theta_k$, or $W = 0$ and $\tau^a \delta_a \Theta_k = 0$.

In the cases of the items c. and d., the MTT is also diffeomorphically LRS II.

This classification is by no means exhaustive since just as we proceeded in (4.13) where we looked at a particular splitting of C, similar conditions can be put on C to obtain further classifications.

Next we use the expression for C for the LRS II class, (4.9), and investigate the MTTs in some well known spacetimes. We obtain some known results like the bounds on the equation of state parameter σ in RW spacetime, which distinguishes timelike, spacelike and null MTTs. We also investigate the stability of the 2-spheres foliating the MTTs, which is determined by conditions on σ.

4.3.3 Relationship between C and the slope to the tangent to the MTT

One of the very first investigations of the evolution of black holes, covariantly, utilizing the $1+1+2$ formalism, was carried out by Ellis *et al.* (2014). This approach was generalized in Sherif *et al.* (2018). Since in this paper we utilize the formalism in Booth *et al.* (2005), but in context of the geometric and thermodynamic variables from the $1+1+2$ formalism, it would make sense to show the relationship between the two approaches. While we do this (and provide examples) for LRS II spacetimes, this extends to the NNF spacetimes.

The approach in Ellis *et al.* (2014) chooses a tangent vector $\Psi^a = \alpha u^a + \beta e^a$, which lies on entirely on the MTT, defined as the curve $\Theta_k = 0$ in the $[u, e]$ plane. Since the vector $\nabla_a \Theta_k$ is normal to the MTT, then $\Psi^a \nabla_a \Theta_k$ is zero, which gives the slope, α/β, of the MTT curve

$$\frac{\alpha}{\beta} = -\frac{\hat{\Theta}_k}{\dot{\Theta}_k}. \tag{4.21}$$

The sign of α/β then determines if the MTT is future outgoing or future ingoing: if $\alpha/\beta < 0$, the MTT is said to be future ingoing, and if $\alpha/\beta > 0$ the MTT is said to be future outgoing. Now from the definition of the function C, lets look at for which values of α/β we can causally characterize the MTT.

We can write the function C from (4.6) as

$$C = \frac{\dot{\Theta}_k + \hat{\Theta}_k}{\dot{\Theta}_k - \hat{\Theta}_k}.$$

Dividing the numerator and denominator of by $\dot{\Theta}_k$ gives the function C in terms of the slope

$$C = \frac{1 - \alpha/\beta}{1 + \alpha/\beta}. \tag{4.22}$$

Multiplying (4.22) by $\left(1+\alpha/\beta\right)/\left(1+\alpha/\beta\right)$ (for $\alpha/\beta \neq -1$) gives

$$C = \frac{1-\alpha^2/\beta^2}{\left(1+\alpha/\beta\right)^2}. \tag{4.23}$$

Then the MTT is spacelike, timelike of null if α^2/β^2 is less than 1, greater than 1 or equal to 1, respectively. For $\alpha/\beta = -1$, we simply multiply (4.22) by $\left(1-\alpha/\beta\right)/\left(1-\alpha/\beta\right)$ and obtain

$$C = \frac{\left(1-\alpha/\beta\right)^2}{1-\alpha^2/\beta^2}. \tag{4.24}$$

The definitions of spacelike, timelike and null MTT follow as before.

4.3.4 Characterization of MTTs and the stability of MTS in some well known spacetimes

We here consider the characterization of MTTs in the Robertson-Walker and Lemaitre-Tolman-Bondi models, as well as the stability of the MTS foliating these MTTs.

4.3.4.1 Robertson-Walker spacetimes

As mentioned earlier, for certain solution types, the form of (4.9), when compared to the expression for C, allows us to express certain quantities in terms of the geometric and thermodynamic variables. Consider the case of a timelike perfect fluid. The tangent to the timelike congruence is given by

$$u^a = Dk^a + (2D)^{-1} l^a. \tag{4.25}$$

The function C is expressed as

$$C = \frac{1}{2D^2}\frac{\rho+p}{\left(\frac{1}{A}\right)+p-\rho}, \tag{4.26}$$

for some function D, Booth *et al.* (2005), where A is the area of the 2-spheres. The function C is thus determined by the sign of

$$\frac{\rho + p}{\left(\frac{1}{A}\right) + p - \rho}. \tag{4.27}$$

We can directly compare (4.26) to (4.9) and write the area A as

$$A \quad = \quad \frac{3}{\rho} D^2 \tag{4.28}$$

(we note that for such solutions $\Lambda, \mathscr{E} = 0$). Consider the Robertson-Walker spacetimes which assume an equation of state of the form $p = \sigma\rho$. Substituting (4.28) in the denominator of (4.27) we obtain

$$\left(\frac{1}{3D^2} + (\sigma - 1)\right)\rho. \tag{4.29}$$

For a given solution (or class of solutions), the denominator of C in (4.9) determines D. For LRS II spacetimes, $D = 1/\sqrt{2}$ (we can check (4.25) against the expressions for k^a and l^a in (4.2)). This gives the area A, of the marginally trapped 2-spheres in the case of timelike perfect fluids as $A = 3/\left(2\rho\right)$. Inserting this in the condition in proposition 4.3.1 and noting that both $\mathscr{E}$ and Λ vanish, we obtain the results in Booth *et al.* (2005), i.e. timelike and spacelike MTTs satisfy the cut-offs $\rho - p > 1/A$ and $\rho - p \le 1/A$ respectively.

For the Robertson-Walker spacetimes, we can explicitly write the quantity C in terms of σ,

$$C = \frac{3\left(\sigma + 1\right)}{\left(3\sigma - 1\right)}. \tag{4.30}$$

Of course, from (4.30), any real value of σ is supposed to fix C everywhere on the MTT. However, in Booth *et al.* (2005) the author showed that in certain cases, for higher values of σ, the value of C may vary on the MTT, in particular for $\sigma = 2$. However, the horizon remains spacelike. We see that the causal characterization of an MTT corresponds to the following bounds on σ: For a timelike MTTs, $-1 < \sigma < \frac{1}{3}$, a spacelike MTTs, $\sigma > \frac{1}{3}$ or $\sigma < -1$, and for null MTTs, $\sigma = -1$ or $\sigma = \frac{1}{3}$. The approach used here relatively easily obtains these results, which agree with those of Ben-Dov (2004) and Senovilla (1998).

The formula in (4.30) could have been directly obtained from (4.9): For timelike fluids C becomes

$$C = \frac{3\left(\rho + p\right)}{\left(3p - \rho\right)}. \tag{4.31}$$

With an equation of state of the form $p = \sigma\rho$, (4.31) is reduced to (4.30). Of interest is the form that A takes. Consider stability of the marginally trapped 2-spheres of the Robertson-Walker spacetimes. By stability we mean that if a marginally trapped 2-surface S is deformed outward, the associated outgoing null expansion scalar is non-negative and somewhere positive on the marginally trapped 2-sphere (see chapters 1 and 3, and references therein). Given a function φ on S, stability of S is determined by the sign of the principal eigenvalue of the stability operator L acting on φ. For spherically symmetric spacetimes, this eigenvalue is given by

$$\lambda = 8\pi\left(\frac{1}{2\mathscr{A}} - T_{ab}k^a l^b\right), \tag{4.32}$$

(see Booth *et al.* (2005)). For timelike fluids, we can then write λ as

$$\lambda = -8\pi\left(\frac{2}{3}\rho + p\right). \tag{4.33}$$

Since energy density is positive, for spacetimes in the class of timelike perfect fluids, the MTS are stable if $\rho \le -\frac{3}{2}p$ and strictly stable if $\rho < -\frac{3}{2}p$. We see that stability of the marginally trapped 2-spheres is only obtained under the condition of negative pressure. For the Robertson-Walker spacetimes, the condition for stability reduces to

$$\sigma \le -\frac{2}{3}. \tag{4.34}$$

From (4.34) it is evident that the marginally trapped 2-spheres of null and contracting, spacelike and expanding MTT, as well as the 2-spheres of MTTs in the dust filled universe are all unstable (this was also recently considered in Sherif *et al.* (2018) in context of the $1+1+2$ semitetrad formalism).

Again, our approach has relatively easily allowed us to completely characterize the marginally trapped 2-spheres of MTTs in these cosmological models in terms of stability. It is also possible to identify the SFOTHS in the Robertson-Walker spacetimes. We have that (4.15) reduces to the condition

$$-\sigma < 1 < 3\sigma. \tag{4.35}$$

It is clear from (4.35) that the SFOTHs in these models are precisely those spacelike MTTs with $\sigma > \frac{1}{3}$, containing unstable marginally trapped 2-spheres.

4.3.4.2 Lemaitre-Tolman-Bondi dust model

By the same token, let us consider the the Lemaitre-Tolman-Bondi (LTB) dust model (Goswami & Joshi, 2004a; Goswami & Joshi, 2004b; Zibin, 2008), a gravitational collapse model violating the cosmic censorship conjecture, with interior metric given by the line element

$$ds^2 = -dt^2 + \frac{R'^2}{1-r^2 b_0}dr^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{4.36}$$

Here $R = R(t,r)$ is the area radius of the collapsing shell and $b_0 = b_0(r)$ is their energy profile. The only non-zero matter and thermodynamic quantities are ρ and $\mathcal{E}$. Evolution of the MTT in the LTB model has been considered in Booth *et al.* (2005). We also consider the evolution of the MTT in this model, but in terms of the geometric and thermodynamic variable as we have done throughout this chapter. The function C for the LTB model is given by

$$C = -\frac{\rho}{\frac{1}{3}\rho + 2\mathcal{E}}, \tag{4.37}$$

where ρ is given by

$$\rho = \frac{\left(r^3\mathscr{M}\right)'}{R^2R'}, \tag{4.38}$$

with $\mathscr{M} = \mathscr{M}(r)$ being the Misner-Sharp mass (Goswami & Joshi, 2004a; Goswami & Joshi, 2004b; Zibin, 2008). Thus the condition for the MTT to be spacelike, timelike, or null is given by

$$\rho < -6\mathscr{E}, \quad \rho > -6\mathscr{E}, \quad \text{or} \quad \rho = -6\mathscr{E},$$

respectively.

The scalar λ, whose sign determines the stability of the MTS, is given by

$$\lambda = -4\pi\left(\frac{1}{3}\rho + 2\mathscr{E}\right). \tag{4.39}$$

From (4.32), stability of the MTS thus requires that

$$\rho < -6\mathscr{E}, \tag{4.40}$$

on the MTS. This implies spacelike MTTs are necessarily foliated by stable MTS, and the MTS foliating timelike MTTs are unstable. The null MTTs are marginally stable.

Let us now consider the relationship between stability and shell crossing. For the LTB model, $\mathscr{E}$ is given by

$$\mathscr{E} = \frac{1}{3}\rho - \frac{r^3\mathscr{M}}{R^3}, \tag{4.41}$$

where R is given as $R = r\left(1-\sqrt{\mathscr{M}}\,t\right)^{2/3}$ (Goswami & Joshi, 2004a; Goswami & Joshi, 2004b; Zibin, 2008). The condition for stability can be expressed as a condition on $\mathscr{M}'$: Upon inserting (4.41) into (4.40), we obtain the condition for stability (which is also an equivalent condition for the MTT to be spacelike) as

$$\rho < \frac{2\mathscr{M}}{\left(1-\sqrt{\mathscr{M}}\,t\right)^2}, \tag{4.42}$$

and from (4.38), (4.42) can be rewritten as

$$\mathscr{M}' \quad < \quad \frac{3r\mathscr{M}\sqrt{\mathscr{M}}\left(1-\sqrt{\mathscr{M}}\,t\right)}{3r^2\left(1-\sqrt{\mathscr{M}}\,t\right)+2t\mathscr{M}}, \tag{4.43}$$

as we move away from the central singularity, ($t_c < t < \infty$), where $t_c = 1/\sqrt{\mathscr{M}}$ is the time for collapse (comoving time) to zero area ($R = 0$). The condition for no shell crossing is given by

$$t_c' = -\frac{\mathscr{M}'}{2\mathscr{M}^{\frac{3}{2}}} \geq 0,$$

which requires $\mathscr{M}' \leq 0$. It is clear that stability of the leaves of the MTT ensures no shell crossing.

In the next section we consider surface gravity calculated on the MTT for various classes of spacetimes.

4.4 Surface gravity of MTTs

Surface gravity is a fundamental notion in the formulation of the laws of black hole mechanics. As such, understanding their evolution has been of keen interest (Bardeen *et al.,* 1973; Brown & York, 1993; Ashtekar *et al.*, 2001; Ashtekar *et al.*, 2000). In this section, using the formulation in (Hayward, 1994; Ashtekar *et al.*, 2001; Ashtekar *et al.*, 2000; Ashtekar *et al.*, 2000) (see also Fairhurst *et al.*, (2004)), coupled with adapting the calculations to quantities from the $1+1+2$ splitting, we calculate the surface gravity for black hole horizons in the LRS II and NNF spacetimes, as well as a general 4-dimensional spacetime. Under this formulation we prove a standard result in black hole mechanics, specifically the third law of black hole thermodynamics, for the LRS II class.

One defines the connection on the normal bundle to the 2 -surface S as

$$\omega_a := -\, l_b \nabla_a k^b, \tag{4.44}$$

with the covariant derivative pulled back to the horizon. For the case of an IH, the surface gravity is given by the contraction of ω_a by the outgoing null normal:

$$\kappa = \omega_a k^a. \tag{4.45}$$

Let us first consider the case of the LRS II class of spacetimes. Computing ω_a we obtain $\omega_a = -\left(A u_a - \left(\frac{1}{3}\Theta + \Sigma\right) e_a\right)$, which gives

$$\kappa_{LRS\,II} = \frac{1}{\sqrt{2}}\left(A + \frac{1}{3}\Theta + \Sigma\right). \tag{4.46}$$

In the time symmetric case, $\left(\frac{1}{3}\Theta + \Sigma\right)$ is zero and the surface gravity proportional to the accelaration A, which is the expected result.

