UNIVERSITY OF KWAZULU-NATAL

LIE SYMMETRIES OF JUNCTION CONDITIONS FOR RADIATING STARS

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Abstract

We consider shear-free radiating spherical stars in general relativity. In particular we study the junction condition relating the pressure to the heat flux at the boundary of the star. This is a nonlinear equation in the metric functions. We analyse the junction condition when the spacetime is conformally flat, and when the particles are travelling in geodesic motion. We transform the governing equation using the method of Lie analysis. The Lie symmetry generators that leave the equation invariant are identified and we generate the optimal system in each case. Each element of the optimal system is used to reduce the partial differential equation to an ordinary differential equation which is further analysed. As a result, particular solutions to the junction condition are presented. These exact solutions can be presented in terms of elementary functions. Many of the solutions found are new and could be useful in the modelling process. Our analysis is the first comprehensive treatment of the boundary condition using a symmetry approach. We have shown that this approach is useful in generating new results.

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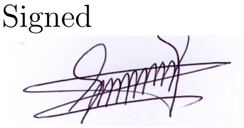

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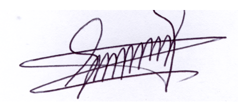

December 2011

To

my late father Zewdie Abebe

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Chapter 1

Introduction

Einstein's theory of general relativity is founded on four-dimensional spacetime and is best expressed in the language of differential geometry. In the four-dimensional spacetime manifold the nonsingular symmetric metric tensor field is a gravitational field carrier. The spacetime is locally similar to flat special relativity but is globally curved. The Einstein tensor which arises from the Riemann tensor contains vital information about the geometry of the spacetime manifold. The gravitational field is described by the Einstein field equations which couple the Einstein tensor to the energy momentum tensor which describes the energy momentum of a small region of spacetime. Results from the theory of general relativity are very important in relativistic astrophysics and cosmology for predicting and describing physical phenomena such as bending of light, gravitational lensing, black hole formation, gravitational red shift, the cosmic microwave background, gravitational waves and so on. In this thesis we solve the junction condition of shear-free radiating spherically symmetric relativistic models by using the method of Lie analysis of differential equations (Sean 1997, Stephani 2004, Wald 1984).

Differential equations connect the problems of real life with calculus and are used as a language to express the laws of nature. Many laws of nature in chemistry, in biology, in engineering and physics find their natural expression in the language of differentials. Differentials were invented as part of differential and integral calculus. Newton solved a differential equation by using an infinite series method in 1676. However, the results were not published until 1693. In the same year, differential equations occurred for the first time in Leibniz's work which was published in 1684 in the account of differential calculus. Sophus Lie introduced the idea of the continuous transformation group for differential equations, which is today known as the Lie group. The theory of Lie groups and Lie algebras have been developed into one of the most important areas of applications in mathematics and physics. One of the most important applications of the Lie theory of symmetry groups for differential equations is the construction of group invariant solutions (Bluman and Kumei 1989, Olver 1993, Miwa 2000). We concentrate on solving the partial differential equations which govern the gravitational behaviour of a radiating star at the stellar surface in relativistic astrophysics.

This thesis has six chapters:

- Chapter One: Introduction.

- Chapter Two: In this chapter we briefly describe one-parameter Lie group transformations in the (x, y) plane. Extended generators and their use are also discussed. The uses of generators discussed in this chapter are reduction of order, transformation of partial differential equations to ordinary differential equations, and extending known solutions of differential equations.

  Lie algebras are an important component of this thesis, and that is briefly discussed in this chapter. One of the uses of Lie algebra is its importance in determining the optimal system which are widely used in chapters four and chapter five by judicious application of the action of adjoint maps.

- Chapter Three: In this chapter we briefly describe differential geometry, curvature and the field equations in the framework of the spacetime of general relativity. We establish the general junction condition equation based on the treatment of Santos (1985). We consider the fundamental equation relating the pressure to the heat flux at the stellar boundary and discuss some of the fundamental results obtained for this model.

- Chapter Four: In this chapter we generate the junction condition equation for conformally flat spacetimes. Several new classes of exact solutions are found using the Lie analysis approach.

- Chapter Five: We generate the junction condition equation for geodesic models. A number of new exact solutions to the boundary condition are found using Lie methods.

- Chapter Six: In this chapter we discuss and summarise the entire work of the thesis.

Chapter 2

Lie symmetry analysis

2.1 Introduction

In this chapter, we briefly outline the technique of Lie symmetry analysis applied to differential equations. We will employ this method in the solution of differential equations (DEs) that arise from relativistic models in later chapters. In particular, the junction condition relating pressure and heat flux in shear-free radiating stars is to be studied. The strength of this approach lies mainly in the ability of the technique to solve DEs in a systematic manner using symmetries. By a symmetry, we mean the generator of a transformation which leaves the form of the DE invariant. The main application of Lie point symmetries is in finding general solutions by the reduction of an n^{th} order ordinary differential equation (ODE) through its symmetries to an $(n-1)^{th}$ order ODE. The hope is that the reduced equation will then be solvable. Another method of solving DEs is via transforming the dependent or independent variable. This can make the resultant DE a simpler equation on substitution of these new variables. When the transformation depends on the variables alone, it is called a point transformation. This is the transformation we shall concern ourselves with here, though other types of transformations exist (Olver 1993, Miwa 2000).

In section 2.2 we introduce a one-parameter group of transformations in the (x, y) plane. The infinitesimal transformation is obtained via Taylor series expansions about $\epsilon = 0$. In section 2.3 we briefly discuss extending generators for one dependent and one independent variable in ODEs and one dependent and k independent variables in partial differential equations (PDEs). We discuss Lie algebras in section 2.4. In this section we also illustrate the Lie bracket relation using the symmetries admitted by the Korteweg-de Vries equation. In section 2.5 we demonstrate uses of Lie symmetries. These are reduction of order of ODEs and group invariant solutions for PDEs. Crucial to group invariant solutions in the concept of an optimal system. We illustrate the optimal system for the symmetries admitted by the Korteweg-de Vries equation.

2.2 Infinitesimal generators

Let us consider an invertible one-parameter group of transformations

$$x_1 = f(x, y, \epsilon) \tag{2.1a}$$

$$y_1 = g(x, y, \epsilon) \tag{2.1b}$$

of the (x, y) plane (Hydon 2000). The functions $f(x, y, \epsilon)$ and $g(x, y, \epsilon)$ are collectively called the global form of the group. The infinitesimal transformations of functions x and y can be approximately estimated, via Taylor series expansion about $\epsilon = 0$, as

$$x_1 \approx x + \epsilon \left(\frac{dx_1}{d\epsilon} \right)_{\epsilon=0} \tag{2.2a}$$

$$y_1 \approx y + \epsilon \left(\frac{dy_1}{d\epsilon} \right)_{\epsilon=0} \tag{2.2b}$$

We call (2.2) the infinitesimal form of the group.

Now if we introduce new functions ξ and η as

$$\left(\frac{dx_1}{d\epsilon}\right)_{\epsilon=0} = \xi(x,y) \tag{2.3a}$$

$$\left(\frac{dy_1}{d\epsilon}\right)_{\epsilon=0} = \eta(x,y) \tag{2.3b}$$

then we obtain

$$x_1 = x + \epsilon\xi(x,y) \tag{2.4a}$$

$$y_1 = y + \epsilon\eta(x,y) \tag{2.4b}$$

Note that we use the equal sign above for convenience. In reality this is an approximation as we have ignored the higher order terms. If we define the operator G as

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{2.5}$$

then we can write the infinitesimal transformations (2.4) as

$$x_1 = (1+\epsilon G)x \tag{2.6a}$$

$$y_1 = (1+\epsilon G)y \tag{2.6b}$$

Note that once we find G we can obtain the global form of the transformation that leaves the equation invariant directly. We merely need to integrate (2.3) (in general not just at $\epsilon = 0$) subject to

$$x_1 = x \tag{2.7a}$$

$$y_1 = y \tag{2.7b}$$

at $\epsilon = 0$. These transformations can be used to generate new solutions to the original DE under analysis (Bluman *et al.* 2010).

2.3 Extended generators

In the above we provided transformations of the variables only. In dealing with DEs we need to consider the transformations of the derivatives too. As a result we need to consider the extension or prolongation of G to accommodate derivatives.

2.3.1 One dependent and one independent variable

If we take (2.4) into account we find that the derivatives transform as

$$\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}x_1} &= y' + \epsilon(\eta' - y'\xi') && (2.8a)\\
\frac{\mathrm{d}^2y_1}{\mathrm{d}x_1^2} &= y'' + \epsilon(\eta'' - 2y''\xi' - y'\xi'') && (2.8b)\\
\vdots \quad & \quad \vdots \\
\frac{\mathrm{d}^n y_1}{\mathrm{d}x_1^n} &= y^n + \epsilon\left(\eta^{(n)} - \sum_{i=0}^{n-1}\begin{pmatrix} n \\ i \end{pmatrix} y^{(i+1)}\xi^{(n-i)}\right) && (2.8c)
\end{aligned}$$

To indicate that a generator G has been extended (prolonged) one can write (Bluman and Cole 1974, Bluman and Kumei 1989, Hill 1992)

$$\begin{aligned}
G^{[1]} &= G + (\eta' - y'\xi')\frac{\partial}{\partial y'} && (2.9a)\\
G^{[2]} &= G^{[1]} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''} && (2.9b)\\
\vdots \quad & \quad \vdots \\
G^{[n]} &= G + \sum_{i=1}^{n}\left(\eta^{(i)} - \sum_{j=0}^{i-1}\begin{pmatrix} i \\ j \end{pmatrix} y^{(j+1)}\xi^{(i-j)}\right)\frac{\partial}{\partial y^{(i)}} && (2.9c)
\end{aligned}$$

Note that

$$\xi' = \frac{\partial \xi}{\partial x} + y'\frac{\partial \xi}{\partial y} \qquad (2.10)$$

in the case of the first derivative, and

$$\xi'' = \frac{\partial^2 \xi}{\partial x^2} + 2y' \frac{\partial^2 \xi}{\partial x \partial y} + y'^2 \frac{\partial^2 \xi}{\partial y^2} + y'' \frac{\partial \xi}{\partial y} \tag{2.11}$$

in the case of the second derivative, and so on. Similar expressions are obtained for the derivatives of η. A differential equation

$$E(x, y, y', y'', ..., y^n) = 0$$

is said to possess (admit) a symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.12}$$

if and only if

$$G^{[n]}E\big|_{E=0} = 0 \tag{2.13}$$

This means the action of the n^{th} extension (prolongation) of G on E is zero when the original equation is satisfied.

