On the linearization of systems of differential equations

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This dissertation is submitted in fulfilment of the academic requirements for the degree of Doctor of Philosophy in Applied Mathematics to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban.

As the candidate's supervisors, we have approved this dissertation for submission.

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Abstract

In this thesis, a complete group classification of the general case of linear systems of two second-order ordinary differential equations is presented. Previous studies which produced distinguished representatives of systems of two linear second-order ordinary differential equations were not exhaustive. We show that the problem of classifying these systems using an algebraic approach leads to the study of a variety of cases in addition to those already obtained in the literature.

Secondly, we provide a new treatment for the linearization of a system of two second-order stochastic ordinary differential equations. We provide the necessary and sufficient conditions for the linearization of these systems. The linearization criteria are given in terms of coefficients of the system followed by some illustrations.

Finally, we consider the underlying group theoretic properties of a system of two linear second-order stochastic ordinary differential equations. For this system we obtained the determining equations and the corresponding equivalent transformations which assist with further classifying the system for some selected cases. This adds to the sparse body of knowledge on this subject.

Preface

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

T G Mkhize

June 2019

Declaration 1 - Plagiarism

I, Thembisile Gloria Mkhize, declare that

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Declaration 2 - Publications

Details of contribution to publications presented in this thesis:

Chapter 2

Mkhize TG, Moyo S and Meleshko SV, Complete group classification of systems of two linear second-order ordinary differential equations: The algebraic approach, *Mathematical Methods of Applied Sciences*, **38** (2015), 1824–1837.

Chapter 3

Mkhize TG, Govinder K, Moyo S and Meleshko SV, Linearization criteria for systems of two second-order stochastic ordinary differential equations, *Applied Mathematics and Computation*, **301** (2017), 25–35.

Chapter 4

Mkhize TG, Oguis GF, Govinder K, Moyo S and Meleshko SV, Group classification of systems of two linear second-order stochastic ordinary differential equations, *AIP Conference Proceedings: Modern Treatment of Symmetries, Differential Equations and Applications*, (accepted).

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Contents

Chapter 1

Mathematical Preliminaries and Outline

1.1 Introduction

Stochastic differential equations (SDEs), and in particular, stochastic ordinary differential equations (SODEs) include a stochastic component which describes the randomness within the differential equations. Thus they have many applications in areas such as engineering, physics, finance and mathematical biology. SDEs in physics describe the motion of a particle in a noise-perturbed force field, in particular in the harmonic oscillator [35]. SDEs are also encountered frequently in the theory of lasers, chemical kinetics and population dynamics [21]. In finance, SDEs are used to model the option price [45]. They also serve as the basic tool for understanding and implementing most important issues in interest rate modeling, and ultimately the analysis of inflation linked products [13, 58].

The stochastic analysis based on Brownian motion, is the best approach for dealing with random effects in SDEs. In 1827, Brown [9], observed the irregular motion of pollen particles suspended in a stationary liquid. He noted that the path of a given particle is very irregular, having a tangent at no point and the motions of two distinct particles appeared to be independent [26]. Unfortunately, Brown died in 1858 without providing any kind of theory to explain what he had observed [14]. In 1900, Louis Bachelier [4] attempted to described fluctuations in stock prices mathematically and combined his reasoning with the Markov[1] property and semigroups. He connected Brownian motion with the heat equation and successfully defined other processes related to Brownian motion, such as the maximum change during a time interval for one-dimensional Brownian motion [26]. His research work was later proved and extended by Einstein.

Albert Einstein published one of his ground-breaking papers in 1905 [16] which was based on the theory of Brownian motion. He showed how the motion of pollen particles in water could be explained by a random processes due to random bombardment of the pollen by water molecules. In his approach, he described the Brownian particle's velocity as an Ornstein-Uhlenbeck process and its position as a driftless Wiener process. He assumed that Brownian motion was a stochastic process with continuous paths, independent increments, and stationary Gaussian increments. He also derived and solved a partial differential equation (PDE) now known as the Fokker-Planck equation [35].

In 1908, a French physicist, Paul Langevin [34], developed a successful description of Brownian motion. Langevin's approach of Brownian motion was slightly more general and more correct than Einstein's. He also described the Brownian particle's velocity as an Ornstein-Uhlenbeck process and its position as the time integral of its velocity. He introduced a stochastic force pushing the Brownian particle in velocity space. He applied Newton's second law to a representative of Brownian particle and in this way he invented the $F = ma$ law of stochastic physics now called the Langevin equation. Langevin's analysis proved to have great utility in describing molecular fluctuations in other systems including non-equilibrium thermodynamics. Einstein and Langevin represented the two main approaches in the modeling of physical systems using the theory of stochastic processes and in particular, diffusion processes and as a result their research work remains current and is widely referenced and discussed [35].

Norbert Wiener [68] proposed a process as a mathematical description of Brownian motion in 1921. He used the idea of measure theory to construct Brownian motion and he proved many properties of the paths of the Brownian motion. He developed two key properties relating to

[1] A Markov process is sometimes referred to as a memory-less stochastic process.

stochastic integration: that, (1) the paths of Brownian motion have nonzero finite quadratic variation; and (2) the paths of Brownian motion have infinite variation on compact time intervals, almost surely. He also constructed a multiple integral, known today as the Multiple Wiener Integral. In recognition of his research work, his construction of Brownian motion is often referred to as the Wiener process. The Wiener process was discovered to have exceptionally complex mathematical properties [26].

The fundamentals of stochastic integration were laid down by Kolmogorov [30]. In his research, he developed a large part of his theory of Markov processes. He demostrated that the continuous Markov processes depended essentially on only two parameters: one for the size of the random part or the diffusion component and the other for the speed of the drift. He then connected the probability distributions of the processes to the solutions of partial differential equations which he solved and are now known as Kolmogorov equations. Since the Itô integrals were unavailable at that time, Kolmogorov relied on the analysis of the semigroup and its infinitesimal generator and the resulting PDEs [26].

Kiyosi Itô used terms and tools from measure theory to develop the theory on a probability space and his first research paper [24] on stochastic integration was published in 1944. In his effort to model the Markov processes, he constructed SDE. Furthermore, Itô [25] stated and proved what is now known as the Itô formula[2]. Itô knew that it was impossible to integrate all continuous stochastic processes. One of his key visions was to restrict his space of integrands to those that were adapted to the underlying filtration of σ-algebras generated by the Brownian motion [26]. Itô developed the so-called Itô calculus which extends the rules and methods from classical calculus to stochastic processes such as a Wiener process.

1.2 Itô calculus

A stochastic process is a mathematical model of a probability experiment that evolves in time and generates a sequence of numerical values. Each numerical value in the sequence can be modeled by a random variable, so a stochastic process is simply a sequence of random variables.

[2]The Itô formula provides the chain rule for differentials in the stochastic context.

This section reviews basic concepts and results on the Brownian motion and stochastic calculus. Some definitions and properties of stochastic processes which can be found in [2, 3, 20, 46, 54, 58] are presented.

1.2.1 Stochastic differential equations

Definition 1.1. A continuous and adapted stochastic process $X_i(t,\omega)$ is called an Itô process if it can be expressed in the form,

$$X_i(t,\omega) = X_i(0,\omega) + \int_0^t f_i(s, X(s,\omega))ds + \int_0^t g_i(s, X(s,\omega))dW(s) \ \ (i = 1, \dots, n). \tag{1.1}$$

In differential form, this is expressed as,

$$dX_i = f_i(t, X(t,\omega))dt + g_i(t, X(t,\omega))dW(t), \tag{1.2}$$

where the drift rate f and the volatility g are given adapted stochastic processes. The Wiener process, $W(t)$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is the stochastic process which provides an adequate model for Brownian motion. Since the Wiener process is of unbounded variation, the second integral in (1.1) is interpreted as a stochastic integral [46]. ■

1.2.2 Stochastic processes

Let Ω be a set of elementary events ω, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure on $\mathcal{F}$. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. The non-decreasing family of σ-algebras $\mathcal{F}_t$ is also called a filtration and the σ-algebra $\mathcal{F}$ is denoted by $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$.

Definition 1.2. A probability measure $\mathbb{P}$ is a function mapping $\mathcal{F}$ into $[0, 1]$ with the following properties [54]:

1. $\mathbb{P}(\Omega)$=1,

2. If $A_1, A_2, \dots$ is a sequence of disjoint sets in $\mathcal{F}$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

■

Definition 1.3. A family $\mathcal{F} = (\mathcal{F}_t)_{t\geq 0}$ of σ-algebras is a filtration, if $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s, t \in T$ with $s < t$ [2]. ■

Definition 1.4. A stochastic process X on $(\Omega,\ \mathcal{F},\ \mathbb{P})$ is a collection of random variables $\{X(t,\omega)\}_{t\in T}$, where T is some index set and Ω is the common sample space of the random variables [3]. ■

A stochastic process $\{X(t)\}_{t\geq 0}$ is said to be adapted to $(\mathcal{F}_t)_{t\geq 0}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each t. Denoting the Borel σ-algebra on $[0,\infty)$ by $\mathcal{B}$, the process X is called measurable if $(t,\omega) \longmapsto X(t,\omega)$ is a $\mathcal{B}\otimes\mathcal{F}$-measurable mapping. The process X is said to be continuous if the trajectories $t \longmapsto X(t,\omega)$ are continuous for almost all $\omega \in \Omega$. It is called progressively measurable if $X : [0,t]\times\Omega \longmapsto \mathbb{R}$ is a $\mathcal{B}([0,t])\otimes\mathcal{F}_t$-measurable mapping $0 \leq t \leq \infty$. Note that a progressively measurable process is measurable and adapted.

Remark: For brevity, we write $X(t)$ or $\{X(t)\}_{t\geq 0}$ instead of $X(t,\omega)$ or $\{X(t,\omega)\}_{(t,\omega)\in\Omega\times T}$.

Definition 1.5. Let $X_1, X_2, \ldots$ be a sequence of random variables. We say that these random variables are independent if for every sequence of sets $A_1 \in \sigma(X_1)$, $A_2 \in \sigma(X_2)\ldots$ and for every positive integer n, $\mathbb{P}(A_1 \cap A_2 \cap \cdots A_n) = \mathbb{P}(A_1)\mathbb{P}(A_2)\cdots\mathbb{P}(A_n)$ [54]. ■

Definition 1.6. A real-valued stochastic process $W(t)$ is called a Wiener process [2, 46, 54] if:

1. the map $s \longmapsto W(s,\omega)$ is continuous almost surely (a.s.),
2. $W(t) - W(s)$ is $N(0, t-s)$ for all $0 \leq s \leq t$ and
3. if for all times $0 < t_1 < t_2 \cdots < t_n$, the random variables $W(t_1), W(t_2) - W(t_1), \cdots, W(t_n) - W(t_{n-1})$ are independent increments. ■

A Wiener process is said to be standard if it satisfies the following properties:

1. $W(0,\omega) = 0$ almost surely,
2. $EW(t) = 0$ for all $t \geq 0$,
3. $EW^2(t) = t$ for all $t \geq 0$.

Definition 1.7. Let $\{X(t)\}_{t\geq 0}$ be a stochastic process adapted to $\mathcal{F}_t$. Assume that $E\{|X(t)|\} < 0$ for $t \in T$. The process X is a martingale with respect to $\mathcal{F}_t$ if $E\{X(t)|\mathcal{F}_s\} = X(s)$ for each $s < t$ [2]. ■

Definition 1.8. A family of integer valued random variable $X = \{X_n\}_{n=0}^{\infty}$ defined on a probability space is a Markov chain if

$$\mathbb{P}\{X_{n+1} = i_{n+1}|X_n = i_n, \ldots, X_0 = i_0\} = \mathbb{P}\{X_{n+1} = i_{n+1}|X_n = i_n\}$$

for all $n \geq 0$ and $i_0, \ldots, i_{n+1} \in \mathbb{Z}$ [2]. ■

The Markov chain property means that the future X_{n+1} of the stochastic process X depends on the history of the process $X_n, \ldots, X_0$, only through the value of the process right now X_n (but not on how that value was obtained).

Some properties of the Itô integral

The stochastic processes X and Y satisfy the following [2, 46]:

1. The Itô integral $\int_0^t X(r)dW(r)$ is well-defined for $0 \leq t \leq T$,
2. The Itô isometry property: $E\left(\int_0^t X(r)dW(r)\right)^2 = \int_0^t E\left(X^2(r)\right) dr$ for $0 \leq t \leq T$,
3. $E\left(\int_0^t X(r)dW(r) \int_0^t Y(r)dW(r)\right) = \int_0^t E\left(X(r)Y(r)\right) dr$ for $0 \leq t \leq T$,
4. $\int_0^t \left(aX(r) + bY(r)\right) dW(r) = a\int_0^t X(r)dW(r) + b\int_0^t Y(r)dW(r)$ for all $a, b \in \mathbb{R}$ and $0 \leq t \leq T$,
5. $\int_0^t X(r)dW(r)$ is $\mathcal{F}_t$-measurable for $0 \leq t \leq T$,
6. $\left\{\int_0^t X(r)dW(r)\right\}_{t\geq 0}$ is continuous and progressively measurable with probability one.

1.2.3 Itô formula

The key tool to solve SDEs is the Itô formula, which is a stochastic version of the chain rule in calculus. Itô formula is often used in stochastic calculus to find differentials of the stochastic process that depends on time. It may be used to obtain closed form solutions for some SDEs.

Theorem 1.1. (The 1-dimensional Itô formula)

Assume $\Phi(t) : \mathbb{R} \times [0, T] \longrightarrow \mathbb{R}$ is continuous and that $\Phi_t, \Phi_x, \Phi_{xx}$ exist and are continuous. Set

$$\{Y(t) = \Phi(t, X(t))\}.$$

Then Y has the stochastic differential

$$dY = L^0\Phi(t, X(t))dt + L^1\Phi(t, X(t))dW(t), \tag{1.3}$$

where the partial differential operators L^0 and L^1 are defined as

$$L^0 = \frac{\partial}{\partial t} + f\frac{\partial}{\partial x} + \frac{1}{2}g^2\frac{\partial^2}{\partial x^2} \tag{1.4}$$

and

$$L^1 = g\frac{\partial}{\partial x}. \tag{1.5}$$

The first two terms in L^0 correspond to the known chain rule in classical calculus and the last term appears in stochastic calculus since the Wiener process is not of bounded variation. This comes from the fact that the Brownian motion moves too quickly and so second order effects are not negligible. According to the rules of Brownian motion, the stochastic term $dW^2(t) = dt$.

1.2.4 Generalized Itô formula

The Itô formula can be written in n-vector form with the stochastic differential

$$d\mathbf{X}(t) = F(t)dt + G(t)d\mathbf{W}(t), \tag{1.6}$$

where $\mathbf{W}(t) = (W_1(t), \cdots, W_m(t))^T$ is an m-dimensional Wiener process with independent components, $F(t) = (f_1(t), \cdots, f_n(t))^T$ where an n-vector function with probability 1 and

$G(t) = \{g_{ij}(t)\}$ is an $n \times m$-matrix function. In component form (1.6) can be written as

$$dx_i(t) = f_i dt + \sum_{j=1}^{m} g_{ij} dW_j. \tag{1.7}$$

Hence, the Itô formula, which is established similarly to the one-dimensional case, becomes

$$d\Phi = \left(\frac{\partial \Phi}{\partial t} + \sum_{i=1}^{n} f_i \frac{\partial \Phi}{\partial x_i} + \sum_{i,j=1}^{n} \sum_{k=1}^{m} \frac{1}{2} \frac{\partial^2 \Phi}{\partial x_i \partial x_j} g_{ik} g_{jk} \right) dt + \sum_{i=1}^{n} \frac{\partial \Phi}{\partial x_i} \sum_{j=1}^{m} g_{ij} dW_j. \tag{1.8}$$

In vector-matrix notation, (1.8) can be written

$$d\Phi = \left(\Phi_t + f\Phi_x + \frac{1}{2} tr(gg^T)\Phi_{xx} \right) dt + g\Phi_x dW, \tag{1.9}$$

where tr denotes the trace operator and $tr(gg^T) = \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} g_{ij}^2$.

1.2.5 Solving SDEs

SDEs are classified into two large groups: linear and nonlinear. Linear SDEs are explicitly solvable. Solving the SDEs requires an existence and uniqueness theorem.

Theorem 1.2. (Existence and Uniqueness) Let $T > 0$ and let f, g ($f : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}, g : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$) be measurable functions satisfying a linear growth condition

$$|f(t,x)| + |g(t,x)| \leq k_1(1 - |x|) \quad \text{for all } x \in \mathbb{R}, t \in [0,T],$$

for some positive constant k_1. To ensure uniqueness, the Lipschitz condition,

$$|f(t,x) - f(t,y)| + |g(t,x) - g(t,y)| \leq k_2 |x - y| \quad \text{for all } x \in \mathbb{R}^n, t \in [0,T],$$

is sufficient for some positive constant k_2. Let X_0 be a random variable which is independent of $\mathcal{F}_T^W$, the σ-algebra generated by $(W(s) : 0 \leq s \leq T)$, and $E|X_0|^2 < \infty$. Then the SDE (1.1) has a unique t-continuous solution $X(t)$ with the properties that

1. $X(t)$ is adapted to $\mathcal{F}_T^W \vee \sigma(X_0)$;
2. $E\left(\int_0^T |X(t)|^2 dt \right) < \infty$.

This ensures that $X(t)$ does not tend to ∞ in finite time. Such an existence and uniqueness theorem also exists in the multidimensional case.

1.2.6 Types of solutions

Definition 1.9. A strong solution of the SDE (1.1) is a stochastic process X with continuous sample paths and the following properties [2]:

1. X is adapted to the filtration $(\mathcal{F})$

2. $\mathbb{P}[X_0 = 0] = 1$

3. $\mathbb{P}\left[\int_0^t \big(|f(s, X(s))| + g^2(s, X(s))\big)\, ds < \infty\right] = 1, \quad 0 \le t < \infty.$ ■

Definition 1.10. A weak solution of the SDE (1.1) is a triple $\mathcal{F}_t$, (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$ where

1. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and $\mathcal{F}_t$ is a filtration of σ-subalgebra of $\mathcal{F}$ satisfying the usual conditions.

2. $X = (X(t), \mathcal{F}_t)_{0 \le t < \infty}$ is a continuous, adapted $\mathbb{R}^n$-valued process and (3) above, the stochastic version of equation (1.2) are satisfied [2]. ■

   Therefore, a solution is a weak solution if it is valid for given coefficients but with an unspecified Wiener process. This implies that its probability law is unique .

Remark: A strong solution is obtained if the driving Wiener process is given in advance as part of the problem such that the obtained solution to the SDE (1.1) is $\mathcal{F}_t$-adapted where $\mathcal{F}_t$ is the σ-algebra generated by a Wiener process. A strong solution is also a weak solution but the converse is not true in general.

1.2.7 Linear SDEs

If the functions $f(t, x)$ and $g(t, x)$ are linear with respect to the variable x, that is, $f(t, x) = f_1(t)X(t) + f_2(t)$ and $g(t, x) = g_1(t)X(t) + g_2(t)$, then

$$dX(t) = [f_1(t)X(t) + f_2(t)]dt + [g_1(t)X(t) + g_2(t)]dW(t) \tag{1.10}$$

is called a linear SDE. A linear SDE is autonomous if all coefficients are constants, homogeneous if $f_2 = g_2 = 0$ and in a narrow sense linear when $g_1 = 0$.

Theorem 1.3. The solution $X(t)$ of the non-homogeneous linear SDE (1.10) can be written as

$$X(t) = \gamma(t)\left[X_0 + \int_0^t (f_2(s) - g_1(s)g_2(s))\,\gamma^{-1}(s)ds + \int_0^t g_2(s)\gamma^{-1}(s)dW(s)\right], \qquad (1.11)$$

where

$$\gamma(t) = \exp\left[\int_0^t \left(f_1(s) - \frac{1}{2}g_1^2(s)\right)ds + \int_0^t g_1(s)dW(s)\right]. \qquad (1.12)$$

In case of linear SDE in the narrow sense, equation (1.1) reduces to

$$X(t) = \gamma(t)\left(X_0 + \int_0^t f_2(s)\gamma^{-1}(s)ds + \int_0^t g_2\gamma^{-1}(s)dW(s)\right) \qquad (1.13)$$

where

$$\gamma(t) = \exp\left(\int_0^t f_1(s)ds\right). \qquad (1.14)$$

1.2.8 Reducing scalar SDEs

Suppose a smooth function $\Phi(t,x)$ has an inverse $\delta(t,y)$ such that $\Phi(t,\delta(t,x)) = x$ and $\delta(t,\Phi(t,x)) = x$. Applying the Itô formula to (1.2) the stochastic process $Y(t) = \Phi(t, X(t))$ satisfies the equation

$$dY(t) = \bar{f}(t, Y(t))dt + \bar{g}(t, Y(t))dW(t), \qquad (1.15)$$

where

$$\bar{f}(t,y) = \left(\Phi_t + f\Phi_x + \frac{1}{2}tr(gg^T)\Phi_{xx}\right)(t, \delta(t,y)) \qquad (1.16)$$

and

$$\bar{g}(t,y) = (g\Phi_x)(t, \delta(t,y)). \qquad (1.17)$$

Equation (1.2) is said to be reducible if such a function Φ can be found such that the functions, $\bar{f}$ and $\bar{g}$ given by (1.16) and (1.17) respectively are independent of y.