For the NNF, ω_a is given by

$$\begin{aligned}
\omega_a &= \left[-A + \frac{\sqrt{2}}{2}\left(A^b l_b + \alpha^b l_b\right)\right] u_a + \left[\frac{1}{3}\Theta + \Sigma + \frac{\sqrt{2}}{2}\left(\Sigma^b l_b + \varepsilon^b_{m}\Omega^m l_b + a^b l_b\right)\right] e_a \\
&\quad + \left[\Sigma_a - \varepsilon_{am}\Omega^m - \frac{\sqrt{2}}{2}\left((\Omega + \xi)\,\varepsilon^b_a + \Sigma^b_a + \zeta^b_a\right) l_b\right] \\
&= -A u_a + \left(\frac{1}{3}\Theta + \Sigma\right) e_a + \Sigma_a - \varepsilon_{am}\Omega^m.
\end{aligned} \tag{4.47}$$

The surface gravity on an IH in the NNF class is then given by

$$\kappa_{NNF} = \frac{1}{\sqrt{2}}\left(A + \frac{1}{3}\Theta + \Sigma\right) = \kappa_{LRS\,II}. \tag{4.48}$$

In general, the surface gravity on a leaf of the foliation, applicable to different horizon types, is given by

$$\kappa^\nu = -l_b X^a \nabla_a X^b, \tag{4.49}$$

where ν is the foliation label and X^a is the vector given in (4.4), which can be expanded as

$$\kappa^\nu = k^a \omega_a + C\left(X^a l_b \nabla_a l^b - l^a \omega_a\right), \tag{4.50}$$

We see that on an isolated horizon ($C = 0$), (4.50) becomes (4.45). In fact for a slowly evolving horizon (see Booth & Fairhurst (2005)) for discussions on slowly evolving horizons), (4.50) gives a splitting of the surface gravity with C being - taken as the evolution rate - sufficiently less than 1, and $k^a\omega_a$ is constant. Then a slowly evolving horizon will be interpreted simply as one satisfying the conditions in Booth & Fairhurst (2005) where ϵ is replaced by C, and so are simply those horizons for which C satisfies $0 < |C| \ll 1$ along with additional conditions.

From (4.50) it is now obvious that, in general, the surface gravity will not be constant with $C\left(X^a l_b \nabla_a l^b - l^a\omega_a\right)$ being the evolving component of κ^ν.

For the LRS II class of spacetimes, $l_b\nabla_a l^b = 0$ and thus, on a leaf, the surface gravity can be expressed as

$$\kappa^\nu_{_{LRS\,II}} = \frac{1}{\sqrt{2}}\left[\left(1 - C_{_{LRS\,II}}\right)A + \left(1 + C_{_{LRS\,II}}\right)\left(\frac{1}{3}\Theta + \Sigma\right)\right]. \tag{4.51}$$

In the case of an NEH with $C_{_{LRS\,II}} = 0$, (4.51) reduces to (4.46).

As an example of the application of (4.51) to horizons that are not NEHs, we again consider the OS collapse case. In this case A, Σ are both vanishing and we have

$$\kappa^\nu \propto -\frac{2}{3}\Theta.$$

Of course since it is known that $\kappa > 0$ we therefore expect $\Theta < 0$ and so the horizon in this case is necessarily collapsing, and thus timelike.

For the NNF class, we have

$$\begin{aligned} X^a l_b \nabla_a l^b &= \frac{(1 + C_{NNF})}{2}\left(\Sigma^b + \varepsilon^b_m\Omega^m + (\Omega - \xi)\, e^a\varepsilon^b_a + e^a\Sigma^b_a - a^b - e^a\zeta^b_a\right) l_b \\ &\quad + \frac{(1 - C_{NNF})}{2}\left(A^b - \alpha^b + u^a\Sigma^b_a - u^a\zeta^b_a + (\Omega - \xi)\, u^a\varepsilon^b_a\right) l_b \\ &= 0, \end{aligned}$$

and

$$l^a \tilde{\omega}_a = \frac{1}{\sqrt{2}} \left[A - \left(\frac{1}{3}\Theta + \Sigma \right) \right].$$

We therefore have (4.50) reducing to

$$\begin{aligned} \kappa^{\nu}_{_{NNF}} &= \frac{1}{\sqrt{2}} \left(A + \frac{1}{3}\Theta + \Sigma \right) + \frac{1}{\sqrt{2}} C_{NNF} \left(-A + \frac{1}{3}\Theta + \Sigma \right) \\ &= \frac{1}{\sqrt{2}} \left[(1 - C_{NNF})\, A + (1 + C_{NNF}) \left(\frac{1}{3}\Theta + \Sigma \right) \right]. \end{aligned} \tag{4.52}$$

Let us consider under what condition the horizon of a black hole spacetime in the LRS II class (and by extension NNF) in general will have a vanishing κ^{ν} on the horizon. Suppose $\kappa^{\nu}_{_{LRS\ II}} = 0$ on H. Then setting (4.52) to zero gives

$$C_{_{LRS\ II}} = \frac{A + \left(\frac{1}{3}\Theta + \Sigma\right)}{A - \left(\frac{1}{3}\Theta + \Sigma\right)}. \tag{4.53}$$

Equating (4.53) to (4.9) we obtain the following constraint equation

$$\left[A - \left(\frac{1}{3}\Theta + \Sigma \right) \right] \left[-\frac{4}{3}\rho - \Pi + 2\,(Q - \mathscr{E}) \right] = 0, \tag{4.54}$$

so either $A = \left(\frac{1}{3}\Theta + \Sigma\right)$ or $-\frac{4}{3}\rho - \Pi + 2\,(Q - \mathscr{E}) = 0$. If $A = \left(\frac{1}{3}\Theta + \Sigma\right)$, then $C_{_{LRS\ II}} = \infty$ from (4.53). But from the definition of $\kappa^{\nu}_{_{LRS\ II}}$, $A = \left(\frac{1}{3}\Theta + \Sigma\right)$ implies $\left(\frac{1}{3}\Theta + \Sigma\right) = 0$ (since $\kappa^{\nu}_{_{LRS\ II}}$ is 0). This then further implies $A = 0$ which makes $C_{_{LRS\ II}}$ indeterminate, and we have a contradiction. We therefore assume $A \neq \left(\frac{1}{3}\Theta + \Sigma\right)$ and consider the second case, $-\frac{4}{3}\rho - \Pi + 2\,(Q - \mathscr{E}) = 0$. This condition implies $C_{_{LRS\ II}} = 1$, which would then imply that

$$\left(\frac{1}{3}\Theta + \Sigma \right) = 0, \tag{4.55}$$

(from either equation (4.52) or (4.53)). Since $A \neq \left(\frac{1}{3}\Theta + \Sigma\right)$ and $\left(\frac{1}{3}\Theta + \Sigma\right) = 0$, $A \neq 0$. We therefore have that $\kappa^{\nu} = 0$ if $A \neq 0$ and $\Theta = -\phi$ on the horizon, recalling that on the horizon,

$$\Theta_k = \frac{2}{3}\Theta - \Sigma + \phi = 0. \tag{4.56}$$

Combining (4.55) and (4.56) gives $\phi = -\Theta$ and $\Sigma = \frac{1}{3}\phi$. But $C_{LRS\,II} = 1$ implies $\Theta > 0$ (expanding). As the 2-surfaces foliating the horizon are marginally trapped, the expansion scalar in the ingoing null direction is negative, i.e. $\Theta_l = \frac{2}{3}\Theta - \Sigma - \phi < 0$. However, we have $\Theta_l = 2\Theta$. So for $\Theta > 0$, we have that $\Theta_l > 0$, which is a contradiction. We conclude that

Proposition 4.4.1. *We cannot have a black hole spacetime in the LRS II class (and by extension the NNF class) with κ^ν vanishing on the horizon,*

which is just the third law of black hole thermodynamics (Bardeen *et al.*, 1973). This is akin to the statement that *the LRS II class of spacetimes admit no extremal black holes.*

We now consider these quantities for a general 4-dimensional spacetime. The outgoing null normal k^a in this case is as defined as in (4.16). Using (4.44), ω_a is calculated as

$$\omega_a = -Au_a + \left(\frac{1}{3}\Theta + \Sigma\right) e_a - \tau_a \Theta_k \Big|_{LRS\,II} + l_b \nabla_a \tau^b + \Sigma_a - \varepsilon_{am}\Omega^m, \tag{4.57}$$

which gives the surface gravity as

$$\kappa = \frac{1}{\sqrt{2}}\left[\left(A + \frac{1}{3}\Theta + \Sigma\right) - l_b\left(\dot{\tau}^b + \hat{\tau}^b + \breve{\tau}^b\right)\right], \tag{4.58}$$

where the notation $\breve{*}$ is introduced to denote the derivative along the direction τ^a. So $\breve{\tau}^b = \tau^a \nabla_a \tau^b$. From (4.58) we see that the surface gravity on a NEH in a general 4-dimensional spacetime coincides with the LRS II class if $l_b \breve{\tau}^b = -l_b\left(\dot{\tau}^b + \hat{\tau}^b\right)$. The $1+1+2$ decomposition of the surface gravity on a leaf of a general horizon type is calculated as

$$\begin{aligned}
\kappa^\nu &= \frac{1}{\sqrt{2}}\left[\left(A + \frac{1}{3}\Theta + \Sigma\right) - l_b\left(\dot{\tau}^b + \hat{\tau}^b + \breve{\tau}^b\right)\right] + \frac{1}{\sqrt{2}}C\left[-A + \left(\frac{1}{3}\Theta + \Sigma\right) + \frac{1}{2}(C-3)\,\dot{\tau}^b l_b\right] \\
&\quad + \frac{1}{\sqrt{2}}C\left[\frac{1}{2}(C+3)\left(\hat{\tau}^b + \breve{\tau}^b\right) l_b + \frac{1}{2}\tau_b\left(V_2^b - V_1^b\right) + \frac{1}{2}(1+C)\,V + \tau_b\left(\Sigma^b - \varepsilon_a^{\ b}\Omega^a\right)\right] \\
&= \frac{1}{\sqrt{2}}\left[(1-C)\,A + (1+C)\left(\frac{1}{3}\Theta + \Sigma + \frac{1}{2}CV\right)\right] + \frac{1}{2\sqrt{2}}\left[C\left(V_2^b - V_1^b\right) + \left(\Sigma^b - \varepsilon_a^{\ b}\Omega^a\right)\right]\tau_b \\
&\quad + \frac{1}{2\sqrt{2}}\left[\left(C^2 - 3C - 1\right)\dot{\tau}^b - \left(C^2 + 3C - 1\right)\left(\hat{\tau}^b + \breve{\tau}^b\right)\right] l_b,
\end{aligned} \tag{4.59}$$

where

$$V_1^b = A^b - \alpha^b,\ V_2^b = a^b - \Sigma^b - \varepsilon_c^{\ b}\Omega^c,\ V = \frac{1}{2}\tau^2\Theta_k\Big|_{LRS\ II}.$$

Again, when $C = 0$, (4.59) reduces to the form of (4.58).

In fact the expression for the surface gravity of IH in the LRS II and NNF classes has consequence for the definition of the temperature of the gravitational field as formulated in the gravitational entropy proposal paper by Clifton *et. al* (2013). Let us define a new one-form ω'_a as

$$\begin{aligned} \omega'_a &= -\frac{1}{2\pi\sqrt{2}}\left(l_b\nabla_a k^b + l_b\nabla_a l^b\right) \\ &= \frac{1}{\pi\sqrt{2}}\left(\omega_a + l_b\nabla_a e^b\right), \end{aligned} \tag{4.60}$$

where ω_a is as defined in (4.44). We can then write the temperature of the gravitational field as

$$\begin{aligned} T_{grav} &= k^a\omega'_a \\ &= \frac{1}{\pi\sqrt{2}}\left(k^a\omega_a + l_b k^a\nabla_a e^b\right) \\ &= \left(\frac{2-\sqrt{2}}{4\pi}\right)\left(A + \frac{1}{3}\Theta + \Sigma\right), \end{aligned} \tag{4.61}$$

(also see Acquaviva *et al.* (2015) for discussion of the temperature of the gravitational field in context of the $1+1+2$ splitting). The first term in the parenthesis of the second line of (4.61) is clearly associated with the horizon and as such, in an LRS II spacetime admitting an IH, the temperature will have contribution from the horizon.

We mention that in general one has to exercise caution when assigning physical meaning to all these quantities precisely because of the fact that the true values are tied to the scaling of the vector k^a.

Also we can give the following geometrical interpretation to T_{grav} (up to scale): T_{grav} is the sum of the accelerations of the null normal vector fields k^a and l^a along the null direction $\l^a$.

4.5 Conclusion

In this work we have considered various aspects of the dynamics of black hole horizons. Work by Booth established that the matter content of the horizon determines the causal character of the horizon (at least for the spherically symmetric case). This work treats this same problem using the formulation by Booth and coauthors in Booth *et al.* (2005), but in context of the geometric and thermodynamic variables from the $1+1+2$ splitting. The form that the function C takes - whose sign determines the causal character of the horizon - indeed establishes that the matter content determines the causal character of the horizon. Stated another way, the causal character of the horizon is determined by the relationship between the matter and thermodynamic variables, even for more general cases. The form of C in our treatment further allows us to recover established results and well as provide us with new insights into the properties of the horizon in a straightforward and relatively simple way. This is demonstrated by the recovery of the bounds of the equation of state parameter σ, for which horizons in the Robertson-Walker spacetimes are timelike, spacelike of non-expanding. We have also been able to determine the values of σ for which MTS in the Robertson-Walker spacetimes are stable as well as classify the SFOTHs. In particular, it is seen that the existence of stable MTS are possible only if the isotropic pressure is negative. We then went on to show that for the LTB model, a relationship between the energy density and electric part of the Weyl curvature $\mathscr{E}$, gives the causal classification of the MTTs. It was further shown that the MTS foliating the spacelike MTTs are necessarily stable, and that this stability guarantees no shell crossing.

