Completion of an Incomplete Market by Quadratic Variation Assets

SW Mgobhozi

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by

SW Mgobhozi

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As the candidate's supervisor, I have/have not approved this thesis/dissertation for submission.

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Dr. Sure Mataramvura (Supervisor)

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Prof.Peter Dankelmann (Co-supervisor)

SCHOOL OF STATISTICS AND ACTUARIAL SCIENCE

WESTVILLE CAMPUS, DURBAN, SOUTH AFRICA

UNIVERSITY OF KWAZULU-NATAL

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Abstract

It is well known that the general geometric Lévy market models are incomplete, except for the geometric Brownian and the geometric Poissonian, but such a market can be completed by enlarging it with power-jump assets as Corcuera and Nualart [12] did on their paper. With the knowledge that an incomplete market due to jumps can be completed, we look at other cases of incompleteness. We will consider incompleteness due to more sources of randomness than tradable assets, transactions costs and stochastic volatility. We will show that such markets are incomplete and propose a way to complete them. By doing this we show that such markets can be completed.
In the case of incompleteness due to more randomness than tradable assets, we will enlarge the market using the market's underlying quadratic variation assets. By doing this we show that the market can be completed. Looking at a market paying transactional costs, which is also an incomplete market model due to indifference between the buyers and sellers price, we will show that a market paying transactional costs as the one given by, Cvitanic and Karatzas [13] can be completed.
Empirical findings have shown that the Black and Scholes assumption of constant volatility is inaccurate (see Tompkins [40] for empirical evidence). Volatility is in some sense stochastic, and is divided into two broad classes. The first class being single-factor models, which have only one source of randomness, and are complete markets models. The other class being the multi-factor models in which other random elements are introduced, hence are an incomplete markets models. In this project we look at some commonly used multi-factor models and attempt to complete one of them.

Keywords

Complete markets, equivalent martingale measure, variation processes

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Chapter 1

Introduction

A market model consisting of investment in a safe asset $X_0(t)$ (bank account) and a single risky asset $X_1(t)$ like stock modeled by a stochastic differential equation (SDE) driven by a two dimensional Brownian motion
$B = \{B_1(t), B_2(t), t \geqslant 0\}$, satisfying the following condition

$$\begin{aligned} dX_0 &= qX_0 dt \\ dX_1 &= \alpha dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) \end{aligned}$$

is an incomplete market which has many equivalent martingale measures. Claims in such markets cannot be hedged by a self financing portfolio. In this dissertation we will suggest to enlarge the market by a series of very special assets related to the quadratic variation process. These processes are related to the power-jump process introduced by Corcuera and Nualart [12]. A martingale representative theorem in terms of these quadratic variation processes leads us to market completion.

There are many reasons for incompleteness, for instance, the above market is incomplete because there are more than one equivalent martingale measures, which gives different prices for the contingent claims under each measure. We also have more sources of risk than tradable asset, it will thus be very difficult to hedge away the risk associated with the second source of randomness. If all uncertainty in a market is generated by independent Brownian motions, then completeness roughly corresponds to the requirement that the number of tradable assets be at least as large as the number of Brownian motions. When enlarging such a market, we are in fact creating a market with at least as much tradable asset as Brownian motion processes. After enlargement, we then create a self financing portfolio which we propose to be a hedging portfolio for a market enlarged with quadratic variation asset. This proposition is the main result of the project we are presenting.

We also consider a market portfolio paying transactional costs, like the one presented by Cvitanić and Karatzas [13]. A market paying transactional costs is by definition an incomplete market model. Hedging in such a market is expensive and thus the buyers price of the (European) contingent claim will conflict with the sellers price. We also show that there are claims in such a market which cannot be hedged, hence, the market is incomplete. In the presence of transactional costs, the usual Black-Scholes style of hedging is no longer riskless, which is the reason why we have so much research, attempting to deal with the problem of transactional costs. An important breakthrough was achieved by Leland [32]. He introduced a method of pricing a call option from the seller's view-point in the presence of transactional costs. His main idea was to increase the volatility in the Black-Scholes PDE to offset the increased risk of the seller. He gave his argument for the call option. However, it works as long as the payoff function is convex. In this dissertation, we are only concerned with the completion of the market model paying transactional costs instead of finding a cheapest or optimal hedging strategy. Our hedging strategy in the presence of transactional costs is of course of no interest to practitioners, as they will be only concerned with an optimal trading strategy.

We then move away from Black and Scholes' assumptions of constant volatility and interest rate. We consider stochastic volatility models introduced by Stein and Stein [38], who developed a stochastic volatility model in which the volatility follows an Ornstein-Uhlinbeck process, which raises the possibility that the volatility $\sigma(t)$ can be negative. Assuming volatility is uncorrelated with the asset price, they derived an exact closed-form solution for the stock price distribution. They also used analytic techniques to develop an approximation to the distribution. Then, they used their results to develop closed form option pricing formulas, and to sketch some links between stochastic volatility and the nature of fat tails in stock price distributions.

The model for stock price introduced by Stein and Stein [38] is incomplete market, due to the extra randomness introduced by stochastic volatility. The volatility coefficient introduces an additional source of randomness for every tradable asset which renders the market incomplete. Other stochastic volatility models are models by Heston [26] who used characteristic functions to derive a closed-form solution for the price of a European call option on an asset with stochastic volatility. He assumed that the spot asset's price is correlated with the volatility and concluded that correlation between the spot asset's price and the volatility is important for explaining return skewness and strike-price biases in the Black-Scholes model. The Cox-Ingersoll-Ross (CIR) model and the Hull and White [28] model. We show that such market models can be completed in the same manner as in the case of having more risk than tradable assets, which makes sense since stochastic volatility models are also incomplete due to more sources of randomness than tradable assets.

The addition of jumps into the stock price process was another method introduced to correct the imperfections of the Black-Scholes model of stock price evolution. The motivation for this model is the fact that stock markets do eventually crash and when they do there is no opportunity to carry out a continuously changing delta hedge. One consequence of this will be the impossibility of perfect hedging. At any given time the stock price can increase slightly or decrease slightly or fall a lot. It is impossible to hedge against all these scenarios simultaneously. Such an impossibility of perfect hedging imply that the market is incomplete. That is, not every option can be replicated by a self-financing portfolio. Chapter 7 of our dissertation therefore looks at incompleteness due to jumps and how Corcuera and Nualart [12] managed to complete a market which is incomplete due to jumps.

The remainder of this dissertation is as follow
Chapter 2 contains background theory as well as necessary instruments for the implementation of the theory contained in the later Chapters. Chapter 3 introduces the famous Black-Scholes option pricing formula which is the frequently used formula to predict future movement of stock prices. We also look at the disadvantages of using such a formula to predict future movement of stock prices. The main objective of this Chapter is to introduce the concept of pricing derivatives in both the complete and incomplete market settings. Even though this dissertation is mainly concerned with the concept of completing an incomplete market, one might also be interested in looking at the price of the derivative in an incomplete market so as to compare with the price after completion. Chapter 4 contains the main Propositions of this dissertation. This Chapter shows how to complete an incomplete market with more sources of randomness than tradable assets. This is used to complete a market which is incomplete due to transactional costs in Chapter 5, when transactional costs are stochastic and to completion a market with stochastic volatility in Chapter 6.