The problem of finding reducible SDEs using the Itô formula is solvable if a function $\Phi(t,x)$ satisfies the conditions:

$$\left(\Phi_t + f\Phi_x + \frac{1}{2}g^2\Phi_{xx}\right)(t,x) = \bar{f}(t), \tag{1.18}$$

and

$$\left(g\Phi_x\right)(t,x) = \bar{g}(t), \tag{1.19}$$

where $g \neq 0$. Re-writing equation (1.19) we obtain

$$\Phi_x = \frac{\bar{g}}{g}. \tag{1.20}$$

Differentiating equation (1.18) and (1.20) with respect to x gives

$$\Phi_{xt} + \frac{\partial}{\partial x}\left(f\Phi_x + \frac{1}{2}g^2\Phi_{xx}\right) = 0, \tag{1.21}$$

and

$$\Phi_{xx} = -\frac{\bar{g}g_x}{g^2}. \tag{1.22}$$

Differentiating equation (1.20) with respect to t gives

$$\Phi_{xt} = \frac{\bar{g}\prime g - \bar{g}g_t}{g^2}. \tag{1.23}$$

Substituting (1.20), (1.22) and (1.23) into (1.21) gives

$$\frac{\bar{g}\prime}{g} - \frac{\bar{g}}{g^2}g_t + \bar{g}\frac{\partial}{\partial x}\left(\frac{f}{g}\right) - \frac{1}{2}\bar{g}g_{xx} = 0.$$

Therefore,

$$\frac{\bar{g}\prime}{\bar{g}} = g\left[\frac{1}{g^2}g_t - \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{1}{2}g_{xx}\right]. \tag{1.24}$$

Since the left side of (1.24) is independent of x, it follows that

$$\frac{\partial}{\partial x}\left[g\left(\frac{1}{g^2}g_t - \frac{\partial}{\partial x}\left(\frac{f}{g}\right) + \frac{1}{2}g_{xx}\right)\right] = 0. \tag{1.25}$$

If (1.25) holds then Φ can be computed from (1.20) and $\bar{g} \neq 0$ can be determined from (1.24). Since (1.24) and (1.21) are equivalent, the function $\bar{f}$ obtained in this case by (1.16) is independent of x. Hence the following theorem holds:

Theorem 1.4.

Equation (1.2) is reducible if and only if the coefficient functions f and g satisfy (1.25).

Remark: Linear equations are in general irreducible. A homogeneous SDE and narrow-sense linear SDEs are reducible if condition (1.25) satisfies

$$g_1' g_2 + g_1(f_1 g_2 - f_2 g_1 - g_2') = 0.$$

Nonhomogeneous linear SDEs can be solved explicitly by a variation of parameters method as in ODEs but in general are also not reducible.

Reducible conditions were used in [62] to find the the invertible transformations that linearize first-order SODEs. The conditions were then extended to second-order SODEs [40] and to jump diffusion second-order SODEs [49].

1.3 Lie group analysis

The Lie group theory is a very general and useful tool for finding analytical solutions of large classes of differential equations. The concept of symmetries of differential equations was introduced by Sophus Lie in 1870. Lie [36] provided a group classification of linear second-order partial differential equations in two independent variables. He [37] further used group theoretical methods to provide a classification of all ordinary differential equations of arbitrary order in terms of their symmetry group. Later this classification was obtained in a different way [47] by directly solving the determining equations and exploiting the equivalence transformations.

The application of Lie group analysis to Itô SODEs has been successfully performed [1, 43, 65, 59, 17, 18, 31, 32]. It has since been applied to stochastic dynamical systems [1, 43] and to the Fokker-Planck equations [19]. However, these approaches considered the restricted cases of point transformations only. An algorithm for obtaining Lie point symmetries for both the first-order and n-th order SODEs can be found in [65]. Some of the symmetries obtained in [65] were not projectable and hence do not belong to the Lie algebra associated with the Fokker-Planck equations. Ünal [59] included a Brownian motion in his transformation. Unfortunately, there was no proof that the transformation of Brownian motion satisfies the properties of the Brownian motion [39]. A Lie group of transformations was constructed that involves both the dependent and the Brownian motion in the transformations in [39]. This work was reviewed and corrected in [18].

1.3.1 Stochastic integrals as time change of Wiener process

We provide the mathematical tools required for defining the transformation of Brownian motion. The constructions below are similar to those in [46].

Let $\eta(t,\omega)$ be the time change rate and $\alpha(t,\omega)$ be a scalar stochastic process satisfying

- $\alpha(0,\omega) = 0$,
- $d\alpha(t,\omega)/dt = 1/\eta(\alpha(t),\omega) \geq 0$, for almost all positive time and almost all $\omega \in \Omega$,
- There is a stochastic process $\beta(t,\omega)$, such that $\alpha(\beta(t,\omega),\omega) = \beta(\alpha(t,\omega),\omega) = t$ for all $(t,\omega) \in T \times \Omega$.

Then, under the (random) time change $\bar{t} = \alpha(t,\omega)$, the Wiener process $W(t)$ is mapped to another process $\bar{W}(\bar{t})$ according to the relation:

$$d\bar{W}(\bar{t}) = \sqrt{\frac{d\alpha(t)}{dt}}dW(t).$$

1.3.2 Determining equations

Consider the infinitesimal group of transformations

$$\bar{t} = \bar{t}(t,\mathbf{x},\varepsilon) \approx t + h(t,\mathbf{x})\varepsilon, \quad \bar{x}_i = \bar{x}_i(t,\mathbf{x},\varepsilon) \approx x_i + \xi_i(t,\mathbf{x})\varepsilon, \tag{1.26}$$

which leaves equation (1.2) and the framework of Itô calculus invariant. The general form of the infinitesimal generator is

$$H = h(t,\mathbf{x})\frac{\partial}{\partial t} + \xi_i(t,\mathbf{x})\frac{\partial}{\partial \mathbf{x}} \ \ (i = 1,\ldots,n). \tag{1.27}$$

The infinitesimal generator is used in the transformation of the drift and diffusion coefficients of equation (1.2). The Lie point theorem symmetry approach for ODEs requires the coefficients of the admitted generator h and ξ_i in its analysis. In the SODEs framework these entities are functionally based on the stochastic process $\{X(t)\}$. The determining equations for the coefficients of the admitted generator $h = h(t)$ and $\xi_i = \xi_i(t, \mathbf{x})$ are [59]:

$$\xi_{i,t} + \xi_{i,j} f_j + \tfrac{1}{2}\xi_{i,jl} g_j g_l - f_{i,t} h - f_{i,j}\xi_j - f_i h' = 0, \tag{1.28}$$

$$\xi_{i,j} g_j - g_{i,t} h - \frac{1}{2} g_i h' - g_{i,j}\xi_j = 0. \tag{1.29}$$

The determining equations (1.28) and (1.29) for an admitted Lie group of transformation were constructed under the assumption that transformations (1.27) transform any solution of equation (1.2) into a solution of the same equation.

1.4 Outline

The rest of this thesis is organized as follows: In Chapter 2, we use algebraic methods and find the group classification of systems of two linear second-order ODEs. The algebraic approach enables one to identify the class of differential equations to which the system presented by particular model belongs. This enables one to construct generalizations based upon the algebraic properties of the original system. In Chapter 3, we present the linearization criteria for systems of two second-order SODEs. One of the main reasons for studying the linearization problem is the possibility of finding the general solution or a class of transformations that can be applied to systems of equations to lead to their general solution(s). Chapter 4, the underlying group theoretic properties of a system of two linear second-order SODEs with constant coefficients is considered. We additionally solved a group classification problem as it applies to stochastic processes. We conclude with a discussion and future research in Chapter 5.

Chapter 2

Complete Group Classification of Systems of Two Linear Second-Order Ordinary Differential Equations: The Algebraic Approach

2.1 Introduction

Systems of second-order ordinary differential equations appear in the modeling of many physical phenomena. A main feature of these differential equations is their symmetry properties. The theory of group analysis has been well studied in the literature. The presence of symmetries allows one to reduce the order of these equations or even find a general solution in quadratures.

Linear equations play a significant role among all ordinary differential equations. They are considered as a first approximation of the model being studied. In applications, linear equations often occur in disguised forms. In the study of the symmetries, it is convenient to rewrite the equations in their simplest equivalent form. We note that equations equivalent with respect to a change of the dependent and independent variables possess similar symmetry properties. This leads to a classification problem.

Systems of two linear second-order ordinary differential equations were studied in [64] where a new canonical form,

$$y'' = a(x)y + b(x)z$$
$$z'' = c(x)y - a(x)z,$$

was obtained. For this canonical form, the number of arbitrary elements is reduced. The group classification problem is usually simpler after reducing the number of arbitrary elements. In this paper, the authors also gave a representation of several admitted Lie groups. In addition it was also proved that a system of two linear second-order ordinary differential equations can have 5, 6, 7, 8 or 15 point symmetries. However the exhaustive list of all distinguished representatives of systems of two linear second-order ordinary differential equations was not obtained there.

The main objective of this paper is to use the algebraic approach where the determining equations presented in [44] are solved up to finding relations between constants defining the admitted generators.

The algebraic approach takes into account algebraic properties of Lie groups admitted by a system of equations: the knowledge of the algebraic structure of admitted Lie algebras allows for significant simplification of the group classification. In particular, the group classification of a single second-order ordinary differential equation, done by the founder of the group analysis method, S. Lie [36, 37], cannot be performed without using the algebraic structure of admitted Lie groups. Recently, the algebraic properties for group classification were applied in [15, 7, 51, 52, 53, 12, 27, 22]. We also note that the use of the algebraic structure of admitted Lie groups completely simplified the group classification of equations describing behavior of fluids with internal inertia in [55, 63].

In the present paper, we obtain a complete group classification of the general case of linear systems of two second-order ordinary differential equations,

$$y'' = F(x, y, z), \quad z'' = G(x, y, z),$$

by using an algebraic approach. The system considered in this case is a generalization of Lie's study [37]. Excluded from our consideration is the studied earlier systems of second-order ordinary differential equations with constant coefficients of the form

$$\mathbf{y}'' = M\mathbf{y}, \tag{2.1}$$

where M is a matrix with constant entries and $\mathbf{y} = \begin{pmatrix} y \\ z \end{pmatrix}$. These cases of systems have been studied in [8, 67, 41, 10, 11]. We also exclude from the analysis the degenerate case given as follows:

$$y'' = F(x, y, z), \;\; z'' = 0. \tag{2.2}$$

It is worth mentioning here that the complete group classification of two linear second-order ordinary differential equations has been done recently in [44]. The following four cases of linear systems of equations with none inconstant coefficients were obtained:

$$F = \alpha_{11}y + e^{x}z, \;\; G = e^{-x}\alpha_{21}y + \alpha_{22}z, \tag{2.3}$$

$$\begin{aligned} F &= y(\sin(x) + c_2) + z(\cos(x) - c_1), \\ G &= y(\cos(x) + c_1) + z(-\sin(x) + c_2), \end{aligned} \tag{2.4}$$

$$\begin{aligned} F &= y(\alpha_{11} + x) + z(\alpha_{12} + (\alpha_{22} - \alpha_{11})x - x^2), \\ G &= y + z(-x + \alpha_{22}), \end{aligned} \tag{2.5}$$

$$F = yc + z, G = -y + zc \tag{2.6}$$

where α_{ij} and c_i $(i, j = 1, 2)$ are constant, $c = c(x)$ and $\alpha_{21}c' \neq 0$. These systems have the following nontrivial admitted generators:

System	Admitted Generator
(2.3)	$\partial_x - z\partial_z$
(2.4)	$2\partial_x + z\partial_y - y\partial_z$
(2.5)	$\partial_x + z\partial_y$
(2.6)	$z\partial_y - y\partial_z$.

We note that the approach used in [44] is different from the approach used in the present paper. Since there is an opinion that the algebraic approach is more efficient, the present paper can be a good example for comparing these two approaches. We show here that for the problem of classifying systems of two linear second-order ordinary differential equations the algebraic approach leads to the study of a variety of cases, although the analysis of these cases is not complicated.

The paper is organized as follows: The first part of the paper deals with the preliminary study of systems of two second-order linear equations followed by the group classification method as applied to linear systems of equations. The subsequent subsections deal with the equivalence transformations, determining equations and the optimal system of subalgebras. The later part lists the different cases with their respective results. This is then followed by the results and conclusion.

2.2 Preliminary study of systems of linear equations

Linear second-order ordinary differential equations have the following form,

$$\mathbf{y}'' = B(x)\mathbf{y}' + A(x)\mathbf{y} + f(x), \tag{2.7}$$

where $A(x)$ and $B(x)$ are $n \times n$ matrices, and $f(x)$ is a vector. Using a particular solution $\mathbf{y}_p(x)$ and the change of variable,

$$\mathbf{y} = \widetilde{\mathbf{y}} + \mathbf{y}_p,$$

one can without loss of generality assume that $f(x) = 0$. Applying the change

$$\mathbf{y} = C(x)\bar{\mathbf{y}},$$

where $C = C(x)$ is a nonsingular matrix, system (2.7) becomes

$$\bar{\mathbf{y}}'' = \bar{B}\bar{\mathbf{y}}' + \bar{A}\bar{\mathbf{y}}, \tag{2.8}$$

where

$$\bar{B} = C^{-1}(BC - 2C'), \quad \bar{A} = C^{-1}(AC + BC' - C'').$$

If one chooses the matrix $C(x)$ such that

$$C' = \frac{1}{2}BC,$$

then

$$\bar{B} = 0, \quad \bar{A} = C^{-1}\left(A + \frac{1}{4}B^2 - \frac{1}{2}B'\right)C. \tag{2.9}$$

The existence of the nonsingular matrix $C(x)$ is guaranteed by the existence of the solution of the Cauchy problem,

$$C' = \frac{1}{2}BC, \quad C(0) = E,$$

where E is the unit $n \times n$ matrix.

If the matrices A and B are constant, then the matrix $\bar{A}$ in (2.9) is constant only for commuting matrices A and B. The complete study of noncommutative constant matrices, A and B, was done recently in [42]. Without loss of generality, up to equivalence transformations in the class of systems of the form (2.7), it suffices to study the systems of the form

$$\mathbf{y}'' = A\mathbf{y}. \tag{2.10}$$

Applying the change of the dependent and independent variables [44]

$$\tilde{x} = \varphi(x), \quad \tilde{\mathbf{y}} = \psi(x)\mathbf{y} \tag{2.11}$$

satisfying the condition

$$\frac{\varphi''}{\varphi'} = 2\frac{\psi'}{\psi}, \tag{2.12}$$

system (2.10) becomes

$$\tilde{\mathbf{y}}'' = \tilde{A}\tilde{\mathbf{y}}, \tag{2.13}$$

where

$$\tilde{A} = \varphi'^{-2}\left(A - \frac{\rho''}{\rho}E\right), \quad \rho = \frac{1}{\psi}.$$

For reducing the number of entries of the matrix $\tilde{A}$, one can choose the function ψ such that[1] $tr\,\tilde{A} = 0$. This condition leads to the equation

$$\rho'' - \frac{tr\,A}{n}\rho = 0. \tag{2.14}$$

[1] This change was used in [64] for the case of $n = 2$

In particular, for matrices with $tr\,A = 0$ choosing $\rho = c_1 x + c_2$, the matrix $\tilde{A}$ still satisfies the condition $tr\,\tilde{A} = 0$. Here,

$$\psi = (c_1 x + c_2)^{-1}, \quad \varphi' = k_0 \psi^2 = k_0 (c_1 x + c_2)^{-2}, \tag{2.15}$$

where k_0 is constant.

2.3 Group classification

For the group classification of systems of two linear second-order ordinary differential equations, we consider a system of equations (2.10) with a matrix

$$A = \begin{pmatrix} a(x) & b(x) \\ c(x) & -a(x) \end{pmatrix}.$$

Any linear system of ordinary differential equations (2.10) admits the following trivial generators

$$y\partial_y + z\partial_z, \tag{2.16}$$

$$h(x)\partial_y + g(x)\partial_z, \tag{2.17}$$

where (2.16) is the homogeneity symmetry and $(h(x), g(x))^t$ is any solution of system (2.10). For the classification problem one needs to study equations which admit generators different from (2.16) and (2.17).

2.3.1 Equivalence transformations

Calculations show that the equivalence Lie group is defined by the generators:

$$X_1^e = x(x\partial_x + y\partial_y + z\partial_z - 4a\partial_a - 4b\partial_b - 4c\partial_c),$$
$$X_2^e = 2x\partial_x + y\partial_y + z\partial_z - 4a\partial_a - 4b\partial_b - 4c\partial_c,$$
$$X_3^e = \partial_x, \quad X_4^e = y\partial_z - b\partial_a + 2a\partial_c, \quad X_5^e = z\partial_y + c\partial_a - 2a\partial_b,$$
$$X_6^e = y\partial_y - z\partial_z - 2b\partial_b + 2c\partial_c, \quad X_7^e = y\partial_y + z\partial_z.$$

The transformations corresponding to the generator X_1^e define transformations of the form (2.15). The transformation corresponding to X_2^e and X_3^e define the dilation and shift of x, respectively. The transformations corresponding to the generator X_4^e, X_5^e,X_6^e and X_7^e correspond to the linear change of the dependent variables $\widetilde{\mathbf{y}} = P\mathbf{y}$ with a constant nonsingular matrix P.

2.3.2 Determining equations

According to the Lie algorithm [47], the generator

$$X = \xi(x, y, z)\frac{\partial}{\partial x} + \eta_1(x, y, z)\frac{\partial}{\partial y} + \eta_2(x, y, z)\frac{\partial}{\partial z}$$

is admitted by system (2.10) if it satisfies the associated determining equations. One can show that the admitted generator has the property that $\xi_y^2 + \xi_z^2 \neq 0$ if and only if system (2.10) is equivalent to free particle equations [44]. Hence, one obtains $\xi = \xi(x)$. The determining equations are

$$\begin{aligned}&b(\xi'z + zq_4 + yq_3) + a(\xi'y + zq_2 + yq_1) + 2(a'y + b'z)\xi - \xi'''y + (3\xi' - q_1)(ay + bz)\\&-q_1(cy - az) = 0,\end{aligned}$$

and

$$\begin{aligned}&-a(\xi'z + zq_4 + yq_3) + c(\xi'y + zq_1 + yq_2) + 2(c'y - a'z)\xi - \xi'''z - q_3(ay + bz)\\&+(3\xi' - q_4)(cy - az) = 0,\end{aligned}$$

where an admitted generator has the form

$$X = 2\xi(x)\partial_x + (y\xi'(x) + q_1y + q_2z)\partial_y + (\xi'z + q_3y + q_4z)\partial_z$$

and q_i, $(i = 1, ..., 4)$ are constant. We exclude the trivial admitted generators (2.17).

Splitting the determining equations with respect to y and z leads to $\xi''' = 0$ and the equations[2]

$$2a'\xi + 4a\xi' + bq_3 - cq_1 = 0, \tag{2.18}$$

$$2b'\xi + 2aq_1 + b(4\xi' + q_4 - q_2) = 0, \tag{2.19}$$

[2]These equations coincide with equations (32)-(34) of [64], where the constants from [64] are $s_0 = q_1,\ r_0 = q_2,\ p_0 = q_3,\ q_0 = q_4$. The difference is that in our study, there is no necessity at this stage of the assumption $b \neq 0$ comparing with [64].

$$2c'\xi - 2aq_3 + c(4\xi' - q_4 + q_2) = 0. \tag{2.20}$$

Since $\xi^{(3)} = 0$ or $\xi = a_1 x^2 + a_2 x + a_3$, the admitted generators have the form

$$X = a_1 X_1 + a_2 X_2 + a_3 X_3 + q_3 X_4 + q_1 X_5 + \frac{q_2 - q_4}{2} X_6 + \frac{q_2 + q_4}{2} X_7,$$

where

$$X_1 = x(x\partial_x + y\partial_y + z\partial_z), \quad X_2 = 2x\partial_x + y\partial_y + z\partial_z, \quad X_3 = \partial_x,$$
$$X_4 = y\partial_z, \quad X_5 = z\partial_y, \quad X_6 = y\partial_y - z\partial_z, \quad X_7 = y\partial_y + z\partial_z.$$

In addition, since the generator X_7 is the trivial admitted generator (2.16), one can assume that $q_4 = -q_2$. The constants a_1, a_2, a_3, q_1, q_2 and q_3 depend on the functions $a(x)$, $b(x)$ and $c(x)$. These relations are defined by equations (2.18)-(2.20), and they present the group classification of linear systems of two second-order ordinary differential equations.

One of the methods for analyzing relations between the constants consists of employing the algorithm developed for the gas dynamics equations [47]. This algorithm allows one to study all possible admitted Lie algebras without omission. Unfortunately, it is difficult to implement for system (2.10). Observe also that in this approach it is difficult to select out equivalent cases with respect to equivalence transformations.

In [53, 27, 22][3] a different approach was applied for group classification. In most applications the algebraic algorithm essentially reduces the study of group classification to a simpler problem. Here, we follow this approach. For further analysis we study the Lie algebra L_6 spanned by the generators $X_1, X_2, ..., X_6$.

[3]See also references therein.

