UNIVERSITY OF KWAZULU-NATAL

EXACT MODELS OF COMPACT STARS WITH EQUATIONS OF STATE

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Abstract

We study exact solutions to the Einstein-Maxwell system of equations and relate them to compact objects. It is well known that there are substantial analytic difficulties in the modelling of self-gravitating, static fluid spheres when the pressure explicitly depends on the matter density. Much simplification in solving the Einstein- Maxwell equations is achieved with the introduction of electric charge and anisotropic matter. In this thesis, in order to obtain analytical solutions, we consider the general situation of anisotropy in the presence of electric charge satisfying a barotropic equation of state. Firstly, a linear equation of state, secondly a quadratic equation of state, and thirdly a polytropic equation of state are analysed. For each of these equations of state the Einstein-Maxwell equations are integrated and exact solutions are found in terms of elementary functions. By choosing specific rational forms for one of the gravitational potential and particular forms for the electric charge, new classes of solutions in static spherically symmetric interior spacetimes are generated in the presence of electric charge. It is interesting to note that, from our new class of solutions with an equation of state, we can regain earlier models. A detailed physical analysis performed indicates that the classes of solutions are physically reasonable. We regain the current accurate observed masses for the binary pulsars PSR J1614-2230 , PSR J1903+327, Vela X-1, SMC X-1 and Cen X-3.

Dedicace

Je dédie humblement cette thèse :

*A **Dieu puissant**, la sublime origine du savoir, profondeur de la sagesse, la lumière qui anime l'univers. C'est dans ce que tu es que je trouve ce que je puis; c'est dans ce que tu as que je vis ce je suis. T'avoir pour Seigneur et Maître est plus que tout ce que l'existence peut me proposer de bien.*

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P. Carmel Mafa

P. Symphony Mafa

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Declaration 2 - Publications

Details of contribution to publications that form part and/or include research presented in this thesis are

Publication 1

Maharaj, S.D., Mafa Takisa, P.: Regular models with quadratic equation of state, *Gen. Relativ. Gravit.* **44**, 1419 (2012).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisor.)

Publication 2

Mafa Takisa, P., Maharaj, S.D.: Compact models with regular charge distributions, *Astrophys. Space Sci.* **343**, 569 (2013).

(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisor.)

Publication 3

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Publication 4

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(There were regular meetings between myself and my supervisor to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisor.)

Contents

CHAPTER 1

Introduction

Compact relativistic stars are one of the most fascinating entities known in the Universe. Particular astrophysical bodies such white dwarfs, quark stars, neutron stars, hyperon stars, hybrid stars and magnetars seem to be relics of big luminous stars which are born from extreme phenomena such as in supernova explosions. To understand the behaviour of these objects, and to explain how stars collapse under gravity, it seems reasonable to investigate what is allowed by the laws of astrophysics within the framework of the theory of General Relativity. Research in the field of relativistic astrophysics has been investigated since the first solution of Einstein's field equation for the interior of a compact object in hydrostatic equilibrium which was found by Schwarzschild (1916a, 1916b). Since the inception of the theory of general relativity, a substantial amount of exact solutions to the Einstein equations and the Einstein-Maxwell system have been found. There are five notable exact solutions representing fundamental breakthroughs:

(a) The exterior Schwarzschild solution (Schwarzschild 1916a) was the first exact solution to the Einstein field equations discovered which describes the gravitational field outside a static spherically symmetric star. This solution is crucial for a discussion of the classical tests of general relativity (d'Inverno 1992, Wald 1984, Will 1981). There have been many attempts to find interior solutions which match to the exterior Schwarzschild solution.

(b) The Schwarzschild interior solution (Schwarzschild 1916b) was the first, and most famous model, which describes the interior gravitational field of a static fluid body. It is a good approximation of dense stars in which the pressures are not too large. The interior and exterior Schwarzschild solutions together provided the first complete relativistic description of the matter distribution and spacetime geometry for a static star.

(c) The Reissner-Nordstr$\ddot{o}$m (Reissner 1916, Nordstr$\ddot{o}$m 1918) solution is the unique exterior metric for a spherically symmetric charged star. In the limit of vanishing electromagnetic field, this solution reduces to the Schwarzschild exterior solution. The interior regular charged perfect fluid solutions are not unique and have been investigated by different authors.

(d) Kerr (1963) surprisingly discovered an exact solution for an uncharged, rotating black hole, called the Kerr metric. It has been proposed that this solution be interpreted as describing the exterior gravitational field arising from a spinning body. Newman and Janis (1965) discovered the natural extension to a charged, rotating black hole, named the Kerr-Newman metric.

(e) Vaidya (1951) discovered the first radiating solution to the Einstein field equations which describes the radial flow of coherent null radiation in the presence of a spherically symmetric gravitational field.

A significant number of interior static, spherically symmetric solutions are known today. These are given by Stephani $et\ al$ (2003), Finch and Skea (1998), Delgaty and Lake (1998), amongst others. Most of these solutions, however, do not satisfy all the tests of physical reality. Some of the exact solutions, that satisfy all the physical requirements include those of Durgapal and Bannerji (1983), Durgapal and Fuloria (1985), Finch and Skea (1989), Tikekar (1990), Maharaj and Leach (1996) and Lake (2003). Many researchers use a variety of techniques to find exact charged solutions. A comprehensive list of Einstein-Maxwell solutions, satisfying a diversity of criteria for physical allowability, is provided by Ivanov (2002). It is interesting to observe that, in the presence of charge, the gravitational collapse of a spherically symmetric distribution of matter to a point singularity may be avoided (Krasinski 1997). As pointed out by Delgaty and Lake (1998), there are very few exact interior solutions of the field equations satisfying the required general physical conditions inside the compact star. Therefore, the study of the interior of a relativistic star is still an active field of research.

In the past years, many investigators have studied anisotropic matter where the radial and tangential pressure components are different. These include the studies of Bowers and Liang (1974), Chaisi and Maharaj (2005, 2006a, 2006b), Dev and Gleiser (2002, 2003), Herrera $et\ al$ (2002, 2004), Maharaj and Chaisi (2006a, 2006b) and Mak and Harko (2002, 2004), amongst others. They showed that anisotropy may affect parameters like the maximum equilibrium mass and the surface redshift. Mak and Harko (2002) and Sharma and Mukherjee (2002) pointed out that the anisotropic matter appears to be an essential ingredient when studying boson stars and strange matter with densities higher than neutron stars. The charged anisotropic fluid spheres have been explored in general relativity since the pioneering work of Bonnor (1965). Solving the Einstein-Maxwell system with anisotropic pressures is a difficult problem.

In recent years, there have been attempts to investigate the Einstein-Maxwell field equations with a linear equation of state when the matter distribution is anisotropic. These include the works of Ivanov (2002), Sharma and Maharaj (2007a), Thirukkanesh and Maharaj (2008) and Mafa Takisa and Maharaj (2013). Thirukkanesh and Ragel (2013) found a linear equation of state for an uncharged anisotropic sphere. The simplification of the field equations for a charged perfect fluid with a linear equation of state was discussed by Ivanov (2002), who showed how to reduce the system involving the most general linear equation of state to a linear differential equation for one metric function. Particular solutions with a quadratic equation of state, relating the radial pressure to the energy density, were found by Feroze and Siddiqui (2011). Maharaj and Mafa Takisa (2012) investigated the quadratic equation of state which is important in brane world models and the study of dark energy. A general approach of dealing with anisotropic charged matter with linear or nonlinear equations of state has been highlighted in the analysis of Varela $et\ al$ (2010). Hitherto, the models with prescribed equation of state remain relatively unexplored.

Theoretical analytical solutions may play an important role in the astrophysical analysis by giving a deeper insight than numerical solutions. Furthermore, they may be used as testing points to check if the numerical scheme is accurate, and they are the first step in comparison between theory and observation. In this thesis, we are concerned with generating exact solutions to the Einstein-Maxwell system. We impose symmetries on the spacetime manifold with isotropic and anisotropic matter distributions in the presence of an electromagnetic field. We emphasize that it is possible to generate a diversity of exact solutions which are physically acceptable. The linear, quadratic and polytropic equations of state, modelling neutron, strange matter and quark matter, are shown to be relevant to matter distributions with charge and anisotropy.

The outline of our research is given by:

- Chapter 1: Introduction.

- Chapter 2: In this chapter we investigate a compact relativistic star with anisotropic pressures in the presence of an electric field and a linear equation of state. New exact solutions of the Einstein-Maxwell equations are generated. A graphical analysis shows that the matter and electric field are well behaved. In particular the proper charge density is regular at the stellar centre contrary to earlier anisotropic models in the presence of charge. We demonstrate that the electric field affects the mass of stellar objects and the observed mass for a particular binary pulsar is regained. For special parameter values our solutions contain previous results of anisotropic charged model of Thirukkanesh and Maharaj (2008).

- Chapter 3: Here we find new exact solutions to the Einstein-Maxwell system of equations which are physically reasonable. We use an equation of state which is quadratic relating the radial pressure to the energy density. Earlier models of Feroze and Siddiqui (2011) and Thirukkanesh and Maharaj (2008), are shown to be contained in our general class of solutions. The new solutions to the Einstein-Maxwell equations are found in terms of elementary functions. A physical analysis of the matter and electromagnetic variables indicates that the model is well behaved and regular. In particular there is no singularity in the proper charge density at the stellar centre unlike previous anisotropic solutions in the presence of an electric field intensity.

- Chapter 4: The Einstein-Maxwell systems with anisotropic pressures and electric field intensity are examined with a polytropic equation of state. New exact solutions to the Einstein-Maxwell field equations are found in terms of elementary functions. Special cases of the uncharged models of Feroze and Siddiqui (2011) and Maharaj and Mafa Takisa (2012) are regained. We also obtained from our general treatment, the exact solutions for a neutral anisotropic gravitating stellar body for a polytrope. Graphically we indicate that the radial pressure profiles are consistent with previous investigations which suggest relevance in describing relativistic compact bodies.

- Chapter 5: In this chapter we apply the results of the chapter 2 to observed astronomical objects. We chose particular parameter values which match with recent mass-radius estimates of five different uncharged compact objects. Then we study the effect of electric field intensity on the stellar structure set that fits the observed stars.

- Chapter 6: Conclusion

CHAPTER 2

Compact models with regular charge distributions

2.1 Introduction

Solutions of the Einstein-Maxwell system of equations for static spherically symmetric interior spacetimes are important in describing charged compact objects in relativistic astrophysics where the gravitational field is strong, as in the case of neutron stars. The detailed analyses of Ivanov (2002) and Sharma $et\ al$ (2001) show that the presence of an electromagnetic field affects the values of redshifts, luminosities and maximum mass of compact objects. The role of the electromagnetic field in describing the gravitational behaviour of stars composed of quark matter has been recently highlighted by Mak and Harko (2004) and Komathiraj and Maharaj (2007a, 2007b). In recent years, many researchers have attempted to introduce different approaches of finding solutions to the field equations. Hansraj and Maharaj (2006) found solutions to the Einstein-Maxwell system with a specified form of the electric field with isotropic pressures. These solutions satisfy a barotropic equation of state and regain the model due to Finch and Skea (1989). Thirukkanesh and Maharaj (2008) found new exact classes of solutions to the Einstein-Maxwell system. They considered anisotropic pressures in the presence of the electromagnetic field with the linear equation of state of strange stars with quark matter. Other recent investigations involving charged relativistic stars include the results of Karmakar $et\ al$ (2007), Maharaj and Komathiraj (2007) and Maharaj and Thirukkanesh (2009). The approach of Esculpi and Aloma (2010) is interesting in that it utilises the existence of a conformal symmetry in the spacetime manifold to find a solution. These exact solutions are relevant in the description of dense relativistic astrophysical objects. Applications of charged relativistic models include studies in cold compact objects by Sharma $et\ al$ (2006), analyses of strange matter and binary pulsars by Sharma and Mukherjee (2001), and analyses of quark-diquark mixtures in equilibrium by Sharma and Mukherjee (2002). Thomas $et\ al$ (2005), Tikekar and Thomas (1998) and Paul and Tikekar (2005) modelled charged core-envelope stellar structures in which the core of the sphere is an isotropic fluid surrounded by a layer of anisotropic fluid. To get more flexibility in solving the Einstein-Maxwell system, Varela $et\ al$ (2010) considered a general approach of dealing with anisotropic charged matter with linear or nonlinear equations of state. Their approach offers a fresh view of relationships between the equation of state, charge distributions and pressure anisotropy. A detailed physical analysis showed the desirability of models with an equation of state. However there are few models in the literature with an equation of state which satisfy the physical criteria.

It is desirable that the criteria for physical acceptability, as given by Delgaty and Lake (1998), be satisfied in a realistic charged stellar model. In particular the proper charge density should be regular at the centre of the sphere. This is an essential requirement for a well behaved electromagnetic field, and this important feature has been highlighted in the treatment of Varela $et\ al$ (2010). In many models found in the past, including the charged anisotropic solution with a linear equation of state of Thirukkanesh and Maharaj (2008), the proper charge density is singular at the centre of the star. For a realistic description this is a limiting feature for the applicability of the model. It is desirable to avoid this singularity if possible. Our object in this chapter is to generate new solutions to the Einstein-Maxwell system which satisfy the physical properties: the gravitational potentials, electric field intensity, charge distribution and matter distribution should be well behaved and regular throughout the star, in particular at the stellar centre. We find new exact solutions for a charged relativistic sphere with anisotropic pressures and a linear equation of state. Previous models are shown to be special cases of our general result. In Section 2.2, we rewrite the Einstein-Maxwell field equations for a static spherically symmetric line element as an equivalent set of differential equations using a transformation due to Durgapal and Bannerji (1983). In Section 2.3, we motivate the choice of the gravitational potential and the electric field intensity that enables us to integrate the field equations. In Section 2.4, we present new exact solutions to the Einstein-Maxwell system, which contain earlier results. The singularity in the charge density may be avoided. In Section 2.5, a physical analysis of the new solutions is performed; the matter variables and the electromagnetic quantities are plotted. Values for the stellar mass are generated for charged and uncharged matter for particular parameter values. The values are consistent with the conclusions of Dey $et\ al$ (1998, 1999a, 1998b) for strange stars. The variation of density for different charged compact structures is discussed in Section 2.6.

2.2 Basic equations

In standard coordinates the line element for a static spherically symmetric fluid has the form

$$\mathrm{d}s^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.1}$$

The Einstein-Maxwell equations take the form

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho + \frac{1}{2}E^2, \tag{2.2}$$

$$-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2, \tag{2.3}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_t + \frac{1}{2}E^2, \tag{2.4}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)', \tag{2.5}$$

where primes represent differentiation with respect to r. It is convenient to introduce a new independent variable x and introduce new functions y and Z:

$$x = Cr^2, \;\; Z(x) = e^{-2\lambda(r)}, \;\; A^2y^2(x) = e^{2\nu(r)}, \tag{2.6}$$

where A and C are constants. We assume a barotropic equation of state

$$p_r = \alpha\rho - \beta, \tag{2.7}$$

relating the radial pressure p_r to the energy density ρ. The quantity p_t is the tangential pressure, *E* represents the electric field intensity, and σ is the proper charged density. Then the equations governing the gravitational behaviour of a charged anisotropic sphere, with a linear equation of state, are given by

$$\begin{aligned}
\frac{\rho}{C} &= \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}, && (2.8a)\\
p_r &= \alpha\rho - \beta, && (2.8b)\\
p_t &= p_r + \Delta, && (2.8c)\\
\Delta &= 4CxZ\frac{\ddot{y}}{y} + 2C\left[x\dot{Z} + \frac{4Z}{(1+\alpha)}\right]\frac{\dot{y}}{y} + \frac{(1+5\alpha)}{(1+\alpha)}C\dot{Z}\\
&\quad -\frac{C(1-Z)}{x} + \frac{2\beta}{(1+\alpha)}, && (2.8d)\\
\frac{E^2}{2C} &= \frac{1-Z}{x} - \frac{1}{(1+\alpha)}\left[2\alpha\dot{Z} + 4Z\frac{\dot{y}}{y} + \frac{\beta}{C}\right], && (2.8e)\\
\frac{\sigma^2}{C} &= \frac{4Z}{x}\left(x\dot{E} + E\right)^2, && (2.8f)
\end{aligned}$$

where $\Delta = p_t - p_r$ is called the measure of anisotropy. Dots represents differentiation with respect to x. The nonlinear system as given in (2.8a)-(2.8f) consists of six independent equations in the eight variables $\rho, p_r, p_t, \Delta, E, \sigma$, *y* and *Z*. To solve (2.8a)-(2.8f) we need to specify two of the quantities involved in the integration process.

2.3 Choice of potentials

We need to solve the Einstein-Maxwell field equations (2.8a)-(2.8f) by choosing specific forms for the gravitational potential Z and the electric field intensity E which are physically reasonable. Then equation (2.8e) becomes a first order equation in the potential y which is integrable.

We make the choice

$$Z = \frac{1+(a-b)x}{1+ax}, \tag{2.9}$$

where a and b are real constants. The quantity Z is regular at the stellar centre and continuous in the interior because of freedom provided by parameters a and b. It is important to realise that this choice for Z is physically reasonable and contains special cases which contain neutron star models. When $a = b = 1$ we regain the form of Z for the charged Hansraj and Maharaj (2006) charged stars. For the value $a = 1$ the potential corresponds to the Maharaj and Komathiraj (2007) compact spheres in electric fields. The choice (2.9) was made by Finch and Skea (1989) to generate stellar models that satisfy all physical criteria for a stellar source. If we set $a = 1$, $b = -3/2$ then we generate the Durgapal and Bannerji (1989) neutron star model. When $a = 7$, $b = 8$ then we generate the gravitational potential of the superdense stars of Tikekar (1990). Thus the form Z chosen is likely to produce physically reasonable models for charged anisotropic spheres with an equation of state.

For the electric field we make the choice

$$\frac{E^2}{C} = \frac{k(3+ax) + sa^2x^2}{(1+ax)^2}, \tag{2.10}$$

which has desirable physical features in the stellar interior. It is finite at the centre of the star and remains bounded and continuous in the interior; for large values of x it approaches a constant value. When $s = 0$ then we regain E studied by Thirukkanesh and Maharaj (2008). However their choice is not suitable as the proper charge density becomes singular at the origin as pointed out by Varela $et\ al$ (2010). Consequently we have adapted the form of E so that the proper charge density remains regular throughout the stellar interior with an equation of state. These features become clear in the analysis that follows.

2.4 New models

On substituting (2.9) and (2.10) into (2.8e) we get the first order equation

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{(1+\alpha)b}{4[1+(a-b)x]} + \frac{\alpha b}{2(1+ax)[1+(a-b)x]} - \frac{\beta(1+ax)}{4C[1+(a-b)x]} \\ &\quad - \frac{(1+\alpha)[k(3+ax)+sa^2x^2]}{8(1+ax)[1+(a-b)x]}. \end{aligned} \tag{2.11}$$

For the integration of equation (2.11) it is convenient to consider three cases: $b = 0, a = b$ and $a \neq b$.

2.4.1 The case $b = 0$

When $b = 0$, (2.11) gives the solution

$$\begin{aligned} y &= D\,(1+ax)^{(-(k-2s)(1+\alpha))/(8a)} \\ &\quad \times \exp\left[\frac{(1+\alpha)[2k - sax(2+ax)]}{8a(1+ax)} - \frac{\beta x}{4C}\right], \end{aligned} \tag{2.12}$$

where D is the constant of integration. The potential y in (2.12) generates a negative density $\rho = -\frac{E^2}{2}$ which is physically undesirable.

2.4.2 The case $b = a$

When $a = b$, (2.11) yields the solution

$$y = D\,(1+ax)^{(4a\alpha-(2k+s)(1+\alpha))/(8a)} \exp\left[F(x)\right], \tag{2.13}$$

where we have let

$$F(x) = \frac{x}{16C}\left[2C(s-k)(1+\alpha) - 4\beta - a(C(-4+sx)(1+\alpha) + 2\beta x)\right], \tag{2.14}$$

and D is the constant of integration. Then we can generate an exact model for the system (2.8a)-(2.8f) in the form

$$e^{2\lambda} = 1 + ax, \tag{2.15a}$$

$$e^{2\nu} = A^2D^2\,(1+ax)^{(2a\alpha-k(1+\alpha))/(2a)}\exp\left[2F(x)\right], \tag{2.15b}$$

$$\frac{\rho}{C} = \frac{(2a-k)}{2}\frac{(3+ax)}{(1+ax)^2}, \tag{2.15c}$$

$$p_r = \alpha\rho - \beta, \tag{2.15d}$$

$$p_t = p_r + \Delta, \tag{2.15e}$$

$$\begin{aligned}\Delta = \; & \frac{1}{16C(1+ax)^3}\{C^2[k^2((1+\alpha)^2x(3+ax)^2+2sx(1+ax)) \\ & +4a^2x(3-8\alpha+9\alpha^2+a^2(1+\alpha)^2x^2+2ax(2+3\alpha+3\alpha^2)) \\ & -4k(12+a^3(1+\alpha)^2x^3+a^2x^2(7+9\alpha+6\alpha^2) \\ & +ax(12+5\alpha+9\alpha^2+4s)) \\ & +s^2(2x(a^2x^2-1)(2k-asx+s)+a^2x^2-6ax+1) \\ & -2s(ax(6\alpha^2+6\alpha+2a(2\alpha^2+2\alpha+1)+k(7-2\alpha-\alpha^2)) \\ & +a^2x^2(\alpha^2+2\alpha-2a(1+\alpha+x)+17)+a(\alpha^2-2\alpha) \\ & -k(1+\alpha)^2+7)]-4Cx(1+ax)^2[(1+\alpha)(2a^2x-3k-2s) \\ & -a\beta(k(1+\alpha)+sax(1+\alpha)-6\alpha-4)]+4\beta^2x(1+ax)^4\},\end{aligned} \tag{2.15f}$$

$$\frac{E^2}{C} = \frac{k(3+ax)+sa^2x^2}{(1+ax)^2}, \tag{2.15g}$$

$$\frac{\sigma^2}{C} = \frac{C\left(\sqrt{k}(a^2x^2+3ax+6)+2\sqrt{s}ax\sqrt{3+ax}(2+ax)\right)^2}{x(3+ax)(1+ax)^5}. \tag{2.15h}$$

The new exact solution (2.15a)-(2.15h) of the Einstein-Maxwell system is presented in terms of elementary functions. When $s = 0$ we regain the first class of charged anisotropic models of Thirukkanesh and Maharaj (2008). For our models the mass function is given by

$$\begin{aligned}M(x) = \; & \frac{1}{8C^{3/2}}\left[\frac{(4a^2-2ak)x^{3/2}}{a(1+ax)}+\frac{s(15+10ax-2a^2x^2)x^{1/2}}{3a(1+ax)}\right. \\ & \left.-\frac{5s\arctan(\sqrt{ax})}{a^{3/2}}\right].\end{aligned} \tag{2.16}$$

The gravitational potentials and matter variables are well behaved in the interior of the sphere. However, as in earlier treatments, the singularity in the charge distribution at the centre is still present in general. In our new solution the singularity can be eliminated when $k = 0$. Then equation (2.15h) becomes

$$\frac{\sigma^2}{C} = \frac{4Csa^2x(2+ax)^2}{(1+ax)^5}. \tag{2.17}$$

At the stellar centre $x = 0$ and the charge density vanishes.

2.4.3 The case $b \neq a$

On integrating (2.11), with $b \neq a$ we obtain

$$\begin{aligned} y &= D(1+ax)^m[1+(a-b)x]^n \\ &\quad \times \exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{8C(a-b)}\right], \end{aligned} \tag{2.18}$$

where *D* is the constant of integration. The constants *m* and *n* are given by

$$\begin{aligned} m &= \frac{4\alpha b-(1+\alpha)(s+2k)}{8b}, \\ n &= \frac{1}{8bC(a-b)^2}[a^2C((1+\alpha)(s+2k)-4\alpha b)-abC(5k(1+\alpha) \\ &\quad -2b(1+5\alpha))+b^2(3kC(1+\alpha)-2bC(1+3\alpha)+2\beta)]. \end{aligned}$$

Then we can generate an exact model for the system (2.8a)-(2.8f) in the form

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+(a-b)x}, &(2.19a)\\
e^{2\nu} &= A^2D^2\left(1+ax\right)^{2m}\left[1+(a-b)x\right]^{2n}\\
&\quad\times\exp\left[-\frac{ax[Cs(1+\alpha)+2\beta]}{4C(a-b)}\right], &(2.19b)\\
\frac{\rho}{C} &= \frac{(2b-k)(3+ax)-sa^2x^2}{2(1+ax)^2}, \qquad \rho>0, &(2.19c)\\
p_r &= \alpha\rho-\beta, &(2.19d)\\
p_t &= p_r+\Delta, &(2.19e)\\
\Delta &= \frac{-bC}{(1+ax)}-\frac{bC(1+5\alpha)}{(1+\alpha)(1+ax)^2}+\frac{2\beta}{1+\alpha}+\frac{Cx[1+(a-b)x]}{(1+ax)}\\
&\quad\times\left[4\left(\frac{a^2m(m-1)}{(1+ax)^2}+\frac{2a(a-b)mn}{(1+ax)[1+(a-b)x]}+\frac{(a-b)^2n(n-1)}{[1+(a-b)x]^2}\right)\right.\\
&\quad-\frac{a[Cs(1+\alpha)+2\beta](a(m+n)[1+(a-b)x]-bn)}{(a-b)C(1+ax)[1+(a-b)x]}\\
&\quad\left.+\frac{a^2[Cs(1+\alpha)+2\beta]^2}{16C^2(a-b)^2}\right]-\frac{4[1+ax(2+(a-b)x)]-b(5+\alpha)x}{4(a-b)(1+\alpha)(1+ax)^3[1+(a-b)x]}\\
&\quad\times[-8b^2Cn+a^3x(-8C(m+n)+[Cs(1+\alpha)+2\beta]x)\\
&\quad+a^2(8C(m+n)(2bx-1)+[Cs(1+\alpha)+2\beta](2-bx)x)\\
&\quad+a(-8b^2C(m+n)x+[Cs(1+\alpha)+2\beta]\\
&\quad+b(8Cm+16Cn-[Cs(1+\alpha)+2\beta]x))], &(2.19f)\\
\frac{E^2}{C} &= \frac{k(3+ax)+sa^2x^2}{(1+ax)^2}, &(2.19g)\\
\frac{\sigma^2}{C} &= [1+(a-b)x](\sqrt{k}(a^2x^2+3ax+6)\\
&\quad+2\sqrt{s}ax\sqrt{3+ax}(2+ax))^2/[x(3+ax)(1+ax)^5]. &(2.19h)
\end{aligned}
$$

The exact solution (2.19a)-(2.19h) of the Einstein-Maxwell system is written in terms of elementary functions. For this case the mass function is given by

$$
\begin{aligned}
M(x) &= \frac{1}{8C^{3/2}}\left[\frac{(4ab-2ak)x^{3/2}}{a(1+ax)}+\frac{s(15+10ax-2a^2x^2)x^{1/2}}{3a(1+ax)}\right.\\
&\quad\left.-\frac{5s\arctan(\sqrt{ax})}{a^{3/2}}\right]. &(2.20)
\end{aligned}
$$

This solution is a generalisation of the second class of charged anisotropic models of Thirukkanesh and Maharaj (2008): When $s = 0$ we obtain their expressions for the gravitational potentials and matter variables. These quantities are well behaved and regular in the interior of the sphere. However in general there is a singularity in the charge density at the centre. This singularity is eliminated when $k = 0$ so that

$$\frac{\sigma^2}{C} = \frac{4Csa^2x[1+(a-b)x](2+ax)^2}{(1+ax)^5}. \tag{2.21}$$

At the centre of the star $x = 0$ and the charge density vanishes.