Chapter 2

The Background and Basic Tools of Mathematics of Finance

2.1 General Probability Theory

The purpose of this section is to introduce basic probability theory and the definitions which we will frequently encounter throughout this dissertation. The following definitions are taken from Shreve [39].

Definition 2.1. If Ω is a given set, then a σ-algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$,

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$,

(iii) $A_1, A_2, \ldots \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 2.2. Let Ω be a nonempty set, and let $\mathcal{F}$ be a σ-algebra of subset of Ω. A probability measure P is a function that, to every set $A \in \mathcal{F}$, assigns a number in $[0,1]$, called the probability of A and written $P(A)$. We require:
(a) $P(\emptyset) = 0$, $P(\Omega) = 1$,
(b) if $(A_{i\in\mathbb{N}})$ is a sequence of mutually disjoint sets in $\mathcal{F}$ then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a probability space.

Definition 2.3. Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0,T]$ there is a σ-algebra $\mathcal{F}(t)$ on Ω. Assume further that if $s \leq t$, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of σ-algebras $\mathcal{F}(t), 0 \leq t \leq T$, a filtration.

A filtration describes the information we will have at future times t. More precisely, when we get to time t, we will know for each set in $\mathcal{F}(t)$ whether the true $\omega \in \Omega$ lies in that set.

Definition 2.4. A sample space or universal sample space, often denoted S or Ω, of an experiment or random trial is the set of all possible outcomes.

For example, if the experiment is tossing a coin, the sample space is the set $\{\text{head,tail}\}$.

Definition 2.5. A Borel set is a set in the σ-field generated by the class of all bounded, semi-closed intervals of the form $[a, b)$ of the real line.

Definition 2.6. Let X be a random variable defined on a nonempty sample space Ω. The σ-algebra generated by X, denoted $\sigma(X)$, is the collection of all subsets of Ω of the form $\{\omega \in \Omega, X(\omega) \in \mathcal{B}\}$, where $\mathcal{B}$ ranges over the Borel subsets of $\mathbb{R}$.

Definition 2.7. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A random variable is a real-valued function X defined on Ω with the property that for every Borel subset $\mathcal{B}$ of $\mathbb{R}$, the subset of Ω given by

$$\{X \in \mathcal{B}\} = \{\omega \in \Omega; X(\omega) \in \mathcal{B}\}$$

is in the σ-algebra $\mathcal{F}$.

We sometimes also permit a random variable to take the values $+\infty$ and $-\infty$.

Definition 2.8. Let X be a random variable defined on a nonempty sample space Ω. Let $\mathcal{G}$ be a σ-algebra of subsets of Ω. If every set in $\sigma(X)$ is also in $\mathcal{G}$, we say that X is $\mathcal{G}$-measurable.

A random variable X is $\mathcal{G}$-measurable if and only if the information in $\mathcal{G}$ is sufficient to determine the value of X. If X is $\mathcal{G}$-measurable, then $f(X)$ is also $\mathcal{G}$-measurable for any measurable function f.

Definition 2.9. Given a probability space $(\Omega, \mathcal{F}, P)$ a stochastic process (or random process) is a parameterized collection of random variables $X(t)_{t\in[0,T]}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ and assuming values in $\mathbb{R}^n$, where $n \in \mathbb{N}$. We say that $X(t)$ is an n-dimensional stochastic process.

In probability theory, a stochastic process which is sometimes referred to as an Itô process, is the counterpart to a deterministic process. Instead of dealing with only one possible reality of how the process might evolve under time (as is the case, for example, for solutions of an ordinary differential equation), in a stochastic or random process there is some indeterminacy in its future evolution described by probability distributions. This means that even if the initial condition is known, there are many possibilities the process might go to, but some paths may be more probable and others less so.

Definition 2.10. Let Ω be a nonempty sample space with a filtration $\mathcal{F}(t), 0 \leq t \leq T$. Let $X(t)$ be a collection of random variables indexed by $t \in [0, T]$. We say this collection of random variables is an $\mathcal{F}(t)$- adapted stochastic process if, for each t, the random variable $X(t)$ is $\mathcal{F}(t)$-measurable.

Definition 2.11. A stochastic process $\{X(t), t \in [0, T]\}$ is a stationary process if the joint distribution of $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ and $\{X_{t_1+k}, X_{t_2+k}, \ldots, X_{t_n+k}\}$ are identical for all the $\{t_1, t_2, \ldots, t_n\} \in [0, T]$ and all $k \in \mathbb{R}$.

Definition 2.12. The increments of a stochastic process $\{X(t), t \in [0, T]\}$ are defined as $\{X_{t+u} - X_t\}$, $\forall t, t+u \in [0, T]$ and $u > 0$.

Definition 2.13. A stochastic process $\{X(t), t \in [0, T]\}$ has independent increments if for all $t \in [0, T]$ and every $u > 0$, the increments $\{X_{t+u} - X_t\}$ is independent of the past of the process $\{X_s, 0 \leq s \leq t\}$, stated differently,

$$X_{t_1} - X_{t_0}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}},$$

are independent random variables for all $t_0 < t_1 < \ldots < t_n, t_i \in [0, T]$ and $\forall i \in \mathbb{N}$.

A stochastic process $\{X(t), t \in [0, T]\}$ is said to have stationary increments if $\{X_{t+s} - X_t\}$ has the same distribution for all $t \in [0, T]$, $s + t \in [0, T]$ and $s \geq 0$.

Definition 2.14. Two stochastic processes on the same probability space X and Y are modifications if $X(t) = Y(t)$ almost surely ($a.s.$,) for each t.
Two stochastic processes on the same probability space X and Y are indistinguishable if for all t, $X(t) = Y(t)$ $a.s.$

Definition 2.15. If $\{X(t), t \in [0, T]\}$ is a stochastic process, a sample path for the process is the function on $[0, T]$ to the range of the process which assigns to each t the value $X_t(\omega)$.

Definition 2.16. A stochastic process $X = \{X(t), t \geq 0\}$ on $(\Omega, \mathcal{F}, P)$ is said to be càdlàg or right continuous with left limits (RCLL) if it a.s., has sample paths which are right continuous, with left limits. That is for each t the limits $X(t_-) = \lim_{s \to t} X(s)$ and $X(t_+) = \lim_{s \leftarrow t} X(s)$ exist for every $s < t$

2.2 Market, Portfolio and Arbitrage

Definition 2.17. (A), A market is an $\mathcal{F}_t^{(m)}$-adapted $(n+1)$-dimensional Itô process
$X(t) = (X_0(t), X_1(t), ..., X_n(t)), \quad 0 \leq t \leq T$, which satisfies the following stochastic differential equation (SDE)

$$dX_0(t) = \rho(t,\omega)X_0(t)dt, \qquad X_0(0) = 1, \tag{2.1}$$

and

$$\begin{aligned} dX_i(t) &= \mu_i(t,\omega)dt + \sum_{j=1}^{m} \sigma_{ij}(t,\omega)dB_j(t) \\ &= \mu_i(t,\omega)dt + \sigma_i(t,\omega)dB(t), \qquad X_i(0) = x_i, \end{aligned} \tag{2.2}$$

where σ_i is row number i of the $n \times m$ matrix $[\sigma_{ij}]$; $\quad 1 \leq i \leq n \in \mathbb{N}$.