2.3.3 Optimal system of subalgebras of L_6

The Lie algebras $L_6 = L_3^{(1)} \oplus L_3^{(2)}$, where $L_3^{(1)} = \{X_1, X_2, X_3\}$, $L_3^{(2)} = \{X_4, X_5, X_6\}$. The commutator table can be split into two tables:

	X_1	X_2	X_3
X_1	0	$-2X_1$	$-X_2$
X_2	$2X_1$	0	$-2X_3$
X_3	X_2	$2X_3$	0

	X_4	X_5	X_6
X_4	0	X_6	$-2X_4$
X_5	$-X_6$	0	$2X_5$
X_6	$2X_4$	$-2X_5$	$0.$

Denoting

$$X_1 = e_1, \quad X_2 = -2e_2, \quad X_3 = e_3,$$

one can show that the commutator table of the algebra $L_3^{(1)}$ becomes

	e_1	e_2	e_3
e_1	0	e_1	$2e_2$
e_2	$-2e_2$	0	e_3
e_3	$-2e_2$	$-e_3$	$0.$

Hence, the Lie algebra [4] $L_3^{(1)}$ is sl(2,R). One also can check that $L_3^{(2)}$ is sl(2,R) by denoting

$$X_4 = -e_1, \quad X_5 = e_3, \quad X_6 = -2e_2.$$

Notice that an optimal system of subalgebras of the Lie algebra sl(2,R) classification was performed in [50] and it consists of the following list:

$$\{e_2\}, \quad \{e_3\}, \quad \{e_1 + e_3\}, \quad \{e_2, e_3\}, \quad \{e_1, e_2, e_3\}. \tag{2.21}$$

Then, the optimal systems of subalgebras of $L_3^{(1)}$ and $L_3^{(2)}$ are

$$\{X_2\}, \quad \{X_3\}, \quad \{X_1 + X_3\}, \quad \{X_2, X_3\}, \quad \{X_1, X_2, X_3\} \tag{2.22}$$

[4] This Lie algebra is Lie algebra of type VIII in the Bianchi classification.

and

$$\{X_5\}, \quad \{X_6\}, \quad \{X_4 - X_5\}, \quad \{X_5, X_6\}, \quad \{X_4, X_5, X_6\}, \tag{2.23}$$

respectively.

2.3.4 Relations between automorphisms and equivalence transformations

Let us consider an operator

$$X = x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6$$

Automorphisms of the Lie algebra L_6 are

$$\begin{aligned}
A_1 :&\quad \bar{x}_1 = x_1 + 2ax_2 + a^2x_3,\ \bar{x}_2 = x_2 + ax_3;\\
A_2 :&\quad \bar{x}_1 = x_1e^a,\ \bar{x}_3 = x_3e^{-a};\\
A_3 :&\quad \bar{x}_2 = x_2 + ax_1,\ \bar{x}_3 = x_3 + 2ax_2 + a^2x_1;\\
A_4 :&\quad \bar{x}_4 = x_4 - 2ax_6 - a^2x_5,\ \bar{x}_6 = x_6 + ax_5;\\
A_5 :&\quad \bar{x}_5 = x_5 + 2ax_6 - a^2x_4,\ \bar{x}_6 = x_6 - ax_4;\\
A_6 :&\quad \bar{x}_4 = x_4e^a,\ \bar{x}_5 = x_5e^{-a}.
\end{aligned}$$

Here and further on, only changeable coordinates are presented.

One can show that actions of equivalence transformations are similar to actions of the automorphisms. These properties allow one to use an optimal system of subalgebras of the Lie algebra L_6 for group classification.

In fact, using the change of the dependent and independent variables corresponding to the equivalence transformation (2.11) with

$$\varphi = \frac{x}{1 - \tau x}, \qquad \psi = (x + \tau)^{-1},$$

the operator

$$X = x_1X_1 + x_2X_2 + x_3X_3 + x_4X_4 + x_5X_5 + x_6X_6$$

becomes

$$X = \bar{x}_1\bar{X}_1 + \bar{x}_2\bar{X}_2 + \bar{x}_3\bar{X}_3 + \bar{x}_4\bar{X}_4 + \bar{x}_5\bar{X}_5 + \bar{x}_6\bar{X}_6,$$

where

$$\bar{X}_1 = \bar{x}(\bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}} + \bar{z}\partial_{\bar{z}}), \quad \bar{X}_2 = 2\bar{x}\partial_{\bar{x}} + \bar{y}\partial_{\bar{y}} + \bar{z}\partial_{\bar{z}}, \quad \bar{X}_3 = \partial_{\bar{x}},$$
$$\bar{X}_4 = \bar{y}\partial_{\bar{z}}, \quad \bar{X}_5 = \bar{z}\partial_{\bar{y}}, \quad \bar{X}_6 = \bar{y}\partial_{\bar{y}} - \bar{z}\partial_{\bar{z}},$$

and the changeable coefficients are

$$\bar{x}_1 = x_1 + 2x_2\tau + x_3\tau^2, \quad \bar{x}_2 = x_2 + x_1\tau.$$

Hence, the change of the dependent and independent variables corresponding to the equivalence transformation X_1^e is similar to the automorphism Aut_1. This property we denote as $X_1^e\ Aut_1$. Similarly, one can check that $X_i^e \cong Aut_i$, $(i = 2, 3, ..., 6)$. Using the two-step algorithm of constructing an optimal system of subalgebras [48], and the optimal systems of subalgebras (2.22) and (2.23), one obtains an optimal system of one-dimensional subalgebras of the Lie algebra L_6 which consists of the following set of subalgebras

1.1.	$X_2 + \gamma(X_4 - X_5)$	3.1.	$X_1 + X_3 + \gamma(X_4 - X_5)$
1.2.	$X_2 + \gamma X_5$	3.2.	$X_1 + X_3 + \gamma X_5$
1.3.	$X_2 + \gamma X_6$	3.3.	$X_1 + X_3 + \gamma X_6$
2.1.	$X_3 + \gamma(X_4 - X_5)$	4.1.	$X_4 - X_5$
2.2.	$X_3 + \gamma X_5$	4.2.	X_5
2.3.	$X_3 + \gamma X_6$	4.3.	X_6

where γ is an arbitrary constant.

2.4 Solutions of the determining equations

We obtained the condition that for the group classification one needs to construct solutions of equations (2.18)-(2.20), where the constants are

$$a_1 = x_1, \ a_2 = x_2, \ a_3 = x_3, \ q_3 = x_4, \ q_1 = x_5, \ q_2 = x_6, \ q_4 = -x_6.$$

Here, x_i $(i = 1, 2, ..., 7)$ are coordinates of the generator

$$X = x_1 X_1 + x_2 X_2 + x_3 X_3 + x_4 X_4 + x_5 X_5 + x_6 X_6$$

chosen from the optimal system of subalgebras.

Note that the subalgebra with the generator X_3 corresponds to equations with constant coefficients. One can also check that using equivalence transformation (2.11) with corresponding function $\bar{x} = \psi(x)$, the generators presented in the optimal system of subalgebras for $\gamma = 0$ are reduced to the generator $\bar{X}_3 = \partial_{\bar{x}}$. Hence, we only need to consider the cases where $\gamma \neq 0$.

Subalgebra 1.1 with generator $X_2 + \gamma(X_4 - X_5)$.

In this case, equations (2.18)-(2.20) become

$$\begin{aligned} 2xa' + 4a + \gamma(b + c) = 0, \\ xb' + 2b - \gamma a = 0, \\ xc' + 2c - \gamma a = 0. \end{aligned} \tag{2.24}$$

Applying the change,

$$a = x^{-2}\bar{a}, \quad b = x^{-2}\bar{b}, \quad c = x^{-2}\bar{c},$$

equation (2.24) is reduced to the equations

$$\begin{aligned} 2x\bar{a}' + \gamma(\bar{b} + \bar{c}) = 0, \\ x\bar{b}' - \gamma\bar{a} = 0, \\ x\bar{c}' - \gamma\bar{a} = 0. \end{aligned} \tag{2.25}$$

From the last two equations of (2.25), one finds that $\bar{b} = \bar{c} + k$, where k is a constant. Denoting $\bar{c} = \tilde{c} - k/2$, the remaining equations of (2.25) become

$$\begin{aligned} x\bar{a}' + \gamma\tilde{c} = 0, \\ x\tilde{c}' - \gamma\bar{a} = 0. \end{aligned} \tag{2.26}$$

Then, from the second equation of (2.26), one finds

$$\bar{a} = \gamma^{-1} x \tilde{c}'.$$

The first equation of (2.26) becomes

$$x^2\bar{a}'' + x\bar{a}' - \gamma^2\bar{a} = 0.$$

This is the Euler type equation with general solution

$$\bar{a} = k_1 \sin(\gamma \ln x) + k_2 \cos(\gamma \ln x).$$

Finally, we obtain

$$a = \frac{k_1 \sin(\gamma \ln x) + k_2 \cos(\gamma \ln x)}{x^2}, \quad b = \frac{k - 2k_1 \cos(\gamma \ln x) + 2k_2 \sin(\gamma \ln x)}{2x^2},$$
$$c = \frac{-k - 2k_1 \cos(\gamma \ln x) + 2k_2 \sin(\gamma \ln x)}{2x^2}.$$

We note that this case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.4).

Subalgebra 1.2 with generator $X_2 + \gamma X_5$.

Equations (2.18)-(2.20) in this case are reduced to

$$\begin{aligned} 2xa' + 4a - \gamma c &= 0, \\ xb' + 4b + \gamma a &= 0, \\ xc' + 2c &= 0. \end{aligned} \tag{2.27}$$

Applying the change

$$a = x^{-2}\bar{a}, \quad b = x^{-2}\bar{b}, \quad c = x^{-2}\bar{c},$$

equations (2.27) become

$$\begin{aligned} 2x\bar{a}' - \gamma k &= 0, \\ x\bar{b}' + \gamma\bar{a} &= 0, \end{aligned} \tag{2.28}$$

where $\bar{c} = k$, k is a constant.

From the first equation of (2.28), we have

$$\bar{a}' = \frac{\gamma k}{2x}.$$

From the second equation of (2.28), one obtains

$$x^2\bar{b}'' + x\bar{b}' + \frac{\gamma^2 k}{2} = 0 \tag{2.29}$$

of which the general solution is

$$\bar{b} = -\frac{\gamma^2 k}{4}(\ln\, x)^2 + k_2 \ln\, x + k_3.$$

Finally,

$$a = \frac{k_1 + \gamma k \ln\, x}{2x^2}, \quad b = \frac{k_3 + 4k_2 \ln\, x - \gamma^2 k(\ln\, x)^2}{4x^2}, \quad c = \frac{k}{x^2}.$$

This case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.5).

Subalgebra 1.3 with the generator $X_2 + \gamma X_6$.

Equations (2.18)-(2.20) can be expressed in this form

$$\begin{aligned} xa' + 2a &= 0, \\ xb' + (2-\gamma)b &= 0, \\ xc' + (\gamma+2)c &= 0. \end{aligned} \tag{2.30}$$

Solving (2.30), one obtains

$$a = \frac{k_1}{x^2}, \quad b = \frac{k_2}{x^{2-\gamma}}, \quad c = \frac{k_3}{x^{2+\gamma}}.$$

It is observed that this case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.3).

Subalgebra 2.1 with the generator $X_3 + \gamma(X_4 - X_5)$.

In this case, we have

$$\begin{aligned} 2a' + \gamma(b+c) &= 0, \\ b' - \gamma a &= 0, \\ c' - \gamma a &= 0. \end{aligned} \tag{2.31}$$

From equations (2.31), we get

$$a'' + \gamma^2 a = 0. \tag{2.32}$$

Then, the solution is

$$a = k_1 \sin(\gamma x) + k_2 \cos(\gamma x), \quad b = k_1 \gamma x \sin(\gamma x) + k_2 \gamma x \cos(\gamma x) + k,$$

$$c = k_1 \gamma x \sin(\gamma x) + k_2 \gamma x \cos(\gamma x) - k,$$

where k_1 and k_2 are constant.

We note that this case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.4).

Subalgebra 2.2 with the generator $X_3 + \gamma X_5$.

We can write equations (2.18)-(2.20) as

$$\begin{aligned} 2a' - \gamma c &= 0, \\ b' - \gamma a &= 0, \\ c' &= 0. \end{aligned} \tag{2.33}$$

From (2.33), we have

$$b'' = \frac{\gamma^2 k}{2},$$

where $c = k$, k is the constant.

Therefore,

$$a = \frac{\gamma k}{2} x + k_1, \quad b = \frac{\gamma^2 k}{4} x^2 + k_1 x + k_2,$$

where k_1 and k_2 are constant.

We note that this case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.5).

Subalgebra 2.3 with the generator $X_3 + \gamma X_6$.

From equations (2.18)-(2.20), one obtains

$$
\begin{aligned}
a' &= 0, \\
b' - \gamma b &= 0, \\
c' + \gamma c &= 0.
\end{aligned}
\tag{2.34}
$$

Therefore, the solutions to (2.34) are

$$
a = k, \quad b = be^{\gamma x}, \quad c = ke^{-\gamma x}.
$$

Hence the case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.3).

Subalgebra 3.1 with the generator $X_1 + X_3 + \gamma(X_4 - X_5)$.

In this case, equations (2.18)-(2.20) become

$$
\begin{aligned}
2a'(x^2+1) + 8ax + \gamma(b+c) &= 0, \\
b'(x^2+1) - \gamma a + 4bx &= 0, \\
c'(x^2+1) - \gamma a + 4cx &= 0.
\end{aligned}
\tag{2.35}
$$

We solve (2.35) by applying the change

$$
a = (x^2+1)^{-2}\bar{a}, \quad b = (x^2+1)^{-2}\bar{b}, \quad c = (x^2+1)^{-2}\bar{c},
$$

and thus, we obtain

$$
\begin{aligned}
(x^2+1)\bar{a}' + \tfrac{\gamma}{2}(\bar{b}+\bar{c}) &= 0 \\
(x^2+1)\bar{b}' - \gamma\bar{a} &= 0 \\
(x^2+1)\bar{c}' - \gamma\bar{a} &= 0.
\end{aligned}
\tag{2.36}
$$

From equations (2.36)

$$
\bar{c} = \bar{b} + k,
$$

where k is a constant and

$$
(x^2+1)\bar{a}' + \frac{\gamma}{2}(2\bar{b} + k) = 0.
$$

Therefore,

$$
\bar{b}' = \frac{\gamma\bar{a}}{x^2+1}.
$$

Hence,

$$(x^2+1)^2\bar{a}''+2x(x^2+1)\bar{a}'+\gamma^2\bar{a}=0 \tag{2.37}$$

and equation (2.37) is reduced to

$$\frac{d^2\bar{a}}{d\bar{x}^2}+\gamma^2\bar{a}=0$$

with solution

$$\bar{a}=k_1\cos(\gamma\arctan x)+k_2\sin(\gamma\arctan x).$$

Therefore,

$$a=(x^2+1)^{-2}\left(k_1\cos(\gamma\arctan x)+k_2\sin(\gamma\arctan x)\right),$$

$$b=(x^2+1)^{-2}\left(2k_1\sin(\gamma\arctan x)-2k_2\cos(\gamma\arctan x)+k\right),$$

$$c=(x^2+1)^{-2}\left(2k_1\sin(\gamma\arctan x)-2k_2\cos(\gamma\arctan x)-k\right).$$

Here the case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.4).

Subalgebra 3.2 with the generator $X_1+X_3+\gamma X_5$.

In this case, equations (2.18)-(2.20) become

$$\begin{gathered}2a'(x^2+1)+8ax-\gamma c=0,\\ b'(x^2+1)+4bx+\gamma a=0,\\ c'(x^2+1)+4cx=0.\end{gathered} \tag{2.38}$$

Applying the change

$$a=(x^2+1)^{-2}\bar{a},\quad b=(x^2+1)^{-2}\bar{b},\quad c=(x^2+1)^{-2}\bar{c},$$

equations (2.38) are reduced to the equations

$$\begin{gathered}(x^2+1)\bar{a}'-\tfrac{\gamma}{2}k=0,\\ (x^2+1)\bar{b}'+\gamma\bar{a}=0,\end{gathered} \tag{2.39}$$

where $\bar{c}=k$, k is a constant. From the first equation of (2.39)

$$\bar{a}'=\frac{\gamma k}{2(x^2+1)}$$

or

$$a = \frac{\gamma k \arctan\, x + k_1}{(x^2+1)^2},$$

and the second equation of (2.39) becomes

$$\bar{b}' = -\gamma(k_1 + \frac{\gamma k}{2}\bar{x}), \tag{2.40}$$

where $\bar{x} = \arctan\, x$. The solution of this equation is

$$\bar{b} = -\frac{\gamma^2 k}{4}(\arctan\, x)^2 - \gamma k_1 \arctan\, x + k_2.$$

Finally,

$$a = \frac{\gamma k \arctan\, x + k_1}{2(x^2+1)^2}, \quad b = -\frac{\gamma^2 k(\arctan\, x)^2 + 4\gamma k_1 \arctan\, x + k_2}{4(x^2+1)^2}, \quad c = \frac{k}{(x^2+1)^2}.$$

We note that this case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.5).

Subalgebra 3.3 with the generator $X_1 + X_3 + \gamma X_6$.

In this case, equations (2.18)-(2.20) become

$$\begin{aligned} a'(x^2+1) + 4ax &= 0, \\ b'(x^2+1) + b(4x-\gamma) &= 0, \\ c'(x^2+1) + c(4x+\gamma) &= 0. \end{aligned} \tag{2.41}$$

The general solution of equations (2.41) is

$$a = \frac{k_1}{(x^2+1)^2}, \quad b = \frac{k_2}{(x^2+1)^2}e^{\gamma \arctan x}, \quad c = \frac{k_3}{(x^2+1)^2}e^{-\gamma \arctan x}.$$

We note that this case of equations (2.10) can be reduced by a point transformation to the equations with arbitrary elements of the form (2.3).

Subalgebra 4.1 with the generator $X_4 - X_5$.

In this case, we solve equations (2.18)-(2.20) to obtain

$$a = 0, \quad b = -c.$$

This case of equations of (2.10) with $c' \neq 0$ belongs to the class of equations of the form (2.6).

Subalgebra 4.2 with the generator X_5.

In this case, we solve equations (2.18)-(2.20) to obtain

$$a = 0, \quad c = 0.$$

In this case, the second equation of (2.10) is reduced to the free particle equation. This case is excluded from our consideration.

Subalgebra 4.3 with the generator X_6.

For this case, we solve equations (2.18)-(2.20) to obtain

$$b = 0, \quad c = 0.$$

This case is also excluded from our consideration.

2.5 Discussion on solving determining equations

We note that the linear combination, where equation (2.18) is multiplied by $q_2 - q_4$, equation (2.19) is multiplied by q_3, and equation (2.20) is multiplied by q_1 gives the integral

$$(h\xi^2)' = 0,$$

where $h = (q_2 - q_4)a + q_3 b + q_1 c$. In particular, for $\xi \neq 0$, this gives

$$(q_2 - q_4)a + q_3 b + q_1 c = k\xi^{-2},$$

where k is constant. Moreover, in this case, the change

$$\bar{x} = \varphi(x), \quad a = \bar{a}\xi^{-2}, \quad b = \bar{b}\xi^{-2}, \quad c = \bar{c}\xi^{-2},$$

where $2\varphi'\xi = 1$ reduces equations (2.18)-(2.20) to the simpler form

$$\frac{d\bar{a}}{d\bar{x}} + \bar{b}q_3 - \bar{c}q_1 = 0,$$
$$\frac{d\bar{b}}{d\bar{x}} + 2(\bar{a}q_1 - \bar{b}q_2) = 0,$$
$$\frac{d\bar{c}}{d\bar{x}} - 2(\bar{a}q_3 - \bar{c}q_2) = 0.$$

The latter system can be rewritten in the matrix form

$$\frac{d}{d\bar{x}}\bar{A} + \bar{A}H - H\bar{A} = 0,$$

where

$$\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & -\bar{a} \end{pmatrix}, H = \begin{pmatrix} q_2 & q_1 \\ q_3 & -q_2 \end{pmatrix}.$$

The general solution of the matrix equation is [6]

$$\bar{A} = e^{\bar{x}H} A_0 e^{-\bar{x}H},$$

where the matrix

$$A_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & -a_0 \end{pmatrix}$$

is a matrix with arbitrary constant entries a_0, b_0 and c_0. The following three particular cases of the matrix H are used earlier:

$$H_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, H_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For these matrices, their corresponding exponential matrices

$$e^{sH_1} = E + sH_1 - \frac{s^2}{2!}E - \frac{s^3}{3!}H_1 + \cdots = \cos(s)E + \sin(s)H_1,$$
$$e^{sH_2} = E + sH_2, \quad e^{sH_3} = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}.$$

Note also that for $\gamma = 0$ the matrix $H = 0$, which means that the matrix $\bar{A}$ is constant.

2.6 Algebras of dimensions $n \geq 2$

Assuming that in the admitted $n \geq 2$ dimensional Lie algebra there exists one generator such that $\xi = 0$, one finds that this generator has to be $X_4 - X_5$ and the system is with $a = 0$ and $c = -b \neq 0$. Substituting these values into (2.18) - (2.20), we find that the other generators can be written in the form,

$$x_1 X_1 + x_2 X_2 + x_3 X_3, \quad (x_1^2 + x_2^2 + x_3^2 \neq 0).$$

As shown earlier, systems (2.10) admitting such generators are equivalent to a system with constant coefficients.

Assuming that in the admitted Lie algebra there are two linearly independent generators with $\xi \neq 0$, one can conclude that a set of basis generators contains the generators,

$$X_2 + x_4 X_4 + x_5 X_5 + x_6 X_6, \;\; X_3 + k(y_4 X_4 + y_5 X_5 + y_6 X_6),$$

where k is some constant chosen for simplicity as will be explained further. Notice also that because for $k = 0$ the matrix A is constant, one has to assume that $k \neq 0$.

Substituting the coefficients of the second generator into system (2.18) - (2.20), where the coefficients y_4, y_5 and y_6 are chosen from the optimal system (2.23), one finds the derivatives a', b' and c'. After the next substitution of the coefficients of the first generator into system (2.18) - (2.20), from equation (2.18) we obtain

$$a = f_1 b + f_2 c,$$

where $f_i(x)$ are some functions. The remaining equations (2.18) - (2.20) compose a system of two algebraic linear homogeneous equations with respect to b and c. If the determinant of this system $\Delta(x)$ is not equal to zero, then $b = 0$, $c = 0$ and $a = 0$. Hence, one needs to study the case where $\Delta(x) = 0$. Because $\Delta(x)$ is a polynomial with respect to x, one can split it. The splitting leads to the case where $k = 0$.

