A study of optimization and fixed point problems in certain geodesic metric spaces

by

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As the candidate's supervisor, I have approved this thesis for submission.

Dr. O. T. Mewomo

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Dedication

This thesis is dedicated to God Almighty and to my lovely mother Anthonia Uchechukwu Izuchukwu.

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Abstract

In this thesis, we study multivalued monotone operators in Hadamard spaces and introduce a new mapping given by a finite family of these operators. We propose a modified Halpern-type algorithm for this mapping and prove that the algorithm converges strongly to a common solution of a finite family of monotone inclusion problems and fixed point problem for a nonexpansive mapping in Hadamard spaces. Furthermore, we study some viscosity approximation techniques for approximating a common solution of a finite family of monotone inclusion problems and fixed point problem for nonexpansive mapping, which is also a unique solution of some variational inequality problems in Hadamard spaces. More so, we propose and study some viscosity-type proximal point algorithms for approximating a common solution of minimization problem and fixed point problem for nonexpansive multivalued mappings, which is also a unique solution of some variational inequality problems in Hadamard spaces. We then progress to propose some iterative algorithms for approximating a common solution of a finite family of minimization, monotone inclusion and fixed point problems for demicontractive-type mappings in Hadamard spaces. In addition, we study equilibrium problems in Hadamard spaces and propose some viscosity-type proximal point algorithms, comprising of a nonexpansive mapping and resolvents of monotone bifunctions. We then prove that the proposed algorithms converge strongly to a common solution of a finite family of equilibrium problems in Hadamard spaces. To generalize the study of equilibrium problems in Hadamard spaces, we introduce a new optimization problem in Hadamard spaces, called the mixed equilibrium problem, and establish the existence of solutions for this problem in Hadamard spaces. We then analyze the asymptotic behavior of the sequence generated by a certain proximal point algorithm for this new optimization problem in Hadamard spaces. We also introduce and study a new class of mappings called the generalized strictly pseudononspreading mappings in Hadamard spaces. We then propose a Mann and Ishikawa-type algorithms for this class of mappings and prove that both algorithms converge strongly to a fixed point of the generalized strictly pseudononspreading mapping. More so, we propose an S-type iteration and a viscosity-type iteration for approximating a fixed point of this mapping, which is also a solution of minimization and monotone inclusion problems in Hadamard spaces. To further generalize the study of optimization and fixed point problems, we study the concept of minimization and fixed point problems for nonexpansive mappings in geodesic metric spaces more general than Hadamard spaces, namely, the p-uniformly convex metric spaces. We introduce the concept of split minimization problems in p-uniformly convex metric spaces and study both Mann and Halpern proximal point algorithms for solving these problems in these spaces. Furthermore, we introduce the classes of asymptotically demicontractive multivalued mappings in Hadamard space, strict asymptotically pseudocontractive-type mappings in p-uniformly convex metric space and generalized strictly pseudononspreading mappings in p-uniformly convex metric spaces. Moreover, we propose several iterative algorithms for approximating a common fixed point of finite family of these mappings. As application of the above study, we solve variational inequality problems and convex feasibility problems in Hadamard spaces. More so, we give several nontrival numerical examples of our results. Using these examples, we carry out various numerical experiments of these results in comparison with other important existing results in the literature. The results of the numerical experiments show that our theoretical results have competitive advantages over

existing results in the literature. In some cases, we see that these numerical results are not applicable in Hilbert and Banach spaces. This means that, established results concerning optimization and fixed point problems in these spaces (Hilbert and Banach) cannot be applied to such examples. Finally, some open problems regarding our results are identified and discussed, which offer many opportunities for future research.

Contents

Declaration

This thesis in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used, proper reference has been made.

Chinedu Izuchukwu

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Contributed papers from the thesis

Papers published/accepted from the thesis.

1. **C. Izuchukwu**, G.C. Ugwunnadi, O.T. Mewomo, A.R. Khan and M. Abbas, Proximal-type algorithms for split minimization problem in p-uniformly convex metric spaces, Numer. Algorithms., **82** (3) (2019), 909-935.
2. **C. Izuchukwu**, A.A. Mebawondu, K.O. Aremu, H.A. Abass and O.T. Mewomo, Viscosity iterative techniques for approximating a common zero of monotone operators in a Hadamard space, Rendiconti del Circolo Matematico di Palermo Series 2, (2019), https://doi.org/10.1007/s12215-019-00415-2.
3. **C. Izuchukwu**, K.O. Aremu, A.A. Mebawondu and O.T. Mewomo, A viscosity iterative technique for equilibrium and fixed point problems in a Hadamard space, Appl. Gen. Topol., **20** (1) (2019), 193-210.
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Chapter 1

General Introduction

1.1 Background of study

Optimization problems which includes minimization problems, variational inequality problems, equilibrium problems, monotone inclusion problems, among others, are known to be very useful in diverse fields such as ecology, physics, economics, computer science and engineering, since many problems arising from these fields can be modeled as an optimization problem. Thus, various methods for solving optimization problems have been developed and studied by numerous authors. Among these methods are the fixed point methods, proximal-like methods, auxiliary principles, decomposition methods, extra-gradient methods and normal map equations (for example, see [5, 7, 91, 113, 117, 118, 192]). In recent years, optimization problems have been extensively studied in both Hilbert and Banach spaces by using the methods mentioned above. One of the most successful and effective methods for solving optimization problems is the fixed point method. As a result of this, a lot of research efforts have been devoted in developing different techniques for finding solutions of optimization problems using the fixed point methods.

Let X be a metric space, a point $x \in X$ is called a fixed point of a nonlinear mapping $T : X \to X$ if $Tx = x$. In general, finding a solution of an optimization problem is equivalent to finding a fixed point for a suitable nonlinear mapping. For instance, a solution of a minimization problem is a fixed point of the resolvent of the convex function associated with the minimization problem. Also, a solution of a monotone inclusion problem is a fixed point of the resolvent of the monotone operator associated with the monotone inclusion problem. Thus, the fixed point method for solving optimization problems is concerned with developing different iterative algorithms for finding fixed points of resolvent of mappings associated with these problems. Therefore, fixed point theory is of paramount importance in the study of optimization problems. It can also be said that fixed point theory is one of the most flourishing areas of research in nonlinear analysis that has enjoyed rapid development over the years, and has continued to attract the interest of many researchers due to its extensive applications in diverse mathematical problems such as inverse problems, signal processing, game theory, fuzzy theory, optimal control problems and many others (see [11, 12, 14, 17, 51, 59, 125, 127, 138, 166, 177, 169, 156, 191] and the references therein). Thus, the theory of fixed point can be considered as the kernel of modern nonlinear and convex analysis.

It is well known that the pivot of the metric fixed point theory is the Banach contraction mapping principle, which states that; a contraction mapping T defined on a complete metric space X always has a unique fixed point, and for any starting point $x_1 \in X$, the sequence defined by the Picard iteration process $x_{n+1} = Tx_n,\ n \geq 1$, converges strongly to that fixed point. This principle was stated by Banach [21] in 1922. It is the most widely applied fixed point theorem in nonlinear analysis, since the contraction condition on the mapping T is easy to check and only the structure of a complete metric space is required. However, for classes of mappings more general than the class of contraction mappings, one may not be able to apply the Banach contraction mapping principle. For instance, there are several examples in the literature (see [49, 86, 107]) which show that for a nonexpansive mapping, its Picard iteration process may not converge to its fixed point, even when the fixed point exists. As a result of this, considerable efforts have been made to approximate fixed points of not only nonexpansive mappings, but more general mappings, by developing different iteration methods. For example, the Mann iteration process introduced by Mann [124], is defined in a real Hilbert space H as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n,\ \forall n \geq 1, \end{cases} \tag{1.1.1}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. It is well known that, if the Mann iterative process converges, then it will converge to a fixed point of a continuous mapping T. However, if T is not continuous, then the Mann iteration process may fail to converge to a fixed point of T even when it converges (see for example, [49, 81, 86, 107]). In 1974, Ishikawa [87] introduced the following generalization of the Mann iteration process, called the Ishikawa iteration process for approximating fixed points of pseudocontractive mappings in Hilbert spaces.

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n,\ \forall n \geq 1, \end{cases} \tag{1.1.2}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$.

Recently, Agarwal *et al.* [2] introduced and studied the following S-iteration process:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n\ \forall n \geq 1, \end{cases} \tag{1.1.3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. It was observed in [2] that iteration process (1.1.3) is independent of (1.1.1) and (1.1.2), and has better convergence rate than (1.1.1) and (1.1.2).

In general, the Picard, Mann, Ishikawa and S-iteration processes only converge weakly. However, in infinite dimensional spaces, strong convergence are more desirable and interesting than weak convergence. For this reason (among others), Halpern [81] introduced the following Halpern iterative process which converges strongly to a fixed point of a nonexpansive mapping in real Hilbert spaces.

$$\begin{cases} u, x_1 \in H, \\ x_{n+1} = \alpha_n u + (1-\alpha_n)Tx_n, \end{cases} \tag{1.1.4}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$. An important generalization of the Halpern iteration process is the viscosity iteration process proposed in real Hilbert spaces by XU [190], as follows:

$$\begin{cases} x_1 \in H, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \end{cases} \tag{1.1.5}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and f is a contractive mapping on X. One important advantage of Algorithm (1.1.5) over the Halpern iteration process (1.1.4) is that it also converges strongly to a unique solution of some variational inequalities associated with the contractive mapping f. Furthermore, the viscosity iteration process is shown to have higher rate of convergence than the Halpern iteration process (see [138, 171]).

For so many years, many researchers have studied the above iteration processes and their modifications to approximate fixed points of nonexpansive mappings and wider classes of mappings in Hilbert and Banach spaces (see, for example [38, 49, 90, 106, 125, 127, 166, 191, 177, 169, 156] and the references therein). These iterative methods and their modifications have been extensively studied in Hilbert and Banach spaces to approximate solutions of optimization problems since in most cases, finding exact solutions of optimization problems is generally very difficult. The study has recently been generalized from these linear spaces (Hilbert and Banach spaces) to nonlinear spaces, precisely, the differentiable manifolds, Hadamard spaces and p-uniformly convex metric spaces. Interestingly, these studies already have more applications in nonlinear spaces than in linear spaces. For instance, many non-convex problems in the linear settings can be viewed as convex problems in the nonlinear spaces (see Example 4.3.9). Also, the minimizers of energy functional (an example of a convex and lower semicontinuous functional on an Hadamard space) are very useful in geometry and analysis. Furthermore, the study of minimization problems in Hadamard spaces have proved to be very useful in computing medians and means of trees, which are very important in computational phylogenetics, difussion tensor imaging, consensus algorithms and modeling of airway systems in human lungs and blood vessels (see [15, 16, 17, 18, 72, 73] for details). Thus, nonlinear spaces are more suitable frameworks for the study of optimization problems. However, this study has not been extensively developed in these spaces (the nonlinear spaces). Therefore, it is our intention in this thesis to further develop and generalize the study of optimization and fixed point problems in nonlinear spaces, particularly, in Hadamard and p-uniformly convex metric spaces.

1.2 Research problems and motivation

In this section, we discuss the research problems and motivation of our study.

1.2.1 Research problems

Let C be a nonempty set and f be any real-valued convex function defined on C. The Minimization Problem (MP) is defined as:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad f(x^*) \leq f(y), \quad \forall y \in C. \tag{1.2.1}$$

We study in this work, the notion of MPs in Hadamard spaces. We then generalize the study to p-uniformly convex metric spaces, and introduce the following two notions of Split Minimization Problem (SMP) in p-uniformly convex metric spaces:

$$\min \ \Psi(x,y) \quad \text{such that} \quad (x,y) \in X \times X, \tag{1.2.2}$$

where $\Psi(x,y) = f(x) + g(y) \ \forall x,y \in X$ and $f,g : X \to (-\infty, +\infty]$ are convex functions, and

$$\min \ \Psi(x,y) \quad \text{such that} \quad (x,y) \in X \times Y, \tag{1.2.3}$$

where X and Y are p-uniformly convex metric spaces (not necessarily equal) and $\Psi : X \times Y \to (-\infty, +\infty]$ is a function defined by $\Psi(x,y) = f(x) + g(y)$; $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ are convex functions. Furthermore, we propose the following backward-backward algorithm

$$\begin{cases} y_n = J^g_{\mu_n} x_n, \\ x_{n+1} = J^f_{\mu_n} y_n, \ \ n \geq 1, \end{cases} \tag{1.2.4}$$

and alternating proximal algorithm

$$\begin{cases} x_{n+1} = \underset{x \in X}{\operatorname{argmin}} \left(\Psi(x, y_n) + \frac{1}{p\mu_n^{p-1}} d(x_n, x)^p \right), \ x \in X, \\ y_{n+1} = \underset{y \in Y}{\operatorname{argmin}} \left(\Psi(x_{n+1}, y) + \frac{1}{p\mu_n^{p-1}} d(y_n, y)^p \right), \ y \in Y, \ n \geq 1, \end{cases} \tag{1.2.5}$$

in p-uniformly convex metric spaces, for solving SMP (1.2.2) and SMP (1.2.3) respectively.

Also, we study an important generalization of the MP (1.2.1) in Hadamard spaces; namely, the following Monotone Inclusion Problem (MIP) (also known as the problem of finding a zero of a monotone operator): Find $x \in D(A)$ such that

$$0 \in A(x), \tag{1.2.6}$$

where A is a multivalued monotone operator and $D(A) := \{x \in C : A(x) \neq \emptyset\}$ is the domain of A. We then extend this study to find a common solution of MPs and MIPs in Hadamard spaces. More so, we introduce a new mapping given by a finite family of multivalued monotone operators in Hadamard spaces.

Another important generalization of the MP is the following Equilibrium Problem (EP), defined as:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \varphi(x^*, y) \geq 0, \ \ \forall y \in C, \tag{1.2.7}$$

where $\varphi : C \times C \to \mathbb{R}$ is a bifunction.
In this work, we study the approximation of a common solution of finite family of EPs and fixed point problems for certain nonlinear mappings in Hadamard spaces. We then generalize this study by introducing a new class of EPs in Hadamard spaces, namely the Mixed Equilibrium Problem (MEP), defined as:

$$\text{Find } \ x^* \in C \ \text{ such that } \ \varphi(x^*, y) + f(y) - f(x^*) \geq 0, \ \ \forall y \in C. \tag{1.2.8}$$

Let C be a nonempty closed and convex subset of a real Hilbert space H and T be a nonlinear mapping defined on C. The Variational Inequality Problem (VIP) is defined in real Hilbert spaces as:

$$\text{Find } x \in C \text{ such that } \langle Tx, y - x \rangle \geq 0 \ \forall y \in C. \tag{1.2.9}$$

The VIP was recently formulated in Hadamard spaces by Khatibzadeh and Ranjbar [99] as:

$$\text{Find } x \in C \text{ such that } \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \ \forall y \in C. \tag{1.2.10}$$

Recall also that the Convex Feasibility Problem (CFP) is defined as:

$$\text{Find } x \in \cap_{i=1}^{N} C_i, \tag{1.2.11}$$

where $C_i, i = 1, 2, \ldots, N$ is a finite family of nonempty closed and convex sets such that $\cap_{i=1}^{N} C_i \neq \emptyset$.

We shall apply some of our results to solve VIPs and CFPs in Hadamard spaces.

Finally, we introduce and study certain classes of mappings in both Hadamard and p-uniformly convex metric spaces. In particular, we introduce a more general class of nonspreading-type mappings, which we called the class of generalized strictly pseudononspreading mappings, and we study fixed point properties for this class of mappings.

1.2.2 Motivation

The motivation of our study will be discussed in two headings; namely, Hadamard spaces and p-uniformly convex metric spaces.

1. **Hadamard spaces:**

   Hadamard spaces also known as complete CAT(0) spaces (which we will discuss in details in Chapter 2) have recently turned out to be a suitable framework for geometric group theory, convex analysis, optimization and nonlinear probability theory. The attractiveness of these spaces for solving optimization problems stems from the fact that some of the recent results relating to optimization problems in these spaces, already have more applications in Hadamard spaces than in Hilbert spaces (see [15, 16, 17, 72, 73]). We have already highlighted in Section 1.1, some of the importance of studying optimization and fixed point problems in Hadamard spaces. However, we will also like to add that, another remarkable application in Hadamard spaces is in the use of gradient flow theorem to counter a conjecture of Donaldson on the asymptotic behavior of the Calabi flow in Kähler geometry (see [15]). Also, the theory of optimization has successfully been applied to finding minimizers of submodular functions on modular lattices (see [15]).

To further increase the motivation of our study in Hadamard spaces, let us consider the following example.

Example 1.2.1. *Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by*

$$d_X(x,y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \ \forall x, y \in \mathbb{R}^2.$$

Then, $(\mathbb{R}^2, d_X)$ is an Hadamard space with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Now define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_1, 2x_1^2 - x_2)$. Clearly, T is not a nonexpansive mapping in the classical sense. However, T is nonexpansive in $(\mathbb{R}^2, d_X)$ (see Example 4.3.9 for details).

Again, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = 100((x_2 + 1) - (x_1 + 1)^2)^2 + x_1^2$. Then f is not convex in the classical sense but convex in $(\mathbb{R}^2, d_X)$ (see [194]).

Thus, existing results on fixed point problems for nonexpansive mappings and on optimization problems for convex functions in Hilbert spaces are not applicable to Example 1.2.1. Therefore, this example (among others) motivates the need to generalize the results on optimization and fixed point problems from Hilbert spaces to Hadamard spaces.

2. **p-uniformly convex metric spaces:**

   Despite the interesting applications of Hadamard spaces, the only Banach spaces which are Hadamard spaces are Hilbert spaces. Thus, there is a need to further generalize the study of optimization and fixed point problems to higher nonlinear spaces which generalizes other Banach spaces and are more applicable than these Banach spaces. For this reason, we studied the concept of minimization problems and fixed point problems for certain nonlinear mappings in p-uniformly convex metric spaces which are natural generalizations of p-uniformly convex Banach spaces.

   Moreover, we note that some recent results obtained in p-uniformly convex metric spaces have already found applications in L^p-Wasserstein spaces, Finsler and metric geometry (see [52, 110, 111, 133, 160] and the references therein). More so, the theory of optimization in p-uniformly convex metric spaces has been applied to obtain solutions of initial boundary value problems for p-harmonic maps (see [110]).

To further inspire the study of optimization and fixed point problems in p-uniformly convex metric spaces, we shall construct in Chapter 2 (see Example 2.1.36), an example of a p-uniformly convex metric space which is not an Hadamard space for $p > 2$.

In summary, despite that the theory of optimization in Hadamard and p-uniformly convex metric spaces are still in the developing stage, they already have interesting applications within and outside mathematics, and they open many new possibilities for further research. Therefore, the need to further develop the study of optimization and fixed point problems in these spaces arises. Thus, we carry out our study in these two spaces.

1.3 Objectives

The main objectives of this work are to:

(i) further develop the study of MIPs using Halpern-type and viscosity-type iteration processes in Hadamard spaces,

(ii) introduce and study various iterative algorithms for approximating common solutions of MPs and MIPs in Hadamard spaces,

(iii) further develop the concept of EPs in Hadamard spaces,

(iv) introduce the concept of MEPs in Hadamard spaces,

(v) apply the results of the above mentioned optimization problems to solve other optimization problems like VIPs and CFPs,

(vi) introduce a new class of nonspreading-type mappings more general than other classes of nonspreading-type mappings, and study the fixed point problems for this class of mappings in both Hadamard and p-uniformly convex metric spaces,

(vii) generalize the study of MPs and fixed point problems from Hadamard spaces to p-uniformly convex metric spaces,

(viii) introduce the classes of asymptotically demicontractive multivalued mappings in Hadamard spaces and strict asymptotically pseudocontractive-type mappings in p-uniformly convex metric space,

(ix) propose several iterative algorithms for approximating a common fixed point of the mappings mentioned in (viii),

(x) give nontrivial numerical experiments of our results in comparison with other important results in the literature in order to illustrate the applicability and the competitive advantages of our results over existing results in the literature,

(xi) highlight and discuss some open problems concerning our study for the purpose of further studies.

1.4 Organization of the thesis

The thesis is organized into nine chapters as follows:

Chapter 1 (General Introduction): In this chapter, we give a brief background of our study. We also discuss the research problems and the motivation for our study. Furthermore, we give the objectives of the study and a comprehensive organization of the thesis.

Chapter 2 (Preliminaries and Literature): In this chapter, we give basic definitions and discuss some concepts, terms and results that are important to our study. We also give detailed literature review of some recent and important past works on optimization and fixed point problems.

Chapter 3 (Contributions to Monotone Inclusion Problems in Hadamard Spaces): The main results of this thesis begins in this chapter. The chapter comprises of five sections.
In Section 3.1, we give a brief introduction of the main results in this chapter.
In Section 3.2, we introduce and prove some new lemmas that will be needed in establishing the main theorems of Chapter 3.
In Section 3.3, we introduce a new mapping given by a finite family of multivalued monotone operators in an Hadamard space. We further propose a modified Halpern-type algorithm for the mapping and prove a strong convergence theorem for approximating a common solution of a finite family of monotone inclusion problems in an Hadamard space. We also applied the results established in this section to solve a finite family of minimization problems in an Hadamard space. A numerical example of our algorithm in nonlinear setting is given to further show the applicability of the main results.
In Section 3.4, we propose and study a Halpern-type PPA for approximating a common solution of a finite family of monotone inclusion problems and fixed point problem for a nonexpansive mapping in an Hadamard space. Numerical example of the result obtained in this section is also given to further show its applicability.
In Section 3.5, we study some viscosity-type proximal point algorithms which comprise of a nonexpansive mapping and a finite sum of resolvents of monotone operators, and prove their strong convergence to a common solution of a finite family of MIPs and fixed point problems for nonexpansive mapping, which is also a unique solution of some variational inequality problems (associated with contraction mappings) in Hadamard spaces. We apply the results obtained in this section to solve convex feasibility problems and variational inequality problem associated with a nonexpansive mapping.

Chapter 4 (Contributions to Minimization and Monotone Inclusion Problems in Hadamard Spaces): This chapter also comprises of five sections organized as follows:
In Section 4.1, we give a brief introduction of our study in Chapter 4.
In Section 4.2, we discuss some lemmas that are associated with convex functions for the minimization problems.
In Section 4.3, we propose and study some viscosity-type proximal point algorithms for approximating a common solution of minimization and fixed point problems for nonexpansive multivalued mappings, which is also a unique solution of some variational inequality problems. Furthermore, we give some numerical examples of our algorithm in order to

show its competitive advantage over existing algorithms in the literature.
In Section 4.4, we propose a Halpern algorithm and prove its strong convergence to a zero of a monotone operator which is also a minimizer of a proper convex and lower semicontinuous function and a fixed point of a demicontractive-type mapping in Hadamard spaces.
In Section 4.5, we extend the results obtained in Section 4.4 to finite family of minimization, monotone inclusion and fixed point problems using a modified Ishikawa iteration process.

Chapter 5 (Contributions to Equilibrium Problems in Hadamard Spaces): This chapter is devoted to the study of equilibrium problems and mixed equilibrium problems in Hadamard spaces. It also comprises of five sections:
In Section 5.1, we give a brief introduction of the main results in this chapter.
In Section 5.2, we discuss some important results that will be needed in this chapter. We also prove some new lemmas that are required to establish the main theorems of the chapter.
In Section 5.3, we propose and study a viscosity-type proximal point algorithm for approximating a common solution of a finite family of equilibrium problems and fixed point problem for a nonexpansive mapping in an Hadamard space. Applications of the results we shall establish in this section to other optimization problems in Hadamard spaces are also discussed.
In Section 5.4, we study the asymptotic behavior of the sequence given by a viscosity-type algorithm and extend the study to approximate a common solution of finite family of equilibrium problems in Hadamard spaces.
In Section 5.5, we introduce and study the concept of mixed equilibrium problems in Hadamard spaces.

Chapter 6 (Generalized Strictly Pseudononspreading Mappings in Hadamard Spaces): This chapter also comprises of 5 sections. Section 6.1 and 6.2 deals with a brief introduction of our study and the discussion of some important results that will be needed in the chapter. In Section 6.3 to 6.5, we introduce and study a new class of mappings called the class of generalized strictly pseudononspreading mappings in Hadamard spaces. We then propose the Mann and Ishikawa-type algorithms for this class of mappings and prove that both algorithms converge to a fixed point of the generalized strictly pseudononspreading mapping. We also propose an S-type iteration and a viscosity-type iteration for approximating a fixed point of the new mapping, which is also a minimizer of a convex function and a zero of a monotone operator.

Chapter 7 (Contributions to Minimization Problems in p-uniformly Convex Metric Spaces): In this chapter, we generalize the study of minimization and fixed point problems from Hadamard spaces to p-uniformly convex metric spaces. Precisely, we introduce and study the Mann and Halpern algorithms for solving minimization and fixed point problems in these spaces. We also introduce and study the notion of split minimization problems in p-uniformly convex metric spaces.

Chapter 8 (Contributions to Fixed Point Problems in Geodesic Metric Spaces): This chapter focuses only on fixed point problems for mappings more general than the ones studied in Chapter 3 to Chapter 7. In particular, we introduce the classes of asymptotically demicontractive multivalued mappings in Hadamard space, strict asymptotically pseudocontractive-type mappings in p-uniformly convex metric space and generalized strictly pseudononspreading mappings in p-uniformly convex metric spaces. We also propose several iterative algorithms for approximating a common fixed point of finite family of these mappings.

Chapter 9 (Conclusion, Contribution to Knowledge and Future Research): In this chapter, we give the conclusion of our study and highlight the contributions of our study to existing knowledge. We also identify and discuss possible areas of future research.

Chapter 2

Preliminaries and Literature Review

In this chapter, we provide definitions of basic terms and concepts that will be useful throughout our research. We also give detailed literature review of some recent and important past works on optimization and fixed point problems. Furthermore, we recall important results that are required in the proofs of the main results of this thesis.

2.1 Preliminaries and definitions

In this section, we give definitions of concepts and discuss some important results.

2.1.1 Geometry of geodesic metric spaces

Many real-world problems such as modeling of airway systems in human lungs and blood vessels, diffusion tensor imaging, computational phylogenetics, inverse problems, signal processing, image recovery, game theory, fuzzy theory and others, naturally occur in metric spaces (for example, see [15, 16, 17, 72, 73]). However, the structure of metric spaces sometimes makes it difficult to apply existing results to solve these problems. Therefore, the need to consider some properties which provides sufficient information that guarantees the applications of such existing results in metric spaces arises. One of such properties is the existence of distance-preserving mapping. This property provides the metric space with a structure that is analogue (in some ways) to the linear structure of a normed linear space. Metric spaces with this property are called geodesic metric spaces. In other words, a metric space (X, d) (or simply X) is called a geodesic space, if every two points x and y in X are joined by a distance-preserving mapping (or an isometry) $c : [0, d(x, y)] \to X$ such that $c(0) = x$, $c(d(x, y)) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x, y)]$. The image of c is called a geodesic segment joining x to y. When it is unique, it is denoted by $[x, y]$. A metric space X is called a uniquely geodesic space, if every two points of X are joined by only one geodesic segment. For all $x, y \in X$ and $t \in [0, 1]$, we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining x and y such that

$$d(x, z) = (1 - t)d(x, y) \quad \text{and} \quad d(z, y) = td(x, y). \tag{2.1.1}$$

We now give another characterization of geodesic metric spaces through the existence of metric midpoints. First, we recall the definition of a metric midpoint.

Definition 2.1.1. *Let X be a metric space and $x, y \in X$. A point $z \in X$ is a metric midpoint of x and y if $d(x,z) = d(y,z) = \frac{1}{2}d(x,y)$.*

Proposition 2.1.2. *(see [17]). Let X be a complete metric space. Then the following are equivalent.*

(i) The space X is a geodesic space.

(ii) For every $x, y \in X$, there exists a point $z \in X$ such that

$$d^2(x,z) + d^2(y,z) = \frac{1}{2}d^2(x,y).$$

(iii) Every pair of points in X has a metric midpoint.

Definition 2.1.3. *Let X be a geodesic metric space. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. In other words, C is convex if for every $x, y \in C$, we have that $tx \oplus (1-t)y \in C$.*

Remark 2.1.4. *(see [31]).*

(i) A geodesic segment in a space that is not uniquely geodesic may not necessarily be convex.

(ii) A subset of a uniquely geodesic metric space which is endowed with the induced metric, is geodesic if and only if it is convex.

Some fundamental examples of geodesic metric spaces

The most common example of a uniquely geodesic metric space is $\mathbb{R}^n$. In this case, the unique geodesic segment joining any two points, say x and y, is the line seqment between them, i.e., the set of points $\{(1-t)x + ty \mid 0 \leq t \leq 1\}$. In view of Remark 2.1.4 (ii), we can see that a round disc in $\mathbb{R}^2$ is a geodesic metric space while a circle in $\mathbb{R}^2$ is not.

More generally, a normed linear space E endowed with the metric $d(x,y) = ||x - y||$ is a geodesic metric space. Here, the distance preserving mapping $t \mapsto tx + (1-t)y$ from $[0,1]$ into X, is a linearly reparameterized geodesic joining x and y in E. E is uniquely geodesic if and only if the unit ball in E is strictly convex (see [31, 49] for the definition of a strictly convex unit ball).

Other fundamental examples of geodesic metric spaces includes the model spaces of constant curvature M_k^n, complete Riemannian manifolds, polyhedral complexes, Busemann spaces, Hadamard spaces, p-uniformly convex metric spaces (see [17, 31, 35, 78, 93] for a detailed treatment of these spaces). However, we shall discuss two of these examples (namely, the Hadamard spaces and p-uniformly convex metric spaces) in the next two subsections, since the main results of this thesis were carried out in these two spaces. But first, we recall an important tool for comparing the geometry of an arbitrary geodesic metric space to that of the Euclidean plane $\mathbb{R}^2$.

Definition 2.1.5. *Let X be a geodesic metric space. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in X consist of three points x_1, x_2, x_3 in X (known as the vertices of Δ) and a geodesic segment between each pair of vertices (known as the edges of Δ).*
A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in X is a triangle $\bar{\Delta}(x_1, x_2, x_3) := \Delta(\bar{x}_1, \bar{x}_2, \bar{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$.

As we shall see in the next subsection, Definition 2.1.5 plays an important role in the characterization of Hadamard spaces.
We now end this subsection, with the following notion of Δ-convergence in geodesic metric spaces, which was first introduced and studied by Lim [119]. We begin with the following definition of asymptotic center.

Definition 2.1.6. *Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by*

$$A(\{x_n\}) = \{\bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v\in X} \limsup_{n\to\infty} d(v, x_n)\}.$$

Definition 2.1.7. *A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta\text{-}\lim_{n\to\infty} x_n = \bar{v}$ (see [67]).*

2.1.2 Geometry of Hadamard spaces

Hadmard spaces which are also referred to as complete uniquely geodesic metric spaces of nonpositive curvature, include Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, complete simply connected Riemannian manifolds of nonpositive sectional curvature (for example, the classical hyperbolic spaces and the manifold of positive definite matrices) [31], nonlinear Lebesgue spaces [17], $\mathbb{R}$-trees [17], Hilbert ball [76], Hyperbolic spaces [158], among others. The geometry of Hadamard spaces can be seen as the nonlinearization of the geometry of Hilbert spaces. The history of Hadamard spaces can be traced to the 1936 paper of Wald [187]. Later (in the 1950s), Alexandrov [3] discovered some interesting characteristics of the space, and as a result of this, Hadamard spaces are sometimes referred to as spaces of nonpositive curvature in the sense of Alexandrov. Since then, Hadamard spaces have proved very useful in the study of geometric group theory. Analytical results first appeared in Hadamard spaces in the late 1990s, and it turns out that Hadamard spaces are appropriate frameworks for the theory of convex and nonlinear analysis. More so, the space has also proved to be an appropriate framework for the study of optimization problems which may be applied to science, economics and engineering. A typical description of such application is an application to computational phylogenetics which can be found in an excellent book of Bačák [17]. Other interesting applications include applications to difussion tensor imaging, consensus algorithms, modeling of airway systems in human lungs and blood vessels, modular lattices, the existence of Lipschitz retractions in finite subsets spaces and Kähler geometry (see [15, 18, 16, 17, 72, 73] for details).

Characterizations of CAT(0) spaces

In what follows, we give various characterizations (definitions) of CAT(0) spaces.

The acronym CAT(0) was coined by Gromov [77] in 1987, where C stands for Cartan, A for Alexandrov, T for Toponogov and 0 is the upper curvature bound. Different means (including geodesic and comparison triangles, see Definition 2.1.5) have been employed to characterize (or define) CAT(0) spaces. We present some of them here under the following headings:

1. Geodesic and comparison triangles.
2. Quadratic and nonquadratic inequalities.
3. Busemann and Ptolemaic spaces.
4. Cauchy-Schwartz inequality.

1. Geodesic and comparison triangles:

Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be its comparison triangle in $\mathbb{R}^2$, then Δ is said to satisfy the CAT(0) inequality if for all points $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.1.2}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z). \tag{2.1.3}$$

Inequality (2.1.3) is known as the CN inequality of Bruhat and Titis [34].

Definition 2.1.8. *A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality.*

2. Quadratic and nonquadratic inequalities:

First, we present the following Theorem which gives equivalent conditions for a complete metric space to be a CAT(0) space via some useful inequalities.

Theorem 2.1.9. *(se [17, Theorem 1.3.2]). Let X be a complete metric space. Then the following are equivalent.*

(i) The space X is a CAT(0) space.

(ii) For every pair of points $x, y \in X$, there exists $m \in X$ such that for each $z \in X$, we have that

$$d^2(m, z) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y).$$

(iii) For every pair of points $x, y \in X$ and $\epsilon > 0$, there exists $m \in X$ such that for each $z \in X$, we have that

$$d^2(m, z) \leq \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - \frac{1}{4}d^2(x, y) + \epsilon.$$

Next, we present the following Theorem which gives equivalent conditions for a geodesic space to be a CAT(0) space.

Theorem 2.1.10. *(see [17, Theorem 1.3.3]). Let X be a geodesic metric space. Then the following are equivalent:*

(i) The space X is a CAT(0) space.

(ii) For every pair of points $x, y, z \in X$, we have

$$d^2(m, x) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(x, z),$$

where m is the midpoint of $[y, z]$.

(iii) For every geodesic $x : [0, 1] \to X$ and every point $p \in X$, we have

$$d^2(p, x_t) \leq (1 - t)d^2(p, x_0) + td^2(p, x_1) - t(1 - t)d^2(x_0, x_1).$$

(iv) For every $x, y, u, v \in X$, we have

$$d^2(x, u) + d^2(y, v) \leq d^2(x, y) + d^2(u, v) + 2d(x, v)d(y, u).$$

(v) For every $x, y, u, v \in X$, we have

$$d^2(x, u) + d^2(y, v) \leq d^2(x, y) + d^2(y, u) + d^2(u, v) + d^2(v, x).$$

3. Busemann and Ptolemaic spaces:

Here, we characterize CAT(0) spaces using the notion of Busemann and Ptolemaic spaces. We begin with the definitions of these two spaces.

Definition 2.1.11. *Let X be a geodesic space. Then X is said to have nonpositive curvature in the sense of Busemann [36] if for every $x, y, z \in X$, we have that $2d(m_1, m_2) \leq d(x, y)$, where m_1 is a midpoint of $[x, z]$ and m_2 is a midpoint of $[y, z]$. A geodesic space with this property is called a Busemann space. A common example of a Busemann space is a strictly convex Banach space. For detailed treatment of Busemann spaces (see [15, 17, 36]).*

Definition 2.1.12. *Let X be a metric space. Then X is called Ptolemaic, if for all $x_1, x_2, x_3, x_4 \in X$,*

$$d(x_1, x_3)d(x_2, x_4) \leq d(x_1, x_2)d(x_3, x_4) + d(x_2, x_3)d(x_4, x_1).$$

For more discussion on Ptolemaic spaces, see [74] and the references therein.

Theorem 2.1.13. *[74] A geodesic metric space is a CAT(0) space if and only if it is Busemann and Ptolemaic.*

4. Cauchy-Schwartz inequality:

To characterize CAT(0) spaces via the Cauchy-Schwartz inequality, we will need the notion of quasilineraization mapping, introduced in CAT(0) spaces by Berg and Nikolaev [26].

Definition 2.1.14. *Let X be a CAT(0) space. Denote the pair $(a,b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle .,. \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by*

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}\left(d^2(a,d) + d^2(b,c) - d^2(a,c) - d^2(b,d)\right) \ \forall a,b,c,d \in X$$

is called a quasilinearization mapping.

It is easily to check that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a,b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a,b,c,d,e \in X$.

Definition 2.1.15. *[26]. A geodesic metric space is a CAT(0) space if it satisfies the following Cauchy-Schwartz inequality:*

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a,b)d(c,d) \ \forall a,b,c,d \in X.$$

Definition 2.1.16. *A complete CAT(0) space is called an Hadamard space.*

Some typical examples of Hadamard spaces

Having studied various characterizations of CAT(0) spaces, we now turn to present some typical examples of complete CAT(0) spaces. Detailed constructions of these examples can be found in [103, 147, 194].

Example 2.1.17. *(see [194]). Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \to [0,\infty)$ defined by*

$$d_X(x,y) = \sqrt{(x_1 - y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \ \forall x,y \in \mathbb{R}^2.$$

Then, $(\mathbb{R}^2, d_X)$ is an Hadamard space with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Example 2.1.18. *(see [184]). Let $X = \{\bar{x} = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ be the Poincaré plane endowed with the Riemannain metric given by*

$$g_{11} = g_{22} = \frac{1}{x_2^2}, \quad g_{12} = 0 \text{ for each point } (x_1,x_2) \in X.$$

Then, X is an Hadamard space.

Example 2.1.19. *Let $Y = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x,y) = d(x,y)$ if x and y are situated on a Euclidean straight line passing through the origin and $d_r(x,y) = d(x,\boldsymbol{0}) + d(y,\boldsymbol{0}) := \|x\| + \|y\|$ otherwise. We put $p = (1,0)$ and $X = B \cup C$, where*

$$B = \{(h,0) : h \in [0,1]\} \text{ and } C = \{(h,k) : h + k = 1, h \in [0,1)\}.$$

Note that X is closed and convex and so, (X, d_r) is an Hadamard space.

2.1.3 Dual space of an Hadamard space

Based on the notion of quasilinearization mapping, the notion of dual space of an Hadamard space X was introduced by Kakavandi and Amini [95] as follows: Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X)$ defined by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \tag{2.1.4}$$

where $C(X)$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwarz inequality implies that $\Theta(t, a, b)$ is a Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$ for all $a, b \in X$, where

$$L(f) = \sup\left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y \right\},$$

is the Lipschiz semi-norm of the function $f : X \to \mathbb{R}$. Now, define the pseudometric $\mathcal{D}$ on $\mathbb{R} \times X \times X$ by

$$\mathcal{D}((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)).$$

$\mathcal{D}((t, a, b), (s, c, d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$ for all $x, y \in X$ (see [94, Lemma 2.1]). For an Hadamard space X, the pseudometric space $(\mathbb{R} \times X \times X, \mathcal{D})$ can be considered as a subspace of the pseudometric space $(\mathrm{Lip}(X, \mathbb{R}), L)$ of all real-valued Lipschitz functions. Also, the metric $\mathcal{D}$ defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of $\overrightarrow{tab} := (t, a, b)$ is given by

$$[\overrightarrow{tab}] = \{\overrightarrow{scd} : t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle \ \forall x, y \in X\}.$$

Thus, $X^* = \{[\overrightarrow{tab}] : (t, a, b) \in \mathbb{R} \times X \times X\}$ is a metric space with $\mathcal{D}$ as the metric.

Definition 2.1.20. *Let (X, d) be an Hadamard space. Then, the pair $(X^*, \mathcal{D})$ is called the dual space of (X, d).*

Throughout this thesis, we shall simply write X^* for the dual space of X.

Remark 2.1.21. *(see [95]).*

(i) The dual space X^* acts on $X \times X$ by $\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle$, where $x^* = [\overrightarrow{tab}] \in X^*$, $x, y, a, b \in X$ and $t \in \mathbb{R}$.

(ii) The dual of a closed and convex subset of a Hilbert space H with nonempty interior is also a Hilbert space and $t(b - a) \equiv [\overrightarrow{tab}]$ for all $t \in \mathbb{R}$, $a, b \in H$.

We now construct a typical (non-trivial) example of the dual space of the Hadamard space defined in Example 2.1.19.

Example 2.1.22. *Let (X, d_r) be as defined in Example 2.1.19. Then, its dual space X^* is the space of elements $[\overrightarrow{tab}]$ such that*

$$[\overrightarrow{tab}] = \begin{cases} \{\overrightarrow{scd} : c, d \in B, s \in \mathbb{R}, \ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a, b \in B, \\ \{\overrightarrow{scd} : c, d \in C \cup \{\boldsymbol{0}\}, s \in \mathbb{R}, \ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a, b \in C \cup \{\boldsymbol{0}\}, \\ \{\overrightarrow{tab}\} & a \in B, \ b \in C. \end{cases}$$

To see this, for each $[t\overrightarrow{ab}] \in X^*$, we calculate its equivalence class as follows: If $[t\overrightarrow{ab}] = [s\overrightarrow{cd}] \neq [\overrightarrow{pp}]$, we must have that $t\langle\overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle\overrightarrow{cd}, \overrightarrow{xy}\rangle$ for all $x, y \in X$, where p is as defined in Example 2.1.19.

Cases:

(I): If $\{a, b, c, d\} \subset C \cup \{\mathbf{0}\}$, we have that $d_r(e, z) = \|e\| + \|z\|$ for all $e \in A$ and $z \in X$. Thus, the equality $t\langle\overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle\overrightarrow{cd}, \overrightarrow{xy}\rangle$ is equivalent to $t(\|b\| - \|a\|) = s(\|d\| - \|c\|)$.

(II): If $\{a, b, c, d\} \subset B$, we obtain by similar argument as in Case I that the equality $t\langle\overrightarrow{ab}, \overrightarrow{xy}\rangle = s\langle\overrightarrow{cd}, \overrightarrow{xy}\rangle$ is equivalent to $t(\|b\| - \|a\|) = s(\|d\| - \|c\|)$.

(III): In Case I and Case II, the equation does not depend on x and y. But in other cases, the equation depends on x and y, i.e., the equality for $x, y \in B$ is different from the equality for $x, y \in C$.

Thus, we conclude that X^* defined above is the dual space of the Hadamard space X defined in Example 2.1.19.

2.1.4 Monotone operators

Thanks to the concept of dual space of an Hadamard space, we can now study the notion of multivalued monotone operators defined on an Hadamard space and valued in the dual space, whose theory is known to be one of the most important theory in optimization, nonlinear and convex analysis.

Definition 2.1.23. *Let X be an Hadamard space and X^* be its dual space. A multivalued operator $A : X \to 2^{X^*}$ with domain $D(A) := \{x \in X : Ax \neq \emptyset\}$ is monotone, if for all $x, y \in D(A)$, $x \neq y$, $x^* \in Ax$ and $y^* \in Ay$, we have*

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq 0.$$

A is called α-strongly monotone, if there exists $\alpha > 0$ such that

$$\langle x^* - y^*, \overrightarrow{yx}\rangle \geq \alpha d^2(x, y) \ \forall x, y \in D(A), x^* \in Ax, \text{ and } y^* \in Ay.$$

Clearly, every α-strongly monotone operator is monotone.

Definition 2.1.24. *[98] Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be any multivalued operator. Then, the resolvent of A of order $\lambda > 0$ is a mapping $J_\lambda^A : X \to X$ defined by*

$$J_\lambda^A(x) := \{z \in X \mid [\frac{1}{\lambda}\overrightarrow{zx}] \in Az\}. \tag{2.1.5}$$

Remark 2.1.25. *(see [98]). A monotone operator A is said to satisfy the range condition if for every $\lambda > 0$, $D(J_\lambda^A) = X$.*

In what follows, we give a detailed construction of a multivalued monotone operator together with its resolvent. The construction will be done in the setting of the Hadamard space defined in Example 2.1.19 and its corresponding dual space defined in Example 2.1.22.

Example 2.1.26. *Let $Y = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x,y) = d(x,y)$ if x and y are situated on a Euclidean straight line passing through the origin and $d_r(x,y) = d(x,\boldsymbol{0}) + d(y,\boldsymbol{0}) := \|x\| + \|y\|$ otherwise. Let $p = (1,0)$ and $X = B \cup C$, where*

$$B = \{(h,0) : h \in [0,1]\} \quad \textit{and} \quad C = \{(h,k) : h + k = 1, h \in [0,1)\}.$$

Then, (X, d_r) is an Hadamard space and X^* which is the space of elements $[t\overrightarrow{ab}]$ such that

$$[t\overrightarrow{ab}] = \begin{cases} \{s\overrightarrow{cd} : c,d \in B, s \in \mathbb{R}, t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a,b \in B, \\ \{s\overrightarrow{cd} : c,d \in C \cup \{\boldsymbol{0}\}, s \in \mathbb{R}, t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a,b \in C \cup \{\boldsymbol{0}\}, \\ \{t\overrightarrow{ab}\} & a \in B,\ b \in C, \end{cases} \tag{2.1.6}$$

is the dual space of X (see Example 2.1.22).

Now, define $A : X \to 2^{X^*}$ by

$$Ax = \begin{cases} \{[\overrightarrow{\boldsymbol{0}p}]\} & x \in B, \\ \{[\overrightarrow{\boldsymbol{0}p}], [\overrightarrow{\boldsymbol{0}x}]\} & x \in C. \end{cases}$$

Then A is a multivalued monotone operator. To see this, we consider the following cases.

Cases:

(I): If $x, y \in B$, then $Ax = Ay = \{[\overrightarrow{\boldsymbol{0}p}]\}$ and $x^* = y^* = [\overrightarrow{\boldsymbol{0}p}]$. So, $\langle x^* - y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(II): If $x, y \in C$, then $Ax = \{[\overrightarrow{\boldsymbol{0}p}], [\overrightarrow{\boldsymbol{0}x}]\}$ and $Ay = \{[\overrightarrow{\boldsymbol{0}p}], [\overrightarrow{\boldsymbol{0}y}]\}$.

- *(i) If $x^* = y^* = [\overrightarrow{\boldsymbol{0}p}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.*
- *(ii) If $x^* = [\overrightarrow{\boldsymbol{0}x}]$ and $y^* = [\overrightarrow{\boldsymbol{0}y}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = d_r^2(x,y) \geq 0$.*
- *(iii) If $x^* = [\overrightarrow{\boldsymbol{0}p}]$ and $y^* = [\overrightarrow{\boldsymbol{0}y}]$, then*

$$\begin{aligned} \langle x^* - y^*, \overrightarrow{yx}\rangle &= \langle \overrightarrow{yp}, \overrightarrow{yx}\rangle \\ &= \frac{1}{2}(d_r^2(y,x) + d_r^2(p,y) - d_r^2(p,x)) \\ &= \frac{1}{2}((\|y\| + \|x\|)^2 + (1 + \|y\|)^2 - (1 + \|x\|)^2) \\ &\geq 0 \ (\textit{since } 1/\sqrt{2} \leq \|x\|, \|y\| \leq 1). \end{aligned}$$

- *(iv) If $x^* = [\overrightarrow{\boldsymbol{0}x}]$ and $y^* = [\overrightarrow{\boldsymbol{0}p}]$, then $\langle x^* - y^*, \overrightarrow{yx}\rangle = \langle \overrightarrow{px}, \overrightarrow{yx}\rangle$, which is similar to (iii).*

(III): If $x \in B, y \in C$, then $Ax = \{[\overrightarrow{\boldsymbol{0}p}]\}$, $Ay = \{[\overrightarrow{\boldsymbol{0}p}], [\overrightarrow{\boldsymbol{0}y}]\}$.

(i) *If* $x^* = y^* = [\overrightarrow{\mathbf{0}p}]$, *then* $\langle x^* - y^*, \overrightarrow{yx}\rangle = 0 \geq 0$.

(ii) *If* $x^* = [\overrightarrow{\mathbf{0}p}]$ *and* $y^* = [\overrightarrow{\mathbf{0}y}]$, *then*

$$\begin{aligned}\langle x^* - y^*, \overrightarrow{yx}\rangle &= \langle \overrightarrow{yp}, \overrightarrow{yx}\rangle \\ &= \frac{1}{2}(d_r^2(y,x) + d_r^2(p,y) - d_r^2(p,x)) \\ &\geq 0 \ (\textit{since } d(p,x) \leq 1 \leq d(p,y)).\end{aligned}$$

Thus, A is monotone.

Now, we compute the resolvent of A as follows:

Cases:

(I) *Let* $x = (h, 0) \in B$.

(i) *If* $z = (k, 0) \in B$ *and* $z \in J_\lambda^A(x)$, *then* $Az = \{[\overrightarrow{\mathbf{0}p}]\}$ *and* $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$. *It follows from (2.1.6) that* $\frac{1}{\lambda}(k - h) = 1$ *or* $k = h - \lambda$.

(ii) *If* $z = (h', k') \in C$ *and* $z \in J_\lambda^A(x)$, *then* $Az = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}z}]\}$ *and* $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$ *or* $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}z}]$. *Using (2.1.6) we see that both of these two cases are impossible.*

(II) *Let* $x = (h, k) \in C$.

(i) *If* $z = (h', 0) \in B$ *and* $z \in J_\lambda^A(x)$, *then* $Az = \{[\overrightarrow{\mathbf{0}p}]\}$ *and* $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$ *which is impossible by (2.1.6).*

(ii) *If* $z = (h', k') \in C$ *and* $z \in J_\lambda^A(x)$, *then* $Az = \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}z}]\}$ *and* $[\frac{1}{\lambda}\overrightarrow{zx}] \in Az$. *The case* $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}p}]$ *is impossible. For the case* $[\frac{1}{\lambda}\overrightarrow{zx}] = [\overrightarrow{\mathbf{0}z}]$. *Using (2.1.6) we see that* $\frac{1}{\lambda}(\|x\| - \|z\|) = \|z\|$ *or* $\|z\| = \frac{1}{1+\lambda}\|x\|$. *Note that there are at most two solutions for z.*

Therefore,

$$J_\lambda^A(x) = \begin{cases} \{z = (h - \lambda, 0)\} & x = (h, 0) \in B, \\ \\ \{z = (h', k') \in C : (1 + \lambda)^2(h'^2 + k'^2) = h^2 + k^2\} & x = (h, k) \in C. \end{cases}$$

We now list other examples of monotone operators existing in the literature.

Example 2.1.27. *(see [188, p.25]). Let X be as defined in Example 2.1.18. For each $\bar{x} \in X$, we have*

$$\langle u, v\rangle_{\bar{x}} = \frac{1}{x_2^2}\langle u, v\rangle \ \ \textit{for any pair } (u, v) \in T_{\bar{x}}X \times T_{\bar{x}}X,$$

where $\langle ., .\rangle_{\bar{x}}$ and $\langle ., .\rangle$ denote the inner product in $T_{\bar{x}}X$ (the tagent space of X at $\bar{x}$) and $\mathbb{R}^2$ respectively.

Then, $A : X \to 2^{TX}$ (where $TX = \underset{\bar{x}\in X}{\cup} T_{\bar{x}}X$) defined by

$$A(\bar{x}) = \begin{cases} (2x_1x_2,\ x_2^2 - x_1^2), & \text{if } x_1^2 + (x_2-2)^2 > 4, \\ \{t(2x_1x_2,\ x_2^2 - x_1^2) :\ t \in [0,1]\}, & \text{if } x_1^2 + (x_2-2)^2 = 4, \\ (0,0), & \text{if if } x_1^2 + (x_2-2)^2 < 4 \end{cases}$$

is monotone.

Example 2.1.28. *Let X be an Hadamard space and X^* be its dual space. Then, the subdifferential $\partial f : X \to 2^{X^*}$ of a proper convex and lower semicontinuous function $f : X \to (-\infty, +\infty]$, defined by*

$$\partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz}\rangle,\ \forall z \in X\}, & \text{if } x \in D(f), \\ \emptyset, & \text{otherwise} \end{cases} \tag{2.1.7}$$

is a monotone operator (see [95]). In particular, for a nonempty, closed and convex subset C of X, the indicator function $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases} \tag{2.1.8}$$

is a proper convex and lower semicontinuous function. Thus, the subdifferential of δ_C,

$$\partial\delta_C(x) = \begin{cases} \{x^* \in X^* : \langle x^*, \overrightarrow{xz}\rangle \leq 0\ \forall z \in C\} \text{ if } x \in C, \\ \emptyset, & \text{otherwise} \end{cases} \tag{2.1.9}$$

is a monotone operator.

Example 2.1.29. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a nonexpansive mapping. Then, the mapping $A : X \to 2^{X^*}$ defined by $Ax := [\overrightarrow{Txx}]$ is monotone.*

In the next theorem, we present the relationship between monotone operators and their resolvents in CAT(0) spaces. We first recall the following definition which we will need in what follows.

Definition 2.1.30. *Let X be an Hadamard space. A nonlinear mapping $T : X \to X$ is said to be firmly nonexpansive (see [98]), if*

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy}\rangle\ \forall x, y \in X.$$

Remark 2.1.31. *From Cauchy-Schwartz inequality, it is clear that every firmly nonexpansive mapping is nonexpasive.*

Theorem 2.1.32. *(see [98]). Let X be a CAT(0) space and J_λ^A be the resolvent of a multivalued operator A of order λ. Then,*

(i) for any $\lambda > 0$, we have that $R(J_\lambda^A) \subset D(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A and $F(J_\lambda^A)$ is the set of fixed points of J_λ^A,

(ii) if A is monotone, then J_λ^A is a singlevalued and firmly nonexpansive mapping,

(iii) if A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu-\lambda}{\mu+\lambda} d^2(x, J_\mu^A x)$.

2.1.5 Geometry of p-uniformly convex metric spaces

In this section, we briefly discuss the notion of p-uniformly convex metric spaces. A detailed discussion of these spaces can be found in [52, 111, 133, 144, 141, 142, 143].

To further generalize established results in other Banach spaces like p-uniformly convex Banach spaces, Noar and Silberman [133] introduced the notion of p-uniformly convex metric spaces in 2011 as follows.

Definition 2.1.33. *(see [133]). Let $1 < p < \infty$, a metric space X is called p-uniformly convex with parameter $c > 0$ if X is a geodesic space and for all $x, y, v \in X$ and $t \in [0,1]$,*

$$d(v,(1-t)x \oplus ty)^p \leq (1-t)d(v,x)^p + td(v,y)^p - \frac{c}{2}t(1-t)d(x,y)^p. \tag{2.1.10}$$

The notion of p-uniformly convex metric space is an obvious generalization of the classical notion of p-uniformly convex Banach space (see [20, 133]). More precisely, L^p-spaces with $p \geq 2$ are typical examples of p-uniformly convex metric spaces. Furthermore, when $p = 2 = c$ in (2.1.10), we obtain the CAT(0) inequality (see [31, 133]). In fact, every CAT(0) space is 2-uniformly convex with parameter $c = 2$ and every CAT(k) space ($k > 0$) with $\mathrm{diam}(X) < \frac{\pi}{2\sqrt{k}}$ is 2-uniformly convex with parameter $c = (\pi - 2\sqrt{k}\epsilon)\tan(\sqrt{k}\epsilon)$ for any $0 < \epsilon \leq \frac{\pi}{2\sqrt{k}} - \mathrm{diam}(X)$ (see [111, 133, 143, 160]).

Remark 2.1.34. *Inequality (2.1.10) ensures that p-uniformly convex metric spaces are uniquely geodesic.*

Proposition 2.1.35. *(see [110]). Let $W_p(t) := t(1-t)^p \oplus (1-t)t^p$, and since $\frac{4}{2^p}t(1-t) \leq W_p(t) \leq t(1-t)$, $t \in [0,1]$, then, one can easily see that a geodesic metric space X is p-uniformly convex with parameter $c \in (0, \frac{8}{2^p}]$ if and only if there exists a constant $k \in (0,1]$ such that for all $x, y, v \in X$ and $t \in [0,1]$,*

$$d(v,(1-t)x \oplus ty)^p \leq (1-t)d(v,x)^p + td(v,y)^p - kW_p(t)d(x,y)^p. \tag{2.1.11}$$

We now give a concrete example of a p-uniformly convex metric space which is not an Hadamard space (for $p > 2$).

Example 2.1.36. *Let $\mathbf{P}(n)$ be the space of an $n \times n$ Hermitian positive definite matrices. For $1 < p < \infty$, the geodesic distance between A and B in $\mathbf{P}(n)$ (also called the p-Schatten distance) $d_p : \mathbf{P}(n) \times \mathbf{P}(n) \to [0, \infty)$ is defined by*

$$\begin{aligned} d_p(A,B) &= \inf\{L(c) \mid c : [0,1] \to \mathbf{P}(n) \textit{ is a smooth curve with } c(0) = A \textit{ and } c(1) = B\} \\ &= ||\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})||_p \\ &= \left(\sum_{i=1}^{m} \log^p \mu_i(A^{-1}B)\right)^{\frac{1}{p}}, \end{aligned}$$

where $\mu_i(A^{-1}B)$ are the eigenvalues of $A^{-1}B$, $L(c) := \int_0^1 ||c(t)^{-\frac{1}{2}}c'(t)c(t)^{-\frac{1}{2}}||_p dt$, $||A||_p := (tr|A|^p)^{\frac{1}{p}}$, tr is the the usual trace functional and $|A| = (A^H A)^{\frac{1}{2}}$ (where A^H is the conjugate *transpose of A). The pair $(\mathbf{P}(n), d_p)$ is a p-uniformly convex metric space with geodesic joining A to B in $\mathbf{P}(n)$ given by*

$$(1-t)x \oplus ty = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}, \ 0 \leq t \leq 1.$$

Other examples of p-uniformly convex metric spaces can be found in [52].

2.1.6 Some operators in geodesic metric spaces

Nonlinear singlevalued mappings

Definition 2.1.37. *Let X be a geodesic metric space and $T : X \to X$ be a nonlinear mapping. Throughout this thesis, we shall denote by $F(T)$, the set of fixed points of T. The mapping T is said to be*

- *L-Lipschitzian, if there exists $L > 0$ such that*

$$d(Tx, Ty) \leq Ld(x, y) \ \forall x, y \in X,$$

if $L \in [0, 1)$, then T is called contraction, while T is called nonexpansive, if $L = 1$;

- *quasinonexpansive, if $F(T) \neq \emptyset$ and*

$$d(Tx, y) \leq d(x, y) \ \forall y \in F(T), \ x \in X;$$

- *k-demicontractive, if $F(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that*

$$d^2(Tx, y) \leq d^2(x, y) + kd^2(Tx, x) \ \forall x \in X, \ y \in F(T);$$

- *uniformly L-Lipschitzian, if there exists $L > 0$ such that*

$$d(T^n x, T^n y) \leq Ld(x, y) \ \forall x, y \in X, n \geq 1;$$

- *asymptotically nonexpansive, if there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that*

$$d(T^n x, T^n y) \leq k_n d(x, y) \ \forall n \geq 1 \text{ and } x, y \in X;$$

- *asymptotically demicontractive, if $F(T) \neq \emptyset$ and there exists a sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$, $\lim\limits_{n\to\infty} k_n = 1$ such that*

$$d^2(T^n x, y) \leq k_n d^2(x, y) + kd^2(x, T^n x)$$

for some $k \in [0, 1)$ and for all $n \geq 1$, $x \in X, y \in F(T)$;

- *$(\{\mu_n\},\{v_n\},\phi)$-total asymptotically demicontractive, if $F(T)\neq\emptyset$ and there exist a constant $k\in[0,1)$ and sequences $\{\mu_n\},\{v_n\}\subset[0,\infty)$ with $\mu_n\to 0,\ v_n\to 0$, and a strictly increasing continuous function $\phi:[0,\infty)\to[0,\infty)$ with $\phi(0)=0$ such that*

$$d^2(T^nx,y)\leq d^2(x,y)+\mu_n\phi\left(d(x,y)\right)+kd^2(x,T^nx)+v_n$$

for all $n\geq 1,\ x\in X, y\in F(T)$;

- *asymptotically regular, if $\lim\limits_{n\to\infty} d(T^nx,T^{n+1}x)=0\ \forall x\in X$.*

Remark 2.1.38. *Besides being an obvious generalization of the class of contraction mappings, the class of nonexpansive mappings is very important for the following reasons, among others (see [32]):*

(i) The class of nonexpansve mappings is closely connected with the monotonicity methods developed since the early 1960's.

(ii) They appear in applications as transition operators for initial value problems of differential inclusion problems of the form $0\in\frac{du}{dt}+T(t)u$, where $T(t)$ is a setvalued dissipation operator which is also minimally continuous.

Remark 2.1.39. *Clearly, nonexpansive mappings (with nonempty fixed points set) $\subset$ quasinonexpansive mappings $\subset$ demicontractive mappings. There are several examples in the literature which show that these inclusions are proper (see for example [9, 50, 90] and the references therein). Furthermore, the class of demicontractive mappings is known to be of central importance in optimization theory since it contains many common types of operators that are useful in solving optimization problems (see [69, 128, 193] and the references therein).*

Nonlinear Multivalued mappings

Let X be a metric space, a subset C of X is called proximinal, if for each $x\in X$, there exists $z\in C$ such that $d(x,z)=\inf\{d(x,y):y\in C\}$. We shall denote by $P(X)$, the family of all nonempty proximinal subsets of X, $CB(X)$ the family of all nonempty closed and bounded subsets of X and 2^X the family of all nonempty subsets of X. Let $\mathcal{H}$ denote the Hausdorff metric induced by the metric d, then for all $A,B\in 2^X$,

$$\mathcal{H}(A,B)=\max\{\sup_{a\in A} d(a,B),\ \sup_{b\in B} d(b,A)\},\tag{2.1.12}$$

where $d(a,B)=\inf\limits_{b\in B} d(a,b)$ is the distance from the point a to the subset B. Let $T:X\to 2^X$ be a multivalued mapping. A point $x\in X$ is called a fixed point of T, if $x\in Tx$ while $x\in X$ is called a strict fixed point of T, if $Tx=\{x\}$. The mapping $T:X\to 2^X$ is called

- *L-Lipschitz,* if there exists $L>0$ such that

$$\mathcal{H}(Tx,Ty)\leq Ld(x,y)\ \forall\ x,y\in X,$$

if $L=1$, then T is called nonexpansive, while T is called a contraction if $L\in(0,1)$;

- *quasinonexpansive,* if $F(T) \neq \emptyset$ and

$$\mathcal{H}(Tx, p) \leq d(x, p) \; \forall \; x \in X \text{ and } p \in F(T);$$

- *uniformly L-Lipschitzian,* if there exists a constant $L > 0$ such that

$$\mathcal{H}(T^n x, T^n y) \leq Ld(x, y) \; \forall \; n \geq 0, x, y \in C.$$

Convex functions

Here, we present a brief and concise study of convex functions that are important to our work.

Definition 2.1.40. *Let X be a geodesic metric space. The domain of a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $D(f) = \{x \in X : f(x) < +\infty\}$.*
The function $f : D(f) \subseteq X \to \mathbb{R} \cup \{+\infty\}$ is said to be

- *proper, if $D(f) \neq \emptyset$;*
- *convex, if*

$$f(tx \oplus (1-t)y) \leq tf(x) + (1-t)f(y) \; \forall x, y \in X, \; t \in (0, 1);$$

- *uniformly convex (see [52]), if there exists a strictly increasing function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$f\left(\frac{1}{2}x \oplus \frac{1}{2}y\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y) - \phi(d(x, y));$$

- *lower semicontinuous at a point $x \in D(f)$, if*

$$f(x) \leq \liminf_{n\to\infty} f(x_n),$$

  for each sequence $\{x_n\}$ in $D(f)$ such that $\lim_{n\to\infty} x_n = x$;

- *upper semicontinuous at a point $x \in D(f)$, if*

$$f(x) \geq \limsup_{n\to\infty} f(x_n),$$

  for each sequence $\{x_n\}$ in $D(f)$ such that $\lim_{n\to\infty} x_n = x$.
  We say that f is lower semicontinuous (or upper semicontinuous) on $D(f)$, if it is lower semicontinuous (or upper semicontinuous) at any point in $D(f)$.

The following is an example of a convex function in an Hadamard space.

Example 2.1.41. *[16] Let X be an Hadamard space. For a finite number of points $a_1, a_2, \ldots, a_N$ and $(w_1, w_2, \ldots, w_N) \in S$ (where S is the convex hull of the canonical basis $e_1, e_2, \ldots, e_N \in \mathbb{R}^N$), the function $f : X \to \mathbb{R}$ defined by $f(x) = \sum_{n=1}^{N} w_n d^2(x, a_n)$ is convex and continuous.*

2.2 Literature review

In this section, we review some recent and important past works on minimization problems, monotone inclusion problems, equilibrium problems and fixed point problems. We shall discuss in details, mainly the works done in Hadamard and p-uniformly convex metric spaces, since these two spaces are our major interest in this thesis. For related important works in other spaces (like the Hilbert spaces, Banach spaces, topological spaces, Hadamard manifolds, Hilbert unit balls), we shall simply refer the readers to them.

2.2.1 Minimization problems

MPs (1.2.1) are very useful in optimization theory, convex and nonlinear analysis. One of the most popular and effective methods for solving MPs is the Proximal Point Algorithm (PPA), introduced by Martinet [126] in 1970 and further developed by Rockafellar [159] in real Hilbert spaces as follows: Let f be a proper convex and lower semicontinuous function defined on a real Hilbert space H. The PPA is defined for arbitrary $x_1 \in H$ by

$$x_{n+1} = \arg\min_{y\in H}\left(f(y) + \frac{1}{2\mu_n}||y - x_n||^2\right),\ n \geq 1, \tag{2.2.1}$$

where $\mu_n > 0$ for all $n \geq 1$. Rockafellar [159] proved that the PPA converges weakly to a minimizer of a proper convex and lower semicontinuous function and raised a very important question as to whether the PPA converges strongly or not. The question was resolved in the negative by Güler [79] who constructed a counterexample showing that the PPA does not necessarily converges strongly (see also [23, 25] for more counterexamples on this subject matter). In other words, except additional conditions are imposed on either the convex function or on the underlying space, only weak convergence results for PPA are expected. In order to obtain strong convergence of the PPA, Kamimura and Takahashi [96] modified the PPA (2.2.1) into a Halpern-type PPA, and proved that it converges strongly to a minimizer of f when f is a proper convex and lower semicontinuous function. Since then, different modifications of the PPA for solving MPs have been introduced, well-developed and extensively studied in both Hilbert and Banach spaces (see [1, 33, 92, 134, 139] and the references therein).

The study of the PPA for solving MPs has recently been generalized from Hilbert spaces to nonlinear spaces, in particular, the Hadamard manifolds and Hilbert unit balls (see for example [62, 71, 116, 151] and the references therein). It is important to note that, although the PPA was first introduced and studied in the linear settings, it is known to have some important metric characteristics (see [18]). Thus, these generalizations (that is, generalizing the study of PPA from Hilbert spaces to nonlinear spaces) are ideal and very important. Motivated by this, Bačák [18] in 2013, further generalized the study of the PPA (for solving MPs) to the setting of Hadamard spaces, as follows: For arbitrary point x_1 in an Hadamard space X, define the sequence $\{x_n\}$ iteratively by

$$x_{n+1} = J^f_{\mu_n}(x_n),\ n \geq 1, \tag{2.2.2}$$

where $\mu_n > 0$ for all $n \geq 1$, and $J_\mu^f : X \to X$ is the Moreau-Yosida resolvent of a proper convex and lower semicontinuous function defined by

$$J_\mu^f(x) = \arg\min_{v\in X}\left(f(v) + \frac{1}{2\mu}d^2(v,x)\right). \tag{2.2.3}$$

Remark 2.2.1. *We note that the mapping J_μ^f is firmly nonexpansive (by Remark 2.1.31, it is nonexpansive) and well defined for all $\mu > 0$ (see [130]). Furthermore, if f is a proper convex and lower semicontinuous function, then $F(J_\mu^f)$ coincides with the set $\underset{y\in X}{\operatorname{argmin}} f(y)$ of minimizers of f (see [130, 174]).*

Bačák [18] proved that the PPA (2.2.2) Δ-converges to a minimizer of f provided that $\sum_{n=1}^{\infty}\mu_n = \infty$ and f has a minimizer in X. More precisely, he proved the following theorem.

Theorem 2.2.2. *Let X be an Hadamard space and $f : X \to (-\infty,\infty]$ be a proper convex and lower semicontinuous function. Suppose that f has a minimizer in X and $\{\mu_n\}$ is a sequence of positive real numbers such that $\sum\limits_{n=1}^{\infty}\mu_n = \infty$. Then, for arbitrary starting point $x_1 \in X$, the sequence $\{x_n\}$ generated by (2.2.2) Δ (weakly)-converges to a minimizer of f.*

Since then, there has been increasing interest in the study of PPA for solving MPs by numerous researchers in Hadamard spaces. For instance, in 2014, Bačák [16] employed a splitting version of the PPA for minimizing the sum of finitely many convex functions in Hadamard spaces. In 2015, Cholamjiak *et. al.* [55] proposed a new algorithm by combining the PPA and the S-type iteration process (1.1.3), resulting into the following S-type PPA for approximating a common solution of MP (1.2.1) and fixed point problem for two nonexpansive mappings in an Hadamard space: For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ by

$$\begin{cases} z_n = \arg\min_{y\in X}\left[f(y) + \frac{1}{2\mu_n}d^2(y,x_n)\right], \\ y_n = (1-\beta_n)x_n \oplus \beta_n T_1 z_n, \\ x_{n+1} = (1-\alpha_n)T_1 x_n \oplus \alpha_n T_2 y_n, \ \forall n \geq 1, \end{cases} \tag{2.2.4}$$

where $f : X \to (-\infty,\infty]$ is a proper, convex and lower semicontinuous function, T_1, T_2 are nonexpansive mappings on X, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ satisfying some conditions, and $\{\mu_n\}$ is a sequence such that $\mu_n \geq \mu > 0$ for all $n \geq 1$. They obtained strong convergence results of the iteration process (2.2.4) under some compactness conditions. Later in 2016, Suparatulatorn *et al.* [174] proposed a new algorithm by combining the PPA and the Halpern iterative process (1.1.4) (resulting into a Halpern-type PPA) to approximate a common solution of MP and fixed point problem for nonexpansive singlvalued mapping in an Hadamard space. They proved the following strong convergence result.

Theorem 2.2.3. *Let X be an Hadamard space and $f : X \to (-\infty,\infty]$ be a proper convex and lower semicontinuous function. Let T be a nonexpansive mapping on X such that $\Omega := F(T) \cap \arg\min\limits_{y\in X} f(y)$ is nonempty. Assume that $\{\mu_n\}$ is a sequence such that*

$\mu_n \geq \mu > 0$ for some μ and for all $n \geq 1$. Suppose that $u, x_1 \in X$ are arbitrarily chosen and $\{x_n\}$ is generated in the following manner:

$$\begin{cases} y_n = argmin_{y \in X}\left[f(y) + \frac{1}{2\mu_n} d^2(y, x_n)\right], \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T y_n, \end{cases} \tag{2.2.5}$$

for each $n \geq 1$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying
(I) $\lim\limits_{n\to\infty} \alpha_n = 0$,
(II) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,
(III) $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,
(IV) $\sum\limits_{n=1}^{\infty} |\mu_n - \mu_{n+1}| < \infty$.
Then $\{x_n\}$ strongly converges to $z \in \Omega$ which is the nearest point of Ω to u.

For more important results on MPs in Hadamard spaces, see [8, 16, 46, 54, 137] and the references therein.

Since Hilbert spaces are the only Banach spaces which are Hadamard spaces, there is a need to further generalize the study of MPs to more general nonlinear spaces which includes other Banach spaces. The study of MPs in such nonlinear spaces, in particular, p-uniformly convex metric spaces, is part of our interest in this work. However, existing results concerning PPA in Hadamard spaces cannot be simply carried into p-uniformly convex metric spaces due to the structure of the space; the smoothness constant c (see inequality (2.1.10)) among others, always serves as a natural obstacle to be overcome in order to extend existing results on PPA to p-uniformly convex metric space.

Let X be a p-uniformly convex metric space. Choi and Ji [52] introduced the notion of p-resolvent mapping of a proper, convex and lower semicontinuous function f in X as follows: For $x \in X$ and $\mu > 0$, $J_\mu^f : X \to X$ is defined by

$$J_\mu^f(x) = \arg\min_{v \in X}\left(f(v) + \frac{1}{2\mu} d(v,x)^p\right). \tag{2.2.6}$$

Clearly, if $p = 2$, then (2.2.6) reduces to the Moreau-Yosida resolvent (2.2.3). Using (2.2.6), they proved that the PPA converges to a minimizer of f in a p-uniformly convex metric space. In fact, they proved the following theorem.

Theorem 2.2.4. *[52, Theorem 3.6] Let X be a p-uniformly convex metric space with parameter $c > 0$ and diameter $\alpha > 0$. Let $f : X \to (-\infty, \infty]$ be a proper uniformly convex, lower semicontinuous function, and $\{\mu_n\}$ be a sequence of positive real numbers such that $\lim\limits_{n\to\infty} \frac{n}{\left(\sum_{i=1}^{n} \mu_i\right)} = 0$. Suppose that the sequence $\{x_n\}$ in X is generated by the following PPA:*

$$x_n = J_{\mu_n}^f(x_{n-1}), \; n \geq 1, \tag{2.2.7}$$

where $J_{\mu_n}^f$ is as defined in (2.2.6). Then, $\{x_n\}$ converges to a minimizer of f.

Kuwae [110] defined the p-resolvent mapping slightly different from the one in (2.2.6) as follows:

$$J_\mu^f(x) = \arg\min_{v\in X}\left(f(v) + \frac{1}{p\mu^{p-1}}d(v,x)^p\right). \tag{2.2.8}$$

Observe that if we set $p = 2$ in $\frac{1}{p\mu^{p-1}}$, then (2.2.8) reduces to (2.2.6). Thus, (2.2.8) is more general than (2.2.6), and known to be applicable in obtaining solutions of initial boundary value problems for p-harmonic maps (see [110] for more details). Kuwae [110] also proved the existence of minimizers of proper lower semicontinuous coercive functions in p-uniformly convex metric spaces.

Remark 2.2.5. *To the best of our knowledge, the results of Kuwae [110], Choi and Ji [52] are the only results on MPs in p-uniformly convex metric spaces. Thus, there are still a lot to be done on MPs in p-uniformly convex metric spaces given their importance in these spaces. Inspired by this, we further develop and generalize the study of MPs in these spaces in Chapter 7.*

2.2.2 Monotone inclusion problems

One of the most important problems in monotone operator theory is the MIP (1.2.6), which is also known as the problem of finding zeros of monotone operators. The quest in developing different techniques for solving this problem by many researchers has become enormous in recent time, since many mathematical problems such as MPs (1.2.1), VIPs (1.2.10), fixed point problems, CFP (1.2.11), saddle point problems, and others, can be modeled as MIP (1.2.6). For instance, the problem of finding a zero of a monotone operator is the problem of finding a solution of MP for a proper convex and lower semicontinuous function. In this case, the monotone operator is the subdifferential of the convex function. Also, a zero of a monotone operator is a solution of a VIP associated to the monotone operator. More so, a zero of a monotone operator is a fixed point of a nonexpansive mapping. In this case, the monotone operator is defined as in Example 2.1.29. Furthermore, it describes the equilibrium or stable state of an evolution system governed by the monotone operator, which is very important in ecology, physics, economics, among others (see [24, 30, 49, 98, 153] and the references therein). One of the most popular techniques for approximating solutions of (1.2.6) is the PPA. MIP (1.2.6) has been extensively studied using the PPA and its modifications by numerious authors in both Hilbert and Banach spaces (see [37, 38, 45, 96, 97, 129, 140, 157, 178] and the references therein).

The study of MIP was recently extended from Hilbert spaces to Hadamard spaces by Khatibzadeh and Ranjbar [98]. In 2016, Khatibzadeh and Ranjbar [98] introduced and studied the following PPA in Hadamard spaces for approximating a solution of MIP (1.2.6), for which they obtained a Δ-convergence result:

$$\begin{cases} x_0 \in X, \\ x_n = J_{\lambda_n}^A x_{n-1}, \end{cases} \tag{2.2.9}$$

where $J_{\lambda_n}^A$ is the resolvent of the monotone mapping A with sequence $\{\lambda_n\} \subset (0,\infty)$ such that $\sum_{n=1}^{\infty}\lambda_n = \infty$. They also obtained a strong convergence result using the above

PPA under the assumption that A is strongly monotone. More precisely, they proved the following theorems.

Theorem 2.2.6. *[98] Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator which satisfies the range condition and $A^{-1}(0) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that $0 < \lambda \leq \lambda_n$. Then, the sequence $\{x_n\}$ generated by (2.2.9) Δ-converges to an element of $A^{-1}(0)$.*

Theorem 2.2.7. *[98] Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued α-strongly monotone operator which satisfies the range condition and $A^{-1}(0) \neq \emptyset$. Let $\{\lambda_n\}$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. Then, the sequence $\{x_n\}$ generated by (2.2.9) converges strongly to the single element x of $A^{-1}(0)$.*

Very recently, Ranjbar and Khatibzadeh [153] proposed the following Mann-type and Halpern-type PPA in Hadamard spaces for approximating solutions of (1.2.6):

$$\begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n) J^A_{\lambda_n} x_n \end{cases} \tag{2.2.10}$$

and

$$\begin{cases} u, x_0 \in X, \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J^A_{\lambda_n} x_n, \end{cases} \tag{2.2.11}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. Under some conditions, they obtained Δ-convergence result using (2.2.10) and strong convergence result using (2.2.11). Motivated by the work of Khatibzadeh and Ranjbar [98], Heydari et.al. [83] modified the PPA (2.2.9) in order to approximate a common solution of a finite family of MIPs in Hadamard spaces. They obtained a Δ-convergence result when the underlying operators are monotone and a strong convergence result when these operators are strongly monotone.

Remark 2.2.8. *Besides the results of Khatibzadeh and Ranjbar [98], Heydari et.al. [83], Ranjbar and Khatibzadeh [153], which mainly motivate our study of MIPs (1.2.6) in the next chapter, we would like to mention that there are few other results on MIPs in Hadamard spaces (see for example, [9, 41, 47, 137, 152, 194]). This then means that, the study of MIPs in Hadamard spaces is still in the developing stage. Therefore, it is important to further develop and generalize this study in Hadamard spaces.*

2.2.3 Equilibrium problems

EP (1.2.7) is another important area of research in mathematics that has attracted the interest of many researchers in recent time. It includes many other optimization and mathematical problems as special cases; namely, MPs, VIPs, complementarity problems, fixed point problems, CFPs, among others. Thus, EPs are of central importance in optimization theory as well as in nonlinear and convex analysis.

Throughout this thesis, we shall denote the solution set of the EP(1.2.7) by EP(φ, C) and call all points in X satisfying (1.2.7), an equilibrium point of the underlying bifunction φ.

EPs have been widely studied in Hilbert, Banach and topological vector spaces by many authors (see [27, 28, 59, 89, 176, 179]), as well as in Hadamard manifolds (see [58, 136]). The EP (1.2.7) was recently studied in Hadamard spaces by Kimura and Kishi [100] under the assumption that the Hadamard space must satisfy the Convex Hull Finite Property (CHFP) (see [100, 108] for the definition of CHFP). In particular, they proved the following theorem.

Theorem 2.2.9. *Let X be an Hadamard space with CHFP and C be a nonempty closed and convex subset of X. Let $\{\alpha_n\}$ be a sequence of real numbers satisfying $0 < a \leq \alpha_n \leq b < 1$. For arbitrary $x_1 \in X$, define the sequence $\{x_n\}$ in X by*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)J^{\varphi}x_n, n \geq 1, \tag{2.2.12}$$

where $J^{\varphi} : X \to C$ is the resolvent of the bifunction φ satisfying some conditions (see [100, Condition 1]). Then, $\{x_n\}$ Δ-converges to an element of $EP(\varphi, C)$.

Very recently, Kumam and Chaipunya [108] studied the EP (1.2.7) in Hadamard spaces without the CHFP assumption. They established the existence of an equilibrium point of a bifunction satisfying some convexity, continuity and coercivity assumptions, and they also established some fundamental properties of the resolvent of the bifunction. Furthermore, they proved that the PPA Δ-converges to an equilibrium point of a monotone bifunction in an Hadamard space. More precisely, they proved the following theorem.

Theorem 2.2.10. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be monotone, Δ-upper semicontinuous in the first variable such that $D(J_\lambda^{\varphi}) \supset C$ for all $\lambda > 0$. Suppose that $EP(\varphi, C) \neq \emptyset$ and for an initial guess $x_0 \in C$, the sequence $\{x_n\} \subset C$ is generated by*

$$x_n := J_{\lambda_n}^{\varphi}(x_{n-1}),\ n \in \mathbb{N},$$

where $\{\lambda_n\}$ is a sequence of positive real numbers bounded away from 0. Then, $\{x_n\}$ Δ-converges to an element of $EP(\varphi, C)$.

In the linear settings (for example, in Hilbert spaces), EPs have been generalized into an important optimization problem called the MEP (1.2.8). The MEP is an important class of optimization problems since it contains many other optimization problems as special cases. For instance, if $\varphi \equiv 0$ in (1.2.8), then the MEP (1.2.8) reduces to MP (1.2.1). Also, if $f \equiv 0$ in (1.2.8), then the MEP (1.2.8) reduces to the EP (1.2.7). Therefore, it is indisputable that MEP is one of the most general and applicable problems in optimization theory. The existence of solutions of MEP (1.2.8) was established in Hilbert spaces by Peng and Yao [148] (see also [40]). Since then, many authors have extensively studied the MEP (1.2.8) in both Hilbert and Banach spaces (see [40, 75, 146, 148] and the references contained therein).

Remark 2.2.11. *The study of EPs in Hadamard spaces is still in the embryonic stage since there are very few results concerning EPs in Hadamard spaces. Thus, it is important* *to further develop its study in these spaces. More so, to the best of our knowledge, MEP* (1.2.8) *has never been studied in the frame work of Hadamard spaces. Since MEPs contain both MPs and EPs as special cases in Hilbert spaces, it is important to extend its study to Hadamard spaces, so as to unify other optimization problems (in particular, MPs and EPs) in Hadamard spaces.*

2.2.4 Fixed point problems

The approximation of solutions of fixed point problems for nonlinear mappings is one of the most flourishing areas of research in mathematics that has enjoyed prosperous development in the last fifty years or so (see [49, 139, 156]). Its extensive applications in diverse mathematical problems such as inverse problems, signal processing, game theory, fuzzy theory and many others (see [51, 127, 156, 59] and the references therein) is of great interest and has been a major source of attraction for researchers in this direction. Furthermore, many mathematical problems emanating from biology, engineering, economics, computer science, are among others which can be modeled as a fixed point problem. As mentioned earlier, the origin of fixed point problems in metric spaces goes back to the Banach contraction mapping principle. Since then, there have been rapid growing interest in this direction. For instance, the work of Kirk [104] was the pioneer work of fixed point theory in Hadamard spaces, after which Dhompongsa and Panyanak [105], Saejung [161], Shi and Chen [167], Wangkeeree et al. [185], Shi et al. [168], among others, continued to obtain interesting fixed point results in Hadamard spaces.

In 2003, Kirk [104] proved that every nonexpansive singlevalued mapping defined on a nonempty closed convex and bounded subset of a CAT(0) space has a fixed point. In 2008, Dhompongsa and Panyanak [105] proved the following theorems for approximating solutions of fixed point problems for nonexpansive mappings in Hadamard spaces.

Theorem 2.2.12. *Let C be a bounded closed and convex subset of X and $T : C \to C$ be a nonexpansive asymptotically regular mapping. Then, for any $x_0 \in C$, the sequence $\{x_n\}$ generated by the Picard iteration process, Δ-converges to an element of $F(T)$.*

Theorem 2.2.13. *Let C be a bounded closed and convex subset of X and $T : C \to C$ be a nonexpansive mapping. Then, for any initial point $x_0 \in C$, the sequence $\{x_n\}$ generated by the following Mann iteration process, Δ-converges to an element of $F(T)$:*

$$x_{n+1} = \alpha_n x_n \oplus (1-\alpha_n)Tx_n, \ n \geq 0, \tag{2.2.13}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$ and $\limsup\limits_{n\to\infty} \alpha_n < 1$.

Theorem 2.2.14. *Let C be a bounded closed and convex subset of X and $T : C \to C$ be a nonexpansive mapping. Then, for any initial point $x_0 \in C$, the sequence $\{x_n\}$ generated by the following Ishikawa iteration process, Δ-converges to an element of $F(T)$:*

$$x_{n+1} = \alpha_n T\left(\beta_n Tx_n \oplus (1-\beta_n)x_n\right) \oplus (1-\alpha_n)x_n, \ n \geq 0, \tag{2.2.14}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $\sum\limits_{n=0}^{\infty} \alpha_n(1-\alpha_n) = \infty$, $\sum\limits_{n=0}^{\infty} \beta_n(1-\alpha_n) < \infty$ and $\limsup\limits_{n\to\infty} \beta_n < 1$.

Later in 2009, Saejung [161] introduced the following Halpern iteration process and proved its strong convergence to a fixed point of a nonexpansive mapping in an Hadamard space: For arbitrary $u, x_1 \in C$, let $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)Tx_n, \ \forall n \geq 1. \tag{2.2.15}$$

In fact, he proved the following theorem.

Theorem 2.2.15. *Let C be a closed and convex subset of an Hadamard space X and $T : C \to C$ be a nonexpansive mapping with nonempty fixed point set $F(T)$. Suppose that $\{x_n\}$ is generated by (2.2.15), where $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying*

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$ *or* $\lim_{n\to\infty} \left(\frac{\alpha_n}{\alpha_{n+1}}\right) = 1$.

Then, $\{x_n\}$ converges to $z \in F(T)$ which is the nearest point of $F(T)$ to u.

In 2012, Shi and Chen [167] studied the following viscosity iteration process for approximating fixed points of a nonexpansive mapping T in Hadamard spaces: Let $x_t \in C$ be a unique fixed point of the contraction $x \mapsto tf(x) \oplus (1-t)Tx$, $t \in (0,1)$; i.e.,

$$x_t = tf(x_t) \oplus (1-t)Tx_t. \tag{2.2.16}$$

Also, for arbitrarily $x_0 \in C$, let $\{x_n\}$ be a sequence generated by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1-\alpha_n)Tx_n, \ n \geq 0, \tag{2.2.17}$$

where $\{\alpha_n\} \subset (0,1)$. They proved that $\{x_t\}$ defined by (2.2.16) converges strongly to $\tilde{x} \in F(T)$ such that $\tilde{x} = P_{F(T)}f(\tilde{x})$ satisfies property $\mathcal{P}$, i.e., if for $x, u, y_1, y_2 \in X$,

$$d(x, P_{[x,y_1]}u)d(x,y_1) \leq d(x, P_{[x,y_2]}u)d(x,y_2) + d(x,u)d(y_1,y_2).$$

Furthermore, they obtained that $\{x_n\}$ defined by (2.2.17) converges strongly to $\tilde{x} \in F(T)$ under appropriate conditions on $\{\alpha_n\}$.

By using the concept of quasilinearization mapping, Wangkeeree and Preechasilp [184] improved Shi and Chen's results. In fact, they proved some strong convergence theorems for the two iterative schemes (2.2.16) and (2.2.17) in an Hadamard space without the property $\mathcal{P}$.

Later, Wangkeeree et al. [185] studied some strong convergence theorems of the viscosity approximation schemes for an asymptotically nonexpansive mapping in Hadamard spaces: Let C be a closed convex subset of an Hadamard space X and $T : C \to C$ be an asymptotically nonexpansive mapping. For a given contraction f on C and $\alpha_n \in (0,1)$, let $x_n \in C$ be a unique fixed point of the contraction $x \mapsto \alpha_n f(x) \oplus (1-\alpha_n)T^n x$; i.e.

$$x_n = \alpha_n f(x_n) \oplus (1-\alpha_n)T^n x_n, \quad n \geq 1. \tag{2.2.18}$$

Also, for arbitrary $x_1 \in C$, let $\{x_n\}$ be a sequence defined by

$$x_{n+1} = \alpha_n f(x_n) \oplus (1 - \alpha_n)T^n x_n, \quad n \geq 1. \tag{2.2.19}$$

They proved that the iterative schemes (2.2.18) and (2.2.19) converge strongly to the same point $\tilde{x} := P_{F(T)} f(\tilde{x})$, which is the unique solution of the variational inequality:

$$\langle \overrightarrow{\tilde{x}f\tilde{x}}, \overrightarrow{x\tilde{x}} \rangle \geq 0, \quad x \in F(T). \tag{2.2.20}$$

In 2013, Kim [102] studied the Ishikawa-type iterative scheme for approximating fixed point of a completely continuous and uniformly L-Lipschitzian asymptotically demicontractive singlevalued mapping in an Hadamard space (see Definition 2.1.37). Under some compactness conditions, Kim [102] obtained a strong convergence result. In 2014, Liu and Chang [120] proved some strong convergence theorems using Mann- and Ishikawa-type iterative schemes for uniformly L-Lipschitzian asymptotically demicontractive singlevalued mapping. Several other authors have also studied fixed point problems for nonlinear mappings in Hadamard spaces (see, for example [8, 9, 27, 43, 44, 50, 68, 94, 99, 105, 104, 181, 186] and the references therein).

Researchers are now beginning to extend the study of fixed point problems for nonlinear mappings from Hadamard spaces to p-uniformly convex metric spaces. To the best of our knowledge, there are only two results on fixed point problems for nonlinear mappings in p-uniformly convex metric spaces (see [53, 160]). However, we shall omit the discussion of these results since they are not of interest to our study on fixed point problems in p-uniformly convex metric spaces.

2.3 Some important lemmas

In this section, we recall important lemmas which will be needed in the poof of our main results. We begin with the following important inequalities.

Lemma 2.3.1. *Let X be a CAT(0) space, $w, x, y, z \in X$ and $t, s \in [0, 1]$. Then*

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$ (see [68]).

(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$ (see [68]).

(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x, z) + (1-t)^2 d^2(y, z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (see [65]).

(iv) $d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w, y) + (1-t)d(x, z)$ (see [31]).

(v) $z = tx \oplus (1-t)y$ implies $\langle \overrightarrow{zy}, \overrightarrow{zw} \rangle \leq t\langle \overrightarrow{xy}, \overrightarrow{zw} \rangle$ (see [65]).

(vi) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s|d(x, y)$ (see [42]).

Lemma 2.3.2. *[184] Let X be a CAT(0) space. For any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$, we have* *(i)* $\langle \overrightarrow{u_t x}, \overrightarrow{u_t y} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{u_t y} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{u_t y} \rangle$;

(ii) $\langle \overrightarrow{u_t x}, \overrightarrow{uy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{uy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{ux} \rangle$;

(iii) $\langle \overrightarrow{u_t x}, \overrightarrow{vy} \rangle \leq t\langle \overrightarrow{ux}, \overrightarrow{vy} \rangle + (1-t)\langle \overrightarrow{vx}, \overrightarrow{vy} \rangle$;

(iii) $d^2(x,u) \leq d^2(y,u) + 2\langle \overrightarrow{xy}, \overrightarrow{xu} \rangle$.

Lemma 2.3.3. *[180] Let X be a CAT(0) space. Let $\{x_i, i = 1, 2, \dots, N\} \subset X$ and $\alpha_i \in [0,1]$, $i = 1, 2, \dots, N$ such that $\sum\limits_{i=1}^{N} \alpha_i = 1$. Then,*

$$d\left(\bigoplus_{i=1}^{N} \alpha_i x_i, z\right) \leq \sum_{i=1}^{N} \alpha_i d(x_i, z), \ \forall z \in X.$$

Remark 2.3.4. *[180]. For a CAT(0) space X, if $\{x_i, i = 1, 2, \dots, N\} \subset X$, and $\alpha_i \in [0,1]$, $i = 1, 2, \dots, N$. Then by induction, we can write*

$$\begin{aligned} \bigoplus_{i=1}^{N} \alpha_i x_i &:= (1-\alpha_N)\left[\frac{\alpha_1}{1-\alpha_N} x_1 \oplus \frac{\alpha_2}{1-\alpha_N} x_2 \oplus \cdots \oplus \frac{\alpha_{N-1}}{1-\alpha_N} x_{N-1}\right] \oplus \alpha_N x_N \\ &= (1-\alpha_N) \bigoplus_{i=1}^{N-1} \frac{\alpha_i}{1-\alpha_N} x_i \oplus \alpha_N x_N. \end{aligned} \tag{2.3.1}$$

The following Lemmas are very important as regards to Δ-convergence.

Lemma 2.3.5. *[68] Every bounded sequence in an Hadamard space always have a $\triangle$-convergent subsequence.*

Lemma 2.3.6. *[114] Let X be an Hadamard space. Then, every bounded sequence in X has a unique asymptotic center.*

Lemma 2.3.7. *[66] If $\{x_n\}$ is a bounded sequence in a closed and convex subset C of an Hadamard space, then the asymptotic center of $\{x_n\}$ is in C.*

Let $\{x_n\}$ be a bounded sequence in a closed and convex subset C of an Hadamard space. We use the notation

$$x_n \rightharpoonup w \Longleftrightarrow \Phi(w) = \inf_{x \in C} \Phi(x),$$

where $\Phi(x) = \limsup\limits_{n\to\infty} d(x_n, x)$. We note that $x_n \rightharpoonup w$ if and only if $A(\{x_n\}) = \{w\}$ (see [132]).

Lemma 2.3.8. *[132] If $\{x_n\}$ is a bounded sequence in a closed and convex subset C of an Hadamard space, then Δ-$\lim\limits_{n\to\infty} x_n = w$ implies that $x_n \rightharpoonup w$.*

Lemma 2.3.9. *[154, Opial's Lemma] Let X be an Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonempty subset F in which* *(i)* $\lim_{n\to\infty} d(x_n, z)$ *exists for every* $z \in F$*, and*

(ii) if $\{x_{n_k}\}$ *is a subsequence of* $\{x_n\}$ *which is* Δ*-convergent to* x*, then* $x \in F$*.*

Then, there is a $p \in F$ *such that* $\{x_n\}$ *is* Δ*-convergent to* p *in* X*.*

Lemma 2.3.10. *[94] Let* X *be an Hadamard space,* $\{x_n\}$ *be a sequence in* X *and* $x \in X$*. Then,* $\{x_n\}$ $\triangle$*-converges to* x *if and only if* $\limsup_{n\to\infty}\langle \overrightarrow{xx_n}, \overrightarrow{xy}\rangle \leq 0$ *for all* $y \in C$*.*

Definition 2.3.11. *Let* X *be a complete convex metric space and* $T : X \to X$ *be any nonlinear mapping.* T *is said to be* Δ*-demiclosed, if for any bounded sequence* $\{x_n\}$ *in* X *such that* $\Delta\text{-}\lim_{n\to\infty} x_n = z$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$*, then* $z = Tz$*.*

Lemma 2.3.12. *[66] Let* X *be an Hadamard space and* $T : X \to X$ *be a nonexpansive mapping, then* T *is* Δ*-demiclosed.*

Lemma 2.3.13. *[29] Let* C *be a closed and convex subset of an Hadamard space* X *and* $T : C \to CB(C)$ *be a nonexpansive multivalued mapping, then the conditions that* $\{x_n\}$ Δ*-converges to* x *and* $\{d(x_n, z_n)\}$ *converges strongly to* 0 *(where* $z_n \in Tx_n$*), imply that* $x \in Tx$*.*

Lemma 2.3.14. *[149] Let* X *be an Hadamard space and* $T : X \to X$ *be a generalized asymptotically nonspreading mapping, then* T *is* Δ*-demiclosed.*

We now recall other important lemmas to our study.

