University of KwaZulu-Natal

New exact solutions for neutral and charged shear-free relativistic fluids

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by

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Submitted in fulfilment of the academic

requirements for the degree of

Doctor of Philosophy

in the

School of Mathematics, Statistics and Computer Science,

University of KwaZulu-Natal,

Durban,

November 2022

As the candidate's supervisors, we have approved this thesis for submission.

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Abstract

We study shear-free gravitating fluids in general relativity. We first analyse the integrability of the Emden-Fowler equation that governs the behaviour of shear-free neutral perfect fluid distributions. We find a new exact solution and generate a new first integral. The first integral is subject to an integrability condition which can be expressed as a third order differential equation whose solution can be expressed in terms of elementary functions and elliptic integrals. We extend this approach to include the effect of the electromagnetic charge. The Einstein-Maxwell system for a charged shear-free matter can be reduced to a generalized Emden-Fowler equation. We integrate this equation and find a new first integral. For this solution to exist two integral equations arise as integrability conditions. The integrability conditions can be solved to find new solutions. In both cases the first integrals are given parametrically. Our investigations suggest that complexity of a self-gravitating fluid is related to the existence of a first integral. For both neutral and charged fluids the general form of the parametric solution depends on a cubic and quartic polynomial respectively. The special case of repeated roots leads to simplification and this regains earlier results. We also study relativistic charged shear-free gravitating fluids in higher dimensions. Two classes of exact solutions to the Einstein-Maxwell equations are found. We obtain these solutions by reducing the Einstein-Maxwell equations to a single second order nonlinear partial differential equation containing two arbitrary functions. This generalizes the condition of pressure isotropy to higher dimensions; the new condition is functionally different from four dimensions. The new exact solutions obtained in higher dimensions reduce to known results in four dimensions. The presence of higher dimensions affects the dynamics of relativistic fluids in general relativity.

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I, Sfundo Cebolenkosi Gumede

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Declaration 2- Publications

Details of contribution to publications that form part of the research presented in this thesis:

Chapter 2

Publication 1

Gumede, S.C.; Govinder, K.S.; Maharaj, S.D. First integrals of shear-free fluids and complexity. *Entropy* **2021**, 23, 1539.

Chapter 3

Publication 2

Gumede, S.C.; Govinder, K.S.; Maharaj, S.D. Charged shear-free fluids and complexity in first integrals. *Entropy* **2022**, 24, 645.

Chapter 4

Publication 3

Gumede, S.C.; Maharaj, S.D.; Govinder, K.S. The role of dimensions in charged relativistic shear-free fluids. *Fortschritte der physik* **2022** (submitted).

Acknowledgements

I would like to express my sincere gratitude to the following people and organisations for their contributions throughout my doctoral journey:

- Professor Sunil Maharaj for his tireless patience, dedication, motivation as well as inspiration. Professor Maharaj did not only become my research supervisor, he also developed my research passion and provided me with numerous advices and opportunities for growth.
- Professor Kesh Govinder for his expert guidance and supervision. His reliable and constructive support and feedback made this research to be a success.
- Administrative staff in the School of Mathematics, Statistics and Computer Science, especially Laila Gurudas for all much needed administrative assistance.
- Mangosuthu University of Technology for their support financially and otherwise.
- My wife (Buyi), my daughter (Courtesy) and my son (Mihla) for always supporting me and understanding that I had to spend less time with them for this study to be successful.
- My mother (Cebisile), my grandmother (MaMncwango) and my whole family for their continuous motivation.

Contents

Chapter 1

Introduction

The Einstein theory of general relativity is very useful in describing gravitational interactions between bodies. Before general relativity was proposed, gravitational interactions between bodies was mainly described using the classical Newtonian theory. However, the Newtonian theory could not explain certain astronomical observations, hence the need to create a new gravitational theory. In general relativity the gravitational field of a body can be described by the curvature of spacetime rather than just a force that attracts objects to one another. Spacetime is considered to be a four dimensional differentiable manifold endowed with a non-degenerate metric tensor field. The spacetime geometry is described using the Einstein tensor, which is defined in terms of the metric tensor, the Ricci tensor and the Ricci scalar. The Einstein tensor is a very important quantity in general relativity as it is used to generate the Einstein field equations. The Einstein field equations are a system of nonlinear partial differential equations that relate spacetime curvature and the matter content. The matter content is expressed in terms of the energy momentum tensor containing the energy density, pressure, heat flux and anisotropy. In the presence of the electromagnetic field, the Einstein field equations can be extended to the Einstein-Maxwell equations. The action integral of general relativity can be extended to include higher order curvature connections and scalar fields.

Exact solutions to the Einstein field equations are used to model many astrophysical and cosmological processes. Well known solutions to the field equations include the Schwarzschid interior solution (Schwarzschild 1916a), the Schwarzschid exterior solution (Schwarzschild 1916b), the Reissner-Nordström solutions (Nordström 1918, Reissner 1916), the Vaidya solution (Vaidya 1951) as well as the Kerr solution (Kerr 1963). Spherical geometry is of specific interest. In general, spherically symmetric spacetimes are expanding, accelerating and shearing. The absence of shear in spherically symmetric spacetimes simplifies the field equations and is a special case of physical interest. Stephani et al. (2009) provide various categories of shear-free spherically symmetric solutions. These solutions include those obtained by Stephani (1983), Srivastava (1987) as well as Maharaj et al. (1996) amongst others. The exact solutions have been obtained using various methods from differential equations including the group theoretical approach.

A primary aim of this study is to seek exact solutions to the Einstein field equations for spherically symmetric shear-free fluid distributions. We achieve this by reducing the system of field equations to the single Emden-Fowler equation

$$y_{xx} = f(x)y^2$$

under a certain transformation. Kustaanheimo and Qvist (1948) were the first to find a general solution to this equation in general relativity. Various techniques have been used to analyse this Emden-Fowler equation. These involve the Lie symmetry analysis (Maharaj et al. 1996) as well the Noether symmetry approach (Wafo Soh and Mahomed 1999). We adopt an $adhoc$ method of Maharaj et al. (1996) of analysing integrability conditions to obtain new solutions. Incorporating the electromagnetic field, the field equations are supplemented by Maxwell's equations to form the Einstein-Maxwell equations. The charged system of field equations reduce to the generalization of the Emden-Fowler equation

$$y_{xx} = f(x)y^2 + g(x)y^3$$

under a specific transformation as given by Faulkes (1969). In this thesis we study the integrability of this equation using the approach of Kweyama et al. (2012) which is a charged generalization of the study of Maharaj et al. (1996).

Another objective of this study is to investigate the role of dimensions in gravitating fluids. Exact solutions to the Einstein field equations have been investigated by a number of researchers in higher dimensional spacetimes. The concept of higher dimensions in general relativity was first introduced by Kaluza (1921) and Klein (1926). The existence of higher dimensions is important in the description of dynamics of stellar objects. A number of exact solutions in four dimensions have been generalized to higher dimensions. For example, the Schwarzschild spacetime was generalized to higher dimensions by Tangherlini (1963). Myers and Perry (1986) studied black hole solutions to the Einstein field equations in higher dimensions, generalizing the Reissner-Nordström as well as the Kerr spacetimes. The Vaidya spacetime was later generalized by Iyer and Vishweshwara (1989) and Chatterjee et al. (1990). Banerjee et al. (1992) provided classes of higher dimensional exact solutions to the Einstein field equations for shear-free uncharged fluids by first reducing the field equations to a single second order partial differential equation. The effect of higher dimensions has been shown to be crucial in the study of modified gravity theories. For example, Brassel et al. (2019) recently studied higher dimension black holes in the Einstein-Gauss-Bonnet (EGB) gravity. Gravitational collapse and the formation of singularities in modified gravity theories depend on spacetime dimension. In this thesis we generalize the approach of Banerjee et al. (1992) by including the effects of the electromagnetic field. We reduce the consequent Einstein-Maxwell equations to a single partial differential equation which depends on the spacetime dimension. We present new classes of exact solutions for charged shear-free relativistic fluids in higher dimensions.

This thesis is organised as follows:

In this chapter we give the general background on general relativity and the Einstein's field equations. We discuss some examples of exact solutions to the Einstein field equations for uncharged matter as well as for the Einstein-Maxwell equations for charged matter. We also provide the outline of this thesis.

In Chapter 2 we show that the Einstein field equations for neutral shear-free fluids reduce to a second order partial differential equation of Emden-Fowler type. We use an approach utilized by Maharaj et al. (1996) to solve this equation. In our treatment we first multiply the Emden-Fowler equation by an integrating factor and apply integration by parts. We generate a new first integral which is subject to an integrability condition that is different from that of Maharaj et al. (1996). We reduce the integrability condition to a third order ordinary differential equation which we solve to obtain new solutions which do not reduce to known solutions.

In Chapter 3 we include the effect of the electromagnetic charge in the field equations. We show that the Einstein-Maxwell equations can be reduced to a single second order differential equation which is a generalization of the Emden-Fowler equation. This equation has been studied extensively using various approaches by Wafo Soh and Mahomed (2000) and Kweyama et al. (2012) among others. We adopt the approach of Kweyama et al. (2012). However, in our treatment we first multiply the master equation by an integrating factor. We obtain a new first integral which is subject to two integrability conditions that we solve, leading to new exact solutions to the Einstein-Maxwell equations for charged shear-free relativistic fluids. Our new solutions do not have an uncharged limit. We conclude the chapter by showing that our new first integral is independent of the Kweyama et al. (2012) result.

In Chapter 4 we analyse the charged condition of pressure isotropy for shear-free relativistic fluids in higher dimensions. We reduce the Einstein-Maxwell equations in higher dimensions to a single second order differential equation which depends on the spacetime dimension N. We present two new classess of exact solutions to the Einstein-Maxwell equations in higher dimensions. We find new gravitational potentials which depend on N. We show that when $N = 4$ the new solutions reduce to the known solutions obtained by Gürses and Heydazarde (2019). We also regain the solutions of Shah and Vaidya (1968). We conclude the chapter by graphically illustrating the effect of dimension in the field equations.

In Chapter 5 we provide a summary of results presented in this thesis.

Chapter 2

First integrals of shear-free fluids and complexity

2.1 Introduction

In many studies the concept of complexity has been applied to topics such as entropy and information. An intriguing approach is to also utilize this concept in self-gravitating systems. Herrera (2018) suggested that complexity in gravity would be studied by the definition of a minimal complexity factor. This approach may also be applied to dissipative fluids in general relativity with applications to compact stars, neutron stars and radiating objects in the strong gravity regime. Several investigations have been initiated involving the concept of complexity in self-gravitating systems in general relativity and some modified theories of gravity (Casadio et al. 2019, Herrera et al. 2018, Herrera et al. 2019, Herrera et al 2020, Herrera et al 2021, Sharif and Butt 2018a, Sharif and Butt 2018b, Sharif and Butt 2019, Sharif and Tariq 2020, Sharif et al. 2019). Jasim et al. (2021) studied a strange star model in a special case of Lovelock theory, namely Einstein-Gauss-Bonnet gravity, and showed that such theories are consistent with the concept of complexity. General matter distributions including dissipative effects are necessary to analyze relativistic self-gravitating fluids. Shear-free matter distributions arise as a special case, and deserve special attention because of their applicability to stellar models, and they have been used to model both static and radiating stars. Therefore in this investigation we consider the behaviour of shear-free fluids in a spherical spacetime. Our results indicate that it is possible to find new first integrals which provide insight into the behaviour of the self-gravitating fluids. Our approach may help in generating a general relationship between first integrals, extended to a general shearing relativistic matter and complexity of a self-gravitating relativistic fluid.

Seeking exact solutions to the Einstein field equations has been the subject of study in many astrophysical and cosmological applications. Such solutions may be used to model inhomogeneous processes in systems of galaxies and the broader universe (Krasinski 2006). Exact solutions to the field equations have also been used to model and investigate properties of observable phenomena such as relativistic stars (Shapiro and Teukolsky 1983) as well as expanding and contracting spherical stars (Santos 1985). Dissipative fluids in general relativity are of particular importance because of various applications in astrophysics and the description of radiating stars. The general framework for the study of physically acceptable dissipating systems in spherical symmetry was undertaken in several studies (Barreto and Da Silva 1999, Barreto et al. 2007, Di Prisco et al. 2007, Herrera et al. 2011, Sharif and Bashir 2012). Some particular exact models have been found using this framework (Mahomed et al. 2020a, Mahomed et al. 2020b, Sharif and Iftikhar 2015, Thirukkanesh and Govender 2013). The special case of vanishing shear provides new insights into the behaviour of gravity, and some particular radiating stellar models have been generated (Charan et al. 2021, Pinheiro and Chan 2013, Shah and Abbas 2018, Sharif and Bhatti 2013). Our approach in this chapter is to find a general result, namely a first integral, in a shear-free fluid without having to specify the gravitational potentials.

When seeking exact solutions to the Einstein field equations, it is usual to assume spherical symmetry for spacetimes and the absence of shear for the matter distribution. These assumptions greatly simplify the field equations while ensuring that the results are still physically meaningful.

Spherically symmetric shear-free solutions have been used to model many physical applications. Some of the classes of these solutions were obtained by Maharaj et al.(1996), Srivastava (1987), Stephani (1983) as well as Sussman (1988a, 1988b). Brassel et al. (2015) found gravitational potentials for shear-free heat conducting fluids in terms of elementary functions in a recent treatment. However, it is important to note that most of the known exact solutions to the Einstein field equations are not shearing (Stephani et al. 2009). This is largely due to the fact that the shear-free condition introduces an additional equation which needs to be solved. The Einstein field equations for spherically symmetric shear-free neutral matter comprise a system of nonlinear partial differential equations. We will show how this system of equations can be further reduced to the single, Emden-Fowler, equation

$$y_{xx} = f(x)y^2.$$

The first general solution to this Emden-Fowler equation in general relativity was found by Kustaanheimo and Qvist (1948) for a specified form of the function $f(x)$. Other classes of solutions were found later by Srivastava (1987) and Stephani (1983). Maharaj et al. (1991), Sussman (1988a) and Wafo Soh and Mahomed (2000) found further classes of solutions by assuming that the spacetime is invariant under a conformal Killing vector. Another recent treatment of this problem was given by Maharaj et al. (1996).

The shear-free condition is often applied in the study of radiating stars, gravitational collapse, and relativistic astrophysical processess. The vanishing shear assumption leads to simplification of the field equations. Note that the homogeneous expansion rate and shear-free condition are the classical analogue of homologous fluids in the Newtonian limit. This means that the shear-free assumption in general relativity is well justified. However, it is important to point out that the shear-free fluids may be unstable due to pertubations arising from anisotropy and dissipation.

Herrera et al. (2010) investigated the conditions when an initial shear-free configuration continues to be shear-free as the system evolves. Pressure anisotropy and dissipation affect the propagation of the relativistic fluid. These quantities play a role in realistic modelling involving gravitational collapse and should be contained in a stable model

In this chapter we analyse the integrability and find exact solutions to the Emden-Fowler equation using an *ad hoc* method that was previously shown to be useful (Maharaj et al. 1996). In Section 2.2 we show how the field equations for the spherically symmetric nonstatic shear-free metric reduce to the equation

$$y_{xx} = f(x)y^2.$$

We obtain its first integral in Section 2.3. This first integral is subject to the integrability condition which we study in Section 2.4. In Section 2.5 we find the functional form the function $f(x)$ and give the corresponding first integral.

2.2 Shear-free fluids

The Einstein field equations follow from variation of the Lagrangian

$$L = \frac{1}{2}R + L_m,$$

where R is the Ricci scalar and L_m represents the matter source. Variation of the Lagrangian L leads to the Einstein field equations

$$R_{ab} - \frac{1}{2}Rg_{ab} = (\mu + p)\,u_a u_b + pg_{ab},$$

for a perfect fluid source with energy density μ and pressure p.