2.5 Physical features

In this section we show that the exact solutions found in Section 2.4, for particular choices of the parameters $a,\ b$ and s, are physically reasonable. A detailed physical analysis for general values of the parameters will be a future investigation. We used the programming language Python to generate these plots for case: $a > b$ ($a = 2.5$, $b = 2.0$), $\alpha = 0.33$, $\beta = \alpha\tilde{\rho} = 0.198$, $C = 1$ and $s = 2.5$, $k = 0$ where $\tilde{\rho}$ is the density at the boundary. The solid lines correspond to $s = 0$ and the dashed lines correspond to $s \neq 0$ in the graphs. We generated the following plots: energy density (Figure 2.1), radial pressure (Figure 2.2), tangential pressure (Figure 2.3), anisotropy measure (Figure 2.4), electric field intensity (Figure 2.5), charge density (Figure 2.6) and mass (Figure 2.7). The energy density ρ is positive, finite and monotonically decreasing. The radial pressure p_r is similar to ρ since p_r and ρ are related by a linear equation of state. In Figure 2.2, it is clear that the numerical values of the radius are $r = 0.85$ for $s = 0$ and $r = 0.72$ for $s \neq 0$. The values of ρ and p_r are lower in the presence of the electric field $E \neq 0$. The tangential pressure is well behaved increasing away from the centre, reaches a maximum and becomes a decreasing function. This is reasonable since the conservation of angular momentum during the quasi-equilibrium contraction of a massive body should lead to high values of p_t in central regions of the star as pointed out by Karmakar $et\ al$ (2007). The anisotropy Δ is increasing in the neighbourhood of the centre, reaches a maximum value and then subsquently decreases. The profile of Δ is similar to the profiles studied by Sharma and Maharaj (2007a) and Tikekar and

Jotania (2009) for strange stars with quark matter.

The form chosen for *E* is physically reasonable and describes a function which is initially small and then increases as we approach the boundary. The charge density in general is continuous, initially increases and then decreases. Note that the singularity at the stellar centre is eliminated since $k = 0$. The mass function is a strictly increasing function which is continuous and finite. We observe that the mass, in the presence of charge, has lower values than that of the corresponding uncharged case. This is consistent as $E \neq 0$ generates lower densities which produces a weaker total field since the electromagnetic field is repulsive. Thus all matter variables, electromagnetic quantities and gravitational potentials are nonsingular and well behaved in a region away from the stellar centre. We emphasize that the electromagnetic quantities are all well behaved close to the stellar centre since there are finite values for the charge density. This is different from other treatments with an equation of state.

The solutions found in this chapter can be used to model realistic stellar bodies. We introduce the transformations:

$$\tilde{a} = aR^2,\ \tilde{b} = bR^2,\ \tilde{\beta} = \beta R^2,\ \tilde{k} = kR^2,\ \tilde{s} = sR^2.$$

Using these transformations the energy density becomes

$$\rho = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}y) - \tilde{s}\tilde{a}^2y^2}{2R^2(1 + \tilde{a}y)^2}. \tag{2.22}$$

The mass contained within a radius *r* is given by

$$M = \frac{r^3(6\tilde{b} - 3\tilde{k} + 5\tilde{s})}{12R^2(1 + \tilde{a}y)} + \frac{\tilde{s}r(15 - 2\tilde{a}y^2)}{24\tilde{a}(1 + \tilde{a}y)} - \frac{5\tilde{s}R\arctan[\sqrt{\tilde{a}y}]}{8\tilde{a}^{3/2}}, \tag{2.23}$$

where we have set $C = 1$ and $y = \frac{r^2}{R^2}$. If $\tilde{k} = 0$, $\tilde{s} = 0$ $(E = 0)$ then we have

$$\rho = \frac{\tilde{b}(3 + \tilde{a}y)}{R^2(1 + \tilde{a}y)^2}, \tag{2.24}$$

$$M = \frac{\tilde{b}r^3}{2R^2(1 + \tilde{a}y)}. \tag{2.25}$$

In this case there is no charge and we obtain the expressions of Sharma and Maharaj (2007a). For the astrophysical importance of our solutions, we try to compare the masses corresponding to the models of this chapter to those found by Sharma and Maharaj (2007a) and Thirukkanesh and Maharaj (2008).

We calculate the masses for the various cases with and without charge. We set $r = 7.07$ km, $R = 43.245$ km, $\tilde{k} = 37.403$ and $\tilde{s} = 0.137$. We tabulate the information in Table 2.1 Note that when $k = 0$, $s = 0$ we have an uncharged stellar body and we regain masses generated by Sharma and Maharaj (2007a). When $k \neq 0$ and $s = 0$ we find masses for a charged relativistic star in the Thirukkanesh and Maharaj (2008) models. We have included those two sets of values for consistency and to demonstrate that our general results contain the special cases considered previously. When $k = 0$ and $s \neq 0$ the masses found correspond to new charged solutions, with nonsingular charge densities at the origin. When $k \neq 0$ and $s \neq 0$ then we have the most general case. In all cases, we obtain stellar masses which are physically reasonable. We observe that the presence of charge generates a lower mass *M* because of the repulsive electromagnetic field which corresponds to a weaker field. Observe that our masses are consistent with the results of Dey $et\ al$ (1998, 1999a, 1999b) with an equation of state for strange matter. When the charge is absent the mass $M = 1.434 M_{\bigodot}$; the presence of charge in the different solutions affects this value. Dey $et\ al$ (1999, 1999a, 1999b) have shown that these values are consistent with observations for the X-ray binary pulsar SAX J1808.4-3658. Consequently these charged general relativistic models have astrophysical significance. A trend in the masses is observable in Table 2.1. It is interesting to observe the smallest masses are attained when $k \neq 0$, $s = 0$ in the presence of the electromagnetic field. When $k \neq 0$, $s = 0$ then the electric field intensity is stronger by (2.10) which negates the attraction of the gravitational field leading to a weaker field. Note that we have also included the value of anisotropy Δ in Table 2.1. We note that larger stellar masses correspond to increasing values of anisotropy.

2.6 Density variation

From (2.22) we observe that

$$\rho_{\Re} = \frac{(2\tilde{b} - \tilde{k})(3 + \tilde{a}\frac{\Re^2}{R^2}) - \tilde{s}\tilde{a}^2\frac{\Re^4}{R^4}}{2R^2(1 + \tilde{a}\frac{\Re^2}{R^2})^2}, \tag{2.26}$$

is the density at the stellar surface. The density at the centre of the star is

$$\rho_c = \frac{3(2\tilde{b} - \tilde{k})}{2R^2}. \tag{2.27}$$

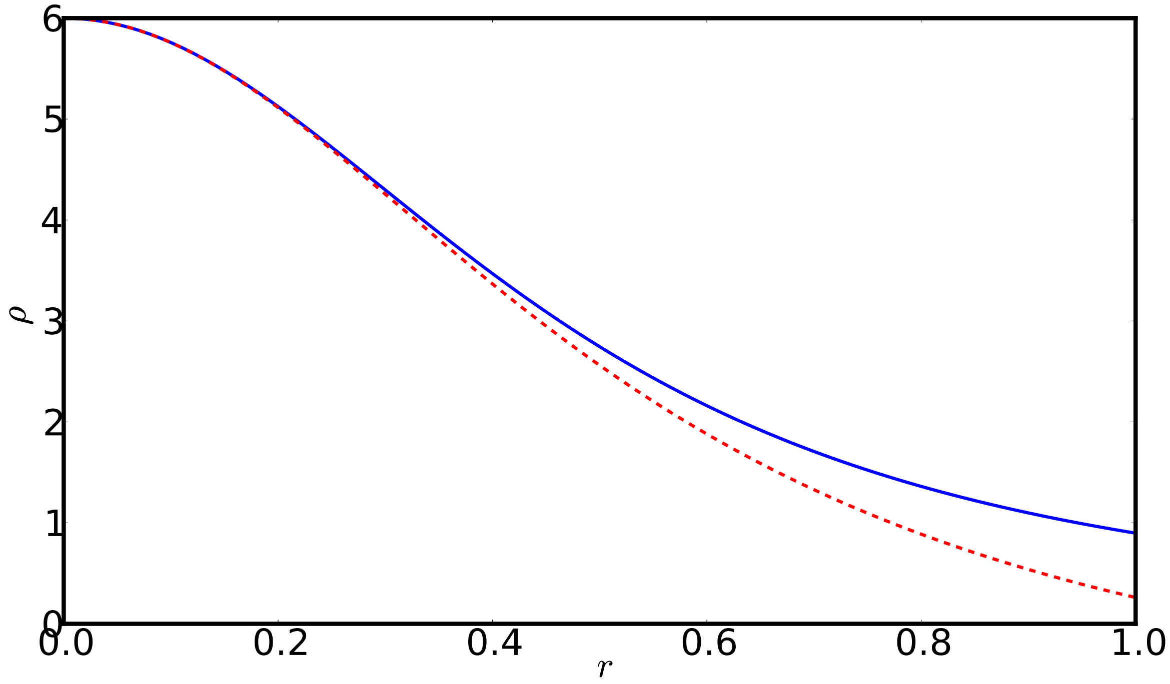

Figure 2.1: Energy density $\rho(r)$ versus radius.

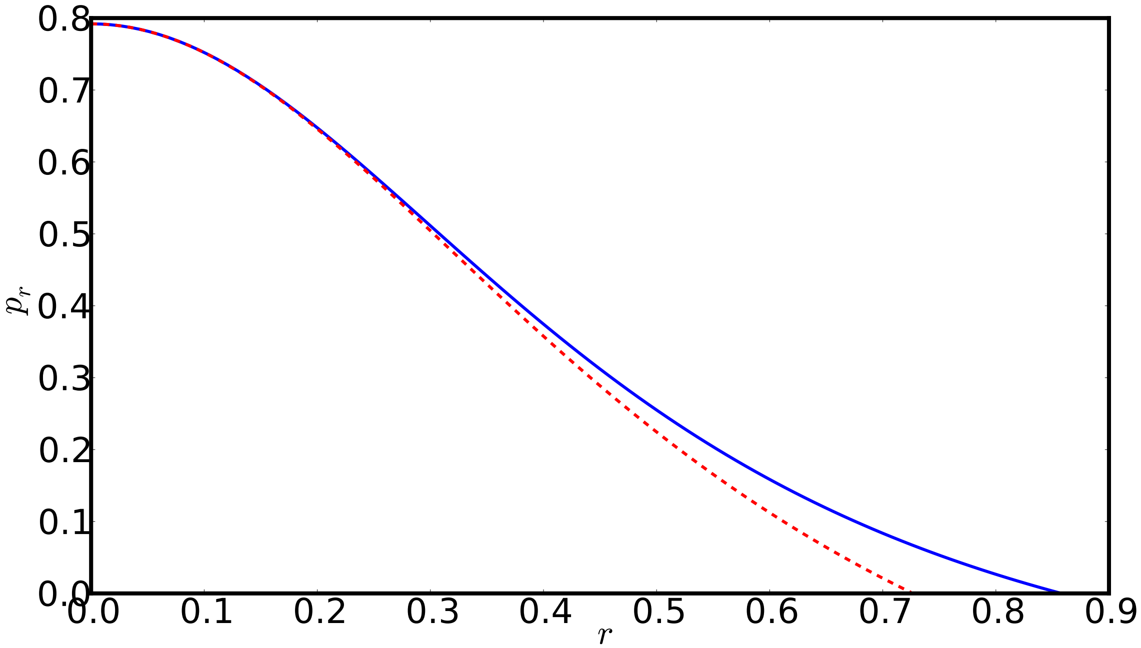

Figure 2.2: Radial pressure $p_r(r)$ versus radius.

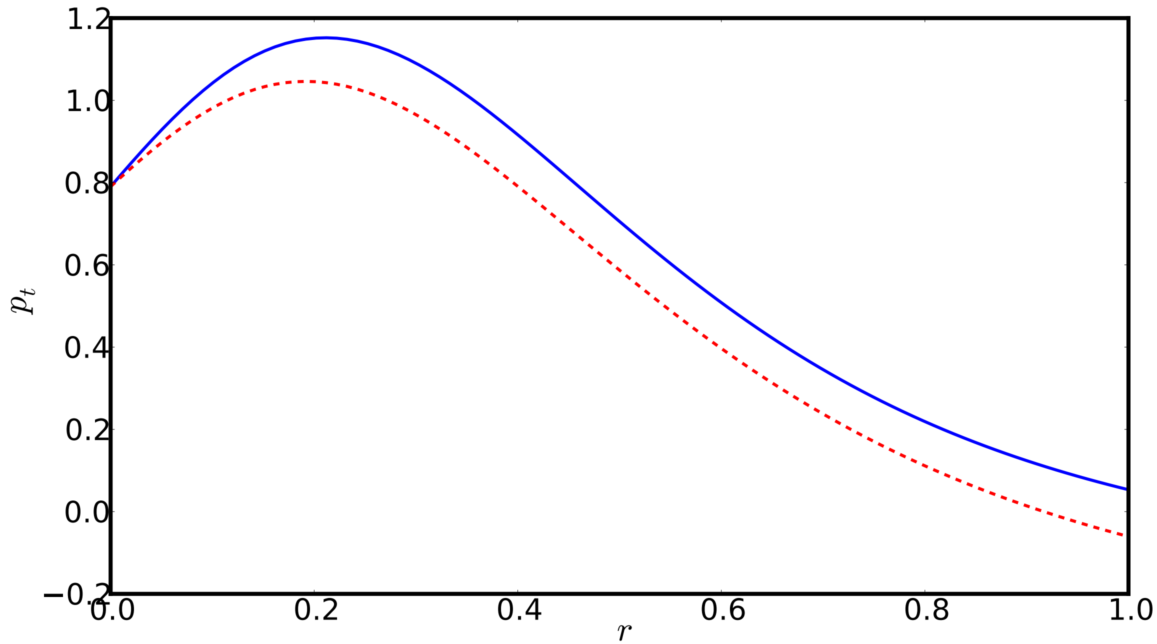

Figure 2.3: Tangential pressure $p_t(r)$ versus radius.

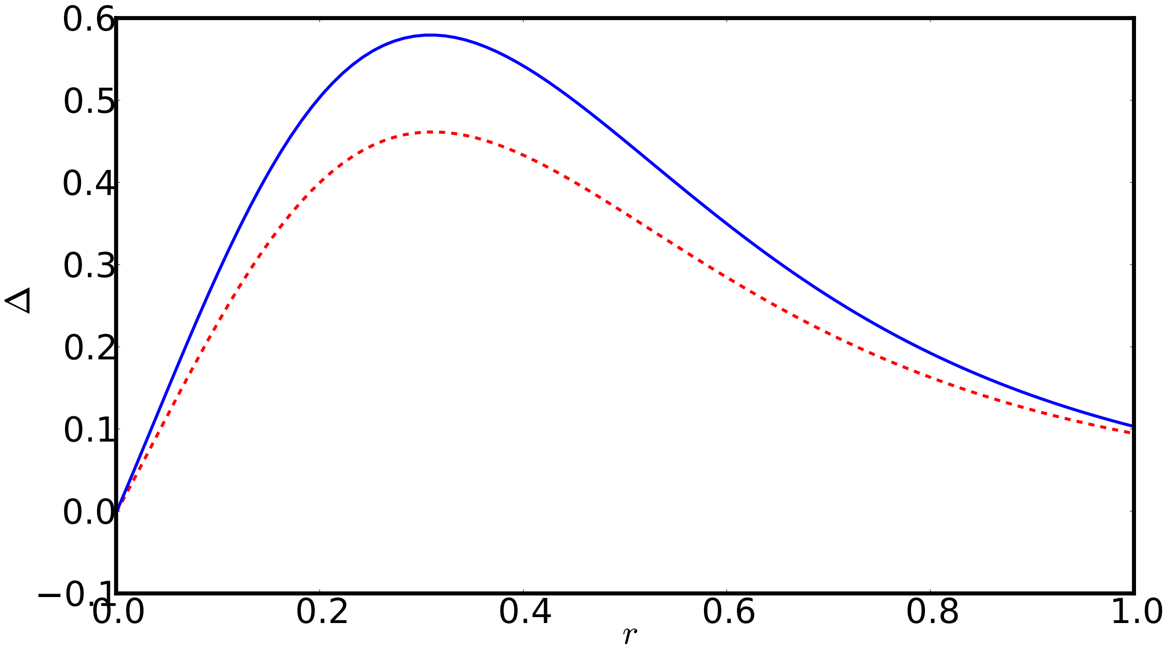

Figure 2.4: Anisotropy $\Delta(r)$ versus radius.

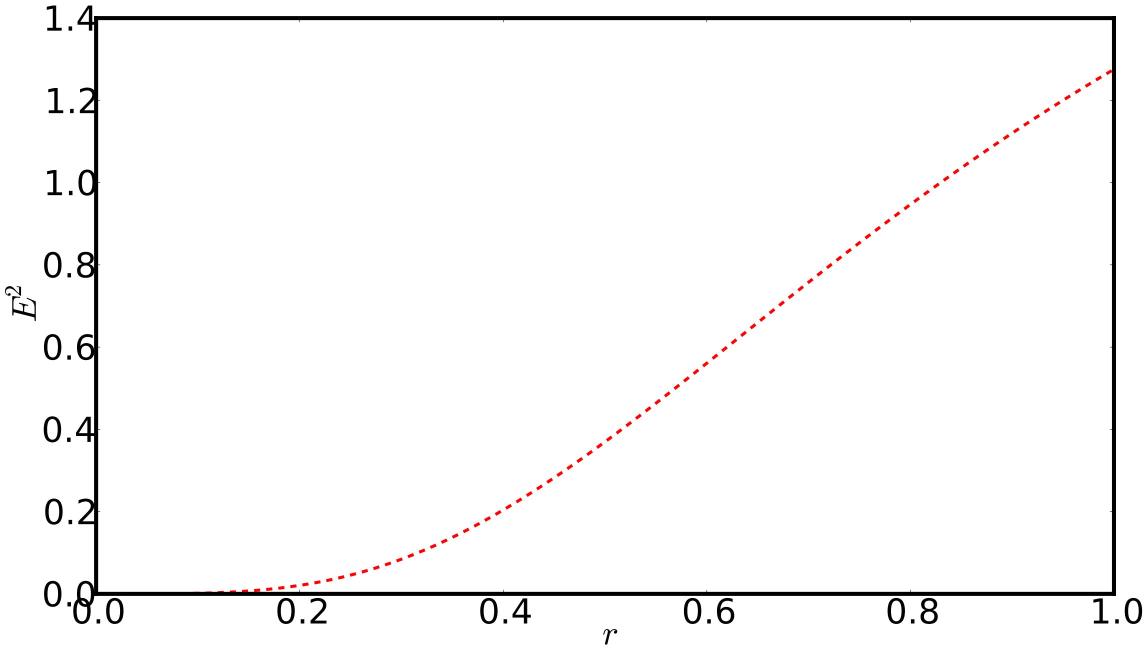

Figure 2.5: Electric field $E^2(r)$ versus radius.

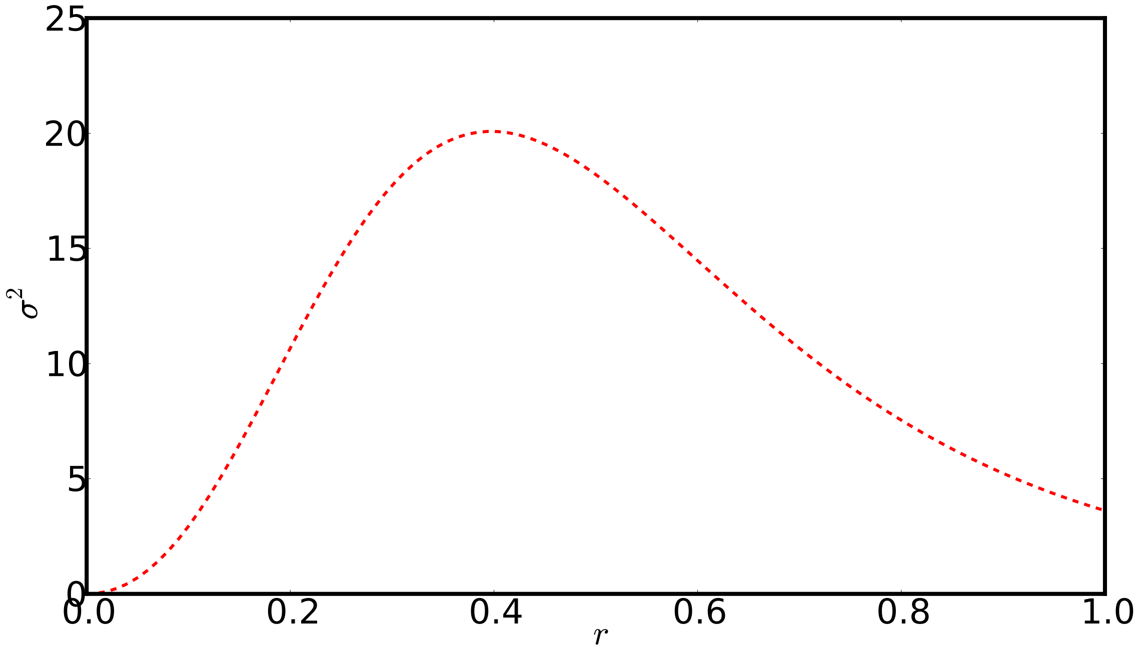

Figure 2.6: Charge density $\sigma^2(r)$ versus radius.

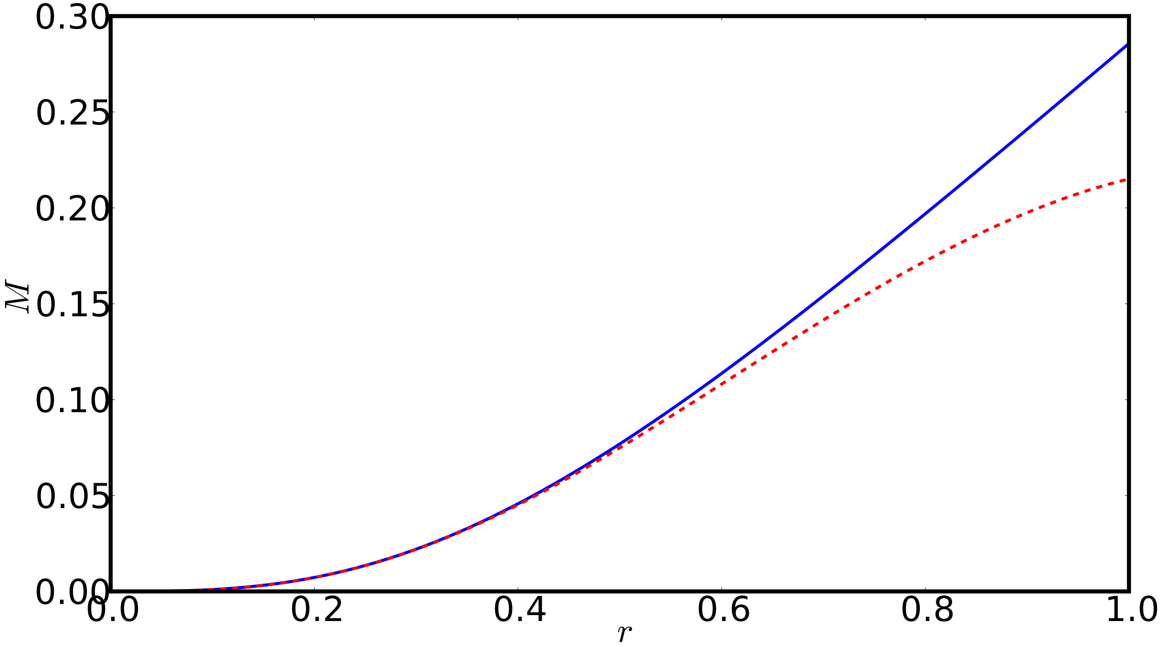

Figure 2.7: Mass $M(r)$ versus radius.

Table 2.1: Central density and mass for different anisotropic stellar models for neutral and charged bodies

$\tilde{b}$	$\tilde{a}$	α	ρ_c ($\times 10^{15}$ g cm^{-3}) $E = 0$	$\frac{M}{M_\odot}$ $E = 0$	ρ_c ($\times 10^{15}$ g cm^{-3}) $E \neq 0$	$\frac{M}{M_\odot}$ $E \neq 0$	Δ
30	23.681	0.401	2.579	1.175	0.971	0.433	0.039
40	36.346	0.400	3.439	1.298	1.831	0.691	0.054
50	48.307	0.424	4.298	1.396	2.691	0.874	0.072
54.34	**53.340**	**0.437**	**4.671**	**1.433**	**3.064**	**0.940**	**0.081**
60	59.788	0.457	5.158	1.477	3.550	1.017	0.094
70	70.920	0.495	6.017	1.546	4.410	1.133	0.119
80	81.786	0.537	6.877	1.606	5.269	1.231	0.146
90	92.442	0.581	7.737	1.659	6.129	1.314	0.177
100	102.929	0.627	8.596	1.705	6.989	1.386	0.207
183	186.163	1.083	15.730	1.959	14.124	1.759	0.593

Then we can generate the ratio

$$\epsilon = \frac{(2\tilde{b}-\tilde{k})(3+\tilde{a}\frac{\Re^2}{R^2}) - \tilde{s}\tilde{a}^2\frac{\Re^4}{R^4}}{3(2\tilde{b}-\tilde{k})(1+\tilde{a}\frac{\Re^2}{R^2})^2}, \tag{2.28}$$

called the density contrast. With the help of (2.28) we can find $\frac{\Re^2}{R^2}$ in terms of ϵ:

$$\begin{aligned} \frac{\Re^2}{R^2} &= \frac{-(2\tilde{b}-\tilde{k})(6\epsilon-1)}{2a(3\epsilon(2\tilde{b}-\tilde{k})+\tilde{s})} \\ &\quad + \frac{\sqrt{(24\epsilon+1)(2\tilde{b}-\tilde{k})^2 + 12\tilde{s}(2\tilde{b}-\tilde{k})(1-\epsilon)}}{2a(3\epsilon(2\tilde{b}-\tilde{k})+\tilde{s})}. \end{aligned} \tag{2.29}$$

Then (2.23) and (2.29) yield the quantity

$$\begin{aligned} \frac{M}{\Re} &= \frac{\Re^2(6\tilde{b}-3\tilde{k}+5\tilde{s})}{12R^2(1+\tilde{a}\frac{\Re^2}{R^2})} + \frac{\tilde{s}\Re(15-2\tilde{a}\frac{\Re^4}{R^4})}{24\tilde{a}(1+\tilde{a}\frac{\Re^2}{R^2})} \\ &\quad - \frac{5\tilde{s}R\arctan[\sqrt{\tilde{a}\frac{Re^2}{R^2}}]}{8\Re a^{3/2}}, \end{aligned} \tag{2.30}$$

which is the compactification factor. Consequently our model is characterised by the surface density $\rho_{\Re}$, the density contrast ϵ, and the compactification factor $\frac{M}{\Re}$ in the presence of charge. These two parameters produce information of astrophysical significance for specific choices of the parameters. The compactification factor classifies stellar objects in various categories depending on the range of $\frac{M}{\Re}$:

- for normal stars $\frac{M}{\Re} \sim 10^{-5}$,
- for white dwarfs $\frac{M}{\Re} \sim 10^{-3}$,
- for neutron stars $\frac{M}{\Re} \sim 10^{-1}$ to $\frac{1}{4}$,
- for ultra-compact stars $\frac{M}{\Re} \sim \frac{1}{4}$ to $\frac{1}{2}$,
- and for black holes $\frac{M}{\Re} \sim \frac{1}{2}$.