Lemma 2.3.15. *(see [115, Lemma 7]) Let* X *be a uniformly convex hyperbolic space with modulus of uniform convexity* η*. For any* $c > 0$*,* $\epsilon \in (0, 2]$*,* $\lambda \in [0, 1]$ *and* $v, x, y \in X$*, we have that*

$$d(x, v) \leq c, d(y, v) \leq c \text{ and } d(x, y) \geq \epsilon c \text{ imply that } d((1-\lambda)x \oplus \lambda y, v) \leq (1-2\lambda(1-\lambda)\eta(c, \epsilon))c.$$

Remark 2.3.16. *If* X *is a CAT(0) space, then* X *is uniformly convex with modulus of uniform convexity* $\eta(c, \epsilon) := \frac{\epsilon^2}{8}$ *(see [115, Proposition 8]).*

Lemma 2.3.17. *[175] Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a metric space of hyperbolic type* X *and* $\{\beta_n\}$ *be a sequence in [0,1] with* $\liminf_{n\to\infty} \beta_n < \limsup_{n\to\infty} \beta_n < 1$*. Suppose that* $x_{n+1} = \beta_n x_n \oplus (1 - \beta_n)y_n$ *for all* $n \geq 0$ *and* $\limsup_{n\to\infty}(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \leq 0$*. Then* $\lim_{n\to\infty} d(y_n, x_n) = 0$*.*

Lemma 2.3.18. *[183] Let* C *be a closed and convex subset of an Hadamard space* X *and* $T : C \to C$ *be a uniformly L-Lipschitzian and* $(\{\mu\}, \{v_n\}, \phi)$*-total asymptotically demicontractive mapping with* $F(T) \neq \emptyset$*. If* $\{x_n\}$ *is a bounded sequence in* C *such that* $\Delta\text{-}\lim_{n\to\infty} x_n = p$ *and* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$*. Then* $Tp = p$*.*

Remark 2.3.19. *If* $\phi(\lambda) = \lambda^2$ *and* $v_n = 0$ *for each* $n \geq 1$*, then in Lemma 2.3.18,* T *is an asymptotically demicontractive mapping.*

Definition 2.3.20. *[65] Let C be a nonempty closed and convex subset of a CAT(0) space X. The metric projection of X onto C is a mapping $P_C : X \to C$ which assigns to each $x \in X$, the unique point $P_C x$ in C such that $d(x, P_C x) = \inf\{d(x, y) : y \in C\}$.*

Lemma 2.3.21. *[65, Theorem 2.4] Let C be a nonempty closed and convex subset of an Hadamard space X, $x \in X$ and $u \in C$. Then, $u = P_C x$ if and only if $\langle \overrightarrow{yu}, \overrightarrow{ux} \rangle \geq 0 \; \forall y \in C$.*

Lemma 2.3.22. *[88] Let X be a metric space and $A, B \in P(X)$. Then, for all $a \in A$, there exists $b \in B$ such that $d(a, b) \leq \mathcal{H}(A, B)$.*

Lemma 2.3.23. *[131] Let X be a complete metric space and T be a mapping from X to $CB(X)$ such that for all $x, y \in X$,*

$$\mathcal{H}(Tx, Ty) \leq \lambda d(x, y),$$

where $0 < \lambda < 1$. Then T has a fixed point.

The following lemmas are very useful to our study in p-uniformly convex metric spaces.

Lemma 2.3.24. *[160],[70]. For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c > 0$. Then,*

(i) every bounded sequence in X has a unique asymptotic center,

(ii) every bounded sequence in X has a Δ-convergent subsequence.

Lemma 2.3.25. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c > 0$, and $T : X \to X$ be a nonexpansive mapping. Then T is Δ-demiclosed. The proof follows easily from the proof of [66, Lemma 2.3]*

We now end this section with the following results which play vital roles in establishing strong convergence results.

Lemma 2.3.26. *[191]. Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n + \gamma_n, \quad n \geq 0,$$

where $\{\alpha_n\}$, $\{\delta_n\}$ and $\{\gamma_n\}$ satisfy the following conditions:

(i) $\{\alpha_n\} \subset [0, 1]$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$,

(ii) $\limsup\limits_{n\to\infty} \delta_n \leq 0$,

(iii) $\gamma_n \geq 0 (n \geq 0)$, $\sum\limits_{n=0}^{\infty} \gamma_n < \infty$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.3.27. *[123] Let $\{a_n\}$ be a sequence of non-negative numbers such that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n,$$

where $\{\delta_n\}$ is a sequence of real numbers bounded from above and $\{\alpha_n\} \subset [0, 1]$ satisfies $\sum \alpha_n = \infty$. Then it holds that

$$\limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} \delta_n.$$

Lemma 2.3.28. *[108] Suppose that* $\{x_n\}$ *is* Δ*-convergent to* q *and there exists* $y \in X$ *such that* $\limsup_{n\to\infty} d(x_n, y) \leq d(q, y)$*, then* $\{x_n\}$ *converges strongly to* q*.*

Lemma 2.3.29. *[122]. Let* $\{a_n\}$ *be a sequence of real numbers such that there exists a subsequence* $\{n_j\}$ *of* $\{n\}$ *such that* $a_{n_j} < a_{n_j+1}$ $\forall j \in \mathbb{N}$*. Then there exists a nondecreasing sequence*$\{m_k\} \subset \mathbb{N}$ *such that* $m_k \to \infty$ *when the following properties are satisfied by all (sufficiently large) numbers* $k \in \mathbb{N}$*:*

$$a_{m_k} \leq a_{m_k+1} \quad \text{and } a_k \leq a_{m_k+1}.$$

In fact, $m_k = \max\{i \leq k : a_i < a_{i+1}\}$*.*

Chapter 3

Contributions to Monotone Inclusion Problems in Hadamard Spaces

3.1 Introduction

The extension of known concepts from Hilbert, Banach and topological vector spaces, as well as differentiable manifolds to Hadamard spaces has been of great interest to many researchers in this subject field. One of such known concept is the theory of monotone operators which is known to be one of the most important notions in optimization theory. Monotone operator theory is an area of research in mathematics that has received a lot of attention over the years. An important problem in monotone operator theory is the MIP (1.2.6). In Section 2.2.2, we reviewed some important works that motivate our study of MIPs in Hadamard spaces. In this chapter, based on Remark 2.2.8, we shall propose and study some strong convergence theorems for approximating solutions of MIPs in Hadamard spaces, and also apply the obtained results to solve other related mathematical problems with numerical examples.

3.2 Preliminaries

In this section, we introduce and prove some new lemmas that will be needed in establishing the proposed strong convergence theorems of this chapter. We begin with the following important inequalities.

Lemma 3.2.1. *Let X be a CAT(0) space, $\{x_i,\ i = 1, 2, \ldots, N\} \subset X$, $\{y_i,\ i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \ldots, N$ such that $\sum\limits_{i=1}^{N} \alpha_i = 1$. Then,*

$$d\left(\bigoplus_{i=1}^{N} \alpha_i x_i, \bigoplus_{i=1}^{N} \alpha_i y_i\right) \leq \sum_{i=1}^{N} \alpha_i d(x_i, y_i). \tag{3.2.1}$$

Proof. (By induction). For $N = 2$, the result follows from Lemma 2.3.1 (iv). Now, assume that (3.2.1) holds for $N = k$, for some $k \geq 2$. Then, we prove that (3.2.1) also holds for $N = k+1$. Indeed, by (2.3.1), Lemma 2.3.1 (iv) and our assumption, we obtain that

$$\begin{aligned}
& d\left(\bigoplus_{i=1}^{k+1} \alpha_i x_i, \bigoplus_{i=1}^{k+1} \alpha_i y_i\right) \\
= \; & d\left((1-\alpha_{k+1})\bigoplus_{i=1}^{k} \frac{\alpha_i}{1-\alpha_{k+1}} x_i \oplus \alpha_{k+1}x_{k+1}, (1-\alpha_{k+1})\bigoplus_{i=1}^{k} \frac{\alpha_i}{1-\alpha_{k+1}} y_i \oplus \alpha_{k+1}y_{k+1}\right) \\
\leq \; & (1-\alpha_{k+1}) d\left(\bigoplus_{i=1}^{k} \frac{\alpha_i}{1-\alpha_{k+1}} x_i, \bigoplus_{i=1}^{k} \frac{\alpha_i}{1-\alpha_{k+1}} y_i\right) + \alpha_{k+1} d(x_{k+1}, y_{k+1}) \\
\leq \; & \sum_{i=1}^{k} \alpha_i d(x_i, y_i) + \alpha_{k+1} d(x_{k+1}, y_{k+1}) \\
= \; & \sum_{i=1}^{k+1} \alpha_i d(x_i, y_i).
\end{aligned}$$

Hence, (3.2.1) holds for $N = k+1$. Therefore, we conclude by induction that (3.2.1) holds for all $N \in \mathbb{N}$. $\square$

Lemma 3.2.2. *Let X be a CAT(0) space, $x_1, x_2, x_3 \in X$ and $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3 \in [0,1]$ such that $\alpha_1 + \alpha_2 + \alpha_3 = 1$ and $\beta_1 + \beta_2 + \beta_3 = 1$. Then,*

$$\begin{aligned}
d\left(\bigoplus_{i=1}^{3} \alpha_i x_i, \bigoplus_{i=1}^{3} \beta_i x_i\right) \leq \; & |\alpha_3 - \beta_3| \left[\frac{\alpha_1}{1-\alpha_3} d(x_1, x_3) + \left(1 - \frac{\alpha_1}{1-\alpha_3}\right) d(x_2, x_3)\right] \\
& + \left|\frac{\alpha_2}{1-\alpha_3} - \frac{\beta_2}{1-\beta_3}\right| (1-\beta_3) d(x_1, x_2). \qquad (3.2.2)
\end{aligned}$$

Proof. From Lemma 2.3.1 (vi), (iv), (i) and (2.3.1), we obtain

$$\begin{aligned}
d\left(\bigoplus_{i=1}^{3} \alpha_i x_i, \bigoplus_{i=1}^{3} \beta_i x_i\right) \leq \; & d\left((1-\alpha_3)\bigoplus_{i=1}^{2} \frac{\alpha_i}{1-\alpha_3} x_i \oplus \alpha_3 x_3, (1-\beta_3)\bigoplus_{i=1}^{2} \frac{\alpha_i}{1-\alpha_3} x_i \oplus \beta_3 x_3\right) \\
& + d\left((1-\beta_3)\bigoplus_{i=1}^{2} \frac{\alpha_i}{1-\alpha_3} x_i \oplus \beta_3 x_3, (1-\beta_3)\bigoplus_{i=1}^{2} \frac{\beta_i}{1-\beta_3} x_i \oplus \beta_3 x_3\right) \\
\leq \; & |\alpha_3 - \beta_3| d\left(\bigoplus_{i=1}^{2} \frac{\alpha_i}{1-\alpha_3} x_i, x_3\right) \\
& + (1-\beta_3) d\left(\bigoplus_{i=1}^{2} \frac{\alpha_i}{1-\alpha_3} x_i, \bigoplus_{i=1}^{2} \frac{\beta_i}{1-\beta_3} x_i\right) + \beta_3 d(x_3, x_3) \\
\leq \; & |\alpha_3 - \beta_3| \left[\frac{\alpha_1}{1-\alpha_3} d(x_1, x_3) + \left(1 - \frac{\alpha_1}{1-\alpha_3}\right) d(x_2, x_3)\right] \\
& + \left|\frac{\alpha_2}{1-\alpha_3} - \frac{\beta_2}{1-\beta_3}\right| (1-\beta_3) d(x_1, x_2).
\end{aligned}$$

$\square$

Remark 3.2.3. *Observe that, if we set $\alpha_3 = \beta_3 = 0$, then Lemma 3.2.2 reduces to Lemma 2.3.1 (vi).*

Lemma 3.2.4. *Let X be an Hadamard space and $A : X \to 2^{X^*}$ be a monotone operator. Then,*

(i) $d^2(u, J_\lambda^A x) + d^2(J_\lambda^A x, x) \leq d^2(u, x)$ for all $u \in F(J_\lambda^A)$, $x \in X$ and $\lambda > 0$,

(ii) $d(J_\lambda^A x, J_\mu^A x) \leq \left(\sqrt{1 - \frac{\lambda}{\mu}}\right) d(x, J_\mu^A x)$, $\forall x \in X$ and for $0 < \lambda \leq \mu$.

Proof. (i) For any $u \in A^{-1}(0)$, $x \in D(J_\lambda^A)$ and $\lambda > 0$, we obtain from Theorem 2.1.32 (i) and (ii), and by the definition of firmly nonexpansive mapping that

$$\begin{aligned} d^2(J_\lambda^A x, u) &\leq \langle \overrightarrow{J_\lambda^A x\ u}, \overrightarrow{xu} \rangle \\ &= \frac{1}{2}\left(d^2(J_\lambda^A x, u) + d^2(u, x) - d^2(J_\lambda^A x, x)\right), \end{aligned}$$

which implies

$$d^2(u, J_\lambda^A x) + d^2(J_\lambda^A x, x) \leq d^2(u, x).$$

(ii) From Theorem 2.1.32 (iii), we obtain that

$$\frac{\mu + \lambda}{\mu} d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu - \lambda}{\mu} d^2(x, J_\mu^A x),$$

which implies that

$$d^2(J_\lambda^A x, J_\mu^A x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(x, J_\mu^A x).$$

That is,

$$d(J_\lambda^A x, J_\mu^A x) \leq \left(\sqrt{1 - \frac{\lambda}{\mu}}\right) d(x, J_\mu^A x).$$

□

Remark 3.2.5. *Observe that the inequality in Lemma 3.2.4 (i) is a property of any firmly nonexpansive mapping. That is, if T is a firmly nonexpansive mapping, then from the definition of quasilinearization mapping (Definition 2.1.14), we obtain*

$$d^2(u, Tx) + d^2(Tx, x) \leq d^2(u, x), \ \forall u \in F(T), \ x \in X.$$

Lemma 3.2.6. *Let X be an Hadmard space and X^* be its dual space. Let $T : X \to X$ be a nonexpansive mapping and for each $i = 1, 2, \ldots N$, let $J_\lambda^{A_i}$ be the resolvent of monotone operators A_i of order $\lambda > 0$ such that $F(T) \cap F(J_\lambda^{A_N}) \cap F(J_\lambda^{A_{N-1}}) \cap \cdots \cap F(J_\lambda^{A_2}) \cap F(J_\lambda^{A_1}) \neq \emptyset$. Then*

$$F(T \circ J_\lambda^{A_N} \circ J_\lambda^{A_{N-1}} \circ \cdots \circ J_\lambda^{A_2} \circ J_\lambda^{A_1}) = F(T) \cap F(J_\lambda^{A_N}) \cap F(J_\lambda^{A_{N-1}}) \cap \cdots \cap F(J_\lambda^{A_2}) \cap F(J_\lambda^{A_1}).$$

Proof. Clearly, $F(T)\cap F(J_\lambda^{A_N})\cap F(J_\lambda^{A_{N-1}})\cap\cdots\cap F(J_\lambda^{A_2})\cap F(J_\lambda^{A_1})\subseteq F(T\circ J_\lambda^{A_N}\circ J_\lambda^{A_{N-1}}\circ\cdots\circ J_\lambda^{A_2}\circ J_\lambda^{A_1})$.

We now show that $F(T\circ J_\lambda^{A_N}\circ J_\lambda^{A_{N-1}}\circ\cdots\circ J_\lambda^{A_2}\circ J_\lambda^{A_1})\subseteq F(T)\cap F(J_\lambda^{A_N})\cap F(J_\lambda^{A_{N-1}})\cap\cdots\cap F(J_\lambda^{A_2})\cap F(J_\lambda^{A_1})$.

Let $\Phi_\lambda^N = J_\lambda^{A_N}\circ J_\lambda^{A_{N-1}}\circ\cdots\circ J_\lambda^{A_2}\circ J_\lambda^{A_1}$ and $\Phi_\lambda^0 = I$, then for any $x\in F(T\circ\Phi_\lambda^N)$ and $y\in F(T)\cap F(J_\lambda^{A_N})\cap F(J_\lambda^{A_{N-1}})\cap\cdots\cap F(J_\lambda^{A_2})\cap F(J_\lambda^{A_1})$, we have that

$$\begin{aligned} d^2(x,y) &= d^2(T\Phi_\lambda^N x, T\Phi_\lambda^N y)\\ &\leq d^2(\Phi_\lambda^N x, \Phi_\lambda^N y)\\ &= d^2(\Phi_\lambda^N x, y). \end{aligned}\tag{3.2.3}$$

From Lemma 3.2.4 (i) and (3.2.3), we have

$$\begin{aligned} d^2(J_\lambda^{A_N}(\Phi_\lambda^{N-1}x),\Phi_\lambda^{N-1}x) &\leq d^2(\Phi_\lambda^{N-1}x,y)-d^2(J_\lambda^{A_N}(\Phi_\lambda^{N-1}x),y)\\ &\ \vdots\\ &\leq d^2(x,y)-d^2(\Phi_\lambda^N x,y)\\ &\leq d^2(\Phi_\lambda^N x,y)-d^2(\Phi_\lambda^N x,y), \end{aligned}$$

which implies

$$\Phi_\lambda^N x = \Phi_\lambda^{N-1}x. \tag{3.2.4}$$

Also, from Lemma 3.2.4 (i) and (3.2.3), we have

$$\begin{aligned} d^2(\Phi_\lambda^{N-1}x,\Phi_\lambda^{N-2}x) &\leq d^2(\Phi_\lambda^{N-2}x,y)-d^2(\Phi_\lambda^{N-1}x,y)\\ &\ \vdots\\ &\leq d^2(x,y)-d^2(\Phi^{N-1}x,y)\\ &\leq d^2(\Phi_\lambda^N x,y)-d^2(\Phi_\lambda^N x,y), \end{aligned}$$

which implies

$$\Phi_\lambda^{N-1}x = \Phi_\lambda^{N-2}x. \tag{3.2.5}$$

Continuing in this manner, we obtain that

$$\Phi_\lambda^N x = \Phi_\lambda^{N-1}x = \Phi_\lambda^{N-2}x = \Phi_\lambda^{N-3}x = \cdots = \Phi_\lambda^2 x = \Phi_\lambda^1 x = \Phi_\lambda^0 x = x. \tag{3.2.6}$$

From (3.2.6), we obtain

$$x = J_\lambda^{A_1}x. \tag{3.2.7}$$

From (3.2.6) and (3.2.7), we obtain

$$x = \Phi_\lambda^2 x = J_\lambda^{A_2}(J_\lambda^{A_1}x) = J_\lambda^{A_2}x. \tag{3.2.8}$$

Continuing in this manner, we obtain

$$x = J_\lambda^{A_1} x = J_\lambda^{A_2} x = \cdots = J_\lambda^{A_{N-1}} x = J_\lambda^{A_N} x. \tag{3.2.9}$$

Finally, from (3.2.6), we get

$$x = T(\Phi_\lambda^N x) = Tx. \tag{3.2.10}$$

Thus, we have from (3.2.9) and (3.2.10) that $F(T \circ J_\lambda^{A_N} \circ J_\lambda^{A_{N-1}} \circ \cdots \circ J_\lambda^{A_2} \circ J_\lambda^{A_1}) \subseteq F(T) \cap F(J_\lambda^{A_N}) \cap F(J_\lambda^{A_{N-1}}) \cap \cdots \cap F(J_\lambda^{A_2}) \cap F(J_\lambda^{A_1})$, which completes the proof. $\square$

Lemma 3.2.7. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators and $T : X \to X$ be a nonexpansive mapping. Then, for $\beta_i \in (0,1)$ with $\sum_{i=0}^N \beta_i = 1$, the mapping S_λ defined by $S_\lambda x := \beta_0 x \oplus \beta_1 J_\lambda^{A_1} x \oplus \beta_2 J_\lambda^{A_2} x \oplus \cdots \oplus \beta_N J_\lambda^{A_N} x$ is nonexpansive and $F(T \circ S_\mu) \subseteq \cap_{i=1}^N F(J_\lambda^{A_i}) \cap F(T)$ for all $x \in X$, $0 < \lambda \leq \mu$.*

Proof. Since A_i is monotone for each $i = 1, 2, \ldots, N$, it follows from Theorem 2.1.32 that $J_\lambda^{A_i}$ is single-valued and nonexpansive for $\lambda > 0$, $i = 1, 2, \ldots, N$. Thus, by Lemma 3.2.1, we obtain

$$\begin{aligned} d(S_\lambda x, S_\lambda y) &\leq \beta_0 d(x,y) + \beta_1 d(J_\lambda^{A_1} x, J_\lambda^{A_1} y) + \cdots + \beta_N d(J_\lambda^{A_N} x, J_\lambda^{A_N} y) \\ &\leq \sum_{i=0}^N \beta_i d(x,y) \\ &= d(x,y). \end{aligned}$$

Hence, S_λ is nonexpansive.

Now, let $x \in F(T \circ S_\mu)$ and $v \in \cap_{i=1}^N F(J_\mu^{A_i}) \cap F(T)$. Then, by Lemma 3.2.1, we obtain

$$\begin{aligned} d(x,v) &\leq d(S_\mu x, v) \\ &\leq \beta_0 d(x,v) + \beta_1 d(J_\mu^{A_1} x, v) + \cdots + \beta_N d(J_\mu^{A_N} x, v) \\ &\leq \sum_{i=0}^{N-1} \beta_i d(x,v) + \beta_N d(J_\mu^{A_N} x, v) \qquad (3.2.11) \\ &\leq d(x,v). \end{aligned}$$

From (3.2.11), we obtain that

$$d(x,v) = \sum_{i=0}^{N-1} \beta_i d(x,v) + \beta_N d(J_\mu^{A_N} x, v) = (1-\beta_N) d(x,v) + \beta_N d(J_\mu^{A_N} x, v),$$

which implies that $d(x,v) = d(J_\mu^{A_N} x, v)$. Similarly, we obtain

$$d(x,v) = d(J_\mu^{A_{N-1}} x, v) = \cdots = d(J_\mu^{A_2} x, v) = d(J_\mu^{A_1} x, v).$$

Thus,

$$d(x,v) = d(J_\mu^{A_N} x, v) = \cdots = d(\beta_0 x \oplus \beta_1 J_\mu^{A_1} x \oplus \beta_2 J_\mu^{A_2} x \oplus \cdots \oplus \beta_N J_\mu^{A_N} x, v). \tag{3.2.12}$$

Now, let $d(x,v)=c$. If $c>0$, and there exist $\epsilon>0$ and $i,j\in\{0,1,2,\dots,N\}$, $i\neq j$ such that $d(J_\mu^{A_i}x, J_\mu^{A_j}x)\geq \epsilon c$ (where $J_\mu^{A_0}=I$), then since X is uniformly convex, we obtain from Lemma 2.3.15 that

$$d(\beta_0 x\oplus \beta_1 J_\mu^{A_1}x\oplus \beta_2 J_\mu^{A_2}x\oplus\cdots\oplus \beta_N J_\mu^{A_N}x, v)<c=d(x,v),$$

and this contradicts (3.2.12). Hence, $c=0$. This implies that $x=v$, hence

$$x=J_\mu^{A_i}x,\ \ i=1,2,\dots,N. \tag{3.2.13}$$

Thus, $d(x,Tx)=d(TS_\mu x,Tx)\leq d(S_\mu x,x)\leq 0$, which implies that $x=Tx$.

Since $0<\lambda\leq\mu$, we obtain from Theorem 2.1.32 (iii) and (3.2.13) that

$$d(x,J_\lambda^{A_i}x)\leq 2d(x,J_\mu^{A_i}x)=0,\ \ i=1,2,\dots,N.$$

Hence, $x=J_\lambda^{A_i}x,\ \ i=1,2,\dots,N$. Therefore, we conclude that $F(T\circ S_\mu)\subseteq \cap_{i=1}^N F(J_\lambda^{A_i})\cap F(T)$. □

3.3 Iterative algorithm for a family of monotone inclusion problems in Hadamard spaces

Here, we introduce a new mapping given by a finite family of multivalued monotone operators in an Hadamard space. We further propose a modified Halpern-type algorithm for the mapping and prove a strong convergence theorem for approximating a common solution of finite family of monotone inclusion problems in an Hadamard space. We also applied our results to solve a finite family of MPs in an Hadamard space. A numerical example of our proposed algorithm in nonlinear setting is given to further show the applicability of the obtained results.

3.3.1 Main results

Definition 3.3.1. *Let X be an Hadamard space and X^* be its dual space. Let $A_i: X\to 2^{X^*}, i=1,2,\dots,N$ be multivalued monotone operators with resolvent operators $J_{\lambda_n}^{(i)}$, $i=1,2,\dots,N$. Then, we define the mapping $T_n: X\to X$ as follows:*

$$\begin{cases} U_n^{(0)}x=x,\\ U_n^{(1)}x=a_n^{(1)}J_{\lambda_n}^{(1)}x\oplus b_n^{(1)}x\oplus c_n^{(1)}x,\\ U_n^{(2)}x=a_n^{(2)}J_{\lambda_n}^{(2)}U_n^{(1)}x\oplus b_n^{(2)}U_n^{(1)}x\oplus c_n^{(2)}U_n^{(1)}x,\\ U_n^{(3)}x=a_n^{(3)}J_{\lambda_n}^{(3)}U_n^{(2)}x\oplus b_n^{(3)}U_n^{(2)}x\oplus c_n^{(3)}U_n^{(2)}x,\\ \vdots\\ U_n^{(N-1)}x=a_n^{(N-1)}J_{\lambda_n}^{(N-1)}U_n^{(N-2)}x\oplus b_n^{(N-1)}U_n^{(N-2)}x\oplus c_n^{(N-1)}U_n^{(N-2)}x,\\ T_nx=U_n^{(N)}x=a_n^{(N)}J_{\lambda_n}^{(N)}U_n^{(N-1)}x\oplus b_n^{(N)}U_n^{(N-1)}x\oplus c_n^{(N)}U_n^{(N-1)}x,\end{cases} \tag{3.3.1}$$

for all $x \in X$, $n \geq 1$, where $\{\lambda_n\}$ is a sequence of real numbers and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}, \{c_n^{(i)}\}$ are sequences in $[0,1]$ such that

$$a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1, \ i = 1, 2, \dots N.$$

For the case where $\lambda_n = \lambda$, $a_n^{(i)} = a^{(i)}$, $b_n^{(i)} = b^{(i)}$ and $c_n^{(i)} = c^{(i)}$ $\forall n \geq 1$, $i = 1, 2, \dots, N$ in (3.3.1), we have the mapping $T : X \to X$ defined as

$$Tx = U^{(N)}x = a^{(N)}J_\lambda^{(N)}U^{(N-1)}x \oplus b^{(N)}U^{(N-1)}x \oplus c^{(N)}U^{(N-1)}x. \tag{3.3.2}$$

Lemma 3.3.2. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}, i = 1, 2, \dots, N$ be multivalued monotone operators that satisfy the range condition. Suppose $\Gamma := \cap_{i=1}^N A_i^{-1}(0) \neq \emptyset$ and the mapping $T_n : X \to X$ is defined by (3.3.1). Then, $U_n^{(1)}, U_n^{(2)}, \dots, U_n^{(N-1)}, T_n$ are nonexpansive, and $\Gamma = F(T_n)$. In particular, T is nonexpansive and $\Gamma = F(T)$.*

Proof. It follows from Theorem 2.1.32 that $J_{\lambda_n}^{(i)}$ is nonexpansive for each $i = 1, 2, \dots, N$. Thus, for each $x, y \in X$, we obtain from Lemma 3.2.1 that

$$\begin{aligned}
d(T_nx, T_ny) &\leq a_n^{(N)}d(J_{\lambda_n}^{(N)}U_n^{(N-1)}x, J_{\lambda_n}^{(N)}U_n^{(N-1)}y) + b_n^{(N)}d(U_n^{(N-1)}x, U_n^{(N-1)}y) \\
&\quad + c_n^{(N)}d(U_n^{(N-1)}x, U_n^{(N-1)}y) \\
&\leq d(U_n^{(N-1)}x, U_n^{(N-1)}y) \\
&\leq a_n^{(N-1)}d(J_{\lambda_n}^{(N-1)}U_n^{(N-2)}x, J_{\lambda_n}^{(N-1)}U_n^{(N-2)}y) + b_n^{(N-1)}d(U_n^{(N-2)}x, U_n^{(N-2)}y) \\
&\quad + c_n^{(N-1)}d(U_n^{(N-2)}x, U_n^{(N-2)}y) \\
&\leq d(U_n^{(N-2)}x, U_n^{(N-2)}y) \\
&\ \ \vdots \\
&\leq d(U_n^{(1)}x, U_n^{(1)}y) \\
&\leq d(x, y).
\end{aligned}$$

Hence, $U_n^{(1)}, U_n^{(2)}, \dots, U_n^{(N-1)}$ and T_n are nonexpansive. We now show that $\Gamma = F(T_n)$. For this, it is obvious that $\Gamma \subset F(T_n)$. So, we only show that $F(T_n) \subset \Gamma$. Let $p \in F(T_n)$ and $z \in \Gamma$, we obtain from (3.3.1) and Lemma 2.3.3 that

$$\begin{aligned}
d(p, z) &\leq a_n^{(N)}d(J_{\lambda_n}^{(N)}U_n^{(N-1)}p, z) + b_n^{(N)}d(U_n^{(N-1)}p, z) + c_n^{(N)}d(U_n^{(N-1)}p, z) \\
&\leq a_n^{(N)}d(J_{\lambda_n}^{(N)}U_n^{(N-1)}p, z) + b_n^{(N)}d(U_n^{(N-1)}p, z) + c_n^{(N)}d(U_n^{(N-1)}p, z) \\
&\leq d(U_n^{(N-1)}p, z) \\
&\leq a_n^{(N-1)}d(J_{\lambda_n}^{(N-1)}U_n^{(N-2)}p, z) + b_n^{(N-1)}d(U_n^{(N-2)}p, z) + c_n^{(N-1)}d(U_n^{(N-2)}p, z) \\
&\leq d(U_n^{(N-2)}p, z) \\
&\ \ \vdots \\
&\leq a_n^{(2)}d(J_{\lambda_n}^{(2)}U_n^{(1)}p, z) + b_n^{(2)}d(U_n^{(1)}p, z) + c_n^{(2)}d(U_n^{(1)}p, z) \qquad (3.3.3) \\
&\leq d(U_n^{(1)}p, z) \\
&\leq a_n^{(1)}d(J_{\lambda_n}^{(1)}p, z) + b_n^{(1)}d(p, z) + c_n^{(1)}d(p, z) \\
&\leq d(p, z).
\end{aligned}$$

Thus, we obtain that

$$d(p,z) = a_n^{(1)} d(J_{\lambda_n}^{(1)} p, z) + b_n^{(1)} d(p,z) + c_n^{(1)} d(p,z),$$

which implies that $d(p,z) = d(J_{\lambda_n}^{(1)} p, z)$. Similarly, we obtain

$$d(p,z) = d(J_{\lambda_n}^{(2)} p, z) = \cdots = d(J_{\lambda_n}^{(N-1)} p, z) = d(J_{\lambda_n}^{(N)} p, z).$$

Thus, by the uniform convexity of X (see Remark 2.3.16), we obtain that $J_{\lambda_n}^{(i)} p = p,\ i = 1, 2, \ldots, N$. Therefore, $F(T_n) \subset \Gamma$ and this completes our proof. $\square$

Lemma 3.3.3. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}, i = 1, 2, \ldots, N$ be multivalued monotone operators that satisfy the range condition. Let $T_n : X \to X$ be defined by (3.3.1) and $T : X \to X$ be defined by (3.3.2), where $\{\lambda_n\}$ is a sequence of real numbers and $\{a_n^{(i)}\},\ \{b_n^{(i)}\}, \{c_n^{(i)}\}$ are sequences in $[0,1]$ such that the following conditions are satisfied*

(i) $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1,\ i = 1, 2, \ldots, N,$

(ii) $\lim\limits_{n\to\infty} a_n^{(i)} = a^{(i)},\ i = 1, 2, \ldots, N,$

(iii) $0 < \lambda_n \leq \lambda\ \forall n \geq 1,$ *and* $\lim\limits_{n\to\infty} \lambda_n = \lambda.$

Then, $\lim\limits_{n\to\infty} d(T_n x, Tx) = 0\ \forall x \in X.$

Proof. Let $x \in X$, then from (3.3.1), (3.3.2), Lemma 3.2.1, Lemma 3.2.2 and Lemma 3.2.4 (ii), we obtain that

$$\begin{aligned}
d(U_n^{(1)} x, U^{(1)} x) &= d(a_n^{(1)} J_{\lambda_n}^{(1)} x \oplus b_n^{(1)} x \oplus c_n^{(1)} x, a^{(1)} J_\lambda^{(1)} x \oplus b^{(1)} x \oplus c^{(1)} x) \\
&\leq d(a_n^{(1)} J_{\lambda_n}^{(1)} x \oplus b_n^{(1)} x \oplus c_n^{(1)} x, a_n^{(1)} J_\lambda^{(1)} x \oplus b_n^{(1)} x \oplus c_n^{(1)} x) \\
&\quad + d(a_n^{(1)} J_\lambda^{(1)} x \oplus b_n^{(1)} x \oplus c_n^{(1)} x, a^{(1)} J_\lambda^{(1)} x \oplus b^{(1)} x \oplus c^{(1)} x) \\
&\leq a_n^{(1)} d(J_{\lambda_n}^{(1)} x, J_\lambda^{(1)} x) + d(a_n^{(1)} J_\lambda^{(1)} x \oplus b_n^{(1)} x \oplus c_n^{(1)} x, a^{(1)} J_\lambda^{(1)} x \oplus b^{(1)} x \oplus c^{(1)} x) \\
&\leq a_n^{(1)} d(J_{\lambda_n}^{(1)} x, J_\lambda^{(1)} x) \\
&\quad + |a_n^{(1)} - a^{(1)}| \left[\frac{b_n^{(1)}}{1 - a_n^{(1)}} d(x, J_\lambda^{(1)} x) + \left(1 - \frac{b_n^{(1)}}{1 - a_n^{(1)}}\right) d(x, J_\lambda^{(1)} x) \right] \\
&\quad + \left| \frac{b_n^{(1)}}{1 - a_n^{(1)}} - \frac{b^{(1)}}{1 - a^{(1)}} \right| (1 - a^{(1)}) d(x, x) \\
&\leq a_n^{(1)} \left(\sqrt{1 - \frac{\lambda_n}{\lambda}} \right) d(x, J_\lambda^{(1)} x) + |a_n^{(1)} - a^{(1)}| d(x, J_\lambda^{(1)} x) \\
&\leq \left(\sqrt{1 - \frac{\lambda_n}{\lambda}} + |a_n^{(1)} - a^{(1)}| \right) d(x, J_\lambda^{(1)} x).
\end{aligned} \tag{3.3.4}$$

By the same argument as above, we obtain that

$$
\begin{aligned}
& d(U_n^{(N)}x, U^{(N)}x) \\
= \; & d(a_n^{(N)}J_{\lambda_n}^{(N)}U_n^{(N-1)}x \oplus b_n^{(N)}U_n^{(N-1)}x \oplus c_n^{(N)}U_n^{(N-1)}x, \\
& a^{(N)}J_{\lambda}^{(N)}U^{(N-1)}x \oplus b^{(N)}U^{(N-1)}x \oplus c^{(N)}U^{(N-1)}x) \\
\leq \; & d(a_n^{(N)}J_{\lambda_n}^{(N)}U_n^{(N-1)}x \oplus b_n^{(N)}U_n^{(N-1)}x \oplus c_n^{(N)}U_n^{(N-1)}x, \\
& a_n^{(N)}J_{\lambda_n}^{(N)}U^{(N-1)}x \oplus b_n^{(N)}U_n^{(N-1)}x \oplus c_n^{(N)}U_n^{(N-1)}x) \\
& + d(a_n^{(N)}J_{\lambda_n}^{(N)}U^{(N-1)}x \oplus b_n^{(N)}U_n^{(N-1)}x \oplus c_n^{(N)}U_n^{(N-1)}x, \\
& a_n^{(N)}J_{\lambda}^{(N)}U^{(N-1)}x \oplus b_n^{(N)}U^{(N-1)}x \oplus c_n^{(N)}U^{(N-1)}x) \\
& + d(a_n^{(N)}J_{\lambda}^{(N)}U^{(N-1)}x \oplus b_n^{(N)}U^{(N-1)}x \oplus c_n^{(N)}U^{(N-1)}x, \\
& a^{(N)}J_{\lambda}^{(N)}U^{(N-1)}x \oplus b^{(N)}U^{(N-1)}x \oplus c^{(N)}U^{(N-1)}x) \\
\leq \; & a_n^{(N)}d(J_{\lambda_n}^{(N)}U_n^{(N-1)}x, J_{\lambda_n}^{(N)}U^{(N-1)}x) + a_n^{(N)}d(J_{\lambda_n}^{(N)}U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
& + b_n^{(N)}d(U_n^{(N-1)}x, U^{(N-1)}x) + c_n^{(N)}d(U_n^{(N-1)}x, U^{(N-1)}x) \\
& + \left|a_n^{(N)} - a^{(N)}\right| \left[\frac{b_n^{(N)}}{1-a_n^{(N)}} d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x)\right. \\
& \left. + \left(1 - \frac{b_n^{(N)}}{1-a_n^{(N)}}\right) d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x)\right] \\
\leq \; & d(U_n^{(N-1)}x, U^{(N-1)}x) + a_n^{(N)}d(J_{\lambda_n}^{(N)}U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
& + \left|a_n^{(N)} - a^{(N)}\right| d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
\leq \; & d(U_n^{(N-1)}x, U^{(N-1)}x) + a_n^{(N)}\left(\sqrt{1 - \frac{\lambda_n}{\lambda}}\right) d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
& + \left|a_n^{(N)} - a^{(N)}\right| d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
\leq \; & d(U_n^{(N-1)}x, U^{(N-1)}x) + \left(\sqrt{1 - \frac{\lambda_n}{\lambda}} + \left|a_n^{(N)} - a^{(N)}\right|\right) d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
\leq \; & d(U_n^{(N-2)}x, U^{(N-2)}x) + \left(\sqrt{1 - \frac{\lambda_n}{\lambda}} + \left|a_n^{(N-1)} - a^{(N-1)}\right|\right) d(U^{(N-2)}x, J_{\lambda}^{(N-1)}U^{(N-2)}x) \\
& + \left(\sqrt{1 - \frac{\lambda_n}{\lambda}} + \left|a_n^{(N)} - a^{(N)}\right|\right) d(U^{(N-1)}x, J_{\lambda}^{(N)}U^{(N-1)}x) \\
= \; & d(U_n^{(N-2)}x, U^{(N-2)}x) \\
& + \sum_{i=1}^{2}\left(\sqrt{1 - \frac{\lambda_n}{\lambda}} + \left|a_n^{(N-i+1)} - a^{(N-i+1)}\right|\right) d(U^{(N-i)}x, J_{\lambda}^{(N-i+1)}U^{(N-i)}x) \\
\leq \; & d(U_n^{(1)}x, U^{(1)}x) + \sum_{i=1}^{N-1}\left(\sqrt{1 - \frac{\lambda_n}{\lambda}} + \left|a_n^{(N-i+1)} - a^{(N-i+1)}\right|\right) d(U^{(N-i)}x, J_{\lambda}^{(N-i+1)}U^{(N-i)}x).
\end{aligned}
$$

Thus from (3.3.4), we obtain that

$$\begin{aligned} d(T_nx,Tx) \;\leq\;& \left(\sqrt{1-\frac{\lambda_n}{\lambda}}+\left|a_n^{(1)}-a^{(1)}\right|\right)d(x,J_\lambda^{(1)}x) \\ &+\sum_{i=1}^{N-1}\left(\sqrt{1-\frac{\lambda_n}{\lambda}}+\left|a_n^{(N-i+1)}-a^{(N-i+1)}\right|\right)d(U^{(N-i)}x,J_\lambda^{(N-i+1)}U^{(N-i)}x). \end{aligned}$$

It then follows from conditions (ii) and (iii) that $\lim\limits_{n\to\infty} d(T_nx,Tx)=0$. □

Lemma 3.3.4. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}, i = 1, 2, \ldots, N$ be multivalued monotone operator that satisfy the range condition. Let $\{x_n\}$ be a bounded sequence in X and the mapping $T_n : X \to X$ be defined by (3.3.1), where $\{\lambda_n\}$ is a sequence of real numbers and $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}, \{c_n^{(i)}\}$ are sequences in $[0,1]$ such that the following conditions are satisfied*

- *(i)* $a_n^{(i)}+b_n^{(i)}+c_n^{(i)}=1,\ i=1,2,\ldots,N,$
- *(ii)* $\sum\limits_{n=1}^{\infty}|a_n^{(i)}-a_{n-1}^{(i)}|<\infty,\ i=1,2,\ldots,N,$
- *(iii)* $0<\lambda_{n-1}\leq\lambda_n\ \forall n\geq 1,$ *and* $\sum\limits_{n=1}^{\infty}\left(\sqrt{1-\frac{\lambda_{n-1}}{\lambda_n}}\right)<\infty.$

Then, $\sum\limits_{n=1}^{\infty} d(T_nx_n,T_{n-1}x_n)<\infty.$

Proof. Following the line of argument in the proof of Lemma 3.3.3, we obtain that

$$\begin{aligned} & d(T_nx_n,T_{n-1}x_n) \\ \leq\;& \sum_{i=1}^{N}\left(\sqrt{1-\frac{\lambda_{n-1}}{\lambda_n}}+\left|a_n^{(N-i+1)}-a_{n-1}^{(N-i+1)}\right|\right)d(U_{n-1}^{(N-i)}x_n,J_{\lambda_{n-1}}^{(N-i+1)}U_{n-1}^{(N-i)}x_n) \\ \leq\;& \left(\sqrt{1-\frac{\lambda_{n-1}}{\lambda_n}}+\left|a_n^{(N)}-a_{n-1}^{(N)}\right|\right)M+\left(\sqrt{1-\frac{\lambda_{n-1}}{\lambda_n}}+\left|a_n^{(N-1)}-a_{n-1}^{(N-1)}\right|\right)M \\ &+\cdots+\left(\sqrt{1-\frac{\lambda_{n-1}}{\lambda_n}}+\left|a_n^{(1)}-a_{n-1}^{(1)}\right|\right)M, \end{aligned} \tag{3.3.5}$$

where $M:=\sup\limits_{n\geq 1}\left\{\sum\limits_{i=1}^{N}d(U_{n-1}^{(N-i)}x_n,J_{\lambda_{n-1}}^{(N-i+1)}U_{n-1}^{(N-i)}x_n)\right\}$. Thus, by conditions (ii) and (iii), we obtain from (3.3.5) that $\sum\limits_{n=1}^{\infty}d(T_nx_n,T_{n-1}x_n)<\infty$. □

Theorem 3.3.5. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfy the range condition.*

Let T_n be defined by (3.3.1) such that conditions (i)-(iii) in Lemma 3.3.4 are satisfied. Suppose that $\Gamma \neq \emptyset$, let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)T_n x_n, \ n \geq 1, \tag{3.3.6}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$, satisfying the following conditions

C1: $\lim_{n\to\infty} \alpha_n = 0$,

C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to an element of Γ.

Proof. Let $p \in \Gamma$, then from (3.3.6), Lemma 2.3.1 (i) and Lemma 3.3.2, we have that

$$\begin{aligned} d(x_{n+1}, p) &= d(\alpha_n u \oplus (1-\alpha_n)T_n x_n, p) \\ &\leq \alpha_n d(u,p) + (1-\alpha_n) d(x_n, p) \\ &\leq \max\{d(u,p), d(x_n,p)\} \\ &\vdots \\ &\leq \max\{d(u,p), d(x_1,p)\}, \end{aligned}$$

which implies that $\{d(x_n,p)\}$ is bounded. Consequently, $\{x_n\}$ and $\{T_n x_n\}$ are also bounded. From (iv) and (vi) of Lemma 2.3.1, we obtain

$$\begin{aligned} d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1-\alpha_n)T_n x_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1})T_{n-1}x_{n-1}) \\ &\leq d(\alpha_n u \oplus (1-\alpha_n)T_n x_n, \alpha_n u \oplus (1-\alpha_n)T_{n-1}x_{n-1}) \\ &\quad + d(\alpha_n u \oplus (1-\alpha_n)T_{n-1}x_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1})T_{n-1}x_{n-1}) \\ &\leq (1-\alpha_n) d(T_n x_n, T_{n-1}x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1}x_{n-1}) \\ &\leq (1-\alpha_n) d(T_{n-1}x_n, T_{n-1}x_{n-1}) + (1-\alpha_n) d(T_n x_n, T_{n-1}x_n) \\ &\quad + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1}x_{n-1}) \\ &\leq (1-\alpha_n) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1}x_{n-1}) + d(T_n x_n, T_{n-1}x_n). \end{aligned}$$

It then follows from conditions C2 and C3, Lemma 3.3.4 and Lemma 2.3.26 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.3.7}$$

Also, from (3.3.6), Lemma 2.3.1 (i) and using T as defined in (3.3.2), we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T_n x_n) + d(T_n x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(\alpha_n u \oplus (1-\alpha_n)T_n x_n, T_n x_n) + d(T_n x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + \alpha_n d(u, T_n x_n) + d(T_n x_n, Tx_n), \end{aligned}$$

which implies from C1, Lemma 3.3.3 and (3.3.7) that

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.3.8}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z$. Thus, it follows from (3.3.8) and Lemma 2.3.12 that $z \in F(T) = \Gamma$.

Moreover, from Lemma 2.3.10, we obtain for arbitrary $u \in X$ that

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \leq 0. \tag{3.3.9}$$

From the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \langle \overrightarrow{uz}, \overrightarrow{T_n x_n\ z} \rangle &= \langle \overrightarrow{uz}, \overrightarrow{T_n x_n\ x_n} \rangle + \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \\ &\leq d(u,z) d(T_n x_n, x_n) + \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle, \end{aligned}$$

which implies from (3.3.8) and (3.3.9) that

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{T_n x_n\ z} \rangle \leq 0. \tag{3.3.10}$$

Thus from condition C1, we get

$$\limsup_{n\to\infty} \Big(\alpha_n d^2(u,z) + 2(1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{T_n x_n z} \rangle \Big) \leq 0. \tag{3.3.11}$$

Finally, we show that $\{x_n\}$ converges strongly to $z \in \Gamma$. From (3.3.6) and Lemma 2.3.1(ii), we obtain

$$d^2(x_{n+1}, z) \leq \alpha_n^2 d^2(u,z) + (1-\alpha_n)^2 d^2(T_n x_n, z) + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{T_n x_n\ z} \rangle,$$

which implies

$$d^2(x_{n+1}, z) \;\leq\; (1-\alpha_n) d^2(x_n, z) + \alpha_n \Big(\alpha_n d^2(u,z) + 2(1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{T_n x_n\ z} \rangle \Big) \tag{3.3.12}$$

Hence, from (3.3.11) and Lemma 2.3.26, we conclude that $\{x_n\}$ converges strongly to $z \in \Gamma$. □

By setting $N = 1,\ a_n^{(1)} = 1,\ b_n^{(1)} = c_n^{(1)} = 0$ in Algorithm (3.3.1) and $a^{(1)} = 1,\ b^{(1)} = c^{(1)} = 0$, we have that $T_n = J_{\lambda_n}^{(1)}$ and $T = J_\lambda^{(1)}$. In this case, we obtain the following result.

Corollary 3.3.6. *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping that satisfies the range condition. Suppose that $A^{-1}(0) \neq \emptyset$, let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J_{\lambda_n} x_n,\ \ n \geq 1, \tag{3.3.13}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$ and $\{\lambda_n\}$ is a sequence of real numbers satisfying the following conditions

C1: $\lim_{n\to\infty} \alpha_n = 0$,

C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$,

C4: $0 < \lambda_{n-1} \leq \lambda_n \ \forall n \geq 1$, *and* $\sum_{n=1}^{\infty} \left(\sqrt{1 - \frac{\lambda_{n-1}}{\lambda_n}}\right) < \infty$.

Then $\{x_n\}$ *converges strongly to an element of* $A^{-1}(0)$.

3.3.2 Application

Here, we apply our results to solve a finite family of MPs in Hadamard space.
Let X be an Hadamard space and X^* be its dual space. Then, the subdifferential ∂f of a proper convex and lower semicontinuous function f (see (2.1.7)) is proved in [95] to have the following properties:

(i) ∂f is a monotone operator,

(ii) ∂f satisfies the range condition. That is, $D(J_\lambda^{\partial f}) = X$ for all $\lambda > 0$,

(iii) f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$.

Now, consider the following MP: Find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{3.3.14}$$

It follows from property (iii) that (3.3.14) can be formulated as follows: Find $x \in X$ such that

$$0 \in \partial f(x).$$

Thus, by properties (i) and (ii), and by setting $A_i = \partial f_i,\ i = 1, 2, \ldots, N$ in Definition 3.3.1 and Theorem 3.3.5, we obtain the following result.

Theorem 3.3.7. *Let X be an Hadamard space and X^* be its dual space. Let $f_i : X \to (-\infty, \infty],\ i = 1, 2, \ldots, N$ be a finite family of proper, convex and lower semicontinuous functions, and T_n be defined by (3.3.1) such that conditions (i)-(iii) in Lemma 3.3.4 are satisfied. Suppose that $\Gamma^* = \cap_{i=1}^N \partial f_i^{-1}(0) \neq \emptyset$, let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T_n x_n,\ n \geq 1, \tag{3.3.15}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$, satisfying the following conditions

C1: $\lim_{n\to\infty} \alpha_n = 0$,

C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ *converges strongly to an element of* Γ^*.

3.3.3 Numerical example

We now recall Example 2.1.26 for the numerical experiments of the results obtained in this section in comparison with the results of Takahashi and Shimoji [177]. Let $Y = \mathbb{R}^2$ be an $\mathbb{R}$-tree with the radial metric d_r, where $d_r(x, y) = d(x, y)$ if x and y are situated on a Euclidean straight line passing through the origin and $d_r(x, y) = d(x, \mathbf{0}) + d(y, \mathbf{0}) := \|x\| + \|y\|$ otherwise. We put $p = (1, 0)$ and $X = B \cup C$, where

$$B = \{(h, 0) : h \in [0, 1]\} \quad \text{and} \quad C = \{(h, k) : h + k = 1, h \in [0, 1)\}.$$

For each $[t\overrightarrow{ab}] \in X^*$, we obtain

$$[t\overrightarrow{ab}] = \begin{cases} \{s\overrightarrow{cd} : c, d \in B, s \in \mathbb{R},\ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a, b \in B, \\ \{s\overrightarrow{cd} : c, d \in C \cup \{\mathbf{0}\}, s \in \mathbb{R},\ t(\|b\| - \|a\|) = s(\|d\| - \|c\|)\} & a, b \in C \cup \{\mathbf{0}\}, \\ \{t\overrightarrow{ab}\} & a \in B,\ b \in C. \end{cases}$$

Now, define $A : X \to 2^{X^*}$ by

$$A(x) = \begin{cases} \{[\overrightarrow{\mathbf{0}p}]\} & x \in B, \\ \\ \{[\overrightarrow{\mathbf{0}p}], [\overrightarrow{\mathbf{0}x}]\} & x \in C. \end{cases}$$

Then, A is monotone and

$$J_\lambda^A(x) = \begin{cases} \{z = (h - \lambda, 0)\} & x = (h, 0) \in B, \\ \\ \{z = (h', k') \in C : (1 + \lambda)^2(h'^2 + k'^2) = h^2 + k^2\} & x = (h, k) \in C. \end{cases}$$

Let $a_n^{(i)} = a^{(i)} = \frac{1}{2}$, $b_n^{(i)} = b^{(i)} = \frac{1}{5}$ and $c_n^{(i)} = c^{(i)} = \frac{3}{10}$, $\forall n \geq 1$, $i = 1, 2, 3$. Then, (3.3.1) becomes:

$$\begin{cases} U_n^{(1)}x = \frac{1}{2}\left(J_\lambda^{(A)}x + x\right), \\ U_n^{(2)}x = \frac{1}{2}\left(J_\lambda^{(A)}U_n^{(1)}x + U_n^{(1)}x\right), \\ T_n x = \frac{1}{2}\left(J_\lambda^{(A)}U_n^{(2)}x + U_n^{(2)}x\right), \ \forall n \geq 1 \end{cases} \tag{3.3.16}$$

while Algorithm (1) of Takahashi and Shimoji [177] becomes:

$$\begin{cases} U_n^{(1)}x = \frac{1}{2}\left(J_\lambda^{(A)}x + x\right), \\ U_n^{(2)}x = \frac{1}{2}\left(J_\lambda^{(A)}U_n^{(1)}x + x\right), \\ T_n x = \frac{1}{2}\left(J_\lambda^{(A)}U_n^{(2)}x + x\right), \ \forall n \geq 1. \end{cases} \tag{3.3.17}$$

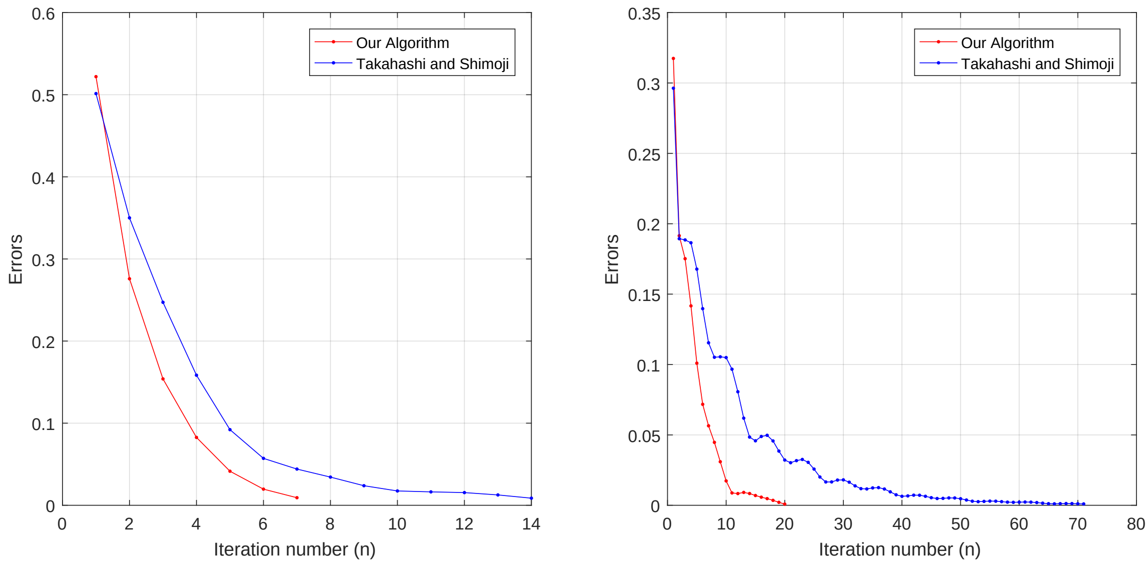

Figure 3.1: Errors vs Iteration number (n): Case I(a) (left); Case I(b) (right).

Now, take $\alpha_n = \frac{1}{2(n+1)}$ $\forall n \geq 1$, then α_n satisfies the conditions in Theorem 3.3.5. Hence, for $u, x_1 \in X$, Algorithm (3.3.6) becomes:

$$x_{n+1} = \frac{u}{2(n+1)} + \frac{2n+1}{2(n+1)} T_n x_n, \;\; n \geq 1. \tag{3.3.18}$$

Case I

(a) Take $x_1 = (-1, \; -0.5)^T$, $u = (-0.5, \; 0.1)^T$ and $\lambda = 0.00004$.

(b) Take $x_1 = (0.3, \; 0.06)^T$, $u = (0.2, \; 0.9)^T$ and $\lambda = 5$.

Case II

(a) Take $x_1 = (1, \; 0.5)^T$, $u = (-1, \; -0.5)^T$ and $\lambda = 0.1$.

(b) Take $x_1 = (1, \; 0.5)^T$, $u = (-1, \; -0.5)^T$ and $\lambda = 3$.

Case III

(a) Take $x_1 = (2, \; -0.5)^T$, $u = (-1, \; 0.5)^T$ and $\lambda = 1$.

(b) Take $x_1 = (-0.5, \; 2)^T$, $u = (0.5, \; -1)^T$ and $\lambda = 1$.

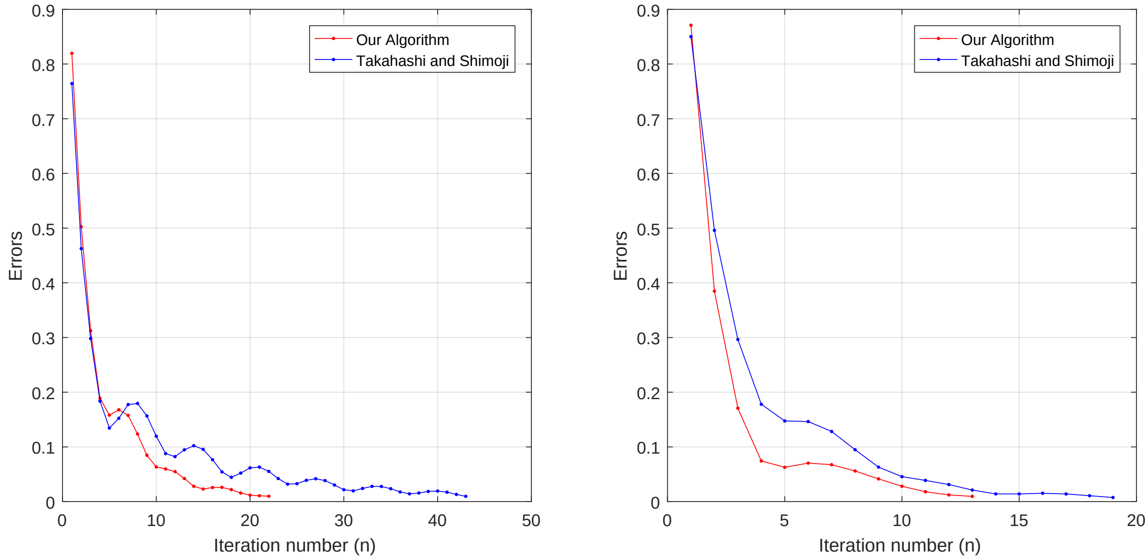

Figure 3.2: Errors vs Iteration number (n): Case II(a) (left); Case II(b) (right).

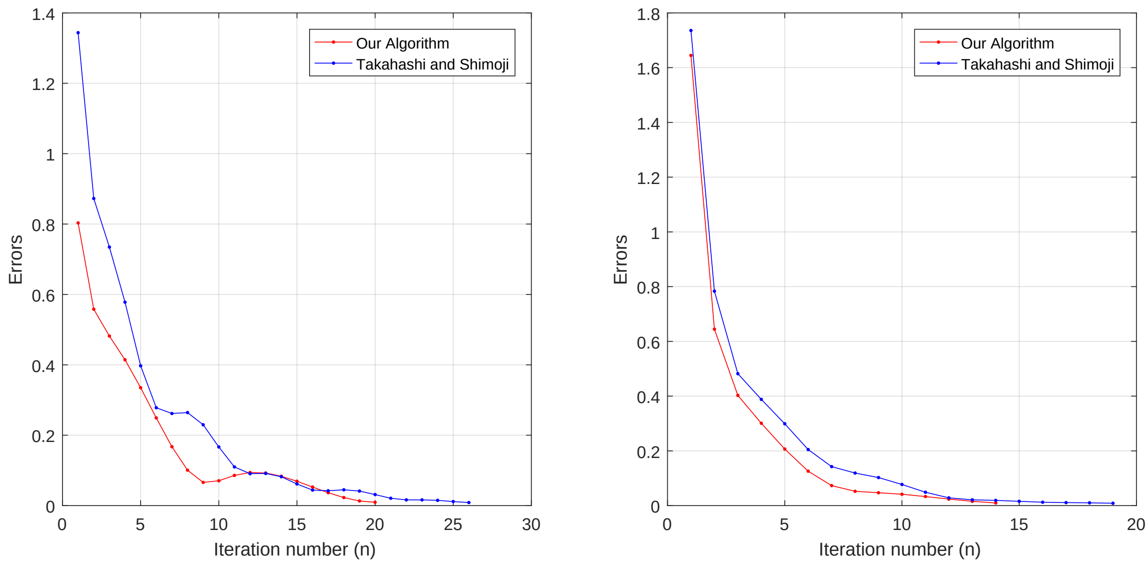

Figure 3.3: Errors vs Iteration number (n): Case III(a) (left); Case III(b) (right).

Remark 3.3.8. *By considering the 3 Cases above, we compare our algorithm with that of Takahashi and Shimoji [177]. The numerical results (see Figures 3.1, 3.2 and 3.3) show that our algorithm converges faster than that of Takahashi and Shimoji [177]. However, an improvement of the example studied in this section would be that of the "proximal-like algorithm", since it performs better than the one given in this section. In fact, the reason (among other possible reasons) our scheme performs better than that of Takahashi and Shimoji [177] is because it is more closer to the "proximal-like algorithm" compared to the one of Takahashi and Shimoji [177] (this observation was brought to our attention by the anonymous reviewer). Therefore, it is ideal to put forward the following important question for further study: Is it possible to find such an example (the "proximal-like algorithm") so as to obtain better numerical results than the one considered in this paper?*

3.4 Halpern iteration process for monotone inclusion and fixed point problems in Hadamard spaces

In this section, motivated by Remark 2.2.8 and the importance of nonexpansive mappings (see Remark 2.1.38), we propose and study a Halpern-type PPA for approximating a common solution of a finite family of monotone inclusion problems and fixed point problem for a nonexpansive mapping in an Hadamard space. We also applied our results to approximate a common solution of a finite family of MPs and fixed point problem for nonexpansive mapping in an Hadamard space. Numerical example of the result obtained in this section is also given to further show its applicability.

3.4.1 Main results

Theorem 3.4.1. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \dots, N$ be multivalued monotone operators that satisfy the range condition and T be a nonexpansive mapping on X. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} A_i^{-1}(0)\right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1(x_n), \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{3.4.1}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$, satisfying the following conditions

C1: $\lim\limits_{n\to\infty} \alpha_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ converges strongly to an element of Γ.

Proof. Let $p \in \Gamma$, $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$, where $\Phi_\lambda^0 = I$. Then from (3.4.1), we have

$$\begin{aligned}
d(x_{n+1}, p) &= d(\alpha_n u \oplus (1-\alpha_n)Ty_n, p) \\
&\leq \alpha_n d(u,p) + (1-\alpha_n) d(Ty_n, p) \\
&\leq \alpha_n d(u,p) + (1-\alpha_n) d(\Phi_\lambda^N x_n, p) \\
&\leq \alpha_n d(u,p) + (1-\alpha_n) d(\Phi_\lambda^{N-1} x_n, p) \\
&\vdots \\
&\leq \alpha_n d(u,p) + (1-\alpha_n) d(x_n, p) \\
&\leq \max\{d(u,p), d(x_n,p)\},
\end{aligned}$$

which implies by mathematical induction that

$$d(x_n, p) \leq \max\{d(u,p), d(x_1,p)\}, \ \forall n \geq 1. \tag{3.4.2}$$

Therefore, $\{d(x_n,p)\}$ is bounded. Consequently, $\{x_n\}, \{y_n\}$ and $\{Ty_n\}$ are all bounded. From (3.4.1) and Lemma 2.3.1, we have

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(\alpha_n u \oplus (1-\alpha_n)Ty_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1})Ty_{n-1}) \\
&\leq d(\alpha_n u \oplus (1-\alpha_n)Ty_n, \alpha_n u \oplus (1-\alpha_n)Ty_{n-1}) \\
&\quad + d(\alpha_n u \oplus (1-\alpha_n)Ty_{n-1}, \alpha_{n-1} u \oplus (1-\alpha_{n-1})Ty_{n-1}) \\
&\leq (1-\alpha_n) d(Ty_n, Ty_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Ty_{n-1}) \\
&\leq (1-\alpha_n) d(y_n, y_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Ty_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, Ty_{n-1}).
\end{aligned} \tag{3.4.3}$$

By applying condition C2 and C3 in (3.4.3), we have from Lemma 2.3.26 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.4.4}$$

Now, observe that

$$\begin{aligned}
d^2(x_n, p) - d^2(x_{n+1}, p) &\leq [d(x_n, x_{n+1}) + d(x_{n+1}, p)]^2 - d^2(x_{n+1}, p) \\
&= d^2(x_n, x_{n+1}) + 2d(x_n, x_{n+1}) d(x_{n+1}, p),
\end{aligned}$$

which implies from (3.4.4) that

$$\lim_{n\to\infty} \left[d^2(x_n, p) - d^2(x_{n+1}, p)\right] = 0. \tag{3.4.5}$$

From Lemma 2.3.1 and Lemma 3.2.4, we have

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2(\alpha_n u \oplus (1-\alpha_n)Ty_n, p) \\
&\leq \alpha_n d^2(u,p) + (1-\alpha_n) d^2(Ty_n, p) - \alpha_n(1-\alpha_n) d^2(u, Ty_n) \\
&\leq \alpha_n d^2(u,p) + (1-\alpha_n) d^2(Ty_n, p) \\
&\leq \alpha_n d^2(u,p) + (1-\alpha_n) d^2(y_n, p) \\
&\leq \alpha_n d^2(u,p) + d^2(y_n, p) \qquad (3.4.6) \\
&\leq \alpha_n d^2(u,p) + d^2(\Phi_\lambda^{N-1} x_n, p) - d^2(\Phi_\lambda^{N-1} x_n, y_n),
\end{aligned}$$

which implies

$$\begin{aligned} d^2(y_n, \Phi_\lambda^{N-1} x_n) &\leq \alpha_n d^2(u,p) + d^2(\Phi_\lambda^{N-1} x_n, p) - d^2(x_{n+1}, p) \\ &\leq \alpha_n d^2(u,p) + d^2(x_n, p) - d^2(x_{n+1}, p) \to 0, \text{ as } n \to \infty. \end{aligned}$$

That is

$$\lim_{n\to\infty} d^2(y_n, \Phi_\lambda^{N-1} x_n) = 0. \tag{3.4.7}$$

From (3.4.6), we have

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \alpha_n d^2(u,p) + d^2(y_n, p) \\ &\leq \alpha_n d^2(u,p) + d^2(\Phi_\lambda^{N-1} x_n, p) \\ &\leq \alpha_n d^2(u,p) + d^2(\Phi_\lambda^{N-2} x_n, p) - d^2(\Phi_\lambda^{N-2} x_n, \Phi_\lambda^{N-1} x_n), \end{aligned}$$

which implies

$$\begin{aligned} d^2(\Phi_\lambda^{N-1} x_n, \Phi_\lambda^{N-2} x_n) &\leq \alpha_n d^2(u,p) + d^2(\Phi_\lambda^{N-2} x_n, p) - d^2(x_{n+1}, p) \\ &\leq \alpha_n d^2(u,p) + d^2(x_n, p) - d^2(x_{n+1}, p) \to 0, \text{ as } n \to \infty. \end{aligned}$$

That is

$$\lim_{n\to\infty} d^2(\Phi_\lambda^{N-1} x_n, \Phi_\lambda^{N-2} x_n) = 0. \tag{3.4.8}$$

Continuing in the same manner, we have that

$$\lim_{n\to\infty} d^2(\Phi_\lambda^{N-2} x_n, \Phi_\lambda^{N-3} x_n) = \cdots = \lim_{n\to\infty} d^2(\Phi_\lambda^{2} x_n, \Phi_\lambda^{1} x_n) = \lim_{n\to\infty} d^2(\Phi_\lambda^{1} x_n, \Phi_\lambda^{0} x_n) = 0 \tag{3.4.9}$$

Thus,

$$d(y_n, x_n) \leq d^2(y_n, \Phi_\lambda^{N-1} x_n) + d^2(\Phi_\lambda^{N-1} x_n, \Phi_\lambda^{N-2} x_n) + \cdots + d^2(\Phi_\lambda^{2} x_n, \Phi_\lambda^{1} x_n) + d^2(\Phi_\lambda^{1} x_n, x_n),$$

which implies from (3.4.7), (3.4.8) and (3.4.9) that

$$\lim_{n\to\infty} d(y_n, x_n) = \lim_{n\to\infty} d(\Phi_\lambda^{N} x_n, x_n) = 0. \tag{3.4.10}$$

From (3.4.1), (3.4.4), (3.4.10), Lemma 2.3.1 (i) and condition C1, we obtain

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Ty_n) + d(Ty_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(\alpha_n u \oplus (1-\alpha_n) Ty_n, Ty_n) + d(y_n, x_n) \\ &\leq d(x_n, x_{n+1}) + \alpha_n d(u, Ty_n) + d(y_n, x_n) \to 0, \text{ as } n \to \infty. \end{aligned}$$

That is

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0. \tag{3.4.11}$$

From (3.4.10) and (3.4.11), we have

$$\begin{aligned} d(x_n, Ty_n) &\leq d(x_n, Tx_n) + d(Tx_n, Ty_n) \\ &\leq d(x_n, Tx_n) + d(x_n, y_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

That is

$$\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, T\Phi_\lambda^N x_n) = 0. \tag{3.4.12}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = \bar{v}$ for some $\bar{v} \in X$. Also, since $T \circ \Phi_\lambda^N$ is the composition of nonexpansive mappings, it implies that $T \circ \Phi_\lambda^N$ is nonexpansive. Thus, it follows from (3.4.12) and Lemma 2.3.12 that $\bar{v} \in F(T \circ \Phi_\lambda^N)$. Hence, by Lemma 3.2.6, we obtain that $\bar{v} \in \Gamma$.

Again, since $\{x_{n_k}\}$ $\triangle-$converges to $\bar{v} \in \Gamma$, it follows from Lemma 2.3.9 that there exists $z \in \Gamma$ such that $\{x_n\}$ $\triangle-$converges to z. Thus, we obtain from Lemma 2.3.10 that

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \leq 0. \tag{3.4.13}$$

By using the quasilinearization properties, we obtain

$$\begin{aligned} \langle \overrightarrow{uz}, \overrightarrow{Ty_n z} \rangle &= \langle \overrightarrow{uz}, \overrightarrow{Ty_n x_n} \rangle + \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle \\ &\leq d(u, z) d(Ty_n, x_n) + \langle \overrightarrow{uz}, \overrightarrow{x_n z} \rangle, \end{aligned}$$

which implies from (3.4.12) and (3.4.13) that

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{Ty_n z} \rangle \leq 0. \tag{3.4.14}$$

By condition C1 and inequality (3.4.14), we get

$$\limsup_{n\to\infty} \Big(\alpha_n d^2(u, z) + 2(1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{Ty_n z} \rangle \Big) \leq 0. \tag{3.4.15}$$

Next, we show that $\{x_n\}$ converges strongly to z. From (3.4.1) and Lemma 2.3.1 (iii), we obtain

$$d^2(x_{n+1}, z) \leq \alpha_n^2 d^2(u, z) + (1-\alpha_n)^2 d^2(Ty_n, z) + 2\alpha_n(1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{Ty_n\ z} \rangle,$$

which implies

$$d^2(x_{n+1}, z) \leq (1-\alpha_n) d^2(x_n, z) + \alpha_n \Big(\alpha_n d^2(u, z) + 2(1-\alpha_n) \langle \overrightarrow{uz}, \overrightarrow{Ty_n\ z} \rangle \Big).$$

It follows from (3.4.15) and Lemma 2.3.26 that $\{x_n\}$ converges strongly to z. □

By setting $N = 1$ in Theorem 3.4.1, we obtain the following result.

Corollary 3.4.2. *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone mapping that satisfies the range condition and T be a nonexpansive mapping on X. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_\lambda(x_n), \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) Ty_n, \ n \geq 1, \end{cases} \tag{3.4.16}$$

where $\lambda \in (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$, satisfying the following conditions

C1: $\lim_{n\to\infty} \alpha_n = 0$,

C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ *converges strongly to an element of* Γ.

By setting $T = I$ (I is the identity mapping on X) in Theorem 3.4.1, we obtain the following result.

Corollary 3.4.3. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ *be multivalued monotone mappings that satisfy the range condition. Suppose that* $\cap_{i=1}^N A_i^{-1}(0) \neq \emptyset$. *Let* $u, x_1 \in X$ *be arbitrary and the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1(x_n), \; n \geq 1, \tag{3.4.17}$$

where $\lambda \in (0, \infty)$ *and* $\{\alpha_n\} \subset [0, 1]$, *satisfying the following conditions*

C1: $\lim_{n\to\infty} \alpha_n = 0$,

C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ *converges strongly to an element of* $\cap_{i=1}^N A_i^{-1}(0)$.

3.4.2 Application

By similar discussion as in Section 3.3.2, we can apply Theorem 3.4.1 to establish the following theorem for approximating a common solution of a finite family of MPs and fixed point problem for nonexpansive mapping.

Theorem 3.4.4. *Let* X *be an Hadamard and* X^* *be its dual space. Let* $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ *be a finite family of proper, lower semicontinuous and convex function and let* T *be a nonexpansive mapping on* X. *Suppose that* $\Gamma^* := F(T) \cap \left(\cap_{i=1}^N \partial f_i^{-1}(0)\right) \neq \emptyset$. *Let* $u, x_1 \in X$ *be arbitrary and the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} y_n = J_\lambda^{\partial f_N} \circ J_\lambda^{\partial f_{N-1}} \circ \cdots \circ J_\lambda^{\partial f_2} \circ J_\lambda^{\partial f_1}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \; n \geq 0, \end{cases} \tag{3.4.18}$$

where $\lambda \in (0, \infty)$ *and* $\{\alpha_n\} \subset [0, 1]$ *satisfying the following conditions*

C1: $\lim_{n\to\infty} \alpha_n = 0$,

C2: $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$.

Then $\{x_n\}$ *converges strongly to an element of* Γ^*.

3.4.3 Numerical example

We give a numerical example in $(\mathbb{R}^2, ||.||_2)$ (where $\mathbb{R}^2$ is the Euclidean plane) to support our main result. Let $N = 2$ in Theorem 3.4.1, then for $i = 1$, we define $A_1 : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$A_1(x) = (x_1 + x_2,\ x_2 - x_1).$$

Then A_1 is a monotone operator.

Recall that $[t\overrightarrow{ab}] \equiv t(b - a)$, for all $t \in \mathbb{R}$ and $a, b \in \mathbb{R}^2$ (see [95]). Using this, we have for each $x \in \mathbb{R}^2$ that

$$\begin{aligned} J_\lambda^1(x) = z &\iff \frac{1}{\lambda}(x - z) = A_1 z \\ &\iff x = (I + \lambda A_1)z \\ &\iff z = (I + \lambda A_1)^{-1}x. \end{aligned}$$

Hence, we compute the resolvent of A_1 as follows:

$$\begin{aligned} J_\lambda^1(x) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda & \lambda \\ -\lambda & \lambda \end{bmatrix} \right)^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} 1+\lambda & \lambda \\ -\lambda & 1+\lambda \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \frac{1}{1 + 2\lambda + 2\lambda^2} \begin{bmatrix} 1+\lambda & -\lambda \\ \lambda & 1+\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \left(\frac{(1+\lambda)x_1 - \lambda x_2}{1 + 2\lambda + 2\lambda^2}, \frac{\lambda x_1 + (1+\lambda)x_2}{1 + 2\lambda + 2\lambda^2} \right). \end{aligned}$$

Thus,

$$J_\lambda^1(x) = \left(\frac{(1+\lambda)x_1 - \lambda x_2}{1 + 2\lambda + 2\lambda^2}, \frac{\lambda x_1 + (1+\lambda)x_2}{1 + 2\lambda + 2\lambda^2} \right).$$

Now, for $i = 2$, let $A_2 : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$A_2(x) = (x_2,\ -x_1).$$

So that by the same argument as in above, we obtain

$$J_\lambda^2(x) = \left(\frac{x_1 - \lambda x_2}{1+\lambda^2}, \; \frac{x_2 + \lambda x_1}{1+\lambda^2} \right).$$

Thus for $i = 1, 2$, we obtain

$$J_\lambda^2(J_\lambda^1 x) = \left(\frac{(1+\lambda-\lambda^2)x_1 - (2\lambda+\lambda^2)x_2}{(1+\lambda^2)(1+2\lambda+2\lambda^2)}, \; \frac{(2\lambda+\lambda^2)x_1 + (1+\lambda-\lambda^2)x_2}{(1+\lambda^2)(1+2\lambda+2\lambda^2)} \right).$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (-x_2, x_1)$. Then T is a nonexpansive mapping. Take $\alpha_n = \frac{1}{n+1}$, then $\{\alpha_n\}$ satisfies the conditions in Theorem 3.4.1.

Hence, for $u, x_1 \in \mathbb{R}^2$, our Algorithm (3.4.1) becomes:

$$\begin{cases} y_n = J_\lambda^2(J_\lambda^1 x_n), \\ x_{n+1} = \frac{u}{n+1} + \frac{n}{2(n+1)} y_n, \;\; n \geq 1. \end{cases} \tag{3.4.19}$$

Case I

(a) Take $x_1 = (0.5, \; 0.25)^T, \; u = (1, \; 0.5)^T$ and $\lambda = 0.001$.

(b) Take $x_1 = (0.5, \; 0.25)^T, \; u = (1, \; 0.5)^T$ and $\lambda = 0.000002$.

Case II

(a) Take $x_1 = (1, \; 0.5)^T, \; u = (-1, \; 0.5)^T$ and $\lambda = 0.0002$.

(b) Take $x_1 = (0.1, \; 0.03)^T, \; u = (0.3, \; 0.1)^T$ and $\lambda = 0.0002$.

Case III

(a) Take $x_1 = (-1, \; -0.5)^T, \; u = (-0.5, \; 0.1)^T$ and $\lambda = 0.00004$.

(b) Take $x_1 = (0.3, \; 0.06)^T, \; u = (0.2, \; 0.9)^T$ and $\lambda = 0.000009$.

Mathlab version R2014a is used to obtain the graphs of errors against number of iterations, execution time against accuracy and number of iterations against accuracy.

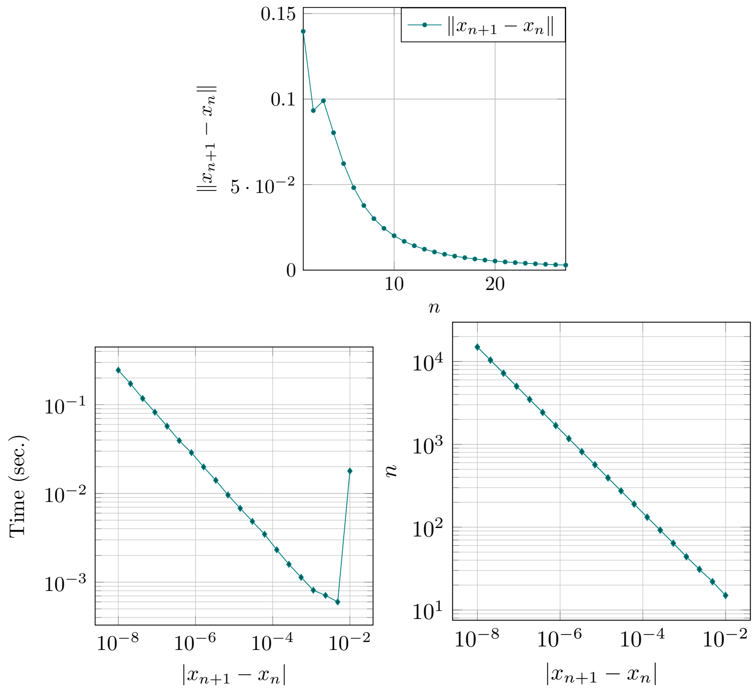

Figure 3.4: Case I(a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

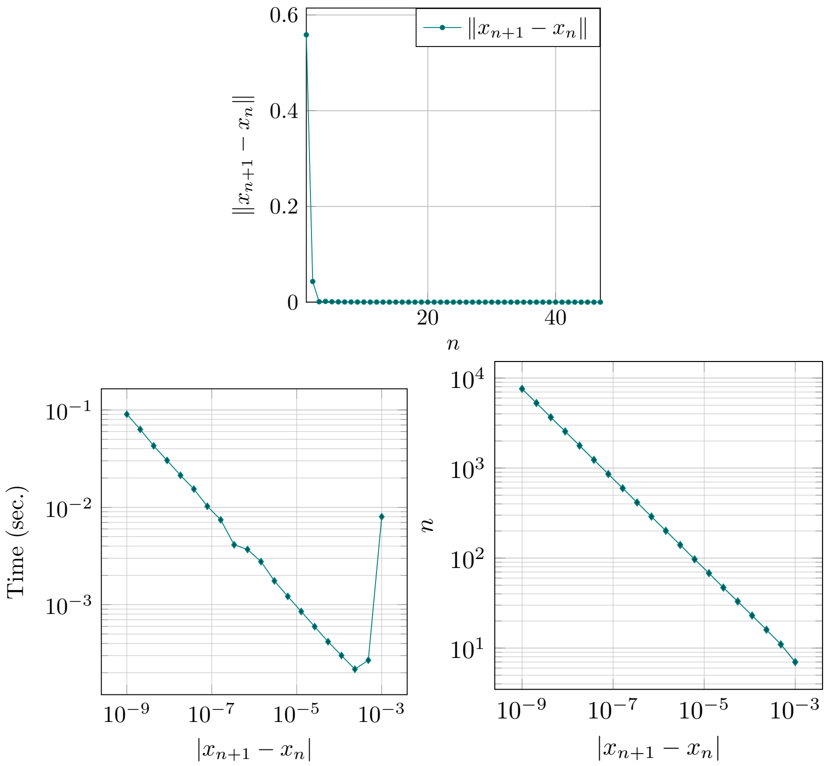

Figure 3.5: Case I(b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

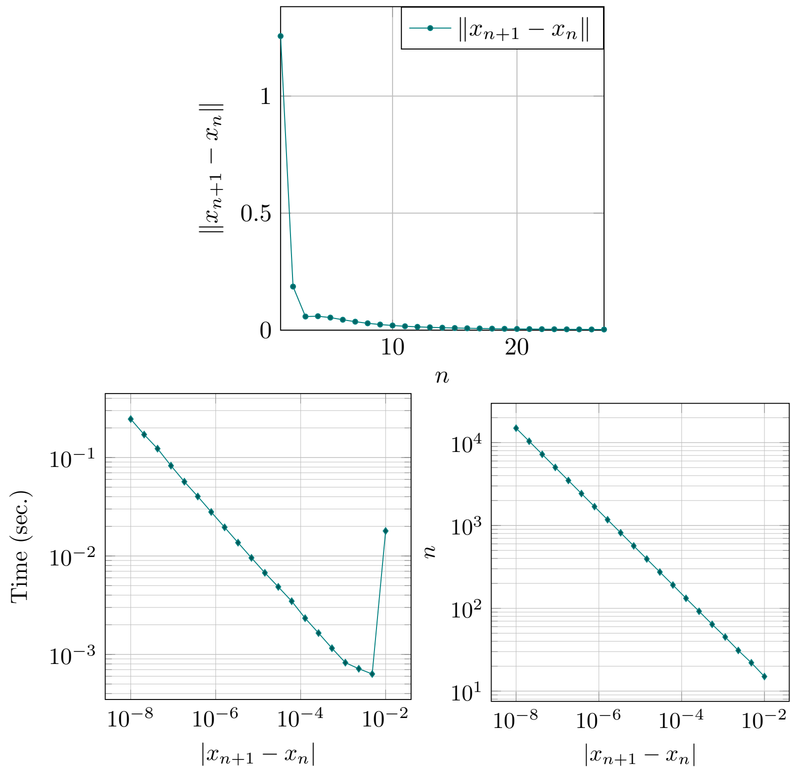

Figure 3.6: Case II(a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

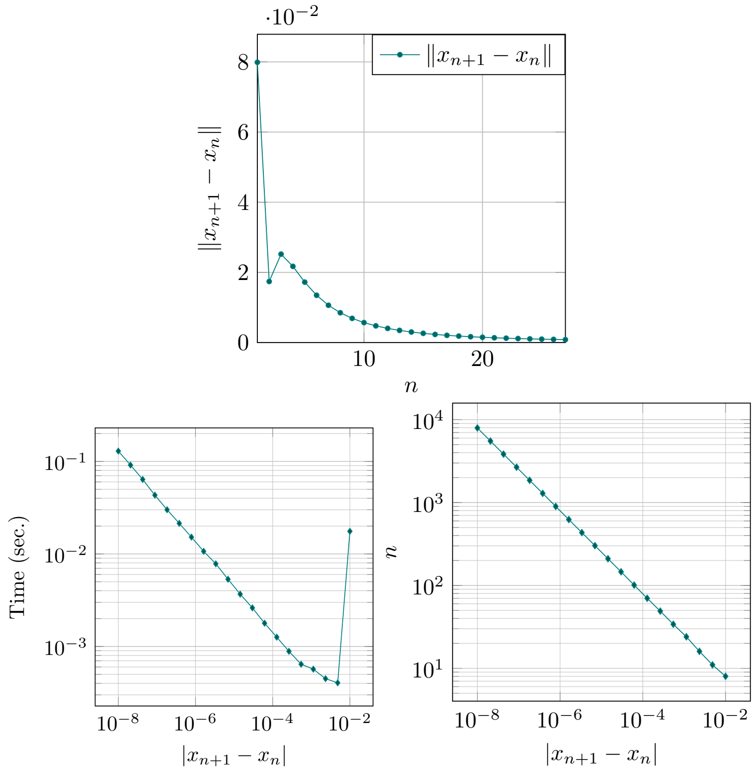

Figure 3.7: Case II(b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

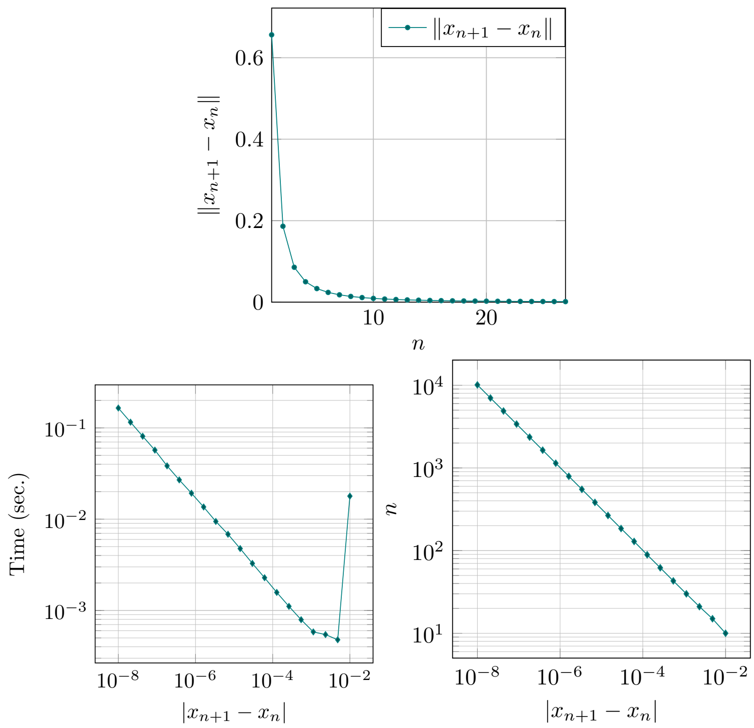

Figure 3.8: Case III(a): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

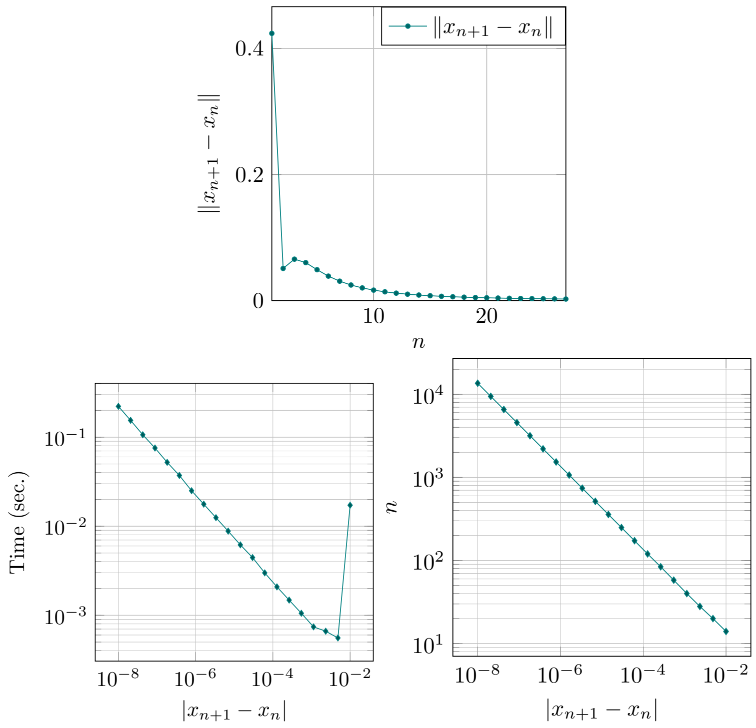

Figure 3.9: Case III(b): errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

3.5 Viscosity iterative techniques for monotone inclusion and fixed point problems in Hadamard spaces

Like the Halpern algorithm, the viscosity algorithm also converges strongly. However, one advantage of the viscosity algorithm over the Halpern iteration process is that it also converges strongly to a unique solution of the VIP (1.2.10) associated with a contraction mapping. Furthermore, the viscosity iteration process has higher rate of convergence than the Halpern iteration process (see [138, Remark 3.7 (iii), (iv)]). Motivated by this, we study in this section, some viscosity-type proximal point algorithms which comprise of a nonexpansive mapping and a finite sum of resolvents of monotone operators, and prove their strong convergence to a common solution of a finite family of MIPs, which is also a fixed point of a nonexpansive mapping and a unique solution of some VIPs (associated with contraction mappings) in an Hadamard space. We apply our results to solve CFPs (1.2.11) and VIPs associated with a nonexpansive mapping.

3.5.1 Main results

Theorem 3.5.1. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the* *range condition. Let T be a nonexpansive mapping on X and h be a contraction mapping on X with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \beta_2 J_{\lambda_n}^{A_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ x_n = \alpha_n h(x_n) \oplus (1-\alpha_n) T y_n, \quad n \geq 1, \end{cases} \tag{3.5.1}$$

where $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ and $\{\alpha_n\}$ is in $(0,1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle \geq 0, \quad \forall u \in \Gamma. \tag{3.5.2}$$

Proof. **Step 1:** We first show that (3.5.1) is well defined. Let $S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \beta_2 J_{\lambda_n}^{A_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n$, then by Lemma 3.2.7, we have that S_{λ_n} is nonexpansive for all $n \geq 1$. Now, define the mapping $T_n^h : X \to X$ as follows:

$$T_n^h x = \alpha_n h(x) \oplus (1-\alpha_n) T S_{\lambda_n} x.$$

Since T is nonexpansive, we obtain from Lemma 2.3.1 (iv) that

$$\begin{aligned} d(T_n^h x, T_n^h y) &\leq \alpha_n d(h(x), h(y)) + (1-\alpha_n) d(TS_n x, TS_n y) \\ &\leq \tau \alpha_n d(x,y) + (1-\alpha_n) d(S_n x, S_n y) \\ &\leq (\tau \alpha_n + (1-\alpha_n))\, d(x,y). \end{aligned}$$

Since $\tau \in (0,1)$, we have that $0 < (\tau\alpha_n + (1-\alpha_n)) < 1$. Hence, T_n^h is a contraction for each $n \geq 1$. Therefore, by Banach contraction mapping principle, there exists a unique fixed point x_n of T_n^h for each $n \geq 1$. Thus, (3.5.1) is well defined.

Step 2: Next, we show that $\{x_n\}$ is bounded. Let $v \in \Gamma$, by (3.5.1) and Lemma 2.3.1 (i), we obtain

$$\begin{aligned} d(x_n, v) &\leq \alpha_n d(h(x_n), v) + (1-\alpha_n) d(Ty_n, v) \\ &\leq \alpha_n \tau d(x_n, v) + \alpha_n d(h(v), v) + (1-\alpha_n) d(y_n, v) \\ &\leq \Big(1 - \alpha_n(1-\tau)\Big) d(x_n, v) + \alpha_n d(h(v), v), \end{aligned}$$

which implies that

$$d(x_n, v) \leq \frac{d(h(v), v)}{1-\tau}. \tag{3.5.3}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ $\{Ty_n\}$ and $\{h(x_n)\}$ are all bounded.

Step 3: We now show that $\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, TS_{\lambda_n}x_n) = 0$ and $\bar{z} \in \Gamma$.

From (3.5.1) and Lemma 2.3.1, we obtain that

$$\begin{aligned} d(x_n, Ty_n) &= d(\alpha_n h(x_n) \oplus (1-\alpha_n)Ty_n, Ty_n) \\ &\leq \alpha_n d(h(x_n), Ty_n). \end{aligned} \tag{3.5.4}$$

Since $\{h(x_n)\}$ and $\{Ty_n\}$ are bounded, we obtain from condition (i) and (3.5.4) that

$$\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, TS_{\lambda_n}x_n) = 0. \tag{3.5.5}$$

Now, by the boundedness of $\{x_n\}$ and the completeness of X, we obtain from Lemma 2.3.5 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = \bar{v}$. Again, since $T \circ S_{\lambda_n}$ is nonexpansive (and every nonexpansive mapping is demiclosed), it follows from (3.5.5), Lemma 3.2.7 and Theorem 2.1.32 (i) that $\bar{v} \in F(T \circ S_{\lambda_n}) \subseteq \cap_{i=1}^{N} F(J_\lambda^{A_i}) \cap F(T) = \Gamma$.

Step 4: We show that $\{x_n\}$ converges strongly to $\bar{z}$. Since $\{x_{n_k}\}$ $\triangle-$converges to $\bar{v} \in \Gamma$ and by (3.5.3), $\lim_{n\to\infty} d(x_n, v)$ exists for all $v \in \Gamma$, we obtain from Lemma 2.3.9 that there exists $\bar{z} \in \Gamma$ such that $\{x_n\}$ $\triangle-$converges to $\bar{z}$. Thus, by Lemma 2.3.10, we obtain that

$$\limsup_{n\to\infty} \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle \leq 0. \tag{3.5.6}$$

Also, by Lemma 2.3.1 (iii) and (3.5.1), we have

$$\begin{aligned} d^2(x_n, \bar{z}) &\leq \alpha_n^2 d^2(h(x_n), \bar{z}) + (1-\alpha_n) d^2(Ty_n, \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{Ty_n\bar{z}} \rangle \\ &\leq \alpha_n^2 d^2(h(x_n), \bar{z}) + (1-\alpha_n) d^2(x_n, \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{Ty_n x_n} \rangle + \langle \overrightarrow{h(x_n)h(\bar{z})}, \overrightarrow{x_n\bar{z}} \rangle \\ &\quad + \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle] \\ &\leq \alpha_n^2 d^2(h(x_n), \bar{z}) + (1-\alpha_n) d^2(x_n, \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{Ty_n x_n} \rangle + \tau d^2(x_n, \bar{z}) + \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle] \\ &\leq \Big[(1-\alpha_n) + 2\tau\alpha_n(1-\alpha_n)\Big] d^2(x_n, \bar{z}) \\ &\quad + \alpha_n\Big[\alpha_n d^2(h(x_n), \bar{z}) + 2(1-\alpha_n) d(Ty_n, x_n)\Big] d(h(x_n), \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle. \end{aligned} \tag{3.5.7}$$

Therefore

$$d^2(x_n, \bar{z}) \leq \frac{[\alpha_n d^2(h(x_n), \bar{z}) + 2(1-\alpha_n) d(Ty_n, x_n)] d(h(x_n), \bar{z})}{[1 - 2\tau(1-\alpha_n)]} + \frac{2(1-\alpha_n)\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle}{[1 - 2\tau(1-\alpha_n)]},$$

which implies from condition (i), (3.5.5) and (3.5.6) that

$$\lim_{n\to\infty} d^2(x_n, \bar{z}) = 0.$$

Therefore, $\lim_{n\to\infty} x_n = \bar{z}$.

Step 5: Finally, we show that $\bar{z}$ is a solution of (3.5.2). From Lemma 2.3.1 (ii) and (3.5.1), we obtain for all $u \in \Gamma$ that

$$\begin{aligned} d^2(x_m, u) &\leq \alpha_m d^2(h(x_m), u) + (1-\alpha_m)d^2(Ty_m, u) \\ &\quad -\alpha_m(1-\alpha_m)d^2(h(x_m), Ty_m) \\ &\leq \alpha_m d^2(h(x_m), u) + (1-\alpha_m)d(x_m, u) \\ &\quad -\alpha_m(1-\alpha_m)d^2(h(x_m), Ty_m), \end{aligned}$$

which implies

$$d^2(x_m, u) \leq d^2(h(x_m), u) - (1-\alpha_m)d^2(h(x_m), Ty_m).$$

Thus, taking limit as $m \to \infty$, we obtain

$$d^2(\bar{z}, u) \leq d^2(h(\bar{z}), u) - d^2(h(\bar{z}), \bar{z}).$$

Hence,

$$\langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle = \frac{1}{2}\Big(d^2(h(\bar{z}), u) - d^2(\bar{z}, u) - d^2(h(\bar{z}), \bar{z}) \Big) \geq 0, \ \forall u \in \Gamma.$$

Therefore, we have that $\bar{z}$ solves the variational inequality (3.5.2).
Now, assume that $\{x_{n_k}\}$ $\triangle-$converges to u. Then, by the same argument, we obtain that $u \in \Gamma$ solves the variational inequality (3.5.2). That is,

$$\langle \overrightarrow{uh(u)}, \overrightarrow{u\bar{z}} \rangle \leq 0. \quad \text{Also} \quad \langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{\bar{z}u} \rangle \leq 0.$$

Now, adding both, we get

$$\begin{aligned} 0 &\geq \langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{\bar{z}u} \rangle - \langle \overrightarrow{uh(u)}, \overrightarrow{\bar{z}u} \rangle \\ &= \langle \overrightarrow{\bar{z}h(u)}, \overrightarrow{\bar{z}u} \rangle + \langle \overrightarrow{h(u)h(\bar{z})}, \overrightarrow{\bar{z}u} \rangle \\ &\quad -\langle \overrightarrow{u\bar{z}}, \overrightarrow{\bar{z}u} \rangle - \langle \overrightarrow{\bar{z}h(u)}, \overrightarrow{\bar{z}u} \rangle \\ &= \langle \overrightarrow{\bar{z}u}, \overrightarrow{\bar{z}u} \rangle - \langle \overrightarrow{h(u)h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle \\ &\geq \langle \overrightarrow{\bar{z}u}, \overrightarrow{\bar{z}u} \rangle - d(h(u)h(\bar{z}))d(u, \bar{z}) \\ &\geq d^2(\bar{z}, u) - \tau d^2(u, \bar{z}) \\ &= (1-\tau)d^2(\bar{z}, u). \end{aligned}$$

which implies that $d(\bar{z}, u) = 0$. Hence, $\bar{z} = u$. Therefore, $\{x_n\}$ converges strongly to $\bar{z}$, which is a solution of the variational inequality (3.5.2). $\square$

By setting $T \equiv I$ (where I is the identity mapping on X) and $h(x) = c$ for arbitrary but fixed $c \in X$ and $\forall x \in X$, we obtain the following corollary which is of Halpern type.

