UNIVERSITY OF KWAZULU-NATAL

UNIVERSITY OF KWAZULU-NATAL

ROLE OF WEYL TENSOR AND SPACETIME SHEAR IN RELATIVISTIC FLUIDS

ROGER MBONGA MAYALA

.

Role of Weyl tensor and spacetime shear in relativistic fluids

Roger Mbonga Mayala

This thesis is submitted to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science, University of KwaZulu-Natal, Durban, in fulfilment of the academic requirements for the degree of Doctor of Philosophy in Applied Mathematics.

As the candidate's supervisors, we have approved this thesis for submission.

Signed: [redacted] Prof S. D. Maharaj Date: 09/ 11/ 2021

Signed: [redacted] Prof R. Goswami Date: 09/ 11/ 2021

Dedication

To my wife Ida Noki Mayala and to my children:

Philippe Mayala, Alpha-Maureen Mayala and Emmanuella-Chantal Mayala,

I dedicate this work.

Acknowledgments

I thank first of all God who gave me the free breath of life in order to realize this work. May glory and praise come back to him.

I would like to thank my supervisors Prof Sunil D Maharaj and Prof Rituparno Goswami for having never ceased to guide me, and to advise me during the preparation of this thesis.

I especially thank Prof Rituparno Goswami, and that he receives here the expression of my gratitude, for having proposed the topic of this thesis, and also for all the help.

Special thanks go also to my beloved wife Dr Ida Noki Mayala, for her love, moral support and patience. And to my son Philippe Mayala, and my daughters, Alpha-Maureen Mayala and Emmanuella-Chantal Mayala, for their patience and love.

My sincere thanks also go to my parents, Mr Philippe Mbonga Ngoma and Mrs Henriette Kasa Mvumbi, to my father-in-law Prof Philippe Noki Vesituluta (that his soul rests in peace), to my uncle Mr Emmanuel Mavinga Nzita, and to my brothers and sister for supporting me spiritually.

I thank the National Research Fondation (NRF), South Africa, for financial support, without which my studies would have been very difficult to complete.

Lastly, I would like to thank all those who, far or near, have encouraged me to complete this work.

Declaration 1 - Plagiarism

I, Roger Mbonga Mayala, student number: 216057850, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

2. This thesis has not been submitted for any degree or examination at any other university.

3. This thesis does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.

4. This thesis does not contain other persons' writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:

    a. their words have been re-written but the general information attributed to them has been referenced;

    b. where their exact words have been used, then their writing has been placed in italics and inside quotation marks, and referenced.

5. This thesis does not contain text, graphics or tables copied and posted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the References sections.

Signed: [redacted] Date: 09/ 11/ 2021

Declaration 2 - Publications

The publications on which the research presented in this thesis is based on the research:

Publication 1

R. M. Mayala, R. Goswami and S. D. Maharaj, Matter shear and vorticity in conformally flat spacetimes, Phys. Rev. D **103**, 044015 (2021)

Publication 2

R. M. Mayala, R. Goswami and S. D. Maharaj, Necessity of spacetime shear for cosmological gravitational waves, Gen. Relativ. Gravit., (submitted) (2021).

Abstract

The main gravitational theory in which we develop this work is general relativity. We study the role of the Weyl tensor in general relativistic fluid motion including the effects of spacetime shear. Firstly we consider conformally flat perturbations on the Friedmann Lemaitre Robertson Walker (FLRW) spacetime containing a general matter field. Working with the linearised field equations, we find some important geometrical properties of matter shear and vorticity, and show how they interact with the thermodynamic quantities in the absence of any free gravity powered by the Weyl curvature. We demonstrate that the matter shear obeys a transverse traceless tensor wave equation and the vorticity obeys a vector wave equation in this linearised regime. These shear and vorticity waves replace the gravitational waves in the sense that they causally carry information about local change in the curvature of these spacetimes. We also study the heat transport equation in this case, and show how this varies from the Newtonian case. Secondly we show that a general but shear-free perturbation of homogeneous and isotropic universes are necessarily silent, without any gravitational waves. We prove this in two steps. First, we establish that a shear-free perturbation of these universes are acceleration-free and the fluid flow geodesics of the background universe map onto themselves in the perturbed universe. This effect then decouples the evolution equations of the electric and magnetic part of the Weyl tensor in the perturbed spacetimes and the magnetic part no longer contains any tensor modes. Although the electric part, that drives the tidal forces, does have tensor modes sourced by the anisotropic stress, these modes have homogeneous oscillations at every point on a time slice without any wave propagation. This analysis shows the critical role of the shear tensor in generating cosmological gravitational waves.

Conventions, important formulas and abbreviations

For sign conventions we follow Ellis (1971) and Ellis and van Elst (1998).

Unless otherwise specified, we use natural units ($c = 8\pi G = 1$) throughout this dissertation.

Latin indices run from 0 to 3.

We use the $(-,\ +\ ,\ +\ ,+)$ signature.

The symmetrisation and the antisymmetrisation over the indexes of a tensor are defined respectively as

$$T_{(ab)} = \frac{1}{2}\left(T_{ab} + T_{ba}\right)\ , \qquad T_{[ab]} = \frac{1}{2}\left(T_{ab} - T_{ba}\right)\ . \tag{1}$$

The symbol ∇ represents covariant derivative and ∂ corresponds to partial differentiation.

The Riemann tensor is defined by

$$R^{a}{}_{bcd} = \Gamma^{a}{}_{bd,c} - \Gamma^{a}{}_{bc,d} + \Gamma^{e}{}_{bd}\Gamma^{a}{}_{ce} - \Gamma^{e}{}_{bc}\Gamma^{a}{}_{de}\ , \tag{2}$$

and $\Gamma^{a}{}_{bd}$ are the Christoffel symbols (i.e. symmetric in the lower indices) defined by

$$\Gamma^{a}{}_{bd} = \frac{1}{2}g^{ae}(g_{be,d} + g_{ed,b} - g_{bd,e}). \tag{3}$$

The Ricci tensor is obtained by contracting the *first* and the *third* indices of the Riemann tensor

$$R_{ab} = g^{cd}R_{cadb}\ . \tag{4}$$

The Hilbert–Einstein action in the presence of matter is given by

$$\mathcal{S} = \frac{1}{2}\int d^4x\sqrt{-g}\left[R - 2\mathcal{L}_m\right]\ , \tag{5}$$

variation of which gives the Einstein's field equations as

$$G_{ab} = T_{ab}\ . \tag{6}$$

The abbreviations below are used in this dissertation

- CDM Cold dark matter

- EFEs Einstein field equations

- FLRW Friedmann-Lemaitre-Robertson-Walker

- GR General relativity

- LRS Local rotational symmetry

- PSTF Projected symmetric and trace-free.

Contents

Chapter 1

Introduction

- Brief introduction of general relativity

General relativity (GR) is the theory of space, time and gravitation formulated by Albert Einstein in 1915 and published in 1916, eleven years after publishing his special theory of relativity. It has been considerably studied since the late 1950s. Furthermore, in the mid 1960s, the modern theory of gravitational collapse, singularities, and black holes have been developed. General relativity is a beautiful physical theory, invented for describing the gravitational field and the equations it obeys. General relativity is phrased in the language of mathematical theory of manifolds, i.e. differential geometry (Boonserm (2006)). General relativity is essentially special relativity on manifolds $\mathcal{M}$ instead of $\mathbb{R}^4$ together with Einstein's field equations.

In general relativity the collection of all events is given by the pair ($\mathcal{M}$, g), where $\mathcal{M}$ is a connected, four-dimensional Hausdorff C^∞ manifold and g is a Lorentz metric on $\mathcal{M}$ (Hawking and Ellis (1973)). Together with the existence of a Lorentz metric, the Hausdorff condition implies that $\mathcal{M}$ is also paracompact (Geroch (1968)). All pairs ($\mathcal{M}'$, g'), which are diffeomorphic to ($\mathcal{M}$, g), are regarded as equivalent and we study ($\mathcal{M}$, g) which represents the entire equivalence class of spacetimes with equivalent physical properties. The spacetime is postulated to follow local causality, i.e. the equations governing the matter field must be such that if $\mathcal{U} \subset \mathcal{M}$ is a convex normal neighborhood and if p, q are points in $\mathcal{U}$, then a signal can be sent in $\mathcal{U}$ between p and q, if and only if, there exists a C^1 curve in $\mathcal{U}$ between p and q, whose tangent is everywhere nonzero and is either timelike or null (Goswami (2005)). It is this postulate which sets the metric g apart from other fields in $\mathcal{M}$, and gives it's distinctive geometrical character. Also, for any matter field on the spacetime, local conservation of energy and momentum is postulated to hold.

The equations governing the matter fields are such that there exists a symmetric tensor T_{ab}, called the energy momentum tensor, which depends upon the fields, their covariant derivatives and the metric tensor. This matter tensor also has important properties, e.g. it vanishes in an open set $\mathcal{U} \subset \mathcal{M}$, if and only if all matter fields vanish on $\mathcal{U}$. Also we must have $T^{ab}{}_{;b} = 0$, which depicts the conservation of energy and momentum of the field.

The geometrical action is derived from a suitable scalar Lagrangian constructed from the metric tensor, and its first and second derivatives. The most abvious choice of such a scalar is the Ricci scalar (Landau and Lifshitz (1975)). The relationship of matter and spacetime geometry is then given via the principle of least action

$$\delta(S_g + S_m) = \delta \int dV \sqrt{-g}(R + \mathcal{L}_m) = 0, \tag{1.1}$$

where S_g and S_m are the gravitational and matter field actions respectively, obtained by integrating over a four volume, R is the *Ricci scalar* of the spacetime and $\mathcal{L}_m$ is the matter Lagrangian. Simplification of the above equation (1.1) gives the Einstein field equations (in units $8\pi G = c = 1$) as

$$G_{ab} \equiv R_{ab} - \frac{1}{2} R g_{ab} = T_{ab}, \tag{1.2}$$

where R_{ab} is the Ricci tensor contracted from the four dimensional Riemann curvature tensor $R^a{}_{bcd}$ of the spacetime. The tensor G_{ab} is called the Einstein tensor. The third postulate of general relativity is that the Einstein equations hold on $\mathcal{M}$.

Solutions result from solving the Einstein field equations (EFE) of general relativity. Solutions are broadly classed as exact or non-exact. Exact solutions are Lorentz metrics that are conformable to a physically realistic matter tensor and are obtained exactly in closed form. Exact solutions in general relativity are hard to come by. Almost all known solutions depend on assuming symmetries of the spacetime. Those solutions that are not exact arise due to the difficulty of solving the EFE in closed form and often take the form of approximations to ideal systems. Many non-exact solutions may be devoid of physical content, but serve as useful counterexamples to theoretical conjectures. Exact solutions are our main way of acquiring intuition about the behaviour of generic solutions to the Einstein field equations. Some important exact solutions are: Schwarzschild and Kerr solutions, the Friedmann-Lemaître-Robertson-Walker (FLRW) solutions, Lemaître-Tolman-Bondi (LTB) solutions, plane waves solutions, and the Taub-NUT family. Apart from these, there are a large number of exact solutions to the Einstein equations which are obtained under various symmetry considerations. The FLRW spacetime and LTB metrics are examples of cosmological exact solutions. Exact solutions of the Einstein field equations were among the first approaches used in studying general relativistic spacetimes. In this dissertation, we focus on the FLRW solutions.

The *Schwarzschild solutions.* Soon after Einstein proposed the field equations, Schwarzschild gave an exact solution to these equations in 1916, which represents the spherically symmetric empty spacetime outside a massive body (Schwarzschild (1916) and Goswami (2005)). In (t,r,θ,ϕ) coordinates, the Schwarzschild solution takes the form

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.3}$$

where $r > 2m$, m is the mass of the central object.

The *FLRW solutions.* These solutions give the geometry of the "standard model" in modern cosmology. These are the solutions with the metric

$$ds^2 = -dt^2 + a^2(t)[dr^2 + \Sigma^2(r,\, k)(d\theta^2 + \sin^2\theta d\phi^2)] \tag{1.4}$$

where $\Sigma^2(r,\, k) = \sin r,\ r,$ or $\sin kr$, respectively, when $k = 1,\ 0$ or -1. Here $\frac{k}{a^2}$ is the curvature scalar of the three-dimensional surfaces $t =$ constant. The major assumptions in the Robertson-Walker geometry are the large scale homogeneity and isotropy of the universe. With these symmetry assumptions, the metric for the space time can also be shown to have the following form

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1.5}$$

where k is a constant which denotes the spatial curvature of the three-space.

Lemaître, Tolman and Bondi relaxed the constrain of homogeneity in the above geometry and discovered an exact cosmological solution to the Einstein's equations for spherically symmetric dust-like matter. The metric, commonly known as LTB metric, can be written as

$$ds^2 = -dt^2 + \frac{R'^2}{1 + r^2 b_0(r)} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1.6}$$

where $R(t,\, r)$ is the area radius of the collapsing shells of dust.

For further details refer to Born (1962), Brassel (2017), Burgess (2004), D'Inverno (2008), Dirac (1996), Shutz (2009), Ryder (2009) and Wald (1984).

- Cosmological models in general relativity

The application of general relativity theory to the study of cosmology gave rise to the first quantitative cosmological models in 1917. A cosmological model is a mathematical representation of the universe at some averaging scale. It describes the geometry of space and time, the distribution and properties of matter in the universe within the framework of some physical theory (most commonly Newtonian or Einsteinian theories of gravity). The cosmological (standard) model is based on the cosmological principle, i.e. a homogeneous and isotropic universe, as well as the assumed existence of a cosmic standard time. This time is intimately connected to the evolution of the universe.

In a relativistic cosmological model space and time is described by a four-dimensional differential manifold $\mathcal{M}$ with a Lorentzian metric g, and matter is given by a symmetric tensor $\mathbf{T}$ - the energy momentum tensor. There are various physical cosmological models classified as follows (Ellis and Elst (1998))

- The *Homogeneous and Isopropic models*: FLRW model and the Einstein static model. The FLRW model will be discussed in the following sections.
- The *Homogeneous and anisopropic models*: The family of *orthogonal* and *tilted Bianchi* models,
- The *Homogeneous and Local Rotational Symmetry (LRS) models*: The family of *Kantowski-Sachs* and *LRS Bianchi* models.
- The *Inhomogeneous and LRS models*: LTB models. The LTB models are a class of spherically symmetric, time-dependent solutions with the right hand side of the Einstein field equations specified by a pressureless fluid, i.e, dust.
- The *Inhomogeneous and anisotropic models.* The *Szekeres' quasi-spherical models,* the *Stephani's conformally flat models* and the *Oleson's type N solutions.* Another interesting family of inhomogeneous models is the *Swiss-Cheese* family of models made by cutting and pasting segments of spherically symmetric models.

- FLRW models

The set of four scientists: Alexander Friedmann, Georges Lemaître, Howard P. Robertson and Arthur Geoffrey Walker are customarily grouped as Friedmann–Lemaître-Robertson–Walker (FLRW). This model is sometimes called the Standard Model of (modern) Cosmology (Bergström and Goobar (2006)). The FLRW model was developed independently by the named authors in the 1920s and 1930s.

The FLRW models are established on the basis of the assumption that the universe is homogeneous and isotropic in all epochs. These models play an important role in Cosmology, and are among the most popular backgrounds in gravitational physics. The current observations give a strong motivation for the adoption of the cosmological principle stating that the Universe at large scales is homogeneous (has spatial translation symmetry) and isotropic (has spatial rotation symmetry) and, hence, its large scale structure is well described by the FLRW metric. We assume that at any given time, the Universe looks exactly the same at every single point in space. Such a spacetime is dubbed to be homogeneous. The homogeneous models are the major models of theoretical cosmology, because they express mathematically the idea of the above cosmological principle. There is another assumption that takes into account the extreme regularity of the Universe and that is the fact that, at any given point in space, the Universe looks very much the same in whatever direction we look. Such a space time is dubbed to be isotropic. In this case, the Weyl curvature tensor vanishes, the kinematical quantities vanish except the expansion Θ. All observations (at every point) are isotropic. Universes satisfying the cosmological principle are described by the FLRW metric (1.5).

- Conformal flatness

Let $\mathcal{M}$ be a (pseudo-) Riemannian manifold. Then $\mathcal{M}$ is *conformally flat* if it can be covered by neighborhoods $\{\mathcal{U}_x\}$ such that there exists a conformal map $\phi_x : \mathcal{U}_x \to \mathbb{R}^n$, where $\mathcal{U}_x$ is a neighborhood $\mathcal{U}$ of x, and $\mathcal{U} \subset \mathcal{M}$ (Kulkarni (1972)). In other words, $\mathcal{M}$ is called *conformally flat* if each point x in $\mathcal{M}$ has a neighborhood that can be mapped to flat space by a conformal transformation. And when referred to just some point x on $\mathcal{M}$, we use the definition of *locally conformally flat.* From the point of view of conformal geometry, conformally flat manifolds are the *core* manifolds. Primary examples of conformally flat manifolds are manifolds with constant sectional curvature. Recall also that every 2-dimensional pseudo-Riemannian manifold is conformally flat. In general relativity, conformally flat manifolds are often used to describe FLRW metrics. Other examples of conformally flat manifolds:

- A 3-dimensional pseudo-Riemannian manifold is conformally flat if and only if the Cotton tensor vanishes. For $n < 3$ the Cotton tensor is identically zero.
- An n-dimensional pseudo-Riemannian manifold for $n \geq 4$ is conformally flat if and only if the Weyl tensor vanishes. In differential geometry, the Weyl curvature tensor is a measure of the curvature of spacetime in a pseudo-Riemannian manifold. Part of the Riemann curvature tensor, the Weyl tensor expresses the tidal force that a body feels when moving along a geodesic. In general relativity, the Weyl tensor is the only part of the curvature that exists in free space, a solution of the vacuum Einstein equations, and it governs the propagation of gravitational waves through regions of space devoid of matter. In dimensions 2 and 3 the Weyl curvature tensor vanishes identically. In dimensions ≥ 4, the Weyl curvature is generally nonzero. If the Weyl tensor vanishes in dimension ≥ 4, then the metric is locally conformally flat: there exists a local coordinate system in which the metric tensor is proportional to the Minkowski metric. In the theory of general relativity, one can split the Weyl tensor into the electric part and the magnetic part, the so-called gravitoelectric-magnetic fields, with some similarity to electrodynamical counterparts (Danehkar (2009)).

A number of conformally flat physically significant spacetimes are known like the Schwarzschild interior solution and the Lemaître cosmological universe. Buchdahl (1959) obtained the conformal flatness of the Schwarzschild interior solution. Singh and Roy (1966) discussed the possibility of existence of electromagnetic fields conformal to some empty spacetime. Singh and Abdussattar (1974) obtained a non-static generalization of Schwarzschild interior solution which is conformal to flat spacetime. Roy and Bali (1978) have obtained the solution of Einstein's field equations representing non-static spherically symmetric perfect fluid distribution which is conformally flat. Pandey and Tiwari (1981) have discussed a conformally flat spherically symmetric charged perfect fluid distribution. Reddy (1979), and Rao and Reddy (1982) discussed static conformally flat solutions in the Brans-Dicke and Nordtvedt-Barker scalar-tensor theories. Shanthi (1989) has shown that the most general conformally flat static vacuum solution in the Nordtvedt-Barker scalar-tensor theory is simply the empty flat spacetime of general relativity. There has been literature from Melfo and Rago (1992), Mannheim (1992), Yadav and Prasad (1993), Endean (1997), Endean (1998), Visser (2015), Obukhov et al (1999), Mak and Harko (2000)) which shows significant interest in the study of conformally flat spacetimes. Several other classes of conformally flat spacetimes have been applied in cosmology, including generalized Friedmann models, generalized Schwarzschild interior models, Bertotti-Robinson models, and radiation fields (Stephani et al (2003)).

Conformal flatness is a condition that is often applied in the study of gravitational interactions, since many of these models characterize spacetimes of physical importance. In a conformally flat spacetime the Weyl tensor vanishes identically (Hawking and Ellis (1973)), and the technique of embedding has proved to be a useful tool in generating a variety of exact solutions (Krasinski (1997), Ellis et al (2007)), with perfect fluids, pure radiation and electromagnetic fields. Conformal flatness is also widely used in studying gravitational collapse for various matter fields. Collapse in the presence of scalar fields was studied in Chakrabarti and Banerjee (2017), with dissipative matter giving rise to radiating stellar configurations (Herrera et al (2004), Herrera et al (2006), Misthry et al (2008), Maharaj and Govender (2005), Sharma et al (2015)). Conformal flatness has also proven to be useful in constructing static anisotropic stars that can represent real pulsars (Ivanov (2018)). In addition to this, vanishing of the Weyl tensor helps to solve the field equations in modified gravity theories, see for example in Chakrabarti et al (2018) for f(R)-theories of gravity, and in Hansraj and Moodly (2020) for Einstein-Gauss-Bonnet gravity.