Thus, there are no systems of equations (2.10), admitting more than one nontrivial generator, which are not equivalent to a constant-coefficient system (2.1).

2.7 Conclusion

We have given a complete group classification of the general case of linear systems of two second-order ordinary differential equations excluding the systems which are equivalent to systems of the type (1) and the degenerate case (2) using the algebraic approach. We were able to apply the algebraic approach because the study is reduced to the analysis of relations between constants. The cause of this possibility is the property $trA = 0$. This condition led us to the equation $\xi^{(3)} = 0$. A complete group classification of the general case is obtained. The delineated list obtained further shows that the problem of classification of systems of two linear second-order ordinary differential equations using the algebraic approach leads to the study of a variety of cases, and this approach can be used as an effective tool to study the group classification of the type of systems studied here. This adds to the body of knowledge in the literature on this subject including the recent results in [44].

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Chapter 3

Linearization Criteria for Systems of Two Second-Order Stochastic Ordinary Differential Equations

3.1 Introduction

Stochastic ordinary differential equations (SODEs) include a stochastic component which describes the randomness within the differential equations. SODEs are in general nonlinear and their solutions are difficult to obtain. Various methods of solving differential equations involve applying a change of variables to transform a given differential equation in to another equation with known properties. The class of linear equations is known to be the simplest class of equations for which it is easier to find a solution, hence, the existence of the problem of transforming a given differential equation into a linear equation. This problem, called a linearization problem, is a particular case of an equivalence problem [5, 21, 40].

Linear SODEs play a role similar to that of linear equations in the deterministic theory of ordinary differential equations (ODEs). However, the change of variables in SODEs differs from that in ODEs due to the Itô formula. The transformation of nonlinear SODEs into linear ones via an invertible stochastic mapping proves to be useful in obtaining the closed form solutions [20, 40, 49, 61]. In this paper, we present a general linearizability criteria for the systems of two second-order SODEs.

We consider the system of two second-order SODEs

$$
\begin{aligned}
d\dot{X} &= f_1(t,X,Y,\dot{X},\dot{Y})\,dt + g_1(t,X,Y,\dot{X},\dot{Y})\,dW \\
d\dot{Y} &= f_2(t,X,Y,\dot{X},\dot{Y})\,dt + g_2(t,X,Y,\dot{X},\dot{Y})\,dW,
\end{aligned} \tag{3.1}
$$

where f_i and g_i, $(i = 1, 2)$ are deterministic functions and dW is the infinitesimal increment of the Wiener process [46]. System (3.1) is said to be linear if the functions f_i and g_i are linear functions with respect to variables X and Y and their respective derivatives. For the linearization problem one considers the class of equations equivalent to linear equations. Thus a linear system of two second-order SODEs has the form,

$$
\begin{aligned}
d\dot{X} =& \Big(\alpha_{11}(t)X + \alpha_{12}(t)Y + \alpha_{13}(t)\dot{X} + \alpha_{14}(t)\dot{Y} + \alpha_{10}(t)\Big)\ dt \\
&+ \Big(\beta_{11}(t)X + \beta_{12}(t)Y + \beta_{13}(t)\dot{X} + \beta_{14}(t)\dot{Y} + \beta_{10}(t)\Big)\ dW, \\
d\dot{Y} =& \Big(\alpha_{21}(t)X + \alpha_{22}(t)Y + \alpha_{23}(t)\dot{X} + \alpha_{24}(t)\dot{Y} + \alpha_{20}(t)\Big)\ dt \\
&+ \Big(\beta_{21}(t)X + \beta_{22}(t)Y + \beta_{23}(t)\dot{X} + \beta_{24}(t)\dot{Y} + \beta_{20}(t)\Big)\ dW.
\end{aligned} \tag{3.2}
$$

We can rewrite (3.2) in the form of first-order SODEs:

$$
\begin{aligned}
d\mathbf{X} &= \dot{\mathbf{X}}dt, \\
d\dot{\mathbf{X}} &= \Big(A\mathbf{X} + B\dot{\mathbf{X}} + \mathbf{a}\Big)\ dt + \Big(F_1\mathbf{X} + F_2\dot{\mathbf{X}} + \mathbf{b}\Big)\ dW,
\end{aligned} \tag{3.3}
$$

where $A(t)$, $B(t)$, F_i $(i = 1, 2)$ are 2×2 matrices; $\mathbf{a(t)}$, $\mathbf{b(t)}$ are vectors and

$$
\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix}.
$$

Similar to the treatment of ordinary differential equations (ODEs), the linearization problem involves finding a change of the dependent variables,

$$
\bar{x} = \varphi(t,x,y),\ \ \bar{y} = \psi(t,x,y),\ \ \Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0
$$

which can transform the system of equations given in (3.1) into linear SODEs (3.2).

Lie [36] laid a foundation for the linearization criteria of the second-order ODEs via an invertible point transformation. He showed that the second-order ODE

$$\ddot{x} = f(t, x, \dot{x}) \tag{3.4}$$

is linearizable by a change of both the independent and dependent variables provided f is a polynomial of the third degree with respect to the first-order derivative,

$$\ddot{x} + F\dot{x}^3 + G\dot{x}^2 + H\dot{x} + L = 0$$

where the coefficients $F(t, x), G(t, x), H(t, x)$ and $L(t, x)$ satisfy the conditions

$$K_1 = 3F_{tt} - 2G_{xt} + H_{xx} - 3F_tH + 3F_xL + 2G_tG - 3H_tF - H_xG + 6L_xF = 0, \tag{3.5}$$

$$K_2 = G_{tt} - 2H_{xt} + 3L_{xx} - 6F_tL + G_tH + 3G_xL - 2H_xH - 3L_tF + 3L_xG = 0.$$

Equation (3.4) is also linearizable by a change of the dependent variable x provided $F = 0$ in which conditions (3.5) become

$$K_1 = (-2G_t + H_x)_x - G(-2G_t + H_x) = 0, \tag{3.6}$$

$$K_2 = (G_t - 2H_x)_t + H(G_t - 2H_x) + 3(L_x + GL)_x = 0.$$

Lie's linearizability criteria for second-order ODEs was extended to the system of second-order ODEs by the authors in [5, 56, 66] and the references therein. In Bagderina [5], a study of the linearization problem of the system of two second-order ODEs was completed.

Modifiying Lie's work for ODEs to SODEs has been done by [62], extended by [40] to the second-order SODEs and in [49, 60, 61], the conditions for the invertible transformations which linearize the jump-diffusion are obtained. The reducibility approach was used in [69] to study the linearization problem of stochastic differential equations (SDEs) with fractional Brownian motion. This work, however, misused the fractional Itô formula to derive the reducibility conditions of nonlinear fractional SDEs to linear fractional SDEs. This was reviewed and corrected in [28].

The rest of the paper is organised as follows: Section 2 discusses an equivalence transformation used to reduce the number of coefficients α_{ij} in system (3.2). In Section 3, the determining equations are derived in an Itô calculus context. These determining equations are non-stochastic. The linearization criteria for a system of two second-order SODEs are given in terms of coefficients of the system. The later part of the paper deals with the β_{ij}s also from equation (3.2) and the analysis of relations for them is given. The main result and Theorem are given in Section 3. Section 4 gives some examples and the conclusion is given in Section 5. To the best of our knowledge this is a new contribution on the linearization problem of systems of two second-order SODEs.

3.2 Equivalence transformation

We consider the transformation

$$\mathbf{X} = C(t)\mathbf{X}_1 + \bar{\mathbf{h}}(t) \tag{3.7}$$

where $C = C(t)$ is a nonsingular matrix and $\bar{h}(t)$ a vector.

Using transformation (3.7), system (3.3) becomes

$$\begin{aligned} d\mathbf{X}_1 &= \dot{\mathbf{X}}_1 dt, \\ d\dot{\mathbf{X}}_1 &= \left(\bar{A}\mathbf{X}_1 + \bar{B}\dot{\mathbf{X}}_1 + \bar{\mathbf{a}}\right) dt + \left(\bar{F}_1\mathbf{X}_1 + \bar{F}_2\dot{\mathbf{X}}_1 + \bar{\mathbf{b}}\right) dW, \end{aligned} \tag{3.8}$$

where

$$\bar{B} = C^{-1}(BC - 2\dot{C}), \quad \bar{A} = C^{-1}(AC + B\dot{C} - \ddot{C}), \quad \bar{\mathbf{a}} = C^{-1}(A\bar{h} + B\dot{\bar{h}} - \ddot{\bar{h}}),$$

$$\bar{F}_1 = CF_1, \quad \bar{F}_2 = CF_2, \quad \bar{\mathbf{b}} = C\mathbf{b}.$$

Choosing C and $\bar{h}$ such that

$$\dot{C} = \frac{1}{2}BC, \quad \ddot{\bar{h}} = B\dot{\bar{h}} + A\bar{h}$$

we have

$$\bar{A} = C^{-1}\left(A + \frac{1}{4}B^2 - \frac{1}{2}\dot{B}\right)C,$$

where we assume that for solving the linearization problem,

$$B = 0, \quad a = 0$$

or

$$\alpha_{10} = \alpha_{13} = \alpha_{14} = \alpha_{20} = \alpha_{23} = \alpha_{24} = 0.$$

Hence the equivalence transformation here is used to reduce the number of coefficients α_{ij} in system (3.2). The rest of the α_{ij}'s will be given in the next section.

3.3 Linearization Criteria for System of Two Second-Order SODEs

Given a system of two second-order SODEs,

$$\begin{aligned} d\dot{X} &= f_1(t, X, Y, \dot{X}, \dot{Y})\,dt + g_1(t, X, Y, \dot{X}, \dot{Y})\,dW, \\ d\dot{Y} &= f_2(t, X, Y, \dot{X}, \dot{Y})\,dt + g_2(t, X, Y, \dot{X}, \dot{Y})\,dW, \end{aligned} \tag{3.9}$$

system (3.9) is reduced to a system of first-order stochastic ordinary differential equations

$$\begin{aligned} dX &= P\,dt \\ d\dot{X} &= f_1(t, X, Y, P, Q)\,dt + g_1(t, X, Y, P, Q)\,dW \\ dY &= Q\,dt \\ d\dot{Y} &= f_2(t, X, Y, P, Q)\,dt + g_2(t, X, Y, P, Q)\,dW. \end{aligned} \tag{3.10}$$

We then apply the change of variables

$$x_1 = \varphi(t, x, y), \quad p_1 = \varphi_2(t, x, y, p, q); \quad y_1 = \psi(t, x, y), \quad q_1 = \psi_2(t, x, y, p, q) \tag{3.11}$$

with

$$\Delta = \varphi_x \psi_y - \varphi_y \psi_x \neq 0,$$

and using the Itô formula [46] for (3.10), we obtain

$$\begin{aligned} dX_1 &= \varphi_2(t, X, Y, P, Q)dt, \\ d\dot{X}_1 &= \tilde{f}_1(t, X, Y, P, Q)dt + \tilde{g}_1(t, X, Y, P, Q)dW, \end{aligned} \tag{3.12}$$

$$dY_1 = \psi_2(t, X, Y, P, Q)dt,$$
$$d\dot{Y}_1 = \tilde{f}_2(t, X, Y, P, Q)dt + \tilde{g}_2(t, X, Y, P, Q)dW,$$

where

$$\varphi_2 = \varphi_t + p\varphi_x + q\varphi_y, \;\; \psi_2 = \psi_t + p\psi_x + q\psi_y,$$

$$\tilde{f}_1(t, x, y, p, q) = \left[\varphi_{2t} + p\varphi_{2x} + q\varphi_{2y} + f_1\varphi_{2p} + f_2\varphi_{2q} + \frac{1}{2}g_1^2\varphi_{2pp} + g_1g_2\varphi_{2pq} + \frac{1}{2}g_2^2\varphi_{2qq}\right](t, x, y, p, q),$$
$$\tilde{g}_1(t, x, y, p, q) = \left[g_1\varphi_{2p} + g_2\varphi_{2q}\right](t, x, y, p, q),$$

and

$$\tilde{f}_2(t, x, y, p, q) = \left[\psi_{2t} + p\psi_{2x} + q\psi_{2y} + f_1\psi_{2p} + f_2\psi_{2q} + \frac{1}{2}g_1^2\psi_{2pp} + g_1g_2\psi_{2pq} + \frac{1}{2}g_2^2\psi_{2qq}\right](t, x, y, p, q),$$
$$\tilde{g}_2(t, x, y, p, q) = \left[g_1\psi_{2p} + g_2\psi_{2q}\right](t, x, y, p, q).$$

Equating $\tilde{f}_1$, $\tilde{g}_1$, $\tilde{f}_2$ and $\tilde{g}_2$ with the linear form (3.2), we obtain four equations,

$$\varphi_{2t}+p\varphi_{2x}+q\varphi_{2y}+f_1\varphi_{2p}+f_2\varphi_{2q}+\frac{1}{2}g_1^2\varphi_{2pp}+\frac{1}{2}g_1g_2\varphi_{2pq}+\frac{1}{2}g_2^2\varphi_{2qq} = \alpha_{11}\varphi+\alpha_{12}\psi+\alpha_{13}\varphi_2+\alpha_{14}\psi_2+\alpha_{10},$$

$$g_1\varphi_{2p} + g_2\varphi_{2q} = \beta_{11}\varphi + \beta_{12}\psi + \beta_{13}\varphi_2 + \beta_{14}\psi_2 + \beta_{10},$$

and

$$\psi_{2t}+p\psi_{2x}+q\psi_{2y}+f_1\psi_{2p}+f_2\psi_{2q}+\frac{1}{2}g_1^2\psi_{2pp}+\frac{1}{2}g_1g_2\psi_{2pq}+\frac{1}{2}g_2^2\psi_{2qq} = \alpha_{21}\varphi+\alpha_{22}\psi+\alpha_{23}\varphi_2+\alpha_{24}\psi_2+\alpha_{20},$$
$$g_1\psi_{2p} + g_2\psi_{2q} = \beta_{21}\varphi + \beta_{22}\psi + \beta_{23}\varphi_2 + \beta_{24}\psi_2 + \beta_{20}.$$

Substituting the functions φ_2 and ψ_2 from (3.11) into the above equations yields the following conditions:

$$\begin{aligned} &\varphi_{xx}p^2 + 2\varphi_{xy}pq + \varphi_{yy}q^2 + 2\varphi_{xt}p + 2\varphi_{yt}q + f_1\varphi_x + f_2\varphi_y + \varphi_{tt} - \alpha_{11}\varphi - \alpha_{12}\psi = 0, \\ &\psi_{xx}p^2 + 2\psi_{xy}pq + \psi_{yy}q^2 + 2\psi_{xt}p + 2\psi_{yt}q + f_1\psi_x + f_2\psi_y + \psi_{tt} - \alpha_{21}\varphi - \alpha_{22}\psi = 0, \end{aligned} \tag{3.13}$$

$$\begin{aligned} g_1\varphi_x + g_2\varphi_y &= \beta_{11}\varphi + \beta_{12}\psi + \beta_{13}(\varphi_t + p\varphi_x + q\varphi_y) + \beta_{14}(\psi_t + p\psi_x + q\psi_y) + \beta_{10}, \\ g_1\psi_x + g_2\psi_y &= \beta_{21}\varphi + \beta_{22}\psi + \beta_{23}(\varphi_t + p\varphi_x + q\varphi_y) + \beta_{24}(\psi_t + p\psi_x + q\psi_y) + \beta_{20}. \end{aligned} \tag{3.14}$$

Thus the two pairs of conditions (3.13) and (3.14) are necessary and sufficient for the SODEs to be linearizable.

The necessary representation of the functions f_1, f_2, g_1 and g_2 are

$$f_1 = a_{11}p^2 + 2a_{12}pq + a_{13}q^2 + a_{14}p + a_{15}q + a_{10},$$
$$f_2 = a_{21}p^2 + 2a_{22}pq + a_{23}q^2 + a_{24}p + a_{25}q + a_{20}, \tag{3.15}$$

$$g_1 = b_{11}p + b_{12}q + b_{10},$$
$$g_2 = b_{21}p + b_{22}q + b_{20}. \tag{3.16}$$

Here the coefficients a_{ij} and b_{ik}, $(i = 1, 2)$, $(j = 0, .., 5)$, $(k = 0, 1, 2)$ are functions of x, y and t. Substituting the representations of the function f_i, $(i = 1, 2)$ in equation (3.15) into (3.13) and splitting them with respect to p and q, we obtain the overdetermined system of equations for the functions φ and ψ, that is,

$$\begin{gathered}\varphi_{xx} = -a_{11}\varphi_x - a_{21}\varphi_y, \;\; \varphi_{yy} = -a_{13}\varphi_x - a_{23}\varphi_y, \;\; \varphi_{xy} = -a_{12}\varphi_x - a_{22}\varphi_y, \\ \varphi_{xt} = -\tfrac{1}{2}\left[a_{14}\varphi_x + a_{24}\varphi_y\right], \varphi_{yt} = -\tfrac{1}{2}\left[a_{15}\varphi_x + a_{25}\varphi_y\right], \\ \varphi_{tt} = -a_{10}\varphi_x - a_{20}\varphi_y + \alpha_{11}\varphi + \alpha_{12}\psi,\end{gathered} \tag{3.17}$$

and

$$\begin{gathered}\psi_{xx} = -a_{11}\psi_x - a_{21}\psi_y, \;\; \psi_{yy} = -a_{13}\psi_x - a_{23}\psi_y, \;\; \psi_{xy} = -a_{12}\psi_x - a_{22}\psi_y, \\ \psi_{xt} = -\tfrac{1}{2}\left[a_{14}\psi_x + a_{24}\psi_y\right], \psi_{yt} = -\tfrac{1}{2}\left[a_{15}\psi_x + a_{25}\psi_y\right], \\ \psi_{tt} = -a_{10}\psi_x - a_{20}\psi_y + \alpha_{21}\varphi + \alpha_{22}\psi.\end{gathered} \tag{3.18}$$

We assume that the coefficients a_{ij} and b_{ik} are given. Compatibility analysis of system (3.17) and system (3.18) gives conditions for these coefficients which are sufficient for linearization. Comparing all mixed derivatives of the third order leads to the following equations:

$$A_{11,1}\varphi_x + A_{11,2}\varphi_y = 0, \quad A_{11,1}\psi_x + A_{11,2}\psi_y = 0, \tag{3.19}$$

$$A_{12,1}\varphi_x + A_{12,2}\varphi_y = 0, \quad A_{12,1}\psi_x + A_{12,2}\psi_y = 0, \tag{3.20}$$

$$A_{13,1}\varphi_x + A_{13,2}\varphi_y = 0, \quad A_{13,1}\psi_x + A_{13,2}\psi_y = 0, \tag{3.21}$$

$$A_{14,1}\varphi_x + A_{14,2}\varphi_y = 0, \quad A_{14,1}\psi_x + A_{14,2}\psi_y = 0, \tag{3.22}$$

$$A_{15,1}\varphi_x + A_{15,2}\varphi_y = 0, \quad A_{15,1}\psi_x + A_{15,2}\psi_y = 0, \tag{3.23}$$

$$A_{16,1}\varphi_x + A_{16,2}\varphi_y = 0, \quad A_{16,1}\psi_x + A_{16,2}\psi_y = 0, \tag{3.24}$$

$$(A_{17,1} - \alpha_{11})\varphi_x + A_{17,2}\varphi_y - \alpha_{12}\psi_x = 0, \quad -\alpha_{21}\varphi_x + (A_{17,1} - \alpha_{22})\psi_x + A_{17,2}\psi_y = 0, \tag{3.25}$$

$$A_{18,1}\varphi_x + (A_{18,2} - \alpha_{11})\varphi_y - \alpha_{12}\psi_y = 0, \quad -\alpha_{21}\varphi_y + A_{18,1}\psi_x + (A_{18,2} - \alpha_{22})\psi_y = 0, \tag{3.26}$$

where

$$A_{11,1} = -a_{11y} + a_{12x} - a_{12}a_{22} + a_{13}a_{21}, \quad A_{11,2} = -a_{21y} + a_{22x} + a_{11}a_{22} - a_{12}a_{21} + a_{21}a_{23} - a_{22}^2,$$

$$A_{12,1} = \frac{1}{2}(a_{14x} - 2a_{11t} - a_{12}a_{24} + a_{15}a_{21}), \quad A_{12,2} = \frac{1}{2}(a_{24x} - 2a_{21t} + a_{11}a_{24} - a_{14}a_{21} + a_{21}a_{25} - a_{22}a_{24}),$$

$$A_{13,1} = \frac{1}{2}(a_{14y} - 2a_{12t} - a_{13}a_{24} + a_{15}a_{22}), \quad A_{13,2} = \frac{1}{2}(a_{24y} - 2a_{22t} + a_{12}a_{24} - a_{14}a_{22} + a_{22}a_{25} - a_{23}a_{24}),$$

$$A_{14,1} = -a_{12y} + a_{13x} - a_{11}a_{13} + a_{12}^2 - a_{12}a_{23} + a_{13}a_{22}, \quad A_{14,2} = -a_{22y} + a_{23x} + a_{12}a_{22} - a_{13}a_{21},$$

$$\begin{aligned}
A_{15,1} &= \frac{1}{2}(-a_{14y} + a_{15x} - a_{11}a_{15} + a_{12}a_{14} - a_{12}a_{25} + a_{13}a_{24}),\\
A_{15,2} &= \frac{1}{2}(-a_{24y} + a_{25x} + a_{14}a_{22} - a_{15}a_{21} - a_{22}a_{25} + a_{23}a_{24}),\\
A_{16,1} &= \frac{1}{2}(a_{15y} - 2a_{13t} - a_{12}a_{15} + a_{13}a_{14} - a_{13}a_{25} + a_{15}a_{23}),\\
A_{16,2} &= \frac{1}{2}(a_{25y} - 2a_{23t} + a_{13}a_{24} - a_{15}a_{22}),\\
A_{17,1} &= \frac{1}{4}(4a_{10x} - 2a_{14t} - 4a_{10}a_{11} - 4a_{12}a_{20} + a_{14}^2 + a_{15}a_{24}),\\
A_{17,2} &= \frac{1}{4}(4a_{20x} - 2a_{24t} - 4a_{10}a_{21} + a_{14}a_{24} - 4a_{20}a_{22} + a_{24}a_{25}),\\
A_{18,1} &= \frac{1}{4}(4a_{10y} - 2a_{15t} - 4a_{10}a_{12} - 4a_{13}a_{20} + a_{14}a_{15} + a_{15}a_{25}),\\
A_{18,2} &= \frac{1}{4}(4a_{20y} - 2a_{25t} - 4a_{10}a_{22} + a_{15}a_{22} - 4a_{20}a_{23} + a_{25}^2).
\end{aligned}$$

Since $\Delta \neq 0$, we can rewrite equation (3.19) in matrix form

$$\begin{pmatrix} \varphi_x & \varphi_y \\ \psi_x & \psi_y \end{pmatrix} \begin{pmatrix} A_{11,1} \\ A_{11,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.27}$$

Then from equations (3.19) - (3.24), it is necessary and sufficient that

$$A_{1i,j} = 0, \tag{3.28}$$

for $(i = 1, .., 6)$ and $(j = 1, 2)$.