Corollary 3.5.2. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the* *range condition. Suppose that* $\Gamma := \cap_{i=1}^N A_i^{-1}(0) \neq \emptyset$ *and for arbitrary* $c,\ x_1 \in X$*, the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \beta_2 J_{\lambda_n}^{A_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ x_n = \alpha_n c \oplus (1-\alpha_n) y_n, \quad n \geq 1, \end{cases} \tag{3.5.8}$$

where $0 < \lambda \leq \lambda_n\ \forall n \geq 1$ *and* $\{\alpha_n\}$ *is in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,*

(ii) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$*.*

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$*.*

By setting $N = 1$ in Theorem 3.5.1, we obtain the following result.

Corollary 3.5.3. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $A : X \to 2^{X^*}$ *be a multivalued monotone operator that satisfies the range condition. Let* T *be a nonexpansive mapping on* X *and* h *be a contraction mapping on* X *with coefficient* $\tau \in (0,1)$*. Suppose that* $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ *and for arbitrary* $x_1 \in X$*, the sequence* $\{x_n\}$ *is generated by*

$$x_n = \alpha_n h(x_n) \oplus (1-\alpha_n) T\left(\beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A} x_n\right), \quad n \geq 1, \tag{3.5.9}$$

where $0 < \lambda \leq \lambda_n\ \forall n \geq 1$ *and* $\{\alpha_n\}$ *is in* $(0,1)$ *satisfying the following conditions:*

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*,*

(ii) $\beta_i \in (0,1),\ i = 0,1$ *with* $\beta_0 + \beta_1 = 1$*.*

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$ *which solves the variational inequality*

$$\langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle \geq 0, \quad \forall u \in \Gamma. \tag{3.5.10}$$

We now present the second theorem for this section.

Theorem 3.5.4. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $A_i : X \to 2^{X^*},\ i = 1,2,\ldots,N$ *be a finite family of multivalued monotone operators that satisfy the range condition. Let* T *be a nonexpansive mapping on* X *and* h *be a contraction mapping on* X *with coefficient* $\tau \in (0,1)$*. Suppose that* $\Gamma := F(T) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ *and for arbitrary* $x_1 \in X$*, the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n, \quad n \geq 1, \end{cases} \tag{3.5.11}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$*, and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n\ \forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,

(iv) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

Proof. **Step 1:** We show that $\{x_n\}$ is bounded. Let $u \in \Gamma$ and set $S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n$, then from (3.5.11), Lemma 2.3.1 (i) and Lemma 3.2.7, we obtain that

$$\begin{aligned}
d(x_{n+1}, u) &\leq \beta_n d(x_n, u) + (1-\beta_n) d(w_n, u) \\
&\leq \beta_n d(x_n, u) + (1-\beta_n)\left[\frac{\alpha_n}{1-\beta_n} d(h(x_n), u) + \frac{\gamma_n}{1-\beta_n} d(Ty_n, u)\right] \\
&\leq \beta_n d(x_n, u) + (1-\beta_n)\left[\frac{\alpha_n}{1-\beta_n} \tau d(x_n, u) + \frac{\alpha_n}{1-\beta_n} d(h(u), u) + \frac{\gamma_n}{1-\beta_n} d(Ty_n, u)\right] \\
&\leq (\beta_n + \tau\alpha_n) d(x_n, u) + \gamma_n d(S_{\lambda_n} x_n, u) + \alpha_n d(h(u), u) \\
&= (1 - \alpha_n(1-\tau)) d(x_n, u) + \alpha_n d(h(u), u) \\
&\leq \max\left\{ d(x_n, u) + \frac{d(h(u), u)}{1-\tau} \right\} \\
&\ \vdots \\
&\leq \max\left\{ d(x_1, u) + \frac{d(h(u), u)}{1-\tau} \right\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{h(x_n)\}$ and $\{T(y_n)\}$ are all bounded.

Step 2: Next, we show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Now, from (3.5.11), Lemma 2.3.1 (iv),(vi) and the nonexpansivity of T, we obtain that

$$\begin{aligned}
d(w_{n+1}, w_n) &= d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} h(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}} Ty_{n+1}, \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} Ty_n\Big) \\
&\leq d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} h(x_{n+1}) \oplus \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) Ty_{n+1}, \\
&\quad \frac{\alpha_{n+1}}{1-\beta_{n+1}} h(x_n) \oplus \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) Ty_n\Big) \\
&+ d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} h(x_n) \oplus \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) Ty_n, \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \big(1 - \frac{\alpha_n}{1-\beta_n}\big) Ty_n\Big) \\
&\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \tau d(x_{n+1}, x_n) + \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) d(y_{n+1}, y_n) \\
&\quad + \Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big| d(h(x_n), Ty_n)
\end{aligned} \tag{3.5.12}$$

Without loss of generality, we may assume that $0 < \lambda_{n+1} \leq \lambda_n \ \forall n \geq 1$. Thus, from (3.5.11), condition (iv), Lemma 3.2.1 and Lemma 3.2.4 (ii), we obtain

$$\begin{aligned}
d(y_{n+1}, y_n) &= d\big(\beta_0 x_{n+1} \oplus \cdots \oplus \beta_N J_{\lambda_{n+1}}^{A_N} x_{n+1},\ \beta_0 x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n\big) \\
&\leq \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_i d(J_{\lambda_{n+1}}^{A_i} x_{n+1}, J_{\lambda_n}^{A_i} x_n) \\
&\leq \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_i d(J_{\lambda_{n+1}}^{A_i} x_{n+1}, J_{\lambda_{n+1}}^{A_i} x_n) + \sum_{i=1}^{N} \beta_i d(J_{\lambda_{n+1}}^{A_i} x_n, J_{\lambda_n}^{A_i} x_n) \\
&\leq d(x_{n+1}, x_n) + \Big(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\Big) M, \qquad (3.5.13)
\end{aligned}$$

where $M := \sup_{n \geq 1}\big\{ \sum_{i=1}^{N} \beta_i d(J_{\lambda_n}^{A_i} x_n, x_n)\big\}$. Substituting (3.5.13) into (3.5.12), we obtain that

$$\begin{aligned}
d(w_{n+1}, w_n) &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \tau d(x_{n+1}, x_n) + \big(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\big) d(x_{n+1}, x_n) \\
&\quad + \Big(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\Big)\Big(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\Big) M \\
&+ \ \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| d(h(x_n), Ty_n) \\
&= \ \big[1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}(1 - \tau)\big] d(x_{n+1}, x_n) + \Big(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\Big)\Big(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\Big) M \\
&+ \ \Big|\frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n}\Big| d(h(x_n), Ty_n).
\end{aligned}$$

Since $\lim_{n\to\infty} \alpha_n = 0$, $\quad \lim_{n\to\infty} \lambda_n = \lambda$ and $\{h(x_n)\}$, $\{Ty_n\}$ are bounded, we obtain that

$$\limsup_{n\to\infty} \big(d(w_{n+1}, w_n) - d(x_{n+1}, x_n)\big) \leq 0.$$

Thus, by Lemma 2.3.17 and condition (ii), we obtain that

$$\lim_{n\to\infty} d(w_n, x_n) = 0. \qquad (3.5.14)$$

Hence, by Lemma 2.3.1, we obtain that

$$d(x_{n+1}, x_n) \leq (1 - \beta_n) d(w_n, x_n) \to 0, \qquad \text{as} \quad n \to \infty. \qquad (3.5.15)$$

Step 3: We now show that $\lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0 = \lim_{n\to\infty} d(w_n, T(S_{\lambda_n})w_n)$. Observe from Remark 2.3.1 that (3.5.11) can be rewritten as

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ x_{n+1} = \alpha_n h(x_n) \oplus (1 - \alpha_n)\Big(\frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big), \quad \geq 1. \end{cases} \qquad (3.5.16)$$

Thus, by Lemma 2.3.1, we obtain that

$$d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1 - \alpha_n)}\Big) \leq \alpha_n d\Big(h(x_n), \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1 - \alpha_n)}\Big) \to 0, \quad \text{as} \ \ n \to \infty. \quad (3.5.17)$$

Also, from (2.1.1), we obtain

$$d\Big(x_n, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) = \frac{\gamma_n}{1-\alpha_n} d(x_n, Ty_n),$$

which implies from (3.5.15) and (3.5.17) that

$$\frac{\gamma_n}{1-\alpha_n} d(x_n, Ty_n) \leq d(x_n, x_{n+1}) + d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) \to 0, \quad \text{as} \quad n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0. \tag{3.5.18}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then by Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim\limits_{k\to\infty} x_{n_k} = \bar{u}$. Again, by the nonexpansivity of $T \circ S_{\lambda_n}$, we obtain from (3.5.18), condition (iii), Lemma 3.2.7 and Theorem 2.1.32 (i) that $\bar{u} \in F(T \circ S_{\lambda_n}) \subseteq \cap_{i=1}^N F(J_\lambda^{A_i}) \cap F(T) = \Gamma$.

Also, by (3.5.14) and (3.5.18), we obtain

$$\begin{aligned} d(w_n, T(S_{\lambda_n})w_n) &\leq d(w_n, x_n) + d(x_n, T(S_{\lambda_n})x_n) + d(T(S_{\lambda_n})x_n, T(S_{\lambda_n})w_n) \\ &\leq 2d(w_n, x_n) + d(x_n, T(S_{\lambda_n}x_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.5.19}$$

Step 4: Next, we show that $\limsup\limits_{n\to\infty} \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle \leq 0$.

If we set $T_m^h x := \beta_m x \oplus (1-\beta_m)w$, where $w = \frac{\alpha_m}{(1-\beta_m)} h(x) \oplus \frac{\gamma_m}{(1-\beta_m)} T(S_{\lambda_m})x$, then by following the same method of proof as in the proof of Theorem 3.5.1, we get that T_m^h is a contraction for each $m \geq 1$. Thus, there exists a unique fixed point z_m of T_m^h $\forall m \geq 1$. That is,
$z_m = \beta_m z_m \oplus (1-\beta_m)w_m$, where $w_m = \frac{\alpha_m}{(1-\beta_m)} h(z_m) \oplus \frac{\gamma_m}{(1-\beta_m)} T(S_{\lambda_m})z_m$. Furthermore, it follows from Theorem 3.5.1 that $\lim\limits_{m\to\infty} z_m = \bar{z} \in \Gamma$. Thus, we obtain that

$$\begin{aligned} d(z_m, w_n) &= d(\beta_m z_m \oplus (1-\beta_m)w_m, w_n) \\ &\leq \beta_m d(z_m, w_n) + (1-\beta_m) d(w_m, w_n), \end{aligned}$$

which implies that

$$d(z_m, w_n) \leq d(w_m, w_n). \tag{3.5.20}$$

From (3.5.20) and Lemma 2.3.1(v), we obtain that

$$\begin{aligned}
d^2(w_m, w_n) &= \langle \overrightarrow{w_m w_n}, \overrightarrow{w_m w_n} \rangle \\
&= \langle \overrightarrow{w_m T(S_{\lambda_m}) z_m}, \overrightarrow{w_m w_n} \rangle + \langle \overrightarrow{T(S_{\lambda_m}) z_m w_n}, \overrightarrow{w_m w_n} \rangle \\
&\le \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m) T(S_{\lambda_m}) z_m}, \overrightarrow{w_m w_n} \rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m) w_n}, \overrightarrow{w_m w_n} \rangle \\
&= \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m) T(S_{\lambda_m} z_m)}, \overrightarrow{w_m z_m} \rangle + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m) w_n}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{w_n T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_m} \rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m) T(S_{\lambda_m} w_n)}, \overrightarrow{w_m w_m} \rangle \\
&+ \langle \overrightarrow{T(S_{\lambda_m} w_m)}, \overrightarrow{w_m w_n} \rangle \\
&\le \frac{\alpha_m}{(1-\beta_m)} d(h(z_m), T(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \rangle + d(T(S_{\lambda_m} z_m), T(S_{\lambda_m} w_n)) d(w_m, w_n) \\
&+ d(T(S_{\lambda_m} w_n), w_n) d(w_m, w_n) \\
&\le \frac{\alpha_m}{(1-\beta_m)} d(h(z_m), T(S_{\lambda_m} z_m)) d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n} \rangle \\
&+ d(z_m, w_m) d(w_m, w_n) + d(T(S_{\lambda_m} w_n), w_n) d(w_n, w_m) \\
&\le \frac{\alpha_m}{(1-\beta_m)} d(h(z_m), T(S_{\lambda_m} z_m)) d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{z_m w_n} \rangle \\
&+ \frac{\alpha_m}{(1-\beta_m)} d(z_m, T(S_{\lambda_m} z_m)) d(w_m, z_m) + d(w_m, w_n) + d(T(S_{\lambda_m} w_n), w_n) d(w_n, w_m),
\end{aligned}$$

which implies that

$$\begin{aligned}
\langle \overrightarrow{h(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \le\; & d(h(z_m), T(S_{\lambda_m}) z_m) d(w_n, z_m) + d(z_m, T(S_{\lambda_m}) z_m) d(z_m, w_m) \\
&+ \frac{(1-\beta_m)}{\alpha_m} d(T(S_{\lambda_n}) w_n, w_n) d(w_m, w_m).
\end{aligned}$$

Thus, taking lim sup as $n \to \infty$ first, then as $m \to \infty$, it follows from (3.5.14), (3.5.18) and (3.5.19) that

$$\limsup_{m\to\infty} \limsup_{n\to\infty} \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \le 0. \tag{3.5.21}$$

Furthermore,

$$\begin{aligned}
\langle \overrightarrow{h(\bar{z}) \bar{z}}, \overrightarrow{x_n \bar{z}} \rangle &= \langle \overrightarrow{h(\bar{z}) h(z_m)}, \overrightarrow{x_n \bar{z}} \rangle + \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{x_n w_n} \rangle + \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \\
&+ \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{z_m \bar{z}} \rangle + \langle \overrightarrow{z_m \bar{z}}, \overrightarrow{x_n \bar{z}} \rangle \\
&\le d(h(\bar{z}), h(z_m)) d(x_n, \bar{z}) + d(h(z_m), z_m) d(x_n, w_n) + \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \\
&+ d(h(z_m), z_m) d(z_m, \bar{z}) + d(z_m, \bar{z}) d(x_n, \bar{z}) \\
&\le (1+\tau) d(z_m, \bar{z}) d(x_n, \bar{z}) + \langle \overrightarrow{h(z_m) z_m}, \overrightarrow{w_n z_m} \rangle \\
&+ [d(x_n, w_n) + d(z_m, \bar{z})] d(h(z_m), z_m),
\end{aligned}$$

which implies from (3.5.14), (3.5.21) and the fact that $\lim_{m\to\infty} z_m = \bar{z}$, that

$$\begin{aligned}\limsup_{n\to\infty}\langle\overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle &= \limsup_{m\to\infty}\limsup_{n\to\infty}\langle\overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle \\ &\le \limsup_{m\to\infty}\limsup_{n\to\infty}\langle\overrightarrow{h(z_m)z_m}, \overrightarrow{w_nz_m}\rangle \le 0. \qquad (3.5.22)\end{aligned}$$

Step 5: Finally, we show that $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

From Lemma 2.3.2, we obtain that

$$\begin{aligned}\langle\overrightarrow{w_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle &\le \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{h(x_n)\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle + \frac{\gamma_n}{(1-\beta_n)}\langle\overrightarrow{T(S_{\lambda_n})x_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle \\ &\le \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{h(x_n)h(\bar{z})}, \overrightarrow{x_n\bar{z}}\rangle \\ &+ \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle + \frac{\gamma_n}{(1-\beta_n)}d(T(S_{\lambda_n})x_n, \bar{z})d(x_n, \bar{z}) \\ &\le \frac{\alpha_n}{(1-\beta_n)}\tau d^2(x_n, \bar{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle + (1 - \frac{\alpha_n}{1-\beta_n})d^2(x_n, \bar{z}) \\ &= \left[\frac{\alpha_n}{(1-\beta_n)}\tau + (1 - \frac{\alpha_n}{1-\beta_n})\right]d^2(x_n, \bar{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle\overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle.\end{aligned}$$

Thus, from Lemma 2.3.1, we have

$$\begin{aligned}d^2(x_{n+1}, \bar{z}) &\le \beta_n d^2(x_n, \bar{z}) + (1-\beta_n)d^2(w_n, \bar{z}) \\ &= \beta_n d^2(x_n, \bar{z}) + (1-\beta_n)\langle\overrightarrow{w_n\bar{z}}, \overrightarrow{w_n\bar{z}}\rangle \\ &= \beta_n d^2(x_n, \bar{z}) + (1-\beta_n)[\langle\overrightarrow{w_n\bar{z}}, \overrightarrow{w_nx_n}\rangle + \langle\overrightarrow{w_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle] \\ &\le [\beta_n + \alpha_n\tau + \gamma_n]d^2(x_n, \bar{z}) + (1-\beta_n)\langle\overrightarrow{w_n\bar{z}}, \overrightarrow{w_nx_n}\rangle + \alpha_n\langle h(\bar{z})\bar{z}, x_n\bar{z}\rangle \\ &\le (1-\alpha_n(1-\tau))d^2(x_n, \bar{z}) + \alpha_n(1-\tau)\left[\frac{1}{1-\tau}\langle\overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\right] \\ &+ (1-\beta_n)d(w_n, x_n)d(w_n, \bar{z}). \qquad (3.5.23)\end{aligned}$$

By (3.5.14) and applying Lemma 2.3.26 to (3.5.23), we obtain that $\{x_n\}$ converges strongly to $\bar{z}$. □

By setting $T \equiv I$ in Theorem 3.5.4, where I is an identity mapping on X, we obtain the following result.

Corollary 3.5.5. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the range condition. Let h be a contraction mapping on X with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := \cap_{i=1}^N A_i^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1}x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N}x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n}h(x_n) \oplus \frac{\gamma_n}{1-\beta_n}y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)w_n, \quad n \ge 1. \end{cases} \qquad (3.5.24)$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,

(iv) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

By setting $N = 1$ in Theorem 3.5.4, we obtain the following result.

Corollary 3.5.6. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $A : X \to 2^{X^*}$ *be a multivalued monotone operator that satisfies the range condition. Let* T *be a nonexpansive mapping on* X *and* h *be a contraction mapping on* X *with coefficient* $\tau \in (0,1)$*. Suppose that* $\Gamma := A^{-1}(0) \cap F(T) \neq \emptyset$ *and for arbitrary* $x_1 \in X$*, the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} w_n = \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T\left(\beta_0 x_n \oplus \beta_1 J^A_{\lambda_n} x_n\right), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n, \quad n \geq 1. \end{cases} \tag{3.5.25}$$

where $\{\alpha_n\}$*,* $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $(0,1)$*, and* $\{\lambda_n\}$ *is a sequence of positive real numbers satisfying the following conditions:*

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,

(iv) $\beta_i \in (0,1), i = 0,1$ *with* $\beta_0 + \beta_1 = 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

3.5.2 Applications

In this subsection, we apply our results to solve variational inequality and convex feasibility problems.

Variational inequality problem

Let us consider the following VIP associated with a nonexpansive mapping T:

$$\text{Find } x \in C \text{ such that } \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \ \forall y \in C. \tag{3.5.26}$$

Recall that the metric projection $P_C : X \to C$ is defined for $x \in X$ by $d(x, P_C x) = \inf_{y \in C} d(x, y)$ and characterized by, $z = P_C x$ if and only if $\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq 0,\ \forall y \in C$ (see [99]). Now, using the characterization of P_C, we obtain that

$$x = P_C T x \Longleftrightarrow \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \ \forall y \in C.$$

Therefore, $x \in F(P_C \circ T)$ if and only if x solves (3.5.26). Recall also that the subdifferential of an indicator function δ_C, given as

$$\partial \delta_C(x) = \begin{cases} \{x^* \in X^* : \langle x^*, \overrightarrow{xz} \rangle \leq 0 \ \forall z \in C\} \text{ if } x \in C, \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases} \tag{3.5.27}$$

is a monotone operator which satisfies the range condition (see Example 2.1.28).

Thus, by (2.1.5), we obtain that

$$z = J_\lambda^{\partial \delta_C} x \Longleftrightarrow [\frac{1}{\lambda}\overrightarrow{zx}] \in \partial \delta_C z \Longleftrightarrow \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq 0,\ \forall\ y \in C \Longleftrightarrow z = P_C x. \tag{3.5.28}$$

Thus, by letting $z = x$, we obtain that $x = P_C x$ if and only if $x \in (\partial \delta_C)^{-1}(0)$. Therefore, we get that

$$x \in (\partial \delta_C)^{-1}(0) \cap F(T) \implies x \in F(P_C) \cap F(T) \implies x \in F(P_C \circ T).$$

Thus, suppose that the solution set of problem (3.5.26) is Ω, then by setting $A = \partial \delta_C$ in Corollary 3.5.6, we apply Corollary 3.5.6 to obtain the following result for approximating solutions of the VIP (3.5.26) in Hadamard spaces.

Theorem 3.5.7. *Let C be a nonempty closed and convex subset of an Hadamard space X and X^* be the dual space of X. Let $T : X \to X$ be a nonexpansive mapping and h be a contraction mapping on X with constant $\tau \in (0, 1)$. Suppose that $\Omega \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by*

$$\begin{cases} w_n = \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T\left(\beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\partial \delta_C} x_n\right), \\ x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) w_n, \quad n \geq 1. \end{cases} \tag{3.5.29}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1 \ \forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n \ \forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,

(iv) $\beta_i \in (0, 1), i = 0, 1$ *with* $\beta_0 + \beta_1 = 1$.

Then, $\{x_n\}$ converges strongly to an element of Ω.

Convex feasibility problem

Let C be a nonempty closed and convex subset of X and $C_i, i = 1, 2, \ldots, N$ be a finite family of nonempty closed and convex subsets of C such that $\cap_{i=1}^{N} C_i \neq \emptyset$. Recall that the convex feasibility problem is defined as:

$$\text{Find } x \in C \text{ such that } x \in \cap_{i=1}^{N} C_i. \tag{3.5.30}$$

Now, observe that (3.5.28) implies that $x = J_{\lambda}^{\partial \delta_{C_i}} x \iff x = P_{C_i} x, \ i = 1, 2, \ldots, N$. Therefore, by setting $A_i = \partial \delta_{C_i}$ in Corollary 3.5.5 and $J_{\lambda_n}^{A_i} = P_{C_i}, \ i = 1, 2, \ldots, N$ in Algorithm (3.5.24), we can apply Corollary 3.5.5 to approximate solutions of (3.5.30).

Chapter 4

Contributions to Minimization Problems and Monotone Inclusion Problems in Hadamard Spaces

4.1 Introduction

As mentioned in Section 1.1, the study of MPs in Hadamard spaces have proved to be very useful within and outside mathematics. The most remarkable of them, is the application to computing of medians and means of trees, which are very important in computational phylogenetics, diffusion tensor imaging, censensus algorithms and modeling of airway systems in human lungs and blood vessels. In this chapter, we shall further investigate the study of MPs in Hadamard spaces. Then, we will study some strong convergence results for approximating a common solution of MPs, MIPs and fixed point problems in Hadamard spaces.

4.2 Preliminaries

In this section, we highlight some lemmas that are associated with convex functions for MPs.

Lemma 4.2.1. *[130]. Let X be a CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. For each $x \in X$ and $\lambda \geq \mu > 0$, the following identity holds:*

$$J_\lambda^f x = J_\mu^f \left(\frac{\lambda - \mu}{\lambda} J_\lambda^f x \oplus \frac{\mu}{\lambda} x \right),$$

where J_λ^f is the Moreau-Yosida resolvent of f defined as in (2.2.3).

Lemma 4.2.2. *[112]. Let X be a CAT(0) space and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Then, for all $x, z \in X$ and $\mu > 0$, we have*

$$\frac{1}{2\mu} d^2(J_\mu^f x, z) - \frac{1}{2\mu} d^2(x, z) + \frac{1}{2\mu} d^2(x, J_\mu^f x) + f(J_\mu^f x) \leq f(z). \tag{4.2.1}$$

Definition 4.2.3. *Let X be a CAT(0) space. A function $f : D(f) \subseteq X \to (-\infty, \infty]$ is said to be Δ-lower semicontinuous at a point $x \in D(f)$, if*

$$f(x) \leq \liminf_{n\to\infty} f(x_n), \tag{4.2.2}$$

for each sequence $\{x_n\}$ in $D(f)$ such that $\Delta\text{-}\lim_{n\to\infty} x_n = x$. We say that f is Δ-lower semicontinuous on $D(f)$ if it is Δ-lower semicontinuous at any point in $D(f)$.

Lemma 4.2.4. *[18] Let X be an Hadamard space and $f : X \to \mathbb{R}$ be a convex and lower semicontinuous function. Then f is Δ-lower semicontinuous.*

Lemma 4.2.5. *Let X be a CAT(0) space and $f : X \to (-\infty, \infty]$ be proper convex and lower semi-continuous function. Then,*
$d^2(J^f_\lambda x, x) \leq d^2(J^f_\mu x, x)$ for $0 < \lambda < \mu$ and $x \in X$.

Proof. Let $x, y \in X$, then we obtain from (2.2.3) that

$$f(J^f_\mu x) + \frac{1}{2\mu}d^2(J^f_\mu x, x) \leq f(y) + \frac{1}{2\mu}d^2(y, x).$$

In particular, we have that

$$f(J^f_\mu x) + \frac{1}{2\mu}d^2(J^f_\mu x, x) \leq f(J^f_\lambda x) + \frac{1}{2\mu}d^2(J^f_\lambda x, x). \tag{4.2.3}$$

Similarly, we obtain

$$f(J^f_\lambda x) + \frac{1}{2\lambda}d^2(J^f_\lambda x, x) \leq f(J^f_\mu x) + \frac{1}{2\lambda}d^2(J^f_\mu x, x) \tag{4.2.4}$$

Adding (4.2.3) and (4.2.4), we obtain that

$$d^2(J^f_\lambda x, x) - \frac{\lambda}{\mu}d^2(J^f_\lambda x, x) \leq d^2(J^f_\mu x, x) - \frac{\lambda}{\mu}d^2(J^f_\mu x, x).$$

That is,

$$\left(1 - \frac{\lambda}{\mu}\right) d^2(J^f_\lambda x, x) \leq \left(1 - \frac{\lambda}{\mu}\right) d^2(J^f_\mu x, x).$$

Since, $0 < \lambda < \mu$, we obtain that

$$d^2(J^f_\lambda x, x) \leq d^2(J^f_\mu x, x).$$

□

Lemma 4.2.6. *Let C be a closed and convex subset of an Hadamard space X and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semi continuous mappings such that $\cap_{i=1}^N \arg\min_{y\in X} f_i(y) \neq \emptyset$. Let $\{u_n\}$ and $\{z_n\}$ be bounded sequences such that*

$$u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)),$$

where $J_{\lambda_n^{(i)}}$ denotes $J^{f_i}_{\lambda_n^{(i)}}$ (for simplicity) and $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \ldots, N$ and $n \geq 1$. If $\lim_{n\to\infty} d(u_n, z_n) = 0$, then $\lim_{n\to\infty} d(J_{\lambda^{(i)}} z_n, z_n) = 0$, for each $i = 1, 2, \ldots, N$.

Proof. Let $p \in \cap_{i=1}^{N} \arg\min_{y\in X} f_i(y)$.

$$\text{Set } w_n^{(i+1)} = J_{\lambda_n^{(i)}} w_n^{(i)}, \text{ for each } i = 1, 2, \ldots, N,$$

where $w_n^{(1)} = z_n$, for all $n \geq 1$. Then,

$w_n^{(2)} = J_{\lambda_n^{(1)}}(z_n), \quad w_n^{(3)} = J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n), \quad \ldots, \quad w_n^{(N+1)} = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n).$

By Lemma 4.2.2, we obtain

$\frac{1}{2\lambda_n^{(i)}} d^2(p, w_n^{(i+1)}) - \frac{1}{2\lambda_n^{(i)}} d^2(p, w_n^{(i)}) + \frac{1}{2\lambda_n^{(i)}} d^2(w_n^{(i)}, w_n^{(i+1)}) + f(w_n^{(i+1)}) \leq f(p).$

As $f(p) \leq f(w_n^{(i+1)})$, so we have that

$$d^2(w_n^{(i)}, w_n^{(i+1)}) \leq d^2(p, w_n^{(i)}) - d^2(p, w_n^{(i+1)}).$$

Taking sum in the above inequality from $i = 1$ to $i = N$, we obtain

$$\begin{aligned}
\sum_{i=1}^{N} d^2(w_n^{(i)}, w_n^{(i+1)}) &\leq d^2(p, z_n) - d^2(p, w_n^{(N+1)}) \\
&\leq d^2(p, z_n) - d^2(p, u_n) \\
&\leq [d(p, u_n) + d(u_n, z_n)]^2 - d^2(p, u_n) \\
&\leq d^2(z_n, u_n) + 2d(z_n, u_n)d(p, u_n) \to 0 \text{ as } n \to \infty,
\end{aligned}$$

which implies

$$\lim_{n\to\infty} d(w_n^{(i)}, w_n^{(i+1)}) = 0, \ i = 1, 2, \ldots, N. \tag{4.2.5}$$

By (4.2.5) and triangle inequality, we obtain for each $i = 1, 2, \ldots, N$, that

$$\lim_{n\to\infty} d(z_n, w_n^{(i+1)}) = \lim_{n\to\infty} d(w_n^{(1)}, w_n^{(i+1)}) = 0. \tag{4.2.6}$$

Also, since $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for all $n \geq 1$, we obtain by Lemma 4.2.1 and (4.2.5) that

$$d\left(w_n^{(i)}, J_{\lambda^{(i)}} w_n^{(i)}\right) \leq d\left(w_n^{(i)}, J_{\lambda_n^{(i)}} w_n^{(i)}\right) \to 0, \text{ as } n \to \infty, \ i = 1, 2, \ldots, N. \tag{4.2.7}$$

Since $J_{\lambda^{(i)}}$ is nonexpansive, we have from (4.2.5) and (4.2.6) that

$$\begin{aligned}
d(J_{\lambda^{(i)}} z_n, J_{\lambda^{(i)}} w_n^{(i)}) &\leq d(J_{\lambda^{(i)}} z_n, J_{\lambda^{(i)}} w_n^{(i+1)}) + d(J_{\lambda^{(i)}} w_n^{(i+1)}, J_{\lambda^{(i)}} w_n^{(i)}) \\
&\leq d(z_n, w_n^{(i+1)}) + d(w_n^{(i+1)}, w_n^{(i)}) \to 0, \text{ as } n \to \infty.
\end{aligned} \tag{4.2.8}$$

By (4.2.5)-(4.2.8), we obtain

$$\begin{aligned}
d\left(J_{\lambda^{(i)}} z_n, z_n\right) &\leq d\left(J_{\lambda^{(i)}} z_n, J_{\lambda^{(i)}} w_n^{(i)}\right) + d\left(J_{\lambda^{(i)}} w_n^{(i)}, w_n^{(i)}\right) + d\left(w_n^{(i)}, w_n^{(i+1)}\right) + d\left(w_n^{(i+1)}, z_n\right) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

That is,

$$\lim_{n\to\infty} d\left(J_{\lambda^{(i)}} z_n, z_n\right) = 0, \quad i = 1, 2, \ldots, N.$$

□

4.3 Viscosity approximation methods for solving minimization problems in Hadamard spaces

In this section, we propose and study some viscosity-type proximal point algorithms for approximating a common solution of MP and fixed point problem for nonexpansive multivalued mappings, which is also a unique solution of some variational inequality problems. Furthermore, numerical examples of our algorithm are given to show its competitive advantage over existing algorithms in the literature.

4.3.1 Main results

In what follows, we propose our implicit iterative net for our first convergence theorem of this section: For each $t \in (0,1]$, let the net $\{x_t\}$ be generated by

$$\begin{cases} u_t = J^f_{\lambda_t}(x_t), \\ x_t = tg(u_t) \oplus (1-t)v_t, \quad v_t \in Tu_t, \end{cases} \tag{4.3.1}$$

where $J^f_{\lambda_t}$ is the resolvent of a proper convex and lower semicontinmuous function f, g is a contraction mapping and T is a nonexpansive multivalued mapping.

The implicit iterative net (4.3.1) clearly generalizes the following implicit iteration studied by Saejung [161]: For $t \in (0,1)$ and fixed $u \in C$, $\{x_t\}$ is defined by

$$x_t = tu \oplus (1-t)Tx_t, \tag{4.3.2}$$

where T is a nonexpansive singlevalued mapping defined on C.
Observe that the implicit iteration (4.3.2) is of Halpern-type, and as mentioned earlier, the rate of convergence of Halpern-type iterations is relatively lower than that of viscosity-type iterations.
Note also that, if $J^f_{\lambda_t} \equiv I$ in (4.3.1) (where I is the identity mapping on X), then (4.3.1) reduces to

$$x_t = tf(x_t) \oplus (1-t)u(x_t), \quad u(x_t) \in T(x_t) \tag{4.3.3}$$

studied by Bo and Yi [29]. Thus, iterative net (4.3.1) extends the implicit iteration of Bo and Yi [29] to an implicit proximal point iteration.

We now present our strong convergence theorem for the implicit proximal point iteration (4.3.1).

Theorem 4.3.1. *Let X be an Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Let $T : X \to P(X)$ be a nonexpansive multivalued mapping such that $Tp = \{p\}$, for each $p \in F(T)$ and $\Gamma := F(T) \cap argmin_{y \in X} f(y)$ is nonempty. Suppose that g is a contraction mapping defined on X with coefficient $\rho \in (0,1)$ and $\lambda_t \geq \lambda > 0$ for some λ. Let the net $\{x_t\}$ be defined by (4.3.1) such that Lemma 2.3.22 holds. Then, the net $\{x_t\}$ converges strongly to $\overline{x} = P_\Gamma g(\bar{x})$ (where P_Γ is the metric projection of X onto Γ) which also solves the variational inequality*

$$\langle \overrightarrow{\overline{x}g(\overline{x})}, \overrightarrow{x\overline{x}} \rangle \geq 0, \ \forall x \in \Gamma. \tag{4.3.4}$$

Proof. By similar argument as in the proof of [29, Page 53], we obtain that (4.3.1) is well-defined.
Let $p \in \Gamma$, then from (4.3.1), Lemma 2.3.1 and Lemma 2.3.22, we obtain

$$\begin{aligned} d(x_t, p) &= d(tg(u_t) \oplus (1-t)v_t, p) \\ &\leq td(g(u_t), p) + (1-t)d(v_t, p) \\ &\leq td(g(u_t), p)) + (1-t)H(Tu_t, Tp) \\ &\leq t(d(g(u_t), g(p)) + d(g(p), p)) + (1-t)d(u_t, p) \\ &\leq t(\rho d(u_t, p)) + d(g(p), p)) + (1-t)d(u_t, p) \\ &= t(\rho d(J^f_{\lambda_t} x_t, p)) + d(g(p), p)) + (1-t)d(J_{\lambda_t} x_t, p) \\ &\leq (t\rho + (1-t))d(x_t, p) + td(g(p), p), \end{aligned}$$

which implies

$$d(x_t, p) \leq \frac{1}{1-\rho} d(g(p), p).$$

Hence, $\{x_t\}$ is bounded. Consequently, $\{u_t\}$, $\{v_t\}$ and $\{g(u_t)\}$ are also bounded.
From (4.3.1), we have

$$\begin{aligned} \lim_{t \to 0} d(x_t, v_t) &= \lim_{t \to 0} d(tg(u_t) \oplus (1-t)v_t, v_t) \\ &\leq \lim_{t \to 0} td(g(u_t), v_t) = 0. \end{aligned} \tag{4.3.5}$$

From Lemma 4.2.2, we have that

$$\frac{1}{2\lambda_t} d^2(u_t, p) - \frac{1}{2\lambda_t} d^2(x_t, p) + \frac{1}{2\lambda_t} d^2(x_t, u_t) \leq f(p) - f(u_t).$$

Since $f(p) \leq f(u_t)$ for all $n \geq 1$, we obtain

$$d^2(u_t, p) \leq d^2(x_t, p) - d^2(x_t, u_t). \tag{4.3.6}$$

Thus, using (4.3.6) and Lemma 2.3.1, we have

$$\begin{aligned} d^2(x_t, p) &= d^2(tg(u_t) \oplus (1-t)v_t, p) \\ &\leq td^2(g(u_t), p) + (1-t)d^2(v_t, p) \\ &\leq td^2(g(u_t), p) + (1-t)H^2(Tu_t, Tp) \\ &\leq td^2(g(u_t), p) + (1-t)d^2(u_t, p) \\ &\leq td^2(g(u_t), p) + d^2(x_t, p) - d^2(x_t, u_t), \end{aligned}$$

which implies that

$$d^2(x_t, u_t) \;\leq\; td^2(g(u_t), p) \to 0, \text{ as } t \to 0.$$

That is,

$$\lim_{t \to 0} d^2(x_t, u_t) = 0. \tag{4.3.7}$$

From (4.3.5) and (4.3.7), we obtain

$$\lim_{t\to 0} d^2(u_t, v_t) = 0. \tag{4.3.8}$$

Since $\lambda_t \geq \lambda > 0$, we obtain from (4.3.7) and Lemma 4.4.1 that

$$\begin{aligned} d(x_t, J^f_\lambda x_t) &\leq d(x_t, J^f_{\lambda_t} x_t) \\ &= d(x_t, u_t) \to 0 \text{ as } t \to 0. \end{aligned} \tag{4.3.9}$$

Let $x_m := x_{t_m}$ for all $m \geq 1$, with $t_m \in (0,1]$ and $t_m \to 0$, as $m \to \infty$. Since $\{x_m\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, we may assume that Δ-$\lim_{m\to\infty} x_m = \bar{x}$. Since T is a nonexpansive multivalued mapping, it follows from (4.3.7), (4.3.8) and Lemma 2.3.13 that $\bar{x} \in F(T)$. Also, since J^f_λ is a nonexpansive mapping, it follows from (4.3.9) and Lemma 2.3.12 that $\bar{x} \in F(J^f_\lambda)$. Therefore $\{x_m\}$ Δ-converges to $\bar{x} \in \Gamma$. Thus, by Lemma 2.3.10, we obtain

$$\limsup_{m\to\infty} \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle \leq 0. \tag{4.3.10}$$

We now show that $\lim_{m\to\infty} x_m = \bar{x}$. From Lemma 2.3.2, we obtain

$$\begin{aligned} d^2(x_m, \bar{x}) &= \langle \overrightarrow{x_m\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m)\langle \overrightarrow{v_m\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d(v_m, \bar{x}) d(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) H(Tu_m, \bar{x}) d(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d(J^f_{\lambda_m} x_m, \bar{x}) d(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d^2(x_m, \bar{x}) \\ &\leq t_m \langle \overrightarrow{g(u_m)g(\bar{x})}, \overrightarrow{x_m\bar{x}} \rangle + t_m \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d^2(x_m, \bar{x}) \\ &\leq t_m \rho d^2(x_m, \bar{x}) + t_m \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle + (1-t_m) d^2(x_m, \bar{x}), \end{aligned}$$

which implies

$$d^2(x_m, \bar{x}) \leq \frac{1}{1-\rho} \langle \overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_m\bar{x}} \rangle. \tag{4.3.11}$$

Thus, from (4.3.10) and (4.3.11), we obtain that

$$\lim_{m\to\infty} x_m = \bar{x}. \tag{4.3.12}$$

Next, we show that $\bar{x} \in \Gamma$ solves the variational inequality (4.3.4).
From (4.3.1), Lemma 2.3.1 and Lemma 2.3.22, we obtain for any $z \in \Gamma$ that

$$\begin{aligned} d^2(x_t, z) &= d^2(tg(u_t) \oplus (1-t)v_t, z) \\ &\leq td^2(g(u_t), z) + (1-t)d^2(v_t, z) - t(1-t)d^2(g(u_t), v_t) \\ &\leq td^2(g(u_t), z) + (1-t)d^2(x_t, z) - t(1-t)d^2(g(u_t), v_t), \end{aligned}$$

which implies

$$d^2(x_t, z) \leq d^2(g(u_t), z) - (1-t)d^2(g(u_t), v_t).$$

So that

$$d^2(x_m, z) \leq d^2(g(u_m), z) - (1-t_m)d^2(g(u_m), v_m). \tag{4.3.13}$$

Taking limit as $m \to \infty$, we obtain from (4.3.5), (4.3.7) and (4.3.13) that

$$d^2(\bar{x}, z) \leq d^2(g(\bar{x}), z) - d^2(g(\bar{x}), \bar{x}). \tag{4.3.14}$$

From (4.3.14) and by the definition of quasilinearization mapping, we obtain

$$\langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{z\bar{x}} \rangle = \frac{1}{2}\left(d^2(g(\bar{x}), z) - d^2(g(\bar{x}), \bar{x}) - d^2(\bar{x}, z)\right) \geq 0, \ \forall z \in \Gamma. \tag{4.3.15}$$

Thus, $\bar{x} \in \Gamma$ solves the variational inequality (4.3.4).

We now show that the net $\{x_t\}$ converges strongly to $\bar{x}$. We may assume that $x_{s_m} \to x^* \in \Gamma$, where $s_m \to 0$ as $m \to \infty$. Then by same argument as above, we obtain that x^* also solves the variational inequality (4.3.4). That is,

$$\langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{\bar{x}x^*} \rangle \leq 0, \quad \langle \overrightarrow{x^*g(x^*)}, \overrightarrow{x^*\bar{x}} \rangle \leq 0.$$

Thus,

$$\begin{aligned}
0 &\geq \langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{x^*g(x^*)}, \overrightarrow{\bar{x}x^*} \rangle \\
&= \langle \overrightarrow{\bar{x}g(x^*)}, \overrightarrow{\bar{x}x^*} \rangle + \langle \overrightarrow{g(x^*)g(\bar{x})}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{x^*\bar{x}}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{\bar{x}g(x^*)}, \overrightarrow{\bar{x}x^*} \rangle \\
&= \langle \overrightarrow{\bar{x}x^*}, \overrightarrow{\bar{x}x^*} \rangle - \langle \overrightarrow{g(x^*)g(\bar{x})}, \overrightarrow{x^*\bar{x}} \rangle \\
&\geq (1-\rho)d^2(\bar{x}, x^*),
\end{aligned}$$

which implies that $d^2(\bar{x}, x^*) = 0$. Thus, $\bar{x} = x^*$. Hence, the net $\{x_t\}$ converges to $\bar{x} \in \Gamma$ which also solves the variational inequality (4.3.4). □

Next, we present the following strong convergence theorem for our proposed viscosity-type PPA.

Theorem 4.3.2. *Let X be an Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Let $T : X \to P(X)$ be a nonexpansive multivalued mapping such that $Tp = \{p\}$, for each $p \in F(T)$ and $\Gamma := F(T) \cap \text{argmin}_{y \in X} f(y)$ is nonempty. Suppose that g is a contraction mapping defined on X with coefficient $\rho \in (0,1)$ and $\lambda_n \geq \lambda > 0$ for some λ. Let $x_1 \in X$ be arbitrarily chosen and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J^f_{\lambda_n}(x_n), \\ x_{n+1} = t_n g(u_n) \oplus (1-t_n)v_n, \quad \textit{where } v_n \in T(u_n) \forall n \geq 1, \end{cases} \tag{4.3.16}$$

such that Lemma 2.3.22 holds and $\{t_n\}$ is a sequence in $(0,1)$ satisfying

(i) $\lim\limits_{n\to\infty} t_n = 0$,

(ii) $\sum_{n=1}^{\infty} t_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$,

(iv) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

Then the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which also solves the variational inequality (4.3.4).

Proof. Let $p \in \Gamma$, then from (4.3.16), Lemma 2.3.1 and Lemma 2.3.22, we obtain

$$\begin{aligned}
d(x_{n+1}, p) &= d(t_n g(u_n) \oplus (1-t_n)v_n, p) \\
&\leq t_n d(g(u_n), p) + (1-t_n) d(v_n, p) \\
&\leq t_n d(g(u_n), p)) + (1-t_n) H(Tu_n, Tp) \\
&\leq t_n (d(g(u_n), g(p)) + d(g(p), p)) + (1-t_n) d(u_n, p) \\
&\leq t_n (\rho d(u_n, p)) + d(g(p), p)) + (1-t_n) d(u_n, p) \\
&\leq (t_n \rho + (1-t_n)) d(x_n, p) + t_n d(g(p), p),
\end{aligned}$$

that is

$$d(x_{n+1}, p) \leq \max\{d(x_n, p), \frac{1}{1-\rho} d(g(p), p)\}.$$

By induction, we obtain that

$$d(x_{n+1}, p) \leq \max\{d(x_1, p), \frac{1}{1-\rho} d(g(p), p)\}.$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{u_n\}$, $\{v_n\}$ and $\{g(u_n)\}$ are also bounded.

Next, we show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Without loss of generality, let us assume that $\lambda_n \geq \lambda_{n-1}$. Since $\lambda_n \geq \lambda > 0$ $\forall n \geq 1$, then from Lemma 4.2.1, we obtain

$$\begin{aligned}
d(u_n, u_{n-1}) &\leq d(u_n, J^f_{\lambda_n} x_{n-1}) + d(J^f_{\lambda_n} x_{n-1}, u_{n-1}) \\
&= d(J^f_{\lambda_n} x_n, J^f_{\lambda_n} x_{n-1}) + d(J^f_{\lambda_n} x_{n-1}, J^f_{\lambda_{n-1}} x_{n-1}) \\
&\leq d(x_n, x_{n-1}) + d\Big(J^f_{\lambda_{n-1}} (\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J^f_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1}), J^f_{\lambda_{n-1}} x_{n-1} \Big) \\
&\leq d(x_n, x_{n-1}) + d\Big(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J^f_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1}, x_{n-1} \Big) \\
&= d(x_n, x_{n-1}) + \frac{\lambda_n - \lambda_{n-1}}{\lambda_n} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big) \\
&= d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big) \\
&\leq d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big). \qquad (4.3.17)
\end{aligned}$$

Also, from (4.3.16) and (4.3.17), we obtain

$$\begin{aligned}
d(x_{n+1}, x_n) &= d(t_n g(u_n) \oplus (1-t_n)v_n, t_{n-1}g(u_{n-1}) \oplus (1-t_{n-1})v_{n-1}) \\
&\leq d(t_n g(u_n) \oplus (1-t_n)v_n, t_n g(u_n) \oplus (1-t_n)v_{n-1}) \\
&+ d(t_n g(u_n) \oplus (1-t_n)v_{n-1}, t_n g(u_{n-1}) \oplus (1-t_n)v_{n-1}) \\
&+ d(t_n g(u_{n-1}) \oplus (1-t_n)v_{n-1}, t_{n-1}g(u_{n-1}) \oplus (1-t_{n-1})v_{n-1}) \\
&\leq (1-t_n)d(v_n, v_{n-1}) + t_n d(g(u_n), g(u_{n-1})) \\
&+ |t_n - t_{n-1}| d(g(u_{n-1}), v_{n-1}) \\
&\leq (1-t_n)d(u_n, u_{n-1}) + t_n d(g(u_n), g(u_{n-1})) \\
&+ |t_n - t_{n-1}| d(g(u_{n-1}), v_{n-1}) \\
&\leq ((1-t_n) + t_n\rho)d(u_n, u_{n-1}) + |t_n - t_{n-1}| d(v_{n-1}, g(u_{n-1})) \\
&\leq ((1-t_n) + t_n\rho)\Big(d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d\big(J^f_{\lambda_n} x_{n-1}, x_{n-1}\big)\Big) \\
&+ |t_n - t_{n-1}| d(v_{n-1}, g(u_{n-1})) \\
&= (1 - t_n(1-\rho))d(x_n, x_{n-1}) \\
&+ (1 - t_n(1-\rho))\frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J^f_{\lambda_n} x_{n-1}, x_{n-1}) \\
&+ |t_n - t_{n-1}| d(v_{n-1}, g(u_{n-1})).
\end{aligned}$$

Using conditions (ii), (iii) and (iv), we obtain by Lemma 2.3.26 that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4.3.18}$$

From Lemma 4.2.2, we have that

$$\frac{1}{2\lambda_n} d^2(u_n, p) - \frac{1}{2\lambda_n} d^2(x_n, p) + \frac{1}{2\lambda_n} d^2(x_n, u_n) \leq f(p) - f(u_n).$$

Since $f(p) \leq f(u_n)$ for all $n \geq 1$, we obtain

$$d^2(u_n, p) \leq d^2(x_n, p) - d^2(x_n, u_n). \tag{4.3.19}$$

Thus, using (4.3.19) and Lemma 2.3.1, we obtain

$$\begin{aligned}
d^2(x_{n+1}, p) &= d^2(t_n g(u_n) \oplus (1-t_n)v_n, p) \\
&\leq t_n d^2(g(u_n), p) + (1-t_n)d^2(v_n, p) \\
&\leq t_n d^2(g(u_n), p) + (1-t_n)d^2(u_n, p) \\
&\leq t_n d^2(g(u_n), p) + d^2(x_n, p) - d^2(x_n, u_n),
\end{aligned}$$

which implies that

$$\begin{aligned}
d^2(x_n, u_n) &\leq t_n d^2(g(u_n), p) + d^2(x_n, p) - d^2(x_{n+1}, p) \\
&\leq t_n d^2(g(u_n), p) + d^2(x_n, x_{n+1}) + 2d(x_n, x_{n+1})d(x_{n+1}, p).
\end{aligned}$$

It then follows from (4.3.18) and condition (i) that

$$\lim_{n\to\infty} d^2(x_n, u_n) = 0. \tag{4.3.20}$$

Since $\lambda_n \geq \lambda > 0$, we obtain from (4.3.20) and Lemma 4.4.1 that

$$\begin{aligned} d(x_n, J_\lambda^f x_n) &\leq d(x_n, J_{\lambda_n}^f x_n) \\ &= d(x_n, u_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{4.3.21}$$

Again,

$$\begin{aligned} d(u_n, v_n) &\leq d(u_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, v_n) \\ &\leq d(u_n, x_n) + d(x_n, x_{n+1}) + d(t_n g(u_n) \oplus (1 - t_n)v_n, v_n) \\ &\leq d(x_n, x_{n+1}) + d(u_n, x_n) + t_n d(g(u_n), v_n), \end{aligned}$$

which implies from (4.3.18), (4.3.20) and condition (i), that

$$\lim_{n\to\infty} d(u_n, v_n) = 0. \tag{4.3.22}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\triangle\text{-}\lim_{k\to\infty} x_{n_k} = \overline{x}$. It follows from (4.3.20) that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\triangle\text{-}\lim_{k\to\infty} u_{n_k} = \overline{x}$. Since T is a nonexpansive multivalued mapping, it follows from (4.3.22) and Lemma 2.3.13 that $\overline{x} \in F(T)$. Also, since J_λ^f is a nonexpansive mapping, it follows from (4.3.21) and Lemma 2.3.12 that $\overline{x} \in F(J_\lambda^f)$. Therefore $\overline{x} \in \Gamma$. Following similar argument as in Theorem 4.3.1, we can show that $\bar{x}$ also solves the variational inequality (4.3.4). Thus, we conclude that $\bar{x} \in \Gamma$ also solves the variational inequality (4.3.4).
Next, we show that

$$\limsup_{n\to\infty} \langle \overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_n\overline{x}} \rangle \leq 0.$$

Observe that

$$\limsup_{n\to\infty} \langle \overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_n\overline{x}} \rangle = \limsup_{k\to\infty} \langle \overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n_k}\overline{x}} \rangle. \tag{4.3.23}$$

Since $\{x_{n_k}\}$ $\Delta-$ converges to $\overline{x}$, by Lemma 2.3.10, we have

$$\limsup_{k\to\infty} \langle \overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_{n_k}\overline{x}} \rangle \leq 0.$$

This together with (4.3.23) gives

$$\limsup_{n\to\infty} \langle \overrightarrow{g(\overline{x})\overline{x}}, \overrightarrow{x_n\overline{x}} \rangle \leq 0. \tag{4.3.24}$$

Finally, we prove that $\{x_n\}$ converges strongly to $\overline{x}$.

For any $n \geq 1$, let $z_n = t_n\bar{x} \oplus (1-t_n)v_n$. Thus, by Lemma 2.3.2, we obtain

$$\begin{aligned}
d^2(x_{n+1},\bar{x}) &\leq d^2(z_n,\bar{x}) + 2\langle\overrightarrow{x_{n+1}z_n}, \overrightarrow{x_{n+1}\bar{x}}\rangle \\
&\leq (1-t_n)^2 d^2(v_n,\bar{x}) + 2(t_n^2\langle\overrightarrow{g(u_n)\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle \\
&\quad + t_n(1-t_n)\langle\overrightarrow{g(u_n)v_n}, \overrightarrow{x_{n+1}\bar{x}}\rangle + t_n(1-t_n)\langle\overrightarrow{v_n\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle) \\
&\leq (1-t_n)^2 d^2(x_n,\bar{x}) + 2(t_n^2\langle\overrightarrow{g(u_n)\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle \\
&\quad + t_n(1-t_n)\langle\overrightarrow{g(u_n)\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle) \\
&= (1-t_n)^2 d^2(x_n,\bar{x}) + 2t_n\langle\overrightarrow{g(u_n)\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle \\
&= (1-t_n)^2 d^2(x_n,\bar{x}) + 2t_n(\langle\overrightarrow{g(u_n)g(\bar{x})}, \overrightarrow{x_{n+1}\bar{x}}\rangle + \langle\overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle) \\
&\leq (1-t_n)^2 d^2(x_n,\bar{x}) + 2t_n\left(\rho d(u_n,\bar{x})d(x_{n+1},\bar{x}) + \langle\overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle\right) \\
&\leq (1-t_n)^2 d^2(x_n,\bar{x}) + 2t_n\langle\overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle \\
&\quad + \rho t_n(d^2(x_n,\bar{x}) + d^2(x_{n+1},\bar{x})),
\end{aligned}$$

which implies that

$$d^2(x_{n+1},\bar{x}) \leq \left(1 - \frac{2t_n(1-\rho)}{1-t_n\rho}\right) d^2(x_n,\bar{x}) + \frac{2t_n}{1-t_n\rho}\langle\overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle + \frac{t_n^2}{(1-t_n\rho)}M,$$

where $M = \sup_{n\geq 1}\{d^2(x_n,\bar{x})\}$. Thus, we have

$$d^2(x_{n+1},\bar{x}) \leq \left(1 - \frac{2t_n(1-\rho)}{1-t_n\rho}\right) d^2(x_n,\bar{x}) + \frac{2t_n(1-\rho)}{1-t_n\rho}\left(\frac{\langle\overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle}{1-\rho} + \frac{t_n M}{2(1-\rho)}\right).$$

If we let $\gamma_n = \frac{2(1-\rho)t_n}{1-t_n\rho}$ and $\delta_n = \frac{1}{1-\rho}\langle\overrightarrow{g(\bar{x})\bar{x}}, \overrightarrow{x_{n+1}\bar{x}}\rangle + \frac{t_n}{2(1-\rho)}M$, we obtain that

$$d^2(x_{n+1},\bar{x}) \leq (1-\gamma_n)d^2(x_n,\bar{x}) + \gamma_n\delta_n. \tag{4.3.25}$$

It then follows from (4.3.18), (4.3.24), (4.3.25) and Lemma 2.3.26 that $\{x_n\}$ converges strongly to $\bar{x}$ which solves the variational inequality (4.3.4). □

Corollary 4.3.3. *Let X be an Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : X \to X$ be a nonexpansive single-valued mapping such that $\Gamma := F(T) \cap argmin_{y\in X} f(y)$ is nonempty. Suppose that g is a contraction mapping defined on X with coefficient $\rho \in (0,1)$ and $\lambda_n \geq \lambda > 0$ for some λ. Let $x_1 \in X$ be arbitrarily chosen and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J_{\lambda_n}^f(x_n), \\ x_{n+1} = t_n g(u_n) \oplus (1-t_n)Tu_n, \end{cases} \tag{4.3.26}$$

for each $n \geq 1$, where $\{t_n\}$ is a sequence in $(0,1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$,
(iv) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.
Then, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$, which also solves the variational inequality

$$\langle \overrightarrow{\overline{x}g(\overline{x})}, \overrightarrow{x\overline{x}} \rangle \geq 0, \ \ x \in \Gamma. \tag{4.3.27}$$

By setting $g(x) = u$ for arbitrary but fixed $u \in X$ and for all $x \in X$, in Corollary 4.3.3, we obtain the following result which coincides with [174, Theorem 3.1].

Corollary 4.3.4. *Let X be an Hadamard space and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semi-continuous function. Let $T : X \to X$ be a nonexpansive single-valued mapping such that $\Gamma := F(T) \cap argmin_{y \in X} f(y)$ is nonempty. Suppose that $\lambda_n \geq \lambda > 0$ for some λ. Let $u, x_1 \in X$ be arbitrarily chosen and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J^f_{\lambda_n}(x_n), \\ x_{n+1} = t_n u \oplus (1 - t_n) T u_n, \end{cases} \tag{4.3.28}$$

for each $n \geq 1$, where $\{t_n\}$ is a sequence in $(0, 1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$,
(iv) $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.
Then the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which also solves

$$\langle \overrightarrow{\overline{x}u}, \overrightarrow{x\overline{x}} \rangle \geq 0, \ \ x \in \Gamma, \tag{4.3.29}$$

which by Lemma 2.3.21 implies that $\bar{x} = P_\Gamma u$. In other words, the sequence $\{x_n\}$ converges strongly to $\overline{x} \in \Gamma$ which is the nearest point of Γ to u.

The following corollary of Theorem 4.3.2 coincides with Theorem 2.3 of [161].

Corollary 4.3.5. *Let X be an Hadamard space and $T : X \to X$ be a nonexpansive single-valued mapping such that $F(T)$ is nonempty. Suppose that $u, x_1 \in X$ are arbitrarily chosen and the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = t_n u \oplus (1 - t_n) T x_n, \tag{4.3.30}$$

for each $n \geq 1$, where $\{t_n\}$ is a sequence in $(0, 1)$ satisfying
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,

(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.
Then, the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in F(T)$ *which is the nearest point of* $F(T)$ *to* u.

The following corollary of Theorem 4.3.2 coincides with Theorem 3.2 of [29].

Corollary 4.3.6. *Let* X *be an Hadamard space and* $T : X \to P(X)$ *be a nonexpansive multivalued mapping such that* $Tp = \{p\}$*, for each* $p \in F(T)$ *and* $F(T) \neq \emptyset$*. Suppose that* g *is a contraction mapping defined on* X *with coefficient* $\rho \in (0, 1)$*. Let* $x_1 \in X$ *be arbitrarily chosen and the sequence* $\{x_n\}$ *be generated by*

$$x_{n+1} = t_n g(x_n) \oplus (1 - t_n)v_n, \quad v_n \in Tx_n, \tag{4.3.31}$$

for each $n \geq 1$*, where* $\{t_n\}$ *is a sequence in* $(0, 1)$ *satisfying*
(i) $\lim_{n\to\infty} t_n = 0$,
(ii) $\sum_{n=1}^{\infty} t_n = \infty$,
(iii) $\sum_{n=1}^{\infty} |t_{n+1} - t_n| < \infty$.
Then the sequence $\{x_n\}$ *converges strongly to* $\bar{x} \in F(T)$ *which also solves the variational inequality*

$$\langle \overrightarrow{\bar{x}g(\bar{x})}, \overrightarrow{x\bar{x}} \rangle \geq 0, \ \ x \in F(T). \tag{4.3.32}$$

Remark 4.3.7. *(1) Our main results generalize and extend the results of Suparatulatorn et al. [174] from approximating a common solution of minimization problem and fixed point problem for singlevalued nonexpansive mapping to approximating a common solution of minimization problem and fixed point problem for multivalued nonexpansive mapping which is also a unique solution of some variational inequalities (see Corollary 4.3.4). Furthermore, our algorithm (Algorithm 4.3.16) has the potential of converging faster than Algorithm* (2.2.5) *studied by Suparatulatorn et al. [174], since our algorithm is of viscosity-type. Examples are given below to further illustrate this (see Figures 1 and 2).*

(2) Our results also extend the results of Bo and Yi [29] from approximating a fixed point of nonexpansive multivalued mapping to approximating a fixed point of nonexpansive multivalued mapping which is also a solution of minimization problem (see Corollary 4.3.6).

(3) Our theorem (Theorem 4.3.2) extends Theorem 2.3 of Saejung [161] (which is a Halpern's convergence theorem) from approximating a fixed point of a singlevalued mapping to approximating a fixed point of a multivalued mapping which is also a minimizer of a proper convex and lower semicontinuous function and a unique solution of some variational inequalities (see Corollary 4.3.5).

4.3.2 Numerical examples

In this subsection, we present two numerical examples of our algorithm (Algorithm 4.3.16) in $\mathbb{R}^2$ and in an Hadamard space to show its advantage over existing algorithms in the

literature.
Throughout this section, we shall take $t_n = \frac{1}{n+1}$ $\forall n \geq 1$ and $g(x) = \frac{1}{2}x$ $\forall x \in X$.

Example 4.3.8. *Let $X = \mathbb{R}^2$ be endowed with the Euclidean norm $||.||_2$. Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (-x_1, x_2)$. Then, T is a nonexpansive mapping.*
Let $f : \mathbb{R}^2 \to (-\infty, +\infty]$ be defined by $f(x) = ||x||_1 + \frac{1}{2}||x||_2^2 + (1, -2)^T x + 8$, then f is a proper convex and lower semi-continuous function. Thus, by using the soft thresholding operator (see [80]) and the proximity opperator (see [60]), we obtain that

$$\begin{aligned} J_1^f(x) &= arg\min_{y\in\mathbb{R}^2}[f(y) + \frac{1}{2}||x-y||^2] \\ &= prox_f x \\ &= prox_{\frac{||.||_1}{2}}\left(\frac{x-(1,-2)^T}{2}\right) \\ &= \Bigg(\max\left\{\frac{|x_1-1|-1}{2}, 0\right\} sgn(x_1-1), \\ &\quad \max\left\{\frac{|x_2+2|-1}{2}, 0\right\} sgn(x_2+2)\Bigg)^T, \end{aligned}$$

where $sgn(.)$ is the signum function of $\alpha \in \mathbb{R}$ defined by

$$sgn(\alpha) = \begin{cases} 1, & if\ \alpha > 0 \\ 0, & if\ \alpha = 0 \\ -1 & if\ \alpha < 0. \end{cases} \tag{4.3.33}$$

Example 4.3.9. *Let $X = \mathbb{R}^2$ be endowed with a metric $d_X : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by*

$$d_X(x,y) = \sqrt{(x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2}\ \forall x, y \in \mathbb{R}^2.$$

Then, $(\mathbb{R}^2, d_X)$ is a complete CAT(0) space (see [194, Example 5.2]) with the geodesic joining x to y given by

$$(1-t)x \oplus ty = \left((1-t)x_1 + ty_1, ((1-t)x_1 + ty_1)^2 - (1-t)(x_1^2 - x_2) - t(y_1^2 - y_2)\right).$$

Now define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x_1, x_2) = (x_1, 2x_1^2 - x_2)$. Clearly, T is not a nonexpansive mapping in the classical sense. However, it is easy to check that T is nonexpansive in $(\mathbb{R}^2, d_X)$. Indeed, for all $x, y \in \mathbb{R}^2$,

$$\begin{aligned} d_X(Tx, Ty) &= \sqrt{(x_1-y_1)^2 + (x_1^2 - (2x_1^2 - x_2) - y_1^2 + (2y_1^2 - y_2))^2} \\ &= \sqrt{(x_1-y_1)^2 + (x_1^2 - x_2 - y_1^2 + y_2)^2} \\ &= d_X(x,y). \end{aligned}$$

Again, define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x_1, x_2) = 100((x_2+1) - (x_1+1)^2)^2 + x_1^2$. Then f is a proper convex and lower semi-continuous function in $(\mathbb{R}^2, d_X)$ but not convex in the classical sense (see [194]).

Using Example 1 and 2, we compare our algorithm (Algorithm (4.3.16)) with Algorithm (2.2.15) of Saejung [161], the algorithm of Bo and Yi (see [29, Algorithm (3.7)]) and Algorithm (2.2.5) of Suparatulatorn *et al.* [174], by considering the following 4 cases (see Figures 4.1 and 4.2):

Case 1: $x_1 = (0.5, -0.25)^T$ and $u = (2, 8)^T$,
Case 2: $x_1 = (1, 3)^T$ and $u = (2, 8)^T$,
Case 3: $x_1 = (-1, -3)^T$ and $u = (0.5, 1)^T$,
Case 4: $x_1 = (-1, -3)^T$ and $u = (-0.5, -1)^T$.

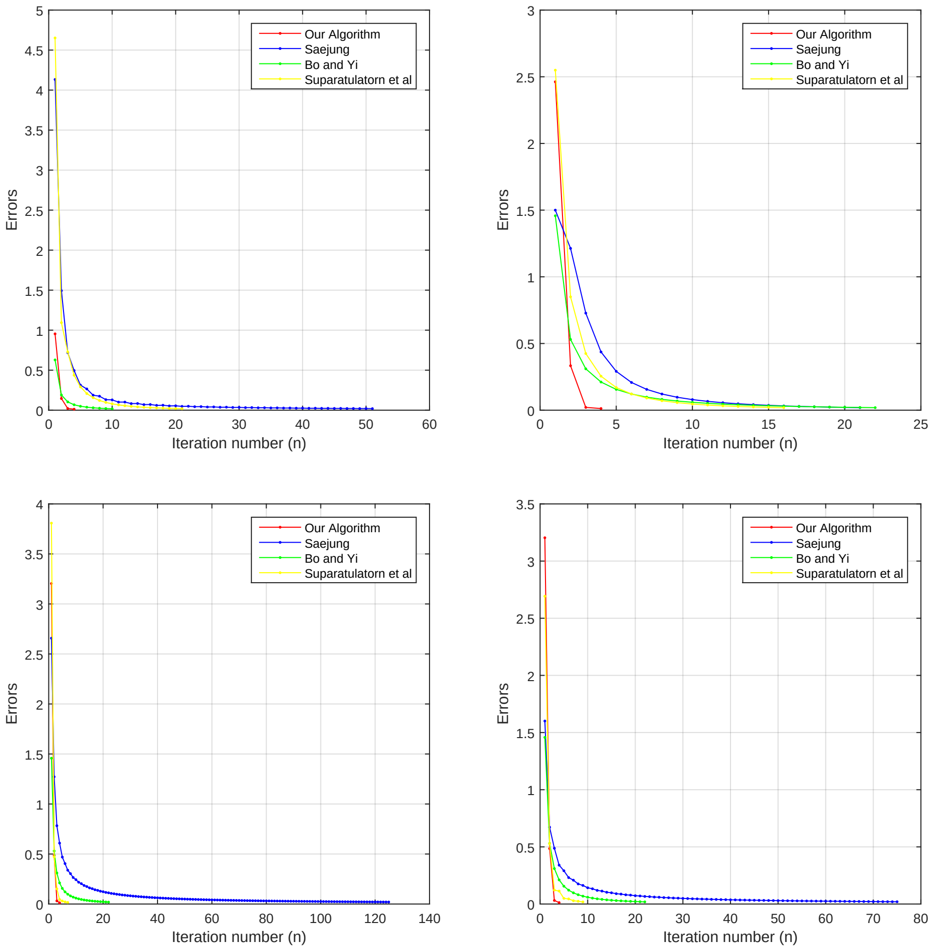

Figure 4.1: Errors vs Iteration numbers for **Example 1**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

Remark 4.3.10. *We can see from the graphs that our viscosity-type algorithm converges faster than the Halpern-type algorithms studied by Saejung [161] and Suparatulatorn et al. [174]. Observe also that, although the algorithm studied by Bo and Yi [29] is also of viscosity-type, our algorithm performs better than it. One possible reason for this could be because of the fact that our viscosity-type iteration is more closer to the proximal point algorithm compared to that of Bo and Yi [29]. In fact, this could also be the reason behind the better performance of Algorithm (2.2.5) of Suparatulatorn et al. [174] compared to Algorithm (2.2.15) of Saejung [161] as shown by the numerical results.*

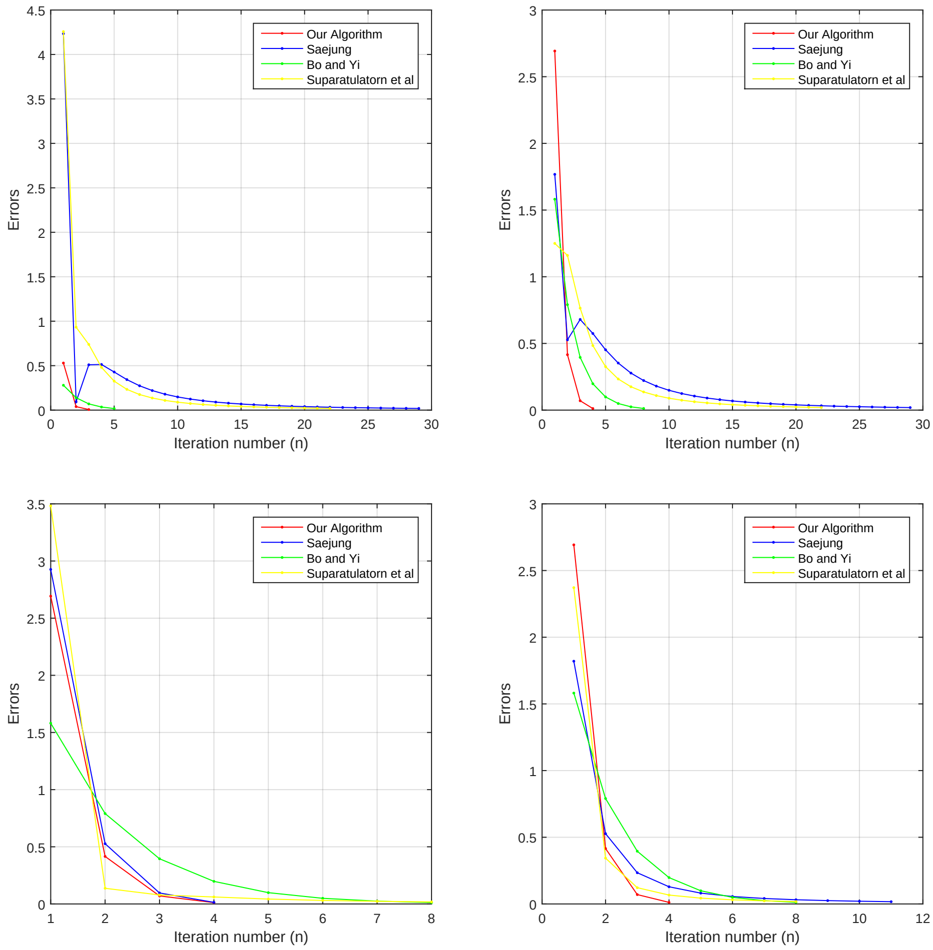

Figure 4.2: Errors vs Iteration numbers for **Example 2**: **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

4.4 A modified Halpern iteration process for solving minimization, monotone inclusion and fixed point problems in Hadamard spaces

In this section, we propose a modified Halpern algorithm and prove its strong convergence to a zero of a monotone operator which is also a minimizer of a proper convex and lower semicontinuous function and a fixed point of a demicontractive-type mapping (see Remark 2.1.39 for the importance of demicontractive-type mappings in optimization theory) in Hadamard spaces. Furthermore, we applied our result to approximate a common solution of MP, MIP and fixed point problem for demimetric mappings (recently introduced in Hadamard spaces [8]) in Hadamard spaces.

4.4.1 Main results

Definition 4.4.1. *Let X be a CAT(0) space. A mapping $T : X \to X$ is called k-demicontractive-type, if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that*

$$d^2(Tx, y) \leq d^2(x, y) + kd^2(Tx, x) \;\forall x \in X,\; y \in F(T). \tag{4.4.1}$$

Proposition 4.4.2. *Let X be a CAT(0) space and $T : X \to X$ be a k-demicontractive-type mapping with $k \in (-\infty, 1)$. Then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to x^*.

Case 1. If $k \in [0, 1)$, then from (4.4.1) we obtain that

$$\begin{aligned} d^2(Tx^*, x_n) &\leq d^2(x^*, x_n) + kd^2(Tx^*, x^*) \\ &\leq [d(x^*, x_n) + \sqrt{k}d(Tx^*, x^*)]^2, \end{aligned}$$

which implies

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_n) + \sqrt{k}d(x_n, Tx^*) \\ &\leq 2d(x^*, x_n) + \sqrt{k}d(x^*, Tx^*). \end{aligned}$$

This implies that $(1 - \sqrt{k})d(x^*, Tx^*) \leq 2d(x^*, x_n) \to 0$ as $n \to \infty$. Since $\sqrt{k} < 1$, it follows that $x^* \in F(T)$.

Case 2. Suppose $k \in (-\infty, 0)$, we obtain from (4.4.1) that

$$\begin{aligned} d^2(Tx^*, x_n) &\leq d^2(x^*, x_n) + kd^2(Tx^*, x^*) \\ &\leq d^2(x^*, x_n). \end{aligned}$$

Taking limits of both sides, we obtain that $d^2(Tx^*, x^*) \leq 0$. Hence $x^* \in F(T)$. Thus, from **Case 1** and **Case 2**, we conclude that $F(T)$ is closed.

Next, we show that F(T) is convex. Let $z = tx \oplus (1-t)y$ for each $x, y \in F(T)$ and $t \in [0,1]$. By Lemma 2.3.1(ii), (2.1.1) and (4.4.1), we obtain

$$\begin{aligned}
d^2(z, Tz) &= d(tx \oplus (1-t)y, Tz) \\
&\le td^2(x, Tz) + (1-t)d^2(y, Tz) - t(1-t)d^2(x, y) \\
&\le t[d^2(x, z) + kd^2(z, Tz)] + (1-t)[d^2(y, z) + kd^2(z, Tz)] - t(1-t)d^2(x, y) \\
&= t[(1-t)^2 d^2(x, y) + kd^2(z, Tz)] + (1-t)[t^2 d^2(x, y) + kd^2(z, Tz)] - t(1-t)d^2(x, y) \\
&= kd^2(z, Tz).
\end{aligned}$$

Since $k < 1$ it follows that $z \in F(T)$ and so $F(T)$ is convex.

□

Lemma 4.4.3. *Let X be a CAT(0) space and $T : X \to X$ be a k-demicontractive-type mapping with $k \in (-\infty, \beta]$ and $\beta \in (0,1)$. Let $T_\beta x = \beta x \oplus (1-\beta)Tx$, then T_β is quasi-nonexpansive and $F(T_\beta) = F(T)$.*

Proof. Let $x \in X$ and $q \in F(T)$. From (4.4.1) and from Lemma 2.3.1 (ii), we have

$$\begin{aligned}
d^2(T_\beta x, q) &= d^2(\beta x \oplus (1-\beta)Tx, q) \\
&\le \beta d^2(x, q) + (1-\beta)d^2(Tx, q) - \beta(1-\beta)d^2(Tx, x) \\
&\le \beta d^2(x, q) + (1-\beta)[d^2(q, x) + kd^2(x, Tx)] - \beta(1-\beta)d^2(Tx, x) \\
&= d^2(q, x) + (1-\beta)(k-\beta)d^2(x, Tx) \\
&\le d^2(x, q).
\end{aligned}$$

Therefore, T_β is quasinonexpansive.
We next show that $F(T_\beta) = F(T)$. Let $x \in F(T_\beta)$, then $x = T_\beta x$. So,

$$\begin{aligned}
d(x, Tx) &= d(\beta x \oplus (1-\beta)Tx, Tx) \\
&\le \beta d(x, Tx),
\end{aligned}$$

which implies that $(1-\beta)d(x, Tx) \le 0$. By the condition on β, we obtain that $d(x, Tx) \le 0$. Therefore, $x \in F(T)$, and thus $F(T_\beta) \subseteq F(T)$.
Similarly, let $x \in F(T)$, then $x = Tx$. By Lemma 2.3.1, we obtain

$$\begin{aligned}
d(x, T_\beta x) &= d(Tx, \beta x \oplus (1-\beta)Tx) \\
&\le \beta d(Tx, x) + (1-\beta)d(Tx, Tx) = 0,
\end{aligned}$$

which implies that $d(x, T_\beta x) = 0$, thus $x \in F(T_\beta)$ and therefore $F(T) \subseteq F(T_\beta)$. Hence, we obtain the desired result. □

Lemma 4.4.4. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a k-demicontractive-type mapping with $k \in (-\infty, \beta]$ and $\beta \in (0,1)$. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Suppose that $F(T_\beta) \cap F(J^A_{\lambda_2}) \cap F(J^f_{\mu_2}) \neq \emptyset$, then for $0 < \lambda_1 \le \lambda_2$ and $0 < \mu_{21} < \mu_2$, we have that*

$$F(T_\beta \circ J^A_{\lambda_2} \circ J^f_{\mu_2}) \subseteq F(T) \cap F(J^A_{\lambda_1}) \cap F(J^f_{\mu_1}),$$

where $T_\beta x = \beta x \oplus (1-\beta)Tx$.

Proof. Let $x \in F(T_\beta \circ J^A_{\lambda_2} \circ J^f_{\mu_2})$ and $y \in F(T_\beta) \cap F(J^A_{\lambda_2}) \cap F(J^f_{\mu_2})$. Then, from Lemma 4.4.3, we obtain that

$$\begin{aligned} d^2(x,y) &= d^2(T_\beta(J^A_{\lambda_2}(J^f_{\mu_2}x)), y) \\ &\leq d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y). \end{aligned} \tag{4.4.2}$$

From Lemma 3.2.5 and (4.4.2), we have

$$\begin{aligned} d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), J^f_{\mu_2}x) &\leq d^2(J^f_{\mu_2}x, y) - d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y) \\ &\leq d^2(x,y) - d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y) \\ &\leq d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y) - d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y), \end{aligned}$$

which implies

$$J^A_{\lambda_2}(J^f_{\mu_2}x) = J^f_{\mu_2}x. \tag{4.4.3}$$

Furthermore, from Lemma 4.2.2, we have

$$\frac{1}{2\mu_2}d^2(J^f_{\mu_2}x, y) - \frac{1}{2\mu_2}d^2(x,y) + \frac{1}{2\mu_2}d^2(x, J^f_{\mu_2}x) + f(J^f_{\mu_2}x) \leq f(y). \tag{4.4.4}$$

Since $f(y) \leq f(J^f_{\mu_2}x)$, we obtain from (4.4.2) that

$$\begin{aligned} d^2(J^f_{\mu_2}x, x) &\leq d^2(x,y) - d^2(J^f_{\mu_2}x, y) \\ &\leq d^2(x,y) - d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y) \\ &\leq d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y) - d^2(J^A_{\lambda_2}(J^f_{\mu_2}x), y), \end{aligned}$$

which implies

$$J^f_{\mu_2}x = x. \tag{4.4.5}$$

From (4.4.3) and (4.4.5), we obtain

$$x = J^A_{\lambda_2}(J^f_{\mu_2}x) = J^A_{\lambda_2}x. \tag{4.4.6}$$

Also, we obtain from Lemma 4.4.3 and (4.4.6) that

$$x = T_\beta(J^A_{\lambda_2}(J^f_{\mu_2}x)) = T_\beta x = Tx. \tag{4.4.7}$$

Furthermore, by setting $z = J^f_{\mu_1}x$ in Lemma 4.2.2, we obtain that

$$f(J^f_{\mu_2}x) + \frac{1}{2\mu_2}d^2(J^f_{\mu_2}x, x) \leq f(J^f_{\mu_1}x) + \frac{1}{2\mu_2}d^2(J^f_{\mu_1}x, x). \tag{4.4.8}$$

Similarly, we obtain

$$f(J^f_{\mu_1}x) + \frac{1}{2\mu_1}d^2(J^f_{\mu_1}x, x) \leq f(J^f_{\mu_2}x) + \frac{1}{2\mu_1}d^2(J^f_{\mu_2}x, x). \tag{4.4.9}$$

Adding (4.4.8) and (4.4.9), we obtain that

$$d^2(J^f_{\mu_1}x, x) - \frac{\mu_1}{\mu_2} d^2(J^f_{\mu_1}x, x) \leq d^2(J^f_{\mu_2}x, x) - \frac{\mu_1}{\mu_2} d^2(J^f_{\mu_2}x, x).$$

That is,

$$\left(1 - \frac{\mu_1}{\mu_2}\right) d^2(J^f_{\mu_1}x, x) \leq \left(1 - \frac{\mu_1}{\mu_2}\right) d^2(J^f_{\mu_2}x, x).$$

Since $0 < \mu_1 < \mu_2$, we obtain that

$$d(J^f_{\mu_1}x, x) \leq d(J^f_{\mu_2}x, x).$$

Thus, from (4.4.5), we obtain that $x \in F(J^f_{\mu_1})$.

Also, from Theorem 2.1.32 (iii) and (4.4.6), we obtain that $x \in F(J^A_{\lambda_1})$. Therefore, we conclude that $F(T_\beta \circ J^A_{\lambda_2} \circ J^f_{\mu_2}) \subseteq F(T) \cap F(J^A_{\lambda_1}) \cap F(J^f_{\mu_1})$. □

Theorem 4.4.5. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a k-demicontractive-type mapping with $k \in (-\infty, \beta]$ and $\beta \in (0, 1)$ such that T is Δ-demiclosed. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator that satisfies the range condition and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \cap \arg\min_{y\in X} f(y) \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $u, x_1 \in X$ by*

$$\begin{cases} y_n = (1 - \alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1 - \gamma_n)y_n \oplus \gamma_n T_\beta \circ J^A_{\lambda_n} \circ J^f_{\mu_n} y_n, \\ x_{n+1} = (1 - \beta_n)y_n \oplus \beta_n z_n, \ \ n \geq 1, \end{cases} \tag{4.4.10}$$

where $T_\beta x = \beta x \oplus (1 - \beta)Tx$, $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $(0, \infty)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $0 < \mu < \mu_n$ and $0 < \lambda \leq \lambda_n$ for all $n \geq 1$,

C2: $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

C3: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Proof. Let $p \in \Gamma$, then from (4.4.10), Lemma 4.4.3 and Lemma 2.3.1 (ii), we obtain

$$\begin{aligned} d^2(z_n, p) &= d^2((1 - \gamma_n)y_n \oplus \gamma_n T_\beta(J^A_{\lambda_n}(J^f_{\mu_n} y_n)), p) \\ &\leq (1 - \gamma_n)d^2(y_n, p) + \gamma_n d^2(T_\beta(J^A_{\lambda_n}(J^f_{\mu_n} y_n)), p) - \gamma_n(1 - \gamma_n)d^2(y_n, T_\beta(J^A_{\lambda_n}(J^f_{\mu_n} y_n))) \\ &\leq d^2(y_n, p) - \gamma_n(1 - \gamma_n)d^2(y_n, T_\beta(J^A_{\lambda_n}(J^f_{\mu_n} y_n))) \\ &\leq d^2(y_n, p). \end{aligned} \tag{4.4.11}$$

From (4.4.10) and (2.1.1), we obtain that

$$\begin{aligned} d(x_{n+1}, y_n) &= d((1-\beta_n)y_n \oplus \beta_n z_n, y_n) \\ &= \beta_n d(z_n, y_n), \end{aligned} \tag{4.4.12}$$

which implies that

$$\begin{aligned} d^2(z_n, y_n) &= \frac{1}{\beta_n^2} d^2(x_{n+1}, y_n) \\ &= \frac{\alpha_n}{\beta_n}\left(\frac{d^2(x_{n+1}, y_n)}{\alpha_n \beta_n}\right). \end{aligned} \tag{4.4.13}$$

Again, from (4.4.10), (4.4.11) and (4.4.13), we obtain

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\beta_n)d^2(y_n, p) + \beta_n d^2(z_n, p) - \beta_n(1-\beta_n)d^2(y_n, z_n) \\ &\leq d^2(y_n, p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &\leq d^2(y_n, p), \end{aligned} \tag{4.4.14}$$

which implies from Lemma 2.3.1 (i) that

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(u, p) \\ &\leq \max\{d(x_n, p), d(u, p)\} \\ &\ \vdots \\ &\leq \max\{d(x_1, p), d(u, p)\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Furthermore, we obtain from (4.4.14) and Lemma 2.3.1 (iii) that

$$\begin{aligned} d^2(x_{n+1}, p) &\leq \alpha_n^2 d^2(u, p) + (1-\alpha_n)^2 d^2(x_n, p) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{x_n p}\rangle \\ &\quad - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n^2 d^2(u, p) - 2\alpha_n(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{px_n}\rangle \\ &\quad - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1}, y_n) \\ &= (1-\alpha_n)d^2(x_n, p) + \alpha_n(-\delta_n), \end{aligned} \tag{4.4.15}$$

$$\text{where } \delta_n = -\alpha_n d^2(u, p) + 2(1-\alpha_n)\langle \overrightarrow{up}, \overrightarrow{px_n}\rangle + \frac{1}{\alpha_n \beta_n}(1-\beta_n)d^2(x_{n+1}, y_n). \tag{4.4.16}$$

Since $\{x_n\}$ is bounded, it is bounded below. Thus, $\{\delta_n\}$ is bounded below, which implies that $\{-\delta_n\}$ is bounded above. Therefore, we obtain from condition C2 of Theorem 4.4.5, and Lemma 2.3.27 that

$$\begin{aligned} \limsup_{n\to\infty} d^2(x_n, p) &\leq \limsup_{n\to\infty}(-\delta_n) \\ &= -\liminf_{n\to\infty} \delta_n, \end{aligned} \tag{4.4.17}$$

which implies that $\liminf_{n\to\infty}\delta_n \leq -\limsup_{n\to\infty} d^2(x_n, p)$. Hence, $\liminf_{n\to\infty}\delta_n$ exists. Thus, we obtain from (4.4.16) and condition C2 that

$$\liminf_{n\to\infty} \delta_n = \liminf_{n\to\infty} \left(2\langle \overrightarrow{up}, \overrightarrow{px_n} \rangle + \frac{1}{\alpha_n \beta_n}(1-\beta_n) d^2(x_{n+1}, y_n) \right).$$

Since $\{x_n\}$ is bounded and X is complete, we obtain from Lemma 2.3.5 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z \in X$ and

$$\liminf_{n\to\infty} \delta_n = \lim_{k\to\infty} \left(2\langle \overrightarrow{up}, \overrightarrow{px_{n_k}} \rangle + \frac{1}{\alpha_{n_k} \beta_{n_k}}(1-\beta_{n_k}) d^2(x_{n_k+1}, y_{n_k}) \right). \tag{4.4.18}$$

Using the fact that $\{x_n\}$ is bounded and $\liminf_{n\to\infty}\delta_n$ exists, we get that $\left\{\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k})d^2(x_{n_k+1}, y_{n_k})\right\}$ is bounded. By condition C3, we obtain that $\frac{1}{\alpha_{n_k}\beta_{n_k}}(1-\beta_{n_k}) \geq \frac{1}{\alpha_{n_k}\beta_{n_k}}(1-b) > 0$. Thus, $\left\{\frac{1}{\alpha_{n_k}\beta_{n_k}}d^2(x_{n_k+1}, y_{n_k})\right\}$ is bounded. Also, from condition C3, we obtain that $0 < \frac{\alpha_{n_k}}{\beta_{n_k}} \leq \frac{\alpha_{n_k}}{a} \to 0,\ k \to \infty$.

Therefore, we obtain from (4.4.13) that

$$\lim_{k\to\infty} d(z_{n_k},\ y_{n_k}) = 0. \tag{4.4.19}$$

From (4.4.12), (4.4.19) and condition C3, we obtain that

$$\lim_{k\to\infty} d(x_{n_k+1}, y_{n_k}) = 0. \tag{4.4.20}$$

From (4.4.11), we obtain

$$\begin{aligned}
\gamma_{n_k}(1-\gamma_{n_k})d^2(y_{n_k}, T_\beta y_{n_k}) &\leq d^2(y_{n_k}, p) - d^2(z_{n_k}, p) \\
&\leq d^2(y_{n_k}, z_{n_k}) + 2d(y_{n_k}, z_{n_k})d(z_{n_k}, p),
\end{aligned}$$

which implies from condition C3 that

$$\lim_{k\to\infty} d(y_{n_k}, T_\beta(J^A_{\lambda_{n_k}}(J^f_{\mu_{n_k}} y_{n_k}))) = 0. \tag{4.4.21}$$

Now, Let $v_{n_k} = J^A_{\lambda_{n_k}}(J^f_{\mu_{n_k}}(y_{n_k}))$. Then, from Lemma 3.2.5 and (4.4.21), we obtain that

$$\begin{aligned}
d^2(v_{n_k}, J^f_{\mu_{n_k}}(y_{n_k})) &\leq d^2(J^f_{\mu_{n_k}}(y_{n_k}), p) - d^2(v_{n_k}, p) \\
&\leq d^2(y_{n_k}, p) - d^2(T_\beta v_{n_k}, p) \\
&\leq d^2(y_{n_k}, T_\beta v_{n_k}) + 2d(y_{n_k}, T_\beta v_{n_k})d(T_\beta v_{n_k}, p) \to 0.
\end{aligned} \tag{4.4.22}$$

Also, from Lemma 4.2.2 and noting that $f(p) \leq f(J^f_{\mu_{n_k}}(y_{n_k}))$, we obtain that

$$\begin{aligned}
d^2(J^f_{\mu_{n_k}}(y_{n_k}), y_{n_k}) &\leq d^2(y_{n_k}, p) - d^2(J^f_{\mu_{n_k}}(y_{n_k}), p) \\
&\leq d^2(y_{n_k}, p) - d^2(T_\beta v_{n_k}, p) \to 0, \text{ as } k \to \infty.
\end{aligned} \tag{4.4.23}$$

From (4.4.22) and (4.4.23), we obtain that

$$\lim_{k\to\infty} d(v_{n_k}, y_{n_k}) = \lim_{k\to\infty} d(J^A_{\lambda_{n_k}}(J^f_{\mu_{n_k}}(y_{n_k})), y_{n_k}) = 0. \tag{4.4.24}$$

Furthermore, from (4.4.21) and (4.4.24), we obtain

$$\lim_{k\to\infty} d(v_{n_k}, T_\beta v_{n_k}) = 0. \tag{4.4.25}$$

Again, from (4.4.10) and condition C2, we obtain

$$d(y_{n_k}, x_{n_k}) \;\; \leq \;\; \alpha_{n_k} d(u, x_{n_k}) \to 0 \text{ as } k \to \infty. \tag{4.4.26}$$

Since $\Delta\text{-}\lim\limits_{k\to\infty} x_{n_k} = z$, we obtain from (4.4.24) and (4.4.26) that $\Delta\text{-}\lim\limits_{k\to\infty} y_{n_k} = z$ and $\Delta\text{-}\lim\limits_{k\to\infty} v_{n_k} = z$. It then follows from Lemma 2.3.12, Lemma 4.4.4, (4.4.24), (4.4.25) and condition C1 that $z \in F(T_\beta \circ J^A_{\lambda_n} \circ J^f_{\mu_n}) \subseteq F(T) \cap F(J^A_\lambda) \cap F(J^f_\mu) = \Gamma$.
Furthermore, since $\Delta\text{-}\lim\limits_{k\to\infty} x_{n_k} = z$, we obtain from Lemma 2.3.10 that

$$\limsup_{k\to\infty} \langle \overrightarrow{uz}, \overrightarrow{zx_{n_k}} \rangle \geq 0 \;\; \text{for arbitrary } u \in X.$$

Thus, from (4.4.18) and (4.4.20), we obtain that

$$\liminf_{n\to\infty} \delta_n = 2 \lim_{k\to\infty} \langle \overrightarrow{uz}, \overrightarrow{zx_{n_k}} \rangle \geq 0. \tag{4.4.27}$$

Hence, from (4.4.17), we obtain that

$$\limsup_{n\to\infty} d^2(x_n, z) \leq -\liminf_{n\to\infty} \delta_n \leq 0.$$

Therefore, $\lim\limits_{n\to\infty} d(x_n, z) = 0$ and this implies that $\{x_n\}$ converges strongly to $z \in \Gamma$. $\square$

Corollary 4.4.6. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a k-demicontractive-type mapping with $k \in (-\infty, \beta]$ and $\beta \in (0, 1)$ such that T is Δ-demiclosed. Suppose the sequence $\{x_n\}$ is generated for arbitrary $u, x_1 \in X$ by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T_\beta y_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \;\; n \geq 1, \end{cases} \tag{4.4.28}$$

where $T_\beta x = \beta x \oplus (1-\beta)Tx$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

C2: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of $F(T)$.

If $k = 0$ and $\{\mu_n\}$, $\{\lambda_n\}$ are constant sequences in Theorem 4.4.5, then we obtain the following result.

Corollary 4.4.7. *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator that satisfies the range condition and $T : X \to X$ be a nonexpansive mapping. Let $f : X \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \cap \arg\min_{y \in X} f(y) \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $u, x_1 \in X$ by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T \circ J_\lambda^A \circ J_\mu^f y_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \ n \geq 1, \end{cases} \tag{4.4.29}$$

where λ and μ are in $(0, \infty)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C2: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Corollary 4.4.8. *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator that satisfies the range condition and $T : X \to X$ be a nonexpansive mapping. Suppose that $\Gamma := A^{-1}(0) \cap F(T) \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $u, x_1 \in X$ by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T \circ J_{\lambda_n}^A y_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \ n \geq 1, \end{cases} \tag{4.4.30}$$

where $\{\lambda_n\}$ is sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $0 < \lambda \leq \lambda_n$ for all $n \geq 1$,

C2: $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

In what follows, we apply our result (Theorem 4.4.5) to approximate a common solution of MP, MIP and fixed point problem for demimetric mappings. We begin with the definition of demimetric mapping.

Definition 4.4.9. *[8] Let X be a CAT(0) space. A mapping $T : X \to X$ is said to be k-demimetric, if $F(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that*

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x, Tx) \ \forall x \in X, y \in F(T). \tag{4.4.31}$$

Theorem 4.4.10. *Let X be an Hadamard space and X^* be its dual space. Let $T : X \to X$ be a k-demimetric mapping with $k \in (-\infty, \beta]$ and $\beta \in (0,1)$ such that T is Δ-demiclosed. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator that satisfies the range condition and $f : X \to (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \cap \arg\min_{y \in X} f(y) \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $u, x_1 \in X$ by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n T_\beta \circ J_{\lambda_n}^A \circ J_{\mu_n}^f y_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n, \ \ n \geq 1, \end{cases} \tag{4.4.32}$$

where $T_\beta x = \beta x \oplus (1-\beta)Tx$, $\{\lambda_n\}$ and $\{\mu_n\}$ are sequences in $(0, \infty)$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$ such that the following conditions are satisfied:

C1: $0 < \mu < \mu_n$ *and* $0 < \lambda \leq \lambda_n$ *for all* $n \geq 1$,

C2: $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C3: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Proof. Since T is k-demimetric, then we obtain from the definition of quasilinearization mapping that

$$\frac{1}{2}\left[d^2(y,x) + d^2(x,Tx) - d^2(y,Tx)\right] \geq \frac{1-k}{2} d^2(x,Tx).$$

That is

$$d^2(y,Tx) \leq d^2(y,x) + kd^2(x,Tx).$$

Thus, applying Theorem 4.4.5, we obtain the desired conclusion. □

4.5 A modified Ishikawa iteration process for a family of minimization, monotone inclusion and fixed point problems in Hadamard spaces

Here, we extend the results obtained in Section 4.4 to finite family of MPs, MIPs and fixed point problem for asymptotically demicontractive mapping in Hadamard spaces using a modified Ishikawa iteration process, and we further gave a numerical example of this iteration process to show its applicability.