The parameter $\frac{M}{\Re}$ in the case of strange stars is in the range of ultra-compact stars with matter densities greater than the nuclear density.

Table 2.2: Variation of density

$\tilde{a}$	$\tilde{b}$	ϵ	$\Re$	$\frac{M}{\Re}$	$\frac{M}{M_\odot}$
1.6	2	0.3	8.30	0.073	0.60
(1.6)	(0.6)	(0.3)	(10.18)	(0.100)	(1.02)
2.4	2.8	0.5	10.67	0.093	0.99
(2.4)	(1.4)	(0.5)	(11.20)	(0.102)	(1.14)
6.0	6.4	0.1	13.59	0.277	3.76
(6.0)	(5.0)	(0.1)	(15.83)	(0.329)	(5.20)

For particular choices of the parameters $\tilde{a}$, $\tilde{b}$, $\tilde{k}$, $\tilde{s}$ and specifying the density contrast ϵ we can find the boundary $\Re$ from (2.29). Note that the parameter R is specified if we take the central density to be $\rho_c = 2 \times 10^{14}$ g cm^{-3} in (2.27). The compactification factor $\frac{M}{\Re}$ then follows from (2.30). For charged matter we take the values $k = 2.8$ and $s = 2$, and for uncharged matter $k = 0$ and $s = 0$. Table 2.2 represents typical values for ϵ and $\frac{M}{\Re}$ for charged matter and uncharged matter; the first set of values are for charged matter and the bracketted values are the corresponding values for neutral matter when $\tilde{k} = \tilde{s} = 0$. For uncharged matter we regain the values of ϵ and $\frac{M}{\Re}$ generated by Tikekar and Jotania (2005) for superdense star models with neutral matter. We observe that the presence of charge has the effect of reducing both the radius $\Re$ and the compactification $\frac{M}{\Re}$ to produce the same value of the density contrast ϵ. This is consistent since the presence of the electric field intensity has the effect of leading to a weaker field. The values that we have generated in Table 2.2 for neutral and charged matter permit configurations typical of neutron stars and strange stars. Note that changing the various parameters will allow for smaller or larger compactification factors. Thus the class models found in this chapter allow for stellar configurations which provide physically viable models of superdense structures.

CHAPTER 3

Regular models with quadratic equation of state

3.1 Introduction

The study of charged relativistic objects in general relativity is achieved by solving the Einstein-Maxwell system of equations and imposing conditions for physical acceptability. This is not easy to achieve because of the nonlinearity of the field equations. The exact solutions found have many applications in relativistic astrophysics. The models generated have been used in the description of neutron stars and black hole formation by Ray $et\ al$ (2003) and de Felice $et\ al$ (1999). Particular models have also helped in the establishment of the absolute stability limit for charged spheres by Giuliani $et\ al$ (2008) and Boehmer and Harko (2007). Several models of charged relativistic matter have been used to study strange stars by Mak and Harko (2002), Farhi and Jaffe (1984), Komathiraj and Maharaj (2007c) and Thirukkanesh and Maharaj (2008). Charged models have been also used in the description of strange quark matter by Discus $et\ al$ (2008), hybrid protoneutron stars by Nicotra $et\ al$ (2006), and bare quark stars by Usov $et\ al$ (2005). A geometric approach is to assume the existence of a group of conformal motions on spacetime; exact solutions have been found by Mak and Harko (2004) for strange quark matter and Esculpi and Aloma (2010) for anisotropic relativistic charged matter by assuming the existence of a conformal Killing vector in static spherically symmetric spacetimes.

Models with an equation of state are desirable in the description of realistic astrophysical matter. However most explicit solutions of the Einstein-Maxwell system that have been found do not satisfy this property. There have been some attempts made recently to find exact analytic solutions of the Einstein-Maxwell system with a linear equation of state. These include the treatments of Ivanov (2002), Sharma and Maharaj (2007a), and Thirukkanesh and Maharaj (2008). Particular solutions with a quadratic equation of state, relating the radial pressure to the energy, were found by Feroze and Siddiqui (2011). This is an important advance since the complexity of the model is greatly increased because of the nonlinearity of the radial pressure in terms of the energy density. However the investigations mentioned above all suffer from the undesirable property of possessing a singularity in the proper charge density at the centre of sphere. An essential requirement for a well behaved electromagnetic field is regularity of the proper charge density throughout the matter distribution, particularly at the stellar centre. The importance of this feature has been highlighted in the analysis of Varela $et\ al$ (2010) whose treatment offers a general approach of dealing with anisotropic charged matter with linear or nonlinear equations of state. It is desirable to eliminate the singularity in the proper charge density for a detailed and complete analysis of physical properties of charged compact objects.

Our results may be helpful in the study of compact stars and gravitational collapse relating to neutron stars and black holes. In this regard we refer to particular papers some of which have static spherical geometry and others are dynamical. Novikov (1967) showed in the case of spherical geometry that collapse of electrically charged matter may be replaced by expansion and infinite densities are avoided. A general treatment of collapsing charged matter was completed by Bekenstein (1971) who showed that nonzero pressure plays a significant role. The analysis of Raychaudhuri (1975) for charged dust distributions showed that conditions for collapse and oscillation depend on the ratio of matter density to charge density. If this ratio is large, corresponding to weakly charged dust spheres, then shell crossings cannot be avoided in gravitational collapse as proved by Ori (1991). Krasinski and Bolejko (2006) showed that there exist initial conditions for a charged dust sphere with finite radius so that a full cycle of pulsation can be completed by the outer layer with no internal singularity. A full and comprehensive analysis of charged, dissipative collapse is provided by Di Prisco $et\ al$ (2007) for the free-streaming and diffusion approximations. A related and detailed analysis in the gravitational collapse of a charged medium was performed by Kouretsis and Tsagas (2010) where the role of Raychaudhuri equation is highlighted. Exact solutions with an equation of state, such as the quadratic case considered in this chapter, are helpful in such studies.

The objective of this chapter is to find new exact solutions of the Einstein-Maxwell field equations with a charged anisotropic matter distribution and a quadratic equation of state. We indicate that particular models found in the past with an equation of state are part of our general analytical framework. Previous solutions with a linear or quadratic equation of state are regained in our treatment. We ensure that the charge density is regular at the centre of the compact body and the physical criteria are satisfied. In Section 3.2, we give the Einstein-Maxwell field equations for a static spherically symmetric line element as an equivalent system of differential equations utilising a transformation due to Durgapal and Bannerji (1983). In Section 3.3, we present new exact solutions to the Einstein-Maxwell system with a quadratic equation of state. The solution is regular at the centre of the compact object. This analysis extends the treatment of Thirukkanesh and Maharaj (2008), and Feroze and Siddiqui (2011). Known solutions with an equation of state are presented in Section 3.4, as particular cases of our new results. In Section 3.5, a physical analysis of the new solutions is performed; the matter variables and the electromagnetic quantities are plotted.

3.2 Field equations

In standard coordinates the line element for a static spherically symmetric spacetime, modelling the interior of the relativistic object, has the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.1}$$

We take the energy momentum tensor to be of the form

$$T_{ij} = \text{diag}(-\rho - \frac{1}{2}E^2,\ p_r - \frac{1}{2}E^2,\ p_t + \frac{1}{2}E^2,\ p_t + \frac{1}{2}E^2), \tag{3.2}$$

where the quantity p_t is the tangential pressure, p_r is the radial pressure, ρ is the density, and E is the electric field intensity. Then the Einstein-Maxwell equations can be written in the form

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho + \frac{1}{2}E^2, \tag{3.3a}$$

$$-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2, \tag{3.3b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_t + \frac{1}{2}E^2, \tag{3.3c}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)', \tag{3.3d}$$

where primes represent differentiation with respect to *r*. The quantity σ represents the proper charge density. We are utilising units where the coupling constant $\frac{8\pi G}{c^2} = 1$ and the speed of light $c = 1$. The mass within a radius *r* of the sphere is

$$M(r) = \frac{1}{2}\int_0^r \omega^2\rho(\omega)d\omega. \tag{3.4}$$

We now introduce a new independent variable *x* and define new functions *y* and *Z* so that

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2y^2(x) = e^{2\nu(r)}, \tag{3.5}$$

where *A* and *C* are constants. We assume an equation of state of the general form $p_r = p_r(\rho)$ for the matter distribution. We take the quadratic form

$$p_r = \gamma\rho^2 + \alpha\rho - \beta, \tag{3.6}$$

relating the radial pressure p_r to the energy density ρ. In the above α, β, and γ are constants. Then the Einstein-Maxwell equations governing the gravitational behaviour of a charged anisotropic sphere, with a quadratic equation of state, are represented by

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}, \tag{3.7a}$$

$$p_r = \gamma\rho^2 + \alpha\rho - \beta, \tag{3.7b}$$

$$p_t = p_r + \Delta, \tag{3.7c}$$

$$\begin{aligned} \frac{\Delta}{C} = {} & 4xZ\frac{\ddot{y}}{y} + 2[x\dot{Z} + 2Z]\frac{\dot{y}}{y} - \alpha\left[\frac{(1-Z)}{x} - 2\dot{Z} - \frac{E^2}{2C}\right] \\ & -C\gamma\left[\frac{(1-Z)}{x} - 2\dot{Z} - \frac{E^2}{2C}\right]^2 + \dot{Z} - \frac{E^2}{2C} + \frac{\beta}{C}, \end{aligned} \tag{3.7d}$$

$$\begin{aligned} \frac{\dot{y}}{y} = {} & \frac{(1-Z)(1+\alpha)}{4Z} - \frac{(1+\alpha)E^2}{8CZ} - \frac{\alpha\dot{Z}}{4Z} - \frac{\beta}{4CZ} \\ & +\frac{C\gamma}{4Z}\left[\frac{(1-Z)}{x} - 2\dot{Z} - \frac{E^2}{2C}\right]^2, \end{aligned} \tag{3.7e}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{3.7f}$$

where dots denote differentiation with respect to the variable x. Equations (3.7a)-(3.7f) are similar to the field equations of Thirukkanesh and Maharaj (2008); however in this case the equation of state is quadratic. The quantity $\Delta = p_t - p_r$ is called the measure of anisotropy and vanishes for isotropic pressures. The nonlinear system as given in (3.7a)-(3.7f) consists of six independent equations in six variables involving the matter and electromagnetic quantities $\rho, p_r, p_t, \Delta, E, \sigma$ and the two gravitational potentials *y* and *Z*. The nonlinearity of the Einstein-Maxwell system (3.7a)-(3.7f) has been increased, when compared with many earlier treatments, become of the appearance of the quadratic term in (3.7b); when $\gamma = 0$ then there is a linear equation of state. In addition, equation (3.7e) now contains terms with E^4 increasing the complexity of system since $\gamma \neq 0$ in general.

3.3 New solutions

To integrate the Einstein-Maxwell system we make the particular choices

$$Z = \frac{1+bx}{1+ax}, \tag{3.8}$$

$$\frac{E^2}{C} = \frac{k(3+ax) + sa^2x^2}{(1+ax)^2}. \tag{3.9}$$

The gravitational potential Z is well behaved and finite at the origin. The electric field intensity E is continuous, regular at the origin and approaches a constant value for increasing values of x. The constants a, b, k and s are real. The general analytic functional forms for Z and E regain particular cases studied in the past with an equation of state.

On substituting (3.8) and (3.9) into (3.7e) we obtain the first order equation

$$\begin{aligned} \frac{\dot{y}}{y} &= \frac{(1+\alpha)(a-b)}{4[1+bx]} + \frac{\alpha(a-b)}{2(1+ax)[1+bx]} - \frac{(1+\alpha)[k(3+ax)+sa^2x^2]}{8(1+ax)[1+bx]} \\ &\quad -\frac{\beta(1+ax)}{4C[1+bx]} + \frac{C\gamma[(3+ax)(2a-2b-k)-sa^2x^2]^2}{16(1+ax)(1+bx)}, \end{aligned} \tag{3.10}$$

for the metric function y. In spite of complexity of equation (3.10) it can be solved in general. On integrating (3.10) we get

$$y = D(1+ax)^m[1+bx]^n \exp\left[F(x)\right], \tag{3.11}$$

where D is the constant of integration. The function $F(x)$ is given explicitly by

$$\begin{aligned} F(x) &= C^2\gamma[2(a-b)-k]^2\left[\frac{2(2b-a)(1+ax)+(b-a)}{8(b-a)^2(1+ax)^2}\right] \\ &\quad -Cs\gamma\left[\frac{(a-b)^2(ax+2)-a(2a+s)(1+ax)}{4(a-b)(1+ax)^2}\right] \\ &\quad -Cs\gamma\left[\frac{(a-b)(4k+s)+(2k(b-3a)+3bs)(1+ax)}{32(a-b)^2(1+ax)^2}\right] \\ &\quad +\frac{ax}{16bC}[C^2s^2\gamma-2Cs(1+\alpha)-4\beta. \end{aligned} \tag{3.12}$$

The constants m and n have the form

$$\begin{aligned} m &= -\frac{(1+\alpha)(s+2k)}{8(b-a)} + \frac{\alpha}{2} \\ &\quad +\gamma[2(a-b)-k]^2\left[\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right] \\ &\quad +\frac{Cs\gamma}{8(a-b)^3}\left[(a-b)[2s(a-b)+a+b]+3ab(k-2b)-b^2k\right. \\ &\quad \left.+2b^3(2a-1)\right], \\ n &= \frac{(1+\alpha)}{8b}[2(a-b)-k] + \frac{(1+\alpha)k-2\alpha(a-b)}{4(b-a)} + \frac{\beta(a-b)}{4Cb^2} \\ &\quad +\gamma[2(a-b)-k]^2\left[\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right] + \frac{Csa^2(1+\alpha)}{8b^2(b-a)} \\ &\quad +\frac{Cs\gamma}{16b^2(b-a)^3}\left[a^4(s+4b)+(k+2b)(6a^2b^2-2a^3b)\right]. \end{aligned}$$

Then we can generate an exact model for the Einstein-Maxwell system (3.7a)-(3.7f) in the form

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ax}{1+bx}, && (3.13a)\\
e^{2\nu} &= A^2D^2\left(1+ax\right)^{2m}\left[1+bx\right]^{2n}\exp[2F(x)], && (3.13b)\\
\frac{\rho}{C} &= \frac{(2a-2b-k)(3+ax)-sa^2x^2}{2(1+ax)^2}, && (3.13c)\\
p_r &= \gamma\rho^2+\alpha\rho-\beta, && (3.13d)\\
p_t &= p_r+\Delta, && (3.13e)\\
\frac{\Delta}{C} &= \frac{4x(1+bx)}{1+ax}\left[\frac{m(m-1)a^2}{(1+ax)^2}+\frac{2mnab}{(1+ax)(1+bx)}+\frac{2ma\dot{F}(x)}{1+ax}+\right.\\
&\left.\frac{b^2n(n-1)}{(1+bx)^2}+\frac{2nb\dot{F}(x)}{1+bx}+\ddot{F}(x)+\dot{F}(x)^2\right]\\
&+\left[-\frac{2(a-b)x}{(1+ax)^2}+\frac{4(1+bx)}{(1+ax)}\right]\left[\frac{am}{1+ax}+\frac{bn}{1+bx}+\dot{F}(x)\right]\\
&-C\gamma\left[\frac{C(2(a-b)-k)(3+ax)-Csa^2x^2}{2(1+ax)^2}\right]^2\\
&-\frac{1}{2(1+ax)^2}[2C(a-b)+k(3+ax)+sa^2x^2-\frac{2\beta}{C}(1+ax)^2]\\
&-\alpha\left[\frac{(2(a-b)-k)(3+ax)-sa^2x^2}{2(1+ax)^2}\right], && (3.13f)\\
\frac{E^2}{C} &= \frac{k(3+ax)+sa^2x^2}{(1+ax)^2}, && (3.13g)\\
\frac{\sigma^2}{C} &= \frac{C[1+bx]\left(\sqrt{k}(a^2x^2+3ax+6)+2\sqrt{s}ax\sqrt{3+ax}(2+ax)\right)^2}{x(3+ax)(1+ax)^5}, && (3.13h)
\end{aligned}
$$

where *F(x)* is given by (3.12). We observe that the exact solution (3.13a)-(3.13h) of the Einstein-Maxwell system has been written solely in terms of elementary functions. For this solution the mass function is given by

$$
\begin{aligned}
M(x) &= \frac{1}{8C^{3/2}}\left[\frac{[(12a(a-b)-6ak)x+s(15+10ax-2a^2x^2)]x^{1/2}}{3a(1+ax)}\right.\\
&\left.-\frac{5s\arctan(\sqrt{ax})}{a^{3/2}}\right]. && (3.14)
\end{aligned}
$$

The gravitational potentials, matter variables and electromagnetic variables are well behaved and regular in the stellar interior. However in general there is a singularity in the charge density at the centre which is evident in (3.13h). This singularity is avoidable when $k = 0$, so that we have

$$\frac{\sigma^2}{C} = \frac{4Csa^2x[1+bx](2+ax)^2}{(1+ax)^5}. \tag{3.15}$$

At the centre of the star $x = 0$ and the charge density vanishes.

3.4 Known solutions

We have found a general class of exact solutions to the Einstein-Maxwell system with a quadratic equation of state. It is interesting to observe that for particular parameter values we can regain uncharged anisotropic and isotropic models $(k = 0,\ s = 0)$ from our general solution (3.13a)-(3.13h). We regain the following particular cases of physical interest.

3.4.1 Feroze and Siddiqui model

This is a special case of our general solution with the quadratic equation of state $p_r = \gamma\rho^2 + \alpha\rho - \beta$. If we set $s = 0$, $C = 1$ and $A^2D^2 = B$, then we regain the line element

$$\begin{aligned} ds^2 &= -B(1+ar^2)^m(1+br^2)^n \exp[2F(r)]dt^2 + \frac{1+ar^2}{1+br^2}dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{3.16}$$

where

$$\begin{aligned} F(x) &= -\frac{\beta ar^2}{4b} + \gamma[2(a-b)-k]^2\left[\frac{2(2b-a)(1+ar^2)+(b-a)}{8(b-a)^2(1+ar^2)^2}\right], \\ m &= \frac{\alpha}{2} - \frac{(1+\alpha)k}{4(b-a)} + \gamma[2(a-b)-k]^2\left[\frac{b^2}{(b-a)^3} + \frac{b}{(b-a)^2} + \frac{1}{4}\right], \\ n &= \frac{(1+\alpha)}{8b}[2(a-b)-k] + \frac{(1+\alpha)k - 2\alpha(a-b)}{4(b-a)} + \frac{\beta(a-b)}{4b^2} \\ &\quad + \gamma[2(a-b)-k]^2\left[\frac{b^2}{(b-a)^3} + \frac{b}{(b-a)^2} + \frac{1}{4}\right]. \end{aligned}$$

The line element (3.16) was found by Feroze and Siddiqui (2011) which was the first model with quadratic equation of state. Some minor misprints in Feroze and Siddiqui (2011) have been corrected in our result. This solution may be used to model a compact body.

3.4.2 Thirukkanesh and Maharaj model

If we set $\gamma = 0$ then we have the linear equation of state $p_r = \alpha\rho - \beta$. Also setting $C = 1$, $s = 0$ and $b = a - \tilde{b}$, we get the line element

$$\begin{aligned} ds^2 &= -A^2D^2(1+ar^2)^{2m}[1+(a-\tilde{b})r^2]^{2n}\exp\left[\frac{-a\beta r^2}{2(a-\tilde{b})}\right]dt^2 \\ &\quad + \frac{1+ar^2}{1+(a-\tilde{b})r^2}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} m &= \frac{2\alpha\tilde{b}-(1+\alpha)k}{4\tilde{b}}, \\ n &= \frac{1}{8\tilde{b}(a-\tilde{b})^2}\Big[2a^2(k(1+\alpha)-2\alpha\tilde{b}) - a\tilde{b}(5k(1+\alpha)-2\tilde{b}(1+5\alpha)) \\ &\quad + \tilde{b}^2(3k(1+\alpha)-2\tilde{b}(1+3\alpha)+2\beta)\Big]. \end{aligned}$$

The metric (3.17) was found by Thirukkanesh and Maharaj (2008). This solution may be used to model realistic charged compact spheres and strange stars with quark matter in the presence of the electromagnetic field.

3.4.3 Sharma and Maharaj model

If we set $\gamma = 0$, $\beta = \alpha\tilde{\rho}$, then we regain the linear equation of state $p_r = \alpha(\rho - \tilde{\rho})$ where $\tilde{\rho}$ is the density at the surface. By setting $k = 0$ $s = 0$, $b = a - \tilde{b}$, $C = 1$ and $A^2D^2 = B$ we find the following form of the line element

$$\begin{aligned} ds^2 &= -B(1+ar^2)^{2m}[1+(a-\tilde{b})r^2]^{2n}\exp\left(\frac{-a\beta r^2}{2(a-\tilde{b})}\right)dt^2 \\ &\quad + \frac{1+ar^2}{1+(a-\tilde{b})r^2}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} m &= \frac{\alpha}{2}, \\ n &= \frac{5a\tilde{b}\alpha - 2a^2\alpha - 3\tilde{b}^2\alpha + a\tilde{b} - \tilde{b}^2 + \tilde{b}\beta}{4(a-\tilde{b})^2}. \end{aligned}$$

The line element (3.18) represents an uncharged anisotropic sphere and was found by Sharma and Maharaj (2007a). It may be used to describe strange stars with a linear equation of state with quark matter.

3.4.4 Lobo model

If we set $\gamma = 0$, $\beta = 0$, then we obtain the linear equation of state $p_r = \alpha\rho$. On setting $k = 0$, $s = 0$, $b = a - \tilde{b}$, $a = 2\tilde{b}$, $C = 1$ and $A^2D^2 = B$ we regain the line element

$$\begin{aligned} ds^2 &= -(1+2\tilde{b}r^2)^{2m}(1+\tilde{b}r^2)^{2n}dt^2 + \left(\frac{1+2\tilde{b}r^2}{1+\tilde{b}r^2}\right)dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} m &= \frac{\alpha}{2}, \\ n &= \frac{1-\alpha}{4}. \end{aligned}$$

The metric (3.19) was first found by Lobo (2006) which represents uncharged anisotropic matter. This solution serves as a stellar interior with $\alpha < -\frac{1}{3}$ and may be matched to the Schwarzchild exterior for dark energy stars.

3.4.5 Isotropic models

We observe that $\Delta \neq 0$ in general and the model remains anisotropic. However, we can show for particular parameter values that $\Delta = 0$ in the general solution (3.13a)-(3.13h). If we set $b = (a-1)$, $k = 0$, $s = 0$, $a = 0$, then we obtain

$$\begin{aligned} m &= \frac{\alpha}{2}, \\ n &= \frac{1}{4C}[\beta - (1+3\alpha)C], \\ \Delta &= \frac{x}{4C(1-x)}[\beta - 3(1+\alpha)C][\beta - (1+3\alpha)C]. \end{aligned} \tag{3.20}$$

Two different cases arise from (3.20) by setting $\Delta = 0$. Firstly, we observe that when $\beta = 0$ and $\alpha = -1$ then $\Delta = 0$. The equation of state becomes $p_r(= p_t) = -\rho$. We get the line element

$$ds^2 = -\left(1 + \frac{r^2}{R^2}\right) dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.21}$$

where we have set $A = D = 1$ and $C = \frac{1}{R^2}$. We mention that the metric (3.21) is the isotropic uncharged de Sitter model. Secondly, we observe that when $\beta = 0$ and $\alpha = -\frac{1}{3}$ then $\Delta = 0$. The equation of state becomes $p_r(= p_t) = -\frac{1}{3}\rho$ with the line element

$$ds^2 = -A^2 dt^2 + \left(1 + \frac{r^2}{R^2}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{3.22}$$

where $D = 1$ and $C = \frac{1}{R^2}$. The metric (3.22) is the isotropic uncharged Einstein model.

3.5 Physical Analysis

We show that the exact solutions of the Einstein-Maxwell system found in Section 3.3 are well behaved by generating graphical plots of matter and electromagnetic variables. We make the particular choices $C = 1$, $a = 2.5$, $b = 2$, $\gamma = 0.01$, $\alpha = 0.33$, $s = 0.017$ and $k = \beta = 0$. We used the programming language Python to generate the plots for the energy density (Figure 3.1), radial pressure (Figure 3.2), electric field intensity (Figure 3.3), charge density (Figure 3.4), mass (Figure 3.5), speed of sound (Figure 3.6), tangential pressure (Figure 3.7) and the measure of anisotropy (Figure 3.8). The energy density ρ is a finite and monotonically decreasing function. The radial pressure p_r is similarly well behaved and continuous. The electric field intensity E is initially small and approaches a maximum value as the boundary is approached. The proper charge density σ is nonsingular at the origin, increases and then decreases after reaching a maximum value. The mass function is a strictly increasing function which is continuous and finite. The speed of sound is less than the speed of light and causality is maintained throughout the stellar interior. The radial pressure is decreasing and does reach a finite value of the radial coordinate. The tangential pressure is also a decreasing function. The measure of anisotropy is a decreasing function as the boundary is approached and remains finite in the interior. Thus all the matter variables, electromagnetic variables and the gravitational potentials are nonsingular and regular in the region containing the stellar centre. In particular the proper charge density σ is finite at the centre unlike earlier treatments.