2.3.2 One dependent and k independent variables

The one-parameter Lie group of transformations

$$\overline{x}_i = x_i + \epsilon\xi_i(x, u) + O(\epsilon^2) \tag{2.14a}$$

$$\overline{u} = u + \epsilon\eta(x, u) + O(\epsilon^2) \tag{2.14b}$$

where $x = (x_1, x_2, \ldots, x_k)$ are independent variables and u is the dependent variable such that $i = 1, 2, \ldots, k$ acting on (x, u) space, has its generator given by

$$G = \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} \tag{2.15}$$

The corresponding n^{th} extended generator is given as

$$\begin{aligned} G^{[n]} &= \xi_i(x,u)\frac{\partial}{\partial x_i} + \eta(x,u)\frac{\partial}{\partial u} + \eta_i^{(1)}(x,u,u_1)\frac{\partial}{\partial u_i} + \cdots \\ &\quad + \eta^{(n)}_{i_1 i_2 \ldots i_n}(x,u,u_1,\ldots,u_n)\frac{\partial}{\partial u_{i_1 i_2 \ldots i_n}} \end{aligned} \tag{2.16}$$

with $n = 1, 2, \ldots$. The explicit formula for the function $\eta^{(n)}$ is given by (Bluman and Anco 2002, Bluman and Kumei 1989)

$$\eta_i^{(1)} = D_i\eta - (D_i\xi_i)u_j, \qquad i = 1, 2, \ldots, n \tag{2.17a}$$

$$\eta^{(n)}_{i_1 i_2 \ldots i_n} = D_{i_n}\eta^{n-1}_{i_1 i_2 \ldots i_{n-1}} - (D_{i_n}\xi_j)u_{i_1 i_2 \ldots i_{n-1} j} \tag{2.17b}$$

where $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, n$ with $n = 2, 3, \ldots$ and D_i is the total derivative defined as

$$D_i = \frac{D}{D_i} = \frac{\partial}{\partial x_i} + u_i\frac{\partial}{\partial u} + u_{ij}\frac{\partial}{\partial u_j} + \cdots + u_{i i_1 i_2 \cdots i_k}\frac{\partial}{\partial u_{i_1 i_2 \cdots i_k}} + \cdots \tag{2.18}$$

For instance, for $i = 1, 2$ we have the generator

$$G = \xi_1 \left(x_1, x_2, u \right) \frac{\partial}{\partial x_1} + \xi_2 \left(x_1, x_2, u \right) \frac{\partial}{\partial x_2} + \eta \left(x_1, x_2, u \right) \frac{\partial}{\partial u} \tag{2.19}$$

and the first prolongation of G becomes

$$G^{[1]} = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + \eta \frac{\partial}{\partial u} + \eta_1^{(1)} \frac{\partial}{\partial u_{x_1}} + \eta_2^{(1)} \frac{\partial}{\partial u_{x_2}} \tag{2.20}$$

where

$$\eta_1^{(1)} = \frac{\partial \eta}{\partial x_1} + u_{x_1} \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi_1}{\partial x_1} \right) - u_{x_2} \frac{\partial \xi_2}{\partial x_1} - u_{x_1}^2 \frac{\partial \xi_1}{\partial u} - u_{x_1} u_{x_2} \frac{\partial \xi_2}{\partial u}$$

and

$$\eta_2^{(1)} = \frac{\partial \eta}{\partial x_2} + u_{x_2} \left(\frac{\partial \eta}{\partial u} - \frac{\partial \xi_2}{\partial x_2} \right) - u_{x_1} \frac{\partial \xi_1}{\partial x_2} - u_{x_2}^2 \frac{\partial \xi_2}{\partial u} - u_{x_1} u_{x_2} \frac{\partial \xi_1}{\partial u}$$

By using the same technique one can find further extended infinitesimal generators for different values of i.

2.4 Lie algebras

The important point to consider in our analysis is that the symmetries of a differential equation form a Lie algebra.

A Lie algebra $\mathcal{L}$ is a vector space over a field $\mathcal{F}$ augmented by a bilinear composition law $[,]$, known as the Lie bracket for $\mathcal{L}$, such that the following properties hold (Pfeifer 2003):

(i) Bilinearity:

If G_1, G_2 and G_3 are in $\mathcal{L}$ then

$$[\alpha G_1 + \beta G_2, G_3] = \alpha[G_1, G_3] + \beta[G_2, G_3] \tag{2.21a}$$

$$[G_1, \alpha G_2 + \beta G_3] = \alpha[G_1, G_2] + \beta[G_1, G_3] \tag{2.21b}$$

where α, β are constants.

(ii) Anti-commutativity (skew-symmetric):

If G_1 and G_2 are in $\mathcal{L}$ then

$$[G_1, G_2] = -[G_2, G_1] \tag{2.22}$$

(iii) The Jacobi identity:

If G_1, G_2 and G_3 are in $\mathcal{L}$ then

$$[[G_1, G_2], G_3] + [[G_2, G_3], G_1] + [[G_3, G_1], G_2] = 0 \tag{2.23}$$

We define the product associated with the Lie algebra as that of commutation

$$[G_i, G_j] = G_i G_j - G_j G_i \tag{2.24}$$

where $i, j = 1, 2, 3, \ldots$ (Hill 1992, Ibragimov 1994, Olver 1993).

For instance, the Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} + u u_x = 0 \tag{2.25}$$

$[G_i, G_j]$	G_1	G_2	G_3	G_4
G_1	0	0	0	G_1
G_2	0	0	G_1	$3G_2$
G_3	0	$-G_1$	0	$-2G_3$
G_4	$-G_1$	$-3G_1$	$2G_3$	0

Table 2.1: Commutation table of symmetries in (2.26)

has the following symmetries

$$G_1 = \frac{\partial}{\partial x} \tag{2.26a}$$

$$G_2 = \frac{\partial}{\partial t} \tag{2.26b}$$

$$G_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \tag{2.26c}$$

$$G_4 = x\frac{\partial}{\partial x} + 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} \tag{2.26d}$$

By using (2.24), one can construct a commutation table for these symmetries (see Table 2.1).

2.5 Use of symmetries

A symmetry group of a system of differential equations is a Lie group acting on the space of independent and dependent variables in such a way that solutions are mapped into other solutions. Knowing the symmetry group allows one to determine some special types of solutions that are invariant under a subgroup of the full symmetry group, and in some cases one can solve the equations completely (Hydon 2000).

2.5.1 Reduction of order

The most important use of symmetries is in the reduction of order of an ODE. The main idea is that one can apply a symmetry to the equation to reduce the order of equation from order n, say, to order $n-1$ (Bluman *et al.* 2010, Boris *et al.* 2009, Hill 1992). If an n^{th} order ODE

$$E(x, y, y', \ldots, y^{(n)}) = y^{(n)} - f(x, y, y, \ldots, y^{(n-1)}) = 0 \tag{2.27}$$

is invariant under the symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2.28}$$

then it follows that E must satisfy the following system of equations

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{(\eta' - y'\xi')} \tag{2.29}$$

The solution of the first equation gives the group invariant $u(x, y)$ while the solution of the remaining equations give the first differential invariant $v(x, y, y')$. This implies that E must be a function of these invariants for it to admit (2.28). In terms of the invariants $u(x,y)$ and $v(x,y,y')$, the original equation E can be written as

$$N(u,v,v',\ldots,v^{(n-1)}) = v^{(n-1)} - g(u,v,v',\ldots,v^{(n-2)}) = 0 \tag{2.30}$$

Thus the variables for the reduction of order are obtained by requiring

$$G^{[1]}z = 0 \tag{2.31}$$

where $z = z(x,y,y')$ is an arbitrary function of its arguments. The operation (2.31) results in the equation

$$\xi\frac{\partial z}{\partial x} + \eta\frac{\partial z}{\partial y} + (\eta' - y'\xi')\frac{\partial z}{\partial y'} = 0 \tag{2.32}$$

and has the invariant surface condition

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{(\eta' - y'\xi')} \tag{2.33}$$

The group invariant u is also called the zeroth order differential invariant, and we call v the first order differential invariant. The terminology follows from

$$Gu(x,y) = 0 \tag{2.34}$$

and

$$G^{[1]}v(x,y,y') = 0 \tag{2.35}$$

It is well known that one can then apply the symmetries successively to the equation to reduce the equation, hopefully from order n to $n-1$, then $n-2$ up to order 0. This gives, by reversing the transformations, the general solution of the DE (Hill 1992).

2.5.2 Group invariant solutions of PDEs

A solution of the system of partial differential equations is said to be G-invariant if it is unchanged by all the group transformations in G. In general, to each s-parameter subgroup H of the full symmetry group g of a system of differential equations, there will correspond a family of group invariant solutions. In this section, we seek to utilise the Lie point symmetries in a systematic manner to generate new solutions. These new solutions are termed group invariant solutions as they will be invariant under the group generated by the symmetry used to find them. The advantage of using Lie point symmetries is that we are guaranteed that the variable combinations obtained will always result in an equation in the new variables. No further consistency conditions are needed. As we are dealing with a partial differential equation in two independent variables here, we will always be able to find an ordinary differential equation in the new variables defined via the symmetries (Olver 1993, Bluman *et al.* 2010)

Since there are almost always a large number of such subgroups, it is not usually feasible to list all possible group invariant solutions to the system. We need an effective systematic means of classifying these solutions, leading to an optimal system of group invariant solutions from which every other solution can be derived (Olver 1993).

Optimal System

Two invariant solutions are equivalent if one can be mapped to the other by a point symmetry of the PDE. Classifying the symmetries of a PDE greatly simplifies the problem of determining all invariant solutions by using the adjoint action representation of symmetries. This is given by simply summing up the Lie series as

$$\begin{aligned} \mathrm{Ad}\left(\exp\left(\epsilon G\right)\right) H &= \sum_{n=0}^{\infty} \frac{\epsilon^n}{n!} \left(\mathrm{Ad}(G)\right)^n H \\ &= H - \epsilon\left[G, H\right] + \frac{\epsilon^2}{2}\left[G, \left[G, H\right]\right] - \frac{\epsilon^3}{3!}\left[G, \left[G, \left[G, H\right]\right]\right] + \cdots \end{aligned} \tag{2.36}$$

for symmetries G and H (Olver 1993).

For instance, from the symmetries in (2.26), the action of the symmetry G_2 on G_4 is given as

$$\begin{aligned} \mathrm{Ad}\left(\exp\left(\epsilon G_2\right)\right) G_4 &= G_4 - \epsilon\left[G_2, G_4\right] + \frac{\epsilon^2}{2}\left[G_2, \left[G_2, G_4\right]\right] \\ &\quad -\frac{\epsilon^3}{3!}\left[G_2, \left[G_2, \left[G_2, G_4\right]\right]\right] + \cdots \\ &= G_4 - 3\epsilon G_2 \end{aligned} \tag{2.37}$$

In this manner, one can construct the adjoint representation table for the symmetries in (2.26) (see Table 2.2).

It is only necessary to find one (general) invariant solution from each class; then the whole class can be constructed by applying the symmetries. This strategy minimizes the effort needed to obtain invariant solutions.

We aim to classify invariant solutions by classifying the associated symmetry generators. Having done so, one generator from each class is used to obtain the desired

Ad	G_1	G_2	G_3	G_4
G_1	G_1	G_2	G_3	$G_4 - \epsilon G_1$
G_2	G_1	G_2	$G_3 - \epsilon G_1$	$G_4 - 3\epsilon G_2$
G_3	G_1	$G_2 + \epsilon G_1$	G_3	$G_4 + 2\epsilon G_3$
G_4	$e^{\epsilon} G_1$	$e^{3\epsilon} G_2$	$e^{-2\epsilon} G_3$	G_4

Table 2.2: Adjoint representation table of symmetries in (2.26)

set of invariant solutions. A set consisting of exactly one generator from each class is called an optimal system of generators.

An optimal system of a Lie algebra is a set of conjugacy equivalent n-dimensional subalgebras such that every other n-dimensional subalgebra is equivalent to a unique element of the set under some element of the adjoint representation (Hydon 2000, Olver 1993).

Let us consider the symmetries (2.26). Consider a nonzero vector

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3 + a_4 G_4 \tag{2.38}$$

Our aim is to simplify as many coefficients a_i as possible through adjoint maps to G.

Suppose first that $a_4 \neq 0$. We can set $a_4 = 1$ without loss of generality. By referring to the adjoint table

$$\tilde{G} = \mathrm{Ad}(\exp(-\frac{1}{2} a_3 G_3))G = a_1' G_1 + a_2' G_2 + G_4 \tag{2.39}$$

Here we have removed G_3. Next we act on $\tilde{G}$ as follows

$$\tilde{\tilde{G}} = \mathrm{Ad}(\exp(\frac{1}{3}a_2' G_2))\tilde{G} = a_1' G_1 + G_4 \tag{2.40}$$

Finally,

$$\tilde{\tilde{\tilde{G}}} = \mathrm{Ad}(\exp(a_1' G_1))\tilde{\tilde{G}} = G_4 \tag{2.41}$$

cancels the remaining symmetry G_1 and G is equivalent to G_4. This means that if we find a solution via G_4 it is equivalent to a solution via G in (2.38).