The form of the surface gravity obtained in our treatment has provided a useful means to verify the third law of black hole thermodynamics for a restricted class of spacetimes. We give an explicit proof of the law for LRS II spacetimes (the proof can also be carried over to NNF spacetimes). For an isolated horizon, we have an expression that allows us to give a geometric interpretation of the temperature of a gravitational field.

Chapter 5

Properties of expansion-free dynamical stars

5.1 Introduction

The aim of this chapter is to investigate the conditions under which there can be trapping in a relativistic expansion-free star. This analysis falls within the scope of stability analysis of self-gravitating systems (Goswami, 2007; Joshi & Goswami, 2011; May & White, 1966; Santos, 1985; Wilson, 1971). We will consider the conditions on the acceleration and radiation quantities that allows for trapping in such stars. It is also an interesting exercise, with all the different quantities acting on such stars, to determine the geometry as these structures evolve. We will make use of the equivalent forms of the field equations from the $1+1+2$ semi-tetrad covariant formulation of general relativity (Greenberg, 1970; Tsamparlis & Mason, 1983; Mason & Tsamparlis, 1985; Tsamparlis, 1992; Clarkson, 2007; Ellis *et al.,* 2014). The semi-tetrad formalism has been a useful approach in displaying geometrical features in a transparent fashion which are difficult to find using other approaches.

Various authors have explored expansion-free models with different considerations. The central theme of the interest in such models is the possibility that they could help explain the existence of voids on cosmological scales. Herrera *et al.* (2009) studied such models with non-zero shear and showed that the appearance of a cavity, with matter which is anisotropic and dissipative, undergoing explosion is inevitable (see Peebles (2001) for more discussion). Herrera *et al.* (2010) in which they ruled out the Skripkin expansion-free model (Skripkin, 1960) - with constant energy density and isotropic pressure - when considering the expansion-free case along with the junction conditions. In another study Herrera *et al.* (2012) studied models collapsing adiabatically, and showed that the instability was independent of the star's stiffness. In particular, it was shown that the instability is entirely governed by the pressures and the radial profile of the energy density.

In section 5.2 we give a short overview of the $1+1+2$ semi-tetrad formulation. In section 5.3 we present the results of the chapter. We conclude with a discussion of the results in section 5.4.

5.2 Preliminaries

To keep track of the actions on the equations, we shall rewrite the field equations here for the LRS II spacetimes. The evolution and propagation equations are given as

- *Evolution (LRS II):*

$$\frac{2}{3}\dot{\Theta}-\dot{\Sigma} = A\phi-2\left(\frac{1}{3}\Theta-\frac{1}{2}\Sigma\right)^2-\frac{1}{3}\left(\rho+3p\right)+\mathcal{E}-\frac{1}{2}\Pi, \tag{5.1}$$

$$\dot{\phi} = \left(\frac{2}{3}\Theta-\Sigma\right)\left(A-\frac{1}{2}\phi\right)+Q, \tag{5.2}$$

$$\begin{aligned}\dot{\mathcal{E}}-\frac{1}{3}\dot{\rho}+\frac{1}{2}\dot{\Pi} = & -\frac{3}{2}\left(\frac{2}{3}\Theta-\Sigma\right)\mathcal{E}-\frac{1}{4}\left(\frac{2}{3}\Theta-\Sigma\right)\Pi+\frac{1}{2}\phi Q \\ & +\frac{1}{2}\left(\rho+p\right)\left(\frac{2}{3}\Theta-\Sigma\right).\end{aligned} \tag{5.3}$$

- *Propagation (LRS II):*

$$\frac{2}{3}\hat{\Theta} - \hat{\Sigma} \;=\; \frac{3}{2}\phi\Sigma + Q, \tag{5.4}$$

$$\hat{\phi} \;=\; \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) - \frac{1}{2}\phi^2 - \frac{2}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi, \tag{5.5}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\rho} + \frac{1}{2}\hat{\Pi} \;=\; -\frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) - \frac{1}{2}\left(\frac{2}{3}\Theta - \Sigma\right)Q. \tag{5.6}$$

- *Propagation/Evolution (LRS II):*

$$\hat{A} - \dot{\Theta} \;=\; -\left(A + \phi\right)A + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\rho + 3p\right), \tag{5.7}$$

$$\hat{Q} + \dot{\rho} \;=\; -\Theta\left(\rho + p\right) - \left(\phi + 2A\right)Q - \frac{3}{2}\Sigma\Pi, \tag{5.8}$$

$$\hat{p} + \hat{\Pi} + \dot{Q} \;=\; -\left(\frac{3}{2}\phi + A\right)\Pi - \left(\frac{4}{3}\Theta + \Sigma\right)Q - \left(\rho + p\right)A. \tag{5.9}$$

The $\hat{*}$ and $\dot{*}$ denote the spatial derivative along e^a and the covariant time derivative along an observer's congruence, respectively.

The outgoing null expansion, whose vanishing necessitates trapping, has been calculated (Ellis *et al.*, 2014; Sherif *et al.*, 2018) as

$$\Theta_k \;=\; \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma + \phi\right). \tag{5.10}$$

The equation of the outgoing null expansion scalar here corresponds to equation (32) of Ellis *et al.* (2014), but we have unitized the energy function for our choice of the outgoing null normal vector field k^a, whose divergence gives Θ_k. It is clear from (5.10) that, even with the vanishing of Θ, it is still possible to have trapping. This is the main focus of this work and is investigated in the next section.

When acting on a scalar ψ, the dot ($\dot{}$) and hat ($\hat{}$) derivatives satisfy the commutation relation (Clarkson, 2007)

$$\dot{\hat{\psi}} - \hat{\dot{\psi}} = -A\dot{\psi} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\psi}. \tag{5.11}$$

This is a useful relation that will be utilized often in our calculations.

5.3 Results

In this section we state and prove the results of the chapter.

5.3.1 Dynamics of expansion-free stars

We state and prove the following theorem.

Theorem 5.3.1. *An expansion-free dynamical star must accelerate and radiate simultaneously.*

Proof. We establish this by fixing both the acceleration and the heat flux to zero, and then by fixing either of the acceleration or the heat flux to zero.

5.3.1.1 Case 1

First, suppose $A = 0$ and $Q = 0$. From (5.7) we have the algebraic constraint equation

$$0 = \frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\rho + 3p\right). \tag{5.12}$$

Here we note that since $\Sigma^2 > 0$, (5.12) implies that the strong energy condition must be violated, i.e. $\rho + 3p < 0$. For $A = \Theta = 0$, (5.11) is simply

$$\dot{\hat{\psi}} - \hat{\dot{\psi}} = \Sigma\hat{\psi}. \tag{5.13}$$

Taking the hat derivative of (5.2) and the dot derivative of (5.5) we obtain respectively

$$\begin{aligned} \dot{\hat{\phi}} &= \frac{1}{2}\hat{\phi}\Sigma + \frac{1}{2}\phi\hat{\Sigma} \\ &= -\left(\phi^2 + \frac{1}{2}\Sigma^2 + \frac{1}{3}\rho + \frac{1}{2}\mathscr{E} + \frac{1}{4}\Pi\right)\Sigma \end{aligned} \tag{5.14}$$

and

$$\begin{aligned} \hat{\dot{\phi}} &= -2\Sigma\dot{\Sigma} - \phi\dot{\phi} - \frac{2}{3}\dot{\rho} - \dot{\mathscr{E}} - \frac{1}{2}\dot{\Pi} \\ &= -\left(\frac{1}{2}\phi^2 + \Sigma^2 + \frac{1}{6}\rho - \frac{1}{2}\mathscr{E} + \frac{3}{2}p\right)\Sigma. \end{aligned} \tag{5.15}$$

Using the commutation relation on (5.14) and (5.15), we obtain

$$\left[\frac{1}{4}\Pi + \frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\rho + 3p\right)\right]\Sigma = 0. \tag{5.16}$$

So either $\Sigma = 0$ or

$$\frac{1}{4}\Pi + \frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\rho + 3p\right) = 0.$$

If $\Sigma = 0$ then the star must be static ($\Theta = \Sigma = 0$), so we assume that $\Sigma \neq 0$ and that

$$\frac{1}{4}\Pi + \frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\rho + 3p\right) = 0. \tag{5.17}$$

From (5.12), (5.17) implies that $\Pi = 0$. Now if we take the dot derivative of (5.12) and substitute for (5.1) and (5.8), we obtain the evolution of p

$$\dot{p} = \left(\Sigma^2 + 2\mathscr{E}\right)\Sigma. \tag{5.18}$$

Taking the hat derivative of (5.18) and the dot derivative of (5.9) we obtain respectively

$$\hat{\dot{p}} = \left(-\frac{9}{2}\phi\Sigma^2 - 6\phi\mathscr{E} + \frac{2}{3}\hat{\rho}\right)\Sigma \tag{5.19}$$

and

$$\dot{\hat{p}} = 0. \tag{5.20}$$

Using the commutation relation on (5.19) and (5.20) we obtain the propagation of ρ

$$\hat{\rho} = 9\phi\left(\frac{3}{4}\Sigma^2 + \mathscr{E}\right). \tag{5.21}$$

Now taking the hat derivative of (5.1) and the dot derivative of (5.4), we obtain respectively

$$\begin{aligned} \hat{\dot{\Sigma}} &= \Sigma\hat{\Sigma} - \hat{\mathscr{E}} \\ &= -\frac{3}{2}\phi\Sigma^2 - \hat{\mathscr{E}} \end{aligned} \tag{5.22}$$

and

$$\begin{aligned} \dot{\hat{\Sigma}} &= -\frac{3}{2}\left(\dot{\phi}\Sigma + \phi\dot{\Sigma}\right) \\ &= -\frac{3}{2}\phi\mathscr{E}. \end{aligned} \tag{5.23}$$

Using the commutation relation on (5.22) and (5.23) we obtain

$$\hat{\mathscr{E}} = \frac{3}{2}\phi\left(\Sigma^2 - \mathscr{E}\right), \tag{5.24}$$

which upon substituting in (5.6) we obtain

$$\hat{\rho} = \frac{9}{2}\phi\Sigma^2. \tag{5.25}$$

Comparing (5.21) and (5.25) we get

$$\phi\left(\mathscr{E} + \frac{1}{4}\Sigma^2\right) = 0. \tag{5.26}$$

Therefore we must have either $\phi = 0$ or

$$\mathscr{E} = -\frac{1}{4}\Sigma^2. \tag{5.27}$$

We show that either case yields $\Sigma = 0$, in which case the star is static. First, suppose $\phi = 0$. Then (5.5) gives the constraint equation

$$\Sigma^2 = -\frac{2}{3}\rho - \mathscr{E}, \tag{5.28}$$

and comparing (5.12) and (5.28) we obtain

$$\mathscr{E} = -\frac{1}{3}\left(\rho - 3p\right). \tag{5.29}$$

Taking the dot derivative of (5.29), using (5.3) and (5.18) we obtain

$$\Sigma\left[\Sigma^2 + \frac{1}{2}\left(\mathscr{E} + \rho + p\right)\right] = 0. \tag{5.30}$$

Again, assuming $\Sigma \neq 0$ we must have

$$\Sigma^2 = -\frac{1}{2}\left(\mathscr{E} + \rho + p\right). \tag{5.31}$$

Taking the dot derivative of (5.31) and using (5.1) we obtain

$$\dot{p} = \Sigma\left(4\mathscr{E} - 2\Sigma^2\right). \tag{5.32}$$

and upon comparing to (5.18) we obtain

$$\Sigma\left(2\mathscr{E} - 3\Sigma^2\right) = 0. \tag{5.33}$$

Since $\Sigma \neq 0$ we have

$$\mathscr{E} = \frac{3}{2}\Sigma^2, \tag{5.34}$$

which, upon using (5.12) and (5.29) we obtain

$$\rho = -\frac{5}{3}p. \tag{5.35}$$

Taking the dot derivative of (5.35) we have $\dot{p} = 0$. Setting (5.18) to zero, while using (5.34) to substitute for $\mathscr{E}$ gives $\Sigma^2 = 0$ which gives $\Sigma = 0$.

Next assume $\phi \neq 0$ and that (5.27) is satisfied. Now taking the dot derivative of (5.27), and using (5.3), (5.1) and (5.27) to simplify we obtain

$$\left[\frac{3}{4}\Sigma^2 + \rho + p\right]\Sigma = 0, \tag{5.36}$$

Assume $\Sigma \neq 0$. We must have

$$\frac{3}{4}\Sigma^2 + \rho + p = 0, \tag{5.37}$$

Using (5.12), (5.37) simplifies to

$$\rho = -\frac{14}{5}p. \tag{5.38}$$

Finally taking the dot derivative of (5.38) we have $\dot{p} = 0$, which upon comparing to (5.18) and substituting for $\mathcal{E}$ using (5.27) we obtain $\Sigma^2 = 0$ which gives $\Sigma = 0$.