The stochastic differential equation above describes the evolution of asset prices in the market. Where the first equation is showing changes in the bank account (which is known as the risk free asset, X_0), with initial amount of one rand in the bank given by $X_0(0)$. The second stochastic differential equation is showing changes in stock prices (which are risky assets due to the presence of their diffusion term). The coefficients $\rho(t,\omega)$ and $\mu_i(t,\omega)$ represent the mean rate of return for the bank account and stocks, respectively.
Note that the bank account, X_0, is riskless due to the absence of the diffusion terms $B(t)$ (although $\rho(t,\omega)$ may depend on ω), hence, we can precisely determine the future value of the bank account.
For example, consider an investment of $R100$ into the bank account or government bond that pays $R10$ per anum, which is the interest on our fixed deposit for the bank account and coupon payment for government bonds, for five years. We are almost sure with probability one that after five years we will have received 150. There are possibilities that the bank will collapse or that the government will renege on its promise to pay, but such possibilities are sufficiently remote and are thus neglected for practical purposes.

(B) The market $\{X(t)\}_{t\in[0,T]}$ is called normalized if $X_0(t) \equiv 1$.

C) A portfolio in the market $\{X(t)\}_{t\in[0,T]}$ is an $(n+1)$ dimensional (t,ω)-measurable and $\mathcal{F}_t^{(m)}$-adapted stochastic process

$$\theta(t,\omega) = (\theta_0(t,\omega), \theta_1(t,\omega), ..., \theta_n(t,\omega)); \qquad 0 \leq t \leq T, \tag{2.3}$$

which represents the number of stocks and bonds an individual investor holds at any time t.

(D) The value at time t of a portfolio $\theta(t)$ is defined by

$$V(t,\omega) = V^{\theta}(t,\omega) = \theta(t).X(t) = \sum_{i=0}^{n} \theta_i(t)X_i(t). \tag{2.4}$$

(E) The portfolio $\theta(t)$ is called self-financing if

$$dV(t) = \theta(t).dX(t) \qquad \text{for} \qquad t \in [0,T], \tag{2.5}$$

that is, if

$$V(t) = V(0) + \int_0^t \theta(s)dX(s). \tag{2.6}$$

A self-financing portfolio is thus a trading strategy without external cash-flow. Any changes in the value of the portfolio are entirely due to changes in the value of the underlying assets and not due to money being put in or out of the portfolio in order to fund assets sales or purchases.

Definition 2.18. An arbitrage opportunity is an admissible strategy θ such that $V_0(\theta) = 0, \quad V_t(\theta) \geq 0$ for all $t \in [0,T]$, and $E[V_T(\theta)] > 0$.

Therefore an arbitrage is a guaranteed profit without exposure to risk.

Example 2.1 (Foreign exchange)
Suppose a $\pounds 1$ is worth $\$1.23$ and a $\$1$ is worth $R8$. If for some reason the exchange rate between the British pound and the South African rand is worth $R11$ for a pound. As an arbitrageur you can do the following

- Sell $\pounds 2$ for rand's and get $R22$
- Use $R16$ from his $R22$ to buy $\$2$ and keep the remaining $R6$ aside
- Use $\$1.25$ from his $\$2$ to repurchase one of his pounds and use $\$0.75$ to buy $R6$ leaving himself with $\pounds 1$ and $R12$
- Keep $R1$ in his pocket (free lunch) from his $R12$ and use the remaining $R11$ to repurchase another pound. Thus leaving him with his initial $\pounds 2$ and a free $R1$ without any risk in his pocket.
  Increasing the amount of pounds the arbitrageur initially sells or continuously following the above process, will increase the amount of free lunch. The above action will drive the pounds/rand rate down to $R10$ for a pound, which is what it should have been initially. Thus the arbitrage opportunity will be short lived. In the real financial market, arbitrage opportunities can exist but they will generally be very small and disappear quickly. An inverse argument can be applied when a pound is worth less than $R10$.

2.3 Martingales

Definition 2.19. Suppose $(\Omega, \mathcal{F}, P)$ is the probability space with filtration $\{\mathcal{F}_t\}$, $t \in [0, T]$. A real-valued adapted stochastic process $\{X_t\}$ is said to be a supermartingale (submartingale) with respect to the filtration $\{\mathcal{F}_t\}$ if
$a)$ $E[X_t] < \infty$ for all $t \in [0, T]$,
$b)$ $E[X_t|\mathcal{F}_s] \leq X_s$ if $s \leq t$, (rep., $E[X_t|\mathcal{F}_s] \geq X_s$)
If $E[X_t|\mathcal{F}_s] = X_s$ for $s \leq t$ then $\{X_t\}$is said to be martingale.

Remark 2.1. The filtration $\mathcal{F}_t$ represent the history of the Brownian motion $B(t)$ (which we will define in section 2.4) up to time t. We require the process $X(t)$ to be adapted to the filtration $\mathcal{F}_t$, meaning that the value of $X(t)$ is determined by the history of the Brownian motion up to time t.

2.3.1 The Doob-Meyer decomposition

The Doob-Meyer decomposition states that any submartingale can be written as a sum of a martingale and an increasing process, that is, if $X(t)$ is a submartingale, then $X(t) = M(t) + A(t)$ where $M(t)$ is a martingale and $A(t)$ is an increasing process. Thus if $X(t)$ is a martingale, then

$$X(t)^2 = M(t) + A(t) \qquad 0 \leq t < \infty \tag{2.7}$$

where $M(t)$ is a martingale and $A(t)$ is a natural increasing process.

Definition 2.20. For $X \in M_2$, we define the quadratic variation of X to be a process $\langle X \rangle_t \equiv A(t)$ where $A(t)$ is the natural increasing process in the Doob-Meyer Decomposition of X^2. In other words, $\langle X \rangle$ is that unique up to indistinguishable adapted, natural increasing process for which $\langle X \rangle_0 = 0$ a.s and $X^2 - \langle X \rangle$ is a martingale.

Where M_2 is a square integrable processes such that whenever $X \in M_2$ then $X(0) = 0$.

2.4 Brownian Motion

The Brownian Motion is the most important example of a continuous time-martingale which was named after Robert Brown, a Scottish botanist.