The metric for a shear-free, perfect fluid in the comoving and isotropic coordinate system $(x^a) = (t, r, \theta, \phi)$ is given by

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\phi^2)\right], \tag{2.2.1}$$

where $e^{2\nu}$ and $e^{2\lambda}$ are the gravitational potentials. The Einstein field equations take the form

$$\mu = 3\frac{{\lambda_t}^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right), \tag{2.2.2a}$$

$$\begin{aligned} p &= \frac{1}{e^{2\nu}}\left(-2\lambda_{tt} - 3\lambda_t^2 + 2\nu_t\lambda_t\right) \\ &\quad + \frac{1}{e^{2\lambda}}\left(\lambda_r^2 + 2\nu_r\lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r}\right), \end{aligned} \tag{2.2.2b}$$

$$\begin{aligned} p &= \frac{1}{e^{2\nu}}\left(-2\lambda_{tt} - 3\lambda_t^2 + 2\nu_t\lambda_t\right) \\ &\quad + \frac{1}{e^{2\lambda}}\left(\nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} + \frac{\lambda_{rr}}{r} + \lambda_{rr}\right), \end{aligned} \tag{2.2.2c}$$

$$0 = \nu_r\lambda_t - \lambda_{tr}. \tag{2.2.2d}$$

The quantities μ and p represent the energy density and pressure, respectively. The subscripts r and t in equations (2.2.2) above represent partial derivatives with respect to r and t, respectively.

The condition of pressure isotropy is obtained by equating (2.2.2b) and (2.2.2c). The resulting equation can be easily integrated once with respect to time which results in an arbitrary function of integration, $g(r)$. We can also integrate (2.2.2d) once with respect to r and obtain another arbitrary function of integration, $h(t)$. Using these simplifications we can now write the system

(2.2.2) in the form

$$\mu = 3e^{2h} - e^{-2\lambda}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right), \tag{2.2.3a}$$

$$p = \frac{1}{\lambda_t}\left[e^{-2\lambda}\left(\lambda_r^2 + \frac{2\lambda_r}{r}\right) - e^{2h}\right]_t, \tag{2.2.3b}$$

$$e^{\nu} = \lambda_t e^{-h}, \tag{2.2.3c}$$

$$e^{\lambda}\left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right) = -g(r). \tag{2.2.3d}$$

The functions h and g need to be specified in order to find exact solutions for the field equations. Thereafter, the metric function λ can be obtained by solving (2.2.3d), while the remaining metric function ν then follows directly from (2.2.3c). Equations (2.2.3a) and (2.2.3b) then define the energy density μ and the isotropic pressure p, respectively. It is clear that the pivotal equation is (2.2.3d).

Using the transformation

$$x = r^2,$$

$$y(x,t) = e^{-\lambda},$$

$$f(x) = \frac{g}{4r^2},$$

equation (2.2.3d) reduces to

$$y_{xx} = f(x)y^2, \tag{2.2.4}$$

as first shown by Kustaanheimo and Qvist (1948). Equation (2.2.4) is the master equation governing the gravitational dynamics of a shear-free fluid in general relativity.

There have been a number of studies seeking solutions of the field equation (2.2.4). However, the solution is known for only a few forms of $f(x)$. The solution with

$$f(x) = \left(a + bx + cx^2\right)^{-5/2},$$

was given by Kustaanheimo and Qvist (1948). Solutions with

$$f(x) = x^{-20/7}, x^{-15/7}, e^x,$$

were found by Stephani (1983). General analyses of the equation (2.2.4) were completed by Maharaj et al. (1996), Stephani et al. (2009) and Wafo Soh and Mahomed (2000). A charged generalization was studied by Kweyama et al. (2012).

Our approach mirrors Maharaj et al. (1996), who extended an idea of Srivastava (1987). We briefly outline that approach here. We can integrate the left hand side of (2.2.4) once directly and the right hand side by repeated applications of 'integration-by-parts'. This eventually yields the first integral (Maharaj et al. 1996)

$$\psi_0(t) = -y_x + f_I y^2 - 2f_{II} y y_x + 2f_{III} y_x^2 + 2[(f f_{II})_I - \tfrac{1}{3}K_0]y^3, \tag{2.2.5}$$

where $\psi_0(t)$ is an arbitrary function of integration, K_0 is an arbitrary constant,

$$f_I = \int f dx \tag{2.2.6}$$

and we have the integrability condition

$$2f f_{III} + 3(f f_{II})_I = K_0. \tag{2.2.7}$$

This equation was then throughly analysed by Maharaj et al. (1996) to find new exact solutions including the explicit forms of $f(x)$ given by

$$f(x) = \frac{48}{343}\left(-\frac{7b}{2}\right)^{6/7}(x - x_0)^{15/7}, \tag{2.2.8}$$

which led to (2.2.5). Note that b and x_0 are arbitrary constants of integration.

2.3 A new first integral

We now apply the approach of Maharaj et al. (1996) but with one important difference. We observe that when (2.2.4) is multiplied by x, it becomes

$$xy_{xx} = xfy^2. \tag{2.3.1}$$

If we now define

$$\bar{f} = xf,$$

then we can write (2.3.1) in the form

$$xy_{xx} = \bar{f}y^2. \tag{2.3.2}$$

We can still integrate the left hand side explicitly once while the right hand side of (2.3.2) is simply the right hand side of (2.2.4) with f relabelled as $\bar{f}$. Thus the Maharaj et al. (1996) integration will apply to the right hand side of (2.3.2) as well. We obtain

$$xy_x - y = \bar{f}_I y^2 - 2\int \bar{f}_I y y_x dx - \phi_1(t), \tag{2.3.3}$$

where for convenience we have used

$$\bar{f}_I = \int \bar{f} dx, \tag{2.3.4}$$

and $\phi_1(t)$ is an arbitary function of integration. Note the subtle difference between (2.2.6) and (2.3.4).

Integrating $\bar{f}_I y y_x$ by parts and using (2.2.4) we obtain

$$xy_x - y = \bar{f}_I y^2 - 2\bar{f}_{II} y y_x + 2\int \bar{f}_{II} y_x^2 dx + 2\int f\bar{f}_{II} y^3 dx - \phi_1(t). \tag{2.3.5}$$

Integrating $f\bar{f}_{II}y^3$ and $\bar{f}_{II}y_x^2$, by parts again, we obtain

$$xy_x - y \;\; = \;\; \bar{f}_I y^2 - 2\bar{f}_{II} y y_x + 2\bar{f}_{III} y_x^2 + 2(f\bar{f}_{II})_I y^3$$

$$-2\left[\int [2f\bar{f}_{III} + 3(f\bar{f}_{II})_I]y^2 y_x dx\right] - \phi_1(t). \tag{2.3.6}$$

The integral on the right hand side of (2.3.6) can be evaluated if

$$2f\bar{f}_{III} + 3(f\bar{f}_{II})_I = K_1, \tag{2.3.7}$$

where K_1 is a constant. This equation can be written as a differential equation which still needs to be solved (see later.).

We now have the expression

$$\begin{aligned} \phi_1(t) &= y - xy_x + \bar{f}_I y^2 - 2\bar{f}_{II} y y_x + 2\bar{f}_{III} y_x^2 \\ &\quad + 2\left[(f\bar{f}_{II})_I - \frac{1}{3}K_1\right] y^3, \end{aligned} \tag{2.3.8}$$

where $\phi_1(t)$ is an arbitrary function of integration. We then observe that (2.3.8) is another, new, first integral of (2.2.4) provided that condition (2.3.7) is satisfied. (It is important to emphasize that (2.3.7) is different from (2.2.7) since $\bar{f}_I = \int xf(x)dx$.)

2.4 Integrability conditions

To complete the analysis we need to determine the form of the function $f(x)$ (or $\bar{f}(x)$). In an attempt to seek the form of the function f, the integral equation (2.3.7) can be transformed into an ordinary differential equation. It is easier to solve the differential equation rather than the integral equation. Differentiating (2.3.7), we obtain

$$2f_x\bar{f}_{III} + 5f\bar{f}_{II} = 0, \tag{2.4.1}$$

which can be written as

$$2\left[\bar{f}_x - \frac{1}{x}\bar{f}\right]\bar{f}_{III} + 5\bar{f}\bar{f}_{II} = 0, \tag{2.4.2}$$

which contains $\bar{f}$ only. Now setting

$$\bar{L} \equiv \bar{f}_{III},$$

we eliminate $\bar{f}$ in (2.4.2) to give the differential equation

$$2\left[\bar{L}_{xxxx} - \frac{1}{x}\bar{L}_{xxx}\right]\bar{L} + 5\bar{L}_x\bar{L}_{xxx} = 0. \tag{2.4.3}$$

We can integrate (2.4.3) once to obtain

$$\bar{L}_{xxx} = C_0 x \bar{L}^{-5/2}, \tag{2.4.4}$$

where C_0 is a constant of integration.

We observe that the third order ordinary differential equation (2.4.4) is equivalent to the integrability condition (2.3.7). Integrating (2.4.4) repeatedly we find $\bar{L}$, and hence $\bar{f}(x)$. Below we indicate how the equation (2.4.4) can be integrated, giving $\bar{L}$.

The nonlinear differential equation (2.4.4) may be written as

$$\bar{L}\bar{L}_{xxx} = C_0 x \bar{L}^{-\frac{3}{2}}.$$

The left hand side can be expressed in the exact form

$$(\bar{L}\bar{L}_{xx})_x - \bar{L}_x\bar{L}_{xx} = (\bar{L}\bar{L}_{xx})_x - \frac{1}{2}(\bar{L}_x^2)_x.$$

Integrating we obtain

$$\bar{L}\bar{L}_{xx} - \frac{1}{2}\bar{L}_x^2 = C_1 + C_0\int x\bar{L}^{-\frac{3}{2}}dx,$$

where C_1 is constant. Again, focussing on the left hand side we can write

$$x\bar{L}^{-\frac{3}{2}}\left(\bar{L}\bar{L}_{xx}-\frac{1}{2}\bar{L}_x^2\right)=x(\bar{L}^{\frac{1}{2}})_{xx}=\left[x(\bar{L}^{\frac{1}{2}})_x\right]_x-(\bar{L}^{\frac{1}{2}})_x,$$

and so we have the equation

$$\left[x(\bar{L}^{\frac{1}{2}})_x\right]_x-(\bar{L}^{\frac{1}{2}})_x=C_1x\bar{L}^{-\frac{3}{2}}+C_0x\bar{L}^{-\frac{3}{2}}\int x\bar{L}^{-\frac{3}{2}}dx,$$

where we have absorbed the factor of $\frac{1}{2}$ in C_0 and C_1. This can be easily integrated to yield

$$x(\bar{L}^{\frac{1}{2}})_x-\bar{L}^{\frac{1}{2}}=C_2+C_1\int x\bar{L}^{-\frac{3}{2}}dx+\frac{1}{2}C_0\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^2, \tag{2.4.5}$$

where C_2 is a new constant. The equation above is not in standard form. However, it is still possible to make progress. When multiplied by a factor $x\bar{L}^{-\frac{3}{2}}$, equation (2.4.5) above can be written as

$$\begin{aligned}\frac{1}{2}x^2\bar{L}^{-2}\bar{L}_x-x\bar{L}^{-1} &= C_2x\bar{L}^{-\frac{3}{2}}+C_1x\bar{L}^{-\frac{3}{2}}\int x\bar{L}^{-\frac{3}{2}}dx\\ &\quad+\frac{1}{2}C_0x\bar{L}^{-\frac{3}{2}}\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^2.\end{aligned}$$

The left hand side can be written as a total derivative, and we have

$$\begin{aligned}\left(-\frac{1}{2}x^2\bar{L}^{-1}\right)_x &= C_2x\bar{L}^{-\frac{3}{2}}+C_1x\bar{L}^{-\frac{3}{2}}\int x\bar{L}^{-\frac{3}{2}}dx\\ &\quad+\frac{1}{2}C_0x\bar{L}^{-\frac{3}{2}}\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^2.\end{aligned}$$

The integral of this equation is

$$\begin{aligned}-\frac{1}{2}x^2\bar{L}^{-1} &= C_3+C_2\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)+\frac{1}{2}C_1\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^2\\ &\quad+\frac{1}{6}C_0\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^3,\end{aligned}$$

where C_3 is constant. This can be simplified to

$$x^2\bar{L}^{-1} = -2C_3 - 2C_2\left(\int x\bar{L}^{-\frac{3}{2}}dx\right) - C_1\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^2$$

$$-\frac{1}{3}C_0\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^3.$$

Redefining the constants in the above equation, we can write it as

$$x^2\bar{L}^{-1} = \bar{C}_3 + \bar{C}_2\left(\int x\bar{L}^{-\frac{3}{2}}dx\right) + \bar{C}_1\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^2$$

$$+\bar{C}_0\left(\int x\bar{L}^{-\frac{3}{2}}dx\right)^3, \tag{2.4.6}$$

where $\bar{C}_3 = -2C_3$, $\bar{C}_2 = -2C_2$, $\bar{C}_1 = -C_1$ and $\bar{C}_0 = -\frac{C_0}{3}$. Therefore the third order equation (2.4.4) has been integrated to yield the solution (2.4.6).

In general we can write the solution parametrically. For convenience, we let

$$u = \int x\bar{L}^{-\frac{3}{2}}dx,$$

so that (2.4.6) becomes

$$x^2u_x = \left(\bar{C}_3 + \bar{C}_2u + \bar{C}_1u^2 + \bar{C}_0u^3\right)^{\frac{3}{2}}.$$

In the above equation the variables separate, and we can write

$$x_0 - \frac{1}{x} = \int \frac{du}{\left(\bar{C}_3 + \bar{C}_2u + \bar{C}_1u^2 + \bar{C}_0u^3\right)^{\frac{3}{2}}}, \tag{2.4.7}$$

where x_0 is constant. Now the function $\bar{f}(x)$ must be found satisfying the integrability condition (2.3.7). In order to find $\bar{f}(x)$ satisfying this integrability condition, it is convenient to express the solution in the parametric form

$$\bar{f}(x) = \bar{L}_{xxx},$$

$$u_x = x\bar{L}^{-\frac{3}{2}},$$

$$x_0 - \frac{1}{x} = p(u),$$

where

$$p(u) = \int \frac{du}{\left(\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3\right)^{\frac{3}{2}}}.$$

The evaluation of the integral is determined by the values of the constants $\bar{C}_0, \bar{C}_1, \bar{C}_2$ and $\bar{C}_3$.

In summary, we have found a new first integral of (2.2.4) given by (2.3.8) where f in (2.2.4) is obtained via L after evaluating the integral in $p(u)$.

2.5 Particular solutions

The evaluation of the integral in (2.4.7) has five cases depending on the nature of the factors of the polynomial $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$. (Since the coefficients are arbitrary, the discriminant does not help us to reduce the options.) The five cases are:

Case I One order-three factor

Case II One order-one factor and one order-two factor

Case III Three order-one (non-repeated) factors

Case IV One linear factor and one quadratic factor

Case V No factors

In order to illustrate the process we provide the details of the calculation for some of these cases.

Case I: One order-three factor

This is the simplest case as the factors are repeated. If $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has one factor repeated three times then we can write

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = (A + Bu)^3,$$

with $B \neq 0$. In this case, the integral in (2.4.7) can be evaluated to give

$$x_0 - \frac{1}{x} = -\frac{2}{7B}(A + Bu)^{-7/2},$$

so that

$$\bar{L} = x^{2/3} {u_x}^{-2/3} = x^2 \left(-\frac{2}{7B}\right)^{-6/7} \left(x_0 - \frac{1}{x}\right)^{6/7}.$$

Differentiating $\bar{L}$ three times we obtain

$$f(x) = \frac{48}{343x^5} \left(-\frac{2}{7B}\right)^{-6/7} \left(x_0 - \frac{1}{x}\right)^{-15/7}. \tag{2.5.1}$$

After reparametrisation, $f(x)$ can be written as

$$f(x) \sim \frac{1}{x^5} \left(1 - \frac{1}{x}\right)^{-15/7}. \tag{2.5.2}$$

Note that in this case the function $f(x)$ can be found explicitly. This functional form is different from (2.2.8) in the approach of Maharaj et al. (1996). Hence the first integral (2.3.8) for this case is a new solution to the Emden-Fowler equation. For applications this form of the solution is probably easier to utilize in modelling as $f(x)$ has a simple explicit form. Now we can write the first integral (2.3.8) in terms of x as follows

$$\begin{aligned} \phi_1(t) &= y - x y_x + 2 \left(-\frac{2}{7B}\right)^{-6/7} \left(x_0 - \frac{1}{x}\right)^{6/7} y^2 \\ &\quad + \frac{12}{7x} \left(-\frac{2}{7B}\right)^{-6/7} \left(x_0 - \frac{1}{x}\right)^{-1/7} y^2 \end{aligned}$$

$$-\frac{6}{49x^2}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-8/7}y^2$$

$$-4x\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{6/7}yy_x$$

$$-\frac{12}{7}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-1/7}yy_x$$

$$+2x^2\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{6/7}y_x^2$$

$$+\left[\frac{192}{343x^4}\left(-\frac{2}{7B}\right)^{-12/7}\left(x_0-\frac{1}{x}\right)^{-9/7}\right]_I y^3$$

$$+\left[\frac{576}{2401}\left(-\frac{2}{7B}\right)^{-12/7}\left(x_0-\frac{1}{x}\right)^{-16/7}\right]_I y^3$$

$$-\frac{2}{3}K_1y^3. \qquad (2.5.3)$$

The subscripts I in the equation (2.5.3) denote a remaining integration which we have omitted for brevity. It can be observed that this first integral is different from the first integral obtained by Maharaj et al. (1996).