5.3.1.2 Case 2

Let us next consider the case $A \neq 0$ and $Q = 0$. The commutation relation (5.11), now becomes

$$\hat{\dot{\psi}} - \dot{\hat{\psi}} = -A\dot{\psi} + \Sigma\hat{\psi}. \tag{5.39}$$

Taking the hat derivative of (5.3) and the dot derivative of (5.6), we obtain respectively

$$\begin{aligned}
\hat{\dot{\Sigma}} &= -\hat{A}\phi - A\hat{\phi} + \Sigma\hat{\Sigma} + \frac{1}{3}\hat{\rho} + \hat{p} - \hat{\mathcal{E}} + \frac{1}{2}\hat{\Pi} \\
&= A^2\phi + \frac{3}{2}A\phi^2 - 3\phi\Sigma^2 - \frac{1}{3}A\rho - \frac{3}{4}\phi\Pi - \frac{1}{2}\phi\rho \\
&\quad -\frac{3}{2}\phi p + A\mathcal{E} - \frac{1}{2}A\Pi - Ap + \frac{3}{2}\phi\mathcal{E}
\end{aligned} \tag{5.40}$$

and

$$\begin{aligned}
\dot{\hat{\Sigma}} &= -\frac{3}{2}\left(\dot{\phi}\Sigma + \phi\dot{\Sigma}\right) \\
&= \frac{3}{2}A\Sigma^2 - \frac{3}{2}\phi\Sigma^2 + \frac{3}{2}A\phi^2 - \frac{1}{2}\phi\rho - \frac{3}{2}\phi p \qquad (5.41) \\
&\quad + \frac{3}{2}\phi\mathcal{E} - \frac{3}{4}\phi\Pi. \qquad (5.42)
\end{aligned}$$

Using the commutation relation on (5.40) and (5.41) we obtain

$$A\Sigma^2 = 0. \tag{5.43}$$

Since $A \neq 0$, we must have $\Sigma = 0$, and thus the star is static.

5.3.1.3 Case 3

Finally, we consider the case $A = 0$ and $Q \neq 0$. From (5.7) we have the constraint equation as (5.12). The commutation relation in this case is (5.13). Taking the hat derivative of (5.2) and the dot derivative of (5.5), we obtain respectively

$$\begin{aligned} \hat{\dot{\phi}} &= \frac{1}{2}\hat{\Sigma}\phi + \frac{1}{2}\Sigma\hat{\phi} \\ &= -\phi^2\Sigma - \frac{1}{2}\phi Q - \frac{1}{2}\Sigma^3 - \frac{1}{3}\Sigma\rho - \frac{1}{2}\Sigma\mathcal{E} - \frac{1}{4}\Sigma\Pi \\ &\quad +\hat{Q} \end{aligned} \tag{5.44}$$

and

$$\begin{aligned} \dot{\hat{\phi}} &= -2\Sigma\dot{\Sigma} - \phi\dot{\phi} - \frac{2}{3}\dot{\rho} - \left(\dot{\mathcal{E}} + \frac{1}{2}\dot{\Pi}\right) \\ &= -\Sigma^3 - \frac{1}{6}\Sigma\rho - \frac{5}{4}\Sigma\Pi + \frac{1}{2}\Sigma\mathcal{E} + \frac{1}{2}\Sigma p - \dot{\rho}. \end{aligned} \tag{5.45}$$

Using the commutation relation on (5.44) and (5.45) we obtain

$$\left[\frac{3}{2}\Sigma^2 + \frac{1}{2}\left(\rho + 3p\right)\right]\Sigma = \phi Q, \tag{5.46}$$

which, upon using (5.12) reduces to

$$\phi Q = 0. \tag{5.47}$$

Since $Q \neq 0$, we must have $\phi = 0$. But from (5.2), this gives $Q = 0$.

From the three cases considered, we therefore must have $A \neq 0$ and $Q \neq 0$ to have an expansion-free star that is evolving. □

5.3.2 Geometry of expansion-free stars

We state and prove the following theorem on the geometry of expansion-free dynamical stars.

Theorem 5.3.2. *An expansion-free dynamical star must be conformally flat.*

Proof. We prove this by checking for additional constraints from the field equations with $A \neq 0$ and $Q \neq 0$. The commutation relation in this case is given by (5.39). Taking the hat derivative of (5.1) we obtain

$$\begin{aligned}\hat{\dot{\Sigma}} &= A^2\phi + \frac{3}{2}A\phi^2 - 3\phi\Sigma^2 + A\Sigma^2 - \frac{1}{2}\phi\rho - \frac{3}{2}\phi p \\ &\quad + \frac{2}{3}A\rho + \frac{1}{2}A\Pi - \frac{3}{2}\Sigma Q + \frac{3}{2}\phi\mathcal{E} + \frac{3}{4}\phi\Pi + \hat{p} \\ &\quad + \hat{\Pi},\end{aligned} \tag{5.48}$$

and taking the dot derivative of (5.4) we have

$$\begin{aligned}\dot{\hat{\Sigma}} &= \frac{3}{2}A\Sigma^2 - \frac{3}{2}\phi\Sigma^2 - \frac{3}{2}\Sigma Q + \frac{3}{2}A\phi^2 - \frac{1}{2}\phi\rho \\ &\quad - \frac{1}{2}\phi p + \frac{3}{2}\phi\mathcal{E} - \frac{3}{4}\phi\Pi - \dot{Q}.\end{aligned} \tag{5.49}$$

Taking the difference of (5.48) and (5.49) and employing the commutation relation (5.39), we obtain

$$\begin{aligned}-A\dot{\Sigma} + \Sigma\hat{\Sigma} &= A^2\phi - \frac{3}{2}\phi\Sigma^2 - \frac{1}{2}A\Sigma^2 + \frac{3}{2}\phi\Pi \\ &\quad + \frac{2}{3}A\rho + \frac{1}{2}A\Pi + \dot{Q} + \hat{p} + \hat{\Pi},\end{aligned} \tag{5.50}$$

which upon using (5.9) and simplifying gives

$$A\mathcal{E} = 0. \tag{5.51}$$

Since $A \neq 0$, we must have $\mathcal{E} = 0$ so that the electric part of the Weyl tensor is vanishing. Fixing $\mathcal{E} = 0$ in the field equations for $A \neq 0, Q \neq 0$, all other commutation relations on pairs of evolution and propagation equations return identities. □

5.4 Discussion

The expansion-free condition in general relativity has received considerable attention in recent years and has been applied to describe physical features of radiating stars. We have utilized the $1+1+2$ semi-tetrad covariant formalism to study such stars in general in spherical symmetry. The analysis shows that expansion-free dynamical stars are severely constrained, and can only exist under very particular conditions. From the set of field equations, we have explicitly shown that a necessary condition for a star with zero expansion to evolve is that the star has non-zero radiation and acceleration. With further analysis of the field equations with $A \neq 0$ and $Q \neq 0$, it is shown that the star is necessarily conformally flat. Proving these results amount to the analysis of the field equations via commutation relations, via which we obtain additional constraints, which further give us additional evolution equations that can be matched against the original set of equations. These results add to the literature on expansion-free dynamical stars, which have been developed over the last decade and half, most notably through works of Herrera and co-authors.

Chapter 6

Geometric results for minimal hypersurfaces in generalized LRS II spacetimes

6.1 Introduction

This chapter considers the geometry of minimal 2-surfaces foliating hypersurfaces in spacetimes. This is carried out in a covariant setting where the formulation of the following references (Senovilla 1998; Senovilla 2003; Senovilla 2003; Bengtsson & Senovilla 2011) defines the various quantities associated with the 2-surfaces. The results by Chern *et al.*, (1970) are employed to prove our results (Fialkow, 1938; Takahashi, 1966; Hsiang, 1967; Calabi, 1967; Simons, 1968; Dwivedi, 2019.)

Section 6.2 gives some preliminaries notes used in this work as well as introduction of definitions. In 6.3 we state and prove the main results of this chapter. We then conclude in section 6.4.

6.2 Preliminaries

In this section we briefly discuss the notion of a co-dimension 2-surface S and define various quantities on S which are used to study the trapping of S. We follow the reference Bengtsson & Senovilla (2011).

We start by defining a co-dimension two surface S in a given spacetime manifold $\left(M, g\right)$. In practice, these surfaces will be the "leaves" that foliate local horizons in the spacetime when they are considered later in the text.

Let M be a 4-dimensional spacetime, and let $\mathfrak{g}$ be a pseudo-Riemannian metric on M with Lorentzian signature $(-,+,+,+)$. A two-surface S in M is a connected submanifold of M which can be represented by an embedding

$$\Phi : S \longrightarrow M$$

via the parametric equations $p^{\mu} = \Phi^{\mu}\left(\lambda^{A}\right)$, where $\{p^{\mu}\}$ are local coordinates in M ($\mu, \nu = 1,2,3,4$) and $\{\lambda^{A}\}$ are local coordinates in S ($A, B = 1,2$). Pull-back and push-forward operations define the metric tensor and tangent vector fields respectively in S, as seen on M. The metric on S is just the projection tensor N^{ab}.

The second fundamental form relative to a normal vector field n on S is given by

$$\chi_{ab}|_{n} \quad = \quad N_a^{\ c} N_b^{\ d} \nabla_d n_c. \tag{6.1}$$

If n is a null vector field normal to S, the null expansion scalar, along n, is given by the projection of (6.1) by N^{ab}:

$$\Theta_n = N^{ab} \chi_{ab}|_{n}. \tag{6.2}$$

The mean curvature vector of S in M (relative to the null normal vector field n) is then given by

$$H^c \quad = \quad N^{ab}\chi_{ab}^{\ \ c}, \tag{6.3}$$

where $\chi_{ab}^{\ \ c}$ is the shape tensor which takes a pair of vector fields to a normal vector to S.

For the LRS II class the two null normals to S, called the outgoing and ingoing null normal vector fields, are respectively given by

$$k_c = \frac{1}{\sqrt{2}}\left(u_c + e_c\right), \quad l_c = \frac{1}{\sqrt{2}}\left(u_c - e_c\right). \tag{6.4}$$

See for example Ellis *et al.* (2014) and references therein for more details. Explicitly computing these quantities we obtain the outgoing and outgoing expansion scalars as (Sherif *et al.*, 2018; Sherif *et al.*, 2019a)

$$\Theta_k = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma + \phi\right) \text{ and } \Theta_l = \frac{1}{\sqrt{2}}\left(\frac{2}{3}\Theta - \Sigma - \phi\right), \tag{6.5}$$

while the mean curvature vector is given by

$$H_c \quad = \quad -\left(\frac{2}{3}\Theta - \Sigma\right)u_c + \phi e_c. \tag{6.6}$$

Definition 6.2.1. *A surface S is said to be minimal if both the outgoing and ingoing null expansions vanish on S. This is equivalent to the requirement that both the u^c and e^c components of the mean curvature vector vanish everywhere on S, i. e.*

$$-u^c H_c = 0 \quad and \quad e^c H_c = 0 \tag{6.7}$$

on S. We will denote these surfaces by MMTS for ***minimal marginally trapped surface****.*

Definition 6.2.2. *An MTT will be called minimal if the MTS foliating MTT are minimal, and will be called a* ***minimal marginally trapped tube*** *(MMTT).*

We now consider the geometry of leaves of the MMTT in these spacetimes.

6.3 Geometry of MMTS in LRS II and NNF spacetimes

The Gaussian curvature of MMTS in LRS II spacetimes is given by (Ellis *et al.,* 2014)

$$K = \frac{1}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi - \frac{1}{4}H^2, \tag{6.8}$$

where $H^2 = H^c H_c$ is the squared norm of the mean curvature vector, and the derivatives of K in the e^a direction and along the observer's congruence are given respectively by

$$\begin{aligned} \dot{K} &= -u^c H_c K, \\ \hat{K} &= -e^c H_c K. \end{aligned} \tag{6.9}$$

Clearly for MMTS (and therefore MMTTs) we have $\dot{K} = \hat{K} = 0$. The Gaussian curvature is then $K = \frac{1}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi$ and is constant by the vanishing of the two derivatives. Therefore, the sign of K determines the geometry of the MMTS.

The set of field equations on the leaves of the MMTT is then

- *Evolution (LRS II):*

$$\frac{2}{3}\dot{\Sigma} - \dot{\Theta} = -\frac{1}{3}\left(\rho + 3p\right) + \mathcal{E} - \frac{1}{2}\Pi, \tag{6.10}$$

$$\dot{\phi} = Q, \tag{6.11}$$

$$\dot{\mathcal{E}} - \frac{1}{3}\dot{\rho} + \frac{1}{2}\dot{\Pi} = 0. \tag{6.12}$$

- *Propagation (LRS II):*

$$\frac{2}{3}\hat{\Theta} - \hat{\Sigma} = Q, \tag{6.13}$$

$$\hat{\phi} = -\frac{2}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi, \tag{6.14}$$

$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\rho} + \frac{1}{2}\hat{\Pi} = 0. \tag{6.15}$$

- *Propagation/Evolution (LRS II):*

$$\hat{A} - \dot{\Theta} = -A^2 + \Theta^2 + \frac{1}{2}\left(\rho + 3p\right), \tag{6.16}$$

$$\hat{Q} + \dot{\rho} = -\Theta\left(\rho + p + \Pi\right) - 2AQ, \tag{6.17}$$

$$\dot{Q} + \hat{p} + \hat{\Pi} = -A\left(\rho + p + \Pi\right) - 2\Theta Q. \tag{6.18}$$

We note that the condition for minimality is to be satisfied at all times on the leaves, and therefore taking the dot derivatives of the two equations in (6.7) also holds. Thus $\phi = 0$ implies

$$Q = 0 \tag{6.19}$$

and

$$-\frac{2}{3}\rho - \mathcal{E} - \frac{1}{2}\Pi = 0, \tag{6.20}$$

from (6.11) and (6.14) respectively. From (6.20) we therefore have that the Gaussian curvature $K = \rho$. Since $K = \rho = constant$, this allows us to conclude that the energy density completely determines the intrinsic geometry of MMTS in LRS II spacetimes.