The remaining subalgebras are spanned by vectors of the above form with $a_4 = 0$. Considering $a_3 \neq 0$ and setting $a_3 = 1$, we act on G in equation (2.38) and so on.

By applying the same technique we can find the optimal system of one dimensional subalgebras

$$\{G_1, G_2, G_3, G_4, G_3 + G_2, G_3 - G_2\} \tag{2.42}$$

for the symmetries in (2.26).

When we take into account the fact that (2.25) is invariant under the involutions $(x \to -x, t \to -t)$ we reduce (2.43) to

$$\{G_1, G_2, G_3, G_4, G_3 + G_2\} \tag{2.43}$$

Chapter 3

Radiating models

3.1 Introduction

In this chapter, we introduce crucial aspects of differential geometry and the field equations which are very important for general relativity. These include tensorial and nontensorial quantities. In section 3.2 we define the metric tensor, the connection coefficients, the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor. The metric tensor contains information about gravity and replaces the Newtonian gravitational field. The connection coefficients are defined in terms of the metric tensor. The Riemann tensor is constructed completely from the connection and its derivatives. In section 3.3 we introduce the energy momentum tensor which describes the matter content. The energy momentum tensor is related to the Ricci tensor via Einstein's equations. For more detailed information the reader is referred to Malcolm (2004), Sean (1997), Stephani (2004) and Wald (1984). In section 3.4 we define a spherically symmetric shear-free spacetime. The Einstein field equations are derived for this metric. The condition for pressure isotropy is derived and presented in several equivalent forms. We introduce the Vaidya (1951) solution, in section 3.5, which describes the exterior spacetime of a radiating body. The matching of the exterior Vaidya solution and the interior spacetime along the boundary of radiating star is discussed. The fundamental junction condition equation, a highly nonlinear partial differential equation, is obtained for a shear-free spherical spacetime. We also review some of the exact models for a relativistic radiating star.

3.2 Differential geometry

We consider spacetime to be a differentiable manifold with real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ such that x^0 is a timelike coordinate and x^1, x^2, x^3 are spacelike coordinates. We also assume that the spacetime manifold is four-dimensional, Hausdorff and connected. It is important to introduce the metric tensor field $\mathbf{g}$ to consider the metrical properties of the manifold. The invariant distance between neighbouring points on a curve in a given manifold is defined by the line element

$$ds^2 = g_{ab} dx^a dx^b \tag{3.1}$$

where $\mathbf{g}$ is the symmetric, nondegenerate metric tensor field. We take the signature to be $(-+++)$. The fundamental theorem of Riemannian geometry implies the existence of a unique metric connection Γ. The metric connection coefficient Γ is defined as

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left(g_{cd,b} + g_{db,c} - g_{bc,d} \right) \tag{3.2}$$

where the commas denote partial differentiation. The quantities $\Gamma^a{}_{bc}$ are also called the Christoffel symbols of the second kind. Equation (3.2) is an important formula for general relativity.

The Riemann tensor $\mathbf{R}$ is a $(1, 3)$ tensor and it is expressed in terms of the connection coefficients and its derivatives. It is defined by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc} \tag{3.3}$$

The Riemann tensor is also called the curvature tensor as it provides a measure of the curvature of the manifold in which the tensor is defined. A manifold is flat if all components of the Riemann tensor vanish; otherwise it is curved. The Riemann tensor is constructed from the Christoffel symbols and its partial derivatives which are nontensorial. However, the Christoffel symbols are arranged so that the final result transforms as a tensor (Edmund 2000, Sean 1997). The Ricci tensor is obtained by contraction of the Riemann tensor. The Ricci tensor is defined by

$$R_{ab} = R^c{}_{acb}$$

Then from (3.3) we have

$$R_{ab} = \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^d{}_{ed}\Gamma^e{}_{ab} - \Gamma^d{}_{eb}\Gamma^e{}_{ad} \tag{3.4}$$

which is a symmetric tensor. The contraction

$$\begin{aligned} R &= g^{ab}R_{ab} \\ &= R^a{}_a \end{aligned} \tag{3.5}$$

is called the Ricci scalar or curvature scalar. The Einstein tensor is defined by

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \tag{3.6}$$

in terms of the Ricci tensor, the Ricci scalar and the metric tensor. The Einstein tensor is a symmetric tensor and has the property of being divergence free. This means that

$$G^{ab}{}_{;b} = 0$$

where semi-colons denote covariant derivatives. This property of the Einstein tensor is sometimes referred to in the literature as the Bianchi identity, and generates conservation laws through the Einstein field equations.

3.3 Matter fields

The matter content is represented by the energy momentum tensor **T**. The energy momentum tensor is given by

$$T_{ab} = (\mu + p)\, u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{3.7}$$

where μ is the energy density, p is the isotropic (kinetic) pressure, **q** is the heat flux vector, and π^{ab} is the anisotropic stress tensor which is trace free ($\pi^a{}_a = 0$). These quantities are measured relative to a fluid-four velocity **u** ($u^a u_a = -1$) such that

$$q^a u_a = 0$$

$$\pi^{ab} u_b = 0$$

For perfect fluids there is no heat conduction ($q^a = 0$) and anisotropic stress ($\pi^{ab} = 0$) terms, and (3.7) becomes

$$T_{ab} = (\mu + p)\, u_a u_b + p g_{ab} \tag{3.8}$$

On physical grounds we can impose a barotropic equation of state $p = p(\mu)$. This is an important relationship in many physical applications in relativistic astrophysics and cosmology.

The energy momentum tensor (3.7) is coupled to the Einstein tensor (3.6) via the Einstein field equations

$$G_{ab} = T_{ab} \tag{3.9}$$

in appropriate units. We use units in which the coupling constant $\frac{8\pi G}{c^4}$ and the speed of light c are unity. The Einstein field equations (3.9) relate the gravitational field to the matter content. The coupling of the Einstein tensor and the energy momentum tensor introduces a system of highly nonlinear partial differential equations which are difficult to integrate in general.

3.4 Shear-free spacetimes

We consider the particular case of spherically symmetric, shear-free spacetimes. This is a reasonable assumption when modelling a relativistic star. In this case there exists coordinates $(x^a) = (t, r, \theta, \phi)$ for which the line element may be expressed in a form that is simultaneously comoving and isotropic. The line element for the interior of a shear-free radiating star is given by

$$ds^2 = -A^2 dt^2 + B^2 \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \tag{3.10}$$

where the metric functions A and B are functions of t and r. It is also possible in general to consider a line element with shear. In our model, we neglect the effect of shear in order to simplify the model.

The nonvanishing connection coefficients (3.2) may be determined for the line element (3.10). These are given by

$$\Gamma^0{}_{00} = \frac{A_t}{A} \qquad \Gamma^0{}_{01} = \frac{A_r}{A}$$

$$\Gamma^0{}_{11} = \frac{BB_t}{A^2} \qquad \Gamma^0{}_{22} = r^2 \frac{BB_t}{A^2}$$

$$\Gamma^0{}_{33} = r^2 \sin^2\theta \frac{BB_t}{A^2} \qquad \Gamma^1{}_{00} = \frac{AA_r}{B^2}$$

$$\Gamma^1{}_{01} = \frac{B_t}{B} \qquad \Gamma^1{}_{11} = \frac{B_r}{B}$$

$$\Gamma^1{}_{22} = -r^2 \left(\frac{B_r}{B} + \frac{1}{r}\right) \qquad \Gamma^1{}_{33} = -r^2 \sin^2\theta \left(\frac{B_r}{B} + \frac{1}{r}\right)$$

$$\Gamma^2{}_{02} = \frac{B_t}{B} \qquad \Gamma^2{}_{12} = \frac{B_r}{B} + \frac{1}{r}$$

$$\Gamma^2{}_{33} = -\sin\theta\cos\theta \qquad \Gamma^3{}_{03} = \frac{B_t}{B}$$

$$\Gamma^3{}_{13} = \frac{B_r}{B} + \frac{1}{r} \qquad\qquad \Gamma^3{}_{23} = \cot\theta$$

where the subscripts denote partial differentiation.

The nonvanishing Ricci tensor components for the line element (3.10) are found from the definition (3.4). They have the following forms

$$R_{00} = \frac{AA_{rr}}{B^2} + AA_r\frac{B_r}{B^3} - 3\frac{B_{tt}}{B} + 3\frac{A_t}{A}\frac{B_t}{B} + \frac{2}{r}\frac{AA_r}{B^2} \tag{3.11a}$$

$$R_{01} = -\frac{2}{B^2}\left(BB_{rt} + B_rB_t - \frac{BA_rB_t}{A}\right) \tag{3.11b}$$

$$\begin{aligned} R_{11} = {} & 2\frac{B_t^2}{A^2} + \frac{A_r}{A}\frac{B_r}{B} - \frac{2}{r}\frac{B_r}{B} - \frac{A_t}{A^3}BB_t - \frac{A_{rr}}{A} \\ & + \frac{BB_{tt}}{A^2} + 2\frac{B_r^2}{B^2} - 2\frac{B_{rr}}{B} \end{aligned} \tag{3.11c}$$

$$\begin{aligned} R_{22} = {} & r^2\frac{BB_{tt}}{A^2} - r^2\frac{A_t}{A^3}BB_t + 2r^2\frac{B_t^2}{A^2} - r^2\frac{A_r}{A}\frac{B_r}{B} - r\frac{A_r}{A} \\ & - 3r\frac{B_r}{B} - r^2\frac{B_{rr}}{B} \end{aligned} \tag{3.11d}$$

$$R_{33} = \sin^2\theta R_{22} \tag{3.11e}$$

By utilising the Ricci tensor components (3.11) and the definition (3.5) we can find the Ricci scalar. It has the form

$$
\begin{aligned}
R &= -2\frac{A_{rr}}{A}\frac{1}{B^2} - \frac{4}{r}\frac{A_r}{A}\frac{1}{B^2} + \frac{6}{A^2}\frac{B_t^2}{B^2} - \frac{8}{r}\frac{B_r}{B^3} + 2\frac{B_r}{B^4} - 2\frac{A_r}{A}\frac{B_r}{B^3} \\
&\quad -4\frac{B_{rr}}{B^3} - 6\frac{A_t}{A^3}\frac{B_t}{B} + \frac{6}{A^2}\frac{B_{tt}}{B} \qquad (3.12)
\end{aligned}
$$

for the line element (3.10). By substituting equation (3.11) and (3.12) into (3.6) the nonvanishing Einstein tensor components are given by

$$
G_{00} = 3\frac{B_t^2}{B^2} - A^2\left(2\frac{B_{rr}}{B^3} - \frac{B_r^2}{B^4} + 4\frac{B_r}{rB^3}\right) \qquad (3.13a)
$$

$$
G_{01} = \frac{2}{B^2}\left(-BB_{rt} + B_rB_t + \frac{BA_rB_t}{A}\right) \qquad (3.13b)
$$

$$
\begin{aligned}
G_{11} &= \frac{1}{A^2}\left(-2BB_{rt} - B_t^2 + 2\frac{A_t}{A}BB_t\right) \\
&\quad + \frac{1}{B^2}\left(B_r^2 + 2\frac{A_r}{A}BB_r + \frac{2}{r}\frac{A_r}{A}B^2 + \frac{2}{r}BB_r\right) \qquad (3.13c)
\end{aligned}
$$

$$
\begin{aligned}
G_{22} &= -2r^2\frac{BB_{tt}}{A^2} + 2r^2\frac{A_t}{A^3}BB_t - r^2\frac{B_t^2}{A^2} + r\frac{A_r}{A} \\
&\quad + r\frac{B_r}{B} + r^2\frac{A_{rr}}{A} - r^2\frac{B_r^2}{B^2} + r^2\frac{B_{rr}}{B} \qquad (3.13d)
\end{aligned}
$$

$$
G_{33} = \sin^2\theta G_{22} \qquad (3.13e)
$$

for the line element (3.10).