In addition to the analysis performed by Maharaj et al. (1996), we substitute the function given in (2.5.1) into the integrability condition (2.3.7) in order to find restriction(s) on the constant K_1. This substitution implies that $K_1 = 0$ in the first integral (2.5.3). Similarly, in the Maharaj et. al (1996) solution, substituting (2.2.8) into the integrability condition (2.2.7) yields $K_0 = 0$ in the first integral (2.2.5).

Case II: One order-one factor and one order-two factor

If $\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3$ has one factor repeated then we can write

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = (A + Bu)(u + C)^2,$$

with $B \neq 0$. In this case, the integral in (2.4.7) can be evaluated using Gradshteyn and Ryzhik (1983) to obtain

$$\begin{aligned} p(u) &= \left(\frac{15B^2}{4(A - BC)^3} + \frac{5B}{4u(A - BC)^2} - \frac{1}{2u^2(A - BC)} \right) \frac{1}{\sqrt{A + Bu - BC}} \\ &\quad + \frac{15B^2}{8(A - BC)^3} \int \frac{du}{u\sqrt{A + Bu - BC}}, \end{aligned} \tag{2.5.4}$$

where the integral can be expressed in terms of elementary functions depending on the sign of $A - BC$. For this case it is not possible to obtain the function $u(x)$ explicitly, as it is not possible to evaluate the integral on the right hand side of (2.5.4). Therefore the solution can only be given parametrically. However, the integral on the right hand side of (2.5.4) can be evaluated in special cases; for example $A = 0$ and $C = 0$. If $A = 0$, we can write

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = Bu(u + C)^2.$$

Using the computer package Maple (Monagan et al. 2005) to evaluate the integral in (2.4.7), we obtain

$$\begin{aligned} p(u) &= -\frac{7}{4}\frac{\sqrt{Bu}}{BC^3(BC + Bu)} - \frac{15}{4}\frac{\arctan\left(\frac{\sqrt{Bu}}{\sqrt{BC}}\right)}{BC^3\sqrt{BC}} - \frac{1}{2}\frac{\sqrt{Bu}}{c^2(BC + Bu)^2} \\ &\quad - \frac{2}{BC^3\sqrt{Bu}}. \end{aligned}$$

If $C = 0$, then we have that

$$\bar{C}_3 + \bar{C}_2 u + \bar{C}_1 u^2 + \bar{C}_0 u^3 = (A + Bu)u^2.$$

We evaluate the integral in (2.4.7) using Mathematica [104] to obtain

$$p(u) = \frac{15B^2u^2 + 5ABu - 2A^2}{4A^3u^2\sqrt{A+Bu}} - \frac{15B^2 \tanh\left(\frac{\sqrt{A+Bu}}{\sqrt{A}}\right)}{4A^{7/2}}.$$

We observe that even for these special cases of A and C, it is not easy to perform the inversion in order to obtain the function $u(x)$ explicitly.

Case III: Three order-one (non-repeated) factors

If $\bar{C}_3 + \bar{C}_2u + \bar{C}_1u^2 + \bar{C}_0u^3$ has three non-repeated factors, then we can write

$$\bar{C}_3 + \bar{C}_2u + \bar{C}_1u^2 + \bar{C}_0u^3 = D(A-u)(B-u)(C-u),$$

with $D \neq 0$. In this case, with the aid of Mathematica (Wolfram 2007), the integral in (2.4.7) can be written in terms of elliptic integrals to obtain

$$\begin{aligned} p(u) \quad &= \quad \frac{2[C(A-C)+B(A-B)-u(2A-C-B)]}{D^{3/2}(A-B)(A-C)(B-C)^2\sqrt{(A-u)(B-u)(C-u)}} \\ &\quad + \frac{2[(B-C)(A+B-2C)F(\alpha,\beta)]}{D^{3/2}(A-B)^2(B-C)^2\sqrt{(A-C)^3}} \\ &\quad - \frac{2[(A^2+B^2+C^2-AB-AB-BC)E(\alpha,\beta)]}{D^{3/2}(A-B)^2(B-C)^2\sqrt{(A-C)^3}}, \end{aligned}$$

where we have set

$$\alpha = \arcsin\sqrt{\frac{A-C}{A-u}},$$

and

$$\beta = \sqrt{\frac{A-B}{A-C}}.$$

In this form of the solution, the quantities $F(\alpha,\beta)$ and $E(\alpha,\beta)$ are elliptic integrals of the first and second kind, respectively. In this case of non-repeated factors, we also cannot obtain $u(x)$

explicitly and hence the solution can only be given in parametric form.

In cases IV and V, the integral (2.4.7) can also be evaluated using elliptic functions. However, the subsequent expressions are lengthy and, since we cannot obtain $f(x)$ explicitly, we omit those results here.

2.6 Discussion

The Emden-Fowler equation $y_{xx} = f(x)y^2$ governs the behaviour of spherically symmetric shear-free uncharged fluid distributions. In this chapter we investigated the integrability of this equation and found a new class of exact solutions. This equation has several applications in general relativity and other areas of mathematical physics. We multiplied the Emden-Fowler equation by an integrating factor x and used integration by parts to obtain the first integral which is given by (2.3.8) subject to the integrability condition (2.3.7). The first integral and the integrability condition are different from the corresponding ones given in Maharaj et al. (1996). We were able to solve the integral equation (2.3.7) by first transforming it to a third order ordinary differential equation (2.4.4) whose solution was given by (2.4.6). For convenience we wrote the solution of (2.4.4) parametrically, which enabled us to find $f(x)$. One form of the function $f(x)$ was given by

$$f(x) \sim \frac{1}{x^5}\left(1 - \frac{1}{x}\right)^{-15/7},$$

in (2.5.2) so that the first integral could be written parametrically as (2.5.3). Remarkably we have obtained a new solution of the Emden-Fowler equation (2.2.4) with a new functional dependence of $f(x)$ given in (2.5.2). Note that the solutions by Maharaj et al. (1996), Srivastava (1987) and Stephani (1983) are not regained from our solution. Thus, our results complement those and together, constitute a more complete analysis of (2.2.4). This first integral may be related to the geometrical structure of the Emden-Fowler equation. The complexity of a self-gravitating relativistic shear-free fluid has been shown to be related to a first integral arising from the integration of the Emden-Fowler equation in our treatment.

Extensions of the approach in this chapter to include charged matter distributions may also lead to useful results. In the presence of charge, the Emden-Fowler equation (2.2.4) becomes

$$y_{xx} = f(x)y^2 + g(x)y^3, \tag{2.6.1}$$

where $g(x)$ is related to the charge distribution (see Wafo Soh and Mahomed (2000)). Equation (2.6.1) arises from the analysis of the Einstein-Maxwell field equations. It may be possible to consider extensions of this work to include anisotropy and dissipation, in addition to the electromagnetic field. For this physical scenario, the generalization of (2.6.1) will involve terms containing the heat flux and anisotropic pressure. The subsequent analysis of the resulting differential equation will involve an extension of the approach developed in this chapter. For a recent analysis of charged fluids with anisotropy and dissipation relevant to radiating stars, see the analysis of Abebe and Maharaj (2019), where the geometric approach of Lie symmetries provided new solutions. The complexity of a self-gravitating relativistically charged, anisotropic, and dissipative fluid will then be related to a first integral arising from the integration of the generalized Emden-Fowler equation; we will show this in the next chapter. This suggests that there may be a deeper connection between general matter fluids, first integrals and complexity. This deserves further investigation.

Chapter 3

Charged shear-free and complexity in first integrals

3.1 Introduction

The concept of complexity was introduced by Herrera (2018) for self-gravitating systems in general relativity. This approach has proved to be useful in studying the behaviour of highly dense stars, neutron stars and radiating stars in strong gravitational fields. Complexity has been studied in spherical systems, cylindrical systems, axial systems and hyperbolic systems by various researchers (Arias et al. 2022, Casadio et al. 2019, Herrera et al. 2019, Herrera et al. 2020, Herrera et al. 2021, Jasim et al. 2021, Maurya and Nag 2022, Maurya et al. 2022, Sharif and Butt 2018a, Sharif and Butt 2018b, Sharif and Butt 2019, Sharif and Tariq 2020) showing its applicability in a variety of applications. Apart from general relativity the concept of complexity has been studied in extended theories of gravity including Einstein-Gauss-Bonnet gravity, Lovelock gravity, $f(R)$ gravity and other generalizations (Abbas and Nazar 2018, Arias et al. 2022, Sharif et al. 2019, Yousaf 2020, Yousaf et al. 2020a, Yousaf et al. 2020b, Zubair and Azmat 2020). It is important to obtain a deeper insight into the behaviour of relativistic self-gravitating fluids, including dissipative effects. Charged shear-free relativistic fluids have been applied to many stellar systems including radiating stars with the Vaidya geometry describing the external atmosphere. In this study we focus on charged shear-free fluids with spherical symmetry. A new first integral is identified. This suggests a deeper connection between first integrals, charged dissipative distributions and the complexity of self-gravitating relativistic fluids in general. Observe that it is difficult to obtain first integrals directly from the field equations for the condition of pressure isotropy. Also note that first integrals are unique. In our treatment we show that a second first integral can be found, and we believe that this should be reflected in the complexity as defined by Herrera (2018), of the charged shear-free gravitating fluid. In a future study we intend to seek a general structure relating complexity to first integrals and gravitating relativistic fluids.

Exact solutions to the Einstein-Maxwell equations are important in relativistic astrophysics and cosmology as they are used to investigate properties of physical phenomena. The Einstein-Maxwell equations may be used to describe charged compact objects with strong electromagnetic effects (Kweyama et al. 2012). There has been substantial research in seeking exact solutions to the Einstein-Maxwell equations. This research include treatments of Ivanov (2002), Kweyama et al. (2012), Sharma et al. (2004) and Srivastava (1992) among others. Assumptions of spherical symmetry in spacetimes and shear-free matter distribution are usually made when seeking exact solutions to the Einstein field equations with uncharged matter. This simplifies the Einstein field equations to a single partial differential equation

$$y_{xx} = f(x)y^2$$

which can be transformed into an ordinary differential equation. Classes of solutions to this ordinary differential equation have been found by Kustaanheimo and Qvist (1948), Maharaj et al. (1996), Srivastava (1987), Stephani (1983) and Stephani et al. (2009). Similarly, when seeking exact solutions to the Einstein-Maxwell equations with charged matter, spherical symmetry and the absence of shear is usually assumed. These assumptions simplify the Einstein-Maxwell equations to a single partial differential equation

$$y_{xx} = f(x)y^2 + g(x)y^3.$$

This equation consists of an additional term $g(x)y^3$ compared to its uncharged counterpart. This term is due to the presence of the electromagnetic field. Kweyama et al. (2012) investigated integrability and found exact solutions to this equation using an approach suggested by Srivastava (1987). Krasinski (2006) provide a review of charged solutions with a Friedmann limit. Sussman (1988a, 1988b) performed a detailed physical analysis of the Einstein-Maxwell equations. The condition of vanishing shear has been applied to different physical applications in cosmology and astrophysics.

Vanishing shear leads to simplification of the Einstein-Maxwell equations. An important reason to consider the shear-free condition and homogeneous expansion rate is the connection to the analogue of homologous fluids in the classical Newtonian limit. This implies that the shear-free restriction has a meaningful basis in general relativity, and other gravity theories. It should be noted that shear-free fluids may become unstable because of pertubations due to anisotropic effects and dissipative effects. The stability of shear-free configurations, and general dissipative matter in relativistic astrophysics, has been studied in the treatments (Herrera 2020, Herrera et al. 2010, Herrera et al. 2011, Herrera et al. 2012, Noureen et al. 2015, Pinheiro and Chan 2013). As observed in these studies pressure anisotropy and dissipation are effects that should be studied, including the stability of the configuration, as the relativistic fluid evolves from the isotropic state. These quantities play an important role in models of gravitational collapse.

In this chapter we investigate the integrability properties and find exact solutions to the charged field equation

$$y_{xx} = f(x)y^2 + g(x)y^3$$

using an *ad hoc* approach adopted in Maharaj et al. (1996). In Section 3.2 we show how the

Einstein-Maxwell equations reduce to this master equation and briefly discuss the results obtained by Kweyama et al. (2012). We obtain our new first integral in Section 3.3. This first integral is subject to two integrability conditions which are integral equations. We solve these integral equations in Section 3.4. We find restrictions on the functions $f(x)$ and $g(x)$ in Section 3.5. Our results indicate that first integrals are obtainable for charged shear-free fluids extending the result of Gumede et al (2021).

3.2 Charged shear-free fluids

The set of the Einstein-Maxwell equations follow from variation of the Lagrangian

$$L = \frac{1}{2}\left(R - \frac{1}{4}F_{ab}F^{ab}\right) + L_m, \tag{3.2.1}$$

where R is the Ricci scalar, F_{ab} is the electromagnetic field tensor and L_m represents the matter source. Variation of the Lagrangian L leads to the Einstein-Maxwell equations

$$\begin{aligned} R_{ab} - \frac{1}{2}Rg_{ab} &= (\mu + p)\, u_a u_b + p g_{ab} \\ &\quad +2\left(F_a{}^c F_{bc} - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right), \end{aligned} \tag{3.2.2a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{3.2.2b}$$

$$F^{ab}{}_{;b} = \frac{1}{2}J^a, \tag{3.2.2c}$$

for a perfect fluid source with energy density μ and pressure p. Note that $J^a = \sigma u^a$ where σ is the proper charge density and u^a is a timelike fluid $4-$velocity.

We consider a spherical spacetime with the metric

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\lambda(t,r)}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\phi^2)\right], \tag{3.2.3}$$

for a charged perfect fluid. The Einstein field equations for the line element (3.2.3), for a shear-free and charged matter distribution, can be written as

$$\mu = 3\frac{{\lambda_t}^2}{e^{2\nu}} - \frac{1}{e^{2\lambda}}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right) - \frac{E^2}{r^4e^{4\lambda}}, \tag{3.2.4a}$$

$$\begin{aligned} p &= \frac{1}{e^{2\nu}}\left(-2\lambda_{tt} - 3\lambda_t^2 + 2\nu_t\lambda_t\right) \\ &+\frac{1}{e^{2\lambda}}\left(\lambda_r^2 + 2\nu_r\lambda_r + \frac{2\nu_r}{r} + \frac{2\lambda_r}{r}\right) + \frac{E^2}{r^4e^{4\lambda}}, \end{aligned} \tag{3.2.4b}$$

$$\begin{aligned} p &= \frac{1}{e^{2\nu}}\left(-2\lambda_{tt} - 3\lambda_t^2 + 2\nu_t\lambda_t\right) \\ &+\frac{1}{e^{2\lambda}}\left(\nu_{rr} + \nu_r^2 + \frac{\nu_r}{r} + \frac{\lambda_{rr}}{r} + \lambda_{rr}\right) - \frac{E^2}{r^4e^{4\lambda}}, \end{aligned} \tag{3.2.4c}$$

$$0 = \nu_r\lambda_t - \lambda_{tr}. \tag{3.2.4d}$$

These quantities are measured relative to the comoving fluid $4-$velocity $u^a = e^{-\nu}{\sigma^a}_0$. The gravitating equations are supplemented with the Maxwell equations

$$E = r^2e^{\lambda-\nu}\Phi_r, \tag{3.2.5a}$$

$$E_r = \sigma r^2 e^{3\lambda}. \tag{3.2.5b}$$

(The subscripts r and t represent partial derivatives with respect to r and t, respectively.) The term

$\Phi_r = F_{10}$ is the only nonzero component of the electromagnetic field tensor $F_{ab} = \phi_{b;a} - \phi_{a;b}$ with $\phi_a = (\Phi(t,r), 0, 0, 0)$. Note that σ is the proper charge density and E is the electric field intensity which represents the total charge of the distribution.