Definition 2.21. Let $(\Omega, \mathcal{F}, P)$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $B(t)$ of $t \geq 0$ that satisfies that $B(0) = 0$ and that depends on ω. Then $B(t)$, $t \geq 0$, is a Brownian motion if for all $0 = t_0 < t_1 < \dots < t_n$ the increments

$$B(t_1) = B(t_1) - B(t_0), B(t_2) - B(t_1), ..., B(t_n) - B(t_{n-1}), \tag{2.8}$$

are independent and each of these increments are normally distributed with

$$E[B(t_{i+1}) - B(t_i)] = 0. \tag{2.9}$$

Theorem 2.1. *Suppose $B(t)$ is a Brownian motion with respect to the filtration $\mathcal{F}_t$, $t \geq 0$. Then B_t is an $\mathcal{F}_t$-martingale and $B_t^2 - t$ is also an $\mathcal{F}_t$-martingale.*

Proof

$$\begin{aligned}
E[B(t)|\mathcal{F}_s] &= E[B(t) - B(s) + B(s)|\mathcal{F}_s] \\
&= E[B(t-s) + B(s)|\mathcal{F}_s] \\
&= E[B(t-s)|\mathcal{F}_s] + E[B(s)|\mathcal{F}_s] \\
&= E[B(s)|\mathcal{F}_s] \qquad \text{independence} \\
&= B(s) \qquad \text{for every } s \leq t,
\end{aligned}$$

and

$$\begin{aligned}
E[B^2(t) - t|\mathcal{F}_s] &= E[(B(t) - B(s) + B(s))^2 - t|\mathcal{F}_s] \\
&= E[(B(t-s) + B(s))^2|\mathcal{F}_s] - t \\
&= E[B^2(t-s) + 2B(t-s)B(s) + B^(s)|\mathcal{F}_s] - t \\
&= E[B^2(t-s)|\mathcal{F}_s] - t + B^2(s) \qquad \text{independence} \\
&= t - s + B^2(s) - t \\
&= B^2(s) - s.
\end{aligned}$$

The first part of our theorem shows that a Brownian motion is a martingale while the second part shows us that the quadratic variation of a Brownian motion is just t, that is $\langle B \rangle_t = t$, where $A_t = t$ is the natural increasing process in the Doob-Meyer Decomposition of B^2.

Also note that if $(M_t)_{t \in \mathbb{R}^+}$ is a continuous martingale such that $M_t^2 - t$ is also a martingale, then M_t is a Brownian motion.

2.5 Itô Integral

Definition 2.22. (The Itô integral) Let $\Delta(t, \omega)$, $0 \leq t \leq T$ be an adapted stochastic process. Then the Itô integral of $\Delta(t, \omega)$ is defined by

$$\int_0^T \Delta(t, \omega) dB_t(\omega) = \lim_{n \to \infty} \int_0^T \phi_n(t, \omega) dB_t(\omega), \tag{2.10}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$E[\int_0^T (\Delta(t, \omega) - \phi_n(t, \omega))^2 dt] \to 0 \quad \text{as } n \to \infty. \tag{2.11}$$

2.5.1 Properties of Itô Integral

- (The Itô Isometry)

$$E[(\int_0^T (\Delta(t,\omega)dB(t))^2] = E[\int_0^T \Delta^2(t,\omega)dt] \quad \text{for all } t \in [0,T], \tag{2.12}$$

- (Mean Zero)

$$E[(\int_0^T (\Delta(t,\omega)dB(t))] = 0, \tag{2.13}$$

- (Linearity)

$$\begin{aligned} & \int_0^T a\Delta(t,\omega) + b\Lambda(t,\omega)dB(t) \\ = \quad & a\int_0^T \Delta(t,\omega)dB(t) + b\int_0^T \Lambda(t,\omega)dB(t), \end{aligned}$$

- (Martingale)

$$\begin{aligned} & E[(\int_0^T (\Delta(t,\omega)dB(t)) \; \mathcal{F}_S(\omega)] \\ = \quad & \int_0^S (\Delta(t,\omega)dB(t)), \end{aligned}$$

- (Local Property)

$$\int_0^T \Delta(t,\omega)dB(t) = 0 \qquad a.s., \tag{2.14}$$

  on the set $G = \{\omega \in \Omega : \int_0^T \Delta^2(t,\omega)dt = 0\}$.

For the proof of the above properties of an Itô integral see Shreve [39], Chapter 4.

2.6 Quadratic Variation

In the Doob-Meyer decomposition, we introduced the concept of quadratic variation and we define the quadratic variation of X to be a process $\langle X \rangle_t \equiv A_t$ where A_t is the natural increasing process in the Doob-Meyer Decomposition of X^2. In this section we will further define this quadratic variation which we will use later on to complete incomplete markets. For the case of Lévy processes like the Brownian motion defined above, their paths are unusual in the sense that their quadratic variation is not zero. This is what makes stochastic calculus different from ordinary calculus.

Definition 2.23. Let $f(t)$ be a function defined on $0 \leq t \leq T$. The quadratic variation of f up to time T is

$$\langle f \rangle_T = \lim_{\|\Pi\| \to 0} \sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2. \tag{2.15}$$

Where $\Pi = t_0, t_1, \ldots, t_n$ and $0 = t_0 < t_1 < \ldots < t_n = T$ is a partition of the interval $[0, T]$. The maximum step size of the partition is denoted by $\|\Pi\| = \max_{j=0,\ldots,n-1}(t_{j+1} - t_j)$.

Suppose the function f has a continuous derivative. Hence

$$\begin{aligned}
\sum_{j=0}^{n-1} [f(t_{j+1}) - f(t_j)]^2 &= \sum_{j=0}^{n-1} |f^{'}(t_j^*)|^2 (t_{j+1} - t_j)^2 \\
&\leq \|\Pi\| \sum_{j=0}^{n-1} |f^{'}(t_j^*)|^2 (t_{j+1} - t_j),
\end{aligned}$$

and

$$\begin{aligned}
\langle f \rangle_T &\leq \lim_{\|\Pi\| \to 0} \left[\|\Pi\| \sum_{j=0}^{n-1} |f^{'}(t_j^*)|^2 (t_{j+1} - t_j) \right] \\
&= \lim_{\|\Pi\| \to 0} \|\Pi\| . \lim_{\|\Pi\| \to \infty} \left[\|\Pi\| \sum_{j=0}^{n-1} |f^{'}(t_j^*)|^2 (t_{j+1} - t_j) \right] \\
&= \lim_{\|\Pi\| \to 0} \|\Pi\| . \int_0^T |f^{'}(t_j^*)|^2 dt = 0.
\end{aligned}$$

Which concludes that in ordinary calculus where f is continuous we have zero quadratic variation.

Lemma 2.1. *Let $\Delta(u)$ and $\Theta(u)$ be adapted stochastic processes. Then the quadratic variation of the Itô processes* (2.10) *is given by*

$$\langle X \rangle_t = \int_0^t \Delta^2(u) du, \tag{2.16}$$

where $X(t) = X(0) + \int_0^t \Delta(u) dB(u) + \int_0^t \Theta(u) du$, with $X(0)$ being nonrandom.

Proof
see Shreve [39], Chapter4, page 143.

Theorem 2.2. *For any continuous martingale M_t, the quadratic variation process $\langle M \rangle_t$ is adapted and a.s., finite, continuous and increasing.*

Proof
see Dempster [8]

Remark 2.2. If M_t is a continuous martingale then on (some enrichment of) $(\Omega, \mathcal{F}, P)$ there exists a Brownian motion $B(t)$ such that, by Dempster [8],

$$M_t = B(\langle M \rangle_t). \tag{2.17}$$

This is one reason why Brownian motion is so important, up to change of time scale, every continuous martingale is a Brownian motion!