Since we consider a spherically symmetric shear-free radiating star, the energy momentum tensor in equation (3.7) becomes

$$T_{ab} = (\mu + p)\, u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{3.14}$$

with no anisotropic stress. The fluid four-velocity $\mathbf{u}$ is comoving and is given by

$$u^a = \frac{1}{A}\delta^a_0$$

The heat flow vector $\mathbf{q}$ takes the form

$$q^a = (0, q, 0, 0)$$

since $q^a u_a = 0$ and the heat is assumed to flow in the radial direction. Then the nonzero components of the energy momentum tensor (3.14) are

$$T_{00} = A^2 \mu \tag{3.15a}$$

$$T_{01} = -B^2 A q \tag{3.15b}$$

$$T_{11} = B^2 p \tag{3.15c}$$

$$T_{22} = r^2 B^2 p \tag{3.15d}$$

$$T_{33} = \sin^2\theta T_{22} \tag{3.15e}$$

for the line element (3.10).

Then (3.13) and (3.15) can be used to generate the Einstein field equations (3.9) for a shear-free spherically symmetric fluid. We generate the system of partial differential equations

$$\mu = \frac{3}{A^2}\frac{B_t^2}{B^2} - \frac{1}{B^2}\left(2\frac{B_{rr}}{B} - \frac{B_r^2}{B^2} + \frac{4B_r}{rB}\right) \tag{3.16a}$$

$$\begin{aligned} p &= \frac{1}{A^2}\left(-2\frac{B_{tt}}{B} - \frac{B_t^2}{B^2} + 2\frac{A_t}{A}\frac{B_t}{B}\right) \\ &\quad +\frac{1}{B^2}\left(\frac{B_r^2}{B^2} + 2\frac{A_r}{A}\frac{B_r}{B} + \frac{2}{r}\frac{A_r}{A} + \frac{2}{r}\frac{B_r}{B}\right) \end{aligned} \tag{3.16b}$$

$$\begin{aligned} p &= -\frac{2}{A^2}\frac{B_{tt}}{B} + 2\frac{A_t}{A^3}\frac{B_t}{B} - \frac{1}{A^2}\frac{B_t^2}{B^2} + \frac{1}{r}\frac{A_r}{A}\frac{1}{B^2} \\ &\quad +\frac{1}{r}\frac{B_r}{B^3} + \frac{A_{rr}}{A}\frac{1}{B^2} - \frac{B_r^2}{B^4} + \frac{B_{rr}}{B^3} \end{aligned} \tag{3.16c}$$

$$q = -\frac{2}{AB^2}\left(-\frac{B_{rt}}{B} + \frac{B_r B_t}{B^2} + \frac{A_r}{A}\frac{B_t}{B}\right) \tag{3.16d}$$

for the line element (3.10). The equations (3.16) describe the gravitational interactions in the interior of a shear-free spherically symmetric star with heat flux and isotropic pressure.

If we eliminate p from equations (3.16b) and (3.16c) we obtain

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} - 2\frac{B_r^2}{B^2} - 2\frac{A_r}{A}\frac{B_r}{B} - \frac{1}{r}\frac{A_r}{A} - \frac{1}{r}\frac{B_r}{B} = 0 \tag{3.17}$$

Equation (3.17) is called the condition of pressure isotropy. The above result can be written in the form

$$\frac{A_{rr}}{A} + \frac{B_{rr}}{B} = \left(2\frac{B_r}{B} + \frac{1}{r}\right)\left(\frac{B_r}{B} + \frac{A_r}{A}\right) \tag{3.18}$$

Also note that if we redefine the radial coordinate r by

$$x = r^2$$

then (3.17) has the form

$$2A\left(B^{-1}\right)_{xx} = \left(AB^{-1}\right)_{xx} \tag{3.19}$$

which is more compact. Note that by setting

$$B^{-1} = L$$

$$AB^{-1} = M$$

equation (3.19) can also be written as

$$2ML_{xx} = LM_{xx} \tag{3.20}$$

Note that the equations (3.17)–(3.20) are equivalent.

3.5 Junction conditions

The boundary of a shear-free spherically symmetric radiating star divides the entire spacetime into two distinct regions: the interior spacetime and the exterior spacetime.

The interior spacetime (3.10) has to be matched along the boundary of the star to the exterior Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2\left(d\theta^2 + \sin^2\theta d\phi\right) \tag{3.21}$$

where $m(v)$ denotes the mass of the star as measured by an observer at infinity. The metric (3.21) was first derived by Vaidya (1951). The metric (3.21) is the unique spherically symmetric solution of the Einstein field equations for radially directed coherent radiation in the form of a null fluid. The Einstein tensor for the line element (3.21) is given by

$$G_{ab} = -\frac{2}{R^2}\frac{dm}{dv}\delta^0{}_a\delta^0{}_b \tag{3.22}$$

The energy momentum tensor for null radiation assumes the form

$$T_{ab} = \epsilon w_a w_b \tag{3.23}$$

where the null four-vector $\mathbf{w}$ is given by $w_a = (1, 0, 0, 0)$. Thus from (3.22) and (3.23) we have that

$$\epsilon = -\frac{2}{R^2}\frac{dm}{dv} \tag{3.24}$$

for the energy density of the null radiation. Since the star is radiating energy to the exterior spacetime we must have

$$\frac{dm}{dv} \leq 0$$

The matching of the line elements (3.10) and (3.21), and the matching of the extrinsic curvature are necessary at the surface of the star. This matching leads to the following junction conditions

$$
\begin{aligned}
Adt &= \left[\left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv\right]_\Sigma && (3.25a) \\
(rB)_\Sigma &= R_\Sigma && (3.25b) \\
m(v) &= \left[\frac{r^3}{2}\left(\frac{BB_t^2}{A^2} - \frac{B_r^2}{B}\right) - r^2 B_r\right]_\Sigma && (3.25c) \\
(p)_\Sigma &= (Bq)_\Sigma && (3.25d)
\end{aligned}
$$

where Σ is the hypersurface that defines the boundary of the radiating sphere. The junction conditions (3.25) were completed by Santos (1985).

The particular junction condition

$$
(p)_\Sigma = (Bq)_\Sigma \tag{3.26}
$$

is an additional differential equation that has to be solved together with the interior field equations (3.16) to complete the model of a relativistic radiating star. This is a nonlinear differential equation which has to be integrated on the boundary Σ of the star. By substituting equations (3.16b) and (3.16d) in (3.26) we have

$$
\begin{aligned}
&2\frac{B_{rt}}{AB^2} + 2\frac{B_{tt}}{A^2B} - 2\frac{A_tB_t}{A^3B} - 2\frac{B_rB_t}{AB^3} - 2\frac{A_rB_r}{AB^3} - 2\frac{A_rB_t}{A^2B^2} - \frac{B_r^2}{B^4} \\
&+\frac{B_t^2}{A^2B^2} - 2\frac{A_r}{rAB^2} - 2\frac{B_r}{rB^3} = 0 \qquad (3.27)
\end{aligned}
$$

at the boundary of shear-free radiating star.

Equation (3.27) is the master equation that governs the evolution of the model. We will attempt to integrate equation (3.27) in subsequent chapters by making simplifying assumptions on the gravitational potentials and using the Lie analysis of differential equations.

3.6 Discussion

Exact models of relativistic radiating stars are very important for investigating physical phenomena such as the cosmic censorship hypothesis and the gravitational collapse of stars. The interior spacetime of the shear-free radiating star should match with the exterior Vaidya solution (3.21), and the junction condition relating pressure with the heat flux at the boundary of the star, must be satisfied. Such models are important in investigating the physical features of radiating stars such as dynamic stability, surface luminosity, relaxation effects, temperature profiles and particle production at the boundary. The model of slow gravitational collapse of an interior, initially static, configuration was proposed by de Olivier *et al.* (1985). It had be shown by Raychaudhuri (1955) that the slowest possible gravitational collapse arises in shear-free spacetimes. With a vanishing Weyl tensor, Herrera *et al.* (2004) proposed a conformally flat relativistic model without solving the junction condition exactly. For this model Herrera *et al.* (2006) and Maharaj and Govender (2005) subsequently generated classes of solutions by solving the junction condition directly in terms of elementary functions. Misthry *et al.* (2008) generated other classes of solution with vanishing shear by transforming the junction condition equation to an Abel equation. These solutions are useful in determining the gravitational behaviour of stars.

The dissipative effects of fluid particles travelling in geodesic motion were discussed by Kolassis *et al.* (1988). In this model, the Friedman dust solution is regained in the absence of heat flow. This exact solution has been widely used in investigating the physical features of stars. The physical investigations include modelling radiating gravitational collapse in spherical geometry with neutrino flux (Grammenos and Kolassis 1992) and analysing astrophysical processes with heat flux (Tomimura and Nunes 1993). The temperature in casual thermodynamics for particles travelling in geodesic motion produce higher central values than the Eckart theory (Govender *et al.* 1998).

The effects of shear and anisotropic pressure were studied by Chan *et al.* (2003), Herrera and Santos (1997) and Herrera *et al.* (2004). Naidu *et al.* (2006) obtained the first exact solution with shear by considering the geodesic motion of fluid particles. Rajah and Maharaj (2008) extended this result and obtained new classes of solutions by transforming the junction condition and solving it. These extended classes of solutions are nonsingular at the origin. Euclidean stars in general relativity may be modelled with nonvanishing shear. For Euclidean stars the areal radius and proper radius are equal. Particular shearing solutions were found by Herrera and Santos (2010) and Govender (2010).

Chapter 4

Conformally flat model

4.1 Introduction

The boundary condition generated by Santos (1985) is crucial in the process of gravitational collapse for radiating stars in astrophysics. Our aim in this chapter is to provide exact solutions to the boundary condition when the interior spacetime is conformally flat. In section 4.2 we present the junction condition for a shear-free radiating star in conformally flat spacetimes. This equation is highly nonlinear and difficult to solve directly using traditional methods. Therefore we utilise a geometric approach to find solutions. In section 4.3 we obtain the Lie point symmetries for the junction boundary condition. Any and all these symmetries (or any linear combination) help us to obtain group invariant solutions. We transform the boundary condition to an ordinary differential equations for each symmetry in the optimal system. Exact solutions are found for each symmetry. We also find an exact solution for a symmetry not in the optimal system. The results are summarised in section 4.4.

4.2 Junction condition

The fundamental differential equation governing the evolution of a relativistic radiating star is

$$
\begin{aligned}
&2\frac{B_{rt}}{AB^2} + 2\frac{B_{tt}}{A^2B} - 2\frac{A_tB_t}{A^3B} - 2\frac{B_rB_t}{AB^3} - 2\frac{A_rB_r}{AB^3} - 2\frac{A_rB_t}{A^2B^2} - \frac{B_r^2}{B^4} \\
&+\frac{B_t^2}{A^2B^2} - 2\frac{A_r}{rAB^2} - 2\frac{B_r}{rB^3} = 0 \qquad (4.1)
\end{aligned}
$$

which was obtained in section 3.5. For conformally flat spacetimes $A = B$, the Weyl tensor vanishes and there are no tidal effects. For this special case the line element (3.10) can be written as

$$ds^2 = B^2\left[-dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right] \qquad (4.2)$$

and equation (4.1) reduces to the simpler form

$$2rBB_{rt} + 2rBB_{tt} - 4rB_rB_t - rB_t^2 - 3rB_r^2 - 4BB_r = 0 \qquad (4.3)$$

The gravitational behaviour of a relativistic radiating star not experiencing tidal forces is determined by (4.3). We need to solve (4.3) exactly to complete the model. This equation is a highly nonlinear partial differential equation and difficult to solve directly. Conformally flat models have been studied by Herrera *et al.* (2004), Herrera *et al.* (2006), Maharaj and Govender (2005) and Misthry *et al.* (2008) who demonstrated that exact solutions in terms of elementary and special functions exist.