Although a lot of work has been done on conformal symmetries in these spacetimes, and also numerous solutions have been found, most of these studies consider to spacetimes with specific symmetries (for example, spherical or cylindrical symmetries) and very specific types of matter fields. The main reason behind this is that the field equations still remain extremely complicated for any general treatment. To overcome this hurdle, we start by taking baby steps. We consider a general, but conformally flat perturbations on FLRW spacetime, and work with linearised field equations up to first order. In other words, the Weyl tensor vanishes identically in the perturbed spacetime. Nevertheless, all the other quantities, that were zero in the background become first order quantities in the perturbed scenario, and we deal with a general form of matter with anisotropic stresses and heat flux to the first order of smallness. The main aim of this investigation is to track the behaviour of geometrical quantities, like matter shear or vorticity in the perturbed conformally flat scenarios. This will definitely give an indication as how these quantities will behave in the most general treatment of the field equations.

- Shear-free models

As is well known, shear plays an important role in general relativistic and cosmological models (Collins and Wainwright (1983), Glass (1979), Herrera and Santos (2003) and references therein). The differential properties of families of geodesics are described by their expansion, rotation and shear in the timelike case (Ehlers (1993), (Ehlers (2009) and Ellis (2009)), and by the null expansion and shear in the case of the irrotational null geodesic congruences that underlie observations (Ehlers and Sachs (1961), Ellis (2011)).

For a given fundamental observer moving with 4-velocity u^a, spacetime decomposes into space and time (Ehlers (1993) and Ellis (2009)). When the shear is zero, the expansion is isotropic; we expect vorticity to tend to generate anisotropy that would break this condition. However a nonzero Weyl tensor could balance this tendency (Ellis (2011)). From the evolution equations for the shear

$$\dot{\sigma}_{<ab>} + \frac{2}{3}\Theta\sigma_{ab} + \sigma_{c<a}\sigma^{c}_{b>} + \omega_{<a}\omega_{b>} + E_{ab} = 0, \tag{1.7}$$

and for the magnetic part of the Weyl tensor

$$H^{ab} = -D^{<a}\omega^{b>} + (\mathrm{curl}\ \sigma)^{ab}, \tag{1.8}$$

on setting the shear to be zero ($\sigma_{ab} = 0$), (1.7) becomes a new constraint equation, along with the old constraint (1.8) determining E and H in terms of ω:

$$E_{ab} = -\omega_{<a}\omega_{b>},\ H^{ab} = -D^{<a}\omega^{b>}. \tag{1.9}$$

Let us now take time derivatives of these constraints and see if the shear-free equations (1.9) are consistent for some non-trivial special cases. Gödel (1952) showed that in this case, a shear-free universe could either expand or rotate, but not both; he did not show how he had obtained the result. In 1957, Schücking derived the Gödel result in detail (Schücking (1957)). Ellis (1967) used an orthonormal tetrad formalism to show that the restriction of special homogeneity was unnecessary:

Dust shear-free theorem: If a dust solution of EFEs (possibly with a cosmological constant) is shear-free in a domain $\mathcal{U}$, it cannot both expand and rotate in $\mathcal{U}$:

$$\{\dot{u}^a = 0,\ \sigma_{ab} = 0\} \implies \omega\Theta = 0. \tag{1.10}$$

Applying the above dust shear-free theorem in the cosmological context, (1.10) shows $\omega_{ab} = 0$. So from the above equations

$$\{\sigma_{ab} = 0,\, \Theta > 0\} \implies E_{ab} = 0,\, H^{ab} = 0,\, D^a\Theta = 0. \tag{1.11}$$

The spacetime is conformally flat and the universe is a FLRW universe (Ellis (2009)). These are thus the only expanding shear-free baryonic plus CDM cosmological solutions, provided both components move with the same 4-velocity (Ellis (2011)). Note that the result (1.11) does not require the energy density to be positive ($\rho > 0$). Further generalizations considered perfect fluids rather than pressure-free matter, so the timelike congruence acceleration was allowed; the result (1.10) remains true in all cases considered so far. In fact Collins (1986) surmised that all shear-free perfect fluids obeying a barotropic equation of state must have either zero expansion or zero vorticity. Senovilla et al (1998) obtained results towards proving this conjecture. Van den Bergh (1999) gave a tetrad based approach for two particular cases require special treatment: $p + 1/3\rho = constant$, $p - 1/9\rho = constant$, also the equation of state $p = (\gamma - 1)\rho + constant$. Van Den Bergh et al (2007) showed that the result is generically true for shear-free perfect fluid solutions of the Einstein field equations where the fluid pressure satisfies a barotropic equation of state and the spatial divergence of the magnetic part of the Weyl tensor is zero. Is it possible that we can obtain models other than FLRW in these cases? Collins (1985) showed that for irrotational shear-free perfect fluids obeying a barotropic equation of state $p = p(\mu)$ and with nonzero acceleration, we get spherically symmetric Wyman solutions, or models that are plane symmetric, which are spatially or temporally homogeneous. In all cases, when the spacetime is sufficiently extended, the fluid exhibits unphysical properties. Consequently shear-free expanding barotropic perfect fluids must either be FLRW, or must be restricted to local regions where these conditions hold. Thus it turns out that the FLRW models are the only shear-free barotropic perfect fluid models in which the matter is physically reasonable globally (Collins (1986), Ellis (2011)). Generally speaking, these results show clearly how restrictive the shear-free result is for plausible fluid models. It will of course not be true for imperfect fluids with arbitrary equations of state. It is of considerable interest whether the result (1.10) holds in the case of linearised perturbations of FLRW universe models. It has recently been shown that it holds in this case too: if a perfect fluid with equation of state $p = k\rho$ in an almost FLRW universe is shear-free, then it must be either expansion-free or rotation-free. Thus linearization does not lose this property. Nzioki et al (2011) deals with a number of interesting properties of shear-free perfect fluids (i.e. $q^a = \pi = 0$) in general relativity. On our part, we consider the case of imperfect fluids (i.e. $0 \neq q^a \neq \pi \neq 0$).

In this work we analyse the role of spacetime shear in generating cosmological gravitational waves in a universe constructed via a general perturbation of homogeneous and isotropic FLRW universe. This is important, as the corresponding Newtonian or quasi Newtonian description of cosmologies do not permit the existence of gravitational waves. It has been well known for some time that we can describe the cosmology in a quasi-Newtonian way (the *silent models* that are devoid of any gravitational waves), for observers which move along geodesics which are both shear-free and irrotational (Matarrese et al (1994)). The key difference between Newtonian and general relativistic cosmologies, in the absence of shear, emerges from the surprising exact result: Shear-free dust cannot rotate and expand simultaneously (Gödel (1952), Ellis (1967) and Ellis et al (2007)), which was also shown to hold in the case of barotropic perfect fluid solutions linearised about a FLRW geometry (Nzioki et al (2011)). Since this result is not true for Newtonian or quasi Newtonian cosmologies, it is not always obvious which behaviour of a perturbed universe (linearised about FLRW geometry) will have a Newtonian counterpart. Further to this, most of the previous results have assumed the form of matter in the perturbed universe to be barotropic (as in the background scenario). Hence the effect of introducing heat flux or anisotropic stress perturbatively in the energy momentum tensor of the matter field still remains the subject of further investigations.

- Gravitational waves

In Einstein's theory of general relativity the geometry of spacetime is a dynamic physical observable that supports wave-like excitations, propagating at the speed of light. These are known as gravitational waves. One of the general relativity principles is that nothing travels faster than light. This means that changes in the gravitational field cannot be felt everywhere instantaneously: they must propagate. In general relativity they propagate at exactly the same speed as vacuum electromagnetic waves: the speed of light. These propagating changes are called gravitational waves, and they actually represent a hot topic, which plays a central role in astrophysics, cosmology and theoretical physics (Ciufolini et al (2001)).

To discuss the generation of gravitational waves on a given background, the covariant way is to consider the Weyl tensor as the free gravitational field and the metric tensor as it's second order potential field. Given a family of timelike observers, one can then split the Weyl tensor into its electric and magnetic parts, which are projected symmetric trace free tensors of rank 2, and this is absolutely analogous to splitting the electromagnetic field tensor into the electric field vector and magnetic field vector. The once-contracted Bianchi identities then gives the evolution equations of these tensors and one can easily see that the evolution of the electric part is coupled to the curl of the magnetic part and vice versa, exactly analogous to the source-free Maxwell equations for electric and magnetic fields. Using these evolution equations and the tensor identities, we can then find closed wave equations for the electric and magnetic part of the Weyl tensor (see Goswami and Ellis (2021) for a detailed explanation). These are the equations for gravitational (tensor) waves containing the tensorial modes of oscillations. Apparently, it may seem that the spacetime shear plays no direct role in generating these waves. However, as we shall see in this work, the shear has a pivotal role in coupling the evolution equations for electric and magnetic components of Weyl. Absence of shear completely destroys this structure and makes the perturbed spacetime silent.

- Outline

This dissertation is organised as follows:

- Chapter 1: In this chapter we discuss general relativity and some exact solutions. We provide a brief overview of the mathematical tools needed to formulate general relativity. These mathematical tools had been developed earlier by Riemann and Gauss. We discuss conformally flat and shear-free models, and why they are important.
- Chapter 2: In this chapter we present the relevant theoretical concepts inherent with the $1+3$ covariant formalism. We use the socalled $1+3$ covariant description of general relativity which has been developed for use in spacetimes in which there is a preferred timelike congruence $\mathbf{u}$. The "$1+3$" refers to the fact that one performs a "time+space" decomposition relative to $\mathbf{u}$ by projecting tensors and tensorial equations parallel to $\mathbf{u}$ and orthogonal to $\mathbf{u}$. The second aspect of the $1+3$ description is to write a tensor as a sum of algebraically simpler parts.
- Chapter 3: We study the conformally flat perturbations of FLRW spacetimes, to see the effects the of Weyl tensor in a general spacetime. We adapt the $1+3$ covariant approach based on Ellis and van Elst (1998) to conformally flat spacetimes, i.e the Weyl curvature tensor is equal to zero, so the "electric" and "magnetic" Weyl curvature parts respectively are equal to zero. We state and prove several important geometrical properties for matter shear and vorticity in the perturbed, conformally flat spacetime. We use a local semitetrad covariant formalism, and hence all the perturbation results are frame invariant and gauge invariant in general. A pioneering work in this regard was done by Bruni et al (1992), where the scalar, vector and tensor modes of the perturbations were treated in a covariant and gauge invariant

way. Further works on cosmological gravitational waves were studied in Dunsby et al (1997) and Maartens et al (1999). Similar techniques were used in Nzioki et al (2011) and Abebe et al (2011), to show that in a shear-free perturbation of FLRW spacetime, matter cannot expand and rotate simultaneously even in the linearized regime. An interesting parallel emerges between our work and the results presented in Herrera et al (2014) and Herrera et al (2016), where the authors performed a general study of dissipative fluids with anisotropic stresses. These works are more restricted in the sense that they assume axial and reflection symmetries. it is important to remember that our study emphasizes the effects of the Weyl tensor on spacetime geometry via negation. This understanding helps us in recognizing the effects of free gravity with better clarity, as changes involving the appearance of vorticity and shear are highly dependent on the Weyl tensor. It is important to note that the work in this chapter has been published in *Physical Review D* (Mayala et al (2021))

- Chapter 4: In this chapter we study the shear-free perturbations on FLRW spacetimes, to see the effect of the shear tensor in general spacetimes. We adapt the $1+3$ covariant approach based on Ellis and van Elst (1998) so that the shear tensor is equal to zero. This work deals with a number of interesting properties of shear-free imperfect fluids. We use a semi tetrad covariant and gauge invariant formalism to obtain frame invariant and gauge invariant results. As we know, in perturbation theory the gauge choice becomes important while mapping the zeroth order quantities from the background manifold to the perturbed one. However, it is also well known that any covariantly defined geometric and thermodynamic quantity, that vanishes in the background, is automatically gauge invariant in the perturbed manifold (Stewart (1991)). Since in the FLRW background the Weyl tensor is identically zero, in the perturbed manifold this will be a first order quantity and naturally gauge invariant. Therefore any result concerning the existence or non-existence of gravitational waves will naturally be gauge invariant in this formalism.

- Chapter 5: We briefly summarise the work done in this thesis.

Chapter 2

The 1+3 covariant formalism

2.1 Introduction

In this chapter we provide some basic notions regarding the $1+3$ covariant formalism which will be required for this thesis. For further details refer to Ellis and van Elst (1998), Ellis et al (2011), van Elst (1996), Nzioki (2013), Betschart (2005), Bertschinger (1999), Islam (2006) and Mongwane (2014). The $1+3$ covariant approach provides a covariant description of spacetime in terms of scalars, $3-$vectors and projected symmetric trace-free (PSTF) $3-$tensors, and equations governing their dynamics, based on the Ricci and Bianchi identities. These quantities have a physical or direct geometrical meaning, which have a natural interpretation for comoving observers. This formalism is based on a local 1+3 threading of the spacetime manifold and has been a very handy tool for understanding many geometrical and physical aspects of relativistic fluid flows, both in general relativity or in the gauge invariant, covariant perturbation formalism (Ellis et al (2011)). We first define a timelike congruence with a unit tangent vector u^a along the fluid flow lines. Then the spacetime is split locally in the form $R \otimes V$ where R denotes the world line along u^a and V is the 3-space perpendicular to u^a. Any vector X^a can then be projected on the 3-space by the projection tensor $h^a{}_b = g^a{}_b + u^a u_b$. The choice of u^a naturally defines two derivatives: the *covariant time derivative* along the observers' worldlines (denoted by a dot), and the fully orthogonally *projected covariant derivative* D on the three dimensional space.

2.2 Projection and differentiation

The non-intersecting timelike family of wordlines (associated with fundamental observers comoving with the cosmological fluid) form a congruence in spacetime $(\mathcal{M},\, g)$ representing the average motion of matter at each point. In each case their 4-velocity is

$$u^a = \frac{dx^a}{d\tau}, \text{ with } u_a u^a = -1, \tag{2.1}$$

where τ is the proper time along the world line of any fundamental observer. Given the 4-velocity u^a, there are uniquely defined projection tensors

$$U^a_b = -u^a u_b, \tag{2.2}$$

$$h^a_b = \delta^a_b + u^a u_b, \tag{2.3}$$

where (2.2) projects parallel to u^a and (2.3) projects onto the rest space orthogonal to u^a. It follows that

$$U^a_c U^c_b = U^a_b, \qquad U^a_b u^b = u^a, \qquad U^a_a = 1, \tag{2.4}$$

$$h^a_c h^c_b = h^a_b, \qquad h^a_b u^b = 0, \qquad h^a_a = 3, \tag{2.5}$$

$$0 = \dot{U}^{<ab>} = \dot{h}^{<ab>}, \qquad 0 = D_a U_{bc} = D_a h_{bc}.$$

The choice of the timelike vector naturally defines two derivatives: the vector u^a is used to define the *covariant time derivative* along the observers' worldline (denoted by dot) for any tensor $Z^{a..b}_{c..d}$, given by

$$\dot{Z}^{a..b}_{c..d} = u^e \nabla_e Z^{a..b}_{c..d}, \tag{2.6}$$

and the tensor h_{ab} is used to define the fully *orthogonally projected covariant derivative* D for any tensor $Z^{a..b}_{c..d}$ by

$$D_e Z^{a..b}_{c..d} = h^a_f h^p_c ... h^b_g h^q_d h^r_e \nabla_r Z^{f..g}_{p..q}, \tag{2.7}$$

with total projection on all the free indices.

Following Maartens (1997), we use angle brackets to denote the orthogonal projections of vectors, the orthogonally projected symmetric trace-free (PSTF) part of tensors, and their time derivatives, i.e.

$$V^{<a>} = h^a_b V^b, \qquad Z^{<ab>} = \left(h_c^{(a} h_d^{b)} - \frac{1}{3} h^{ab} h_{cd}\right) Z^{cd}, \tag{2.8}$$

$$\dot{V}^{<a>} = h^a_b \dot{V}^b, \qquad \dot{Z}^{<ab>} = \left(h_c^{(a} h_d^{b)} - \frac{1}{3} h^{ab} h_{cd}\right) \dot{Z}^{cd}. \tag{2.9}$$

Volume element

The effective *volume element* for the rest space of the comoving observer is given as follows

$$\epsilon_{abc} = \eta_{abcd} u^d = -\epsilon_{defg} h^d_a h^e_b h^f_c u^g = u^g \epsilon_{gdef} h^d_a h^e_b h^f_c, \tag{2.10}$$

where

$$\begin{aligned} &\epsilon_{abc} = \epsilon_{[abc]} \text{ and } \epsilon_{abc} u^c = 0, \\ &0 = \dot{\epsilon}_{<abc>}, \qquad 0 = D_a \epsilon_{bcd}. \end{aligned} \tag{2.11}$$

Here η_{abcd} is the four-dimensional volume element ($\eta_{abcd} = \sqrt{|det\ g|}\delta^0_{[a}\delta^1_b\delta^2_c\delta^3_{d]}$) so that

$$\eta_{abcd} = 2u_{[a}\epsilon_{b]cd} - 2\epsilon_{ab[c}u_{d]}. \tag{2.12}$$

Since η_{abcd} is totally skew-symmetric $\eta_{abcd} = \eta_{[abcd]}$, it follows that the following contractions hold

$$
\begin{aligned}
\epsilon_{abc}\epsilon^{def} &= 3! h^d_{[a} h^e_b h^f_{c]}, \\
\epsilon_{abc}\epsilon^{dec} &= 2 h^d_{[a} h^e_{b]}, \\
\epsilon_{abc}\epsilon^{dbc} &= 2 h^d_a, \\
\epsilon_{abc}\epsilon^{abc} &= 6.
\end{aligned}
\tag{2.13}
$$

Covariant derivative of u (kinematical variables):

The covariant derivative of the timelike vector u^a can now be decomposed into the irreducible part as

$$
\nabla_a u_b = -u_a \dot{u}_b + D_a u_b = \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \epsilon_{abc}\omega^c - u_a \dot{u}_b, \tag{2.14}
$$

where $\dot{u}_b$ is the acceleration, Θ is the expansion, σ_{ab} is the shear tensor and ω^a is the vorticity vector, they are all defined respectively as

$$
\begin{aligned}
\dot{u} &= u^b \nabla_b u^a, \\
\Theta &= D_a u^a, \\
\sigma_{ab} &= D_{<a} u_{b>}, \\
\omega^a &= \epsilon^{abc} D_b u_c.
\end{aligned}
\tag{2.15}
$$

Weyl curvature variables:

The Weyl curvature tensor can be decomposed irreducibly into the gravito-electric and gravito-magnetic parts as

$$
\begin{aligned}
E_{ab} &= C_{abcd}u^c u^d = E_{<ab>}, \\
H_{ab} &= \frac{1}{2}\epsilon_{acd}C^{cd}{}_{be}u^e = H_{<ab>},
\end{aligned} \tag{2.16}
$$

which allow for a covariant description of tidal forces and gravitational radiation.

Matter variables:

The energy momentum tensor for a general matter field can be similarly decomposed as follows

$$
T_{ab} = \mu u_a u_b + q_a u_b + q_b u_a + p h_{ab} + \pi_{ab} \tag{2.17}
$$

where μ is the energy density, $q_a = q_{<a>}$ is the 3-vector defining the heat flux, p is the isotropic pressure and $\pi_{ab} = \pi_{<ab>}$ is the anisotropic stress. And all defined respectively as follows

$$
\begin{aligned}
\mu &= T_{ab}u^a u^b, \\
q^a &= -h^{ca}T_{cb}u^b, \\
p &= \frac{1}{3}h^{ab}T_{ab}, \\
\pi_{ab} &= h^c_{<a}h^d_{b>}T_{cd}.
\end{aligned} \tag{2.18}
$$

Riemann curvature tensor

In the geometric description of gravity, spacetime curvature is encoded in the Riemann tensor R_{abcd}. This tensor is defined through the Ricci identity (Mongwane 2014) by

$$
\nabla_{[a}\nabla_{b]}u_c = R_{abcd}u^d. \tag{2.19}
$$

A useful $1+3$ decomposition of Riemann curvature tensor R_{abcd} is given as follows (Ellis and van Elst (1998))

$$
\begin{aligned}
R^{ab}{}_{cd} &= R^{ab}_{E\ cd} + R^{ab}_{I\ cd} + R^{ab}_{H\ cd} + R^{ab}_{Pcd}, \\
R^{ab}_{E\ cd} &= 4u^{[a}u_{[c}E^{b]}_{d]} + 4h^{[a}_{[c}E^{b]}_{d]}, \\
R^{ab}_{I\ cd} &= -2u^{[a}h^{b]}_{[c}q_{d]} - 2u_{[c}h^{[a}_{d]}q^{b]} - 2u^{[a}u_{[c}\pi^{b]}_{d]} + 2h^{[a}_{[c}\pi^{b]}_{d]}, \\
R^{ab}_{H\ cd} &= 2\epsilon^{abe}u_{[c}H_{d]e} + 2\epsilon_{cde}u^{[a}H^{b]e}, \\
R^{ab}_{Pcd} &= \frac{2}{3}(\mu+3p)u^{[a}u_{[c}h^{b]}_{d]} + \frac{2}{3}\mu h^{a}_{[c}h^{b}_{d]},
\end{aligned}
\tag{2.20}
$$

where I represents the imperfect fluid part, P the perfect fluid part; E and H are parts due to the electric and magnetic Weyl tensor, respectively.