Theorem 6.3.1. *In LRS II spacetimes, if the MMTT is geometrically S^3, then the MMTS foliating the MMOTS are S^2.*

Proof. The shape tensor on the leaves of the MMTT in LRS II spacetimes can be calculated as

$$\chi_{abc} \quad = \quad -\frac{1}{2}N_{ab}\left[\left(\frac{2}{3}\Theta-\Sigma\right)u_c-\phi e_c\right], \tag{6.21}$$

and is zero identically on every leaf of the MMOTS, under the minimality condition. Thus we have the squared norm as

$$\chi_{abc}\chi^{abc}=0. \tag{6.22}$$

We have $\chi_{abc}\chi^{abc}<2=dim(S)$ (where S denotes a leaf of the MMTT), and therefore S must be isometric to the totally geodesic S^2. Since $K=const.$, the scalar curvature $R=2K$ is constant (Ellis *et al.*, 2014). This is due to a more general results (Chern *et al.*, 1970), which may be summarized as: if a closed minimal surface of dimension n of constant scalar curvature is immersed in the $n+1$ dimensional sphere, and if the squared norm of the shape operator $\chi_{abc}\chi^{abc}$ is less than or equal to n, then either $\chi_{abc}\chi^{abc}=0$, in which case the surface is isometrically S^2 or $\chi_{abc}\chi^{abc}=n$, in which case the surface is isometrically a Clifford torus. □

This result actually shows that all MMTTs in LRS II spacetimes foliated by minimal 2-surfaces will have S^3 geometry, i.e. are spherically symmetric. It is not difficult to see that the leaves of MMTTs in NNF spacetimes would necessarily assume the geometry of those of LRS II spacetimes which follows from standard results in foliation theory that a diffeomorphism f between two manifolds M and N with respective foliations $\mathscr{F}$ and $\mathscr{F}'$, induces an isomorphism from $\mathscr{F}$ to $\mathscr{F}'$ given by $L'=f(L)$, for leaves $L\in\mathscr{F}$ and $L'\in\mathscr{F}'$.

Since the MMTT in NNF spacetimes are diffeomorphically LRS II, we have an isomorphism of foliations. Their leaves are therefore diffeomorphic.

We note, however, that while in the first case we explicitly showed that the Gaussian curvature is constant and ρ-dependent only, which allowed us to classify the geometry, the converse may not necessarily be the case. That we know the geometry does not necessarily mean that the Gaussian is constant and/or ρ-dependent, and this might require a full analysis of the general field equations.

In fact theorem 6.3.1 generalizes to any 4-dimensional spacetime:

Theorem 6.3.2. *In any* 4*-dimensional spacetime, if the MMTT is geometrically S^3, then the* 2*-surfaces foliating the MMOTS are S^2.*

Proof. This generalization follows from the fact that for 4-dimensional spacetime, the shape tensor can be decomposed as (Bengtsson & Senovilla, 2011)

$$\chi_{ab}^{\ \ c} = -k^d \chi_{abd} l^c - l^d \chi_{abd} k^c,$$

where $k^d \chi_{abd}$ and $l^d \chi_{abd}$ are the second fundamental forms. The minimality condition is equivalent to the simultaneous vanishing of $k^d \chi_{abd}$ and $l^d \chi_{abd}$, which gives $\chi_{abc} = 0$.

Next we just need to check that the Gaussian curvature is constant (which would imply the constancy of the scalar curvature). In general we can write the square of the mean curvature vector as K as (see the form of H_c in terms of the expansion scalars in reference Bengtsson & Senovilla (2011))

$$H_c H^c = -2\Theta_k \Theta_l = -\frac{1}{K^2} \nabla_c K \nabla^c K. \tag{6.23}$$

Thus, the condition of being minimal, equivalent to the condition $\Theta_k = \Theta_l = 0$, implies the vanishing of the gradient of K, which implies that K is constant. □

In general the squared norm of the shape tensor can written as

$$\chi_{abc} \chi^{abc} = -2k_d l^s \chi_{ab}^{\ \ d} \chi^{ab}_{\ \ s}. \tag{6.24}$$

Thus, if there is trapping, say $\Theta_k = 0$ (which would imply that $k_d \chi_{ab}^{\ \ d} = 0$), then (6.24) is zero. Of course, from (6.23) K is constant. While the results in Chern *et al.* (1970) has minimality as a necessary condition, if the result holds when only one of the expansion scalars vanishes, there is an obvious consequence. Such generalization would then imply that, in an arbitrary spherically symmetric spacetime, the leaves of a codimension-1 foliation of the boundary of the trapped region are necessarily topological spheres since in arbitrary spherically symmetric spacetimes, the trapped region and its boundary, assuming the boundary is non-empty, have spherical symmetry (Bengtsson & Senovilla, 2011).

6.4 Discussion

In this chapter we have given a complete geometric characterization of cross sections of minimal horizons in LRS II spacetimes. It is shown that the geometry of the hypersurface in LRS II foliated by minimal 2-surfaces constrains the geometry of these 2-surfaces. This result generalizes to any 4-dimensional spacetimes. This was achieved by analyzing the equivalent formulation of the field equations in the $1+1+2$ covariant setting, and showing that the energy density alone determines the geometry of the leaves of these horizons. The fact that the Gaussian curvature K is constant allows us to use standard results from classical minimal surface theory to classify the minimal horizons in LRS II spacetimes.

Chapter 7

A T_2-separable g boundary and its relation to the a boundary

7.1 Introduction

In this chapter, we construct a modified g boundary that is T_2-separable and whose completion of the spacetime retains separation of regular spacetime points from singular points. We also explore the relationship between this boundary and the a boundary, which was identified as an important open problem in the original paper (Scott & Szekeres, 1994) constructing the a boundary. If structures on the g boundary can be carried over to this modified g boundary, then such relationship could in principle allow us to construct on the a boundary, these various structures on the g boundary such as differentiable, causal and metric structures.

In section 7.2, we give an overview of the g and a boundaries. Section 7.3 presents our new construction using notions from the construction of the attached point topology on the a boundary as well as a natural Hausdorff topology on the tangent bundle. In section 7.4, we consider the example in Geroch *et al.* (1982) relative to our construction to determine if this modified g boundary suffers the topological separation problem of the original g and b boundaries. In section 7.5 we summarize our results and anticipate future avenues of research.

7.2 Some notes on the g and a boundaries

In this section we provide some background on the g and a boundary constructions, as well as the topologies that can be placed on these constructions. We follow the following standard references Geroch, (1968b), Hawking & Elliis, (1973), Scott & Szekeres, (1994), Ashley, (2002) and Barry & Scott (2011).

7.2.1 The g boundary

Let M be a geodesically complete spacetime manifold, and let Σ be a co-dimension 1 submanifold of M such that $M = M_1 \sqcup \Sigma \sqcup M_2$. In other words, Σ divides M into disjoint subsets M_1 and M_2. Suppose one is given one of the subsets, say M_1. A natural question arises as to how much information about Σ can be obtained from what we know about M_1. The idea is to use the information about the incomplete geodesics in M_1 - those in M_1 which, when extended in M, pass through Σ - to recover "parts" of Σ. One then groups these geodesics as follows: Let γ be an incomplete geodesic in M_1 and generate a family of geodesics by allowing small variations in the initial conditions - a point $p \in \gamma$ and a tangent vector at p - of γ. This family of geodesics traces out a 4-dimensional tube called a "thickening" of γ. Another incomplete geodesic γ' is said to be related to γ if γ' enters and remains in every thickening of γ. However, since finding Σ may not always suffice, the above construction must be generalized (Geroch, 1968b).

Let $G = TM \setminus \{0\}$ be the reduced tangent bundle on a spacetime manifold M, made up of the non-zero vectors in TM. The set $G \subset TM$ is an 8-dimensional manifold whose elements are the pairs $\left(p, \xi^{\alpha}\right)$. Each $\left(p, \xi^{\alpha}\right) \in G$ uniquely determines the geodesic γ, satisfying the geodesic equation, for which $\gamma(0) = p$ and $\gamma'(0) = \xi^{\alpha}$. Such a geodesic satisfies the following properties:

- has one specified end point;
- has been extended as far as possible in some direction from the end point;
- has a given affine parameter; and
- the affine parameter vanishes at the end point and is positive elsewhere on the curve.

The g boundary is constructed from the subset $G_I \subset G$, where G_I is obtained via the construction of a field φ that identifies points of G with the total affine length of the associated geodesics uniquely determined by those points. Elements of G_I are then those points $\left(p, \xi^{a}\right) \in G$ such that $\varphi\left(p, \xi^{\alpha}\right)$ is finite.

One constructs a 9-dimensional manifold $H = G \times (0, \infty)$. Define the folowing subsets H_0 and H_+ of H as

$$H_0 = \{\left(p, \xi^{\alpha}, d\right) | \varphi\left(p, \xi^{\alpha}\right) = d\}, \tag{7.1}$$

$$H_+ = \{\left(p, \xi^{\alpha}, d\right) | \varphi\left(p, \xi^{\alpha}\right) > d\}. \tag{7.2}$$

Then there is a natural map $\Psi : H_+ \longrightarrow M$, that assigns to each point $\left(p, \xi^{\alpha}, d\right) \in H_+$ a point $p' \in M$ obtained by moving a distance d along the geodesic uniquely associated to $\left(p, \xi^{\alpha}\right)$. One defines a topology on G_I as follows: Let O be an open set of M, and define open subsets $S(O)$ as consisting of those points $\left(p, \xi^{\alpha}\right) \in G_I$ such that there exists an open set $U \subseteq H$ containing the point $\left(p, \xi^{\alpha}, \varphi\left(p, \xi^{\alpha}\right)\right)$ of H_0, and $\Psi(U \cap H_+) \subseteq O$.

Given two open sets O_1, O_2 in M, it can be shown that $S(O_1) \cap S(O_2) = S(O_1 \cap O_2)$, (Geroch, 1968b). These open sets therefore form the basis of a topology on G_I. Equivalence classes of elements of G_I can now be constructed by requiring that two elements $\alpha, \beta \in G_I$ are equivalent if they always appear in the same open set. The collection of such equivalence classes forms the g boundary, where the induced topology on the g boundary is the quotient topology, and is T_0 separable (Hocking & Young, 1961; Geroch, 1968b). However, the ideal separation property would be the T_2 separation property. Geroch did provide a recipe for constructing a g boundary that is T_2 separable, but this requires the construction of all T_2 separable equivalence relations on G_I, which in general would not be feasible.

A new manifold (called the spacetime with g boundary) can now be constructed from the disjoint union of the spacetime manifold and the g boundary: A subset of the form (O, U), where O is an open set of M and U is an open set of g, will be called open in M^* if $S(O) \supset U$. These open sets on M^* form a basis for a topology on M^*. It can be checked that an intersection of two such open sets is open. If $U = S(O)$, then (O, U) will be called a *full open* set of M^*. For more details and discussions see Geroch (1968b), from which we have also adopted the notation for open sets of the topology on the manifold completion throughout the rest of the paper.

Geroch's original construction only considered the set G_I of incomplete geodesics, and only allows for the identification of singular boundary points. Since we are interested in relating the g and a boundaries, in what follows we will consider the set G which allows one to treat a wider class of curves, and, hence, of boundary points. This allows for the modification of the g boundary which we shall exploit in order to establish a relationship with the a boundary (see section 7.3)

7.2.2 The a boundary

The construction of the a boundary relies on the existence of open embeddings into manifolds of the same dimension, or *envelopments*, (Scott & Szekeres, 1994). An advantage of the a boundary construction is that it can be applied to any manifold M and it is independent of both the affine connection on M and the chosen family of curves in M. If we specify a family of curves $\mathscr{C}$ in M, satisfying the bounded parameter property (to be formally defined in section 7.3) then the *a* boundary points can be classified as *regular, points at infinity, unapproachable points*, or *singularities*. This subsection summarizes arguments developed in the reference Scott & Szekeres (1994), unless otherwise cited.

Let M be a spacetime manifold, and let $\phi : M \longrightarrow \hat{M}$ be an envelopment of M into $\hat{M}$ where $dim(M) = dim(\hat{M})$.

Definition 7.2.1. *A **boundary point** of M is a point $p \in \partial_\phi(M)$ in the topological boundary of $\phi(M) \subseteq \hat{M}$. A **boundary set** is a non-empty subset B of $\partial_\phi M$, comprised of boundary points.*

Let $\phi' : M \longrightarrow \hat{M}'$ be a second envelopment of M into $\hat{M}'$, and let $B' \subseteq \partial_{\phi'} M$. We define a *covering relation* as follows:

Definition 7.2.2. *A boundary set $B \subseteq \partial_\phi M$ in $\hat{M}$ **covers** a boundary set $B' \subseteq \partial_{\phi'} M$ in $\hat{M}'$ if for every open neighborhood U of B in $\hat{M}$, there exists an open neighborhood U' of B' in $\hat{M}'$ such that*

$$\phi \circ \phi'^{-1}\left(U' \cap \phi'(M)\right) \quad \subseteq \quad U. \tag{7.3}$$

Definition 7.2.3. *A boundary set B is **equivalent** to a boundary set B' if B covers B' and B' covers B.*

The covering relation defines an equivalence relation on the set of all boundary sets induced by all possible envelopments of M. An equivalence class $[B]$ of boundary sets is called an *abstract boundary set.*

Definition 7.2.4. *An **abstract boundary point** is an abstract boundary set that has a singleton p as a representative element. The set of all abstract boundary points is called the **abstract boundary** or simply the a boundary.*

Let O be an open set in M and let $B \in \partial_\phi M$ be a boundary set in the topological boundary of an envelopment $\phi: M \longrightarrow \hat{M}$ of M into $\hat{M}$. A boundary point $p \in B$ (respectively, a boundary set B) is attached to the open set O of M if for every open neighborhood U in $\hat{M}$ of p (resp., of B), we have that $U \cap \phi(O) \neq \emptyset$. For a boundary set to be attached to the open set O, we require at least one boundary point $p \in B$ to be attached to O . An abstract boundary point $[p]$ is attached to an open set $O \subseteq M$ if the boundary point p is attached to O (see the reference (Barry & Scott, 2011) for more details on the attached point topology).