4.3 Lie symmetry analysis

We use the Lie analysis in an attempt to find new solutions to (4.3). Utilising `PROGRAM LIE` (Head 1993), we can demonstrate that (4.3) admits the following Lie point symmetries:

$$G_1 = \frac{\partial}{\partial t} \tag{4.4a}$$

$$G_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{4.4b}$$

$$G_3 = B\frac{\partial}{\partial B} \tag{4.4c}$$

with the nonzero Lie bracket relationship

$$[G_1, G_2] = G_1 \tag{4.5}$$

The generators (4.4) permit us to obtain the commutation table of symmetries (see Table 4.1).

4.3.1 Optimal system

Given that equation (4.3) has the three symmetries (4.4), we can find group invariant solutions using each symmetry individually, or any linear combination of symmetries. However, taking all possible combinations into account is overly excessive. To avoid this problem we consider a subspace of this vector space. We use the subalgebraic structure of the symmetries (4.4) of the equation (4.3) to construct an optimal system

$[G_i, G_j]$	G_1	G_2	G_3
G_1	0	G_1	0
G_2	$-G_1$	0	0
G_3	0	0	0

Table 4.1: Commutation table for symmetries in (4.4)

of one-dimensional subgroups. All group invariant solutions can be transformed to those obtained via this optimal system. The process of obtaining this subalgebra is algorithmic.

According to the adjoint action representation definition (2.36), we can develop the adjoint representation for the symmetries (4.4). These are given by

$$\mathrm{Ad}\left(\exp\left(\epsilon G_1\right)\right) G_1 = G_1 \qquad \mathrm{Ad}\left(\exp\left(\epsilon G_1\right)\right) G_2 = -\epsilon G_1 + G_2$$

$$\mathrm{Ad}\left(\exp\left(\epsilon G_1\right)\right) G_3 = G_3 \qquad \mathrm{Ad}\left(\exp\left(\epsilon G_2\right)\right) G_1 = e^{\epsilon} G_1$$

$$\mathrm{Ad}\left(\exp\left(\epsilon G_2\right)\right) G_2 = G_2 \qquad \mathrm{Ad}\left(\exp\left(\epsilon G_2\right)\right) G_3 = G_3$$

$$\mathrm{Ad}\left(\exp\left(\epsilon G_3\right)\right) G_1 = G_1 \qquad \mathrm{Ad}\left(\exp\left(\epsilon G_3\right)\right) G_2 = G_2$$

$$\mathrm{Ad}\left(\exp\left(\epsilon G_3\right)\right) G_3 = G_3$$

We may also construct the adjoint representation table for the symmetries (4.4) (see Table 4.2).

To determine the inequivalent subalgebras we begin with the following nonzero

Ad	G_1	G_2	G_3
G_1	G_1	$-\epsilon G_1 + G_2$	G_3
G_2	$e^{\epsilon} G_1$	G_2	G_3
G_3	G_1	G_2	G_3

Table 4.2: Adjoint representation table for symmetries in (4.4)

vector

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3 \tag{4.6}$$

We try to remove as many of the coefficients, a_i of G, as possible through judicious applications of adjoint maps to G.

We first choose $a_3 \neq 0$ and set $a_3 = 1$.
Thus, we have

$$G = a_1 G_1 + a_2 G_2 + G_3 \tag{4.7}$$

We operate on G with $\mathrm{Ad}(\exp(\epsilon G_2))$ to obtain

$$\begin{aligned} \hat{G} &= \mathrm{Ad}(\exp(\epsilon G_2))(a_1 G_1 + a_2 G_2 + G_3) \\ &= \hat{a}_1 G_1 + \hat{a}_2 G_2 + G_3 \end{aligned} \tag{4.8}$$

Our choice of acting on G with $\mathrm{Ad}(\exp(\epsilon G_2))$ does not reduce the coefficients in G.

Next we act on G with $\mathrm{Ad}(\exp(\epsilon G_1))$ to obtain

$$\begin{aligned}\tilde{G} &= \mathrm{Ad}(\exp(\epsilon G_1))(a_1 G_1 + a_2 G_2 + G_3) \\ &= \tilde{a}_2 G_2 + G_3 \qquad (4.9)\end{aligned}$$

with reduced coefficient by setting $\epsilon = \frac{a_1}{a_2}$. We cannot reduce this operator any further.

Similarly, by setting $a_3 = 0$, $a_2 \neq 0$ and moreover $a_2 = 1$, G in (4.6) becomes

$$G = a_1 G_1 + G_2 \qquad (4.10)$$

By acting on G in (4.10) with $\mathrm{Ad}(\exp(\epsilon G_1))$ we have

$$\begin{aligned}\overline{G} &= \mathrm{Ad}(\exp(\epsilon G_1))(a_1 G_1 + G_2) \\ &= G_2 \qquad (4.11)\end{aligned}$$

by setting $\epsilon = a_1$.

Finally, by setting $a_3 = a_2 = 0$, $a_1 \neq 0$ and $a_1 = 1$, we obtain

$$\overrightarrow{G} = G_1 \qquad (4.12)$$

As a result, we have

$$\{G_1, G_2, aG_2 + G_3\} \qquad (4.13)$$

are the subalgebras of the symmetries in (4.4). Such an optimal system of subgroups is determined by classifying the orbits of the infinitesimal adjoint representation of a Lie group on its Lie algebra obtained by using its infinitesimal generators. All group invariant solutions can be transformed to those obtained via this optimal system.

4.3.2 Invariance under G_1

Using the generator

$$G_1 = \frac{\partial}{\partial t} \tag{4.14}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{dB}{0} \tag{4.15}$$

We obtain the invariants

$$x = r \tag{4.16a}$$

$$B = y(x) \tag{4.16b}$$

for the generator G_1.

With this transformation equation (4.3) is reduced to

$$3xy' + 4y = 0 \tag{4.17}$$

Equation (4.17) is a first order, separable ordinary differential equation. This ordinary differential equation has the solution

$$y = \frac{C}{x^{4/3}} \tag{4.18}$$

where C is an arbitrary constant of integration. Hence

$$B(r,t) = \frac{1}{r^{4/3}} \tag{4.19}$$

is a particular solution for the partial differential equation (4.3). We have set $C = 1$ without any lose of generality. Since the gravitational potential B in (4.19) is independent of time, t, it cannot be applied to a radiating star.

4.3.3 Invariance under G_2

Using the generator

$$G_2 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{4.20}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{t} = \frac{dr}{r} = \frac{dB}{0} \tag{4.21}$$

The invariants are given by

$$x = \frac{t}{r} \tag{4.22a}$$

$$B = y\,(x) \tag{4.22b}$$

for the generator G_2.

For this transformation equation (4.3) is reduced to

$$(2x-2)\,yy'' + (2-4x)\,yy' + \left(1-4x+3x^2\right) y'^2 = 0 \tag{4.23}$$

Equation (4.23) is a second order nonlinear ordinary differential equation. The integration of (4.23) is not easy to complete using traditional methods. However, using the computer package `MATHEMATICA` (Wolfram Research, Inc. 2008) we can show that equation (4.23) has the solution

$$y = C_2 \exp\left(\int_1^x \frac{8e^{2z}\,(z-1)}{C_1 + 3e^{2z} - 10ze^{2z} + 6z^2e^{2z}} dz\right) \tag{4.24}$$

Hence we have that

$$B(r,t) = C_2 \exp\left(\int_1^{(t/r)} \frac{8e^{2z}\,(z-1)}{C_1 + 3e^{2z} - 10ze^{2z} + 6z^2e^{2z}} dz\right) \tag{4.25}$$

is a solution for equation (4.3) where C_1 and C_2 are arbitrary constants of integration. Note that the solution (4.24) is a general solution. The integral part of the exponent cannot be simplified further when $C_1 \neq 0$ and $C_2 \neq 0$.

We can simplify (4.24) for particular parameter values by setting $C_1 = 0$. We get the result

$$
\begin{aligned}
y &= C_2 \exp\left(\int_1^x \frac{8\,(z-1)}{3-10z+6z^2}dz\right) \\
&= C_2 \exp\left(\frac{14-2\sqrt{7}}{21}\int_1^x \frac{1}{\sqrt{7}+5-6z}dz - \frac{14+2\sqrt{7}}{21}\int_1^x \frac{1}{\sqrt{7}-5+6z}dz\right) \quad (4.26)
\end{aligned}
$$

These integrals can be evaluated and we find

$$
\begin{aligned}
y &= C_2 \exp\left\{\frac{2}{21}\left[\left(\sqrt{7}-7\right)\log\left(\frac{\sqrt{7}-1}{\sqrt{7}+5-6x}\right)\right.\right. \\
&\quad \left.\left. - \left(\sqrt{7}+7\right)\log\left(\frac{\sqrt{7}+1}{\sqrt{7}-5+6x}\right)\right]\right\} \quad (4.27)
\end{aligned}
$$

The solution (4.27) can be written in the compact form

$$
y = C_2\left(\frac{\sqrt{7}-1}{\sqrt{7}+5-6x}\right)^{\left(2\sqrt{7}-14\right)/21}\left(\frac{\sqrt{7}-5+6x}{\sqrt{7}+1}\right)^{\left(2\sqrt{7}+14\right)/21} \quad (4.28)
$$

Hence (4.28) is a particular solution for the ordinary differential equation (4.23). We have established that

$$
B(r,t) = \left(\frac{\left(\sqrt{7}-1\right)r}{\left(\sqrt{7}+5\right)r-6t}\right)^{\left(2\sqrt{7}-14\right)/21}\left(\frac{\left(\sqrt{7}-5\right)r+6t}{\left(\sqrt{7}+1\right)r}\right)^{\left(2\sqrt{7}+14\right)/21} \quad (4.29)
$$

is a particular solution of (4.3). We have set $C_2 = 1$ without any loss of generality.

We emphasize that the result (4.29) is a new exact solution to the boundary condition (3.26) for a radiating star with a shear-free matter distribution. It is not contained in any of the classes of solution found in previous investigations. This new solution is given in terms of a self-similar variable $x = t/r$. The appearance of the self-similar variable implies the existence of a homothetic Killing vector. In shearing spherically symmetric spacetimes a homothetic vector was found by Wagh and Govinder (2006). The elementary form of the solution in (4.29) will assist in studying the physical features of a conformally flat radiating star. We intended to study the physical behaviour of this model in future research.

4.3.4 Invariance under $aG_2 + G_3$

By using the generator

$$aG_2 + G_3 = a\left(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\right) + B\frac{\partial}{\partial B} \tag{4.30}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{at} = \frac{dr}{ar} = \frac{dB}{B} \tag{4.31}$$

We obtain the invariants

$$x = \frac{t}{r} \tag{4.32a}$$

$$B = y\,(x)\,r^{1/a} \tag{4.32b}$$

in this case.

With this transformation equation (4.3) is reduced to

$$2a^2(x-1)\,yy'' + 2a(1+a-3x-2ax)\,yy' + a^2\left(1-4x+3x^2\right)y'^2$$

$$+(3+4a)\,y^2 = 0 \tag{4.33}$$

which is a second order nonlinear ordinary differential equation. However, this equation is quite difficult to solve since it has no symmetry in general for further reduction.

Using the special value $a=0$, the symmetry $G = aG_2 + G_3$ becomes

$$G = G_3 = B\frac{\partial}{\partial B} \tag{4.34}$$

The invariant surface condition for symmetry (4.34) is

$$\frac{dt}{0} = \frac{dr}{0} = \frac{dB}{B} \tag{4.35}$$

which gives the invariants

$$x = t \tag{4.36a}$$

$$y(x) = r \tag{4.36b}$$

Using this transformation, we cannot generate an ordinary differential equation.