Therefore we can write the Riemann curvature tensor as

$$
\begin{aligned}
R^{ab}{}_{cd} = 4u^{[a}u_{[c}E^{b]}_{d]} + 4h^{[a}_{[c}E^{b]}_{d]} - 2u^{[a}h^{b]}_{[c}q_{d]} - 2u_{[c}h^{[a}_{d]}q^{b]} - 2u^{[a}u_{[c}\pi^{b]}_{d]} + 2h^{[a}_{[c}\pi^{b]}_{d]} \\
+ 2\epsilon^{abe}u_{[c}H_{d]e} + 2\epsilon_{cde}u^{[a}H^{b]e} + \frac{2}{3}(\mu+3p)u^{[a}u_{[c}h^{b]}_{d]} + \frac{2}{3}\mu h^{a}_{[c}h^{b}_{d]}.
\end{aligned}
\tag{2.21}
$$

2.3 The field equations

An arbitrary spacetime, in the $1+3$ formulation, may be completely characterised by the following irreductible set of geometrical quantities:

$$\{\Theta,\ \dot{u}_a,\ \sigma_{ab},\ \omega_{ab},\ E_{ab},\ H_{ab}\}, \tag{2.22}$$

together with the irreductible set of matter variables

$$\{\mu,\ p,\ q^a,\ \pi_{ab}\}, \tag{2.23}$$

provided an equation of state is prescribed.

2.3.1 Ricci identities

Time derivative equations

Evolution equations for the kinematic variables $\{\Theta,\, \omega_a,\, \sigma_{ab}\}$ are obtained by separating out the parallel projected part of the Ricci identity (2.19) into its irreducible components. For an outline of the derivation we refer the reader to Ellis (1971). The results are given as follows

1. The *Raychaudhuri equation* (Raychaudhuri (1955))

$$\dot{\Theta} - D_a\dot{u}^a = -\frac{1}{3}\Theta^2 + \dot{u}_a\dot{u}^a - \sigma_{ab}\sigma^{ab} + 2\omega_a\omega^a - \frac{1}{2}(\mu + 3p), \tag{2.24}$$

2. The *shear propagation equation*

$$\dot{\sigma}^{<ab>} - D^{<a}\dot{u}^{b>} = -\frac{2}{3}\Theta\sigma^{ab} + \dot{u}^{<a}\dot{u}^{b>} - \sigma_c^{<a}\sigma^{b>c} - \omega^{<a}\omega^{b>} - (E^{ab} - \frac{1}{2}\pi^{ab}), \tag{2.25}$$

3. The *vorticity propagation equation*

$$\dot{\omega}^{<a>} - \frac{1}{2}\epsilon^{abc}D_b\dot{u}_c = -\frac{2}{3}\Theta\omega^a + \sigma_b^a\omega^b. \tag{2.26}$$

Constraint equations

The following *constraint equations* are obtained by first projecting the Ricci identities (2.19) orthogonal to the 4−velocity u^a, and we find

4. The (0α)−equation

$$0 = (C_1)^a = D_b\sigma^{ab} - \frac{2}{3}D^a\Theta + \epsilon^{abc}(D_b\omega_c + 2\dot{u}_b\omega_c) + q^a, \tag{2.27}$$

5. The vorticity divergence identity

$$0 = (C_2) = D_a\omega^a - \dot{u}_a\omega^a, \tag{2.28}$$

6. The H_{ab}−equation

$$0 = (C_3)^{ab} = H^{ab} + 2\dot{u}^{<a}\omega^{b>} + D^{<a}\omega^{b>} - \epsilon^{cd<a}D_c\sigma_d^{b>}. \tag{2.29}$$

2.3.2 (Contracted) Second Bianchi identities

The following equations (2.31) and (2.32) arise from the *Bianchi identities* (Ellis and van Elst (1998))

$$\nabla_{[a}R_{bc]de} = 0. \tag{2.30}$$

Time derivative equations

From the Bianchi identities, one recovers the following propagation equations for the electric and magnetic parts of the Weyl tensor

$$\begin{aligned}(\dot{E}^{<ab>} + \frac{1}{2}\dot{\pi}^{<ab>}) - \epsilon^{cd<a}D_c H_d^{b>} + \frac{1}{2}D^{<a}q^{b>} =& -\frac{1}{2}(\mu+p)\sigma^{ab} - \Theta(E^{ab} + \frac{1}{6}\pi^{ab}) \\ &+ 3\sigma_c^{<a}(E^{b>c} - \frac{1}{6}\pi^{b>c}) - \dot{u}^{<a}q^{b>} \\ &+ \epsilon^{cd<a}[2\dot{u}_c H_d^{b>} + \omega_c(E^{b>} + \frac{1}{2}\pi_d^{b>})],\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}\dot{H}^{<ab>} + \epsilon^{cd<a}D_c(E_d^{b>} - \frac{1}{2}\pi_d^{b>}) =& -\Theta H^{ab} + 3\sigma_c^{<a}H^{b>c} + \frac{3}{2}\omega^{<a}q^{b>} \\ &- \epsilon^{cd<a}(2\dot{u}_c E_d^{b>} - \frac{1}{2}\sigma_c^{b>}q_d - \omega_c H_d^{b>}),\end{aligned} \tag{2.32}$$

respectively.

The propagation equations for the matter variables are derived from the conservation of matter $\nabla_a Z^{ab} = 0$. Projecting along and orthogonal to the 4−velocity u^a results in

the energy conservation equation

$$\dot{\mu} + D_a q^a = -\Theta(\mu+p) - 2(\dot{\mu}_a q^a) - \sigma_{ab}\pi^{ab}, \tag{2.33}$$

and

the momentum conservation equation

$$\dot{q}^{<a>} + D^a p + D_b \pi^{ab} = -\frac{4}{3}\Theta q^a - \sigma^a_b q^b - (\mu + p)\dot{u}^a - \dot{u}_b \pi^{ab} - \epsilon^{abc}\omega_b q_c. \tag{2.34}$$

Constraint equations

The constraint equations are

$$\begin{aligned} 0 = (C_4)^a = D_b(E^{ab} + \frac{1}{2}\pi^{ab}) - \frac{1}{3}D^a\mu + \frac{1}{3}\Theta q^a - \frac{1}{2}\sigma^a_b q^b \\ -3\omega_b H^{ab} - \epsilon^{abc}\left(\sigma_{bd}H^d_c - \frac{3}{2}\omega_b q_c\right), \end{aligned} \tag{2.35}$$

$$\begin{aligned} 0 = (C_5)^a = D_b H^{ab} + (\mu + p)\omega^a + 3\omega_b\left(E^{ab} - \frac{1}{6}\pi^{ab}\right) \\ +\epsilon^{abc}\left[\frac{1}{2}D_b q_c + \sigma_{bd}\left(E^d_c + \frac{1}{2}\pi^d_c\right)\right]. \end{aligned} \tag{2.36}$$

2.3.3 Propagation of constraints along timelike congruence

Propagating the constraints (2.27)-(2.29), (2.35) and (2.36) along u^a (Maartens (1997) and van Elst (1996)), we get the following system of equations

$$\begin{aligned} (\dot{C_1})^{<a>} = -\Theta(C_1)^a - \frac{3}{2}\sigma^a_b(C_1)^b + \frac{1}{2}\epsilon^{abc}\omega_b(C_1)_c - \frac{8}{3}\omega^a(C_2) \\ -\epsilon^{abc}\sigma_{bd}(C_3)^d_c - 3\omega_b(C_3)^{ab} - (C_4)^a, \end{aligned} \tag{2.37}$$

$$(\dot{C_2}) = -\Theta(C_2), \tag{2.38}$$

$$\begin{aligned} (\dot{C_3})^{<ab>} = -\Theta(C_3)^{ab} + 3\sigma^{<a}_c(C_3)^{b>c} + \epsilon^{cd<a}\omega_c(C_3)^{b>}_d + \frac{1}{2}\epsilon^{cd<a}\sigma^{b>}_c(C_1)_d \\ +\frac{3}{2}\omega^{<a}(C_1)^{b>}, \end{aligned} \tag{2.39}$$

$$(\dot{C_4})^{<a>} - \frac{1}{2}\epsilon^{abc}D_b(C_5)_c = -\frac{4}{3}\Theta(C_4)^a + \frac{1}{2}\sigma^a_b(C_4)^b - \frac{1}{2}\epsilon^{abc}\omega_b(C_4)_c - \frac{1}{2}(\mu+p)(C_1)^a$$
$$-\frac{1}{2}\pi^a_b(C_1)^b + 2\epsilon^{abc}E_{bd}(C_3)^d_c + \frac{3}{2}\epsilon^{abc}\dot{u}_b(C_5)_c, \tag{2.40}$$

$$(\dot{C_5})^{<a>} + \frac{1}{2}\epsilon^{abc}D_b(C_4)_c = -\frac{4}{3}\Theta(C_5)^a + \frac{1}{2}\sigma^a_b(C_5)^b - \frac{1}{2}\epsilon^{abc}\omega_b(C_5)_c - \frac{1}{2}\epsilon^{abc}q_b(C_1)_c$$
$$+\frac{2}{3}q^a(C_2) + 2\epsilon^{abc}H_{bd}(C_3)^d_c - \frac{3}{2}\epsilon^{abc}\dot{u}_b(C_4)_c. \tag{2.41}$$

2.4 Commutation relations and notations

In general the two derivatives '$\dot{}$' and 'D' introduced in §2.2 do not commute and therefore give rise to various commutator relations which play an integral part in all partial frame formalisms. This is a manifestation of spacetime curvature which is derived from the Ricci identities for spacetime scalars f, 3-vectors V^a and rank-2 tensors A^{ab}, respectively (Betschart (2005)):

$$\nabla_{[a}\nabla_{b]}f = 0, \tag{2.42}$$

$$2\nabla_{[a}\nabla_{b]}V_c = R_{abcd}V^d, \tag{2.43}$$

$$2\nabla_{[a}\nabla_{b]}f_{cd} = -R_{abec}A^e_d - R_{abed}A^e_c. \tag{2.44}$$

Below we have the commutation relations given by van Elst (1998).

2.4.1 3-scalar derivatives

For scalar function f one obtains

- *Time-space derivative commutator:*

$$[D^{<a>}f]^{\cdot} \equiv h^a_b[D^b f]^{\cdot} = [D^a + \dot{u}^a][\dot{f}] - \left[\frac{1}{3}\Theta h^{ab} + \sigma^{ab} + \epsilon^{abc}\omega_c\right] D_b f, \tag{2.45}$$

- *Space-space derivative commutator:*

$$D_{[a}D_{b]}f = \epsilon_{abc}\omega^c \dot{f} \Longleftrightarrow \epsilon^{abc}D_bD_cf = 2\omega^a\dot{f}. \tag{2.46}$$

2.4.2 3-vector derivatives

For the 3-vectors V^a, the following holds

- *Time-space derivative commutator:*

$$\begin{aligned}[D^{<a}V^{b>}]^{\cdot} \equiv h^a_c h^b_d [D^c V^d]^{\cdot} = [D^a + \dot{u}^a][\dot{V}^{<b>}] \\ - [\frac{1}{3}\Theta h^{ac} + \sigma^{ac} + \epsilon^{acd}\omega_d] \times [D_c V^b - 2h^{[b}_c h^{e]}_f \dot{u}^f V_e] \\ [H^a_c - (C_3)^a_c]\epsilon^{bcd}V_d - \frac{1}{2}[q^b - (C_1)^b]V^a + \frac{1}{2}h^{ab}[q_c - (C_1)_c]V^c.\end{aligned} \tag{2.47}$$

- *Space-space derivative commutator:*

$$\begin{aligned}D_{[a}D_{b]}V_c = \left[(E_{c[a} + \frac{1}{2}\pi_{c[a}) - \frac{1}{3}\Theta\sigma_{c[a} + \frac{1}{3}\Theta\omega^d\epsilon_{dc[a} + \omega_c\omega_{[a} + \frac{1}{3}(\mu - \frac{1}{3}\Theta^2 - 3\omega_d\omega^d)h_{c[a}\right] V_{b]} \\ + [h_{c[a}(E_{b]d} + \frac{1}{2}\pi_{b]d}) - \frac{1}{3}\Theta h_{c[a}\sigma_{b]d} - \sigma_{c[a}\sigma_{b]d} - \frac{1}{3}\Theta h_{c[a}\epsilon_{b]de}\omega^e - \sigma_{c[a}\epsilon_{b]de}\omega^e \\ + \sigma_{d[a}\epsilon_{b]ce}\omega^e + h_{c[a}\omega_{b]}\omega_d]V^d + \epsilon_{abd}\omega^d\dot{V}_{<c>}.\end{aligned} \tag{2.48}$$

- *Evolution of spatial divergence terms:*

$$\begin{aligned}[D_aV^a]^{\cdot} = [D_a + \dot{u}_a][\dot{V}^{<a>}] - \frac{1}{3}\Theta D_aV^a - \sigma^a_b D_aV^b + \omega_a\epsilon^{abc}D_bV_c + \frac{2}{3}\Theta\dot{u}_aV^a \\ - \sigma_{ab}\dot{u}^aV^b + \epsilon_{abc}\dot{u}^a\omega^bV^c + [q_a - (C_1)_a]V^a.\end{aligned} \tag{2.49}$$

2.4.3 3-tensor derivatives

For the second-rank 3-tensors A_{ab}, the following holds:

- Space-space derivative commutator:

$$
\begin{aligned}
D_{[a}D_{b]}A^{cd} = {} & 2\left[\left(E^{(c}_{[a} + \frac{1}{2}\pi^{(c}_{[a}\right) - \frac{1}{3}\Theta\sigma^{(c}_{[a} - \frac{1}{3}\Theta\omega^{e}\epsilon^{(c}_{e[a} + \omega^{(c}\omega_{[a}\right] A^{d)}_{b]} \\
& + 2\left[\frac{1}{3}(\mu - \frac{1}{3}\Theta^2 - 3\omega_e\omega^e)h^{(c}_{[a}\right] A^{d)}_{b]} + 2[h^{(c}_{[a}\left(E_{b]e} + \frac{1}{2}\pi_{b]e}\right) \\
& - \frac{1}{3}\Theta h^{(c}_{[a}\sigma_{b]e} - \sigma^{(c}_{[a}\sigma_{b]e} - \frac{1}{3}\Theta h^{(c}_{[a}\epsilon_{b]ef}\omega^{f} - \sigma^{(c}_{[a}\epsilon_{b]ef}\omega^{f} \\
& - \omega^{f}\epsilon^{(c}_{f[a}\sigma_{b]e} + h^{(c)}_{[a}\omega_{b]}\omega_{e}]A^{d)e} + \epsilon_{abe}\omega^{e}\dot{A}^{<cd>}.
\end{aligned}
\tag{2.50}
$$

- Evolution of spacial rotation terms:

$$
\begin{aligned}
h^{a}_{c}h^{b}_{d}[\epsilon^{ef<c}D_{e}A^{d>}_{f}]^{\cdot} = {} & \epsilon^{dc<a}[D_c + \dot{u}_c][h_{de}\dot{A}^{<b>e>}] - \frac{1}{3}\Theta\epsilon^{cd<a}D_{c}A^{b>}_{d} - \epsilon^{cd<a}\sigma^{|e|}_{c}(D_{e}A^{b>}_{d}) \\
& + \omega_{c}D^{<a}A^{b>c} - \omega^{<a}D_{c}A^{b>c} + \frac{1}{3}\Theta\epsilon^{cd<a}\dot{u}_{c}A^{b>}_{d} + \epsilon^{cd<a}\sigma^{b>}\dot{u}_{e}A^{e}_{d} \\
& - \epsilon^{cd<a}\sigma^{|e|}_{c}\dot{u}_{d}A^{b>}_{e} - \epsilon^{cd<a}\dot{u}^{b>}\sigma_{ce}A^{e}_{d} + (\dot{u}_{c}\omega^{c})A^{ab} \\
& + 2\dot{u}_{c}\omega^{<a}A^{b>c} + \dot{u}^{<a}\omega_{c}A^{b>c} + 3[H^{<a}_{c} - (C_3)^{<a}_{c}]A^{b>c} \\
& + \frac{1}{2}\epsilon^{cd<a}[q_c - (C_1)_c]A^{b>}_{d}.
\end{aligned}
\tag{2.51}
$$

In terms of the fully projected spatial derivatives, we can define the usual differential operators of vector calculus as follows: For any projected 3-vector V and second rank 3-tensor A_{ab}, we write

$$\operatorname{div} V = D_a V^a, \tag{2.52}$$

$$(\operatorname{curl} V)_a = \epsilon_{abc} D^b V^c, \tag{2.53}$$

$$(\operatorname{div} A)_a = D^b A_{ab}, \tag{2.54}$$

$$(\operatorname{curl} A)_{ab} = \epsilon_{cd\langle a} D^c A^d{}_{b\rangle}. \tag{2.55}$$

Chapter 3

Conformally flat perturbations of FLRW universe

3.1 Introduction

In this chapter we consider conformally flat perturbations on FLRW spacetimes containing a general matter field. Working with the linearised field equations, we unearth some important geometrical properties for the matter shear and vorticity, and show how they interact with the thermodynamic quantities in the absence of any free gravity powered by the Weyl curvature. Most interestingly, we demonstrate that the matter shear obeys a transverse traceless tensor wave equation and the vorticity obeys a vector wave equation in this linearised regime. These shear and vorticity waves replace the gravitational waves in the sense that they causally carry the information about local change in the curvature of these spacetimes. We also look at the heat transport equation in this case, and indicate how this varies from the Newtonian case.

3.2 Perturbations of FLRW universe

To see in a transparent manner how the absence of the Weyl tensor affects other geometrical and thermodynamical quantities, we consider a conformally flat perturbation of the FLRW manifold. In other words, the background metric is given as

$$ds^2 = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + r^2 a^2(t)(d\theta^2 + \sin^2\theta d\phi^2)\ . \tag{3.1}$$

One can easily see that the nonzero geometric and thermodynamic quantities for the background are

$$\mathcal{D}_0 = \{\Theta,\, \mu,\, p\}\ . \tag{3.2}$$

We now perturb this background spacetime in such a way that the perturbed spacetime still remains conformally flat, that is the Weyl tensor remains identically zero. In that case the quantities that are of first order smallness in the perturbed spacetime is given as

$$\mathcal{D}_1 = \left\{\dot{u}_{\langle a\rangle},\, \omega_{\langle a\rangle},\, \sigma_{\langle ab\rangle},\, q_{\langle a\rangle},\, \pi_{\langle ab\rangle}\right\}\ . \tag{3.3}$$

The Riemann tensor of the perturbed spacetime can now be completely specified in terms of the matter variables as follows

$$\begin{aligned} R^{ab}{}_{cd} &= -2\left(u^{[a}h^{b]}_{[c}q_{d]} + u_{[c}h^{[a}_{d]}q^{b]} + u^{[a}u_{[c}\pi^{b]}_{d]} - h^{[a}_{[c}\pi^{b]}_{d]}\right) \\ &\quad +\frac{2}{3}\left[(\mu+3p)u^{[a}u_{[c}h^{b]}_{d]} + \mu h^{a}_{[c}h^{b}_{d]}\right]. \end{aligned} \tag{3.4}$$

We now use this form of the Riemann tensor to get the Ricci identities of the vector u^a and doubly contracted Bianchi identities (linearised by setting any higher power of the first order quantities to zero). These equations can be further classified into evolution (time derivative) equations and constraints on 3-space. Solutions to these equation will then completely specify the dynamics of the perturbed spacetimes to linear order, and furthermore all these equations remain gauge invariant by the Stewart and Walker (1974) lemma.

3.2.1 Linearised field equations about FLRW backround

In the linearisation procedure, we neglect all products of first order quantities in (2.24)-(2.29), (2.31)-(2.36) and (2.45)-(2.51), and since we consider conformally flat perturbations, the Weyl tensor vanishes identically ($E_{ab} = 0$ and $H_{ab} = 0$).