Again, one wants a topology on the union of the spacetime manifold M and the abstract boundary a: $\bar{M} \equiv M \sqcup a$. Subsets of $\bar{M}$ of the form $(O \cup B, C)$, where O is a nonempty open set of M, B is the set of all abstract boundary points which are attached to O, and C is some subset of the abstract boundary, and where the collection of every C set is the set of all subsets of the abstract boundary will be called open in $\bar{M}$. These open sets form a basis for a topology on $\bar{M}$, which was shown to be Hausdorff (Barry & Scott, 2011). This is known as the attached point topology, and the open sets of the topology induced on a from the attached point topology are precisely the C sets.

By construction, the identification of an abstract boundary point $[\hat{p}]$, eg. a singularity, is completely determined by the open set of M to which $[\hat{p}]$ is attached. This is one of the primary motivations for boundary constructions. What this means is that one has an envelopment ϕ of a manifold M into, say, $\hat{M}$, with $\hat{p} \in \partial_\phi M$ and $\hat{p}$ is attached to O (which implies $[\hat{p}]$ is attached to O). Then the image $\phi(O)$ extends (in $\hat{M}$) to $\hat{p} \in \hat{M}$. One therefore has knowledge as to an open set in M that is close to the singularity, and thus a means of locating them.

7.3 A new construction

As mentioned in the previous section, we intend our construction to be applicable to a wider class of curves - those satisfying what is called the bounded parameter property - as in the case of the a boundary. In this section we first briefly discuss the class of curves satisfying the bounded parameter property, of which geodesics form a subclass. We then proceed to the construction of the modified g boundary, and its relationship with the a boundary, before concluding with a few illustrative examples.

7.3.1 The bounded parameter property

The a boundary considers a broad class of curves satisfying the *bounded parameter property* (BPP), (Scott & Szekeres, 1994).

Let $\gamma : [a,b) \longrightarrow M$ (with $a < b \leq \infty$) be a parametrized and regular (the tangent vector $\dot{\gamma}$ is nowhere vanishing on the interval $[a,b)$) curve in M, with $\gamma(a) = p$ as the starting point of γ. A curve $\gamma' : [a', b') \longrightarrow M$ is a subcurve of γ if $a \leq a' < b' \leq b$ and $\gamma' = \gamma|_{[a',b')}$. If $a = a'$ and $b > b'$ we say that γ is an extension of γ'.

Definition 7.3.1 (Change of parameter)**.** *A* ***change of parameter*** *is a monotone increasing* C^1 *function*

$$s : [a,b) \longrightarrow [a', b'),$$

which maps a to a' *and b to* b'*, and* $\frac{ds}{d\lambda} > 0$ *for* $\lambda \in [a,b)$*. The curve* γ' *is obtained from* γ *via the* *change of parameter s if the following diagram commutes:*

$$\begin{tikzcd} [a,b) \arrow[r, "\gamma"] \arrow[dr, "s"'] & M \\ & {[a',b')} . \arrow[u, "\gamma'"'] \end{tikzcd} \tag{7.4}$$

Definition 7.3.2 (Bounded parameter property (BPP))**.** *A family of parametrized curves $\mathscr{C}$ in M is said to have the bounded parameter property if*

- *for any $p \in M$, $\exists$ at least one $\gamma \in \mathscr{C}$ such that $\gamma(\lambda) = p$ for some $\lambda \in [a,b)$;*
- *if $\gamma \in \mathscr{C}$, so is every subcurve γ' of γ; and*
- *for any pair $\gamma, \gamma' \in \mathscr{C}$ for which there exists an s such that the diagram in (7.4) commutes, we have either the parameter on both γ and γ' is bounded, or the parameter on both γ and γ' is unbounded.*

Families of curves satisfying the BPP include geodesics with affine parameter, differentiable curves with generalized affine parameter and timelike geodesics parametrized by proper time. This generality carries over to the generality of the a boundary construction, as well as our modification of the g boundary. As with the a boundary, our construction applies to a smooth manifold of any dimension which is endowed with an affine connection and BPP curves. While this is still modest compared to the a boundary which exists for any manifold independent of an affine connection and a family of curves, our construction can equally be applied to theories such as Einstein-Cartan, Kaluza-Klein, Yang-Mills, etc. For the remainder of this paper all curves will satisfy the b.p.p., unless otherwise stated.

We conclude this subsection with several standard definitions of utility in the sequel.

Definition 7.3.3 (Limit point)**.** *Let $\gamma : [a,b) \longrightarrow M$ be a curve. A point $\hat{p}$ is a limit point of γ if there exists an increasing infinite sequence of real numbers $t_i \in [a,b)$ with $t_i \to b$ such that $\gamma(t_i) \to \hat{p}$.*

This implies that for every subcurve $\gamma' : [a', b) \longrightarrow M$ with $a \le a' < b$, $\gamma(t)$ enters every neighborhood U of $\hat{p}$.

Definition 7.3.4 (Endpoint)**.** *Let $\gamma : [a, b) \longrightarrow M$ be a curve. A point $\hat{p}$ is an endpoint of γ if $\gamma(t) \to \hat{p}$ as $t \to b$. For Hausdorff manifolds, $\hat{p}$ is unique.*

Definition 7.3.5 (Approach by a curve)**.** *Let $\phi : M \longrightarrow \hat{M}$ be an envelopment of M into $\hat{M}$, and let $\gamma : [a, b) \longrightarrow M$ be a curve in M. We say γ **approaches** a boundary set $B \in \partial_\phi M$ if there exists a $\hat{p} \in B$ such that $\hat{p}$ is a limit point of $\phi \circ \gamma$.*

7.3.2 A T_2 separable g boundary

In this subsection we define open sets on G making use of the attachment relation from the a boundary and a natural topology on the tangent bundle. This allows us to define an equivalence relation on G, and the collection of the associated equivalence classes defines a g boundary which we shall denote $\tilde{g}$.

In what follows, all manifolds are assumed to be smooth, Hausdorff, connected and paracompact.

Given a manifold M, we recall that the tangent bundle on M can be given a natural topology that is Hausdorff (for example see the reference Curtis & Miller (1985)). This is done as follows: suppose (O_1, φ_1) is a chart on M, and let $\{f^i\}_{i\in\{1,\ldots,n\}}$ be the coordinate functions of φ_1. Define a map

$$\tilde{\varphi}_1 : \pi^{-1}(O_1) \longrightarrow \mathbb{R}^{2n},$$

by

$$\xi^i \frac{\partial}{\partial f^i}\Big|_p \mapsto \left(f^1(p), \ldots, f^n(p), \xi^1, \ldots, \xi^n\right),$$

where ξ^i are vector fields in $T_pM\Big|_{O_1}$, and π is the bundle projection. Then $\tilde{\varphi}_1\left(\pi^{-1}(O_1)\right) = \varphi_1(O_1)\times\mathbb{R}^n$ is open in $\mathbb{R}^{2n}$, and $\tilde{\varphi}_1$ is a bijection with inverse

$$\left(f^1(p),\ldots,f^n(p),\xi^1,\ldots,\xi^n\right)\mapsto \xi^i\frac{\partial}{\partial f^i}\Big|_{\varphi^{-1}(f)}.$$

Given a second chart $\left(O_2,\varphi_2\right)$ on M, both of

$$\tilde{\varphi}_j\left(\pi^{-1}(O_1)\cap\pi^{-1}(O_2)\right) = \varphi_j(O_1\cap O_2)\times\mathbb{R}^n,$$

for $j = 1,2$, are open in $\mathbb{R}^{2n}$, and the transition map

$$\tilde{\varphi}_2\circ\tilde{\varphi}_1^{-1} : \varphi_1(O_1\cap O_2)\times\mathbb{R}^n \longrightarrow \varphi_2(O_1\cap O_2)\times\mathbb{R}^n$$

is smooth, for $\pi^{-1}(O_1)\cap\pi^{-1}(O_2)\neq\varnothing$. We therefore have that $\tilde{\varphi}_i$ are diffeomorphisms. Thus, we have a smooth atlas $\mathscr{A}_{TM} = \{\pi^{-1}(O_i),\tilde{\varphi}_i\}_{i\in\{1,\ldots,n\}}$ on TM, the tangent bundle on M, which makes TM a smooth manifold, with the topology given by the open sets $\varphi_i(O)\times\mathbb{R}^n$ of $\mathbb{R}^{2n}$. It is not difficult to check that TM endowed with this topology is Hausdorff: any two points in the same fiber of π will lie in the same chart. Any two points $\left(p,\xi^i\right)$ and $\left(q,\xi^j\right)$, with $p\neq q$, lying in different fibers can be separated by disjoint open neighborhoods $\pi^{-1}(O_1)$ and $\pi^{-1}(O_2)$ respectively, by the choice that neighborhoods O_1 of $p\in M$ and O_2 of $q\in M$ are disjoint.

Now let O be an open subset of M and let $\left(\pi^{-1}(O),\tilde{\varphi}\right)$ be a chart on TM. Furthermore let γ_p be the curve associated with a point $\left(p,\xi^\alpha\right)\in G\subset TM$ (whenever we write γ_p *associated with the point* $\left(p,\xi^\alpha\right)$, we will mean a BPP curve $\gamma_p : [a,b)\longrightarrow M$ starting at the manifold point $\gamma(a) = p$). Suppose $\phi : M\longrightarrow\hat{M}$ is an envelopment from M into $\hat{M}$, and let $\left[\hat{p}\right]$ be an abstract boundary point attached to O, with $\hat{p}$ in the topological boundary $\partial_\phi M$. We define open sets on G as follows:

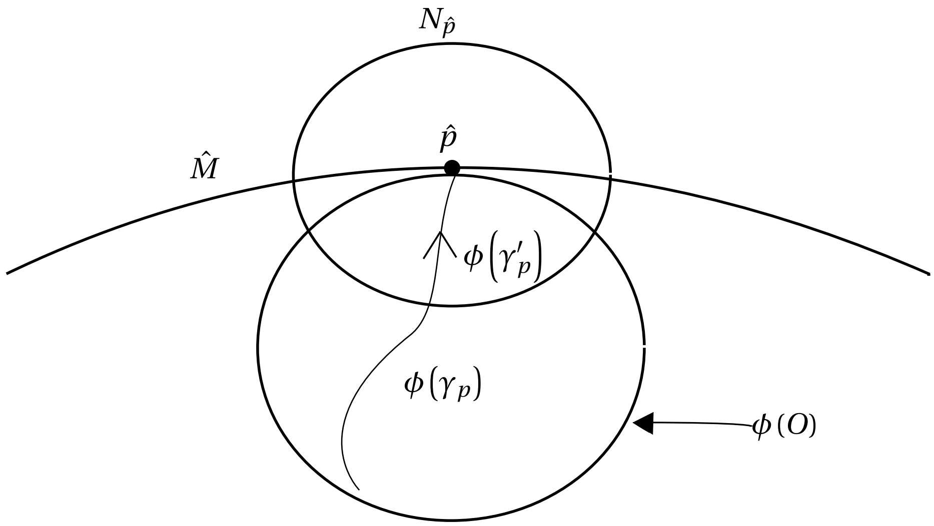

Figure 7.1: Depiction of the open set $\phi(O)$ containing $\phi\left(\gamma_p\right)$ to which $\hat{p}$ is attached, and the portion γ'_p of γ_p lying in the intersection $\phi(O) \cap N_{\hat{p}}$.

$$\begin{aligned} S(O) \quad = \quad & \{\left(p, \xi^{\alpha}\right) \in G \mid \phi\left(\gamma_p(\lambda)\right) \in \phi(O), \forall\ \lambda \in [a, b) \text{ and} \\ & \phi\left(\gamma_p\right) \supset \phi\left(\gamma'_p\right) \subset \phi(O) \cap N_{\hat{p}}, \text{ for a subcurve } \phi\left(\gamma'_p\right) \text{ of } \phi\left(\gamma_p\right) \\ & \text{for all open neighborhoods } N_{\hat{p}} \text{ of } \hat{p}, \text{ over all possible} \\ & \text{envelopments } \phi\}. \end{aligned} \tag{7.5}$$

Of course, $S(O)$ is empty if no boundary set is attached to O. It is also easily seen that $S(O_1) \cap S(O_2) = S(O_1 \cap O_2)$. We therefore have that the $S(O)$ sets form a basis for a topology on G. (See figure 7.1 for a depiction of the attachment of $\hat{p}$ to O and the portion of the curve in the intersection.)

Definition 7.3.6. *Let $\left(p, \xi^{\alpha}\right)$ and $\left(q, \xi^{\beta}\right)$ be points in G. We define an* ***equivalence relation****, denoted E_1, on G as follows: $\left(p, \xi^{\alpha}\right) \sim \left(q, \xi^{\beta}\right)$ iff the curves associated with $\left(p, \xi^{\alpha}\right)$ and $\left(q, \xi^{\beta}\right)$ approach the same abstract boundary point.*

The proof that E_1 is indeed an equivalence relation follows immediately from the definition of an abstract boundary point.

We stress that the use of the abstract boundary point in definition 7.3.6 means that we are considering equivalence classes of boundary sets of envelopment via definition 7.2.3. If two curves γ_1 and γ_2 both start at the same point, then from (7.5) it is clear that they will both approach the same abstract boundary point.