4.5.1 Main results

Theorem 4.5.1. *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \dots N$ be multivalued monotone operators that satisfy the range condition and $f_j : X \to (-\infty, \infty]$, $j = 1, 2, \dots, m$ be proper convex and lower semi continuous functions. Let $T : X \to X$ be a uniformly L-Lipschitzian and asymptotically demicontractive mapping with constant $k \in (0, 1)$ and sequence $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n^2 - 1) < \infty$. Suppose that $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} A_i^{-1}(0)\right) \cap \left(\cap_{j=1}^{m} arg \min\limits_{y \in X} f_j(y)\right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = \Phi_\lambda^N \circ \Psi_\mu^m(v_n), \\ y_n = (1 - \beta_n)u_n \oplus \beta_n T^n u_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n T^n y_n, \ n \geq 1, \end{cases} \tag{4.5.1}$$

where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$, $\Phi_\lambda^0 = I$, $\Psi_\mu^m = J_\mu^m \circ J_\mu^{m-1} \circ \cdots \circ J_\mu^2 \circ J_\mu^1$, $\Phi_\mu^0 = I$, $\lambda, \mu \in (0, \infty)$ and $\{t_n\}, \{\beta_n\}, \{\alpha_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

C1: $\lim\limits_{n \to \infty} t_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} t_n = \infty$,

C3: $0 < \epsilon \leq \alpha_n \leq k\beta_n < \beta_n \leq b < \frac{2}{\left(\frac{k+\delta}{k}\right) + \sqrt{\left(\frac{k+\delta}{k}\right)^2 + 4L^2}}$, *where* $\delta := \sup\limits_{n \geq 1} k_n$,

C4: $\lim\limits_{n \to \infty} \frac{k_n^2 - 1}{t_n} = 0$.

Then $\{x_n\}$ converges strongly to an element of Γ.

Proof. We first observe that since T is uniformly L-Lipschitzian, we obtain from (4.5.1) that

$$\begin{aligned} d(T^n u_n, T^n y_n) &\leq Ld(u_n, y_n) \\ &= Ld(u_n, (1 - \beta_n)u_n \oplus \beta_n T^n u_n) \\ &\leq L\beta_n d(u_n, T^n u_n). \end{aligned} \tag{4.5.2}$$

Let $p \in \Gamma$, then from (4.5.1) and (4.5.2), we have

$$\begin{aligned} d^2(T^n y_n, p) &\leq k_n d^2(y_n, p) + kd^2(y_n, T^n y_n) \\ &= k_n d^2((1 - \beta_n)u_n \oplus \beta_n T^n u_n, p) + kd^2(y_n, T^n y_n) \\ &\leq k_n(1 - \beta_n)d^2(u_n, p) + k_n\beta_n d^2(T^n u_n, p) - k_n\beta_n(1 - \beta_n)d^2(u_n, T^n u_n) \\ &\quad + kd^2((1 - \beta_n)u_n \oplus \beta_n T^n u_n, T^n y_n) \\ &\leq k_n(1 - \beta_n)d^2(u_n, p) + k_n\beta_n d^2(T^n u_n, p) - k_n\beta_n(1 - \beta_n)d^2(u_n, T^n u_n) \\ &\quad + k(1 - \beta_n)d^2(u_n, T^n y_n) + k\beta_n d^2(T^n u_n, T^n y_n) - k\beta_n(1 - \beta_n)d^2(T^n u_n, u_n) \\ &\leq k_n(1 - \beta_n)d^2(u_n, p) + k_n\beta_n d^2(T^n u_n, p) - k_n\beta_n(1 - \beta_n)d^2(u_n, T^n u_n) \\ &\quad + k(1 - \beta_n)d^2(u_n, T^n y_n) + kL^2\beta_n^3 d^2(u_n, T^n u_n) - k\beta_n(1 - \beta_n)d^2(T^n u_n, u_n) \end{aligned}$$

$$\begin{aligned}
&\leq k_n(1-\beta_n)d^2(u_n,p)+k_n\beta_n\left[k_nd^2(u_n,p)+kd^2(u_n,T^nu_n)\right]\\
&\quad -k_n\beta_n(1-\beta_n)d^2(u_n,T^nu_n)+k(1-\beta_n)d^2(u_n,T^ny_n)\\
&\quad +kL^2\beta_n^3d^2(u_n,T^nu_n)-k\beta_n(1-\beta_n)d^2(T^nu_n,u_n)\\
&= (k_n-k_n\beta_n+k_n^2\beta_n)d^2(u_n,p)\\
&\quad -\beta_n\left[(1-\beta_n)(k_n+k)-kL^2\beta_n^2-kk_n\right]d^2(u_n,T^nu_n)\\
&\quad +k(1-\beta_n)d^2(u_n,T^ny_n)\\
&\leq \left[k_n^2(1-\beta_n)+k_n^2\beta_n\right]d^2(u_n,p)+k(1-\beta_n)d^2(u_n,T^ny_n)\\
&\quad -\beta_n\left[(1-\beta_n)(k_n+k)-kL^2\beta_n^2-kk_n\right]d^2(u_n,T^nu_n)\\
&= k_n^2d^2(u_n,p)+k(1-\beta_n)d^2(u_n,T^ny_n)\\
&\quad -\beta_n\left[(1-\beta_n)(k_n+k)-kL^2\beta_n^2-kk_n\right]d^2(u_n,T^nu_n). \qquad (4.5.3)
\end{aligned}$$

Also, from (4.5.1), (4.5.3) and condition C3 , we obtain

$$\begin{aligned}
d^2(x_{n+1},p) &\leq (1-\alpha_n)d^2(u_n,p)+\alpha_nd^2(T^ny_n,p)-\alpha_n(1-\alpha_n)d^2(u_n,T^ny_n)\\
&\leq (1-\alpha_n)d^2(u_n,p)+\alpha_nk_n^2d^2(u_n,p)+k\alpha_n(1-\beta_n)d^2(u_n,T^ny_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(k_n+k)-kL^2\beta_n^2-kk_n\right]d^2(u_n,T^nu_n)\\
&\quad -\alpha_n(1-\alpha_n)d^2(u_n,T^ny_n)\\
&\leq k_n^2d^2(u_n,p)-\alpha_n\left[(1-k)+(k\beta_n-\alpha_n)\right]d^2(u_n,T^ny_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(k_n+k)-kL^2\beta_n^2-kk_n\right]d^2(u_n,T^nu_n)\\
&\leq k_n^2d^2(u_n,p)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(k_n+k)-kL^2\beta_n^2-kk_n\right]d^2(u_n,T^nu_n) \qquad (4.5.4)\\
&\leq \left[1+(k_n^2-1)\right]d^2(u_n,p) \qquad (4.5.5)\\
&= \left[1+(k_n^2-1)\right]d^2(\Phi_\lambda^N\Psi_\mu^m v_n,p)\\
&\leq \left[1+(k_n^2-1)\right]d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n,p)\\
&\;\;\vdots\\
&\leq \left[1+(k_n^2-1)\right]d^2(\Psi_\mu^m v_n,p)\\
&\leq \left[1+(k_n^2-1)\right]d^2(\Psi_\mu^{m-1} v_n,p)\\
&\;\;\vdots\\
&\leq \left[1+(k_n^2-1)\right]d^2(v_n,p)\\
&\leq \left[1+(k_n^2-1)\right]\left[(1-t_n)d^2(x_n,p)+t_nd^2(u,p)\right]\\
&\leq \left[1+(k_n^2-1)\right]\max\{d^2(x_n,p),d^2(u,p)\}\\
&\;\;\vdots\\
&\leq \prod_{r=1}^{n}\left[1+(k_r^2-1)\right]\max\{d^2(x_1,p),d^2(u,p)\}.
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n^2-1)<\infty$, then we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\},\{v_n\}$ are all bounded.

From (4.5.1) and condition C1, we have that

$$\lim_{n\to\infty} d(v_n,x_n)\leq \lim_{n\to\infty} t_nd(u,x_n)=0. \qquad (4.5.6)$$

We now divide our proof into two cases.

Case 1: Suppose that $\{d^2(x_n,p)\}$ is monotonically non-increasing, then $\lim_{n\to\infty}\{d^2(x_n,p)\}$ exists. Consequently,

$$\lim_{n\to\infty}\left[d^2(x_{n+1},p)-d^2(x_n,p)\right]=0. \tag{4.5.7}$$

From Lemma 3.2.4 (i) and (4.5.4), we obtain

$$\begin{aligned} d^2(u_n,\Phi_\lambda^{N-1}\Psi^m v_n) &= d^2(J_\lambda^N(\Phi_\lambda^{N-1}\Psi^m)v_n,\Phi_\lambda^{N-1}\Psi^m v_n)\\ &\leq d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n,p)-d^2(J_\lambda^N(\Phi_\lambda^{N-1}\Psi^m)v_n,p)\\ &\vdots\\ &\leq d^2(v_n,p)-\frac{1}{k_n^2}d^2(x_{n+1},p)\\ &\leq d^2(v_n,x_n)+d^2(x_n,p)+2d(v_n,x_n)d(x_n,p)-\frac{1}{k_n^2}d^2(x_{n+1},p)\\ &= d^2(v_n,x_n)+2d(v_n,x_n)d(x_n,p)\\ &\quad+\left[d^2(x_n,p)-\frac{1}{k_n^2}d^2(x_{n+1},p)\right]. \end{aligned} \tag{4.5.8}$$

Since $\lim_{n\to\infty}k_n=1$ and $\lim_{n\to\infty}d(x_n,p)$ exists, we have from (4.5.6) and (4.5.8) that

$$\lim_{n\to\infty}d^2(u_n,\Phi_\lambda^{N-1}\Psi^m v_n)=0. \tag{4.5.9}$$

Again, from Lemma 2.3.1 and Lemma 3.2.4 (i), we obtain

$$\begin{aligned} d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n,\Phi_\lambda^{N-2}\Psi_\mu^m v_n) &\leq d^2(\Phi_\lambda^{N-2}\Psi_\mu^m v_n,p)-d^2(\Phi_\lambda^{N-1}\Psi^m v_n,p)\\ &\vdots\\ &\leq d^2(v_n,p)-d^2(u_n,p)\\ &\leq d^2(v_n,p)-\frac{1}{k_n^2}d^2(x_{n+1},p)\\ &\leq d^2(v_n,x_n)+2d(v_n,x_n)d(x_n,p)\\ &\quad+\left[d^2(x_n,p)-\frac{1}{k_n^2}d^2(x_{n+1},p)\right]\to 0. \end{aligned} \tag{4.5.10}$$

Continuing in the same manner, we have that

$$\lim_{n\to\infty}d^2(\Phi_\lambda^{N-2}\Psi_\mu^m v_n,\Phi_\lambda^{N-3}\Psi_\mu^m v_n)=\cdots=\lim_{n\to\infty}d^2(\Phi_\lambda^1\Psi_\mu^m v_n,\Psi_\mu^m v_n)=0. \tag{4.5.11}$$

Also, from Lemma 4.2.2, we have

$$\frac{1}{2\mu}d^2(\Psi_\mu^m v_n,p)-\frac{1}{2\mu}d^2(\Psi_\mu^{m-1}v_n,p)+\frac{1}{2\mu}d^2(\Psi_\mu^{m-1}v_n,\Psi_\mu^m p)+f(\Psi_\mu^m v_n)\leq f(p).$$

Since $f(p) \leq f(\Psi_\mu^m v_n)$, we have that

$$\begin{aligned}
d^2(\Psi_\mu^m v_n, \Psi_\mu^{m-1} v_n) &\leq d^2(\Psi_\mu^{m-1} v_n, p) - d^2(\Psi_\mu^m v_n, p) \\
&\vdots \\
&\leq d^2(v_n, p) - d^2(u_n, p) \\
&\leq d^2(v_n, p) - \frac{1}{k_n^2} d^2(x_{n+1}, p) \\
&\leq d^2(v_n, x_n) + 2d(v_n, x_n)d(x_n, p) \\
&\quad + \left[d^2(x_n, p) - \frac{1}{k_n^2} d^2(x_{n+1}, p)\right] \to 0, \text{ as } n \to \infty. \quad (4.5.12)
\end{aligned}$$

Using similar argument as above, we can show that

$$\lim_{n\to\infty} d^2(\Psi_\mu^{m-1} v_n, \Psi_\mu^{m-2} v_n) = \cdots = \lim_{n\to\infty} d^2(\Psi_\mu^2 v_n, \Psi_\mu^1 v_n) = \lim_{n\to\infty} d^2(\Psi_\mu^1 v_n, v_n) = 0. \quad (4.5.13)$$

Thus,

$$\begin{aligned}
d(u_n, v_n) &\leq d(u_n, \Phi_\lambda^{N-1}\Psi_\mu^m v_n) + d(\Phi_\lambda^{N-1}\Psi_\mu^m v_n, \Phi_\lambda^{N-2}\Psi_\mu^m v_n) \\
&\quad + \cdots + d(\Phi_\lambda^1 \Psi_\mu^m v_n, \Psi_\mu^m v_n) + d(\Psi_\mu^m v_n, \Psi_\mu^{m-1} v_n) \\
&\quad + d(\Psi_\mu^{m-1} v_n, \Psi_\mu^{m-2} v_n) + \cdots + d(\Psi_\mu^1 v_n, v_n),
\end{aligned}$$

which implies from (4.5.9), (4.5.10), (4.5.11), (4.5.12) and (4.5.13), that

$$\lim_{n\to\infty} d(u_n, v_n) = \lim_{n\to\infty} d(\Phi_\lambda^N \Psi_\mu^m v_n, v_n) = 0. \quad (4.5.14)$$

From (4.5.6) and (4.5.14), we obtain

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \quad (4.5.15)$$

Since $\{d^2(x_n, p)\}$ and $\{k_n\}$ are bounded, then there exists $M > 0$ such that

$$M := \sup_{n\geq 1}\{d^2(x_n, p), k_n^2\}.$$

Thus, from (4.5.4), we obtain

$$\begin{aligned}
& d^2(x_{n+1}, p) - d^2(x_n, p) \\
\leq\ & k_n^2 d^2(u_n, p) - d^2(x_n, p) \\
& -\alpha_n\beta_n\left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n\right] d^2(u_n, T^n u_n) \\
\leq\ & k_n^2 d^2(u_n, x_n) + k_n^2 d^2(x_n, p) + 2k_n^2 d(u_n, x_n) d(x_n, p) - d^2(x_n, p) \\
& -\alpha_n\beta_n\left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n\right] d^2(u_n, T^n u_n) \\
\leq\ & M d^2(u_n, x_n) + 2M^2 d(u_n, x_n) + (k_n^2 - 1)M \\
& -\alpha_n\beta_n\left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n\right] d^2(u_n, T^n u_n). \quad (4.5.16)
\end{aligned}$$

Since $\sum_{n=1}^\infty (k_n^2 - 1) < \infty$, then $\lim_{n\to\infty}(k_n^2 - 1) = 0$. Thus, letting

$$\delta_n = \alpha_n\beta_n\left[(1-\beta_n)(k_n + k) - kL^2\beta_n^2 - kk_n\right],$$

we obtain from (4.5.16) and (4.5.15) that

$$\begin{aligned}\delta_n d^2(u_n, T^n u_n) &\leq Md^2(u_n, x_n) + 2M^2 d(u_n, x_n) + M(k_n^2 - 1) \\ &\quad + d^2(x_n, p) - d^2(x_{n+1}, p) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{4.5.17}$$

From condition C3, we obtain that $2k - b(k+\delta) > kb\sqrt{\left(\frac{k+\delta}{k}\right)^2 + 4L^2}$. This implies that

$$[2k - b(k+\delta)]^2 > 4L^2k^2b^2 + b^2(k+\delta)^2,$$

which further implies that

$$k - bk - b\delta - kL^2b^2 > 0.$$

Thus,

$$\begin{aligned}\delta_n &= \alpha_n\beta_n\left[(k_n + k) - kk_n - \beta_n(k + k_n) - kL^2\beta_n^2\right] \\ &> \epsilon^2\left[(k_n + k) - k_n - \beta_n(k + k_n) - kL^2\beta_n^2\right] \\ &> \epsilon^2\left[k - bk - bk_n - kL^2b^2\right] \\ &> \epsilon^2\left[k - bk - b\delta - kL^2b^2\right] > 0.\end{aligned}$$

Hence, from (4.5.17), we obtain

$$\lim_{n\to\infty} d(u_n, T^n u_n) = 0. \tag{4.5.18}$$

From (4.5.1) and condition C3, we obtain

$$\lim_{n\to\infty} d(y_n, u_n) \leq \lim_{n\to\infty} \beta_n d(T^n u_n, u_n) = 0. \tag{4.5.19}$$

Since T is uniformly L-Lipschitzian, we obtain from (4.5.18) and (4.5.19) that

$$\begin{aligned}d(x_{n+1}, y_n) &\leq (1-\alpha_n)d(u_n, y_n) + \alpha_n d(T^n y_n, y_n) \\ &\leq (1-\alpha_n)d(u_n, y_n) + \alpha_n d(T^n y_n, T^n u_n) \\ &\quad + \alpha_n\left[d(T^n u_n, u_n) + d(y_n, u_n)\right] \\ &\leq (1-\alpha_n)d(u_n, y_n) + \alpha_n L d(y_n, u_n) \\ &\quad + \alpha_n\left[d(T^n u_n, u_n) + d(y_n, u_n)\right] \to 0, \text{ as } n \to \infty.\end{aligned} \tag{4.5.20}$$

From (4.5.15), (4.5.19) and (4.5.20), we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4.5.21}$$

Again, since T is uniformly L-Lipschitzian, we obtain

$$\begin{aligned}d(u_n, Tu_n) &\leq d(u_n, T^n u_n) + d(T^n u_n, Tu_n) \\ &\leq d(u_n, T^n u_n) + Ld(T^{n-1}u_n, u_n) \\ &\leq d(u_n, T^n u_n) + Ld(T^{n-1}u_n, T^{n-1}u_{n-1}) \\ &\quad + Ld(T^{n-1}u_{n-1}, u_{n-1}) + Ld(u_{n-1}, u_n) \\ &\leq d(u_n, T^n u_n) + L^2 d(u_n, u_{n-1}) \\ &\quad + Ld(T^{n-1}u_{n-1}, u_{n-1}) + Ld(u_{n-1}, u_n) \\ &\leq d(u_n, T^n u_n) + Ld(T^{n-1}u_{n-1}, u_{n-1}) \\ &\quad + (L^2 + L)\left[d(u_n, x_n) + d(x_n, x_{n-1}) + d(x_{n-1}, u_{n-1})\right],\end{aligned}$$

which implies from (4.5.15), (4.5.18) and (4.5.21) that

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0. \tag{4.5.22}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z$. It follows from (4.5.15) that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} u_{n_k} = z$. Thus, from (4.5.22) and Lemma 2.3.18, we obtain that $z \in F(T)$. Also, since $\Phi_\lambda^N \circ \Psi_\mu^m$ is the composition of nonexpansive mappings, it implies that $\Phi_\lambda^N \circ \Psi_\mu^m$ is nonexpansive. Thus, it follows from (4.5.14) and Lemma 2.3.12 that $z \in F(\Phi_\lambda^N \circ \Psi_\mu^m)$. It then follows from Lemma 4.4.4 that $z \in \left(\cap_{i=1}^N A_i^{-1}(0)\right) \cap \left(\cap_{j=1}^m \arg\min_{y\in X} f_j(y)\right)$. Therefore, $z \in \Gamma$.

Furthermore, for arbitrary $u \in X$, we have from Lemma 2.3.10 that

$$\limsup_{n\to\infty} \langle \overrightarrow{uz}, \overrightarrow{x_n z}\rangle \leq 0, \tag{4.5.23}$$

which implies from condition C1 that

$$\limsup_{n\to\infty} \left(t_n d^2(u,z) + 2(1-t_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z}\rangle\right) \leq 0. \tag{4.5.24}$$

Next, we show that $\{x_n\}$ converges strongly to z. Since $\{u_n\}$ is bounded, there exists $M' > 0$ such that $d^2(u_n, z) \leq M'\ \forall\ n \geq 1$. Thus, from (4.5.5) and Lemma 2.3.1 (iii), we obtain

$$\begin{aligned}
d^2(x_{n+1}, z) &\leq d^2(u_n, z) + (k_n^2 - 1)d^2(u_n, z)\\
&\leq d^2(u_n, z) + (k_n^2 - 1)M'\\
&\leq d^2(\Phi_\lambda^{N-1}\Psi_\mu^m v_n, z) + (k_n^2 - 1)M'\\
&\ \vdots\\
&\leq d^2(\Psi_\mu^m v_n, z) + (k_n^2 - 1)M'\\
&\ \vdots\\
&\leq d^2(v_n, z) + (k_n^2 - 1)M'\\
&\leq (1-t_n)^2 d^2(x_n, z) + t_n^2 d^2(u,z) + 2t_n(1-t_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z}\rangle + (k_n^2 - 1)M'\\
&\leq (1-t_n)d^2(x_n, z) + t_n\left(t_n d^2(u,z) + 2(1-t_n)\langle \overrightarrow{uz}, \overrightarrow{x_n z}\rangle\right)\\
&\quad +(k_n^2 - 1)M'.
\end{aligned} \tag{4.5.25}$$

Since $\sum_{n=1}^\infty (k_n^2 - 1)M' < \infty$, it then follows from (4.5.24), (4.5.25) and Lemma 2.3.26 that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{d^2(x_{n_i}, p)\}$ of $\{d^2(x_n, p)\}$ such that $d^2(x_{n_i}, p) < d^2(x_{n_i+1}, p)$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.3.29, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$d^2(x_{m_k}, p) \leq d^2(x_{m_k+1}, p) \text{ and } d^2(x_k, p) \leq d^2(x_{m_k+1}, p)\ \forall k \in \mathbb{N}.$$

Thus, with this and (4.5.5), we obtain

$$
\begin{aligned}
0 &\leq \lim_{k\to\infty}\left(d^2(x_{m_k+1},p)-d^2(x_{m_k},p)\right)\\
&\leq \limsup_{n\to\infty}\left(d^2(x_{n+1},p)-d^2(x_n,p)\right)\\
&\leq \limsup_{n\to\infty}\left(d^2(u_n,p)+(k_n^2-1)d^2(u_n,p)-d^2(x_n,p)\right)\\
&\leq \limsup_{n\to\infty}\left((1-t_n)d^2(x_n,p)+t_nd^2(u,p)+(k_n^2-1)d^2(u_n,p)-d^2(x_n,p)\right)\\
&\leq \limsup_{n\to\infty}\left[t_n\left(d^2(u,p)-d^2(x_n,p)\right)+(k_n^2-1)M'\right]=0,
\end{aligned}
$$

which implies

$$
\lim_{k\to\infty}\left(d^2(x_{m_k+1},p)-d^2(x_{m_k},p)\right)=0. \tag{4.5.26}
$$

Following the same line of argument as in Case 1, we can verify that

$$
\lim_{k\to\infty}\left(t_{m_k}d^2(u,z)+2(1-t_{m_k})\langle\overrightarrow{uz},\overrightarrow{x_nz}\rangle\right)\leq 0. \tag{4.5.27}
$$

Also from (4.5.25), we have

$d^2(x_{m_k+1},z)\leq(1-t_{m_k})d^2(x_{m_k},z)+t_{m_k}\left(t_{m_k}d^2(u,z)+2(1-t_{m_k})\langle\overrightarrow{uz},\overrightarrow{x_{m_k}z}\rangle\right)+(k_{m_k}^2-1)M'.$

Since $d^2(x_{m_k},z)\leq d^2(x_{m_k+1},z)$, we have

$$
d^2(x_{m_k},z)\leq\left(t_{m_k}d^2(u,z)+2(1-t_{m_k})\langle\overrightarrow{uz},\overrightarrow{x_{m_k}z}\rangle\right)+\frac{(k_{m_k}^2-1)M'}{t_{m_k}},
$$

which implies from (4.5.27) and condition C4 that

$$
\lim_{k\to\infty}d^2(x_{m_k},z)=0. \tag{4.5.28}
$$

Since $d^2(x_k,z)\leq d^2(x_{m_k+1},z)$, we obtain from (4.5.28) and (4.5.26) that $\lim\limits_{k\to\infty}d^2(x_k,z)=0$. Thus, from Case 1 and Case 2, we conclude that $\{x_n\}$ converges to $z\in\Gamma$. $\square$

If T is a uniformly L-Lipschitzian and asymptotically nonexpansive mapping defined on X, then we obtain the following result.

Corollary 4.5.2. *Let X be an Hadamard space and X^* be its dual space. Let $A_i: X\to 2^{X^*}$, $i=1,2,\dots N$ be multivalued monotone operators that satisfy the range condition and $f_j: X\to(-\infty,\infty]$, $j=1,2,\dots,m$ be proper convex and lower semi continuous functions. Let $T: X\to X$ be a uniformly L-Lipschitzian and asymptotically nonexpansive mapping. Suppose that $\Gamma := F(T)\cap\left(\cap_{i=1}^N A_i^{-1}(0)\right)\cap\left(\cap_{j=1}^m arg\min\limits_{y\in X} f_j(y)\right)\neq\emptyset$. Let $u,x_1\in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$
\begin{cases}
v_n=(1-t_n)x_n\oplus t_nu,\\
u_n=\Phi_\lambda^N\circ\Psi_\mu^m(v_n),\\
y_n=(1-\beta_n)u_n\oplus\beta_nT^nu_n,\\
x_{n+1}=(1-\alpha_n)u_n\oplus\alpha_nT^ny_n,\ n\geq 1,
\end{cases} \tag{4.5.29}
$$

where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$, $\Phi_\lambda^0 = I$, $\Psi_\mu^m = J_\mu^N \circ J_\mu^{m-1} \circ \cdots \circ J_\mu^2 \circ J_\mu^1$, $\Phi_\mu^0 = I$, $\lambda, \mu \in (0, \infty)$ *and* $\{t_n\}$ *is a sequence in* $(0,1)$ *satisfying the following conditions:*

C1: $\lim\limits_{n\to\infty} t_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} t_n = \infty$,

C3: $0 < \epsilon \leq \alpha_n \leq b < 1$ *and* $0 < \epsilon \leq \beta_n \leq b < 1$,

C4: $\lim\limits_{n\to\infty} \frac{k_n^2 - 1}{t_n} = 0$.

Then $\{x_n\}$ *converges strongly to an element of* Γ.

By setting $N = m = 1$ in Theorem 4.5.1, we obtain the following result.

Corollary 4.5.3. *Let* X *be an Hadamard space and* X^* *be its dual space. Let* $A : X \to 2^{X^*}$ *be multivalued monotone operator that satisfies the range condition and* $f : X \to (-\infty, \infty]$ *be proper convex and lower semi continuous function. Let* $T : X \to X$ *be a uniformly L-Lipschitzian and asymptotically demicontractive mapping with constant* $k \in (0,1)$ *and sequence* $\{k_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ *such that* $\sum\limits_{n=1}^{\infty} (k_n^2 - 1) < \infty$. *Suppose that* $\Gamma := F(T) \cap A^{-1}(0) \cap \arg\min\limits_{y \in X} f(y) \neq \emptyset$. *Let* $u, x_1 \in X$ *be arbitrary and the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = J_\lambda \circ J_\mu(v_n), \\ y_n = (1 - \beta_n)u_n \oplus \beta_n T^n u_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n T^n y_n, \ \ n \geq 1, \end{cases} \tag{4.5.30}$$

where $\lambda, \mu \in (0, \infty)$ *and* $\{t_n\}, \{\beta_n\}, \{\alpha_n\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

C1: $\lim\limits_{n\to\infty} t_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} t_n = \infty$,

C3: $0 < \epsilon \leq \alpha_n \leq k\beta_n < \beta_n \leq b < \frac{2}{\left(\frac{k+\delta}{k}\right) + \sqrt{\left(\frac{k+\delta}{k}\right)^2 + 4L^2}}$, *where* $\delta := \sup\limits_{n\geq 1} k_n$,

C4: $\lim\limits_{n\to\infty} \frac{k_n^2 - 1}{t_n} = 0$.

Then $\{x_n\}$ *converges strongly to an element of* Γ.

By setting $N = m = 1$ and $T = I$ (I is the identity mapping on X) in Theorem 4.5.1, we obtain the following result.

Corollary 4.5.4. *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be multivalued monotone operator that satisfies the range condition and $f : X \to (-\infty, \infty]$ be proper convex and lower semi continuous function. Suppose that $\Gamma := A^{-1}(0) \cap \arg\min_{y \in X} f(y) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} v_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = (J_\lambda \circ J_\mu(v_n)), \\ y_n = (1 - \beta_n)u_n \oplus \beta_n u_n, \\ x_{n+1} = (1 - \alpha_n)u_n \oplus \alpha_n y_n, \; n \geq 1, \end{cases} \tag{4.5.31}$$

where $\lambda, \mu \in (0, \infty)$ and $\{t_n\}, \{\beta_n\}, \{\alpha_n\}$ are sequences in $(0, 1)$ satisfying the following conditions:

C1: $\lim_{n \to \infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$.

Then $\{x_n\}$ converges strongly to an element of Γ.

4.5.2 Numerical example

In this subsection, we give a numerical example of Theorem 4.5.1 to illustrate the applicability of our main result. Let $X = \mathbb{R}^2$ be endowed with the euclidean norm $||.||_2$. Let $B : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $B(x) = (2x_1 + x_2, \; x_1 + 3x_2)$ and $b^T = (0, 0)$. Let $m = 1$ in Theorem 4.5.1, we define $f : \mathbb{R}^2 \to (-\infty, \infty]$ by

$$f(x) = \frac{1}{2}||B(x) - b||^2.$$

Then, f is a proper convex and lower semi continuous function, since B is a continuous linear mapping (see [126]). Thus for $\mu = 1$, we obtain from [126] that

$$\begin{aligned} J_1(x) = \text{Prox}_f(x) &= \arg\min_{y \in \mathbb{R}^2} \left[f(y) + \frac{1}{2}||y - x||^2 \right] \\ &= (I + B^T B)^{-1}(x + B^T b). \end{aligned}$$

So that

$$\begin{aligned} J_1(x) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= \left(\frac{11x_1 - 5x_2}{41}, \frac{-5x_1 + 6x_2}{41} \right). \end{aligned}$$

Let $N = 1$ in Theorem 4.5.1, we define $A : \mathbb{R}^2 \to \mathbb{R}^2$ by $A(x) = (x_1, x_1 - x_2)$. Then, A is a monotone mapping.

We note that $[t\overrightarrow{ab}] \equiv t(b-a)$, for all $t \in \mathbb{R}$ and $a, b \in \mathbb{R}^2$ (see [95]). Thus, we have for each $x \in \mathbb{R}^2$ that

$$\begin{aligned} J_\lambda(x) = z &\iff \frac{1}{\lambda}(x-z) \in Az \\ &\iff x = (I+\lambda A)z \\ &\iff z = (I+\lambda A)^{-1}x. \end{aligned}$$

So that for $\lambda = 2$, we compute the resolvent of A as follows:

$$\begin{aligned} J_2(x) &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\right)^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \left(\frac{x_1}{5}, \ \frac{2x_1 - 3x_2}{5}\right). \end{aligned}$$

Thus, for $N = m = 1, \ \lambda = 2$ and $\mu = 1$, we obtain

$$J_2(J_1 x) = \left(\frac{11x_1 - 5x_2}{205}, \ \frac{37x_1 - 28x_2}{205}\right).$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x) = \frac{-3x}{2}$. Then T is a demicontractive mapping with $k = \frac{1}{5}$. This implies that T is an asymptotically demicontractive mapping with $k = \frac{1}{5}$ and $k_n = 1$ for all $n \geq 1$. Thus, we obtain that $\delta = 1$. It is easy to see that T is also uniformly L-Lipschitzian with $L = \frac{3}{2}$. Now, if we take $b = \frac{1}{10}$ and $\epsilon = \frac{1}{1000}$, then condition C3 becomes:

$$0 < \frac{1}{1000} \leq \alpha_n \leq \frac{\beta_n}{5} < \beta_n \leq \frac{1}{10} < \frac{2}{6+\sqrt{45}}.$$

So that, if we choose $\beta_n = \frac{1}{10+\frac{1}{n}}$, $\alpha_n = \frac{1}{50+\frac{5}{n}}$ and $t_n = \frac{1}{2n+1}$, then t_n, β_n and α_n satisfy the conditions in Theorem 4.5.1.

Hence, for $u, x_1 \in \mathbb{R}^2$, our Algorithm (4.5.1) becomes:

$$\begin{cases} v_n = \frac{2n}{2n+1}x_n + \frac{u}{2n+1}, \\ u_n = J_2(J_1 v_n), \\ y_n = \left(1 - \frac{1}{10+\frac{1}{n}}\right)u_n - \frac{3}{20+\frac{2}{n}}u_n, \\ x_{n+1} = \left(1 - \frac{1}{50+\frac{5}{n}}\right)u_n - \frac{3}{100+\frac{10}{n}}y_n, \ n \geq 1. \end{cases} \tag{4.5.32}$$

Case I Take $x_1 = (0.1, \ 0.5)^T$ and $u = (0.1, \ 0.5)^T$.

Case II Take $x_1 = (0.1, \ 0.5)^T$ and $u = (1, \ 1.5)^T$.

Case III Take $x_1 = (-1, \ -0.5)^T$ and $u = (-1, \ 1.5)^T$.

The Mathlab version used is R2014a and the figures are as follows:

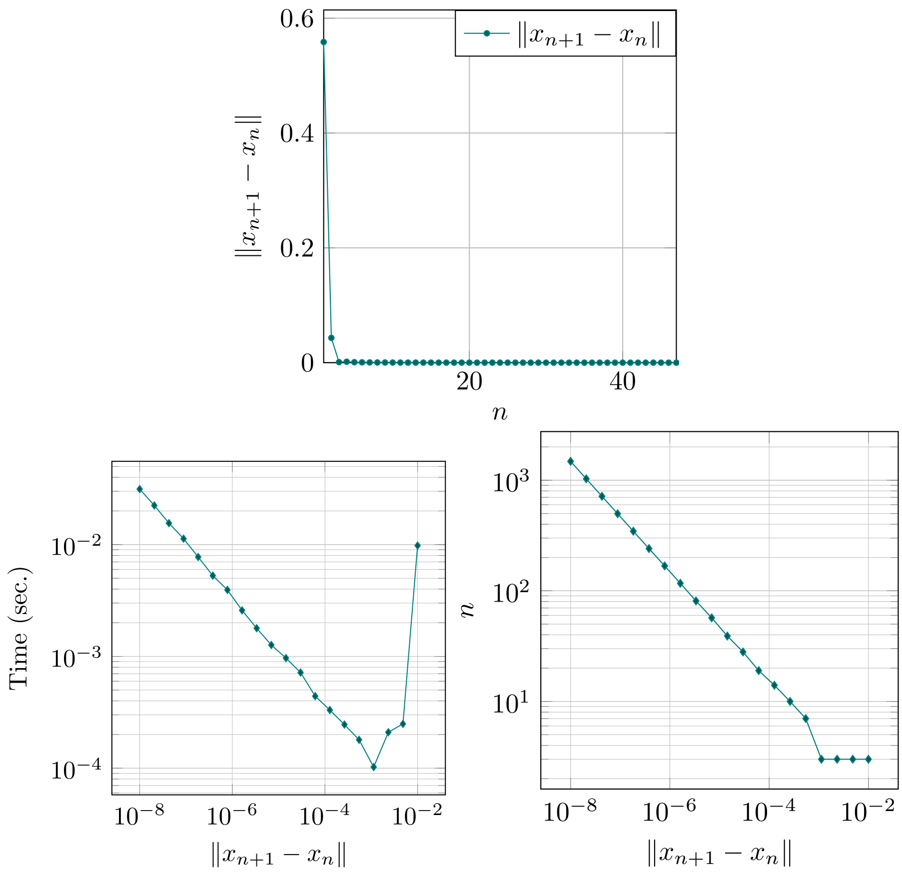

Figure 4.3: Case I: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

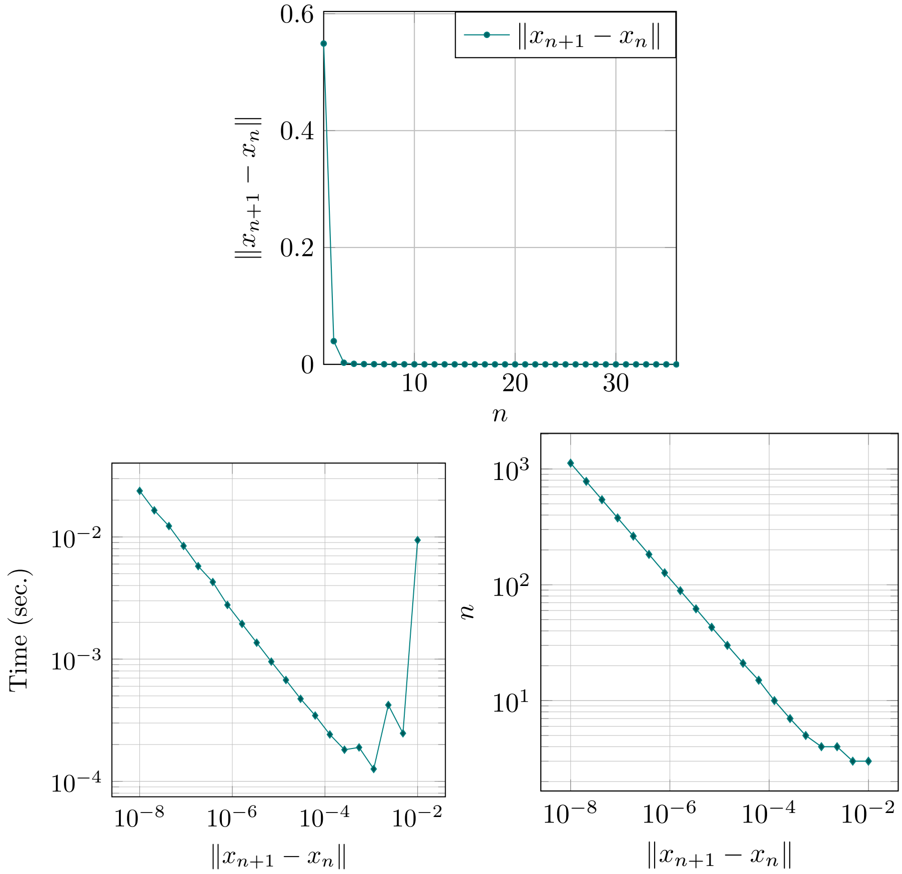

Figure 4.4: Case II: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

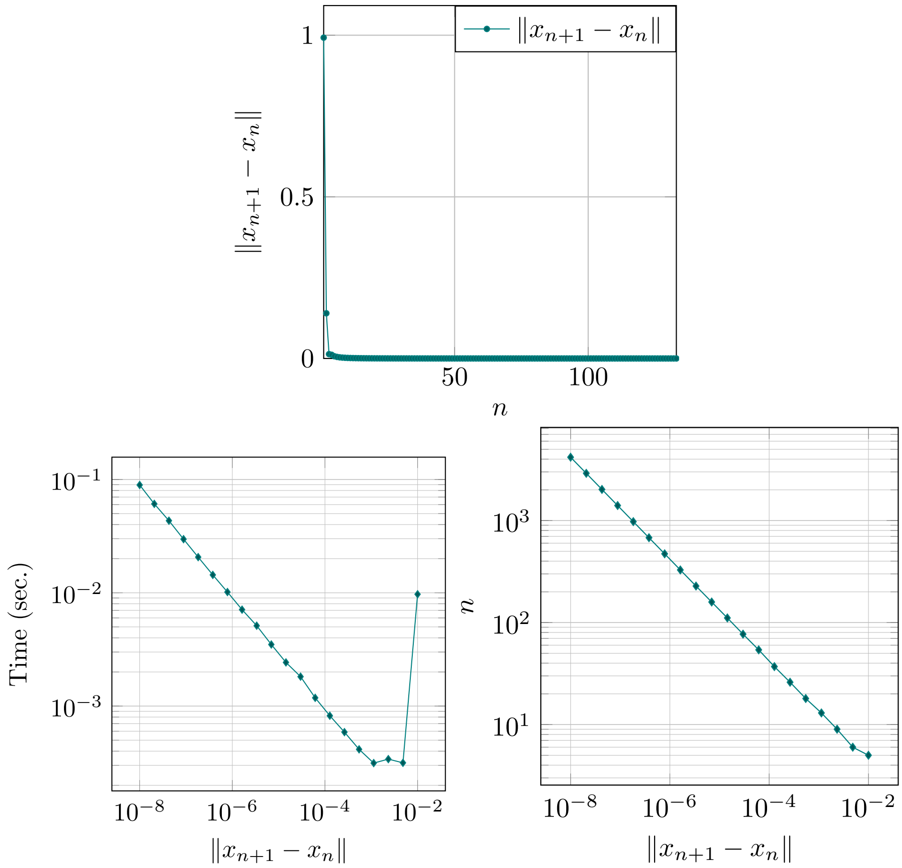

Figure 4.5: Case III: errors vs number of iterations (top); execution time vs accuracy (bottom left); number of iterations vs accuracy (bottom right).

Chapter 5

Contributions to Equilibrium Problems in Hadamard Spaces

5.1 Introduction

EPs include other optimization problems like MPs, VIPs, CFPs, among others. Thus, they are known to be of central importance in optimization theory as well as in nonlinear and convex analysis. However, as mentioned in Remark 2.2.11, the study of EPs in Hadamard spaces is still in the embryonic stage since there are very few results concerning EPs in Hadamard spaces. Thus, it is important to further develop its study in Hadamard spaces. In this chapter, we shall further develop and generalize the study of EPs in Hadamard spaces. Then, we extend the study to the study of MEP (1.2.8) in Hadamard spaces, and also apply our results to solve other optimization problems like the MPs, VIPs and CFPs.

5.2 Preliminaries

Here, we discuss some important results that will be needed in this chapter, which includes the study of the existence of resolvent operators and solution of equilibrium problems.

For a nonempty subset C of X, we denote by $conv(C)$, the convex hull of C. That is, the smallest convex subset of X containing C. Recall that the convex hull of a finite set is the set of all convex combinations of its points.

Theorem 5.2.1. *(The KKM Principle) (see [108, Theorem 3.3]). Let C be a nonempty, closed and convex subset of an Hadamard space X and $G : C \to 2^C$ be a setvalued mapping with closed values. Suppose that for any finite subset $\{x_1, x_2, \dots, x_n\}$ of C,*

$$conv(\{x_1, x_2, \dots, x_m\}) \subset \bigcup_{i=1}^{m} G(x_i). \tag{5.2.1}$$

Then, the family $\{G(x)\}_{x\in C}$ has the finite intersection property. Moreover, if $G(x_0)$ is compact for some $x_0 \in C$, then $\bigcap_{x\in C} G(x) \neq \emptyset$.

Theorem 5.2.2. *[108, Theorem 4.1] Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying the following:*

(A1) $\varphi(x, x) \geq 0$ *for each* $x \in C$,

(A2) for every $x \in C$, *the set* $\{y \in C : \varphi(x, y) < 0\}$ *is convex,*

(A3) for every $y \in C$, *the function* $x \mapsto \varphi(x, y)$ *is upper semicontinuous,*

(A4) there exists a compact subset $L \subset C$ *containing a point* $y_0 \in L$ *such that* $\varphi(x, y_0) < 0$ *whenever* $x \in C \backslash L$.

Then, problem (1.2.7) *has a solution.*

In [108], the authors introduce the resolvent of the bifunction φ associated with the EP (1.2.7). They defined a perturbed bifunction $\bar{F}_{\bar{x}} : C \times C \to \mathbb{R}$ ($\bar{x} \in X$) of φ by

$$\bar{F}_{\bar{x}}(x, y) := \varphi(x, y) - \langle \overrightarrow{x\bar{x}}, \overrightarrow{xy} \rangle, \ \forall x, y \in C. \tag{5.2.2}$$

The perturbed bifunction $\bar{F}$ has a unique equilibrium called the resolvent operator $J^\varphi : X \to 2^C$ of the bifunction φ (see [108]), that is

$$J^\varphi(x) := EP(\bar{F}_x, C) = \{z \in C : \varphi(z, y) - \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \geq 0, \ y \in C\}, \ x \in X. \tag{5.2.3}$$

It was established in [108] that J^φ is well defined.

Definition 5.2.3. *Let X be a CAT(0) space and C be a nonempty closed and convex subset of X. A function $\varphi : C \times C \to \mathbb{R}$ is called monotone if $\varphi(x, y) + \varphi(y, x) \leq 0$ for all $x, y \in C$.*

Lemma 5.2.4. *[108, Proposition 5.4] Suppose that φ is monotone and $D(J^\varphi) \neq \emptyset$. Then, the following properties hold.*

(i) J^φ *is singlevalued.*

(ii) If $D(J^\varphi) \supset C$, *then* J^φ *is nonexpansive restricted to* C.

(iii) If $D(J^\varphi) \supset C$, *then* $F(J^\varphi) = EP(\varphi, C)$.

Theorem 5.2.5. *[108, Theorem 5.2] Suppose that φ has the following properties*

(i) $\varphi(x, x) = 0$ *for all* $x \in C$,

(ii) φ *is monotone,*

(iii) for each $x \in C$, $y \mapsto \varphi(x, y)$ *is convex and lower semicontinuous.*

(iv) for each $y \in C$, $\varphi(x, y) \geq \limsup_{t \downarrow 0} \varphi((1 - t)x \oplus tz, y)$ *for all* $x, z \in C$.

Then $D(J^\varphi) = X$ and J^φ single-valued.

Remark 5.2.6. *It follows from (5.2.3) that the resolvent J_λ^φ of the bifunction φ and order $\lambda > 0$ is given as*

$$J_\lambda^\varphi(x) := EP(\bar{F}_x, C) = \{z \in C : \varphi(z, y) + \frac{1}{\lambda}\langle \overrightarrow{xz}, \overrightarrow{zy}\rangle \geq 0,\ y \in C\},\ x \in X, \tag{5.2.4}$$

where $\bar{F}$ is defined in this case as

$$\bar{F}_{\bar{x}}(x, y) := \varphi(x, y) + \frac{1}{\lambda}\langle \overrightarrow{\bar{x}x}, \overrightarrow{xy}\rangle,\ \forall x, y \in C,\ \bar{x} \in X. \tag{5.2.5}$$

Lemma 5.2.7. *Let C be a nonempty, closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be a monotone bifunction such that $C \subset D(J_\lambda^\varphi)$ for $\lambda > 0$. Then, the following hold:*

(i) J_λ^φ is firmly nonexpansive restricted to C.

(ii) If $F(J_\lambda^\varphi) \neq \emptyset$, then

$$d^2(J_\lambda^\varphi x, x) \leq d^2(x, v) - d^2(J_\lambda^\varphi x, v)\ \forall x \in C,\ v \in F(J_\lambda^\varphi).$$

(iii) If $0 < \lambda \leq \mu$, then $d(J_\mu^\varphi x, J_\lambda^\varphi x) \leq \sqrt{1 - \frac{\lambda}{\mu}}d(x, J_\mu^\varphi x)$, which implies that $d(x, J_\lambda^\varphi x) \leq 2d(x, J_\mu^\varphi x)\ \forall x \in C$.

Proof. (i) Let $x, y \in C$, then by Lemma 5.2.4 (i) and the definition of J_λ^φ, we have

$$\varphi(J_\lambda^\varphi x, J_\lambda^\varphi y) + \frac{1}{\lambda}\langle \overrightarrow{xJ_\lambda^\varphi x}, \overrightarrow{J_\lambda^\varphi xJ_\lambda^\varphi y}\rangle \geq 0 \tag{5.2.6}$$

and

$$\varphi(J_\lambda^\varphi y, J_\lambda^\varphi x) + \frac{1}{\lambda}\langle \overrightarrow{yJ_\lambda^\varphi y}, \overrightarrow{J_\lambda^\varphi yJ_\lambda^\varphi x}\rangle \geq 0. \tag{5.2.7}$$

Adding (5.2.6) and (5.2.7), and noting that φ is monotone, we obtain

$$\frac{1}{\lambda}\left(\langle \overrightarrow{xJ_\lambda^\varphi x}, \overrightarrow{J_\lambda^\varphi xJ_\lambda^\varphi y}\rangle + \langle \overrightarrow{yJ_\lambda^\varphi y}, \overrightarrow{J_\lambda^\varphi yJ_\lambda^\varphi x}\rangle\right) \geq 0,$$

which implies that

$$\langle \overrightarrow{xy}, \overrightarrow{J_\lambda^\varphi xJ_\lambda^\varphi y}\rangle \geq \langle \overrightarrow{J_\lambda^\varphi xJ_\lambda^\varphi y}, \overrightarrow{J_\lambda^\varphi xJ_\lambda^\varphi y}\rangle.$$

That is,

$$\langle \overrightarrow{xy}, \overrightarrow{J_\lambda^\varphi xJ_\lambda^\varphi y}\rangle \geq d^2(J_\lambda^\varphi x, J_\lambda^\varphi y). \tag{5.2.8}$$

(ii) It follows from (5.2.8) and the definition of quasilinearization that

$$d^2(x, J_\lambda^\varphi x) \leq d^2(x, v) - d^2(v, J_\lambda^\varphi x)\ \forall x \in C,\ v \in F(J_\lambda^\varphi).$$

(iii) Let $x \in C$ and $0 < \lambda \leq \mu$, then we have that

$$\varphi(J_\lambda^\varphi x, J_\mu^\varphi x) + \frac{1}{\lambda}\langle \overrightarrow{xJ_\lambda^\varphi x}, \overrightarrow{J_\lambda^\varphi x J_\mu^\varphi x}\rangle \geq 0 \tag{5.2.9}$$

and

$$\varphi(J_\mu^\varphi x, J_\lambda^\varphi x) + \frac{1}{\mu}\langle \overrightarrow{xJ_\mu^\varphi x}, \overrightarrow{J_\mu^\varphi x J_\lambda^\varphi x}\rangle \geq 0. \tag{5.2.10}$$

Adding (5.2.9) and (5.2.10), and by the monotonicity of φ, we obtain that

$$\langle \overrightarrow{J_\lambda^\varphi x x}, \overrightarrow{J_\mu^\varphi x J_\lambda^\varphi x}\rangle \geq \frac{\lambda}{\mu}\langle \overrightarrow{J_\mu^\varphi x x}, \overrightarrow{J_\mu^\varphi x J_\lambda^\varphi x}\rangle.$$

By the definition of quasilinearization, we obtain that

$$\left(\frac{\lambda}{\mu}+1\right) d^2(J_\mu^\varphi x, J_\lambda^\varphi x) \leq \left(1-\frac{\lambda}{\mu}\right) d^2(x, J_\mu^\varphi x) + \left(\frac{\lambda}{\mu}-1\right) d^2(x, J_\lambda^\varphi x).$$

Since $\frac{\lambda}{\mu} \leq 1$, we obtain that

$$\left(\frac{\lambda}{\mu}+1\right) d^2(J_\mu^\varphi x, J_\lambda^\varphi x) \leq \left(1-\frac{\lambda}{\mu}\right) d^2(x, J_\mu^\varphi x),$$

which implies

$$d(J_\mu^\varphi x, J_\lambda^\varphi x) \leq \sqrt{1-\frac{\lambda}{\mu}} d(x, J_\mu^\varphi x). \tag{5.2.11}$$

Furthermore, by triangle inequality and (5.2.11), we obtain

$$d(x, J_\lambda^\varphi x) \leq 2d(x, J_\mu^\varphi x).$$

$\square$

Remark 5.2.8. *We note here that, if the bifunction φ satisfies assumption (i)-(iv) of Theorem 5.2.5, then the conclusions of Lemma 5.2.7 hold in the whole space X.*

Lemma 5.2.9. *Let C be a nonempty, closed and convex subset of an Hadamard space X and T be a nonexpansive mapping on C. Let $\varphi_i : C \times C \to \mathbb{R}$, $i = 1, 2, \dots, N$ be a finite family of monotone bifunctions such that $C \subset D(J_\lambda^{\varphi_i})$ for $\lambda > 0$. Then, for $\beta_i \in (0, 1)$ with $\sum_{i=0}^N \beta_i = 1$, the mapping $S_\lambda : C \to C$ defined by $S_\lambda x := \beta_0 x \oplus \beta_1 J_\lambda^{\varphi_1} x \oplus \beta_2 J_\lambda^{\varphi_2} x \oplus \cdots \oplus \beta_N J_\lambda^{\varphi_N} x$ for all $x \in C$, is nonexpansive and $F(T \circ S_{\lambda_2}) \subseteq \cap_{i=1}^N F(J_{\lambda_1}^{\varphi_i}) \cap F(T)$ for $0 < \lambda_1 \leq \lambda_2$, where $S_{\lambda_2} : C \to C$ is defined by $S_{\lambda_2} x := \beta_0 x \oplus \beta_1 J_{\lambda_2}^{\varphi_1} x \oplus \beta_2 J_{\lambda_2}^{\varphi_2} x \oplus \cdots \oplus \beta_N J_{\lambda_2}^{\varphi_N} x$ for all $x \in C$.*

Proof. It follows easily from the proof of Lemma 3.2.7. $\square$

5.3 A viscosity-type proximal point algorithm for equilibrium and fixed point problems in Hadamard spaces

In this section, we propose and study a viscosity-type PPA, comprising of a nonexpansive mapping and a finite sum of resolvent operators associated with monotone bifunctions. A strong convergence of the proposed algorithm to a common solution of a finite family of EPs and fixed point problem for a nonexpansive mapping is established in an Hadamard space, and applications of the established results to solve other optimization problems (like, EPs, MPs, and CFPs) and fixed point problems in Hadamard spaces are discussed.

Remark 5.3.1. *We will like to emphasize that, approximating a common solution of EPs (or VIPs) and fixed point problems are very applicable in practice. It has been established in [85] that finding a common solution of such problems have some applications to mathematical models whose constraints can be expressed as EPs (or VIPs) and fixed point problems. In fact, this happens in practical problems like signal processing, network resource allocation, image recovery, among others.*

5.3.1 Main results

Theorem 5.3.2. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi_i : C \times C \to \mathbb{R}$, $i = 1, 2, \dots, N$ be a finite family of monotone and upper semicontinuous bifunctions such that $C \subset D(J_\lambda^{\varphi_i})$ for $\lambda > 0$. Let $T : C \to C$ be a nonexpansive mapping and $g : C \to C$ be a contraction mapping with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := \cap_{i=1}^N EP(\varphi_i, C) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\varphi_1} x_n \oplus \beta_2 J_{\lambda_n}^{\varphi_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{\varphi_N} x_n, \\ x_{n+1} = \alpha_n g(x_n) \oplus \beta_n x_n \oplus \gamma_n T y_n, \quad n \geq 1, \end{cases} \tag{5.3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1 \ \forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n \ \forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,

(iv) $\beta_i \in (0,1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

Proof. **Step 1:** We show that $\{x_n\}$ is bounded. Let $u \in \Gamma$, then by Lemma 2.3.3, we obtain that

$$\begin{aligned} d(x_{n+1}, u) &\leq \alpha_n d(g(x_n), u) + \beta_n d(x_n, u) + \gamma_n d(Ty_n, u) \\ &\leq \alpha_n \tau d(x_n, u) + \alpha_n d(g(u), u) + \beta_n d(x_n, u) + \gamma_n d(y_n, u) \\ &\leq \alpha_n \tau d(x_n, u) + (\alpha_n + \beta_n) d(x_n, u) + \alpha_n d(g(u), u) \\ &= (1 - \alpha_n(1-\tau)) d(x_n, u) + \alpha_n d(g(u), u) \\ &\leq \max\left\{ d(x_n, u) + \frac{d(g(u), u)}{1-\tau} \right\} \\ &\vdots \\ &\leq \max\left\{ d(x_1, u) + \frac{d(g(u), u)}{1-\tau} \right\}. \end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{g(x_n)\}$ and $\{T(y_n)\}$ are all bounded.

Step 2: We show that $\lim\limits_{n\to\infty} d(x_{n+1}, x_n) = 0$. Observe from Remark 2.3.4, that (5.3.1) can be rewritten as

$$\begin{cases} y_n = S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\varphi_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{\varphi_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n, \quad n \geq 1. \end{cases} \tag{5.3.2}$$

Now, from (5.3.2), Lemma 2.3.1 (iv),(vi) and the nonexpansivity of T, we obtain that

$$\begin{aligned} d(w_{n+1}, w_n) &= d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} g(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}} T y_{n+1}, \frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T y_n\Big) \\ &\leq d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} g(x_{n+1}) \oplus \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) T y_{n+1}, \frac{\alpha_{n+1}}{1-\beta_{n+1}} g(x_n) \oplus \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) T y_n\Big) \\ &+ d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}} g(x_n) \oplus \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) T y_n, \frac{\alpha_n}{1-\beta_n} g(x_n) \oplus \big(1 - \frac{\alpha_n}{1-\beta_n}\big) T y_n\Big) \\ &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}} \tau d(x_{n+1}, x_n) + \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big) d(y_{n+1}, y_n) \\ &\quad + \Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big| d(g(x_n), T y_n) \end{aligned} \tag{5.3.3}$$

Without loss of generality, we may assume that $0 < \lambda_{n+1} \leq \lambda_n \ \forall n \geq 1$. Thus, from (5.3.2), condition (iv), Lemma 3.2.1 and Lemma 5.2.7 (iii), we obtain

$$\begin{aligned} d(y_{n+1}, y_n) &= d\big(\beta_0 x_{n+1} \oplus \beta_1 J_{\lambda_{n+1}}^{\varphi_1} x_{n+1} \oplus \cdots \oplus \beta_N J_{\lambda_{n+1}}^{\varphi_N} x_{n+1},\ \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\varphi_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{\varphi_N} x_n\big) \\ &\leq \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^N \beta_i d(J_{\lambda_{n+1}}^{\varphi_i} x_{n+1}, J_{\lambda_n}^{\varphi_i} x_n) \\ &\leq \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^N \beta_i d(J_{\lambda_{n+1}}^{\varphi_i} x_{n+1}, J_{\lambda_{n+1}}^{\varphi_i} x_n) + \sum_{i=1}^N \beta_i d(J_{\lambda_{n+1}}^{\varphi_i} x_n, J_{\lambda_n}^{\varphi_i} x_n) \\ &\leq d(x_{n+1}, x_n) + \Big(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\Big) \sum_{i=1}^N \beta_i d(J_{\lambda_{n+1}}^{\varphi_i} x_n, x_n) \\ &\leq d(x_{n+1}, x_n) + \Big(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\Big) \bar{M}, \end{aligned} \tag{5.3.4}$$

where $\bar{M} := \sup_{n\geq 1}\{\sum_{i=1}^{N}\beta_i d(J_{\lambda_{n+1}}^{\varphi_i}x_n, x_n)\}$. Substituting (5.3.4) into (5.3.3), we obtain that

$$\begin{aligned}
d(w_{n+1}, w_n) &\leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\tau d(x_{n+1}, x_n) + \big(1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)d(x_{n+1}, x_n) \\
&\quad +\Big(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\Big)\Big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big)M \\
&\quad + \Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big| d(g(x_n), Ty_n) \\
&= \big[1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}}(1-\tau)\big]d(x_{n+1}, x_n) + \Big(\sqrt{1-\frac{\lambda_{n+1}}{\lambda_n}}\Big)\Big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\Big)M \\
&\quad + \Big|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\Big| d(g(x_n), Ty_n).
\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0, \quad \lim_{n\to\infty}\lambda_n = \lambda$ and $\{g(x_n)\}$, $\{Ty_n\}$ are bounded, we obtain that

$$\limsup_{n\to\infty}\big(d(w_{n+1}, w_n) - d(x_{n+1}, x_n)\big) \leq 0.$$

Thus, by Lemma 2.3.17 and condition (ii), we obtain that

$$\lim_{n\to\infty} d(w_n, x_n) = 0. \tag{5.3.5}$$

Hence, by Lemma 2.3.1 we obtain that

$$d(x_{n+1}, x_n) \leq (1-\beta_n)d(w_n, x_n) \to 0, \qquad \text{as} \quad n \to \infty. \tag{5.3.6}$$

Step 3: We show that $\lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0 = \lim_{n\to\infty} d(w_n, T(S_{\lambda_n})w_n)$. Notice also that (5.3.1) can be rewritten as

$$x_{n+1} = \alpha_n g(x_n) \oplus (1-\alpha_n)\Big(\frac{\beta_n x_n \oplus \gamma_n Ty_n}{(1-\alpha_n)}\Big), \qquad y_n = S_{\lambda_n}x_n.$$

Thus, by Lemma 2.3.1, we obtain that

$$d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n Ty_n}{(1-\alpha_n)}\Big) \leq \alpha_n d\Big(g(x_n), \frac{\beta_n x_n \oplus \gamma_n Ty_n}{(1-\alpha_n)}\Big) \to 0, \quad \text{as} \quad n\to\infty. \tag{5.3.7}$$

Also, from (2.1.1), we obtain

$$d\Big(x_n, \frac{\beta_n x_n \oplus \gamma_n Ty_n}{(1-\alpha_n)}\Big) = \frac{\gamma_n}{1-\alpha_n}d(x_n, Ty_n),$$

which implies from (5.3.6) and (5.3.7) that

$$\frac{\gamma_n}{1-\alpha_n}d(x_n, Ty_n) \leq d(x_n, x_{n+1}) + d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n Ty_n}{(1-\alpha_n)}\Big) \to 0, \quad \text{as} \quad n\to\infty.$$

Hence,

$$\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0. \tag{5.3.8}$$

Now, since $\{x_n\}$ is bounded and X is a complete CAT(0) space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim\limits_{k\to\infty} x_{n_k} = \bar{z}$. Again, since $T \circ S_{\lambda_n}$ is nonexpansive (and every nonexpansive mapping is demiclosed), it follows from (5.3.8), condition (iii), Lemma 3.2.7 and Lemma 5.2.4 (iii) that $\bar{z} \in F(T \circ S_{\lambda_n}) \subseteq \cap_{i=1}^{N} F(J_{\lambda}^{\varphi_i}) \cap F(T) = \Gamma$.

Also, we have

$$\begin{aligned} d(w_n, T(S_{\lambda_n})w_n) &\leq d(w_n, x_n) + d(x_n, T(S_{\lambda_n})x_n) + d(T(S_{\lambda_n})x_n, T(S_{\lambda_n})w_n) \\ &\leq 2d(w_n, x_n) + d(x_n, T(S_{\lambda_n}x_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{5.3.9}$$

Step 4: We show that $\limsup\limits_{n\to\infty}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle \leq 0$.

Now, define $T_n x = \beta_n x \oplus (1-\beta_n)w$, where $w = \frac{\alpha_n}{(1-\beta_n)} g(x) \oplus \frac{\gamma_n}{(1-\beta_n)} T(S_{\lambda_n})x$, then T_n is a contractive mapping for each $n \geq 1$. Thus, there exists a unique fixed point z_n of T_n $\forall n \geq 1$. That is,
$z_m = \beta_m z_m \oplus (1-\beta_m)w_m$, where $w_m = \frac{\alpha_m}{(1-\beta_m)} g(z_m) \oplus \frac{\gamma_m}{(1-\beta_m)} T(S_{\lambda_m})z_m$.

Moreover, $\lim\limits_{m\to\infty} z_m = \bar{z} \in \Gamma$ (see Theorem 3.5.1).

Thus, we obtain that

$$\begin{aligned} d(z_m, w_n) &= d(\beta_m z_m \oplus (1-\beta_m)w_m, w_n) \\ &\leq \beta_m d(z_m, w_n) + (1-\beta_m)d(w_m, w_n), \end{aligned}$$

which implies that

$$d(z_m, w_n) \leq d(w_m, w_n). \tag{5.3.10}$$

From (5.3.10) and Lemma 2.3.1(v), we obtain that

$$\begin{aligned} d^2(w_m, w_n) &= \langle \overrightarrow{w_m w_n}, \overrightarrow{w_m w_n}\rangle \\ &= \langle \overrightarrow{w_m T(S_{\lambda_m})z_m}, \overrightarrow{w_m w_n}\rangle + \langle \overrightarrow{T(S_{\lambda_m})z_m w_n}, \overrightarrow{w_m w_n}\rangle \\ &\leq \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{g(z_m)T(S_{\lambda_m})z_m}, \overrightarrow{w_m w_n}\rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m)w_n}, \overrightarrow{w_m w_n}\rangle \\ &= \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{g(z_m)T(S_{\lambda_m} z_m)}, \overrightarrow{w_m z_m}\rangle + \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{g(z_m)w_n}, \overrightarrow{z_m w_n}\rangle \\ &+ \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{w_n T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_m}\rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m)T(S_{\lambda_m} w_n)}, \overrightarrow{w_m w_m}\rangle \\ &+ \langle \overrightarrow{T(S_{\lambda_m} w_m)}, \overrightarrow{w_m w_n}\rangle \\ &\leq \frac{\alpha_m}{(1-\beta_m)} d(g(z_m), T(S_{\lambda_m} z_m))d(w_m, z_m) + \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{z_m w_n}\rangle \\ &+ \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n}\rangle + d(T(S_{\lambda_m} z_m), T(S_{\lambda_m} w_n))d(w_m, w_n) \\ &+ d(T(S_{\lambda_m} w_n), w_n)d(w_m, w_n) \\ &\leq \frac{\alpha_m}{(1-\beta_m)} d(g(z_m), T(S_{\lambda_m} z_m))d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{z_m w_n}\rangle \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n}\rangle + d(z_m, w_m)d(w_m, w_n) + d(T(S_{\lambda_m} w_n), w_n)d(w_n, w_m)\\
&\leq \frac{\alpha_m}{(1-\beta_m)} d(g(z_m), T(S_{\lambda_m} z_m))d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{z_m w_n}\rangle\\
&+ \frac{\alpha_m}{(1-\beta_m)} d(z_m, T(S_{\lambda_m} z_m))d(w_m, z_m) + d(w_m, w_n) + d(T(S_{\lambda_m} w_n), w_n)d(w_n, w_m),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m}\rangle &\leq d(g(z_m), T(S_{\lambda_m})z_m)d(w_n, z_m) + d(z_m, T(S_{\lambda_m})z_m)d(z_m, w_m)\\
&+ \frac{(1-\beta_m)}{\alpha_m} d(T(S_{\lambda_n})w_n, w_n)d(w_m, w_m).
\end{aligned}
$$

Thus, taking lim sup as $n \to \infty$ first, then as $m \to \infty$, it follows from (5.3.5),(5.3.8) and (5.3.9) that

$$
\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m}\rangle \leq 0. \tag{5.3.11}
$$

Furthermore,

$$
\begin{aligned}
\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle &= \langle \overrightarrow{g(\bar{z})g(z_m)}, \overrightarrow{x_n \bar{z}}\rangle + \langle \overrightarrow{g(z_m)z_m}, \overrightarrow{x_n w_n}\rangle + \langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m}\rangle\\
&+ \langle \overrightarrow{g(z_m)z_m}, \overrightarrow{z_m \bar{z}}\rangle + \langle \overrightarrow{z_m \bar{z}}, \overrightarrow{x_n \bar{z}}\rangle\\
&\leq d(g(\bar{z}), g(z_m))d(x_n, \bar{z}) + d(g(z_m), z_m)d(x_n, w_n) + \langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m}\rangle\\
&+ d(g(z_m), z_m)d(z_m, \bar{z}) + d(z_m, \bar{z})d(x_n, \bar{z})\\
&\leq (1+\tau)d(z_m, \bar{z})d(x_n, \bar{z}) + \langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m}\rangle + [d(x_n, w_n) + d(z_m, \bar{z})]d(g(z_m), z_m),
\end{aligned}
$$

which implies from (5.3.5), (5.3.11) and the fact that $\lim_{m\to\infty} z_m = \bar{z}$, that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle &= \limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle\\
&\leq \limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{g(z_m)z_m}, \overrightarrow{w_n z_m}\rangle \leq 0.
\end{aligned} \tag{5.3.12}
$$

Step 5: Lastly, we show that $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

From Lemma 2.3.2, we obtain that

$$
\begin{aligned}
\langle \overrightarrow{w_n \bar{z}}, \overrightarrow{x_n \bar{z}}\rangle &\leq \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{g(x_n)\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle + \frac{\gamma_n}{(1-\beta_n)}\langle \overrightarrow{T(S_{\lambda_n})x_n \bar{z}}, \overrightarrow{x_n \bar{z}}\rangle\\
&\leq \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{g(x_n)g(\bar{z})}, \overrightarrow{x_n \bar{z}}\rangle + \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle\\
&+ \frac{\gamma_n}{(1-\beta_n)} d(T(S_{\lambda_n})x_n, \bar{z})d(x_n, \bar{z})\\
&\leq \frac{\alpha_n}{(1-\beta_n)}\tau d^2(x_n, \bar{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle + (1 - \frac{\alpha_n}{1-\beta_n})d^2(x_n, \bar{z})\\
&= \left[\frac{\alpha_n}{(1-\beta_n)}\tau + (1 - \frac{\alpha_n}{1-\beta_n})\right]d^2(x_n, \bar{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n \bar{z}}\rangle.
\end{aligned}
$$

Thus, from Lemma 2.3.1, we have

$$
\begin{aligned}
d^2(x_{n+1},\bar{z}) &\le \beta_n d^2(x_n,\bar{z}) + (1-\beta_n)d^2(w_n,\bar{z}) \\
&= \beta_n d^2(x_n,\bar{z}) + (1-\beta_n)\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{w_n\bar{z}}\rangle \\
&= \beta_n d^2(x_n,\bar{z}) + (1-\beta_n)[\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{w_n x_n}\rangle + \langle \overrightarrow{w_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle] \\
&\le [\beta_n + \alpha_n\tau + \gamma_n]d^2(x_n,\bar{z}) + (1-\beta_n)\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{w_n x_n}\rangle + \alpha_n\langle g(\bar{z})\bar{z}, x_n\bar{z}\rangle \\
&\le (1-\alpha_n(1-\tau))d^2(x_n,\bar{z}) + \alpha_n(1-\tau)\left[\frac{1}{1-\tau}\langle \overrightarrow{g(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\right] \\
&+ (1-\beta_n)d(w_n,x_n)M. \qquad (5.3.13)
\end{aligned}
$$

By (5.3.5) and applying Lemma 2.3.26 to (5.3.13), we obtain that $\{x_n\}$ converges strongly to $\bar{z}$. □

Corollary 5.3.3. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi_i : C \times C \to \mathbb{R}$, $i = 1, 2, \dots, N$ be a finite family of monotone and upper semicontinuous bifunctions such that $C \subset D(J_\lambda^{\varphi_i})$ for $\lambda > 0$. Let $g : C \to C$ be a contraction mapping with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := \cap_{i=1}^N EP(\varphi_i, C) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$
\begin{cases}
y_n = S_{\lambda_n}x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\varphi_1}x_n \oplus \beta_2 J_{\lambda_n}^{\varphi_2}x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{\varphi_N}x_n, \\
x_{n+1} = \alpha_n g(x_n) \oplus \beta_n x_n \oplus \gamma_n y_n, \quad n \ge 1,
\end{cases} \qquad (5.3.14)
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf\limits_{n\to\infty} \beta_n \le \limsup\limits_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \ge 1$,

(iii) $0 < \lambda \le \lambda_n$ $\forall n \ge 1$ *and* $\lim\limits_{n\to\infty} \lambda_n = \lambda$,

(iv) $\beta_i \in (0,1)$ *with* $\sum\limits_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

Corollary 5.3.4. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be a monotone and upper semicontinuous bifunction such that $C \subset D(J_\lambda^\varphi)$ for $\lambda > 0$. Let $T : C \to C$ be a nonexpansive mapping and $g : C \to C$ be a contraction mapping with coefficient $\tau \in (0,1)$. Suppose that $\Gamma := EP(\varphi, C) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$
\begin{cases}
y_n = J_{\lambda_n}^f x_n, \\
x_{n+1} = \alpha_n g(x_n) \oplus \beta_n x_n \oplus \gamma_n T y_n, \quad n \ge 1,
\end{cases} \qquad (5.3.15)
$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1$ $\forall n \geq 1$,

(iii) $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

5.3.2 Applications

In this subsection, we give application of our results to solve MPs, VIPs and CFPs in Hadamard spaces.

Minimization problem

Let $h : X \to \mathbb{R}$ be a proper convex and lower semicontimnuous function. Consider the bifunction $\varphi_h : C \times C \to \mathbb{R}$ defined by

$$\varphi_h(x, y) = h(y) - h(x), \ \forall x, y \in C.$$

Then, φ_h is monotone and upper semicontinuous (see [108]). Moreover, $\mathrm{EP}(\varphi_h, C) = \arg\min_C h$, $\ J^{\varphi_h} = \mathrm{prox}^h$ and $D(\mathrm{prox}^h) = X$ (see [108]). Now, consider the following finite family of MP and fixed point problem:

$$\text{Find } x \in F(T) \text{ such that } h_i(x) \leq h_i(y), \ \forall y \in C, \ i = 1, 2 \ldots, N, \tag{5.3.16}$$

where T is a nonexpansive mapping. Thus, by setting $J_{\lambda_n}^{\varphi_i} = \mathrm{prox}_{\lambda_n}^{h_i}$ in Algorithm (5.3.1), we can apply Theorem 5.3.2 to approximate solutions of problem (5.3.16).

Variational inequality problem

Let $S : C \to C$ be a nonexpansive mapping. Now define the bifunction $\varphi_S : C \times C \to \mathbb{R}$ by $\varphi_S(x, y) = \langle \overrightarrow{Sxx}, \overrightarrow{xy} \rangle$. Then, φ_S is monotone and $J^{\varphi_S} = J^S$ (see [19, 99]). Consider the following finite family of variational inequality and fixed point problems:

$$\text{Find } x \in F(T) \text{ such that } \langle \overrightarrow{S_i x x}, \overrightarrow{xy} \rangle \geq 0, \ \forall y \in C, \ i = 1, 2 \ldots, N, \tag{5.3.17}$$

where T is a nonexpansive mapping on C. Thus, by setting $J_{\lambda_n}^{\varphi_i} = J_{\lambda_n}^{S_i}$ in Algorithm (5.3.1), we can apply Theorem 5.3.2 to approximate solutions of problem (5.3.17).

Convex feasibility problem

Let $C_i, i = 1, 2, \ldots, N$ be a finite family of nonempty closed and convex subsets of C such that $\cap_{i=1}^N C_i \neq \emptyset$. Now, consider the following CFP:

$$\text{Find } x \in F(T) \text{ such that } x \in \cap_{i=1}^N C_i. \tag{5.3.18}$$

We know that the indicator function δ_C is a proper convex and lower semicontinuous function. By letting $\delta_C = h$ and following similar argument as in the case of MP, we obtain that φ_{δ_C} is monotone and upper semicontinuous, and $J^{\varphi_{\delta_C}} = \text{prox}^{\delta_C} = P_C$. Therefore, by setting $J^{\varphi_i} = P_{C_i},\ i = 1, 2, \ldots, N$ in Algorithm (5.3.1), we can apply Theorem 5.3.2 to approximate solutions of (5.3.18).

5.4 Asymptotic behavior of viscosity-type proximal point algorithm

In this section, we study the asymptotic behavior of the sequence given by the following viscosity-type PPA and extend the study to approximate a common solution of finite family of EPs in Hadamard spaces. For $x_1 \in C$, define the sequence $\{x_n\} \subset C$ by

$$x_{n+1} = \alpha_n g(x_n) \oplus (1 - \alpha_n) J^{\varphi}_{\lambda_n} x_n, \tag{5.4.1}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$, $\{\lambda_n\}$ is in $(0, \infty)$, g is a contraction on C and φ is a bifunction from $C \times C$ into $\mathbb{R}$.

5.4.1 Main results

Lemma 5.4.1. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Theorem 5.2.5. Then, for $\lambda, \mu > 0$ and $x, y \in C$, we have the following inequalities:*

$$d^2(J^{\varphi}_{\lambda} x, J^{\varphi}_{\mu} y) \leq 2\lambda\varphi(J^{\varphi}_{\lambda} x, J^{\varphi}_{\mu} y) + d^2(x, J^{\varphi}_{\mu} y) - d^2(x, J^{\varphi}_{\lambda} x) \tag{5.4.2}$$

and

$$(\lambda + \mu)d^2(J^{\varphi}_{\lambda} x, J^{\varphi}_{\mu} y) + \mu d^2(J^{\varphi}_{\lambda} x, x) + \lambda d^2(J^{\varphi}_{\mu} y, y) \leq \lambda d^2(J^{\varphi}_{\lambda} x, y) + \mu d^2(J^{\varphi}_{\lambda} y, x). \tag{5.4.3}$$

Proof. We first prove (5.4.2). Let $\lambda,\ \mu > 0$ and $x, y \in C$. Then, by the definition of the resolvent, we obtain that

$$\varphi(J^{\varphi}_{\lambda} x, z) + \frac{1}{\lambda}\langle \overrightarrow{x J^{\varphi}_{\lambda} x}, \overrightarrow{J^{\varphi}_{\lambda} x z} \rangle \geq 0\ \forall\ z \in C,$$

which implies that

$$\begin{aligned} 0 &\leq 2\lambda\varphi(J^{\varphi}_{\lambda} x, z) + 2\langle \overrightarrow{x J^{\varphi}_{\lambda} x}, \overrightarrow{J^{\varphi}_{\lambda} x z} \rangle \\ &= 2\lambda\varphi(J^{\varphi}_{\lambda} x, z) + d^2(x, z) - d^2(x, J^{\varphi}_{\lambda}) - d^2(J^{\varphi}_{\lambda} x, z) \\ &\leq 2\lambda\varphi(J^{\varphi}_{\lambda} x, z) + d^2(x, z) - d^2(x, J^{\varphi}_{\lambda} x). \end{aligned} \tag{5.4.4}$$

Now, set $z = tJ_\mu^\varphi y \oplus (1-t)J_\lambda^\varphi x$ for all $t \in (0,1)$ in (5.4.4). Since φ satisfies conditions (i) and (iii) of Theorem 5.2.5, we obtain that

$$\begin{aligned} d^2(x, J_\lambda^\varphi x) &\leq 2\lambda\Big(t\varphi(J_\lambda^\varphi x, J_\mu^\varphi y) + (1-t)\varphi(J_\lambda^\varphi x, J_\lambda^\varphi x)\Big) \\ &\quad + td^2(x, J_\mu^\varphi y) + (1-t)d^2(x, J_\lambda^\varphi x) - t(1-t)d^2(J_\mu^\varphi y, J_\lambda^\varphi x) \\ &= 2\lambda t\varphi(J_\lambda^\varphi x, J_\mu^\varphi y) + td^2(x, J_\mu^\varphi y) + (1-t)d^2(x, J_\lambda^\varphi x) - t(1-t)d^2(J_\mu^\varphi y, J_\lambda^\varphi x), \end{aligned} \tag{5.4.5}$$

which implies that

$$d^2(x, J_\lambda^\varphi x) \leq 2\lambda\varphi(J_\lambda^\varphi x, J_\mu^\varphi y) + d^2(x, J_\mu^\varphi y) - (1-t)d^2(J_\mu^\varphi y, J_\lambda^\varphi x). \tag{5.4.6}$$

Thus, taking limit as $t \to 0$, we obtain

$$d^2(J_\lambda^\varphi x, J_\mu^\varphi y) \leq 2\lambda\varphi(J_\lambda^\varphi x, J_\mu^\varphi y) + d^2(x, J_\mu^\varphi y) - d^2(x, J_\lambda^\varphi x). \tag{5.4.7}$$

Next, we prove (5.4.3). From (5.4.7), we obtain that

$$\mu d^2(J_\lambda^\varphi x, J_\mu^\varphi y) \leq 2\lambda\mu\varphi(J_\lambda^\varphi x, J_\mu^\varphi y) + \mu d^2(x, J_\mu^\varphi y) - \mu d^2(x, J_\lambda^\varphi x).$$

Similarly, we have

$$\lambda d^2(J_\mu^\varphi y, J_\lambda^\varphi x) \leq 2\mu\lambda\varphi(J_\mu^\varphi y, J_\lambda^\varphi x) + \lambda d^2(y, J_\lambda^\varphi x) - \lambda d^2(y, J_\mu^\varphi y).$$

Adding both inequalities and using condition (ii) of Theorem 5.2.5, we get

$$(\lambda+\mu)d^2(J_\lambda^\varphi x, J_\mu^\varphi y) + \mu d^2(x, J_\lambda^\varphi x) + \lambda d^2(y, J_\mu^\varphi y) \leq \mu d^2(x, J_\mu^\varphi y) + \lambda d^2(y, J_\lambda^\varphi x).$$

□

Lemma 5.4.2. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Theorem 5.2.5. Let $\{\lambda_n\}$ be a sequence in $(0,\infty)$ and $\bar{v}$ be an element of C. Suppose that $\lim_{n\to\infty} \lambda_n = \infty$ and $A(\{J_{\lambda_n}^\varphi x_n\}) = \{\bar{v}\}$ for some bounded sequence $\{x_n\}$ in X, then $\bar{v} \in EP(\varphi, C)$.*

Proof. From (5.4.3), we obtain that

$$(\lambda_n+1)d^2(J_{\lambda_n}^\varphi x_n, J^\varphi\bar{v}) + d^2(J_{\lambda_n}^\varphi x_n, x_n) + \lambda_n d^2(J^\varphi\bar{v}, \bar{v}) \leq d^2(J^\varphi\bar{v}, x_n) + \lambda_n d^2(J_{\lambda_n}^\varphi x_n, \bar{v}),$$

which implies that

$$d^2(J_{\lambda_n}^\varphi x_n, J^\varphi\bar{v}) \leq \frac{1}{\lambda_n}d^2(J^\varphi\bar{v}, x_n) + d^2(J_{\lambda_n}^\varphi x_n, \bar{v}).$$

Since $\lim_{n\to\infty} \lambda_n = \infty$, $\{x_n\}$ is bounded and $A(\{J_{\lambda_n}^\varphi x_n\}) = \{\bar{v}\}$, we obtain that

$$\begin{aligned} \limsup_{n\to\infty} d(J_{\lambda_n}^\varphi x_n, J^\varphi\bar{v}) &\leq \limsup_{n\to\infty} d(J_{\lambda_n}^\varphi x_n, \bar{v}) \\ &= \inf_{y\in X}\limsup_{n\to\infty} d(J_{\lambda_n}^\varphi x_n, y), \end{aligned}$$

which by Lemma 2.3.6 and Lemma 5.2.4(iii) implies that $\bar{v} \in F(J^\varphi) = EP(\varphi, C)$. □

Theorem 5.4.3. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (i)-(iv) of Theorem 5.2.5. Let g be a contraction on C with coefficient $\gamma \in (0,1)$ and $\{x_n\}$ be a sequence defined by (5.4.1), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$. Then, we have the following:*
(i) The sequence $\{J^{\varphi}_{\lambda_n} x_n\}$ is bounded if and only if $EP(\varphi, C)$ is nonempty
(ii) If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $EP(\varphi, C) \neq \emptyset$, then $\{x_n\}$ and $\{J^{\varphi}_{\lambda_n} x_n\}$ converge strongly to an element of $EP(\varphi, C)$.

Proof. (i) Suppose that $\{J^{\varphi}_{\lambda_n} x_n\}$ is bounded. Then by Lemma 2.3.24, there exists $\bar{v} \in X$ such that $A(\{J^{\varphi}_{\lambda_n} x_n\}) = \{\bar{v}\}$. Since $\alpha_n, \gamma \in (0,1)$, we obtain from (5.4.1) that

$$\begin{aligned}
d(x_{n+1}, \bar{v}) &\leq \alpha_n d(g(x_n), \bar{v}) + (1-\alpha_n) d(J^{\varphi}_{\lambda_n} x_n, \bar{v}) \\
&\leq \alpha_n \gamma d(x_n, \bar{v}) + \alpha_n d(g(\bar{v}), \bar{v}) + (1-\alpha_n) d(J^{\varphi}_{\lambda_n} x_n, \bar{v}) \\
&\leq d(x_n, \bar{v}) + \alpha_n d(g(\bar{v}), \bar{v}) + d(J^{\varphi}_{\lambda_n} x_n, \bar{v}) \\
&\leq \alpha_{n-1} \gamma d(x_{n-1}, \bar{v}) + \alpha_{n-1} d(g(\bar{v}), \bar{v}) + (1-\alpha_{n-1}) d(J^{\varphi}_{\lambda_{n-1}} x_{n-1}, \bar{v}) \\
&\quad + \alpha_n d(g(\bar{v}), \bar{v}) + d(J^{\varphi}_{\lambda_n} x_n, \bar{v}) \\
&\leq d(x_{n-1}, \bar{v}) + \alpha_{n-1} d(g(\bar{v}), \bar{v}) + d(J^{\varphi}_{\lambda_{n-1}} x_{n-1}, \bar{v}) + \alpha_n d(g(\bar{v}), \bar{v}) + d(J^{\varphi}_{\lambda_n} x_n, \bar{v}).
\end{aligned}$$

Thus, by induction and the fact that $\{J^{\varphi}_{\lambda_n} x_n\}$ is bounded for all $n \geq 1$, we get that $\{x_n\}$ is bounded. Also, since $\lim_{n\to\infty} \lambda_n = \infty$ and $A(\{J^{\varphi}_{\lambda_n} x_n\}) = \{\bar{v}\}$, we obtain by Lemma 5.4.2 that $\bar{v} \in EP(\varphi, C)$. Hence, $EP(\varphi, C)$ is nonempty.

Conversely, let $EP(\varphi, C)$ be nonempty. Then, there exists a point say $\bar{v} \in C$ such that $\bar{v} \in EP(\varphi, C)$. Thus by (5.4.1), we obtain that

$$\begin{aligned}
d(x_{n+1}, \bar{v}) &\leq \alpha_n d(g(x_n), \bar{v}) + (1-\alpha_n) d(J^{\varphi}_{\lambda_n} x_n, \bar{v}) \\
&\leq \alpha_n \gamma d(x_n, \bar{v}) + \alpha_n d(g(\bar{v}), \bar{v}) + (1-\alpha_n) d(J^{\varphi}_{\lambda_n} x_n, \bar{v}) \\
&\leq (1 - \alpha_n(1-\gamma)) d(x_n, \bar{v}) + \alpha_n d(g(\bar{v}), \bar{v}) \\
&\leq \max\{d(x_n, \bar{v}), \frac{d(g(\bar{v}), \bar{v})}{1-\gamma}\} \\
&\vdots \\
&\leq \max\{d(x_1, \bar{v}), \frac{d(g(\bar{v}), \bar{v})}{1-\gamma}\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{J^{\varphi}_{\lambda_n} x_n\}$ is also bounded.

(ii) Since $EP(\varphi, C)$ is nonempty, we obtain from part (i) that $\{x_n\}$ and $\{J^{\varphi}_{\lambda_n} x_n\}$ are bounded. Now, let $v_n = J^{\varphi}_{\lambda_n} x_n$ for all $n \geq 1$ and $\bar{v} \in EP(\varphi, C)$, then we obtain from

Lemma 5.3.5 (iii) that

$$\begin{aligned}
d^2(x_{n+1},\bar{v}) &\leq (1-\alpha_n)^2 d^2(v_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)\bar{v}},\overrightarrow{v_n\bar{v}}\rangle + \alpha_n^2 d^2(g(x_n),\bar{v})\\
&\leq (1-\alpha_n)^2 d^2(x_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(x_n)\bar{v}},\overrightarrow{v_n\bar{v}}\rangle + \alpha_n^2 d^2(g(x_n),\bar{v})\\
&\leq (1-\alpha_n)^2 d^2(x_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\left(\langle\overrightarrow{g(x_n)g(\bar{v})},\overrightarrow{v_n\bar{v}}\rangle + \langle\overrightarrow{g(\bar{v})\bar{v}},\overrightarrow{v_n\bar{v}}\rangle\right)\\
&+ \alpha_n^2 d^2(g(x_n),\bar{v})\\
&\leq (1-\alpha_n)^2 d^2(x_n,\bar{v}) + 2\alpha_n(1-\alpha_n)\left(\gamma d^2(x_n,\bar{v}) + \langle\overrightarrow{g(\bar{v})\bar{v}},\overrightarrow{v_n\bar{v}}\rangle\right)\\
&+ \alpha_n^2 d^2(g(x_n),\bar{v})\\
&\leq (1-2\alpha_n(1-\gamma))d^2(x_n,\bar{v}) + 2\alpha_n^2(1-\gamma)d^2(x_n,\bar{v})\\
&+ 2\alpha_n(1-\alpha_n)\langle\overrightarrow{g(\bar{v})\bar{v}},\overrightarrow{v_n\bar{v}}\rangle + \alpha_n^2 d^2(g(x_n),\bar{v})\\
&= (1-2\alpha_n(1-\gamma))d^2(x_n,\bar{v}) + 2\alpha_n(1-\gamma)\delta_n, \qquad (5.4.8)
\end{aligned}$$

$$\text{where } \delta_n = \frac{(1-\alpha_n)}{(1-\gamma)}\langle\overrightarrow{g(\bar{v})\bar{v}},\overrightarrow{v_n\bar{v}}\rangle + \alpha_n\left(d^2(x_n,\bar{v}) + \frac{1}{2(1-\gamma)}d^2(g(x_n),\bar{v})\right), \qquad (5.4.9)$$

for all $\bar{v} \in EP(\varphi, C)$. Furthermore, since $\{v_n\}$ is bounded, we obtain from Lemma 2.3.5 that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that Δ-converges to some $\hat{v} \in C$. Thus, by Lemma 2.3.6, we obtain that $A(\{v_{n_k}\}) = \{\hat{v}\}$. Moreover, $\lim_{k\to\infty}\lambda_{n_k} = \infty$ and $\{x_{n_k}\}$ is bounded. Hence, by Lemma 5.4.2, we obtain that $\hat{v} \in EP(\varphi, C)$.

Next, we show that $\{x_n\}$ converges strongly to an element of $EP(\varphi, C)$. Since the subsequence $\{v_{n_k}\}$ of $\{v_n\}$ Δ-converges to $\hat{v} \in EP(\varphi, C)$, we obtain from Lemma 2.3.9 that there exists $\bar{z} \in EP(\varphi, C)$ such that $\{v_n\}$ Δ-converges to $\bar{z}$. Thus, by Lemma 2.3.10, we obtain that

$$\limsup_{n\to\infty}\langle\overrightarrow{g(\bar{z})\bar{z}},\overrightarrow{v_n\bar{z}}\rangle \leq 0, \qquad (5.4.10)$$

which by setting $\bar{v} = \bar{z}$ in (5.4.9), implies that $\limsup_{n\to\infty}\delta_n \leq 0$. Therefore, applying Lemma 2.3.26 to (5.4.8), gives that $\{x_n\}$ converges strongly to $\bar{z} \in EP(\varphi, C)$. It then follows that $\{J_{\lambda_n}^{\varphi}x_n\}$ also converges strongly to $\bar{z} \in EP(\varphi, C)$. □

We are now going to apply Theorem 5.4.3 to approximate a common solution of finite family of equilibrium problems. We begin with the following lemma.

Lemma 5.4.4. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi_j : C\times C \to \mathbb{R}$, $j = 1, 2, \ldots, m$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 5.2.5. Then, for $\lambda > 0$, we have $F\left(\prod_{j=1}^m J_\lambda^{\varphi_j}\right) = \cap_{j=1}^m F\left(J_\lambda^{\varphi_j}\right)$, where $\prod_{j=1}^m J_\lambda^{\varphi_j} = J_\lambda^{\varphi_1}\circ J_\lambda^{\varphi_2}\circ\cdots\circ J_\lambda^{\varphi_{m-1}}\circ J_\lambda^{\varphi_m}$.*

Proof. It follows from the proof of Lemma 3.2.6. □

Theorem 5.4.5. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi_j : C\times C \to \mathbb{R},\ j = 1, 2, \ldots, m$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 5.2.5. Let g be a contraction mapping on C with coefficient $\gamma \in (0, 1)$. Suppose that for arbitrary $x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n g(x_n) \oplus (1 - \alpha_n) \prod_{j=1}^{m} J_{\lambda_n}^{\varphi_j} x_n,\ \ n \geq 1, \tag{5.4.11}$$

where $\prod_{j=1}^{m} J_{\lambda_n}^{\varphi_j} = J_{\lambda_n}^{\varphi_1} \circ J_{\lambda_n}^{\varphi_2} \circ \cdots \circ J_{\lambda_n}^{\varphi_{m-1}} \circ J_{\lambda_n}^{\varphi_m}$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := \cap_{i=1}^{N} EP(\varphi_i, C) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element of Γ.

Proof. By Theorem 5.4.3 (ii) and Lemma 5.2.4 (iii), we obtain that $\{x_n\}$ converges strongly to an element of $F\left(\prod_{j=1}^{m} J_{\lambda}^{\varphi_j}\right)$. Therefore, we conclude by Lemma 5.4.4 (ii) and Lemma 5.2.4 (iii) that $\{x_n\}$ converges strongly to an element of Γ. □

By setting $g(x) = u$ for all $x \in C$ and for arbitrary but fixed $u \in C$, we obtain the following corollary.

Corollary 5.4.6. *Let C be a nonempty closed and convex subset of an Hadamard space X and $\varphi_j : C\times C \to \mathbb{R},\ j = 1, 2, \ldots, m$ be a finite family of bifunctions satisfying assumptions (i)-(iv) of Theorem 5.2.5. Let g be a contraction mapping on C with coefficient $\gamma \in (0, 1)$. Suppose that for arbitrary $u, x_1 \in C$, the sequence $\{x_n\}$ is generated by*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) \prod_{j=1}^{m} J_{\lambda_n}^{\varphi_j} x_n,\ \ n \geq 1, \tag{5.4.12}$$

where $\prod_{j=1}^{m} J_{\lambda_n}^{\varphi_j} = J_{\lambda_n}^{\varphi_1} \circ J_{\lambda_n}^{\varphi_2} \circ \cdots \circ J_{\lambda_n}^{\varphi_{m-1}} \circ J_{\lambda_n}^{\varphi_m}$, $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}$ is a sequence in $(0, \infty)$ such that $\lim_{n\to\infty} \lambda_n = \infty$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := \cap_{i=1}^{N} EP(\varphi_i, C) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element of Γ.

5.5 Mixed equilibrium problems in Hadamard spaces

Motivated by Remark 2.2.11, it is our intention in this section, to further generalize the study of EPs in Hadamard spaces. In particular, we introduce and study mixed equilibrium problems in Hadamard spaces. First, we establish the existence of solutions of the mixed equilibrium problem, and the unique existence of the resolvent operator for the problem. We then prove a strong convergence of the resolvent and a Δ-convergence of the PPA to a solution of the mixed equilibrium problem under some suitable conditions. Furthermore, we study the asymptotic behavior of the sequence generated by a Halpern-type PPA for the mixed equilibrium problem.

5.5.1 Existence of solutions for mixed equilibrium problems

Theorem 5.5.1. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $f : C \to \mathbb{R}$ be a real-valued function and $\varphi : C \times C \to \mathbb{R}$ be a bifunction such that the following assumptions hold:*

(A1) $\varphi(x, x) = 0 \ \ \forall x \in C$,

(A2) for every $x \in C$, the set $\{y \in C : \varphi(x, y) + f(y) - f(x) < 0\}$ is convex,

(A3) there exists a compact subset $D \subset C$ containing a point $y_0 \in D$ such that $\varphi(x, y_0) + f(y_0) - f(x) < 0$ whenever $x \in C \backslash D$.

Then, the MEP (1.2.8) has a solution.

Proof. For each $y \in C$, define the setvalued mapping $G : C \to 2^C$ by

$$G(y) := \{x \in C : \varphi(x, y) + f(y) - f(x) \geq 0\}. \tag{5.5.1}$$

By (A1), we obtain that, for each $y \in C$, $G(y) \neq \emptyset$ since $y \in G(y)$. Also, we obtain from (A2) that $G(y)$ is a closed subset of C for all $y \in C$.
We claim that G satisfies the inclusion (5.2.1). Suppose for contradiction that this is not true, then there exist a finite subset $\{y_1, y_2, \cdots, y_m\}$ of C and $\alpha_i \geq 0, \ \forall \ i = 1, 2, \cdots, m$ with $\sum\limits_{i=1}^{m} \alpha_i = 1$ such that $y^* = \sum\limits_{i=1}^{m} \alpha_i y_1 \notin G(y_i)$ for each $i = 1, 2, \cdots, m$. That is, there exists $y^* \in conv(\{y_1, y_2, \cdots, y_m\})$ such that $y^* \notin G(y_i)$, for each $1, 2, \cdots, m$. By (5.5.1), we obtain for each $i = 1, 2, \cdots, m$ that

$$\varphi(y^*, y_i) + f(y_i) - f(y^*) < 0.$$

Thus, for each $i = 1, 2, \ldots, m$, $y_i \in \{y \in C : \varphi(y^*, y) + f(y) - f(y^*) < 0\}$, which is convex by (A2). Since $conv(\{y_1, y_2, \cdots, y_m\})$ is the smallest convex set containing $y_1, y_2, \ldots, y_m$, we have that $conv(\{y_1, y_2, \cdots, y_m\}) \subset \{y \in C : \varphi(y^*, y) + f(y) - f(y^*) < 0\}$, which implies that $y^* \in \{y \in C : \varphi(y^*, y) + f(y) - f(y^*) < 0\}$. That is, $0 = \varphi(y^*, y^*) + f(y^*) - f(y^*) < 0$, which is a contradiction. Therefore, G satisfies the inclusion (5.2.1).