Evolution equations

The evolution equations (2.24)-(2.26), (2.31), (2.33) and (2.34) take the following form

$$\dot{\Theta} - D_a\dot{u}^a = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu+3p), \tag{3.5}$$

$$\dot{\sigma}^{\langle ab\rangle} - D^{\langle a}\dot{u}^{b\rangle} = \frac{1}{2}\pi^{ab} - \frac{2}{3}\Theta\sigma^{ab}, \tag{3.6}$$

$$\dot{\omega}^{\langle a\rangle} - \frac{1}{2}\epsilon^{abc}D_b\dot{u}_c = -\frac{2}{3}\Theta\omega^a, \tag{3.7}$$

$$\dot{\pi}^{<ab>} + D^{<a}q^{b>} = -(\mu+p)\sigma^{ab} - \frac{\Theta}{3}\pi^{ab}, \tag{3.8}$$

$$\dot{q}^{\langle a\rangle} + D^a p + D_b\pi^{ab} = -\frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a, \tag{3.9}$$

$$\dot{\mu} + D_a q^a = -\Theta(\mu+p). \tag{3.10}$$

Constraint equations

The constraint equatons, (2.23)-(2.29), (2.35) and (2.36), on a given spatial 3−surface can be written as

$$(C_1)^a = D_b\sigma^{ab} - \frac{2}{3}D^a\Theta + \epsilon^{abc}D_b\omega_c + q^a = 0, \tag{3.11}$$

$$(C_2) = D_a\omega^a = 0, \tag{3.12}$$

$$\begin{aligned} (C_3)^{ab} &= D^{<a}\omega^{b>} - \epsilon^{cd<a}D_c\sigma_d^{b>} = 0, \\ (C_3)^{ab} &\equiv D^{<a}\omega^{b>} = \epsilon^{cd<a}D_c\sigma_d^{b>} = 0, \end{aligned} \tag{3.13}$$

$$(C_4)^a = \frac{1}{2}D_b\pi^{ab} - \frac{1}{3}D^a\mu + \frac{1}{3}\Theta q^a = 0, \tag{3.14}$$

$$(C_5)^a = (\mu+p)\omega^a + \frac{1}{2}\epsilon^{abc}D_b q_c = 0, \tag{3.15}$$

$$(C_6)^{ab} = \epsilon^{cd<a}D_c\pi_d^{b>} = 0. \tag{3.16}$$

We note that the last constraint (3.16) is not an original constraint of the field equations. We get this constraint by forcing the perturbed spacetime to be conformally flat. A consistent evolution of this constraint (so that this is valid at all epochs) will give further restrictions on different geometrical and thermodynamic quantities as we shall see in the next sections.

Commutation relations

For a linearised model about an FLRW spacetime, we have the following commutation relations between the derivative operators. For any scalar function f,

$$D_{[a}D_{b]}f = \epsilon_{abc}\omega^c\dot{f}, \tag{3.17}$$

$$\epsilon^{abc}D_bD_cf = 2\omega^a\dot{f}, \tag{3.18}$$

$$h^a_b[D^bf]^{\cdot} = D^a\dot{f} - \frac{1}{3}\Theta D^af. \tag{3.19}$$

Also for any first order 3-vector V^a, we have

$$[D^{<a}V^{b>}]^{\cdot} \equiv h^a_ch^b_d[D^cV^d]^{\cdot} = D^a\dot{V}^{<b>} - \frac{1}{3}\Theta D^aV^b, \tag{3.20}$$

$$(D_aV^a)^{\cdot} = D_a\dot{V}^{<a>} - \frac{1}{3}\Theta D_aV^a, \tag{3.21}$$

$$D_{[a}D_{b]}V_c = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)h_{c[a}V_{b]}, \tag{3.22}$$

$$h^a_b(\epsilon^{bcd}D_cV_d)^{\cdot} = \epsilon^{abc}D_b\dot{V}_{<c>} - \frac{1}{3}\Theta\epsilon^{abc}D_bV_c, \tag{3.23}$$

$$D_a[\epsilon^{abc}D_bV_c] = 2\omega_a\dot{V}^{<a>}. \tag{3.24}$$

Similarly, for any first order second rank 3-tensor A^{ab}, we have

$$D_{[a}D_{b]}A^{cd} = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)h^{(c}_{[a}A^{d)}_{b]}, \tag{3.25}$$

$$\begin{aligned} h^a_ch^b_d[\epsilon^{ef<c}D_eA^{d>}_f]^{\cdot} &= \epsilon^{dc<a}D_c\dot{A}^{<b>e>}h_{de} \\ &\quad -\frac{1}{3}\Theta\epsilon^{cd<a}D_cA^{b>}_d. \end{aligned} \tag{3.26}$$

Further to the above, and relations (2.52)-(2.55), there are some important identities of first order vectors and tensors in perturbed FLRW spacetime, which we list below (Maartens and Bassett (1998))

$$D^a(\mathrm{curl}\, V)_a = 0, \tag{3.27}$$

$$D^b(\mathrm{curl}\,A)_{ab} = \frac{1}{2}\mathrm{curl}\,(D^b A_{ab}), \tag{3.28}$$

$$\begin{aligned}(\mathrm{curl}\,\mathrm{curl}\,V)_a &= -D^2 V_a + D_a(D^b V_b) \\ &\quad + \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) V_a, \end{aligned} \tag{3.29}$$

$$\begin{aligned}(\mathrm{curl}\,\mathrm{curl}\,A)_{ab} &= -D^2 A_{ab} + \frac{3}{2} D_{\langle a} D^c A_{b\rangle c} \\ &\quad + \left(\mu - \frac{1}{3}\Theta^2\right) A_{ab}. \end{aligned} \tag{3.30}$$

3.2.2 Consistency of the new constraints

By imposition of $C^a{}_{bcd} = 0$ we get a few new constraints that have to be obeyed at all epochs. We check the consistancy by time evolving these new constraints.

From relation (3.16), we know

$$(\dot{C_6}) = 0,$$

$$[\epsilon^{cd<a} D_c \pi_d^{b>}]^{\cdot} = 0,$$

$$h_c^a h_d^b [\epsilon^{ef<c} D_e \pi_f^{d>}]^{\cdot} = 0.$$

By relation (3.26) we have

$$\epsilon^{dc<a} D_c \dot{\pi}^{<b>e>} h_{de} - \frac{1}{3}\Theta \epsilon^{cd<a} D_c \pi_d^{b>} = 0 \Rightarrow \epsilon^{dc<a} D_c \dot{\pi}^{<b>e>} h_{de} = 0,$$

and relation (3.8) gives

$$\epsilon^{dc<a} D_c[-D^e q^{b>} - (\mu + p)\sigma^{be} - \frac{\Theta}{3}\pi^{be}] h_{de} = 0,$$

or

$$\epsilon^{dc<a} D_c D^e q^{b>} h_{de} + (\mu + p)\epsilon^{dc<a} D_c \sigma^{b>e} h_{de} + \frac{\Theta}{3}\epsilon^{dc<a} D_c \pi^{b>e} h_{de} = 0,$$

or else

$$\epsilon^{dc<a}D_cD_dq^{b>} + (\mu+p)\epsilon^{dc<a}D_c\sigma_d^{b>} + \frac{\Theta}{3}\epsilon^{dc<a}D_c\pi_d^{b>} = 0$$

$$\Rightarrow \epsilon^{dc<a}D_cD_dq^{b>} + (\mu+p)\epsilon^{dc<a}D_c\sigma_d^{b>} = 0.$$

Using relation (3.13) we can write

$$\epsilon^{dc<a}D_cD_dq^{b>} + (\mu+p)D^{<a}\omega^{b>} = 0.$$

Finally we obtain

$$\epsilon^{dc<a}D_cD_dq^{b>} = -(\mu+p)D^{<a}\omega^{b>} = 0. \tag{3.31}$$

From relation (3.22), we know that

$$D_{[c}D_{d]}V_b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)h_{b[c}V_{d]},$$

which implies

$$\frac{1}{2}(D_cD_d - D_dD_c)V_b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\frac{1}{2}(h_{bc}V_d - h_{bd}V_c),$$

or

$$(D_cD_d - D_dD_c)V_b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(h_{bc}V_d - h_{bd}V_c). \tag{3.32}$$

Acting with ϵ^{abc} on both sides of equation (3.32) above, we have

$$\epsilon^{adc}(D_cD_d - D_dD_c)V_b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{adc}(h_{bc}V_d - h_{bd}V_c),$$

or

$$2\epsilon^{adc}D_cD_dV_b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(\epsilon^{adc}h_{bc}V_d - \epsilon^{adc}h_{bd}V_c),$$

or

$$\epsilon^{adc}D_cD_dV_b = \frac{1}{6}\left(\mu - \frac{1}{3}\Theta^2\right)(\epsilon^{adc}h_{bc}V_d - \epsilon^{adc}h_{bd}V_c),$$

which implies

$$\epsilon^{adc} D_c D_d V^b = \frac{1}{6}\left(\mu - \frac{1}{3}\Theta^2\right)(\epsilon^{adc}\delta^b_c V_d - \epsilon^{adc}\delta^b_d V_c)$$
$$= \frac{1}{6}\left(\mu - \frac{1}{3}\Theta^2\right)(\epsilon^{adb} V_d - \epsilon^{abc} V_c)$$
$$= \frac{1}{6}\left(\mu - \frac{1}{3}\Theta^2\right)(\epsilon^{adb} V_d - \epsilon^{abd} V_d) = \frac{1}{6}\left(\mu - \frac{1}{3}\Theta^2\right)(-\epsilon^{abd} V_d - \epsilon^{abd} V_d).$$

Finally we obtain

$$\epsilon^{adc} D_c D_d V^b = -\frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{abd} V_d.$$

Therefore we find

$$\epsilon^{dc<a} D_c D_d V^{b>} = -\frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{<ab>d} V_d, \tag{3.33}$$

$$\epsilon^{dc<a} D_c D_d q^{b>} = -\frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{<ab>d} q_d. \tag{3.34}$$

From relations (3.31) and (3.34) we have

$$(\mu + p) D^{<a}\omega^{b>} = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{<ab>d} q_d, \tag{3.35}$$

so that

$$\epsilon^{dc<a} D_c (D_d q^{b>}) = -(\mu + p) D^{<a}\omega^{b>} = -\frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{<ab>d} q_d = 0. \tag{3.36}$$

From relation (3.13) we have

$$[\epsilon^{cd<a} D_c \sigma_d^{b>}]^{\cdot} = 0,$$

$$h^a_c h^b_d [\epsilon^{ef<c} D_e \sigma_f^{d>}]^{\cdot} = 0.$$

Using relations (3.26) and (3.6) respectively, we get

$$\epsilon^{dc<a}D_c\dot{\sigma}^{<b>e}h_{de} - \frac{1}{3}\Theta\epsilon^{dc<a}D_c\sigma_d^{b>} = 0 \Rightarrow \epsilon^{dc<a}D_c\dot{\sigma}^{<b>e}h_{de} = 0,$$

$$\epsilon^{dc<a}D_c(D^e\dot{u}^{b>} + \frac{1}{2}\pi^{b>e} - \frac{2}{3}\Theta\sigma^{b>e})h_{de} = 0,$$

$$\Rightarrow \epsilon^{dc<a}D_cD^e\dot{u}^{b>}h_{de} + \frac{1}{2}\epsilon^{dc<a}D_c\pi^{b>e}h_{de} - \frac{2}{3}\epsilon^{dc<a}D_c\Theta\sigma^{b>e}h_{de} = 0,$$

$$\Rightarrow \epsilon^{dc<a}D_cD_d\dot{u}^{b>} + \frac{1}{2}\epsilon^{dc<a}D_c\pi_d^{b>} - \frac{2}{3}\epsilon^{dc<a}D_c\Theta\sigma_d^{b>} = 0 \Rightarrow \epsilon^{dc<a}D_cD_d\dot{u}^{b>} = 0.$$

Therefore

$$\epsilon^{dc<a}D_cD_d\dot{u}^{b>} = 0. \tag{3.37}$$

From (3.9), we have

$$\dot{q}^{<b>} + D^bp + D_f\pi^{bf} + \frac{3}{4}\Theta q^b = -(\mu+p)\dot{u}^b,$$

and acting with $\epsilon^{cd<a}D_cD_d$ on both sides, we get

$$\begin{aligned}\epsilon^{cd<a}D_cD_d\dot{q}^{b>} + \epsilon^{cd<a}D_cD_dD^{b>}p + \epsilon^{cd<a}D_cD_dD_f\pi^{b>f} + \frac{3}{4}\epsilon^{cd<a}D_cD_d\Theta q^{b>} \\ = -\epsilon^{cd<a}D_cD_d(\mu+p)\dot{u}^{b>}.\end{aligned} \tag{3.38}$$

In (3.38), using (3.31) and (3.37), we have

$$\epsilon^{cd<a}D_cD_d\dot{q}^{b>} = 0,\ \epsilon^{cd<a}D_cD_d\Theta q^{b>} = 0 \text{ and } \epsilon^{cd<a}D_cD_d(\mu+p)\dot{u}^{b>} = 0.$$

This gives

$$\epsilon^{cd<a}D_cD_dD^{b>}p + \epsilon^{cd<a}D_cD_dD_f\pi^{b>f} = 0. \tag{3.39}$$

From (3.14) we get

$$D_f\pi^{bf} = \frac{2}{3}(D^b\mu - \Theta q^b), \tag{3.40}$$

and

$$\epsilon^{cd<a}D_cD_dD_f\pi^{b>f} = \frac{2}{3}\epsilon^{cd<a}D_cD_dD^{b>}\mu - \frac{2}{3}\epsilon^{cd<a}D_cD_d\Theta q^{b>}, \tag{3.41}$$

on using (3.31), $\epsilon^{cd<a}D_cD_d\Theta q^{b>} = 0$, so that

$$\epsilon^{cd<a}D_cD_dD_f\pi^{b>f} = \frac{2}{3}\epsilon^{cd<a}D_cD_dD^{b>}\mu. \tag{3.42}$$

Substituting (3.42) in (3.39) we get

$$\epsilon^{cd<a}D_cD_d(D^{b>}p + \frac{2}{3}D^{b>}\mu) = 0. \tag{3.43}$$

Now, by (3.14) we have

$$D_b\pi^{ab} = \frac{2}{3}(D^a\mu - \Theta q^a), \tag{3.44}$$

and

$$[D_b\pi^{ab}]^{\cdot} - \frac{2}{3}[D^a\mu]^{\cdot} + \frac{2}{3}[\Theta q^a]^{\cdot} = 0. \tag{3.45}$$

From relation (3.21), we have

$$[D_b\pi^{ab}]^{\cdot} = D_b\dot{\pi}^{<ab>} - \frac{1}{3}\Theta D_b\pi^{ab},$$

using (3.8) we write

$$\begin{aligned}
[D_b\pi^{ab}]^{\cdot} &= D_b[-D^{<a}q^{b>} - (\mu+p)\sigma^{ab} - \frac{1}{3}\Theta\pi^{ab}] - \frac{1}{3}\Theta D_b\pi^{ab} \\
&= -D_bD^{<a}q^{b>} - D_b[(\mu+p)\sigma^{ab}] - \frac{1}{3}D_b(\Theta\pi^{ab}) - \frac{1}{3}\Theta D_b\pi^{ab} \\
&= -D_bD^{<a}q^{b>} - (\mu+p)D_b\sigma^{ab} - \sigma^{ab}D_b(\mu+p) \\
&- \frac{1}{3}\Theta D_b\pi^{ab} - \frac{1}{3}\pi^{ab}D_b\Theta - \frac{1}{3}\Theta D_b\pi^{ab} \\
&= -D_bD^{<a}q^{b>} - (\mu+p)D_b\sigma^{ab} - \frac{2}{3}\Theta D_b\pi^{ab} \\
&\text{(because } \sigma^{ab}D_b(\mu+p) = 0 \text{ and } \pi^{ab}D_b\Theta = 0\text{)},
\end{aligned}$$

and using relation (3.44) we get

$$[D_b\pi^{ab}]^{\cdot} = -D_bD^{<a}q^{b>} - (\mu+p)D_b\sigma^{ab} - \frac{2}{3}\Theta.\frac{2}{3}(D^a\mu - \Theta q^a).$$

Therefore we have

$$[D_b\pi^{ab}]^{\cdot} = -D_bD^{<a}q^{b>} - (\mu+p)D_b\sigma^{ab} - \frac{4}{9}\Theta D^a\mu + \frac{4}{9}\Theta^2q^a. \tag{3.46}$$

From relation (3.19) we have

$$\begin{aligned}
[D^a\mu]^{\cdot} &= D^a\dot{\mu} - \frac{1}{3}\Theta D^a\mu \overset{(3.10)}{=} D^a[-D_bq^b - \Theta(\mu+p)] - \frac{1}{3}\Theta D^a\mu \\
&= -D^aD_bq^b - D^a[\Theta(\mu+p)] - \frac{1}{3}\Theta D^a\mu,
\end{aligned}$$

and this implies

$$[D^a\mu]^{\cdot} = -D^aD_bq^b - \Theta D^a(\mu+p) - (\mu+p)D^a\Theta - \frac{1}{3}\Theta D^a\mu. \tag{3.47}$$

Now we have

$$[\Theta q^a]^{\cdot} = \Theta \dot{q}^a + \dot{\Theta} q^a.$$

Using relations (4.2) and (3.9) the above becomes

$$[\Theta q^a]^{\cdot} = \Theta[-D^a p - D_b \pi^{ab} - \frac{4}{3}\Theta q^a - (\mu + p)\dot{u}^a] + [D_a \dot{u}^a - \frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p)]q^a$$

$$= -\Theta D^a p - \Theta D_b \pi^{ab} - \frac{4}{3}\Theta^2 q^a - \Theta(\mu + p)\dot{u}^a + q^a \text{div } \dot{u} - \frac{1}{3}\Theta^2 q^a - \frac{1}{2}(\mu + 3p)q^a,$$

and using (3.44) we have

$$[\Theta q^a]^{\cdot} = -\Theta D^a p - \Theta.\frac{2}{3}(D^a \mu - \Theta q^a) - \frac{5}{3}\Theta^2 q^a - \Theta(\mu + p)\dot{u}^a + q^a \text{div } \dot{u} - \frac{1}{2}(\mu + 3p)q^a$$

$$= -\Theta D^a p - \frac{2}{3}\Theta D^a \mu + \frac{2}{3}\Theta^2 q^a - \frac{5}{3}\Theta^2 q^a - \Theta(\mu + p)\dot{u}^a + q^a \text{div } \dot{u} - \frac{1}{2}(\mu + 3p)q^a$$

$$= -\Theta D^a p - \frac{2}{3}\Theta D^a \mu - \Theta^2 q^a - \Theta(\mu + p)\dot{u}^a - \frac{1}{2}(\mu + 3p)q^a \text{ because } q^a \text{div } \dot{u} = 0.$$

Therefore we obtain

$$[\Theta q^a]^{\cdot} = -\Theta D^a p - \frac{2}{3}\Theta D^a \mu - \Theta^2 q^a - \Theta(\mu + p)\dot{u}^a - \frac{1}{2}(\mu + 3p)q^a. \tag{3.48}$$

Substituting relations (3.46), (3.47) and (3.48) into relation (3.45) we have

$$
\begin{aligned}
&- D_b D^{<a} q^{b>} - (\mu+p) D_b \sigma^{ab} - \frac{4}{9}\Theta D^a \mu + \frac{4}{9}\Theta^2 q^a - \frac{2}{3}[-D^a D_b q^b - \Theta D^a(\mu+p) \\
&- (\mu+p) D^a \Theta - \frac{1}{3}\Theta D^a \mu] + \frac{2}{3}[-\Theta D^a p - \frac{2}{3}\Theta D^a \mu - \Theta^2 q^a - \Theta(\mu+p)\dot{u}^a \\
&- \frac{1}{2}(\mu+3p) q^a] = 0, \\
&\Rightarrow -D_b D^{<a} q^{b>} - (\mu+p) D_b \sigma^{ab} - \frac{4}{9}\Theta D^a \mu + \frac{4}{9}\Theta^2 q^a + \frac{2}{3} D^a D_b q^b \\
&+ \frac{2}{3}\Theta D^a(\mu+p) + \frac{2}{3}(\mu+p) D^a \Theta + \frac{2}{9}\Theta D^a \mu - \frac{2}{3}\Theta D^a p \\
&- \frac{4}{9}\Theta D^a \mu - \frac{2}{3}\Theta^2 q^a - \frac{2}{3}\Theta(\mu+p)\dot{u}^a - \frac{1}{3}(\mu+3p) q^a = 0, \\
&\Rightarrow \frac{2}{3} D^a D_b q^b - D_b D^{<a} q^{b>} - (\mu+p) D_b \sigma^{ab} - \frac{2}{3}\Theta D^a \mu - \frac{2}{9}\Theta^2 q^a + \frac{2}{3}\Theta D^a \mu + \frac{2}{3}\Theta D^a p \\
&+ \frac{2}{3}(\mu+p) D^a \Theta - \frac{2}{3}\Theta D^a p - \frac{2}{3}\Theta(\mu+p)\dot{u}^a - \frac{1}{3}(\mu+3p) q^a = 0, \\
&\Rightarrow \frac{2}{3} D^a D_b q^b - D_b D^{<a} q^{b>} - (\mu+p) D_b \sigma^{ab} - \frac{2}{9}\Theta^2 q^a + \frac{2}{3}(\mu+p) D^a \Theta \\
&- \frac{2}{3}\Theta(\mu+p)\dot{u}^a - \frac{1}{3}(\mu+3p) q^a = 0,
\end{aligned}
$$