By inspection we observe that (4.33) is simplified when $a = -\frac{3}{4}$. For this value of a equation (4.33) becomes

$$6(x-1)yy'' + 2(6x-1)yy' + 3(1-4x+3x^2)y'^2 = 0 \tag{4.37}$$

Even though (4.37) has been reduced to a simpler form it remains highly nonlinear and it is not easy to solve with the standard methods. With the help of the software package `MATLAB` (MathWorks, Inc. 2009) equation (4.37) has the general solution

$$y = \exp\left(2\int \frac{1}{\exp(2x)\left(\int \frac{3x+1}{\exp(2x)(x-1)^{5/3}}dx + C_1\right)}dx + C_2\right) \tag{4.38}$$

where C_1 and C_2 are arbitrary constants of integration.

By using the invariants in equation (4.32), the solution in equation (4.38) and the special value of $a = -\frac{3}{4}$ we have

$$B(r,t) = \exp\left(2\int_1^{(t/r)} \frac{1}{\exp(2u)\left(\int_1^{t/r} \frac{3u+1}{\exp(2u)(u-1)^{5/3}}du + C_1\right)}du + C_2\right) r^{-4/3} \tag{4.39}$$

is the general solution for equation (4.3).

4.3.5 Invariance under $bG_1 + G_3$

It is well known that any solution obtained by using any symmetry which is not in the optimal system can transformed to the solution obtained by using the symmetry which is in the optimal system. For the special value $a = 0$ the symmetry $aG_2 + G_3$, which is in the optimal system does not give invariants. As a result we consider symmetries which are not in the optimal system. The optimal system (4.13) contains one symmetry and combination of two symmetries. By using this fact, we also consider one symmetry and the combination of two symmetries. Without loss of generality we take these symmetries to be

$$\{G_3, aG_1 + G_2, bG_1 + G_3\} \tag{4.40}$$

The results for these symmetries are:

- **Symmetry** G_3: The symmetry G_3 does not generate an ODE.
- **Symmetry** $aG_1 + G_2$: The solution obtained by using the symmetry $aG_1 + G_2$ can be transformed to the solution obtained by using symmetry G_2. Symmetry G_2 is in the optimal system.
- **Symmetry** $bG_1 + G_3$:

$$b\frac{\partial}{\partial t} + B\frac{\partial}{\partial B} \tag{4.41}$$

  Here b is a nonzero arbitrary constant. This combination leads to a new solution.

By using the symmetry (4.41), we determine the invariant surface condition

$$\frac{dt}{b} = \frac{dr}{0} = \frac{dB}{B} \tag{4.42}$$

These equations yield the invariants

$$
\begin{aligned}
x &= r \qquad (4.43a)\\
B &= \exp\left(\frac{t}{b}\right) y(x) \qquad (4.43b)
\end{aligned}
$$

for $bG_1 + G_3$.

By using this transformation, equation (4.3) is reduced to the first order ordinary differential equation

$$3b^2 x y'^2 + 2b(2b+x)yy' - xy^2 = 0 \tag{4.44}$$

Equation (4.44) is a nonlinear equation that can be integrated using `MATHEMATICA` (Wolfram Research, Inc. 2008) to give two special solutions

$$y = C\left(\frac{\sqrt[3]{\left(2b+x+2\sqrt{b^2+bx+x^2}\right)^2}}{\exp\left(\frac{x+2\sqrt{b^2+bx+x^2}}{3b}\right)\left(\sqrt[3]{x^4\left[b+2\left(x+\sqrt{b^2+bx+x^2}\right)\right]}\right)}\right) \tag{4.45a}$$

and

$$y = D\left(\frac{\left(\sqrt[3]{b+2\left(x+\sqrt{b^2+bx+x^2}\right)}\right)}{\exp\left(\frac{x-2\sqrt{b^2+bx+x^2}}{3b}\right)\sqrt[3]{\left(2b+x+2\sqrt{b^2+bx+x^2}\right)^2}}\right) \tag{4.45b}$$

where C and D are arbitrary constant of integration. Hence particular solutions to

(4.3) have the form

$$B(r,t) = \frac{\exp(t/b)\sqrt[3]{(2b+r+\alpha(r))^2}}{\exp\left(\frac{r+\alpha(r)}{3b}\right)\left(\sqrt[3]{r^4\,(b+2r+\alpha(r))}\right)} \tag{4.46a}$$

and

$$B(r,t) = \frac{\exp(t/b)\left(\sqrt[3]{b+2r+\alpha(r)}\right)}{\exp\left(\frac{r-\alpha(r)}{3b}\right)\sqrt[3]{(2b+r+\alpha(r))^2}} \tag{4.46b}$$

where $\alpha(r) = 2\sqrt{b^2+br+r^2}$ and we have set $C = D = 1$. Note that in the above solutions $b \neq 0$ for the symmetry to exist.

We have found two new solutions to the boundary condition (3.26) for a radiating star. The solution (4.46a) and (4.46b) have been generated using invariance under $bG_1 + G_3$ which is not in the optimal system. We point out the fact that the new solutions are given in terms of elementary functions and this will help in the analysis of the physical features of a stellar model. If we set $r = 0$ in (4.46b) then the potential B remains finite. Thus the metric function does not have a singularity at the centre of the radiating star. Thus far very few solutions without singularities have been obtained in the literature. This solution can be used to describe the entire interior spacetime of a shear-free radiating star in conformally flat spacetime.

4.3.6 Extending known solutions

We apply the procedure from chapter two which should be followed to extend our previously found solutions to possibly new solutions.

By considering the symmetry (4.4a) we have

$$\xi_1 = 1 \qquad \xi_2 = 0 \qquad \eta = 0 \tag{4.47}$$

From equations (2.3) and (4.47)

$$\frac{dt_1}{d\epsilon_1} = 1 \qquad \frac{dr_1}{d\epsilon_1} = 0 \qquad \frac{dB_1}{d\epsilon_1} = 0 \tag{4.48}$$

By integrating equation (4.48) we obtain

$$t_1 = \epsilon_1 + C_1 \qquad r_1 = C_2 \qquad B_1 = C_3 \tag{4.49}$$

and from the initial conditions (2.7), at $\epsilon_1 = 0$, we have

$$t = t_1|_{\epsilon=0} = C_1 \qquad r = r_1|_{\epsilon=0} = C_2 \qquad B = B_1|_{\epsilon=0} = C_3 \tag{4.50}$$

Finally, by substituting equation (4.50) into equation (4.49), we have

$$t_1 = t + \epsilon_1 \tag{4.51a}$$

$$r_1 = r \tag{4.51b}$$

$$B_1 = B \tag{4.51c}$$

Similarly, via symmetry (4.4b) we have

$$t_2 = te^{\epsilon_2} \tag{4.52a}$$

$$r_2 = re^{\epsilon_2} \tag{4.52b}$$

$$B_2 = B \tag{4.52c}$$

and via symmetry (4.4c) we have

$$t_3 = t \tag{4.53a}$$

$$r_3 = r \tag{4.53b}$$

$$B_3 = Be^{\epsilon_3} \tag{4.53c}$$

We observe that the forms in equation (4.51), (4.52) and (4.53) can be expressed as the general global form

$$\bar{t} = tK_2 + K_1 \tag{4.54a}$$

$$\bar{r} = rK_2 \tag{4.54b}$$

$$\overline{B} = BK_3 \tag{4.54c}$$

where $K_1 = \epsilon_1$, $K_2 = e^{\epsilon_2}$ and $K_3 = e^{\epsilon_3}$ are arbitrary constants and $K_2, K_3 > 0$. The transformation (4.54) leaves equation (4.3) invariant (Bluman *et al.* 2010).

Thus all the solutions previously found in this chapter can be transformed to potentially new solutions via (4.54).

4.4 Discussion

Utilizing `PROGRAM LIE` we found symmetries for equation (4.3). We then determined an optimal system of one-dimensional subalgebras for this set of symmetries. Each of the subalgebras was used to reduce the highly nonlinear partial differential equation (4.3) to an ordinary differential equation. For completeness we also considered symmetries not in the optimal system. Exact solutions to the boundary condition were found corresponding to the symmetries and we believe that these results are new. For convenience and easy reference we have summarised the results of this chapter in Table 4.3.

Generators	ODEs	Solutions $B(r,t)$ for the PDE
G_1	$3xy' + 4y = 0$	$r^{-\frac{4}{3}}$
G_2	$(2x-2)\,yy'' + (2-4x)\,yy'$ $+\left(1-4x+3x^2\right)y'^2 = 0$	$\left(\frac{\left(\sqrt{7}-1\right)r}{\left(\sqrt{7}+5\right)r-6t}\right)^{\frac{\left(2\sqrt{7}-14\right)}{21}} \left(\frac{\left(\sqrt{7}-5\right)r+6t}{\left(\sqrt{7}+1\right)r}\right)^{\frac{\left(2\sqrt{7}+14\right)}{21}}$
$aG_2 + G_3$ $(a = -\frac{3}{4})$	$6(x-1)yy'' + 2(6x-1)yy'$ $+3(1-4x+3x^2)y'^2 = 0$	$\exp\left(2\int_1^{\frac{t}{r}} \frac{1}{\exp(2u)\left(\int_1^{\frac{t}{r}} \frac{3u+1}{\exp(2u)(u-1)^{5/3}}du\right)}du\right) r^{-4/3}$
$bG_2 + G_3$ $(b \neq 0)$	$3b^2xy'^2 + 2b(2b+x)yy'$ $-xy^2 = 0$	$\frac{\exp(t/b)\sqrt[3]{(2b+r+\alpha)^2}}{\exp\left(\frac{r+\alpha}{3b}\right)\left(\sqrt[3]{r^4(b+2r+\alpha)}\right)}$ where $\alpha = 2\sqrt{b^2+br+r^2}$
		$\frac{\exp(t/b)\left(\sqrt[3]{b+2r+\alpha}\right)}{\exp\left(\frac{r-\alpha}{3b}\right)\sqrt[3]{(2b+r+\alpha)^2}}$ where $\alpha = 2\sqrt{b^2+br+r^2}$

Table 4.3: Generators, ODEs and solutions for the PDE

Chapter 5

Geodesic model

5.1 Introduction

Studying relativistic radiating stars when the fluid particles are in geodesic motion is physically acceptable and has been used by other investigators to describe realistic astrophysical processes. Again, the junction condition formulated by Santos (1985) is crucial to obtain exact models. Our aim in this chapter is to provide new exact solutions to the boundary condition when the fluid particles are in geodesic motion. The first exact geodesic solution, satisfying the boundary conditions, was obtained by Kolassis *et al.* (1988). Several investigators have subsequently used this assumption. A comprehensive treatment of the geodesic condition was undertaken by Thirukkanesh and Maharaj (2009). In section 5.2 we derive the junction condition for a shear-free radiating star when the fluid particles are in geodesic motion. The equation we present is a highly nonlinear partial differential equation which is difficult to solve directly using traditional methods. Thus we use the Lie symmetry approach to transform the partial differential to ordinary differential equations. In section 5.3 we obtain the Lie point symmetries admitted by the junction condition. The optimal system for these symmetries is presented. We transform the boundary condition to ordinary differential equations for each symmetry in the optimal system. Explicit and implicit solutions are obtained for each symmetry in the optimal system. Assuming separability, we obtain a new exact solution to the boundary condition. The results are summarised in section 5.4.