Solving equations (3.25) and (3.26) one finds α_{ij}, $i = 1, 2$; $j = 1, 2$ as follows:

$$\begin{aligned}
\alpha_{11} &= \Delta^{-1}(-A_{18,1}\varphi_x\psi_x + A_{17,1}\varphi_x\psi_y - A_{18,2}\varphi_y\psi_x + A_{17,2}\varphi_y\psi_y),\\
\alpha_{12} &= \Delta^{-1}(A_{18,1}\varphi_x^2 + (-A_{17,1} + A_{18,2})\varphi_x\varphi_y - A_{17,2}\varphi_y^2),\\
\alpha_{21} &= \Delta^{-1}(-A_{18,1}\psi_x^2 + (A_{17,1} - A_{18,2})\psi_x\psi_y + A_{17,2}\psi_y^2),\\
\alpha_{22} &= \Delta^{-1}(A_{18,1}\varphi_x\psi_x + A_{18,2}\varphi_x\psi_y - A_{17,1}\varphi_y\psi_x - A_{17,2}\varphi_y\psi_y),
\end{aligned} \tag{3.29}$$

where

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x.$$

Differentiating α_{ij} in (3.29) with respect to x and y, we have the conditions

$$\begin{aligned}
&A_{17,1x} = a_{12}A_{17,2} - a_{21}A_{18,1}; \quad A_{17,2x} = -a_{11}A_{17,2} + a_{21}(A_{17,1} - A_{18,2}) + a_{22}A_{17,2},\\
&A_{18,1x} = a_{11}A_{18,1} - a_{12}(A_{17,1} - A_{18,2}) - a_{22}A_{18,1}; \quad A_{18,2x} = -a_{12}A_{17,2} + a_{21}A_{18,1},\\
&A_{17,1y} = a_{13}A_{17,2} - a_{22}A_{18,1}; \quad A_{17,2y} = -a_{12}A_{17,2} + a_{22}(A_{17,1} - A_{18,2}) + a_{23}A_{17,2},\\
&A_{18,1y} = a_{12}A_{18,1} - a_{13}(A_{17,1} - A_{18,2}) - a_{23}A_{18,1}; \quad A_{18,2y} = -a_{13}A_{17,2} + a_{22}A_{18,1}.
\end{aligned} \tag{3.30}$$

The next step involves finding the β_{ij} and relations for them. To do this we split equations (3.14) and (3.16) with respect to p and q, to obtain

$$-b_{10}\varphi_x - b_{20}\varphi_y + \beta_{10} + \beta_{11}\varphi + \beta_{12}\psi + \beta_{13}\varphi_t + \beta_{14}\psi_t = 0 \tag{3.31}$$

$$-b_{11}\varphi_x - b_{21}\varphi_y + \beta_{13}\varphi_x + \beta_{14}\psi_x = 0, \tag{3.32}$$

$$-b_{12}\varphi_x - b_{22}\varphi_y + \beta_{13}\varphi_y + \beta_{14}\psi_y = 0, \tag{3.33}$$

and

$$-b_{10}\psi_x - b_{20}\psi_y + \beta_{20} + \beta_{21}\varphi + \beta_{22}\psi + \beta_{23}\varphi_t + \beta_{24}\psi_t = 0 \tag{3.34}$$

$$-b_{11}\psi_x - b_{21}\psi_y + \beta_{23}\varphi_x + \beta_{24}\psi_x = 0, \tag{3.35}$$

$$-b_{12}\psi_x - b_{22}\psi_y + \beta_{23}\varphi_y + \beta_{24}\psi_y = 0. \tag{3.36}$$

From equations (3.31) - (3.36) we find

$$\begin{aligned}
\beta_{13} &= \Delta^{-1}(-b_{12}\varphi_x\psi_x + b_{11}\varphi_x\psi_y - b_{22}\varphi_y\psi_x + b_{21}\varphi_y\psi_y),\\
\beta_{14} &= \Delta^{-1}(b_{12}\varphi_x^2 + (-b_{11}+b_{22})\varphi_x\varphi_y - b_{21}\varphi_y^2),\\
\beta_{23} &= \Delta^{-1}(-b_{12}\psi_x^2 + (b_{11}-b_{22})\psi_x\psi_y + b_{21}\psi_y^2),\\
\beta_{24} &= \Delta^{-1}(b_{12}\varphi_x\psi_x + b_{22}\varphi_x\psi_y - b_{11}\varphi_y\psi_x - b_{21}\varphi_y\psi_y),\\
\beta_{10} &= \Delta^{-1}(B_{10,1}\varphi_t + B_{10,2}\psi_t + b_{10}\Delta\varphi_x + b_{20}\Delta\varphi_y - \beta_{11}\Delta\varphi - \beta_{12}\Delta\psi),\\
\beta_{20} &= \Delta^{-1}(B_{20,1}\varphi_t + B_{20,2}\psi_t + b_{10}\Delta\psi_x + b_{20}\Delta\psi_y - \beta_{21}\Delta\varphi - \beta_{22}\Delta\psi),
\end{aligned} \tag{3.37}$$

where

$$B_{10,1} = b_{12}\varphi_x\psi_x - b_{11}\varphi_x\psi_y + b_{22}\varphi_y\psi_x - b_{21}\varphi_y\psi_y, \quad B_{10,2} = -b_{12}\varphi_x^2 + (b_{11}-b_{22})\varphi_x\varphi_y + b_{21}\varphi_y^2,$$

$$B_{20,1} = b_{12}\psi_x^2 + (-b_{11}+b_{22})\psi_x\psi_y - b_{21}\psi_y^2, \quad B_{20,2} = -b_{12}\varphi_x\psi_x - b_{22}\varphi_x\psi_y + b_{11}\varphi_y\psi_x + b_{21}\varphi_y\psi_y.$$

The equations $(\beta_{i0})_x = 0$ and $(\beta_{i0})_y = 0$, $i = 1,2$ compose an algebraic system of linear equations with respect to β_{ij}, $(i = 1,2; j = 1,2)$ with a none vanishing determinant. Hence one can find β_{11}, β_{12}, β_{21}, β_{22} from this system. For this representation we introduce the notations

$$\begin{aligned}
&\xi_1 = 2b_{10y} - 2a_{12}b_{10} - 2a_{13}b_{20} + a_{15}b_{11} + a_{25}b_{12}, && \xi_2 = 2b_{10x} - 2a_{11}b_{10} - 2a_{12}b_{20} + a_{14}b_{11} + a_{24}b_{12},\\
&\xi_3 = 2b_{20y} + a_{15}b_{21} - 2a_{22}b_{10} - 2a_{23}b_{20} + a_{25}b_{22}, && \xi_4 = 2b_{20x} + a_{14}b_{21} - 2a_{21}b_{10} - 2a_{22}b_{20} + a_{24}b_{22},\\
&\xi_5 = b_{12y} - a_{12}b_{12} + a_{13}b_{11} - a_{13}b_{22} + a_{23}b_{12}, && \xi_6 = b_{11x} - a_{12}b_{21} + a_{21}b_{12},\\
&\xi_7 = b_{22y} + a_{13}b_{21} - a_{22}b_{12}, && \xi_8 = b_{21x} + a_{11}b_{21} - a_{21}b_{11} + a_{21}b_{22} - a_{22}b_{21},\\
&\xi_9 = b_{12x} - a_{11}b_{12} + a_{12}b_{11} - a_{12}b_{22} + a_{22}b_{12}, && \xi_{10} = b_{21y} + a_{12}b_{21} - a_{22}b_{11} + a_{22}b_{22} - a_{23}b_{21},\\
&\xi_{11} = b_{11y} - a_{13}b_{21} + a_{22}b_{12}, && \xi_{12} = b_{22x} + a_{12}b_{21} - a_{21}b_{12}.
\end{aligned} \tag{3.38}$$

The final representation of all β_{ij} and relations are presented in the appendix.

Notice that from the definitions of ξ_i and found relations we obtain the following corollaries:

$$
\begin{aligned}
\xi_{2y} &= \xi_{1x} - a_{11}\xi_1 + a_{12}(\xi_2 - \xi_3) + a_{13}\xi_4 + a_{14}\xi_{11} - a_{15}\xi_6 + a_{24}\xi_5 - a_{25}\xi_9, \\
\xi_{4y} &= \xi_{3x} + a_{14}\xi_{10} - a_{15}\xi_8 - a_{21}\xi_1 + a_{22}(\xi_2 - \xi_3) + a_{23}\xi_4 + a_{24}\xi_7 - a_{25}\xi_{12}, \\
\xi_{6y} &= \xi_{11x} - a_{12}\xi_{10} + a_{13}\xi_8 + a_{21}\xi_5 - a_{22}\xi_9 \\
\xi_{8y} &= \xi_{10x} + a_{11}\xi_{10} - a_{12}\xi_8 + a_{21}(\xi_7 - \xi_{11}) + a_{22}(\xi_6 - \xi_{10} - \xi_{12}) + a_{23}\xi_8 \\
\xi_{9y} &= \xi_{5x} - a_{11}\xi_5 - a_{12}(\xi_7 - \xi_9 - \xi_{11}) - a_{13}(\xi_6 - \xi_{12}) + a_{22}\xi_5 - a_{23}\xi_9 \\
\xi_{12y} &= \xi_{7x} + a_{12}\xi_{10} - a_{13}\xi_8 - a_{21}\xi_5 + a_{22}\xi_9.
\end{aligned} \tag{3.39}
$$

The equations $(\beta_{ij})_x = 0$, $(i = 1, 2; j = 1, 2)$ compose an algebraic system of linear equations with respect to ξ_{1x}, ξ_{2x}, ξ_{3x}, ξ_{4x} with a none vanishing determinant. Hence one can find ξ_{1x}, ξ_{2x}, ξ_{3x}, ξ_{4x} from this system. We omit their expressions here.

The equations $(\beta_{ij})_x = 0$, $(i = 1, 2; j = 3, 4)$ compose an algebraic system of linear equations with respect to ξ_6, ξ_8, ξ_9, ξ_{12} with a none vanishing determinant. Hence one can find ξ_6, ξ_8, ξ_9, ξ_{12} from this system:

$$
\xi_6 = 0, \quad \xi_8 = 0, \quad \xi_9 = 0, \quad \xi_{12} = 0. \tag{3.40}
$$

The equations $(\beta_{ij})_y = 0$, $(i = 1, 2; j = 3, 4)$ compose an algebraic system of linear equations with respect to ξ_5, ξ_7, ξ_{10}, ξ_{11} with none vanishing determinant. Hence one can find ξ_5, ξ_7, ξ_{10}, ξ_{11} from this system:

$$
\xi_5 = 0, \quad \xi_7 = 0, \quad \xi_{10} = 0, \quad \xi_{11} = 0. \tag{3.41}
$$

From the equations $(\beta_{11})_y = 0$ and $(\beta_{21})_y = 0$; one finds

$$
\xi_{1y} = a_{12}\xi_1 - a_{13}(\xi_2 - \xi_3) - a_{23}\xi_1, \quad \xi_{3y} = -a_{13}\xi_4 + a_{22}\xi_1. \tag{3.42}
$$

In order to obtain sufficiency conditions, assume that all the identities listed above are satisfied. Hence we have proven the following theorem:

Theorem

A system of two second-order stochastic ordinary differential equations,

$$
d\dot{X} = f_1(t, X, Y, \dot{X}, \dot{Y})\,dt + g_1(t, X, Y, \dot{X}, \dot{Y})\,dW,
$$

$$d\dot{Y} = f_2(t, X, Y, \dot{X}, \dot{Y})\, dt + g_2(t, X, Y, \dot{X}, \dot{Y})\, dW,$$

is linearizable by an invertible point transformation if and only if,

$$f_1 = a_{11}\dot{x}^2 + 2a_{12}\dot{x}\dot{y} + a_{13}\dot{y}^2 + a_{14}\dot{x} + a_{15}\dot{y} + a_{10},$$

$$f_2 = a_{21}\dot{x}^2 + 2a_{22}\dot{x}\dot{y} + a_{23}\dot{y}^2 + a_{24}\dot{x} + a_{25}\dot{y} + a_{20},$$

$$g_1 = b_{11}\dot{x} + b_{12}\dot{y} + b_{10}, \;\; g_2 = b_{21}\dot{x} + b_{22}\dot{y} + b_{20},$$

where f_i, g_i, $a_{ij}, (i = 1, 2; j = 0, ..., 5)$ and $b_{ij}, (i = 1, 2; j = 0, 1, 2)$ satisfy the conditions

$$A_{1i,j} = 0, (i = 1, .., 6); (j = 1, 2).$$

Here the $A_{1i,j}$ are listed as follows:

$$A_{17,1x} = a_{12}A_{17,2} - a_{21}A_{18,1}; \quad A_{17,2x} = -a_{11}A_{17,2} + a_{21}(A_{17,1} - A_{18,2}) + a_{22}A_{17,2},$$

$$A_{18,1x} = a_{11}A_{18,1} - a_{12}(A_{17,1} - A_{18,2}) - a_{22}A_{18,1}; \quad A_{18,2x} = -a_{12}A_{17,2} + a_{21}A_{18,1},$$

$$A_{17,1y} = a_{13}A_{17,2} - a_{22}A_{18,1}; \quad A_{17,2y} = -a_{12}A_{17,2} + a_{22}(A_{17,1} - A_{18,2}) + a_{23}A_{17,2},$$

$$A_{18,1y} = a_{12}A_{18,1} - a_{13}(A_{17,1} - A_{18,2}) - a_{23}A_{18,1}; \quad A_{18,2y} = -a_{13}A_{17,2} + a_{22}A_{18,1}.$$

In addition,

$$\xi_{1y} = (a_{12} - a_{23})\xi_1 - a_{13}(\xi_2 - \xi_3); \quad \xi_{2y} = \xi_{1x} - a_{11}\xi_1 + a_{12}(\xi_2 - \xi_3) + a_{13}\xi_4,$$

$$\xi_{3y} = -a_{13}\xi_4 + a_{22}\xi_1; \;\; \xi_{4y} = \xi_{3x} - a_{21}\xi_1 + a_{22}(\xi_2 - \xi_3) + a_{23}\xi_4,$$

$$b_{12y} - a_{12}b_{12} + a_{13}b_{11} - a_{13}b_{22} + a_{23}b_{12} = 0; \quad b_{11x} - a_{12}b_{21} + a_{21}b_{12} = 0,$$

$$b_{22y} + a_{13}b_{21} - a_{22}b_{12} = 0; \quad b_{21x} + a_{11}b_{21} - a_{21}b_{11} + a_{21}b_{22} - a_{22}b_{21} = 0,$$

$$b_{12x} - a_{11}b_{12} + a_{12}b_{11} - a_{12}b_{22} + a_{22}b_{12} = 0; \quad b_{21y} + a_{12}b_{21} - a_{22}b_{11} + a_{22}b_{22} - a_{23}b_{21} = 0,$$

$$b_{11y} - a_{13}b_{21} + a_{22}b_{12} = 0, \quad b_{22x} + a_{12}b_{21} - a_{21}b_{12} = 0,$$

where

$$\xi_1 = 2b_{10y} - 2a_{12}b_{10} - 2a_{13}b_{20} + a_{15}b_{11} + a_{25}b_{12}; \;\; \xi_2 = 2b_{10x} - 2a_{11}b_{10} - 2a_{12}b_{20} + a_{14}b_{11} + a_{24}b_{12},$$

$$\xi_3 = 2b_{20y} + a_{15}b_{21} - 2a_{22}b_{10} - 2a_{23}b_{20} + a_{25}b_{22}; \;\; \xi_4 = 2b_{20x} + a_{14}b_{21} - 2a_{21}b_{10} - 2a_{22}b_{20} + a_{24}b_{22}.$$

3.4 Examples

In this section we apply the obtained Theorem to some selected examples to illustrate how to linearize a system of two second-order SODEs. For checking whether a system of two second-order SODEs is linearizable we develop a code using REDUCE [23]. We demonstrate the use of this code using three illustrative examples.

3.4.1 Example 1

Consider a nonlinear system of two second-order SODEs

$$d\dot{x} = x\,dW, \;\; d\dot{y} = -2x^{-1}pqdt, \tag{3.43}$$

where the first equation is linear, and the second equation is a nonlinear one without Itô's integral.

First we check that all sufficient conditions for linearization (3.19) - (3.42) are satisfied. To find a linearization transformation we obtain the overdetermined system of equations:

$$\varphi_{xx} = 0, \;\; \varphi_{yy} = 0, \;\; \varphi_{xt} = 0, \;\; \varphi_{yt} = 0, \;\; \varphi_{tt} = 0, \;\; \varphi_{xy} = x^{-1}\varphi_y \tag{3.44}$$

and

$$\psi_{xx} = 0, \;\; \psi_{yy} = 0, \;\; \psi_{xt} = 0, \;\; \psi_{yt} = 0, \;\; \psi_{tt} = 0, \;\; \psi_{xy} = x^{-1}\psi_y. \tag{3.45}$$

Notice that this overdetermined system of equations is compatible. Hence we can find a solution of this system of equations.

The general solution of equations (3.44) and (3.45) is

$$\varphi = c_1xy + c_2x + c_3t + c_4, \;\; \psi = c_5xy + c_6x + c_7t + c_8, \tag{3.46}$$

where c_i, $(i = 1, 2, ..., 8)$ are constant. We choose the constants c_i such that $\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0$, for example,

$$c_1 = 0, \;\; c_2 = 1, \;\; c_3 = 0, \;\; c_4 = 0, \;\; c_5 = 1, \;\; c_6 = 0, \;\; c_7 = 0, \;\; c_8 = 0,$$

which give the transformation

$$\varphi = x, \ \ \psi = xy.$$

The latter change maps the nonlinear system (3.43) into the linear system

$$d\dot{x} = x\,dW, \ \ d\dot{y} = y\,dW. \tag{3.47}$$

3.4.2 Example 2

Consider a nonlinear system of two second-order SODEs

$$\begin{aligned} d\dot{x} &= \frac{1}{\vartheta^2}\left(2\vartheta'\vartheta p - x(2(\vartheta')^2 - \vartheta''\vartheta)\right)dt + \frac{\vartheta^3}{x^2}\left(y\vartheta p + xq\right)dW, \\ d\dot{y} &= \frac{1}{x^2\vartheta^2}\left(-2y\vartheta^2 p^2 + 2x\vartheta^2 pq + 4xy\vartheta'\vartheta p - 2x^2\vartheta'\vartheta q - 2x^2 y(\vartheta')^2\right)dt \\ &\quad + \frac{1}{x^3\vartheta}\left((x^2 + y\vartheta^4)p - xy\vartheta^4 q\right)dW, \end{aligned} \tag{3.48}$$

where $\vartheta = \vartheta(t) \neq 0$. First we check that all sufficient conditions for linearization (3.19) - (3.42) are satisfied. To find a linearization transformation we obtain the overdetermined system of equations:

$$\varphi_{xx} = \frac{2y}{x^2}\varphi_y, \ \ \varphi_{xy} = -\frac{1}{x}\varphi_y, \ \ \varphi_{yy} = 0, \varphi_{xt} = -\frac{\vartheta'}{2x\vartheta}(x\varphi_x + 4y\varphi_y), \ \ \varphi_{yt} = \frac{\vartheta'}{\vartheta}\varphi_y, \tag{3.49}$$

$$\varphi_{tt} = \frac{1}{\vartheta'}\left[x(2(\vartheta')^2 - \vartheta''\vartheta)\varphi_x + 2y(\vartheta')^2\varphi_y\right], \tag{3.50}$$

and

$$\psi_{xx} = \frac{2y}{x^2}\psi_y, \ \ \psi_{xy} = -\frac{1}{x}\psi_y, \ \ \psi_{yy} = 0, \psi_{xt} = -\frac{\vartheta'}{2x\vartheta}(x\psi_x + 4y\psi_y), \ \ \psi_{yt} = \frac{\vartheta'}{\vartheta}\psi_y, \tag{3.51}$$

$$\psi_{tt} = \frac{1}{\vartheta'}\left[x(2(\vartheta')^2 - \vartheta''\vartheta)\psi_x + 2y(\vartheta')^2\psi_y\right]. \tag{3.52}$$

Notice that this overdetermined system of equations is compatible. Hence we can find a solution of this system of equations.