Now using relation (3.11) we have

$$
\begin{aligned}
&D_b D^{<a} q^{b>} - \frac{2}{3} D^a D_b q^b + (\mu+p)[\frac{2}{3} D^a \Theta - \epsilon^{abc} D_b \omega_c - q^a] + \frac{2}{9}\Theta^2 q^a - \frac{2}{3}(\mu+p) D^a \Theta \\
&+ \frac{2}{3}\Theta(\mu+p)\dot{u}^a + \frac{1}{3}(\mu+3p) q^a = 0, \\
&\Rightarrow D_b D^{<a} q^{b>} - \frac{2}{3} D^a D_b q^b + \frac{2}{3}(\mu+p) D^a \Theta - (\mu+p)\epsilon^{abc} D_b \omega_c - (\mu+p) q^a \\
&+ \frac{2}{9}\Theta^2 q^a - \frac{2}{3}(\mu+p) D^a \Theta + \frac{2}{3}\Theta(\mu+p)\dot{u}^a + \frac{1}{3}(\mu+3p) q^a = 0,
\end{aligned}
$$

$$\Rightarrow D_b D^{<a}q^{b>} - \frac{2}{3}D^a D_b q^b - (\mu+p)\epsilon^{abc}D_b\omega_c - \mu q^a - pq^a + \frac{2}{9}\Theta^2 q^a$$

$$+ \frac{2}{3}\Theta(\mu+p)\dot{u}^a + \frac{1}{3}\mu q^a + pq^a = 0,$$

$$\Rightarrow D_b D^{<a}q^{b>} - \frac{2}{3}D^a D_b q^b - (\mu+p)\epsilon^{abc}D_b\omega_c - \frac{2}{3}\mu q^a + \frac{2}{9}\Theta^2 q^a + \frac{2}{3}\Theta(\mu+p)\dot{u}^a = 0.$$

It follows that

$$D_b D^{<a}q^{b>} - \frac{2}{3}D^a D_b q^b - (\mu+p)\epsilon^{abc}D_b\omega_c - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)q^a + \frac{2}{3}\Theta(\mu+p)\dot{u}^a = 0,$$

and finally,

$$2D^a D_b q^b - 3D_b D^{<a}q^{b>} = 2\Theta(\mu+p)\dot{u}^a - 3(\mu+p)\epsilon^{abc}D_b\omega_c - 2(\mu - \frac{1}{3}\Theta^2)q^a. \qquad (3.49)$$

Now let us compute $D^{<a}q^{b>}$ and then $D_b D^{<a}q^{b>}$ so that

$$\begin{aligned}
D^{<a}q^{b>} &= (h_c^{(a}h_d^{b)} - \frac{1}{3}h^{ab}h_{cd})D^c q^d \\
&= \left[\frac{1}{2}(h_c^a h_d^b + h_c^b h_d^a) - \frac{1}{3}h^{ab}h_{cd}\right] D^c q^d \\
&= \frac{1}{2}h_c^a h_d^b D^c q^d + \frac{1}{2}h_c^b h_d^a D^c q^d - \frac{1}{3}h^{ab}h_{cd}D^c q^d \\
&= \frac{1}{2}h_d^b D^a q^d + \frac{1}{2}h_d^a D^b q^d - \frac{1}{3}h^{ab}D_d q^d,
\end{aligned}$$

therefore

$$D_b D^{<a} q^{b>} = \frac{1}{2} D_b h^b_d D^a q^d + \frac{1}{2} D_b h^a_d D^b q^d - \frac{1}{3} D_b h^{ab} \text{div } q$$

$$= \frac{1}{2} D_d D^a q^d + \frac{1}{2} D_b h^a_d D^b q^d - \frac{1}{3} D^a \text{div } q$$

$$= \frac{1}{2} D_b D^a q^b + \frac{1}{2} D^b h^a_b \text{div } q - \frac{1}{3} D^a D_b q^b$$

$$= \frac{1}{2} D_b D^a q^b + \frac{1}{2} D_b D^a q^b - \frac{1}{3} D^a D_b q^b.$$

Therefore we find

$$D_b D^{<a} q^{b>} = D_b D^a q^b - \frac{1}{3} D^a D_b q^b. \tag{3.50}$$

Substituting (3.50) into (3.49), we get

$$2D^a D_b q^b - 3D_b D^a q^b + D^a D_b q^b = 2\Theta(\mu + p)\dot{u}^a - 3(\mu + p)\epsilon^{abc} D_b \omega_c - 2(\mu - \frac{1}{3}\Theta^2) q^a,$$

and

$$3(D_a D_b - D_b D_a) q^b = 2\Theta(\mu + p)\dot{u}_a - 3(\mu + p)\epsilon_a^{\ bc} D_b \omega_c - 2(\mu - \frac{1}{3}\Theta^2) q_a. \tag{3.51}$$

From relation (3.32), we know that

$$(D_a D_b - D_b D_a) V_c = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(h_{ca} V_b - h_{cb} V_a),$$

$$\Rightarrow (D_a D_b - D_b D_a) V^c = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(h^c_a V_b - h^c_b V_a),$$

$$\Rightarrow (D_a D_b - D_b D_a) V^b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(h^b_a V_b - h^b_b V_a) = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(V_a - 3V_a)$$

$$= -\frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) V_a.$$

Therefore we find

$$(D_a D_b - D_b D_a)V^b = -\frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)V_a, \tag{3.52}$$

$$(D_a D_b - D_b D_a)q^b = -\frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)q_a. \tag{3.53}$$

Substituting (3.53) into (3.51) we have

$$\begin{aligned}
&-2(\mu - \frac{1}{3}\Theta^2)q_a = 2\Theta(\mu+p)\dot{u}_a - 3(\mu+p)\epsilon_a^{\ bc}D_b\omega_c - 2(\mu - \frac{1}{3}\Theta^2)q_a \\
&\Rightarrow 2\Theta(\mu+p)\dot{u}^a - 3(\mu+p)\epsilon^{abc}D_b\omega_c = 0.
\end{aligned}$$

This gives

$$\epsilon^{abc}D_b\omega_c = \frac{2}{3}\Theta\dot{u}^a. \tag{3.54}$$

Now from equation (3.15) we have

$$\begin{aligned}
&\epsilon^{abc}D_b q_c = -2(\mu+p)\omega^a \\
&\Rightarrow D_a[\epsilon^{abc}D_b q_c] = -2[(\mu+p)D_a\omega^a + \omega^a D_a(\mu+p)] \stackrel{(46)}{=} -2\omega^a D_a(\mu+p),
\end{aligned}$$

so that

$$D_a[\epsilon^{abc}D_b q_c] = -2\omega^a D_a(\mu+p). \tag{3.55}$$

And from relation (3.24) we have

$$D_a[\epsilon^{abc}D_bq_c] = 2\omega_a\dot{q}^{<a>},$$

and using (3.9)

$$\begin{aligned}
D_a[\epsilon^{abc}D_bq_c] &= 2\omega_a[-D^ap - D_b\pi^{ab} - \frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a] \\
&= -2\omega_aD^ap - 2\omega_aD_b\pi^{ab} - \frac{8}{3}\Theta\omega_aq^a - 2\omega_a(\mu+p)\dot{u}^a \\
&= -2\omega_aD^ap - 2\omega_aD_b\pi^{ab}, \text{ (because } \frac{8}{3}\Theta\omega_aq^a = 0 \text{ and } 2\omega_a(\mu+p)\dot{u}^a = 0), \\
&\overset{(3.44)}{=} -2\omega_aD^ap - 2\omega_a.\frac{2}{3}(D^a\mu - \Theta q^a) \\
&= -2\omega_aD^ap - \frac{4}{3}\omega_aD^a\mu + \frac{4}{3}\omega_a\Theta q^a \\
&= -2\omega_aD^a(p + \frac{2}{3}\mu) \text{ (knowing that } \omega_a\Theta q^a = 0).
\end{aligned}$$

Hence we have

$$D_a[\epsilon^{abc}D_bq_c] = -2\omega_aD^a(\frac{2}{3}\mu + p). \tag{3.56}$$

From (3.55) and (3.56) we have

$$\begin{aligned}
&-2\omega^aD_a(\mu+p) = -2\omega_aD^a(\frac{2}{3}\mu + p), \\
&\Rightarrow \omega^aD_a(\mu+p) = \omega^aD_a(\frac{2}{3}\mu + p), \\
&\Rightarrow \omega^aD_a[(\mu+p) - (\frac{2}{3}\mu + p)] = 0,
\end{aligned}$$

so that

$$\omega^aD_a\mu = 0. \tag{3.57}$$

From relation (3.12) we have

$$[D_a\omega^a]^{\cdot} = 0,$$

using relations (3.21), (3.12) and (3.7) we have respectively

$$D_a\dot{\omega}^{<a>} - \frac{1}{3}\Theta D_a\omega^a = 0,$$

$$D_a\dot{\omega}^{<a>} = 0,$$

$$D_a[\frac{1}{2}\epsilon^{abc}D_b\dot{u}_c - \frac{2}{3}\Theta\omega^a] = 0.$$

This gives

$$\frac{1}{2}D_a[\epsilon^{abc}D_b\dot{u}_c] = \frac{2}{3}D_a(\Theta\omega^a) = \frac{2}{3}\omega^a D_a\Theta + \frac{2}{3}\Theta D_a\omega^a = 0,$$

so that

$$D_a[\epsilon^{abc}D_b\dot{u}_c] = 0. \tag{3.58}$$

Now from (3.11) we have

$$[D_b\sigma^{ab}]^{\cdot} - \frac{2}{3}[D^a\Theta]^{\cdot} + [\epsilon^{abc}D_b\omega_c]^{\cdot} + \dot{q}^a = 0.$$

Using relations (3.9), (3.19) and (3.21) we have

$$D_b\dot{\sigma}^{<ab>} - \frac{1}{3}\Theta D_b\sigma^{ab} - \frac{2}{3}(D^a\dot{\Theta} - \frac{1}{3}\Theta D^a\Theta) + [\epsilon^{abc}D_b\omega_c]^{\cdot} - D^a p - D_b\pi^{ab} - \frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a = 0,$$

using (3.6) and (4.2) we get respectively

$$D_b(D^{<a}\dot{u}^{b>} + \frac{1}{2}\pi^{ab} - \frac{2}{3}\Theta\sigma^{ab}) - \frac{1}{3}\Theta D_b\sigma^{ab} - \frac{2}{3}D^a\dot{\Theta} + \frac{2}{9}\Theta D^a\Theta + [\epsilon^{abc}D_b\omega_c]^{\cdot}$$

$$- D^a p - D_b\pi^{ab} - \frac{4}{3}\Theta q^a - (\mu + p)\dot{u}^a = 0,$$

and

$$D_b D^{<a}\dot{u}^{b>} + \frac{1}{2}D_b\pi^{ab} - \frac{2}{3}D_b(\Theta\sigma^{ab}) - \frac{1}{3}\Theta D_b\sigma^{ab} - \frac{2}{3}D^a[D_b\dot{u}^b - \frac{1}{3}\Theta^2 - \frac{1}{2}(\mu+3p)]$$

$$+ \frac{2}{9}\Theta D^a\Theta + [\epsilon^{abc}D_b\omega_c]^{\cdot} - D^a p - D_b\pi^{ab} - \frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a = 0.$$

This gives

$$D_b D^{<a}\dot{u}^{b>} - \frac{1}{2}D_b\pi^{ab} - \frac{2}{3}\Theta D_b\sigma^{ab} - \frac{2}{3}\sigma^{ab}D_b\Theta - \frac{1}{3}\Theta D_b\sigma^{ab} - \frac{2}{3}D^a D_b\dot{u}^b + \frac{4}{9}\Theta D^a\Theta$$

$$+ \frac{1}{3}D^a(\mu+3p) + \frac{2}{9}\Theta D^a\Theta + [\epsilon^{abc}D_b\omega_c]^{\cdot} - D^a p - \frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a = 0,$$

or

$$D_b D^{<a}\dot{u}^{b>} - \frac{1}{2}D_b\pi^{ab} - \Theta D_b\sigma^{ab} - \frac{2}{3}D^a D_b\dot{u}^b + \frac{2}{3}\Theta D^a\Theta$$

$$+ \frac{1}{3}D^a(\mu+3p) + [\epsilon^{abc}D_b\omega_c]^{\cdot} - D^a p - \frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a = 0 \quad (\text{because } \frac{2}{3}\sigma^{ab}D_b\Theta = 0),$$

or, using (3.14)

$$D_b D^{<a}\dot{u}^{b>} - \frac{1}{3}(D^a\mu - \Theta q^a) - \Theta D_b\sigma^{ab} - \frac{2}{3}D^a D_b\dot{u}^b + \frac{2}{3}\Theta D^a\Theta$$

$$+ \frac{1}{3}D^a(\mu+3p) + [\epsilon^{abc}D_b\omega_c]^{\cdot} - D^a p - \frac{4}{3}\Theta q^a - (\mu+p)\dot{u}^a = 0.$$

This gives

$$D_b D^{<a}\dot{u}^{b>} - \frac{2}{3}D^a D_b\dot{u}^b + [\epsilon^{abc}D_b\omega_c]^{\cdot} - \Theta D_b\sigma^{ab} + \frac{2}{3}\Theta D^a\Theta - \Theta q^a - (\mu+p)\dot{u}^a = 0,$$

using relation (3.11) the above becomes

$$D_b D^{<a}\dot{u}^{b>} - \frac{2}{3}D^a D_b\dot{u}^b + [\epsilon^{abc}D_b\omega_c]^{\cdot} - \Theta(\frac{2}{3}D^a\Theta - \epsilon^{abc}D_b\omega_c - q^a) + \frac{2}{3}\Theta D^a\Theta - \Theta q^a$$

$$- (\mu+p)\dot{u}^a = 0,$$

or

$$D_b D^{<a}\dot{u}^{b>} - \frac{2}{3}D^a D_b\dot{u}^b + [\epsilon^{abc}D_b\omega_c]^{\cdot} + \Theta\epsilon^{abc}D_b\omega_c - (\mu+p)\dot{u}^a = 0.$$

Using (3.50) we have

$$D_bD^a\dot{u}^b - \frac{1}{3}D^aD_b\dot{u}^b - \frac{2}{3}D^aD_b\dot{u}^b + [\epsilon^{abc}D_b\omega_c]^{\cdot} + \Theta\epsilon^{abc}D_b\omega_c - (\mu+p)\dot{u}^a = 0,$$

or

$$D_bD^a\dot{u}^b - D^aD_b\dot{u}^b + [\epsilon^{abc}D_b\omega_c]^{\cdot} + \Theta\epsilon^{abc}D_b\omega_c - (\mu+p)\dot{u}^a = 0,$$

and using relation (3.53) we get

$$\frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a + [\epsilon^{abc}D_b\omega_c]^{\cdot} + \Theta\epsilon^{abc}D_b\omega_c - (\mu+p)\dot{u}^a = 0.$$

It follows that

$$[\epsilon^{abc}D_b\omega_c]^{\cdot} = (\mu+p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a - \Theta\epsilon^{abc}D_b\omega_c,$$

we have

$$[\epsilon^{bcd}D_c\omega_d]^{\cdot} = (\mu+p)\dot{u}^b - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^b - \Theta\epsilon^{bcd}D_c\omega_d.$$

Acting with h^a_b on both sides of the above equation we get

$$h^a_b[\epsilon^{bcd}D_c\omega_d]^{\cdot} = (\mu+p)h^a_b\dot{u}^b - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)h^a_b\dot{u}^b - \Theta h^a_b\epsilon^{bcd}D_c\omega_d,$$

and using (3.23) we have

$$\epsilon^{abc}D_b\dot{\omega}_{<c>} - \frac{1}{3}\Theta\epsilon^{abc}D_b\omega_c = (\mu+p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a - \Theta\epsilon^{acd}D_c\omega_d,$$

or

$$\epsilon^{abc}D_b\dot{\omega}_{<c>} + \frac{2}{3}\Theta\epsilon^{abc}D_b\omega_c = (\mu+p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a.$$

Now using equation (3.7) we get

$$\epsilon^{abc}D_b(\frac{1}{2}\epsilon_{cde}D^d\dot{u}^e - \frac{2}{3}\Theta\omega_c) + \frac{2}{3}\Theta\epsilon^{abc}D_b\omega_c = (\mu+p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a,$$

or

$$\frac{1}{2}\epsilon^{abc}D_b[\epsilon_{cde}D^d\dot{u}^e] - \frac{2}{3}\epsilon^{abc}D_b(\Theta\omega_c) + \frac{2}{3}\Theta\epsilon^{abc}D_b\omega_c = (\mu+p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a,$$

which gives

$$\frac{1}{2}\epsilon^{abc}D_b[\epsilon_{cde}D^d\dot{u}^e] - \frac{2}{3}\epsilon^{abc}\omega_c D_b\Theta - \frac{2}{3}\epsilon^{abc}\Theta D_b\omega_c + \frac{2}{3}\Theta\epsilon^{abc}D_b\omega_c$$

$$= (\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a,$$

or

$$\frac{1}{2}\epsilon^{abc}D_b[\epsilon_{cde}D^d\dot{u}^e] = (\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a \text{ (because } \omega_c D_b\Theta = 0).$$

This implies

$$\epsilon^{abc}\epsilon_{dec}D_b[D^d\dot{u}^e] = 2[(\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a],$$

or

$$2h^a_{[d}h^b_{e]}D_b[D^d\dot{u}^e] = 2[(\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a],$$

or

$$(h^a_d h^b_e - h^a_e h^b_d)D_b[D^d\dot{u}^e] = 2[(\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a],$$

we find

$$h^a_d h^b_e D_b[D^d\dot{u}^e] - h^a_e h^b_d D_b[D^d\dot{u}^e] = 2[(\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a],$$

or

$$h^a_d D_e[D^d\dot{u}^e] - h^a_e D_d[D^d\dot{u}^e] = 2[(\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a],$$

or

$$3D_e[D^a\dot{u}^e] - D_e[D^a\dot{u}^e] = 2[(\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a],$$

or

$$D_b[D^a\dot{u}^b] = (\mu + p)\dot{u}^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a.$$

And this implies that

$$D_b[D_a\dot{u}^b] = (\mu + p)\dot{u}_a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}_a.$$

Using relation (3.53) we get

$$D_b D_a\dot{u}^b = (\mu + p)\dot{u}_a + D_a D_b\dot{u}^b - D_b D_a\dot{u}^b,$$

giving the result

$$(2D_b D_a - D_a D_b)\dot{u}^b = (\mu + p)\dot{u}_a. \tag{3.59}$$

We would like to point out that the relations (3.31), (3.36), (3.37), (3.43),(3.54), (3.57) and (3.59) are the results we found from in consistency of the new constraints.

3.3 Some geometrical results on shear and vorticity

In this section we will state and prove several important geometrical properties of matter shear and vorticity in the perturbed, conformally flat spacetime. We would like to emphasise that throughout this work we consider the matter field satisfying the weak energy conditions, and that would imply $(\mu+p)$ is strictly greater than zero. We start by proving a lemma for a general perturbed FLRW spacetime, which will be used for other results.

Lemma 1 *For a perturbed FLRW spacetime, the spatial variation tensor D_aV_b for any first order 3-vector V^a is curl free in the linearised regime.*

Proof From the commutation relation (3.22), we know that

$$(D_cD_d - D_dD_c)V_b = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)(h_{bc}V_d - h_{bd}V_c). \tag{3.60}$$

Acting with ϵ^{adc} on both sides of the above equation, symmetrising on the indices a, b and subtracting the trace, we get

$$\epsilon^{dc<a}D_cD_dV^{b>} = -\frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{<ab>d}V_d. \tag{3.61}$$

The right hand side of the above equation is identically zero as ϵ^{abd} is completely antisymmetric. Hence we get the required result, $(\mathrm{curl}\, D_aV_b) = 0$. $\square$

We note that although the above result is a constraint on a given hypersurface, it is consistently time propagated. To show this, we note that if V^a is a first order 3-vector, then so is $\dot{V}^{\langle a\rangle}$. Then using the commutation relation (3.23), we see that $(\mathrm{curl}\, D_aV_b)^{\cdot}$ vanishes identically.