Example 2.2
Consider the following Ornstein-Uhlinbeck process given by

$$dX(t) = \beta(a - X(t))dt + \sigma dB(t). \tag{2.18}$$

We can easily solve the above equation quite explicitly by considering its integrating factor, $e^{\beta t}$, which gives,

$$\begin{aligned} d[e^{\beta t}(X(t) - a)] &= e^{\beta t}[dX(t) + \beta(X(t) - a)dt] \\ &= \sigma e^{\beta t} dB(t). \end{aligned}$$

So that

$$e^{\beta t}(X(t) - a) - (X(0) - a) = \int_0^t \sigma e^{\beta s} dB(s),$$

and

$$\begin{aligned} X(t) &= 1 - e^{-\beta t}a + e^{-\beta t}X(0) + \int_0^t \sigma e^{\beta(s-t)} dB(s) \\ &= 1 - e^{-\beta t}a + e^{-\beta t}X(0) + e^{-\beta t}\sigma^2 W\left(\frac{e^{(2\beta t - 1)}}{2\beta}\right), \end{aligned}$$

for some Brownian motion W, in light of the key result of remark (2.2).

2.7 The Girsanov's Theorem

Girsanov's theorem is fundamental in the general theory of stochastic analysis. It is also very important in many applications, for example in economics. Basically the Girsanov's theorem says that if we change the drift coefficient of a given Itô process, then the law of the process will not change dramatically. In fact the law of the new process will be absolutely continuous w.r.t the law of the original process.

2.7.1 Heuristic Introduction of Derivation

This introduction is taken from Foellmer [20]. It makes use of elementary facts of independent normally distributed random variables which leads to the Doléans-Dade exponential as a new density under a change of measure for the Brownian motions. This by Foellmer [20] is called the heuristic derivation of

the Girsanov transformation for the 1-dimensional Brownian motion based on elementary probability concepts. Such an heuristic derivation will help us understand what happens under the Girsanov transformation.
Let X be a standard normally distributed random variable on the probability space $(\Omega, \mathcal{F}, P)$ for some real world measure P. That is $X \sim N(0,1)$ for standard normal distribution. Then

$$\begin{aligned} & P[X \leq a] \qquad a \in \mathbb{R} \\ = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}x^2} dx \\ = & \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} f(x) dx \\ = & N(a). \end{aligned}$$

Now consider another density function given by

$$\begin{aligned} \widehat{f}(x) &= e^{(\mu x - \frac{1}{2}\mu^2)} f(x) \\ &= e^{-\frac{1}{2}(x^2 - 2\mu x + \mu^2)} \\ &= e^{-\frac{1}{2}(x-\mu)^2}. \end{aligned}$$

Then for some probability $Q = \widehat{P}$ we have

$$\begin{aligned} Q[X \leq a] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}(x-\mu)^2} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \widehat{f} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-\frac{1}{2}\widehat{x}^2} d\widehat{x} \\ &= N(a), \end{aligned}$$

where $\widehat{X} = X - \mu$.
Hence the distribution of X under P is equivalent to the distribution of $\widehat{X}$ under Q, that is $P_X \equiv Q_{\widehat{X}}$ and $\widehat{X} \sim N(0,1)$. Also note that under Q, $X \sim N(\mu, 1)$. The Radon-Nikodym derivative for this change of measure is given by

$$\frac{dQ_X}{dP_X}(x) = e^{(\mu x - \frac{1}{2}\mu^2)}. \tag{2.19}$$

Now suppose $X \sim N(0, \sigma^2)$ under P and $X \sim N(\mu, \sigma^2)$ under Q, that is $\widehat{X} = (X - \mu) \sim N(0, \sigma^2)$. It then follows that the Radon-Nikodym derivative for this change of measure is given by

$$\frac{dQ_X}{dP_X}(x) = e^{\frac{1}{\sigma^2}(\mu x - \frac{1}{2}\mu^2)}. \tag{2.20}$$

Application to Brownian Motion $B(t)$ $t \in [0,1]$
Let $B(t)$ be a Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Then $B(t) \sim$

$N(0,t)$ under P and $\Delta B(t) = (B(t+\Delta t) - B(t)) \sim N(0,\Delta t)$, independent of $B(t)$. Now consider a Brownian motion with drift given by

$$\widehat{B}(t) = B(t) - \int_0^t \phi(s)ds,$$

for some stochastic process $\phi(s)$. Now the question is, under which measure Q is $\widehat{B}(t)$ again a Brownian motion without the drift term?
We discretize the unit interval $[0,1]$ by $t_i = \frac{i}{n}$ $(i = 0,\dots,n)$.
It follows that

$$\begin{aligned} B(\frac{j}{n}) &= \sum_{i=1}^{j} B\left(\frac{i}{n}\right) - B\left(\frac{i-1}{n}\right) \qquad j = 1,\dots,n \\ &= \sum_{i=1}^{j} X_i \sim N(0,\Delta t), \end{aligned}$$

under P.
The $\prod_{i=1}^{n} N(0,\Delta t)$ is a joint distribution of the random variables $(X_1,\dots,X_n)$ under P. If we define $\widehat{X}_i = X_i - \phi(\frac{i-1}{n}).\Delta t$, then $\widehat{X}_i \sim N(0,\Delta t)$ under Q with the Radon-Nikodym-derivative given by

$$\frac{dQ_X}{dP_X}(x) = e^{\frac{1}{\Delta t}(\mu x_i - \frac{1}{2}\mu_i^2)}, \tag{2.21}$$

under P, where $\mu_i = \phi(\frac{i-1}{n}).\Delta t$.
The joint distribution of $\widehat{X}_i$ under Q is thus given by

$$\begin{aligned} dQ^{(n)} &= \prod_{i=1}^{n} \exp\left\{\frac{1}{\Delta t}\left(\phi\left(\frac{i-1}{n}\right).\Delta t X_i - \frac{1}{2}\phi^2\left(\frac{i-1}{n}\right).\Delta t^2\right)\right\} dP^{(n)} \\ &= \exp\left\{\sum_{i=1}^{n} \phi\Big(\frac{i-1}{n}\Big)\Big(B\Big(\frac{i}{n}\Big) - B\Big(\frac{i-1}{n}\Big)\Big) - \frac{1}{2}\phi^2\Big(\frac{i-1}{n}\Big).\Delta t\right\} dP^{(n)} \end{aligned}$$

where

$$\begin{aligned} &\lim_{n\to\infty} \exp\left\{\sum_{i=1}^{n} \phi\Big(\frac{i-1}{n}\Big)\Big(B\Big(\frac{i}{n}\Big) - B\Big(\frac{i-1}{n}\Big)\Big) - \frac{1}{2}\phi^2\Big(\frac{i-1}{n}\Big).\Delta t\right\} dP^{(n)} \\ &= \exp\left\{\int_0^1 \phi(s)dB(s) - \frac{1}{2}\int_0^1 \phi^2(s)ds\right\} dP. \end{aligned}$$

Hence, with $L_1 = \int_0^1 \phi(s)dB(s)$, we have

$$\begin{aligned} dQ &= Z(L_1)dP \\ &= \exp\{L_1 - \frac{1}{2}\langle L_1\rangle\} dP, \end{aligned}$$

where $Z(L_1) = \exp\{L_1 - \frac{1}{2}\langle L_1\rangle\}$ is called the stochastic exponent or the Doléans-Dade exponential. Note that the limit process above is only a heuristic argument. The actual derivation is given by the Girsanov transformation given below.