Definition 7.3.7. *The collection of all equivalence classes in G under the equivalence relation E_1 is a modified g boundary which we will denote by $\tilde{g}$.*

The $\tilde{g}$ boundary naturally inherits the coinduced topology from G under the continuous canonical quotient map q from G to $\tilde{g}$ (In section 3.3 we make use of this as part of our discussion about explicitly embedding the $\tilde{g}$ boundary a boundary) (Munkres, 2000). We emphasize that our construction relies on the existence of open embeddings. This is not an overly restrictive requirement as all theories specified via a metric in some coordinate system (eg. general relativity, Einstein-Gauss-Bonnet, etc.), have an implicit envelopment. (See the discussion in the con lusion of Scott & Szekeres, 1994.) Geroch noted (Geroch, 1968b) that there is no unique way of defining the g boundary, and that his particular construction was ad hoc. However, under the condition that envelopments of the spacetime manifold exist, the a boundary provides us with a natural way of defining open sets on G. It also allows for the definition of a desired quotient on G via definition 7.3.7.

Theorem 7.3.8. *The modified g boundary, $\tilde{g}$, is Hausdorff separable in the coinduced topology from G.*

Proof. Let σ_1 and σ_2 be two points in $\tilde{g}$, and let

$$\bigcup_{i\in I} S(O_i) \text{ and } \bigcup_{j\in J} S(O_j)$$

be open neighborhoods of σ_1 and σ_2 respectively. Choose open subsets of M such that $O_m \cap O_n = \varnothing$, for any pair m, n such that $m \in I$ and $n \in J$. Now fix an envelopment $\phi : M \longrightarrow \hat{M}$. Curves associated with the points σ_1 and σ_2 approach different abstract boundary points, and therefore approach different boundary points (or sets) in $\partial_\partial M$, say $\hat{p}_1$ and $\hat{p}_2$ attached to O_m and O_n respectively (for fixed m and n), that are not equivalent under the abstract boundary equivalence. In the topology induced by $\hat{M}$ on $\partial_\phi M$, we can always choose disjoint open neighborhoods $N_{\hat{p}_1}$ and $N_{\hat{p}_1}$ of $\hat{p}_1$ and $\hat{p}_2$ respectively. Since $O_m \cap O_n = \varnothing$, and since curves in O_m and O_n enter and stay in $O_m \cap N_{\hat{p}_1}$ and $O_n \cap N_{\hat{p}_2}$ respectively, $S(O_m) \cap S(O_n) = \varnothing$. We therefore have that

$$\left(\bigcup_{i \in I} S(O_i)\right) \cap \left(\bigcup_{j \in J} S\left(O_j\right)\right) = \varnothing.$$

$\square$

We note that G is not Hausdorff separable in the topology with basis given by the open sets $S(O)$. This is because any two points $\left(p, \xi^\alpha\right), \left(p, \xi^\beta\right) \in G$ will always lie in the same open set since surves associated to the two points start at the same manifold point and thus remain in the open set of the manifold containing the point. However, we have shown that $\tilde{g}$ is Hausdorff separable since under the equivalence relation E_1 these points are identified.

Theorem 7.3.9. *Let M^* be the manifold formed by the completion of the spacetime manifold M with $\tilde{g}$: $M^* \equiv M \sqcup \tilde{g}$, and let a basis for a topology on M^* be given by open subsets of the form*

$$\left(O, \tilde{U} = \bigcup_{i \in I} S(O_i)\right), \tag{7.6}$$

where $\tilde{U}$ are the open sets of $\tilde{g}$, coinduced by the open sets on G, and $O = O_k$ for some $k \in I$ is an open set in M. Then M^* is Hausdorff.

Proof. That M^* is Hausdorff follows from the fact that disjoint union preserves the Hausdorff property: for points $p \in M$ and $\sigma \in \tilde{g}$, open neighborhoods $O_k \ni p$ and

$$\tilde{U} = \bigcup_{i \in I} S(O_i) \ni \sigma$$

are disjoint. □

One may notice that for our choice of basis for a topology on M^*, (7.6), the open set O of M^* restriction to M is simply required to be a choice of one of the $O_i^{\prime} s$ defining the open set $\tilde{U}$ of M^* restriction to $\tilde{g}$. This may be done because boundary points (sets) attached to all of the $O_i s$ are identified under the abstract boundary equivalence.

We stress again that, in the entirety of this subsection's construction we have considered BPP curves γ_p associated to $\left(p, \xi^{\alpha}\right) \in G$. In general they need not be geodesics. However, an explicit reduction to a construction more similar to that of Geroch is obtained by insisting that they are geodesics. This restricted case remains a generalization of Geroch's g boundary in the sense that we have an additional equivalence relation E_1, which is necessary in order to yield T_2 separability.

The key notion here is a rephrasing of the a boundary using BPP curves explicitly related to the tangent bundle. This is what facilitates the understanding of the relationship between the a boundary and other boundary constructions, in this case, a construction similar to the g boundary. This relationship is considered in more detail in the next subsection.

7.3.3 Relation to the a boundary

In this section we now show how the $\tilde{g}$ boundary can be embedded into the a boundary via explicit mappings. We first define an equivalence relation on $\phi(M)$.

Definition 7.3.10. *Let $\phi\left(p_1\right), \phi\left(p_2\right) \in \phi(M)$. The* ***equivalence relation*** *on $\phi(M)$, denoted E_2, is defined as follows: $\phi\left(p_1\right) \sim \phi\left(p_2\right)$ iff* *a.* $\phi(p_1) = \phi \circ \pi(p_1, \xi_1^\alpha)$ *and* $\phi(p_2) = \phi \circ \pi(p_2, \xi_2^\alpha)$*, where* (p_i, ξ_i^α) *are in open sets of* G*, and*

b1. $\phi(p_1)$ *and* $\phi(p_2)$ *lie on the same curve, or*

b2. $\phi(p_1)$ *and* $\phi(p_2)$ *lie on curves approaching the same abstract boundary point.*

This yields equivalence classes of curves with limit points being the same abstract boundary point, the collection $\phi(M)/E_2$, which we shall denote by $\phi(M)_{geo}$. It is not difficult to show that E_2 is indeed an equivalence relation. Reflexivity and symmetry follow immediately from the definition. To show transitivity, let $\phi(p_1) \sim \phi(p_2)$. Then either $\phi(p_1)$ and $\phi(p_2)$ lie on the same curve γ or $\phi(p_1)$ and $\phi(p_2)$ lie on curves approaching the same abstract boundary point. Suppose $\phi(p_1)$ and $\phi(p_2)$ both lie on γ. If $\phi(p_2)$ and $\phi(p_3)$ lie on γ, then $\phi(p_1) \sim \phi(p_3)$. Otherwise, $\phi(p_3)$ lies on some curve $\tilde{\gamma}$ such that $\tilde{\gamma}$ approaches the same abstract boundary point as γ. Since $\phi(p_1)$ lies on γ, this would imply $\phi(p_1) \sim \phi(p_3)$. Now suppose $\phi(p_1)$ and $\phi(p_2)$ lie on curves γ and $\tilde{\gamma}$ respectively, both approaching the same abstract boundary point. If $\phi(p_3)$ lies on either of γ or $\tilde{\gamma}$, then $\phi(p_3) \sim \phi(p_1)$ or $\phi(p_3) \sim \phi(p_2)$ which would imply $\phi(p_3) \sim \phi(p_2)$ or $\phi(p_3) \sim \phi(p_1)$ respectively. Otherwise, $\phi(p_3)$ lies on some curve $\hat{\gamma}$ (different from γ and $\tilde{\gamma}$) approaching the same abstract boundary point as γ and $\tilde{\gamma}$, which would imply that $\phi(p_3) \sim \phi(p_1)$, $\phi(p_1) \sim \phi(p_2)$ and $\phi(p_3) \sim \phi(p_2)$.

Let the map

$$q : G \longrightarrow \tilde{g}, \tag{7.7}$$

be the canonical quotient map defined by

$$(p, \xi^\alpha) \mapsto [(p, \xi^\alpha)]_{E_1},$$

which sends points in G to its equivalence class under the equivalence relation E_1. Define a map

$$\kappa : G \longrightarrow \phi(M)_{geo}, \tag{7.8}$$

by

$$\left(p,\xi^{\alpha}\right)\mapsto\left[\phi\left(p\right)\right]_{E_2},$$

which maps a point of $\left(p,\xi^{\alpha}\right)$ of G to the equivalence class (under the equivalence relation E_2) containing the image of the associated curve under the composition $\phi\circ\pi$. The map q is a quotient map and thus has the natural inverse map, q^{-1}, which sends an equivalence class to the set of its elements. Clearly κ is constant on the set $q^{-1}\left(\left[\left(p,\xi^{\alpha}\right)\right]_{E_1}\right)$, for $\left[\left(p,\xi^{\alpha}\right)\right]_{E_1}\in\tilde{g}$, since all elements in $q^{-1}\left(\left[\left(p,\xi^{\alpha}\right)\right]_{E_1}\right)$ are sent to $\left[\phi\left(p\right)\right]_{E_2}$. The map κ thus induces a map (see Theorem 22.2 of Munkres (2000))

$$r:\tilde{g}\longrightarrow\phi\left(M\right)_{geo}, \tag{7.9}$$

such that $r\circ q=\kappa$. Hence, the diagram

$$\begin{tikzcd} G \arrow[r, "q"] \arrow[dr, "\kappa"'] & \tilde{g} \arrow[d, "r"] \\ & \phi\left(M\right)_{geo}. \end{tikzcd} \tag{7.10}$$

commutes. The map r sends points of $\tilde{g}$ to the appropriate equivalence class under the equivalence relation E_2.

We now introduce the notion of the limit operator (Goreham, 2004; Flores *et al.*, 2011; Flores *et al.*, 2010), which allows us to attach limit points/endpoints to curves from $\phi\left(M\right)$. Let X be any set and let $S\left(X\right)$ denote the set of sequences in X. Let $P\left(X\right)$ denote the set of parts of X. One defines the limit operator as a map $l:S\left(X\right)\longrightarrow P\left(X\right)$ which sends sequences in $S\left(X\right)$ to their limit points in $P\left(X\right)$, satisfying the compatibility condition of subsequences: if $\sigma_1,\sigma_2\in S\left(X\right)$, and σ_2 is a subsequence of σ_1, then $l\left(\sigma_1\right)\subset l\left(\sigma_2\right)$.

As has been mentioned, all manifolds considered here are metrizable. Let t_i be an increasing infinite sequence of real numbers. Then a curve $\phi\left(\gamma_p\right)\in\phi\left(M\right)$ can be written as the sequence

$\phi\left(\gamma_p(t_i)\right)$. The limit operator

$$L : \phi(M)_{geo} \longrightarrow \phi(M)^l_{geo},$$

where $\phi(M)^l_g$ consists of equivalence classes of limit points of the sequences in $\phi(M)$, takes equivalence classes of curves in $\phi(M)_{geo}$ to equivalence classes of limit points in $\partial_\phi M$ under E_2. The equivalence classes are elements of the abstract boundary. The composition

$$L \circ r : \tilde{g} \longrightarrow \phi(M)^l_{geo} \tag{7.11}$$

given by

$$\left[\left(p, \xi^\alpha\right)\right]_{E_1} \mapsto [B], \tag{7.12}$$

sending the $\tilde{g}$ boundary point $\left[\left(p, \xi^\alpha\right)\right]_{E_1}$ to the associated equivalence class $[B]$, for some boundary set B in $\partial_\phi M$, of limit points in $\phi(M)^l_{geo}$. As B is a representative set of an abstract boundary point (from the definition of $\phi(M)_{geo}$), we can write the composite $L \circ r$ as the map

$$L \circ r : \tilde{g} \longrightarrow \bar{a},$$

defined by

$$\left[\left(p, \xi^\alpha\right)\right]_{E_1} \mapsto \left[\hat{p}\right], \tag{7.13}$$

where $\left[\hat{p}\right] \ni B$ and $\bar{a} \subseteq a$ is some subset of the abstract boundary a. The map $L \circ r$ is a bijection, and gives us a desired map from $\tilde{g}$ to a.

Theorem 7.3.11. *The composite map $L \circ r$ is a homeomorphism from $\tilde{g}$ onto its image in a.*

Proof. Clearly, $L \circ r$ is a bijection onto its image $L \circ r\left(\tilde{g}\right)$: any two equivalent points in $\left[\hat{p}\right]$ are endpoints/limit points of curves associated with equivalent points in G (under E_1). Since κ is a quotient map, κ is continuous. By Theorem 22.2 of Munkres (2000) r is also continuous. The limit operator L can be considered as a map sending collection of points in $\phi(M)$ (in this case those lying on a curve) to a point in the boundary of $\phi(M)$ in $\hat{M}$ (in this case the limit point of the curve on which those points lie). Then L is a quotient map and therefore continuous. Since the composition of continuous maps is continuous, $L \circ r$ is therefore continuous.

Since the boundary of a bounded set is compact, and the spacetime completion can be written as the union of M and $\tilde{g}$, we know that $\tilde{g}$ is compact. We also know that the abstract boundary a (and every subset of a) is Hausdorff. By theorem (26.6) of Munkres (2000), a continuous bijection from a compact space to a Hausdorff space is a homeomorphism. We therefore have that $L \circ r$ is a homeomorphism, i.e. $L \circ r$ embeds $\tilde{g}$ into a. $\square$

That this embedding might be a proper embedding is an important point to note since there might also be *unapproachable* boundary points (Scott & Szekeres, 1994) that are not approached by any curve. There might also be cases where the image $L \circ r\left(\tilde{g}\right) = \bar{a}$ coincides with the a boundary, and future work could consider under what conditions this happens. We stress that this construction can be made for any family of BPP curves (not just geodesics) yielding a generalization of Geroch's g boundary.