Now, observe that (A3) implies that, there exists a compact subset D of C containing $y_0 \in D$ such that for any $x \in C \backslash D$, we have

$$\varphi(x, y_0) + f(y_0) - f(x) < 0,$$

which further implies that

$$G(y_0) = \{x \in C : \varphi(x, y_0) + f(y_0) - f(x) \geq 0\} \subset D.$$

Thus, $G(y_0)$ is compact. It then follows from Theorem 5.3.12 that $\bigcap\limits_{y \in C} G(y) \neq \emptyset$. This implies that there exists $x^* \in C$ such that

$$\varphi(x^*, y) + f(y) - f(x^*) \ \geq \ 0 \ \forall \ y \in C.$$

That is, MEP (1.2.8) has a solution. □

5.5.2 Existence and uniqueness of resolvent operators

Definition 5.5.2. *Let X be an Hadamard space and C be a nonempty subset of X. Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction, $f : C \to \mathbb{R}$ be a real-valued function, $\bar{x} \in X$ and $\lambda > 0$, we define the perturbation $\tilde{F}_{\bar{x}} : C \times C \to \mathbb{R}$ of φ and f, by*

$$\tilde{F}_{\bar{x}}(x, y) := \varphi(x, y) + f(y) - f(x) + \frac{1}{\lambda}\langle \overrightarrow{xy}, \overrightarrow{\bar{x}x} \rangle \quad \forall\ x, y \in C. \tag{5.5.2}$$

In the next theorem, we shall prove the existence and uniqueness of solution of the following auxiliary problem: Find $x^* \in C$ such that

$$\tilde{F}_{\bar{x}}(x^*, y) \geq 0 \quad \forall\ y \in C, \tag{5.5.3}$$

where $\tilde{F}_{\bar{x}}$ is as defined in (5.5.2). The proof for existence is similar to the proof of Theorem 5.5.1. But for completeness, we shall give the proof here.

Theorem 5.5.3. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $f : C \to \mathbb{R}$ be a convex function and $\varphi : C \times C \to \mathbb{R}$ be a bifunction such that the following assumptions hold:*

(A1) $\varphi(x, x) = 0 \quad \forall x \in C,$

(A2) φ is monotone, i.e., $\varphi(x, y) + \varphi(y, x) \leq 0 \quad \forall\ x, y, \in C,$

(A3) $\varphi(x, .) : C \to \mathbb{R}$ *is convex* $\forall\ x \in C,$

(A4) for each $\bar{x} \in X$ and $\lambda > 0$, there exists a compact subset $D_{\bar{x}} \subset C$ containing a point $y_{\bar{x}} \in D_{\bar{x}}$ such that $\varphi(x, y_{\bar{x}}) + f(y_{\bar{x}}) - f(x) + \frac{1}{\lambda}\langle \overrightarrow{xy_{\bar{x}}}, \overrightarrow{\bar{x}x} \rangle < 0$ whenever $x \in C \backslash D_{\bar{x}}$.

Then (5.5.3) has a unique solution.

Proof. Let $\bar{x}$ be a point in X. For each $y \in C$, defined the setvalued mapping $G : C \to 2^C$ by

$$G(y) = \{x \in C : \varphi(x, y) + f(y) - f(x) + \frac{1}{\lambda}\langle \overrightarrow{xy}, \overrightarrow{\bar{x}x} \rangle \geq 0\}.$$

Then, it is easy to see that $G(y)$ is a nonempty closed subset of C. As in the proof of Theorem 5.5.1, we claim that G satisfies the inclusion (5.2.1). Suppose for contradiction that this is not true, then there exists $y^* = \sum_{i=1}^{m} \alpha_i y_i \in conv(\{y_1, y_2, \cdots, y_m\})$ such that

$$\varphi(y^*, y_i) + f(y_i) - f(y^*) + \frac{1}{\lambda}\langle \overrightarrow{y^*y_i}, \overrightarrow{\bar{x}y^*} \rangle < 0, \quad i = 1, 2, \ldots, m.$$

By (A3) and the convexity of f, we obtain that

$$\begin{aligned} 0 &= \varphi(y^*, y^*) + f(y^*) - f(y^*) + \frac{1}{\lambda}\langle \overrightarrow{y^*y^*}, \overrightarrow{\bar{x}y^*} \rangle \\ &\leq \sum_{i=1}^{m} \alpha_i \big(\varphi(y^*, y_i) + f(y_i) - f(y^*)\big) + \frac{1}{\lambda}\Big(\sum_{i=1}^{m} \alpha_i \langle \overrightarrow{y^*y_i}, \overrightarrow{\bar{x}y^*} \rangle\Big) < 0, \end{aligned}$$

which is a contradiction. Therefore, G satisfies the inclusion (5.2.1). By (A4), we obtain that $G(y_{\bar{x}}) \subset D_{\bar{x}}$. Thus, $G(y_{\bar{x}})$ is compact and by Theorem 5.3.12, we get that $\bigcap_{y\in C} G(y) \neq \emptyset$. Therefore (5.5.3) has a solution.
Next, we show that this solution is unique. Suppose that x and x^* solve (5.5.3). Then,

$$0 \leq \tilde{F}_{\bar{x}}(x, x^*) = \varphi(x, x^*) + f(x^*) - f(x) + \frac{1}{\lambda}\langle \overrightarrow{\bar{x}x}, \overrightarrow{xx^*} \rangle$$

and

$$0 \leq \tilde{F}_{\bar{x}}(x^*, x) = \varphi(x^*, x) + f(x) - f(x^*) + \frac{1}{\lambda}\langle \overrightarrow{\bar{x}x^*}, \overrightarrow{x^*x} \rangle.$$

Adding both inequalities and noting that φ is monotone, we obtain that

$$\begin{aligned} 0 &\leq -\frac{1}{\lambda}\left(\langle \overrightarrow{x\bar{x}}, \overrightarrow{xx^*} \rangle + \langle \overrightarrow{\bar{x}x^*}, \overrightarrow{xx^*} \rangle\right) \\ &= -\frac{1}{\lambda} d(x, x^*)^2, \end{aligned}$$

which implies that $x = x^*$. □

Definition 5.5.4. *Let X be an Hadamard space and C be a nonempty closed and convex subset of X. Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction and $f : C \to \mathbb{R}$ be a convex function. Assume that (5.5.3) has a unique solution for each $\lambda > 0$ and $x \in X$. This unique solution is denoted by $J^f_{\lambda\varphi} x$ and it is called the resolvent operator associated with φ and f of order $\lambda > 0$ and at $x \in X$. In other words, the resolvent operator associated with φ and f, is the setvalued mapping $J^f_{\lambda\varphi} : X \to 2^C$ defined by*

$$J^f_{\lambda\varphi}(x) := EP(\tilde{F}_x, C) = \{z \in C : \varphi(z, y) + f(y) - f(z) + \frac{1}{\lambda}\langle \overrightarrow{zy}, \overrightarrow{xz} \rangle \geq 0, \ \forall y \in C\} \quad (5.5.4)$$

for all x in X.

Under the assumptions of Theorem 5.5.3, we have the unique existence of $J^f_{\lambda\varphi}(x)$. Therefore, $J^f_{\lambda\varphi}$ is well-defined.

5.5.3 Fundamental properties of resolvent operators

Theorem 5.5.5. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let $f : C \to \mathbb{R}$ be a convex function and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying assumptions (A1)-(A4) of Theorem 5.5.3. For $\lambda > 0$, we have that $J^f_{\lambda\varphi}$ is singlevalued. Moreover, if $C \subset D(J^f_{\lambda\varphi})$, then*

(i) $J^f_{\lambda\varphi}$ is firmly nonexpansive restricted to C,

(ii) for $F(J^f_{\lambda\varphi}) \neq \emptyset$, we have

$$d^2(J^f_{\lambda\varphi}x, x) \leq d^2(x, v) - d^2(J^f_{\lambda\varphi}x, v) \quad \forall x \in C, \quad \forall\, v \in F(J^f_{\lambda\varphi}),$$

(iii) *for* $0 < \lambda \leq \mu$, *we have* $d(J^f_{\mu\varphi}x, J^f_{\lambda\varphi}x) \leq \sqrt{1 - \frac{\lambda}{\mu}}d(x, J^f_{\mu\varphi}x)$, *which implies that* $d(x, J^f_{\lambda\varphi}x) \leq 2d(x, J^f_{\mu\varphi}x) \ \ \forall\ x \in C$,

(iv) $F(J^f_{\lambda\varphi}) = MEP(\varphi, f, C)$.

Proof. For each $x \in D(J^f_{\lambda\varphi})$ and $\lambda > 0$, let $z_1, z_2 \in J^f_{\lambda\varphi}x$. Then from (5.5.4), we have

$$\varphi(z_1, z_2) + f(z_2) - f(z_1) + \frac{1}{\lambda}\langle\overrightarrow{z_1z_2}, \overrightarrow{xz_1}\rangle \geq 0$$

and

$$\varphi(z_2, z_1) + f(z_1) - f(z_2) + \frac{1}{\lambda}\langle\overrightarrow{z_2z_1}, \overrightarrow{xz_2}\rangle \geq 0.$$

Adding both inequalities and using assumption (A2), we obtain that

$$\langle\overrightarrow{z_2z_1}, \overrightarrow{z_1z_2}\rangle \geq 0,$$

which implies that $d^2(z_1, z_2) \leq 0$. This further implies that $z_1 = z_2$. Therefore, $J^f_{\lambda\varphi}$ is single valued.

(i) Let $x, y \in C$, then

$$\varphi(J^f_{\lambda\varphi}x, J^f_{\lambda\varphi}y) + f(J^f_{\lambda\varphi}y) - f(J^f_{\lambda\varphi}x) + \frac{1}{\lambda}\langle\overrightarrow{J^f_{\lambda\varphi}xJ^f_{\lambda\varphi}y}, \overrightarrow{xJ^f_{\lambda\varphi}x}\rangle \geq 0 \tag{5.5.5}$$

and

$$\varphi(J^f_{\lambda\varphi}y, J^f_{\lambda\varphi}x) + f(J^f_{\lambda\varphi}x) - f(J^f_{\lambda\varphi}y) + \frac{1}{\lambda}\langle\overrightarrow{J^f_{\lambda\varphi}yJ^f_{\lambda\varphi}x}, \overrightarrow{yJ^f_{\lambda\varphi}y}\rangle \geq 0. \tag{5.5.6}$$

Adding (5.5.5) and (5.5.6), and noting that φ is monotone, we obtain

$$\frac{1}{\lambda}\left(\langle\overrightarrow{xJ^f_{\lambda\varphi}x}, \overrightarrow{J^f_{\lambda\varphi}xJ^f_{\lambda\varphi}y}\rangle + \langle\overrightarrow{yJ^f_{\lambda\varphi}y}, \overrightarrow{J^f_{\lambda\varphi}yJ^f_{\lambda\varphi}x}\rangle\right) \geq 0,$$

which implies that

$$\langle\overrightarrow{xy}, \overrightarrow{J^f_{\lambda\varphi}xJ^f_{\lambda\varphi}y}\rangle \geq \langle\overrightarrow{J^f_{\lambda\varphi}xJ^f_{\lambda\varphi}y}, \overrightarrow{J^f_{\lambda\varphi}xJ^f_{\lambda\varphi}y}\rangle.$$

That is,

$$\langle\overrightarrow{xy}, \overrightarrow{J^f_{\lambda\varphi}xJ^f_{\lambda\varphi}y}\rangle \geq d^2(J^f_{\lambda\varphi}x, J^f_{\lambda\varphi}y). \tag{5.5.7}$$

(ii) It follows from (5.5.7) and the definition of quasilinearization that

$$d^2(x, J^f_{\lambda\varphi}x) \leq d^2(x, v) - d^2(v, J^f_{\lambda\varphi}x)\ \forall x \in C,\ \ v \in F(J^f_{\lambda\varphi}).$$

(iii) Let $x \in C$ and $0 < \lambda \leq \mu$, then we have that

$$\varphi(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}x) + f(J^f_{\mu\varphi}x) - f(J^f_{\lambda\varphi}x) + \frac{1}{\lambda}\langle\overrightarrow{xJ^f_{\lambda\varphi}x}, \overrightarrow{J^f_{\lambda\varphi}xJ^f_{\mu\varphi}x}\rangle \geq 0 \tag{5.5.8}$$

and

$$\varphi(J^f_{\mu\varphi}x, J^f_{\lambda\varphi}x) + f(J^f_{\lambda\varphi}x) - f(J^f_{\mu\varphi}x) + \frac{1}{\mu}\langle \overrightarrow{xJ^f_{\mu\varphi}x}, \overrightarrow{J^f_{\mu\varphi}xJ^f_{\lambda\varphi}x}\rangle \geq 0. \tag{5.5.9}$$

Adding (5.5.8) and (5.5.9), and using the monotonicity of φ, we obtain that

$$\langle \overrightarrow{J^f_{\lambda\varphi}xx}, \overrightarrow{J^f_{\mu\varphi}xJ^f_{\lambda\varphi}x}\rangle \geq \frac{\lambda}{\mu}\langle \overrightarrow{J^f_{\mu\varphi}xx}, \overrightarrow{J^f_{\mu\varphi}xJ^f_{\lambda\varphi}x}\rangle.$$

By quasilinearization, we obtain that

$$\left(\frac{\lambda}{\mu}+1\right) d^2(J^f_{\mu\varphi}x, J^f_{\lambda\varphi}x) \leq \left(1-\frac{\lambda}{\mu}\right) d^2(x, J^f_{\mu\varphi}x) + \left(\frac{\lambda}{\mu}-1\right) d^2(x, J^f_{\lambda\varphi}x).$$

Since $\frac{\lambda}{\mu} \leq 1$, we obtain that

$$\left(\frac{\lambda}{\mu}+1\right) d^2(J^f_{\mu\varphi}x, J^f_{\lambda\varphi}x) \leq \left(1-\frac{\lambda}{\mu}\right) d^2(x, J^f_{\mu\varphi}x),$$

which implies that

$$d(J^f_{\mu\varphi}x, J^f_{\lambda\varphi}x) \leq \sqrt{1-\frac{\lambda}{\mu}}d(x, J^f_{\mu\varphi}x). \tag{5.5.10}$$

Moreover, we obtain by triangle inequality and (5.5.10) that

$$d(x, J^f_{\lambda\varphi}x) \leq 2d(x, J^f_{\mu\varphi}x).$$

(iv) Observe that

$$\begin{aligned} x \in F(J^f_{\lambda\varphi}) &\Leftrightarrow \varphi(x,y) + f(y) - f(x) + \frac{1}{\lambda}\langle \overrightarrow{xx}, \overrightarrow{xy}\rangle \geq 0 \ \forall y \in C \\ &\Leftrightarrow \varphi(x,y) + f(y) - f(x) \geq 0 \ \forall y \in C \\ &\Leftrightarrow x \in MEP(\varphi, f, C). \end{aligned}$$

□

Remark 5.5.6. *Since firmly nonexpansive mappings are nonexpansive and the set of fixed points of nonexpansive mappings is closed and convex, we obtain from (i) and (iv) of Theorem 5.5.5 that $MEP(\varphi, f, C)$ is closed and convex.*

5.5.4 Convergence of resolvents

For the rest of this chapter, we shall assume that $D(J^f_{\lambda\varphi}) \supset C$, where C is a nonempty closed and convex subset of an Hadamard space X.

In the following theorem, we shall prove that $\{J^f_{\lambda\varphi}x\}$ converges strongly to a solution of MEP (1.2.8) as $\lambda \to 0$.

Theorem 5.5.7. *Let $f : C \to \mathbb{R}$ be a convex and lower semicontinuous function and $\varphi : C \times C \to \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)-(A4) of Theorem 5.5.3. If $MEP(\varphi, f, C) \neq \emptyset$, then $\{J^f_{\lambda\varphi}x\}$ converges strongly to $q \in MEP(\varphi, f, C)$, which is the nearest point of $MEP(\varphi, f, C)$ to x as $\lambda \to 0$.*

Proof. Let $v \in MEP(\varphi, f, C)$, since $J^f_{\lambda\varphi}$ is nonexpansive (see Theorem 5.5.5 (i) and Remark 5.5.6), we obtain that $\{J^f_{\lambda\varphi}x\}$ is bounded. Let $\{\lambda_n\}$ be a sequence that converges to 0 as $n \to \infty$. Then $\{J^f_{\lambda_n\varphi}x\}$ is bounded. Thus, by Lemma 2.3.24 (i), there exists a subsequence $\{J^f_{\lambda_{nk}\varphi}x\}$ of $\{J^f_{\lambda_n\varphi}x\}$ that Δ-converges to $q \in C$.

Now, observe that, by the definition of $J^f_{\lambda\varphi}$, the Δ-upper semicontinuity of φ, lower semicontinuous of f and Lemma 4.2.4, we obtain that

$$\varphi(q, y) + f(y) - f(q) \geq 0.$$

Therefore, $q \in MEP(\varphi, f, C)$. Hence, we obtain from Theorem 5.5.5(ii) that

$$d^2(J^f_{\lambda_{nk}\varphi}x, x) \leq d^2(x, v) \ \ \forall v \in MEP(\varphi, f, C).$$

Since $d^2(., x)$ is Δ-lower semicontinuous, we obtain that

$$d^2(q, x) \leq \liminf_{k\to\infty} d^2(J^f_{\lambda_{nk}\varphi}x, x) \leq d^2(x, v) \ \ \forall v \in MEP(\varphi, f, C),$$

which implies that

$$d(q, x) \leq d(x, v) \ \ \forall v \in MEP(\varphi, f, C).$$

Thus, $q = P_\Gamma x$, where P_Γ is the metric projection of X onto Γ, and $\Gamma = MEP(\varphi, f, C)$. Therefore, by taken $\lambda_{nk} = \lambda$, we have that $\{J^f_{\lambda\varphi}x\}$ Δ-converges to $q = P_\Gamma x$ as $\lambda \to 0$. Now, observe also that Theorem 5.5.5(ii) implies that

$$d(J^f_{\lambda\varphi}x, x) \leq d(q, x).$$

It then follows from Lemma 2.3.28 that $\{J^f_{\lambda\varphi}x\}$ converges strongly to $q = P_\Gamma x$ as $\lambda \to 0$. $\square$

By setting $f \equiv 0$ in Theorem 5.5.7, we obtain the following result.

Corollary 5.5.8. *Let $\varphi : C \times C \to \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)-(A4) of Theorem 5.5.3. If $EP(\varphi, C) \neq \emptyset$, then $\{J_{\lambda\varphi}x\}$ converges strongly to $q \in EP(\varphi, C)$, which is the nearest point of $EP(\varphi, C)$ to x as $\lambda \to 0$.*

5.5.5 Proximal point algorithm

In this section, we study the Δ-convergence of the sequence generated by the following PPA for approximating solutions of MEP(1.2.8): For an initial starting point x_1 in C, define the sequence $\{x_n\}$ in C by

$$x_{n+1} = J^f_{\lambda_n\varphi}x_n, \ n \geq 1, \tag{5.5.11}$$

where $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $\varphi : C \times C \to \mathbb{R}$ is a bifunction and $f : C \to \mathbb{R}$ is a convex function.
Recall that the PPA does not converge strongly in general without additional assumptions even for the case where $\varphi \equiv 0$. See for example [159], the question of interest raised by Rockafellar as to whether the PPA can be improved from weak convergence (an analogue of Δ-convergence) to strong convergence in Hilbert space settings. Several counterexamples have been constructed to resolve this question in the negative (see [23, 25, 79]). Therefore, only weak convergence of the PPA is expected without additional assumptions. Thus, we propose the following Δ-convergence theorem for the PPA (5.5.11).

Theorem 5.5.9. *Let $f : C \to \mathbb{R}$ be a convex and lower semicontinuous function and $\varphi : C \times C \to \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)-(A4) of Theorem 5.5.3. Let $\{\lambda_n\}$ be a sequence in $(0, \infty)$ such that $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$. Suppose that $MEP(\varphi, f, C) \neq \emptyset$, then, the sequence given by (5.5.11) Δ-converges to an element of $MEP(\varphi, f, C)$.*

Proof. Let $v \in MEP(\varphi, f, C)$. Then, by Remark 5.5.6 and Theorem 5.5.5(iv), we obtain that

$$d(v, x_{n+1}) = d(v, J^f_{\lambda_n \varphi} x_n) \leq d(v, x_n),$$

which implies that $\lim\limits_{n\to\infty} d(x_n, v)$ exists for all $v \in MEP(\varphi, f, C)$. Hence $\{x_n\}$ is bounded. It then follows from Theorem 5.5.5(ii) that

$$d^2(x_{n+1}, x_n) \leq d^2(x_n, v) - d^2(x_{n+1}, v) \to 0, \text{ as } n \to \infty.$$

That is,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{5.5.12}$$

Since $\{x_n\}$ is bounded, then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ that Δ-converges to a point, say $q \in C$. From (5.5.11) and (5.5.4), we obtain that

$$\begin{aligned}
\varphi(x_{nk+1}, y) + f(y) - f(x_{nk+1}) &\geq -\frac{1}{\lambda_{nk}} \langle \overrightarrow{x_{nk} x_{nk+1}}, \overrightarrow{x_{nk+1} y} \rangle \\
&\geq -\frac{1}{\lambda_{nk}} d(x_{nk+1}, x_{nk}) d(x_{nk+1}, y).
\end{aligned} \tag{5.5.13}$$

Since $0 < \lambda \leq \lambda_{nk}$, $\{x_n\}$ is bounded, φ is Δ-upper semicontinuous in the first argument and f is lower semicontinuous, we obtained from (5.5.12) and (5.5.13) that

$$\begin{aligned}
\varphi(q, y) + f(y) - f(q) &\geq \limsup_{k\to\infty} \big(\varphi(x_{nk+1}, y) + f(y)\big) - \liminf_{k\to\infty} f(x_{nk+1}) \\
&\geq -\frac{M}{\lambda} \limsup_{k\to\infty} d(x_{nk+1}, x_{nk}) = 0,
\end{aligned} \tag{5.5.14}$$

for some $M > 0$ and for all $y \in C$. This implies that $q \in MEP(\varphi, f, C)$.
It then follows from Lemma 2.3.9 that $\{x_n\}$ Δ-converges to an element of $MEP(\varphi, f, C)$.
$\square$

By setting $f \equiv 0$ in Theorem 5.5.9, we obtain the following result which coincides with [108, Theorem 7.3].

Corollary 5.5.10. *Let $\varphi : C \times C \to \mathbb{R}$ be Δ-upper semicontinuous in the first argument which satisfies assumptions (A1)-(A4) of Theorem 5.5.3 and $\{\lambda_n\}$ be a sequence in $(0, \infty)$ such that $0 < \lambda \leq \lambda_n \ \forall \ n \geq 1$. Suppose that $EP(\varphi, C) \neq \emptyset$, then, the sequence given for $x_1 \in C$ by*

$$x_{n+1} = J_{\lambda_n \varphi} x_n, \ n \geq 1$$

Δ-converges to an element of $EP(\varphi, C)$.

By setting $\varphi \equiv 0$ in Theorem 5.5.9, we obtain the following corollary which is similar to [18, Theorem 1.4].

Corollary 5.5.11. *Let $f : C \to \mathbb{R}$ be a convex and lower semicontinuous function and $\{\lambda_n\}$ be a sequence in $(0, \infty)$ such that $0 < \lambda \leq \lambda_n \ \forall \ n \geq 1$. Suppose that $\arg\min_{y \in C} f(y) \neq \emptyset$, then, the sequence given for $x_1 \in C$ by*

$$x_{n+1} = J^f_{\lambda_n} x_n, \ n \geq 1 \tag{5.5.15}$$

Δ-converges to an element of $\arg\min_{y \in C} f(y)$.

5.5.6 Assymptotic behavior of Halpern's algorithm

To obtain strong convergence result, we modify the PPA into the following Halpern-type PPA and study the asymptotic behavior of the sequence generated by it: For $x_1, u \in C$, define the sequence $\{x_n\} \subset C$ by

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J^f_{\lambda_n \varphi} x_n, \tag{5.5.16}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n\}, \varphi$ and f are as defined in (5.5.11).
We begin by establishing the following lemmas which will be very useful to our study.

Lemma 5.5.12. *Let $f : C \to \mathbb{R}$ be a convex and lower semicontinuous function and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) of Theorem 5.5.3. If $\lambda, \mu > 0$ and $x, y \in C$, then the following inequalities hold:*

$$d^2(J^f_{\lambda\varphi} x, J^f_{\mu\varphi} y) \leq 2\lambda\varphi(J^f_{\lambda\varphi} x, J^f_{\mu\varphi} y) + 2\lambda(f(J^f_{\mu\varphi} y) - f(J^f_{\lambda\varphi} x)) + d^2(x, J^f_{\mu\varphi} y) - d^2(x, J^f_{\lambda\varphi} x) \tag{5.5.17}$$

and

$$(\lambda + \mu) d^2(J^f_{\lambda\varphi} x, J^f_{\mu\varphi} y) + \mu d^2(J^f_{\lambda\varphi} x, x) + \lambda d^2(J^f_{\mu\varphi} y, y) \leq \lambda d^2(J^f_{\lambda\varphi} x, y) + \mu d^2(J^f_{\lambda\varphi} y, x). \tag{5.5.18}$$

Proof. We first prove (5.5.17). Let $\lambda, \ \mu > 0$ and $x, y \in C$. Then, by (5.5.4), we obtain that

$$\varphi(J^f_{\lambda\varphi} x, z) + f(z) - f(J^f_{\lambda\varphi} x) + \frac{1}{\lambda} \langle \overrightarrow{x J^f_{\lambda\varphi} x}, \overrightarrow{J^f_{\lambda\varphi} x z} \rangle \geq 0 \ \forall \ z \in C,$$

which implies that

$$
\begin{aligned}
2\lambda f(J^f_{\lambda\varphi}x) &\le 2\lambda\varphi(J^f_{\lambda\varphi}x, z) + 2\lambda f(z) + 2\langle\overrightarrow{xJ^f_{\lambda\varphi}x}, \overrightarrow{J^f_{\lambda\varphi}xz}\rangle \\
&= 2\lambda\varphi(J^f_{\lambda\varphi}x, z) + 2\lambda f(z) + d^2(x, z) - d^2(x, J^f_{\lambda\varphi}) - d^2(J^f_{\lambda\varphi}x, z) \\
&\le 2\lambda\varphi(J^f_{\lambda\varphi}x, z) + 2\lambda f(z) + d^2(x, z) - d^2(x, J^f_{\lambda\varphi}x). \qquad (5.5.19)
\end{aligned}
$$

Now, set $z = tJ^f_{\mu\varphi}y \oplus (1-t)J^f_{\lambda\varphi}x$ for all $t \in (0, 1)$ in (5.5.19). Since f is convex and φ satisfies conditions (A1) and (A3) of Theorem 5.5.3, we obtain that

$$
\begin{aligned}
2\lambda f(J^f_{\lambda\varphi}x) + d^2(x, J^f_{\lambda\varphi}x) &\le 2\lambda\Big(t\varphi(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}y) + (1-t)\varphi(J^f_{\lambda\varphi}x, J^f_{\lambda\varphi}x)\Big) \\
&\quad + 2\lambda\Big(tf(J^f_{\mu\varphi}y) + (1-t)f(J^f_{\lambda\varphi}x)\Big) \\
&\quad + td^2(x, J^f_{\mu\varphi}y) + (1-t)d^2(x, J^f_{\lambda\varphi}x) - t(1-t)d^2(J^f_{\mu\varphi}y, J^f_{\lambda\varphi}x) \\
&= 2\lambda t\varphi(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}y) + 2\lambda\Big(tf(J^f_{\mu\varphi}y) + (1-t)f(J^f_{\lambda\varphi}x)\Big) \\
&\quad + td^2(x, J^f_{\mu\varphi}y) + (1-t)d^2(x, J^f_{\lambda\varphi}x) - t(1-t)d^2(J^f_{\mu\varphi}y, J^f_{\lambda\varphi}x), \\
&\qquad (5.5.20)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
2\lambda f(J^f_{\lambda\varphi}x) + d^2(x, J^f_{\lambda\varphi}x) &\le 2\lambda\varphi(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}y) + 2\lambda f(J^f_{\mu\varphi}y) \\
&\quad + d^2(x, J^f_{\mu\varphi}y) - (1-t)d^2(J^f_{\mu\varphi}y, J^f_{\lambda\varphi}x). \qquad (5.5.21)
\end{aligned}
$$

As $t \to 0$ in (5.5.21), we obtain (5.5.17).

Next, we prove (5.5.18). From (5.5.17), we obtain that

$$\mu d^2(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}y) \le 2\lambda\mu\big[\varphi(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}y) + f(J^f_{\mu\varphi}y) - f(J^f_{\lambda\varphi}x)\big] + \mu d^2(x, J^f_{\mu\varphi}y) - \mu d^2(x, J^f_{\lambda\varphi}x).$$

Similarly, we have

$$\lambda d^2(J^f_{\mu\varphi}y, J^f_{\lambda\varphi}x) \le 2\mu\lambda\big[\varphi(J^f_{\mu\varphi}y, J^f_{\lambda\varphi}x) + f(J^f_{\lambda\varphi}x) - f(J^f_{\mu\varphi}y)\big] + \lambda d^2(y, J^f_{\lambda\varphi}x) - \lambda d^2(y, J^f_{\mu\varphi}y).$$

Adding both inequalities and noting that φ is monotone, we get

$$(\lambda+\mu)d^2(J^f_{\lambda\varphi}x, J^f_{\mu\varphi}y) + \mu d^2(x, J^f_{\lambda\varphi}x) + \lambda d^2(y, J^f_{\mu\varphi}y) \le \mu d^2(x, J^f_{\mu\varphi}y) + \lambda d^2(y, J^f_{\lambda\varphi}x).$$

□

Lemma 5.5.13. *Let $f : C \to \mathbb{R}$ be a convex and lower semicontinuous function and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) of Theorem 5.5.3. Let $\{\lambda_n\}$ be a sequence in $(0, \infty)$ and $\bar{v}$ be an element of C. Suppose that $\lim_{n\to\infty} \lambda_n = \infty$ and $A(\{J^f_{\lambda_n}x_n\}) = \{\bar{v}\}$ for some bounded sequence $\{x_n\}$ in X, then $\bar{v} \in MEP(\varphi, f, C)$.*

Proof. From (5.5.18), we obtain that

$$(\lambda_n + 1)d^2(J^f_{\lambda_n\varphi}x_n, J^f_{\varphi}\bar{v}) + d^2(J^f_{\lambda_n\varphi}x_n, x_n) + \lambda_n d^2(J^f_{\varphi}\bar{v}, \bar{v}) \le d^2(J^f_{\varphi}\bar{v}, x_n) + \lambda_n d^2(J^f_{\lambda_n\varphi}x_n, \bar{v}),$$

which implies that

$$d^2(J^f_{\lambda_n\varphi}x_n, J^f_\varphi\bar{v}) \le \frac{1}{\lambda_n}d^2(J^f_\varphi\bar{v}, x_n) + d^2(J^f_{\lambda_n\varphi}x_n, \bar{v})^2.$$

Since $\lim_{n\to\infty}\lambda_n = \infty$, $\{x_n\}$ is bounded and $A(\{J^f_{\lambda_n}x_n\}) = \{\bar{v}\}$, we obtain that

$$\begin{aligned}\limsup_{n\to\infty} d(J^f_{\lambda_n\varphi}x_n, J^f_\varphi\bar{v}) &\le \limsup_{n\to\infty} d(J^f_{\lambda_n\varphi}x_n, \bar{v})\\ &= \inf_{y\in X}\limsup_{n\to\infty} d(J^f_{\lambda_n\varphi}x_n, y),\end{aligned}$$

which by Lemma 2.3.24 (ii) and Theorem 5.5.5(iv) implies that $\bar{v} \in F(J^f_\varphi) = MEP(\varphi, f, C)$. □

Theorem 5.5.14. *Let $f : C \to \mathbb{R}$ be a convex and lower semicontinuous function and $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A4) of Theorem 5.5.3. Let $\{x_n\}$ be a sequence defined by (5.5.16), where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\to\infty}\lambda_n = \infty$. Then, we have the following:*
(i) The sequence $\{J^f_{\lambda_n\varphi}x_n\}$ is bounded if and only if $MEP(\varphi, f, C) \neq \emptyset$.
(ii) If $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\Gamma := MEP(\varphi, f, C) \neq \emptyset$, then $\{x_n\}$ and $\{J^\Psi_{\lambda_n\varphi}x_n\}$ converge to $\bar{v} = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ.

Proof. (i) Suppose that $\{J^f_{\lambda_n}x_n\}$ is bounded. Then by Lemma 2.3.24 (ii), there exists $\bar{v} \in X$ such that $A(\{J^f_{\lambda_n}x_n\}) = \{\bar{v}\}$. From (5.5.16) and Lemma 2.3.1 (i), we obtain that

$$d(x_{n+1}, \bar{v}) \;\le\; \alpha_n d(u, \bar{v}) + (1-\alpha_n)d(J^f_{\lambda_n\varphi}x_n, \bar{v}),$$

which implies that $\{x_n\}$ is bounded. Also, since $\lim_{n\to\infty}\lambda_n = \infty$ and $A(\{J^f_{\lambda_n\varphi}x_n\}) = \{\bar{v}\}$, we obtain by Lemma 5.5.13 that $MEP(\varphi, f, C) \neq \emptyset$.

Conversely, let $MEP(\varphi, f, C) \neq \emptyset$. Then, we may assume that $\bar{v} \in MEP(\varphi, f, C) \neq \emptyset$. Thus by (5.5.16) and Lemma 2.3.1, we obtain that

$$\begin{aligned}d(x_{n+1}, \bar{v}) &\le \alpha_n d(u, \bar{v}) + (1-\alpha_n)d(J^f_{\lambda_n\varphi}x_n, \bar{v})\\ &\le \alpha_n d(u, \bar{v}) + (1-\alpha_n)d(x_n, \bar{v})\\ &\le \max\{d(u, \bar{v}), d(x_n, \bar{v})\},\end{aligned}$$

which implies by induction that

$$d(x_n, \bar{v}) \;\le\; \max\{d(u, \bar{v}), d(x_1, \bar{v})\}\ \forall n \ge 1. \tag{5.5.22}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{J^f_{\lambda_n\varphi}x_n\}$ is also bounded.

(ii) Since $\Gamma := MEP(\varphi, f, C) \neq \emptyset$, we obtain from (5.5.22) that $\{x_n\}$ and $\{J^f_{\lambda_n\varphi}x_n\}$ are bounded. Furthermore, we obtain from Lemma 2.3.1 (ii) that

$$\begin{aligned}d^2(x_{n+1}, \bar{v}) &\le \alpha_n d^2(u, \bar{v}) + (1-\alpha_n)d^2(J^f_{\lambda_n\varphi}x_n, \bar{v}) - \alpha_n(1-\alpha_n)d^2(u, J^f_{\lambda_n\varphi}x_n)\\ &\le \alpha_n d^2(u, \bar{v}) + (1-\alpha_n)d^2(x_n, \bar{v}) - \alpha_n(1-\alpha_n)d^2(u, J^f_{\lambda_n\varphi}x_n)\\ &= (1-\alpha_n)d^2(x_n, \bar{v}) + \alpha_n\delta_n\ \forall n \ge 1,\end{aligned} \tag{5.5.23}$$

where $\delta_n = d^2(u,\bar{v}) + (\alpha_n - 1)d^2(u, J^f_{\lambda_n\varphi}x_n)$. Now, set $v_n = J^f_{\lambda_n\varphi}x_n \ \forall n \geq 1$. Then, by the boundedness of $\{v_n\}$ and Lemma 2.3.24 (i), we obtain that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that Δ-converges to some $\hat{v} \in C$. Thus, by Lemma 2.3.24 (ii), we obtain that $A(\{v_{n_k}\}) = \{\hat{v}\}$. Moreover, $\lim\limits_{k\to\infty} \lambda_{n_k} = \infty$ and $\{x_{n_k}\}$ is bounded. Hence, by Lemma 5.5.13, we obtain that $\hat{v} \in MEP(\varphi, f, C)$.

Next, we show that $\{x_n\}$ converges to $\hat{v}$. By the Δ-lower semicontinuity of $d^2(u,.)$, we obtain that

$$d^2(u,\hat{v}) \leq \liminf_{k\to\infty} d^2(u, v_{n_k}) = \lim_{k\to\infty} d^2(u, v_{n_k}) = \liminf_{n\to\infty} d^2(u, v_n). \tag{5.5.24}$$

Since $\delta_n = d^2(u,\bar{v}) + (\alpha_n - 1)d^2(u, v_n)$, $\lim\limits_{n\to\infty} \alpha_n = 0$, $\bar{v} = P_\Gamma u$ and $\hat{v} \in \Gamma$, we obtain from the definition of P_Γ and (5.5.24) that

$$\begin{aligned} \limsup_{n\to\infty} \delta_n &\leq d^2(u,\bar{v}) - \liminf_{n\to\infty} d^2(u, v_n) \\ &\leq d^2(u,\hat{v}) - \liminf_{n\to\infty} d^2(u, v_n) \leq 0. \end{aligned}$$

Thus, applying Lemma 2.3.26 to (5.5.23), gives that $\{x_n\}$ converges to $\bar{v} = P_\Gamma u$. It then follows that $\{J^f_{\lambda_n\varphi}x_n\}$ is convergent to $\bar{v} = P_\Gamma u$. □

By setting $f \equiv 0$ in Theorem 5.5.14, we obtain the following result for equilibrium problem in an Hadamard space.

Corollary 5.5.15. *Let $\varphi : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)-(A3) of Theorem 5.5.3 and $\{x_n\}$ be a sequence defined for $u, x_1 \in C$, by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)J_{\lambda_n\varphi}x_n, \tag{5.5.25}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim\limits_{n\to\infty} \lambda_n = \infty$. Then, we have the following:
(i) The sequence $\{J_{\lambda_n\varphi}x_n\}$ is bounded if and only if $EP(\varphi, C) \neq \emptyset$.
(ii) If $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := EP(\varphi, C) \neq \emptyset$, then $\{x_n\}$ and $\{J_{\lambda_n\varphi}x_n\}$ converge to $\bar{v} = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ.

By setting $\varphi \equiv 0$ in Theorem 5.5.14, we obtain the following result which coincides with [101, Theorem 5.1].

Corollary 5.5.16. *Let $f : C \to C$ be a proper convex and lower semicontinuous function and $\{x_n\}$ be a sequence defined for $u, x_1 \in C$, by*

$$x_{n+1} = \alpha_n u \oplus (1-\alpha_n)J^f_{\lambda_n}x_n, \tag{5.5.26}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,\infty)$ such that $\lim\limits_{n\to\infty} \lambda_n = \infty$. Then, we have the following:
(i) The sequence $\{J^f_{\lambda_n}x_n\}$ is bounded if and only if $\arg\min\limits_{y\in C} f(y) \neq \emptyset$.
(ii) If $\lim\limits_{n\to\infty} \alpha_n = 0$, $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := \arg\min\limits_{y\in C} f(y) \neq \emptyset$, then $\{x_n\}$ and $\{J^\Psi_{\lambda_n}x_n\}$ converge to $\bar{v} = P_\Gamma u$, where P_Γ is the metric projection of X onto Γ.

Remark 5.5.17. *To establish the above results, we assumed that* $D(J^f_{\lambda\varphi}) \supset C$ *(see the assumption at the beginning of Subsection 5.5.4 and in the results of Section 5.5). However, we do not know if any of these results (the results established in Section 5.5) can be obtained without this assumption.*

Chapter 6

Generalized Strictly Pseudononspreding Mappings in Hadamard Spaces

6.1 Introduction

Let X be an Hadamard space and T be a nonlinear mapping on X. T is called

- *nonspreading* (see [106]), if

$$2d^2(Tx,Ty) \leq d^2(Tx,y) + d^2(Ty,x)\ \forall x,y \in C;$$

- *k-strictly pseudononspreading* (see [150]), if

$$\begin{aligned} & (2-k)d^2(Tx,Ty) \\ \leq\ & kd^2(x,y) + (1-k)d^2(y,Tx) + (1-k)d^2(x,Ty) + kd^2(x,Tx) + kd^2(y,Ty) \end{aligned}$$

  for all $x, y \in C$;

- *generalized asymptotically nonspreading* (see [149]), if there exist two mappings $f, g : C \to [0,\gamma],\ \gamma < 1$ such that

$$d^2(T^n x, T^n y) \leq f(x)d^2(T^n x, y) + g(x)d^2(T^n y, x)\ \forall x,y \in C, \quad n \in \mathbb{N},$$

  and

$$0 < f(x) + g(x) \leq 1\ \forall x \in C,$$

  if $n = 1$, then T is called (f,g)*-generalized (or simply generalized) nonspreading.*

It is clear that nonspreading mappings with nonempty fixed point sets are quasinonexpansive. Also, if T is a generalized nonspreading mapping and $f(x) = \frac{1}{2} = g(x)\ \forall x \in C$, then T reduces to a nonspreading mapping. It is also clear that every nonspreading mapping is 0-strictly pseudononspreading. The class of k-strictly pseudononspreading mappings was first introduced and studied in Hilbert spaces by Osilike and Isiogugu [145], and was later extended to $\mathbb{R}$-Trees settings in [150, 162]. In general, the class of nonspreading-type mappings (that is, the class of nonspreading, k-strictly pseudononspreading and generalized nonspreading mappings) are known to be very useful in solving mean ergodic problems (see for example, [109, 106, 145, 189]). Thus, numerous researchers have studied these classes of mappings (both singlevalued and multivalued) in Hilbert spaces and spaces more general than Hilbert spaces (see, for example [121, 150, 162, 189] and the references therein).

Now, consider the following two examples (see Section 6.2 for proofs).

Example 6.1.1. *Let $T : [0, \infty) \to [0, \infty)$ be defined by*

$$Tx = \begin{cases} \frac{1}{x+\frac{1}{10}}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then, T is a generalized nonspreading mapping with $f, g : [0, \infty) \to [0, 0.9]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \geq 1, \\ 0.9, & \text{if } x \in [0, 1) \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \frac{1}{(x+\frac{1}{10})^2}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

However, T is not k-strictly pseudononspreading. To see this, take $x = 1$ and $y = 0.5$.

Example 6.1.2. *Let $T : [0, 1] \to \mathbb{R}$ be defined by $Tx = -3x$. Then, T is a k-strictly pseudononspreading mapping but not a generalized nonspreading mapping.*

Examples 6.1.1 and 6.1.2 show that the class of generalized nonspreading mappings and the class of k-strictly pseudononspreading mappings are independent. Since both classes of mappings are independent and at the same time, very useful (especially in solving mean ergodic problems), it is important and of great interest to consider a class of mappings that will bridge the gap between these classes of mappings. Motivated by this, we shall introduce a new class of nonspreading-type mappings that will serve as a bridge in connecting the two classes of mappings (the class of k-strictly pseudononspreading mappings and the class of generalized nonspreading mappings). We shall call this class of mappings, the class of (f, g)-*generalized (or simply generalized) k-strictly pseudononspreading mappings*. Furthermore, we shall discuss some fixed point properties of this class of mappings and prove some strong convergence results for it.

6.2 Preliminaries

We first introduce our new class of nonspreading-type mappings.

Definition 6.2.1. *Let X be a metric space. We say that a mapping $T : D(T) \subseteq X \to X$ is (f, g)-generalized (or simply generalized) k-strictly pseudononspreading if there exist two functions $f, g : D(T) \subseteq X \to [0, \gamma]$, $\gamma < 1$ and $k \in [0, 1)$ such that*

$$(1-k)d^2(Tx, Ty) \leq kd^2(x, y) + [f(x) - k]\, d^2(Tx, y) + [g(x) - k]\, d^2(x, Ty) + kd^2(x, Tx) + kd^2(y, Ty)$$

$\forall x, y \in D(T)$, and

$$0 < f(x) + g(x) \leq 1 \ \forall x \in D(T).$$

Remark 6.2.2.

(i) Clearly, every generalized nonspreading mapping is a generalized 0*-strictly pseudononspreading mapping.*

(ii) Every k*-strictly pseudononspreading mapping is a generalized* k*-strictly pseudononspreading mapping. Indeed, if* T *is a* k*-strictly pseudononspreading mapping, then for all* $x, y \in D(T)$*, there exists* $k \in [0,1)$ *such that*

$$(2-k)d^2(Tx,Ty) \leq kd^2(x,y) + (1-k)d^2(Tx,y) + (1-k)d^2(x,Ty) + kd^2(x,Tx) + kd^2(y,Ty),$$

which implies

$$\begin{aligned}\left(1-\frac{k}{2}\right)d^2(Tx,Ty) \ &\leq \ \frac{k}{2}d^2(x,y) + \left(\frac{1}{2}-\frac{k}{2}\right)d^2(y,Tx) + \left(\frac{1}{2}-\frac{k}{2}\right)d^2(x,Ty) \\ &\quad + \frac{k}{2}d^2(x,Tx) + \frac{k}{2}d^2(y,Ty).\end{aligned}$$

That is

$$\begin{aligned}(1-k')\,d^2(Tx,Ty) \ &\leq \ k'd^2(x,y) + (f(x)-k')\,d^2(Tx,y) + (g(x)-k')\,d^2(x,Ty) \\ &\quad + k'd^2(x,Tx) + k'd(y,Ty),\end{aligned}$$

where $f(x) = g(x) = \frac{1}{2}$*,* $\forall x \in D(T)$ *and* $k' = \frac{k}{2} \in [0,1)$*. Hence,* T *is a generalized* k*-strictly pseudononspreading mapping.*

The following examples show that the class of k-strictly pseudononspreading mappings and the class of generalized nonspreading mappings are properly contained in the class of generalized k-strictly pseudononspreading mappings.

First, we give an example of a generalized k-strictly pseudononspreading mapping which is not k-strictly pseudononspreading.

Example 6.2.3. *Let* $T : [0,\infty) \to [0,\infty)$ *be defined by*

$$Tx = \begin{cases} \frac{1}{x+\frac{1}{10}}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0,1). \end{cases}$$

Then, T *is not* k*-strictly pseudononspreading. In fact, if we take* $x = 1$ *and* $y = 0.5$*, then* $|Tx-Ty|^2 = 0.82644$*,* $|x-y|^2 = 0.25$*,* $k|x-Tx-(y-Ty)|^2 = 0.16736k$*,* $2\langle x-Tx, y-Ty\rangle = 0.09091$*.*
Hence,

$|Tx-Ty|^2 = 0.82644 > 0.34091 + 0.16736k = |x-y|^2 + k|x-Tx-(y-Ty)|^2 + 2\langle x - Tx, y-Ty\rangle$*,*
for all $k \in [0,1)$*.*

However, T *is a generalized* k*-strictly pseudononspreading mapping with* $k = 0$*. To see this,*
let $f, g : [0,\infty) \to [0, 0.9]$ *be defined by*

$$f(x) = \begin{cases} 0, & \text{if } x \geq 1, \\ 0.9, & \text{if } x \in [0,1) \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{(x+\frac{1}{10})^2}, & \text{if } x \geq 1, \\ 0, & \text{if } x \in [0,1). \end{cases}$$

Case 1: *If $x \geq 1$ and $y \in [0,1)$, then $Tx = \frac{1}{x+\frac{1}{10}}$, $Ty = 0$, $f(x) = 0$ and $g(x) = \frac{1}{(x+\frac{1}{10})^2}$. Thus, we obtain*

$$|Tx - Ty|^2 = \frac{1}{(x+\frac{1}{10})^2} \leq 0 + g(x)x^2 = f(x)|y - Tx|^2 + g(x)|x - Ty|^2.$$

Case 2: *If $x \in [0,1)$ and $y \geq 1$, then $Tx = 0$, $Ty = \frac{1}{y+\frac{1}{10}}$, $f(x) = 0.9$ and $g(x) = 0$. Thus, we obtain*

$$|Tx - Ty|^2 = \frac{1}{(y+\frac{1}{10})^2} < f(x)y^2 + 0 = f(x)|y - Tx|^2 + g(x)|x - Ty|^2.$$

Case 3: *If $x \geq 1$ and $y \geq 1$, then $Tx = \frac{1}{x+\frac{1}{10}}$, $Ty = \frac{1}{y+\frac{1}{10}}$, $f(x) = 0$ and $g(x) = \frac{1}{(x+\frac{1}{10})^2}$. Thus, we obtain*

$$|Tx - Ty|^2 = \Big|\frac{1}{x+\frac{1}{10}} - \frac{1}{y+\frac{1}{10}}\Big|^2 = \frac{(x-y)^2}{(x+\frac{1}{10})^2(y+\frac{1}{10})^2}$$

and

$$f(x)|y - Tx|^2 + g(x)|x - Ty|^2 = \frac{1}{(x+\frac{1}{10})^2}\Big|x - \frac{1}{y+\frac{1}{10}}\Big|^2 = \frac{(1 - xy - \frac{x}{10})^2}{(x+\frac{1}{10})^2(y+\frac{1}{10})^2}.$$

Since $(x-y)^2 - (1 - xy - \frac{x}{10})^2 < 0$, we conclude that $|Tx - Ty|^2 < f(x)|y - Tx|^2 + g(x)|x - Ty|^2$,
for all $x \geq 1$ and $y \geq 1$.

For the case where $x, y \in [0,1)$, we have that $|Tx - Ty|^2 = 0 < f(x)|y - Tx|^2 + g(x)|x - Ty|^2$. Thus,

$$|Tx - Ty|^2 \leq f(x)|y - Tx|^2 + g(x)|x - Ty|^2 \ \forall x, y \in [0, \infty).$$

Hence, T is a generalized nonspreading mapping. It then follows that T is a generalized k-strictly pseudononspreading mapping with $k = 0$.

We now give an example of a generalized k-strictly pseudononspreading mapping which is neither k-strictly pseudononspreading nor generalized nonspreading.

Example 6.2.4. *Let $T : [0, \infty) \to \mathbb{R}$ defined by*

$$Tx = \begin{cases} -3x, & \text{if } x \in [0,1], \\ \frac{1}{x}, & \text{if } x \in (1, \infty). \end{cases}$$

We first show that T is not k-strictly pseudononspreading. Indeed, if $x = \frac{11}{10}$ and $y = \frac{1}{3}$, then
$|Tx - Ty|^2 = 3.64463$, $|x - y|^2 = 0.58778$, $k|x - Tx - (y - Ty)|^2 = 1.30513k$, $2\langle x - Tx, y -$

$Ty\rangle = 0.50909$.
Hence,

$|Tx - Ty|^2 = 3.64463 > 1.09687 + 1.30513k = |x-y|^2 + k|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle \ \forall k \in [0,1)$.
Therefore, T is not k-strictly pseudononspreading.

Next, we show that T is not generalized nonspreading. Suppose for contradiction that T is a generalized nonspreading mapping, then we can always find two functions $f, g : [0,\infty) \to [0,\gamma],\ \gamma < 1$ such that

$$|Tx - Ty|^2 \leq f(x)|Tx - y|^2 + g(x)|Ty - x|^2 \ \forall x, y \in [0,\infty)$$

and

$$0 < f(x) + g(x) \leq 1 \ \forall x \in [0,\infty).$$

In particular, for $x = 0$ and $y = 1$, we have that

$$9 = |Tx - Ty|^2 \leq f(x)|Tx - y|^2 + g(x)|Ty - x|^2 = f(x) + 9g(x).$$

That is,

$$9 \leq f(x) + 9g(x). \tag{6.2.1}$$

If $f(x) = 0$, then we have that $9 \leq 9g(x) < 9$ and this is a contradiction. Now, suppose $f(x) \neq 0$, then we obtain from (6.2.1) that $f(x) \geq 9(1-g(x)) \geq 9f(x)$ (since $f(x)+g(x) \leq 1$). This implies that $1 \geq 9$ and this is a contraction. Therefore, T is not generalized nonspreading.

Finally, we show that T is a generalized k-strictly pseudononspreading mapping with $k = \frac{9}{10}$. To see this,
let $f, g : [0,\infty) \to [0, 0.9]$ be defined by

$$f(x) = \begin{cases} \frac{9}{10}, & \text{if } x \in [0,1], \\ \frac{1}{10}, & \text{if } x \in (1,\infty) \end{cases}$$

and

$$g(x) = \begin{cases} \frac{1}{10}, & \text{if } x \in [0,1], \\ \frac{9}{10}, & \text{if } x \in (1,\infty). \end{cases}$$

Case 1: If $x, y \in [0,1]$, then $Tx = -3x,\ Ty = -3y,\ f(x) = \frac{9}{10}$ and $g(x) = \frac{1}{10}$. So that,

$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{9}{10}|x-y|^2$, $[f(x)-\frac{9}{10}]|Tx-y|^2 = 0$, $[g(x)-\frac{9}{10}]|x-Ty|^2 = \frac{-8x^2-48xy-72y^2}{10}$, $\frac{9}{10}|x - Tx|^2 = \frac{144x^2}{10}$, $\frac{9}{10}|y - Ty|^2 = \frac{144y^2}{10}$.

Hence,

$$\begin{aligned} & \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\ & +\frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\ = \; & \frac{9}{10}|x-y|^2 + \frac{136x^2+72y^2-48xy}{10} \\ \geq \; & \frac{9}{10}|x-y|^2 = \left(1-\frac{9}{10}\right)|Tx-Ty|^2. \end{aligned}$$

Case 2: If $x, y \in (1,\infty)$, then $Tx = \frac{1}{x},\ Ty = \frac{1}{y},\ f(x) = \frac{1}{10}$ and $g(x) = \frac{9}{10}$. So that,

$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{x^2+y^2-2xy}{10x^2y^2}$, $\frac{9}{10}|x-y|^2 = \frac{9x^2+9y^2-18xy}{10}$, $[f(x)-\frac{9}{10}]|Tx-y|^2 = \frac{-8x^2y^2-8+16xy}{10x^2}$,
$[g(x)-\frac{9}{10}]|x-Ty|^2 = 0$, $\frac{9}{10}|x-Tx|^2 = \frac{9x^4-18x^2+9}{10x^2}$, $\frac{9}{10}|y-Ty|^2 = \frac{9y^4-18y^2+9}{10y^2}$.
Hence,

$$\begin{aligned} & \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 \\ & +\frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\ = \; & \frac{18x^4y^2+10x^2y^4+16xy^3+y^2+9x^2-18x^3y^3-36x^2y^2}{10x^2y^2}. \end{aligned}$$

Observe that
$\frac{18x^4y^2+10x^2y^4+16xy^3+y^2+9x^2-18x^3y^3-36x^2y^2}{10x^2y^2} - \frac{x^2+y^2-2xy}{10x^2y^2}$

$= \frac{18x^4y^2+10x^2y^4+16xy^3+8x^2+2xy-18x^3y^3-36x^2y^2}{10x^2y^2} \geq 0$ for all $x, y \in (1,\infty)$.
Hence, we conclude that
$(1-\frac{9}{10})|Tx-Ty|^2 \leq \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 + \frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2$,
for all $x, y \in (1,\infty)$.

Case 3: If $x \in (1,\infty)$ and $y \in [0,1]$, then $Tx = \frac{1}{x},\ Ty = -3y,\ f(x) = \frac{1}{10}$ and $g(x) = \frac{9}{10}$. So that,

$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{1+6xy+9x^2y^2}{10x^2}$, $\frac{9}{10}|x-y|^2 = \frac{9x^2+9y^2-18xy}{10}$, $[f(x)-\frac{9}{10}]|Tx-y|^2 = \frac{-8x^2y^2-8+16xy}{10x^2}$,
$[g(x)-\frac{9}{10}]|x-Ty|^2 = 0$, $\frac{9}{10}|x-Tx|^2 = \frac{9x^4-18x^2+9}{10x^2}$, $\frac{9}{10}|y-Ty|^2 = \frac{144y^2}{10}$.
Hence,

$$\begin{aligned} & \frac{9}{10}|x-y|^2 + [f(x)-\frac{9}{10}]|Tx-y|^2 + [g(x)-\frac{9}{10}]|x-Ty|^2 + \frac{9}{10}|x-Tx|^2 + \frac{9}{10}|y-Ty|^2 \\ = \; & \frac{18x^4+145x^2y^2+16xy+1-18x^3y-18x^2}{10x^2}. \end{aligned}$$

Observe that
$\frac{18x^4+145x^2y^2+16xy+1-18x^3y-18x^2}{10x^2} - \frac{1+6xy+9x^2y^2}{10x^2}$
$= \frac{18x^4+136x^2y^2+10xy-18x^3y-18x^2}{10x^2} \geq 0$ for all $x \in (1, \infty)$ and $y \in [0, 1]$.

Hence, we conclude that

$(1-\frac{9}{10})|Tx-Ty|^2 \leq \frac{9}{10}|x-y|^2+[f(x)-\frac{9}{10}]|Tx-y|^2+[g(x)-\frac{9}{10}]|x-Ty|^2+\frac{9}{10}|x-Tx|^2+\frac{9}{10}|y-Ty|^2$
for all $x \in (1, \infty)$ and $y \in [0, 1]$.

Case 4: If $x \in [0, 1]$ and $y \in (1, \infty)$, then $Tx = -3x,\ Ty = \frac{1}{y},\ f(x) = \frac{9}{10}$ and $g(x) = \frac{1}{10}$. So that,

$(1-\frac{9}{10})|Tx-Ty|^2 = \frac{1+6xy+9x^2y^2}{10y^2}$, $\frac{9}{10}|x-y|^2 = \frac{9x^2+9y^2-18xy}{10}$, $[f(x)-\frac{9}{10}]|Tx-y|^2 = 0$,
$[g(x)-\frac{9}{10}]|x-Ty|^2 = \frac{-8x^2y^2-8+16xy}{10y^2}$, $\frac{9}{10}|x-Tx|^2 = \frac{144x^2}{10}$, $\frac{9}{10}|y-Ty|^2 = \frac{9y^4-18y^2+9}{10y^2}$.

Hence,

$$\begin{aligned}
& \frac{9}{10}|x-y|^2+[f(x)-\frac{9}{10}]|y-Tx|^2+[g(x)-\frac{9}{10}]|x-Ty|^2 \\
& +\frac{9}{10}|x-Tx|^2+\frac{9}{10}|y-Ty|^2 \\
= & \frac{18y^4+145x^2y^2+16xy+1-18xy^3-18y^2}{10y^2}.
\end{aligned}$$

By similar argument as in **Case 3**, we obtain that

$(1-\frac{9}{10})|Tx-Ty|^2 \leq \frac{9}{10}|x-y|^2+[f(x)-\frac{9}{10}]|Tx-y|^2+[g(x)-\frac{9}{10}]|x-Ty|^2+\frac{9}{10}|x-Tx|^2+\frac{9}{10}|y-Ty|^2$
for all $x \in [0, 1]$ and $y \in (1, \infty)$.

Therefore, T is a generalized k-strictly pseudononspreading mapping with $k = \frac{9}{10}$.

Proposition 6.2.5. *The class of k-strictly pseudononspreading mappings and the class of generalized nonspreading mappings are independent. That is, the class of generalized nonspreading mappings is not a subclass of the class of k-strictly pseudononspreading mappings, and the class of k-strictly pseudononspreading mappings is not a subclass of the class of generalized nonspreading mappings.*

Proof. First, we recall that the mapping defined in Example 6.2.3 is a generalized nonspreading mapping but not a k-strictly pseudononspreading mapping.

However, if we consider the mapping $T : [0, 1] \to \mathbb{R}$ defined by $Tx = -3x$. Then, T is k-strictly pseudononspreading but not generalized nonspreading. To see that T is k-strictly pseudononspreading, observe that
$|Tx-Ty|^2 = 9|x-y|^2$, $|x-Tx-(y-Ty)|^2 = 16|x-y|^2$ and $2\langle x-Tx, y-Ty\rangle = 32xy$.

Thus,

$$\begin{aligned}
|Tx-Ty|^2 &= |x-y|^2+8|x-y|^2 \\
&= |x-y|^2+\frac{8}{16}|x-Tx-(y-Ty)|^2 \\
&\leq |x-y|^2+\frac{1}{2}|x-Tx-(y-Ty)|^2+2\langle x-Tx, y-Ty\rangle,
\end{aligned}$$

since $32xy \geq 0\ \forall x,y \in [0,1]$.

But, if we take $x=0$ and $y=1$, by the same argument as in Example 6.2.4, we obtain that T is not a generalized nonspreading mapping. Hence, our proof is complete. $\square$

Remark 6.2.6. *Observe that if T is (f,g)-generalized k-strictly pseudononspreading with $F(T) \neq \emptyset$ and $f(p) \neq 0\ \forall p \in F(T)$, then for each $p \in F(T)$ and $y \in D(T)$, we have*

$$d^2(p,Ty) \leq f(p)d^2(p,y)+g(p)d^2(p,Ty)+kd^2(y,Ty),$$

which implies

$$(1-g(p))d^2(p,Ty) \leq f(p)d^2(p,y)+kd^2(y,Ty).$$

Since $f(p)+g(p) \leq 1$, we obtain

$$d^2(p,Ty) \leq d^2(p,y)+\frac{k}{f(p)}d^2(y,Ty). \tag{6.2.2}$$

Proposition 6.2.7. *Let C be a nonempty closed and convex subset of an Hadamard space X and $T: C \to C$ be (f,g)-generalized k-strictly pseudononspreading mapping with $k \in [0,1)$, where $f,g: C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x)+g(x) \leq 1$ for all $x \in C$. Suppose that $F(T) \neq \emptyset$ and $f(p) \neq 0$, with $\frac{k}{f(p)} \leq \beta < 1$ for each $p \in F(T)$, then $F(T)$ is closed and convex.*

Proof. We first show that $F(T)$ is closed. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\{x_n\}$ converges to $x^* \in C$. Since T is (f,g)-generalized k-strictly pseudononspreading mapping, then from (6.2.2), we obtain

$$\begin{aligned}
d^2(x_n,Tx^*) &\leq d^2(x_n,x^*)+\frac{k}{f(x_n)}d^2(x^*,Tx^*) \\
&\leq \left[d(x_n,x^*)+\sqrt{\frac{k}{f(x_n)}}d(x^*,Tx^*)\right]^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
d(x^*,Tx^*) &\leq d(x^*,x_n)+d(x_n,Tx^*) \\
&\leq 2d(x^*,x_n)+\sqrt{\frac{k}{f(x_n)}}d(x^*,Tx^*) \\
&\leq 2d(x^*,x_n)+\sqrt{\beta}d(x^*,Tx^*), \qquad (6.2.3)
\end{aligned}$$

which implies

$$1-\sqrt{\beta}d(x^*,Tx^*)\le 2d(x^*,x_n)\to 0,\ \text{ as } n\to\infty.$$

Since $\sqrt{\beta}<1$, it follows that $d(x^*,Tx^*)=0$. Therefore, $x^*\in F(T)$.

Next, we show that $F(T)$ is convex. Let $z=tx\oplus(1-t)y$ for each $x,y\in F(T)$ and $t\in[0,1]$, then from
Lemma 2.3.1, (2.1.1) and (6.2.2), we obtain

$$\begin{aligned}
d^2(z,Tz) &= d^2(tx\oplus(1-t)y,Tz)\\
&\le td^2(x,Tz)+(1-t)d^2(y,Tz)-t(1-t)d^2(x,y)\\
&\le t\left[d^2(x,z)+\frac{k}{f(x)}d^2(z,Tz)\right]\\
&\quad +(1-t)\left[d^2(y,z)+\frac{k}{f(y)}d^2(z,Tz)\right]-t(1-t)d^2(x,y)\\
&= t\left[(1-t)^2d^2(x,y)+\frac{k}{f(x)}d^2(z,Tz)\right]\\
&\quad +(1-t)\left[t^2d^2(x,y)+\frac{k}{f(y)}d^2(z,Tz)\right]-t(1-t)d^2(x,y)\\
&= \left(t\frac{k}{f(x)}+(1-t)\frac{k}{f(y)}\right)d^2(z,Tz)\\
&\le Md^2(z,Tz),
\end{aligned}$$

where $M:=\max\left\{\frac{k}{f(x)},\frac{k}{f(y)}\right\}$. Thus,

$$(1-M)d^2(z,Tz)\le 0.$$

Since $M<1$, it follows that $z\in F(T)$, which completes our proof. $\square$

Lemma 6.2.8. *Let C be a nonempty closed and convex subset of an Hadamard space X and $T:C\to C$ be (f,g)-generalized k-strictly pseudononspreading mapping with $k\in[0,1)$, where $f,g:C\to[0,\gamma]$, $\gamma<1$ and $0<f(x)+g(x)\le 1$ for all $x\in C$. Suppose $k<\frac{1-f(x)}{2}$ for all $x\in C$, and $\{x_n\}$ is a bounded sequence in C such that $\Delta\text{-}\lim\limits_{n\to\infty}x_n=z$ and $\lim\limits_{n\to\infty}d(x_n,Tx_n)=0$. Then $z\in F(T)$.*

Proof. Since $\Delta\text{-}\lim\limits_{n\to\infty}x_n=z$, we have from Lemma 2.3.8 that $x_n\rightharpoonup z$. Thus, by Lemma 2.3.7, we obtain that $A(\{x_n\})=\{z\}$. Hence, since $\lim\limits_{n\to\infty}d(x_n,Tx_n)=0$, we have that $\Phi(z):=\limsup\limits_{n\to\infty}d^2(x_n,z)=\limsup\limits_{n\to\infty}d^2(Tx_n,z)$, which implies that $\Phi(Tz)=\limsup\limits_{n\to\infty}d^2(x_n,Tz)=\limsup\limits_{n\to\infty}d^2(Tx_n,Tz)$. Now, since T is (f,g)-generalized k-strictly pseudononspreading, we obtain

$$\begin{aligned}
& (1-k)d^2(x_n, Tz) \\
\leq\ & (1-k)\left(d(x_n, Tx_n) + d(Tz, Tx_n)\right)^2 \\
=\ & (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) + (1-k)d^2(Tz, Tx_n) \\
\leq\ & (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) + kd^2(z, x_n) \\
& +[f(z)-k]d^2(Tz, x_n) + [g(z)-k]d^2(z, Tx_n) + kd^2(z, Tz) + kd^2(x_n, Tx_n) \\
\leq\ & (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) + kd^2(z, x_n) \\
& +[f(z)-k]d^2(Tz, x_n) + [g(z)-k]\left(d(z, x_n) + d(x_n, Tx_n)\right)^2 \\
& +kd^2(z, Tz) + kd^2(x_n, Tx_n) \\
=\ & (1-k)d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) + kd^2(z, x_n) \\
& +[f(z)-k]d^2(Tz, x_n) + [g(z)-k]d^2(z, x_n) + 2[g(z)-k]d(z, x_n)d(x_n, Tx_n) \\
& +[g(z)-k]d^2(x_n, Tx_n) + kd^2(z, Tz) + kd^2(x_n, Tx_n),
\end{aligned}$$

which implies

$$\begin{aligned}
& (1-f(z))d^2(x_n, Tz) \\
\leq\ & d^2(x_n, Tx_n) + 2(1-k)d(x_n, Tx_n)d(Tz, Tx_n) + g(z)d^2(z, x_n) \\
& +2[g(z)-k]d(z, x_n)d(x_n, Tx_n) + [g(z)-k]d^2(x_n, Tx_n) + kd^2(z, Tz) \\
\leq\ & g(z)d^2(z, x_n) + kd^2(z, Tz) + d^2(x_n, Tx_n) + 2M(1-k)d(x_n, Tx_n) \\
& +2M[g(z)-k]d(x_n, Tx_n) + [g(z)-k]d^2(x_n, Tx_n),
\end{aligned}$$

where $M := \sup_{n\geq 1}\{d(x_n, z), d(Tx_n, Tz)\}$. Taking $\limsup$ on both sides of the inequality above, we obtain

$$\limsup_{n\to\infty}(1-f(z))d^2(x_n, Tz) \leq \limsup_{n\to\infty}\left[g(z)d^2(x_n, z) + kd^2(z, Tz)\right]. \tag{6.2.4}$$

That is,

$$(1-f(z))\Phi(Tz) \leq g(z)\Phi(z) + kd^2(z, Tz). \tag{6.2.5}$$

Now, by letting $t = \frac{1}{2}$ in Lemma 2.3.1 (ii), we obtain

$$d^2\left(x_n, \frac{z \oplus Tz}{2}\right) \leq \frac{1}{2}d^2(x_n, z) + \frac{1}{2}d^2(x_n, Tz) - \frac{1}{4}d^2(z, Tz).$$

Taking $\limsup$ on both sides of the inequality above and noting that $A(\{x_n\}) = \{z\}$, we obtain

$$\Phi(z) \leq \Phi\left(\frac{z \oplus Tz}{2}\right) \leq \frac{1}{2}\Phi(z) + \frac{1}{2}\Phi(Tz) - \frac{1}{4}d^2(z, Tz).$$

That is,

$$d^2(z, Tz) \leq 2\Phi(Tz) - 2\Phi(z). \tag{6.2.6}$$

From (6.2.5) and (6.2.6), we obtain

$$d^2(z, Tz) \leq \frac{2g(z)}{1-f(z)}\Phi(z) + \frac{2k}{1-f(z)}d^2(z, Tz) - 2\Phi(z),$$

which implies

$$\frac{1-f(z)-2k}{1-f(z)}d^2(z,Tz) \leq \frac{2(g(z)+f(z)-1)}{1-f(z)}\Phi(z). \tag{6.2.7}$$

Since $g(z)+f(z) \leq 1$, we obtain from (6.2.7) that

$$(1-f(z)-2k)\ d^2(z,Tz) \leq 0.$$

Since $k < \frac{1-f(z)}{2}$, it follows that $z \in F(T)$. Hence, our proof is complete. $\square$

6.3 Convergence theorems for the class of generalized strictly pseudononspreading mappings in Hadamard spaces

Here, we state and prove some strong convergence theorems of the Mann-type and Ishikawa-type algorithms for approximating fixed points of generalized strictly pseudononspreading mappings in Hadamard spaces.

6.3.1 Main results

The Mann-type strong convergence theorem.

Theorem 6.3.1. *Let C be a nonempty closed and convex subset of an Hadamard space X and T be (f,g)-generalized k-strictly pseudononspreading mapping on C with constant $k \in [0,1)$, where $f,g : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x)+g(x) \leq 1$ for all $x \in C$. Suppose $F(T) \neq \emptyset$ and $k < \min\{f(x), \frac{1-f(x)}{2}\}$ for each $x \in C$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = (1-t_n)x_n \oplus t_n u, \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T y_n, \ \ n \geq 1, \end{cases} \tag{6.3.1}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions

C1: $\lim\limits_{n\to\infty} t_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq 1 - \frac{k}{f(p)}$ *for each* $p \in F(T)$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Proof. Let $p \in F(T)$, then from (6.2.2), (6.3.1) and Lemma 2.3.1, we obtain

$$\begin{aligned}
d^2(p, x_{n+1}) &\leq (1-\alpha_n)d^2(p, y_n) + \alpha_n d^2(p, Ty_n) - \alpha_n(1-\alpha_n)d^2(y_n, Ty_n) \\
&\leq (1-\alpha_n)d^2(p, y_n) + \alpha_n \left[d^2(p, y_n) + \frac{k}{f(p)}d^2(y_n, Ty_n)\right] \\
&\quad -\alpha_n(1-\alpha_n)d^2(y_n, Ty_n) \\
&= d^2(p, y_n) - \alpha_n \left[\left(1 - \frac{k}{f(p)}\right) - \alpha_n\right] d^2(y_n, Ty_n) \qquad (6.3.2) \\
&\leq d^2(p, (1-t_n)x_n \oplus t_n u) \qquad (6.3.3) \\
&\leq (1-t_n)d^2(p, x_n) + t_n d^2(p, u) \\
&\leq \max\{d^2(p, x_n), d^2(p, u)\} \\
&\ \vdots \\
&\leq \max\{d^2(p, x_1), d^2(p, u)\}.
\end{aligned}$$

Therefore, $\{d^2(p, x_n)\}$ is bounded. Consequently, $\{x_n\}$ and $\{y_n\}$ are bounded.

Again, from (6.3.1), we obtain

$$\lim_{n\to\infty} d(x_n, y_n) \leq \lim_{n\to\infty} t_n d(x_n, u) = 0. \qquad (6.3.4)$$

We divide our proof into two cases.

Case 1: Suppose that $\{d^2(x_n, p)\}$ is monotonically non-increasing, then $\lim_{n\to\infty}\{d^2(p, x_n)\}$ exists. Consequently,

$$\lim_{n\to\infty} \left[d^2(p, x_n) - d^2(p, x_{n+1})\right] = 0. \qquad (6.3.5)$$

Thus, from (6.3.2), we have

$$\begin{aligned}
\alpha_n \left[\left(1 - \frac{k}{f(p)}\right) - \alpha_n\right] d^2(y_n, Ty_n) &\leq d^2(p, y_n) - d^2(p, x_{n+1}) \\
&\leq (1-t_n)d^2(p, x_n) + t_n d^2(p, u) - d^2(p, x_{n+1}) \\
&= d^2(p, x_n) - d^2(p, x_{n+1}) \\
&\quad + t_n \left[d^2(p, u) - d^2(p, x_n)\right] \to 0, \text{ as } n \to \infty.
\end{aligned}$$

By condition C3, we obtain that

$$\lim_{n\to\infty} d^2(y_n, Ty_n) = 0. \qquad (6.3.6)$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z$. It follows from (6.3.4) that there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} y_{n_k} = z$. Thus, from (6.3.6) and Lemma 6.2.8, we obtain that $z \in F(T)$.

Furthermore, for arbitrary $u \in X$, we have from Lemma 2.3.10 that

$$\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \leq 0, \qquad (6.3.7)$$

which implies from condition C1 that

$$\limsup_{n\to\infty}\left(t_n d^2(z,u) + 2(1-t_n)\langle\overrightarrow{zu},\overrightarrow{zx_n}\rangle\right) \leq 0. \tag{6.3.8}$$

We now show that $\{x_n\}$ converges strongly to z. From (6.3.3) and Lemma 2.3.1, we obtain

$$\begin{aligned} d^2(z,x_{n+1}) &\leq d^2(z,(1-t_n)x_n \oplus t_n u) \\ &\leq (1-t_n)^2 d^2(z,x_n) + t_n^2 d^2(z,u) + 2t_n(1-t_n)\langle\overrightarrow{zx_n},\overrightarrow{zu}\rangle \\ &\leq (1-t_n)d^2(z,x_n) + t_n\left(t_n d^2(z,u) + 2(1-t_n)\langle\overrightarrow{zx_n},\overrightarrow{zu}\rangle\right). \end{aligned} \tag{6.3.9}$$

Hence, from (6.3.8) and Lemma 2.3.26, we conclude that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d^2(x_n,p)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{p,d^2(x_{n_i})\}$ of $\{p,d^2(x_n)\}$ such that $d^2(p,x_{n_i}) < d^2(p,x_{n_i+1})$ for all $i\in\mathbb{N}$. Thus, by Lemma 2.3.29, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ such that $m_k\to\infty$, and

$$d^2(p,x_{m_k}) \leq d^2(p,x_{m_k+1}) \text{ and } d^2(p,x_k) \leq d^2(p,x_{m_k+1})\ \forall k\in\mathbb{N}. \tag{6.3.10}$$

Thus, from (6.3.3) and (6.3.10), we obtain

$$\begin{aligned} 0 &\leq \lim_{k\to\infty}\left(d^2(p,x_{m_k+1}) - d^2(p,x_{m_k})\right) \\ &\leq \limsup_{n\to\infty}\left(d^2(p,x_{n+1}) - d^2(p,x_n)\right) \\ &\leq \limsup_{n\to\infty}\left((1-t_n)d^2(p,x_n) + t_n d^2(p,u) - d^2(p,x_n)\right) \\ &\leq \limsup_{n\to\infty}\left[t_n\left(d^2(p,u) - d^2(p,x_n)\right)\right] = 0, \end{aligned}$$

which implies

$$\lim_{k\to\infty}\left(d^2(p,x_{m_k+1}) - d^2(p,x_{m_k})\right) = 0. \tag{6.3.11}$$

Following the same line of argument as in **Case 1**, we can show that

$$\lim_{k\to\infty}\left(t_{m_k} d^2(z,u) + 2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right) \leq 0. \tag{6.3.12}$$

Also, from (6.3.9) we have

$$d^2(z,x_{m_k+1}) \leq (1-t_{m_k})d^2(z,x_{m_k}) + t_{m_k}\left(t_{m_k} d^2(z,u) + 2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right).$$

Since $d^2(z,x_{m_k}) \leq d^2(z,x_{m_k+1})$, we obtain

$$d^2(z,x_{m_k}) \leq \left(t_{m_k} d^2(z,u) + 2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right).$$

Thus, from (6.3.12) we get

$$\lim_{k\to\infty} d^2(z,x_{m_k}) = 0. \tag{6.3.13}$$

It then follows from (6.3.10), (6.3.11) and (6.3.13) that $\lim_{k\to\infty} d^2(z,x_k) = 0$. Therefore, we conclude from **Case 1** and **Case 2** that $\{x_n\}$ converges to $z\in F(T)$. □

In view of Remark 6.2.2, we obtain the following corollaries.

Corollary 6.3.2. *Let C be a nonempty closed and convex subset of an Hadamard space X and T be a generalized nonspreading mapping on C with $F(T) \neq \emptyset$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = (1-t_n)x_n \oplus t_n u, \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T y_n, \ \ n \geq 1, \end{cases} \tag{6.3.14}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions

C1: $\lim\limits_{n\to\infty} t_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq b < 1$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Corollary 6.3.3. *Let C be a nonempty closed and convex subset of an Hadamard space X and T be a k-strictly pseudononspreading mapping on C with $F(T) \neq \emptyset$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = (1-t_n)x_n \oplus t_n u, \\ x_{n+1} = (1-\alpha_n)y_n \oplus \alpha_n T y_n, \ \ n \geq 1, \end{cases} \tag{6.3.15}$$

where $\{t_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions

C1: $\lim\limits_{n\to\infty} t_n = 0$,

C2: $\sum\limits_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq 1-k$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

The Ishikawa-type strong convergence theorem.

Theorem 6.3.4. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let T be an L-Lipschitzian and (f,g)-generalized k-strictly pseudononspreading mapping on C with constant $k \in (0,1)$, where $f, g : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x) + g(x) \leq 1$ for all $x \in C$. Suppose $F(T) \neq \emptyset$ and $k < \min\{f(x), \frac{1-f(x)}{2}\}$ for each $x \in C$. Let $u, x_1 \in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} u_n = (1-t_n)x_n \oplus t_n u, \\ y_n = (1-\beta_n)u_n \oplus \beta_n T u_n, \\ x_{n+1} = (1-\alpha_n)u_n \oplus \alpha_n T y_n, \ \ n \geq 1, \end{cases} \tag{6.3.16}$$

where $\{t_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \le \alpha_n \le \frac{k}{f(p)}\beta_n < \beta_n \le b < \frac{2}{(1+\frac{f(p)}{k})+\sqrt{(1+\frac{f(p)}{k})^2+4L^2}}$, *for each* $p \in F(T)$.

Then $\{x_n\}$ *converges strongly to an element of* $F(T)$.

Proof. Let $p \in F(T)$, since T is L-Lipschitzian and generalized k-strictly pseudononspreading, we obtain from (6.2.2), (6.3.16) and Lemma 2.3.1 that

$$\begin{aligned}
d^2(p,Ty_n) &\le d^2(p,y_n)+\frac{k}{f(p)}d^2(y_n,Ty_n)\\
&= d^2(p,(1-\beta_n)u_n\oplus\beta_nTu_n)+\frac{k}{f(p)}d^2((1-\beta_n)u_n\oplus\beta_nTu_n,Ty_n)\\
&\le (1-\beta_n)d^2(p,u_n)+\beta_nd^2(p,Tu_n)-\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&\quad +\frac{k}{f(p)}\left[(1-\beta_n)d^2(u_n,Ty_n)+\beta_nd^2(Tu_n,Ty_n)-\beta_n(1-\beta_n)d^2(u_n,Tu_n)\right]\\
&\le (1-\beta_n)d^2(p,u_n)+\beta_nd^2(p,Tu_n)-\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&\quad +\frac{k}{f(p)}(1-\beta_n)d^2(u_n,Ty_n)+\frac{k}{f(p)}L^2\beta_n^3d^2(u_n,Tu_n)\\
&\quad -\frac{k}{f(p)}\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&\le (1-\beta_n)d^2(p,u_n)+\beta_n\left[d^2(p,u_n)+\frac{k}{f(p)}d^2(u_n,Tu_n)\right]\\
&\quad -\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&\quad +\frac{k}{f(p)}(1-\beta_n)d^2(u_n,Ty_n)+\frac{k}{f(p)}L^2\beta_n^3d^2(u_n,Tu_n)\\
&\quad -\frac{k}{f(p)}\beta_n(1-\beta_n)d^2(u_n,Tu_n)\\
&= d^2(p,u_n)+\frac{k}{f(p)}(1-\beta_n)d^2(u_n,Ty_n)\\
&\quad -\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n).
\end{aligned} \tag{6.3.17}$$

Also, from (6.3.16), (6.3.17) and condition C3, we obtain

$$\begin{aligned}
d^2(p,x_{n+1}) &\le (1-\alpha_n)d^2(p,u_n)+\alpha_nd^2(p,Ty_n)-\alpha_n(1-\alpha_n)d^2(u_n,Ty_n)\\
&\le (1-\alpha_n)d^2(p,u_n)+\alpha_nd^2(p,u_n)+\frac{k}{f(p)}\alpha_n(1-\beta_n)d^2(u_n,Ty_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n)\\
&\quad -\alpha_n(1-\alpha_n)d^2(u_n,Ty_n)
\end{aligned}$$

$$\begin{aligned}
&\leq d^2(p,u_n)-\alpha_n\left[(1-\frac{k}{f(p)})+(\frac{k}{f(p)}\beta_n-\alpha_n)\right]d^2(u_n,Ty_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n)\\
&\leq d^2(p,u_n)\\
&\quad -\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]d^2(u_n,Tu_n) \qquad (6.3.18)\\
&\leq d^2(p,(1-t_n)x_n\oplus t_nu) \qquad (6.3.19)\\
&\leq (1-t_n)d^2(p,x_n)+t_nd^2(p,u)\\
&\leq \max\{d^2(p,x_n),d^2(p,u)\}\\
&\vdots\\
&\leq \max\{d^2(p,x_1),d^2(p,u)\}.
\end{aligned}$$

Therefore, $\{d^2(p,x_n)\}$ is bounded. Consequently, $\{x_n\}$, $\{u_n\}$ and $\{y_n\}$ are all bounded. From (6.3.16) and condition C1, we obtain that

$$\lim_{n\to\infty} d(u_n,x_n)\leq \lim_{n\to\infty} t_nd(u,x_n)=0. \qquad (6.3.20)$$

We now consider two cases for our proof.

Case 1: Suppose that $\{d^2(p,x_n)\}$ is monotonically non-increasing, then $\lim_{n\to\infty} d^2(p,x_n)$ exists. Hence,

$$\lim_{n\to\infty}\left[d^2(p,x_{n+1})-d^2(p,x_n)\right]=0. \qquad (6.3.21)$$

Let $P_n=\alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)})-\frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]$, then we obtain from (6.3.18) that

$$\begin{aligned}
P_nd^2(u_n,Tu_n) &\leq d^2(p,u_n)-d^2(p,x_{n+1})\\
&\leq (1-t_n)d^2(p,x_n)+t_nd^2(p,u)-d^2(p,x_{n+1})\\
&= d^2(p,x_n)-d^2(p,x_{n+1})\\
&+ t_n\left[d^2(p,u)-d^2(p,x_n)\right]\to 0,\ \text{as } n\to\infty. \qquad (6.3.22)
\end{aligned}$$

From condition C3, we obtain that $2-b\left(1+\frac{f(p)}{k}\right)>b\sqrt{\left(1+\frac{f(p)}{k}\right)^2+4L^2}$. Which implies that

$$2\frac{k}{f(p)}-b\left(1+\frac{k}{f(p)}\right)>b\frac{k}{f(p)}\sqrt{\left(1+\frac{f(p)}{k}\right)^2+4L^2}.$$

That is,

$$\left[2\frac{k}{f(p)}-b\left(1+\frac{k}{f(p)}\right)\right]^2>4\left(\frac{k}{f(p)}\right)^2b^2L^2+b^2\left(1+\frac{k}{f(p)}\right)^2,$$

which after simplification yields

$$\frac{k}{f(p)}-b\frac{k}{f(p)}-\frac{k}{f(p)}b^2L^2-b>0.$$

Thus,

$$\begin{aligned}
P_n &= \alpha_n\beta_n\left[(1-\beta_n)(1+\frac{k}{f(p)}) - \frac{k}{f(p)}\left(1+L^2\beta_n^2\right)\right]\\
&> a^2\left[(1+\frac{k}{f(p)}) - \frac{k}{f(p)} - \beta_n(1+\frac{k}{f(p)}) - \frac{k}{f(p)}L^2\beta_n^2\right]\\
&> a^2\left[(1+\frac{k}{f(p)}) - 1 - b(1+\frac{k}{f(p)}) - \frac{k}{f(p)}L^2b^2\right] > 0.
\end{aligned}$$

Hence, we obtain from (6.3.22) that

$$\lim_{n\to\infty} d(u_n, Tu_n) = 0. \tag{6.3.23}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then from Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim_{k\to\infty} x_{n_k} = z$. It then follows from (6.3.20), (6.3.23) and Lemma 6.2.8, that $z \in F(T)$.

Furthermore, for arbitrary $u \in X$, we have from Lemma 2.3.10 that

$$\limsup_{n\to\infty}\langle\overrightarrow{zu}, \overrightarrow{zx_n}\rangle \le 0, \tag{6.3.24}$$

which implies from condition C1 that

$$\limsup_{n\to\infty}\left(t_nd^2(z,u) + 2(1-t_n)\langle\overrightarrow{zu}, \overrightarrow{zx_n}\rangle\right) \le 0. \tag{6.3.25}$$

Next, we show that $\{x_n\}$ converges strongly to z. From (6.3.19) and Lemma 2.3.1, we obtain

$$\begin{aligned}
d^2(z, x_{n+1}) &\le d^2(z, (1-t_n)x_n \oplus t_nu)\\
&\le (1-t_n)^2d^2(z,x_n) + t_n^2d^2(z,u) + 2t_n(1-t_n)\langle\overrightarrow{zx_n}, \overrightarrow{zu}\rangle\\
&\le (1-t_n)d^2(z,x_n) + t_n\left(t_nd^2(z,u) + 2(1-t_n)\langle\overrightarrow{zx_n}, \overrightarrow{zu}\rangle\right). \qquad (6.3.26)
\end{aligned}$$

Hence, from (6.3.25) and Lemma 2.3.26, we obtain that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d^2(x_n,p)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{p, d^2(x_{n_i})\}$ of $\{p, d^2(x_n)\}$ such that $d^2(p, x_{n_i}) < d^2(p, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.3.29, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$d^2(p, x_{m_k}) \le d^2(p, x_{m_k+1}) \text{ and } d^2(p, x_k) \le d^2(p, x_{m_k+1})\ \forall k \in \mathbb{N}. \tag{6.3.27}$$

Thus, from (6.3.19) and (6.3.27), we obtain

$$\begin{aligned}
0 &\le \lim_{k\to\infty}\left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k})\right)\\
&\le \limsup_{n\to\infty}\left(d^2(p, x_{n+1}) - d^2(p, x_n)\right)\\
&\le \limsup_{n\to\infty}\left((1-t_n)d^2(p,x_n) + t_nd^2(p,u) - d^2(p,x_n)\right)\\
&\le \limsup_{n\to\infty}\left[t_n\left(d^2(p,u) - d^2(p,x_n)\right)\right] = 0,
\end{aligned}$$

which implies

$$\lim_{k\to\infty}\left(d^2(p,x_{m_k+1})-d^2(p,x_{m_k})\right)=0. \tag{6.3.28}$$

Following the same line of argument as in **Case 1**, we can show that

$$\lim_{k\to\infty}\left(t_{m_k}d^2(z,u)+2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right)\leq 0. \tag{6.3.29}$$

Also, from (6.3.26) we have

$$d^2(z,x_{m_k+1})\leq(1-t_{m_k})d^2(z,x_{m_k})+t_{m_k}\left(t_{m_k}d^2(z,u)+2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right).$$

Since $d^2(z,x_{m_k})\leq d^2(z,x_{m_k+1})$, we obtain

$$d^2(z,x_{m_k})\leq\left(t_{m_k}d^2(z,u)+2(1-t_{m_k})\langle\overrightarrow{zu},\overrightarrow{zx_{m_k}}\rangle\right).$$

Thus, from (6.3.29) we get

$$\lim_{k\to\infty}d^2(z,x_{m_k})=0. \tag{6.3.30}$$

It then follows from (6.3.27), (6.3.28) and (6.3.30) that $\lim\limits_{k\to\infty}d^2(z,x_k)=0$. Therefore, we conclude from **Case 1** and **Case 2** that $\{x_n\}$ converges to $z\in F(T)$. □

Also, by Remark 6.2.2, we obtain the following corollaries.

Corollary 6.3.5. *Let C be a nonempty closed and convex subset of an Hadamard space X and T be a generalized nonspreading mapping on C with $F(T)\neq\emptyset$. Let $u,x_1\in C$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases}u_n=(1-t_n)x_n\oplus t_nu,\\ y_n=(1-\beta_n)u_n\oplus\beta_nTu_n,\\ x_{n+1}=(1-\alpha_n)u_n\oplus\alpha_nTy_n,\ \ n\geq 1,\end{cases} \tag{6.3.31}$$

where $\{t_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions

C1: $\lim\limits_{n\to\infty}t_n=0$,

C2: $\sum\limits_{n=1}^{\infty}t_n=\infty$,

C3: $0<a\leq\alpha_n\leq b<1$ *and* $0<a\leq\beta_n\leq b<1$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Corollary 6.3.6. *Let C be a nonempty closed and convex subset of an Hadamard space X. Let T be an L-Lipschitzian and k-strictly pseudononspreading mapping on C with constant $k\in(0,1)$. Suppose $F(T)\neq\emptyset$ and for arbitrary $u,x_1\in C$, the sequence $\{x_n\}$ be generated by*

$$\begin{cases}u_n=(1-t_n)x_n\oplus t_nu,\\ y_n=(1-\beta_n)u_n\oplus\beta_nTu_n,\\ x_{n+1}=(1-\alpha_n)u_n\oplus\alpha_nTy_n,\ \ n\geq 1,\end{cases} \tag{6.3.32}$$

where $\{t_n\}$, $\{\beta_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$, satisfying the following conditions

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n \leq k\beta_n < \beta_n \leq b < \frac{2}{\left(\frac{k+1}{k}\right)+\sqrt{\left(\frac{k+1}{k}\right)^2+4L^2}}$.

Then $\{x_n\}$ converges strongly to an element of $F(T)$.

Remark 6.3.7. *We now put forward the following important questions concerning the class of generalized strictly pseudononspreading mappings introduced and studied in this Section.*

1. *Is it possible to give an example of this class of mappings in a setting more general than $\mathbb{R}$?*

2. *Is it possible to extend the mapping from singlevalued to multivalued?*

3. *We do not know if our class of mappings generalizes the class of generalized asymptotically nonspreading mappings introduced in [149]. Thus, is it possible to introduce a class of mappings which is more general than our class of mappings and at the same time, more general than the class of asymptotically nonspreading mappings? In other words, is it possible to introduce the class of generalized asymptotically strictly pseudononspreading mappings?*

6.4 S-iteration for minimization and fixed point problems for two families of generalized strictly pseudononspreading mappings in Hadamard spaces

Here, using a modified S-type iteration process (1.1.3), we introduce a modified PPA for approximating a common solution of a finite family of MPs and fixed point problems for two finite families of generalized k-strictly pseudononspreading mappings. Numerical example in support of our main result is given to illustrate its applicability.

6.4.1 Main results

Lemma 6.4.1. *Let C be a closed and convex subset of an Hadamard space X and $T : C \to C$ be (f,g)-generalized k-strictly pseudononspreading mapping with $k \in [0,1)$ such that $F(T) \neq \emptyset$, where $f, g : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f(x) + g(x) \leq 1$ for all $x \in C$. Let $T_\beta : C \to C$ be defined by $T_\beta x = \beta x \oplus (1-\beta)Tx\ \forall x \in C$, where $\frac{k}{f(p)} \leq \beta < 1$ with $f(p) \neq 0$ for each $p \in F(T)$. Then,*

(a) $F(T_\beta) = F(T)$,

(b) T_β is quasinonexpansive.

Proof. (a) If $\beta = 0$, then $T_\beta = T$. Thus, $F(T) = F(T_\beta)$. Now, let $\beta \neq 0$. For each $p \in F(T_\beta)$, we have that $p = T_\beta p$ and by Lemma 2.3.1 (i), we have

$d(p, Tp) \leq \beta d(p, Tp)$, which implies $(1-\beta)d(p, Tp) \leq 0$.

Since $\beta < 1$, it follows that $p \in F(T)$. Thus, $F(T_\beta) \subseteq F(T)$.

We now show that $F(T) \subseteq F(T_\beta)$. Let $p \in F(T)$, then $Tp = p$ and by Lemma 2.3.1 (i) we have

$d(p, T_\beta p) = d(p, \beta p \oplus (1-\beta)p) \leq 0$, which implies that $p \in F(T_\beta)$. Thus,

$F(T) \subseteq F(T_\beta)$. Therefore, $F(T_\beta) = F(T)$.