5.2 Junction condition

The master differential equation governing the evolution of a relativistic radiating star was obtained in section 3.5 as

$$
\begin{aligned}
& 2\frac{B_{rt}}{AB^2} + 2\frac{B_{tt}}{A^2 B} - 2\frac{A_t B_t}{A^3 B} - 2\frac{B_r B_t}{AB^3} - 2\frac{A_r B_r}{AB^3} - 2\frac{A_r B_t}{A^2 B^2} - \frac{B_r^2}{B^4} \\
& + \frac{B_t^2}{A^2 B^2} - 2\frac{A_r}{rAB^2} - 2\frac{B_r}{rB^3} = 0
\end{aligned}
\tag{5.1}
$$

The assumption that the fluid particles of radiating stars are in geodesic motion is reasonable for models in relativistic astrophysics. For this special case we have $A = 1$ and the line element (3.10) can be written as

$$
ds^2 = -dt^2 + B^2 \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2 \right) \right] \tag{5.2}
$$

Then (5.1) can be written in the simpler form

$$
2rB^2 B_{rt} + 2rB^3 B_{tt} - 2rBB_r B_t - rB_r^2 + rB^2 B_r^2 - 2BB_r = 0 \tag{5.3}
$$

The gravitational behaviour of a relativistic radiating star when the fluid particles are not accelerating is determined by (5.3). We need to solve (5.3) exactly to complete the model. This equation is a highly nonlinear partial differential equation and difficult to solve directly. Geodesic models have been studied by Herrera *et al.* (2002), Kolassis *et al.* (1988), Naidu *et al.* (2006), Rajah and Maharaj (2008) and Thirukkanesh and Maharaj (2009).

5.3 Lie symmetry analysis

We use the Lie analysis to seek a solution to (5.3). Utilising `PROGRAM LIE` (Head 1993), we can demonstrate that (5.3) admits the following Lie point symmetries:

$$G_1 = \frac{\partial}{\partial t} \tag{5.4a}$$

$$G_2 = t\frac{\partial}{\partial t} + B\frac{\partial}{\partial B} \tag{5.4b}$$

$$G_3 = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{5.4c}$$

with the nonzero Lie bracket relationships

$$[G_1, G_2] = G_1 \tag{5.5a}$$

$$[G_1, G_3] = G_1 \tag{5.5b}$$

The generators (5.4) permit us to generate the commutation table of symmetries (see Table 5.1).

$[G_i, G_j]$	G_1	G_2	G_3
G_1	0	G_1	G_1
G_2	$-G_1$	0	0
G_3	$-G_1$	0	0

Table 5.1: Commutation table for symmetries in (5.4)

5.3.1 Optimal system

We proceed here as in the previous chapter.

According to the adjoint action representation definition (2.36), we develop the adjoint representation for symmetries (5.4) as

$$\mathrm{Ad}\,(\exp\,(\epsilon G_1))\,G_1 = G_1 \qquad \mathrm{Ad}\,(\exp\,(\epsilon G_1))\,G_2 = -\epsilon G_1 + G_2$$

$$\mathrm{Ad}\,(\exp\,(\epsilon G_1))\,G_3 = -\epsilon G_1 + G_3 \qquad \mathrm{Ad}\,(\exp\,(\epsilon G_2))\,G_1 = e^{\epsilon} G_1$$

$$\mathrm{Ad}\,(\exp\,(\epsilon G_2))\,G_2 = G_2 \qquad \mathrm{Ad}\,(\exp\,(\epsilon G_2))\,G_3 = G_3$$

$$\mathrm{Ad}\,(\exp\,(\epsilon G_3))\,G_1 = e^{\epsilon} G_1 \qquad \mathrm{Ad}\,(\exp\,(\epsilon G_3))\,G_2 = G_2$$

$$\mathrm{Ad}\,(\exp\,(\epsilon G_3))\,G_3 = G_3$$

We also construct the adjoint representation table for the symmetries (5.4) (see Table 5.2).

Ad	G_1	G_2	G_3
G_1	G_1	$-\epsilon G_1 + G_2$	$-\epsilon G_1 + G_3$
G_2	$e^{\epsilon} G_1$	G_2	G_3
G_3	$e^{\epsilon} G_1$	G_2	G_3

Table 5.2: Adjoint representation table for symmetries in (5.4)

In determining the inequivalent subalgebras, whose adjoint representation is determined by the adjoint table above, we begin with the following nonzero vector

$$G = a_1 G_1 + a_2 G_2 + a_3 G_3 \tag{5.6}$$

We attempt to remove as many of the coefficients, a_i of G, as possible through careful application of adjoint maps to G.

We first choose $a_3 \neq 0$ and set $a_3 = 1$. Thus, we have

$$G = a_1 G_1 + a_2 G_2 + G_3 \tag{5.7}$$

We operate on G with $\mathrm{Ad}(\exp(\epsilon G_2))$ to obtain

$$\begin{aligned} \hat{G} &= \mathrm{Ad}(\exp(\epsilon G_2))(a_1 G_1 + a_2 G_2 + G_3) \\ &= \hat{a}_1 G_1 + \hat{a}_2 G_2 + G_3 \end{aligned} \tag{5.8}$$

Our choice of acting on G with $\mathrm{Ad}(\exp(\epsilon G_2))$ does not reduce the coefficients in G.

Next we act on G with $\mathrm{Ad}(\exp(\epsilon G_1))$ to obtain

$$\begin{aligned} \tilde{G} &= \mathrm{Ad}(\exp(\epsilon G_1))(a_1 G_1 + a_2 G_2 + G_3) \\ &= \tilde{a}_2 G_2 + G_3 \end{aligned} \tag{5.9}$$

by setting $\epsilon = \frac{a_1}{a_2+1}$. No further simplification is possible.

Similarly, by setting $a_3 = 0$, $a_2 \neq 0$ and moreover $a_2 = 1$, G in (5.6) becomes

$$G = a_1 G_1 + G_2 \tag{5.10}$$

and by acting on G in (5.10) with $\mathrm{Ad}(\exp(\epsilon G_1))$ we have

$$\begin{aligned} \overline{G} &= \mathrm{Ad}(\exp(\epsilon G_1))(a_1 G_1 + G_2) \\ &= G_2 \end{aligned} \tag{5.11}$$

by setting $\epsilon = a_1$.

Finally, by setting $a_3 = a_2 = 0$, $a_1 \neq 0$ and $a_1 = 1$, we obtain

$$\overrightarrow{G} = G_1 \tag{5.12}$$

Hence,

$$\{G_1, G_2, aG_2 + G_3\} \tag{5.13}$$

are the subalgebras of the symmetries in (5.4). Such an optimal system of subgroups is determined by classifying the orbits of the infinitesimal adjoint representation of a Lie group on its Lie algebra obtained by using its infinitesimal generators. All group invariant solutions can be transformed to those obtained via this optimal system.

5.3.2 Invariance under G_1

Using the generator

$$G_1 = \frac{\partial}{\partial t} \tag{5.14}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{1} = \frac{dr}{0} = \frac{dB}{0} \tag{5.15}$$

This condition gives the invariants

$$x = r \tag{5.16a}$$

$$B = y(x) \tag{5.16b}$$

for the generator G_1.

Using this transformation, equation (5.3) is reduced to the first order ordinary differential equation

$$xy'^2 + 2y'y = 0 \tag{5.17}$$

This equation has the solution

$$y = \frac{C}{x^2} \tag{5.18}$$

where C is arbitrary constant of integration. Hence,

$$B(r, t) = \frac{1}{r^2} \tag{5.19}$$

is a particular solution for equation (5.3) where we have set $C = 1$ without loss of generality. The gravitational potential B in (5.19) depends in r only. As result, (5.19) cannot be applied to a radiating star.

5.3.3 Invariance under G_2

Using the generator

$$G_2 = t\frac{\partial}{\partial t} + B\frac{\partial}{\partial B} \tag{5.20}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{t} = \frac{dr}{0} = \frac{dB}{B} \tag{5.21}$$

We obtain the invariants

$$x = r \tag{5.22a}$$

$$B = y(x)\,t \tag{5.22b}$$

for the generator G_2.

With this transformation equation (5.3) is transformed to

$$xy'^2 + 2yy' - xy^4 = 0 \tag{5.23}$$

Equation (5.23) is first order nonlinear ordinary differential equation. It is difficult to integrate using traditional methods. However, by using `MATLAB` (MathWorks, Inc. 2009) equation (5.23) can be integrated to give the solution

$$y = \frac{4C}{C^2x^2 - 4} \tag{5.24}$$

Hence we have

$$B(r,t) = \frac{4Ct}{C^2r^2 - 4} \tag{5.25}$$

is a particular solution for equation (5.3).

5.3.4 Invariance under $aG_2 + G_3$

Using the generator

$$aG_2 + G_3 = (a+1)t\frac{\partial}{\partial t} + aB\frac{\partial}{\partial B} + r\frac{\partial}{\partial r} \tag{5.26}$$

we determine the invariants from the invariant surface condition

$$\frac{dt}{(a+1)t} = \frac{dr}{r} = \frac{dB}{aB} \tag{5.27}$$

The invariants are given by

$$x = \frac{t^{\frac{1}{1+a}}}{r} \tag{5.28a}$$

$$B = y\,(x)\,r^a \tag{5.28b}$$

With this transformation equation (5.3) is reduced to

$$x^{1-2a}y^2(2(1+a)x^a y' - y'^2 + (a+1)x^a(ax^a(2+3a+a^2) + 2xy'')$$

$$-2(1+a)x^2yy'((1+a)^2 + x^{-a}y' + 2x^{-2a}y^3(ay' - xy''))$$

$$+(1+a)^2x^3y'^2 = 0 \tag{5.29}$$

which is a second order nonlinear ordinary differential equation. Clearly this equation is quite difficult to solve. `PROGRAM LIE` suggests that the special value of $a = -1$ may be helpful.

By using the special value $a = -1$ the symmetry $G = aG_2 + G_3$ becomes

$$G = r\frac{\partial}{\partial r} - B\frac{\partial}{\partial B} \tag{5.30}$$

Using the symmetry in (5.30) we determine the invariants from the invariant surface condition

$$\frac{dt}{0} = \frac{dr}{r} = -\frac{dB}{B} \tag{5.31}$$

This surface condition gives invariants

$$x = t \tag{5.32a}$$

$$B = \frac{y(x)}{r} \tag{5.32b}$$

for the symmetry G.

With this transformation equation (5.3) is reduced to

$$2yy'' + y'^2 + 1 = 0 \tag{5.33}$$

Equation (5.33) is a second order nonlinear ordinary differential equation. It can be integrated implicitly to give

$$\pm\frac{1}{2}C_1 \arctan\left(\frac{y-\frac{1}{2}C_1}{\sqrt{-y^2+C_1 y}}\right) = x + C_2 \pm \sqrt{-y^2+C_1 y} \tag{5.34}$$

where C_1 and C_2 are arbitrary constants. Since $y = rB(r,t)$ we have

$$\pm\frac{1}{2}C_1 \arctan\left(\frac{rB-\frac{1}{2}C_1}{\sqrt{-r^2B^2+C_1 rB}}\right) = t + C_2 \pm \sqrt{-r^2B^2+C_1 rB} \tag{5.35}$$

which are implicit solutions for equation (5.3).

The result (5.35) is a new solution to the fundamental boundary condition at the stellar surface. This solution does not appear in any other published works in relativistic astrophysics. Using the symmetry aG_2+G_3 we have found the new implicit solution (5.35) to the governing equation (5.3). This solution is obtained for the special value $a=-1$. We intend to pursue the study of the physical features of this model in future research.