It is desirable to study comprehensively the stability of our new models; this is a objective for the future reseach. The solutions generated may be matched to the exterior Reissner-Nordstrom spacetime

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 \\ &\quad + r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{3.23}$$

across the boundary $r = \Re$. This generates the following conditions

$$1 - \frac{2M}{\Re} + \frac{Q^2}{\Re^2} = A^2 y^2 (C\Re^2), \tag{3.24}$$

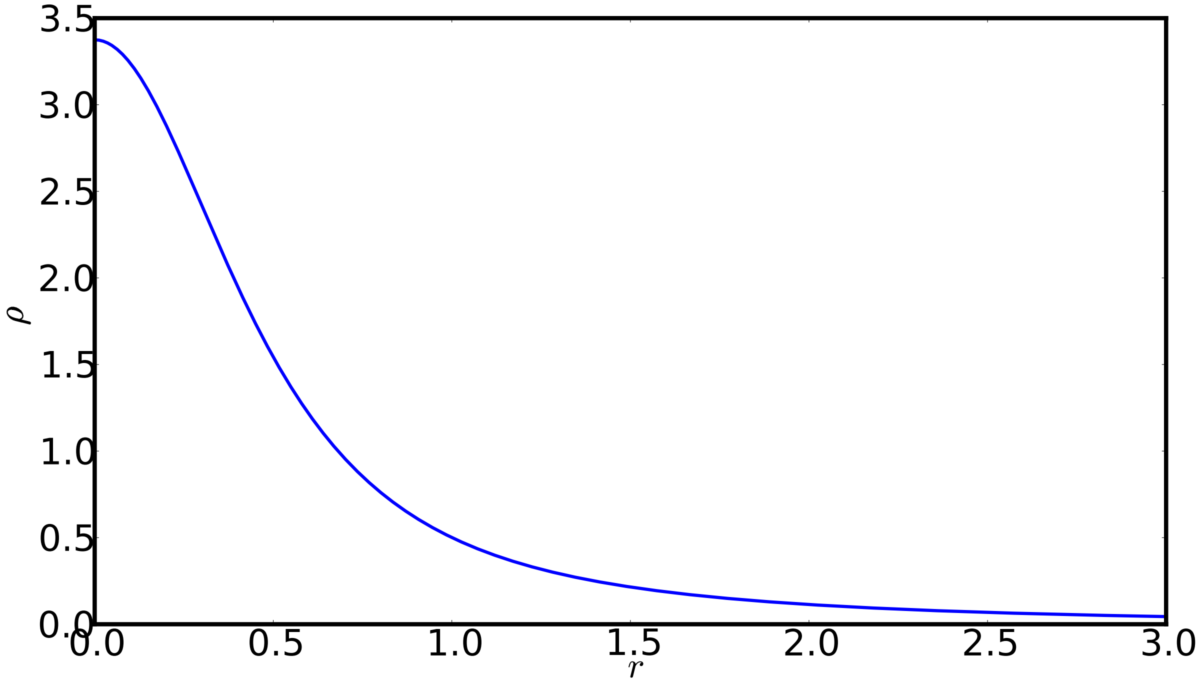

Figure 3.1: Energy density $\rho(r)$.

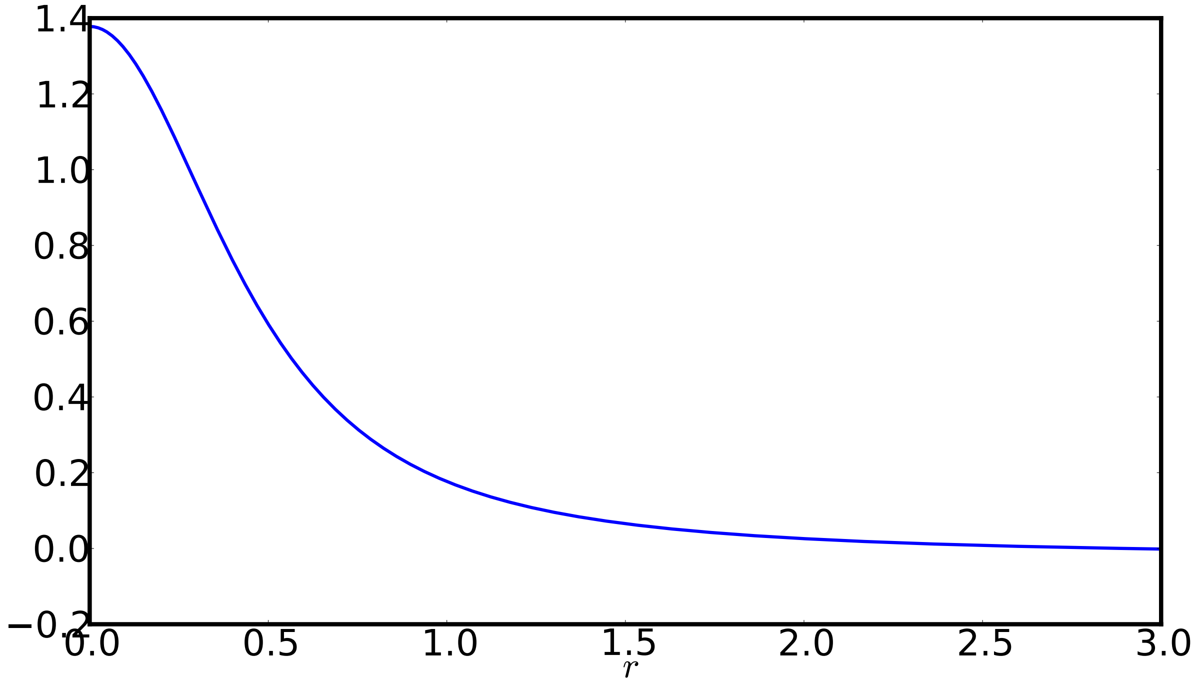

Figure 3.2: Radial pressure $p_r(r)$.

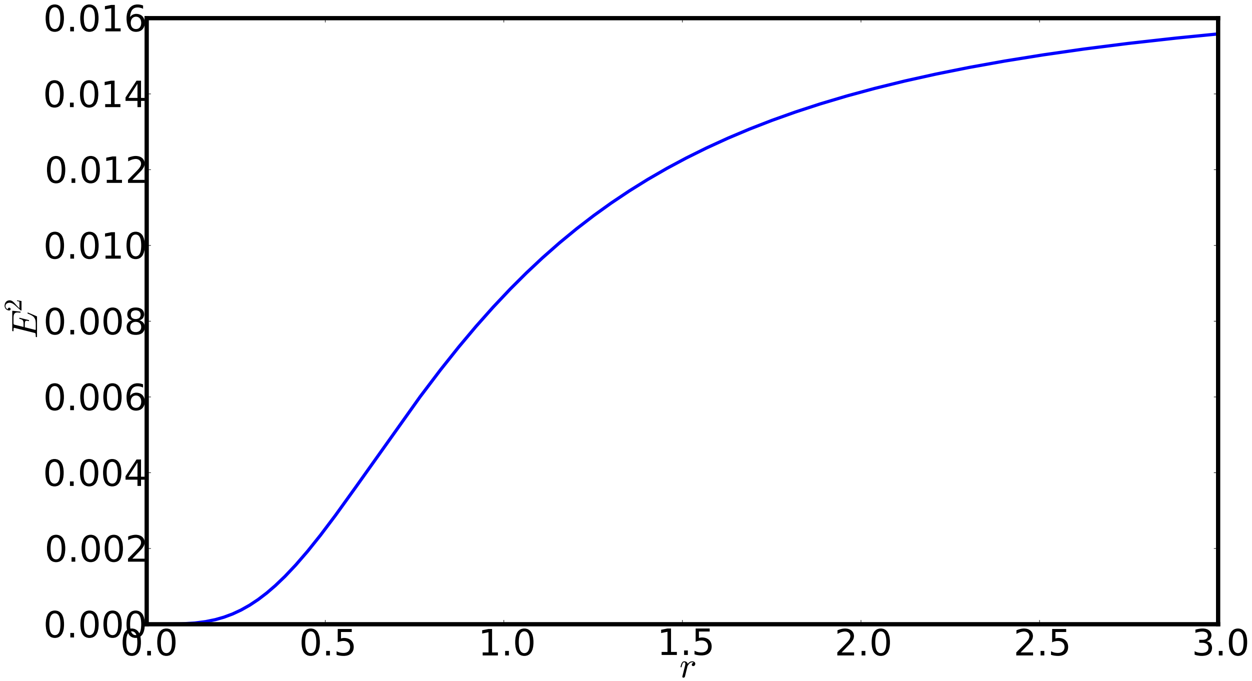

Figure 3.3: Electric field intensity $E^2(r)$.

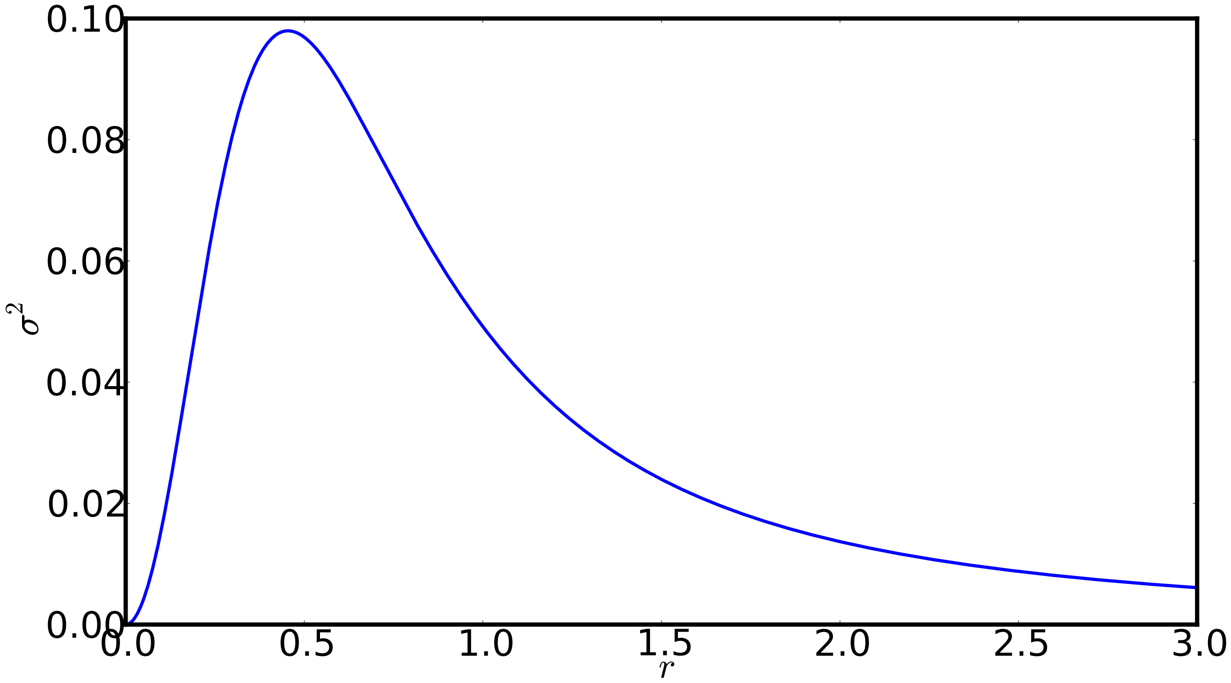

Figure 3.4: Charge density σ^2.

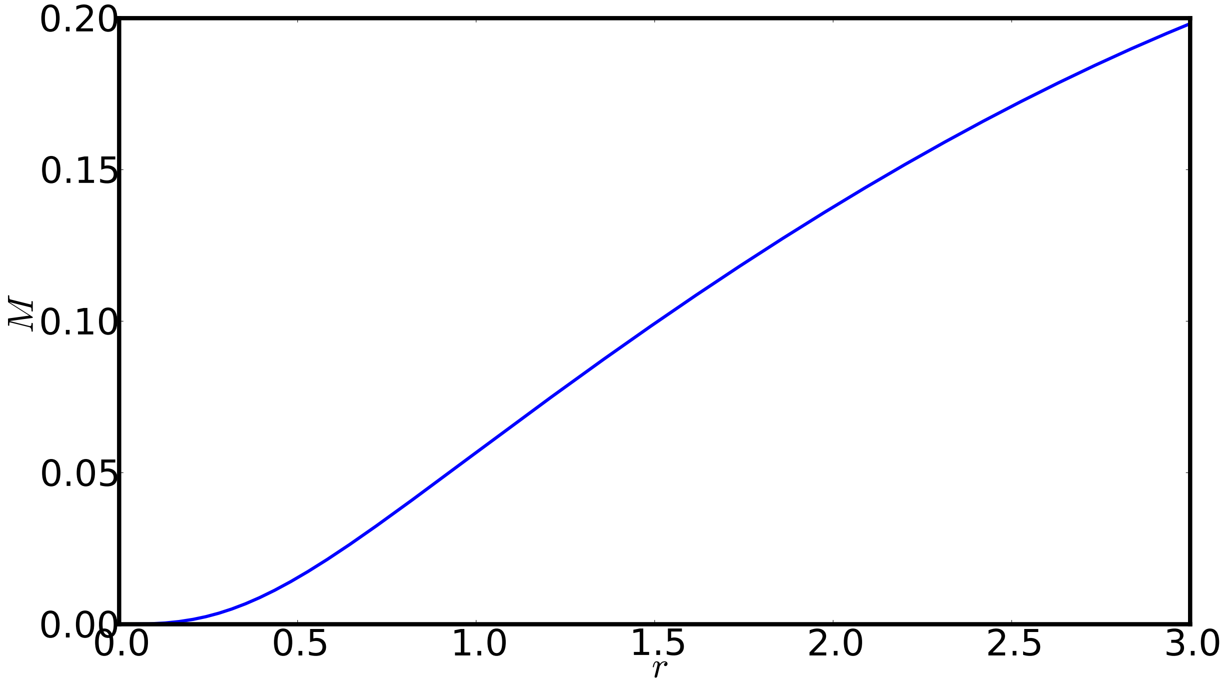

Figure 3.5: Mass $M(r)$.

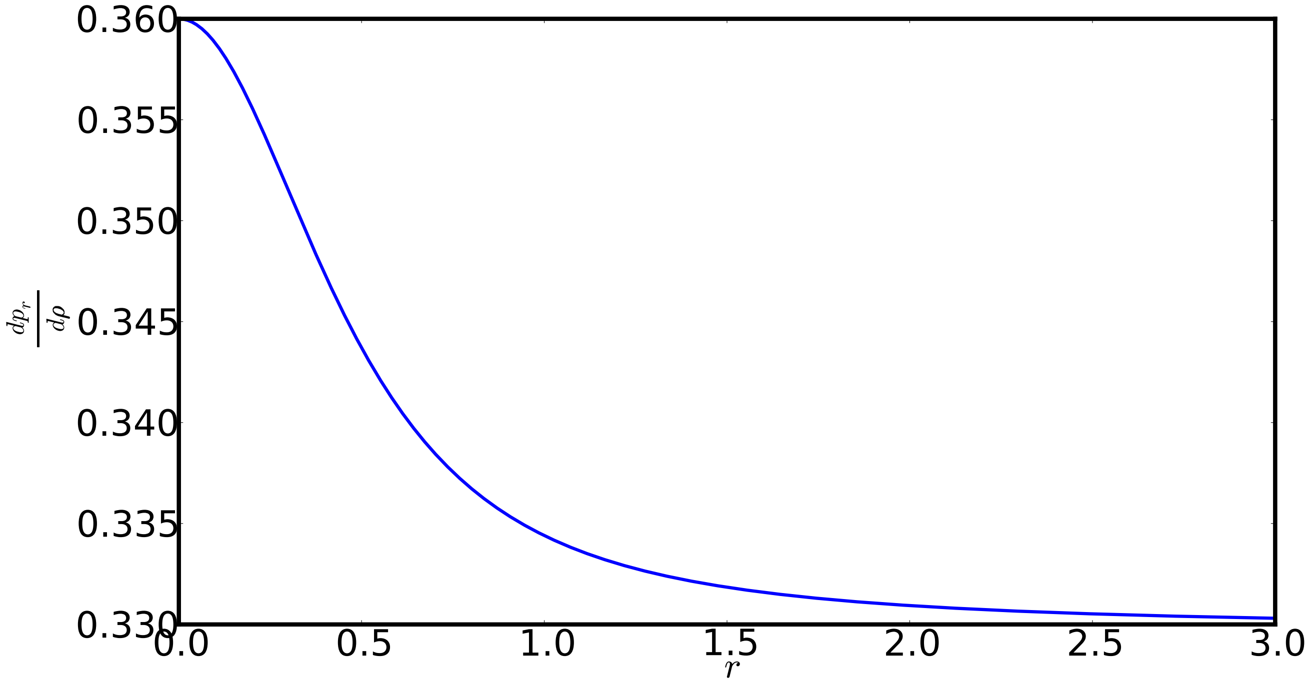

Figure 3.6: Speed of sound $\frac{dp_r}{d\rho}(r)$.

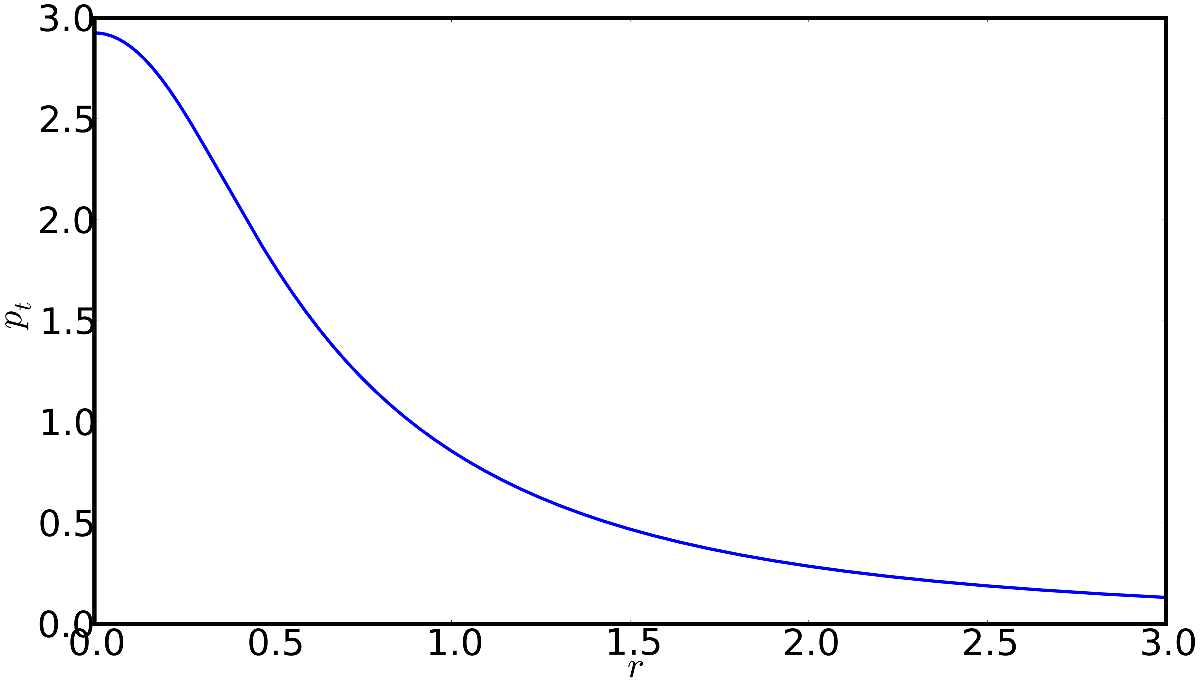

Figure 3.7: Tangential pressure $p_t(r)$.

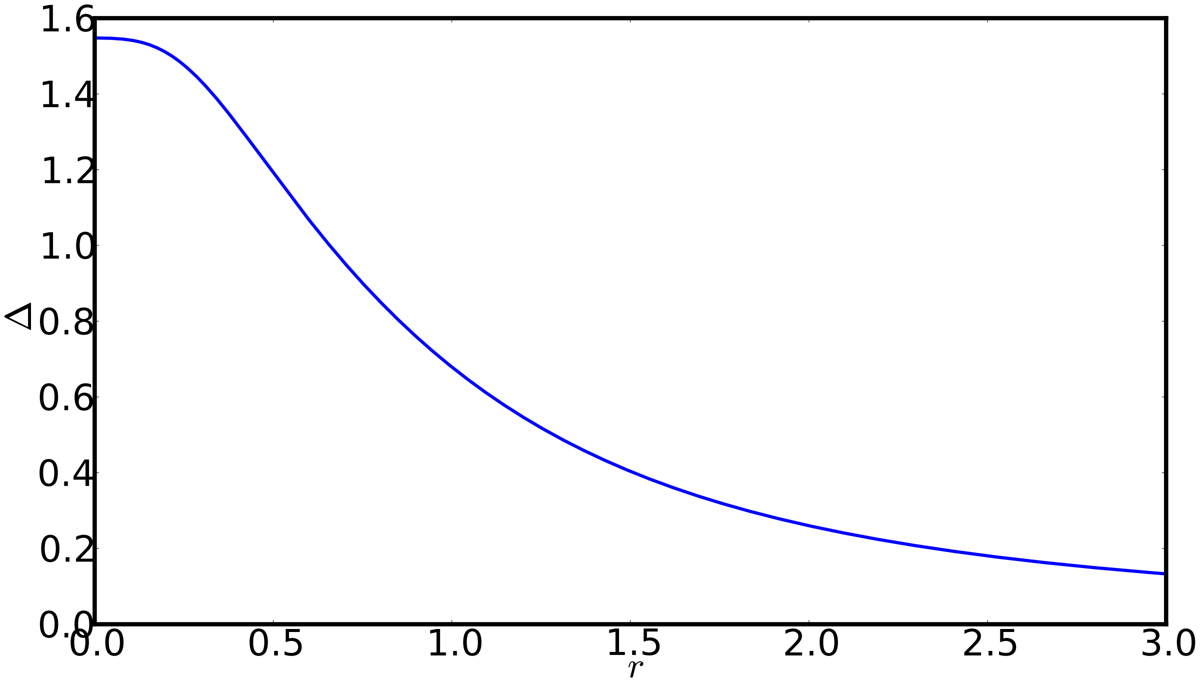

Figure 3.8: Measure of anisotropy $\Delta(r)$.

and

$$\left(1 - \frac{2M}{\Re} + \frac{Q^2}{\Re^2}\right)^{-1} = \frac{1 + aC\Re^2}{1 + bC\Re^2}, \tag{3.25}$$

relating the constants *a, b, A, C,* α, β and γ. There is a sufficient number of free parameters to ensure the continuity of the metric coefficients across the boundary of the star. Also note that the radial pressure vanishes at the boundary. It is possible to study the astrophysical significance of the exact solutions to the Einstein-Maxwell equations found in this chapter. This is the object of future research. We point out that for suitable parameter values we regain the mass $M = 1.433M_{\bigodot}$ of Dey $et\ al$ (1998, 1999a, 1999b) corresponding to a strange star model when there is no electromagnetic field. Therefore the solutions found in this chapter may be used to generalise earlier results and to model charged relativistic strange and quark stars.

Our aim in this chapter was to find new regular exact solutions to the Einstein-Maxwell system for spherically symmetric gravitational field with an equation of state. In particular we selected a quadratic equation of state relating the energy density to the radial pressure. The new models presented in this chapter may be used to model relativistic compact objects in astrophysics.

CHAPTER 4

Some charged polytropic models

4.1 Introduction

In this chapter we are concerned with anisotropic, charged fluids in general relativity theory satisfying the Einstein-Maxwell system. The canonical approach to study such a model is to specify initially the properties of matter in terms of an equation of state. Then the model may be simplified by imposing symmetries on the spacetime manifold which eases the task of solving the field equations. The resulting family of solutions should be studied to confirm their physical relevance. For neutral gravitating spheres, Delgaty and Lake (1998) discuss the relevant physical requirements and they show that only a restricted family of models satisfy the physical tests. In our approach, we impose the requirement that the spacetime is static and spherically symmetric, specify an equation of state relating the radial pressure to the density, and choose forms for one of the metrics variables and the electric field. This line of approach is different from the canonical approach but has the advantage of simplifying the integration process. It does produce exact solutions which may be useful examples for stellar models.

The modelling of dense charged gravitating objects in strong gravitational fields has generated much interest in recent times because of its relevance to relativistic astrophysics. Gupta and Maurya (2011a, 2011b, 2011c), Kiess (2012), Maurya and Gupta (2011a, 2011b, 2011c) and Pant $et\ al$ (2011) have generated specific charged models with desirable physical features. These investigations require an exact solution of the Einstein-Maxwell system. The presence of charge produces values for the redshift, luminosity and maximum mass which are different from neutral matter. Applications of dense charged gravitating spheres include describing quarks stars, spheres with strange equation of state, hybrid protoneutron stars, bare quark stars and the accreting process onto a compact object where the matter is likely to acquire large amounts of electric charge as pointed out by Esculpi and Aloma (2010), Sharma and Maharaj (2007a), Sharma and Mukherjee (2001, 2002) and Sharma $et\ al$ (2001) amongst others.

A considerable number of exact solutions to the Einstein-Maxwell system has been generated by Ivanov (2002), Komathiraj and Maharaj (2007a, 2007b) and Thirukkanesh and Maharaj (2008) by choosing a generalised form for one of the gravitational potentials. The solutions are represented as an infinite series in closed form in general; polynomial and algebraic functions are possible for particular parameter values and previously known models are regained in the appropriate limit. However these models do not satisfy a barotropic equation of state, relating the radial pressure to the energy density in general. The importance of an equation of state in a stellar model has been emphasized by Varela $et\ al$ (2010) who provided a mechanism of dealing with anisotropic matter in a general approach. Some solutions of the Einstein-Maxwell system found recently do in fact satisfy an equation of state. The models of Thirukkanesh and Maharaj (2008), Mafa Takisa and Maharaj (2013), Thirukkanesh and Ragel (2013) possess a linear equation of state for a charged anisotropic sphere. The solution of Hansraj and Maharaj (2006) satisfies a complicated nonlinear barotropic equation of state with isotropic pressures. The models of Feroze and Siddiqui (2011) and Maharaj and Mafa Takisa (2012) satisfy a quadratic equation of state which is important in brane world models and the study of dark energy. Models with a polytropic equation of state are rare. Thirukkanesh and Ragel (2012) have recently obtained particular uncharged models by specifying the polytropic index leading to masses and energy densities which are consistent with observations.