Proposition 1 *For a linear and conformally flat perturbation of FLRW spacetime, the shear tensor is curl free.*

Proof To prove this, we demand that the new constraint (3.16), due to the absence of the Weyl tensor, be consistently time propagated. In other words, we must have $(\mathrm{curl}\,\pi)^{\cdot}_{\langle ab\rangle} = 0$. Using the commutation relation (3.26), we then see that $(\mathrm{curl}\,\pi)^{\cdot}_{\langle ab\rangle}$ must vanish identically. Now we use the evolution equation (3.8), to get

$$(\mathrm{curl}\, D_{\langle a}q_{b\rangle}) + (\mu + p)(\mathrm{curl}\,\sigma)_{\langle ab\rangle} = 0. \tag{3.62}$$

The first term in the left hand side is zero by Lemma 1. Since the weak energy condition demands that $(\mu+p)$ is strictly greater than zero, we see that the shear tensor must be curl free. Again, this is a result on a given hypersurface. We time propagate this constraint and using (3.26) and (3.6), we see that $(\mathrm{curl}\,\sigma)^{\cdot}_{\langle ab\rangle} = 0$ is identically satisfied. Hence the shear tensor being curl free is true at all epochs. □

Proposition 2 *For a linear and conformally flat perturbation of FLRW spacetime*

1. *For non-vanishing vorticity, the heat flux vector is purely axial on a given hypersurface, and consistent time propagation of this constraint gives an implicit equation of state relating the density and isotropic pressure.*

2. *The vorticity is purely generated by the curl of the heat flux vector for all epochs.*

Proof To prove the first point, we take the curl of constraint (3.14) to get

$$\frac{1}{2}(\mathrm{curl}\,\mathrm{div}\,\pi)^a - \frac{1}{3}(\mathrm{curl}\,D^a\mu) + \frac{1}{3}\Theta(\mathrm{curl}\,q)^a = 0\ . \tag{3.63}$$

The first term of the above equation can be written as $2D_b(\mathrm{curl}\,\pi)^{ab}$ (by commutation (3.28)), which vanishes because of the new constraint (3.16) . Hence the above equation becomes

$$\frac{1}{3}(\mathrm{curl}\,D^a\mu) - \frac{1}{3}\Theta(\mathrm{curl}\,q)^a = 0\ . \tag{3.64}$$

Now by the relation (3.18) we see that $(\mathrm{curl}\,D^a\mu) = 2\omega^a\dot{\mu}$. Further, using (3.15), we get $(\mathrm{curl}\,q)^a = -2(\mu+p)\omega^a$. Putting all of these in the above equation, and noting that the vorticity is not vanishing, we get

$$\dot{\mu} + \Theta(\mu+p) = 0\ . \tag{3.65}$$

Comparing this with the evolution equation (3.10), we can easily see that

$$D_a q^a = 0\ . \tag{3.66}$$

In other words the heat flux vector has vanishing divergence and hence it is purely axial. To check that this constraint is consistently time propagated, we impose the condition

$$(D_a q^a)^{\cdot} = 0\ . \tag{3.67}$$

Using the commutation relation (3.21), we see that this gives $D_a\dot{q}^{\langle a\rangle} = 0$. Now using the evolution equation (3.9) we get

$$D^2p + D_aD_b\pi^{ab} + (\mu+p)D_a\dot{u}^a = 0\ . \tag{3.68}$$

Now if we take the divergence of the constraint (3.14), we get

$$D_a D_b \pi^{ab} = \frac{2}{3} D^2 \mu \ . \tag{3.69}$$

Substituting the above in equation (3.68) we get the implicit equation of state as a second order differential equation

$$D^2 \left(p + \frac{2}{3}\mu \right) + (\mu + p) D_a \dot{u}^a = 0 \ . \tag{3.70}$$

In other words, for the vorticity to remain nonzero at all epochs, the above equation of state relating the energy density and isotropic pressure must be satisfied.

The proof of the second point is obvious from the constraint equation (3.15). To see whether this constraint is identically satisfied at all epochs, we take a dot of this constraint and use the density evolution equation (3.10), vorticity evolution equation (3.7) and heat flux evolution equation (3.9). One can then easily check that this constraint is consistently time propagated and hence the vorticity is purely generated by the curl of heat flux vector for all epochs. □

Proposition 3 *If the spacetime is perturbed in a conformally flat way about the FLRW background, the spatial variation tensor of the vorticity is purely antisymmetric at all epochs and the curl of the vorticity is generated by the heat flux vector and it's Laplacian.*

Proof From the constraint equations (3.12), (3.13) and using the result from Proposition 1, we can immediately see that $D_a \omega^a$ and $D^{<a}\omega^{b>}$ vanish identically on a given epoch. These relations can then be time propagated to check that they remain true for all epochs. Since both the trace and the trace-free symmetric part of the spatial variation tensor of the vorticity vanish at all epochs, this tensor must be purely antisymmetric.
We now take the curl of the constraint equation (3.15), to get

$$(\mu + p)(\mathrm{curl}\,\omega)^a = -\frac{1}{2}(\mathrm{curl}\,\mathrm{curl}\,q)^a. \tag{3.71}$$

Now using the identity (3.29) and the result from Proposition 2, we get

$$(\mu + p)(\mathrm{curl}\,\omega)^a = \frac{1}{2} D^2 q^a - \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2 \right) q^a. \tag{3.72}$$

□

It is interesting to see that there exists a class of spacetime with nonzero heat flux, which are solutions of the following second order differential equation

$$\frac{1}{2}D^2 q^a - \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right) q^a = 0, \tag{3.73}$$

for which the vorticity can be nonzero but curl-free.

3.4 Alternatives to gravitational waves

We note that for gravitational waves to exist in a given spacetime, both the electric and magnetic parts of the Weyl tensor, E_{ab} and H_{ab} must be nonzero with nonzero curl. These quantities generate a tensor wave with a closed wave equation, in a similar fashion as the electric field and the magnetic field with nonzero curl generating electromagnetic vector waves in electromagnetism. Therefore in any spacetime where either of E_{ab} or H_{ab}, or their curl vanishes identically, will be devoid of any gravitational waves. Such cosmologies are commonly termed as *silent universes* (Ellis et al (2011)), as any information about change in local curvature of the manifold cannot causally travel via gravitational waves.

Obviously, conformally flat spacetimes fall in the category of *silent universes*, as in this case the complete Weyl tensor is identically zero, and the Riemann tensor is entirely specified by the matter variables. Therefore any information about local change of curvature must causally travel via propagation of matter disturbances. The question is: *Can we quantify the process via which any information about local change of spacetime curvature causally travels in conformally flat models?* In this section we transparently demonstrate two such processes, a closed tensor wave equation for matter shear and a closed vector wave equation for vorticity, that carries such information causally.

Proposition 4 *In a conformally flat perturbation of FLRW spacetime, the shear tensor obeys a closed and transverse-traceless tensor wave equation, which is given by*

$$\begin{aligned} \Box\sigma^{\langle ab\rangle} \equiv\ & \ddot{\sigma}^{\langle ab\rangle} - D^2\sigma^{\langle ab\rangle} = -\Theta\dot{\sigma}^{\langle ab\rangle} \\ & + \left(\frac{1}{3}\Theta^2 - \frac{7}{6}\mu + \frac{1}{2}p\right)\sigma^{\langle ab\rangle}\ . \end{aligned} \tag{3.74}$$

Proof Since in this case we are only concerned with the tensor modes, we use the standard procedure of neglecting all first order vector perturbations, namely the gradient of background scalars together with the acceleration, heat flux and vorticity. Since the shear tensor is curl free, $(\mathrm{curl}\,\mathrm{curl}\,\sigma)_{ab}=0$, and equation (3.30) becomes

$$D^2\sigma_{ab}=\frac{3}{2}D_{\langle a}D^c\sigma_{b\rangle c}+\left(\mu-\frac{1}{3}\Theta^2\right)\sigma_{ab}\ . \tag{3.75}$$

Now the first term in right hand side is linked to vorticity, heat flux and gradient of expansion by constraint (3.11). Neglecting that term we have

$$D^2\sigma_{ab}=\left(\mu-\frac{1}{3}\Theta^2\right)\sigma_{ab}\ . \tag{3.76}$$

Furthermore, taking the dot of shear evolution equation (3.6) and neglecting the acceleration term, we get

$$\ddot{\sigma}^{\langle ab\rangle}=\frac{1}{2}\dot{\pi}^{\langle ab\rangle}-\frac{2}{3}\dot{\Theta}\sigma^{ab}-\frac{2}{3}\Theta\left(\frac{1}{2}\pi^{ab}-\frac{2}{3}\Theta\sigma^{ab}\right). \tag{3.77}$$

Using evolution equation (3.8) and neglecting the heat flux term, we have

$$\dot{\pi}^{<ab>}=-(\mu+p)\sigma^{ab}-\frac{\Theta}{3}\pi^{ab}. \tag{3.78}$$

Plugging this in (3.77), and noting that $\pi^{ab}=2\dot{\sigma}^{\langle ab\rangle}+\frac{4}{3}\Theta\sigma^{ab}$, we get

$$\ddot{\sigma}^{\langle ab\rangle}=-\frac{1}{6}(\mu-3p)\sigma^{ab}-\Theta\dot{\sigma}^{\langle ab\rangle}. \tag{3.79}$$

Subtracting equation (3.76) from (3.79), we get the required tensor wave equation (5.1). □

It is interesting to note that a similar shear wave exists, even when the perturbations are not conformally flat, but the matter is taken to be perfect fluid, as proved in Dunsby et al (1997). What we showed here is that these waves do not go away, when we take a general form of matter perturbation and restrict the Weyl tensor to be identically zero. Also, when the expansion of the spacetime is positive, these waves gets damped as they move towards the causal future.

Proposition 5 *In a conformally flat perturbation of FLRW spacetime, if the acceleration is curl free, then the vorticity vector obeys a closed vorticity wave equation, given as*

$$\Box\omega^{\langle a\rangle}\equiv\ddot{\omega}^{\langle a\rangle}-D^2\omega^{\langle a\rangle}=\left(\mu+p+\frac{4}{9}\Theta^2\right)w^a. \tag{3.80}$$

Proof The proof of this proposition crucially depends on the result of Proposition 1, that is the shear tensor is curl-free. In that case we can use (3.28) to write

$$0 = D^b(\mathrm{curl}\,\sigma)_{ab} = \frac{1}{2}\mathrm{curl}\,(D^b\sigma_{ab})\ , \tag{3.81}$$

which can be further simplified using the constraint (3.11), and we get

$$\frac{1}{3}(\mathrm{curl}\,D_a\Theta) - \frac{1}{2}(\mathrm{curl}\,\mathrm{curl}\,\omega)_a - \frac{1}{2}(\mathrm{curl}\,q)_a = 0. \tag{3.82}$$

Now using the commutation (3.18) for the first term in the left hand side, the identity (3.29) for the second term and the constraint (3.15) for the third term, we get

$$D^2\omega^a = \left(\frac{2}{9}\Theta^2 - \frac{2}{3}\mu\right)\omega^a. \tag{3.83}$$

Furthermore when the curl of the acceleration term vanishes we have

$$\dot{\omega}^{\langle a\rangle} = -\frac{2}{3}\Theta\omega^a\ . \tag{3.84}$$

Taking the dot of the above equation and using the evolution equation (4.2), we get

$$\ddot{\omega}^{\langle a\rangle} = \frac{1}{3}\left(2\Theta^2 + \mu + 2p\right)\omega^a\ . \tag{3.85}$$

Subtracting (3.83) from (3.85), we get the required result. □

3.5 Heat transport equation modification

Until now, we discussed in detail how the heat flux and the aniosotropic stress gets affected by the absence of the Weyl tensor in an almost FLRW but conformally flat spacetime. However, it is well understood that the inclusion of the heat flux term in the system of field equations is devoid of any physical meaning until a specific heat transport equation is assumed. This is indeed a tricky issue as when we use the standard Eckart theory, that makes the simplest possible assumption about the entropy being a linear function of the dissipative quantities, we get the usual form of the heat conduction equation, which is a parabolic second order partial differential equation. As any parabolic equation, this corresponds to infinite speed of prapagation of heat flow. Apart from this causality violation, the Eckart theory has in addition the pathology of unstable equilibrium states.

Hence we require a heat transport equation that is derived from a causal dissipative theory (Herrera et al (2014), and all the references therein). The basic idea here is to obtain a hyperbolic equation that obeys the causality condition, and also the relaxation time required for the system to come back to a steady state cannot be neglected. Therefore we use the hyperbolic Cattaneo-type equation for heat transfer, which is given as

$$\tau\dot{q}^{<a>} = -k(D^aT + T\dot{u}^a) - \frac{1}{2}kT^2\nabla_b\left(\frac{\tau u^b}{kT^2}\right)q^a, \tag{3.86}$$

where τ, k and T are the relaxation time, thermal conductivity and the temperature respectively. Using the equation (3.9), we get for a conformally flat almost FLRW spacetime

$$D^ap + D_b\pi^{ab} + \frac{4}{3}\Theta q^a + \left(\mu + p - \frac{k}{\tau}T\right)\dot{u}^a = \frac{k}{\tau}D^aT + \frac{1}{2}\frac{k}{\tau}T^2\nabla_b\left(\frac{\tau u^b}{kT^2}\right)q^a. \tag{3.87}$$

We are now in the position to state and prove the following important proposition:

Proposition 6 *In a conformally flat perturbation of FLRW spacetime, if the vorticity is strictly nonzero, then on the small enough neighbourhood (where the relaxation time and heat conductivity can be taken to be constants) in constant time 3-spaces, the temperature obeys Poisson's equation*

$$D^2T + (D_a\dot{u}^a)T = 0. \tag{3.88}$$

Proof We note that for a small enough neighbourhood on a constant time slice, where the relaxation time and heat conductivity can be taken to be constants, the last term in the heat transport equation can be simplified as

$$\frac{1}{2}\frac{k}{\tau}T^2\nabla_b\left(\frac{\tau u^b}{kT^2}\right)q^a = \frac{1}{2}\Theta q^a - \frac{\dot{T}}{T}q^a. \tag{3.89}$$

Furthermore, since the background spacetime is homogeneous and isotropic we have T and $\dot{T}$ as the zeroth order quantities while D^aT and $D^a\dot{T}$ are the first order quantities. Now since the vorticity is strictly nonzero, the heat flux vector and also it's time derivative projected onto the 3−surface are purely axial (by Proposition 2), that is

$$D_aq^a = D_a\dot{q}^{<a>}. \tag{3.90}$$

Hence we can take the divergence of the heat transport equation (3.86), and neglecting the second order quantities we obtain the required Poisson's equation (3.88) for the temperature variation on the time slice. □

It is interesting to note that, if the acceleration vector is also purely axial (divergence free) on this neighbourhood, then the temperature variation follows the Laplace equation $D^2 = 0$, just like the non-relativistic case.

3.6 Discussion

We illustrated the features of the 1+3 covariant approach by considering conformally flat perturbations of FLRW universes to see the effect of the Weyl tensor in general spacetimes. By imposition of $C^a{}_{bcd} = 0$ we obtained new constraints that have to be obeyed at all epochs. We checked the consistency by time evolving these new constraints. We found that the vorticity must be perpendicular to the gradient of the energy density.

Chapter 4

Shear-free perturbations of FLRW universe

4.1 Introduction

We show that a general but shear-free perturbation of homogeneous and isotropic universes are necessarily silent, without any gravitational waves. We prove this in two steps. First, we establish that a shear-free perturbation of these universes are acceleration-free and the fluid flow geodesics of the background universe map onto themselves in the perturbed universe. This effect then decouples the evolution equations of the electric and magnetic parts of the Weyl tensor in the perturbed spacetimes, and the magnetic part no longer contains any tensor modes. Although the electric part, that drives the tidal forces, do have tensor modes sourced by the anisotropic stress, these modes have homogeneous oscillations at every point on a time slice without any wave propagation. This analysis shows the critical role of the shear tensor in generating cosmological gravitational waves.

4.2 Shear free perturbation around FLRW spacetime: Linearised field equations

Our background spacetime is homogeneous and isotropic FLRW universe. Therefore the only nonzero (zeroth order) geometric and thermodynamic quantities in the background are

$$\mathcal{D}_0 = \{\Theta,\, \mu,\, p\}\ . \tag{4.1}$$

Since the background spacetime is homogeneous and isotropic, the projected spatial derivatives of the above quantities will vanish identically, and the evolution equations that govern the temporal evolution of these quantities are given by

$$\dot{\Theta} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p), \tag{4.2}$$

$$\dot{\mu} = -\Theta(\mu + p)\ . \tag{4.3}$$

The above two equations, with a given equation of state of the form $p = p(\mu)$, will then completely solve the background system. Let us now perturb the above system by considering all the quantities that vanish in the background be of first order smallness in the perturbed spacetime. Along with perturbing the geometrical quantities, we also perturb the energy momentum tensor of the matter by introducing a small amount of heat flux and anisotropic stress. In other words the matter in the perturbed spacetime is no longer barotropic. However, we take the shear tensor to be identically zero in the perturbed manifold. That is, we would like to show the importance of shear by the method of negation. In that case the quantities, of first order smallness in the perturbed spacetime, are given as

$$\mathcal{D}_1 = \{E_{\langle ab\rangle},\ H_{\langle ab\rangle},\ \dot{u}_{\langle a\rangle},\ \omega_{\langle a\rangle},\ q_{\langle a\rangle},\ \pi_{\langle ab\rangle}\}\ . \tag{4.4}$$

Apart from the above set, any spatial derivative of zeroth order background quantities are also first order. As mentioned earlier, all these first order quantities will be gauge invariant. The Riemann tensor of the perturbed spacetime can now be completely specified in terms of the matter variables and the Weyl variables as follows

$$\begin{aligned} R^{ab}{}_{cd} &= 2\left(2u^{[a}u_{\ [c}E^{b]}{}_{d]} + 2h^{[a}{}_{[c}E^{b]}{}_{d]} - u^{[a}h^{b]}{}_{[c}q_{d]} - u_{[c}h^{[a}{}_{d]}q^{b]}\right) \\ &- 2\left(u^{[a}u_{[c}\pi^{b]}{}_{d]} - h^{[a}_{[c}\pi^{b]}{}_{d]} - \epsilon^{abe}u_{[c}H_{d]e} - \epsilon_{cde}u^{[a}H^{b]e}\right) \\ &+ \frac{2}{3}\left[(\mu + 3p)u^{[a}u_{[c}h^{b]}{}_{d]} + \mu h^{a}{}_{[c}h^{b}{}_{d]}\right]. \end{aligned} \tag{4.5}$$

Using the above, we can write the Ricci identities of the vector u^a, and once and twice contracted Bianchi identities, project them along u^a and on the instantaneous spatial 3-surface of the observer, and finally linearize them by keeping the terms only to first order smallness, to get the following set of linearised evolution equations and constraints.

4.2.1 Evolution equations

$$\dot{\Theta} - \text{div}\ \dot{u} = -\frac{1}{3}\Theta^2 - \frac{1}{2}(\mu + 3p), \tag{4.6}$$

$$\dot{\omega}^{<a>} - \frac{1}{2}(\text{curl}\ \dot{u})^a = -\frac{2}{3}\Theta\omega^a, \tag{4.7}$$

$$\begin{aligned} \dot{E}^{<ab>} &= (\text{curl}\ H)^{ab} - \Theta\left(E^{ab} + \frac{1}{6}\pi^{ab}\right) \\ &- \frac{1}{2}\left(\dot{\pi}^{<ab>} + D^{<a}q^{b>}\right), \end{aligned} \tag{4.8}$$

$$\dot{H}^{<ab>} = -(\text{curl}\ E)^{ab} + \frac{1}{2}(\text{curl}\ \pi)^{ab} - \Theta H^{ab}, \tag{4.9}$$

$$\dot{q}^{\langle a\rangle} + D^a p + D_b\pi^{ab} = -\frac{4}{3}\Theta q^a - (\mu + p)\dot{u}^a, \tag{4.10}$$

$$\dot{\mu} + D_a q^a = -\Theta(\mu + p). \tag{4.11}$$

4.2.2 Constraints

$$(C_0)^{ab} \equiv D^{<a}\dot{u}^{b>} - E^{ab} + \frac{1}{2}\pi^{ab} = 0, \tag{4.12}$$

$$(C_1)^a \equiv q^a - \frac{2}{3}D^a\Theta + \epsilon^{abc}D_b\omega_c = 0, \tag{4.13}$$

$$(C_2) \equiv D_a\omega^a = 0, \tag{4.14}$$

$$(C_3)^{ab} \equiv H^{ab} + D^{<a}\omega^{b>} = 0, \tag{4.15}$$

$$(C_4)^a \equiv D_b\left(E^{ab} + \frac{1}{2}\pi^{ab}\right) - \frac{1}{3}D^a\mu + \frac{1}{3}\Theta q^a = 0, \tag{4.16}$$

$$(C_5)^a \equiv D_b H^{ab} + (\mu + p)\omega^a + \frac{1}{2}(\text{curl}\ q)^a = 0. \tag{4.17}$$

We note here three important points:

1. First of all, the above system of equations is not closed. To close the system we must supply a thermodynamic relation between isotropic pressure, fluid energy density, heat flux and anisotropic stress in the form of $F(\mu,\, p,\, q^a,\, \pi^{ab}) = 0$.