Solving the first five sets of equations in (3.49) and (3.51) we obtain

$$\varphi = \vartheta\lambda_1\frac{y}{x} + \lambda_3 x + \lambda_{10}, \ \ \psi = \vartheta\lambda_2\frac{y}{x} + \lambda_4 x + \lambda_{20}, \tag{3.53}$$

where λ_i, $(i = 1, ..., 4)$ are constant and $\lambda_{10}(t)$ and $\lambda_{20}(t)$ are arbitrary functions. Substituting (3.53) into (3.50) and (3.52) we obtain $\lambda_{10}'' = 0$ and $\lambda_{20}'' = 0$. We can choose the trivial solution of these equations $\lambda_{10} = 0$, $\lambda_{20} = 0$. Since

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0,$$

we can also choose a particular set of constants such that $\lambda_1 = \lambda_4 = 1$ and $\lambda_2 = \lambda_3 = 0$. This gives us the transformation

$$\varphi = \frac{\vartheta}{x}y, \quad \psi = \frac{x}{\vartheta},$$

which maps the nonlinear system (3.48) into the linear system

$$d\dot{x} = (\vartheta' y + \vartheta q)dW, \quad d\dot{y} = (\vartheta' x + \vartheta p)dW. \tag{3.54}$$

3.4.3 Example 3

For the third example, we have

$$d\dot{x} = -\frac{1}{2(x^2-y^2)}\left[(2x(p^2+q^2) - 4ypq - x^3 + xy^2)dt + y(x^2-y^2)dW\right],$$

$$d\dot{y} = \frac{1}{2(x^2-y^2)}\left[(2y(p^2+q^2) - 4xpq + x^2y - y^3)dt + x(x^2-y^2)dW\right]. \tag{3.55}$$

Equation (3.55) satisfies all the sufficient conditions (3.19) - (3.42) and hence the compatible overdetermined system of equations is given by

$$\varphi_{xx} = \varphi_{yy} = \frac{1}{x^2-y^2}\left(x\varphi_x - y\varphi_y\right), \quad \varphi_{xy} = -\frac{1}{x^2-y^2}\left(y\varphi_x - x\varphi_y\right), \quad \varphi_{xt} = \varphi_{yt} = 0, \tag{3.56}$$

$$\varphi_{tt} = -\frac{1}{2}\left(x\varphi_x + y\varphi_y - 2\varphi\right), \tag{3.57}$$

and

$$\psi_{xx} = \psi_{yy} = \frac{1}{x^2-y^2}\left(x\psi_x - y\psi_y\right), \quad \psi_{xy} = -\frac{1}{x^2-y^2}\left(y\psi_x - x\psi_y\right), \quad \psi_{xt} = \psi_{yt} = 0. \tag{3.58}$$

$$\psi_{tt} = -\frac{1}{2}\left(x\psi_x + y\psi_y - 2\psi\right). \tag{3.59}$$

Solving the system of equations (3.56) and (3.58) we get

$$\begin{aligned}\varphi &= x^2(C_1+C_2)+2xy(C_1-C_2)+y^2(C_1+C_2)+C_{10},\\ \psi &= x^2(C_3+C_4)+2xy(C_3-C_4)+y^2(C_3+C_4)+C_{20},\end{aligned} \tag{3.60}$$

where C_i, $(i = 1, ..., 4)$ are constant and $C_{10}(t)$ and $C_{20}(t)$ are arbitrary functions. After substituting (3.60) into (3.62) and (3.59) we obtain $C_{10}''' = 0$ and $C_{20}''' = 0$. For simplicity we choose the trivial solution of these equations $C_{10} = 0$, $C_{20} = 0$. Since

$$\Delta = \varphi_x\psi_y - \varphi_y\psi_x \neq 0,$$

we also choose a particular set of constants such that $C_1 = C_2 = C_3 = \frac{1}{2}$ and $C_4 = -\frac{1}{2}$. Hence the transformation

$$\varphi = x^2 + y^2, \quad \psi = 2xy \tag{3.61}$$

linearizes the equation (3.55) into

$$d\dot{x} = xdt + ydW, \quad d\dot{y} = ydt + xdW. \tag{3.62}$$

3.5 Conclusion

In this paper, we have completely solved the linearization problem of systems of two second-order stochastic ordinary differential equations. Necessary and sufficient conditions for linearization by an invertible transformation are given in terms of coefficients of the system. The result is given in terms of a Theorem with three examples. In addition, we have also shown that the system of two nonlinear second-order stochastic ordinary differential equations is linearizable via an invertible transformation when certain conditions are satisfied. Moreover, we have developed a code using REDUCE for checking whether a system of two second-order stochastic ordinary differential equations is linearizable. Certain nonlinear second-order stochastic ordinary differential equations appeared to be linearizable via invertible transformations.

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Appendix

Final conditions are

$$A_{17,1x}=a_{12}A_{17,2}-a_{21}A_{18,1};\quad A_{17,2x}=-a_{11}A_{17,2}+a_{21}(A_{17,1}-A_{18,2})+a_{22}A_{17,2},$$

$$A_{18,1x}=a_{11}A_{18,1}-a_{12}(A_{17,1}-A_{18,2})-a_{22}A_{18,1};\quad A_{18,2x}=-a_{12}A_{17,2}+a_{21}A_{18,1},$$

$$A_{17,1y}=a_{13}A_{17,2}-a_{22}A_{18,1};\quad A_{17,2y}=-a_{12}A_{17,2}+a_{22}(A_{17,1}-A_{18,2})+a_{23}A_{17,2},$$

$$A_{18,1y}=a_{12}A_{18,1}-a_{13}(A_{17,1}-A_{18,2})-a_{23}A_{18,1};\quad A_{18,2y}=-a_{13}A_{17,2}+a_{22}A_{18,1}.$$

$$\begin{aligned}
\beta_{11}&=\Delta^{-1}(-\xi_1\varphi_x\psi_x+\xi_2\varphi_x\psi_y-\xi_3\varphi_y\psi_x+\xi_4\varphi_y\psi_y),\\
\beta_{12}&=\Delta^{-1}(\xi_1\varphi_x^2-(\xi_2-\xi_3)\varphi_x\varphi_y-\xi_4\varphi_y^2),\\
\beta_{13}&=\Delta^{-1}(-b_{12}\varphi_x\psi_x+b_{11}\varphi_x\psi_y-b_{22}\varphi_y\psi_x+b_{21}\varphi_y\psi_y),\\
\beta_{14}&=\Delta^{-1}(b_{12}\varphi_x^2-(b_{11}-b_{22})\varphi_x\varphi_y-b_{21}\varphi_y^2),\\
\beta_{21}&=\Delta^{-1}(-\xi_1\psi_x^2+(\xi_2-\xi_3)\psi_x\psi_y+\xi_4\psi_y^2),\\
\beta_{22}&=\Delta^{-1}(\xi_1\varphi_x\psi_x+\xi_3\varphi_x\psi_y-\xi_2\varphi_y\psi_x-\xi_4\varphi_y\psi_y),\\
\beta_{23}&=\Delta^{-1}(-b_{12}\psi_x^2+(b_{11}-b_{22})\psi_x\psi_y+b_{21}\psi_y^2),\\
\beta_{24}&=\Delta^{-1}(b_{12}\varphi_x\psi_x+b_{22}\varphi_x\psi_y-b_{11}\varphi_y\psi_x-b_{21}\varphi_y\psi_y),\\
\beta_{10}&=\Delta^{-1}(B_{10,1}\varphi_t+B_{10,2}\psi_t+b_{10}\Delta\varphi_x+b_{20}\Delta\varphi_y-\beta_{11}\Delta\varphi-\beta_{12}\Delta\psi),\\
\beta_{20}&=\Delta^{-1}(B_{20,1}\varphi_t+B_{20,2}\psi_t+b_{10}\Delta\psi_x+b_{20}\Delta\psi_y-\beta_{21}\Delta\varphi-\beta_{22}\Delta\psi),
\end{aligned}$$

where

$$B_{10,1}=b_{12}\varphi_x\psi_x-b_{11}\varphi_x\psi_y+b_{22}\varphi_y\psi_x-b_{21}\varphi_y\psi_y,\quad B_{10,2}=-b_{12}\varphi_x^2+(b_{11}-b_{22})\varphi_x\varphi_y+b_{21}\varphi_y^2,$$

$$B_{20,1}=b_{12}\psi_x^2+(-b_{11}+b_{22})\psi_x\psi_y-b_{21}\psi_y^2,\quad B_{20,2}=-b_{12}\varphi_x\psi_x-b_{22}\varphi_x\psi_y+b_{11}\varphi_y\psi_x+b_{21}\varphi_y\psi_y,$$

$$\xi_1=-2b_{10y}+2a_{12}b_{10}+2a_{13}b_{20}-a_{15}b_{11}-a_{25}b_{12},\quad \xi_2=2b_{10x}-2a_{11}b_{10}-2a_{12}b_{20}+a_{14}b_{11}+a_{24}b_{12},$$

$$\xi_3=-2b_{20y}-a_{15}b_{21}+2a_{22}b_{10}+2a_{23}b_{20}-a_{25}b_{22},\quad \xi_4=2b_{20x}+a_{14}b_{21}-2a_{21}b_{10}-2a_{22}b_{20}+a_{24}b_{22}.$$

$$b_{10y} = (2a_{12}b_{10} + 2a_{13}b_{20} - a_{15}b_{11} - a_{25}b_{12} + \xi_1)/2,$$

$$b_{10x} = (2a_{11}b_{10} + 2a_{12}b_{20} - a_{14}b_{11} - a_{24}b_{12} + \xi_2)/2,$$

$$b_{11y} = a_{13}b_{21} - a_{22}b_{12}, \qquad b_{11x} = a_{12}b_{21} - a_{21}b_{12},$$

$$b_{12y} = a_{12}b_{12} - a_{13}b_{11} + a_{13}b_{22} - a_{23}b_{12}, \qquad b_{12x} = a_{11}b_{12} - a_{12}b_{11} + a_{12}b_{22} - a_{22}b_{12},$$

$$b_{21y} = -a_{12}b_{21} + a_{22}b_{11} - a_{22}b_{22} + a_{23}b_{21}, \qquad b_{21x} = -a_{11}b_{21} + a_{21}b_{11} - a_{21}b_{22} + a_{22}b_{21},$$

$$b_{22y} = -a_{13}b_{21} + a_{22}b_{12}, \qquad b_{11x} = -a_{12}b_{21} + a_{21}b_{12},$$

$$b_{20y} = (-a_{15}b_{21} + 2a_{22}b_{10} + 2a_{23}b_{20} - a_{25}b_{22} + \xi_3)/2,$$

$$b_{20x} = (-a_{14}b_{21} + 2a_{21}b_{10} + 2a_{22}b_{20} - a_{24}b_{22} + \xi_4)/2,$$

$$\xi_{1x} = a_{11}\xi_1 - a_{12}(\xi_2 - \xi_3) - a_{22}\xi_1, \qquad \xi_{2x} = a_{12}\xi_4 - a_{21}\xi_1,$$

$$\xi_{3x} = -a_{12}\xi_4 + a_{21}\xi_1, \qquad \xi_{4x} = -a_{11}\xi_4 + a_21(\xi_2 - \xi_3) + a_{22}\xi_4,$$

$$\xi_{1y} = a_{12}\xi_1 - a_{13}(\xi_2 - \xi_3) - a_{23}\xi_1, \qquad \xi_{3y} = -a_{13}\xi_4 + a_{22}\xi_1.$$

Chapter 4

Group Classification of Systems of Two Linear Second-Order Stochastic Ordinary Differential Equations

4.1 Introduction

Stochastic ordinary differential equations (SODEs) are in general nonlinear and hence their solutions can be difficult to obtain. The most famous and well established method for obtaining exact solutions of differential equations is the Lie group analysis method which in its classical form uses the concept of admitted Lie groups to find corresponding transformations for any given equation or system of equations amenable to the analysis. The admitted Lie group by a system of equations is a Lie group for which the coefficients of the corresponding generator satisfy the subsequent determining equations. The application of Lie group analysis to SODEs was successfully performed in [43, 1, 38, 59, 57, 18, 21, 32].

In this study, we consider the group classification of systems of two linear second-order SODEs with constant coefficients. The system of two second-order SODEs is given by

$$d\dot{X} = F_1(t, X, Y, \dot{X}, \dot{Y})\, dt + G_1(t, X, Y, \dot{X}, \dot{Y})\, dW, \tag{4.1}$$

$$d\dot{Y} = F_2(t, X, Y, \dot{X}, \dot{Y})\, dt + G_2(t, X, Y, \dot{X}, \dot{Y})\, dW,$$

where F_i and G_i $(i = 1, 2)$, are deterministic functions and dW is the infinitesimal increment of the Wiener process [46]. System (4.1) is said to be linear if the functions F_i and G_i are linear functions with respect to variables X and Y and their respective derivatives. Thus a linear system of two second-order SODEs has the form:

$$\begin{aligned} d\dot{X} &= \Big(\alpha_{21}(t)X + \alpha_{22}(t)\dot{X}, + \alpha_{23}(t)Y + \alpha_{24}(t)\dot{Y} + \alpha_{20}(t)\Big)\ dt \\ &\quad + \Big(\beta_{21}(t)X + \beta_{22}(t)\dot{X}, + \beta_{23}(t)Y + \beta_{24}(t)\dot{Y} + \beta_{20}(t)\Big)\ dW, \\ d\dot{Y} &= \Big(\alpha_{41}(t)X + \alpha_{42}(t)\dot{X}, + \alpha_{43}(t)Y + \alpha_{44}(t)\dot{Y} + \alpha_{40}(t)\Big)\ dt \\ &\quad + \Big(\beta_{41}(t)X + \beta_{42}(t)\dot{X}, + \beta_{43}(t)Y + \beta_{44}(t)\dot{Y} + \beta_{40}(t)\Big)\ dW. \end{aligned} \tag{4.2}$$

System (4.2) can be rewritten in the form of the first-order SODEs:

$$\begin{aligned} dx_1 = x_2 dt,\ \ dx_2 = f_2(t, x_1, x_2, x_3, x_4)\, dt + g_2(t, x_1, x_2, x_3, x_4)\, dW, \\ dx_3 = x_4 dt,\ \ dx_4 = f_4(t, x_1, x_2, x_3, x_4)\, dt + g_4(t, x_1, x_2, x_3, x_4)\, dW. \end{aligned} \tag{4.3}$$

We introduce the following notations:

$$f_1 = x_2,\ \ f_2 = f_2(t, x_1, x_2, x_3, x_4),\ \ f_3 = x_4,\ \ f_4 = f_4(t, x_1, x_2, x_3, x_4),$$

$$g_1 = 0,\ \ g_2 = g_2(t, x_1, x_2, x_3, x_4),\ \ g_3 = 0,\ \ g_4 = g_4(t, x_1, x_2, x_3, x_4).$$

The generators of an admitted Lie group of equations (4.3) are considered in the form

$$h\partial_t + \xi_1\partial_{x_1} + \xi_2\partial_{x_2} + \xi_3\partial_{x_3} + \xi_4\partial_{x_4}, \tag{4.4}$$

where

$$h = h(t),\ \ \xi_1 = \xi_1(t, x_1, x_3),\ \ \xi_3 = \xi_3(t, x_1, x_3).$$

The coefficients ξ_2 and ξ_4 in (4.4) are obtained by the prolongation formulae:

$$\begin{aligned} \xi_2 &= \xi_{1,t} + \xi_{1,x_1} x_2 + \xi_{1,x_3} x_4 - h' x_2, \\ \xi_4 &= \xi_{3,t} + \xi_{3,x_1} x_2 + \xi_{3,x_3} x_4 - h' x_4. \end{aligned} \tag{4.5}$$

From here on, the following notation:

$$_{,t} = \frac{\partial}{\partial t},\ \ _{,j} = {}_{,x_j} = \frac{\partial}{\partial x_j}$$

is used so that the subsequent equations are written in a more compact form. The admitted Lie group can be defined by using the determining equations. The determining equations for the coefficients of the admitted generator in (4.4), given by h, ξ_1 and ξ_3 are [59, 57, 18]:

$$\xi_{i,t} + \xi_{i,j} f_j + \tfrac{1}{2}\xi_{i,jl} g_j g_l - f_{i,t} h - f_{i,j}\xi_j - f_i h' = 0, \tag{4.6}$$

$$\xi_{i,j} g_j - g_{i,t} h - \frac{1}{2} g_i h' - g_{i,j}\xi_j = 0. \tag{4.7}$$

Here $i = 1, 2, 3, 4$ and summation with respect to j is from 1 to 4. Notice that equations $(4.6)_{i=1}$ and $(4.6)_{i=3}$ coincide with equations (4.5), respectively.

The objective here is to find a generator (4.4) satisfying the determining equations (4.6), (4.7).

The paper is organized as follows: The subsequent sections deal with the determining equations derived in an Itô stochastic calculus context, equivalent transformations used to reduce the number of coefficients α_{ij} in system (4.2), the group classification of systems of two second-order SODEs given in terms of coefficients of the system and the latter part of the paper deals with the coefficients β_{ij} of system (4.2) and the analysis of their respective relations for them.

4.2 Determining equations

For the linear system (4.3) the determining equations (4.6) and (4.7) become

$$\xi_{1,x_3} x_4 + \xi_{1,x_1} x_2 + \xi_{1,t} - h' x_2 - \xi_2 = 0, \tag{4.8a}$$

$$\begin{gathered} -2f_{2,x_4}\xi_4 - 2f_{2,x_3}\xi_3 - 2f_{2,x_2}\xi_2 - 2f_{2,x_1}\xi_1 - 2f_{2,t}h + 2\xi_{2,x_4x_2} g_4 g_2 + \xi_{2,x_4x_4} g_4^2 + \\ 2\xi_{2,x_4} f_4 + 2\xi_{2,x_3} x_4 + \xi_{2,x_2,x_2} g_2^2 + 2\xi_{2,x_2} f_2 + 2\xi_{2,x_1} x_2 + 2\xi_{2,t} - 2h' f_2 = 0, \end{gathered} \tag{4.8b}$$

$$\xi_{3,x_3} x_4 + \xi_{3,x_1} x_2 + \xi_{3,t} - h' x_4 - \xi_4 = 0, \tag{4.8c}$$

$$\begin{gathered} -2f_{4,x_4}\xi_4 - 2f_{4,x_3}\xi_3 - 2f_{4,x_2}\xi_2 - 2f_{4,x_1}\xi_1 - 2f_{4,t}h + 2\xi_{4,x_4x_2} g_4 g_2 + \xi_{4,x_4,x_4} g_4^2 + \\ 2\xi_{4,x_4} f_4 + 2\xi_{4,x_3} x_4 + \xi_{4,x_2,x_2} g_2^2 + 2\xi_{4,x_2} f_2 + 2\xi_{4,x_1} x_2 + 2\xi_{4,t} - 2h' f_4 = 0, \end{gathered} \tag{4.8d}$$

$$-2g_{2,x_4}\xi_4 - 2g_{2,x_3}\xi_3 - 2g_{2,x_2}\xi_2 - 2g_{2,x_1}\xi_1 - 2g_{2,t}h + 2\xi_{2,x_4} g_4 + 2\xi_{2,x_2} g_2 - h' g_2 = 0, \tag{4.8e}$$

$$-2g_{4,x_4}\xi_4 - 2g_{4,x_3}\xi_3 - 2g_{4,x_2}\xi_2 - 2g_{4,x_1}\xi_1 - 2g_{4,t}h + 2\xi_{4,x_4} g_4 + 2\xi_{4,x_2} g_2 - h' g_4 = 0. \tag{4.8f}$$

Substituting the prolongation formula (4.5) into equations (4.8a - 4.8f), equations (4.8a) and (4.8c) are satisfied, and the remaining determining equations become:

$$\begin{aligned}&-f_{2,x_4}\xi_{3,x_3}x_4 - f_{2,x_4}\xi_{3,x_1}x_2 - f_{2,x_4}\xi_{3,t} + f_{2,x_4}h'x_4 - f_{2,x_3}\xi_3 - f_{2,x_2}\xi_{1,x_3}x_4\\ &-f_{2,x_2}\xi_{1,x_1}x_2 - f_{2,x_2}\xi_{1,t} + f_{2,x_2}h'x_2 - f_{2,x_1}\xi_1 - f_{2,t}h + 2\xi_{1,x_3x_1}x_4x_2 + 2\xi_{1,x_3t}x_4\\ &+\xi_{1,x_3x_3}x_4^2 + \xi_{1,x_3}f_4 + 2\xi_{1,x_1t}x_2 + \xi_{1,x_1x_1}x_2^2 + \xi_{1,x_1}f_2 + \xi_{1,tt} - h''x_2 - 2h'f_2 = 0,\end{aligned} \tag{4.9a}$$

$$\begin{aligned}&-f_{4,x_4}\xi_{3,x_3}x_4 - f_{4,x_4}\xi_{3,x_1}x_2 - f_{4,x_4}\xi_{3,t} + f_{4,x_4}h'x_4 - f_{4,x_3}\xi_3 - f_{4,x_2}\xi_{1,x_3}x_4\\ &-f_{4,x_2}\xi_{1,x_1}x_2 - f_{4,x_2}\xi_{1,t} + f_{4,x_2}h'x_2 - f_{4,x_1}\xi_1 - f_{4,t}h + 2\xi_{3,x_3x_1}x_4x_2 + 2\xi_{3,x_3t}x_4\\ &+\xi_{3,x_3x_3}x_4^2 + \xi_{3,x_3}f_4 + 2\xi_{3,x_1t}x_2 + \xi_{3,x_1x_1}x_2^2 + \xi_{3,x_1}f_2 + \xi_{3,tt} - h''x_4 - 2h'f_4 = 0,\end{aligned} \tag{4.9b}$$

$$\begin{aligned}&-2g_{2,x_4}\xi_{3,x_3}x_4 - 2g_{2,x_4}\xi_{3,x_1}x_2 - 2g_{2,x_4}\xi_{3,t} + 2g_{2,x_4}h'x_4 - 2g_{2,x_3}\xi_3 - 2g_{2,x_2}\xi_{1,x_3}x_4\\ &-2g_{2,x_2}\xi_{1,x_1}x_2 - 2g_{2,x_2}\xi_{1,t} + 2g_{2,x_2}h'x_2 - 2g_{2,x_1}\xi_1 - 2g_{2,t}h + 2\xi_{1,x_3}g_4 + 2\xi_{1,x_1}g_2\\ &-3h'g_2 = 0,\end{aligned} \tag{4.9c}$$

$$\begin{aligned}&-2g_{4,x_4}\xi_{3,x_3}x_4 - 2g_{4,x_4}\xi_{3,x_1}x_2 - 2g_{4,x_4}\xi_{3,t} + 2g_{4,x_4}h'x_4 - 2g_{4,x_3}\xi_3 - 2g_{4,x_2}\xi_{1,x_3}x_4\\ &-2g_{4,x_2}\xi_{1,x_1}x_2 - 2g_{4,x_2}\xi_{1,t} + 2g_{4,x_2}h'x_2 - 2g_{4,x_1}\xi_1 - 2g_{4,t}h + 2\xi_{3,x_3}g_4 + 2\xi_{3,x_1}g_2\\ &-3h'g_4 = 0.\end{aligned} \tag{4.9d}$$