Definition 2.24. Consider the probability P under the probability space $(\Omega, \mathcal{F}, P)$. Let Q be another probability measure, then

$$\begin{aligned} &(i) Q \ll P \Leftrightarrow P[A] = 0 \Rightarrow Q[A] = 0 \qquad \forall A \in \mathcal{F}, \\ &(ii) Q \sim P \Leftrightarrow Q \ll P \quad \text{and} \quad P \ll Q. \end{aligned} \tag{2.22}$$

Theorem 2.3. *(Radon-Nikodym)*
Let $Q \ll P$. Then there exist a strictly positive $\mathcal{F}$-measurable function $Z(\omega)$ with $Q = ZP$, that is

$$\begin{aligned} &Q[A] = \int_A Z(\omega) P(d\omega) \qquad \forall A \in \mathcal{F}, \\ &Z = \frac{dQ}{dP}, \\ &\Longrightarrow Z(t) = E_P[Z|\mathcal{F}_t] = E_P[\frac{dQ}{dP}|\mathcal{F}_t]. \end{aligned} \tag{2.23}$$

Then $Z(t)$ is a right continuous martingale, and $E_P[Z(t)] \equiv 1$. Furthermore

$$\begin{aligned} &(i) Z(t,\omega) > 0 \qquad Q-a.s., \\ &(ii) Z(t) = \frac{dQ_t}{dP_t} \qquad on \quad \mathcal{F}_t. \end{aligned}$$

Proof
see Föllmer [20].

Equation (2.23) is called the Radon-Nikodym derivative.

Theorem 2.4. *(The General Girsanov Theorem)*
Let $Y(t) \in \mathbb{R}^n$ be an Itô process of the form

$$dY(t) = \beta(t,\omega)dt + \theta(t,\omega)dB(t) \qquad t \leq T, \tag{2.24}$$

where $B(t) \in \mathbb{R}^m$ and $\theta(t,\omega) \in \mathbb{R}^{n\times m}$. Suppose there exist processes $u(t,\omega)$ and $\alpha(t,\omega)$ such that

$$\theta(t,\omega)u(t,\omega) = \beta(t,\omega) - \alpha(t,\omega), \tag{2.25}$$

and assume that $u(t,\omega)$ satisfies Nivikov's condition

$$E[\exp(\frac{1}{2}\int_0^T u^2(s,\omega)ds)] < \infty. \tag{2.26}$$

Put

$$M_t = \exp\,(-\int_0^t u(s,\omega)dB(s) - \frac{1}{2}\int_0^t u^2(s,\omega)ds); \qquad t \leq T, \tag{2.27}$$

and

$$dQ(\omega) = M_T(\omega)dP(\omega) \qquad \text{on } \mathcal{F}_T^{(m)}. \tag{2.28}$$

Then

$$\widehat{B}(t) \equiv \int_0^t u(s,\omega)ds + B(t); \qquad t \leq T, \tag{2.29}$$

is a Brownian motion with respect to Q and in terms of $\widehat{B}(t)$ the process $Y(t)$ has the stochastic integral representation

$$dY(t) = \alpha(t,\omega)dt + \theta(t,\omega)d\widehat{B}(t). \tag{2.30}$$

Proof: see Øksendal [37] Chapter 8 page 156.

2.8 Attainability and Completeness of Contingent T-Claims

2.8.1 Contingent Claims

A contingent claim (derivative) is a claim that can be paid only if one or more specified outcomes occur. It is a contract between two parties that defines rights and obligations of the parties. The value of the contract depends on the terms of the contract. Depending on the underlying asset and the terms of the contract a derivative may take on many forms. Some of the most widely spread "standard" derivatives today include those whose value is determined by the value of one or more underlying variable. The analysis of such claims, and their pricing in particular, forms a large part of the modern theory of finance. Decisions about the prices appropriate for such claims are made contingent on the price behavior of these underlying securities and the theory of derivatives markets is primarily concerned with these relationships. Construction of mathematical models for this analysis often involves very sophisticated mathematical concepts. In this dissertation we are more concerned with continuous models based on diffusions and Itô processes. The main type of financial instruments currently traded are forwards, futures, swaps and options.

2.8.2 Options

There are two basic types of options. A call option which gives its holder the right to buy an asset by a certain date for a certain price. A put option gives the holder the right to sell an asset by a certain date for certain price. The date specified in the contract is known as the expiration date on the maturity date. The price specified is known as the exercise price or strike price. Options can be either American or European. American options can be exercised at any time up to expiration date, whereas European options can be exercise only on the expiration date itself.

Call Option

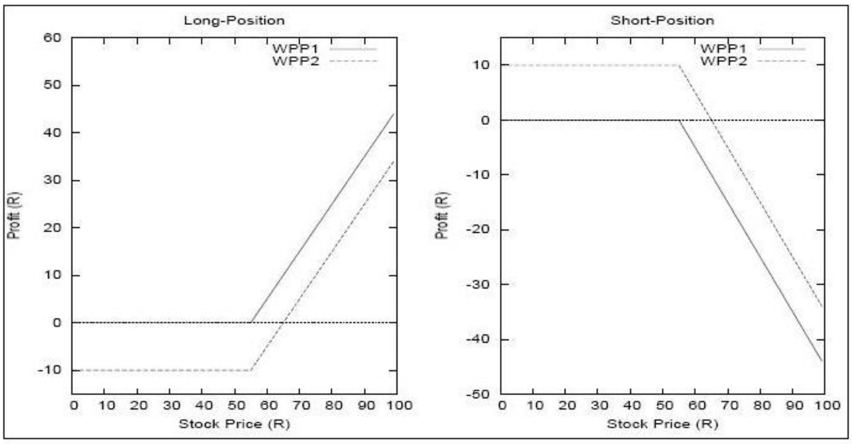

Figure 2.1: Profit profile of a call option

Suppose the pre-specified date for the underlying security is T and the pre-specified price (strike price) is K. Then the call option is exercised if the stock price at terminal date $S(T) \geqslant K$, otherwise it is abandoned. The pay off $g(x)$ at expiry date T for a European call option is

$$g(S(T)) = [S(T) - K]^{+}, \tag{2.31}$$

that is

$$g(S(T)) = \begin{cases} [S(t) - K] & \text{if } S(t) > K, \\ 0 & \text{if } S(t) \leq K. \end{cases}$$

In any option contract, there are two parties involved. An investor that buys an option (that is, an option's holder) and an investor that sells an option (that is, an option's writer). The option's holder is said to take a long position while the option's writer is said to take a short position. In order to purchase an option contract an option holder needs to pay an option price or premium to the second party called the seller or the writer. The payment of the price is done at the initial date when the contract is entered into. The payoff and profit diagram for buying European call option is given by figure 2.1 above where WPP1 and WPP2 denote Without-Premium-Paid and With-Premium-Paid respectively. If the cost of the call is c (premium) and we ignore the effects of discounting, then the profit to the buyer of a European call option is

$$P(S(T)) = \begin{cases} [S(t) - K - c] & \text{if } S(t) > K, \\ -c & \text{if } S(t) \leq K. \end{cases}$$