The Einstein-Maxwell system of equations (3.2.4) and (3.2.5) can also be written in the equivalent form

$$\mu = 3e^{2h} - e^{-2\lambda}\left(2\lambda_{rr} + \lambda_r^2 + \frac{4\lambda_r}{r}\right) - \frac{E^2}{r^4 e^{4\lambda}}, \tag{3.2.6a}$$

$$p = \frac{1}{\lambda_t e^{3\lambda}}\left[e^{\lambda}\left(\lambda_r^2 + \frac{2\lambda_r}{r}\right) - e^{3\lambda+2h} - \frac{E^2}{r^4 e^{4\lambda}}\right]_t, \tag{3.2.6b}$$

$$e^{\nu} = \lambda_t e^{-h}, \tag{3.2.6c}$$

$$e^{\lambda}\left(\lambda_{rr} - \lambda_r^2 - \frac{\lambda_r}{r}\right) = -\rho(r) - \frac{E^2}{r^4 e^{4\lambda}}, \tag{3.2.6d}$$

$$\sigma = r^{-2} e^{-3\lambda} E_r, \tag{3.2.6e}$$

where $h = h(t)$ and $\rho = \rho(r)$ are arbitrary functions of integration. The functions h and ρ need to be specified in order to find exact solutions for the field equations. The quantity $E = E(r)$ is also a function of integration. The metric function λ is obtained from the condition of pressure isotropy (3.2.6d) which has been generalized to include electromagnetic effects. The remaining metric function ν then follows from (3.2.6c). The energy density μ and the isotropic pressure p can be calculated using equations (3.2.6a) and (3.2.6b). Using the transformation

$$x = r^2,$$

$$y(x,t) = e^{-\lambda},$$

and setting

$$f(x) = \frac{\rho}{4r^2},$$

$$g(x) = \frac{E^2}{2r^6},$$

we can rewrite (3.2.6d) as

$$y_{xx} = f(x)y^2 + g(x)y^3. \tag{3.2.7}$$

The partial differential equation (3.2.7) is the master equation governing the gravitational dynamics of a shear-free charged fluid in general relativity. Since there are no temporal derivatives in (3.2.7) we can treat it as an ordinary differential equation but note that the arbitrary quantities that arise from integration are functions of t. If the function $g = 0$, then the equation reduces to $y_{xx} = f(x)y^2$ for a neutral fluid. The neutral case has been studied by many researchers including Gumede et al. (2021), Kustaanheimo (1948), Maharaj et al. (1996), Stephani (1983), Stephani et al. (2009) and Wafo Soh and Mahomed (1999).

A recent analysis of the master equation (3.2.7) was performed by Kweyama et al. (2012) where they found its first integral by directly integrating this equation using integration by parts. They found the first integral of (3.2.7) to be

$$\begin{aligned} \tau_0(t) &= -y_x + f_I y^2 + g_I y^3 - 2f_{II} y y_x + 2f_{III} y_x^2 \\ &\quad +2\left[(ff_{II})_I - \frac{1}{3}C_0\right] y^3 + [2(gf_{II})_I - C_1]y^4, \end{aligned} \tag{3.2.8}$$

subject to the integrability conditions

$$C_0 = 2ff_{III} + 3(ff_{II})_I + \frac{3}{2}g_I, \tag{3.2.9a}$$

$$C_1 \quad = \quad gf_{III} + 2(gf_{II})_I, \tag{3.2.9b}$$

where C_0 and C_1 are constants, $\tau_0(t)$ is an arbitrary function of integration, $f_I = \int f dx$ and $g_I = \int g dx$. The form (3.2.9) is difficult to analyse as they are integral equations. They can be converted to nonlinear differential equations. Solving the integral equations (3.2.9) gives specific forms of $f(x)$ and $g(x)$ in terms of elementary functions. In one instance, these functions are given by

$$f(x) = \frac{24}{75}(5b)^{4/5}(x - x_0)^{-11/5},$$

and

$$g(x) = C_0(5b)^{-12/5}(x - x_0)^{-12/5},$$

where b is an arbitrary constant and x_0 is a constant of integration.

We use a similar approach to obtain a new first integral of the charged generalization (3.2.7) subject to different integrability conditions to obtain different forms of $f(x)$ and $g(x)$ in the next section.

3.3 A first integral

In order to obtain (3.2.8), Kweyama et al. (2012) adopted a method first suggested by Srivastava (1987), and subsequently extended by Maharaj et al. (2012). The approach was simple to apply – the left hand side of (3.2.7) was integrated directly and the right hand side integrated by parts. However, note that the difficulty that arises is that the process yields integral equations which need to be solved to complete an exact solution. Observe that it is difficult to explicitly find first integrals in practice. There is no algorithm that generates them systematically. Here we show that a new first integral arises, in a simple approach, by adapting a previous method. We multiply equation (3.2.7) by a function to generate a new differential equation, eventually leading to a new first integral. There is no guarantee that this approach will work in general; we find that this simple idea is effective for a relativistic charged gravitating fluid.

We multiply (3.2.7) by x to obtain

$$xy_{xx} = \bar{f}y^2 + \bar{g}y^3, \tag{3.3.1}$$

where for convenience we have let

$$\bar{f} = xf,$$

and

$$\bar{g} = xg.$$

We observe that the left hand side of (3.3.1) can still be integrated directly, and we can apply integration by parts to the right hand side. This yields

$$xy_x - y = \bar{f}_I y^2 + \bar{g}_I y^3 - 2\int \bar{f}_I y y_x dx - 3\int \bar{g}_I y^2 y_x dx - \psi_1(t), \tag{3.3.2}$$

where we have let

$$\int \bar{f} dx = \int xf dx = \bar{f}_I,$$

and

$$\int \bar{g} dx = \int xg dx = \bar{g}_I,$$

for convenience, and $\psi_1(t)$ is a function of integration. Integrating $\bar{f}_I y y_x$ and using (3.2.7), we obtain

$$xy_x - y \;\; = \;\; \bar{f}_I y^2 + \bar{g}_I y^3 - 2\bar{f}_{II} y y_x + 2\int \bar{f}_{II} y_x^2 dx + 2\int f\bar{f}_{II} y^3 dx$$

$$+2\int g\bar{f}_{II}y^4dx - 3\int \bar{g}_I y^2 y_x dx - \psi_1(t). \tag{3.3.3}$$

Integrating $f\bar{f}_{II}y^3$, $g\bar{f}_{II}y^4$ and $\bar{f}_{II}y_x^2$ and substituting in (3.3.3), we obtain

$$\begin{aligned} xy_x - y &= \bar{f}_I y^2 + \bar{g}_I y^3 - 2\bar{f}_{II}yy_x + 2(f\bar{f}_{II})_I y^3 + 2\left(g\bar{f}_{II}\right)_I y^4 + 2\bar{f}_{III}y_x^2 \\ &\quad -\frac{2}{3}\int \left\{ \left[2f\bar{f}_{III} + 3(f\bar{f}_{II})_I + \frac{3}{2}\bar{g}_I\right]\left(\frac{d(y^3)}{dx}\right)\right\}dx \\ &\quad -\int \left\{ \left[g\bar{f}_{III} + 2(g\bar{f}_{II})_I\right]\left(\frac{d(y^4)}{dx}\right)\right\}dx - \psi_1(t). \end{aligned} \tag{3.3.4}$$

The integrals in (3.3.4) can be evaluated if

$$K_0 = 2f\bar{f}_{III} + 3(f\bar{f}_{II})_I + \frac{3}{2}\bar{g}_I, \tag{3.3.5a}$$

$$K_1 = g\bar{f}_{III} + 2(g\bar{f}_{II})_I, \tag{3.3.5b}$$

where K_0 and K_1 are arbitrary constants. A first integral of (2.2.4) is then given by

$$\begin{aligned} \psi_1(t) &= y - xy_x + \bar{f}_I y^2 + \bar{g}_I y^3 - 2\bar{f}_{II}yy_x + 2\bar{f}_{III}y_x^2 \\ &\quad +2\left[(f\bar{f}_{II})_I - \frac{1}{3}K_0\right]y^3 + \left[2(g\bar{f}_{II})_I - K_1\right]y^4, \end{aligned} \tag{3.3.6}$$

subject to the integral Equations (3.3.5). Note that (3.3.6) is a new first integral of (3.2.7) subject to new integrability conditions. Thus, the first integral exists for new functions $f(x)$ and $g(x)$ for a charged shear-free matter distribution. We show in Section 3.6 that this new first integral is independent of the charged first integral found by Kweyama et al. (2012).

3.4 Integral equations

The two equations in (3.3.5) are integral equations that need to be solved. To complete the analysis we need to determine the form of the functions $f(x)$ and $g(x)$. In an attempt to seek the form of the functions f and g, we rewrite the integral equations (3.3.5) as ordinary differential equations as these are (usually) easier to solve. Setting

$$\bar{L} = \bar{f}_{III}$$

and differentiating (3.3.5b) we obtain

$$\left(g\bar{L}\right)_x + 2g\bar{L}_x = 0,$$

whose solution is given by

$$g = K_2\bar{L}^{-3}. \tag{3.4.1}$$

In the equation above, K_2 is a constant of integration.

Differentiating (3.3.5a) and using (3.4.1) we obtain

$$f_x\bar{L} + \frac{5}{2}f\bar{L}_x = -\frac{3}{4}K_2x\bar{L}^{-3}$$

which can be written as the fourth order differential equation

$$\left(\frac{1}{x}\bar{L}^{5/2}\bar{L}_{xxx}\right)_x = -\frac{3}{4}K_2x\bar{L}^{-3/2}, \tag{3.4.2}$$

since $f = \frac{1}{x}\bar{f} = \frac{1}{x}\bar{L}_{xxx}$.

Below we show how (3.4.2) can be integrated repeatedly to obtain $\bar{L}$.

Integrating (3.4.2) once we obtain

$$\frac{1}{x}\bar{L}^{5/2}\bar{L}_{xxx} = K_3 - \frac{3}{4}K_2\int x\bar{L}^{-3/2}dx.$$

Multiplying this equation by $x\bar{L}^{-3/2}$ and writing the left hand side as a total derivative we obtain

$$\left(\bar{L}\bar{L}_{xx}\right)_x - \frac{1}{2}\left(\bar{L}_x^2\right)_x = K_3x\bar{L}^{-3/2} - \frac{3}{4}K_2x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx,$$

which integrates to

$$\bar{L}\bar{L}_{xx} - \frac{1}{2}\bar{L}_x^2 = K_4 + K_3\int x\bar{L}^{-3/2}dx - \frac{3}{8}K_2\left(\int x\bar{L}^{-3/2}\right)^2.$$

Multiplying this equation by $x\bar{L}^{-3/2}$ we can rewrite it as

$$\begin{aligned} x\left(\bar{L}^{1/2}\right)_{xx} &= K_4x\bar{L}^{-3/2} + K_3x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx \\ &\quad -\frac{3}{8}K_2x\bar{L}^{-3/2}\left(\int x\bar{L}^{-3/2}\right)^2. \end{aligned} \tag{3.4.3}$$

Since

$$x\left(\bar{L}^{1/2}\right)_{xx} = \left[x\left(\bar{L}^{1/2}\right)_x\right]_x - \left(\bar{L}^{1/2}\right)_x,$$

equation (3.4.3) can be written as

$$\begin{aligned} \left[x\left(\bar{L}^{1/2}\right)_x\right]_x - \left(\bar{L}^{1/2}\right)_x &= K_4x\bar{L}^{-3/2} + K_3x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx \\ &\quad -\frac{3}{8}K_2x\bar{L}^{-3/2}\left(\int x\bar{L}^{-3/2}\right)^2, \end{aligned}$$

and integrated to obtain

$$\begin{aligned} \left(\bar{L}^{1/2}\right)_x - \bar{L}^{1/2} &= K_4x\bar{L}^{-3/2} + K_3x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx \\ &\quad -\frac{3}{8}K_2x\bar{L}^{-3/2}\left(\int x\bar{L}^{-3/2}\right)^2. \end{aligned}$$

Multiplying the equation above by $x\bar{L}^{-3/2}$ and writing the left hand side as a total derivative we obtain

$$\begin{aligned}\left(-\frac{1}{2}x^2\bar{L}^{-1}\right)_x &= K_4 x\bar{L}^{-3/2} + K_3 x\bar{L}^{-3/2}\int x\bar{L}^{-3/2}dx \\ &\quad -\frac{3}{8}K_2 x\bar{L}^{-3/2}\left(\int x\bar{L}^{-3/2}\right)^2.\end{aligned}$$

Integrating yields

$$\begin{aligned}x^2\bar{L}^{-1} &= K_6 + K_5\int x\bar{L}^{-3/2}dx + \frac{K_4}{2}\left(\int x\bar{L}^{-3/2}\right)^2 \\ &\quad +\frac{K_3}{6}\left(\int x\bar{L}^{-3/2}\right)^3 - \frac{K_2}{32}\left(\int x\bar{L}^{-3/2}\right)^4, \qquad (3.4.4)\end{aligned}$$

where K_3, K_4, K_5 and K_6 are constants of integration and we absorbed a factor of $-\frac{1}{2}$ into the K_i's.

The solution of (3.4.4) can be written parametrically in general. The constant K_2 is related to the charge. For neutral fluids $K_2 = 0$ and the polynomial in $p(u)$ is third order. For charged fluids $K_2 \neq 0$ and the polynomial in $p(u)$ is fourth order. Hence the presence of the electromagnetic field changes the nature of the exact solutions that are permitted when compared to neutral matter.

It is convenient to define

$$u = \int x\bar{L}^{-3/2}dx,$$

so that (3.4.4) becomes

$$x^2 u_x = \left(K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4\right)^{3/2}.$$

This equation is separable and can be integrated to obtain

$$x_0 - \frac{1}{x} = \int \frac{du}{\left(K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4\right)^{3/2}}, \tag{3.4.5}$$

where x_0 is a constant of integration. The evaluation of the integral on the right hand side of (3.4.5) above depends on the nature of the roots of the polynomial $K_6+K_5u+\frac{1}{2}K_4u^2+\frac{1}{6}K_3u^3-\frac{1}{32}K_2u^4$. In order to find $\bar{f}(x)$ and $\bar{g}(x)$ satisfying the integrability conditions (3.3.5), it is convenient to express the solution in the parametric form

$$\begin{aligned}
\bar{f}(x) &= \bar{L}_{xxx}, \\
g &= K_2 \bar{L}^{-3} \\
u_x &= x\bar{L}^{-\frac{3}{2}}, \\
x_0 - \frac{1}{x} &= p(u),
\end{aligned}$$

where

$$p(u) = \int \frac{du}{\left(K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4\right)^{3/2}}. \tag{3.4.6}$$

3.5 Particular solutions

The evaluation of the integral in (3.4.5) can be reduced to nine cases depending on the nature of the factors of the polynomial

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4$$

that appear in $p(u)$. The nine cases correspond to

Case I: One order-four linear factor

Case II: One order-three linear factor

Case III: One order-two linear factor and one order-one quadratic factor

Case IV: One order-two linear factor and two order-one linear factors

Case V: Two order-two linear factors

Case VI: Four non-repeated linear factors

Case VII: One order-two quadratic factor

Case VIII: Two order-one quadratic factors

Case IX: One order-one cubic factor.

We discuss these cases below.