The latter equations are deterministic (non-stochastic). We then proceed in classifying equations (4.3). The functions f_j and g_j $(j = 2, 4)$ take the form:

$$\begin{aligned}&f_2 = \alpha_{21}x_1 + \alpha_{22}x_2 + \alpha_{23}x_3 + \alpha_{24}x_4 + \alpha_{20}, \quad f_4 = \alpha_{41}x_1 + \alpha_{42}x_2 + \alpha_{43}x_3 + \alpha_{44}x_4 + \alpha_{40},\\ &g_2 = \beta_{21}x_1 + \beta_{22}x_2 + \beta_{23}x_3 + \beta_{24}x_4 + \beta_{20}, \quad g_4 = \beta_{41}x_1 + \beta_{42}x_2 + \beta_{43}x_3 + \beta_{44}x_4 + \beta_{40},\end{aligned} \tag{4.10}$$

where α_{ij}'s and β_{ij}'s $(i = 2, 4;\ j = 0, .., 4)$ depend on t. Notice that equation (4.3) with equation (4.10) can be rewritten in the form:

$$\begin{aligned}&d\mathbf{X} = \dot{\mathbf{X}}dt,\\ &d\dot{\mathbf{X}} = (A\mathbf{X} + B\dot{\mathbf{X}} + \mathbf{a})dt + (B_1\mathbf{X} + B_2\dot{\mathbf{X}} + \mathbf{b})dW,\end{aligned} \tag{4.11}$$

where $A = \begin{pmatrix} \alpha_{21} & \alpha_{23} \\ \alpha_{41} & \alpha_{43} \end{pmatrix}$, $B = \begin{pmatrix} \alpha_{22} & \alpha_{24} \\ \alpha_{42} & \alpha_{44} \end{pmatrix}$, $\mathbf{a} = \begin{pmatrix} \alpha_{20} \\ \alpha_{40} \end{pmatrix}$, $B_1 = \begin{pmatrix} \beta_{21} & \beta_{23} \\ \beta_{41} & \beta_{43} \end{pmatrix}$, $B_2 = \begin{pmatrix} \beta_{22} & \beta_{24} \\ \beta_{42} & \beta_{44} \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} \beta_{20} \\ \beta_{40} \end{pmatrix}$ with $\mathbf{X} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$ and $\dot{\mathbf{X}} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$.

Here, A, B, B_i, $(i = 1,\ 2)$, $\mathbf{a}$ and $\mathbf{b}$ depend on t.

4.3 Equivalent transformations

In this section, we analyze the determining equations by considering the transformation,

$$\mathbf{X} = C(t)\mathbf{X}_1 + \bar{\mathbf{h}}(t), \tag{4.12}$$

where $C = C(t)$ is a nonsingular matrix and $\bar{\mathbf{h}}$ is a vector. Using transformation (4.12), system (4.11) becomes

$$\begin{aligned} d\mathbf{X}_1 &= \dot{\mathbf{X}}_1 dt, \\ d\dot{\mathbf{X}}_1 &= \left(\bar{A}\mathbf{X}_1 + \bar{B}\dot{\mathbf{X}}_1 + \bar{\mathbf{a}}\right) dt + \left(\bar{F}_1\mathbf{X}_1 + \bar{F}_2\dot{\mathbf{X}}_1 + \bar{\mathbf{b}}\right) dW, \end{aligned} \tag{4.13}$$

where

$$\bar{B} = C^{-1}(BC - 2\dot{C}), \quad \bar{A} = C^{-1}(AC + B\dot{C} - \ddot{C}), \quad \bar{\mathbf{a}} = C^{-1}(A\bar{h} + B\dot{\bar{h}} - \ddot{\bar{h}}),$$

$$\bar{B}_1 = CB_1, \quad \bar{B}_2 = CB_2, \quad \bar{\mathbf{b}} = C\mathbf{b}.$$

Choosing C and m such that

$$\dot{C} = \frac{1}{2}BC, \quad \ddot{\bar{h}} = B\dot{\bar{h}} + A\bar{h},$$

we have

$$\bar{A} = C^{-1}\left(A + \frac{1}{4}B^2 - \frac{1}{2}\dot{B}\right) C.$$

This allows us to assume that

$$B = 0, \quad \mathbf{a} = 0$$

or

$$\alpha_{20} = \alpha_{22} = \alpha_{24} = \alpha_{40} = \alpha_{42} = \alpha_{44} = 0.$$

The system (4.11) can now be rewritten as

$$\begin{aligned} d\mathbf{X} &= \dot{\mathbf{X}} dt, \\ d\dot{\mathbf{X}} &= A\mathbf{X} dt + (B_1\mathbf{X} + B_2\dot{\mathbf{X}} + \mathbf{b}) dW. \end{aligned} \tag{4.14}$$

In the present paper, we study the case where A, B_i, $(i = 1, 2)$ and $\mathbf{b}$ are constant.

4.4 Group classification

For the group classification of systems of two second-order SODEs, we consider the functions f_j and g_j, $(j = 2, 4)$ to be linear with constant coefficients, that is, according to equations (4.10) and the equivalence transformations:

$$\begin{aligned} f_2 &= \alpha_{21}x_1 + \alpha_{23}x_3, \quad f_4 = \alpha_{41}x_1 + \alpha_{43}x_3, \\ g_2 &= \beta_{21}x_1 + \beta_{22}x_2 + \beta_{23}x_3 + \beta_{24}x_4 + \beta_{20}, \\ g_4 &= \beta_{41}x_1 + \beta_{42}x_2 + \beta_{43}x_3 + \beta_{44}x_4 + \beta_{40}. \end{aligned} \tag{4.15}$$

Substituting the functions (4.15) into determining equations (4.9a-4.9d), we obtain the following:

$$\begin{aligned} &2\xi_{1,x_3x_1}x_4x_2 + \xi_{1,x_3x_3}x_4^2 + \xi_{1,x_3}(\alpha_{41}x_1 + \alpha_{43}x_3) + 2\xi_{1,x_1t}x_2 + \xi_{1,x_1}(\alpha_{23}x_3 + \alpha_{21}x_1) - 2\alpha_{23}h'x_3 \\ &-\alpha_{23}\xi_3 - 2\alpha_{21}h'x_1 - \alpha_{21}\xi_1 + 2\xi_{1,x_3t}x_4 + \xi_{1,x_1x_1}x_2^2 + \xi_{1,tt} - h''x_2 = 0, \end{aligned} \tag{4.16a}$$

$$\begin{aligned} &2\xi_{3,x_3x_1}x_4x_2 + 2\xi_{3,x_3t}x_4 + \xi_{3,x_3x_3}x_4^2 + \xi_{3,x_3}(\alpha_{41}x_1 + \alpha_{43}x_3) + 2\xi_{3,x_1t}x_2 + \xi_{3,x_1x_1}x_2^2 + \xi_{3,x_1}(\alpha_{23}x_3 \\ &+\alpha_{21}x_1) - 2\alpha_{41}h'x_1 - \alpha_{41}\xi_1 - 2\alpha_{43}h'x_3 - \alpha_{43}\xi_3 + \xi_{3,tt} - h''x_4 = 0, \end{aligned} \tag{4.16b}$$

$$\begin{aligned} &-2\xi_{3,x_3}\beta_{24}x_4 - 2\xi_{3,x_1}\beta_{24}x_2 + 2\xi_{1,x_3}(\beta_{44}x_4 + \beta_{43}x_3 + \beta_{42}x_2 + \beta_{41}x_1 + \beta_{40} - \beta_{22}x_4) \\ &+2\xi_{1,x_1}(\beta_{24}x_4 + \beta_{23}x_3 + \beta_{21}x_1 + \beta_{20}) - 2\xi_{1,t}\beta_{22} - 2\beta_{24}\xi_{3,t} - \beta_{24}h'x_4 - 3\beta_{23}h'x_3 \\ &-2\beta_{23}\xi_3 - \beta_{22}h'x_2 - 3\beta_{21}h'x_1 - 2\beta_{21}\xi_1 - 3\beta_{20}h' = 0, \end{aligned} \tag{4.16c}$$

$$\begin{aligned} &2\xi_{3,x_3}(\beta_{43}x_3 + \beta_{42}x_2 + \beta_{41}x_1 + \beta_{40}) + 2\xi_{3,x_1}(-\beta_{44}x_2 + \beta_{24}x_4 + \beta_{23}x_3 + \beta_{22}x_2 + \beta_{21}x_1 \\ &+\beta_{20}) - 2\xi_{1,x_3}\beta_{42}x_4 - 2\xi_{1,x_1}\beta_{42}x_2 - 2\xi_{1,t}\beta_{42} - 2\beta_{44}\xi_{3,t} - \beta_{44}h'x_4 - 3\beta_{43}h'x_3 \\ &-2\beta_{43}\xi_3 - \beta_{42}h'x_2 - 3\beta_{41}h'x_1 - 2\beta_{41}\xi_1 - 3\beta_{40}h' = 0. \end{aligned} \tag{4.16d}$$

Splitting equation (4.16a) with respect to x_2 and x_4 results in

$$\xi_1 = \gamma_2 + \gamma_1 x_1 + k_1 x_3, \tag{4.17}$$

$$h' = 2\gamma_1 + k_2, \tag{4.18}$$

where γ_1 and γ_2 are functions of t and k_1 and k_2 are constant. Splitting equation (4.16b) with respect to x_2 and x_4 gives

$$\xi_3 = \gamma_3 + k_3 x_1 + (k_4 + \gamma_1)x_3, \tag{4.19}$$

where γ_1 and γ_3 are functions of t and k_3 and k_4 are constant.

These conditions lead to

$$\xi_2 = \gamma_2' + \gamma_1' x_1 - (k_2 + \gamma_1)x_2 + k_1 x_4, \quad \xi_4 = \gamma_3' + k_3 x_2 + \gamma_1' x_3 + (k_4 - k_2 - \gamma_1)x_4. \tag{4.20}$$

After substitution of the conditions (4.17),

$$-\alpha_{23}\gamma_3 - \alpha_{21}\gamma_2 + \gamma_2'' = 0, \tag{4.21a}$$

$$-4\gamma_1\alpha_{21} + \alpha_{41}k_1 - \alpha_{23}k_3 - 2\alpha_{21}k_2 + \gamma_1'' = 0, \tag{4.21b}$$

$$-4\gamma_1\alpha_{23} - 2\alpha_{23}k_2 - \alpha_{23}k_4 + (\alpha_{43} - \alpha_{21})k_1 = 0. \tag{4.21c}$$

For splitting (4.16b), one has the following equations:

$$-\alpha_{41}\gamma_2 - \alpha_{43}\gamma_3 + \gamma_3'' = 0, \tag{4.22a}$$

$$-4\gamma_1\alpha_{41} - 2\alpha_{41}k_2 + \alpha_{41}k_4 + (\alpha_{21} - \alpha_{43})k_3 = 0, \tag{4.22b}$$

$$-4\gamma_1\alpha_{43} - \alpha_{41}k_1 + \alpha_{23}k_3 - 2\alpha_{43}k_2 + \gamma_1'' = 0. \tag{4.22c}$$

Note that from equations (4.21a) and (4.22a), γ_2 and γ_3 are solutions of the system of two linear second order equations

$$\gamma_2'' = \alpha_{21}\gamma_2 + \alpha_{23}\gamma_3, \quad \gamma_3'' = \alpha_{41}\gamma_2 + \alpha_{43}\gamma_3. \tag{4.23}$$

Observe also that equations (4.21b) and (4.22c) give two conditions:

$$\gamma_1'' = h'(\alpha_{21} + \alpha_{43}), \tag{4.24a}$$

$$h'(\alpha_{21} - \alpha_{43}) - \alpha_{41}k_1 + \alpha_{23}k_3 = 0, \tag{4.24b}$$

where $h' = 2\gamma_1 + k_2$. Further simplifications are related with simplifications of the matrix A. The matrix A can be presented in Jordan form. For a real 2×2 matrix A, the Jordan matrix is one of the following 3 types:
$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}$, $J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ where a, b, c are real numbers and $c > 0$. Before studying the three cases of A, it is worth listing down the determining equations

related with β_{ij}'s. From equations (4.16c) and (4.16d) and the above computations, the split determining equations related with the β_{ij}'s are as follows:

$$2\gamma_1' \begin{pmatrix} \beta_{22} \\ \beta_{24} \\ \beta_{42} \\ \beta_{44} \end{pmatrix} + 3h' \begin{pmatrix} \beta_{21} \\ \beta_{23} \\ \beta_{41} \\ \beta_{43} \end{pmatrix} + 2k_1 \begin{pmatrix} -\beta_{41} \\ \beta_{21} - \beta_{43} \\ 0 \\ \beta_{41} \end{pmatrix} + 2k_3 \begin{pmatrix} \beta_{23} \\ 0 \\ \beta_{43} - \beta_{21} \\ -\beta_{23} \end{pmatrix} + 2k_4 \begin{pmatrix} 0 \\ \beta_{23} \\ -\beta_{41} \\ 0 \end{pmatrix} = 0, \tag{4.25a}$$

$$h' \begin{pmatrix} \beta_{22} \\ \beta_{24} \\ \beta_{42} \\ \beta_{44} \end{pmatrix} + 2k_1 \begin{pmatrix} -\beta_{42} \\ \beta_{22} - \beta_{44} \\ 0 \\ \beta_{42} \end{pmatrix} + 2k_3 \begin{pmatrix} \beta_{24} \\ 0 \\ \beta_{44} - \beta_{22} \\ -\beta_{24} \end{pmatrix} + 2k_4 \begin{pmatrix} 0 \\ \beta_{24} \\ -\beta_{42} \\ 0 \end{pmatrix} = 0, \tag{4.25b}$$

$$\begin{aligned} &4\gamma_1 \begin{pmatrix} \beta_{20} \\ \beta_{40} \end{pmatrix} + 2\gamma_3' \begin{pmatrix} \beta_{24} \\ \beta_{44} \end{pmatrix} + 2\gamma_2' \begin{pmatrix} \beta_{22} \\ \beta_{42} \end{pmatrix} + 2\gamma_3 \begin{pmatrix} \beta_{23} \\ \beta_{43} \end{pmatrix} + 2\gamma_2 \begin{pmatrix} \beta_{21} \\ \beta_{41} \end{pmatrix} \\ &-2k_1 \begin{pmatrix} \beta_{40} \\ 0 \end{pmatrix} + 3k_2 \begin{pmatrix} \beta_{20} \\ \beta_{40} \end{pmatrix} - 2k_3 \begin{pmatrix} 0 \\ \beta_{20} \end{pmatrix} - 2k_4 \begin{pmatrix} 0 \\ \beta_{40} \end{pmatrix} = 0. \end{aligned} \tag{4.25c}$$

Here we consider the cases of the Jordan matrices associated with A.

1. **Case:** $A = J_1$. For this case, apart from the determining equations related with the β_{ij}'s, one has the following conditions to study:

$$(b - a) \begin{pmatrix} k_1 \\ k_3 \\ 2\gamma_1 + k_2 \end{pmatrix} = 0, \tag{4.26}$$

$$\gamma_1'' = (2\gamma_1 + k_2)(a + b), \tag{4.27}$$

$$\gamma_2'' = a\gamma_2, \quad \gamma_3'' = b\gamma_3. \tag{4.28}$$

Equations (4.26) give two cases: $b - a = 0$ or $b - a \neq 0$.

(a) Case: $b - a = 0$. For this case, one has $b = a$. The general solution for γ_2 depends on three cases, that is, $a = 0$, $a > 0$ and $a < 0$.

i. Case: $a = 0$. This also satisfies equation (4.27). For this case, $\gamma_1 = \gamma_{11}t + \gamma_{10}$, $\gamma_2 = \gamma_{21}t + \gamma_{20}$ and $\gamma_3 = \gamma_{31}t + \gamma_{30}$ where γ_{ij} $(i = 1,\ 2,\ 3,\ j = 0, 1)$ are constant. The generators obtained for this case are as follows:

$$\begin{aligned}
&X_0 = \partial_t, \quad X_1 = x_3\partial_{x_1} + x_4\partial_{x_2}, \quad X_2 = t\partial_t - x_2\partial_{x_2} - x_4\partial_{x_4},\\
&X_3 = x_1\partial_{x_3} + x_2\partial_{x_4}, \quad X_4 = x_3\partial_{x_3} + x_4\partial_{x_4},\\
&X_5 = t(t\partial_t + x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4}) + x_1\partial_{x_2} + x_3\partial_{x_4},\\
&X_6 = 2t\partial_t + x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4},\\
&X_7 = t\partial_{x_1} + \partial_{x_2}, \quad X_8 = \partial_{x_1}, \quad X_9 = t\partial_{x_3} + \partial_{x_4}, \quad X_{10} = \partial_{x_3}.
\end{aligned}$$

ii. Case: $a > 0$. Solving the equations (4.27-4.28), we obtain the following forms of γ's: $\gamma_1 = \gamma_{11}e^{2r_1t} + \gamma_{10}e^{2r_2t} - r_1r_2k_2t^2$, $\gamma_2 = \gamma_{21}e^{r_1t} + \gamma_{20}e^{r_2t}$ and $\gamma_3 = \gamma_{31}e^{r_1t} + \gamma_{30}e^{r_2t}$ where $r_2 = -r_1$ and $r_1 = \sqrt{a}$. The generators obtained for this case are as follows:

$$\begin{aligned}
&X_0 = \partial_t, \quad X_1 = x_3\partial_{x_1} + x_4\partial_{x_2},\\
&X_2 = (t - 2/3r_1r_2t^3)\partial_t - r_1r_2t[t(x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4}) - 2(x_1\partial_{x_2} + x_3\partial_{x_4})]\\
&-(x_2\partial_{x_2} + x_4\partial_{x_4}), \quad X_3 = x_1\partial_{x_3} + x_2\partial_{x_4}, \quad X_4 = x_3\partial_{x_3} + x_4\partial_{x_4},\\
&X_5 = e^{2r_1t}[1/r_1\partial_t + x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4} + 2r_1(x_1\partial_{x_2} + x_3\partial_{x_4})],\\
&X_6 = e^{2r_2t}[1/r_2\partial_t + x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4} + 2r_2(x_1\partial_{x_2} + x_3\partial_{x_4})],\\
&X_7 = e^{r_1t}(\partial_{x_1} + r_1\partial_{x_2}), \quad X_8 = e^{r_2t}(\partial_{x_1} + r_2\partial_{x_2}),\\
&X_9 = e^{r_1t}(\partial_{x_3} + r_1\partial_{x_4}), \quad X_{10} = e^{r_2t}(\partial_{x_3} + r_2\partial_{x_4}).
\end{aligned}$$

iii. Case: $a < 0$. Solving the equations (4.27-4.28), we obtain the following forms of γ's: $\gamma_1 = \gamma_{11}\sin(2r_1t) + \gamma_{10}\cos(2r_2t) - r_1r_2k_2t^2$, $\gamma_2 = \gamma_{21}\sin(r_1t) + \gamma_{20}\cos(r_2t)$ and $\gamma_3 = \gamma_{31}\sin(r_1t) + \gamma_{30}\cos(r_2t)$ where $r_2 = -r_1$ and $r_1 = \sqrt{a}$. The generators obtained for this case are as follows:

$$
\begin{aligned}
&X_0 = \partial_t, \quad X_1 = x_3\partial_{x_1} + x_4\partial_{x_2},\\
&X_2 = (t - 2/3r_1r_2t^3)\partial_t - r_1r_2t[t(x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4}) - 2(x_1\partial_{x_2} + x_3\partial_{x_4})]\\
&-(x_2\partial_{x_2} + x_4\partial_{x_4}), \quad X_3 = x_1\partial_{x_3} + x_2\partial_{x_4}, \quad X_4 = x_3\partial_{x_3} + x_4\partial_{x_4},\\
&X_5 = \sin(2r_1t)(x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4}) - 2r_1\cos(2r_1t)(1/2r_1^2\partial_t - x_1\partial_{x_2} - x_3\partial_{x_4}),\\
&X_6 = \cos(2r_2t)(x_1\partial_{x_1} - x_2\partial_{x_2} + x_3\partial_{x_3} - x_4\partial_{x_4}) + 2r_2\sin(2r_2t)(1/2r_2^2\partial_t - x_1\partial_{x_2} - x_3\partial_{x_4}),\\
&X_7 = \sin(r_1t)\partial_{x_1} + r_1\cos(r_1t)\partial_{x_2}, \quad X_8 = \cos(r_2t)\partial_{x_1} - r_2\sin(r_2t)\partial_{x_2},\\
&X_9 = \sin(r_1t)\partial_{x_3} + r_1\cos(r_1t)\partial_{x_4}, \quad X_{10} = \cos(r_2t)\partial_{x_3} - r_2\sin(r_2t)\partial_{x_4}.
\end{aligned}
$$