Via its deep relationship to the a boundary and the manner in which that boundary construction may be used (given some metric) to classify boundary points as *regular/singular/point at infinity/unapproachable*, our construction similarly allows for the classification of all approachable boundary points and not just singular ones, which could prove useful for future work where we might want to look at other boundary constructions built from the tangent bundle structure. We emphasize that, for the purposes of this paper, the focus is on maps of the form (7.13) since we are primarily interested in curves as they approach their associated limit points.

7.3.4 Simple illustrative examples

Simple examples that illustrate the subtle differences between our construction and Geroch's original construction include the Schwarzschild spacetime and Misner's simplified version of the Taub-NUT spacetime. For the Schwarzschild spacetime, the $\tilde{g}$ boundary is a single point, α, consisting of points in G whose associated geodesics all approach the surface $r = 0$, which is a point in the a boundary. By contrast, the g boundary is made up of two parts, each topologically $\mathbb{S}^2 \times \mathbb{R}$, for different approaches of the geodesics to $r = 0$ (Geroch, 1968b).

For Misner's simplified version of the Taub-NUT spacetime, $\tilde{g}$ consists of just one point, β, which is the equivalence class consisting of those points in G associated with all geodesics that approach $t = 0$. In contrast, the g boundary consists of three points, $\tilde{g}$, and two circles, C and C', (Geroch, 1968b), with a point on the circles representing those geodesics from both halves of the cylinder striking the circles at exactly one point.

Consider another simple example. Let ϕ embed the unit interval $(0,1)$ into $\mathbb{R}$ via the inclusion map. The boundary set $B = \{0,1\}$ is the boundary set of this envelopment. Now let a second envelopment, ϕ', embed $(0,1)$ in the unit circle via the map $\theta = 2\pi t$. The boundary points of the first envelopment are both identified with the boundary point 0 of the second envelopment and so 0 and 1 of the first envelopment are equivalent (under the equivalence relation defining the abstract boundary) to 0 of the second envelopment. The boundary set B is disconnected and so takes the discrete topology. Therefore, the singletons $\{0\}$ and $\{1\}$ are open sets containing each of the boundary points.

Suppose we are presented with just the envelopment ϕ. The boundary points 0 and 1 may wrongly be associated with different equivalence classes in the g boundary. With the knowledge of the second envelopment, it becomes clear that they are equivalent (since they are both equivalent to 0 in the second envelopment). We thus have 0 and 1 as elements of an equivalence class in the $\tilde{g}$ boundary.

A subtle point regarding the equivalence classes taken over all possible envelopments is illustrated by the following example from reference (Anonymous referee, 2019). Let L^2 be the 2-dimensional Minkowski space and define the subset $M = \{(x, y) \in L^2 : y < 0\}$ of L^2. Consider the inclusion $\phi : M \longrightarrow L^2$ as an envelopment, and take $(0,0) \in \partial_\phi M$. Let $O = I^-(0,0) \cap M$ (where I^- is taken in L^2), which is an open set. If one is presented with only this envelopment then the set $S(O)$ is the set of pairs (p, ξ) with $p \in O$ and ξ pointing to $(0,0)$ which is open since all points of $S(O)$ are identified. In fact the $\tilde{g}$ is made up of one point, $\partial_\phi M$ in L^2. This is a very simple example for which this construction would seem trivial. However, a crucial motivation of this construction (like that of the a boundary) would be missed. A single envelopment has been considered. Considering all possible envelopments would produce a richer $\tilde{g}$ boundary, as well as allowing for the classification of boundary points.

Let us state this in general terms, relative to the above example, an important reason for considering not just a single envelopment. Suppose we have a manifold M, and let ϕ, ϕ' be two envelopments of M. Let $\partial_\phi M = B$ be a regular boundary set (comprise of regular boundary points), and suppose $\hat{p} \in \partial_{\phi'} M$ is a singular boundary point which is equivalent to B under the abstract boundary equivalence. Then the point $[\hat{p}] \in \tilde{g}$ of M contains $\hat{p}$ and B.

Now, suppose one is *only* presented with the second envelopment ϕ'. One might wrongly conclude that the manifold M is singular. However, since B is regular and $B \sim \hat{p}$, by the classification of the different types of boundary points, the singularity of $\hat{p}$ is an artefact of the choice of envelopment.

7.4 The $\tilde{g}$ boundary appears robust

There are many potential applications of the relationship between the a and g boundary constructions (see the conclusion), and most will involve considerable work. As a first application, we consider one important outstanding question, namely whether all boundary constructions are necessarily pathological for a suitably contrived example, such as that presented in Geroch *et al.* (1982).

The example spacetime is constructed as follows: in the two dimensional Minkowski spacetime N with the usual 2-metric, let s be the endpoint of a future directed timelike geodesic γ_p which starts at p, and let r be a point in N which lies on a future directed null geodesic in N. Take the cross product of $N \setminus \{s\}$ and the 2-dimensional spacelike plane.

Now, let M be a spacetime and let $\phi : M \longrightarrow \hat{M}$ embed M into $\hat{M}$. Furthermore, let $\overline{\phi(M)} = \phi(M) \cup \partial_\phi M_I$ be a spacetime in which $\phi(M)$ is dense, where $\partial_\phi M_I$ is a subset of $\partial_\phi M$ containing points that are endpoints/limit points of incomplete geodesics in $\phi(M)$.

Suppose we have a singular boundary construction satisfying the following conditions:

a. For every incomplete geodesic γ_p associated with a point $\left(p, \xi^\alpha\right) \in G$, there exists a point $\hat{p} \in \partial_\phi M_I$ such that $\phi\left(\gamma_p\right)$ limits to $\hat{p}$,

b. the extension $\overline{exp} : \bar{U} \longrightarrow \overline{\phi(M)}$ of the exponential map $exp : U \longrightarrow \phi(M)$, which take points of G associated with incomplete geodesics of affine length exactly one to their endpoints/limit points in $\overline{\phi(M)}$, is continuous.

Choosing a sequence of curves from p to r that converge to γ_p, the authors in Geroch *et* *al.* (1982) showed that r limits to s and thus any open neighborhood of the singular point s will contain the regular point r. The justification is also dependent on certain smoothness condition on open neighborhoods around the "bends" where the sequences of curves are near s. The choice is to have the open neighborhoods "smooth everywhere except s where it accumulates".

Now, let us consider the $\tilde{g}$ boundary and this example. Restrict G to points associated with incomplete geodesics and let the topology on G be that for which the $S(O)$ sets are a basis. Moreover, let the topology on this subset of G be the induced topology from G. Then clearly our construction satisfies the above two conditions: of course, by our construction, every point $\hat{p} \in \partial_\phi M_I$ is an endpoint/limit point of an incomplete geodesic $\phi(\gamma_p)$. Also, the inverse image of $(\phi(O), V)$ - where V is open in $\partial_\phi M_I$ - under $\overline{exp}$ is just $S(O)$, which is open in G. The T_2-separability of the manifold completion with our construction ensures that there are open neighborhoods of the regular point r and the singular point s such that r and s remain T_2-separated which implies that this example is therefore T_1 separated. This seems to suggest that the non-T_1 separation problem of this example experienced by the Geroch's g boundary and the b boundary constructions is inherited from the topological separation problems inherent in these boundary constructions themselves, rather than be symptomatic of *all* boundary constructions satisfying the two conditions.

7.5 Conclusion

The aim of this paper was to construct a T_2-separable g boundary, $\tilde{g}$, and establish a relationship between this g boundary and the a boundary. This important open problem (Scott & Szekeres, 1994) has been solved. Our construction, in essence, has rephrased the a boundary in terms of the reduced tangent bundle. We have built a boundary construction using the tangent bundle which inherits many of the generalizations from the a boundary, i.e. applicability to smooth manifolds of any dimension endowed with an affine connection and BPP family of curves. Moreover, we potentially have a means of generalizing other constructions built from the tangent bundle structure. Specializing to geodesics we obtain a narrower generalization of Geroch boundary. We have also shown that regular spacetime points remain separated from points in $\tilde{g}$ upon completing the spacetime with $\tilde{g}$. In order to achieve this, properties of the attached point topology on the a boundary were used to define new open sets on G. It was then shown that the new boundary $\tilde{g}$, constructed from equivalences classes defined on the set G, is Hausdorff in this topology. In the process of obtaining the definition of the equivalence relation for our construction, all other identifications we attempted were non-Hausdorff.

We also applied our construction to the example of Geroch *et al.* (1982), and showed that the $\tilde{g}$-boundary is free of the topological separation problem experienced by other constructions, thus refuting their conjecture that such problems are endemic to all boundary constructions of a certain class. It may be that $\tilde{g}$ is the only pertinent subset of the a boundary with respect to the two conditions of Geroch *et al* (1982). If that is the case, this might provide a route to establishing a relationship between the b and a boundaries.

Now that we have a relationship between the $\tilde{g}$ and the a boundary, there are many further interesting problems that come to mind. Among these, we identify the following as exciting possible future work:

1. Can the causal, metric and differentiable structures defined on g (Geroch, 1968b) be extended to $\tilde{g}$?

2. If the answer to 1. is yes, can our construction then provide a way to construct causal, metrical and differentiable structures on the a boundary?

3. Again, if the answer to 2. is in the affirmative, what new information, if any, can we obtain

from these structures on the a boundary?

Finally, an obvious problem to consider is the relationship between the boundaries considered in this paper to other boundary constructions.

Chapter 8

Conclusion

We have dealt with general properties of black hole spacetimes and spacetime singularities in this thesis. We have also considered stars with zero expansion, as well as the geometry of what we named minimal horizons (as they are foliated by minimal 2-surfaces). We detail the conclusions of the chapters of the thesis.

In chapters 3 and 4 we substantially extended earlier work (Ellis *et al.,* 2014) on the employment of the $1+1+2$ semitetrad covariant formalism to study a range of properties of marginally trapped surfaces and the hypersurfaces they foliate in black hole spacetimes. We have established a result on the equivalence, up to diffeomorphism, of marginally trapped surfaces (and the 3-surfaces they foliate) in LRS II spacetimes and a broader class of spacetimes which generalizes LRS II spacetimes (which we notate as NNF). From generalizing the determination of the sign of the intrinsic metric on the horizon, we proposed a causal classification scheme, that relates (causally) 3-surfaces foliated by marginally trapped surfaces in the LRS II, NNF, and a general 4-dimensional spacetimes. We then went on to provide a detailed study of stability of the 2-surfaces in terms of well defined geometric and thermodynamic variables, while providing explicit examples to demonstrate the usefulness of our approach. We showed that the cross sections of the event horizon in the Schwarzschild spacetime are stable, and those of the inner MTT in the Oppenheimer-Snyder dust model are unstable. Stability of these 2-surfaces in the Robertson-Walker spacetimes is characterized. We also considered the surface gravity κ on the horizon. For isolated horizons, it is seen that the surface gravity is invariant under the diffeomorphism between LRS II and NNF spacetimes. This is not the case in general since we do not have a general causal equivalence between LRS II and NNF spacetimes (in the nonisolated case the surface gravity depends on the causal signature of the horizon). The classification scheme, however, does provide us with a way to determine the conditions under which κ is invariant in general cases.

In chapter 5 we have explicitly shown that a necessary condition for a star with zero expansion to evolve is that the star has non-zero radiation and acceleration. With further analysis of the field equations for $A \neq 0$ and $Q \neq 0$, it is shown that the star is necessarily conformally flat. The analyses of the field equations were carried out using the commutation relation of the hat and dot derivatives, via which we obtain additional constraints, which further give us additional evolution equations that can be matched against the original set of equations. These results add to the literature on expansion-free dynamical stars, which have been developed over the last decade and half, most notably through the works of Herrera and co-authors.

In chapter 6 we studied the geometry of hypersurfaces in LRS II spacetimes and a generalization of LRS II spacetimes, the NNF spacetimes, foliated by minimal 2-surfaces, i.e. surfaces of vanishing mean curvature vector. Resolving the field equations, we showed that the Gaussian curvature K is just the energy density and that it is constant, This allowed us to classify the 2-surfaces. Since under the minimality condition the shape tensor is also identically zero on the 2-surfaces, from the result in Chern *et al.*, (1970), we obtain the result that all minimal horizons in LRS II spacetimes are geometrically S^3, i.e. have spherical symmetry. This result extends to NNF spacetimes.

In chapter 7 we solved an outstanding problem proposed in Scott & Szekeres (1994), which has to do with establishing a relationship between the a-boundary and other boundary constructions. The main difficulty was that different approaches are used in constructing the different boundaries, and therefore relating any two constructions would required relating the approaches. We also saw this as a very important problem since in many instances each construction has its own unique benefits of application, and if we could relate any two, these properties could in principle be carried over to the other. We went on to construct a proper embedding of a modified g-boundary, which we denoted by $\tilde{g}$, into the a-boundary by redefining the open sets forming the basis of a topology on the reduced tangent bundle, making use of properties of the attached point topology of the a-boundary. The method of constructing $\tilde{g}$ to embed it into the a-boundary naturally meant that $\tilde{g}$ was Hausdorff, and this was explicitly proved. Once such relationship is established we are led to additional follow up questions, and these we setup up as possible future avenues for research.

1. Can the causal, metric and differentiable structures defined on g (Geroch, 1968b) be extended to $\tilde{g}$?

2. If the answer to 1. is yes, can our construction then provide a way to construct causal, metrical and differentiable structures on the a boundary?

3. Again, if the answer to 2. is in the affirmative, what new information, if any, can we obtain from these structures on the a boundary?

4. Can the method employed here be used to establish relationships between other constructions?

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