Case I: One order-four linear factor

If $K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4$ has one linear factor repeated four times then we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + bu)^4,$$

with $b \neq 0$. The integral in (3.4.5) or (3.4.6) can be evaluated to obtain

$$p(u) = -\frac{1}{5b}\,(a + bu)^{-5}\,,$$

so that

$$\bar{L} = x^2 \left(-\frac{1}{5b}\right)^{-4/5} \left(x_0 - \frac{1}{x}\right)^{4/5}.$$

Differentiating $\bar{L}$ three times and using (3.4.1) we obtain

$$f(x) = \frac{24}{125x^5}\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0 - \frac{1}{x}\right)^{-11/5}, \qquad (3.5.1a)$$

$$g(x) = \frac{K_2}{x^6}\left(-\frac{1}{5b}\right)^{12/5}\left(x_0 - \frac{1}{x}\right)^{-12/5}. \qquad (3.5.1b)$$

Hence the functions $f(x)$ and $g(x)$ can be found explicitly in this Case I. After reparametrisation we can write

$$f(x) \quad \sim \quad \frac{1}{x^5}\left(1-\frac{1}{x}\right)^{-11/5}, \tag{3.5.2a}$$

$$g(x) \quad \sim \quad \frac{1}{x^6}\left(1-\frac{1}{x}\right)^{-12/5}. \tag{3.5.2b}$$

This is the simplest form. The first integral (3.3.6) becomes

$$\begin{aligned}
&\psi_1(t) = y - xy_x + 2\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0-\frac{1}{x}\right)^{4/5}y^2 \\
&+\frac{8}{5x}\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0-\frac{1}{x}\right)^{-1/5}y^2 - \frac{4}{25x^2}\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0-\frac{1}{x}\right)^{-6/5}y^2 \\
&+\left[\frac{K_2}{x^5}\left(-\frac{1}{5b}\right)^{12/5}\left(x_0-\frac{1}{x}\right)^{-12/5}\right]_I y^3 - 4x\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0-\frac{1}{x}\right)^{4/5}yy_x \\
&+\frac{8}{5}\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0-\frac{1}{x}\right)^{-1/5}yy_x + 2x^2\left(-\frac{1}{5b}\right)^{-4/5}\left(x_0-\frac{1}{x}\right)^{4/5}y_x^2 \\
&+2\left[\frac{48}{125x^4}\left(-\frac{1}{5b}\right)^{-8/5}\left(x_0-\frac{1}{x}\right)^{-7/5} + \frac{96}{625x^2}\left(-\frac{1}{5b}\right)^{-8/5}\left(x_0-\frac{1}{x}\right)^{-12/5}\right]_I y^3 \\
&+2\left[\frac{2K_2}{x^5}\left(-\frac{1}{5b}\right)^{8/5}\left(x_0-\frac{1}{x}\right)^{-8/5} + \frac{4K_2}{5x^6}\left(-\frac{1}{5b}\right)^{8/5}\left(x_0-\frac{1}{x}\right)^{-13/5}\right]_I y^4 \\
&-\frac{2}{3}K_0y^3 - K_1y^4,
\end{aligned} \tag{3.5.3}$$

where the subscripts I denote the remaining integration. This first integral is a new solution to the Einstein-Maxwell equations for the functions f and g given in (3.5.2). It corresponds to a shear-free spherically symmetric charged fluid. Interestingly, there is no corresponding neutral solution as we must have $b \neq 0$ (equivalently $K_2 \neq 0$) otherwise the polynomial in (3.4.6) is not fourth order. This means that charge is always present.

As a final check on our results, we substitute the forms (3.5.1a) and (3.5.1b) into the integrability conditions (3.3.5) in order to find any restrictions on the constants K_0 and K_1. In this case, we find that these constants are both equal to zero. We note that the same restriction occurs in the Kweyama et al. (2012) model though this was not observed at that time.

Case II: One order-three linear factor

If $K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4$ has one order-three linear factor, then we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + bu)(u + c)^3.$$

We can evaluate the integral in (3.4.5) with the help of the package Mathematica (Wolfram 2007), to obtain

$$\begin{aligned} p(u) &= \frac{2\sqrt{(a+bu)(u+c)}}{35(a-bc)^5}\left[\frac{35b^4}{a+bu} + \frac{93b^3}{u+c} - \frac{29b^2(a-bc)}{(u+c)^2}\right] \\ &\quad + \frac{2\sqrt{(a+bu)(u+c)}}{35(a-bc)^5}\left[\frac{13b(a-bc)^2}{(u+c)^3} - \frac{5(a-bc)^3}{(u+c)^4}\right]. \end{aligned}$$

We observe that in this case, the integral in (3.4.5) can be expressed in terms of elementary functions. However, it is not straightforward to perform the inversion to find $u(x)$, and find $f(x)$ and $g(x)$ explicitly as in the previous case.

If we let $g=0$, $K_2=0$ and $b=0$, then

$$p(u) \;=\; \frac{2}{7}a^{-3/2}\,(u+c)^{-7/2}\,,$$

$$\bar{L} \;=\; a^{2/7}\left(\frac{2}{7}\right)^{-4/2}\left(-\frac{2}{7}\right)^{-2/3}x^2\left(x_0-\frac{1}{x}\right)^{6/7}.$$

After reparametrisation, $f(x)$ can be written as

$$f(x)\sim\frac{1}{x^5}\left(1-\frac{1}{x}\right)^{-15/7},$$

which was found previously in the case of a shear-free spherically symmetric uncharged fluid (Gumede et al. 2021). The corresponding uncharged first integral is given by

$$\begin{aligned}
\psi_1(t) \;=\;& y-xy_x+2\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{6/7}y^2\\
&+\frac{12}{7x}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-1/7}y^2\\
&-\frac{6}{49x^2}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-8/7}y^2\\
&-4x\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{6/7}yy_x\\
&-\frac{12}{7}\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{-1/7}yy_x\\
&+2x^2\left(-\frac{2}{7B}\right)^{-6/7}\left(x_0-\frac{1}{x}\right)^{6/7}y_x^2\\
&+\left[\frac{192}{343x^4}\left(-\frac{2}{7B}\right)^{-12/7}\left(x_0-\frac{1}{x}\right)^{-9/7}\right]_I y^3\\
&+\left[\frac{576}{2401}\left(-\frac{2}{7B}\right)^{-12/7}\left(x_0-\frac{1}{x}\right)^{-16/7}\right]_I y^3
\end{aligned}$$

$$-\frac{2}{3}K_1 y^3. \tag{3.5.4}$$

as established earlier in Chapter 2.

Case III: One order-two linear factor and one order-one quadratic factor

If $K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4$ has one order-two linear factor and one order-one quadratic factor, then we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + bu + cu^2)(u + d)^2,$$

with $b^2 - 4ac < 0$. We evaluate the integral in (3.4.5) with the aid of equations (2.266) and (2.269.6) in Gradshteyn and Rhyzhik (1983) to obtain

$$p(u) =$$

$$\left\{ \frac{15(b-2cd)^4 - 62c(b-2cd)^2(a-bd+cd^2) + 24c^2(a-bd+cd^2)^2}{2(a-bd+cd^2)[4c(a-bd+cd^2)-(b-2cd)^2]} \right.$$

$$+\frac{c(b-2cd)[15(b-2cd)^2 - 52c(a-bd+cd^2)]u}{2(a-bd+cd^2)\Delta}$$

$$\left. -\frac{1}{(a-bd+cd^2)u^2} - \frac{5(b-2cd)}{2(a-bd+cd^2)u} \right\} \frac{1}{2\sqrt{(a-bd+cd^2)+(b-2cd)u+cu^2}}$$

$$+\frac{15(b-2cd)^2 - 12c(a-bd+cd^2)}{8(a-bd+cd^2)^3} \int \frac{du}{u\sqrt{(a-bd+cd^2)+(b-2cd)u+cu^2}}, \tag{3.5.5}$$

where $\Delta = 4(a-bd+cd^2)c - (b-2cd)^2$. The exact form of the integral on the right hand side of (3.5.5) depends on the signs of Δ and $a - bd + cd^2$.

Specific examples of the constants a, b or d make the integral on the right hand side of (3.4.5) easier to write in terms of elementary functions. For example, if $a = 0$, then we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (bu + cu^2)(u + d)^2,$$

which can be evaluated using [equation (2.269)](Gradshteyn and Ryzhik 1983) to obtain

$$\begin{aligned} p(u) &= \frac{2}{7}\left\{ -\frac{1}{(b-2cd)u^3} + \frac{8c}{5(b-2cd)u^2} - \frac{16c^2}{5(b-2cd)^3 u} + \frac{64c^3}{5(b-2cd)^4} \right. \\ & \left. + \frac{128c^4 u}{5(b-2cd)^5} \right\} \frac{1}{\sqrt{(b-2cd)u + cu^2}}. \end{aligned}$$

As a second example if $b = 0$, then we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + cu^2)(u + d)^2.$$

Using Mathematica (Wolfram 2007) and evaluating the integral in (3.4.5) yields

$$\begin{aligned} p(u) &= \frac{1}{2}\left\{ \frac{-a^3 + 2c^3 d^3 u(d+u)^3 - a^2 c(10d^2 + 11du + 3u^2)}{a(a+cd^2)^3 (d+u)^2 \sqrt{a + cu^2}} \right. \\ & + \frac{ac^2 d(6d^3 + 6d^2 u - 14du^2 - 13u^3)}{a(a+cd^2)^3 (d+u)^2 \sqrt{a+cu^2}} - \frac{3c(a - 4cd^2)\log[d+u]}{(a+cd^2)^{7/2}} \\ & \left. + \frac{3c(a-4cd^2)\log[a - cdu + \sqrt{(a+cd^2)(a+cu^2)}]}{(a+cd^2)^{7/2}} \right\}. \end{aligned}$$

Thirdly if $d = 0$, then we have that

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + bu + cu^2)u^2.$$

With the aid of Mathematica (Wolfram 2007), we evaluate the integral in (3.4.5) to obtain

$$p(u) = \frac{1}{8a^{7/2}(-b^2 + 4ac)u^2\sqrt{a + u(b + cu)}}\left\{ -2\sqrt{a}[8a^3 c + 15b^3 u^2 (b + cu) \right.$$

$$+abu(5b^2 - 62bcu - 52c^2u^2) - 2a^2(b^2 + 10bcu - 12c^2u^2)]$$

$$-3(5b^4 - 24ab^2c + 16a^2c^2)u^2\sqrt{a + u(b + cu)}\log[u]$$

$$+3(5b^2 - 24ab^2c + 16a^2c^2)u^2\sqrt{a + u(b + cu)}\log[2a + bu + 2\sqrt{a}\sqrt{a + u(bu + c)}]\Big\},$$

expressed in terms of elementary functions. As a fourth example if $d = b = 0$, then (3.4.5) becomes

$$p(u) = \frac{-\sqrt{a}(a + 3cu^2) - 3cu^2\sqrt{a + cu^2}\left(\log[u] - \log[a + \sqrt{a}\sqrt{a + cu^2}]\right)}{2a^{5/2}u^2\sqrt{a + cu^2}}.$$

Finally, we study the case where $a = d = 0$. In this case, evaluating the integral in (3.4.5) yields

$$p(u) = \frac{2(-5b^4 + 8b^3cu - 16b^2c^2u^2 + 64bc^3u^3 + 128c^4u^4)}{35b^5u^3\sqrt{u(b + cu)}}.$$

We observe that if the polynomial $K_6 + K_5u + \frac{1}{2}K_4u^2 + \frac{1}{6}K_3u^3 - \frac{1}{32}K_2u^4$ has one order-two linear factor and one order-one quadratic factor, it is difficult to obtain $f(x)$ and $g(x)$ explicitly as the expressions for p cannot be inverted to obtain u.

Case IV: One order-two linear factor and two order-one linear factors

If the polynomial $K_6 + K_5u + \frac{1}{2}K_4u^2 + \frac{1}{6}K_3u^3 - \frac{1}{32}K_2u^4$ has one order-two linear factor and two order-one linear factors, then we have

$$K_6 + K_5u + \frac{1}{2}K_4u^2 + \frac{1}{6}K_3u^3 - \frac{1}{32}K_2u^4 = (a + bu)(c + du)(u + e)^2.$$

In this case, the integral in (3.4.5) can be expressed completely in terms of elementary functions and can be obtained using Mathematica (Wolfram 2007). However we do not include it here due to its length, and the fact that u cannot be obtained explicitly.

Case V: Two order-two linear factors

With two order-one linear factors, we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a+bu)^2(u+c)^2.$$

The integral in (3.4.5) may be evaluated to obtain

$$\begin{aligned} p(u) &= \frac{3b^2}{(a-bc)^4(a+bu)} + \frac{6b^2\log[u+c]}{(a-bc)^5} + \frac{1}{2(a-bc)^3(a+bu)^2} \\ & -\frac{6b^2\log[a+u]}{(a-bc)^5} - \frac{1}{2(a-bc)^3(u+c)^2} + \frac{3b}{(a-bc)^4(u+c)}. \end{aligned}$$

Setting the constants $a = 0$ or $c = 0$ simplifies the result. For example, if $c = 0$, then we have

$$\begin{aligned} p(u) &= \frac{3b^2}{a^4(a+bu)} + \frac{6b^2\log[u]}{a^5} + \frac{b^2}{2a^3(a+bu)^2} - \frac{6b^2\log[a+bu]}{a^5} \\ & +\frac{3b}{a^4 u} - \frac{1}{2a^3u^2}, \end{aligned}$$

while for $a = 0$ we have (Monagan et al. 2005)

$$\begin{aligned} p(u) &= \frac{6\log[u]}{b^3c^5} - \frac{6\log[u+c]}{b^3c^5} + \frac{3}{b^3c^4u} + \frac{1}{2b^3c^3(u+c)^2} \\ & +\frac{3}{b^3c^4(u+c)} - \frac{1}{2b^3c^3u^2}. \end{aligned}$$

However, due to the combination of logarithmic terms and powers of u, one cannot invert in order to obtain $u(x)$ explicitly.

Case VI: Four non-repeated linear factors

With four non-repeated linear factors, we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = e(a+u)(b+u)(c+u)(d+u)$$

with $e \neq 0$. The integral in (3.4.5) is given by

$$p(u) = \frac{2e^{-3/2}}{(a-b)\sqrt{(a+u)(b+u)(c+u)(d+u)}}$$

$$\times \left[\frac{(a+u)(b+u)}{(b-c)(a-d)}\right] \left[\frac{2}{(b-d)^2} \frac{1}{(b-d)(c-d)} + \frac{1}{(a-c)(c-d)}\right]$$

$$+ \left[\frac{(a+u)(b+u)}{(b-c)(a-d)}\right] \left[\left[\frac{2(d+u)(b+u)}{(a-b)(a-d)^2} \frac{1}{(b-c)(b-d)(a-c)}\right] - \frac{b+u}{(b-c)(b-d)(a-c)}\right]$$

$$- \frac{4e^{-3/2}}{(a-b)\sqrt{b-d}} \left[\frac{1}{(a-d)^2(c-d)\sqrt{a-c}} + \frac{\sqrt{a-c}}{(a-b)(b-d)(b-c)^2}\right] E(\alpha, p)$$

$$- \frac{4e^{-3/2}}{(a-b)\sqrt{b-d}} \left[\frac{a-b-c+d}{(b-c)(c-d)^2(a-c)^{3/2}}\right] E(\alpha, p)$$

$$+ \frac{2e^{-3/2}}{(a-d)(b-c)(a-c)^{3/2}(b-d)^{3/2}} \times \left[\frac{2(a+b-c-d)^2}{(b-c)(a-d)}\right] F(\alpha, p)$$

$$+ \left[\frac{(a-b-c+d)^2}{(a-b)(c-d)}\right] F(\alpha, p), \qquad (0 < d < c < b < a). \tag{3.5.6}$$

In the above

$$\alpha = \arcsin\sqrt{\frac{(a-c)(u+d)}{(a-d)(u+c)}}, \qquad p = \frac{(b-c)(a-d)}{(a-c)(b-d)},$$

and $F(\alpha, p)$ and $E(\alpha, p)$ are the elliptic integrals of the first and second kind, respectively (see also Kweyama et al. (2012)). We did not include the special cases in this case because they do not simplify the integral, which are still in terms of elliptic integrals and they do not have an uncharged limit.

Case VII: One order-two quadratic factor

With one order-two quadratic factor, we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + bu + cu^2)^2.$$

Using equation (2.173.2) in Gradshteyn and Ryzhik (1983), the integral in (3.4.5) may be evaluated to obtain

$$\begin{aligned} p(u) &= \frac{b + 2cu}{4ac - b^2}\left[\frac{1}{2(a + bu + cu^2)^2} + \frac{3c}{(4ac - b^2)(a + bu + cu^2)}\right] \\ &\quad + \frac{6c^2}{(4ac - b^2)^2}\int \frac{du}{a + bu + cu^2}, \end{aligned}$$

where the integral on the right hand side depends on the sign of $4ac - b^2$. In the special case of $a = 0$ we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (bu + cu^2)^2,$$

so that

$$p(u) = \frac{4b^2cu - b^3 + 18bc^2u^2 + 12c^3u^3}{2b^4u^2(b + cu)^2} + 6c^2\frac{\log[u] - \log[b + cu]}{b^5}.$$

For $b = 0$ we have

$$K_6 + K_5 u + \frac{1}{2}K_4 u^2 + \frac{1}{6}K_3 u^3 - \frac{1}{32}K_2 u^4 = (a + cu^2)^2,$$

which yields

$$p(u) = \frac{5au + 3cu^3}{8a^2(a + cu^2)^2} + \frac{3\arctan[\frac{u\sqrt{c}}{\sqrt{a}}]}{8a^{5/2}\sqrt{c}}.$$

Cases VIII and IX

In cases VIII and IX, the integral in (3.4.5) can also be evaluated using Mathematica (Wolfram 2007). The solution can be expressed in terms of elementary functions as well elliptic integrals but are omitted due to space considerations.