In this chapter we consider the general situation of anisotropic matter in the presence of an electromagnetic field satisfying a polytropic equation of state. Our objective is to find exact solutions to the Einstein-Maxwell system. We ensure that the charge density is regular throughout the sphere and finite at the centre. The gravitational potential selected has a functional form which has produced physically viable models in the past. An advantage of our approach is that we can automatically produce a new uncharged anisotropic model, with a polytropic equation of state, when the charge vanishes. In Section 4.2, we express the Einstein-Maxwell system as an equivalent set of differential equations using a transformation due to Durgapal and Bannerji (1983). In Section 4.3, we motivate the choice of the gravitational potential and the electric field intensity that allow us to integrate the field equations. The range of polytropic indices are considered in Section 4.4. We obtain a family of exact solutions to the Einstein-Maxwell system for particular polytropic indices in this section. Uncharged models are also obtained. In Section 4.5, we discuss the physical features of the model and generate graphical plots for the matter quantities.

4.2 Field equations

In standard coordinates the line element for a static spherically symmetric fluid in the stellar interior has the form

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{4.1}$$

We are considering an anisotropic fluid in the presence of electromagnetic field; the energy momentum tensor is given by

$$T_{ij} = \text{diag}\left[-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2\right], \tag{4.2}$$

where ρ is the energy density, p_r is the radial pressure, p_t is the tangential pressure and E is the electric field intensity. The Einstein-Maxwell equations take the form

$$\frac{1}{r^2}\left[r(1-e^{-2\lambda})\right]' = \rho + \frac{1}{2}E^2, \tag{4.3a}$$

$$-\frac{1}{r^2}(1-e^{-2\lambda}) + \frac{2\nu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}E^2, \tag{4.3b}$$

$$e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r} - \nu'\lambda' - \frac{\lambda'}{r}\right) = p_t + \frac{1}{2}E^2, \tag{4.3c}$$

$$\sigma = \frac{1}{r^2}e^{-\lambda}(r^2E)', \tag{4.3d}$$

where primes represent differentiation with respect to *r*, and the quantity σ represents the proper charge density.

The fundamental equations describing the underlying gravitating model for an anisotropic charged spherically symmetric relativistic fluid are given by the system (4.3a)-(4.3d). When the charge is absent then (4.3a)-(4.3d) is a system of three equations in five unknowns $(\nu, \lambda, \rho, p_r, p_t)$. An uncharged solution may be generated by specifying forms for two unknowns or supplementing the system with two equations of state relating the matter variables as pointed out by Barraco $et\ al$ (2003). In the presence of charge (4.3a)-(4.3d) is a system of four equations in six unknowns ($\nu, \lambda, \rho, p_r, p_t, E$ or σ). Note that if we choose a form of the electric field E then the system (4.3a)-(4.3d) becomes a system of three equations in four unknowns. A charged solution may be found by specifying forms for three unknowns or any combination of unknowns and equations of state relating the matter variables. The equations of state should be chosen on physical grounds. We note that the equations (4.3a)-(4.3d) imply

$$p_r' = \frac{2}{r}(p_t - p_r) - r(\rho + p_r)\nu' + \frac{E}{r^2}\left(r^2E\right)', \tag{4.4}$$

which is the generalised Bianchi identity representing hydrostatic equilibrium of the charged anisotropic fluid. Equation (4.4) indicates that the anisotropy and charge influence the gradient of the pressure. These quantities may drastically affect quantities of physical importance such as surface tension as established by Sharma and Maharaj (2007b) in the generalised Tolman-Oppenheimer equation (4.4). The specific forms of p_t and E in particular models studied will determine the nature of profiles of p_r'.

We assume a polytropic equation of state relating the radial pressure p_r to the energy density ρ given by

$$p_r = \kappa \rho^{\Gamma}, \tag{4.5}$$

where $\Gamma = 1 + (1/\eta)$ and η is the polytropic index.

It is convenient to introduce a new independent coordinate *x* and introduce new metric functions *y* and *Z*:

$$x = Cr^2, \quad Z(x) = e^{-2\lambda(r)}, \quad A^2 y^2(x) = e^{2\nu(r)}, \tag{4.6}$$

where *A* and *C* are constants. Then the equations governing the gravitational behaviour of a charged anisotropic sphere, with nonlinear polytropic equation of state, are given by

$$\frac{\rho}{C} = \frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}, \tag{4.7a}$$

$$p_r = \kappa \rho^{1+(1/\eta)}, \tag{4.7b}$$

$$p_t = p_r + \Delta, \tag{4.7c}$$

$$\frac{\Delta}{C} = 4xZ\frac{\ddot{y}}{y} + \dot{Z}\left[1 + 2x\frac{\dot{y}}{y}\right] + \frac{1-Z}{x} - \frac{E^2}{C}, \tag{4.7d}$$

$$\frac{\dot{y}}{y} = \frac{1-Z}{4xZ} - \frac{E^2}{8CZ} + \frac{\kappa C^{1+(1/\eta)}}{4Z}\left[\frac{1-Z}{x} - 2\dot{Z} - \frac{E^2}{2C}\right]^{1+(1/\eta)}, \tag{4.7e}$$

$$\frac{\sigma^2}{C} = \frac{4Z}{x}\left(x\dot{E} + E\right)^2, \tag{4.7f}$$

where $\Delta = p_t - p_r$ is called the measure of anisotropy. The analogue of the system (4.7a)-(4.7f), with a linear equation of state, was pursued by Thirukkanesh and Maharaj (2008). The Einstein-Maxwell equations, with a quadratic equation of state, was studied by Feroze and Siddiqui (2011) and Maharaj and Mafa Takisa (2012). The system (4.7a)-(4.7f), representing gravitating matter with a polytropic equation of state, is physically more relevant, and the model is of importance in relativistic astrophysics. However the polytropic equation of state is the most difficult to study because of the nonlinearity introduced through the polytropic index η. The transformed form of the Einstein-Maxwell equations simplifies the integration to produce exact solutions.

4.3 Integration

We solve the Einstein-Maxwell field equations by choosing specific forms for the gravitational potential Z and the electric field intensity E which are physically reasonable. The model depends on obtaining a solution to (4.7e). Equation (4.7e) becomes a first order equation in the potential y which is integrable.

We make the choice

$$Z = \frac{1+bx}{1+ax} \qquad a \neq b, \quad b \neq 0, \tag{4.8}$$

where a and b are real constants. The quantity Z is regular at the stellar centre and continuous in the interior because of the freedom provided by the parameters a and b. It is important to realise that this choice for Z is physically reasonable and contains special cases of known relativistic star models. The choice (4.8) was made by Maharaj and Mafa Takisa (2012) to generate stellar models that satisfy physical criteria for a stellar source with a quadratic equation of state. Charged stellar models were also found by John and Maharaj (2011), Thirukkanesh and Maharaj (2008), Maharaj and Komathiraj (2007), and Feroze and Siddiqui (2011) with this form of Z. A detailed study of the Einstein-Maxwell system, for isotropic matter distributions, was performed by Thirukkanesh and Maharaj (2009). Neutral stellar models in general relativity have been found for special cases of the potential Z. If we set $a = 1$, $b = 1/2$ then we generate the Durgapal and Bannerji (1983) neutron star model. When $a = 7$, $b = -1$ then we generate the gravitational potential of Tikekar (1990) for superdense stars. Thus the form Z chosen is likely to produce physically reasonable models for charged anisotropic spheres with a polytropic equation of state.

For the electric field we make the choice

$$\frac{E^2}{2C} = \frac{\varepsilon x}{(1+ax)^2}, \tag{4.9}$$

which has desirable physical features in the stellar interior. It is finite at the centre of the star and remains bounded and continuous in the interior; for large values of x it approaches zero. A similar form of the electric field was studied by Hansraj and Maharaj (2006) which reduces to the uncharged Finch and Skea (1989) model. Finch and Skea stars satisfy all the requirements for physical acceptability. Therefore the choice (4.9) is likely to produce charged anisotropic models with a polytropic equation of state.

By substituting (4.8) and (4.9) into (4.7e) we obtain the result

$$\begin{aligned}\frac{\dot{y}}{y} &= \frac{a-b}{4(1+bx)} - \frac{\varepsilon x}{4(1+ax)(1+bx)} \\ &\quad + \frac{\kappa C^{1+(1/\eta)}(1+ax)}{4(1+bx)}\left[\frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}\right]^{1+(1/\eta)}. \end{aligned} \tag{4.10}$$

This is a first order equation but the presence of the polytropic index η makes it difficult to solve. The right hand side of (4.10) and its first derivative must be continuous to ensure integrability; clearly this is possible for a wide range of the parameters a, b, ε and η. We can integrate (4.10) in terms of elementary functions for particular values of η as shown in the next section.

In summary the potential Z and the electric field E have been specified. Then the charge density must have the form

$$\frac{\sigma^2}{C^2} = \frac{2\varepsilon(1+bx)(3+2ax)^2}{(1+ax)^5}. \tag{4.11}$$

The energy density is given by

$$\frac{\rho}{C} = \frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}. \tag{4.12}$$

On integrating (4.10) we can find the gravitational potential y. As Z and E are now known quantities, we can find the measure of anisotropy Δ by simple substitution in (4.7d). The tangential pressure p_t then follows from (4.7c). Thus we must find an analytic form for y to complete the integration.

4.4 Polytropic models

Newtonian polytropic models have been studied for over a hundred years. Early results have been extensively described by Chandrasekhar (1939). Particular polytropic indices have been shown to be consistent with neutron stars, main sequence stars, convective stellar cores of red giants and brown dwarfs, and relativistic degenerate cores of white dwarfs. When $\eta = 5$ then the polytrope has an infinite radius, and when the index $\eta \to \infty$ the isothermal sphere is generated. Polytropes have also been studied in the context of general relativity. It is important to note that in Newtonian theory polytropes with certain exponents correspond to adiabates. The physical interpretation of the distribution in relativity is more difficult since the adiabates obey different equations of state as indicated in treatment of Tooper (1965). Some numerical results have been found by Tooper (1964) who studied the structure of polytropic fluid spheres for $\eta = 1, 3/2, 5/2$ and $\eta = 3$. Pandey $et\ al$ (1991) presented an exhaustive study of relativistic polytropes in the range $1/2 \leq \eta \leq 3$. de Felice $et\,al$ (1999) considered the structure and energy of singular general relativistic polytropes in the range $0 \leq \eta \leq 4.5$. Recently Thirukkanesh and Ragel (2012) found uncharged exact solutions with a polytropic equation of state for $\eta = 1$ and $\eta = 2$. Nilsson and Uggla (2001) demonstrated numerically that general relativistic perfect fluid models have finite radius for the polytropic index $0 \leq \eta \leq 3.339$. Subsequently Heinzle $et\ al$ (2003) performed a comprehensive dynamical systems treatment for perfect fluids that are asymptotically polytropic. The mass-radius ratio for anisotropic matter configurations is bounded for a compact general relativistic object as given by Boehmer and Harko (2006) and Andreasson and Boehmer (2009) for general matter distributions, and they are consequently applicable for polytropes. In this chapter we consider the polytropic index ranging over the four cases $\eta = 1/2, 2/3, 1, 2$ for strong gravitational fields when anisotropy and the electromagnetic field are present. The values of η chosen produce finite models that correspond to physically acceptable matter distributions as shown in the analyses of Pandey $et\ al$ (1991) and Thirukkanesh and Ragel (2012).

4.4.1 The case $\eta = 1$

When $\eta = 1$, the equation of state (4.5) becomes

$$p_r = \kappa\rho^2. \tag{4.13}$$

On integrating (4.10) we get

$$y = B(1+ax)^k[1+bx]^l \exp\left[F(x)\right], \tag{4.14}$$

where B is the constant of integration. The variable $F(x)$, the constants k and l are given by

$$\begin{aligned}
F(x) &= \frac{C^2\kappa[2(2b-a)(1+ax)+(b-a)]}{2(b-a)^2(1+ax)^2} \\
&\quad -\frac{C^2\kappa\varepsilon[4a(a-b)+\varepsilon]}{8a^2(a-b)(1+ax)} - \frac{C^2\kappa\varepsilon[2a(a^2-2\varepsilon)+b(2ab-\varepsilon)]}{4a^2(a-b)^2(1+ax)}, \\
k &= C^2\kappa[2(a-b)]^2\left[\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right] \\
&\quad -\frac{2\varepsilon[(a-b)^2+C^2\kappa a\varepsilon]}{a} - 4C^2\kappa a\varepsilon[1+b(4-3b)], \\
l &= \frac{(a-b)}{4b} + C^2\kappa[2(a-b)]^2\left[\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right] \\
&\quad -\frac{2\varepsilon[(a-b)^2+C^2\kappa b\varepsilon]}{b} - 4C^2\kappa\varepsilon[(a-b)(a-3b)].
\end{aligned} \tag{4.15}$$

The tangential pressure and the measure of anisotropy are given by

$$\begin{aligned}
p_t &= \frac{4xC(1+bx)}{1+ax}\left[\frac{k(k-1)a^2}{(1+ax)^2}+\frac{2klab}{(1+ax)(1+bx)}\right. \\
&\quad +\frac{2ka\dot{F}(x)}{1+ax}+\frac{b^2l(l-1)}{(1+bx)^2}+\frac{2lb\dot{F}(x)}{1+bx} \\
&\quad \left.+\ddot{F}(x)+\dot{F}(x)^2\right] + 2xC\left[\frac{ak}{1+ax}+\frac{b}{1+bx}+\dot{F}(x)\right] \\
&\quad +\frac{C(a-b)ax-2\varepsilon x}{(1+ax)^2}+\kappa C^2\left[\frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}\right]^2,
\end{aligned} \tag{4.16a}$$

$$\begin{aligned}
\Delta &= \frac{4xC(1+bx)}{1+ax}\left[\frac{k(k-1)a^2}{(1+ax)^2}+\frac{2klab}{(1+ax)(1+bx)}\right. \\
&\quad +\frac{2ka\dot{F}(x)}{1+ax}+\frac{b^2l(l-1)}{(1+bx)^2}+\frac{2lb\dot{F}(x)}{1+bx} \\
&\quad \left.+\ddot{F}(x)+\dot{F}(x)^2\right] + 2xC\left[\frac{ak}{1+ax}+\frac{b}{1+bx}+\dot{F}(x)\right] \\
&\quad +\frac{C(a-b)ax-2\varepsilon x}{(1+ax)^2}.
\end{aligned} \tag{4.16b}$$

If we set $A^2B^2 = D$ and $C = 1$ then the line element has the form

$$\begin{aligned}
ds^2 &= -D\left(1+ar^2\right)^{2k}(1+br^2)^{2l}\exp[2F(r^2)]dt^2 + \frac{1+ar^2}{1+br^2}dr^2 \\
&\quad +r^2(d\theta^2+\sin^2\theta d\phi^2),
\end{aligned} \tag{4.17}$$

in this case.

Therefore we have obtained a new charged anisotropic model corresponding to the polytropic index η. Observe that it is possible to set $\varepsilon = 0$ in this solution so that $E = 0$ and there is no charge. Thus our approach automatically generates an uncharged model. The uncharged polytrope with $\eta = 1$ is given by the metric

$$
\begin{aligned}
ds^2 \;=\; & -D\left(1+ar^2\right)^{2\kappa[2(a-b)]^2\left[\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right]} \\
& \times\left(1+br^2\right)^{2\frac{(a-b)}{4b}+\kappa[2(a-b)]^2\left[\frac{b^2}{(b-a)^3}+\frac{b}{(b-a)^2}+\frac{1}{4}\right]} \\
& \times\exp\left[\frac{\kappa(2(2b-a)(1+ax)+(b-a))}{(b-a)^2(1+ax)^2}\right]dt^2 \\
& +\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2).
\end{aligned}
\tag{4.18}
$$

Note that the metric (4.17), with $\varepsilon = 0$, is contained in the models of Feroze and Siddiqui (2011) and Maharaj and Mafa Takisa (2012). They considered the quadratic equation of state $p_r = \gamma\rho^2 + \alpha\rho + \beta$. If we set $\gamma = \kappa$, $\alpha = 0$, $\beta = 0$ and $E = 0$ then we find that their solutions are equivalent to our uncharged metric (4.18).

4.4.2 The case $\eta = 2$

When $\eta = 2$, the equation of state (4.5) becomes

$$p_r = \kappa\rho^{3/2}. \tag{4.19}$$

On integrating (4.10) we obtain

$$
\begin{aligned}
y \;=\; & B\frac{[1+bx]^{\frac{(a-b)^2+\varepsilon}{4b(a-b)}}}{[1+ax]^{\frac{-\varepsilon}{4a(a-b)}}}\left[\frac{\sqrt{2a(a-b)+\varepsilon}-\sqrt{b}\sqrt{(3+ax)(a-b)-\varepsilon x}}{\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}\sqrt{(3+ax)(a-b)-\varepsilon x}}\right]^{m+w} \\
& \times\exp[G(x)],
\end{aligned}
\tag{4.20}
$$

where B is the constant of integration. The variable $G(x)$, the constants m and w are given by

$$\begin{aligned} G(x) &= -\frac{C^{3/2}\kappa}{2(1+ax)} - \frac{C^3\kappa\varepsilon\sqrt{(3+ax)(a-b)-\varepsilon x}}{4a(a-b)(1+ax)}, \\ m &= \frac{C^{3/2}\kappa[(a-b)(3b-a)+\varepsilon]^{3/2}}{2\sqrt{b}(a-b)}, \\ w &= \frac{C^{3/2}\kappa[2a^2(a-b)(3a+7b)-a\varepsilon(3a+5b)]-\varepsilon^2(b-3a)}{4a^{3/2}(a-b)\sqrt{2a(a-b)+\varepsilon}}. \end{aligned} \tag{4.21}$$

The tangential pressure and measure of anisotropy have the forms

$$\begin{aligned} p_t &= \frac{4xC(1+bx)}{1+ax}\left[\frac{d}{dx}\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\ &\quad \left.\left. - \frac{(m+w)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right) + \frac{\dot{y}^2}{y^2}\right] + \frac{C(b-a)}{(1+ax)^2} \\ &\quad \times \left[1 + 2x\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\ &\quad \left.\left. - \frac{(m+w)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right)\right] \\ &\quad + \frac{(a-b)(1+ax)-2\varepsilon x}{(1+ax)^2} + \kappa C^{3/2}\left[\frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}\right]^{3/2}, \end{aligned} \tag{4.22a}$$

$$\begin{aligned} \Delta &= \frac{4xC(1+bx)}{1+ax}\left[\frac{d}{dx}\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\ &\quad \left.\left. - \frac{(m+w)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right) + \frac{\dot{y}^2}{y^2}\right] + \frac{C(b-a)}{(1+ax)^2} \\ &\quad \times \left[1 + 2x\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\ &\quad \left.\left. - \frac{(m+w)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right)\right] + \frac{(a-b)(1+ax)-2\varepsilon x}{(1+ax)^2}, \end{aligned} \tag{4.22b}$$

where $T = \sqrt{(3+ax)(a-b)-\varepsilon x}$.

By setting $A^2B^2 = D$ and $C = 1$ the line element takes the form

$$
\begin{aligned}
ds^2 &= -D\left[\frac{\sqrt{2a(a-b)+\varepsilon}-\sqrt{b}\sqrt{(3+ar^2)(a-b)-\varepsilon r^2}}{\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}\sqrt{(3+ar^2)(a-b)-\varepsilon r^2}}\right]^{2(m+w)} \\
&\quad \times \left(1+br^2\right)^{\frac{(a-b)^2+2\varepsilon}{2b(a-b)}}\left(1+ar^2\right)^{\frac{\varepsilon}{2a(a-b)}}\exp[2G(r^2)]dt^2 \\
&\quad +\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2),
\end{aligned}
\tag{4.23}
$$

for this case.

Setting $\varepsilon = 0$ implies $E = 0$ and we find the uncharged polytropic model with $\eta = 2$. The corresponding line element is given by

$$
\begin{aligned}
ds^2 &= -D\left[\frac{\sqrt{2a(a-b)}-\sqrt{b}\sqrt{(3+ar^2)(a-b)}}{\sqrt{2a(a-b)}+\sqrt{b}\sqrt{(3+ar^2)(a-b)}}\right]^{\frac{\kappa(3b-a)\sqrt{(a-b)(3b-a)}}{\sqrt{b}}+\frac{\kappa\sqrt{a}(3a+7b)}{\sqrt{2a(a-b)}}} \\
&\quad \times \left(1+br^2\right)^{\frac{(a-b)^2}{2b(a-b)}}\exp\left[\frac{-\kappa}{(1+ar^2)}\right]dt^2 \\
&\quad +\frac{1+ar^2}{1+br^2}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2),
\end{aligned}
\tag{4.24}
$$

which is a new solution to the Einstein-Maxwell equations with this polytropic index.

4.4.3 The case $\eta = 2/3$

When $\eta = 2/3$, the equation of state (4.5) is

$$
p_r = \kappa\rho^{5/2}. \tag{4.25}
$$

On integrating (4.10) we find

$$
\begin{aligned}
y &= B\frac{[1+bx]^{\frac{(a-b)^2+\varepsilon}{4b(a-b)}}}{[1+ax]^{\frac{-\varepsilon}{4a(a-b)}}}\left[\frac{\sqrt{2a(a-b)+\varepsilon}-\sqrt{b}\sqrt{(3+ax)(a-b)-\varepsilon x}}{\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}\sqrt{(3+ax)(a-b)-\varepsilon x}}\right]^{p+q} \\
&\quad \times\exp[H(x)],
\end{aligned}
\tag{4.26}
$$

where B is the constant of integration. The variable $H(x)$, the constants p and q are given by

$$
\begin{aligned}
H(x) &= -\frac{C^{5/2}\kappa(2a(a-b)+\varepsilon)^2\mathcal{A}}{12a^2(a-b)(1+ax)^3} \\
&\quad -\frac{C^{5/2}\kappa(2a(a-b)+\varepsilon)((a-b)(13a^2-25ab)+\varepsilon(13a+7b))\mathcal{A}}{48a^2(a-b)^2(1+ax)^3} \\
&\quad -\frac{C^{5/2}\kappa[(a-b)(8a^3(a^2-\varepsilon)+4a\varepsilon(\varepsilon-1))]\mathcal{A}}{32a^2(a-b)^3(1+ax)} \\
&\quad -\frac{C^{5/2}\kappa[-9a^2b^3+206a^3b^2(a-b^3)+70a^2b^2(b^2-\varepsilon)]\mathcal{A}}{32a^2(a-b)^3(1+ax)} \\
&\quad -\frac{C^{5/2}\kappa[a^3(3a^6-b^3)-a\varepsilon(8a^2-7\varepsilon)\mathcal{A}}{32a^2(a-b)^3(1+ax)}, \\
p &= \frac{C^{5/2}\kappa\sqrt{b}[(a-b)(3b-a)+\varepsilon]^{5/2}}{2(a-b)}, \\
q &= \frac{C^{5/2}\kappa[51a^2b^4\varepsilon+30a^3b\varepsilon^2+1468a^5b^3]}{32a^{5/2}(a-b)^3\sqrt{2a(a-b)+\varepsilon}} \\
&\quad +\frac{C^{5/2}\kappa[(a+b)(498a^4b\varepsilon+15a^6s+5ab\varepsilon^3-5a^8-15a^4\varepsilon^2)]}{32a^{5/2}(a-b)^4\sqrt{2a(a-b)+\varepsilon}} \\
&\quad +\frac{C^{5/2}\kappa\varepsilon[-b(42b+75a^4)-\varepsilon^2(2a+5b)+6b^2(a^3-3b^3)]}{16a^{1/2}(a-b)^4\sqrt{2a(a-b)+\varepsilon}} \\
&\quad +\frac{C^{5/2}\kappa[\varepsilon^3(a^3+b^3)+ab\varepsilon^2(9b^3+15a^3)+535b^4(a^5+b^5)]}{32a^{5/2}(a-b)^4\sqrt{2a(a-b)+\varepsilon}} \\
&\quad -\frac{C^{5/2}\kappa[a^4b^5(353ab-1354)+a^5b(16b^3-85a^3)]}{32a^{5/2}(a-b)^4\sqrt{2a(a-b)+\varepsilon}},
\end{aligned}
$$

where $\mathcal{A} = \sqrt{(3+ax)(a-b)-\varepsilon x}$. The tangential pressure and the measure of anisotropy are given by

$$
\begin{aligned}
p_t &= \frac{4xC(1+bx)}{1+ax}\left[\frac{d}{dx}\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\
&\quad \left.\left. - \frac{(p+q)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right) + \frac{\dot{y}^2}{y^2}\right] + \frac{C(b-a)}{(1+ax)^2} \\
&\quad \times \left[1 + 2x\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\
&\quad \left.\left. - \frac{(p+q)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T(x))}\right)\right] \\
&\quad + \frac{(a-b)(1+ax)-2\varepsilon x}{(1+ax)^2} + \kappa C^{5/2}\left[\frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}\right]^{5/2}, \qquad (4.27a) \\
\Delta &= \frac{4xC(1+bx)}{1+ax}\left[\frac{d}{dx}\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\
&\quad \left.\left. - \frac{(p+q)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right) + \frac{\dot{y}^2}{y^2}\right] + \frac{C(b-a)}{(1+ax)^2} \\
&\quad \times \left[1 + 2x\left(\frac{b((a-b)^2+\varepsilon)}{4b(a-b)(1+bx)} - \frac{a\varepsilon}{4a(a-b)(1+ax)}\right.\right. \\
&\quad \left.\left. - \frac{(p+q)\sqrt{b}(a(a-b)\varepsilon)}{2T(\sqrt{2a(a-b)+\varepsilon}+\sqrt{b}T)}\right)\right] + \frac{(a-b)(1+ax)-2\varepsilon x}{(1+ax)^2}. \qquad (4.27b)
\end{aligned}
$$

If we set $A^2B^2 = D$ and $C = 1$ then the line element assumes the form

$$
\begin{aligned}
ds^2 &= -D\left[\frac{\sqrt{2a(a-b)+\varepsilon} - \sqrt{b}\sqrt{(3+ar^2)(a-b)-\varepsilon r^2}}{\sqrt{2a(a-b)+\varepsilon} + \sqrt{b}\sqrt{(3+ar^2)(a-b)-\varepsilon r^2}}\right]^{2(p+q)} \\
&\quad \times \left(1+br^2\right)^{\frac{(a-b)^2+2\varepsilon}{2b(a-b)}} \left(1+ar^2\right)^{\frac{\varepsilon}{2a(a-b)}} \exp[2H(r^2)]dt^2 \\
&\quad + \frac{1+ar^2}{1+br^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.28)
\end{aligned}
$$

in this case.

If we set $\varepsilon = 0$ then $E = 0$, and we get the uncharged polytropic model with $\eta = 2/3$. The uncharged line element has the form

$$
\begin{aligned}
ds^2 &= -D[1+br^2]^{\frac{(a-b)^2}{2b(a-b)}}\left[\frac{\sqrt{2a(a-b)} - \sqrt{b}\sqrt{(3+ar^2)(a-b)}}{\sqrt{2a(a-b)} + \sqrt{b}\sqrt{(3+ar^2)(a-b)}}\right]^{2(p+q)} \\
&\quad \times \exp[2H(r^2)]dt^2 + \frac{1+ar^2}{1+br^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \qquad (4.29)
\end{aligned}
$$

which is another new model for the index $\eta = 2/3$.