Put Option

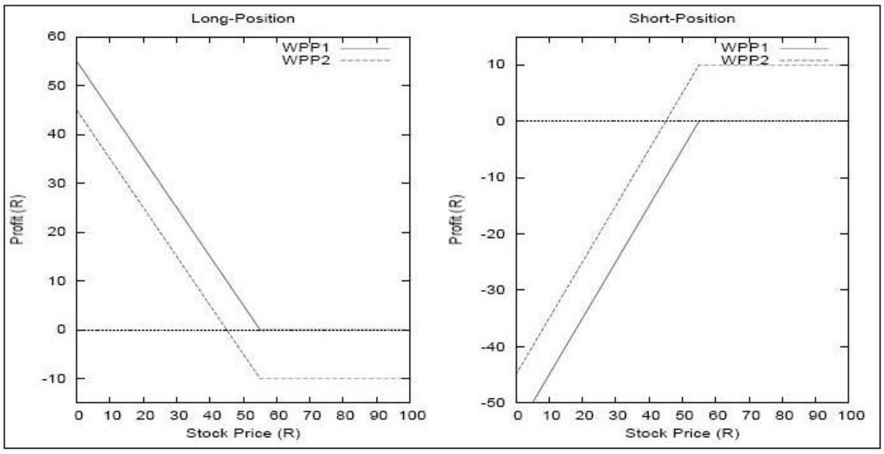

Figure 2.2: Profit profile of a put option

Whereas the purchaser of a call option is hoping that the stock price will increase, the purchaser of a put option is hoping that it will decrease. The European put option is exercised if and only if the stock price at terminal date $S(T) < K$, otherwise it is also abandoned. The payoff $f(x)$ at expiry date T for a European put option is given by

$$f(S(T)) = [K - S(T)]^{+}, \tag{2.32}$$

that is

$$f(S(T)) = \begin{cases} [K - S(t)] & \text{if } S(t) < K, \\ 0 & \text{if } S(t) \geq K. \end{cases}$$

The profit and payoff diagram for a European put option is given by figure 2.2 above. The method of pricing such derivatives will depend on whether the stochastic differential equation driving the stock price process $S(t)$ is a complete market model or not. In the complete market model setting, the price of the European call and put option is given by the Black-Scholes formula which we will derive in the following Chapter. There is no unique price for such derivatives in the incomplete market settings but there are numerical methods of pricing such derivatives when the market is incomplete.

Definition 2.25. (a) A (European) contingent T-claim is a lower bounded $\mathcal{F}_T^{(m)}$-measurable random variable $F(\omega)$.

(b) We say that the claim $F(\omega)$ is attainable in the market $X(t)_{t\in[0,T]}$ if there exists an admissible portfolio $\theta(t)$ and a real number z such that

$$F(\omega) = V_z^\theta(T) \equiv z + \int_0^t \theta(t)dX(t) \qquad a.s. \tag{2.33}$$

If such a $\theta(t)$ exist we call it a replicating or hedging portfolio for F.

(c) The market $X(t)_{t\in[0,T]}$ is called complete if every bounded T-claim is attainable.

In other words, a claim $F(\omega)$ is attainable if there exists a real number z such that if we start with z as our initial fortune we can find an admissible portfolio $\theta(t)$ which generates a value $V_z^\theta(t)$ at time T which a.s. equals F:

$$V_z^\theta(T,\omega) = F(\omega) \qquad \text{for a.a. } \omega. \tag{2.34}$$

Lemma 2.2. *Suppose $u(t,\omega)$ satisfies the condition that*

$$E\left[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)ds\right)\right] < \infty. \tag{2.35}$$

Define the measure $Q = Q_u$ on $\mathcal{F}_T^{(m)}$, where m is the number of diffusion terms for the assets $X_i(t)$, by

$$dQ(\omega) = \exp\left(-\int_0^T u(t,\omega)dB(t) - \frac{1}{2}\int_0^T u^2(t,\omega)dt\right)dP(\omega). \tag{2.36}$$

Then

$$\widetilde{B}(t) := \int_0^t u(s,\omega)ds + B(t), \tag{2.37}$$

is an $\mathcal{F}_t^{(m)}$-martingale (and hence an $\mathcal{F}_t^{(m)}$-Brownian motion) with respect to with respect to Q and any $F \in L^2(\mathcal{F}_T^{(m)}, Q)$ has a unique representation

$$F(\omega) = E_Q[F] + \int_0^T \phi(t,\omega)d\widetilde{B}(t). \tag{2.38}$$

Where $\phi(t,\omega)$ is an $\mathcal{F}_t^{(m)}$-adapted, (t,ω)-measurable $\mathbb{R}^m$-valued process such that

$$E_Q[\int_0^T \phi^2(t,\omega)dt] < \infty. \tag{2.39}$$

Lemma 2.3. *Let $\overline{X}(t) = X_0^{-1}(t)X(t)$ be a normalized price process, where $X_0(t) = \exp\left(\int_0^t \rho(s,\omega)ds\right)$. Then the market $\{X(t)\}$ is complete if and only if the normalized market $\{\overline{X}(t)\}$ is complete.*

Lemma 2.4. *Suppose there exists an m-dimensional process $u(t,\omega)$ such that, with $\widehat{X}(t,\omega) = (X_1(t,\omega),...,X_n(t,\omega))$,*

$$\sigma(t,\omega)u(t,\omega) = \mu(t,\omega) - \rho(t,\omega)\widehat{X}(t,\omega) \qquad \text{for a.a. } (t,\omega), \tag{2.40}$$

and

$$E[\exp\left(\frac{1}{2}\int_0^T u^2(s,\omega)ds\right)] < \infty. \tag{2.41}$$

Define the measure $Q = Q_u$ and the process $\widetilde{B}(t)$ as in the Girsanov Theorem. Then in terms of $\widetilde{B}(t)$ we have the following representation of the normalized market $\overline{X}(t) = X_0^{-1}(t)X(t)$

$$\begin{aligned} d\overline{X}_0(t) &= 0, \\ d\overline{X}_i(t) &= X_0^{-1}\sigma_i(t)d\widetilde{B}(t), \qquad 1 \leq i \leq n. \end{aligned} \tag{2.42}$$

In particular, if $\int_0^T E_Q[X_0^{-2}(t)\sigma_i^2(t)]dt < \infty$, then Q is an equivalent martingale measure.

Theorem 2.5. *Suppose (2.40) and (2.41) holds. Then the market $\{X(t)\}$ is complete if and only if $\sigma(t,\omega)$ has a left inverse $\Lambda(t,\omega)$ for a.a (t,ω), i.e., there exists an $\mathcal{F}_t^{(m)}$-adapted matrix valued process $\Lambda(t,\omega) \in \mathbb{R}^{(m\times n)}$ such that*

$$\Lambda(t,\omega)\sigma(t,\omega) = I_m \qquad \textit{for a.a. } (t,\omega), \tag{2.43}$$

which is equivalent to the property that

$$\text{rank}\{\sigma(t,\omega)\} = m \qquad \text{for a.a. } (t,\omega). \tag{2.44}$$

Proof
see Øksendal [37] Chapter 12 page 262

Corollary 2.1. *Suppose (2.40) and (2.41) holds.*

(a) If $n = m$ then the market is complete if and only if $\sigma(t,\omega)$ is invertible for $a.a.(t,\omega)$.