We therefore conclude that it is only in the case where $K_6 + K_5u + \frac{1}{2}K_4u^2 + \frac{1}{6}K_3u^3 - \frac{1}{32}K_2u^4$ has one linear factor repeated four times where we can easily use the integral in (3.4.5) to obtain specific functional forms of $f(x)$ and $g(x)$ that satisfy the integrability conditions (3.3.5).

3.6 Independence of the first integrals

In this section, we explore the possibility of both our first integral (3.3.5) and that of Kweyama et al. (2012) existing simultaneously. We note that those two first integrals exist subject to the integrability conditions (3.3.5) and (3.2.9). Differentiating equations (3.2.9b) and (3.3.5b) leads to

$$2gf_{II} + (gf_{III})_x = 0, \tag{3.6.1a}$$

$$2g\bar{f}_{II} + \left(g\bar{f}_{III}\right)_x = 0. \tag{3.6.1b}$$

The general solution of (3.6.1a) is given by

$$g = K_4\left(f_{III}\right)^{-3}. \tag{3.6.2}$$

Now, if we substitute (3.6.2) into (3.6.1b) we obtain the fourth order integral equation

$$3f_{IIII}f_{II} - 2\left(f_{III}\right)^2 = 0, \tag{3.6.3}$$

whose solution is

$$f_{IIII} = \frac{1}{27}\left(K_7x + K_8\right)^3. \tag{3.6.4}$$

Differentiating f_{IIII} four times leads to $f = 0$. In order to find the form of g that corresponds to $f = 0$, we substitute $f = 0$ in (3.2.7) to obtain

$$y_{xx} = gy^3, \tag{3.6.5}$$

whose first integral is given by

$$\begin{aligned} y_x &= g_I y^3 - 3\int g_I y^2 y_x dx \\ &= g_I y^3 - 3\int g_I \frac{1}{3}\left(\frac{dy^3}{dx}\right). \end{aligned} \tag{3.6.6}$$

The integral on the right hand side of (3.6.6) can be evaluated if $g_I = \bar{C}_0$, hence $g(x) = 0$. Similarly, if we let $f = 0$ in (3.3.1), the resulting first integral can be evaluated if $\bar{g}_I = \bar{C}_1$; that is if $g = 0$ as before.

Thus, the requirements of both sets of integrability conditions, arising for (3.2.9) and (3.3.5) force $f = g = 0$. This implies that the first integrals (3.2.9) and (3.3.5) are independent of each other.

3.7 Discussion

In this chapter we studied the equation $y_{xx} = f(x)y^2 + g(x)y^3$ which is a charged generalization of the Emden-Fowler equation. This equation is a consequence of the Einstein-Maxwell system of field equations, and it is important for describing the evolution of a relativistic charged shear-free matter distribution. We multiplied the charged Emden-Fowler equation by an integrating factor and obtained a new first integral (3.3.6) which is subject to consistency conditions (3.3.5). We emphasize that the conditions (3.3.5) are integral equations. Note that earlier charged first integrals are not contained in this solution. In particular we do not regain the result of Kweyama et al.

(2012). Thus our results complement existing treatments, and provide an independent analysis of the charged Emden-Fowler equation (3.2.7). Therefore charged shear-free fluids display desirable features of complexity in our treatment.

We summarize the results that have been obtained for the equation (3.2.7) in terms of first integrals. For neutral matter with $g(x) = 0$ interesting results were obtained by Gumede et al. (2021), Maharaj et al. (1996), Srivastava (1987), Stephani (1983) and Wafo Soh and Mahomed (1999). Some simple forms of the function $f(x)$ that were identified correspond to

$$f(x) \sim x^{-15/7},$$

and

$$f(x) \sim \frac{1}{x^5}\left(1 - \frac{1}{x}\right)^{-15/7}.$$

For charged matter $g(x) \neq 0$ and first integrals were obtained by Kweyama et al. (2012) and the results in this paper. The functional forms of $f(x)$ and $g(x)$ are given by

$$f(x) \sim \left(1 - \frac{1}{x}\right)^{-11/5}, \ g(x) \sim \left(1 - \frac{1}{x}\right)^{-12/5},$$

and

$$f(x) \sim \frac{1}{x^5}\left(1 - \frac{1}{x}\right)^{-11/5}, \ g(x) \sim \frac{1}{x^6}\left(1 - \frac{1}{x}\right)^{-12/5}.$$

The charged solutions arise as repeated roots of a fourth order polynomial. Note that the charged models do not have an uncharged limit since the polynomial then becomes a cubic which is a contradiction. Our results indicate that complexity of the system is affected by the presence of the electromagnetic field. In future work it would be interesting to investigate complexity in general dissipative fluids, including electromagnetic effects, and to consider geometries with less symmetry such as cylindrical and axial spacetimes.

Chapter 4

The role of dimensions in charged relativistic shear-free fluids

4.1 Introduction

Investigations of gravitating bodies in an electromagnetic field in general relativity include several early studies (Chatterjee 1984, Faulkes 1969, Nduka 1976, Shah and Vaidya 1968, Vaidya 1967) in spherical symmetry. Solutions to the Einstein-Maxwell equations for charged spherically symmetric bodies are important in studies of self-gravitating spheres, gravitational collapse, the formation of singularities and many other astrophysical processes. General methods have been developed to generate self-gravitating spheres, including static and nonstatic potentials. Some interesting approaches to find interior metrics in charged perfect fluid spheres are given by Fatema and Murad (2013), Ivanov (2002, 2021), Kiess (2012) and Murad and Fatema (2013, 2015). Choosing a generalized form of one of the metric functions leads to series solutions in terms of elementary and special functions. This approach has been utilized by Komathiraj and Maharaj (2007) and Thirukkanesh and Maharaj (2009). On physical grounds a barotropic equation of state is often imposed relating the isotropic pressure to the energy density. Over the years models have been found with several types of equations of states: linear (Thirukkanesh and Maharaj 2008), quark (Murad 2016), quadratic (Mafa Takisa et al. 2014) and polytropic (Mafa Takisa and Maharaj 2013, Mardan et al. 2020, Noureen et al. 2009, Ray et al. 2003). Nonlinear and linear equations of state were studied by Varela et al. (2010). The role of charge in superdense stars with several layers, forming core-envelope and multi-layered astronomical objects, has been studied in recent times. These studies indicate that charged layered objects are stable and physically acceptable (Lighuda et al. 2021a, Lighuda et al. 2021b, Mardan et al. 2021). Exact solutions to the Einsten-Maxwell system of differential equations are therefore of critical importance to study physical features of relativistic fluids, the spacetime geometry and physical processes in strong gravity fields.

If the spacetime is shear-free with isotropic pressures, then the solution of the Einstein-Maxwell system reduces to a single nonlinear differential equation, the charged condition of pressure isotropy. In the presence of charge this differential equation has been studied by Srivastava (1992) and Sussman (1988a) in four dimensions. Gürses and Heydazarde (2019) recently found simple forms of exact solutions using elementary methods. Lie and Noether symmetries were analysed, in relation to the condition of pressure isotropy, by Kweyama et al. (2011). First integrals were also generated, using a simple integration technique, by Gumede et al. (2022) and Kweyama et al. (2012).

The condition of pressure isotropy can also be found in N dimensions. The existence of higher dimensions is important in modelling charged stars in general relativity. For static stars the mass of a relativistic star changes with dimension as established by Paul (2001). Wafo Soh and Mahomed (2000) studied the higher dimensional condition of pressure isotropy equation with Noether symmetries which contains earlier results. Banerjee et al. (1992) found particular exact solutions in simple form. Maharaj and Brassel (2021) showed that the boundary condition for a nonstatic radiating star changes in different dimensions. The consequences of extra dimensions in the Einstein-Maxwell equations on some physical phenomena such as on the structure and a mass of a star have also been discussed extensively in Liddle et al. (1990). The effects of extra dimensions has also been studied in modified gravity theories such as the $f(R)$ theory (Buchdahl 1970), Lovelock gravity theory (Lovelock 1971), supergravity (Nath and Arnowitt 1975), Jackwick-Teitelboim gravity (Mann et al. 1990) and de Rham-Gabadadze-Tolley (dRGT) massive gravity (de Rahm and Gabadadze 2010). It is therefore necessary to consider the role of dimension in the charged condition of pressure isotropy.

In this chapter we analyse the charged condition of pressure isotropy for shear-free fluids in higher dimensions N. We show that the form of the master equation governing the behavior depends on the spacetime dimension N. This feature has been neglected in some earlier treatments. Exact solutions of the Einstein-Maxwell equations are presented, and we point out those cases which reduce to known $N = 4$ models. We derive a new form of the charged condition of pressure isotropy in N dimensions. The form is related to that of Gürses and Heydazarde (2019) when $N = 4$. New solutions in higher dimensions are found. The effect of dimensions on the new solutions is demonstrated graphically.

4.2 Field equations in higher dimensions

The Einstein-Maxwell field equations can be written in the form

$$G_{ab} = \kappa_N T_{ab}, \tag{4.2.1a}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0, \tag{4.2.1b}$$

$$F^{ab}{}_{;b} = \mathcal{A}_{N-2} J^a, \tag{4.2.1c}$$

where **G** is the Einstein tensor, **T** is the energy momentum tensor, **F** is the Faraday tensor and **J** denotes the current. In the above the quantity

$$\kappa_N = \frac{2\,(N-2)\,\pi^{(N-1)/2} G}{c^4\,(N-3)\left(\frac{N-1}{2}-1\right)!}$$

is the coupling constant, and

$$\mathcal{A}_{N-2} = \frac{2\pi^{\frac{N-1}{2}}}{\Gamma\left(\frac{N-1}{2}\right)},$$

is the surface area of the $(N-2)$-sphere. Note that both κ_N and $\mathcal{A}_{N-2}$ depend on the dimension N. Consequently the nature of the solutions of the field equations (4.2.1) are dependant on N. When $N = 4$, we have $\kappa_4 = \frac{8\pi G}{c^4}$ and $\mathcal{A}_2 = 4\pi$.

The matter distribution is a combination of barotropic and charged components. The total energy momentum tensor is given by

$$T_{ab} = (\rho + p)\, u_a u_b + p g_{ab} + E_{ab}.$$

The energy density ρ and the isotropic pressure p are measured relative to **u**. The vector **u** is a unit, timelike comoving N-velocity. The electromagnetic energy momentum tensor E_{ab} is defined by

$$E_{ab} = \frac{1}{\mathcal{A}_{N-2}}\left(F_a{}^c F_{bc} - \frac{1}{4} F^{cd} F_{cd}\right). \tag{4.2.2}$$

A choice has to be made, on physical grounds, for the electromagnetic potential Φ_a which then generates E_{ab} in (4.2.2) through the Faraday tensor $F_{ab} = \Phi_{b;a} - \Phi_{a;b}$. For the current we have

$$J^a = \xi u^a, \tag{4.2.3}$$

where ξ is the proper charge density.

We consider the N-dimensional shear-free nonstatic spacetime. The spacetime metric has the form

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}\left[dr^2 + r^2\left(d\Omega_{N-2}\right)^2\right], \tag{4.2.4}$$

in the comoving and isotropic coordinate system. We are utilizing spacetime coordinates $(x^a) = (t, r, \theta_1, \theta_2, ..., \theta_{N-2})$. The $(N-2)$-sphere is

$$\begin{aligned}
\left(d\Omega_{N-2}\right)^2 &= \left(d\theta_1\right)^2 + \sin^2\theta_1\left(d\theta_2\right)^2 + \sin^2\theta_1\sin^2\theta_2\left(d\theta_3\right)^2 + \cdots \\
&\quad + \sin^2\theta_1\sin^2\theta_2\sin^2\theta_3\cdots\sin^2\theta_{N-3}\left(d\theta_{N-2}\right)^2 \\
&= \sum_{i=1}^{N-2}\left[\prod_{j=1}^{i-1}\sin^2(\theta_j)\right]\left(d\theta_i\right)^2.
\end{aligned}$$

The metric functions have the dependence $\nu = \nu(t,r)$ and $\lambda = \lambda(t,r)$. For a spherically symmetric charged fluid the comoving N-velocity has the form

$$u^a = \left(\frac{1}{e^\nu}, 0, 0, \cdots, 0\right). \tag{4.2.5}$$

The total charge Q within the $(N-2)$-sphere of radius r is given by

$$Q = \mathcal{A}_{N-2}\int^r \xi e^{(N-1)\lambda}dr. \tag{4.2.6}$$

Note that $Q = Q(r)$ and is independent of the timelike coordinate t, and satisfies the conservation of current requirement $J^a{}_{;a} = 0$.

Then the Einstein-Maxwell equations for the shear-free line element (4.2.4) take the form

$$\kappa_N\rho + \frac{\kappa_N}{2\mathcal{A}_{N-2}}\cdot\frac{Q^2}{\left(re^\lambda\right)^{2N-4}} = \frac{(N-2)}{e^{2\nu}}\dot{\lambda}^2 + \left[\frac{(N-3)(N-2)}{2}\right]\left(\frac{1}{r^2e^{2\lambda}} + \frac{\dot{\lambda}^2}{e^{2\nu}}\right)$$

$$- \frac{(N-2)}{e^{2\lambda}} \left[\lambda'' + \lambda'^2 + \frac{2}{r}\lambda' + \left[\frac{N-3}{2}\right] \left(\frac{1}{r^2} + \frac{2}{r}\lambda' + \lambda'^2 \right) \right]$$

$$- \left[\frac{N-3}{2}\right] \left(\frac{\lambda'}{r} - \lambda'^2 \right), \tag{4.2.7a}$$

$$\kappa_N p - \frac{\kappa_N}{2\mathcal{A}_{N-2}} \cdot \frac{Q^2}{(re^\lambda)^{2N-4}} = \frac{(N-2)}{e^{2\nu}} \left[-\dot{\lambda}^2 - \ddot{\lambda} - \frac{(N-3)}{2}\dot{\lambda}^2 + \dot{\lambda}\dot{\nu} \right]$$

$$+ \frac{(N-2)}{e^{2\lambda}} \left[(N-3)\frac{\lambda'}{r} - \frac{(N-3)}{2}\lambda'^2 + \frac{\nu'}{r} + \lambda'\nu' \right], \tag{4.2.7b}$$

$$\kappa_N p + \frac{\kappa_N}{2\mathcal{A}_{N-2}} \cdot \frac{Q^2}{(re^\lambda)^{2N-4}} = \frac{(N-2)}{e^{2\nu}} \left(-\ddot{\lambda} + \dot{\lambda}\dot{\nu} \right) + \frac{(N-3)}{e^{2\nu}} \left(\frac{\lambda'}{r} + \lambda'' + \frac{\nu'}{r} \right)$$

$$+ \frac{(N-3)}{e^{2\nu}} 2\dot{\lambda}^2 + \frac{1}{e^{2\nu}}\dot{\lambda}^2 - \frac{(N-4)}{e^{2\nu}} \left[\frac{(N-3)}{2}\dot{\lambda}^2 \right]$$

$$+ \frac{(N-4)}{e^{2\lambda}} \left[\lambda'\nu' + \frac{(N-3)}{2}\lambda'^2 \right] + (N-3)\frac{\lambda'}{r}, \tag{4.2.7c}$$

$$\dot{\lambda}\nu' - \dot{\lambda}' = 0, \tag{4.2.7d}$$

$$\xi = \frac{Q'}{\mathcal{A}_{N-2} (re^\lambda)^{N-2}}. \tag{4.2.7e}$$

The primes and dots in equations (4.2.7) above represent partial derivatives with respect to r and t, respectively. This is a system of nonlinear equations in ν, λ, ρ, p and Q (or ξ). It is possible to obtain a single differential equation containing λ and Q from the system (4.2.7).