(b) First, observe that if T is (f,g)-generalized k-strictly pseudononspreading mapping, then for each $p \in F(T) = F(T_\beta)$ and $x \in C$, we obtain from Lemma 2.3.1 (ii) and (6.2.2) that

$$\begin{aligned}
d^2(T_\beta p, T_\beta x) &= d^2(p, \beta x \oplus (1-\beta)Tx) \\
&\leq \beta d^2(p,x) + (1-\beta)d^2(p,Tx) - \beta(1-\beta)d^2(x,Tx) \\
&\leq \beta d^2(p,x) + (1-\beta)\left[d^2(p,x) + \frac{k}{f(p)}d^2(x,Tx)\right] \\
&\quad -\beta(1-\beta)d^2(x,Tx) \\
&= d^2(p,x) + (1-\beta)\left(\frac{k}{f(p)} - \beta\right)d^2(x,Tx) \\
&\leq d^2(p,x).
\end{aligned}$$

Therefore, T_β is quasinonexpansive. □

Theorem 6.4.2. *Let C be a closed and convex subset of an Hadamard space X and $h_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semi continuous mappings. For each $j = 1, 2, \ldots, m$, let $T_j : C \to C$ be a finite family of (f_j, g_j)-generalized k_j-strictly pseudononspreading mapping with $k_j \in [0,1)$, where $f_j, g_j : C \to [0,\gamma]$, $\gamma < 1$ and $0 < f_j(x) + g_j(x) \leq 1$ for all $x \in C$, and $S_j : C \to X$ be a finite family of (f'_j, g'_j)-generalized k'_j-strictly pseudononspreading mapping with $k'_j \in [0,1)$, where $f'_j, g'_j : C \to [0,\gamma']$, $\gamma' < 1$ and $0 < f'_j(x) + g'_j(x) \leq 1$ for all $x \in C$. Suppose that $\Gamma := \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} h_i(y)\right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases}
z_n = (1-t_n)x_n \oplus t_n u, \\
u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\
y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_{(\beta,1)} u_n \oplus \beta_n^{(2)} T_{(\beta,2)} u_n \oplus \cdots \oplus \beta_n^{(m)} T_{(\beta,m)} u_n, \\
x_{n+1} = \alpha_n^{(0)} T_{(\beta,m)} u_n \oplus \alpha_n^{(1)} S_{(\alpha,1)} u_n \oplus \alpha_n^{(2)} S_{(\alpha,2)} u_n \oplus \cdots \oplus \alpha_n^{(m)} S_{(\alpha,m)} y_n, \ n \geq 1,
\end{cases} \tag{6.4.1}$$

where $T_{(\beta,j)}x = \beta x \oplus (1-\beta)T_j x$ and $S_{(\alpha,j)}x = \alpha x \oplus (1-\alpha)S_j x$, $j = 1, 2, \ldots, m$, for all $x \in C$ such that $T_{(\beta,j)}$ and $S_{(\beta,j)}$ are Δ-demiclosed with $\frac{k_j}{f_j(p)} \leq \beta < 1$, $f_j(p) \neq 0$

and $\frac{k'_j}{f'_j(p)} \leq \alpha < 1$, $f'_j(p) \neq 0$ *respectively, for each* $j = 1, 2, \dots, m$ *and for each* $p \in \big(\cap_{j=1}^m F(T_j)\big) \cap \big(\cap_{j=1}^m F(S_j)\big)$, $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(j)}\}$ *and* $\{\alpha_n^{(j)}\}$ *are sequences in* $(0, 1)$ *satisfying the following conditions:*

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)},\ \beta_n^{(j)} \leq b < 1,\ j = 0, 1, 2, \dots, m$ *such that* $\sum_{j=0}^{m} \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^{m} \beta_n^{(j)} = 1$ *for all* $n \geq 1$,

C4: $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1$, $i = 1, 2, \dots, N$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

Proof. Let $p \in \Gamma$. Then for each $j = 1, 2, \dots, m$, we have by Lemma 6.4.1 that $p = T_{(\beta,j)}p = S_{(\alpha,j)}p$, and $T_{(\beta,j)}$ and $S_{(\alpha,j)}$ are quasi-nonexpansive mappings. Set $\Phi_{\lambda_n}^N = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}$, where $\Phi_{\lambda_n}^0 = I$. Now by (6.4.1) and Lemma 2.3.3, we have

$$\begin{aligned}
d(p, x_{n+1}) &\leq \alpha_n^{(0)} d(p, T_{(\beta,m)}u_n) + \alpha_n^{(1)} d(p, S_{(\alpha,1)}u_n) + \alpha_n^{(2)} d(p, S_{(\alpha,2)}u_n) \\
&\quad + \cdots + \alpha_n^{(m)} d(p, S_{(\alpha,m)}y_n) \\
&\leq \alpha_n^{(0)} d(p, u_n) + \alpha_n^{(1)} d(p, u_n) + \alpha_n^{(2)} d(p, u_n) + \cdots + \alpha_n^{(m)} d(p, y_n) \\
&\leq \sum_{j=0}^{m-1} \alpha_n^{(j)} d(p, u_n) + \alpha_n^{(m)} [\beta_n^{(0)} d(p, u_n) + \beta_n^{(1)} d(p, T_{(\beta,1)}u_n) \\
&\quad + \beta_n^{(2)} d(p, T_{(\beta,2)}u_n) + \cdots + \beta_n^{(m)} d(p, T_{(\beta,m)}u_n)] \\
&\leq \sum_{j=0}^{m-1} \alpha_n^{(j)} d(p, u_n) + \alpha_n^{(m)} d(p, u_n) \\
&= d(p, u_n) \qquad (6.4.2) \\
&\leq d(p, \Phi_{\lambda_n}^N z_n) \\
&\leq d(p, z_n) \qquad (6.4.3) \\
&\leq (1 - t_n) d(p, x_n) + t_n d(u, p) \\
&\leq \max\{d(p, x_n), d(p, u)\} \\
&\leq \max\{d(p, x_1), d(p, u)\}.
\end{aligned}$$

Therefore, $\{d(p, x_n)\}$ is bounded. Hence, $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{y_n\}$ are all bounded. From (6.4.1), Lemma 2.3.1 (i) and condition C1, we obtain that

$$d(z_n, x_n) \leq t_n d(u, x_n) \to 0, \text{ as } n \to \infty. \qquad (6.4.4)$$

We need to consider two cases for our proof.

Case 1: Suppose that $\{d(p,x_n)\}$ is monotonically non-increasing. Then $\lim_{n\to\infty} d(p,x_n)$ exists. Without loss of generality, we may assume that

$$\lim_{n\to\infty} d(p,x_n) = c \geq 0. \tag{6.4.5}$$

Since, P_C is firmly nonexpansive, therefore, we have

$$d^2(p,u_n) \leq \langle \overleftarrow{u_n p}, \overleftarrow{\Phi^N_{\lambda_n} z_n\ p} \rangle = \frac{1}{2}\left(d^2(p,u_n) + d^2(p,\Phi^N_{\lambda_n} z_n) - d^2(u_n,\Phi^N_{\lambda_n} z_n)\right),$$

which together with (6.4.2), (6.4.3), (6.4.4) and (6.4.5), implies that

$$\begin{aligned} d^2(u_n,\Phi^N_{\lambda_n} z_n) &\leq d^2(p,\Phi^N_{\lambda_n} z_n) - d^2(p,u_n)\\ &\leq d^2(p,\Phi^N_{\lambda_n} z_n) - d^2(p,x_{n+1})\\ &\vdots\\ &\leq d^2(p,z_n) - d^2(p,x_{n+1})\\ &\leq d^2(p,x_n) + 2d(p,x_n)d(x_n,z_n) + d^2(x_n,z_n)\\ &\quad -d^2(p,x_{n+1}) \to 0 \text{ as } n\to\infty. \end{aligned} \tag{6.4.6}$$

We now show that $\lim_{n\to\infty} d(u_n, J_{\lambda^{(i)}} u_n) = 0,\ i = 1,2,\ldots,N.$
Indeed, it follows from Lemma 4.2.2 that

$$\frac{1}{2\lambda_n} d^2(p,\Phi^N_{\lambda_n} z_n) - \frac{1}{2\lambda_n} d^2(p,\Phi^{N-1}_{\lambda_n} z_n) + \frac{1}{2\lambda_n} d^2(\Phi^N_{\lambda_n} z_n,\Phi^{N-1}_{\lambda_n} z_n) + f\left(\Phi^N_{\lambda_n}\right) \leq f(p).$$

Since $f(p) \leq f\left(\Phi^N_{\lambda_n}\right)$, we have by (6.4.2), (6.4.5) and (6.4.4) that

$$\begin{aligned} d^2(\Phi^N_{\lambda_n} z_n,\Phi^{N-1}_{\lambda_n} z_n) &\leq d^2(p,\Phi^{N-1}_{\lambda_n} z_n) - d^2(p,\Phi^N_{\lambda_n} z_n)\\ &\leq d^2(p,\Phi^{N-1}_{\lambda_n} z_n) - d^2(p,x_{n+1})\\ &\vdots\\ &\leq d^2(p,z_n) - d^2(p,x_{n+1})\\ &\leq d^2(z_n,x_n) + 2d(z_n,x_n)d(p,x_n)\\ &\quad + \left[d^2(p,x_n) - d^2(p,x_{n+1})\right] \to 0, \text{ as } n\to\infty. \end{aligned} \tag{6.4.7}$$

Similarly, we obtain by Lemma 4.2.2, (6.4.2), (6.4.5) and (6.4.4) that

$$\begin{aligned} d^2(\Phi^{N-1}_{\lambda_n} z_n,\Phi^{N-2}_{\lambda_n} z_n) &\leq d^2(p,\Phi^{N-2}_{\lambda_n} z_n) - d^2(p,\Phi^{N-1}_{\lambda_n} z_n)\\ &\leq d^2(p,\Phi^{N-2}_{\lambda_n} z_n) - d^2(p,\Phi^N_{\lambda_n} z_n)\\ &\leq d^2(p,\Phi^{N-2}_{\lambda_n} z_n) - d^2(p,x_{n+1})\\ &\vdots\\ &\leq d^2(p,z_n) - d^2(p,x_{n+1}) \to 0, \text{ as } n\to\infty. \end{aligned} \tag{6.4.8}$$

Continuing in this manner, we can show that

$$\lim_{n\to\infty} d^2(\Phi^{N-2}_{\lambda_n} z_n,\Phi^{N-3}_{\lambda_n} z_n) = \cdots = \lim_{n\to\infty} d^2(\Phi^2_{\lambda_n} z_n,\Phi^1_{\lambda_n} z_n) = \lim_{n\to\infty} d^2(\Phi^1_{\lambda_n} z_n, z_n) = 0. \tag{6.4.9}$$

Thus,

$$d(u_n, z_n) \leq d(u_n, \Phi^N_{\lambda_n} z_n) + d(\Phi^N_{\lambda_n} z_n, \Phi^{N-1}_{\lambda_n} z_n) + d(\Phi^{N-1}_{\lambda_n} z_n, \Phi^{N-2}_{\lambda_n} z_n) + \cdots + d(\Phi^1_{\lambda_n} z_n, z_n),$$

which implies by (6.4.6), (6.4.7), (6.4.8) and (6.4.9) that

$$\lim_{n\to\infty} d(u_n, z_n) = 0. \tag{6.4.10}$$

It follows from (6.4.10) and Lemma 4.2.6 that

$$\begin{aligned} d(J_{\lambda^{(i)}} u_n, u_n) &\leq d(J_{\lambda^{(i)}} u_n, J_{\lambda^{(i)}} z_n) + d(J_{\lambda^{(i)}} z_n, z_n) + d(z_n, u_n) \\ &\leq 2d(u_n, z_n) + d(J_{\lambda^{(i)}} z_n, z_n) \to 0, \text{ as } n \to \infty, \ i = 1, 2, \ldots, N. \end{aligned}$$

That is,

$$\lim_{n\to\infty} d(u_n, J_{\lambda^{(i)}} u_n) = 0, \text{ for each } i = 1, 2, \ldots, N. \tag{6.4.11}$$

Next, we show that $\lim_{n\to\infty} d(u_n, x_n) = 0$ and $\lim_{n\to\infty} d(p, y_n) = c$. By (6.4.4) and (6.4.10), we obtain

$$\lim_{n\to\infty} d(u_n, x_n) = 0. \tag{6.4.12}$$

Again, by (6.4.1), we have

$$\begin{aligned} d(p, x_{n+1}) &\leq \alpha^{(0)}_n d(p, T_{(\beta,m)} u_n) + \alpha^{(1)}_n d(p, S_{(\alpha,1)} u_n) + \alpha^{(2)}_n d(p, S_{(\alpha,2)} u_n) \\ &\quad + \cdots + \alpha^{(m)}_n d(p, S_{(\alpha,m)} y_n) \\ &\leq \alpha^{(0)}_n d(p, u_n) + \alpha^{(1)}_n d(p, u_n) + \alpha^{(2)}_n d(p, u_n) + \cdots + \alpha^{(m)}_n d(p, y_n) \\ &= (1 - \alpha^{(m)}_n) d(p, u_n) + \alpha^{(m)}_n d(p, y_n) \\ &\vdots \\ &\leq (1 - \alpha^{(m)}_n) d(p, z_n) + \alpha^{(m)}_n d(p, y_n) \\ &\leq (1 - \alpha^{(m)}_n) \left[(1 - t_n) d(p, x_n) + t_n d(p, u)\right] + \alpha^{(m)}_n d(p, y_n) \\ &\leq (1 - \alpha^{(m)}_n) d(p, x_n) + t_n (1 - \alpha^{(m)}_n) d(p, u) + \alpha^{(m)}_n d(p, y_n), \end{aligned}$$

which implies

$$d(p, x_n) \leq \frac{1}{\alpha^{(m)}_n} \left[d(p, x_n) - d(p, x_{n+1}) + (1 - \alpha^{(m)}_n) t_n d(u, p)\right] + d(p, y_n).$$

It then follows from (6.4.5) and conditions C1 and C3 that

$$c = \liminf_{n\to\infty} d(p, x_n) \leq \liminf_{n\to\infty} d(p, y_n). \tag{6.4.13}$$

Also, by (6.4.1), we have

$$\begin{aligned} d(p, y_n) &\leq \beta^{(0)}_n d(p, u_n) + \beta^{(1)}_n d(p, T_{(\beta,1)} u_n) + \beta^{(2)}_n d(p, T_{(\beta,2)} u_n) \\ &\quad + \cdots + \beta^{(m)}_n d(p, T_{(\beta,m)} u_n) \\ &\leq d(p, u_n) \qquad (6.4.14) \\ &\leq d(p, z_n) \\ &\leq d(p, x_n) + t_n \left[d(p, u) - d(p, x_n)\right], \end{aligned}$$

which implies that

$$\limsup_{n\to\infty} d(p,y_n) \le \limsup_{n\to\infty} \left(d(p,x_n) + t_n\left[d(p,u) - d(p,x_n)\right]\right) = c. \tag{6.4.15}$$

Thus, by (6.4.13) and (6.4.15), we have

$$\lim_{n\to\infty} d(p,y_n) = c. \tag{6.4.16}$$

We now show that $\lim_{n\to\infty} d(u_n, T_{(\beta,j)}u_n) = 0$, for each $j = 1, 2, \ldots, m$ and $\lim_{n\to\infty} d(u_n, y_n) = 0$. Indeed, by (6.4.1), Lemma 2.3.3 and Lemma 6.4.1, we have

$$\begin{aligned} d^2(p,y_n) &\le \beta_n^{(0)} d^2(p,u_n) + \sum_{j=1}^{m} \beta_n^{(j)} d^2(p, T_{(\beta,j)}u_n) - \sum_{j=1}^{m} \beta_n^{(0)}\beta_n^{(j)} d^2(u_n, T_{(\beta,j)}u_n) \\ &\quad - \sum_{j,r=1, j\neq r}^{m} \beta_n^{(j)}\beta_n^{(r)} d^2(T_{(\beta,j)}u_n, T_{(\beta,r)}u_n) \\ &\le d^2(p,u_n) - \sum_{j=1}^{m} \beta_n^{(0)}\beta_n^{(j)} d^2(u_n, T_{(\beta,j)}u_n) - \sum_{j,r=1, j\neq r}^{m} \beta_n^{(j)}\beta_n^{(r)} d^2(T_{(\beta,j)}u_n, T_{(\beta,r)}u_n), \end{aligned}$$

which implies

$$\begin{aligned} \sum_{j=1}^{m} \beta_n^{(0)}\beta_n^{(j)} d^2(u_n, T_{(\beta,j)}u_n) &\le d^2(p,u_n) - d^2(p,y_n) \\ &\le d^2(u_n,x_n) + 2d(u_n,x_n)d(p,x_n) + d^2(p,x_n) - d^2(p,y_n). \end{aligned}$$

By (6.4.5), (6.4.12), (6.4.16) and condition C3, we obtain that

$$\lim_{n\to\infty} d(u_n, T_{(\beta,j)}u_n) = 0,\ j = 1, 2, \ldots, m. \tag{6.4.17}$$

Thus, by (6.4.1), (6.4.17) and Lemma 2.3.3, we have

$$\begin{aligned} d(u_n,y_n) &\le \beta_n^{(0)} d(u_n,u_n) + \beta_n^{(1)} d(u_n, T_{(\beta,1)}u_n) + \beta_n^{(2)} d(u_n, T_{(\beta,2)}u_n) \\ &\quad + \cdots + \beta_n^{(m)} d(u_n, T_{(\beta,m)}u_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{6.4.18}$$

Next, we show that $\lim_{n\to\infty} d(u_n, S_{(\alpha,j)}u_n) = 0$, for each $j = 1, 2, \ldots, m-1$, and $\lim_{n\to\infty} d(y_n, S_{(\alpha,m)}y_n) = 0$.

By (6.4.1), (6.4.14), Lemma 2.3.3 and Lemma 6.4.1, we obtain

$$\begin{aligned} d^2(p,x_{n+1}) &\le \alpha_n^{(0)} d^2(p, T_{(\beta,m)}u_n) + \sum_{j=1}^{m-1} \alpha_n^{(j)} d^2(p, S_{(\alpha,j)}u_n) + \alpha_n^{(m)} d^2(p, S_{(\alpha,m)}y_n) \\ &\quad - \sum_{j=1}^{m-1} \alpha_n^{(0)}\alpha_n^{(j)} d^2(T_{(\beta,m)}u_n, S_{(\alpha,j)}u_n) - \alpha_n^{(0)}\alpha_n^{(m)} d^2(T_{(\beta,m)}u_n, S_{(\alpha,m)}y_n) \\ &\quad - \sum_{j=1}^{m-1} \alpha_n^{(m)}\alpha_n^{(j)} d^2(S_{(\alpha,m)}y_n, S_{(\alpha,j)}u_n) - \sum_{j,r=1, j\neq r}^{m-1} \alpha_n^{(j)}\alpha_n^{(r)} d^2(S_{(\alpha,j)}u_n, S_{(\alpha,r)}u_n) \end{aligned}$$

$$
\begin{aligned}
&\leq \; d^2(p,u_n) - \sum_{j=1}^{m-1}\alpha_n^{(0)}\alpha_n^{(j)}d^2(T_{(\beta,m)}u_n, S_{(\alpha,j)}u_n) - \alpha_n^{(0)}\alpha_n^{(m)}d^2(T_{(\beta,m)}u_n, S_{(\alpha,m)}y_n) \\
&\quad - \sum_{j=1}^{m-1}\alpha_n^{(m)}\alpha_n^{(j)}d^2(S_{(\alpha,m)}y_n, S_{(\alpha,j)}u_n) - \sum_{j,r=1,j\neq r}^{m-1}\alpha_n^{(j)}\alpha_n^{(r)}d^2(S_{(\alpha,j)}u_n, S_{(\alpha,r)}u_n),
\end{aligned}
$$

which implies by (6.4.5) and (6.4.12) that

$$
\begin{aligned}
\sum_{j=1}^{m-1}\alpha_n^{(0)}\alpha_n^{(j)}d^2(T_{(\beta,m)}u_n, S_{(\alpha,j)}u_n) + \alpha_n^{(0)}\alpha_n^{(m)}d^2(T_{(\beta,m)}u_n, S_{(\alpha,m)}y_n) \;&\leq\; d^2(p,u_n) - d^2(p,x_{n+1}) \\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
$$

This together with condition C3, implies that

$$
\lim_{n\to\infty} d(T_{(\beta,m)}u_n, S_{(\alpha,j)}u_n) = 0,\ j = 1,2,\ldots,m-1 \tag{6.4.19}
$$

and

$$
\lim_{n\to\infty} d(T_{(\beta,m)}u_n, S_{(\alpha,m)}y_n) = 0. \tag{6.4.20}
$$

By (6.4.17), (6.4.19) and triangle inequality, we obtain

$$
\lim_{n\to\infty} d(u_n, S_{(\alpha,j)}u_n) = 0,\ j = 1,2,\ldots,m-1. \tag{6.4.21}
$$

Furthermore,

$$
d(y_n, S_{(\alpha,m)}y_n) \leq d(y_n,u_n) + d(u_n, T_{(\beta,m)}u_n) + d(T_{(\beta,m)}u_n, S_{(\alpha,m)}y_n),
$$

which implies by (6.4.17), (6.4.18) and (6.4.20) that

$$
\lim_{n\to\infty} d(y_n, S_{(\alpha,m)}y_n) = 0. \tag{6.4.22}
$$

Moreover, as $\{x_n\}$ is bounded and X is a Hadamard space, so by Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim\limits_{k\to\infty} x_{n_k} = z \in C$. It follows from (6.4.12) and (6.4.18) that there exist subsequences $\{u_{n_k}\}$ of $\{u_n\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $\Delta\text{-}\lim\limits_{k\to\infty} u_{n_k} = z = \Delta\text{-}\lim\limits_{k\to\infty} y_{n_k}$. Since $T_{(\beta,j)}$ and $S_{(\alpha,j)}$ are Δ-demiclosed, it follows from (6.4.17), (6.4.21), (6.4.22) and Lemma 6.4.1 that $z \in \left(\cap_{j=1}^m F(T_{(\beta,j)})\right) \cap \left(\cap_{j=1}^m F(S_{(\beta,i)})\right) = \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right)$. Also, since $J_{\lambda^{(i)}}$ is nonexpansive for each $i = 1,2,\ldots,N$, we obtain by (6.4.11) and Lemma 2.3.12 that $z \in \cap_{i=1}^N F(J_{\lambda^{(i)}}) = \left(\cap_{i=1}^N \arg\min_{y\in X} f_i(y)\right)$. Hence $z \in \Gamma$.

Furthermore, for arbitrary $u \in X$, we have by Lemma 2.3.10 that

$$
\limsup_{n\to\infty} \langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle \leq 0, \tag{6.4.23}
$$

which implies by condition C1 that

$$
\limsup_{n\to\infty} \left(t_n d^2(z,u) + 2(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n} \rangle\right) \leq 0. \tag{6.4.24}
$$

We now show that $\{x_n\}$ converges strongly to z. By (6.4.4) and Lemma 2.3.1, we obtain

$$\begin{aligned} d^2(z, x_{n+1}) &\leq d^2(z, z_n) \\ &\leq (1-t_n)^2 d^2(z, x_n) + t_n^2 d^2(z, u) + 2t_n(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n}\rangle \\ &\leq (1-t_n) d^2(z, x_n) + t_n \left(t_n d^2(z, u) + 2(1-t_n)\langle \overrightarrow{zu}, \overrightarrow{zx_n}\rangle\right). \end{aligned} \tag{6.4.25}$$

Hence, by (6.4.24) and Lemma 2.3.26, we conclude that $\{x_n\}$ converges strongly to z.

Case 2: Suppose that $\{d^2(p, x_n)\}$ is monotonically non-decreasing. Then, there exists a subsequence $\{d^2(p, x_{n_i})\}$ of $\{d^2(p, x_n)\}$ such that $d^2(p, x_{n_i}) < d^2(p, x_{n_i+1})$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.3.29, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$, and

$$d^2(p, x_{m_k}) \leq d^2(p, x_{m_k+1}) \text{ and } d^2(p, x_k) \leq d^2(p, x_{m_k+1}) \ \forall k \in \mathbb{N}. \tag{6.4.26}$$

Thus, by (6.4.3), (6.4.26) and Lemma 2.3.1, we obtain

$$\begin{aligned} 0 &\leq \lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k})\right) \\ &\leq \limsup_{n\to\infty} \left(d^2(p, x_{n+1}) - d^2(p, x_n)\right) \\ &\leq \limsup_{n\to\infty} \left(d^2(p, z_n) - d^2(p, x_n)\right) \\ &\leq \limsup_{n\to\infty} \left((1-t_n) d^2(p, x_n) + t_n d^2(p, u) - d^2(p, x_n)\right) \\ &= \limsup_{n\to\infty} \left[t_n \left(d^2(p, u) - d^2(p, x_n)\right)\right] = 0, \end{aligned}$$

which implies that

$$\lim_{k\to\infty} \left(d^2(p, x_{m_k+1}) - d^2(p, x_{m_k})\right) = 0. \tag{6.4.27}$$

Following the arguments as in **Case 1**, we can show that

$$\lim_{k\to\infty} \left(t_{m_k} d^2(z, u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right) \leq 0. \tag{6.4.28}$$

Also, by (6.4.25) we have

$$d^2(z, x_{m_k+1}) \leq (1-t_{m_k}) d^2(z, x_{m_k}) + t_{m_k} \left(t_{m_k} d^2(z, u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right).$$

Since $d^2(z, x_{m_k}) \leq d^2(z, x_{m_k+1})$, we obtain

$$d^2(z, x_{m_k}) \leq \left(t_{m_k} d^2(z, u) + 2(1-t_{m_k})\langle \overrightarrow{zu}, \overrightarrow{zx_{m_k}}\rangle\right).$$

Thus, by (6.4.28) we get

$$\lim_{k\to\infty} d^2(z, x_{m_k}) = 0. \tag{6.4.29}$$

It then follows from (6.4.26), (6.4.27) and (6.4.29) that $\lim_{k\to\infty} d^2(z, x_k) = 0$. Therefore, we conclude by **Case 1** that $\{x_n\}$ converges to $z \in \Gamma$. $\square$

By setting $N = 2 = m$ in Theorem 6.4.2, we obtain the following result.

Corollary 6.4.3. *Let C be a closed and convex subset of an Hadamard space X and $h_i : X \to (-\infty, \infty]$, $i = 1, 2$ be a finite family of proper convex and lower semi continuous mappings. For each $j = 1, 2$, let $T_j : C \to C$ be a finite family of (f_j, g_j)-generalized k_j-strictly pseudononspreading mapping with $k_j \in [0, 1)$, where $f_j, g_j : C \to [0, \gamma]$, $\gamma < 1$ and $0 < f_j(x)+g_j(x) \leq 1$ for all $x \in C$, and $S_j : C \to X$ be a finite family of (f'_j, g'_j)-generalized k'_j-strictly pseudononspreading mapping with $k'_j \in [0, 1)$, where $f'_j, g'_j : C \to [0, \gamma']$, $\gamma' < 1$ and $0 < f'_j(x) + g'_j(x) \leq 1$ for all $x \in C$. Suppose that $\Gamma := \left(\cap_{j=1}^2 F(T_j)\right) \cap \left(\cap_{j=1}^2 F(S_j)\right) \cap \left(\cap_{i=1}^2 \arg\min_{y \in X} h_i(y)\right) \neq \emptyset$. Let $u, x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be generated by*

$$\begin{cases} z_n = (1 - t_n)x_n \oplus t_n u, \\ u_n = P_C(J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\ y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_{(\beta,1)} u_n \oplus \beta_n^{(2)} T_{(\beta,2)} u_n, \\ x_{n+1} = \alpha_n^{(0)} T_{(\beta,2)} u_n \oplus \alpha_n^{(1)} S_{(\alpha,1)} u_n \oplus \alpha_n^{(2)} S_{(\alpha,2)} y_n, \ n \geq 1, \end{cases} \tag{6.4.30}$$

where $T_{(\beta,j)}x = \beta x \oplus (1-\beta)T_j x$ and $S_{(\alpha,j)}x = \alpha x \oplus (1-\alpha)S_j x$, $j = 1, 2$, for all $x \in C$ such that $T_{(\beta,j)}$ and $S_{(\beta,j)}$ are Δ-demiclosed with $\frac{k_j}{f_j(p)} \leq \beta < 1$, $f_j(p) \neq 0$ and $\frac{k'_j}{f'_j(p)} \leq \alpha < 1$, $f'_j(p) \neq 0$ respectively, for each $j = 1, 2$ and for each $p \in \left(\cap_{j=1}^2 F(T_j)\right) \cap \left(\cap_{j=1}^2 F(S_j)\right)$, $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(j)}\}$ and $\{\alpha_n^{(j)}\}$ are sequences in $(0, 1)$ satisfying the following conditions:

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)}$, $\beta_n^{(j)} \leq b < 1$, $j = 0, 1, 2$ *such that* $\sum_{j=0}^{2} \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^{2} \beta_n^{(j)} = 1$ *for all* $n \geq 1$,

C4: $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1$, $i = 1, 2$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

In view of Remark 6.2.2, we obtain the following corollaries which extend and improve the main results of Osilike and Isiogugu [145], Bačák [16] and Bačák [18].

Corollary 6.4.4. *Let C be a closed and convex subset of an Hadamard space X and $h_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semi continuous mappings. For each $j = 1, 2, \ldots, m$, let $T_j : C \to C$ be a finite family of (f_j, g_j)-generalized nonspreading mapping, where $f_j, g_j : C \to [0, \gamma]$, $\gamma < 1$, $0 < f_j(x) + g_j(x) \leq 1$ for all $x \in C$, and $S_j : C \to X$ be a finite family of (f'_j, g'_j)-generalized nonspreading mapping, where $f'_j, g'_j : C \to [0, \gamma']$, $\gamma' < 1$, $0 < f'_j(x) + g'_j(x) \leq 1$ for all $x \in C$.*

Suppose that $\Gamma := \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right) \cap \left(\cap_{i=1}^N arg\min_{y\in X} h_i(y)\right) \neq \emptyset$*. Let* $u, x_1 \in X$ *be arbitrary and the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} z_n = (1-t_n)x_n \oplus t_n u, \\ u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\ y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_1 u_n \oplus \beta_n^{(2)} T_2 u_n \oplus \cdots \oplus \beta_n^{(m)} T_m u_n, \\ x_{n+1} = \alpha_n^{(0)} T_m u_n \oplus \alpha_n^{(1)} S_1 u_n \oplus \alpha_n^{(2)} S_2 u_n \oplus \cdots \oplus \alpha_n^{(m)} S_m y_n,\ n \geq 1, \end{cases} \tag{6.4.31}$$

where $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(j)}\}$ *and* $\{\alpha_n^{(j)}\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)},\ \beta_n^{(j)} \leq b < 1,\ j = 0, 1, 2, \ldots, m$ *such that* $\sum_{j=0}^{m} \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^{m} \beta_n^{(j)} = 1$ *for all* $n \geq 1$,

C4: $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1,\ i = 1, 2, \ldots, N$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

Corollary 6.4.5. *Let* C *be a closed and convex subset of an Hadamard space* X *and* $h_i : X \to (-\infty, \infty],\ i = 1, 2, \ldots, N$ *be a finite family of proper convex and lower semi continuous mappings. For each* $j = 1, 2, \ldots, m$*, let* $T_j : C \to C$ *and* $S_j : C \to X$ *be finite family of* k_j*-strictly pseudononspreading mappings with* $k_j \in [0,1)$ *and finite family of* k'_j*-strictly pseudononspreading mappings with* $k_j \in [0,1)$ *respectively. Suppose that* $\Gamma := \left(\cap_{j=1}^m F(T_j)\right) \cap \left(\cap_{j=1}^m F(S_j)\right) \cap \left(\cap_{i=1}^N arg\min_{y\in X} h_i(y)\right) \neq \emptyset$*. Let* $u, x_1 \in X$ *be arbitrary and the sequence* $\{x_n\}$ *be generated by*

$$\begin{cases} z_n = (1-t_n)x_n \oplus t_n u, \\ u_n = P_C(J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(z_n)), \\ y_n = \beta_n^{(0)} u_n \oplus \beta_n^{(1)} T_{(\beta,1)} u_n \oplus \beta_n^{(2)} T_{(\beta,2)} u_n \oplus \cdots \oplus \beta_n^{(m)} T_{(\beta,m)} u_n, \\ x_{n+1} = \alpha_n^{(0)} T_{(\beta,m)} u_n \oplus \alpha_n^{(1)} S_{(\alpha,1)} u_n \oplus \alpha_n^{(2)} S_{(\alpha,2)} u_n \oplus \cdots \oplus \alpha_n^{(m)} S_{(\beta,m)} y_n,\ n \geq 1, \end{cases} \tag{6.4.32}$$

where $T_{(\beta,j)}x = \beta x \oplus (1-\beta)T_j x$ *and* $S_{(\alpha,j)}x = \alpha x \oplus (1-\alpha) S_j x,\ j = 1, 2, \ldots, m,$ *for all* $x \in C$ *such that* $k_j \leq \beta < 1$ *and* $k'_j \leq \alpha < 1$*. For each* $i, j = 0, 1, 2, \ldots, m,$ $\{t_n\}, \{\lambda_n^{(i)}\}, \{\beta_n^{(i)}\}$ *and* $\{\alpha_n^{(i)}\}$ *are sequences in* $(0,1)$ *satisfying the following conditions:*

C1: $\lim_{n\to\infty} t_n = 0$,

C2: $\sum_{n=1}^{\infty} t_n = \infty$,

C3: $0 < a \leq \alpha_n^{(j)},\ \beta_n^{(j)} \leq b < 1$ *such that* $\sum_{j=0}^{m} \alpha_n^{(j)} = 1$ *and* $\sum_{j=0}^{m} \beta_n^{(j)} = 1$ *for all* $n \geq 1$,

C4: $\{\lambda_n^{(i)}\}$ *is a sequence such that* $\lambda_n^{(i)} > \lambda^{(i)}$ *for all* $n \geq 1,\ i = 1, 2, \ldots, N$ *and some* $\lambda^{(i)} > 0$.

Then, $\{x_n\}$ *converges strongly to an element of* Γ.

6.4.2 Numerical example

Let $X = \mathbb{R}$, be endowed with usual metric and $C = [0, 100]$. Then,

$$P_C(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } x \in [0,\ 100], \\ 100, & \text{if } x > 100 \end{cases}$$

is a metric projection onto C. For $m = 1$, we define $S : C \to \mathbb{R}$ by

$$Sx = \begin{cases} -3x, & \text{if } x \in [0, 1], \\ \frac{1}{x}, & \text{if } x \in (1,\ 100]. \end{cases}$$

Then, S is an (f', g')-generalized k'-strictly pseudononspreading mapping with $k' = \frac{9}{10}$ and $f', g' : [0,\ 100] \to [0,\ \frac{10}{11}]$ defined by

$$f'(x) = \begin{cases} \frac{10}{11}, & \text{if } x \in [0, 1], \\ \frac{1}{11}, & \text{if } x \in (1, 100] \end{cases} \qquad \text{and} \qquad g'(x) = \begin{cases} \frac{1}{11}, & \text{if } x \in [0, 1], \\ \frac{10}{11}, & \text{if } x \in (1, 100]. \end{cases}$$

Also, we define $T : C \to C$ by

$$Tx = \begin{cases} \frac{1}{x + \frac{1}{10}}, & \text{if } x \in [1, 100], \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then, T is an (f, g)-generalized k-strictly pseudocontractive mapping with $k = 0$ and $f, g : [0, 100] \to [0, \frac{9}{10}]$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in [1, 100], \\ \frac{9}{10}, & \text{if } x \in [0, 1) \end{cases} \qquad \text{and} \qquad g(x) = \begin{cases} \frac{1}{(x + \frac{1}{10})^2}, & \text{if } x \in [1,\ 100], \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Clearly, $F(T) \cap F(S) = \{0\}$. Thus, we can choose $\alpha = \frac{k'}{f'(0)} = \frac{99}{100}$ and $\beta = 0$. Then, $S_\alpha x = \frac{99}{100}x + \left(1 - \frac{99}{100}\right) Sx$ and $T_\beta x = Tx$.
Let $N = 2$. Then for $i = 1, 2$, we define $h_1, h_2 : \mathbb{R} \to (-\infty, \infty]$ by $h_1(x) = \frac{1}{2}|B_1(x) - b_1|^2$ and $h_2(x) = \frac{1}{2}|B_2(x) - b_2|^2$, where $B_1(x) = 2x,\ B_2(x) = 5x$ and $b_1 = b_2 = 0$. Since B_i is a

continuous and linear mapping, so for each $i = 1, 2$, h_i is a proper convex and lower semi continuous mapping (see [126]). Thus, for $\lambda_n = 1$, we have that (see [126])

$$\begin{aligned} J_{1^{(i)}}(x) &= \text{Prox}_{h_i} x = \arg\min_{y \in C} \left(h_i(y) + \frac{1}{2}|y - x|^2 \right) \\ &= (I + B_i^T B_i)^{-1}(x + B_i^T b_i). \end{aligned}$$

Take $t_n = \frac{1}{4n+3}$, $\alpha_n^{(0)} = \frac{n}{3n+5}$, $\alpha_n^{(1)} = \frac{2n+5}{3n+5}$, $\beta_n^{(0)} = \frac{n}{2n+1}$ and $\beta_n^{(1)} = \frac{n+1}{2n+1}$. Now, conditions C1-C4 in Theorem 6.4.2 are satisfied.

Hence, for $u, x_1 \in \mathbb{R}$, our Algorithm (6.4.1) becomes:

$$\begin{cases} z_n = (1 - t_n)x_n + t_n u, \\ u_n = P_C\left(J_{1^{(2)}}(J_{1^{(1)}}(z_n))\right), \\ y_n = \beta_n^{(0)} u_n + \beta_n^{(1)} T_\beta u_n, \\ x_{n+1} = \alpha_n^{(0)} T_\beta u_n + \alpha_n^{(1)} S_\alpha y_n, \; n \geq 1. \end{cases} \tag{6.4.33}$$

Case I: Take $x_1 = 1$ and $u = 0.1$.

Case II: Take $x_1 = 0.5$ and $u = 0.1$.

Case III: Take $x_1 = 0.5$ and $u = 2$.

The following table shows results of our numerical experiment based on Mathlab version R2016a software.

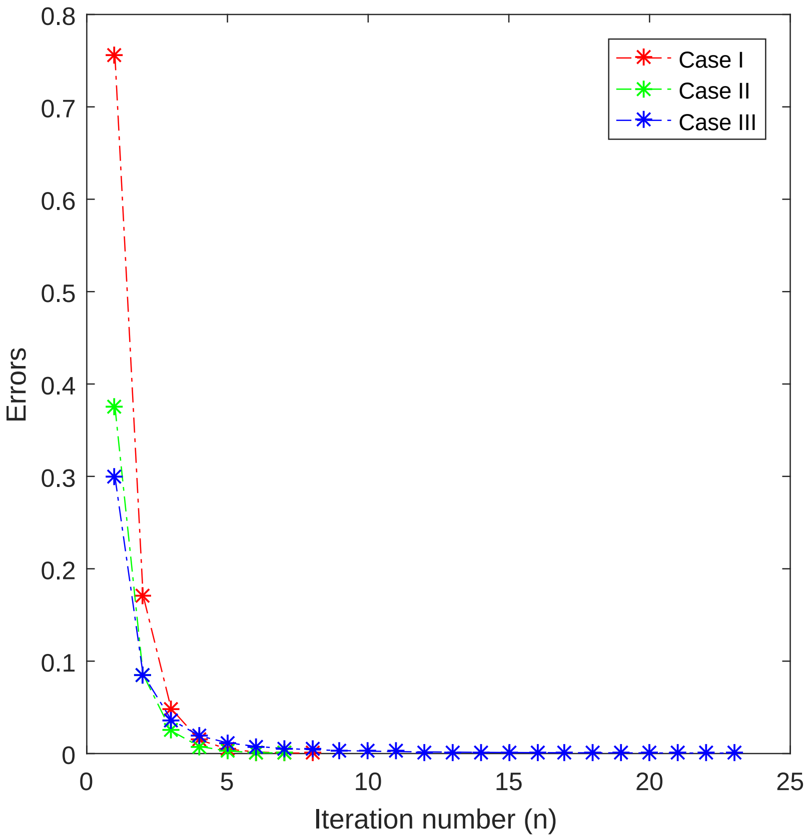

Figure 6.1: Errors vs number of iterations for **Case I**, **Case II** and **Case III**.

Iteration Numbers	**<u>Errors for Case I</u>** u=0.1	**<u>Errors for Case II</u>** u=0.1	**<u>Errors for Case III</u>** u=2
1	1.0000	0.5000	0.5000
2	0.7560	0.3760	0.3000
3	0.1715	0.0857	0.0858
4	0.0485	0.0246	0.0367
5	0.0149	0.0077	0.0192
6	0.0049	0.0027	0.0116
7	0.0017	0.0010	0.0077
8	0.0007	0.0005	0.0055
9	0.0003	0.0003	0.0041
10	0.0002	0.0002	0.0032
11	0.0001	0.0001	0.0026
12	0.0001	0.0001	0.0021
13	0.0001	0.0001	0.0018
14	0.0001	0.0001	0.0015
15	0.0001	0.0001	0.0013

TABLE 1. Showing numerical results for **Case I**, **Case II** and **Case III**.

6.5 Iterative algorithm for monotone inclusion problems and fixed point problem for a family of generalized strictly pseudononspreading mappings in Hadamard spaces

In this section, we introduce a new mapping given by a finite family of a generalized strictly pseudononspreading mappings. Also, we introduce a viscosity-type proximal point algorithm and prove its strong convergence to a common solution of a finite family of monotone inclusion problems and fixed point problem for the new mapping in an Hadamard space. A numerical example of our algorithm to show its applicability is presented. Our numerical experiment shows that our algorithm converges faster than other related algorithms in the literature.

6.5.1 Main results

Definition 6.5.1. *Let C be a nonempty closed and convex subset of a CAT(0) space X and $T_i : C \to C,\ i = 1, 2, \ldots, N$ be finite family of (f_i, g_i)-generalized k_i-strictly pseudononspreading mappings. Then, we define the mapping $W_n : C \to C$ as follows:*

$$\begin{cases} U_n^{(0)}x = x, \\ U_n^{(1)}x = a_n^{(1)}S_1x \oplus b_n^{(1)}x \oplus c_n^{(1)}x, \\ U_n^{(2)}x = a_n^{(2)}S_2U_n^{(1)}x \oplus b_n^{(2)}U_n^{(1)}x \oplus c_n^{(2)}x, \\ U_n^{(3)}x = a_n^{(3)}S_3U_n^{(2)}x \oplus b_n^{(3)}U_n^{(2)}x \oplus c_n^{(3)}x, \\ \vdots \\ U_n^{(N-1)}x = a_n^{(N-1)}S_{N-1}U_n^{(N-2)}x \oplus b_n^{(N-1)}U_n^{(N-2)}x \oplus c_n^{(N-1)}x, \\ W_nx = U_n^{(N)}x = a_n^{(N)}S_NU_n^{(N-1)}x \oplus b_n^{(N)}U_n^{(N-1)}x \oplus c_n^{(N)}x, \end{cases} \tag{6.5.1}$$

where $S_ix := \delta_n^{(i)}x \oplus (1 - \delta_n^{(i)})T_ix,\ i = 1, 2, \ldots, N$ for all $x \in C$, $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$, $\{c_n^{(i)}\}$ and $\{\delta_n^{(i)}\}$ are sequences in $(0, 1)$ such that $a_n^{(i)} + b_n^{(i)} + c_n^{(i)} = 1,\ i = 1, 2, \ldots, N$.

Lemma 6.5.2. *Let C be a nonempty closed and convex subset of a CAT(0) space X and $T_i : C \to C,\ i = 1, 2, \ldots, N$ be a finite family of (f_i, g_i)-generalized k_i-strictly pseudononspreading mappings with $k_i \in [0, 1)$, where $f_i, g_i : C \to [0, \gamma],\ \gamma < 1$ and $0 < f_i(x) + g_i(x) \leq 1$ for all $x \in C,\ i = 1, 2, \ldots, N$. Suppose that $\cap_{i=1}^N F(T_i) \neq \emptyset$ and W_n is defined by (6.5.1), where $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$ and $\{c_n^{(i)}\}$ are in $[\epsilon, 1 - \epsilon]$, for all $n \geq 1,\ i = 1, 2, \ldots, N$ and for some $\epsilon \in (0, 1)$. For $i = 1, 2, \ldots, N$, let S_i be as defined in (6.5.1), where $\frac{k_i}{f_i(p)} \leq \delta_n^{(i)} < 1,\ n \geq 1$ with $f_i(p) \neq 0$ for each $p \in \cap_{i=1}^N F(T_i)$. Then,*

(a) $d(p, S_ix) \leq d(p, x)\ \forall x \in C,\ p \in \cap_{i=1}^N F(T_i),\ i = 1, 2, \ldots, N,$

(b) $\cap_{i=1}^N F(T_i) = \cap_{i=1}^N F(S_i) = F(W_n),$

(c) W_n is quasinonexpansive.

Proof. (a) Since T_i is (f_i, g_i)-generalized k_i-strictly pseudononspreading mapping, then for each $p \in \cap_{i=1}^N F(T_i)$ and $x \in C$, we obtain for each $i = 1, 2, \ldots, N$ that

$$d^2(p, T_i x) \leq f_i(p) d^2(p, x) + g_i(p) d^2(p, Tx) + k d^2(x, T_i x),$$

which implies

$$(1 - g_i(p)) d^2(p, T_i x) \leq f_i(p) d^2(p, x) + k d^2(x, T_i x).$$

Since $f_i(p) + g_i(p) \leq 1$ for $i = 1, 2, \ldots, N$, we obtain that

$$d^2(p, T_i x) \leq d^2(p, x) + \frac{k_i}{f_i(p)} d^2(x, T_i x). \tag{6.5.2}$$

Thus, by Lemma 2.3.1 (ii), we have for each $x \in C$, $p \in \cap_{i=1}^N F(T_i)$ and $i = 1, 2, \ldots, N$ that

$$\begin{aligned} d^2(p, S_i x) &= d^2(p, \delta_n^{(i)} x \oplus (1 - \delta_n^{(i)}) T_i x) \\ &\leq \delta_n^{(i)} d^2(p, x) + (1 - \delta_n^{(i)}) d^2(p, T_i x) - \delta_n^{(i)} (1 - \delta_n^{(i)}) d^2(x, T_i x) \\ &\leq \delta_n^{(i)} d^2(p, x) + (1 - \delta_n^{(i)}) \left[d^2(p, x) + \frac{k_i}{f_i(p)} d^2(x, T_i x) \right] - \delta_n^{(i)} (1 - \delta_n^{(i)}) d^2(x, T_i x) \\ &= d^2(p, x) + (1 - \delta_n^{(i)}) \left(\frac{k_i}{f_i(p)} - \delta_n^{(i)} \right) d^2(x, T_i x) \\ &\leq d^2(p, x), \end{aligned}$$

which implies that $d(p, S_i x) \leq d(p, x)$ for all $x \in C$, $p \in \cap_{i=1}^N F(T_i)$ and for each $i = 1, 2, \ldots, N$.

(b) First, we show that $\cap_{i=1}^N F(S_i) = \cap_{i=1}^N F(T_i)$. For $i = 1$, observe that if $\delta_n^{(1)} = 0$ for any $n \geq 1$, then $S_1 = T_1$. Hence, $F(S_1) = F(T_1)$. Now, assume that $\delta_n^{(1)} \neq 0$ for all $n \geq 1$. Then, for each $p \in F(S_1)$, we have from Lemma 2.3.1 (i) that $d(p, T_1 p) = d(\delta_n^{(1)} p \oplus (1 - \delta_n^{(1)}) T_1 p, T_1 p) \leq \delta_n^{(1)} d(p, T_1 p)$, which implies that $(1 - \delta_n^{(1)}) d(p, T_1 p) \leq 0$. Since $\delta_n^{(1)} < 1$ $\forall n \geq 1$, it follows that $p \in F(T_1)$. Hence, $F(S_1) \subseteq F(T_1)$. Thus, by repeating the same argument for $i = 2, 3 \ldots, N$, we can show that $F(S_i) \subseteq F(T_i)$ for each $i = 2, 3, \ldots, N$.

On the other hand, let $p \in F(T_i)$, then from Lemma 2.3.1 (i), we have for each $i = 1, 2, \ldots, N$ that

$d(p, S_i p) = d(p, \delta_n^{(i)} p \oplus (1 - \delta_n^{(i)}) p) \leq 0$. Hence, $p \in F(S_i)$. Thus, $F(T_i) \subseteq F(S_i)$ for each $i = 1, 2, \ldots, N$.
Therefore, $F(S_i) = F(T_i)$, $i = 1, 2, \ldots, N$, which implies that $\cap_{i=1}^N F(T_i) = \cap_{i=1}^N F(S_i)$.

Next, we show that $\cap_{i=1}^N F(S_i) = F(W_n)$. It is obvious that $\cap_{i=1}^N F(S_i) \subseteq F(W_n)$. Therefore, we will only show that $F(W_n) \subseteq \cap_{i=1}^N F(S_i)$.

Let $p \in F(W_n)$, then for any $z \in \cap_{i=1}^{N} F(T_i) = \cap_{i=1}^{N} F(S_i)$, we have from (6.5.1), Lemma 6.5.2(a) and Lemma 2.3.3 that

$$\begin{aligned}
d(z,p) &\leq a_n^{(N)} d(z, S_N U_n^{(N-1)} p) + b_n^{(N)} d(z, U_n^{(N-1)} p) + c_n^{(N)} d(z,p) \\
&\leq a_n^{(N)} d(z, U_n^{(N-1)} p) + b_n^{(N)} d(z, U_n^{(N-1)} p) + c_n^{(N)} d(z,p) \\
&= (1 - c_n^{(N)}) \left[d(z, a_n^{(N-1)} S_{N-1} U_n^{(N-2)} p \oplus b_n^{(N-1)} U_n^{(N-2)} p \oplus c_n^{(N-1)} p) \right] \\
&\quad + \left[1 - (1 - c_n^{(N)}) \right] d(z,p) \\
&\leq (1 - c_n^{(N)}) \left[a_n^{(N-1)} d(z, S_{N-1} U_n^{(N-2)} p) + b_n^{(N-1)} d(z, U_n^{(N-2)} p) + c_n^{(N-1)} d(z,p) \right] \\
&\quad + \left[1 - (1 - c_n^{(N)}) \right] d(z,p) \\
&\leq (1 - c_n^{(N)})(1 - c_n^{(N-1)}) d(z, U_n^{(N-2)} p) + \left[1 - (1 - c_n^{(N)})(1 - c_n^{(N-1)}) \right] d(z,p) \\
&\leq (1 - c_n^{(N)})(1 - c_n^{(N-1)}) \Big[a_n^{(N-2)} d(z, S_{N-2} U_n^{(N-3)} p) + b_n^{(N-2)} d(z, U_n^{(N-3)} p) \\
&\quad + c_n^{(N-2)} d(z,p) \Big] + \left[1 - (1 - c_n^{(N)})(1 - c_n^{(N-1)}) \right] d(z,p) \\
&\leq (1 - c_n^{(N)})(1 - c_n^{(N-1)})(1 - c_n^{(N-2)}) d(z, U_n^{(N-3)} p) \\
&\quad + \left[1 - (1 - c_n^{(N)})(1 - c_n^{(N-1)})(1 - c_n^{(N-2)}) \right] d(z,p) \\
&= \prod_{i=0}^{2} \left[1 - c_n^{(N-i)} \right] d(z, U_n^{(N-3)} p) + \left[1 - \prod_{i=0}^{2} (1 - c_n^{(N-i)}) \right] d(z,p) \\
&= \prod_{i=0}^{N-(N-2)} \left[1 - c_n^{(N-i)} \right] d(z, U_n^{(N-3)} p) + \left[1 - \prod_{i=0}^{N-(N-2)} (1 - c_n^{(N-i)}) \right] d(z,p) \\
&\ \vdots \\
&\leq \prod_{i=0}^{N-3} \left[1 - c_n^{(N-i)} \right] d(z, U_n^{(2)} p) + \left[1 - \prod_{i=0}^{N-3} (1 - c_n^{(N-i)}) \right] d(z,p) \\
&\leq \prod_{i=0}^{N-3} \left[1 - c_n^{(N-i)} \right] \left[a_n^{(2)} d(z, S_2 U_n^{(1)} p) + b_n^{(2)} d(z, U_n^{(1)} p) + c_n^{(2)} d(z,p) \right] \\
&\quad + \left[1 - \prod_{i=0}^{N-3} (1 - c_n^{(N-i)}) \right] d(z,p) \\
&\leq \prod_{i=0}^{N-2} \left[1 - c_n^{(N-i)} \right] d(z, U_n^{(1)} p) + \left[1 - \prod_{i=0}^{N-2} (1 - c_n^{(N-i)}) \right] d(z,p) \qquad (6.5.3) \\
&\leq \prod_{i=0}^{N-2} \left[1 - c_n^{(N-i)} \right] \left[a_n^{(1)} d(z, S_1 p) + b_n^{(1)} d(z,p) + c_n^{(1)} d(z,p) \right] \\
&\quad + \left[1 - \prod_{i=0}^{N-2} (1 - c_n^{(N-i)}) \right] d(z,p) \qquad (6.5.4) \\
&\leq \prod_{i=0}^{N-1} \left[1 - c_n^{(N-i)} \right] d(z,p) + \left[1 - \prod_{i=0}^{N-1} (1 - c_n^{(N-i)}) \right] d(z,p) \\
&= d(z,p).
\end{aligned}$$

From (6.5.4), we obtain that

$$\begin{aligned} d(z,p) &= \prod_{i=0}^{N-2}\left[1-c_n^{(N-i)}\right]\left[a_n^{(1)}d(z,S_1p)+b_n^{(1)}d(z,p)+c_n^{(1)}d(z,p)\right] \\ &\quad+\left[1-\prod_{i=0}^{N-2}(1-c_n^{(N-i)})\right]d(z,p), \end{aligned}$$

which implies that

$$d(z,p)=a_n^{(1)}d(z,S_1p)+b_n^{(1)}d(z,p)+c_n^{(1)}d(z,p).$$

That is,

$$d(z,p)=a_n^{(1)}d(z,S_1p)+(1-a_n^{(1)})d(z,p).$$

Thus, $d(z,p)=d(z,S_1p)$. Hence, $S_1p=p$.

Again, from (6.5.3), we obtain that

$$d(z,p)=\prod_{i=0}^{N-2}\left[1-c_n^{(N-i)}\right]d(z,U_n^{(1)}p)+\left[1-\prod_{i=0}^{N-2}(1-c_n^{(N-i)})\right]d(z,p).$$

This implies that $d(z,p)=d(z,U_n^{(1)}p)$. Hence,

$$U_n^{(1)}p=p. \tag{6.5.5}$$

Similarly, we obtain that

$$\begin{aligned} d(z,p) &= \prod_{i=0}^{N-3}\left[1-c_n^{(N-i)}\right]\left[a_n^{(2)}d(z,S_2U_n^{(1)}p)+b_n^{(2)}d(z,U_n^{(1)}p)+c_n^{(2)}d(z,p)\right] \\ &\quad+\left[1-\prod_{i=0}^{N-3}(1-c_n^{(N-i)})\right]d(z,p), \end{aligned}$$

which implies that $d(z,p)=a_n^{(2)}d(z,S_2U_n^{(1)}p)+b_n^{(2)}d(z,U_n^{(1)}p)+c_n^{(2)}d(z,p)$. Thus, we obtain from (6.5.5) that

$$d(z,p)=a_n^{(2)}d(z,S_2p)+(1-a_n^{(2)})d(z,p).$$

That is, $d(z,p)=d(z,S_2p)$. Hence,

$$S_2p=p. \tag{6.5.6}$$

Continuing in this manner, we can show that $S_ip=p,\ i=3,4,\dots,N$. Hence, $F(W_n)\subseteq \cap_{i=1}^N F(S_i)$. Therefore, $F(W_n)=\cap_{i=1}^N F(S_i)$.

(c) Let $p\in F(W_n)$ and $x\in C$, then from (6.5.1), Lemma 6.5.2(a),(b) and Lemma 2.3.3, we obtain

$$\begin{aligned}
d(p, U_n^{(N-1)}x) &\leq a_n^{(N-1)} d(p, S_{N-1}U_n^{(N-2)}x) + b_n^{(N-1)} d(p, U_n^{(N-2)}x) + c_n^{(N-1)} d(p, x) \\
&\leq a_n^{(N-1)} d(p, U_n^{(N-2)}x) + b_n^{(N-1)} d(p, U_n^{(N-2)}x) + c_n^{(N-1)} d(p, x) \\
&\leq (1 - c_n^{(N-1)}) d(p, U_n^{(N-2)}x) + \left[1 - (1 - c_n^{(N-1)})\right] d(p, x) \\
&\leq \prod_{i=1}^{2} \left[1 - c_n^{(N-i)}\right] d^2(p, U_n^{N-3}x) + \left[1 - \prod_{i=1}^{2}(1 - c_n^{(N-i)})\right] d(p, x) \\
&\ \ \vdots \\
&\leq \prod_{i=1}^{N-1} \left[1 - c_n^{(N-i)}\right] d(p, x) + \left[1 - \prod_{i=1}^{N-1}(1 - c_n^{(N-i)})\right] d(p, x) \\
&\leq d(p, x). \qquad (6.5.7)
\end{aligned}$$

Also, we obtain from (6.5.1) and (6.5.7) that

$$\begin{aligned}
d(p, W_n x) &= d(p, a_n^{(N)} S_N U_n^{(N-1)} x \oplus b_n^{(N)} U_n^{(N-1)} x \oplus c_n^{(N)} x) \\
&\leq a_n^{(N)} d(p, S_N U_n^{(N-1)} x) + b_n^{(N)} d(p, U_n^{(N-1)} x) + c_n^{(N)} d(p, x) \\
&\leq a_n^{(N)} d(p, U_n^{(N-1)} x) + b_n^{(N)} d(p, U_n^{(N-1)} x) + c_n^{(N)} d(p, x) \\
&\leq a_n^{(N)} d(p, x) + b_n^{(N)} d(p, x) + c_n^{(N)} d(p, x) \\
&= d(p, x). \qquad (6.5.8)
\end{aligned}$$

$\square$

Theorem 6.5.3. *Let C be a nonempty closed and convex subset of an Hadamard space X and $T_i : C \to C$, $i = 1, 2, \ldots, N$ be a finite family of (f_i, g_i)-generalized k_i-strictly pseudononspreading mappings with $k_i \in [0, 1)$, where $f_i, g_i : C \to [0, \gamma]$, $\gamma < 1$ and $0 < f_i(x) + g_i(x) \leq 1$ for all $x \in C$, $i = 1, 2, \ldots, N$. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfy the range condition and f be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := \cap_{i=1}^N F(T_i) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and W_n is as defined in (6.5.1) such that W_n is Δ-demiclosed for each $n \geq 1$, where $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$ and $\{c_n^{(i)}\}$ are in $[\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$, $\frac{k_i}{f_i(p)} \leq \delta_n^{(i)} < 1$ with $f_i(p) \neq 0$ for each $p \in \cap_{i=1}^N F(T_i)$, $i = 1, 2, \ldots, N$ and for all $n \geq 1$. Let $x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be defined by*

$$\begin{cases}
y_n = (1 - \alpha_n) x_n \oplus \alpha_n f(x_n), \\
z_n = (1 - \gamma_n) y_n \oplus \gamma_n W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)), \\
x_{n+1} = (1 - \beta_n) y_n \oplus \beta_n z_n, \ n \geq 1,
\end{cases} \qquad (6.5.9)$$

where $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

C2: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Proof. First, we show that $\{x_n\}$ is bounded.
Let $p \in \Gamma$, then from (6.5.9), Lemma 6.5.2 and Lemma 2.3.1 (ii), we obtain

$$\begin{aligned} d^2(z_n,p) &= d^2((1-\gamma_n)y_n \oplus \gamma_n W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)),p) \\ &\leq (1-\gamma_n)d^2(y_n,p) + \gamma_n d^2(W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)),p) \\ &- \gamma_n(1-\gamma_n)d^2(y_n, W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n))) \\ &\leq d^2(y_n,p) - \gamma_n(1-\gamma_n)d^2(y_n, W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n))) \\ &\leq d^2(y_n,p). \end{aligned} \tag{6.5.10}$$

Also, from (6.5.9) and (2.1.1), we obtain that

$$\begin{aligned} d(x_{n+1},y_n) &= d((1-\beta_n)y_n \oplus \beta_n z_n, y_n) \\ &= \beta_n d(z_n,y_n). \end{aligned} \tag{6.5.11}$$

Again, from (6.5.9), (6.5.10) and (6.5.11), we obtain

$$\begin{aligned} d^2(x_{n+1},p) &\leq (1-\beta_n)d^2(y_n,p) + \beta_n d^2(z_n,p) - \beta_n(1-\beta_n)d^2(y_n,z_n) \\ &\leq d^2(y_n,p) - \frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1},y_n) \qquad (6.5.12) \\ &\leq d^2(y_n,p), \end{aligned}$$

which implies from Lemma 2.3.1 (i) that

$$\begin{aligned} d(x_{n+1},p) &\leq d(y_n,p) \\ &\leq (1-\alpha_n)d(x_n,p) + \alpha_n d(f(x_n),p) \\ &\leq (1-\alpha_n(1-\tau))d(x_n,p) + \alpha_n d(f(p),p) \\ &\leq \max\{d(x_n,p), \frac{d(f(p),p)}{1-\tau}\} \\ &\vdots \\ &\leq \max\{d(x_1,p), \frac{d(f(p),p)}{1-\tau}\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{z_n\}$ and $\{f(x_n)\}$ are all bounded.
From (6.5.12), (6.5.9) and Lemma 2.3.1 (iii), we obtain that

$$\begin{aligned} d^2(x_{n+1},p) &\leq (1-\alpha_n)^2 d^2(x_n,p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(x_n)p}, \overrightarrow{x_n p}\rangle + \alpha_n^2 d^2(f(x_n),p) \\ &\leq (1-\alpha_n)^2 d^2(x_n,p) + 2\alpha_n(1-\alpha_n)\left(\langle\overrightarrow{f(x_n)f(p)}, \overrightarrow{x_n p}\rangle + \langle\overrightarrow{f(p)p}, \overrightarrow{x_n p}\rangle\right) \\ &+ \alpha_n^2 d^2(f(x_n),p) \\ &\leq (1-\alpha_n)^2 d^2(x_n,p) + 2\alpha_n(1-\alpha_n)\left(\tau d^2(x_n,p) + \langle\overrightarrow{f(p)p}, \overrightarrow{x_n p}\rangle\right) + \alpha_n^2 d^2(f(x_n),p) \\ &\leq (1-2\alpha_n(1-\tau))d^2(x_n,p) \\ &+ 2\alpha_n^2(1-\tau)d^2(x_n,p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{f(p)p}, \overrightarrow{x_n p}\rangle + \alpha_n^2 d^2(f(x_n),p) \\ &= (1-2\alpha_n(1-\tau))d^2(x_n,p) + 2\alpha_n(1-\tau)T_n, \end{aligned} \tag{6.5.13}$$

$$\text{where } T_n = \frac{(1-\alpha_n)}{(1-\tau)}\langle\overrightarrow{f(p)p}, \overrightarrow{x_n p}\rangle + \alpha_n\left(d^2(x_n,p) + \frac{1}{2(1-\tau)}d^2(f(x_n),p)\right) \quad (6.5.14)$$

Also, from (6.5.9), we obtain that

$$d(y_n, x_n) \leq \alpha_n d(f(x_n), x_n) \to 0, \text{ as } n \to \infty. \quad (6.5.15)$$

Case 1: Suppose that $\{d^2(x_n,p)\}$ is monotone decreasing. Then, $\lim_{n\to\infty} d^2(x_n,p)$ exists. Consequently, we obtain that

$$\lim_{n\to\infty}\left(d^2(x_n,p) - d^2(x_{n+1},p)\right) = 0 = \lim_{n\to\infty}\left(d^2(x_{n+1},p) - d^2(x_n,p)\right). \quad (6.5.16)$$

Thus, we obtain from (6.5.12) and (6.5.15) that

$$\begin{aligned}\frac{1}{\beta_n}(1-\beta_n)d^2(x_{n+1},y_n) &\leq d^2(y_n,p) - d^2(x_{n+1},p)\\ &\leq d^2(y_n,x_n) + 2d(y_n,x_n)d(x_n,p) + d^2(x_n,p) - d^2(x_{n+1},p)\\ &\quad \to 0, \text{ as } n\to\infty,\end{aligned}$$

which implies from condition C2 of Theorem 6.5.3 that

$$\lim_{n\to\infty} d(x_{n+1},y_n) = 0. \quad (6.5.17)$$

Thus, we obtain from (6.5.11) that

$$\lim_{n\to\infty} d(z_n, \ y_n) = 0. \quad (6.5.18)$$

Also, from (6.5.10) and (6.5.18), we obtain that

$$\begin{aligned}\gamma_n(1-\gamma_n)d^2(y_n, W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n))) &\leq d^2(y_n,p) - d^2(z_n,p)\\ &\leq d^2(y_n,z_n) + 2d(y_n,z_n)d(z_n,p)\\ &\quad + d^2(z_n,p) - d^2(z_n,p) \to 0.\end{aligned}$$

This, together with condition C2 implies

$$\lim_{n\to\infty} d(y_n, W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n))) = 0. \quad (6.5.19)$$

Now, let $u_n = P_C(\Phi_\lambda^N y_n)$, where $\Phi_\lambda^N = J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1$ with $\Phi_\lambda^0 = I$. Since, P_C is firmly nonexpansive, we obtain from Remark 3.2.5 and (6.5.19) that

$$\begin{aligned}d^2(u_n, \Phi_\lambda^N y_n) &\leq d^2(p, \Phi_\lambda^N y_n) - d^2(p, u_n)\\ &\leq d^2(p, y_n) - d^2(p, W_n u_n)\\ &\leq d^2(p, W_n u_n) + 2d(p, W_n u_n)d(W_n u_n, y_n)\\ &\quad + d^2(W_n u_n, y_n) - d^2(p, W_n u_n) \to 0 \text{ as } n\to\infty. \end{aligned} \quad (6.5.20)$$

Similarly, since J_λ^N is firmly nonexpansive, we obtain that

$$\begin{aligned} d^2(\Phi_\lambda^N y_n, \Phi_\lambda^{N-1} y_n) &\leq d^2(p, \Phi_\lambda^{N-1} y_n) - d^2(p, \Phi_\lambda^N y_n) \\ &\leq d^2(p, y_n) - d^2(p, u_n) \\ &\leq d^2(p, y_n) - d^2(p, W_n u_n) \\ &\leq d^2(p, W_n u_n) + 2d(p, W_n u_n) d(W_n u_n, y_n) \\ &\quad + d^2(W_n u_n, y_n) - d^2(p, W_n u_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{6.5.21}$$

Continuing in this manner, we can show that

$$\lim_{n\to\infty} d^2(\Phi_\lambda^{N-1} y_n, \Phi_\lambda^{N-2} y_n) = \cdots = \lim_{n\to\infty} d^2(\Phi_\lambda^1 y_n, y_n) = 0. \tag{6.5.22}$$

Thus,

$$d(u_n, y_n) \leq d(u_n, \Phi_\lambda^N y_n) + d(\Phi_\lambda^N y_n, \Phi_\lambda^{N-1} y_n) + d(\Phi_\lambda^{N-1} y_n, \Phi_\lambda^{N-2} y_n) + \cdots + d(\Phi_\lambda^1 y_n, y_n),$$

which implies from (6.5.20), (6.5.21) and (6.5.22) that

$$\lim_{n\to\infty} d(u_n, y_n) = \lim_{n\to\infty} d(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) y_n, y_n) = 0. \tag{6.5.23}$$

Furthermore, from (6.5.19) and (6.5.23), we obtain

$$\lim_{n\to\infty} d(u_n, W_n u_n) = 0. \tag{6.5.24}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then by Lemma 2.3.5, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta\text{-}\lim\limits_{k\to\infty} x_{n_k} = z$. Thus, we obtain from (6.5.15) and (6.5.23) that $\Delta\text{-}\lim\limits_{k\to\infty} y_{n_k} = z$ and $\Delta\text{-}\lim\limits_{k\to\infty} u_{n_k} = z$ for some subsequences $\{y_{n_k}\}$ and $\{u_{n_k}\}$ of $\{y_n\}$ and $\{u_n\}$ respectively. Thus, by the demicloseness of W_n, (6.5.24) and Lemma 6.5.2(b), we obtain that $z \in F(W_n) = \cap_{i=1}^N F(T_i)$. Again, since P_C and J_λ^i, $i = 1, 2, \ldots, N$ are nonexpansive mappings, and the composition of nonexpansive mappings is nonexpansive, we obtain from Lemma 2.3.12, Lemma 3.2.6 and (6.5.23) that $z \in F(P_C \circ J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1) = F(P_C) \cap F(J_\lambda^N) \cap F(J_\lambda^{N-1}) \cap \cdots \cap F(J_\lambda^2) \cap F(J_\lambda^1)$. Hence, $z \in \Gamma$.

Next we show that $\{x_n\}$ converges strongly to $v \in \Gamma$. Since $\{x_{n_k}\}$ $\triangle-$converges to $z \in \Gamma$, it follows from Lemma 2.3.9 that there exists $v \in \Gamma$ such that $\{x_n\}$ $\triangle$-converges to v. Thus, by Lemma 2.3.10, we obtain that

$$\limsup_{n\to\infty} \langle \overrightarrow{f(v)v}, \overrightarrow{x_n v} \rangle \leq 0.$$

Hence, we obtain from (6.5.14) and condition C1 of Theorem 6.5.3 that $\limsup\limits_{n\to\infty} T_n \leq 0$.

Also, we obtain from (6.5.13) that

$$d^2(x_{n+1}, v) \leq (1 - 2\alpha_n(1-\tau)) d^2(x_n, v) + 2\alpha_n(1-\tau) T_n. \tag{6.5.25}$$

Therefore, by Lemma 2.3.26 and condition C1 of Theorem 6.5.3, we obtain that $\{x_n\}$ converges strongly to $v \in \Gamma$.

Case 2: Suppose that $\{d^2(x_n, p)\}$ is not monotone decreasing. Then, there exists a subsequence $\{d^2(x_{n_i}, p)\}$ of $\{d^2(x_n, p)\}$ such that $d^2(x_{n_i}, p) < d^2(x_{n_i+1}, p)$ for all $i \in \mathbb{N}$. Thus, by Lemma 2.3.29, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and

$$d^2(x_{m_k}, p) \leq d^2(x_{m_k+1}, p) \text{ and } d^2(x_k, p) \leq d^2(x_{m_k+1}, p)\ \forall k \in \mathbb{N}.$$

Thus, we have

$$\begin{aligned}
0 &\leq \lim_{k\to\infty} \big(d^2(x_{m_k+1}, p) - d^2(x_{m_k}, p)\big) \\
&\leq \limsup_{n\to\infty} \big(d^2(x_{n+1}, p) - d^2(x_n, p)\big) \\
&\leq \limsup_{n\to\infty} \big(d^2(y_n, p) - d^2(x_n, p)\big) \\
&\leq \limsup_{n\to\infty} \big(\alpha_n d^2(f(x_n), p) + (1-\alpha_n)d^2(x_n, p) - d^2(x_n, p)\big) \\
&\leq \limsup_{n\to\infty} \alpha_n \big(d^2(f(x_n), p) - d^2(x_n, p)\big) = 0,
\end{aligned}$$

which implies

$$\lim_{k\to\infty} \big(d^2(x_{m_k+1}, p) - d^2(x_{m_k}, p)\big) = 0. \tag{6.5.26}$$

Following the same line of argument as in Case 1, we can verify that

$$\lim_{k\to\infty} \langle \overrightarrow{f(v)v}, \overrightarrow{x_{m_k}v} \rangle \leq 0 \quad \text{and} \quad \lim_{k\to\infty} T_{m_k} \leq 0. \tag{6.5.27}$$

Also from (6.5.25), we have

$$d^2(x_{m_k+1}, v) \leq (1 - 2\alpha_{m_k}(1-\tau))d^2(x_{m_k}, v) + 2\alpha_{m_k}(1-\tau)T_{m_k}. \tag{6.5.28}$$

Since $d^2(x_{m_k}, v) \leq d^2(x_{m_k+1}, v)$, we have

$$d^2(x_{m_k}, v) \leq T_{m_k},$$

which implies from (6.5.27) that

$$\lim_{k\to\infty} d^2(x_{m_k}, v) = 0. \tag{6.5.29}$$

Since $d^2(x_k, z) \leq d^2(x_{m_k+1}, v)$, we obtain from (6.5.29) and (6.5.26) that $\lim\limits_{k\to\infty} d^2(x_k, v) = 0$. Thus, from Case 1 and Case 2, we conclude that $\{x_n\}$ converges to $v \in \Gamma$. $\square$

By letting the mapping W_n defined by (6.5.1), to be generated by a finite family of generalized nonspreading mappings and k-strictly pseudononspreading mappings, we obtain the following corollaries.

Corollary 6.5.4. *Let C be a nonempty closed and convex subset of an Hadamard space X and $T_i : C \to C,\ i = 1, 2, \ldots, N$ be a finite family of generalized nonspreading mappings. Let $A_i : X \to 2^{X^*},\ i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfy the range condition and f be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := \cap_{i=1}^N F(T_i) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$, and for arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n f(x_n), \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n,\ n \geq 1, \end{cases} \tag{6.5.30}$$

where $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

C2: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

Corollary 6.5.5. *Let C be a nonempty closed and convex subset of an Hadamard space X and $T_i : C \to C,\ i = 1, 2, \ldots, N$ be a finite family of k_i-strictly pseudononspreading mappings with $k_i \in [0, 1)$. Let $A_i : X \to 2^{X^*},\ i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfy the range condition and f be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := \cap_{i=1}^N F(T_i) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and W_n is defined by (6.5.1) such that W_n is Δ-demiclosed for each $n \geq 1$, where $\{a_n^{(i)}\}$, $\{b_n^{(i)}\}$ and $\{c_n^{(i)}\}$ are in $[\epsilon, 1-\epsilon]$ for some $\epsilon \in (0, 1)$, $k_i \leq \delta_n^{(i)} < 1$ for $i = 1, 2, \ldots, N$ and $n \geq 1$. Let $x_1 \in X$ be arbitrary and the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n f(x_n), \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n,\ n \geq 1, \end{cases} \tag{6.5.31}$$

where $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

C2: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

By letting the mapping W_n defined by (6.5.1), to be generated by a finite family of nonexpansive mappings, and by setting $f(x) = u$ for arbitrary but fixed $u \in X$ and for all $x \in X$, we obtain the following corollary (whose algorithm is of Halpern-type).

Corollary 6.5.6. *Let C be a nonempty closed and convex subset of an Hadamard space X and $T_i : C \to C,\ i = 1, 2, \ldots, N$ be a finite family of nonexpansive mappings. Let $A_i : X \to 2^{X^*},\ i = 1, 2, \ldots, N$ be multivalued monotone mappings that satisfy the range condition. Suppose that $\Gamma := \cap_{i=1}^N F(T_i) \cap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$, and for arbitrary $u, x_1 \in X$, let the sequence $\{x_n\}$ be defined by*

$$\begin{cases} y_n = (1-\alpha_n)x_n \oplus \alpha_n u, \\ z_n = (1-\gamma_n)y_n \oplus \gamma_n W_n(P_C(J_\lambda^N \circ J_\lambda^{N-1} \circ \cdots \circ J_\lambda^2 \circ J_\lambda^1 y_n)), \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n z_n,\ n \geq 1, \end{cases} \tag{6.5.32}$$

where $\lambda > 0$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

C1: $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,

C2: $0 < a \leq \beta_n, \gamma_n \leq b < 1$.

Then, $\{x_n\}$ converges strongly to an element of Γ.

6.5.2 Numerical example

Let $X = \mathbb{R}$ be endowed with the usual metric and $C = [0, 100]$. Then,

$$P_C(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } x \in [0,\ 100], \\ 100, & \text{if } x > 100 \end{cases}$$

is a metric projection onto C. Let $N = 2$, then for $i = 1$, define T_1 by

$$T_1 x = \begin{cases} \frac{1}{x+\frac{1}{10}}, & \text{if } x \in [1, 100], \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Then, T_1 is an (f_1, g_1)-generalized k_1-strictly pseudononspreading mapping with $k_1 = 0$ and $f_1, g_1 : [0, 100] \to [0, \frac{9}{10}]$ defined by

$$f_1(x) = \begin{cases} 0, & \text{if } x \in [1, 100], \\ \frac{9}{10}, & \text{if } x \in [0, 1) \end{cases} \quad \text{and} \quad g_1(x) = \begin{cases} \frac{1}{(x+\frac{1}{10})^2}, & \text{if } x \in [1,\ 100], \\ 0, & \text{if } x \in [0, 1). \end{cases}$$

Also, for $i = 2$, we define T_2 by

$$T_2 x = \begin{cases} -3x, & \text{if } x \in [0, 1], \\ \frac{1}{x}, & \text{if } x \in (1,\ 100]. \end{cases}$$

Then, T_2 is an (f_2, g_2)-generalized k_2-strictly pseudononspreading mapping with $k_2 = \frac{9}{10}$ and $f_2, g_2 : [0,\ 100] \to [0,\ \frac{10}{11}]$ defined by

$$f_2(x) = \begin{cases} \frac{10}{11}, & \text{if } x \in [0, 1], \\ \frac{1}{11}, & \text{if } x \in (1, 100] \end{cases} \quad \text{and} \quad g_2(x) = \begin{cases} \frac{1}{11}, & \text{if } x \in [0, 1], \\ \frac{10}{11}, & \text{if } x \in (1, 100]. \end{cases}$$

Clearly, $\cap_{i=1}^{2}F(T_i)=\{0\}$. Now, observe that $\frac{k_2}{f_2(0)}=\frac{99}{100}\leq\frac{99n+\frac{1}{2}}{100n}<1$. Thus, we can choose $\delta_n^{(2)}=\frac{99n+\frac{1}{2}}{100n}$ and $\delta_n^{(1)}=0$ for all $n\geq 1$. Also, let $a_n^{(i)}=\frac{in}{3in+3}$, $b_n^{(i)}=\frac{in+1}{3in+3}$, $c_n^{(i)}=\frac{in+2}{3in+3}$ for all $n\geq 1,\ i=1,2$. Then, the mapping W_n defined by (6.5.1) becomes:

$$\begin{cases} U_n^{(0)}x=x,\\ U_n^{(1)}x=\frac{n}{3n+3}T_1x+\frac{n+1}{3n+3}x+\frac{n+2}{3n+3}x,\\ W_nx=U_n^{(2)}x=\frac{2n}{6n+3}\left[\frac{99n+\frac{1}{2}}{100n}U_n^{(1)}x+\left(1-\frac{99n+\frac{1}{2}}{100n}\right)T_2U_n^{(1)}x\right]+\frac{2n+1}{6n+3}U_n^{(1)}x+\frac{2n+1}{6n+3}x,\ n\geq 1\end{cases}\tag{6.5.33}$$

and the mapping W_n of Takahashi and Shimoji [177, Algorithm (1)] becomes:

$$\begin{cases} U_n^{(0)}x=x,\\ U_n^{(1)}x=\frac{n}{3n+3}T_1x+\left(1-\frac{n}{3n+3}\right)x,\\ W_nx=U_n^{(2)}x=\frac{2n}{6n+3}T_2U_n^{(1)}x+\left(1-\frac{2n}{6n+3}\right)x,\ n\geq 1.\end{cases}\tag{6.5.34}$$

Let $A_i:\mathbb{R}\to\mathbb{R}$ be defined by $A_i(x)=2ix$. Then A_i is monotone for each $i=1,2$. Now, recall that $[t\overrightarrow{ab}]\equiv t(b-a)$, for all $t\in\mathbb{R}$ and $a,b\in\mathbb{R}$ (see [153]). Thus, $J_\lambda^i(x)=(I+\lambda A_i)^{-1}x$. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(x)=\frac{3}{4}x$. Take $\alpha_n=\frac{1}{n+1}$, $\beta_n=\frac{3n}{5n+1}$ and $\gamma_n=\frac{n+1}{4n+2}$ for all $n\geq 1$. Then, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfy conditions C1-C2 in Theorem 6.5.3. Hence, Algorithm (6.5.9) becomes:

$$\begin{cases} y_n=\frac{n}{n+1}x_n+\frac{3}{4(n+1)}x_n,\\ z_n=\frac{3n+1}{4n+2}y_n+\frac{n+1}{4n+2}W_n(P_C(J_\lambda^2(J_\lambda^1y_n))),\\ x_{n+1}=\frac{2n+1}{5n+1}y_n+\frac{3n}{5n+1}z_n,\ n\geq 1\end{cases}\tag{6.5.35}$$

and Algorithm (3.4.1) of Ugwunnadi *et. al* [182] becomes:

$$\begin{cases} u,x_1\in X,\\ y_n=J_\lambda^2(J_\lambda^1(x_n)),\\ x_{n+1}=\frac{1}{n+1}u+(1-\frac{1}{n+1})W_ny_n,\ n\geq 1,\end{cases}\tag{6.5.36}$$

where W_n is defined by (6.5.33).

Case I Take $x_1=0.5$ and $\lambda=0.5$.

Case II Take $x_1=0.5$ and $\lambda=2$.

Case III Take $x_1=1$ and $\lambda=2$

Using Algorithm (6.5.35), we compared the W_n mapping defined by (6.5.33) and the W_n mapping defined by (6.5.34) as shown in Figure 6.2 for Cases 1, 2 and 3. Also, we compared Algorithm (6.5.35) with Algorithm (6.5.36) using the same W_n mapping (defined by (6.5.33)) as shown in Figure 6.3 for Cases 1, 2 and 3. The graphs below show that our algorithm converges faster than that proposed by Ugwunnadi *et. al.* [182], Takahashi and Shimoji [177].

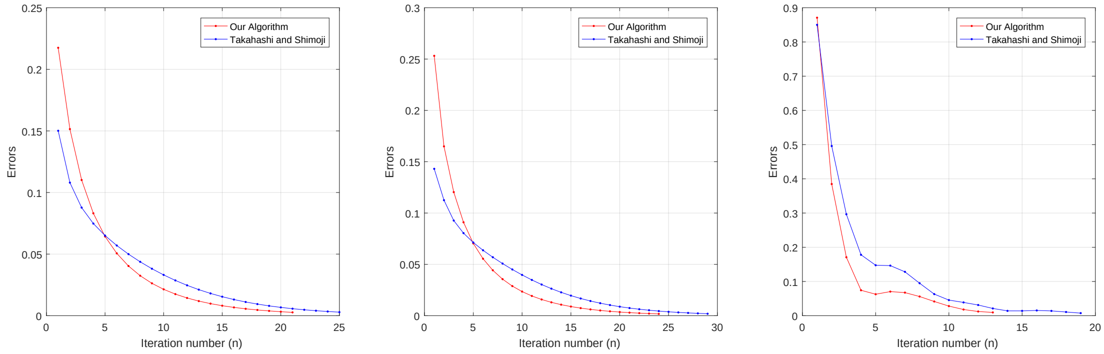

Figure 6.2: Errors vs Iteration number (n): Case I (left); Case 2 (middle); Case 3 (right).

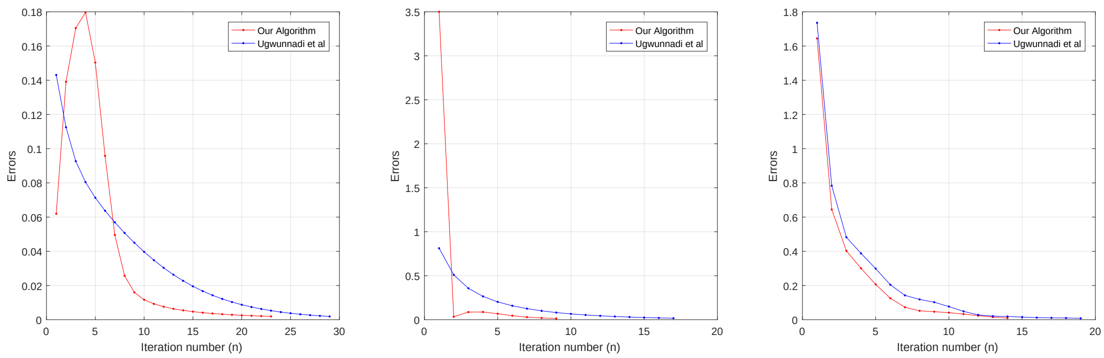

Figure 6.3: Errors vs Iteration number (n): Case I (left); Case 2 (middle); Case 3 (right).

Chapter 7

Contributions to Minimization Problems in p-uniformly Convex Metric Spaces

7.1 Introduction

In an attempt to generalize the study of optimization and fixed point problems from linear spaces (mainly Hilbert spaces) to nonlinear spaces, we discussed in the previous chapters (Chapter 3-6), our contributions to the study of these problems in Hadamard spaces. In this chapter, we shall further generalize the study of these problems to more general nonlinear spaces. In particular, we shall study MPs and fixed point problems in p-uniformly convex metric spaces which are natural generalizations of p-uniformly convex Banach space. Moreover, as mentioned in Remark 2.2.5 and Section 2.2.4, very few results on MPs and fixed point problems exists in p-uniformly convex metric spaces. Thus, it is very necessary to further develop the study of these problems in these spaces.

7.2 Preliminaries

In this section, we introduce and prove some new important results in p-uniformly convex metric spaces that are useful in establishing our theorems in this chapter. We begin with the following important definition.

Definition 7.2.1. *Let X and Y be two complete p-uniformly convex metric spaces. Then the Cartesian product $X \times Y$ is a complete p-uniformly convex metric space endowed with the metric $d : (X \times Y) \times (X \times Y) \to [0, \infty)$ defined by*

$$d((x_1, y_1), (x_2, y_2)) = [d_X(x_1, x_2)^p + d_Y(y_1, y_2)^p]^{\frac{1}{p}}, \ \forall x_1, x_2 \in X, \ y_1, y_2 \in Y. \quad (7.2.1)$$

Lemma 7.2.2. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function.*

Then, for all $a, b, c, d \in X$, we have

$$d(a,b)^p + d(c,d)^p \leq \frac{2}{c}\left(d(a,c)^p + d(a,d)^p + d(b,c)^p + d(b,d)^p\right).$$

Proof. From (2.1.10), we obtain that

$$\begin{aligned} 0 &\leq d\left(\frac{1}{2}a \oplus \frac{1}{2}b, \frac{1}{2}c \oplus \frac{1}{2}d\right)^p \\ &\leq \frac{1}{4}\left[d(a,c)^p + d(a,d)^p + d(b,c)^p + d(b,d)^p - \frac{c}{2}\left(d(c,d)^p + d(a,b)^p\right)\right], \end{aligned}$$

which implies

$$d(a,b)^p + d(c,d)^p \leq \frac{2}{c}\left(d(a,c)^p + d(a,d)^p + d(b,c)^p + d(b,d)^p\right).$$

□

7.2.1 Unique existence of resolvent of convex functions

Here, we discuss the unique existence of the resolvent of a proper convex and semicontinuous function defined in (2.2.8).

Proposition 7.2.3. *[173] Let X be a geodesic space and $f : X \to (-\infty, +\infty]$ be a proper uniformly convex and lower semicontinuous function. Then, there exists a unique minimizer $\bar{v} \in X$ of f (that is $\bar{v} := \underset{v \in X}{\operatorname{argmin}} f(v)$).*

Proposition 7.2.4. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for any $\mu > 0$ and $x \in X$, there exists a unique point, say $J_\mu^f(x) \in X$ (see (2.2.8)) such that*

$$f(J_\mu^f(x)) + \frac{1}{p\mu^{p-1}} d(J_\mu^f(x), x)^p = \inf_{v \in X}\left(f(v) + \frac{1}{p\mu^{p-1}} d(v,x)^p\right). \tag{7.2.2}$$

Proposition 7.2.4 which is also known as the unique existence of resolvent of a proper convex and lower semicontinuous function, is proved in [110, Proposition 3.26] under the following assumption:

Assumption (see [110, Assumption 3.21]): Fix $c > 0$, then for $\lambda \in \mathbb{R}$, $q := \frac{p}{p-1}$, assume that for any $z \in X$, $x, y \in D(f)$, there exists a geodesic $\gamma : [0, l] \to X$ with $\gamma(0) = x$ and $\gamma(l) = y$ such that $t \longmapsto f(\gamma(t)) + \frac{1}{p\mu^{p-1}} d(z, \gamma(t))$, $t \in [0,1]$ is p-uniformly $\left(\frac{c}{p\mu^{p-1}} + \lambda\right)$-convex for each $\mu \in (0, (c/p\lambda^{-1})^{q-1})$,

$$\begin{aligned} f(\gamma(t)) + \frac{1}{p\mu^{p-1}} d(z,\gamma(t))^p \;&\leq\; (1-t)f(\gamma(0)) + tf(\gamma(l)) + \frac{1-t}{p\mu^{p-1}} d(z,\gamma(0))^p + \frac{t}{p\mu^{p-1}} d(z,\gamma(l))^p \\ &\quad - \frac{1}{2}\left(\frac{c}{p\mu^{p-1}} + \lambda\right) t(1-t) d(\gamma(0),\gamma(l))^p \end{aligned}$$

and the geodesic γ satisfies $d(x,\gamma(t)) \le td(x,y)\ \forall t \in [0,1]$.

Using Proposition 7.2.3, we prove Proposition 7.2.4 without this assumption. Furthermore, our method of proof is shorter and easier to read.

Proof. Let $G_\mu^f(v) := f(v) + \frac{1}{p\mu^{p-1}} d(v,x)^p$. Clearly, G_μ^f is a proper and lower semicontinuous mapping. Also, G_μ^f is uniformly convex. For this, let $v = tv_1 \oplus (1-t)v_2$ for all $v_1, v_2 \in X$ and $t \in [0,1]$ (in particular, $t = \frac{1}{2}$), we obtain from the convexity of f and (2.1.10) that

$$\begin{aligned} G_\mu^f(\frac{1}{2}v_1 \oplus \frac{1}{2}v_2) &\le \frac{1}{2}\left(f(v_1) + \frac{1}{p\mu^{p-1}} d(v_1,x)^p\right) \\ &\quad + \frac{1}{2}\left(f(v_2) + \frac{1}{p\mu^{p-1}} d(v_2,x)^p\right) - \frac{c}{8p\mu^{p-1}} d(v_1,v_2)^p \\ &= \frac{1}{2} G_\mu^f(v_1) + G_\mu^f(v_2) - \frac{c}{8p\mu^{p-1}} d(v_1,v_2)^p, \end{aligned}$$

which implies that G_μ^f is uniformly convex. Hence, by Proposition 7.2.3, we obtain the desired conclusion. □

7.2.2 Fundamental properties of resolvent of convex functions

We now obtain some basic properties of the resolvent of a proper convex and lower semicontinuous function.

Lemma 7.2.5 (Firmly nonexpansive-type property). *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for all $x_1, x_2 \in X$, we have*

$$d(J_\mu^f x_1, J_\mu^f x_2)^p \le \frac{1}{c}\left[d(J_\mu^f x_1, x_2) + d(J_\mu^f x_2, x_1)^p - d(J_\mu^f x_1, x_1)^p - d(J_\mu^f x_2, x_2)^p\right].$$

Proof. From (2.2.8) (or (7.2.2)), we obtain that

$$f(J_\mu^f x) + \frac{1}{p\mu^{p-1}} d(J_\mu^f x, x)^p \le f(z) + \frac{1}{p\mu^{p-1}} d(z,x)^p \quad \forall z \in X.$$

Now, set $z = (1-t)v \oplus tJ_\mu^f x,\ t \in [0,1)$. Then, we obtain from the convexity of f and the inequality (2.1.10) that

$$\begin{aligned} f(J_\mu^f x) + \frac{1}{p\mu^{p-1}} d(J_\mu^f x, x)^p &\le (1-t)f(v) + tf(J_\mu^f x) + \frac{(1-t)}{p\mu^{p-1}} d(v,x)^p \\ &\quad + \frac{t}{p\mu^{p-1}} d(J_\mu^f x, x)^p - \frac{ct(1-t)}{2p\mu^{p-1}} d(v, J_\mu^f x)^p, \end{aligned}$$

which implies (since $t \ne 1$) that

$$p\mu^{p-1} f(J_\mu^f x) + d(J_\mu^f x, x)^p \le p\mu^{p-1} f(v) + d(v,x)^p - \frac{ct}{2} d(v, J_\mu^f x)^p. \qquad (7.2.3)$$

As $t \to 1$ in (7.2.3), we obtain

$$p\mu^{p-1}f(J_\mu^f x) + d(J_\mu^f x, x)^p \leq p\mu^{p-1}f(v) + d(v,x)^p - \frac{c}{2}d(v, J_\mu^f x)^p. \tag{7.2.4}$$

Now, for $x_1, x_2 \in X$, we obtain from (7.2.4) that

$$\begin{aligned} & p\mu^{p-1}f(J_\mu^f x_1) + d(J_\mu^f x_1, x_1)^p \\ \leq\ & p\mu^{p-1}f(J_\mu^f x_2) + d(J_\mu^f x_2, x_1)^p - \frac{c}{2}d(J_\mu^f x_2, J_\mu^f x_1)^p \end{aligned} \tag{7.2.5}$$

and

$$\begin{aligned} & p\mu^{p-1}f(J_\mu^f x_2) + d(J_\mu^f x_2, x_2)^p \\ \leq\ & p\mu^{p-1}f(J_\mu^f x_1) + d(J_\mu^f x_1, x_2)^p - \frac{c}{2}d(J_\mu^f x_1, J_\mu^f x_2)^p. \end{aligned} \tag{7.2.6}$$

Adding (7.2.5) and (7.2.6), we obtain

$$d(J_\mu^f x_1, J_\mu^f x_2)^p \leq \frac{1}{c}\left[d(J_\mu^f x_1, x_2) + d(J_\mu^f x_2, x_1)^p - d(J_\mu^f x_1, x_1)^p - d(J_\mu^f x_2, x_2)^p\right].$$

□

Remark 7.2.6. *(a) Observe that if $c \geq 2$ and $p = 2$ in Lemma 7.2.5, then by the definition of quasilinearization mapping (see Definition 2.1.14), one obtains that J_μ^f is a firmly nonexpansive mapping in a CAT(0) space. That is,*

$$d(J_\mu^f x_1, J_\mu^f x_2)^2 \leq \langle \overrightarrow{J_\mu^f x_1 J_\mu^f x_2}, \overrightarrow{x_1 x_2}\rangle \ \forall x_1, x_2 \in X,$$

which by Cauchy-Schwartz inequality gives that J_μ^f is nonexpansive in a CAT(0) space.

(b) From (7.2.4), we obtain that

$$d(v, J_\mu^f x)^p \leq \frac{2}{c}\left[d(v,x)^p - d(J_\mu^f x, x)^p - p\mu^{p-1}\left(f(J_\mu^f x) - f(v)\right)\right], \ \forall v \in X.$$

(c) If we replace convexity of f with uniform convexity in Lemma 7.2.5, then (b) becomes

$$d(v, J_\mu^f x)^p \leq \frac{2}{c}\left[d(v,x)^p - d^p(J_\mu^f x, x) - p\mu^{p-1}\left(\psi(d(v, J_\mu^f x)) + f(J_\mu^f x) - f(v)\right)\right],$$

for all $v \in X$.