4.4.4 The case $\eta = 1/2$

When $\eta = 1/2$, the equation of state (4.5) is

$$p_r = \kappa\rho^3. \tag{4.30}$$

On integrating (4.10) we find

$$y = B(1+ax)^s[1+bx]^u \exp\left[I(x)\right], \tag{4.31}$$

where B is the constant of integration. The variable $I(x)$, the constants s, and u are given by

$$
\begin{aligned}
I(x) &= -\frac{C^3\kappa(2a(a-b)+\varepsilon)^3}{16a^3(a-b)(1+ax)^4} - \frac{C^3\kappa((a-b)(a-3b)-\varepsilon)^3}{4(a-b)^4(1+ax)} \\
&\quad -\frac{C^3\kappa(2a(a-b)+\varepsilon)^2[(a-b)(a(3a-5b)-2\varepsilon)-a\varepsilon]}{12a^3(a-b)^2(1+ax)^3} \\
&\quad -\frac{C^3\kappa[6a^4(a^3+6b\varepsilon)+4a^4b^2(29a+10b)+3a\varepsilon^3]}{8a^3(a-b)^2(1+ax)^2} \\
&\quad -\frac{C^3\kappa b\varepsilon[b\eta^2+3a(\varepsilon(b^2+3a^2)+ab(b^2-a^2))]}{8a^3(a-b)^3(1+ax)^2} \\
&\quad -\frac{C^3\kappa[36a^4b^3(a^3-b^3)+12a^2b^2(\varepsilon(ab-1)-a^2b^2)]}{8a^3(a-b)^3(1+ax)^2} \\
&\quad -\frac{C^3\kappa a^2(3\varepsilon(3a-b^2)+14b^4)}{8a^3(a-b)^3(1+ax)^2}, \\
s &= -\frac{\varepsilon[(a^2-b^2)^2-4b(a^2(a-b)+b^2)]}{4a(a-b)^5} \\
&\quad +\frac{C^3\kappa[a^2b^2(a^2+b^2)(136b^2+11a^2)]}{4a(a-b)^5} \\
&\quad -\frac{C^3\kappa[9ab^5(3a^2-19b^2)+3b^2\varepsilon^2(4a^2+3b)]}{4(a-b)^5} \\
&\quad +\frac{C^3\kappa[3ab^4\varepsilon(4a+9b)+ab\varepsilon(a^3b+\varepsilon^2)]}{4(a-b)^5} \\
&\quad -\frac{C^3\kappa a^2b[a^2+17ab^2-22b\varepsilon]}{4(a-b)^4} \\
&\quad -\frac{C^3\kappa[3a^3b\varepsilon(a^2+1)+40a^3b^3(a^2-\varepsilon)]}{4(a-b)^5}, \\
u &= \frac{(a^2+b^2)(4ab\varepsilon+15a^2b2)-\varepsilon(a^4+b^4)}{4b(a-b)^5} \\
&\quad +\frac{(a^3-b^3)-6ab(a^4+b^4+ab(3+\varepsilon))}{4b(a-b)^5} \\
&\quad +\frac{C^3\kappa[12a^3b^3(a^2-b^2)+b^2\varepsilon(6ab-\varepsilon^2)]}{4b(a-b)^4} \\
&\quad +\frac{C^3\kappa[27b^3(1+b^3)+a^3b^2(a^3-4b^3)]}{4b(a-b)^4} \\
&\quad +\frac{C^3\kappa[ab^4(57a^2+108b^2)-3b^2\varepsilon^2(1+3b)-3ab^3(21ab^2+22b\varepsilon)]}{4b(a-b)^4},
\end{aligned}
$$

in this case. The tangential pressure and the measure of anisotropy are given by

$$\begin{aligned} p_t &= \frac{4xC(1+bx)}{1+ax}\left[\frac{s(s-1)a^2}{(1+ax)^2}+\frac{2suab}{(1+ax)(1+bx)}\right.\\ &\quad +\frac{2sa\dot{I}(x)}{1+ax}+\frac{b^2u(u-1)}{(1+bx)^2}+\frac{2ub\dot{I}(x)}{1+bx}\\ &\quad \left.+\ddot{I}(x)+\dot{I}(x)^2\right]+2xC\left[\frac{as}{1+ax}+\frac{b}{1+bx}+\dot{I}(x)\right]\\ &\quad +\frac{C(a-b)ax-2\varepsilon x}{(1+ax)^2}+\kappa C^2\left[\frac{(a-b)(3+ax)-\varepsilon x}{(1+ax)^2}\right]^3, \end{aligned} \tag{4.32a}$$

$$\begin{aligned} \Delta &= \frac{4xC(1+bx)}{1+ax}\left[\frac{s(s-1)a^2}{(1+ax)^2}+\frac{2suab}{(1+ax)(1+bx)}\right.\\ &\quad +\frac{2sa\dot{I}(x)}{1+ax}+\frac{b^2l(u-1)}{(1+bx)^2}+\frac{2lb\dot{I}(x)}{1+bx}\\ &\quad \left.+\ddot{I}(x)+\dot{I}(x)^2\right]+2xC\left[\frac{as}{1+ax}+\frac{b}{1+bx}+\dot{I}(x)\right]\\ &\quad +\frac{C(a-b)ax-2\varepsilon x}{(1+ax)^2}. \end{aligned} \tag{4.32b}$$

If we set $A^2B^2 = D$ and $C = 1$ then the line element is given by

$$\begin{aligned} ds^2 &= -D(1+ar^2)^{2s}(1+br^2)^{2u}\exp[2I(r^2)]dt^2+\frac{1+ar^2}{1+br^2}dr^2\\ &\quad +r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned} \tag{4.33}$$

for this case.

Setting $\varepsilon = 0$ implies $E = 0$, and we generate the uncharged polytropic model with $\eta = 1/2$. The corresponding line element is given by

$$\begin{aligned} ds^2 &= -D(1+ar^2)^{2s}(1+br^2)^{2u}\exp[2I(r^2)]dt^2+\frac{1+ar^2}{1+br^2}dr^2\\ &\quad +r^2(d\theta^2+\sin^2\theta d\phi^2), \end{aligned} \tag{4.34}$$

which is a new solution to the Einstein-Maxwell equations.

4.5 Physical Analysis

In this section we indicate that the exact polytropic solutions found in Section 4.4 are physically reasonable. The gravitational potential Z is regular at the centre and well behaved in the interior. The potentials y presented for various cases in Section 4.4 are given in terms of simple elementary functions. They are regular at the stellar centre and continuous in the interior. The potentials Z and y reduce for particular values of parameters to relativistic stellar models studied previously which have been shown to possess desirable physical features. Clearly the choice of the electric field E in (4.9) is physically acceptable as shown by Hansraj and Maharaj (2006). The choice of E leads to forms of charge density σ in (4.11) and the energy density ρ in (4.12) given in terms of rational functions. The quantities E, σ and ρ become decreasing functions for large values of x.

We used the programming language Python to generate two sets of plots for the radial pressure p_r, the tangential pressure p_t, and the anisotropy Δ for the polytropic indices $\eta = 1/2, 2/3, 1, 2$. These represent profiles for charged anisotropic matter with $\varepsilon \neq 0$ for $a = 5.5$, $b = 3.0$, $\varepsilon = 1$, the boundary $r = 4$, $C = 1$ and κ given by the causality condition $\frac{dp_r}{d\rho} \leq 1$ for each case. In the first set of figures, we have plotted p_r, p_t and Δ against the radial coordinate r: Figure 4.1 represents the radial pressure, Figure 4.2 represents the tangential pressure, and Figure 4.3 represents the anisotropy. The radial pressure is a finite and decreasing function in Figure 4.1. The tangential pressure in Figure 4.2 initially increases, reaches a maximum and then decreases. The anisotropy in Figure 4.3 also reaches a maximum in the interior and then decreases. These profiles are similar to other studies. The high values of p_t in central regions of a star is reasonable as pointed out by Karmakar $et\ al$ (2007) because of conservation of angular momentum in quasi-equilibrium contraction of a compact body. The profile of Δ is similar to the profiles generated in studies of strange stars with quark matter by Sharma and Maharaj (2007a) and Tikekar and Jotania (2009). In the second set of figures, we have plotted p_r, p_t and Δ against the density ρ: Figure 4.4 represents radial pressure, Figure 4.5 represents the tangential pressure, and Figure 4.6 represents the anisotropy. The various plots in Figure 4.4 appear to be straight lines; this arises because of the interval chosen. We have utilised the forms for p_r, p_t and Δ from Section 4.4. The radial pressure remains an increasing function in Figure 4.4. The tangential pressure p_t increases to a maximum and then becomes a decreasing function in Figure 4.5. This feature is to be expected as we commented above about the expected higher values of p_t in the central regions. In the same way, in Figure 4.6, the anisotropy reaches a maximum in the interior and then decreases. Our profiles are similar to those given by Ray $et\ al$ (2003) who showed that the presence of electric charge has a significant effect on the phenomenology of compact stars with intense gravitational fields. Observe that the profiles of the radial pressure p_r increases as a function of the energy density in Figure 4.4 for each polytropic index. The gradient is larger as the polytropic index increases; the behaviour is consistent with the physical requirements of Pandey $et\ al$ (1991). We observe the same behaviour for the profiles for p_t and Δ. Finally in Figure 4.7 we have plotted the speed of sound $\frac{dp_r}{d\rho}$. This quantity is always less than unity and the causality is maintained which is a requirement for a physical object as indicated by Delgaty and Lake (1989).

In this chapter we have generated new exact solutions to the Einstein-Maxwell system of equations with a polytropic equation of state. These solutions may be used to model compact objects which are anisotropic and charged. Note that the solutions are expressed in terms of elementary functions which facilitate a physical study. A graphical analysis shows that the gravitational potentials and matter variables are regular at the centre and well behaved in the interior. It would be interesting to relate these new solutions to particular astronomical objects such as SAX J1804.4-3658 as was done by Dey $et\ al$ (1998, 1999a, 1999b), in the absence of charge, and Mafa Takisa and Maharaj (2013), in the presence of charge. Such a study will reinforce the astrophysical significance of the models in this chapter. We point out that our approach automatically leads to new uncharged anisotropic solutions when the electric field $E = 0$.

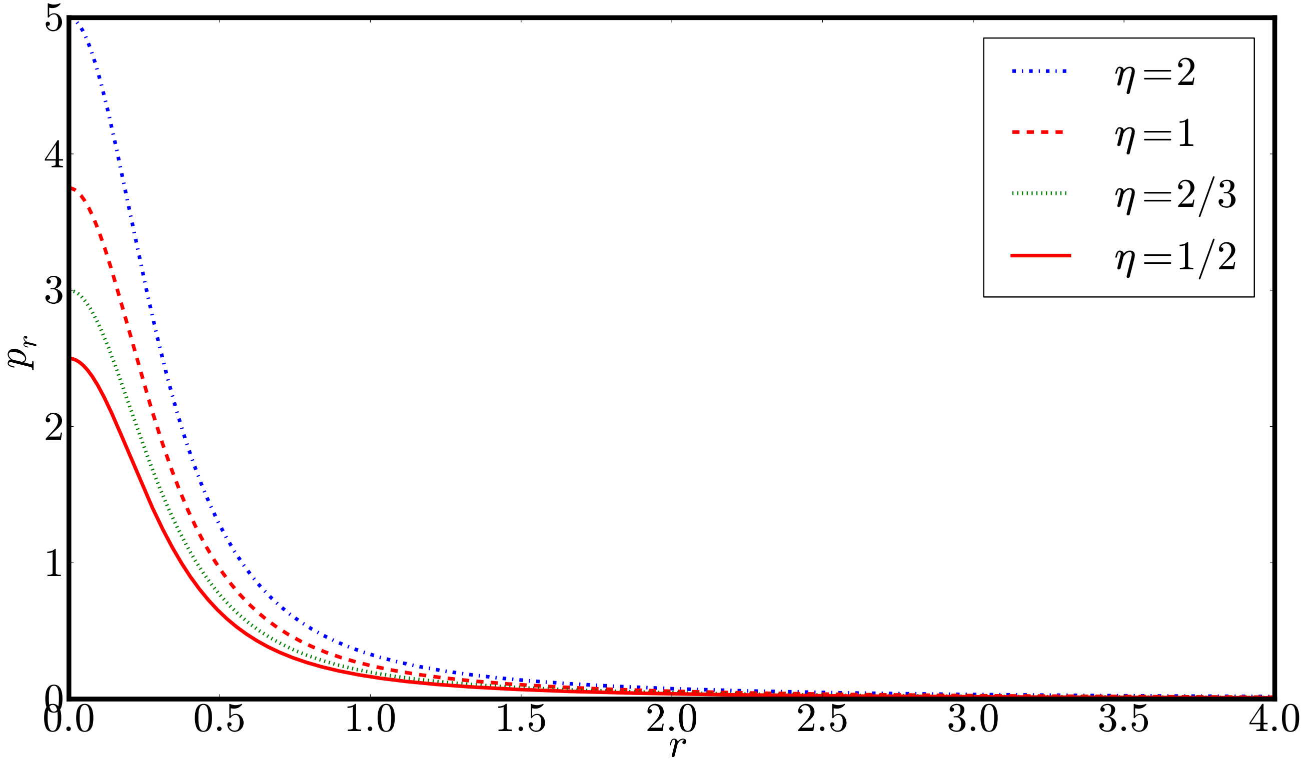

Figure 4.1: Radial pressure $p_r(r)$.

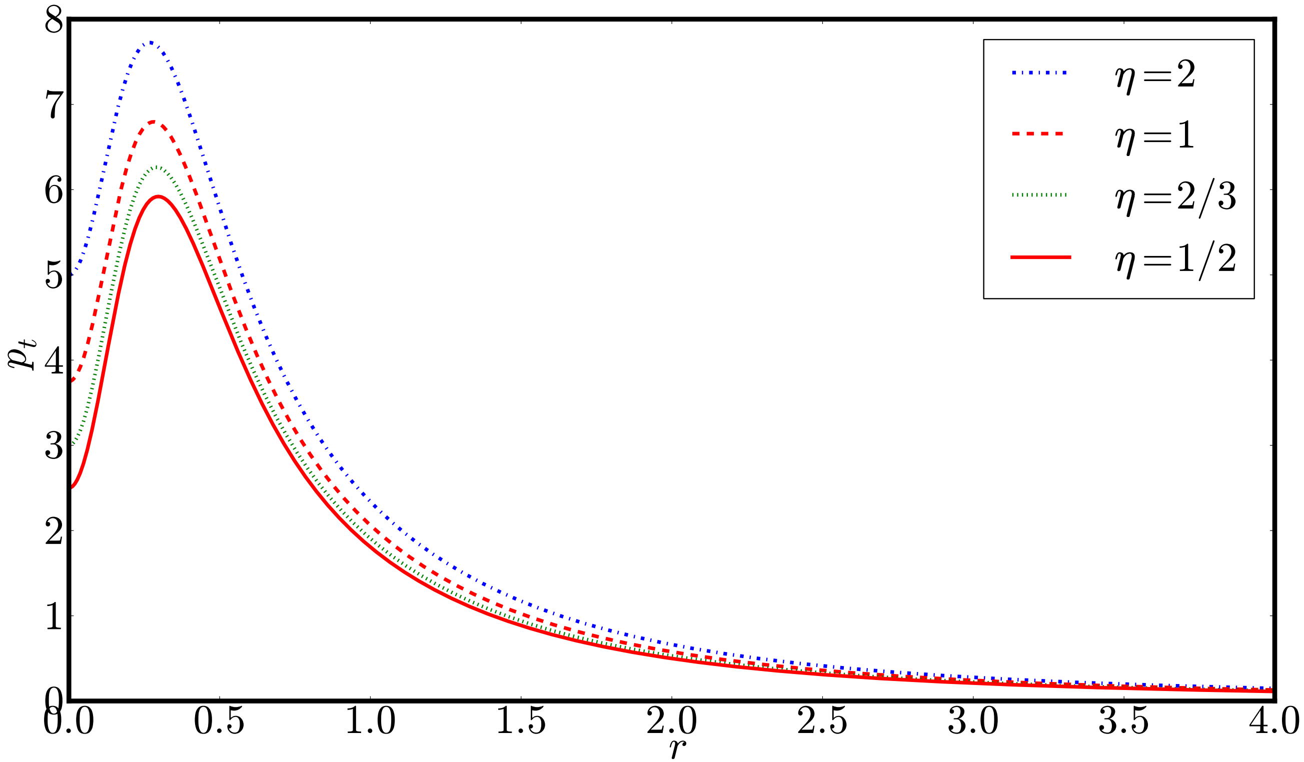

Figure 4.2: Tangential pressure $p_t(r)$.

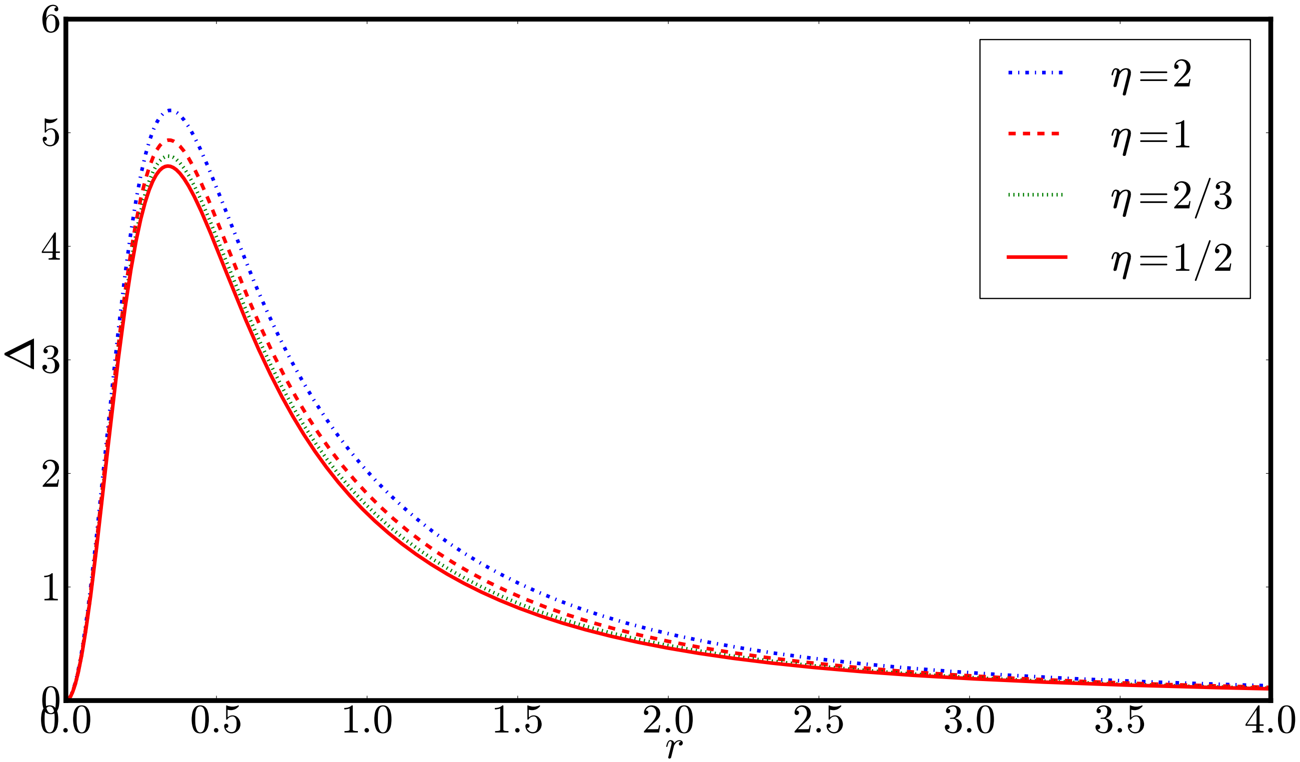

Figure 4.3: Anisotropy $\Delta(r)$

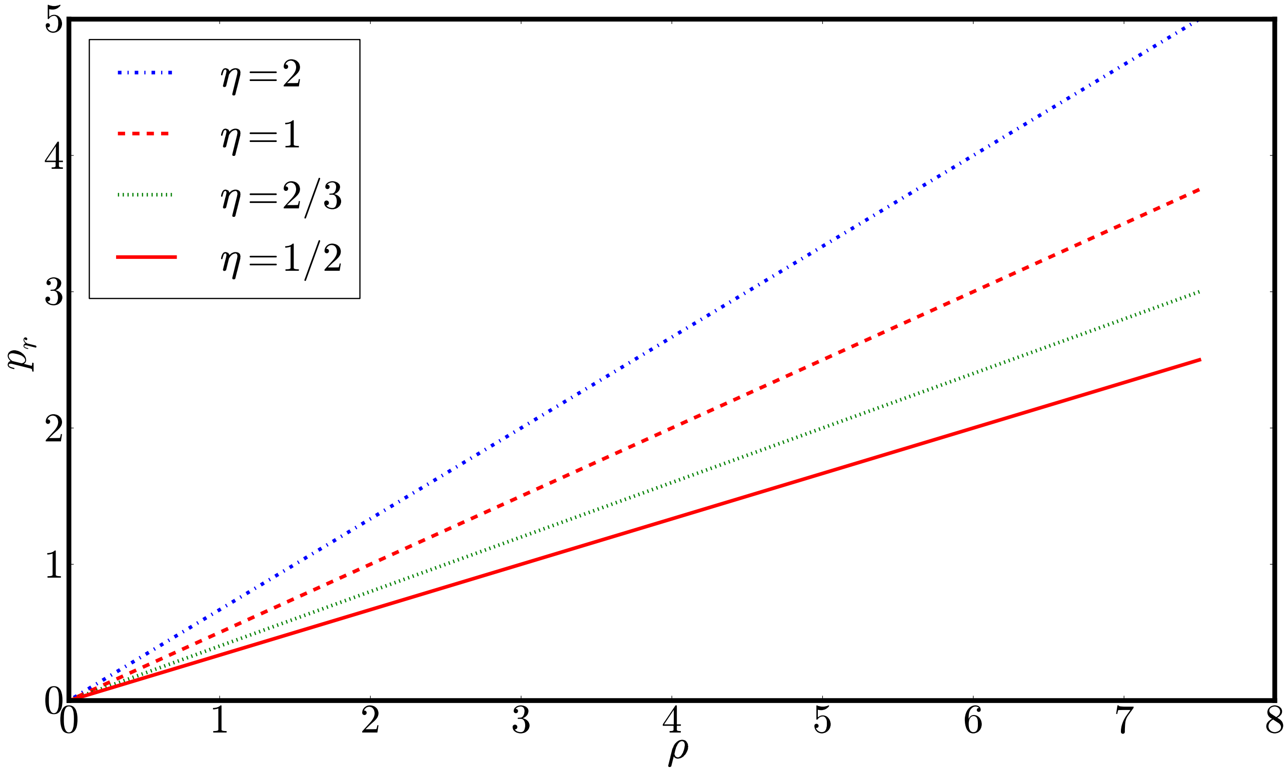

Figure 4.4: Radial pressure $p_r(\rho)$.

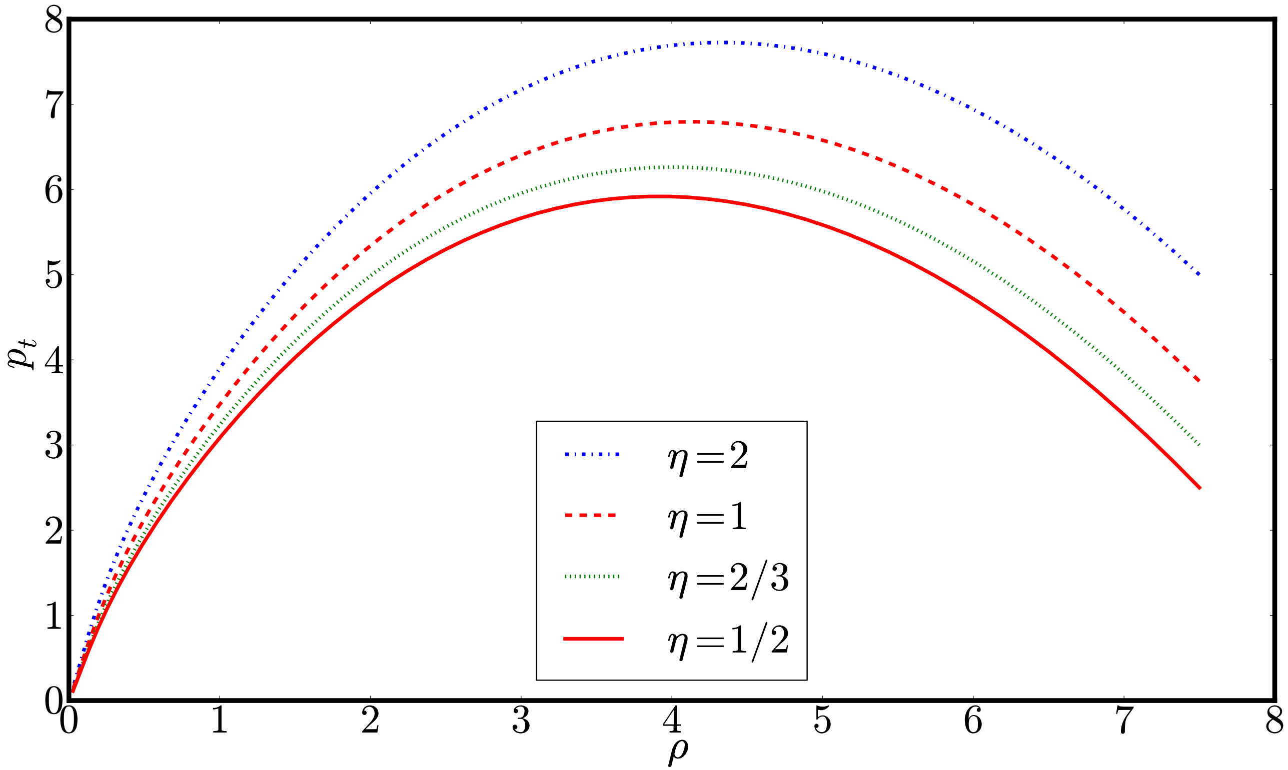

Figure 4.5: Tangential pressure $p_t(\rho)$.