From (4.2.7d), we obtain

$$\nu' = \frac{\dot{\lambda}'}{\dot{\lambda}}. \tag{4.2.8}$$

Equating (4.2.7b) and (4.2.7c), we eliminate p to find

$$\frac{1}{e^{2\lambda}}\left[\frac{\nu'}{r} - \nu'^2 - \nu'' + (N-3)\frac{\lambda'}{r} + (N-3)\lambda'^2 - (N-3)\lambda''\right]$$

$$= \quad -\frac{\kappa_N}{2\mathcal{A}_{N-2}} \cdot \frac{Q^2}{(re^{\lambda})^{2N-4}}. \tag{4.2.9}$$

Using (4.2.8) in (4.2.9) yields

$$\frac{\dot{\lambda}'}{\dot{\lambda}}\frac{1}{r} - \frac{\left(\dot{\lambda}'\right)^2}{\dot{\lambda}^2} - \frac{\dot{\lambda}''}{\dot{\lambda}} + \frac{\left(\dot{\lambda}'\right)^2}{\dot{\lambda}^2} + 2\frac{\dot{\lambda}'\lambda'}{\dot{\lambda}} + (N-3)\frac{\lambda'}{r} + (N-3)\lambda'^2 - (N-3)\lambda''$$

$$= \quad -\frac{\kappa_N}{\mathcal{A}_{N-2}} \cdot \frac{Q^2}{(re^{\lambda})^{2N-4}}, \tag{4.2.10}$$

which contains only the potential λ. Equation (4.2.10) is a third order nonlinear differential equation. It can be integrated to yield a second order differential equation. This is the master equation governing the evolution of a charged gravitating relativistic shear-free fluid in N dimensions.

4.3 Pressure isotropy

Equation (4.2.10) is called the charged condition of pressure isotropy. It can be written in simpler form as described below.

We multiply equation (4.2.10) above by $-\dot{\lambda}e^{(N-3)\lambda}$ to obtain

$$\frac{\kappa_N}{\mathcal{A}_{N-2}} \cdot \frac{Q^2}{r^{2N-4}} \cdot \dot{\lambda}e^{(N-3)\lambda} \quad = \quad e^{(N-3)\lambda}\dot{\lambda}'' + (N-3)e^{(N-3)\lambda}\dot{\lambda}\lambda''$$

$$- \frac{1}{r}(N-3)e^{(N-3)\lambda}\dot{\lambda}\lambda' - \frac{1}{r}(N-3)e^{(N-3)\lambda}\dot{\lambda}'$$

$$- 2e^{(N-3)\lambda}\lambda'\dot{\lambda}' - (N-3)e^{(N-3)\lambda}\dot{\lambda}\lambda'^2. \tag{4.3.1}$$

This is a third order nonlinear equation; however it can be written as a total derivative with respect to t yielding

$$\frac{\partial}{\partial t}\left[e^{(N-3)\lambda}\left(\lambda'' - \lambda'^2 - \frac{\lambda'}{r}\right) + \frac{\kappa_N}{\mathcal{A}_{N-2}} \cdot \frac{Q^2}{r^{2N-4}} \cdot \frac{1}{N-3} \cdot e^{-(N-3)\lambda}\right] = 0. \tag{4.3.2}$$

Integrating (4.3.2) with respect to t and multiplying by the factor $e^{-(N-3)\lambda}$, we obtain

$$\lambda'' - \lambda'^2 - \frac{\lambda'}{r} = -\tilde{F}e^{-(N-3)\lambda} + \frac{\kappa_N}{\mathcal{A}_{N-2}} \cdot \frac{Q^2}{r^{2N-4}} \cdot \frac{1}{N-3} \cdot e^{-2(N-3)\lambda}, \tag{4.3.3}$$

where $\tilde{F} = \tilde{F}(r)$ is a constant of integration. Equation (4.3.3) is now multiplied by $-e^{-(N-3)\lambda}\frac{1}{4r^2}$ to yield

$$\begin{aligned} \frac{1}{4r^2}\left(\lambda'^2 + \frac{\lambda'}{r} - \lambda''\right)e^{-(N-3)\lambda} &= \frac{\kappa_N}{\mathcal{A}_{N-2}} \cdot \frac{Q^2}{r^{2N-2}} \cdot \frac{1}{N-3} \cdot e^{-3(N-3)\lambda} \\ &\quad + \frac{\tilde{F}}{4r^2}e^{-2(N-3)\lambda}. \end{aligned} \tag{4.3.4}$$

We now introduce a set of new variables that transform λ. Using the transformation

$$x = r^2, \tag{4.3.5a}$$

$$y = e^{-(N-3)\lambda}, \tag{4.3.5b}$$

$$f(x) = \frac{(N-3)\tilde{F}}{4r^2}, \tag{4.3.5c}$$

$$g(x) \quad = \quad \frac{\kappa_N}{\mathcal{A}_{N-2}} \cdot \frac{Q^2}{r^{2N-2}}, \tag{4.3.5d}$$

(4.3.4) becomes

$$y_{xx} - \frac{(N-4)}{(N-3)}\frac{1}{y}y_x^2 = f(x)y^2 + g(x)y^3. \tag{4.3.6}$$

Note that equation (4.3.6) is similar to the form that arises in four dimensions with an additional term $\frac{(N-4)}{(N-3)}\frac{1}{y}y_x^2$ which shows that the value of N influences the dynamics. The functions $f(x)$ and $g(x)$ depend on N. This form of the charged condition of pressure isotropy is new. When $N = 4$, we obtain

$$y_{xx} = f(x)y^2 + g(x)y^3, \tag{4.3.7}$$

which has been widely studied. In Section 4.4 we find exact solutions to (4.3.6) by assuming that the gravitational potential is a series of separable functions, which is a method that was utilized by Gürses and Heydazarde (2019) in their four-dimensional treatment. The extension of the method in Gürses and Heydazarde (2019) to higher dimensions is not obvious because of the appearance of the term containing $\frac{1}{y}y_x^2$. The resulting differential equation (4.3.6) is functionally different when $N \geq 5$, and cannot be mapped to the form contained in (4.3.7). Equation (4.3.6) has to be solved separately.

4.4 Higher dimensional solutions

It is difficult to solve equation (4.3.6) in general. We demonstrate that simple classes of exact solutions are possible. We seek solutions with a particular analytic form: a finite sum of separable functions in the variables t and x. We suppose that

$$e^{(N-3)\lambda} = \frac{1}{y} = \sum_{m=0}^{M} \alpha_m(x)\beta_m(t), \tag{4.4.1}$$

where $\alpha_m(x)$ and $\beta_m(t)$ are independent functions of x and t, respectively, and M is a finite natural number. Then substituting y, y_x and y_{xx} in (4.3.6) and multiplying by $\left(\sum_{m=0}^{M}\alpha_m(x)\beta_m(t)\right)^3$ gives

$$f(x)\left(\sum_{m=0}^{M}\alpha_m(x)\beta_m(t)\right)+g(x) = -\left(\sum_{m=0}^{M}\alpha_m(x)\beta_m(t)\right)\left(\sum_{m=0}^{M}\alpha_m(x)\beta_m(t)\right)_{xx}$$

$$+\eta\left[\left(\sum_{m=0}^{M}\alpha_m(x)\beta_m(t)\right)_x\right]^2. \qquad (4.4.2)$$

Note that we have set

$$\eta = 2-\frac{N-4}{N-3},$$

for convenience. Equation (4.4.2) can be solved for specific values of M. Below we present solutions that correspond to these values.

4.4.1 Case I: $M = 0$

With $M = 0$, equation (4.4.2) reduces to

$$\alpha_0 f(x)\beta_0 + g(x) = \left[-\alpha_0\alpha_0'' + \eta\left(\alpha_0''\right)^2\right]\beta_0^2, \qquad (4.4.3)$$

which leads to trivial solutions.

4.4.2 Case II: $M = 1$

If $M = 1$, then equation (4.4.2) becomes

$$\alpha_0 f(x)\beta_0 + \alpha_1 f(x)\beta_1 + g(x) = \left[-\alpha_1\alpha_1'' + \eta\left(\alpha_1'\right)^2\right]{\beta_1}^2 + \left[-\alpha_0\alpha_0'' + \eta\left(\alpha_0'\right)^2\right]{\beta_0}^2$$

$$+\left[-\alpha_0\alpha_1'' - \alpha_1\alpha_0'' + 2\eta\alpha_0'\alpha_1'\right]\beta_0\beta_1. \qquad (4.4.4)$$

If we try to solve (4.4.4) in full generality we are forced to set $f = g = 0$. To avoid this situation, we let $\beta_0 = 1$ in (4.4.4) and equate coefficients of powers of β_1 to obtain

$$\beta_1{}^2 : \;\; 0 \;\; = \;\; -\alpha_1 \alpha_1{}'' + \eta \left(\alpha_1{}'\right)^2 , \tag{4.4.5a}$$

$$\beta_1 : \;\; \alpha_1 f(x) \;\; = \;\; -\alpha_0 \alpha_1{}'' - \alpha_1 \alpha_0{}'' + 2\eta \alpha_0{}' \alpha_1{}', \tag{4.4.5b}$$

$$\beta_1{}^0 : \;\; \alpha_0 f(x) + g(x) \;\; = \;\; -\alpha_0 \alpha_0{}'' + \eta \left(\alpha_0{}'\right)^2 . \tag{4.4.5c}$$

From (4.4.5a), we have the solution

$$\alpha_1 = B \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}} , \tag{4.4.6}$$

where C_1 and C_2 are arbitrary constants of integration and

$$B = (1-\eta)^{\frac{1}{1-\eta}} .$$

From (4.4.5b), we find the function $f(x)$ to be

$$f(x) = \frac{2\eta\alpha_0'}{1-\eta} \frac{C_1}{C_1 x + C_2} - \frac{\eta \alpha_0 C_1^2}{(1-\eta)^2} \left(C_1 x + C_2\right)^{\frac{2\eta-1}{1-\eta}} - \alpha_0''. \tag{4.4.7}$$

From (4.4.5c), we obtain

$$g(x) = \eta \left(\alpha_0'\right)^2 - \frac{2\eta\alpha_0\alpha_0'}{1-\eta} \frac{C_1}{C_1 x + C_2} + \frac{\eta \left(\alpha_0\right)^2 C_1^2}{(1-\eta)^2} \left(C_1 x + C_2\right)^{\frac{2\eta-1}{1-\eta}} . \tag{4.4.8}$$

The gravitational potentials e^{λ} and e^{ν} for the line element (4.2.4) are then given by

$$e^{\lambda} \;\; = \;\; \left[\alpha_0 + B\beta_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}\right]^{\frac{1}{N-3}} , \tag{4.4.9a}$$

$$e^{\nu} \;\; = \;\; \frac{q_3(t)}{N-3} \left[\alpha_0 + B\beta_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}\right]^{-1}$$

$$\times B\dot{\beta}_1 (C_1 x + C_2)^{\frac{1}{1-\eta}}, \quad (4.4.9b)$$

respectively, where $q_3(t)$ is an arbitrary function of integration. The functions $\alpha_0(x)$, β_1 and $q_3(t)$ are arbitrary functions. We observe that the potentials ν and λ are affected by dimension N. If we set $N = 4$, then we get

$$e^{\lambda} = \alpha_0 - \frac{\beta_1}{C_1 x + C_2}, \quad (4.4.10a)$$

$$e^{\nu} = q_3(t) \frac{\dot{\beta}_1}{\beta_1 - \alpha_0 (C_1 x + C_2)}, \quad (4.4.10b)$$

which was obtained by Gürses and Heydazarde (2019) for charged shear-free fluids.

4.4.3 Case III: $M = 2$

If $M = 2$, then equation (4.4.2) becomes

$$\begin{aligned}\alpha_0 f(x)\beta_0 + \alpha_1 f(x)\beta_1 + \alpha_2 f(x)\beta_2 + g(x) &= \left[-\alpha_1 \alpha_1'' + \eta (\alpha_1')^2\right] {\beta_1}^2 \\ &+ \left[-\alpha_0 \alpha_0'' + \eta (\alpha_0')^2\right] {\beta_0}^2 \\ &+ \left[-\alpha_2 \alpha_2'' + \eta (\alpha_2')^2\right] {\beta_2}^2 \\ &+ \left[-\alpha_0 \alpha_1'' - \alpha_1 \alpha_0'' + 2\eta \alpha_0' \alpha_1'\right] \beta_0 \beta_1 \\ &+ \left[-\alpha_0 \alpha_2'' - \alpha_2 \alpha_0'' + 2\eta \alpha_0' \alpha_2'\right] \beta_0 \beta_2\end{aligned}$$

$$+\left[-\alpha_1\alpha_2'' - \alpha_2\alpha_1'' + 2\eta\alpha_1'\alpha_2'\right]\beta_1\beta_2.$$

(4.4.11)

We let $\beta_0 = 1$ and $\beta_2 = \frac{1}{\beta_1}$ in (4.4.11) and equate coefficients of powers of β_1 to obtain

$$\beta_1{}^2: \;\; 0 = -\alpha_1\alpha_1'' + \eta\left(\alpha_1'\right)^2, \tag{4.4.12a}$$

$$\frac{1}{\beta_1{}^2}: \;\; 0 = -\alpha_2\alpha_2'' + \eta\left(\alpha_2'\right)^2, \tag{4.4.12b}$$

$$\beta_1: \;\; \alpha_1 f(x) = -\alpha_0\alpha_1'' - \alpha_1\alpha_0'' + 2\eta\alpha_0'\alpha_1', \tag{4.4.12c}$$

$$\frac{1}{\beta_1}: \;\; \alpha_2 f(x) = -\alpha_0\alpha_2'' - \alpha_2\alpha_0'' + 2\eta\alpha_0'\alpha_2', \tag{4.4.12d}$$

$$\beta_1{}^0: \;\; \alpha_0 f(x) + g(x) = -\alpha_0\alpha_0'' + \eta\left(\alpha_0'\right)^2 - \alpha_1\alpha_2'' - \alpha_2\alpha_1'' + 2\eta\alpha_1'\alpha_2'. \tag{4.4.12e}$$

From (4.4.12a), we have the solution

$$\alpha_1 = B\left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}, \tag{4.4.13}$$

where C_1 and C_2 are arbitrary constants of integration. From (4.4.12b), we have the solution

$$\alpha_2 = B\left(C_3 x + C_4\right)^{\frac{1}{1-\eta}}, \tag{4.4.14}$$

where C_3 and C_4 are arbitrary constants of integration. We eliminate $f(x)$ from (4.4.12c) and (4.4.12d) and integrate to obtain the function α_0. It has the form

$$\alpha_0 = \delta\sqrt{\left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}}\sqrt{\left(C_3 x + C_4\right)^{\frac{1}{1-\eta}}}, \tag{4.4.15}$$

where

$$\delta = \sqrt[2\eta]{A\,(1-\eta)^{1-2\eta}\,(C_1C_4 - C_2C_3)},$$

and A is a constant of integration with $A\,(1-\eta)^{1-2\eta}\,(C_1C_4 - C_2C_3) > 0$. From (4.4.12c), we obtain the function $f(x)$ to be

$$\begin{aligned}
f(x) \;=\; & -\frac{\delta\,(2C_1C_3x + C_2C_3 + C_1C_4)^2}{4\,(1-\eta)^2}\sqrt{(C_1x+C_2)^{\frac{4\eta-3}{1-\eta}}\,(C_3x+C_4)^{\frac{4\eta-3}{1-\eta}}}\\
& -\delta\frac{C_1C_2}{1-2\eta}\sqrt{(C_1x+C_2)^{\frac{4\eta-3}{1-\eta}}\,(C_3x+C_4)^{\frac{2\eta-1}{1-\eta}}}\\
& +\frac{\delta\eta C_1}{1-\eta}\sqrt{(C_1x+C_2)^{\frac{4\eta-3}{1-\eta}}\,(C_3x+C_4)^{\frac{2\eta-1}{1-\eta}}}\,(2C_1C_3x + C_2C_3 + C_1C_4)\\
& -\frac{\delta\eta C_1^2}{(1-\eta)^2}\,(C_1x+C_2)^{\frac{2\eta-1}{1-\eta}}\sqrt{(C_1x+C_2)^{\frac{1}{1-\eta}}\,(C_3x+C_4)^{\frac{1}{1-\eta}}}. \qquad (4.4.16)
\end{aligned}$$

From (4.4.12e) and (4.4.16), the function $g(x)$ is given by

$$\begin{aligned}
g(x) \;=\; & \frac{\eta\delta^2\,(2C_1C_3x + C_2C_3 + C_1C_4)^2}{4\,(1-\eta)^2}\,(C_1x+C_2)^{\frac{2\eta-1}{1-\eta}}\,(C_3x+C_4)^{\frac{2\eta-1}{1-\eta}}\\
& -\frac{\eta B^2}{(1-\eta)^2}\left[C_1\,(C_1x+C_2)^{\frac{\eta}{1-\eta}} - C_3\,(C_3x+C_4)^{\frac{\eta}{1-\eta}}\right]^2\\
& -\frac{\eta C_1\delta^2}{(1-\eta)^2}\,(C_1x+C_2)^{\frac{2\eta-1}{1-\eta}}\,(C_3x+C_4)^{\frac{\eta}{1-\eta}}\,(2C_1C_3x + C_2C_3 + C_1C_4)\\
& +\frac{\eta\delta^2C_1^2}{(1-\eta)^2}\,(C_1x+C_2)^{\frac{2\eta}{1-\eta}}\,(C_3x+C_4)^{\frac{\eta}{1-\eta}}\,. \qquad (4.4.17)
\end{aligned}$$

The gravitational potentials e^{λ} and e^{ν} for the line element (4.2.4) are then given by

$$e^{\lambda} = \left[\alpha_0 + B\beta_1 (C_1 x + C_2)^{\frac{1}{1-\eta}} + \frac{B}{\beta_1} (C_3 x + C_4)^{\frac{1}{1-\eta}}\right]^{\frac{1}{N-3}}, \tag{4.4.18a}$$

$$e^{\nu} = \frac{q_4(t)}{N-3} \left[\alpha_0 + B\beta_1 (C_1 x + C_2)^{\frac{1}{1-\eta}} + \frac{B}{\beta_1} (C_3 x + C_4)^{\frac{1}{1-\eta}}\right]^{-1}$$

$$\times \left[B\dot{\beta}_1 (C_1 x + C_2)^{\frac{1}{1-\eta}} - \frac{B\dot{\beta}_1}{\beta_1^2} (C_3 x + C_4)^{\frac{1}{1-\eta}}\right], \tag{4.4.18b}$$

respectively, where $q_4(t)$ is a function of integration. The functions α_0, β_1 and $q_3(t)$ are arbitrary functions. The dimension N influences the possible forms of the potentials ν and λ.