Lemma 7.2.7 (Nonexpansive property). *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, the resolvent J_μ^f of f is nonexpansive. That is, for all $x_1, x_2 \in X$, we have*

$$d(J_\mu^f x_1, J_\mu^f x_2) \leq d(x_1, x_2).$$

Proof. By Lemma 7.2.2 and Lemma 7.2.5 (note that $c \geq 2$), we obtain that

$$\begin{aligned} d(J_\mu^f x_1, J_\mu^f x_2)^p &\leq \frac{1}{c}\Big[\frac{2}{c}\Big(d(J_\mu^f x_1, J_\mu^f x_2)^p + d(J_\mu^f x_1, x_1) + d(J_\mu^f x_2, x_2)^p + d(x_1, x_2)^p\Big) \\ &\quad - d(J_\mu^f x_1, x_1)^p - d(J_\mu^f x_2, x_2)^p\Big] \\ &\leq \frac{1}{2}\left[d(J_\mu^f x_1, J_\mu^f x_2)^p + d(x_1, x_2)^p\right], \end{aligned}$$

which yields the desired conclusion. □

Lemma 7.2.8 (Monotonicity of resolvent)**.** *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for $0 < \mu_1 < \mu_2$, we have*

$$d(J_{\mu_1}^f x, x) \leq d(J_{\mu_2}^f x, x) \ \forall x \in X.$$

Proof. Let $x \in X$. We obtain from (2.2.8) (or (7.2.2)) that

$$f(J_{\mu_2}^f x) + \frac{1}{p\mu_2^{p-1}} d(J_{\mu_2}^f x, x)^p \leq f(J_{\mu_1}^f x) + \frac{1}{p\mu_2^{p-1}} d(J_{\mu_1}^f x, x)^p. \tag{7.2.7}$$

Similarly, we obtain

$$f(J_{\mu_1}^f x) + \frac{1}{p\mu_1^{p-1}} d(J_{\mu_1}^f x, x)^p \leq f(J_{\mu_2}^f x) + \frac{1}{p\mu_1^{p-1}} d(J_{\mu_2}^f x, x)^p. \tag{7.2.8}$$

Adding (7.2.7) and (7.2.8), we obtain that

$$\left(1 - \frac{\mu_1^{p-1}}{\mu_2^{p-1}}\right) d(J_{\mu_1}^f x, x)^p \leq \left(1 - \frac{\mu_1^{p-1}}{\mu_2^{p-1}}\right) d(J_{\mu_2}^f x, x)^p.$$

Since, $0 < \mu_1 < \mu_2$, therefore $1 - \left(\frac{\mu_1}{\mu_2}\right)^{p-1} > 0$. Thus, we obtain that

$$d(J_{\mu_1}^f x, x) \leq d(J_{\mu_2}^f x, x).$$

□

Lemma 7.2.9 (Monotonicity of resolvent)**.** *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper convex and lower semicontinuous function. Then, for $0 < \mu_1 \leq \mu_2$, we have*

$$d(J_{\mu_1}^f x, J_{\mu_2}^f x) \leq \left[\frac{2}{c}\left(1 - \frac{\mu_1^{p-1}}{\mu_2^{p-1}}\right)\right]^{\frac{1}{p}} d(x, J_{\mu_2}^f x) \ \forall x \in X.$$

Proof. From (2.2.8), we obtain that

$$f(J_{\mu_2}^f x) + \frac{1}{p{\mu_2}^{p-1}} d(J_{\mu_2}^f x, x)^p \leq f(v) + \frac{1}{p{\mu_2}^{p-1}} d(v, x)^p \quad \forall v \in X.$$

Let $v = (1-t)J^f_{\mu_1}x \oplus tJ^f_{\mu_2}x,\ t \in [0,1)$. Then, we obtain from the convexity of f and the inequality (2.1.10) that

$$\begin{aligned} f(J^f_{\mu_2}x) + \frac{1}{p{\mu_2}^{p-1}} d(J^f_{\mu_2}x, x)^p \ &\leq \ (1-t)f(J^f_{\mu_1}x) + tf(J^f_{\mu_2}x) + \frac{(1-t)}{p{\mu_2}^{p-1}} d(J^f_{\mu_1}x, x)^p \\ &\quad + \frac{t}{p{\mu_2}^{p-1}} d(J^f_{\mu_2}x, x)^p - \frac{ct(1-t)}{2p{\mu_2}^{p-1}} d(J^f_{\mu_1}x, J^f_{\mu_2}x)^p, \end{aligned}$$

which implies that

$$\begin{aligned} & f(J^f_{\mu_2}x) + \frac{1}{p{\mu_2}^{p-1}} d(J^f_{\mu_2}x, x)^p \\ \leq \ & f(J^f_{\mu_1}x) + \frac{1}{p{\mu_2}^{p-1}} d(J^f_{\mu_1}x, x)^p - \frac{ct}{2p{\mu_2}^{p-1}} d(J^f_{\mu_1}x, J^f_{\mu_2}x)^p. \end{aligned} \tag{7.2.9}$$

Letting $t \to 1$ in (7.2.9), we obtain that

$$\frac{c}{2p{\mu_2}^{p-1}} d(J^f_{\mu_1}x, J^f_{\mu_2}x)^p \leq f(J^f_{\mu_1}x) - f(J^f_{\mu_2}x) + \frac{1}{p{\mu_2}^{p-1}} \left[d(J^f_{\mu_1}x, x)^p - d(J^f_{\mu_2}x, x)^p\right] \tag{7.2.10}$$

Similarly, we obtain that

$$\frac{c}{2p{\mu_1}^{p-1}} d(J^f_{\mu_2}x, J^f_{\mu_1}x)^p \leq f(J^f_{\mu_2}x) - f(J^f_{\mu_1}x) + \frac{1}{p{\mu_1}^{p-1}} \left[d(J^f_{\mu_2}x, x)^p - d(J^f_{\mu_1}x, x)^p\right] \tag{7.2.11}$$

Adding (7.2.10) and (7.2.11), and noting that $\mu_1 \leq \mu_2$, we obtain that

$$\begin{aligned} \frac{c}{2p}\left(\frac{1}{{\mu_1}^{p-1}} + \frac{1}{{\mu_2}^{p-1}}\right) d(J^f_{\mu_1}x, J^f_{\mu_2}x)^p \ &\leq \ \frac{1}{p}\left(\frac{1}{{\mu_2}^{p-1}} - \frac{1}{{\mu_1}^{p-1}}\right) d(J^f_{\mu_1}x, x)^p \\ &\quad + \frac{1}{p}\left(\frac{1}{{\mu_1}^{p-1}} - \frac{1}{{\mu_2}^{p-1}}\right) d(J^f_{\mu_2}x, x)^p \\ &\leq \ \frac{1}{p}\left(\frac{1}{{\mu_1}^{p-1}} - \frac{1}{{\mu_2}^{p-1}}\right) d(J^f_{\mu_2}x, x)^p, \end{aligned}$$

which after further simplification implies that

$$d(J^f_{\mu_1}x, J^f_{\mu_2}x) \leq \left[\frac{2}{c}\left(1 - \frac{\mu_1^{p-1}}{\mu_2^{p-1}}\right)\right]^{\frac{1}{p}} d(x, J^f_{\mu_2}x).$$

□

Lemma 7.2.10. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f : X \to (-\infty, +\infty]$ be a proper, convex and lower semicontinuous function. For $\mu_1, \mu_2 > 0$ and $x_1, x_2 \in X$, the following inequality holds:*

$$\begin{aligned} & \frac{c}{2}(\mu_1^{p-1} + \mu_2^{p-1}) d(J^f_{\mu_1}x_1, J^f_{\mu_2}x_2)^p + \mu_2^{p-1} d(J^f_{\mu_1}x_1, x_1)^p + \mu_1^{p-1} d(J^f_{\mu_2}x_2, x_2)^p \\ & \leq \mu_1^{p-1} d(J^f_{\mu_1}x_1, x_2)^p + \mu_2^{p-1} d(J^f_{\mu_2}x_2, x_1)^p. \end{aligned}$$

Proof. Put $x = x_1$ and $v = J^f_{\mu_2}x_2$ in (7.2.4) to obtain

$$p{\mu_1}^{p-1}f(J^f_{\mu_1}x_1) + d(J^f_{\mu_1}x_1, x_1)^p \quad \leq \quad p{\mu_1}^{p-1}f(J^f_{\mu_2}x_2) + d(J^f_{\mu_2}x_2, x_1)^p - \frac{c}{2}d(J_{\mu_2}x_2, J^f_{\mu_1}x_1)^p.$$

That is,

$$\frac{c}{2}d(J^f_{\mu_1}x_1, J^f_{\mu_2}x_2)^p + d(J^f_{\mu_1}x_1, x_1)^p + p\mu_1^{p-1}\left(f(J^f_{\mu_1}x_1) - f(J^f_{\mu_2}x_2)\right) \leq d(J^f_{\mu_2}x_2, x_1)^p,$$

from which we obtain that

$$\begin{aligned} &p\mu_2^{p-1}\left[\frac{c}{2}d(J^f_{\mu_1}x_1, J^f_{\mu_2}x_2)^p + d(J^f_{\mu_1}x_1, x_1)^p + p\mu_1^{p-1}\left(f(J^f_{\mu_1}x_1) - f(J^f_{\mu_2}x_2)\right)\right] \\ &\leq p\mu_2^{p-1}d(J^f_{\mu_2}x_2, x_1)^p. \end{aligned} \tag{7.2.12}$$

Similarly, we obtain

$$\begin{aligned} &p\mu_1^{p-1}\left[\frac{c}{2}d(J^f_{\mu_2}x_2, J^f_{\mu_1}x_1)^p + d(J^f_{\mu_2}x_2, x_2)^p + p\mu_2^{p-1}\left(f(J^f_{\mu_2}x_2) - f(J^f_{\mu_1}x_1)\right)\right] \\ &\leq p\mu_1^{p-1}d(J^f_{\mu_1}x_1, x_2)^p. \end{aligned} \tag{7.2.13}$$

Adding (7.2.12) and (7.2.13), we obtain the desired conclusion. □

Lemma 7.2.11. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, +\infty]$ be proper, convex and lower semicontinuous function such that for $\mu > 0$ $F(J^f_\mu) \neq \emptyset$ (where $F(J^f_\mu)$ denotes the set of fixed points of J^f_μ). Then, $F(J^f_\mu) = \underset{y\in X}{\operatorname{argmin}} f(y)$.*

Proof. Let $\bar{v} \in F(J^f_\mu)$. Then, by (2.2.8), we obtain that

$$f(\bar{v}) \leq f(v) + \frac{1}{p\mu^{p-1}}d(v, \bar{v})^p.$$

Let $v = (1-t)y \oplus t\bar{v}$ for all $y \in X$ and $t \in [0, 1)$. Then, by the convexity of f and (2.1.10), we obtain that

$$(1-t)f(\bar{v}) \leq (1-t)f(y) + \frac{(1-t)}{p\mu^{p-1}}d(y, \bar{v})^p + \frac{t}{p\mu^{p-1}}d(\bar{v}, \bar{v})^p - \frac{ct(1-t)}{2p\mu^{p-1}}d(y, \bar{v})^p.$$

Since $c \geq 2$, therefore we obtain that

$$\frac{t(1-t)}{p\mu^{p-1}}d(y, \bar{v})^p \leq (1-t)\left(f(y) - f(\bar{v})\right) + \frac{(1-t)}{p\mu^{p-1}}d(y, \bar{v})^p,$$

which implies that

$$td(y, \bar{v})^p \leq p\mu^{p-1}\left(f(y) - f(\bar{v})\right) + d(y, \bar{v})^p.$$

As $t \to 1$, we obtain that

$$0 \leq f(y) - f(\bar{v}) \ \forall y \in X.$$

Hence, $\bar{v} \in \underset{y\in X}{\operatorname{argmin}} f(y)$.

Conversely, suppose that $\bar{v} \in \operatorname{argmin}_{y \in X} f(y)$. Then, we obtain by (2.2.8) that

$$f(J_\mu^f \bar{v}) + \frac{1}{p\mu^{p-1}} d(J_\mu^f \bar{v}, \bar{v})^p \leq f(v) + \frac{1}{p\mu^{p-1}} d(v, \bar{v})^p.$$

Let $v = (1-t)\bar{v} \oplus t J_\mu^f \bar{v}$, for $t \in [0,1)$. Then, we obtain by the convexity of f and (2.1.10) that

$$\begin{aligned} \frac{1}{p\mu^{p-1}} d(J_\mu^f \bar{v}, \bar{v})^p &\leq (1-t) f(\bar{v}) - (1-t) f(J_\mu^f \bar{v}) + \frac{1}{p\mu^{p-1}} d((1-t)\bar{v} \oplus t J_\mu^f \bar{v}, \bar{v})^p \\ &\leq \frac{(1-t)}{p\mu^{p-1}} d(\bar{v}, \bar{v})^p + \frac{t}{p\mu^{p-1}} d(J_\mu^f \bar{v}, \bar{v})^p - \frac{ct(1-t)}{2p\mu^{p-1}} d(J_\mu^f \bar{v}, \bar{v})^p, \end{aligned}$$

which implies that

$$\left(1 + \frac{ct(1-t)}{2} - t\right) d(J_\mu^f \bar{v}, \bar{v})^p \leq 0.$$

Since $t \neq 1$, we obtain that $\bar{v} \in F(J_\mu^f)$. Hence, $F(J_\mu^f) = \operatorname{argmin}_{y \in X} f(y)$. $\square$

Lemma 7.2.12. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, +\infty]$ a proper, convex and lower semicontinuous function. Then, for $\mu > 0$, we have the following:*

(i) $d(x^*, J_\mu^f x)^p + d(J_\mu^f x, x)^p \leq d(x^*, x)^p$ for all $x \in X$ and $x^* \in F(J_\mu^f)$;

(ii) $F\left(\prod_{j=1}^m J_\mu^{(j)}\right) = \cap_{j=1}^m F\left(J_\mu^{(j)}\right)$, where $\prod_{j=1}^m J_\mu^{(j)} = J_\mu^{f_1} \circ J_\mu^{f_2} \circ \cdots \circ J_\mu^{f_{m-1}} \circ J_\mu^{f_m}$.

Proof. (i) Let $x \in X$ and $x^* \in F(J_\mu^f)$. Then by setting $v = x^*$ in (7.2.4), we obtain that

$$\frac{c}{2} d(J_\mu^f x, x^*)^p \leq p\mu^{p-1} \left(f(x^*) - f(J_\mu^f x)\right) + d(x^*, x)^p - d(J_\mu^f x, x)^p.$$

Since $x^* \in F(J_\mu^f)$, therefore by Lemma 7.2.11 we obtain that $f(x^*) \leq f(J_\mu^f x)$. Hence, we obtain that

$$d(x^*, J_\mu^f x)^p + d(J_\mu^f x, x)^p \leq d(x^*, x)^p.$$

(ii) Clearly, $\cap_{j=1}^m F\left(J_\mu^{(j)}\right) \subseteq F\left(\prod_{j=1}^m J_\mu^{(j)}\right)$. Thus, we only have to show that $F\left(\prod_{j=1}^m J_\mu^{(j)}\right) \subseteq \cap_{j=1}^m F\left(J_\mu^{(j)}\right)$. For this, let $x \in F\left(\prod_{j=1}^m J_\mu^{(j)}\right)$ and $y \in \cap_{j=1}^m F\left(J_\mu^{(j)}\right)$, we obtain by Lemma 7.2.7 that

$$\begin{aligned} d(x,y)^p &= d\left(\prod_{j=1}^m J_\mu^{(j)} x, \prod_{j=1}^m J_\mu^{(j)} y\right)^p \\ &\leq d\left(\prod_{j=2}^m J_\mu^{(j)} x, y\right)^p. \end{aligned} \tag{7.2.14}$$

Furthermore, we obtain by (i), Lemma 7.2.7 and (7.2.14) that

$$\begin{aligned} d\left(\prod_{j=2}^{m} J_{\mu}^{(j)}x, \prod_{j=1}^{m} J_{\mu}^{(j)}x\right)^{p} &\leq d\left(\prod_{j=2}^{m} J_{\mu}^{(j)}x, y\right)^{p} - d\left(\prod_{j=1}^{m} J_{\mu}^{(j)}x, y\right)^{p} \\ &\vdots \\ &\leq d(x,y)^{p} - d\left(\prod_{j=1}^{m} J_{\mu}^{(j)}x - y\right)^{p} \\ &= d\left(\prod_{j=1}^{m} J_{\mu}^{(j)}x, y\right)^{p} - d\left(\prod_{j=1}^{m} J_{\mu}^{(j)}x - y\right)^{p}, \end{aligned}$$

which implies

$$\prod_{j=1}^{m} J_{\mu}^{(j)}x = \prod_{j=2}^{m} J_{\mu}^{(j)}x. \tag{7.2.15}$$

Similarly, we obtain that

$$\begin{aligned} d\left(\prod_{j=3}^{m} J_{\mu}^{(j)}x, \prod_{j=2}^{m} J_{\mu}^{(j)}x\right)^{p} &\leq d\left(\prod_{j=3}^{m} J_{\mu}^{(j)}x, y\right)^{p} - d\left(\prod_{j=2}^{m} J_{\mu}^{(j)}x, y\right)^{p} \\ &\vdots \\ &\leq d(x,y)^{p} - d\left(\prod_{j=2}^{m} J_{\mu}^{(j)}x, y\right)^{p} \\ &\leq d\left(\prod_{j=1}^{m} J_{\mu}^{(j)}x, y\right)^{p} - d\left(\prod_{j=1}^{m} J_{\mu}^{(j)}x - y\right)^{p}, \end{aligned}$$

which implies

$$\prod_{j=2}^{m} J_{\mu}^{(j)}x = \prod_{j=3}^{m} J_{\mu}^{(j)}x. \tag{7.2.16}$$

Continuing in this manner, we can show that

$$\prod_{j=3}^{m} J_{\mu}^{(j)}x = \prod_{j=4}^{m} J_{\mu}^{(j)}x = \cdots = \prod_{j=m-1}^{m} J_{\mu}^{(j)}x = J_{\mu}^{(m)}x = x. \tag{7.2.17}$$

From (7.2.17), we have

$$x = J_{\mu}^{f_m}x. \tag{7.2.18}$$

From (7.2.17) and (7.2.18), we obtain

$$x = \prod_{j=m-1}^{m} J_{\mu}^{(j)}x = J_{\mu}^{f_{m-1}} J_{\mu}^{f_m}x = J_{\mu}^{f_{m-1}}x. \tag{7.2.19}$$

Continuing in this manner, we obtain from (7.2.15)-(7.2.19) that

$$x = J_{\mu}^{f_{m-2}}x = \cdots = J_{\mu}^{f_2}x = J_{\mu}^{f_1}x. \tag{7.2.20}$$

That is,

$$J_{\mu}^{f_1}x = J_{\mu}^{f_2}x = \cdots = J_{\mu}^{f_{m-1}}x = J_{\mu}^{f_m}x = x. \tag{7.2.21}$$

Hence, we obtain the desired conclusion. □

7.3 Mann-type algorithm for minimization and fixed point problems in p-uniformly convex metric spaces

In this section, we propose a modified Mann-type PPA involving nonexpansive mapping and prove that the sequence generated by the algorithm Δ-converges to a common solution of a finite family of minimization problem which is also a fixed point of a nonexpansive mapping in the frame work of a complete p-uniformly convex metric space.

7.3.1 Main results

Lemma 7.3.1. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and let $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} arg \min\limits_{y \in X} f_i(y)\right) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J_{\lambda_n^{(N)}} \circ J_{\lambda_n^{(N-1)}} \circ \cdots \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n)Ty_n, \ n \geq 1, \end{cases} \tag{7.3.1}$$

where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \ldots, N$, $n \geq 1$ and $\{\alpha_n\}$ is a sequence in $[a, b]$, for some $a, b \in (0, 1)$.
Then,
(a) $\lim\limits_{n\to\infty} d^p(x_n, z)$ exists for all $z \in \Gamma$;
(b) $\lim\limits_{n\to\infty} d(Tx_n, x_n) = 0 = \lim\limits_{n\to\infty} d(J_{\lambda^{(i)}} w_n^{(i)}, w_n^{(i)})$, where $w_n^{(i+1)} = J_{\lambda_n^{(i)}} w_n^{(i)}$ and $w_n^{(1)} = x_n$ for each $i = 1, 2, \ldots, N$ and $n \geq 1$.

Proof. Let $z \in \Gamma$. Since

$$w_n^{(i+1)} = J_{\lambda_n^{(i)}} w_n^{(i)}, \text{ for each } i = 1, 2, \ldots, N,$$

where $w_n^{(1)} = x_n$, for all $n \geq 1$. Then,

$w_n^{(2)} = J_{\lambda_n^{(1)}}(x_n),\ \ w_n^{(3)} = J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(x_n),\ \ \ldots,\ \ w_n^{(N)} = J_{\lambda_n^{(N-1)}} \circ J_{\lambda_n^{(2)}} \circ J_{\lambda_n^{(1)}}(x_n),\ \ w_n^{(N+1)} = y_n$.

Thus, we obtain from (2.1.10) and Lemma 7.2.7 and Lemma 7.2.11 that

$$\begin{aligned} d^p(x_{n+1},z) &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(Ty_n,z) - \frac{c}{2}\alpha_n(1-\alpha_n)d^p(x_n,Ty_n) \\ &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(w_n^{(N+1)},z) - \frac{c}{2}\alpha_n(1-\alpha_n)d^p(x_n,Ty_n) \quad (7.3.2) \\ &= \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(J_{\lambda_n^{(N)}}w_n^{(N)},z) - \frac{c}{2}\alpha_n(1-\alpha_n)d^p(x_n,Ty_n) \\ &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(w_n^{(N)},z) - \frac{c}{2}\alpha_n(1-\alpha_n)d^p(x_n,Ty_n) \\ &\ \ \vdots \\ &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(w_n^{(1)},z) - \frac{c}{2}\alpha_n(1-\alpha_n)d^p(x_n,Ty_n) \quad (7.3.3) \\ &\leq d^p(x_n,z), \end{aligned}$$

which implies that $\lim\limits_{n\to\infty} d^p(x_n,z)$ exists for all $z \in \Gamma$.

Thus, $\{x_n\}$ is bounded and hence, $\{y_n\}$ and $\{Ty_n\}$ are also bounded. It then follows from (7.3.3) that

$$\frac{c}{2}\alpha_n(1-\alpha_n)d^p(x_n,Ty_n) \to 0, \text{ as } n \to \infty.$$

By the condition on α_n, we obtain that

$$\lim_{n\to\infty} d^p(x_n,Ty_n) = 0. \qquad (7.3.4)$$

By Lemma 7.2.12, we obtain for each $i = 1,2,\ldots,N$ that

$$d^p(w_n^{(i+1)},z) \leq d^p(w_n^{(i)},z) - d^p(w_n^{(i)},w_n^{(i+1)}). \qquad (7.3.5)$$

By setting $i = N$ in (7.3.5), we obtain from (7.3.2) that

$$\begin{aligned} d^p(x_{n+1},z) &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(w_n^{(N+1)},z) \\ &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(w_n^{(N)},z) - (1-\alpha_n)d^p(w_n^{(N)},w_n^{(N+1)}) \\ &\ \ \vdots \\ &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(w_n^{(1)},z) - (1-\alpha_n)d^p(w_n^{(N)},w_n^{(N+1)}) \\ &\leq \alpha_n d^p(x_n,z) + (1-\alpha_n)d^p(x_n,z) - (1-\alpha_n)d^p(w_n^{(N)},w_n^{(N+1)}) \\ &= d^p(x_n,z) - (1-\alpha_n)d^p(w_n^{(N)},w_n^{(N+1)}), \end{aligned}$$

which implies from Lemma 7.3.1 (a) that

$$(1-\alpha_n)d^p(w_n^{(N)},w_n^{(N+1)}) \to 0 \text{ as } n \to \infty.$$

By the condition on α_n, we obtain that

$$\lim_{n\to\infty} d^p(w_n^{(N)},w_n^{(N+1)}) = 0. \qquad (7.3.6)$$

Similarly, we obtain for $i = N-1$, (7.3.2) and (7.3.5) that

$$\begin{aligned}
d^p(x_{n+1}, z) &\leq \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(w_n^{(N)}, z) \\
&\leq \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(w_n^{(N-1)}, z) - (1-\alpha_n) d^p(w_n^{(N-1)}, w_n^{(N)}) \\
&\vdots \\
&\leq \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(w_n^{(1)}, z) - (1-\alpha_n) d^p(w_n^{(N-1)}, w_n^{(N)}) \\
&= \alpha_n d^p(x_n, z) + (1-\alpha_n) d^p(x_n, z) - (1-\alpha_n) d^p(w_n^{(N-1)}, w_n^{(N)}) \\
&= d^p(x_n, z) - (1-\alpha_n) d^p(w_n^{(N-1)}, w_n^{(N)}),
\end{aligned}$$

which implies from Lemma 7.3.1 (a) and the condition on α_n that

$$\lim_{n\to\infty} d^p(w_n^{(N-1)}, w_n^{(N)}) = 0. \tag{7.3.7}$$

Continuing in this manner, we can show that

$$\lim_{n\to\infty} d(w_n^{(i)}, w_n^{(i+1)}) = 0, \ i = 1, 2, \ldots, N-2. \tag{7.3.8}$$

This, together with (7.3.6) and (7.3.7), gives

$$\lim_{n\to\infty} d(w_n^{(i)}, w_n^{(i+1)}) = 0, \ i = 1, 2, \ldots, N. \tag{7.3.9}$$

From (7.3.9), and applying triangle inequality, we obtain for each $i = 1, 2, \ldots, N+1$, that

$$\lim_{n\to\infty} d(x_n, w_n^{(i)}) = \lim_{n\to\infty} d(w_n^{(1)}, w_n^{(i)}) = 0. \tag{7.3.10}$$

Since $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ for all $n \geq 1$, we obtain from Lemma 7.2.8 and (7.3.9) that

$$d\left(w_n^{(i)}, J_{\lambda^{(i)}} w_n^{(i)}\right) \leq d\left(w_n^{(i)}, J_{\lambda_n^{(i)}} w_n^{(i)}\right) \to 0, \text{ as } n \to \infty, \ i = 1, 2, \ldots, N. \tag{7.3.11}$$

Moreover, we obtain from (7.3.4) and (7.3.10) that

$$\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, Ty_n) + d(Ty_n, Tx_n) \\
&\leq d(x_n, Ty_n) + d(y_n, x_n) \to 0 \text{ as } n \to \infty. \qquad (7.3.12)
\end{aligned}$$

□

Theorem 7.3.2. *For $1 < p < \infty$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and let $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y\in X} f_i(y)\right) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by (7.3.1), where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \ldots, N$, $n \geq 1$ and $\{\alpha_n\}$ is a sequence in $[a, b]$, for some $a, b \in (0, 1)$. Then, $\{x_n\}$ Δ-converges to some $x^* \in \Gamma$.*

Proof. Since $\{x_n\}$ is bounded and X is a complete p-uniformly convex metric space, then by Lemma 2.3.24 (i), $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{x^*\}$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{u\}$. Then, by (7.3.12), we have that $\lim_{n\to\infty} d(x_{n_k}, Tx_{n_k}) = 0$. Thus, by Lemma 2.3.24 (ii) and Lemma 2.3.25, we obtain that $u \in F(T)$. Since $J_{\lambda^{(i)}}$ is nonexpansive for each $i = 1, 2, \dots, N$, it follows from Lemma 2.3.24 (ii), (7.3.10) and (7.3.11) that $u \in \cap_{i=1}^{N} F(J_{\lambda^{(i)}})$. Hence from Lemma 7.2.11, we have that $u \in \Gamma$.

Furthermore, from Lemma 7.3.1 (a), we obtain that $\lim_{n\to\infty} d(x_n, u)$ exists. Thus, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, u) &\leq \limsup_{k\to\infty} d(x_{n_k}, x^*) \\
&\leq \limsup_{n\to\infty} d(x_n, x^*) \\
&\leq \limsup_{n\to\infty} d(x_n, u) \\
&= \lim_{n\to\infty} d(x_n, u) \\
&= \limsup_{k\to\infty} d(x_{n_k}, u),
\end{aligned}$$

which implies that $x^* = u$. Therefore, $\{x_n\}$ Δ-converges to $x^* \in \Gamma$. $\square$

Recall that Hadamard spaces are p-uniformly convex metric spaces with $p = 2$ and parameter $c = 2$. Also, if $p = 2$, the p-resolvent reduces to the Moreau-Yosida resolvent mapping in Hadamard spaces. Therefore, if we let $p = 2$ and $c = 2$ in Theorem 7.3.2, we obtain the following result.

Corollary 7.3.3. *Let X be an Hadamard space and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \dots, N$ be a finite family of proper convex and lower semi continuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^{N} \arg\min_{y\in X} f_i(y)\right) \neq \emptyset$. For arbitrary $x_1 \in X$, let the sequence $\{x_n\}$ be generated by (7.3.1), where $\{\lambda_n^{(i)}\}$, $i = 1, 2, \dots, N$ is a sequence such that $\lambda_n^{(i)} > \lambda^{(i)} > 0$ for each $i = 1, 2, \dots, N$, $n \geq 1$ and $\{\alpha_n\}$ is a sequence in $[a,\ b]$, for some $a, b \in (0, 1)$. Then, $\{x_n\}$ Δ-converges to some $x^* \in \Gamma$.*

Remark 7.3.4. *In general, existing results on PPA in Hadamard spaces (that is, the case where $p = c = 2$) cannot be simply carried into p-uniformly convex metric spaces due to the structure of the space. For example, the smoothness constant (parameter) $c \in (0, \infty)$ is a natural obstacle one has to overcome in order to extend such results to p-uniformly convex metric spaces. The results of this section are established under the assumption that $c \in [2, \infty)$. However, we do not know whether these results still hold if we consider $c \in (0, 2)$.*

7.4 Halpern-type algorithm for minimization and fixed point problems in p-uniformly convex metric spaces

In the previous section, we obtained a Δ-convergence result using the modified Mann iteration process. However, in infinite dimensional spaces, it is more desirable to study strong convergence results. Therefore, it is our interest in this section, to study Halpern-type algorithms for the purpose of establishing strong convergence results for minimization and fixed point problems in p-uniformly convex metric spaces. We shall also discuss some numerical experiments in p-uniformly convex metric spaces to show the applicability of our results in this space.

7.4.1 Main results

Lemma 7.4.1. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $S : X \to X$ be a nonexpansive mapping. Let $u \in X$ be fixed, then for each $t \in (0,1)$, the mapping $f_t : X \to X$ defined by*

$$f_t x = tu \oplus (1-t)Sx \quad \forall x \in X, \tag{7.4.1}$$

has a unique fixed point $x_t \in X$. That is,

$$x_t = f_t x_t = tu \oplus (1-t)Sx_t. \tag{7.4.2}$$

Proof. From (7.4.1) and (2.1.10), we obtain

$$\begin{aligned} d(f_tx, f_ty)^p &\leq td(f_tx,u)^p + (1-t)d(f_tx,Sy)^p - \frac{c}{2}t(1-t)d(u,Sy)^p \\ &\leq t^2 d(u,u)^p + t(1-t)d(Sx,u)^p - \frac{c}{2}t^2(1-t)d(u,Sx)^p + t(1-t)d(u,Sy)^p \\ &\quad +(1-t)^2 d(Sx,Sy)^p - \frac{c}{2}t(1-t)^2 d(u,Sx)^p - \frac{c}{2}t(1-t)d(u,Sy)^p \\ &\leq (1-t)^2 d(x,y)^p, \end{aligned}$$

Thus, f_t is a contraction with constant $(1-t)^{\frac{2}{p}}$, and by Banach contraction mapping principle, we obtain the desired conclusion. □

Definition 7.4.2. *[161, 169] A continuous linear functional μ defined on l_∞ (where l_∞ is the Banach space of bounded real sequences) is called a Banach limit, if*

$$||\mu|| = \mu(1,1,\dots) = 1 \quad and \quad \mu_n(a_n) = \mu_n(a_{n+1}) \ \forall a_n \in l_\infty.$$

Lemma 7.4.3. *[161, 169] Let $(a_1, a_2, \dots) \in l_\infty$ be such that $\mu_n(a_n) \leq 0$ for all Banach limits μ, and $\limsup_{n\to\infty}(a_{n+1} - a_n) \leq 0$. Then, $\limsup_{n\to\infty} a_n \leq 0$.*

Lemma 7.4.4. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $S : X \to X$ be a nonexpansive mapping. Then $F(S) \neq \emptyset$ if and only if $\{x_t\}$ defined by (7.4.2) is bounded as $t \to 0$. Furthermore, we have the following* *(i)* $\{x_t\}$ *converges to* $v = P_{F(S)}u$, *where* $P_{F(S)}$ *is the the nearest point map (projection) of* X *onto* $F(S)$.

(ii) $d(u,v)^p \leq \frac{2}{c}\mu_n\left(d(u,x_n)^p\right)$ *for all Banach limits* μ *and all bounded sequences* $\{x_n\}$ *with* $\{x_n - Sx_n\}$ *converging strongly to* 0.

Proof. Let $F(S) \neq \emptyset$, then it is easy to show that $\{x_t\}$ is bounded. Now suppose that $\{x_t\}$ is bounded, we prove that $F(S) \neq \emptyset$. Let $\{t_n\}$ be a sequence in $(0,1)$ such that $\lim_{n\to\infty} t_n = 0$. Then, $\{x_{t_n}\}$ is bounded. Thus, by Lemma 2.3.24 (i), there exists $v \in X$ such that $A(\{x_{t_n}\}) = \{v\}$. That is

$$\limsup_{n\to\infty} d(v, x_{t_n}) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_{t_n}). \tag{7.4.3}$$

Using the nonexpansivity of S and (7.4.2), we obatin

$$\begin{aligned}
\limsup_{n\to\infty} d(x_{t_n}, Sv)^p &\leq \limsup_{n\to\infty} \left(d(x_{t_n}, Sx_{t_n}) + d(Sx_{t_n}, Sv)\right)^p \\
&\leq \limsup_{n\to\infty} \left(t_n d(u, Sx_{t_n}) + d(x_{t_n}, v)\right)^p \\
&= \limsup_{n\to\infty} d(x_{t_n}, v)^p. \qquad (7.4.4)
\end{aligned}$$

Setting $t = \frac{1}{2}$ in (2.1.10), we obtain

$$d(x_{t_n}, \frac{1}{2}Sv \oplus \frac{1}{2}v)^p \leq \frac{1}{2}d(x_{t_n}, Sv)^p + \frac{1}{2}d(x_{t_n}, v) - \frac{c}{8}d(Sv, v)^p. \tag{7.4.5}$$

Since $A(\{x_{t_n}\}) = \{v\}$, then by setting $y = \frac{1}{2}Sv \oplus \frac{1}{2}v$ in (7.4.3), we obtain from (7.4.5) that

$$\begin{aligned}
\limsup_{n\to\infty} d(x_{t_n}, v)^p &\leq \limsup_{n\to\infty} d(x_{t_n}, \frac{1}{2}Sv \oplus \frac{1}{2}v)^p \\
&\leq \frac{1}{2}\limsup_{n\to\infty} d(x_{t_n}, Sv)^p + \frac{1}{2}\limsup_{n\to\infty} d(x_{t_n}, v) - \frac{c}{8}d(Sv, v)^p,
\end{aligned}$$

which implies from (7.4.4) that

$$\begin{aligned}
d(Sv, v)^p &\leq 2\limsup_{n\to\infty} d(x_{t_n}, Sv)^p - 2\limsup_{n\to\infty} d(x_{t_n}, v)^p \\
&\leq 2\limsup_{n\to\infty} d(x_{t_n}, v)^p - 2\limsup_{n\to\infty} d(x_{t_n}, v)^p = 0. \qquad (7.4.6)
\end{aligned}$$

Hence, $v \in F(S)$. Therefore, $F(S) \neq \emptyset$.

We now give the proofs of (i) and (ii).
(i) Let $v = P_{F(S)}u$, then $\{x_t\}$ is bounded. Since $c \geq 2$, we obtain from (7.4.2) and (2.1.10) that

$$\begin{aligned}
d(v, x_t)^p &= d(v, tu \oplus (1-t)Sx_t)^p \\
&\leq td(v,u)^p + (1-t)d(v, Sx_t)^p - \frac{c}{2}t(1-t)d(u, Sx_t)^p \\
&\leq td(v,u)^p + (1-t)d(v, x_t) - \frac{c}{2}t(1-t)d(u, Sx_t)^p,
\end{aligned}$$

which implies that

$$d(v, x_t)^p \;\; \leq \;\; d(v, u)^p + (t-1)d(u, Sx_t)^p. \qquad (7.4.7)$$

Now, let $\{t_m\}$ be any sequence in $(0,1)$ such that $t_m \to 0$, as $m \to \infty$. Since $\{x_{t_m}\}$ is bounded, it follows from Lemma 2.3.24 (ii) that there exists a subsequence $\{x_{t_{m_k}}\}$ of $\{x_{t_m}\}$ that Δ-converges to $q \in X$. Thus, by Lemma 2.3.24 (i), we obtain that $A(\{x_{t_m}\}) = \{q\}$. By the same argument as in (7.4.3)-(7.4.6), we obtain that $q \in F(S)$.
Now, observe that $d(x_{t_m}, Sx_{t_m}) \to 0$, as $m \to \infty$. Thus, we may assume that the subsequence $\{Sx_{t_{m_k}}\}$ of $\{Sx_{t_m}\}$ Δ-converges to $q \in F(S)$ and

$$\lim_{k\to\infty} d(u, Sx_{t_{m_k}})^p = \liminf_{m\to\infty} d(u, Sx_{t_m})^p. \qquad (7.4.8)$$

It then follows from the Δ-lower semicontinuity of $d(u,.)^p$ and (7.4.8) that

$$d(u,q)^p \leq \liminf_{k\to\infty} d(u, Sx_{t_{m_k}})^p = \lim_{k\to\infty} d(u, Sx_{t_{m_k}})^p = \liminf_{m\to\infty} d(u, Sx_{t_m})^p. \qquad (7.4.9)$$

Since $v = P_{F(S)}u$ and $q \in F(S)$, then we have that $d(v,u) \leq d(q,u)$. Thus, letting $d_{t_m} = d(v,u)^p + (t_m - 1)d(u, Sx_{t_m})^p$, we obtain from (7.4.9) that

$$\begin{aligned} \limsup_{m\to\infty} d_{t_m} &= d(v,u)^p + \limsup_{m\to\infty} \left(-d(u, Sx_{t_m})^p\right) \\ &\leq d(q,u)^p - \liminf_{m\to\infty} d(u, Sx_{t_m})^p \leq 0. \end{aligned} \qquad (7.4.10)$$

From (7.4.7) and (7.4.10), we obtain that $\limsup_{m\to\infty} d(v, x_{t_m})^p \leq 0$. Therefore, $\lim_{m\to\infty} d(v, x_{t_m}) = 0$, and this implies that $\{x_{t_m}\}$ converges strongly to v. Hence, it follows that $\{x_t\}$ converges strongly to $v = P_{F(S)}u$.

(ii) Let $\{x_{t_m}\}$ be a sequence defined by (7.4.2), where $\{t_m\}$ is as defined in (i). Then by (i), $\lim_{m\to\infty} x_{t_m} = v$, where $v = P_{F(S)}u$.
From (7.4.2) and (2.1.10), we obtain that

$$\begin{aligned} d(x_n, x_{t_m})^p &\leq t_m d(x_n, u)^p + (1-t_m)d(x_n, Sx_{t_m})^p - \frac{c}{2}t_m(1-t_m)d(u, Sx_{t_m})^p \\ &\leq t_m d(x_n, u)^p + (1-t_m)\left[d(x_n, Sx_n) + d(Sx_n, Sx_{t_m})\right]^p \\ &\quad -\frac{c}{2}t_m(1-t_m)d(u, Sx_{t_m})^p \\ &\leq t_m d(x_n, u)^p + (1-t_m)\left[d(x_n, Sx_n) + d(x_n, x_{t_m})\right]^p \\ &\quad -\frac{c}{2}t_m(1-t_m)d(u, Sx_{t_m})^p. \end{aligned} \qquad (7.4.11)$$

Since μ is a Banach limit, then (7.4.11) becomes

$$\mu_n\left(d(x_n, x_{t_m})^p\right) \;\; \leq \;\; t_m\mu_n\left(d(x_n, u)^p\right) + (1-t_m)\mu_n\left(d(x_n, x_{t_m})^p\right) - \frac{c}{2}t_m(1-t_m)d(u, Sx_{t_m})^p,$$

which implies

$$\mu_n\left(d(x_n, x_{t_m})^p\right) \;\; \leq \;\; \mu_n\left(d(x_n, u)^p\right) - \frac{c}{2}(1-t_m)d(u, Sx_{t_m})^p.$$

By letting $m \to \infty$ in (7.4.11), we obtain

$$\mu_n\left(d(x_n, v)^p\right) \;\; \leq \;\; \mu_n\left(d(x_n, u)^p\right) - \frac{c}{2}d(u,v)^p,$$

which gives the desired conclusion. $\square$

Lemma 7.4.5. *For $p > 1$, let X be a p-uniformly convex metric space with parameter $c \geq 2$ and $S : X \to X$ be a nonexpansive mapping. Let $T_i : X \to X$, $i = 1, 2, \ldots N$ be a finite family of firmly nonexpansive-type mappings such that $F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1) \neq \emptyset$. Then*

$$F(S \circ T_N \circ T_{N-1} \circ \cdots \circ T_2 \circ T_1) = F(S) \cap F(T_N) \cap F(T_{N-1}) \cap \cdots \cap F(T_2) \cap F(T_1).$$

Proof. The proof follows easily from the proof of Lemma 7.2.12 (ii). □

Theorem 7.4.6. *For $1 < p < \infty$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} f_i(y)\right) \neq \emptyset$. For arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \ n \geq 1, \end{cases} \tag{7.4.12}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence in $(0, \infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,

(iii) $\sum\limits_{n=1}^{\infty} \left(1 - \frac{(\lambda_{n-1}^{(i)})^{p-1}}{(\lambda_n^{(i)})^{p-1}}\right)^{\frac{1}{p}} < \infty$, $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$, where P_Γ is the nearest point map (projection) of X onto Γ.

Proof. First, we show that $\{x_n\}$ is bounded.
Set $w_n^{(i+1)} = J^{f_i}_{\lambda_n^{(i)}} w_n^{(i)}$, $i = 1, 2, \ldots, N$, where $w_n^{(1)} = x_n$, for all $n \geq 1$. Then,

$w_n^{(2)} = J^{f_i}_{\lambda_n^{(1)}}(x_n)$, $w_n^{(3)} = J^{f_i}_{\lambda_n^{(2)}} \circ J^{f_i}_{\lambda_n^{(1)}}(x_n)$, $\ldots$, $w_n^{(N)} = J^{f_i}_{\lambda_n^{(N-1)}} \circ J^{f_i}_{\lambda_n^{(2)}} \circ J^{f_i}_{\lambda_n^{(1)}}(x_n)$ and $w_n^{(N+1)} = y_n$.
Now, let $v \in \Gamma$, then we obtain from (2.1.10), (7.4.12), Lemma 7.2.11 and Lemma 7.2.7 that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(u, v)^p + (1 - \alpha_n) d(Ty_n, v)^p \\ &\leq \alpha_n d(u, v)^p + (1 - \alpha_n) d(w_n^{(N+1)}, v)^p \\ &= \alpha_n d(u, v)^p + (1 - \alpha_n) d(J^{f_N}_{\lambda_n^{(N)}} w_n^{(N)}, v)^p \\ &\ \vdots \\ &\leq \alpha_n d(u, v)^p + (1 - \alpha_n) d(x_n, v)^p \\ &\leq \max\{d(u, v)^p, d(x_n, v)^p\}, \end{aligned} \tag{7.4.13}$$

which implies by induction that

$$d(x_n,v)^p \le \max\{d(u,v)^p, d(x_1,v)^p\}\ \forall n \ge 1. \tag{7.4.14}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ and $\{Ty_n\}$ are also bounded.
Next, we show that $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = 0$.
We may assume without loss of generality that $\lambda_{n-1}^{(i)} \le \lambda_n^{(i)},\ i = 1,2,\ldots,N,\ \ n \ge 1$. Thus, from (7.4.12) and Lemma 7.2.9, we obtain

$$\begin{aligned}
d\left(w_n^{(i+1)}, w_{n-1}^{(i+1)}\right) &\le d\left(J^{f_i}_{\lambda_n^{(i)}} w_n^{(i)}, J^{f_i}_{\lambda_n^{(i)}} w_{n-1}^{(i)}\right) + d\left(J^{f_i}_{\lambda_n^{(i)}} w_{n-1}^{(i)}, J^{f_i}_{\lambda_{n-1}^{(i)}} w_{n-1}^{(i)}\right)\\
&\le d\left(w_n^{(i)}, w_{n-1}^{(i)}\right) + \left[\frac{2}{c}\left(1 - \frac{\left(\lambda_{n-1}^{(i)}\right)^{p-1}}{\left(\lambda_n^{(i)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(i)}, J^{f_i}_{\lambda_n^{(i)}} w_{n-1}^{(i)}\right)\\
&\le d\left(J^{f_{i-1}}_{\lambda_n^{(i-1)}} w_n^{(i-1)}, J^{f_{i-1}}_{\lambda_n^{(i-1)}} w_{n-1}^{(i-1)}\right) + d\left(J^{f_{i-1}}_{\lambda_n^{(i-1)}} w_{n-1}^{(i-1)}, J^{f_{i-1}}_{\lambda_{n-1}^{(i-1)}} w_{n-1}^{(i-1)}\right)\\
&\quad + \left[\frac{2}{c}\left(1 - \frac{\left(\lambda_{n-1}^{(i)}\right)^{p-1}}{\left(\lambda_n^{(i)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(i)}, J^{f_i}_{\lambda_n^{(i)}} w_{n-1}^{(i)}\right)\\
&\le d\left(w_n^{(i-1)}, w_{n-1}^{(i-1)}\right) + \left[\frac{2}{c}\left(1 - \frac{\left(\lambda_{n-1}^{(i-1)}\right)^{p-1}}{\left(\lambda_n^{(i-1)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(i-1)}, J^{f_{i-1}}_{\lambda_n^{(i-1)}} w_{n-1}^{(i-1)}\right)\\
&\quad + \left[\frac{2}{c}\left(1 - \frac{\left(\lambda_{n-1}^{(i)}\right)^{p-1}}{\left(\lambda_n^{(i)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(i)}, J^{f_i}_{\lambda_n^{(i)}} w_{n-1}^{(i)}\right)\\
&\le d\left(w_n^{(i-(N-1))}, w_{n-1}^{(i-(N-1))}\right)\\
&\quad + \sum_{j=0}^{N-1}\left[\frac{2}{c}\left(1 - \frac{\left(\lambda_{n-1}^{(i-j)}\right)^{p-1}}{\left(\lambda_n^{(i-j)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(i-j)}, J^{f_{i-j}}_{\lambda_n^{(i-j)}} w_{n-1}^{(i-j)}\right).
\end{aligned} \tag{7.4.15}$$

Since $c \ge 2$, we obtain from (7.4.12) and (2.1.10) that

$$\begin{aligned}
&d(x_{n+1},x_n)^p\\
= \;& d(\alpha_n u \oplus (1-\alpha_n)Ty_n, \alpha_{n-1}u \oplus (1-\alpha_{n-1})Ty_{n-1})^p\\
\le \;& \alpha_{n-1} d(\alpha_n u \oplus (1-\alpha_n)Ty_n, u)^p + (1-\alpha_{n-1}) d(\alpha_n u \oplus (1-\alpha_n)Ty_n, Ty_{n-1})^p\\
& -\frac{c}{2}\alpha_{n-1}(1-\alpha_{n-1}) d(u, Ty_{n-1})^p\\
\le \;& \alpha_{n-1}(1-\alpha_n) d(Ty_n, u)^p - \frac{c}{2}\alpha_{n-1}\alpha_n(1-\alpha_n) d(u,Ty_n)^p + \alpha_n(1-\alpha_{n-1}) d(u, Ty_{n-1})^p\\
& +(1-\alpha_{n-1})(1-\alpha_n) d(Ty_n, Ty_{n-1})^p - \frac{c}{2}(1-\alpha_{n-1})\alpha_n(1-\alpha_n) d(u, Ty_n)^p\\
& -\frac{c}{2}\alpha_{n-1}(1-\alpha_{n-1}) d(u, Ty_{n-1})^p
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_{n-1})(1-\alpha_n)d(y_n,y_{n-1})^p + \left[\left(\alpha_n - \frac{c\alpha_{n-1}}{2}\right)(1-\alpha_{n-1})\right] d(u,Ty_{n-1})^p \\
&\quad + \left[\left(\alpha_{n-1} - \frac{c}{2}\alpha_{n-1}\alpha_n - \frac{c}{2}\alpha_n(1-\alpha_{n-1})\right)(1-\alpha_n)\right] d(u,Ty_n)^p \\
&\leq (1-\alpha_{n-1})(1-\alpha_n)d(w_n^{(N+1)}, w_{n-1}^{(N+1)})^p \\
&\quad +|\alpha_n-\alpha_{n-1}|d(u,Ty_{n-1})^p + |\alpha_n-\alpha_{n-1}|d(u,Ty_n)^p. \qquad (7.4.16)
\end{aligned}$$

For $i=N$, we obtain from (7.4.15) and (7.4.16) that

$$\begin{aligned}
d(x_{n+1},x_n)^p &\leq (1-\alpha_{n-1})(1-\alpha_n)\Bigg[d(w_n^{(1)}, w_{n-1}^{(1)}) \\
&\quad + \sum_{j=0}^{N-1}\left[\frac{2}{c}\left(1-\frac{\left(\lambda_{n-1}^{(N-j)}\right)^{p-1}}{\left(\lambda_n^{(N-j)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(N-j)}, J_{\lambda_n^{(N-j)}}^{f_{N-j}} w_{n-1}^{(N-j)}\right)\Bigg]^p \\
&\quad +|\alpha_n-\alpha_{n-1}|\left[d(u,Ty_{n-1})^p + d(u,Ty_n)^p\right] \\
&= (1-\alpha_{n-1})(1-\alpha_n)\Bigg[d(x_n,x_{n-1}) \\
&\quad + \sum_{j=0}^{N-1}\left[\frac{2}{c}\left(1-\frac{\left(\lambda_{n-1}^{(N-j)}\right)^{p-1}}{\left(\lambda_n^{(N-j)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(N-j)}, J_{\lambda_n^{(N-j)}}^{f_{N-j}} w_{n-1}^{(N-j)}\right)\Bigg]^p \\
&\quad +|\alpha_n-\alpha_{n-1}|\left[d(u,Ty_{n-1})^p + d(u,Ty_n)^p\right],
\end{aligned}$$

which implies that

$$\begin{aligned}
d(x_{n+1},x_n) &\leq (1-\alpha_n)d(x_n,x_{n-1}) \\
&\quad + \sum_{j=0}^{N-1}\left[\frac{2}{c}\left(1-\frac{\left(\lambda_{n-1}^{(N-j)}\right)^{p-1}}{\left(\lambda_n^{(N-j)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} d\left(w_{n-1}^{(N-j)}, J_{\lambda_n^{(N-j)}}^{f_{N-j}} w_{n-1}^{(N-j)}\right) \\
&\quad +|\alpha_n-\alpha_{n-1}|\left[d(u,Ty_{n-1}) + d(u,Ty_n)\right]
\end{aligned}$$

$$\begin{aligned}
&\leq (1-\alpha_n)d(x_n,x_{n-1}) + \left[\frac{2}{c}\left(1-\frac{\left(\lambda_{n-1}^{(N)}\right)^{p-1}}{\left(\lambda_n^{(N)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} M \\
&\quad + \left[\frac{2}{c}\left(1-\frac{\left(\lambda_{n-1}^{(N-1)}\right)^{p-1}}{\left(\lambda_n^{(N-1)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} M + \cdots + \left[\frac{2}{c}\left(1-\frac{\left(\lambda_{n-1}^{(1)}\right)^{p-1}}{\left(\lambda_n^{(1)}\right)^{p-1}}\right)\right]^{\frac{1}{p}} M \\
&\quad +|\alpha_n-\alpha_{n-1}|\left[d(u,Ty_{n-1}) + d(u,Ty_n)\right], \qquad (7.4.17)
\end{aligned}$$

where $M := \sup_{n\geq 1}\left\{\sum_{j=0}^{N-1} d\left(w_{n-1}^{(N-j)}, J_{\lambda_n^{(N-j)}}^{f_{N-j}} w_{n-1}^{(N-j)}\right)\right\}$. Therefore, using conditions (ii), (iii) of Theorem 7.4.6, and applying Lemma 2.3.26 in (7.4.17), we obtain that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \qquad (7.4.18)$$

We now show that $\lim_{n\to\infty} d(x_n, TJ^{f_i}_{\lambda^{(i)}} x_n) = 0,\ i, 1, 2\ldots, N.$
From Lemma 7.2.12, we obtain for each $i = 1, 2, \ldots, N$ that

$$d(w_n^{(i+1)}, v)^p \leq d(w_n^{(i)}, v)^p - d(w_n^{(i)}, w_n^{(i+1)})^p. \tag{7.4.19}$$

By setting $i = N$ in (7.4.19), we obtain from (7.4.13) that

$$\begin{aligned} d(x_{n+1}, v)^p &\leq \alpha_n d(u, v)^p + (1-\alpha_n) d(w_n^{(N+1)}, v)^p \\ &\leq \alpha_n d(u, v)^p + (1-\alpha_n) d(w_n^{(N)}, v)^p - (1-\alpha_n) d(w_n^{(N)}, w_n^{(N+1)})^p \\ &\leq \alpha_n d(u, v)^p + (1-\alpha_n) d(x_n, v)^p - (1-\alpha_n) d(w_n^{(N)}, w_n^{(N+1)})^p, \end{aligned}$$

which implies that

$$\begin{aligned} (1-\alpha_n) d(w_n^{(N)}, w_n^{(N+1)})^p &\leq \alpha_n [d(u, v)^p - d(x_n, v)^p] + d(x_n, v)^p - d(x_{n+1}, v)^p \\ &\leq \alpha_n [d(u, v)^p - d(x_n, v)^p] \\ &\quad + [d(x_n, x_{n+1}) + d(x_{n+1}, v)]^p - d(x_{n+1}, v)^p. \end{aligned} \tag{7.4.20}$$

Using (7.4.18) and condition (i) of Theorem 7.4.6 in (7.4.20), we obtain

$$\lim_{n\to\infty} d(w_n^{(N)}, w_n^{(N+1)}) = 0. \tag{7.4.21}$$

Also, by setting $i = N - 1$ in (7.4.19), and following the steps as above, we obtain

$$\lim_{n\to\infty} d(w_n^{(N-1)}, w_n^{(N)}) = 0. \tag{7.4.22}$$

Repeating the same process, we can show that

$$\lim_{n\to\infty} d(w_n^{(i)}, w_n^{(i+1)}) = 0,\ i = 1, 2, \ldots, N-2. \tag{7.4.23}$$

This, together with (7.4.21) and (7.4.22), gives

$$\lim_{n\to\infty} d(w_n^{(i)}, w_n^{(i+1)}) = 0,\ i = 1, 2, \ldots, N. \tag{7.4.24}$$

From (7.4.24), and applying triangle inequality, we obtain for each $i = 1, 2, \ldots, N+1$, that

$$\lim_{n\to\infty} d(x_n, w_n^{(i)}) = \lim_{n\to\infty} d(w_n^{(1)}, w_n^{(i)}) = 0. \tag{7.4.25}$$

Since $0 < \lambda^{(i)} \leq \lambda_n^{(i)}$ for all $n \geq 1$, we obtain from Lemma 7.2.8 and (7.4.24) that

$$d\left(w_n^{(i)}, J^{f_i}_{\lambda^{(i)}} w_n^{(i)}\right) \leq d\left(w_n^{(i)}, J^{f_i}_{\lambda_n^{(i)}} w_n^{(i)}\right) \to 0, \text{ as } n \to \infty,\ i = 1, 2, \ldots, N. \tag{7.4.26}$$

Furthermore, we obtain from (2.1.1), (7.4.18) and (7.4.25) that

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, Ty_n) + d(Ty_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, Ty_n) + d(y_n, x_n) \\ &= d(x_n, x_{n+1}) + \alpha_n d(u, Ty_n) + d(y_n, x_n) \to 0,\ n \to \infty. \end{aligned} \tag{7.4.27}$$

Also, from (7.4.25) and (7.4.27), we obtain that

$$\begin{aligned} d(x_n, Ty_n) &\le d(x_n, Tx_n) + d(Tx_n, Ty_n) \\ &\le d(x_n, Tx_n) + d(x_n, y_n) \to 0, \; n \to \infty. \end{aligned} \tag{7.4.28}$$

Again, we obtain from (7.4.25), (7.4.26) and (7.4.28) that

$$\begin{aligned} d(x_n, TJ^{f_i}_{\lambda^{(i)}} x_n) &\le d(x_n, Ty_n) + d(Ty_n, TJ^{f_i}_{\lambda^{(i)}} x_n) \\ &\le d(x_n, Ty_n) + d(w_n^{(N+1)}, J^{f_i}_{\lambda^{(i)}} w_n^{(1)}) \\ &\le d(x_n, Ty_n) + d(w_n^{(N+1)}, w_n^{(1)}) + d(w_n^{(1)}, w_n^{(i)}) \\ &\quad + d(w_n^{(i)}, J^{f_i}_{\lambda^{(i)}} w_n^{(i)}) + d(J^{f_i}_{\lambda^{(i)}} w_n^{(i)}, J^{f_i}_{\lambda^{(i)}} w_n^{(1)}) \\ &\le d(x_n, Ty_n) + d(w_n^{(N+1)}, w_n^{(1)}) \\ &\quad + 2d(w_n^{(1)}, w_n^{(i)}) + d(w_n^{(i)}, J^{f_i}_{\lambda^{(i)}} w_n^{(i)}) \to 0, \; n \to \infty. \end{aligned} \tag{7.4.29}$$

Next, we show that $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$.
Let $\bar{v} = \lim\limits_{t \to 0} x_t$, where $\{x_t\}$ is as defined in (7.4.2) with $S = T \circ J^{f_N}_{\lambda^{(N)}} \circ J^{f_{N-1}}_{\lambda^{(N-1)}} \circ \cdots \circ J^{f_1}_{\lambda^{(1)}}$.
Then, by Lemma 7.4.4, Lemma 7.4.5 and Lemma 7.2.7, we obtain that $\bar{v} = P_{F(S)} u = P_\Gamma u$.
Thus, from (2.1.10), we obtain

$$\begin{aligned} d(x_{n+1}, \bar{v})^p &\le \alpha_n d(u, \bar{v})^p + (1-\alpha_n) d(Ty_n, \bar{v})^p - \frac{c}{2}\alpha_n(1-\alpha_n) d(u, Ty_n)^p \\ &\le (1-\alpha_n) d(x_n, \bar{v})^p + \alpha_n \left(d(u, \bar{v})^p - \frac{c}{2}(1-\alpha_n) d(u, Ty_n)^p \right). \end{aligned} \tag{7.4.30}$$

By Lemma 7.4.4 (ii), we obtain that $\mu_n \left(d(u, \bar{v})^p - \frac{2}{c} d(u, x_n)^p \right) \le 0$ for all Banach limits μ. Also, we obtain from (7.4.18) that

$$\limsup_{n \to \infty} \left([d(u, \bar{v})^p - \frac{2}{c} d(u, x_{n+1})^p] - [d(u, \bar{v})^p - \frac{2}{c} d(u, x_n)^p] \right) \le 0.$$

Hence, it follows from Lemma 7.4.3 that

$$\limsup_{n \to \infty} \left(d(u, \bar{v})^p - \frac{2}{c} d(u, x_n)^p \right) \le 0. \tag{7.4.31}$$

From (7.4.28) and (7.4.31), we obtain that

$$\limsup_{n \to \infty} \left(d(u, \bar{v})^p - \frac{2}{c}(1-\alpha_n) d(u, Ty_n)^p \right) \le 0. \tag{7.4.32}$$

Using (7.4.32) and applying Lemma 2.3.26 in (7.4.30), we obtain that $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$. □

The following corollary of Theorem 7.4.6 generalizes the results of [174, Theorem 3.1] and [161, Theorem 2.3] in Hadamard spaces.

Corollary 7.4.7. *Let X be a complete 2-uniformly convex metric space with parameter $c = 2$ (in particular, an Hadamard space) and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $T : X \to X$ be a nonexpansive mapping and $\Gamma := F(T) \cap \left(\cap_{i=1}^N \arg\min_{y \in X} f_i(y) \right) \neq \emptyset$. For arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T y_n, \; n \geq 1, \end{cases} \tag{7.4.33}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence in $(0, \infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that

(i) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,

(iii) $\sum_{n=1}^{\infty} \left(\sqrt{1 - \frac{(\lambda_{n-1}^{(i)})}{(\lambda_n^{(i)})}} \right) < \infty$, $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$, where P_Γ is the nearest point map (projection) of X onto Γ.

By setting $T \equiv I$ in Theorem 7.4.6 (where I is the identity mapping on X), we obtain the following new result in p-uniformly convex metric space.

Corollary 7.4.8. *For $p > 1$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper convex and lower semicontinuous functions. Let $\Gamma := \cap_{i=1}^N \arg\min_{y \in X} f_i(y) \neq \emptyset$. For arbitrary $u, x_1 \in X$, the sequence $\{x_n\}$ be generated by*

$$\begin{cases} y_n = J^{f_N}_{\lambda_n^{(N)}} \circ J^{f_{N-1}}_{\lambda_n^{(N-1)}} \circ \cdots \circ J^{f_2}_{\lambda_n^{(2)}} \circ J^{f_1}_{\lambda_n^{(1)}}(x_n), \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) y_n, \; n \geq 1, \end{cases} \tag{7.4.34}$$

where $\{\alpha_n\}$ is a sequence in $(0, 1)$ and $\{\lambda_n^{(i)}\}$, $i = 1, 2, \ldots, N$ is a sequence in $(0, \infty)$ with $\lambda_n^{(i)} \geq \lambda^{(i)} > 0$ such that

(i) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$,

(iii) $\sum_{n=1}^{\infty} \left(1 - \frac{(\lambda_{n-1}^{(i)})^{p-1}}{(\lambda_n^{(i)})^{p-1}} \right)^{\frac{1}{p}} < \infty$, $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ converges strongly to $\bar{v} = P_\Gamma u$, where P_Γ is the nearest point map (projection) of X onto Γ.

Remark 7.4.9. *Since viscosity-type algorithms have higher rate of convergence than the Halpern-type, and Halpern-type convergence theorems imply viscosity-type convergence theorems. It is natural to ask whether it is possible to replace the Halpern algorithm in Theorem 7.4.6 with a viscosity algorithm and obtain similar result? In other words, is it possible to study viscosity-type algorithms for minimization and fixed point problems in p-uniformly convex metric spaces?*

Asymptotic behavior of Halpern-type algorithm for minimization problems in p-uniformly convex metric space

We now study the asymptotic behaviour of the sequence $\{x_n\}$ generated by the following Halpern-type PPA:

$$\begin{cases} u, x_1 \in X, \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) J_{\mu_n}^f x_n, \end{cases} \tag{7.4.35}$$

where $\{\alpha_n\}$ and $\{\mu_n\}$ are sequences in $[0,1)$ and $(0,\infty)$ respectively, and $f : X \to (-\infty, +\infty]$ is a proper convex and lower semicontious function. We also extend our study to examine the behaviour of the sequence given by the following Halpern-type PPA involving finite composition of resolvents of proper convex and lower semicontinuous functions:

$$\begin{cases} u, x_1 \in X, \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) \prod_{j=1}^m J_{\mu_n}^{f_j} x_n, \ n \geq 1, \end{cases} \tag{7.4.36}$$

where $\prod_{j=1}^m J_{\mu_n}^{f_j} = J_{\mu_n}^{f_1} \circ J_{\mu_n}^{f_2} \circ \cdots \circ J_{\mu_n}^{f_{m-1}} \circ J_{\mu_n}^{f_m}$, $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\mu_n\}$ is a sequence in $(0,\infty)$.

Lemma 7.4.10. *For $1 < p < \infty$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f : X \to (-\infty, +\infty]$ a proper, convex and lower semicontinuous function. Let $\{\mu_n\}$ be a sequence of positive real numbers. Suppose $\lim\limits_{n\to\infty} \mu_n = \infty$ and $A(\{J_{\mu_n}^f x_n\}) = \{\bar{v}\}$ for some bounded sequence $\{x_n\}$ of X. Then $\bar{v}$ is a minimizer of f, that is, $\bar{v} \in \underset{y\in X}{argmin} f(y)$.*

Proof. By Lemma 7.2.10, we obtain that

$$\frac{c}{2}(\mu_n^{p-1}+1)d(J_{\mu_n}^f x_n, J^f\bar{v})^p + d(J_{\mu_n}^f x_n, x_n)^p + \mu_n^{p-1} d(J^f\bar{v}, \bar{v})^p \leq d(J^f\bar{v}, x_n)^p + \mu_n^{p-1} d(J_{\mu_n}^f x_n, \bar{v})^p,$$

which implies

$$\frac{c}{2} d(J_{\mu_n}^f x_n, J^f\bar{v})^p \leq \frac{1}{\mu_n^{p-1}} d(J^f\bar{v}, x_n)^p + d(J_{\mu_n}^f x_n, \bar{v})^p.$$

By $\lim_{n\to\infty}\mu_n = \infty$ and $\{x_n\}$ is bounded, we obtain that

$$\frac{c}{2}\limsup_{n\to\infty} d(J^f_{\mu_n}x_n, J^f\bar{v})^p \le \limsup_{n\to\infty} d(J^f_{\mu_n}x_n, \bar{v})^p.$$

Furthermore, since $A(\{J^f_{\mu_n}x_n\}) = \{\bar{v}\}$ and $c \ge 2$, we obtain that

$$\limsup_{n\to\infty} d(J^f_{\mu_n}x_n, J^f\bar{v}) \le \limsup_{n\to\infty} d(J^f_{\mu_n}x_n, \bar{v}) = \inf_{y\in X}\limsup_{n\to\infty} d(J^f_{\mu_n}x_n, y). \tag{7.4.37}$$

By (7.4.37), Lemma 2.3.24 (i) and Lemma 7.2.11, we obtain that $\bar{v} \in F(J^f) = \operatorname{argmin}_{y\in X} f(y)$. $\square$

Theorem 7.4.11. *For $1 < p < \infty$, let X be a complete p-uniformly convex metric space with parameter $c \ge 2$ and $f : X \to (-\infty, +\infty]$ a proper, convex and lower semicontinuous function. Let $\{x_n\}$ be the sequence defined by (7.4.35), where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\mu_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\to\infty}\mu_n = \infty$. Then, the following hold:*

(i) The sequence $\{J^f_{\mu_n}x_n\}$ is bounded if and only if $\operatorname{argmin}_{y\in X} f(y) \neq \emptyset$.

(ii) If $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\operatorname{argmin}_{y\in X} f(y) \neq \emptyset$, then $\{x_n\}$ and $\{J^f_{\mu_n}x_n\}$ converge to an element of $\operatorname{argmin}_{y\in X} f(y)$.

Proof. (i) Suppose that $\{J^f_{\mu_n}x_n\}$ is bounded. Then by Lemma 2.3.24 (i), there exists $\bar{v} \in X$ such that $A(\{J^f_{\mu_n}x_n\}) = \{\bar{v}\}$. Thus, from (7.4.35), we obtain that

$$d(x_{n+1}, \bar{v})^p \le \alpha_n d(u, \bar{v})^p + (1-\alpha_n) d(J^f_{\mu_n}x_n, \bar{v})^p,$$

which implies that $\{x_n\}$ is bounded. Also, since $\lim_{n\to\infty}\mu_n = \infty$ and $A(\{J^f_{\mu_n}x_n\}) = \{\bar{v}\}$, we obtain by Lemma 7.4.10 that $\bar{v}$ is a minimizer of f. Hence, $\operatorname{argmin}_{y\in X} f(y) \neq \emptyset$.

Conversely, let $\operatorname{argmin}_{y\in X} f(y) \neq \emptyset$. Then, we may assume that $\bar{v}$ is a minimizer of f. Thus by (7.4.35) and Lemma 7.2.7, we obtain that

$$\begin{aligned} d(x_{n+1}, \bar{v})^p &\le \alpha_n d(u, \bar{v})^p + (1-\alpha_n) d(J^f_{\mu_n}x_n, \bar{v})^p \\ &\le \alpha_n d(u, \bar{v})^p + (1-\alpha_n) d(x_n, \bar{v})^p \\ &\le \max\{d(u, \bar{v})^p, d(x_n, \bar{v})^p\}, \end{aligned}$$

which implies by induction that

$$d(x_n, \bar{v})^p \le \max\{d(u, \bar{v})^p, d(x_1, \bar{v})^p\}\ \forall n \ge 1. \tag{7.4.38}$$

Therefore, $\{x_n\}$ is bounded. Consequently, $\{J^f_{\mu_n}x_n\}$ is also bounded.

(ii) Since $\underset{y\in X}{\operatorname{argmin}} f(y) \neq \emptyset$, we obtain from (7.4.38) that $\{x_n\}$ and $\{J^f_{\mu_n} x_n\}$ are bounded. Furthermore, we obtain by (2.1.10) and Lemma 7.2.7 that

$$\begin{aligned} d(x_{n+1}, \bar{v})^p &\leq \alpha_n d(u, \bar{v})^p + (1-\alpha_n) d(J^f_{\mu_n} x_n, \bar{v})^p - \frac{\alpha_n(1-\alpha_n)c}{2} d(u, J^f_{\mu_n} x_n)^p \\ &\leq \alpha_n d(u, \bar{v})^p + (1-\alpha_n) d(x_n, \bar{v})^p - \alpha_n(1-\alpha_n) d(u, J^f_{\mu_n} x_n)^p \\ &= (1-\alpha_n) d(x_n, \bar{v})^p + \alpha_n \delta_n \ \forall n \geq 1, \end{aligned} \tag{7.4.39}$$

where $\delta_n = d(u, \bar{v})^p + (\alpha_n - 1) d(u, J^f_{\mu_n} x_n)^p$. Now, set $v_n = J^f_{\mu_n} x_n \ \forall n \geq 1$. Then, by the boundedness of $\{J^f_{\mu_n} x_n\}$, we obtain by Lemma 2.3.24 (ii) that there exists a subsequence $\{v_{n_k}\}$ of $\{v_n\}$ that Δ-converges to some $\hat{v} \in X$. Thus, by Lemma 2.3.24 (i), we obtain that $A(\{v_{n_k}\}) = \{\hat{v}\}$. Moreover, $\lim_{k\to\infty} \mu_{n_k} = \infty$ and $\{x_{n_k}\}$ is bounded. Hence, by Lemma 7.4.10, we obtain that $\hat{v}$ is a minimizer of f.

Next, we show that $\{x_n\}$ converges to $\hat{v}$. Observe that

$$d(u, \hat{v})^p \leq \liminf_{k\to\infty} d(u, v_{n_k})^p = \lim_{k\to\infty} d(u, v_{n_k})^p = \liminf_{n\to\infty} d(u, v_n)^p.$$

Thus,

$$\limsup_{n\to\infty} \delta_n \leq d(u, \hat{v})^p - \liminf_{n\to\infty} d(u, v_n)^p \leq 0.$$

Now, Lemma 2.3.26 applied to (7.4.39), gives that $\{x_n\}$ converges to $\hat{v}$. □

In what follows, we shall apply Theorem 7.4.11 to establish the convergence of Halpern-type PPA (7.4.36) involving finite composition of resolvents of f.

Theorem 7.4.12. *For $1 < p < \infty$, let X be a complete p-uniformly convex metric space with parameter $c \geq 2$ and $f_j : X \to (-\infty, +\infty]$ be proper, convex and lower semicontinuous functions. Let $\{x_n\}$ be a sequence generated by (7.4.36), where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\mu_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} \mu_n = \infty$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := \cap_{j=1}^m \underset{y\in X}{\operatorname{argmin}} f_j(y) \neq \emptyset$, then the sequence $\{x_n\}$ converges to an element of Γ.*

Proof. By Theorem 7.4.11 (ii) and Lemma 7.2.11, we obtain that $\{x_n\}$ converges to an element of $F\left(\prod_{j=1}^m J^{f_j}_{\mu}\right)$. Therefore, we conclude by Lemma 5.4.4 (ii) and Lemma 7.2.11 that $\{x_n\}$ converges to an element of Γ. □

Corollary 7.4.13. *Let X be a complete 2-uniformly convex metric space (in particular, an Hadamard space) and $f_j : X \to (-\infty, +\infty]$ be proper, convex and lower semicontinuous functions. Let $\{x_n\}$ be a sequence generated by (7.4.36), where $\{\alpha_n\}$ is a sequence in $[0,1)$ and $\{\mu_n\}$ is a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} \mu_n = \infty$. If $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\Gamma := \cap_{j=1}^m \underset{y\in X}{\operatorname{argmin}} f_j(y) \neq \emptyset$, then the sequence $\{x_n\}$ converge to an element of Γ.*

Proof. Take $p = 2 = c$ in Theorem 7.4.12. □

7.4.2 Numerical example

In this subsection, we discuss some numerical experiments of Algorithm (7.4.12) (in comparison with the algorithm studied by Okeke and Izuchukwu [137] and Suparatulatorn *et al.* [174]) in a p-uniformly convex metric space, to show its applicability and advantage.

Let $\mathbf{P}(n)$ be the space of $n \times n$ Hermitian positive definite matrices. For $1 < p < \infty$, the geodesic distance between A and B in $\mathbf{P}(n)$ (also called the p-Schatten distance) $d_p : \mathbf{P}(n) \times \mathbf{P}(n) \to [0, \infty)$ is defined by (see [61], [135, Chapter 2] and [53, Example 5.2])

$$\begin{aligned} d_p(A, B) &= \inf\{L(c) \mid c : [0, 1] \to \mathbf{P}(n) \text{ is a smooth curve with } c(0) = A \text{ and } c(1) = B\} \\ &= || \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} ||_p \\ &= \left(\sum_{i=1}^{m} \log^p \mu_i(A^{-1}B) \right)^{\frac{1}{p}}, \end{aligned}$$

where $\mu_i(A^{-1}B)$ are the eigenvalues of $A^{-1}B$, $L(c) := \int_0^1 ||c(t)^{-\frac{1}{2}} c'(t) c(t)^{-\frac{1}{2}}||_p dt$, $||A||_p := (\text{tr}|A|^p)^{\frac{1}{p}}$, tr is the the usual trace functional and $|A| = (A^H A)^{\frac{1}{2}}$ (where A^H is the conjugate transpose of A). The pair $(\mathbf{P}(n), d_p)$ is a p-uniformly convex metric space with geodesic joining A to B in $\mathbf{P}(n)$ given by (see [48, 61, 135])

$$(1 - t)x \oplus ty = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \right)^t A^{\frac{1}{2}}, \ 0 \le t \le 1.$$

Now, define $T : \mathbf{P}(n) \to \mathbf{P}(n)$ by $TA = D^H A D$, where $D \in GL(n)$ (the set of $n \times n$ invertible matrices). Then T is a nonexpansive mapping (see [135, Chapter 2]). Also, define $f_1 : \mathbf{P}(n) \to \mathbb{R}$ by $f_1 A = \left(\sum_{i=1}^{m} \log^p \mu_i(A^{-1} e^A) \right)^{\frac{1}{p}}$, where $\mu_i(A^{-1}e^A)$ are the eigenvalues of $A^{-1}e^A$. Then f_1 is convex and lower semicontinuous (see [6]). Again, define $f_2, f_3, f_4 : \mathbf{P}(n) \to \mathbb{R}$ by $f_2 A = -\log det A$, $f_3 A = \text{tr}(A)$ and $f_4 A = \text{tr}(e^A)$ respectively, then f_i is convex and lower semicontiunous for each $i = 2, 3, 4$ (see [6, 172]).

Take $\alpha_n = \frac{1}{3n+1}$ and $\lambda_n^{(i)} = \frac{in-1}{in}$, $i = 1, 2, 3, 4$, for all $n \ge 1$. Hence, Algorithm (7.4.12) becomes

$$\begin{cases} z_n = \arg\min_{v \in X} \left(f_1(v) + \left(\frac{1}{p\lambda_n^{p-1}} \right) d_X(v, x_n)^p \right), \\ w_n = \arg\min_{v \in X} \left(f_2(v) + \left(\frac{1}{p\lambda_n^{p-1}} \right) d_X(v, z_n)^p \right), \\ v_n = \arg\min_{v \in X} \left(f_3(v) + \left(\frac{1}{p\lambda_n^{p-1}} \right) d_X(v, w_n)^p \right), \\ y_n = \arg\min_{v \in X} \left(f_4(v) + \left(\frac{1}{p\lambda_n^{p-1}} \right) d_X(v, v_n)^p \right), \\ x_{n+1} = \frac{u}{3n+1} \oplus (\frac{3n}{3n+1}) T y_n, \ n \ge 1, \end{cases} \tag{7.4.40}$$

the algorithm studied by Okeke and Izuchukwu [137] becomes

$$\begin{cases} w_n = \arg\min_{v \in X} \left(f(v) + \left(\frac{1}{p\lambda^{p-1}} \right) d_X(v, x_n)^p \right), \\ y_n = \arg\min_{v \in X} \left(f(v) + \left(\frac{1}{p\lambda^{p-1}} \right) d_X(v, w_n)^p \right), \\ x_{n+1} = \frac{u}{3n+1} \oplus (\frac{3n}{3n+1}) T y_n, \ n \ge 1, \end{cases} \tag{7.4.41}$$

and the algorithm studied by Suparatulatorn *et al.* [174] becomes

$$\begin{cases} y_n = \arg\min_{v\in X}\left(f(v) + \left(\frac{1}{p\lambda_n^{p-1}}\right) d_X(v,x_n)^p\right), \\ x_{n+1} = \frac{u}{3n+1} \oplus (\frac{3n}{3n+1})Ty_n,\ n \geq 1. \end{cases} \tag{7.4.42}$$

We now consider the following 4 cases for our numerical experiments.

Case I: $x_1 = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$ and $u = \begin{bmatrix} 5 & 1+i \\ 1-i & 4 \end{bmatrix}$,

Case II: $x_1 = \begin{bmatrix} 2 & 2-i \\ 2+i & 4 \end{bmatrix}$ and $u = \begin{bmatrix} 5 & 1+i \\ 1-i & 4 \end{bmatrix}$,

Case III: $x_1 = \begin{bmatrix} 2 & 2-i \\ 2+i & 4 \end{bmatrix}$ and $u = \begin{bmatrix} 1 & 4+i \\ 4-i & 3 \end{bmatrix}$,

Case IV: $x_1 = \begin{bmatrix} 3 & -3-i \\ -3+i & 4 \end{bmatrix}$ and $u = \begin{bmatrix} 2 & -i \\ i & 2 \end{bmatrix}$.

Remark 7.4.14. *Using different choices of the initial matrices x_1 and u (that is,* **Case I-Case IV***), we obtain the numerical results shown in Figure 7.1, Table 1 and Table 2. We see in the figures that the error values converge to 0, suggesting that by choosing arbitrary starting vectors, the sequence $\{x_n\}$ converges to the common minimizer of f_i, $i = 1,2,3,4$ which is also a fixed point of T. In all our comparisons (see the table and graphs), we see that our algorithm performs better than the algorithms studied in [137] and [174].*

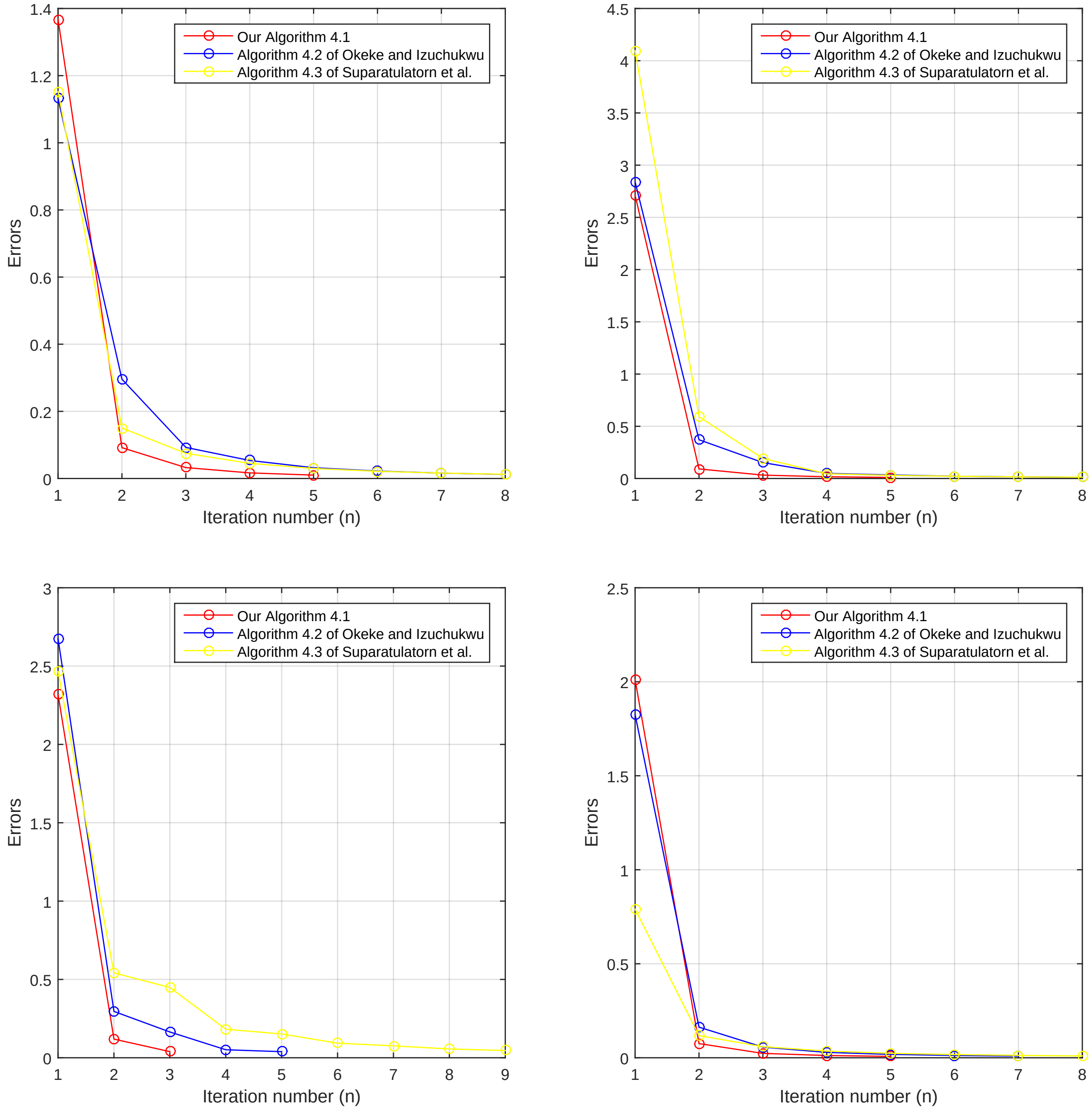

Figure 7.1: Errors vs Iteration numbers: **Case I** (top left); **Case II** (top right); **Case III** (bottom left); **Case IV** (bottom right).

Table 1. Showing CPU time and iteration number

Initial Vectors with Tol=10^{-2}	Our Algorithm 7.4.35 Iter	CPU	Algorithm 7.4.36 Iter	CPU	Algorithm 7.4.37 Iter	CPU
Case I	6	1.0670	11	1.3310	15	1.3910
Case II	6	1.0530	11	1.3010	15	1.3820
Case III	7	1.0650	18	1.4310	25	1.8230
Case IV	5	1.0260	8	1.0910	11	1.3200

Table 2. Showing CPU time and iteration number

Initial Vectors with Tol=10^{-3}	Our Algorithm 7.4.35 Iter	CPU	Algorithm 7.4.36 Iter	CPU	Algorithm 7.4.37 Iter	CPU
Case I	7	1.2290	10	1.4330	11	1.3980
Case II	7	1.2390	10	1.4340	11	1.3980
Case III	7	1.2290	11	1.4330	21	2.0240
Case IV	6	1.0010	8	1.2920	9	1.2990

In the tables above, Iter denotes iteration number, CPU denotes the CPU time in seconds and Tol denotes tolerance (stopping criterion).

7.5 Proximal-type algorithms for split minimization problems in p-uniformly convex metric spaces

In this section, we study the strong convergence of some proximal-type algorithms. Recall that the PPA only converges weakly even in Hilbert spaces, unless additional assumption(s) are imposed on either the convex function or on the underlying space. Since our interest here is to obtain strong convergence results, we shall assume that the proper lower semicontinuous function f is uniformly convex. More precisely, we study the strong convergence of the backward-backward algorithm and the alternating proximal algorithm to a solution of SMPs in complete p-uniformly convex metric spaces.

7.5.1 Backward-backward algorithm

The backward-backward algorithm is defined for an initial point $x_1 \in X$ as:

$$\begin{cases} y_n = J^g_{\mu_n} x_n, \\ x_{n+1} = J^f_{\mu_n} y_n, \ \ n \geq 1, \end{cases} \tag{7.5.1}$$

where $\{\mu_n\}$ is a sequence of positive real numbers and $f, g : X \to (-\infty, \infty]$ are two proper, convex and lower semicontinuous functions (see [22] for a related work in the frame work of Hadamard spaces). In what follows, we shall study the strong convergence of Algorithm (7.5.1) to a solution of the following SMP:

$$\min \ \ \Psi(x,y) \ \ \text{such that} \ \ (x,y) \in X \times X, \ \text{where } \Psi(x,y) = f(x) + g(y) \ \forall x, y \in X. \tag{7.5.2}$$

We begin with the following lemma.

Lemma 7.5.1. *For $1 < p < \infty$, let X be a p-uniformly convex metric space with parameter $c > 0$ and $f, g : X \to (-\infty, +\infty]$ be two proper, convex and lower semicontinuous functions. Let $\{x_n\}$ and $\{y_n\}$ be defined by (7.5.1), where $\{\mu_n\}$ is a sequence of positive real numbers. Then, for any $v = (x,y) \in X \times X$, we have*

$$\Psi(v_n) - \Psi(v) \leq \frac{\sum_{i=1}^{n-1} d(v, v_i)^p - \frac{c}{2}\sum_{i=2}^{n} d(v, v_i)^p}{p\sum_{i=1}^{n-1} \mu_i^{p-1}}, \tag{7.5.3}$$

where $v_n = (x_n, y_n) \in X \times X$.

Proof. By (7.5.1) and (2.2.8), we obtain that

$$g(y_n) + \frac{1}{p\mu_n^{p-1}} d(y_n, x_n)^p \leq g(y) + \frac{1}{p\mu_n^{p-1}} d(x_n, y)^p \tag{7.5.4}$$

and

$$f(x_{n+1}) + \frac{1}{p\mu_n^{p-1}} d(x_{n+1}, y_n)^p \leq f(x) + \frac{1}{p\mu_n^{p-1}} d(y_n, x)^p \tag{7.5.5}$$

Adding (7.5.4) and (7.5.5), we obtain for all $x, y \in X$ that

$$\begin{aligned} & f(x_{n+1}) + g(y_n) + \frac{1}{p\mu_n^{p-1}} \left[d(x_{n+1}, y_n)^p + d(y_n, x_n)^p \right] \\ \leq \quad & f(x) + g(y) + \frac{1}{p\mu_n^{p-1}} \left[d(y_n, x)^p + d(x_n, y)^p \right]. \end{aligned} \tag{7.5.6}$$

In particular, for $y = y_n$, we obtain that

$$\begin{aligned} & f(x_{n+1}) + \frac{1}{p\mu_n^{p-1}} \left[d(x_{n+1}, y_n)^p + d(y_n, x_n)^p \right] \\ \leq \quad & f(x) + \frac{1}{p\mu_n^{p-1}} \left[d(y_n, x)^p + d(x_n, y_n)^p \right]. \end{aligned} \tag{7.5.7}$$

Now, by interchanging f and g, and starting the iteration process at y_1 in (7.5.1), then by an argument similar to above, we obtain that

$$\begin{aligned} & g(y_{n+1}) + f(x_n) + \frac{1}{p\mu_n^{p-1}} \left[d(y_{n+1}, x_n)^p + d(x_n, y_n)^p \right] \\ \leq \quad & g(y) + f(x) + \frac{1}{p\mu_n^{p-1}} \left[d(x_n, y)^p + d(y_n, x)^p \right]. \end{aligned} \tag{7.5.8}$$

By setting $x = x_n$ in (7.5.8), we obtain

$$\begin{aligned} & g(y_{n+1}) + \frac{1}{p\mu_n^{p-1}} \left[d(y_{n+1}, x_n)^p + d(x_n, y_n)^p \right] \\ \leq \quad & g(y) + \frac{1}{p\mu_n^{p-1}} \left[d(x_n, y)^p + d(y_n, x_n)^p \right]. \end{aligned} \tag{7.5.9}$$

Adding (7.5.7) and (7.5.9), we obtain

$$\begin{aligned} f(x_{n+1}) + g(y_{n+1}) + \frac{1}{p\mu_n^{p-1}} \left[d(x_{n+1}, y_n)^p + d(y_{n+1}, x_n)^p \right] \quad \leq \quad & f(x) + g(y) \\ & + \frac{1}{p\mu_n^{p-1}} \left[d(x_n, y)^p + d(y_n, x)^p \right], \end{aligned}$$

which gives by (7.2.1) that

$$\Psi(v_{n+1}) + \frac{1}{p\mu_n^{p-1}} d(v_{n+1}, v_n)^p \leq \Psi(v) + \frac{1}{p\mu_n^{p-1}} d(v_n, v)^p. \tag{7.5.10}$$

Thus, by Remark 7.2.6 (b), (or inequality (7.2.4)), we obtain that

$$p\mu_n^{p-1} \left(\Psi(v_{n+1}) - \Psi(v) \right) \leq d(v, v_n)^p - \frac{c}{2} d(v, v_{n+1})^p. \tag{7.5.11}$$

By letting $v = v_n$ in (7.5.10), we obtain that

$$\Psi(v_{n+1}) + \frac{1}{p\mu_n^{p-1}} d(v_{n+1}, v_n)^p \leq \Psi(v_n),$$

which implies that $\{\Psi(v_n)\}$ is monotone non-increasing. Thus, we obtain from (7.5.11) that

$$\begin{aligned} p\left(\Psi(v_n)-\Psi(v)\right)\sum_{i=1}^{n-1}\mu_i^{p-1} &\leq p\sum_{i=1}^{n-1}\mu_i^{p-1}\left(\Psi(v_{i+1})-\Psi(v)\right) \\ &\leq \sum_{i=1}^{n-1}d(v,v_i)^p-\frac{c}{2}\sum_{i=2}^{n}d(v,v_i)^p, \end{aligned} \tag{7.5.12}$$

which yields the desired conclusion. $\square$

Theorem 7.5.2. *For $1<p<\infty$, let X be a complete p-uniformly convex metric space with parameter $c>0$ such that the diameter of $X\times X$ is $K>0$. Let $f,g:X\to(-\infty,+\infty]$ be two proper, uniformly convex and lower semicontinuous functions and $\{x_n\}$, $\{y_n\}$ be sequences defined by (7.5.1), where $\{\mu_n\}$ is a sequence of positive real numbers such that $\lim\limits_{n\to\infty}\frac{n}{\sum_{i=1}^{n}\mu_i^{p-1}}=0$. Then, $\{(x_n,y_n)\}$ converges to a solution of (7.5.2).*

Proof. Since the diameter of $X\times X$ is $K>0$, therefore we obtain from (7.5.3) that

$$\begin{aligned} \Psi(v_n)-\Psi(v) &\leq \frac{\sum_{i=1}^{n-1}d(v,v_i)^p-\frac{c}{2}\sum_{i=2}^{n}d(v,v_i)^p}{p\sum_{i=1}^{n-1}\mu_i^{p-1}} \\ &\leq \frac{(n-1)K^p}{p\sum_{i=1}^{n-1}\mu_i^{p-1}}\to 0,\ \text{ as } n\to\infty. \end{aligned} \tag{7.5.13}$$

That is, $\lim\limits_{n\to\infty}\Psi(v_n)\leq\Psi(v)$ for all $v\in X\times X$, which implies that

$$\lim_{n\to\infty}\Psi(v_n)=\inf_{v\in(X\times X)}\Psi(v). \tag{7.5.14}$$

Furthermore, we obtain by Proposition 7.2.3 that, there exists a unique minimizer $\bar{v}\in(X\times X)$ of Ψ. Thus, by (7.5.14), we obtain that

$$\lim_{n\to\infty}\Psi(v_n)=\Psi(\bar{v}). \tag{7.5.15}$$

Also, using the uniform convexity of Ψ, we obtain that there exists a function $\psi:\mathbb{R}_+\to\mathbb{R}_+$ with $\psi(t)=0\iff t=0$ such that

$$\Psi\left(\frac{1}{2}v_n\oplus\frac{1}{2}v_m\right)\leq\frac{1}{2}(\Psi(v_n)+\Psi(v_m))-\psi(d(v_n,v_m)),\ \forall n,m\geq 1.$$

Since $\psi(t)=0\iff t=0$, we obtain from (7.5.15) that $d(v_n,v_m)\to 0$, as $n,m\to\infty$. Thus, $\{v_n\}$ is a Cauchy sequence in $X\times X$. As X is complete, so $X\times X$ is also complete. Thus, $\{v_n\}$ converges to a point say $\hat{v}\in X\times X$. It follows from the lower semicontinuity of Ψ (since f and g are lower semicontinuous functions) and (7.5.3) that $\Psi(\hat{v})=\inf\limits_{v\in X\times X}\Psi(v)$. Therefore, we conclude that $\{v_n\}=\{(x_n,y_n)\}$ converges to a solution of (7.5.2). $\square$

Remark 7.5.3. *If X is a complete 2-uniformly convex metric space in Theorem 7.5.2 with parameter $c = 2$ for $X \times X$, then (7.5.13) becomes*

$$\begin{aligned}\Psi(v_n) - \Psi(v) &\leq \frac{\sum_{i=1}^{n-1} d(v, v_i)^2 - \sum_{i=2}^{n} d(v, v_i)^2}{2\sum_{i=1}^{n-1} \mu_i} \\ &\leq \frac{d(v, v_1)^2}{2\sum_{i=1}^{n-1} \mu_i},\end{aligned}$$

which implies that $\lim_{n\to\infty} \Psi(v_n) = \inf_{v\in(X\times X)} \Psi(v)$, provided $\lim_{n\to\infty} \sum_{i=1}^{n-1} \mu_i = \infty$. In this case, we do not need the assumption that $X \times X$ has a diameter $K > 0$. Thus, we obtain the following result from Theorem 7.5.2.

Corollary 7.5.4. *Let X be a complete 2-uniformly convex metric space (in particular, an Hadamard space) and $f, g : X \to (-\infty, +\infty]$ be two proper, uniformly convex and lower semicontinuous functions. Suppose that $\{x_n\}$ and $\{y_n\}$ are sequences defined by (7.5.1), where $\{\mu_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \mu_n = \infty$. Then, $\{(x_n, y_n)\}$ converges to a solution of (7.5.2).*

7.5.2 Alternating proximal algorithm

In problem (7.5.2), the functions f and g are defined on the same space X. In this subsection, we shall consider the SMP for the case where f and g are defined on two different p-uniformly convex metric spaces, say X and Y respectively. That is, we consider the following SMP:

$$\min\ \Psi(x, y) \ \text{ such that }\ (x, y) \in X \times Y, \tag{7.5.16}$$

where X and Y are two different p-uniformly convex metric spaces and $\Psi : X \times Y \to (-\infty, +\infty]$ is a function defined by $\Psi(x, y) = f(x) + g(y)$; $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ are two proper convex and lower semicontinuous functions.

To solve problem (7.5.16), we define the following algorithm called the alternating proximal algorithm: For arbitrary point $v_1 = (x_1, y_1)$ in $X \times Y$, the sequence $\{v_n\} = \{(x_n, y_n)\}$ in $X \times Y$ is defined as follows:

$$(x_n, y_n) \to (x_{n+1}, y_n) \to (x_{n+1}, y_{n+1}),$$
$$\begin{cases} x_{n+1} = \operatorname*{argmin}_{x\in X} \left(\Psi(x, y_n) + \frac{1}{p\mu_n^{p-1}} d(x_n, x)^p\right), \ x \in X, \\ y_{n+1} = \operatorname*{argmin}_{y\in Y} \left(\Psi(x_{n+1}, y) + \frac{1}{p\mu_n^{p-1}} d(y_n, y)^p\right), \ y \in Y, \ n \geq 1, \end{cases} \tag{7.5.17}$$

where $\{\mu_n\}$ is a sequence of positive numbers. We remark here that, in each iteration, we have to solve the following subproblems:

$$\min \Psi(x, y_n) + \frac{1}{p\mu_n^{p-1}} d^2(x_n, x), \quad \text{where } x \in X \tag{7.5.18}$$

and

$$\min \Psi(x_{n+1}, y) + \frac{1}{p\mu_n^{p-1}} d^2(y_n, y), \quad \text{where } y \in Y. \tag{7.5.19}$$

In order to solve the subproblem (7.5.18) or (7.5.19), we employ the following PPA: For arbitrary $x_1 \in X$, $\{x_n\}$ is generated by

$$x_{n+1} = \arg\min_{x \in X} \left(f(x) + \frac{1}{p\mu_n^{p-1}} d(x_n, x)^p \right), \ n \geq 1, \tag{7.5.20}$$

where $f(x) = \Psi(x, y_n)$. This process has been studied in several settings. For instance, in Euclidean spaces (see [11, 14]), Hilbert spaces (see [12, 39]), Hadamard manifolds (see [64]) and Hadamard spaces (see [63]).

Algorithm (7.5.17) has many applications, for instance, it has applications in decision science ([11]), game theory ([12, 64]), PDE's and many other disciplines (see [12, 63]). Furthermore, unlike Algorithm (7.5.1), Algorithm (7.5.17) allows us to check or monitor what happens in each space of action after a given iteration (see [63]).

Therefore, it is of practical importance to study problems of the form (7.5.16) using Algorithm (7.5.17). To this end, we present the following convergence result for problem (7.5.16).

Theorem 7.5.5. *For $1 < p < \infty$, let X and Y be two complete p-uniformly convex metric spaces with parameter $c > 0$ and such that the diameter of $X \times Y$ is $K > 0$. Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be two proper, uniformly convex and lower semicontinuous functions and $\{(x_n, y_n)\}$ be the sequence defined by (7.5.17), where $\{\mu_n\}$ is a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{n}{\sum_{i=1}^{n} \mu_i^{p-1}} = 0$. Then, $\{(x_n, y_n)\}$ converges to a solution of (7.5.16).*

Proof. By (7.5.17) (also see (7.5.20)), we obtain that

$$f(x_{n+1}) + g(y_n) + \frac{1}{p\mu^{p-1}} d(x_n, x_{n+1})^p \leq f(x) + g(y_n) + \frac{1}{p\mu^{p-1}} d(x_n, x)^p \tag{7.5.21}$$

and

$$g(y_{n+1}) + f(x_{n+1}) + \frac{1}{p\mu^{p-1}} d(y_n, y_{n+1})^p \leq g(x) + f(x_{n+1}) + \frac{1}{p\mu^{p-1}} d(y_n, y)^p. \tag{7.5.22}$$

Adding above two inequalities, we obtain that

$$\begin{aligned} & f(x_{n+1}) + g(y_{n+1}) + \frac{1}{p\mu^{p-1}} \left[d(x_n, x_{n+1})^p + d(y_n, y_{n+1})^p \right] \\ \leq \ & f(x) + g(y) + \frac{1}{p\mu^{p-1}} \left[d(x_n, x)^p + d(y_n, y)^p \right], \end{aligned}$$

which gives by (7.2.1) that

$$\begin{aligned} & \Psi(x_{n+1}, y_{n+1}) + \frac{1}{p\mu_n^{p-1}} d((x_n, y_n), (x_{n+1}, y_{n+1}))^p \\ & \leq \Psi(x, y) + \frac{1}{p\mu_n^{p-1}} d((x_n, y_n), (x, y))^p. \end{aligned} \tag{7.5.23}$$

Set $v = (x, y)$ and $v_n = (x_n, y_n)$ in (7.5.23), to get

$$\Psi(v_{n+1}) + \frac{1}{p\mu_n^{p-1}} d(v_n, v_{n+1})^p \leq \Psi(v) + \frac{1}{p\mu_n^{p-1}} d(v_n, v)^p. \tag{7.5.24}$$

As in the proof of (7.5.10) -(7.5.12), we can show that that

$$\Psi(v_n) - \Psi(v) \leq \frac{\sum_{i=1}^{n-1} d(v, v_i)^p - \frac{c}{2} \sum_{i=2}^{n} d(v, v_i)^p}{p \sum_{i=1}^{n-1} \mu_i^{p-1}}. \tag{7.5.25}$$

Hence, by a proof similar to that of Theorem 7.5.2, we obtain the desired conclusion. □

Corollary 7.5.6. *Let X and Y be two complete 2-uniformly convex metric spaces (in particular, Hadamard spaces). Let $f : X \to (-\infty, +\infty]$ and $g : Y \to (-\infty, +\infty]$ be two proper, uniformly convex and lower semicontinuous functions. Suppose that $\{(x_n, y_n)\}$ is a sequence defined by (7.5.17), where $\{\mu_n\}$ is a sequence of positive real numbers such that $\sum_{n=1}^{\infty} \mu_n = \infty$. Then, $\{(x_n, y_n)\}$ converges to a solution of (7.5.16).*

Proof. It follows from Theorem 7.5.5 and Remark 7.5.3. □

Chapter 8

Contributions to Fixed Point Problems in Geodesic Metric Spaces

8.1 Introduction

Besides optimization problems, we also studied fixed point problems for nonlinear mappings like nonexpansive (both singlevalued and multivalued), quasinonexpansive, demicontractive, demimetric and nonspreading-type mappings in previous chapters. In this chapter, we shall focus only on fixed point problems for nonlinear mappings more general than the ones we had previously studied. In particular, we shall introduce and study the classes of asymptotically demicontractive multivalued mappings in Hadamard spaces, strict asymptotically psuedocontractive-type mappings in p-uniformly convex metric spaces and generalized strictly pseduononspreading mappings in p-uniformly convex metric spaces.