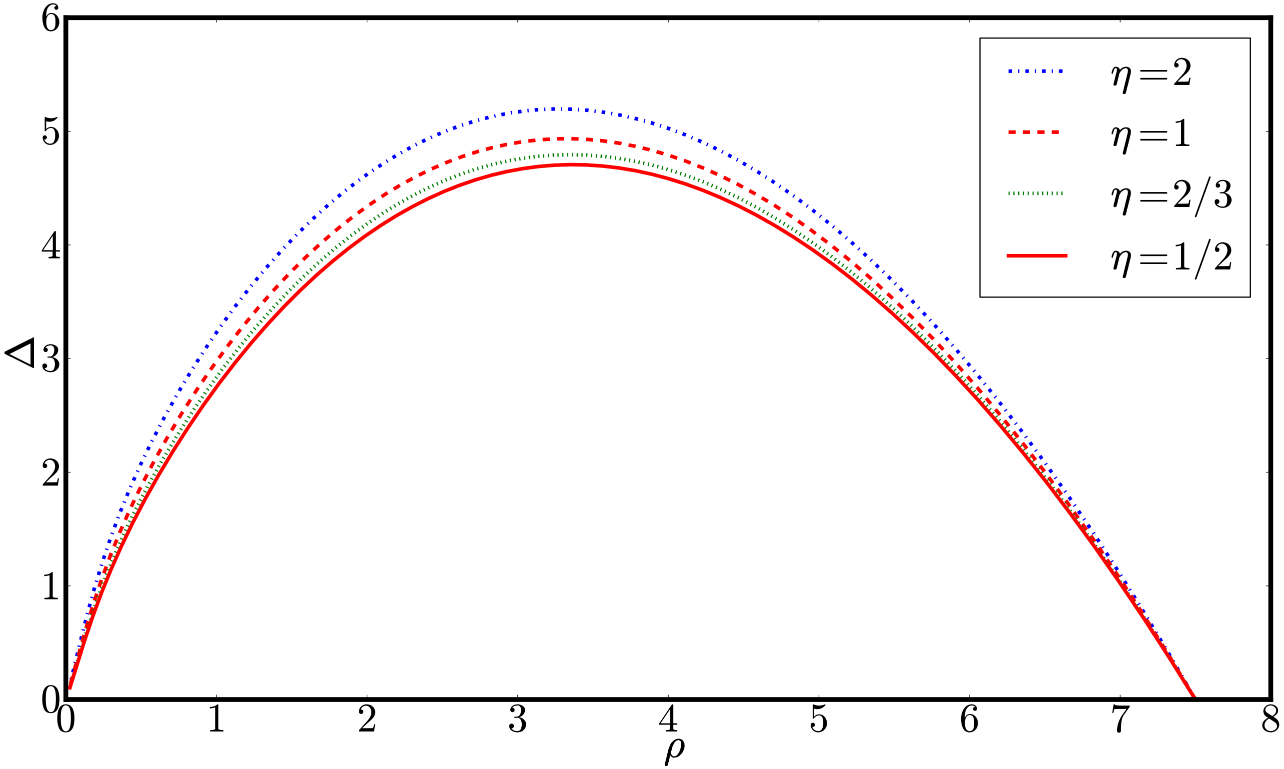

Figure 4.6: Anisotropy $\Delta(\rho)$.

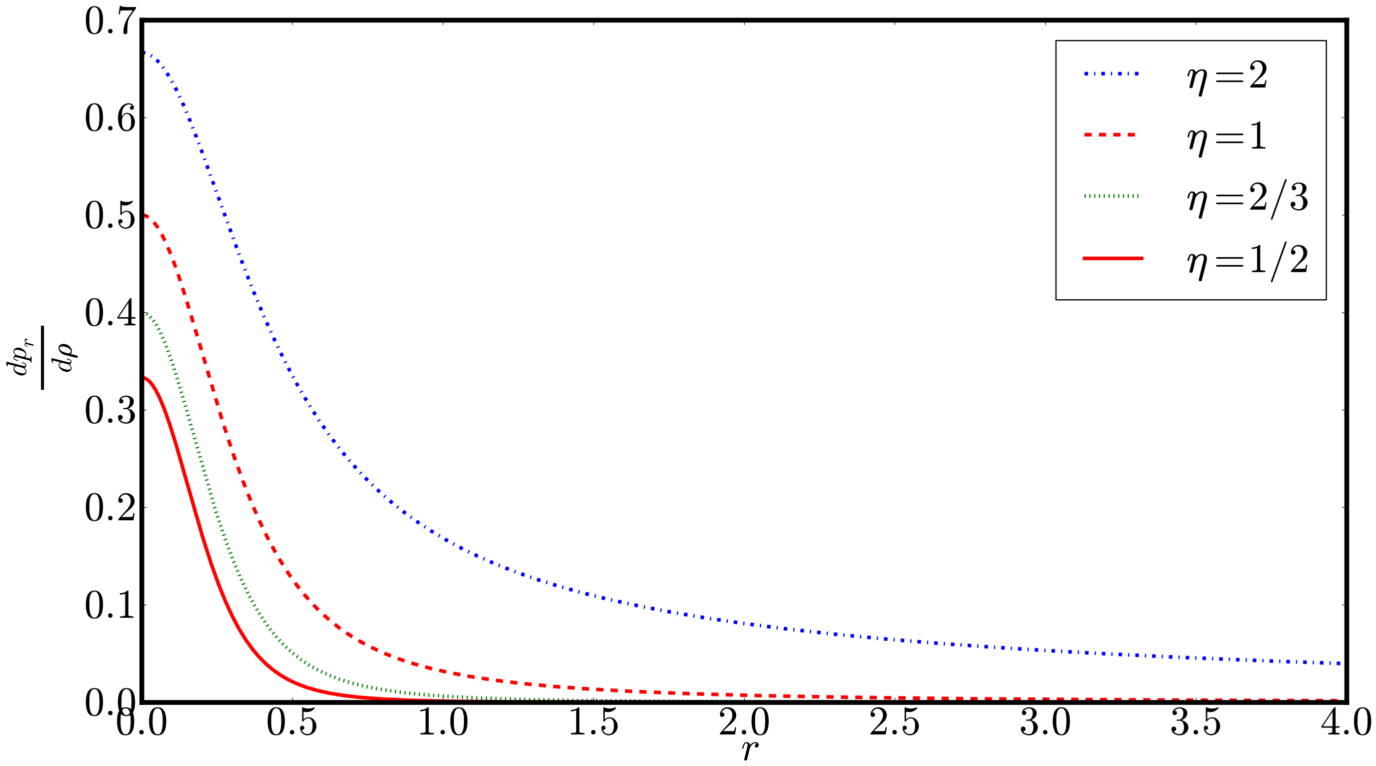

Figure 4.7: Speed of sound $\frac{dp_r}{d\rho}(r)$.

CHAPTER 5

Charged compact objects in the linear regime

5.1 Introduction

Exact solutions of the Einstein-Maxwell system are of vital importance in relativistic astrophysics. Bonnor (1965) demonstrated that the electric charge plays a crucial role in the equilibrium of large bodies which can possibly halt gravitational collapse. The challenge in astrophysics is to find stable equilibrium solutions for charged fluid spheres, and to construct models of various astrophysical objects of immense gravity by considering the relevant matter distributions. Such models may successfully describe the characteristics of compact stellar objects like, neutron stars, quark stars, etc.

Astrophysical compact stars are generally considered to be neutron stars. Over the past two decades, there has been considerable development in observations of the compact stars. Although the mass of many of the compact stars are determined with a fair precision, the main problem comes in determining its radius. In some recent papers, improved techniques give accurate mass and radius of a few compact stars. The improved observational information about such compact stars have invoked considerable interest about the internal composition and consequent geometry of such objects.

As an alternative to neutron star models, strange stars have been suggested in the studies of compact relativistic astrophysical bodies. Similar to neutron stars, strange stars are considered likely to form from the core collapse of a massive star during a supernova explosion, or during a primordial phase transition where quarks clump together. Another hypothesis is that an accreting neutron star in a binary system, can accrete enough mass to induce a phase transition at the centre or the core, to become a strange star. In the literature, there are numerous models of neutron stars and strange stars. In this chapter we use the most widely studied strange star model, namely the MIT Bag model. Studies of strange stars have been mostly performed within the framework of the Bag model as the physics of high densities is still not very clear. Chodos *et al* (1974) used the phenomenological MIT Bag model, where they assumed that the quark confinement is caused by a universal bag pressure at the boundary of any region containing quarks, namely the hadrons. The equation of state describing the strange matter in the bag model has a simple linear form (Witten 1984). Weber (2005) have shown that for a stable quark matter, the bag constant B is restricted to a particular range.

It is remarkable to note that a strange matter equation of state seems to explain the observed compactness of many astrophysical bodies such as Her X-1, 4U 1820-30, SAX J 1808.4-3658, 4U 1728-34, PSR 0943+10, and RX J185635 as pointed out by Rahaman $et\ al$ (2012). Dey $et\ al$ (1998) studied a new approach for strange stars by assuming an interquark vector potential originating from gluon exchange and a density dependent scalar potential which restores chiral symmetry at a high density. In the Dey *et al* (1998) formulation, the equation of state can also be approximated to a linear form. If pulsars are modelled as strange stars, the linear equation of state appears to be a feature in the composition of such objects (Sharma and Maharaj 2007a). For a given central density or pressure, the conservation equations can be integrated to compute the macroscopic features like the mass and radius of the star.

In situations where the densities inside the stars are beyond nuclear matter density, the anisotropy can play a crucial role, as the conservation equations are modified. Usov (2004) suggested the consideration of anisotropy in modelling strange stars in the presence of strong electric field. The analysis of static spherically symmetric anisotropic fluid spheres is important in relativistic astrophysics. Since the first study of Bowers and Liang (1974) there has been much research in the study of anisotropic relativistic matter in general relativity. It has been pointed out that nuclear matter may be anisotropic in high density ranges of order 10^{15}g cm^{-3}, where nuclear interactions have to be treated relativistically (Ruderman, 1972). It has been noted that anisotropy can arise from different kinds of phase transitions (Sokolov, 1980) or pion condensation (Sawyer, 1972). The role of charge in a relativistic quark star was considered by Mak and Harko (2004).

In the present work we review the regular exact model of Mafa Takisa and Maharaj (2013) by testing the consistency and compatibility with observations of this model. We use this model to find the maximum mass and physical parameters of observed compact objects, namely PSR J1614-2230, PSR J1903+327, Vela X-1, SMC X-1 and Cen X-3, which has been recently identified by Gangopadhyay *et al* (2013) to be strange stars. In Section 5.2, Einstein-Maxwell field equations are briefly reviewed and the Mafa Takisa and Maharaj (2013) model is revisited. Recent observations are presented in Section 5.3. In Section 5.4, we present and discuss our results obtained for the uncharged case and compare them to values of masses derived from current accurate observations of compact objects. In Section 5.5, we apply finite charge to the uncharged systems presented in Section 5.4, and observe the changes. We discuss and conclude our results in Section 5.6.

5.2 The model

The metric of the static spherically symmetric spacetime in curvature coordinates reads

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{5.1}$$

where $\nu = \nu(r)$ and $\lambda = \lambda(r)$. The energy momentum tensor for an anisotropic charged imperfect fluid sphere is of the form

$$T^{ij} = \text{diag}(-\rho - \frac{1}{2}E^2, p_r - \frac{1}{2}E^2, p_t + \frac{1}{2}E^2, p_t + \frac{1}{2}E^2), \tag{5.2}$$

where ρ, p_r, p_t and E are density, radial pressure, tangential pressure and electric field intensity respectively. For a physically realistic relativistic star we expect that the matter distribution should satisfy a barotropic equation of state $p_r = p_r(\rho)$; the linear case is given by

$$p_r = \alpha\rho - \beta, \tag{5.3}$$

where $\beta = \alpha\rho_\varepsilon$. The constant α is determined by the sound speed causality condition ($\alpha = \frac{dp_r}{dr} \leq 1$) and ρ_ε represents the density at the surface $r = \varepsilon$. The total energy momentum tensor ***T*** is the sum of ***M*** and ***E*** for a charged gravitating fluid. This is given by

$$T^{ij} = M^{ij} + E^{ij}.$$

The gravitational interactions on matter and electromagnetic fields are governed by a relevant set of field equations. These interactions are contained in the Einstein-Maxwell system

$$\begin{aligned} G^{ij} &= 8\pi T^{ij} \\ &= 8\pi(M^{ij} + E^{ij}), && (5.4a) \\ F_{ij;k} + F_{jk;i} + F_{ki;j} &= 0, && (5.4b) \\ F^{ij}{}_{;j} &= 4\pi J^i, && (5.4c) \end{aligned}$$

where the coupling constant $k = 8\pi$ $(G = c = 1)$ in geometrized units. The system above is a highly nonlinear system of coupled, partial differential equations governing the behaviour of the gravitating system in the presence of an electromagnetic field.

For static, charged anisotropic matter with the line element (5.1), the Einstein-Maxwell system takes the form

$$\begin{aligned} 8\pi\rho + \frac{1}{2}E^2 &= \frac{1}{r^2}\left[r(1 - e^{-2\lambda})\right]', && (5.5a) \\ 8\pi p_r - \frac{1}{2}E^2 &= -\frac{1}{r^2}\left(1 - e^{-2\lambda}\right) + \frac{2\nu'}{r}e^{-2\lambda}, && (5.5b) \\ 8\pi p_t + \frac{1}{2}E^2 &= e^{-2\lambda}\left(\nu'' + \nu'^2 + \frac{\nu'}{r}\lambda' - \frac{\lambda'}{r} - \nu\right), && (5.5c) \\ \sigma &= \frac{1}{4\pi r^2}e^{-\lambda}(r^2E)', && (5.5d) \end{aligned}$$

where $\sigma = \sigma(r)$ is called proper charge density and primes denote differentiation with respect to r. We note that equations (5.5a)-(5.5d) imply

$$\frac{dp_r}{dr} = \frac{2}{r}(p_t - p_r) - r(\rho + p_r)\nu' + \frac{E}{4\pi r^2}\left(r^2E\right)', \tag{5.6}$$

which is the Bianchi identity representing hydrostatic equilibrium of the charged anisotropic fluid. Equation (5.6) indicates that the anisotropy and charge influence the gradient of the pressure. These quantities may drastically affect quantities of physical importance such as surface tension as established by Sharma and Maharaj (2007b) in the generalised Tolman-Oppenheimer equation (5.6). We define the gravitational mass to be

$$m(r) = 4\pi \int_0^r \left(\rho(\omega) + \frac{E^2}{8\pi} \right) d\omega, \tag{5.7}$$

in the presence of charge.

In this Chapter, we utilise the results of Chapter 2, with a linear equation of state. The motivation for this is that those results were consistent with the observed X-ray binary pulsar SAX J1808.4-3658. It is likely that the exact solutions of Chapter 2 may be applicable to other observed astronomical bodies. With the equation of state (5.3), the Einstein-Maxwell system

(5.8a)-(5.8i) can be written as

$$
\begin{aligned}
e^{2\lambda} &= \frac{1+ar^2}{1+(a-b)r^2}, && (5.8a)\\
e^{2\nu} &= A^2(1+ar^2)^{2t}[1+(a-b)r^2]^{2n} \times \exp\left[-\frac{ar^2[s(1+\alpha)+2\beta]}{4(a-b)}\right], && (5.8b)\\
\rho &= \frac{2b(3+ar^2)-sa^2r^4}{16\pi(1+ar^2)^2}, && (5.8c)\\
p_r &= \frac{\alpha[2b(3+ar^2)-sa^2r^4]}{16\pi(1+ar^2)^2}-\beta, && (5.8d)\\
p_t &= p_r+\Delta, && (5.8e)\\
8\pi\Delta &= \frac{-b}{(1+ar^2)}-\frac{b(1+5\alpha)}{(1+\alpha)(1+ar^2)^2}+\frac{2\beta}{1+\alpha}+\frac{r^2[1+(a-b)r^2]}{(1+ar^2)}\\
&\quad \times\{\frac{4a^2t(t-1)}{(1+ar^2)^2}+\frac{8a(a-b)tn}{(1+ar^2)[1+(a-b)r^2]}\\
&\quad \frac{-a[s(1+\alpha)+2\beta](a(t+n)[1+(a-b)r^2]-bn)}{(a-b)(1+ar^2)[1+(a-b)r^2]}\\
&\quad +\frac{a^2[s(1+\alpha)+2\beta]^2}{16(a-b)^2}+\frac{4(a-b)^2n(n-1)}{[1+(a-b)r^2]^2}\}\\
&\quad -\frac{4[1+ar^2(2+(a-b)r^2)]-b(5+\alpha)r^2}{4(a-b)(1+\alpha)(1+ar^2)^3[1+(a-b)r^2]}\\
&\quad \times[-8b^2n+a^3r^2(-8(t+n)+[s(1+\alpha)+2\beta]r^2)\\
&\quad +a^2(8(t+n)(2br^2-1)+[s(1+\alpha)+2\beta](2-br^2)r^2)\\
&\quad +a(-8b^2(t+n)r^2+[s(1+\alpha)+2\beta]\\
&\quad +b(8t+16n-[s(1+\alpha)+2\beta]r^2))], && (5.8f)\\
E^2 &= \frac{sa^2r^4}{(1+ar^2)^2}, && (5.8g)\\
\sigma^2 &= \frac{sa^2r^2[1+(a-b)r^2](2+ar^2)^2}{\pi(1+ar^2)^5}, && (5.8h)\\
m(r) &= \frac{1}{8}\left[\frac{s(-15-10ar^2+2a^2r^4)r}{3a(1+ar^2)}+\frac{5s\arctan(\sqrt{ar^2})}{a^{3/2}}\right.\\
&\quad \left.+\frac{4br^3}{(1+ar^2)}\right], && (5.8i)
\end{aligned}
$$

which has been adopted to the conventions of this Chapter for consistency. In the above equations the constants *m* and *n* are given by

$$
\begin{aligned}
t &= \frac{4\alpha b - (1+\alpha)s}{8b}, \\
n &= \frac{1}{8b(a-b)^2}[a^2((1+\alpha)s - 4\alpha b) \\
& \quad 2ab^2(1+5\alpha) + b^2(-2b(1+3\alpha)+2\beta)].
\end{aligned}
$$

The constants a, b, s have the dimension of $[length]^{-2}$. We make the following transformations for simplicity in numerical calculations:

$$\tilde{a} = a\Re^2,\ \tilde{b} = b\Re^2,\ \tilde{s} = s\Re^2,$$

where $\Re$ is a parameter which has the dimension of $[length]$. For physical reasonableness, following Delgaty and Lake (1998), we impose particular restrictions on our model. The values of $\tilde{a}$, $\tilde{b}$, $\tilde{s}$ should be chosen so that

- the energy density ρ remains positive inside the star,
- the radial pressure p_r should vanish at the boundary of the star ($p_r(\varepsilon) = 0$),
- the tangential pressure p_t should be positive within the interior of star,
- the gradient of pressure $\frac{dp_r}{dr} < 0$ in the interior of the star,
- At the centre $p_r(r=0) = p_t(r=0)$ and $\Delta(r=0) = 0$,
- The metric functions $e^{2\lambda}$, $e^{2\nu}$ and the electric field intensity E should be positive and nonsingular in the interior of the star
- The density $\rho(r=0) = \rho_c$ must be finite
- across the boundary $r = \varepsilon$:

$$
\begin{aligned}
e^{2\nu(\varepsilon)} &= 1 - \frac{2M}{\varepsilon} + \frac{Q^2}{\varepsilon^2}, \\
e^{2\lambda(\varepsilon)} &= \left(1 - \frac{2M}{\varepsilon} + \frac{Q^2}{\varepsilon^2}\right)^{-1}, \\
m(\varepsilon) &= M.
\end{aligned}
$$

5.3 Recent observations

For a pulsar in a binary system, Jacobi *et al* (2005) and Verbiest *et al* (2008) used detection of the general relativistic Shapiro delay to infer the masses of both the neutron star and its binary companion to high precision. Based on this approach Demorest *et al* (2010) presented radio timing observations of the binary millisecond pulsar PSR J1614-2230, which showed a strong Shapiro delay signature. The implied pulsar mass of $(1.97 \pm 0.08 M_{\odot})$ is by far the highest yet measured with accurate precision.

Freire *et al* (2011) utilised the Arecibo and Green Bank radio timing observations and included a full determination of the relativistic Shapiro delay, a very precise measurement of the apsidal motion and new constraints of the orbital orientation of the system. Through a detailed analysis, they derived new constraints on the mass of the pulsar and its companion and determined the accurate mass for PSR J1903+0327 $(1.667 \pm 0.02 M_{\odot})$.

Recently Rawls *et al* (2011) have found an improved method for determining the mass of neutron stars such as (Vela X-1, SMC X-1, Cen X-3) in eclipsing X-ray pulsar binaries. They used a numerical code based on Roche geometry with various optimizers to analyze the published data for these systems, which they supplemented with new spectroscopic and photometric data for 4U 1538-52. This allowed them to model the eclipse duration more accurately, and they calculated an improved value for the neutron star masses. Their derived values are $(1.77 \pm 0.08 M_{\odot})$ for Vela X-1, $(1.29 \pm 0.05 M_{\odot})$ for LMC X-4 and $(1.29 \pm 0.08 M_{\odot})$ for Cen X-3.

There has been similar observations for other stars also, but for our present work, we restrict ourselves to these five stars only.

5.4 Uncharged stars

In this section, we use the analytical solutions (5.8a)-(5.8i) to calculate the mass and radius of five different compact stars (PSR J1614-2230, PSR J1903+327, Vela X-1, SMC X-1, Cen X-3), and compare the output to the recent accurate maximum mass values as mentioned in previous section. We consider the equation of state of strange stars and choose the central density ρ_c

Table 5.1: Mass and radius of different stars in the uncharged case $(\tilde{s} = 0.0)$

STAR	α	$\tilde{b}$	$\frac{M}{M_\odot}$	$\frac{M}{M_\odot R}$	R (km)	ρ_c $(\times 10^{15}$g cm$^{-3})$
PSR J1614-2230	0.33	40.11	1.97	0.191	10.30	3.45
PSR J1903+327	0.33	36.48	1.667	0.170	9.82	3.14
Vela X-1	0.33	37.77	1.77	0.177	9.99	3.25
SMC X-1	0.33	31.68	1.29	0.141	9.13	2.72
Cen X-3	0.33	34.29	1.49	0.157	9.51	2.95

in the range of 2.2×10^{15}g cm^{-3} $\leq \rho_c \leq 5.5 \times 10^{15}$g cm^{-3}, $\tilde{a} = 53.34$, $\Re = 43.245$ km, $\alpha = 0.33$, $\rho_\varepsilon = 0.5 \times 10^{15}$g cm^{-3}. Then the model yields the above stars which are represented in Table 5.1. We regain the accurate mass and corresponding radius for each star. For PSR J1614-2230, with $\rho_c = 3.45 \times 10^{15}$g cm^{-3}, leads to $M = 1.97M_\odot$ and $R = 10.30$ km. For PSR J1903+327, $\rho_c = 3.14 \times 10^{15}$g cm^{-3} and we obtain $M = 1.667M_\odot$ and $R = 9.82$ km. By taking $\rho_c = 3.25 \times 10^{15}$g cm^{-3}, we get the mass and radius of Vela X-1 $M = 1.77M_\odot$ and $R = 9.99$ km. The value 2.72×10^{15} g cm^{-3} leads to $M = 1.29M_\odot$ and $R = 9.13$ km which corresponds to SMC X-1. Finally for Cen X-3, we take $\rho_c = 2.95 \times 10^{15}$ g cm^{-3}, and obtain the accurate mass and the radius ($M = 1.49M_\odot$ and $R = 9.51$ km).

5.5 Charged stars

The parameters used in Section 5.4. have generated results which are consistent with observational data. Consequently, we use these values to study charged bodies. We take the central density in the range of 2.2×10^{15}g cm^{-3} $\leq \rho \leq 5.5 \times 10^{15}$g cm^{-3}, $\tilde{a} = 53.34$, $\Re = 43.245$ km, $\alpha = 0.33$, $\rho_\varepsilon = 0.5 \times 10^{15}$g cm^{-3}, $\tilde{s} = 0.0, 7.5, 14.5$ and $M_\odot = 1.477$. Then the corresponding results are given in Table 5.2. It is clear that the presence of electric charge leads to a considerable increase in the mass of a stellar object obeying the linear equation of state. On the other hand the radius of different charged congurations ($\tilde{s} = 7.5, 14.5$) is smaller than the maximum radius of the uncharged case ($\tilde{s} = 0.0$). A similar situation arises in the analysis of Mak and Harko (2004).

To illustrate the behavior of physical parameters at the interior of different stars, we have plotted the energy density ρ, radial pressure p_r, tangential pressure p_t and the measure of anisotropy Δ. Figures 5.1, 5.2, 5.3, 5.4, 5.5 represent PSR J1614-2230, SMC X-1, PSR J1903+327, Cen X-3 and Vela X-1 respectively. The density profiles are positive and well behaved inside all stars. The effect of electric charge is more significant near the surface of stars; this situation is consistent with the form of the electric field of Mafa Takisa and Maharaj (2013) in (5.8e) which vanishes at the centre $E(0) = 0$. We note that the interior profile of radial pressure p_r, tangential pressure p_r and the measure of anisotropy Δ profiles of PSR J1614-2230, PSR J1903+327, Vela X-1, SMC X-1 and Cen X-3 stars are completely unaffected by the electric charge layer, since the latter is located in a thin, spherical shell close the surface. A similar statement has also been made by Negreiros *et al* (2009). The tangential pressure p_t profiles for all studied stars are well behaved, increasing in the vicinity of the centre, reaches a maximum, and becomes a decreasing function. This is reasonable since the conservation of angular momentum during the quasi-equilibrium contraction of a massive body should lead to high values of p_t in central regions of the star, as pointed out by Karmakar *et al* (2007). The anisotropy is increasing in the neighborhood of the centre, reaches a maximum value, then starts decreasing up to the boundary. The density profile is similar to Sharma and Maharaj (2007a).