(b) Case: $b - a \neq 0$. For this case, one obtains $k_1 = k_3 = 0$ and $\gamma_1 = -k_2/2$. This gives $h = const$. The general solution for γ's depends on 2 cases: $a = 0$; $b > 0$ and $a = 0$; $b < 0$.

i. Case: $a = 0$ and $b > 0$. For this case, $\gamma_2 = \gamma_{21}t + \gamma_{20}$ and $\gamma_3 = \gamma_{31}e^{r_1t} + \gamma_{30}e^{r_2t}$ where $r_2 = -r_1$ and $r_1 = \sqrt{b}$. The generators obtained for this case are as follows:

$$
\begin{aligned}
&X_0 = \partial_t, \quad X_1 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4},\\
&X_2 = x_3\partial_{x_3} + x_4\partial_{x_4}, \quad X_3 = t\partial_{x_1} + \partial_{x_2}, \quad X_4 = \partial_{x_1},\\
&X_5 = e^{r_1t}(\partial_{x_3} + r_1\partial_{x_4}), \quad X_6 = e^{r_2t}(\partial_{x_3} + r_2\partial_{x_4}).
\end{aligned}
$$

ii. $a = 0$ and $b < 0$. For this case, one obtains: $\gamma_2 = \gamma_{21}t + \gamma_{20}$ and $\gamma_3 = \gamma_{31}\sin(r_1t) + \gamma_{30}\cos(r_2t)$ where $r_2 = -r_1$ and $r_1 = \sqrt{b}$. The generators obtained for this case are as follows:

$$
\begin{aligned}
&X_0 = \partial_t, \quad X_1 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4},\\
&X_2 = x_3\partial_{x_3} + x_4\partial_{x_4}, \quad X_3 = t\partial_{x_1} + \partial_{x_2}, \quad X_4 = \partial_{x_1},\\
&X_5 = \sin(r_1t)\partial_{x_3} + r_1\cos(r_1t)\partial_{x_4}, \quad X_6 = \cos(r_2t)\partial_{x_3} - r_2\sin(r_2t)\partial_{x_4}.
\end{aligned}
$$

2. **Case:** $A = J_2$. For this case, along with the determining equations related with the β_{ij}'s, one needs to analyze the following:

$$c(4\gamma_1 + 2k_2 + k_4) = 0, \tag{4.29}$$

$$c(4\gamma_1 + 2k_2 - k_4) = 0, \tag{4.30}$$

$$c(k_1 + k_3) = 0, \tag{4.31}$$

$$\gamma_1'' = 2a(2\gamma_1 + k_2), \tag{4.32}$$

$$\gamma_2'' = a\gamma_2 + c\gamma_3, \quad \gamma_3'' = -c\gamma_2 + a\gamma_3. \tag{4.33}$$

As $c > 0$, equations (4.29-4.32) yield $k_1 = -k_3$, $k_4 = 0$ and $\gamma_1 = -k_2/2$. The general solution for γ_2 and γ_3 depend on cases: $a = 0$ and $a \neq 0$.

(a) Case: $a = 0$. Solving the equations (4.33), we obtain the following forms of γ_2 and γ_3:

$$\begin{aligned} \gamma_2 &= (\gamma_{21}e^{q_1 t} + \gamma_{20}e^{-q_1 t})\cos t + (\gamma_{31}e^{q_1 t} + \gamma_{30}e^{-q_1 t})\sin t, \\ \gamma_3 &= (-\gamma_{21}e^{q_1 t} + \gamma_{20}e^{-q_1 t})\sin t + (\gamma_{31}e^{q_1 t} - \gamma_{30}e^{-q_1 t})\cos t, \end{aligned} \tag{4.34}$$

where $q_1 = c/2$. The generators obtained for this case are as follows:

$$\begin{aligned} &X_0 = \partial_t, \quad X_1 = x_3\partial_{x_1} + x_4\partial_{x_2} - x_1\partial_{x_3} - x_2\partial_{x_4}, \\ &X_2 = t(x_1\partial_{x_1} + x_2\partial_{x_2} - x_3\partial_{x_3} + x_4\partial_{x_4}) + x_1\partial_{x_2} + x_3\partial_{x_4}, \\ &X_3 = x_1\partial_{x_1} + x_2\partial_{x_2} - x_3\partial_{x_3} + x_4\partial_{x_4}, \\ &X_4 = e^{q_1 t}[\cos t(\partial_{x_1} + q_1\partial_{x_2} - \partial_{x_4}) - \sin t(\partial_{x_2} + \partial_{x_3} + q_1\partial_{x_4})], \\ &X_5 = e^{-q_1 t}[\cos t(\partial_{x_1} - q_1\partial_{x_2} + \partial_{x_4}) - \sin t(\partial_{x_2} - \partial_{x_3} + q_1\partial_{x_4})], \\ &X_6 = e^{q_1 t}[\sin t(\partial_{x_1} + q_1\partial_{x_2} - \partial_{x_4}) + \cos t(\partial_{x_2} + \partial_{x_3} + q_1\partial_{x_4})], \\ &X_7 = e^{-q_1 t}[\sin t(\partial_{x_1} - q_1\partial_{x_2} - \partial_{x_4}) + \cos t(\partial_{x_2} + \partial_{x_3} - q_1\partial_{x_4})]. \end{aligned}$$

(b) Case: $a \neq 0$. Solving the equations (4.33), we obtain the following forms of γ_2 and γ_3:

$$\begin{aligned} \gamma_2 &= (\gamma_{21}e^{q_1 t} + \gamma_{20}e^{-q_1 t})\cos(q_2 t) + (\gamma_{31}e^{q_1 t} + \gamma_{30}e^{-q_1 t})\sin(q_2 t) \\ \gamma_3 &= (-\gamma_{21}e^{q_1 t} + \gamma_{20}e^{-q_1 t})\sin(q_2 t) + (\gamma_{31}e^{q_1 t} - \gamma_{21}e^{-q_1 t})\cos(q_2 t), \end{aligned} \tag{4.35}$$

where $q_1 = \dfrac{c}{2q_2}$ and $q_2 = \dfrac{-a + \sqrt(a^2 + c^2)}{c}$. The generators obtained for this case are as follows:

$$X_0 = \partial_t, \quad X_1 = x_3\partial_{x_1} + x_4\partial_{x_2} - x_1\partial_{x_3} - x_2\partial_{x_4},$$
$$X_2 = t(x_1\partial_{x_1} + x_2\partial_{x_2} - x_3\partial_{x_3} + x_4\partial_{x_4}) + x_1\partial_{x_2} + x_3\partial_{x_4},$$
$$X_3 = x_1\partial_{x_1} + x_2\partial_{x_2} - x_3\partial_{x_3} + x_4\partial_{x_4},$$
$$X_4 = e^{q_1 t}[\cos(q_2 t)(\partial_{x_1} + q_1\partial_{x_2} - q_2\partial_{x_4}) - \sin(q_2 t)(q_2\partial_{x_2} + \partial_{x_3} + q_1\partial_{x_4})],$$
$$X_5 = e^{-q_1 t}[\cos(q_2 t)(\partial_{x_1} - q_1\partial_{x_2} + q_2\partial_{x_4}) - \sin(q_2 t)(q_2\partial_{x_2} - \partial_{x_3} + q_1\partial_{x_4})],$$
$$X_6 = e^{q_1 t}[\sin(q_2 t)(\partial_{x_1} + q_1\partial_{x_2} - q_2\partial_{x_4}) + \cos(q_2 t)(q_2\partial_{x_2} + \partial_{x_3} + q_1\partial_{x_4})],$$
$$X_7 = e^{-q_1 t}[\sin(q_2 t)(\partial_{x_1} - q_1\partial_{x_2} - q_2\partial_{x_4}) + \cos(q_2 t)(q_2\partial_{x_2} + \partial_{x_3} - q_1\partial_{x_4})].$$

3. **Case:** $A = J_3$. For this case, apart from the determining equations related with the β_{ij}'s, the following has to be considered:

$$4\gamma_1 + 2k_2 + k_4 = 0, \tag{4.36}$$

$$k_3 = 0, \tag{4.37}$$

$$\gamma_1'' = 2a(2\gamma_1 + k_2), \tag{4.38}$$

$$\gamma_2'' = a\gamma_2 + \gamma_3, \quad \gamma_3'' = a\gamma_3. \tag{4.39}$$

From equation (4.36), one gets $k_4 = -2(2\gamma_1 + k_2)$. Here, we have two cases to study: $a = 0$ and $a \neq 0$.

(a) Case $a = 0$. For this case, $\gamma_1 = \gamma_{11}t + \gamma_{10}$, $\gamma_3 = \gamma_{31}t + \gamma_{30}$ and $\gamma_2 = \gamma_{31}t^3/6 + \gamma_{30}t^2/2 + \gamma_{21}t + \gamma_{20}$ where γ_{ij} ($i = 1, 2, 3$, $j = 0, 1$) are constant. The generators obtained for this case are as follows:

$$X_0 = \partial_t, \quad X_1 = x_3\partial_{x_1} + x_4\partial_{x_2}, \quad X_2 = t\partial_t - x_2\partial_{x_2} - 2x_3\partial_{x_3} - 3x_4\partial_{x_4},$$
$$X_3 = t(t\partial_t + x_1\partial_{x_1} - x_2\partial_{x_2} - 3x_3\partial_{x_3} - 5x_4\partial_{x_4}) + x_1\partial_{x_2} + x_3\partial_{x_4},$$
$$X_4 = 2t\partial_t + x_1\partial_{x_1} - x_2\partial_{x_2} - 3x_3\partial_{x_3} - 5x_4\partial_{x_4}, \quad X_5 = t\partial_{x_1} + \partial_{x_2},$$
$$X_6 = \partial_{x_1}, \quad X_7 = t^3/6\partial_{x_1} + t^2/2\partial_{x_2} + t\partial_{x_3} + \partial_{x_4}, \quad X_8 = t^2/2\partial_{x_1} + t\partial_{x_2} + \partial_{x_3}.$$

(b) Case $a \neq 0$. For this case, one assume that $h' = 0$. The general solution for γ_3 depends on these cases: $a > 0$ and $a < 0$.

i. Case $a > 0$. Solving γ's, one obtains: $\gamma_3 = \gamma_{31} e^{r_1 t} + \gamma_{30} e^{r_2 t}$ and $\gamma_2 = \gamma_{21} e^{r_1 t} + \gamma_{20} e^{r_2 t} + \gamma_{31}/r_1^2 e^{r_1 t} + \gamma_{30}/r_2^2 e^{r_2 t}$. The generators obtained for this case are as follows:

$$X_0 = \partial_t, \quad X_1 = x_3 \partial_{x_1} + x_4 \partial_{x_2}, \quad X_2 = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4},$$
$$X_3 = e^{r_1 t}(\partial_{x_1} + r_1 \partial_{x_2}), \quad X_4 = e^{r_2 t}(\partial_{x_1} + r_2 \partial_{x_2}),$$
$$X_5 = e^{r_1 t}(1/r_1^2 \partial_{x_1} + 1/r_1 \partial_{x_2} + \partial_{x_3} + r_1 \partial_{x_4}),$$
$$X_6 = e^{r_2 t}(1/r_2^2 \partial_{x_1} + 1/r_2 \partial_{x_2} + \partial_{x_3} + r_2 \partial_{x_4}).$$

ii. Case $a < 0$. Solving γ's for this case we obtain: $\gamma_3 = \gamma_{31} \sin(r_1 t) + \gamma_{30} \cos(r_2 t)$ and $\gamma_2 = (\gamma_{21} - \gamma_{31}/r_1^2) \sin(r_1 t) + (\gamma_{20} - \gamma_{30}/r_2^2) \cos(r_2 t)$. The generators obtained for this case are as follows:

$$X_0 = \partial_t, \quad X_1 = x_3 \partial_{x_1} + x_4 \partial_{x_2}, \quad X_2 = x_1 \partial_{x_1} + x_2 \partial_{x_2} + x_3 \partial_{x_3} + x_4 \partial_{x_4}$$
$$X_3 = \sin(r_1 t) \partial_{x_1} + r_1 \cos(r_1 t) \partial_{x_2}, \quad X_4 = \cos(r_2 t) \partial_{x_1} - r_2 \sin(r_2 t) \partial_{x_2},$$
$$X_5 = -\sin(r_1 t)(1/r_1^2 \partial_{x_1} - \partial_{x_3}) - \cos(r_1 t)(1/r_1 \partial_{x_2} - r_1 \partial_{x_4}),$$
$$X_6 = -\cos(r_2 t)(1/r_2^2 \partial_{x_1} - \partial_{x_3}) + \sin(r_2 t)(1/r_2 \partial_{x_2} - r_2 \partial_{x_4}).$$

For each of these 3 cases of matrix A, the study of the determining equations related with the β_{ij}'s was carried out and the relationships between them were found. However the details for this computation have been left due to the journal restriction on the number of pages per paper.

4.5 Conclusion

We considered a system of two linear second-order stochastic ordinary differential equations with constant coefficients and found the determining equations which are non-stochastic. We illustrated the cases where the α_{ij}'s for the determining equations were found and used to simplify the original system of stochastic ordinary differential equations.

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Chapter 5

Conclusion

There are several methods of finding analytical solutions. One way is to guess a solution and use the Itô calculus to verify that it is a solution for the SODE under the consideration. For some classes of SODEs, analytical formulae exist to find the solutions. However, most SODEs, especially nonlinear SODEs, do not have analytical solutions. A class of solvable SODEs is the linear class, for which both the drift and volatility functions are linear. In this thesis, we used a group analysis approach to find solutions for certain classes of linear differential equations.

Firstly, we considered systems of two linear second-order ODEs without a stochastic component. The algebraic approach was used to solve the group classification problem for systems of two linear second-order ODEs. We have shown that using the algebraic approach leads to the study of a variety of cases in addition to those already obtained in the literature. We illustrated that this approach can be used as an effective tool to study the group classification of the type of systems studied in this thesis.

Since most SODEs are nonlinear, we have presented the linearization criteria for systems of two nonlinear second-order SODEs. We provided the necessary and sufficient conditions for linearization by an invertible transformation. In addition, we also showed that a system of two nonlinear second-order SODEs is linearizable via an invertible transformation when certain conditions are satisfied. Moreover, a code was developed using REDUCE for checking whether a system of two second-order SODEs is linearizable. Certain nonlinear second-order SODEs appeared to be linearizable via invertible transformations. The result was given in terms of a Theorem with three examples. This thesis gives a new treatment for the linearization of two second-order SODEs.

Lastly, we considered the underlying group theoretic properties of a system of two linear second-order SODEs with constant coefficients. For this system we obtained the determining equations and the corresponding equivalent transformations which assist with further classifying the system for some selected cases. This system with constant coefficients was solved as part of this thesis.

Future Research

In chapter four, we only considered the determining equations for the α_{ij}'s due to the journal restriction on the number of pages. However, one selected case of the determining equations involving β_{ij}'s is considered here, with the rest of the cases being left for a future research project which is underway.

Case: $A = J_1$.

From equations (4.26), one obtains $k_1 = k_3 = 0$ and $\gamma_1 = -k_2/2$. After substituting these conditions, the remaining determining equations with respect to β_{ij} are as follows:

$$2\beta_{24}\gamma_3' + 2\beta_{23}\gamma_3 + 2\beta_{22}\gamma_2' + 2\beta_{21}\gamma_2 + \beta_{20}k_2 = 0, \tag{5.1a}$$

$$2\beta_{44}\gamma_3' + 2\beta_{43}\gamma_3 + 2\beta_{42}\gamma_2' + 2\beta_{41}\gamma_2 + \beta_{40}(k_2 - 2k_4) = 0, \tag{5.1b}$$

$$k_4 \begin{pmatrix} \beta_{23} \\ \beta_{24} \\ \beta_{41} \\ \beta_{42} \end{pmatrix} = 0. \tag{5.1c}$$

From this case, one can further study the following two cases:

1. the case where there exists at least one generator with $k_4 \neq 0$,
2. the case with all generators having $k_4 = 0$.

We consider the case for $k_4 \neq 0$ for illustrative purposes.

1. For case 1, with $k_4 \neq 0$, we have $\beta_{23} = \beta_{24} = \beta_{41} = \beta_{42} = 0$. All determining equations are satisfied except for equations (4.28) and these equations become:

$$2\beta_{22}\gamma_2' + 2\beta_{21}\gamma_2 + \beta_{20}k_2 = 0,$$
$$2\beta_{44}\gamma_3' + 2\beta_{43}\gamma_3 + \beta_{40}(k_2 - 2k_4) = 0.$$

The general solution for the γ's depends on three cases: $a \neq 0$; $b \neq 0$, $a = 0$; $b \neq 0$ and $a \neq 0$; $b = 0$. The cases are considered as follows:

   (a) Case A: $a \neq 0$ and $b \neq 0$. For this case, one obtains: $\gamma_2 = \gamma_{21}e^{-p_1 t} + \gamma_{20}e^{p_1 t}$ and $\gamma_3 = \gamma_{31}e^{-p_2 t} + \gamma_{30}e^{p_2 t}$, where $p_1^2 = a$ and $p_2^2 = b$. If $\beta_{21} = -\beta_{22}p_1$ and $\beta_{43} = -\beta_{44}p_2$. Hence, two cases to consider are: $\beta_{40} \neq 0$ and $\beta_{40} = 0$.

   Case A.1: If $\beta_{40} \neq 0$, then $k_2 = 2k_4$. The generators obtained for this case are:

$$X_0 = \partial_t, \quad X_1 = x_1\partial_{x_1} + x_2\partial_{x_2}, \quad X_2 = e^{p_1 t}(\partial_{x_1} + p_1\partial_{x_2}), \quad X_3 = e^{p_2 t}(\partial_{x_3} + p_2\partial_{x_4}).$$

   Case A.2: In this case, $\beta_{40} = 0$. Hence the two sub-cases to study are: $\beta_{20} \neq 0$ and $\beta_{20} = 0$.

   Case A.2.1: Here $\beta_{20} \neq 0$, so that $k_2 = 0$. The generators obtained for this case are:

$$X_0 = \partial_t, \quad X_1 = x_3\partial_{x_3} + x_4\partial_{x_4}, \quad X_2 = e^{p_1 t}(\partial_{x_1} + p_1\partial_{x_2}), \quad X_3 = e^{p_2 t}(\partial_{x_3} + p_2\partial_{x_4}).$$

   Case A.2.2: For this case, $\beta_{20} = 0$. The generators obtained in this case are:

$$X_0 = \partial_t, \quad X_1 = x_3\partial_{x_3} + x_4\partial_{x_4}, \quad X_2 = e^{p_1 t}(\partial_{x_1} + p_1\partial_{x_2}), \quad X_3 = e^{p_2 t}(\partial_{x_3} + p_2\partial_{x_4}),$$
$$X_4 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4}.$$

   (b) Case B: $a = 0$ and $b \neq 0$. For this case, one obtains $\gamma_2 = \gamma_{21}t + \gamma_{20}$ and $\gamma_3 = \gamma_{31}e^{p_2 t} + \gamma_{30}e^{-p_2 t}$. The cases to consider are: $\beta_{40} \neq 0$ and $\beta_{21} \neq 0$.

   Case B.1: For $\beta_{40} \neq 0$, we further, consider two sub-cases: $\beta_{21} \neq 0$ and $\beta_{21} = 0$.

   Case B.1.1: For $\beta_{21} \neq 0$, this lead to $\beta_{20} = -\gamma_{20}\beta_{21}/k_4$. For simplicity, we let $\beta_{21}/k_4 = 1$. The generators obtained for this case are as follows:

$$X_0 = \partial_t, \quad X_1 = -x_1\partial_{x_1} - x_2\partial_{x_2}, \quad X_2 = \partial_{x_1}, \quad X_3 = e^{p_2 t}(\partial_{x_3} + p_2\partial_{x_4}).$$

(c) Case C: $a \neq 0$ and $b = 0$. For this case, we obtain: $\gamma_2 = \gamma_{21}e^{p_2 t} + \gamma_{20}e^{-p_2 t}$ and $\gamma_3 = \gamma_{31}t + \gamma_{30}$. The generators obtained for this case are as follows:

$$X_0 = \partial_t, \quad X_1 = x_1\partial_{x_1} + x_2\partial_{x_2} + x_3\partial_{x_3} + x_4\partial_{x_4},$$
$$X_2 = x_3\partial_{x_3} + x_4\partial_{x_4}, \quad X_3 = t\partial_{x_3} + \partial_{x_4}, \quad X_4 = \partial_{x_3},$$
$$X_5 = e^{p_2 t}(\partial_{x_1} + p_2\partial_{x_2}), \quad X_6 = e^{-p_2 t}(\partial_{x_1} - p_2\partial_{x_2}).$$

The rest of the study of the analysis of the determining equations related to β_{ij}'s has been left for a future research project. The rest of the cases would consider $A = J_2$ and $A = J_3$.

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