It is remarkable that simple analytic forms of the potentials ν and λ are possible in higher dimensions. It is important to note that for all values of M, the potentials e^{λ} and e^{ν} reduce to those of Gürses and Heydazarde (2019) in four dimensions. If we set $N = 4$ in (4.4.18) we get

$$e^{\lambda} = \frac{\delta}{\sqrt{C_1 x + C_2}\sqrt{C_3 x + C_4}} - \frac{\beta_1}{C_1 x + C_2} - \frac{1}{\beta_1 (C_3 x + C_4)}, \tag{4.4.19a}$$

$$e^{\nu} = \frac{\dot{\beta}_1 q_4(t) \left[(C_1 x + C_2) - \beta_1^2 (C_3 x + C_4)\right]}{\left[\beta_1^2 \delta \sqrt{(C_1 x + C_2)(C_3 x + C_4)} - \beta_1^3 (C_3 x + C_4) - \beta_1 (C_1 x + C_2)\right]}, \tag{4.4.19b}$$

where $\delta = \sqrt[4]{C_2 C_3 - C_1 C_4}$. The result in (4.4.19) was obtained by Gürses and Heydazarde (2019) in their analysis of charged shear-free fluids. Note that when $C_1 = 0, C_3 \neq 0$ in (4.4.19) then we get

$$e^{\lambda} = \frac{\delta}{\sqrt{C_2 (C_3 x + C_4)}} - \frac{\beta_1}{C_2} - \frac{1}{\beta_1 (C_3 x + C_4)}, \tag{4.4.20a}$$

$$e^{\nu} = \frac{\dot{\beta}_1 q_4(t) \left[C_2 - \beta_1^2 (C_3 x + C_4)\right]}{\left[\beta_1^2 \delta \sqrt{C_2 (C_3 x + C_4)} - \beta_1^3 (C_3 x + C_4) - \beta_1 C_2\right]}, \tag{4.4.20b}$$

which was first obtained by Shah and Vaidya (1968) in four dimensions. Alternatively we could have set $C_3 = 0, C_1 \neq 0$ to obtain the result of Shah and Vaidya (1968).

It seems that we could extend this approach for other values of M. However for $M \geq 3$, we obtain an overdetermined system of equations which lead to inconsistencies. Hence, this approach yields solutions only for $M = 1$ and $M = 2$.

4.5 Dimension

Our results show that the gravitational potentials ν and λ in (4.4.9) and (4.4.18) depend critically on the spacetime dimension N. The matter variables ρ, p and ξ are also affected by dimension N. Other physical quantities and parameters can take on different values as the dimension N changes. In static spheres, Paul (2001) showed that the mass-radius ratio, determining the compactness of a uniform density star, increases, reaches a maximum for $N = 9$, and then decreases. Dimension also affects the evolution of a charged shear-free fluid. We demonstrate the effect of changing N on the quantities $f(x)$ and $g(x)$ in (4.3.5c) and (4.3.5d). Note that $f(x)$ affects the gravitational potential λ and $g(x)$ depends on the charge Q as pointed out by Srivastava (1992). We plot $f(x)$ and $g(x)$ for dimensions $N = 4, 5, 6, 7$ for increasing values of r. The sketches are given below in Figure 4.1 and Figure 4.2 respectively. We have fixed $\tilde{F} = Q^2 = 1$ as they do not depend on N. From Figure 4.1 we observe that $f(x)$ is a decreasing function for all values of N. The values of $f(x)$ increase as N increases. The behaviour of $f(x)$ is determined by $f \sim r^{-2}$ in (4.3.5c). From Figure 4.2 we note that $g(x)$ is a decreasing function for all values of N. The values of $g(x)$ decrease as N increases. The quantity $g(x)$ becomes rapidly smaller because of $g(x) \sim r^{2-2N}$ in (4.3.5d) which dominates the behaviour. It is clear that the dimension N affects the profile of the physically relevant quantities.

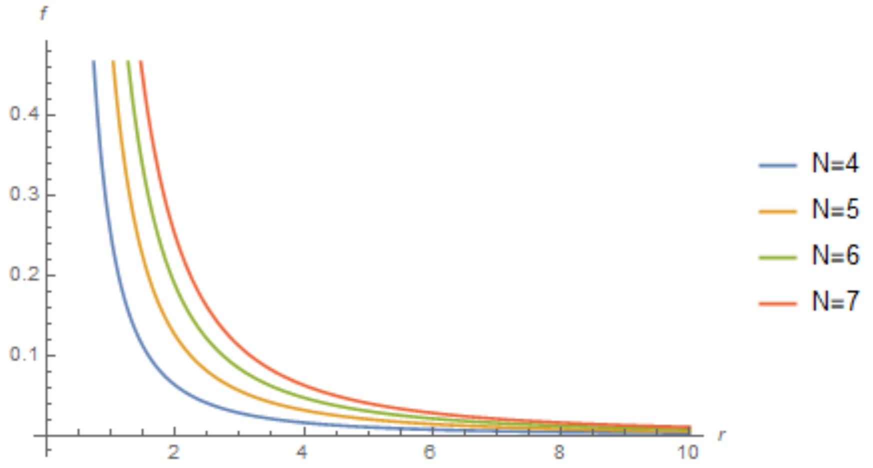

Figure 4.1: Graphs of the function $f(x)$ for $N = 4, 5, 6, 7$.

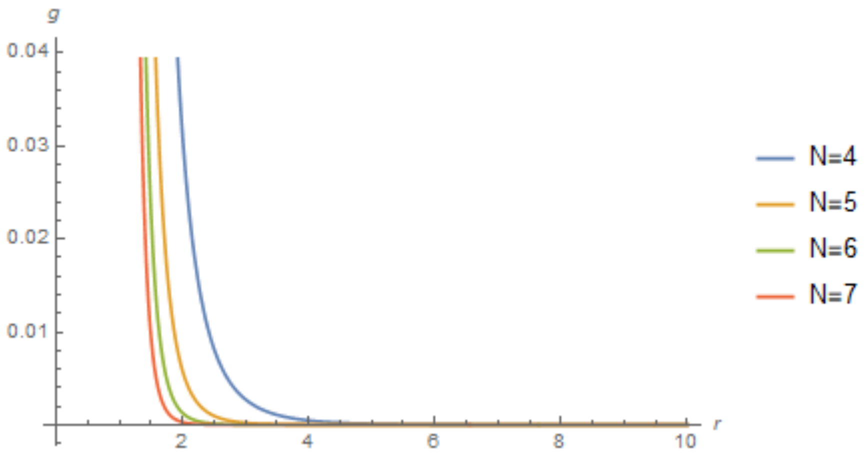

Figure 4.2: Graphs of the function $g(x)$ for $N = 4, 5, 6, 7$.

4.6 Discussion

In this chapter we found new classes of exact solutions to the Einstein-Maxwell equations for charged shear-free fluid distributions in an N-dimensional spacetime. We first showed that the Einstein-Maxwell equations can be reduced to the single second order partial differential equation

$$y_{xx} - \frac{(N-4)}{(N-3)} \frac{1}{y} y_x^2 = f(x) y^2 + g(x) y^3.$$

This shows that the dynamics of the spacetime depends on the dimension N. We solved this equation by assuming that one of the gravitational potentials is a sum of products of separable functions. Two classes of exact solutions were found. Explicit forms of the gravitational potentials were found in terms of elementary functions. Our N-dimensional results reduce to the 4-dimensional results obtained by Gürses and Heydazarde (2019) and Shah and Vaidya (1968). The effect of dimensions on the functions $f(x)$ and $g(x)$ was demonstrated graphically. In future work we intend to apply our results to a charged star in general relativity. It would also be interesting to consider shear-free fluids in modified gravity theories such as Einstein-Gauss-Bonnet gravity or general Lovelock gravity models. In such theories both the dimension N and new contributions from the curvature should lead to interesting dynamics.

Chapter 5

Conclusion

The aim of this thesis was to generate new exact solutions to the Einstein field equations for shear-free uncharged relativistic fluids, as well as exact solutions to the Einstein-Maxwell equations for charged shear-free fluids in a four dimensional spacetime. We also investigated the role of dimension in the field equations for charged shear-free fluids. We performed this latter investigation by finding new exact solutions to the Einstein-Maxwell equations in higher dimensions. We used different techniques in generating solutions. Standard methods for solving nonlinear differential equations do not apply to the Emden-Fowler type equations that arise in this study.

We now summarize the contents of this thesis and highlight the results obtained. In Chapter 1 we provided background and some history of Einstein's theory of general relativity and the field equations. We discussed some well known examples of exact solutions to the Einstein field equations and the Einstein-Maxwell equations with emphasis on the spherically symmetric shear-free solutions.

In Chapter 2 we studied integrability of and found new classes of exact solutions to the Emden-Fowler equation

$$y_{xx} = f(x)y^2,$$

which is a master equation that governs the behaviour of spherically symmetric shear-free uncharged relativistic fluids. It has several applications in different areas of mathematical physics. We multiplied the Emden-Fowler equation by an integrating factor x and used integration by parts to obtain the first integral (2.3.8). This first integral is subject to the integrability condition (2.3.7), which is an integral equation. In order to solve the integral equation (2.3.7), we first transformed it to a third order ordinary differential equation (2.4.4) whose solution is given by (2.4.6). For convenience we wrote the solution of (2.4.4) parametrically, which enabled us to find a new form of the function $f(x)$. One form of the function $f(x)$ was given by

$$f(x) \sim \frac{1}{x^5}\left(1-\frac{1}{x}\right)^{-15/7},$$

in (2.5.2) so that the first integral could be written parametrically as (2.5.3). This form of the function is different from those obtained previously. The existence of different first integrals suggests a connection to complexity in self-gravitating systems in general relativity. We noted that the complexity of a self-gravitating relativistically neutral fluid is related to a first integral that arises from the integration process of the Emden-Fowler equation.

In Chapter 3 we extended the approach used in Chapter 2 by including the effect of the electromagnetic charge. In the presence of the electromagnetic charge, the Emden-Fowler equation (3.2.7) becomes

$$y_{xx} = f(x)y^2 + g(x)y^3,$$

where $g(x)$ relates to the charge distribution to the Einstein-Maxwell equations for charged shear-free relativistic fluids. As in Chapter 2 we multiplied the charged Emden-Fowler equation by an integrating factor and obtained a new first integral (3.3.6). This first integral is subject to integrability conditions (3.3.5) which are integral equations. This approach led to new functional forms of $f(x)$ and $g(x)$ which are given by

$$f(x) \sim \frac{1}{x^5}\left(1-\frac{1}{x}\right)^{-11/5}, \quad g(x) \sim \frac{1}{x^6}\left(1-\frac{1}{x}\right)^{-12/5},$$

which are different from those obtained previously. However, if $g = 0$, then the first integral (3.3.6) reduces to the first integral (2.3.8) obtained in Chapter 2. We demonstrated that the new first integral obtained is independent of that previously obtained by Kweyama et al. (2012). We concluded that the complexity of a self-gravitating fluid is affected by the electromagnetic charged and is related to the first integral for charged shear-free fluids.

In Chapter 4 we investigated the role of dimension in the behaviour of charged relativistic shear-free fluids. We found new classes of exact solutions to the Einstein-Maxwell equations in a higher dimensional spacetime. We first showed that the field equations can reduced to a second order differential equation

$$y_{xx} - \frac{(N-4)}{(N-3)}\frac{1}{y}y_x^2 = f(x)y^2 + g(x)y^3.$$

This is a new result. This form of the equation show explicitly that the dimension N affects the dynamics of the spacetime in general relativity. To find solutions we used the method of separation of variables by assuming that the gravitational potential is a sum of products of separable functions

$$e^{(N-3)\lambda} = \frac{1}{y} = \sum_{m=0}^{M} \alpha_m(x)\beta_m(t),$$

which is a generalization of the approach used by Gürses and Heydazarde (2019) in four dimensions. We found new classes of solutions for different values of M. For $M = 1$, we found the potentials to be

$$e^{\lambda} = \left[\alpha_0 + B\beta_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}\right]^{\frac{1}{N-3}},$$

$$e^{\nu} = \frac{q_3(t)}{N-3}\left[\alpha_0 + B\beta_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}\right]^{-1} B\dot{\beta}_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}}.$$

If $M = 2$, then the potentials are given by

$$e^{\lambda} = \left[\alpha_0 + B\beta_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}} + \frac{B}{\beta_1}\left(C_3 x + C_4\right)^{\frac{1}{1-\eta}}\right]^{\frac{1}{N-3}},$$

$$e^{\nu} = \frac{q_4(t)}{N-3}\left[\alpha_0 + B\beta_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}} + \frac{B}{\beta_1}\left(C_3 x + C_4\right)^{\frac{1}{1-\eta}}\right]^{-1}$$

$$\times \left[B\dot{\beta}_1 \left(C_1 x + C_2\right)^{\frac{1}{1-\eta}} - \frac{B\dot{\beta}_1}{\beta_1^2}\left(C_3 x + C_4\right)^{\frac{1}{1-\eta}}\right].$$

We found that $M \geq 3$ leads to a system of overdetermined equation, hence exact solutions can be found only for $M = 1$ and $M = 2$. Our new classes of solutions reduce to those obtained by Gürses and Heydazarde (2019) for $N = 4$. The earlier model of Shah and Vaidya (1968) is regained as a special case. A graphical analysis indicates that the model is well behaved.

The results of our analysis are of relevance in gravitational physics and mathematical physics. Firstly, the role of dimension affects the dynamics of relativistic fluids. Our work shows that the master equations governing the dynamics of the fluid have to be derived with dimension N as additional nonlinear terms arise. The dynamical evolution of the system changes. This also applies to applications in cosmology. Even in the simple case of a homogeneous and isotropic spacetime the evolution of the model in higher dimensions is different from the case when $N = 4$. This needs to be studied in spacetimes with less symmetry than Robertson-Walker metrics. In relativistic astrophysics the spacetime dimension affects the evolution of the stellar boundary as demonstrated in a number of recent works. The implication of this feature needs to be pursued in relation to luminosity, temperature profiles, stability and other quantities of physical importance.

In conclusion the exact solutions to the Einstein-Maxwell equations found in this thesis have several interesting features. They can be related to complexity of self-gravitating systems as first suggested by Herrera (2018). The presence of higher dimensions influences the dynamical behaviour of the fluid.

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