8.2 Iterative algorithm for a finite family of asymptotically demicontractive multivalued mappings in Hadamard Spaces

In this section, motivated and inspired by the concept of asymptotically demicontractive singlevalued mappings introduced in Hadamard spaces by Liu and Change [120] (see Section 2.2.4), we introduce the following concept of asymptotically demicontractive multivalued mapping as follows:

Let C be a nonempty subset of an Hadamard space X. A mapping $T : C \subseteq X \to P(X)$ is said to be *asymptotically demicontractive multivalued mapping,* if there exist a constant $k \in [0, 1)$ and a sequence $\{u_n\} \in [0, \infty)$ with $\lim_{n\to\infty} u_n = 0$ such that for all $x \in C,\ p \in F(T),\ n \geq 1$ and $w_n \in T^n x$, we have

$$\mathcal{H}^2(T^n x, p) \leq (1 + u_n)d^2(x, p) + kd^2(x, w_n), \tag{8.2.1}$$

where $\mathcal{H}$ denotes the Hausdorff metric defined by (2.1.12).

In what follows, we prove the strong convergence of a modified Mann iteration to a common fixed point of a finite family of the new class of mappings in the frame work of Hadamard spaces. Furthermore, we give a numerical example of our iterative method to show its applicability.

8.2.1 Main results

Lemma 8.2.1. *Let C be a closed and convex subset of a CAT(0) space X and $T_i : C \subset X \to P(X), (i = 1, 2, \cdots, m)$ be a finite family of uniformly L_i-Lipschitzian and asymptotically demicontractive multivalued mappings such that $k_i \in [0, 1)$ with sequences $\{\rho_{n(i)}\} \subset [0, \infty)$ such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{a_{n(i)}\}_{i=1}^m$ be sequences in $(0, 1)$ such that $a_{n(i)} \leq 1 - k_i$, $i = 1, 2, \ldots, m$, and S_n be a mapping generated by $T_1, T_2, T_3, ..., T_m$ and $a_{n(1)}, a_{n(2)}, a_{n(3)}, ..., a_{n(m)}$ as follows:*

$$\begin{aligned} U_{n(1)}x &= (1 - a_{n(1)})U_{n(0)}x \oplus a_{n(1)}z_{n(1)}, \\ U_{n(2)}x &= (1 - a_{n(2)})U_{n(1)}x \oplus a_{n(2)}z_{n(2)}, \\ &\vdots \\ U_{n(m-1)}x &= (1 - a_{n(m-1)})U_{n(m-2)}x \oplus a_{n(m-1)}z_{n(m-1)}, \\ S_n x = U_{n(m)}x &= (1 - a_{n(m)})U_{n(m-1)}x \oplus a_{n(m)}z_{n(m)}, \end{aligned} \tag{8.2.2}$$

where $U_{n(0)} = I$ and $z_{n(i)} \in T_i^n U_{n(i-1)}x$ for each $i = 1, 2, \cdots m$ and $x \in X$. Then, the following hold:

(i) $d(S_n x, p) \leq (1 + \lambda_n)d(x, p)\ \forall p \in \bigcap_{i=1}^m F(T_i)$, where $\lambda_n = (1 + \max\limits_{1\leq i\leq m} a_{n(i)}\rho_{n(i)})^m - 1$,

(ii) $d(S_n x, S_n y) \leq \bar{L} d(x, y)$, where $\bar{L} := \max\limits_{1\leq i\leq m} L_i^m$.

Proof. For any $p \in \bigcap_{i=1}^m F(T_i), x \in C$ and $z_{n(i)} \in T_i^n U_{n(i-1)}x$ for each $i = 1, 2, \cdots, m$ using Lemma 2.3.1, we obtain

$$\begin{aligned} d^2(S_n x, p) &= d^2((1 - a_{n(m)})U_{n(m-1)}x \oplus a_{n(m)}z_{n(m)}, p) \\ &\leq (1 - a_{n(m)})d^2(U_{n(m-1)}x, p) + a_{n(m)}d^2(z_{n(m)}, p) \\ &\quad -a_{n(m)}(1 - a_{n(m)})d^2(U_{n(m-1)}x, z_{n(m)}) \\ &\leq (1 - a_{n(m)})d^2(U_{n(m-1)}x, p) + a_{n(m)}\mathcal{H}^2(z_{n(m)}, p) \\ &\quad -a_{n(m)}(1 - a_{n(m)})d^2(U_{n(m-1)}x, z_{n(m)}) \\ &\leq (1 - a_{n(m)})d^2(U_{n(m-1)}x, p) \\ &\quad +a_{n(m)}[(1 + \rho_{n(m)})d^2(U_{n(m-1)}x, p) + k_m d^2(U_{n(m-1)}x, z_{n(m)})] \\ &\quad -a_{n(m)}(1 - a_{n(m)})d^2(U_{n(m-1)}x, z_{n(m)}) \\ &= [1 + a_{n(m)}\rho_{n(m)}]d^2(U_{n(m-1)}x, p) \\ &\quad -a_{n(m)}(1 - a_{n(m)} - k_m)d^2(U_{n(m-1)}x, z_{n(m)}) \\ &= [1 + a_{n(m)}\rho_{n(m)}]\Big\{d^2((1 - a_{n(m-1)})U_{n(m-2)}x \oplus a_{n(m-1)}z_{n(m-1)}, p)\Big\} \\ &\quad -a_{n(m)}(1 - a_{n(m)} - k_m)d^2(U_{n(m-1)}x, z_{n(m)}) \end{aligned}$$

$$
\begin{aligned}
\leq\ & [1+a_{n(m)}\rho_{n(m)}]\Big\{(1-a_{n(m-1)})d^2(U_{n(m-2)}x,p) \\
& +a_{n(m-1)}d^2(z_{n(m-1)},p)-a_{n(m-1)}(1-a_{n(m-1)})d^2(U_{n(m-2)}x,z_{n(m-1)})\Big\} \\
& -a_{n(m)}(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x,z_{n(m)}) \\
\leq\ & [1+a_{n(m)}\rho_{n(m)}]\Big\{(1-a_{n(m-1)})d^2(U_{n(m-2)}x,p) \\
& +a_{n(m-1)}\mathcal{H}^2(z_{n(m-1)},p)-a_{n(m-1)}(1-a_{n(m-1)})d^2(U_{n(m-2)}x,z_{n(m-1)})\Big\} \\
& -a_{n(m)}(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x,z_{n(m)}) \\
\leq\ & [1+a_{n(m)}\rho_{n(m)}]\Big\{(1-a_{n(m-1)})d^2(U_{n(m-2)}x,p) \\
& +a_{n(m-1)}[(1+\rho_{n(m-1)})d^2(U_{n(m-2)}x,p)+k_{m-1}d^2(U_{n(m-2)}x,z_{n(m-1)})] \\
& -a_{n(m-1)}(1-a_{n(m-1)})d^2(U_{n(m-2)}x,z_{n(m-1)})\Big\} \\
& -a_{n(m)}(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x,z_{n(m)}) \\
=\ & [1+a_{n(m)}\rho_{n(m)}][1+a_{n(m-1)}\rho_{n(m-1)}]d^2(U_{n(m-2)}x,p) \\
& -a_{n(m)}(1-a_{n(m-1)}-k_{m-1})[1+a_{n(m)}\rho_{n(m)}]d^2(U_{n(m-2)}x,z_{n(m-1)}) \\
& -a_{n(m)}(1-a_{n(m)}-k_m d^2(U_{n(m-1)}x,z_{n(m)}) \\
\leq\ & [1+a_{n(m)}\rho_{n(m)}][1+a_{n(m-1)}\rho_{n(m-1)}][1+a_{n(m-2)}\rho_{n(m-2)}]d^2(U_{n(m-3)}x,p) \\
& -a_{n(m-2)}(1-a_{n(m-2)}-k_{m-2})[1+a_{n(m-1)}\rho_{n(m-1)}] \\
& \quad \times[1+a_{n(m)}\rho_{n(m)}]d^2(U_{n(m-2)}x,z_{n(m-1)}) \\
& \quad -a_{n(m-1)}(1-a_{n(m-1)}-k_{m-1})[1+a_{n(m)}\rho_{n(m)}]d^2(U_{n(m-2)}x,z_{n(m-1)}) \\
& \quad -a_{n(m)}(1-a_{n(m)}-k_m d^2(U_{n(m-1)}x,z_{n(m)}) \\
=\ & \prod_{i=0}^{2}[1+a_{n(i)}\rho_{n(i)}]d^2(x_n,p) \\
& -a_n(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x,z_{n(m)}) \\
& -\sum_{j=1}^{2}\prod_{j=0}^{i-1}a_{n(m-i)}(1-a_{n(m-i)}-k_{m-i}) \\
& \times[1+a_{n(m-j)}\rho_{n(m-j)}]d^2(U_{n(m-i-1)}x,z_{n(m-i)}) \\
\vdots\ & \\
\leq\ & \prod_{i=0}^{m-1}[1+a_{n(i)}\rho_{n(i)}]d^2(x_n,p) \\
& -a_n(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x,z_{n(m)}) \\
& -\sum_{j=1}^{m-1}\prod_{j=0}^{i-1}a_{n(m-i)}(1-a_{n(m-i)}-k_{m-i}) \\
& \times[1+a_{n(m-j)}\rho_{n(m-j)}]d^2(U_{n(m-i-1)}x,z_{n(m-i)}) \\
\leq\ & (1+\max_{1\leq i\leq m}a_{n(i)}\rho_{n(i)})^m d^2(x,p) \\
& -a_n(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x,z_{n(m)})
\end{aligned}
$$

$$\begin{aligned}
& -\sum_{j=1}^{m-1}\prod_{j=0}^{i-1} a_{n(m-i)}(1-a_{n(m-i)}-k_{m-i}) \\
& \times[1+a_{n(m-j)}\rho_{n(m-j)}]d^2(U_{n(m-i-1)}x, z_{n(m-i)}) \\
= \;& (1+\lambda_n)d^2(x,p) \\
& -a_n(1-a_{n(m)}-k_m)d^2(U_{n(m-1)}x, z_{n(m)}) \\
& -\sum_{j=1}^{m-1}\prod_{j=0}^{i-1} a_{n(m-i)}(1-a_{n(m-i)}-k_{m-i}) \\
& \times[1+a_{n(m-j)}\rho_{n(m-j)}]d^2(U_{n(m-i-1)}x, z_{n(m-i)}) \qquad (8.2.3)\\
\leq \;& (1+\lambda_n)d^2(x,p) \qquad (8.2.4)
\end{aligned}$$

where $\lambda_n := (1+\max\limits_{1\leq i\leq m} a_{n(i)}\rho_{n(i)})^m - 1$

(ii) Let $x,y\in C$, then from (8.2.2), if $m=1$ the result follows. Assume $m\neq 1$, then for any $i\in\{1,2,\cdots,m\}$, $z_{n(i)}\in T_i^nU_{n(i-1)}x$ and $w_{n(i)}\in T_i^nU_{n(i-1)}y$, we obtain from Lemma 2.3.1 that

$$\begin{aligned}
d(S_nx,S_ny) \;\leq\;& (1-a_{n(i)})d(U_{n(i-1)}x,U_{n(i-1)}y)+a_{n(i)}d(z_{n(i)},w_{n(i)}) \\
\leq\;& (1-a_{n(i)})d(U_{n(i-1)}x,U_{n(i-1)}y)+a_{n(i)}\mathcal{H}(T_i^nU_{n(i-1)}x,T_i^nU_{n(i-1)}y) \\
\leq\;& (1-a_{n(i)})d(U_{n(i-1)}x,U_{n(i-1)}y)+a_{n(i)}L_id(U_{n(i-1)}x,U_{n(i-1)}y) \\
\leq\;& [1+a_{n(i)}(L_i-1)]\Big\{(1-a_{n(i-1)})d(U_{n(i-2)}x,U_{n(i-2)}y) \\
& +a_{n(i-1)}d(z_{n(i-1)},w_{n(i-1)})\Big\} \\
\leq\;& [1+a_{n(i)}(L_i-1)]\Big\{(1-a_{n(i-1)})d(U_{n(i-2)}x,U_{n(i-2)}y) \\
& +a_{n(i-1)}\mathcal{H}(T_i^nU_{n(i-1)}x,T_i^nU_{n(i-1)}y)\Big\} \\
\leq\;& [1+a_{n(i)}(L_i-1)]\Big\{(1-a_{n(i-1)})d(U_{n(i-2)}x,U_{n(i-2)}y) \\
& +a_{n(i-1)}L_{i-1}d(U_{n(i-1)}x,U_{n(i-1)}y)\Big\} \\
=\;& [1+a_{n(i)}(L_i-1)][1+a_{n(i-1)}(L_{i-1}-1)]d(U_{n(i-2)}x,U_{n(i-2)}y) \\
\vdots\; & \\
\leq\;& [1+a_{n(i)}(L_i-1)][1+a_{n(i-1)}(L_{i-1}-1)]\times\cdots \\
& \times[1+a_{n(2)}(L_2-1)][1+a_{n(1)}(L_1-1)]d(U_{n(0)}x,U_{n(0)}y) \\
=\;& \prod_{j=1}^{i}[1+a_{n(j)}(L_j-1)]d(x,y) \\
\leq\;& [1+\max_{1\leq j\leq i}a_{n(j)}(L_j-1)]^id(x,y) \\
\leq\;& [1+\max_{1\leq j\leq i}(L_j-1)]^id(x,y) \\
=\;& [\max_{1\leq j\leq i}L_j]^id(x,y).
\end{aligned}$$

Letting $\bar{L} := [\max_{1\le j\le i} L_j]^i$, we obtain

$$d(S_n x, S_n y) \le \bar{L} d(x,y). \tag{8.2.5}$$

□

Theorem 8.2.2. *Let C be a closed and convex subset of an Hadamard space X and $T_i : C \subset X \to P(X), (i = 1,2,\cdots,m)$ be a finite family of uniformly L_i-Lipschitzian and asymptotically demicontractive multivalued mappings and $\triangle$-demiclosed such that $k_i \in [0,1)$ with sequences $\{\rho_{n(i)}\} \subset [0,\infty)$ such that $\bigcap_{i=1}^m F(T_i) \neq \emptyset$. Let S_n be as in Lemma 8.2.1, where $t_{n(i)} \in T_i x_n$ with $d(x_n, t_{n(i)}) = d(x_n, T_i x_n)$ for all $i = 1,2,\cdots,m$. Let $\{\alpha_n\}_{n=1}^\infty$, $\{\beta_n\}_{n=1}^\infty$ and $\{a_{n(i)}\}_{i=1}^m$ be sequences in (0,1) satisfying the conditions:*

(c1) $\lim_{n\to\infty} \alpha_n = 0$;

(c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(c3) $\lim_{n\to\infty} \frac{\lambda_n}{\alpha_n} = 0$; *where* $\lim_{n\to\infty} \lambda_n = 0$;

(c4) $0 < c \le a_{n(i)} \le 1 - k_i \ \forall n \ge 1, \ i = 1,2,\cdots m$;

(c5) $0 < a \le \beta_n \le b < 1 \ \forall n \ge 1$.

Then, the sequence $\{x_n\}_{n=1}^\infty$ defined iteratively for arbitrary $x_1 \in C$, by

$$\begin{cases} y_n = (1-\alpha_n)x_n, \\ x_{n+1} = (1-\beta_n)y_n \oplus \beta_n S_n y_n, \end{cases} \tag{8.2.6}$$

converges strongly to $p \in \bigcap_{i=1}^m F(T_i)$.

Proof. Let $\delta_n := (1+\beta_n\lambda_n)\alpha_n$. Since there exists $N_0 > 0$ such that $\frac{\lambda_n}{\alpha_n} \le \frac{\epsilon(1+\beta_n\lambda_n)}{\beta_n}$ for all $n \ge N_0$ and for some $\epsilon > 0$ satisfying $0 \le (1-\epsilon)\delta_n \le 1$, then for any $p \in \bigcap_{i=1}^m F(T_i)$ and $n \ge N_0$, we obtain from Lemma 8.2.1 (i) and (8.2.6) that

$$\begin{aligned} d^2(y_n,p) &= d^2((1-\alpha_n)x_n, p) \\ &= d^2(\alpha_n(0) \oplus (1-\alpha_n)x_n, p) \\ &\le \alpha_n d^2(0,p) + (1-\alpha_n)d^2(x_n,p) - \alpha_n(1-\alpha_n)d^2(x_n,0) \end{aligned} \tag{8.2.7}$$

and

$$\begin{aligned} d^2(x_{n+1},p) &\le (1-\beta_n)d^2(y_n,p) + \beta_n d^2(S_n y_n, p) - \beta_n(1-\beta_n)d^2(y_n, S_n y_n) \\ &\le (1-\beta_n)d^2(y_n,p) + \beta_n(1+\lambda_n)d^2(y_n,p) - \beta_n(1-\beta_n)d^2(y_n,S_n y_n) \\ &= [1-\beta_n+\beta_n(1+\lambda_n)]d^2(y_n,p) - \beta_n(1-\beta_n)d^2(y_n,S_n y_n) \qquad (8.2.8) \\ &\le [1+\beta_n\lambda_n]d^2(y_n,p) \\ &\le [1+\beta_n\lambda_n]\Big(\alpha_n d^2(0,p) + (1-\alpha_n)d^2(x_n,p)\Big) \\ &\le [1-(1-\epsilon)\delta_n]d^2(x_n,p) + \delta_n d^2(0,p) \\ &= [1-(1-\epsilon)\delta_n]d^2(x_n,p) + (1-\epsilon)\delta_n[(1-\epsilon)^{-1}]d^2(0,p) \\ &\le \max\Big\{d^2(x_n,p), (1-\epsilon)^{-1}d^2(0,p)\Big\}. \end{aligned}$$

By induction, we have

$$d^2(x_n, p) \leq \max\left\{d^2(x_{N_0}, p), (1-\epsilon)^{-1}d^2(0, p)\right\}, \; n \geq N_0.$$

Thus, $\{x_n\}_{n=1}^{\infty}$ is bounded, and hence $\{y_n\}_{n=1}^{\infty}$ is bounded. Furthermore from the recursion formula (8.2.6), we obtain

$$\begin{aligned} d(y_n, x_n) &= d(\alpha_n(0) \oplus (1-\alpha_n)x_n, x_n) \\ &\leq \alpha_n d(0, x_n) + (1-\alpha_n)d(x_n, x_n) \to 0 \;\text{ as }\; n \to \infty. \end{aligned} \tag{8.2.9}$$

Since $\{y_n\}_{n=1}^{\infty}$ is bounded, then for some $D > 0$, $d(y_n, p) \leq D$. From (8.2.8), we obtain

$$\begin{aligned} d^2(x_{n+1}, p) \;&\leq\; d^2(x_n, p) + \beta_n\lambda_n D \\ &\quad -\beta_n(1-\beta_n)d^2(y_n, S_n y_n). \end{aligned} \tag{8.2.10}$$

Moreover from (8.2.6) and Lemma 2.3.1 (iii), we obtain

$$d^2(y_n, p) \leq \alpha_n^2 d^2(p, 0) + (1-\alpha_n)^2 d^2(x_n, p) + 2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p}, \overrightarrow{x_n p}\rangle, \tag{8.2.11}$$

which implies from (8.2.10) that

$$\begin{aligned} d^2(x_{n+1}, p) \;&\leq\; \alpha_n^2 d^2(0, p) + (1-\alpha_n)d^2(x_n, p) \\ &\quad +2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p}, \overrightarrow{x_n p}\rangle + \beta_n\lambda_n D \\ &\quad -\beta_n(1-\beta_n)d^2(y_n, S_n y_n) & (8.2.12) \\ &\leq\; (1-\alpha_n)d^2(x_n, p) + \alpha_n^2 d^2(0, p) + \beta_n\lambda_n D \\ &\quad +2\alpha_n(1-\alpha_n)\langle\overrightarrow{0p}, \overrightarrow{x_n p}\rangle. & (8.2.13) \end{aligned}$$

Since $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ are bounded, then there exists $D_1 > 0$ such that

$$\alpha_n d^2(0, p) + (1-\alpha_n)\langle\overrightarrow{0p}, \overrightarrow{x_n p}\rangle \leq D_1 \;\forall n \geq 1. \tag{8.2.14}$$

Then from (8.2.12) and (8.2.14), we obtain

$$\begin{aligned} \beta_n(1-\beta_n)d^2(y_n, S_n y_n) \;&\leq\; d^2(x_n, p) - d^2(x_{n+1}, p) \\ &\quad +\alpha_n[D_1 + \beta_n\lambda_n D - d^2(x_n, p)]. \end{aligned} \tag{8.2.15}$$

We consider two cases to complete the proof.
Case 1. Assume that for any $n_0 \in \mathbb{N}$ such that $\{d(x_n, p)\}_{n=1}^{\infty}$ is nonincreasing. Since $\{x_n\}_{n=1}^{\infty}$ is bounded, $\{d(x_n, p)\}_{n=1}^{\infty}$ is convergent. Then from (8.2.15) and condition (c5), we obtain

$$\beta_n(1-\beta_n)d^2(y_n, S_n y_n) \to 0 \;\text{ as }\; n \to \infty,$$

which implies that

$$d(y_n, S_n y_n) \to 0 \;\text{ as }\; n \to \infty. \tag{8.2.16}$$

Thus,

$$d(x_n, S_n y_n) \leq d(x_n, y_n) + d(y_n, S_n y_n) \to 0 \tag{8.2.17}$$

as $n \to \infty$. Also from (8.2.6), (8.2.9) and (8.2.17), we obtain

$$d(x_{n+1}, x_n) \leq (1-\beta_n)d(y_n, x_n) + \beta_n d(S_n y_n, x_n) \to 0, \text{ as } n \to \infty. \tag{8.2.18}$$

In what follows, we prove that $\lim_{n\to\infty} d(x_n, t_{n(i)}) = 0$ for each $i = 1, 2, \cdots, m$ and $t_{n(i)} \in T_i x_n$. But from (ii) of Lemma 8.2.1, we obtain

$$\begin{aligned} d(x_n, S_n x_n) &\leq d(x_n, S_n y_n) + d(S_n y_n, S_n x_n) \\ &\leq d(x_n, S_n y_n) + \bar{L} d(y_n, x_n). \end{aligned}$$

Then from (8.2.9) and (8.2.17), we obtain

$$d(x_n, S_n x_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{8.2.19}$$

Since

$$d(x_n, p) - d(x_n, S_n x_n) \leq d(x_n, S_n x_n) \leq d(x_n, p) + d(x_n, S_n, x_n),$$

it implies that

$$\lim_{n\to\infty} d(x_n, p) = \lim_{n\to\infty} d(S_n x_n, p). \tag{8.2.20}$$

From (8.2.3), we obtain

$$\begin{aligned} d^2(S_n x_n, p) &\leq (1+\lambda_n)d^2(x_n, p) - \prod_{j=0}^{m-1} a_{n(1)}(1 - a_{n(1)} - k_1) \\ &\quad \times (1 + a_{n(m-j)}\rho_{n(m-j)})d^2(U_{n(0)}x_n, z_{n(1)}) \\ &\leq (1+\lambda_n)d^2(x_n, p) - a_{n(1)}(1 - a_{n(1)} - k_1)d(x_n, z_{n(1)}), \end{aligned}$$

which implies that

$$a_{n(1)}(1 - a_{n(1)} - k_1)d(x_n, z_{n(1)}) \leq d^2(x_n, p) - d^2(S_n x_n, p) + \lambda_n d^2(x_n, p).$$

From (8.2.20) and condition (c4), we obtain

$$\lim_{n\to\infty} d(x_n, z_{n(1)}) = 0. \tag{8.2.21}$$

Thus, from (8.2.2), we obtain

$$d(U_{n(1)}x_n, x_n) = a_{n(1)}d(x_n, z_{n(1)}) \to 0 \quad \text{as} \quad n \to \infty. \tag{8.2.22}$$

Letting $w_{n(1)} \in T_1^n x_n$, we obtain

$$d(w_{n(1)}, z_{n(1)}) \leq \mathcal{H}(T_1^n x_n, T_1^n U_{n(0)} x_n) \leq L_1 d(x_n, x_n) \to 0 \text{ as } n \to 0,$$

hence

$$d(x_n, w_{n(1)}) \leq d(x_n, z_{n(1)}) + d(z_{n(1)}, w_{n(1)}) \to 0 \text{ as } n \to 0.$$

Also from (8.2.3), we obtain

$$\begin{aligned} d^2(S_n x_n, p) &\leq (1+\lambda_n)d^2(x_n, p) - \prod_{j=0}^{m-1} a_{n(2)}(1 - a_{n(2)} - k_2) \\ &\quad \times (1 + a_{n(m-j)}\rho_{n(m-j)})d^2(U_{n(1)}x_n, z_{n(2)}) \\ &\leq (1+\lambda_n)d^2(x_n, p) - a_{n(2)}(1 - a_{n(2)} - k_2)d(U_{n(1)}x_n, z_{n(2)}), \end{aligned}$$

which implies

$$a_{n(2)}(1 - a_{n(2)} - k_2)d(U_{n(1)}x_n, z_{n(2)}) \leq d^2(x_n, p) - d^2(S_n x_n, p) + \lambda_n d^2(x_n, p).$$

From (8.2.20), we obtain

$$\lim_{n\to\infty} d(U_{n(1)}x_n, z_{n(2)}) = 0. \tag{8.2.23}$$

From (8.2.2), we obtain

$$d(U_{n(2)}x_n, x_n) \leq (1 - a_{n(2)})d(U_{n(1)}x_n, x_n) + a_{n(2)}d(z_{n(1)}, z_n),$$

then

$$\lim_{n\to\infty} d(U_{n(2)}x_n, x_n) = 0. \tag{8.2.24}$$

Letting $w_{n(2)} \in T_2^n x_n$, then

$$d(z_{n(2)}, w_{n(2)}) \leq \mathcal{H}(T_2^n U_{n(1)}x_n, T_2^n x_n) \leq L_2 d(U_{n(1)}x_n, x_n) \to 0 \text{ as } n \to \infty,$$

hence

$$\begin{aligned} d(x_n, w_{n(2)}) &\leq d(x_n, U_{n(1)}x_n) + d(U_{n(1)}x_n, z_{n(2)}) \\ &\quad + d(z_{n(2)}, w_{n(2)}) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Repeating these steps, we obtain

$$\lim_{n\to\infty} d(U_{n(2)}x_n, z_{n(3)}) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(w_{n(3)}, z_{n(3)}) = 0.$$

Thus

$$\begin{aligned} d(w_{n(3)}, x_n) &\leq d(w_{n(3)}, z_{n(3)}) + d(z_{n(3)}, U_{n(2)}x_n) \\ &\quad + d(U_{n(2)}x_n, x_n) \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

By continuing in this way, we can show that

$$\lim_{n\to\infty} d(w_{n(i)}, x_n) = 0, \; i = 1, 2, \cdots, m, \quad \text{where} \quad w_{n(i)} \in T_i^n x_n. \tag{8.2.25}$$

Since $w_{n(i)} \subset T_i^n x_n$ for any $i = 1, 2, \cdots, m$, it follows that $T_i w_{n(i)} \subset T_i^{n+1} x_n$. Now, let $v_{n+1(i)} \in T_i w_{n(i)}$, then we have that $v_{n+1(i)} \in T_i^{n+1} x_n$. Then, since T_i is uniformly L_i Lipschtzian, for each $i = 1, 2, \cdots, m$, we obtain

$$\begin{aligned} d(v_{n+1(i)}, w_{n(i)}) &\leq d(v_{n+1(i)}, w_{n+1(i)}) + d(w_{n+1(i)}, x_{n+1}) \\ &\quad d(x_{n+1}, x_n) + d(x_n, w_{n(i)}) \\ &\leq \mathcal{H}(T_i^{n+1} x_n, T_i^{n+1} x_{n+1}) + d(w_{n+1(i)}, x_{n+1}) \\ &\quad d(x_{n+1}, x_n) + d(x_n, w_{n(i)}) \\ &\leq (L_i + 1) d(x_{n+1}, x_n) + d(w_{n+1(i)}, x_{n+1}) \\ &\quad + d(x_n, w_{n(i)}). \end{aligned}$$

From this together with (8.2.18) and (8.2.25), we obtain that

$$\lim_{n \to \infty} d(v_{n+1(i)}, w_{n(i)}) = 0, \ \ i = 1, 2, \cdots, m. \tag{8.2.26}$$

Letting $t_{n(i)} \in T_i x_n$ for each $i = 1, 2, \cdots, m$, then from (8.2.25) and (8.2.26), we obtain

$$\begin{aligned} d(t_{n(i)}, x_n) &\leq d(t_{n(i)}, v_{n+1(i)}) + d(v_{n+1(i)}, w_{n(i)}) + d(w_{n(i)}, x_n) \\ &\leq \mathcal{H}(T_i x_n, T_i^{n+1} x_n) + d(v_{n+1(i)}, w_{n(i)}) + d(w_{n(i)}, x_n) \\ &\leq L_i d(x_n, T_i^n x_n) + d(v_{n+1(i)}, w_{n(i)}) + d(w_{n(i)}, x_n) \\ &= L_i d(x_n, w_{n(i)} x_n) + d(v_{n+1(i)}, w_{n(i)}) + d(w_{n(i)}, x_n) \\ &= (L_i + 1) d(x_n, w_{n(i)} x_n) + d(v_{n+1(i)}, w_{n(i)}). \end{aligned}$$

Then from (8.2.25) and (8.2.26), we obtain

$$\lim_{n \to \infty} d(t_{n(i)}, x_n) = 0, \ \ i = 1, 2, \cdots, m. \tag{8.2.27}$$

Moreover, since $\{x_n\}$ is bounded and X is an Hadamard space, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\triangle - \lim x_{n_i} = p$. Then, from (8.2.27) and the demiclosedness of T_i for each $i = 1, 2, \ldots, m$, we obtain that $p \in \bigcap_{i=1}^m F(T_i)$. Also, from Lemma 2.3.10, we have $\limsup\limits_{n \to \infty} \langle \overrightarrow{0p}, \overrightarrow{x_n p} \rangle \leq 0$.

Thus, from inequality (8.2.13), we get that, for $n \geq N_0$

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1 - \alpha_n) d^2(x_n, p) + \alpha_n^2 d^2(0, p) + \beta_n \lambda_n D \\ &\quad + 2\alpha_n (1 - \alpha_n) \langle \overrightarrow{0p}, \overrightarrow{x_n p} \rangle \\ &= (1 - \alpha_n) d^2(x_n, p) + \alpha_n \left[\alpha_n d^2(0, p) + \frac{\lambda_n}{\alpha_n} \beta_n D + 2(1 - \alpha_n) \langle \overrightarrow{0p}, \overrightarrow{x_n p} \rangle \right]. \end{aligned}$$

It then follows from Lemma 2.3.26 that $d(x_n, p) \to 0$ as $n \to \infty$. Consequently, $x_n \to p$.

Case 2. Suppose that for each $N_0 \in \mathbb{N}$, $\{d(x_n, p)\}_{n \geq N_0}$ is not a decreasing sequence. Then, there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$d(x_{n_k}, p) < d(x_{n_k+1}, p)$$

for all $k \in \mathbb{N}$. Then, by Lemma 2.3.29, there exists an increasing sequence $\{r_s\}_{s\geq N_0}$ such that $r_s \to \infty$, $d(p, x_{r_s}) \leq d(p, x_{r_s+1})$ and $d(p, x_s) \leq d(p, x_{r_s+1})$ for all $s \geq N_0$. Then from (8.2.12) and the fact that $\alpha_n \to 0$, we get

$$\begin{aligned}\beta_{r_s}(1-\beta_{r_s})d^2(y_{r_s}, S_{r_s}y_{r_s}) &\leq d^2(x_{r_s}, p) - d^2(x_{r_s+1}, p)\\ &+\alpha_{r_s}\Big(\alpha_{r_s}d^2(0,p) + \beta_{r_s}\frac{\lambda_{r_s}}{\alpha_{r_s}}D - d^2(x_{r_s}, p)\Big)\\ &+2\alpha_{r_s}(1-\alpha_{r_s})\langle\overrightarrow{0p}, \overrightarrow{x_{r_s}p}\rangle.\end{aligned}$$

This implies $d(y_{r_s}, S_{r_s}y_{r_s}) \to 0$ as $s \to \infty$. Thus, as in Case 1, we obtain that $d(x_{r_s}, t_{r_s(i)}) \to 0$ as $s \to \infty$ for each $i = 1, 2, \cdots, m$ and also following the same argument in Case 1, we get $\limsup\limits_{n\to\infty}\langle\overrightarrow{0p}, \overrightarrow{x_{r_s}b}\rangle \leq 0$. Then (8.2.12), we obtain,

$$\begin{aligned}d^2(x_{r_s+1}, p) &\leq (1-\alpha_{r_s})d^2(x_{r_s}, p) + \alpha_{r_s}^2 d^2(0,p) + \beta_{r_s}\lambda_{r_s}D\\ &\quad +2\alpha_{r_s}(1-\alpha_{r_s})\langle\overrightarrow{0p}, \overrightarrow{x_{r_s}p}\rangle.\end{aligned} \tag{8.2.28}$$

Since $d^2(x_{r_s}, p) \leq d^2(x_{r_s+1}, p)$, (8.2.28) implies that

$$\begin{aligned}\alpha_{r_s}d^2(x_{r_s}, p) &\leq d^2(x_{r_s}, p) - d^2(x_{r_s+1}, p) + \alpha_{r_s}^2 d^2(0,p)\\ &\quad +\beta_{r_s}\lambda_{r_s}D + 2\alpha_{r_s}(1-\alpha_{r_s})\langle\overrightarrow{0p}, \overrightarrow{x_{r_s}p}\rangle\\ &\leq \alpha_{r_s}^2 d^2(0,p) + \beta_{r_s}\lambda_{r_s}D + 2\alpha_{r_s}(1-\alpha_{r_s})\langle\overrightarrow{0,p}, \overrightarrow{x_{r_s}p}\rangle.\end{aligned}$$

In particular, since $\alpha_{r_s} > 0$, we get

$$\begin{aligned}d^2(x_{m_j}, p) &\leq \alpha_{r_s}d^2(0,p) + \beta_{r_s}\frac{\lambda_{r_s}}{\alpha_{r_s}}D\\ &\quad +2(1-\alpha_{r_s})\langle\overrightarrow{0p}, \overrightarrow{x_{r_s}p}\rangle.\end{aligned}$$

Then, since $\limsup\limits_{n\to\infty}\langle\overrightarrow{0p}, \overrightarrow{x_{r_s}p}\rangle \leq 0$ and $\frac{\lambda_{r_s}}{\alpha_{r_s}} \to 0$ as $s \to \infty$, we obtain that $d(x_{r_s}, p) \to 0$ as $s \to \infty$. This together with (8.2.28) give $d(x_{r_s+1}, p) \to 0$ as $s \to \infty$. But $d(x_s, p) \leq d(x_{r_s+1}, p)$, for all $s \geq N_0$, thus we obtain that $x_s \to p$. Therefore, from the above two cases, we can conclude that $\{x_n\}_{n=1}^{\infty}$ converges strongly to an element of $\bigcap_{i=1}^m F(T_i)$ and the proof is complete.

$\square$

8.2.2 Numerical example

In this subsection, we give a numerical example to show that our proposed iterative method can be implemented. Let $X = \mathbb{R}$, endowed with the usual metric. For each $i = 1, 2, \ldots, m$, define $T_i : [0, \infty) \to P(X)$ by

$$T_i x = [-\frac{1}{2^i}x, -\frac{1}{3^i}x], \ \forall x \in [0, \infty).$$

Then, for each $i = 1, 2, \ldots, m$, T_i is asymptotically demicontractive multivalued mapping, with $F(T_i) = \{0\}$. Indeed, for all $x \in [0, \infty)$ and for each $i = 1, 2, \ldots, m$,

$$\begin{aligned} \mathcal{H}^2(T_i^n x, 0) &= \max\left\{ |\frac{1}{2^{in}}x|^2, \ |\frac{1}{3^{in}}x|^2 \right\} \\ &= \left|\frac{1}{2^{in}}x\right|^2 \leq \left(1 + \frac{1}{2^{in}}\right)\left|x - 0\right|^2 + \left|x - 0\right|^2. \end{aligned} \tag{8.2.29}$$

Also, for each $w_n^i \in T_i^n x$, $w_n^i = -\alpha^{in}x$, where $\frac{1}{3^{in}} \leq \alpha^{in} \leq \frac{1}{2^{in}}$, we have for each $i = 1, 2, \ldots, m$ that

$$|w_n^i - x|^2 = |-\alpha^{in}x - x|^2 = \left(1 + \alpha^{in}\right)^2 |x - 0|^2. \tag{8.2.30}$$

From (8.2.29) and (8.2.30), we obtain that

$$\begin{aligned} \mathcal{H}^2(T_i^n x, 0) &\leq \left(1 + \frac{1}{2^{in}}\right)\left|x - 0\right|^2 + \frac{1}{(1 + \alpha^{in})^2}\left|w_n^i - x\right|^2 \\ &\leq \left(1 + \frac{1}{2^{in}}\right)\left|x - 0\right|^2 + \frac{1}{(1 + \frac{1}{3^i})^2}\left|w_n^i - x\right|^2 \\ &= \left(1 + \frac{1}{2^{in}}\right)\left|x - 0\right|^2 + \frac{9^i}{(1 + 3^i)^2}\left|w_n^i - x\right|^2. \end{aligned}$$

Hence, T_i is asymptotically demicontractive multivalued mapping for each $i = 1, 2, \ldots, m$.

We also check that T_i is uniformly L_i-Lipschitzian for each $i = 1, 2, \ldots, m$. Indeed, for each $x, y \in [0, \infty)$ and for each $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \mathcal{H}(T_i x, T_i y) &= \max\left\{ |-\frac{1}{3^{in}}x + \frac{1}{3^{in}}y|, \ |-\frac{1}{2^{in}}x + \frac{1}{2^{in}}y| \right\} \\ &= \frac{1}{2^{in}}|x - y| \\ &\leq \frac{1}{2^i}|x - y|. \end{aligned}$$

Therefore, T_i is uniformly $\frac{1}{2^i}$-Lipschitzian for each $i = 1, 2, \ldots, m$.

For $m = 3$, let $a_{n(i)} = \frac{55in}{784in+5}$ $\forall n \geq 1, \ i = 1, 2, 3$. Then, (8.2.2) becomes:

$$\begin{cases} U_{n(1)}x = \left(1 - \frac{55n}{784n+5}\right)x + \frac{55n}{784n+5}z_{n(1)}, \\ U_{n(2)}x = \left(1 - \frac{110n}{1568n+5}\right)U_{n(1)}x + \frac{110n}{1568n+5}z_{n(2)}, \\ S_n x = \left(1 - \frac{165n}{2352n+5}\right)U_{n(2)}x + \frac{165n}{2352n+5}z_{n(3)}, \ \forall n \geq 1, \end{cases} \tag{8.2.31}$$

where $z_{n(i)} \in \left[-\frac{1}{2^{in}}U_{n(i-1)}x, \ -\frac{1}{3^{in}}U_{n(i-1)}x\right]$ for all $n \geq 1, \ i = 1, 2, 3$.

Now, take $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{n}{2n+3}$ for all $n \geq 1$, then all the conditions in Theorem 8.2.2 are satisfied. Thus, (8.2.6) becomes:

$$\begin{cases} y_n = \frac{n}{n+1}x_n, \\ x_{n+1} = \frac{n+3}{2n+3}y_n + \frac{n}{2n+3}S_n y_n, \ n \geq 1. \end{cases} \tag{8.2.32}$$

We now consider the following cases for our numerical experiments.

Case 1: $x_1 = -1$ and **Case 2:** $x_1 = 1$.

Case 3: $x_1 = -20$ and **Case 4:** $x_1 = 20$.

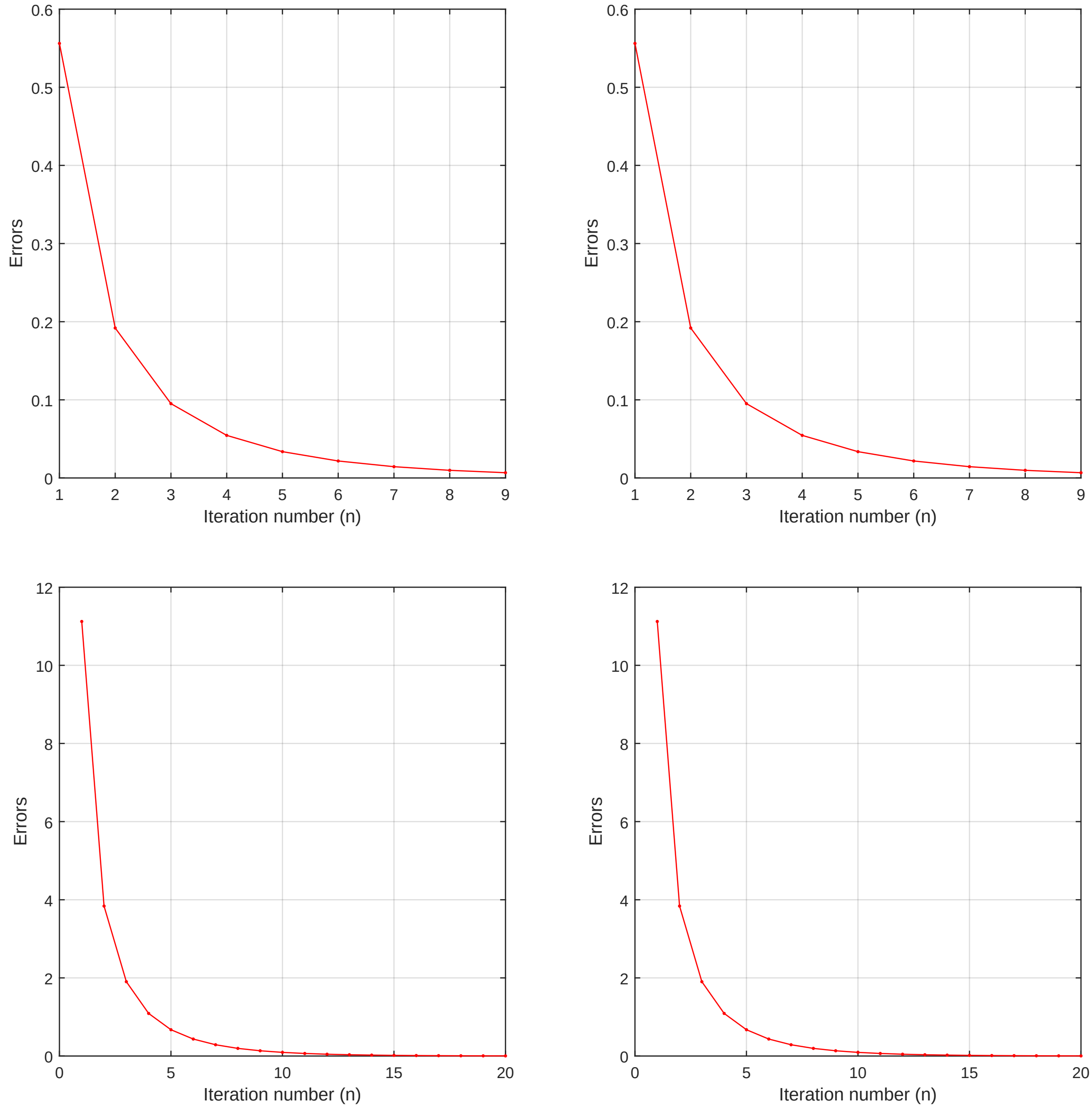

Figure 8.1: Errors vs Iteration numbers(n): **Case 1** (top left); **Case 2** (top right); **Case 3** (bottom left); **Case 4** (bottom right).

8.3 Iterative algorithm for finite family of strict asymptotically pseudocontractive mappings in p-uniformly convex metric spaces

Marino and Xu [125] proved that the Mann iterative sequence converges weakly to a fixed point of a k-strictly pseudocontractive mapping in Hilbert spaces. However, according to them, it is not known if their result (see [125, Theorem 3.1]) also holds in uniformly convex Banach spaces with a Fréchet differentiable norm. In other words, they posed the following question: Can Riech's theorem involving nonexpansive mapping in uniformly convex Banach space (see [156, Theorem 2]) be extended to k-strictly pseudocontractive mapping in the same space?

Hu and Wang [84] provided a partial answer to the above question of Marino and Xu [125], by proving some weak convergence theorems for approximating fixed points of k-strictly pseudocontractive mapping with respect to p, which they defined as follows in p-uniformly convex Banach spaces: Let D be a nonempty subset of a real Banach space E and $T : D \to D$ be any nonlinear mapping. Then, the mapping T is said to be a *k-strictly pseudocontractive mapping with respect to p*, if there exists a constant $k \in [0, 1)$ such that

$$||Tx - Ty||^p \leq ||x - y||^p + k||(x - Tx) - (y - Ty)||^p \ \forall x, y \in D.$$

Motivated by the results of Hu and Wang [84], we introduce and study a new class of mappings more general than that studied by Hu and Wang [84]. Furthermore, we shall study the demicloseness principle for the newly introduced class of mappings. We shall further introduce and study a new scheme in p-uniformly convex metric spaces and establish its Δ-convergence to a common fixed point of a finite family of these mappings in the frame work of complete p-uniformly convex metric spaces.

8.3.1 Main results

Definition 8.3.1. *Let D be a nonempty subset of a p-uniformly convex metric space X. A mapping $T : D \to D$ is said to be k-strict asymptotically pseudocontractive with respect to p, if there exist a constant $k \in [0, 1)$ and a sequence $\{u_n\}_{n=1}^{\infty} \subseteq [1, \infty)$ with $\lim\limits_{n\to\infty} u_n = 1$ such that*

$$d(T^n x, T^n y)^p \leq u_n d(x, y)^p + k\left(d(x, T^n x) + d(y, T^n y)\right)^p \ \ \forall x, y \in D, \ n \geq 1.$$

Lemma 8.3.2. *(Demicloseness Principle) Let D be a nonempty closed and convex subset of a complete p-uniformly convex metric space X with $p > 1$ and parameter $c > 0$. Let $T : D \to D$ be a uniformly L-Lipschitzian and k-strict asymptotically pseudocontractive mapping with respect to p with $k < \min\{1, \frac{c}{4}\}$ and sequence $\{u_n\}_{n=1}^{\infty} \subseteq [1, \infty)$. Suppose $\{x_n\}$ is a bounded sequence in D such that $\Delta\text{-}\lim\limits_{n\to\infty} x_n = z$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. Then $z \in F(T)$.*

Proof. Since T is uniformly L-Lipschitzian and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we obtain for any fixed integer $r \geq 1$ that

$$\begin{aligned} d(x_n, T^r x_n) &\leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots + d(T^{r-1}x_n, T^r x_n) \\ &\leq rLd(x_n, Tx_n) \to 0, \text{ as } n \to \infty. \end{aligned}$$

Also, since $\{x_n\}$ is a bounded sequence in X, we have from Lemma 2.3.24 (i) that $\{x_n\}$ has a unique asymptotic center in X. Thus, using the hypothesis that $\Delta\text{-}\lim_{n\to\infty} x_n = z$, it follows that $A(\{x_n\}) = \{z\}$. Let $\Psi(z) := \limsup_{n\to\infty} d(x_n, z)$. Since $\lim_{n\to\infty} d(x_n, T^r x_n) = 0$, for each $r \geq 1$, we obtain that $\Psi(z) = \limsup_{n\to\infty} d(T^r x_n, z)$. Hence, we get that $\Psi(T^r z) = \limsup_{n\to\infty} d(x_n, T^r z) = \limsup_{n\to\infty} d(T^r x_n, T^r z)$. Thus, we obtain from Definition 8.3.1 that

$$d(T^r x_n, T^r z)^p \leq u_r d(x_n, z)^p + k\left(d(x_n, T^r x) + d(z, T^r z)\right)^p.$$

Taking lim sup on both sides, we obtain that

$$\Psi(T^r z)^p \leq \Psi(z)^p + kd(z, T^r z)^p. \tag{8.3.1}$$

By letting $t = \frac{1}{2}$ in (2.1.10), we have that

$$d(x_n, \frac{z \oplus T^r z}{2})^p \leq \frac{1}{2} d(x_n, z)^p + \frac{1}{2} d(x_n, T^r z)^p - \frac{c}{8} d(z, T^r z)^p.$$

Taking lim sup on both sides of the above inequality and noting that $A(\{x_n\}) = \{z\}$, we obtain that

$$\Psi(z)^p \leq \Psi\left(\frac{z \oplus T^r z}{2}\right)^p \leq \frac{1}{2}\Psi(z)^p + \frac{1}{2}\Psi(T^r z)^p - \frac{c}{8} d(z, T^r z)^p,$$

which implies

$$cd(z, T^r z)^p \leq 4\Psi(T^r z)^p - 4\Psi(z)^p. \tag{8.3.2}$$

From (8.3.1) and (8.3.2), we obtain that
$(c - 4k)d(z, T^r z)^p \leq 0$. From the condition on k, we obtain that $d(z, T^r z) = 0$ for each $r \geq 1$. Hence, we obtain that

$$\begin{aligned} d(z, Tz) &\leq d(z, T^r z) + d(T^r z, Tz) \\ &\leq d(z, T^r z) + Ld(T^{r-1} z, z) = 0, \end{aligned}$$

thus we obtain that $z \in F(T)$. □

Lemma 8.3.3. *Let D be a nonempty closed and convex subset of a p-uniformly convex metric space X with $p > 1$ and parameter $c > 0$. For each $i = 1, 2, \ldots, l$, let $T_i : D \to D$ be uniformly L_i-Lipschitzian and k_i-strict asymptotically pseudocontractive mapping with respect to p, with $k \in [0, 1)$, $k = \max\{k_i,\ i = 1, 2, \ldots, l\}$, $k_i \in [0, 1)$, $i = 1, 2, \ldots, l$ and* *sequence* $\{u_{in}\}_{n=1}^{\infty} \subseteq [1,\infty)$. *Suppose that* $\Gamma := \cap_{i=1}^{l} F(T_i) \neq \emptyset$ *and for arbitrary* $x_1 \in D$, *the sequence* $\{x_n\}$ *is generated by*

$$\begin{cases} x_{n+1} = (1-\alpha_{ln})y_{(l-1)n} \oplus \alpha_{ln}T_l^n y_{(l-1)n}, \\ y_{(l-1)n} = (1-\alpha_{(l-1)n})y_{(l-2)n} \oplus \alpha_{(l-1)n}T_{l-1}^n y_{(l-2)n}, \\ \vdots \\ y_{2n} = (1-\alpha_{2n})y_{1n} \oplus \alpha_{2n}T_2^n y_{1n}, \\ y_{1n} = (1-\alpha_{1n})y_{0n} \oplus \alpha_{1n}T_1^n y_{0n}, \end{cases} \tag{8.3.3}$$

where $y_{0,n} = x_n\ \forall n \geq 1$, *and the following conditions are satisfied:*
$0 < a \leq \alpha_{in} \leq 1-2k_i$, $\sum_{n=1}^{\infty}\left(\max_{1\leq i\leq l} u_{in} - 1\right) < \infty$ *and* $L = \max\{L_i,\ i = 1,2,\ldots,l\}$.
Then, (a) $\lim_{n\to\infty} d(x_n,z)^p$ *exists for all* $z \in \Gamma$ *and (b)* $\lim_{n\to\infty} d(x_n,T_ix_n) = 0,\ i = 1,2,\ldots,l$.

Proof. (a) Let $z \in \Gamma$, then we obtain from (2.1.10) and Definition 8.3.1 that

$$\begin{aligned} & d(y_{(l-1)n},z)^p \\ \leq\ & (1-\alpha_{(l-1)n})d(y_{(l-2)n},z)^p + \alpha_{(l-1)n}d(T_{l-1}^n y_{(l-2)n},z)^p \\ & -\frac{c\alpha_{(l-1)n}(1-\alpha_{(l-1)n})}{2}d(y_{(l-2)n},T_{l-1}^n y_{(l-2)n})^p \\ \leq\ & (1-\alpha_{(l-1)n})d(y_{(l-2)n},z)^p + \alpha_{(l-1)n}\left[u_{(l-1)n}d(y_{(l-2)n},z)^p + k_{(l-1)}d(T_{l-1}^n y_{(l-2)n},y_{(l-2)n})^p\right] \\ & -\frac{c\alpha_{(l-1)n}(1-\alpha_{(l-1)n})}{2}d(y_{(l-2)n},T_{l-1}^n y_{(l-2)n})^p \\ \leq\ & \left[u_{(l-1)n}(1-\alpha_{(l-1)n}) + u_{(l-1)n}\alpha_{(l-1)n}\right]d(y_{(l-2)n},z)^p \\ & -\alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right]d(y_{(l-2)n},T_{l-1}^n y_{(l-2)n})^p \\ =\ & u_{(l-1)n}d(y_{(l-2)n},z)^p - \alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right]d(y_{(l-2)n},T_{l-1}^n y_{(l-2)n})^p. \end{aligned} \tag{8.3.4}$$

Similarly, we obtain

$$\begin{aligned} d(y_{(l-2)n},z)^p \leq\ & u_{(l-2)n}d(y_{(l-3)n},z)^p \\ & -\alpha_{(l-2)n}\left[\frac{c(1-\alpha_{(l-2)n})}{2} - k_{(l-2)}\right]d(y_{(l-3)n},T_{l-2}^n y_{(l-3)n})^p. \end{aligned} \tag{8.3.5}$$

From (8.3.4) and (8.3.5), we obtain

$$\begin{aligned} & d(y_{(l-1)n},z)^p \\ \leq\ & u_{(l-1)n}u_{(l-2)n}d(y_{(l-3)n},z)^p - u_{(l-1)n}\alpha_{(l-2)n}\left[\frac{c(1-\alpha_{(l-2)n})}{2} - k_{(l-2)}\right]d(y_{(l-3)n},T_{l-2}^n y_{(l-3)n})^p \\ & -\alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right]d(y_{(l-2)n},T_{l-1}^n y_{(l-2)n})^p \\ =\ & \prod_{i=1}^{2}u_{(l-i)n}d(y_{(l-3)n},z)^p - u_{(l-1)n}\alpha_{(l-2)n}\left[\frac{c(1-\alpha_{(l-2)n})}{2} - k_{(l-2)}\right]d(y_{(l-3)n},T_{l-2}^n y_{(l-3)n})^p \\ & -\alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right]d(y_{(l-2)n},T_{l-1}^n y_{(l-2)n})^p \end{aligned}$$

$$
\begin{aligned}
&\leq \prod_{i=1}^{3} u_{(l-i)n} d(y_{(l-4)n}, z)^p - \prod_{i=1}^{2} u_{(l-i)n}\alpha_{(l-3)n}\left[\frac{c(1-\alpha_{(l-3)n})}{2} - k_{(l-3)}\right] d(y_{(l-4)n}, T_{l-3}^n y_{(l-4)n})^p \\
&\quad - u_{(l-1)n}\alpha_{(l-2)n}\left[\frac{c(1-\alpha_{(l-2)n})}{2} - k_{(l-2)}\right] d(y_{(l-3)n}, T_{l-2}^n y_{(l-3)n})^p \\
&\quad - \alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right] d(y_{(l-2)n}, T_{l-1}^n y_{(l-2)n})^p \\
&\ \vdots \\
&\leq \prod_{i=1}^{l-1} u_{(l-i)n} d(y_{0n}, z)^p - \prod_{i=1}^{l-2} u_{(l-i)n}\alpha_{1n}\left[\frac{c(1-\alpha_{1n})}{2} - k_1\right] d(y_{0n}, T_1^n y_{0n})^p \\
&\quad - \cdots - \prod_{i=1}^{2} u_{(l-i)n}\alpha_{(l-3)n}\left[\frac{c(1-\alpha_{(l-3)n})}{2} - k_{(l-3)}\right] d(y_{(l-4)n}, T_{l-3}^n y_{(l-4)n})^p \\
&\quad - u_{(l-1)n}\alpha_{(l-2)n}\left[\frac{c(1-\alpha_{(l-2)n})}{2} - k_{(l-2)}\right] d(y_{(l-3)n}, T_{l-2}^n y_{(l-3)n})^p \\
&\quad - \alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right] d(y_{(l-2)n}, T_{l-1}^n y_{(l-2)n})^p \qquad (8.3.6)
\end{aligned}
$$

Again, we obtain from (8.3.3), (8.3.6), (2.1.10) and Definition 8.3.1 that

$$
\begin{aligned}
d(x_{n+1}, z)^p &\leq (1-\alpha_{ln}) d(y_{(l-1)n}, z)^p + \alpha_{(l)n} d(T_{l-1}^n y_{(l-1)n}, z)^p \\
&\quad - \frac{c\alpha_{ln}(1-\alpha_{ln})}{2} d(y_{(l-1)n}, T_l^n y_{(l-1)n})^p \\
&\leq (1-\alpha_{ln}) d(y_{(l-1)n}, z)^p + \alpha_{ln}\left[u_{ln} d(y_{(l-1)n}, z)^p + k_l d(T_l^n y_{(l-1)n}, y_{(l-1)n})^p\right] \\
&\quad - \frac{c\alpha_{ln}(1-\alpha_{ln})}{2} d(y_{(l-1)n}, T_l^n y_{(l-1)n})^p \\
&\leq u_{ln} d(y_{(l-1)n}, z)^p - \alpha_{ln}\left[\frac{c(1-\alpha_{ln})}{2} - k_l\right] d(y_{(l-1)n}, T_l^n y_{(l-1)n})^p \\
&\leq \prod_{i=1}^{l-1} u_{(l-i)n} u_{ln} d(y_{0n}, z)^p - \prod_{i=1}^{l-2} u_{(l-i)n} u_{ln}\alpha_{1n}\left[\frac{c(1-\alpha_{1n})}{2} - k_1\right] d(y_{0n}, T_1^n y_{0n})^p \\
&\quad - \cdots - \prod_{i=1}^{2} u_{(l-i)n} u_{ln}\alpha_{(l-3)n}\left[\frac{c(1-\alpha_{(l-3)n})}{2} - k_{(l-3)}\right] d(y_{(l-4)n}, T_{l-3}^n y_{(l-4)n})^p \\
&\quad - u_{(l-1)n} u_{ln}\alpha_{(l-2)n}\left[\frac{c(1-\alpha_{(l-2)n})}{2} - k_{(l-2)}\right] d(y_{(l-3)n}, T_{l-2}^n y_{(l-3)n})^p \\
&\quad - u_{ln}\alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right] d(y_{(l-2)n}, T_{l-1}^n y_{(l-2)n})^p \\
&\quad - \alpha_{ln}\left[\frac{c(1-\alpha_{ln})}{2} - k_l\right] d(y_{(l-1)n}, T_l^n y_{(l-1),n})^p
\end{aligned}
$$

$$
\begin{aligned}
&= \prod_{i=1}^{l} u_{(l-i+1)n} d(x_n, z)^p - \prod_{i=1}^{l-1} u_{(l-i+1)n}\alpha_{1n}\left[\frac{c(1-\alpha_{1n})}{2} - k_1\right] d(y_{0n}, T_1^n y_{0n})^p \\
&\quad - \cdots - u_{ln}\alpha_{(l-1)n}\left[\frac{c(1-\alpha_{(l-1)n})}{2} - k_{(l-1)}\right] d(y_{(l-2)n}, T_{l-1}^n y_{(l-2)n})^p \\
&\quad -\alpha_{ln}\left[\frac{c(1-\alpha_{ln})}{2} - k_l\right] d(y_{(l-1)n}, T_l^n y_{(l-1),n})^p \qquad (8.3.7)\\
&\le \prod_{i=1}^{l} u_{(l-i+1)n} d(x_n, z)^p \\
&\le \left[1 + \left(\max_{1\le i\le l}\{u_{in}\}\right)^l - 1\right] d(x_n, z)^p.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty}\left(\max_{1\le i\le l} u_{in} - 1\right) < \infty$, it follows that $\lim_{n\to\infty} d(x_n, z)^p$ exists. Hence $\{x_n\}$ is bounded.

(b) Since $\lim_{n\to\infty} u_{in} = 1,\ i = 1, 2, \ldots, l$, we obtain from (8.3.7) and the condition on $\alpha_{i,n}$ that

$$
\lim_{n\to\infty} d(y_{(i-1)n}, T_i^n y_{(i-1)n})^p = 0,\ i = 1, 2, \ldots, l. \qquad (8.3.8)
$$

From (8.3.3), (2.1.10) and (8.3.8), and noting that $y_{0,n} = x_n$, we obtain

$$
d(y_{1n}, x_n)^p \le \alpha_{1n} d(T_1^n x_n, x_n)^p \to 0,\ \text{as } n\to\infty. \qquad (8.3.9)
$$

Now, observe that from Definition 8.3.1, we obtain that

$$
d(T^n x, T^n y)^p \le \left[(u_n)^{\frac{1}{p}} d(x,y) + k^{\frac{1}{p}}\left(d(x, T^n x) + d(y, T^n y)\right)\right]^p,
$$

for each $x, y \in D$. That is,

$$
d(T^n x, T^n y) \le (u_n)^{\frac{1}{p}} d(x,y) + k^{\frac{1}{p}}\left(d(x, T^n x) + d(y, T^n y)\right). \qquad (8.3.10)
$$

Thus,

$$
\begin{aligned}
d(T_2^n x_n, x_n) &\le d(T_2^n x_n, T_2^n y_{1n}) + d(T_2^n y_{1n}, y_{1n}) + d(y_{1n}, x_n) \\
&\le \left[1 + (u_{2n})^{\frac{1}{p}}\right] d(x_n, y_{1n}) + (k_2)^{\frac{1}{p}} d(x_n, T_2^n x_n) + \left[1 + (k_2)^{\frac{1}{p}}\right] d(T_2^n y_{1n}, y_{1n}).
\end{aligned}
$$

Since $1 - (k_2)^{\frac{1}{p}} > 0$, we obtain from (8.3.8) and (8.3.9) that

$$
\lim_{n\to\infty} d(T_2^n x_n, x_n) = 0. \qquad (8.3.11)
$$

Again, from (8.3.8) and (8.3.9), we obtain

$$
d(T_2^n y_{1n}, x_n) \le d(x_n, y_{1n}) + d(y_{1n}, T_2^n y_{1n}) \to 0,\ \text{as } n\to\infty. \qquad (8.3.12)
$$

Thus, we obtain from (8.3.3) and (8.3.9) that

$$
d(y_{2n}, x_n)^p \le (1-\alpha_{2n}) d(y_{1n}, x_n)^p + \alpha_{2n} d(T_2^n y_{1n}, x_n)^p \to 0,\ \text{as } n\to\infty. \qquad (8.3.13)
$$

From (8.3.10), we obtain

$$\begin{aligned} d(T_3^n x_n, x_n) &\le d(T_3^n x_n, T_3^n y_{2n}) + d(T_3^n y_{2n}, y_{2n}) + d(y_{2n}, x_n) \\ &\le \left[1 + (u_{3n})^{\frac{1}{p}}\right] d(x_n, y_{2n}) + (k_3)^{\frac{1}{p}} d(x_n, T_3^n x_n) + \left[1 + (k_3)^{\frac{1}{p}}\right] d(T_3^n y_{2n}, y_{2n}). \end{aligned}$$

Since $1 - (k_3)^{\frac{1}{p}} > 0$, we obtain from (8.3.8) and (8.3.13) that

$$\lim_{n\to\infty} d(T_3^n x_n, x_n) = 0. \tag{8.3.14}$$

Continuing in this manner, we can show that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0,\ \ i = 4, 5, \ldots, l. \tag{8.3.15}$$

Hence, we have that

$$\lim_{n\to\infty} d(x_n, T_i^n x_n) = 0,\ \ i = 1, 2, \ldots, l. \tag{8.3.16}$$

Also, using similar argument as the one used in obtaining (8.3.9) and (8.3.13), we can show that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{8.3.17}$$

Since T is uniformly L-Lipschitzian, we obtain

$$\begin{aligned} d(x_n, T_i x_n) &\le d(x_n, T_i^n x_n) + d(T_i^n x_n, T_i x_n) \\ &\le d(x_n, T_i^n x_n) + L_i d(T_i^{n-1} x_n, x_n) \\ &\le d(x_n, T_i^n x_n) + L d(T_i^{n-1} x_n, T_i^{n-1} x_{n-1}) + L d(T_i^{n-1} x_{n-1}, x_{n-1}) + L d(x_{n-1}, x_n) \\ &\le d(x_n, T_i^n x_n) + (L^2 + L) d(x_n, x_{n-1}) + L d(T_i^{n-1} x_{n-1}, x_{n-1}). \end{aligned}$$

It follows from (8.3.16) and (8.3.17) that

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0,\ \ i = 1, 2, \ldots, l. \tag{8.3.18}$$

□

Theorem 8.3.4. *Let D be a nonempty closed and convex subset of a complete p-uniformly convex metric space X with $p > 1$ and parameter $c > 0$. For each $i = 1, 2, \ldots, l$, let $T_i : D \to D$ be uniformly L_i-Lipschitzian and k_i-strict asymptotically pseudocontractive mapping with respect to p, with $k < \min\{1, \frac{c}{4}\}$, $k = \max\{k_i,\ i = 1, 2, \ldots, l\}$ and sequence $\{u_{in}\}_{n=1}^{\infty} \subseteq [1, \infty)$. Suppose that $\Gamma := \cap_{i=1}^{l} F(T_i) \neq \emptyset$ and for arbitrary $x_1 \in D$, the sequence $\{x_n\}$ is generated by Algorithm (8.3.3), where $y_{0,n} = x_n\ \forall n \ge 1$, and the following conditions are satisfied:*

$0 < a \le \alpha_{in} \le 1 - 2k_i$, $\sum_{n=1}^{\infty}\left(\max\limits_{1\le i\le l} u_{in} - 1\right) < \infty$ *and* $L = \max\{L_i,\ i = 1, 2, \ldots, l\}$.

Then $\{x_n\}$ Δ-converges to $v \in \Gamma$.

Proof. Since $\{x_n\}$ is bounded, then by Lemma 2.3.24 (i), $\{x_n\}$ has a unique asymptotic center. That is, $A(\{x_n\}) = \{v\}$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{u\}$. Then, by (8.3.18), we have that $\lim_{k\to\infty} d(x_{n_k}, T_i x_{n_k}) = 0, i = 1, 2, \ldots, l$. Thus, by Lemma 2.3.24 (ii) and Lemma 8.3.2, we obtain that $u \in \Gamma$.

Moreover, from Lemma 8.3.3 (a), we obtain that $\lim_{n\to\infty} d(x_n, u)$ exists. Thus, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{k\to\infty} d(x_{n_k}, u) &\leq \limsup_{k\to\infty} d(x_{n_k}, v) \\
&\leq \limsup_{n\to\infty} d(x_n, v) \\
&\leq \limsup_{n\to\infty} d(x_n, u) \\
&= \lim_{n\to\infty} d(x_n, u) \\
&= \limsup_{k\to\infty} d(x_{n_k}, u),
\end{aligned}$$

which implies that $v = u$. Therefore, $\{x_n\}$ Δ-converges to $v \in \Gamma$. □

By setting $p = 2$ and $c = 2$ in Theorem 8.3.4, we obtain the following result.

Corollary 8.3.5. *Let X be an Hadamard space and D be a nonempty closed and convex subset of X. For each $i = 1, 2, \ldots, l$, let $T_i : D \to D$ be uniformly L_i-Lipschitzian and k_i-strict asymptotically pseudocontractive mapping, with $k <\in [0, \frac{1}{2})$, $k = \max\{k_i,\ i = 1, 2, \ldots, l\}$ and sequence $\{u_{in}\}_{n=1}^{\infty} \subseteq [1, \infty)$. Suppose that $\Gamma := \cap_{i=1}^{l} F(T_i) \neq \emptyset$ and for arbitrary $x_1 \in D$, the sequence $\{x_n\}$ is generated by Algorithm (8.3.3), where $y_{0,n} = x_n\ \forall n \geq 1$, and the following conditions are satisfied:*
$0 < a \leq \alpha_{in} \leq 1 - k_i$, $\sum_{n=1}^{\infty} \left(\max_{1\leq i\leq l} u_{in} - 1 \right) < \infty$ and $L = \max\{L_i,\ i = 1, 2, \ldots, l\}$.
Then $\{x_n\}$ Δ-converges to $v \in \Gamma$.

8.4 Iterative algorithm for finite family of generalized k-strictly pseudononspreading mappings in p-uniformly convex metric spaces

In this section, motivated by the study of the class of generalized strictly pseudononspreading mappings in Hadamard spaces, discussed in Chapter 6 of this thesis, we introduce and study this class of mappings in p-uniformly convex metric spaces. Furthermore, using Algorithm (8.3.3), we establish a Δ-convergence result for approximating a common fixed point of a finite family of this class of mappings in p-uniformly convex metric spaces.

8.4.1 Main results

Definition 8.4.1. *Let X be a p-uniformly convex metric space with $p > 1$. A mapping $T : D(T) \subseteq X \to X$ is said to be (f, g)-generalized (or simply generalized) k-strictly* *pseudononspreading with respect to p, if there exist two functions $f, g : D(T) \subseteq X \to [0, \gamma]$, $\gamma < 1$ and $k \in [0, 1)$ such that*

$$\begin{aligned} & (1-k)d(Tx,Ty)^p \\ \leq \; & kd(x,y)^p + [f(x)-k]\,d(Tx,y)^p + [g(x)-k]\,d(x,Ty)^p + kd(x,Tx)^p + kd(y,Ty)^p, \end{aligned}$$

$\forall x, y \in D(T)$ and

$$0 < f(x) + g(x) \leq 1 \;\forall x \in D(T).$$

We next study the demicloseness principle for generalized k-strictly pseudononspreading mappings with respect to p.

Lemma 8.4.2. *(Demicloseness Principle) Let D be a nonempty closed and convex subset of a complete p-uniformly convex metric space X with $p > 1$ and parameter $c > 0$. Let $T : D \to D$ be (f, g)-generalized k-strictly pseudononspreading mapping with respect to p, where $k \in [0, 1)$, $f, g : D \to [0, \gamma]$, $\gamma < 1$ and $0 < f(x) + g(x) \leq 1$ for all $x \in D$. Suppose $k < \frac{c(1-f(x))}{4}$ for all $x \in D$, and $\{x_n\}$ is a bounded sequence in D such that $\Delta\text{-}\lim\limits_{n\to\infty} x_n = \bar{v}$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$. Then $\bar{v} \in F(T)$.*

Proof. Since $\{x_n\}$ is a bounded sequence in X, we have from Lemma 2.3.24 (i) that $\{x_n\}$ has a unique asymptotic center in X. Thus, by the hypothesis that $\Delta\text{-}\lim\limits_{n\to\infty} x_n = \bar{v}$, it follows that $A(\{x_n\}) = \{\bar{v}\}$. Let $\Psi(\bar{v}) := \limsup\limits_{n\to\infty} d(\bar{v}, x_n)$. Since $\lim\limits_{n\to\infty} d(x_n, Tx_n) = 0$, we obtain that $\Psi(\bar{v}) = \limsup\limits_{n\to\infty} d(\bar{v}, Tx_n)$. Hence, we get that $\Psi(T\bar{v}) = \limsup\limits_{n\to\infty} d(x_n, T\bar{v}) = \limsup\limits_{n\to\infty} d(Tx_n, T\bar{v})$. Also, we have from Definition 8.4.1 that

$$\begin{aligned} & (1-k)d(T\bar{v},Tx_n)^p \\ \leq \; & kd(\bar{v},x_n)^p + [f(\bar{v})-k]d(T\bar{v},x_n)^p + [g(\bar{v})-k]d(\bar{v},Tx_n)^p + kd(\bar{v},T\bar{v})^p + kd(x_n,Tx_n)^p. \end{aligned}$$

Now, taking limsup on both sides of the above inequality, we obtain that

$$(1-f(\bar{v}))\Psi(T\bar{v})^p \leq g(\bar{v})\Psi(\bar{v})^p + kd^2(\bar{v},T\bar{v}). \tag{8.4.1}$$

By letting $t = \frac{1}{2}$ in (2.1.10), we obtain

$$d\left(\frac{\bar{v} \oplus T\bar{v}}{2}, x_n\right)^p \leq \frac{1}{2}d(\bar{v},x_n)^p + \frac{1}{2}d(T\bar{v},x_n)^p - \frac{c}{8}d(\bar{v},T\bar{v})^p. \tag{8.4.2}$$

Taking $\limsup$ on both sides of (8.4.2) and noting that $A(\{x_n\}) = \{\bar{v}\}$, we obtain that

$$\Psi(\bar{v})^p \leq \Psi\left(\frac{\bar{v} \oplus T\bar{v}}{2}\right)^p \;\leq\; \frac{1}{2}\Psi(\bar{v})^p + \frac{1}{2}\Psi(T\bar{v})^p - \frac{c}{8}d(\bar{v},T\bar{v})^p,$$

which implies that

$$cd(\bar{v},T\bar{v})^p \leq 4\Psi(T\bar{v})^p - 4\Psi(\bar{v})^p. \tag{8.4.3}$$

Substituting (8.4.1) into (8.4.3), we obtain

$$cd(\bar{v}, T\bar{v})^p \leq \frac{4g(\bar{v})}{1-f(\bar{v})}\Psi(\bar{v})^p + \frac{4k}{1-f(\bar{v})}d(\bar{v}, T\bar{v})^p - 4\Psi(\bar{v})^p,$$

which implies

$$\frac{c(1-f(\bar{v}))-4k}{1-f(\bar{v})}d(\bar{v}, T\bar{v})^p \leq \frac{4(g(\bar{v})+f(\bar{v})-1)}{1-f(\bar{v})}\Psi(\bar{v})^p. \tag{8.4.4}$$

Since $g(\bar{v}) + f(\bar{v}) \leq 1$, we obtain from (8.4.4) that

$$(c(1-f(\bar{v}))-4k)\ d(\bar{v}, T\bar{v})^p \leq 0.$$

It then follows from the condition $k < \frac{c(1-f(\bar{v}))}{4}$, that $\bar{v} \in F(T)$. □

Lemma 8.4.3. *Let D be a nonempty closed and convex subset of a p-uniformly convex metric space X with $p > 1$ and parameter $c > 0$. For $i = 1, 2, \ldots, r$, let $T_i : D \to D$ be a finite family of (f_i, g_i)-generalized k_i-strictly pseudononspreading mapping with respect to p, where $k_i \in [0, 1)$, $f_i, g_i : D \to [0, \gamma]$, $\gamma < 1$ and $0 < f_i(x) + g_i(x) \leq 1$ for all $x \in D$. Suppose $\Gamma := \cap_{i=1}^r F(T) \neq \emptyset$ and for arbitrary $x_1 \in D$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} x_{n+1} = (1-\beta_{rn})w_{(r-1)n} \oplus \beta_{rn}T_r w_{(r-1)n}, \\ w_{(r-1)n} = (1-\beta_{(r-1)n})w_{(r-2)n} \oplus \beta_{(r-1)n}T_{r-1}w_{(r-2)n}, \\ \vdots \\ w_{2n} = (1-\beta_{2n})w_{1n} \oplus \beta_{2n}T_2 w_{1n}, \\ w_{1n} = (1-\beta_{1n})w_{0n} \oplus \beta_{1n}T_1 w_{0n}, \end{cases} \tag{8.4.5}$$

where $w_{0n} = x_n$ $\forall n \geq 1$, and $0 < a \leq \beta_{in} < 1 - 2\frac{k_i}{f_i(v)}$, $f_i(v) \neq 0$, for each $v \in \Gamma$. Then, $\lim_{n\to\infty} d(v, x_n)^p$ exists for all $v \in \Gamma$.

Proof. Let $v \in \Gamma$, then we obtain from Definition 8.4.1 that

$$(1-g(v))d(v, Ty)^p \leq f(v)d(v, y)^p + kd(y, Ty)^p \quad \forall y \in D.$$

Since $f(v) + g(v) \leq 1$, we obtain that

$$d(v, Ty)^p \leq d(v, y)^p + \frac{k}{f(v)}d(y, Ty)^p. \tag{8.4.6}$$

Now, from (2.1.10), (8.4.5) and (8.4.6), we obtain

$$\begin{aligned}
d(v,w_{(r-1)n})^p &\leq (1-\beta_{(r-1)n})d(v,w_{(r-2)n})^p+\beta_{(r-1)n}d(v,T_{r-1}w_{(r-2)n})^p\\
&\quad -\frac{c\beta_{(r-1)n}(1-\beta_{(r-1)n})}{2}d(w_{(r-2)n},T_{r-1}w_{(r-2)n})^p\\
&\leq (1-\beta_{(r-1)n})d(v,w_{(r-2)n})^p\\
&\quad +\beta_{(r-1)n}\left[d(v,w_{(r-2)n})^p+\frac{k_{(r-1)}}{f_{(r-1)}(v)}d(T_{r-1}w_{(r-2)n},w_{(r-2)n})^p\right]\\
&\quad -\frac{c\beta_{(r-1)n}(1-\beta_{(r-1)n})}{2}d(w_{(r-2)n},T_{r-1}w_{(r-2)n})^p\\
&= d(v,w_{(r-2)n})^p\\
&\quad -\beta_{(r-1)n}\left[\frac{c(1-\beta_{(r-1)n})}{2}-\frac{k_{(r-1)}}{f_{(r-1)}(v)}\right]d(w_{(r-2)n},T_{r-1}w_{(r-2)n})^p
\end{aligned} \tag{8.4.7}$$

Following the same process as above, we obtain

$$\begin{aligned}
&d(v,w_{(r-2)n})^p\\
&\leq d(v,w_{(r-3)n})^p\\
&-\beta_{(r-2)n}\left[\frac{c(1-\beta_{(r-2)n})}{2}-\frac{k_{(r-2)}}{f_{(r-2)}(v)}\right]d(w_{(r-3)n},T_{r-2}w_{(r-3)n})^p.
\end{aligned} \tag{8.4.8}$$

Again, from (2.1.10), (8.4.5), (8.4.6), (8.4.7) and (8.4.8), we obtain

$$\begin{aligned}
d(v,x_{n+1})^p &\leq (1-\beta_{rn})d(v,w_{(r-1)n})^p+\beta_{(r)n}d(v,T_rw_{(r-1)n})^p\\
&\quad -\frac{c\beta_{rn}(1-\beta_{rn})}{2}d(w_{(r-1)n},T_rw_{(r-1)n})^p\\
&\leq d(v,w_{(r-1)n})^p-\beta_{rn}\left[\frac{c(1-\beta_{rn})}{2}-\frac{k_{(r)}}{f_{(r)}(v)}\right]d(w_{(r-1)n},T_rw_{(r-1)n})^p\\
&\leq d(v,w_{(r-3)n})^p-\beta_{(r-2)n}\left[\frac{c(1-\beta_{(r-2)n})}{2}-\frac{k_{(r-2)}}{f_{(r-2)}(v)}\right]d(w_{(r-3)n},T_{r-2}w_{(r-3)n})^p\\
&\quad -\beta_{(r-1)n}\left[\frac{c(1-\beta_{(r-1)n})}{2}-\frac{k_{(r-1)}}{f_{(r-1)}(v)}\right]d(w_{(r-2)n},T_{r-1}w_{(r-2)n})^p\\
&\quad -\beta_{rn}\left[\frac{c(1-\beta_{rn})}{2}-\frac{k_{(r)}}{f_{(r)}(v)}\right]d(w_{(r-1)n},T_rw_{(r-1)n})^p\\
&= d(v,w_{(r-3)n})^p-\sum_{i=r-2}^{r}\beta_{in}\left[\frac{c(1-\beta_{in})}{2}-\frac{k_{(i)}}{f_{(i)}(v)}\right]d(w_{(i-1)n},T_iw_{(i-1)n})^p\\
&\leq d(v,w_{(r-4)n})^p-\sum_{i=r-3}^{r}\beta_{in}\left[\frac{c(1-\beta_{in})}{2}-\frac{k_{(i)}}{f_{(i)}(v)}\right]d(w_{(i-1)n},T_iw_{(i-1)n})^p\\
&\vdots\\
&\leq d(v,w_{0n})^p-\sum_{i=1}^{r}\beta_{in}\left[\frac{c(1-\beta_{in})}{2}-\frac{k_{(i)}}{f_{(i)}(v)}\right]d(w_{(i-1)n},T_iw_{(i-1)n})^p\\
&= d(v,x_n)^p-\sum_{i=1}^{r}\beta_{in}\left[\frac{c(1-\beta_{in})}{2}-\frac{k_{(i)}}{f_{(i)}(v)}\right]d(w_{(i-1)n},T_iw_{(i-1)n})^p.
\end{aligned} \tag{8.4.9}$$

By the condition on $\{\beta_{in}\},\ i = 1, 2, \ldots, r$, we obtain that $\lim_{n\to\infty} d(v, x_n)^p$ exists. $\square$

Theorem 8.4.4. *Let D be a nonempty closed and convex subset of a complete p-uniformly convex metric space X with $p > 1$ and parameter $c > 0$. For $i = 1, 2, \ldots, r$, let $T_i : D \to D$ be a finite family of (f_i, g_i)-generalized k_i-strictly pseudononspreading mapping with respect to p, where $k_i \in [0, 1)$, $f_i, g_i : D \to [0, \gamma]$, $\gamma < 1$ and $0 < f_i(x) + g_i(x) \leq 1$ for all $x \in D$. Suppose $\Gamma := \cap_{i=1}^{r} F(T) \neq \emptyset$ and for arbitrary $x_1 \in D$, the sequence $\{x_n\}$ is generated by Algorithm (8.4.5), where $w_{0n} = x_n\ \forall n \geq 1$, and the following conditions are satisfied: $k_i < \frac{c(1-f_i(x))}{4}$ and $0 < a \leq \beta_{in} < 1 - 2\frac{k_i}{f_i(v)},\ f_i(v) \neq 0$, for all $v \in \Gamma$.*

Then $\{x_n\}$ Δ-converges to an element of Γ.

Proof. Since $\lim_{n\to\infty} d(v, x_n)^p$ exists, then we obtain from (8.4.9) and the condition on $\{\beta_{in}\},\ i = 1, 2, \ldots, r$ that

$$\lim_{n\to\infty} d(w_{(i-1)n}, T_i w_{(i-1)n})^p = 0,\ i = 1, 2, \ldots, r. \tag{8.4.10}$$

From (2.1.10), (8.4.5), (8.4.10), and noting that $w_{0,n} = x_n$, we obtain

$$d(w_{1n}, x_n)^p \leq \beta_{1n} d(T_1 x_n, x_n)^p \to 0,\ \text{as } n \to \infty. \tag{8.4.11}$$

Also, from (2.1.10) (8.4.5) and (8.4.10), we obtain that

$$d(w_{2n}, w_{1n})^p \leq \beta_{2n} d(T_2 w_{1n}, w_{1n})^p \to 0,\ \text{as } n \to \infty. \tag{8.4.12}$$

It then follows from (8.4.11) and (8.4.12) that

$$\lim_{n\to\infty} d(w_{2n}, x_n) = 0. \tag{8.4.13}$$

Following the same process as in (8.4.11)-(8.4.13), we can show that

$$\lim_{n\to\infty} d(w_{(i-1)n}, x_n) = 0,\ i = 1, 2, \ldots, r. \tag{8.4.14}$$

Now, from Lemma 8.4.3, we obtain that $\{x_n\}$ is bounded. Thus, by Lemma 2.3.24 (i), $\{x_n\}$ has a unique asymptotic center, say $\bar{v}$. That is, $A(\{x_n\}) = \{\bar{v}\}$. Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $A(\{x_{n_k}\}) = \{\bar{u}\}$. Then, by (8.4.14), we obtain for each $i = 1, 2, \ldots, r$ that, $A(\{w_{(i-1)n_k}) = \{\bar{u}\}$. Also, we obtain from (8.4.10) that $\lim_{n\to\infty} d(w_{(i-1)n_k}, T_i w_{(i-1)n_k}) = 0,\ i = 1, 2, \ldots, r$. Thus, by Lemma 2.3.24 (ii) and Lemma 8.4.2, we obtain that $\bar{u} \in \Gamma$.

Moreover, from Lemma 8.4.3, we obtain that $\lim_{n\to\infty} d(\bar{u}, x_n)$ exists. Thus, by the uniqueness of asymptotic centers, we have

$$\begin{aligned}
\limsup_{k\to\infty} d(\bar{u}, x_{n_k}) &\leq \limsup_{k\to\infty} d(\bar{v}, x_{n_k}) \\
&\leq \limsup_{n\to\infty} d(\bar{v}, x_n) \\
&\leq \limsup_{n\to\infty} d(\bar{u}, x_n) \\
&= \lim_{n\to\infty} d(\bar{u}, x_n) \\
&= \limsup_{k\to\infty} d(\bar{u}, x_{n_k}),
\end{aligned}$$

which implies that $\bar{v} = \bar{u}$. Therefore, $\{x_n\}$ Δ-converges to $\bar{v} \in \Gamma$. $\square$

By setting $p = 2$ and $c = 2$ in Theorem 8.4.4, we obtain the following result.

Corollary 8.4.5. *Let D be a nonempty closed and convex subset of an Hadamard space. For $i = 1, 2, \ldots, r$, let $T_i : D \to D$ be a finite family of (f_i, g_i)-generalized k_i-strictly pseudononspreading mapping with constant $k_i \in [0, 1)$, where $f_i, g_i : D \to [0, \gamma]$, $\gamma < 1$ and $0 < f_i(x) + g_i(x) \le 1$ for all $x \in D$. Suppose $\Gamma := \cap_{i=1}^{r} F(T) \neq \emptyset$ and for arbitrary $x_1 \in D$, the sequence $\{x_n\}$ is generated by Algorithm (8.4.5), where $w_{0n} = x_n$ $\forall n \ge 1$, and the following conditions are satisfied:*
$k_i < \frac{1-f_i(x)}{2}$ and $0 < a \le \beta_{in} < 1 - \frac{k_i}{f_i(v)}$, $f_i(v) \neq 0$, for each $v \in \Gamma$.

Then $\{x_n\}$ Δ-converges to an element of Γ.

Chapter 9

Conclusion, Contribution to Knowledge and Future Research

In this chapter, we conclude the study of this thesis and highlight the contributions of our study to knowledge. We also identify and discuss possible areas of future research.

9.1 Conclusion

This thesis presented a systematic and comprehensive study of optimization and fixed point problems in both Hadamard and p-uniformly convex metric spaces. Some of these studies are generalizations of existing results from Hilbert and Banach spaces to these spaces, and others are completely new results even in Hilbert and Banach spaces. We have presented our study in a coherent manner, first by giving in Chapter 1, a brief background of our study for which we highlighted some of the importance of optimization and fixed point problems in general. We also highlighted some of the successful methods used for solving these problems, and the relationships between optimization problems and fixed point problems. We then discussed the research problems studied in this thesis, mainly the ones studied in Chapter 3 to Chapter 8 of this thesis. Afterwards, we highlighted the motivation behind the study of these problems, and the objectives of our study. We then progressed to Chapter 2, to define some basic terms and concepts that was useful throughout our study, gave a detailed literature review of past works that motivated our study and lastly recalled a number of results that were very important to our study. As seen in Chapter 3 to Chapter 8 (which are the main results of this thesis), our results provided important insight of our contribution to the study of optimization and fixed point problems in both Hadamard and p-uniformly convex metric spaces. Chapter 3 to Chapter 6 were devoted to the study of optimization problems and fixed point problems in Hadamard spaces. Chapter 7 was devoted to the study of minimization and fixed point problems in p-uniformly convex metric spaces, while Chapter 8 was all about fixed point problems of higher nonlinear mappings in both Hadamard and p-uniformly convex metric spaces. In each of these chapters, several numerical experiments of our results in comparison with other results in the literature were given to illustrate the applicability and advantages of our results over the existing results in the literature. In some cases, we see that these numerical results are not applicable in the linear settings (Hilbert and Banach spaces). This means that, established results on optimization and fixed point problems in these spaces cannot be applied to such examples. Furthermore, some open problems concerning our established results were also presented (for instance, see Remark 3.3.8, 5.5.17, 6.3.7, 7.3.4 and 7.4.9).

9.2 Contribution to knowledge

In general, the results of this thesis are generalizations of existing results from Hilbert and Banach spaces to Hadamad and p-uniformly convex metric spaces. In many cases, they are completely new results even in Hilbert and Banach spaces. In particular, we made the following contributions, among others.

In chapter 3, we generalize the results of Khatibzadeh and Ranjbar [98], Ranjbar and Khatibzadeh [153] (see Theorems 2.2.6 and 2.2.7) from approximating a solution of a monotone inclusion problem to approximating a common solution of a finite family of monotone inclusion problems, which is also a fixed point of a nonexpansive mapping and a unique solution of some variational inequality problems in Hadamard spaces. Furthermore, to obtain strong convergence results for monotone inclusion problems in Hadamard spaces, Heydari et.al. [83] assumed that the underlying operator, be strongly monotone, while we obtained our strong convergence results without the strong monotonicity assumption. Also, the authors in [95] introduced the concept of dual space of an Hadamard space and the authors in [98] introduced the concept of resolvent of a monotone operator defined on an Hadamard space and valued in the dual space. However, examples of these concepts were not given to motivate the study of monotone inclusion problems in Hadamard spaces, and also to be certain that these sets (the dual space and the resolvents) are not empty. In Subsection 3.3.2, we gave examples of these concepts (see also Section 2.1.3 and Section 2.1.4 for a detailed discussion of these examples).

In Chapter 4, we generalize the results of Bačák [18] (see Theorem 2.2.2) from Δ-convergence results for approximating a solution of minimization problem to strong convergence results for approximating a solution of minimization problem which is also a fixed point of a multivalued nonexpansive mapping and a unique solution of some variational inequality problems in Hadamard spaces. Furthermore, we generalize the work of Suparatulatorn *et. al.* [174] from solving a minimization problem and fixed point problems for nonexpansive mapping to solving a finite family of minimization problem, monotone inclusion problem and fixed point problems for mappings more general than nonexpansive mappings. We also gave numerical examples of our results in Hadamard spaces.

The results established in Chapter 5 of this thesis served as a continuation of the works of Kimura and Kishi [100], Kumam and Chaipunya [108]. They also extend related results from Hilbert spaces to Hadamard spaces. In particular, the results of Section 5.3 and 5.4 generalized the results of Kimura and Kishi [100] (see Theorem 2.2.9, Kumam and Chaipunya [108] (see Theorem 2.2.10) from Δ-convergence results for equilibrium problems to strong convergence results for finite family of equilibrium problems and fixed point problems. Furthermore, the results in Section 5.5 extends the results on mixed equilibrium problems from the frame work of real Hilbert spaces to Hadamard spaces. They also extends the results of Kimura and Kishi [100], Kumam and Chaipunya [108] from equilibrium problems to mixed equilibrium problems.

The class of mappings introduced in Chapter 6 is more general than the class of mappings introduced in [106],[145] and several other classes of nonspreading-type mappings. Furthemore, the results obtained in Chapter 6 generalize and improve the results of Osilike and Isiogugu [145] from fixed point problems for strictly pseudononspreading mappings in Hilbert spaces to fixed point problems for generalized strictly pseudononspreading mappings in Hadamard spaces. Also, in [149, Theorem 3.12], the author proved a Δ-convergence of the Mann-type iteration to a fixed point of a generalized asymptotically nonspreading mapping while in Chapter 6 of this thesis, we prove some strong convergence of Mann-type, Ishikawa-type, S-type and viscosity-type algorithms to a common fixed point of a finite family of generalized strictly pseudononspreading mappings which is also a common solution of a finite family of minimization and monotone inclusion problems in Hadamard spaces. Therefore, our results generalize, improve and complement the results in [145], [149] and host of other related results in this direction.

In Chapter 7, we further developed the study of minimization problems and fixed point problems in p-uniformly convex metric spaces since very few results have been studied in these spaces. First, we improved on the proof of Kuwae [110] for the existence of resolvent of convex function, by removing Assumption 3.21 imposed by Kuwae [110] and by given a more shorter and comprehensive method of proof (see Proposition 7.2.4). We further developed several properties of the resolvent operators. The main results in Section 7.3 and 7.4 improve and generalize the main results of Choi and Ji [52] from solving minimization problems in p-uniformly convex metric spaces to solving finite family of minimization problems and fixed point problems in p-uniformly convex metric spaces. Also, the results in Section 7.5 generalize results on split minimization problems from the frame work of real Hilbert and Banach spaces, as well as Hadamard spaces to p-uniformly convex metric spaces. From application point of view and to further motivate the study in p-uniformly convex metric spaces, we gave a typical example of this space, as well as examples of convex functions and nonexpansive mappings in the space (see Subsection 7.4.2).

The class of mappings introduced and studied in Section 8.2 is more general than that introduced and studied by Liu and Chang [120]. Also, the class of mappings introduced and studied in Section 8.3 of this thesis is more general than that introduced and studied by Hu and Wang [84]. Furthermore, we generalized the results of Hu and Wang [84] from the frame work of p-uniformly convex Banach spaces to the frame work of p-uniformly convex metric spaces. More so, the results in Section 8.4 generalize the results on nonspreading mappings and strictly pseudononspreading mappings in p-uniformly convex Banach spaces.

In addition to these contributions, the open problems identified and discussed in this thesis, offer many opportunities for future research.

9.3 Future research

As mentioned in the previous section, the questions identified and discussed in this thesis (see for example, Remark 3.3.8, 5.5.17, 6.3.7, 7.3.4 and 7.4.9), offer many opportunities for future research. Part of our future research is to try and answer these questions.

Very recently, Alizadeh *et. al.* [4] solved the variational inequality problem (1.2.10) in Hadamard spaces when the cost operator T is an inverse strongly monotone mapping. However, we know that monotone and pseudomonotone mappings are more general and more applicable than inverse strongly monotone mappings. But it is yet unknown if the VIP (1.2.10) can be solved in Hadamard spaces when the cost operator T is either monotone or pseudomonotone. Part of our future research will be to solve the VIP (1.2.10) in Hadamard spaces when the mapping T is atleast monotone. Furthermore, in our future research, we shall try to formulate the VIP, when the cost operator is defined on an Hadamard space and valued in the dual space, which will have an obvious relationship with the classical VIP (1.2.9) in Banach spaces and offcourse, will improve the formulation of Khatibzadeh and Ranjbar [99].

In Section 5.5, we solved the mixed equilibrium problem (1.2.8) in Hadamard spaces. A generalization of this problem called the Generalized Mixed Equilibrium Problem (GMEP) has been studied in Hilbert spaces (see [40, 75, 146] and the references contained therein), which is defined as:

$$\text{Find } \; x^* \in C \; \text{ such that } \; \varphi(x^*, y) + f(y) - f(x^*) + \langle Tx^*, y - x^* \rangle \geq 0, \quad \forall y \in C, \tag{9.3.1}$$

where φ is a bifunction satisfing some monotonicity and convexity assumptions, f is a convex and lower semicontinuous functions and T is a nonlinear operator. It is indisputable that GMEP (9.3.1) is one of the most general and applicable problems in optimization theory since it includes MPs (1.2.1), EPs (1.2.7) and MEPs (1.2.8) as special cases. Thus, we dare to believe that the GMEP (9.3.1) will prove very useful in Hadamard spaces. Therefore, we intend in future, to generalize the GMEP (9.3.1) from the frame work of real Hilbert spaces to Hadamard spaces.

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