2. Secondly, the constraints C_1, C_2, $\cdots$, C_5 are the constraint equations for general matter motion, which are known to be consistently *time propagated* along u^a locally. However the absence of shear gives a new constraint C_0, which was the shear evolution equation in the original system. This new constraint must be *spatially compatible* with the original constraints, and furthermore it should be consistently time propagated locally.

3. And finally, although the spatial derivative operators D^a are orthogonal to u^a (that is $u^a D_a = 0$), if the vorticity is nonvanishing then these operators do not span a three dimensional surface as the commutators of these spatial directional derivatives acting on a scalar do not vanish. In that case the instantaneous rest frame of an observer is not a genuine 3-surface but rather a collection of tangent planes.

4.2.3 Commutations

We note that unlike the partial derivatives, the projected covariant derivatives on the 3-space do not commute with each other, and neither do they commute with the time derivative. Given any scalar function f, we have

$$D_{[a}D_{b]}f = \epsilon_{abc}\omega^c \dot{f}, \tag{4.18}$$

$$\epsilon^{abc}D_bD_cf = 2\omega^a\dot{f}, \tag{4.19}$$

$$[D^{<a>}f]^{\cdot} \equiv h^a_b[D^bf]^{\cdot} = D^a\dot{f} - \frac{1}{3}\Theta D^a f. \tag{4.20}$$

Further to this, if V^a is a projected first order 3-vector on the perturbed manifold, about a FLRW background, then the linearised commutation relations are given as

$$\left[D^{<a}V^{b>}\right]^{\cdot} = D^a\dot{V}^{<b>} - \frac{1}{3}\Theta D^aV^b, \tag{4.21}$$

$$(\mathrm{div}\, V)^{\cdot} = \mathrm{div}\, \dot{V} - \frac{1}{3}\Theta(\mathrm{div}\, V), \tag{4.22}$$

$$D_{[a}D_{b]}V_c = \frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)h_{c[a}V_{b]}, \tag{4.23}$$

$$[(\mathrm{curl}\, V)_a]^{\cdot} = (\mathrm{curl}\, \dot{V})_a - \frac{1}{3}\Theta(\mathrm{curl}\, V)_a. \tag{4.24}$$

Similarly, for any first order second rank 3-tensor A^{ab}, the following linearised relation holds

$$D_{[a}D_{b]}A^{cd} = \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) h^{(c}{}_{[a}A^{d)}{}_{b]}, \tag{4.25}$$

$$[(\mathrm{curl}\, A)_{ab}]^{\cdot} = (\mathrm{curl}\, \dot{A})_{ab} - \frac{1}{3}\Theta(\mathrm{curl}\, A)_{ab}. \tag{4.26}$$

Apart from these there are several other linearised relations for first order vectors and tensors in a perturbed spacetime about a FLRW background, that we list here (Maartens and Bassett (1998))

$$D^a(\mathrm{curl}\, V)_a = 0\,, \tag{4.27}$$

$$D^b(\mathrm{curl}\, A)_{ab} = \frac{1}{2}\mathrm{curl}\,(D^b A_{ab})\,, \tag{4.28}$$

$$(\mathrm{curl}\,\mathrm{curl}\, V)^a = D_b D^a V^b - D^2 V^a, \tag{4.29}$$

$$\begin{aligned}(\mathrm{curl}\,\mathrm{curl}\, A)_{ab} &= -D^2 A_{ab} + \frac{3}{2}D_{\langle a}(\mathrm{div}\, A)_{b\rangle} \\ &\quad + \left(\mu - \frac{1}{3}\Theta^2\right) A_{ab}.\end{aligned} \tag{4.30}$$

In what follows, we will be using these relations repeatedly to extract the results for a shear-free perturbation of the FLRW universe.

4.3 Spatial consistency of the new constraint: An important theorem

In this section, we will state and prove the following important theorem on the 4-acceleration of matter for a perturbed spacetime linearised about a FLRW background.

Theorem 1. If a homogeneous and isotropic spacetime is perturbed in a shear-free way, then the 4-acceleration of matter in the perturbed spacetime is necessarily zero, if the matter obeys the strong energy condition. In other words, the geodesics of matter flow line in the background map onto themselves in the perturbed manifold.

Proof. This theorem can be proved by checking the spatial consistency of the new constraint (4.12), that is, if this constraint is consistent with the existing constraints of the field equations. Contracting the commutation relation (4.23) we get

$$D_b D^a V^b = D^a(\text{div V}) + \frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right) V^a, \tag{4.31}$$

and by definition we know

$$D^{<a}V^{b>} = \frac{1}{2}\left(D^a V^b + D^b V^a\right) - \frac{1}{3}h^{ab}(\text{div V}). \tag{4.32}$$

Now, from our new constraint (4.12), we see that

$$D_b E^{ab} = D_b(D^{<a}\dot{u}^{b>}) + \frac{1}{2}D_b\pi^{ab}. \tag{4.33}$$

Using relation (4.32) we get

$$D_b E^{ab} = \frac{1}{2}D_b D^a \dot{u}^b + \frac{1}{2}D^2\dot{u}^a - \frac{1}{3}D^a(\text{div }\dot{u}) + \frac{1}{2}D_b\pi^{ab}. \tag{4.34}$$

But by relation (4.29) we have

$$\frac{1}{2}D^2\dot{u}^a = \frac{1}{2}D_b D^a\dot{u}^b - \frac{1}{2}(\text{curl curl }\dot{u})^a. \tag{4.35}$$

Substituting relation (4.35) into (4.34) and using (4.31) we finally have

$$D_b E^{ab} = \frac{2}{3}D^a(\text{div }\dot{u}) + \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a - \frac{1}{2}(\text{curl curl }\dot{u})^a + \frac{1}{2}D_b\pi^{ab}. \tag{4.36}$$

To calculate $(\text{curl curl }\dot{u})^a$, we note that the field equation (4.7) gives

$$\frac{1}{2}(\text{curl }\dot{u})^a = \dot{\omega}^{<a>} + \frac{2}{3}\Theta\omega^a. \tag{4.37}$$

Taking curl on both sides and using the commutation relation (4.24) we have

$$\frac{1}{2}(\text{curl curl }\dot{u})^a = [(\text{curl }\omega)^a]^{\cdot} + \Theta(\text{curl }\omega)^a. \tag{4.38}$$

To find the expression for $[(\text{curl }\omega)^a]^{\cdot}$, we use the field equation (4.13) yielding

$$(\text{curl }\omega)^a = \frac{2}{3}D^a\Theta - q^a\ . \tag{4.39}$$

Taking the dot of the above equation and using relation (4.20) we get

$$[(\text{curl } \omega)^a]^{\cdot} = \frac{2}{3}D^a\dot{\Theta} - \frac{1}{9}D^a\Theta^2 - \dot{q}^a \ . \tag{4.40}$$

We now use the field equation (4.6) and the above gets simplified to

$$[(\text{curl } \omega)^a]^{\cdot} = \frac{2}{3}D^a(\text{div } \dot{u}) - \frac{1}{3}D^a\Theta^2 - \frac{1}{3}D^a(\mu + 3p) - \dot{q}^a. \tag{4.41}$$

Substituting relations (4.39) and (4.41) into (4.38), we obtain

$$\frac{1}{2}(\text{curl curl } \dot{u})^a = \frac{2}{3}D^a(\text{div } \dot{u}) - \frac{1}{3}D^a(\mu + 3p) - \dot{q}^a - \Theta q^a. \tag{4.42}$$

Using (4.42) in (4.36) we get

$$D_bE^{ab} = \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\dot{u}^a + \frac{1}{3}D^a(\mu + 3p) + \dot{q}^a + \Theta q^a + \frac{1}{2}D_b\pi^{ab}. \tag{4.43}$$

From the field equation (4.16) we know

$$D_bE^{ab} + \frac{1}{2}D_b\pi^{ab} - \frac{1}{3}D^a\mu + \frac{1}{3}\Theta q^a = 0. \tag{4.44}$$

Using (4.43) we get

$$\frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\dot{u}^a + D^ap + \dot{q}^a + \frac{4}{3}\Theta q^a + D_b\pi^{ab} = 0. \tag{4.45}$$

Again from the field equation (4.10) we have

$$\dot{q}^a + \frac{4}{3}\Theta q^a + D^ap + D_b\pi^{ab} = -(\mu + p)\dot{u}^a. \tag{4.46}$$

Therefore the relation (4.45) becomes

$$\frac{2}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\dot{u}^a - (\mu + p)\dot{u}^a = 0 \tag{4.47}$$

or

$$\left[\frac{1}{3}\Theta^2 + \frac{1}{2}(\mu + 3p)\right]\dot{u}^a = 0. \tag{4.48}$$

If the matter obeys the strong energy conditions then we have

$$(\mu + 3p) \geq 0 \text{ and } \Theta^2 \geq 0.$$

Therefore the term in the square bracket of the above equation is strictly non-negative, and only vanishes for the non-interesting case of a Minkowski spacetime which we exclude here. Therefore the matter acceleration must identically vanish for the perturbed spacetime and the geodesics of matter flow lines in the background, mapping onto themselves in the perturbed manifold. □

A direct corollary of the above result is given below

Corollary 1 *If a homogeneous and isotropic spacetime is perturbed in a shear-free way and the matter obeys the strong energy conditions, then the electric part of the Weyl tensor is half of the anisotropic stress.*

Another straightforward corollary of this theorem is

Corollary 2 *If a homogeneous and isotropic spacetime is perturbed in a shear-free way with the matter obeying the strong energy conditions and the vorticity is zero at any given instant on a observer's worldline, then it continues to be zero in the entire worl line (as* $\omega^a = 0 \Rightarrow \dot{\omega}^a = 0$*).*

Thus we showed that if the new constraint emerging due to vanishing of shear has to be spatially consistent with the original constraints, the matter acceleration must identically vanish, and the electric part of the Weyl is completely specified by the anisotropic stress. However this does not interfere with the definition of magnetic part of Weyl, given by constraint (4.15). This can be shown in the following way: Using relation (4.32) in (4.15) we have

$$H^{ab} = -\frac{1}{2}D^a\omega^b - \frac{1}{2}D^b\omega^a + \frac{1}{3}h^{ab}(\text{div }\omega), \tag{4.49}$$

and

$$D_bH^{ab} = -\frac{1}{2}D_bD^a\omega^b - \frac{1}{2}D^2\omega^a + \frac{1}{3}h^{ab}(\text{div }\omega). \tag{4.50}$$

But by (4.14) the above becomes

$$D_bH^{ab} = -\frac{1}{2}D_bD^a\omega^b - \frac{1}{2}D^2\omega^a. \tag{4.51}$$

Now using (4.29) we know

$$-\frac{1}{2}D^2\omega^a = -\frac{1}{2}(\text{curl curl }\omega)^a - \frac{1}{2}D_bD^a\omega^b. \tag{4.52}$$

And (4.51) becomes

$$D_b H^{ab} = -D_b D^a \omega^b + \frac{1}{2}(\text{curl curl } \omega)^a. \tag{4.53}$$

But using (4.31) and (4.14) gives

$$D_b D^a \omega^b = \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\omega^a, \tag{4.54}$$

and equation (4.53) becomes

$$D_b H^{ab} = \frac{1}{2}(\text{curl curl } \omega)^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\omega^a. \tag{4.55}$$

To find the expression for (curl curl $\omega)^a$ we use the field equation (4.13) that gives

$$(\text{curl } \omega)^a = \frac{2}{3}D^a\Theta - q^a \ . \tag{4.56}$$

Taking the curl on both sides, and simplifying using (4.19), we then have

$$(\text{curl curl } \omega)^a = \frac{4}{3}\omega^a\dot{\Theta} - (\text{curl } q)^a. \tag{4.57}$$

Substituting (4.57) into (4.55) we get

$$D_b H^{ab} = \frac{2}{3}\omega^a\dot{\Theta} - \frac{1}{2}(\text{curl } q)^a - \frac{2}{3}(\mu - \frac{1}{3}\Theta^2)\omega^a. \tag{4.58}$$

But from the field equation (4.17) we have

$$D_b H^{ab} + (\mu + p)\omega^a + \frac{1}{2}(\text{curl } q)^a = 0. \tag{4.59}$$

Plugging in the value of $D_b H^{ab}$ from (4.58) and simplifying we finally get the constraint

$$\frac{2}{3}\omega^a(\text{div } \dot{u}) = 0. \tag{4.60}$$

We see that the above constraint is identically satisfied if the matter acceleration vanishes.

4.4 Non existence of gravitational waves

As discussed in detail in Goswami and Ellis (2021), Maartens and Bassett (1998) and Dunsby et al (1997), the general way to look for the close form wave equation for the electric or magnetic part of the Weyl tensor, is to take the curl of the evolution equation (4.8) and dot of (4.9) and then use the commutation relations (4.23) and (4.30). However for this to happen, the equations (4.8) and (4.9) must be coupled to each other, just like their electromagnetic Maxwell equation analogue. Interestingly, when the shear tensor is identically zero in the perturbed spacetime, we see by Corollary 1, the magnetic Weyl evolution decouples itself from the electric Weyl evolution, and (4.9) now becomes

$$\dot{H}^{<ab>} = -\Theta H^{ab}, \tag{4.61}$$

and on any given world line the magnetic Weyl has an oscillatory nature given by

$$\ddot{H}^{<ab>} = -(\dot{\Theta} H^{ab} + \Theta \dot{H}^{ab}) = -[\dot{\Theta} H^{ab} + \Theta(-\Theta H^{ab})]. \tag{4.62}$$

Using (4.6) the above equation becomes

$$\ddot{H}^{<ab>} = \left[\frac{4}{3}\Theta^2 + \frac{1}{2}(\mu + 3p)\right] H^{ab}. \tag{4.63}$$

which is very similar to the vorticity vector. This is hardly surprising, as the magnetic Weyl doesn't contain any tensor modes anymore, and it is purely driven by the vector mode of the vorticity vector. The electric Weyl, however, still contains tensor modes due to the presence of anisotropic stress. To see how these tensor modes behave, we use the usual technique of making all the scalar and the vector modes vanish. That is, we take the vorticity, heat flux and the gradient of all the scalars to be zero. In that case we see that the magnetic part of the Weyl vanishes and we get

$$\dot{E}^{<ab>} = -\Theta(E^{ab} + \frac{1}{6}\pi^{ab}) - \frac{1}{2}\dot{\pi}^{<ab>}, \tag{4.64}$$

and

$$\pi^{ab} = 2E^{ab}. \tag{4.65}$$

This then implies

$$\dot{\pi}^{ab} = 2\dot{E}^{ab}. \tag{4.66}$$

So (4.64) becomes

$$\dot{E}^{<ab>} = -\frac{2}{3}\Theta E^{<ab>}, \tag{4.67}$$

$$\ddot{E}^{<ab>} = -\frac{2}{3}\dot{\Theta} E^{<ab>} - \frac{2}{3}\Theta \dot{E}^{<ab>}. \tag{4.68}$$

Finally using relations (4.6) and (4.67) in (4.68) above we can show that the electric part of the Weyl obeys the following tensor oscillatory equation

$$\ddot{E}^{<ab>} = \frac{1}{3}\left[2\Theta^2 + (\mu + 3p)\right] E^{ab}. \tag{4.69}$$

We see that this is just a homogeneous oscillation of all points at a given time slice and hence there is no gravitational wave propagations. It is interesting to note that, although there is no tensor wave propagation, there exists a vector wave as described in the following proposition.

Proposition 7 *In a general but shear free perturbation of homogeneous and isotropic universes the vorticity vector obeys a wave equation, that is sourced by the curl of the heat flux and is given by*

$$\Box\omega^a \equiv \ddot{\omega}^{<a>} - D^2\omega^a = \frac{1}{3}\omega^a(\mu - 3p) - (\text{curl } q)^a. \tag{4.70}$$

Proof. By (4.7) we have

$$\dot{\omega}^{<a>} = -\frac{2}{3}\Theta\omega^a, \tag{4.71}$$

and

$$\ddot{\omega}^{<a>} = \frac{1}{3}\omega^a[2\Theta^2 + (\mu + 3p)]. \tag{4.72}$$

By (4.29) we have

$$D^2\omega^a = D_b D^a \omega^b - (\text{curl curl } \omega)^a. \tag{4.73}$$

We know by (4.57) that

$$(\text{curl curl } \omega)^a = \frac{4}{3}\omega^a\dot{\Theta} - (\text{curl } q)^a, \tag{4.74}$$

or using (4.6) we find

$$(\text{curl curl } \omega)^a = -\frac{1}{3}\omega^a\left[\frac{4}{3}\Theta^2 + 2(\mu + 3p)\right] - (\text{curl } q)^a. \tag{4.75}$$

Substituting (4.75) into (4.73) we obtain

$$D^2\omega^a = D_b D^a \omega^b + \frac{1}{3}\omega^a\left[\frac{4}{3}\Theta^2 + 2(\mu + 3p)\right] + (\text{curl } q)^a. \tag{4.76}$$

Subtracting (4.76) from (4.72) we obtain the result. $\Box$

4.5 Discussion

We used the 1+3 covariant approach based on Ellis and van Elst (1998) considering shear tensor to be equal to zero. Therefore in the linearisation procedure, we got the following new constraint

$$D^{<a}\dot{u}^{b>} - E^{ab} + \frac{1}{2}\pi^{ab} = 0.$$

In respect of consistency of the new constraints we found the following relations

$$\left[\frac{1}{3}\Theta^2 + \frac{1}{2}(\mu + 3p)\right]\dot{u}^a = 0,$$

or

$$\left[(\text{div } \dot{u}) - \dot{\Theta}\right]\dot{u}^a = 0,$$

and

$$\frac{2}{3}\omega^a(\text{div } \dot{u}) = 0.$$

These situations pointed out to us that, for shear-free perturbation of FLRW spacetime, either the acceleration vanishes (i.e $\dot{u} = 0$) or , $\dot{\Theta} = \text{div } \dot{u}$ and either the vorticity vanishes (i.e $\omega^a = 0$) or, div $\dot{u} = 0$.

Chapter 5

Conclusion

This thesis was based on work first started by Ellis and van Elst (1998). We considered two cases, namely conformally flat, and shear-free spacetimes. In this dissertation we have studied the role of the Weyl tensor in general relativistic fluid motion, including spacetime shear. Since the framework of our thesis was focussed in general relativity, we have started by providing a brief overview of the mathematical tools needed to formulate general relativity. We discussed conformally flat and shear-free models, and showed why they are important. We now provide an overview of the main results obtained during the course of our investigations:

In chapter 2, we presented the relevant theoretical concepts inherent with the $1+3$ covariant formalism. We used the socalled $1+3$ covariant description of general relativity which has been developed for use in spacetimes in which there is a preferred timelike congruence $\mathbf{u}$. The "$1+3$" refers to the fact that one performs a "time+space" decomposition relative to $\mathbf{u}$ by projecting tensors and tensorial equations parallel to $\mathbf{u}$ and orthogonal to $\mathbf{u}$. The second aspect of the $1+3$ description is to write a tensor as a sum of algebraically simpler parts, i.e. to give an algebraic decomposition.