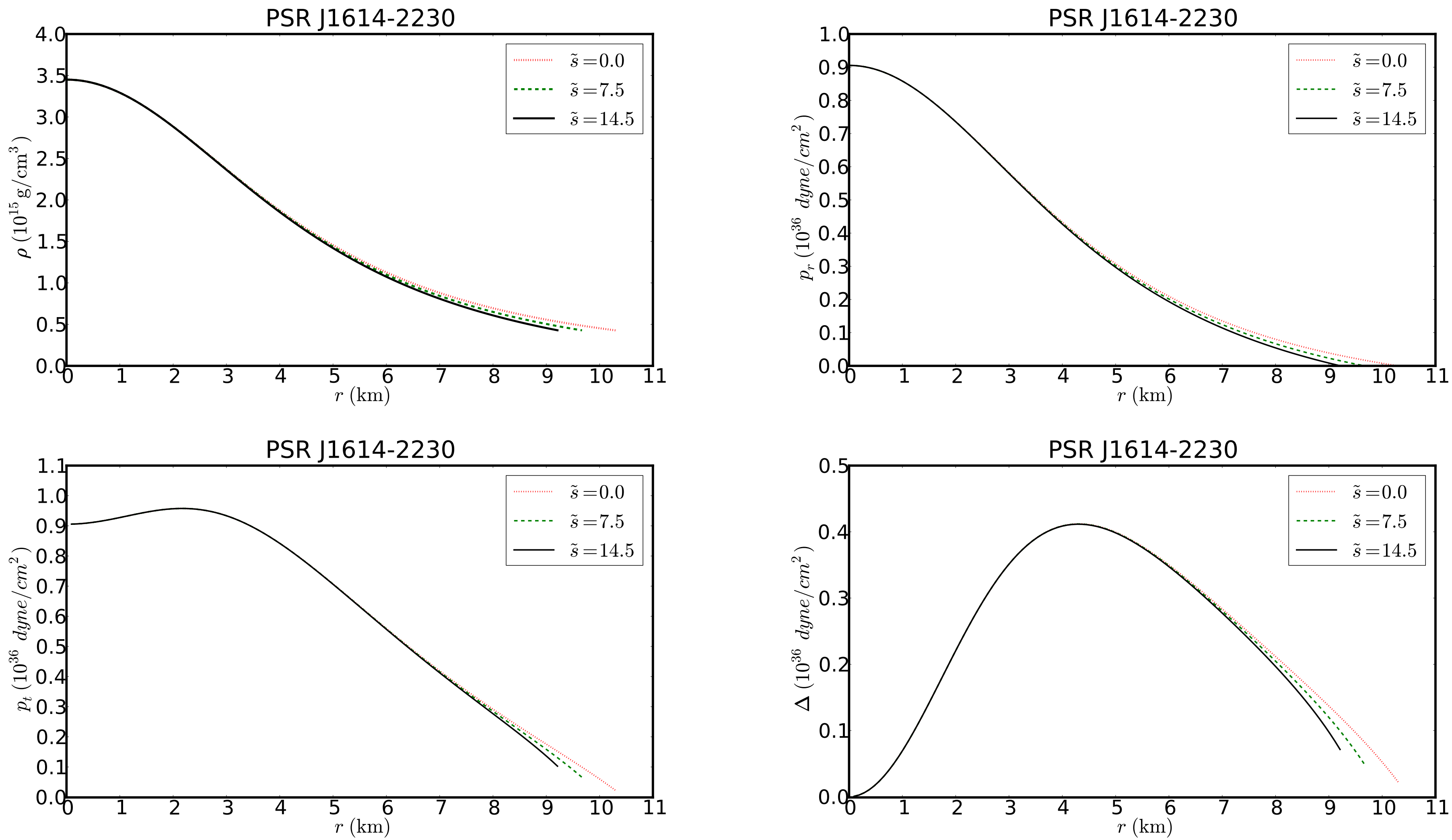

Figure 5.1: Matter variables ρ, p_r, p_t for uncharged and charged cases of PSR J1614-2230.

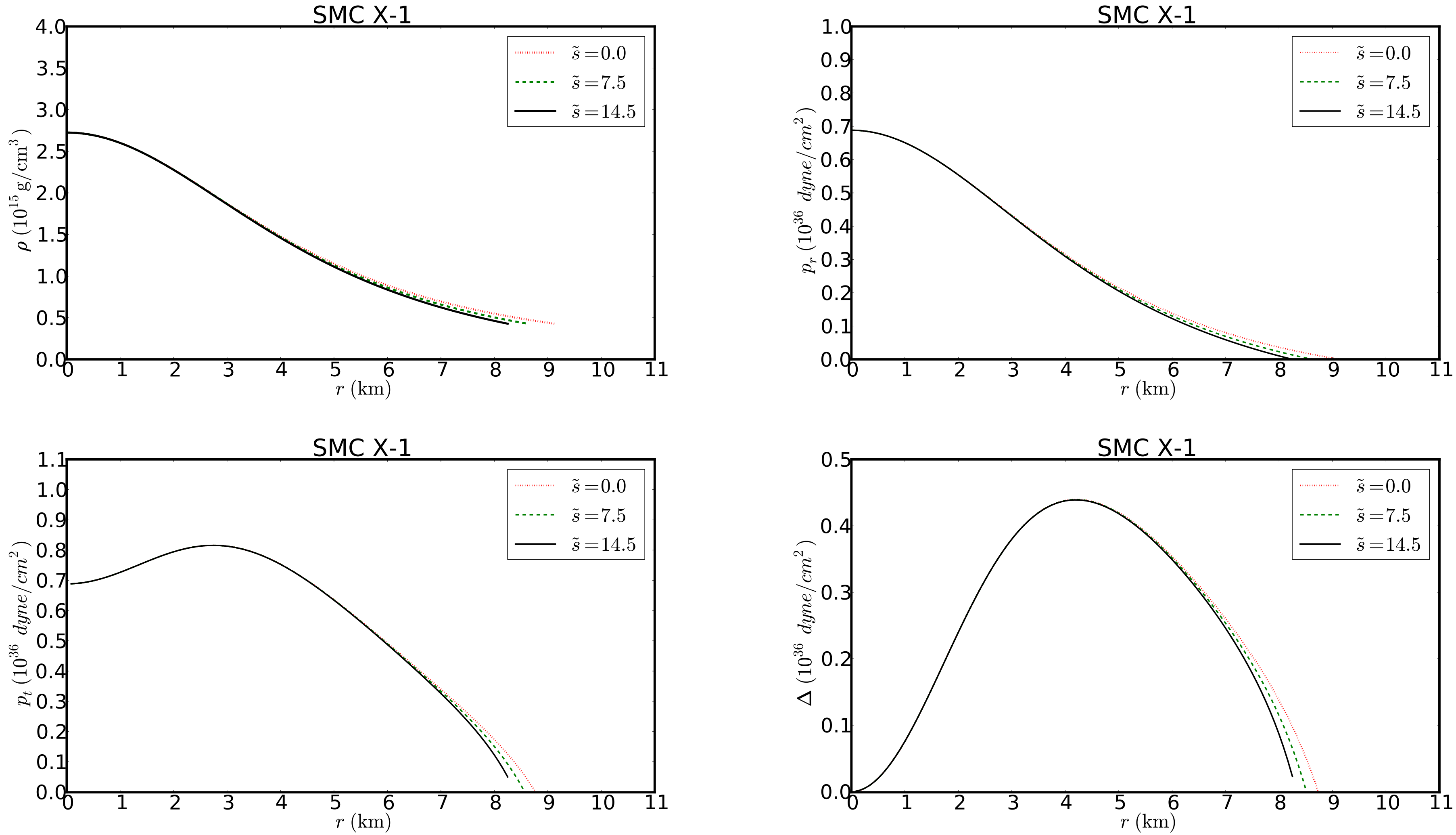

Figure 5.2: Matter variables ρ, p_r, p_t for uncharged and charged cases of SMC X-1.

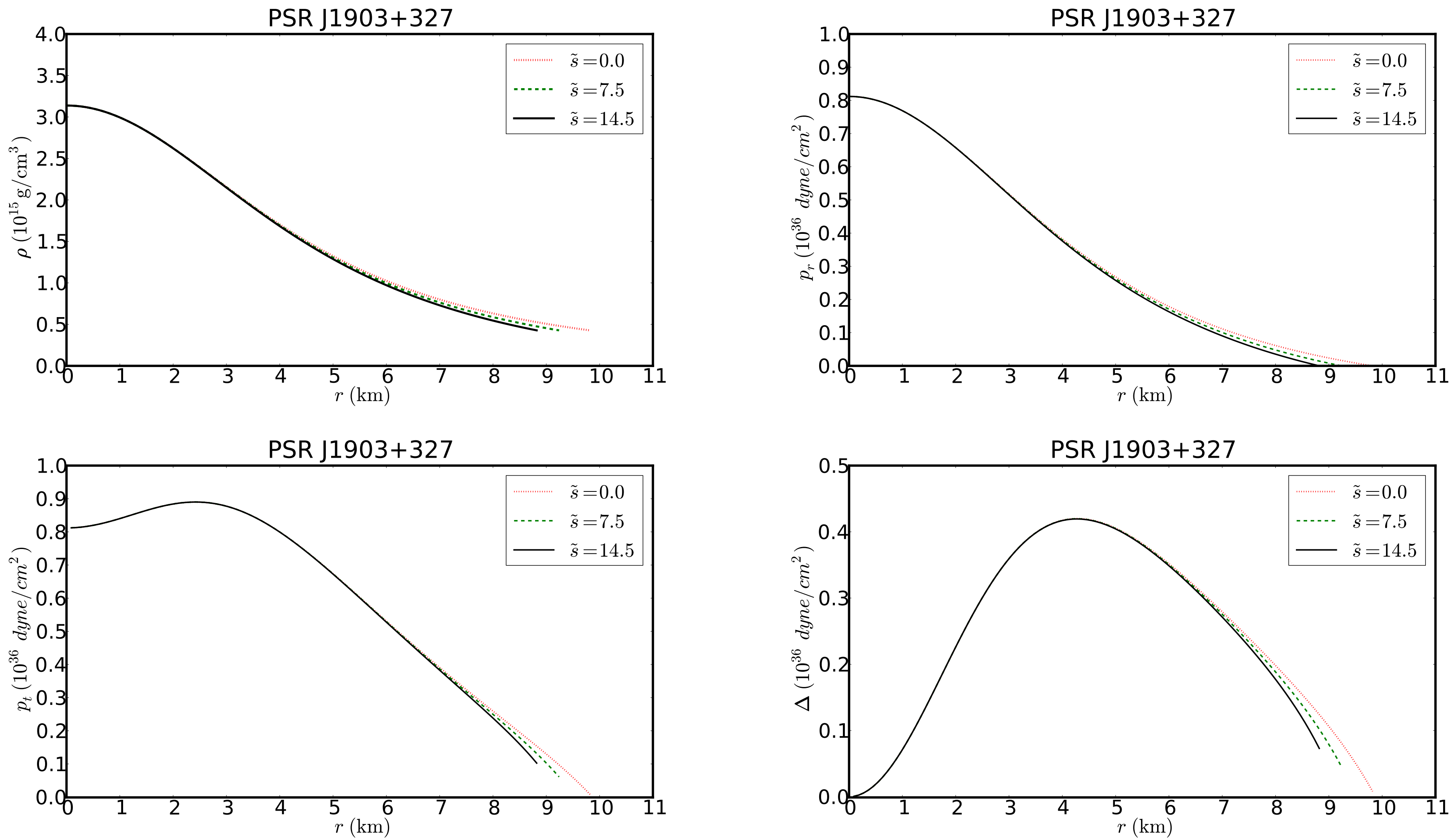

Figure 5.3: Matter variables ρ, p_r, p_t for uncharged and charged cases of PSR J1903+327.

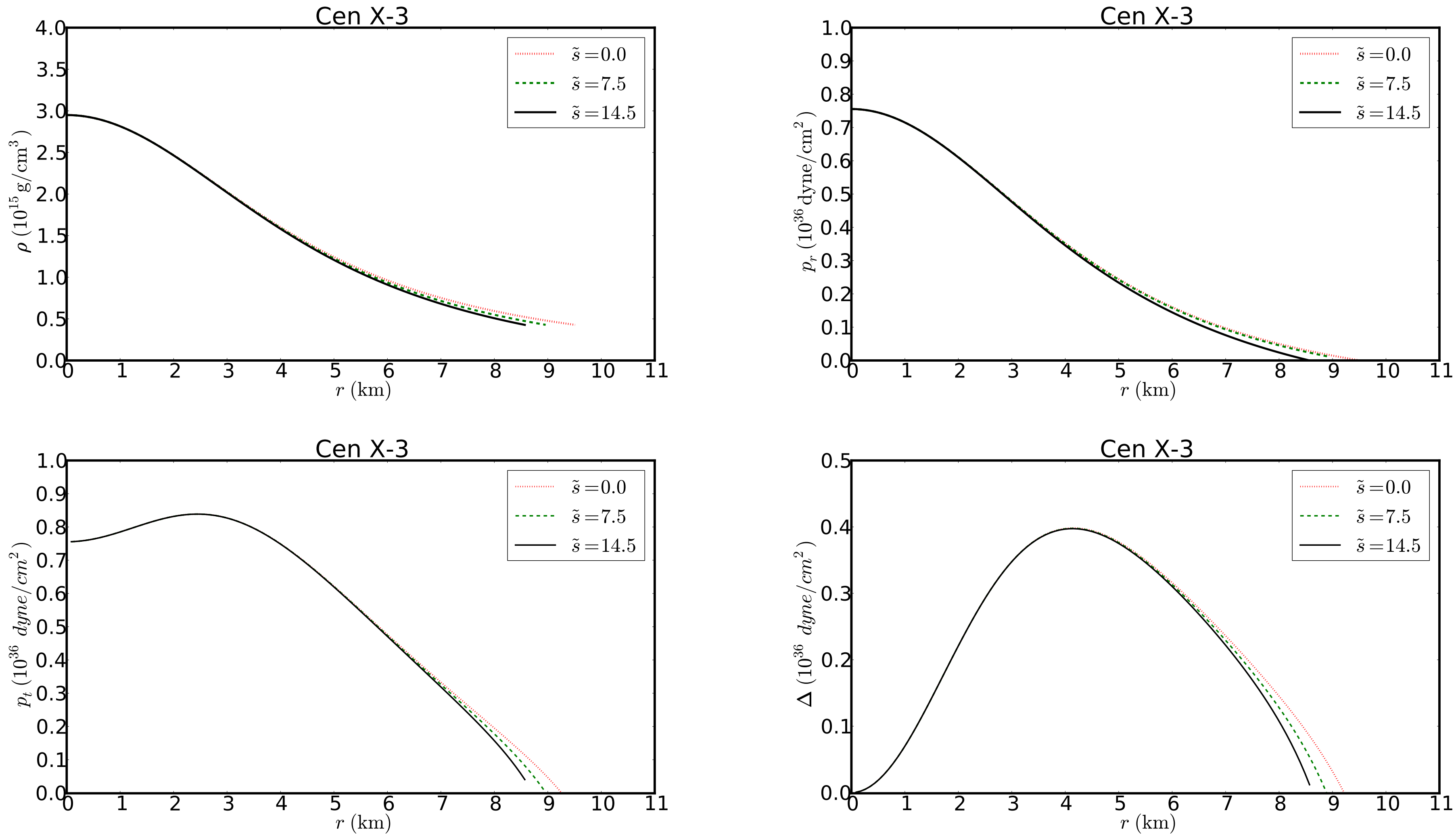

Figure 5.4: Matter variables ρ, p_r, p_t for uncharged and charged cases of Cen X-3.

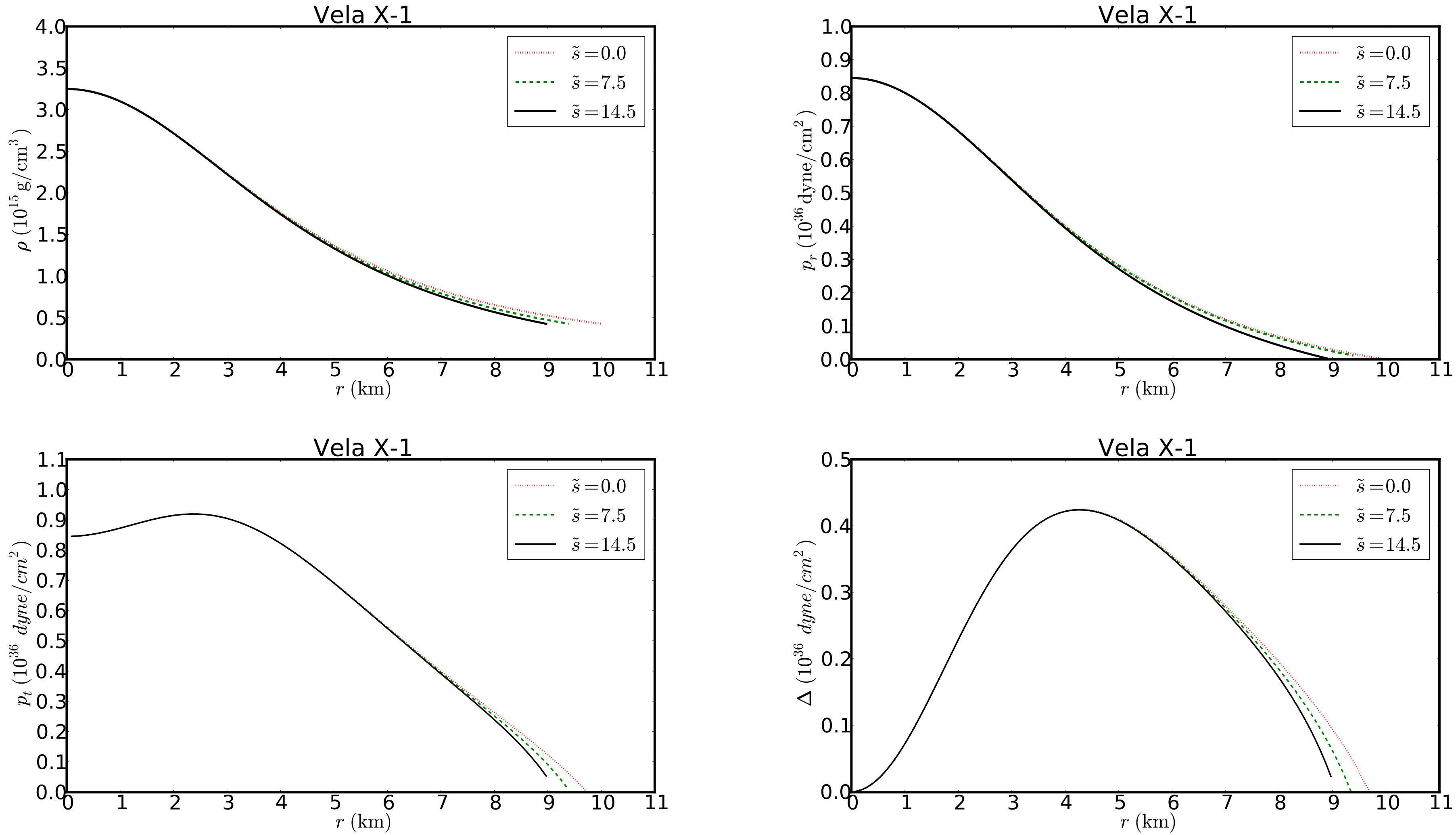

Figure 5.5: Matter variables ρ, p_r, p_t for uncharged and charged cases of Vela X-1.

Table 5.2: Mass and radius of different stars for the charged case ($\tilde{s} \neq 0$). For $\tilde{s} = 0.1$, the results are similar to the uncharged case $\tilde{s} = 0.0$. A small difference appears at $\tilde{s} = 7.5$ and the effect of charge becomes significant at $\tilde{s} = 14.5$.

Charge parameter	STAR	α	$\tilde{b}$	$\frac{M}{M_\odot}$	R (km)	ρ_c ($\times 10^{15}$ g cm^{-3})
$\tilde{s} = 0.1$	PSR J1614-2230	0.33	40.11	1.97	10.30	3.45
	PSR J1903+327	0.33	36.48	1.667	9.82	3.14
	Vela X-1	0.33	37.77	1.77	9.99	3.25
	SMC X-1	0.33	31.68	1.29	9.13	2.72
	Cen X-3	0.33	34.29	1.49	9.51	2.95
$\tilde{s} = 7.5$	PSR J1614-2230	0.33	40.11	1.98	9.67	3.45
	PSR J1903+327	0.33	36.48	1.674	9.24	3.14
	Vela X-1	0.33	37.77	1.78	9.39	3.25
	SMC X-1	0.33	31.68	1.30	8.62	2.72
	Cen X-3	0.33	34.29	1.50	8.96	2.95
$\tilde{s} = 14.5$	PSR J1614-2230	0.33	40.11	2.13	9.21	3.45
	PSR J1903+327	0.33	36.48	1.81	8.82	3.14
	Vela X-1	0.33	37.77	1.92	8.96	3.25
	SMC X-1	0.33	31.68	1.40	8.25	2.72
	Cen X-3	0.33	34.29	1.62	8.57	2.95

5.6 Discussion

We have used the Mafa Takisa and Maharaj model (2013) to model compact stars. In our investigation, we have considered a constant slope $\alpha = 1/3$ in the equation of state, and the surface density $\rho_s = 0.5 \times 10^{15}$g cm^{-3}. The surface density chosen in this work is approximately close to $4B = 0.45 \times 10^{15}$g cm^{-3} of Alcock *et al* (1986). It shows that, for particular parameters values, the model can be used to describe the observed compact stars (PSR J1614-2230, PSR J1903+327, Vela X-1, SMC X-1, Cen X-3). The recent measurement of the mass of PSR J1614-2230 provides one of the strongest observational constraints on the equation of state so far. In our present result we have found the mass value of $M = 1.97M_{\odot}$, $\rho_c = 3.45 \times 10^{15}$g cm^{-3} and $R = 10.30$ km as corresponding radius for the pulsar PSR J1614-2230. As accurate and reliable radius measurements of this star are not yet available, our theoretical result may be useful in future investigations. From the general relativistic structure equations and according to Buchdahl (1959), the maximum allowable compactness (mass-radius ratio) for an uncharged star is set by $\frac{2M}{R} < \frac{8}{9}$. The compactness values for all stars shown in Table 5.1, shows the acceptability of our model. Unlike others models (Thirukkanesh and Maharaj 2008, Mafa Takisa and Maharaj 2013), the masses for charged case $\tilde{s} \neq 0$ increases. For the maximum charge case $\tilde{s} = 14.5$, it has been observed that our class of solutions gives us a maximum mass of PSR J1614-2230 $M = 2.13M_{\odot}$, with electric field $E = 4.91059 \times 10^{20}$ V/m which leads to a 10$\%$ increase. Our results are in agreement with the work done by Negreiros *et al* (2009), who have demonstrated that the presence of electric fields of similar magnitude, generated by charge distributions located near the surfaces of strange quark stars, may increase the stellar mass by up to 15$\%$; this helps in the interpretation of massive compact stars, with masses of around $M = 2.0M_{\odot}$. We conclude by pointing out that such solutions may be used to construct a suitable model of a superdense object with both uncharged and charged matter.

CHAPTER 6

Conclusion

The main goal of this dissertation was to generate new exact solutions to the Einstein-Maxwell equations with different equations of state, which may be useful in modelling uncharged and charged anisotropic compact stars. Specifically we considered linear, quadratic and polytropic equations of state relating the energy density to the radial pressure. We showed that the new solutions to the Einstein-Maxwell systems generated are physically relevant. Earlier models are shown to be special cases of our general results. We believe that a detailed physical analysis of solutions found could be helpful in modeling realistic bodies in general relativity.

We now offer an overview of the principal results obtained during the course of our probes:

- In Chapter 2 we made specific choices of the gravitational potential and the electric field intensity that enabled us to integrate the field equations. We made the choice for the gravitational potential Z and the electric field intensity E:

$$\begin{aligned} Z &= \frac{1+(a-b)x}{1+ax}, \\ \frac{E^2}{C} &= \frac{k(3+ax)+sa^2x^2}{(1+ax)^2}. \end{aligned}$$

We used the linear equation of state

$$p_r = \alpha\rho - \beta,$$

in this Chapter. For specific values of a, b, we recover the models of Finch and Skea (1989), Durgapal and Bannerji (1983), Tikekar (1990), Hansraj and Maharaj (2006), and Maharaj and Komathiraj (2007). The electric field intensity E has been adapted so that the proper charge density remains nonsingular within the star. Charged relativistic solutions to the Einstein-Maxwell equations were presented. A graphical analysis showed that the matter and electromagnetic variables are physically relevant. Note that when $k = 0$ the proper charge density becomes regular at the centre of the star. This is an improvement on previous models which possessed a singularity at the stellar origin. The presence of anisotropy has a significant effect on stellar masses as shown in Table 2.1. The values that we have generated in Table 2.2, for neutral and charged matter, permit configurations typical of neutron stars and strange stars. We obtain stellar mass values generated by Tikekar and Jotania (2005) for superdense stars models with neutral matter. Therefore the class of solutions found in this chapter allow for stellar configurations which provide physically feasible models of superdense structures.

- In chapter 3 we presented a general framework for the Einstein-Maxwell equations with a quadratic equation of state that models the interior of a dense compact star with anisotropic charged stellar matter. To investigate the system we made the assumptions

$$\begin{aligned} Z &= \frac{1+bx}{1+ax}, \\ \frac{E^2}{C} &= \frac{k(3+ax)+sa^2x^2}{(1+ax)^2}. \end{aligned}$$

The quadratic equation of state chosen was of the form

$$p_r = \gamma\rho^2 + \alpha\rho - \beta.$$

A general class of exact solutions to the Einstein-Maxwell system with the quadratic equation of state was found. Particular models found previously were regained. In particular we extended the treatment of Thirukkanesh and Maharaj (2008) and Feroze and Siddiqui (2011) which are contained in our general class of solutions. These solutions may be used to model charged relativistic strange and quark stars.

- The purpose of Chapter 4 was to generate new exact models to the Einstein-Maxwell system with a polytropic equation of state. We made the choices

$$\begin{aligned} Z &= \frac{1+bx}{1+ax}, \\ \frac{E^2}{2C} &= \frac{\varepsilon x}{(1+ax)^2}. \end{aligned}$$

  The polytropic equation of state was of the form

$$p_r = \kappa \rho^{1+(1/\eta)}.$$

  We chose the polytropic index ranging over the four cases $\eta = 1/2, 2/3, 1, 2$ for strong gravitational fields when anisotropy and the electric field intensity are present. Several families of exact solutions to the Einstein-Maxwell system were presented. For $\eta = 1$ and $\varepsilon = 0$, we regained the models of Feroze and Siddiqui (2011) and Maharaj and Mafa Takisa (2012). Therefore it is possible to relate these new solutions to particular astronomical objects such as SAX J1804.4-3658, as was done by Dey $et\ al$ (1998, 1999a, 1999b) in the uncharged case, and Mafa Takisa and Maharaj (2013) for the charged case. This can also be extended to other compact relativistic objects. Such an investigation will reinforce the astrophysical significance of the models in this chapter. We point out that our approach automatically leads to new uncharged anisotropic solutions with a polytropic equation of state when the electric field $E = 0$.

- In chapter 5 we reinforced the astrophysical significance of the solutions of chapter 2 by considering particular astronomical objects. We obtained the accurate value of the mass and radius of five compact stars:

  - For PSR J1614-2230: $M = 1.97 M_{\odot}$ and $R = 10.30$ km.
  - For PSR J1903+327: $M = 1.667 M_{\odot}$ and $R = 9.82$ km.

- For Vela X-1: $M = 1.77M_{\odot}$ and $R = 9.99$ km.
- For SMC X-1 $M = 1.29M_{\odot}$ and $R = 9.13$ km.
- For Cen X-3: $M = 1.49M_{\odot}$ and $R = 9.51$ km.

Since reliable radius measurements of these stars are not yet available, our theoretical model may be useful in future surveys. Note that the presence of charge leads to an increase in the stellar mass by 10%. This increase is comparable to the surveys done by Negreiros *et al* (2009), who showed that electric fields of this magnitude, generated by charge distributions located near the surfaces of strange quark stars, can increase the stellar mass by up to 15%.

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