5.3.5 Separable solutions

The solution obtained in equation (5.25) suggests that the master equation (5.3) admits solutions which are separable. We let

$$B(r,t) = f(r)g(t) \tag{5.36}$$

where f and g are arbitrary. Then (5.3) yields

$$\frac{(2f(r)+rf'(r))\,f'(r)}{rf^4(r)} = 2\ddot{g}(t)g(t) + \dot{g}^2(t) \tag{5.37}$$

and the variables have been separated. The method of separation of variables is normally applied to linear equations; here we have shown that it leads to simplification for the nonlinear equation (5.3). We can write

$$\frac{(2f(r) + rf'(r))\, f'(r)}{rf^4(r)} = p \tag{5.38a}$$

$$2\ddot{g}(t)g(t) + \dot{g}^2(t) = p \tag{5.38b}$$

where p is a constant.

Equations (5.38) comprise a nonlinear system of ordinary differential equations. Equation (5.38a) can be solved in terms of real functions in general for all values of p. Equation (5.38b) can be integrated implicitly for different values of p, and admits explicit solutions in terms of a complex variable. For the special value of $p = 0$ we can find solutions in terms of real functions:

$$f(r) = \frac{\alpha}{r^2} \tag{5.39a}$$

$$g(t) = \sqrt[3]{(\beta t + \gamma)^2} \tag{5.39b}$$

where α, β and γ are arbitrary constants. Then we can write

$$B(r,t) = \frac{\sqrt[3]{(C_1 t + C_2)^2}}{r^2} \tag{5.40}$$

which is a solution for the master equation (5.3) where we set $C_1 = \alpha^{3/2}\beta$ and $C_2 = \alpha^{3/2}\gamma$ for convenience.

The result (5.40) is a new solution to the boundary condition. Using separability of the governing equation (5.3) we have found this new exact solution. This solution has been obtained without using symmetries to transform equation (5.3) to ordinary differential equation. This solution will help in the analysis of the physical features of a radiating star in geodesic motion in astrophysics since it has a simple form.

5.3.6 Extending known solutions

By using the symmetries (5.4), we have the following global form of general point transformations that leaves equation (5.3) invariant:

$$\bar{t} = tK_2K_3 + K_1 \tag{5.41a}$$

$$\bar{r} = rK_3 \tag{5.41b}$$

$$\overline{B} = BK_2 \tag{5.41c}$$

where $K_1 = \epsilon_1$, $K_2 = e^{\epsilon_2}$ and $K_3 = e^{\epsilon_3}$ are arbitrary constants. In this transformation $K_2, K_3 > 0$.

Thus all the solutions previously found in this chapter can be transformed to potentially new solutions via (5.41).

5.4 Discussion

Utilizing `PROGRAM LIE` we found symmetries for equation (5.3). We then determined an optimal system of one-dimensional subalgebras for this set of symmetries. Each of the subalgebras was used to reduce the highly nonlinear partial differential equation (5.3) to an ordinary differential equation. We obtained solution for each ODE. Motivated by the form of these solutions, we also assumed that the PDE was separable. The separability of this equation led us to a new solution. The generators, the transformed ODEs and the solutions for the boundary condition in this chapter are summarised in Table 5.3.

Generators	ODEs	Solutions for the PDE
G_1	$y'\left(2y+y'\right)=0$	$B(r,t)=r^{-2}$
G_2	$xy^4-2yy'-xy'^2=0$	$B(r,t)=\frac{4Ct}{C^2r^2-4}$
aG_2+G_3 $(a=-1)$	$1+y'^2+2yy''=0$	$\pm\frac{1}{2}C_1\arctan\left(\frac{rB-\frac{1}{2}C_1}{\sqrt{-r^2B^2+C_1rB}}\right)$ $=t+C_2\pm\sqrt{-r^2B^2+C_1rB}$
No generator	$\frac{(2f(r)+rf'(r))f'(r)}{rf^4(r)}=\text{constant}$ $2\ddot{g}(t)g(t)+\dot{g}^2(t)=\text{constant}$	$B(r,t)=\frac{\sqrt[3]{(C_1t+C_2)^2}}{r^2}$

Table 5.3: Generators, ODEs and solutions for the PDE

Chapter 6

Conclusion

Most of the physically important partial differential equations are nonlinear. While there is no existing general theory for solving such equations the method of group analysis is a powerful tool. Knowing the symmetry group allowed us to determine some special types of solutions that are invariant under a subgroup of the full symmetry group, and in some cases we could solve the equations completely. The primary purpose of this dissertation has been to provide exact solutions to the boundary condition of a relativistic radiating stellar model, by using the Lie analysis of differential equations. We now provide an overview of the main results obtained during the course of our investigations:

- In chapter two, we briefly discussed the Lie group analysis approach of DEs. This method was then illustrated using the example of the KdV equation. The optimal system of the symmetries admitted by the KdV equation was also discussed. We indicated how to reduce the order of an equation, find its group invariant solutions and extend known solutions of ordinary differential equations by using symmetries.

- In chapter three, we discussed important aspects of differential geometry and defined the field equations of the interior spacetime of a shear-free radiating star based on its line element. We generated the junction condition relating the pressure to the heat flux at the boundary of a radiating star, which was first proved by Santos (1985).

- In chapter four, we considered a shear-free radiating star in conformally flat spacetime. We established the junction condition equation which relates the pressure to heat flux as the master equation of this chapter. We demonstrated that the master equation admits three Lie point symmetries. By using these symmetries, we obtained an optimal system of symmetries. This was used to reduce the governing highly nonlinear partial differential equation to ordinary differential equations. We also used a symmetry which is not in the optimal system. By solving the reduced ordinary differential equations and transforming to the original variables we obtained exact solutions for the master equation. All the solutions obtained in this chapter are explicit solutions for the governing partial differential equations. We believe that these are new solutions for the boundary conditions.

- In chapter five, we considered a shear-free radiating star in geodesic motion. We established the junction condition equation as the master equation in this chapter. Similar to chapter four, we demonstrated that the master equation admits three Lie point symmetries. We developed the optimal system using these symmetries. We used the optimal system to reduce the partial differential equation to ordinary differential equations in the hope of finding solutions to the master equation. The results obtained in this chapter were explicit and implicit solutions for the governing partial differential equation. Another solution was obtained by observing that the master equation is separable even though it is nonlinear. We believe that these results are new.

We emphasize that the solutions obtained in this thesis are not contained in the literature. They represent new solutions to the boundary condition relating the pressure to the heat flux. The physical feature of our new results are likely to throw more light on the evolution of the relativistic radiating star. This will be pursued in future work.

References

[1] Bluman G W and Anco S C, Symmetries and integration methods for differential equations (Springer-Verlag: New York) (2002).

[2] Bluman G W and Cole J D, Similarity methods for differential equations (Springer-Verlag: New York) (1974).

[3] Bluman G W and Kumei S, Symmetries and differential equations (Springer-Verlag: New York) (1989).

[4] Bluman G W, Cheviakov A F and Anco S C, Applications of symmetry methods to partial differential equations (Springer-Verlag: New York) (2010).

[5] Chan R, Da Silva M F A and Da Rocha J F V, Gravitational collapse of self-similar and shear-free fluid with heat flow, *Int. J. Mod. Phys.* D **12**, 347 (2003).

[6] de Oliviera A K G, Santos N O and Kolassis C A, Collapse of a radiating star, *Mon. Not. R. Astron. Soc.* **216**, 1001 (1985).

[7] Govender G, Govender M and Govinder K S, Thermal behaviour of Euclidean stars, *Int. J. Mod. Phys. D* **19**, 1773 (2010).

[8] Govender M and Maharaj S D, Collapse of a charged radiating star with shear, *Pramana-J. Phys.* **54**, 715 (2000).

[9] Govender M, Maharaj S D and R Maartens, A causal model of radiating stellar collapse, *Class. Quantum Grav.* **15**, 323 (1998).

[10] Grammenos T and Kolassis C, An analytic model of radiating spherical gravitational collapse, *Phys. Lett. A* **169**, 5 (1992).

[11] Head A K, LIE, a PC program for Lie analysis of differential equations, *Comp. Phys. Comm.* **71**, 241 (1993).

[12] Herrera L and Santos N O, Thermal evolution of compact objects and relaxation time, *Mon. Not. R. Astron. Soc.* **287**, 161 (1997).

[13] Herrera L, le Denmat G, Santos N O and Wang A, Shear-free radiating collapse and conformal flatness, *Int. J. Mod, Phys. D* **13**, 583 (2004).

[14] Herrera L and Santos N O, Collapsing spheres satisfying an "Euclidean condition", *Gen. Relativ. Gravit.* **42**, 2383 (2010).

[15] Herrera L, di Prisco A and Ospino J, Some analytical models of radiating collapsing spheres, *Phys. Rev. D* **74**, 044001 (2006).

[16] Herrera L, Martin J and Ospino J, Anisotropic geodesic fluid spheres in general relativity, *J. Math. Phys.* **43**, 4889 (2002).

[17] Hill J M, Differential equations and group methods for scientists and engineers (CRC Press: Florida) (1992).

[18] Hydon P E, Symmetry methods for differential equations (Cambridge University Press: Cambridge) (2000).

[19] Ibragimov N H, CRC handbook of Lie group analysis of differential equations, (CRC Press: Florida) (1994).

[20] Kolassis C A, Santos N O and Tsoubelis D, Friedmann-like collapsing model of a radiating sphere with heat flow, *Astrophys. J.* **327**, 755 (1988).

[21] Maharaj S D and Govender M, Radiating collapse with vanishing Weyl stresses, *Int. J. Mod. Phys. D* **14**, 667 (2005).

[22] Malcolm L, General Relativity: Geometric approach (Cambridge University Press: Cambridge) (2004).

[23] MathWorks, Inc., `MATLAB` version 7.14.0.41010 (MathWorks, Inc.: Boston) (2009).

[24] Misthry S S, Maharaj S D and Leach P G L, Nonlinear shear-free radiative collapse, *Math. Meth. Appl. Sci.* **31**, 363 (2008)., Leach 1995

[25] Miwa T, Solitons: Differential equations, symmetries and infinite dimensional algebras (Cambridge University Press: Cambridge) (2000).

[26] Naidu N F, Govender M and Govinder K S, Thermal evolution of a radiating anisotropic star with shear, *Int. J. Mod. Phys. D* **15**, 1053 (2006).

[27] Olver P J, Applications of Lie groups to differential equations (Springer-Verlag: New York) (1993).

[28] Pfeifer W, The Lie algebra su(N): An introduction, (Birkhäuser Verlag: Basel) (2003).

[29] Rajah S S and Maharaj S D, A Riccati equation in radiative stellar collapse, *J. Math. Phys.* **49**, 012501 (2008).

[30] Raychaudhuri A K, Relativistic cosmology I, *Phys. Rev.* **98**, 1123 (1955).

[31] Santos N O, Non-adiabatic radiating collapse, *Mon. Not. R. Astron. Soc.* **216**, 403 (1985).

[32] Sean M C, Lecture Notes on General Relativity, *arXiv: [gr-qc]* 9712019v1 (1997).

[33] Stephani H, Relativity: An Introduction to special and general relativity (Cambridge University Press: Cambridge) (2004).

[34] Thirukkanesh S and Maharaj S D, Mixed potentials in radiative stellar collapse, *J. Math. Phys.* **51**, 072502 (2010).

[35] Thirukkanesh S and Maharaj S D, Radiating relativistic matter in geodesic motion, *J. Math. Phys.* **50**, 022502 (2009).

[36] Tomimura N A and Nunes F C, Radiating spherical collapse with shear and heat flow, *Astrophys. Space Sci.* **199**, 215 (1993).

[37] Vaidya P C, The gravitational field of a radiating star, *Proc. Ind. Acad. Sci. A* **33**, 264 (1951).

[38] Wagh S and Govinder K S, Spherically symmetric, self-similar spacetimes *Gen. Relativ. Gravit.* **38**, 1253 (2006).

[39] Wald R, General relativity (University of Chicago Press: Chicago) (1984).

[40] Wolfram Research, Inc., `MATHEMATICA`, version 8.0 (Wolfram Research, Inc., Champaign: Illinios) (2008).