In chapter 3 we studied conformally flat perturbations of FLRW spacetimes, to see the effect of Weyl tensor in general spacetimes. Since we considered conformally flat perturbation, the Weyl tensor vanishes identically. In the linearisation procedure, we neglected all products of first order quantities in Ricci and (contracted) second Bianchi identities giving the following new constraint

$$(\mathrm{curl}\ \pi)^{ab} = 0.$$

The following relations are the results we found in respect of consistency of the new constraints:

$$\epsilon^{dc<a}D_cD_dq^{b>} = -(\mu+p)D^{<a}\omega^{b>} = 0,$$

$$\epsilon^{dc<a}D_c(D_dq^{b>}) = -(\mu+p)D^{<a}\omega^{b>} = -\frac{1}{3}\left(\mu - \frac{1}{3}\Theta^2\right)\epsilon^{<ab>d}q_d = 0,$$

$$\epsilon^{dc<a}D_cD_d\dot{u}^{b>} = 0,$$

$$\epsilon^{cd<a}D_cD_d(D^{b>}p + \frac{2}{3}D^{b>}\mu) = 0,$$

$$\epsilon^{abc}D_b\omega_c = \frac{2}{3}\Theta\dot{u}^a, \text{ or } (curl\ \omega)^a = \frac{2}{3}\Theta\dot{u}^a,$$

$$\omega^aD_a\mu = 0,$$

$$(2D_bD_a - D_aD_b)\dot{u}^b = (\mu+p)\dot{u}_a.$$

We stated and proved the following important geometrical properties of matter shear and vorticity in the perturbed, conformally flat spacetime:

- For a perturbed FLRW spacetime, the spatial variation tensor D_aV_b for any first order 3-vector V^a is curl free in the linearised regime, i.e $(\mathrm{curl}\, D_aV_b) = 0$.

- For a linear and conformally flat perturbation of FLRW spacetime, the shear tensor is curl free, i.e. $(\mathrm{curl}\,\sigma)_{\langle ab\rangle} = 0$.
- For a linear and conformally flat perturbation of FLRW spacetime,
  1. For nonvanishing vorticity, the heat flux vector is purely axial on a given hypersurface, and consistent time propagation of this constraint gives an implicit equation of state relating the density and isotropic pressure.
  2. The vorticity is generated purely by the curl of heat flux vector for all epochs.
- If the spacetime is perturbed in a conformally flat way about the FLRW background, the spatial variation tensor of the vorticity is purely antisymmetric at all epochs, and the curl of the vorticity is generated by the heat flux vector and it's Laplacian.

- In a conformally flat perturbation of FLRW spacetime, the shear tensor obeys a closed and transverse-traceless tensor wave equation, which is given by

$$\begin{aligned}\Box\sigma^{\langle ab\rangle} \equiv \ddot{\sigma}^{\langle ab\rangle} - D^2\sigma^{\langle ab\rangle} &= -\Theta\dot{\sigma}^{\langle ab\rangle} \\ &+ \left(\frac{1}{3}\Theta^2 - \frac{7}{6}\mu + \frac{1}{2}p\right)\sigma^{\langle ab\rangle}\ .\end{aligned}$$

- In a conformally flat perturbation of FLRW spacetime, if the acceleration is curl-free, then the vorticity vector obeys a closed vorticity wave equation

$$\Box\omega^{\langle a\rangle} \equiv \ddot{\omega}^{\langle a\rangle} - D^2\omega^{\langle a\rangle} = \left(\mu + p + \frac{4}{9}\Theta^2\right) w^a.$$

- In a conformally flat perturbation of FLRW spacetime, if the vorticity is strictly nonzero, then on the small enough neighbourhood (where the relaxation time and heat conductivity can be taken to be constants) in constant time 3-spaces, the temperature obeys Poisson's equation

$$D^2T + (D_a\dot{u}^a)T = 0.$$

Geometrical properties of general conformally flat spacetimes are still under active investigations, to understand transparently how different geometrical and thermodynamic quantities of the spacetime interact in the absence of free gravity. This understanding will definitely help us to recognise the effects of free gravity with better clarity.

Working in the linearised regime, we demonstrated some interesting features of matter shear and vorticity and how they are powered by different thermodynamic quantities of matter, like energy density, heat flux, isotropic pressure and anisotropic stress. Although these results are only valid in the linearised regime, they give an indication as to how these quantities behave in a more general setting of conformally flat spacetimes. The most important point that emerged from this investigation, is that both the matter shear and the vorticity obey a transverse traceless tensor wave equation and a vector wave equation respectively. These shear and vorticity waves actually replaces the gravitational waves, that these spacetimes are devoid of, in the sense that any information about local change in the curvature of the spacetime can be propagated causally via these waves. Presence of these waves makes the dynamics of relativistic and conformally flat fluid flows extremely interesting and can shed new light on the general conformally flat solutions of Einstein field equations. Furthermore, we worked out explicity, how the causal Cattaneo-type heat transport equation gets modified in this scenario of conformally flat almost FLRW spacetimes, and showed that in small enough neighbourhoods of constant time slices, the temperature obeys a Poisson's equation.

In chapter 4 we stated and demonstrated the following property:

In a general but shear-free perturbation of homogeneous and isotropic universes the vorticity vector obeys a wave equation, that is sourced by the curl of the heat flux and is given as

$$\Box\omega^a \equiv \ddot{\omega}^{<a>} - D^2\omega^a = \frac{1}{3}\omega^a(\mu - 3p) - (\text{curl } q)^a.$$

We established the important effects of spacetime shear via the method of negation, by perturbing a homogeneous and isotropic universe in a shear-free fashion. The key points that emerged from our analysis are as follows

1. Our analysis once again emphasised the relation of spacetime shear with inhomogeneity and anisotropy that was initially pointed out in Joshi et al (2004). Inhomogeneity and anisotropy in a spacetime is directly manifested via pressure gradients, anisotropic stresses and heat flux, which in turn generates the matter acceleration. We showed that vanishing of shear necessarily makes the acceleration vanish in the perturbed spacetimes, that makes the matter geodesics in the background to map onto themselves in the perturbed manifold.

2. We further proved the crucial role of shear in coupling the evolution of electric and magnetic parts of the Weyl tensor, that puts the gravito-electromagnetism on the same footing as normal electromagnetism. The absence of shear breaks this coupling and furthermore the magnetic part of the Weyl ceases to have any tensor modes (that are generated purely by the shear). Hence the spacetime does not contain any tensorial waves. It is interesting to note that in general, the shear tensor itself obeys a wave equation in a perturbed FLRW background (Dunsby et al (1997)), and it is very clear that shear waves mediate the interction between the electric and magnetic Weyl, to produce gravitational waves.

3. Although the shear free perturbation of FLRW universes, do not contain any gravitational waves, it was shown that these contain vorticity waves sourced by the curl of the heat flux. If the matter in the perturbed universe continues to be a perfect fluid, then these waves will die down. However if a heat flux with non-zero curl is present in the perturbed manifold, then this will act as a forcing term and the

vector waves can grow indefinitely resulting in structure formation.

Thus we established the validity of the results obtained in Matarrese et al (1994), even when we allow for the general matter perturbation.

References

Abebe A., Goswami R. and Dunsby P. K. S., Shear-free perturbations of f(R) gravity, Phys. Rev. D **84**, 124027 (2011).

van den Bergh N., The shear-free perfect fluid conjecture, Class. Quantum Grav. **16**, 113 (1999).

van den Bergh N., Carminati J. and Karimian H. R., Shearfree perfect fluids with solenoidal magnetic curvature and a gamma-law equation of state, Class. Quantum Grav. **24**, 3735 (2007).

Bergström L. and Goobar A., Cosmology and particle astrophysics, Springer, Berlin (2006).

Bertschinger E., Introduction to tensor calculus for general relativity, Springer, Berlin (1999).

Betschart G., General relativistic electrodynamics with applications in cosmolology and astrophysics, PhD thesis, University of Cape Town, Cape Town (2005)

Boonserm P., Some exact solutions in general relativity, MSc thesis, Victoria University of Wellington, Wellington (2006).

Born M., Einstein's theory of relativity, Dover, New York (1962).

Brassel B. P., Dynamics of radiating stars in the strong gravity regime, PhD Thesis, University of KwaZulu-Natal, Durban (2017).

Bruni M., Dunsby P. K. S. and Ellis G. F. R., Cosmological perturbations and the physical meaning of gauge-invariant variables, Astrophys. J. **395**, 34 (1992).

Bruni M., Ellis G. F. R. and Dunsby P. K. S., Gauge-invariant perturbations in a scalar field dominated universe, Class. Quantum Grav. **9**, 921 (1992).

Buchdahl H. A., Reciprocal static metrics and scalar fields in the general theory of relativity, Phys. Rev. **115**, 1325 (1959).

Burgess C. P., Quantum gravity in everyday life: General relativity as an effective field theory, Living Reviews in Relativity **7**, 1 (2004).

Chakrabarti S. and Banerjee N., Scalar field collapse in a conformally flat spacetime, Eur. Phys. J. C. **77**, 166 (2017).

Chakrabarti S., Goswami R., Maharaj S. D. and Banerjee N., Conformally flat collapsing stars in f(R) gravity, Gen. Relativ. Gravit. **50**, 11 (2018).

Ciufolini I., Gorini V., Moshella U. and Fre P., Gravitational Waves, C R C Press, Boca Raton (2001).

Collins C.B. and Wainwright J., Role of shear in general relativistic cosmological and stellar models, Phys. Rev. D **27**, 1209 (1983).

Collins C. B., Global aspects of shear-free perfect fluids in general relativity, J. Math. Phys. **26**, 2009 (1985).

Collins C.B., Shear-free fluids in general relativity, Can. J. Phys. **64**, 191 (1986).

Danehkar A., On the significance of the Weyl curvature in a relativistic cosmological model, Mod. Phys. Lett. A **24**, 3113 (2009).

D'Inverno R., Introducing Einstein's relativity, Oxford University Press, Oxford (2008).

Dirac P., General theory relativity, Princeton University Press, Princeton (1996).

Dunsby P. K. S., Bruni M. and Ellis G. F. R., Covariant perturbations in a multifluid cosmological medium, Astrophys. J. **395**, 54 (1992).

Dunsby P. K. S., Bassett B. and Ellis G. F. R., Covariant analysis of gravitational waves in a cosmological context, Class. Quantum Grav. **14**, 1215 (1997).

Ehlers J., Contributions to the relativistic mechanics of continuous media, Akad. Wiss. Lit. Mainz Abhandl. Mat. Nat. Kl, Nr. 11, 1961 (pp. 792837) Reprinted as Golden Oldie: Gen. Relativ. Gravit. **25**, 1225 (1993).

Ehlers J., Jordan P. and Kundt W., Strenge Lsungen der Feldgleichungen der allgemeinen Relativittstheorie Akad. Wiss. Lit. Mainz Abhandl. Mat. Nat. Kl Nr 2, 21-105 [Reprinted as Golden Oldie: Gen. Relativ. Gravit. **41**, 2191 (2009)].

Ehlers J. and Sachs R., Exact solutions of the field equations of general relativity, II: Contributions to the theory of pure gravitational radiation, Akad. Wiss. Lit. Mainz Abhandl. Mat. Nat. Kl. Nr 1, 1-62, (1961).

Ellis G. F. R., The dynamics of pressure-free matter in general relativity, J. Math. Phys. **8**, 1171 (1967).

Ellis G. F. R. and Bruni M., Covariant and gauge-invariant approach to cosmological density fluctuations, Phys. Rev. D **40**, 1804 (1989).

Ellis G. F. R., Bruni M. and Hwang J., Density-gradient-vorticity relation in perfect-fluid Robertson-Walker perturbations, Phys. Rev. D **42**, 1035 (1990).

Ellis G. F. R. and van Elst H., Cosmological Models, Cargèse Lectures 1998, in Theoretical and Observational Cosmology, Ed. M Lachze-Rey, Kluwer, Dordrecht (1999).

Ellis G. F. R., Relativistic cosmology. In General Relativity and Cosmology, Proc. Int. School of Physics *Enrico Fermi* (Varenna), Course XLVII. Ed. R K Sachs (Academic Press, 1971), 104-179. [Reprinted as Golden Oldie: Gen. Relativ. Gravit. **41**, 581 (2009)].

Ellis G. F. R., Roy Maartens and Malcolm A. H. MacCallum, Relativistic Cosmology, Cambridge University Press, Cambridge (2007). "Almost Birkhoff Theorem in General Relativity", Gen. Relativ. Grav. **43**, 2157 (2011).

Ellis G. F. R., Shear-free solutions in general relativity theory, Gen. Relativ. Gravit. **43**, 3253 (2011).

van Elst H., Extension and applications of 1+3 decomposition methods in general relativistic cosmological modelling, PhD thesis, University of London, London (1996).

Endean G., Cosmology in conformally flat spacetime, Astrophys. J. **479**, 40 (1997).

Endean G., A note on Tauber's expanding universe in conformally flat, J. Math. Phys. **39**, 1551 (1998).

Geroch R. P., What is a singularity in general relativity?, J. Math. Phys. **9**, 1739 (1968).

Glass E.N., Shear-free gravitational collapse, J. Math. Phys. **20**, 1508 (1979).

Gödel K., Proceedings of the International Congress of Mathematicians, Cambridge, Mass., **1**, 175- 181 (Amer. Math. Soc., R.I.), (1952).

Goswami R., An investigation in gravitation theory, PhD Thesis, Tata Institute of Fundamental Research, Mumbai (2005).

Goswami R. and Ellis G. F.R., Tidal forces are gravitational waves, Class. Quantum Grav. **38**, 085023 (2021).

Hansraj S. and Moodly L., Stellar modelling of isotropic Einstein–Maxwell perfect fluid spheres of embedding class one, Eur. Phys. J. C. **80**, 496 (2020).

Hawking S. W. and Ellis G. F. R., The large scale structure of spacetime, Cambridge University Press, Cambridge (1973).

Herrera L. and Santos N. O., Shear-free and homology conditions for self-gravitating dissipative fluids, Mon. Not. R. Astron. Soc. **343**, 1207 (2003).

Herrera L., Le Denmat G., Santos N. O. and Wang A., Shear-free radiating collapse and conformal flatness, Int. J. Mod. Phys. D **13**, 583 (2004).

Herrera L., Di Prisco A. and Ospino J., Some analytical models of radiating collapsing spheres, Phys. Rev. D **74**, 044001 (2006).

Herrera L., Di Prisco A., Ibanez J. and Ospino J., Dissipative collapse of axially symmetric, general relativistic sources: a general framework and some applications, Phys. Rev. D **89**, 084034 (2014).

Herrera L., Di Prisco A., Ospino J. and Carot J., Earliest stages of the nonequilibrium in axially symmetric, self-gravitating, dissipative fluids, Phys. Rev. D **94**, 064072 (2016)

Islam N., Tensors and their Applications, New Age, New Delhi (2006).

Ivanov B. V., A conformally flat realistic anisotropic model for a compact star, Eur. Phys. J. C. **78**, 332 (2018).

Joshi P. S. , Goswami R. and Dadhich N., Why do naked singularities form in gravitational collapse? II Phys. Rev. D **70**, 087502 (2004).

Krasinski A., Inhomogeneous cosmological models, Cambridge University Press, Cambridge (1997).

Kulkarni R. S., Conformally flat manifolds, Proc. Nat. Acad. Sci., **69**, 2675 (1972).

Landau L. D. and Lifshitz E. M., The classical theory of fields, Pergamon Press, Oxford (1975).

Maartens R., Linearization instability of gravity waves, Phys. Rev. D **55**, 463 (1997).

Maartens R. and Triginer J., Density perturbations with relativistic thermodynamics, Phys. Rev. D **56**, 4640 (1997).

Maartens R. and Bassett B., Gravitational electromagnetic duality, Class. Quantum Grav. **15**, 705 (1998).

Maartens R., Gebbie and Ellis G. F. R., Cosmic microwave background anisotropies: Nonlinear dynamics, Phys. Rev. D **59**, 083506 (1999).

Mak M. K. and Harko T., Causal bulk viscous cosmologies in conformally flat spacetime, Int. J. Mod. Phys. D **9**, 475 (2000).

Maharaj S. D. and Govender M., Radiating collapse with vanishing Weyl stresses, Int. J. Mod. Phys. D **14**, 667 (2005).

Mannheim P. D., Conformal gravity and the flatness problem, Astrophys. J. **391**, 429 (1992).

Matarrese S., Pantano O. and Saez D., General relativistic dynamics of irrotational dust: Cosmological implications, Phys. Rev. Lett. **72**, 320 (1994).

Mayala R. M., Goswami R. and Maharaj S. D., Matter shear and vorticity in conformally flat spacetimes, Phys. Rev. D **103**, 044015 (2021).

Melfo A. and Rago H., Conformally flat solutions to the Einstein-Maxwell equations, Astrophys. Space Sci. **193**, 9 (1992).

Mewalal N., Relativistic astropysical models of perfect and radiating fluids, PhD Thesis, University of KwaZulu-Natal, Durban (2019).

Misthry S. S., Maharaj S. D. and Leach P. G. L., Nonlinear shear-free radiative collapse, Math. Meth. Appl. Sci. **31**, 363 (2008).

Mkenyeleye M. D., Investigation of gravitational collapse of generalized Vaidya spacetimes, PhD Thesis, University of KwaZulu-Natal, Durban (2015).

Mongwane B., Problems in cosmology and numerical relativity, PhD Thesis, University of Cape Town, Cape Town (2014).

Nzioki A. M., Goswami R., Dunsby P. K. S. and Ellis G. F. R., Shear-free perturbation of Friedmann-Lemaitre-Roberston-Walker universe, Phys. Rev. D **84**, 124028 (2011)

Nzioki A.M., A study of solutions and perturbations of spherically symmetric spacetimes in fourth order gravity, PhD Thesis, University of Cape Town, Cape Town (2013).

Obukhov V. V., Odintsov S. D. and Granda L. N., Conformally flat universe as quantum back reaction of dilaton-coupled matter, Europhys. Lett. **46**, 268 (1999).

Pandey S. N. and Tiwari R. , Conformally flat spherically symmetric charged perfect fluid distribution in general relativity, Indian J. Pure Appl. Math. **12**, 261 (1981).

Petrov A. Z., Centrally-symmetric gravitational fields, Soviet Physics JETP, **17**, 5 (1963).

Rao V. U. M. and Reddy D. R. K., Static conformally flat solutions in a general scalar-tensor theory of gravitation, Gen. Relativ. Gravit. **14**, 1017 (1982).

Raychaudhuri A., Relativistic cosmology. I, Phys. Rev. **98**, 1123 (1955).

Reddy D. R. K., Static conformally flat solution in a scalar-tensor theory of gravitation, J. Math. Phys. **20**, 23 (1979).

Roy S. R. and Bali R., Conformally flat non-static spherically symmetric perfect fluid distributions in general relativity, Indian J. Pure Appl. Math. **9**, 871 (1978).

Ryder L., Introduction to general relativity, Cambridge University Press, Cambridge (2009).

Samsonov V. M. and Petrov E.K., On the physical interpretation of the central-symmetric Gravitational-Field Singularities, Physics of Particles and Nuclei Letters **8**, 1 (2011).

Schücking E., Homogene scherungsfreie weltmodelle in der relativistischen Kosmologie, Naturwiss **19**, 507 (1957).

Schutz B., Geometrical methods of mathematical physics, Cambridge University Press, Cambridge (1982).

Schutz B., A first course in general relativity, Cambridge uniniversity Press, Cambridge (2009).

Schwarzschild K., Sitzungsberichte Königlich Preuss. Akad. Wiss. Physik-Math. KI., 189, (1916).

Senovilla J. M. M., Sopuerta C. F. and Szekeres P., Theorems on shear-free perfect fluids with their Newtonian analogues, Gen. Relativ. Gravit. **30**, 389 (1998).

Shanthi K., Conformally flat static space-time in the general scalar-tensor theory of gravitation, Astrophys. Space Sci. **162**, 163 (1989).

Sharma R., Das S. and Tikekar R., A class of conformally flat solutions for systems undergoing radiative gravitational collapse, Gen. Relativ. Gravit. **47**, 25 (2015).

Singh K. P. and Roy S. R., Electromagnetic behaviour in spacetimes conformal to some wellknown empty spacetimes, Proc. Nat. Inst. Sci. India. A**32**, 223 (1966).

Singh K. P. and Abdussattar, A conformally non-static perfect-fluid distribution, Gen. Relativ. Gravit. **5**, 115 (1974).

Smith G. D., The Bianchi identity and the Ricci curvature equation, PhD Thesis, University of Queensland, Brisbane (2016).

Stephani H., Kramer D., MacCallum M. A. H., Hoenselaers C. and Herlt E., Exact solutions to Einstein's field equations, Cambridge University Press, Cambridge (2003).

Stewart J.M. and Walker M., Perturbations of spacetimes in general relativity, Pro. R. Soc. A **341**, 49 (1974).

Stewart J. Advanced general relativity, Cambridge University Press, Cambridge (1991).

Ullrich P., Exact and perturbed Friedmann-Lemaître cosmologies, MSc Thesis, University of Waterloo, Waterloo (2007).

Visser M., Conformally Friedmann-Lemaître-Robertson-Walker cosmologies, Class. Quantum Grav., **32**, 13 (2015).

Wald R. M., General relativity, University of Chicago Press, Chicago (1984).

Yadav R. B. S. and Prasad U., Non-static conformally flat spherically symmetric perfect fluid distribution in Einstein-Cartan theory, Astrophys. Space Sci. **203**, 37 (1993).