(b) If the market is complete, then

$$\text{rank}\{\sigma(t,\omega)\} = m \qquad \text{for a.a. } (t,\omega), \tag{2.45}$$

in particular, $n \geq m$.
Moreover, the process $u(t,\omega)$ satisfying (2.43) is unique.

Example 2.3
A market which is an $\mathcal{F}_t^{(m)}$-adapted 4-dimensional Itô process
$X(t) = (X_0(t), X_1(t), ..., X_3(t))$; $0 \leq t \leq T$ which has a differential form

$$\begin{aligned} dX_0(t) &= 0, \\ dX_1(t) &= dt + dB_1(t), \\ dX_2(t) &= 2dt + dB_2(t), \\ dX_3(t) &= 3dt + dB_1(t) + dB_2(t), \end{aligned}$$

is a complete market.
Proof
Since this is a normalized market with $X_0(t) = 1$, we have $\rho = 0$ and from (2.40) we have that

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

which has a unique solution $u_1 = 1$, $u_2 = 2$. Since u is constant, it is clear that (2.41) holds. The rank$\sigma = 2$, so equation (2.44) holds. Moreover, since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

which shows that

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is a left inverse of σ. We then conclude by Theorem 2.5 above that this is a complete market.
Now consider our market, which is a market given in equation (1.1)

Example 2.4
A market, satisfying the conditions that

$$\begin{aligned} dX_0 &= qX_0 dt, \\ dX_1 &= \alpha dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t), \end{aligned}$$

is an incomplete market.
Proof
Let our market be given by

$$\begin{aligned} dX_0 &= qX_0 dt, \\ dX_1 &= \alpha dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t). \end{aligned}$$

We can show that (see Chapter 4, section 4.2) the above market can be written as

$$dX_1 = qX_1(t)dt + \sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t), \tag{2.46}$$

after applying Girsanov's theorem to change measure.
So we have $\mu = qX_1$, and

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_2 \end{pmatrix} \in \mathbb{R}^{1\times 2}.$$

So $n = 1 < 2 = m$. Hence this market is incomplete, by Corollary 2.1. Thus there exist bounded T-claims which cannot be hedged.
The corresponding value process is given by

$$V_z^\theta = z + \int_0^t \theta_1(s)(\sigma_1 d\widetilde{B}_1(s) + \sigma_2 d\widetilde{B}_2(s)). \tag{2.47}$$

Thus if θ hedges a T-claim $F(\omega)$ we have

$$F(\omega) = z + \int_0^T \theta_1(s)(\sigma_1 d\widetilde{B}_1(s) + \sigma_2 d\widetilde{B}_2(s)). \tag{2.48}$$

Choose $F(\omega) = g(\widetilde{B}_1(T))$, where $g : \mathbb{R} \to \mathbb{R}$ is bounded. Then by the Itô representation theorem applied to the two-dimensional Brownian motion $\widetilde{B}(t) = (\widetilde{B}_1(t), \widetilde{B}_2(t))$ there is a unique $\phi(t,\omega) = (\phi_1(t,\omega), \phi_2(t,\omega))$ such that

$$g(\widetilde{B}_1(T)) = E_Q[g(\widetilde{B}_1(T))] + \int_0^T \phi_1(s)d\widetilde{B}_1(s) + \phi_2(s)d\widetilde{B}_2(s), \tag{2.49}$$

and by the Itô representation theorem applied to $\widetilde{B}_1(t)$, we must have $\phi_2 = 0$. That is

$$g(\widetilde{B}_1(T)) = E_Q[g(\widetilde{B}_1(T))] + \int_0^T \phi_1(s)d\widetilde{B}_1(s). \tag{2.50}$$

Comparing this with (2.48) we see that no such θ_1 exists. So $F(\omega) = g(\widetilde{B}_1(T))$ cannot be hedged.

2.9 Pricing of European Options

Consider a European contingent claim as defined in definition 2.25. Suppose you were offered a guarantee to be paid such a contingent claim given by the amount $F(\omega)$ at time $t = T$. The question is, how much would you be willing to pay at time $t = 0$ for such a guarantee?
You could argue as follows:
If I-the buyer-pay the price y for this guarantee, then I have an initial fortune

$-y$ in my investment strategy. With this initial fortune (debt) it must be possible to hedge at time T a value $V^{\theta}_{-y}(T,\omega)$ which, if the guaranteed payoff $F(\omega)$ is added, gives me a nonnegative result such that

$$V^{\theta}_{-y}(T,\omega) + F(\omega) \geq 0 \quad a.s.$$

Thus the maximal price $p = p(F)$ the buyer is willing to pay is
(Buyer's price of the (European) contingent claim F)

$$\begin{aligned} p(F) \quad = \quad & \sup\{y; \text{There exists an admissible portfolio } \theta \text{ such that} \\ & V^{\theta}_{-y}(T,\omega) := -y + \int_0^T \theta(s)dX(s) \geq -F(\omega) \quad a.s.\}. \end{aligned}$$

On the other hand, the seller of this guarantee could argue as follows:
If I-the seller-receive the price z for this guarantee, then I can use this as the initial value in an investment strategy. With this initial fortune is must be possible to hedge to time T a value $V^{\theta}_{z}(T,\omega)$ which is not less than the amount $F(\omega)$ that I have promised to pay the buyer such that

$$V^{\theta}_{z}(T,\omega) \geq F(\omega) \quad a.s.$$

Thus the minimal price $q = q(F)$ the seller is willing to accept is
(Seller's price of the (European) contingent claim F)

$$\begin{aligned} q(F) \quad = \quad & \inf\{z; \text{There exists an admissible portfolio } \theta \text{ such that} \\ & V^{\theta}_{z}(T,\omega) := z + \int_0^T \theta(s)dX(s) \geq F(\omega) \quad a.s.\}. \end{aligned}$$

Definition 2.26. If $p(F) = q(f)$ we call this common value the price (at $t = 0$) of the (European) T-contingent claim $F(\omega)$

Theorem 2.6. *a) Suppose* (2.40) *and* (2.41) *hold and let Q be as in* (2.36). *Let F be a (European) T-claim such that $E_Q[X_0^{-1}(T)F] < \infty$. Then*

$$\textit{ess inf}\{F(\omega)\} \leq p(F) \leq E_Q[\xi(T)F] \leq q(F) \leq \infty. \tag{2.51}$$

Where the essential infimum of F, ess supp F, is the largest essential lower bound.

b) Suppose in addition to condition in a), that the market $\{X(t)\}$ is complete. Then the price of the (European) T-claim F is

$$p(F) = E_Q[\xi(T)F] = q(F). \tag{2.52}$$

Where

$$\xi(t) = X_0^{-1}(t) = \exp\left(- \int_0^t \rho(s,\omega)ds \right) \quad \text{for all } t \in [0,T]. \tag{2.53}$$

Proof
see Øksendal [37], Chapter 12

Remark 2.3. The above theorem implies that if a market is incomplete, then the buyers price $p(F)$ and the sellers price $q(F)$ are different. In fact any price in the interval $\{p(F), q(F)\}$ is a possible price admissible to both buyer and seller when the market is incomplete. Hence, we have infinitely many prices for contingent claims in an incomplete market and equally infinitely many equivalent martingale measures associated with each price. So when we look for the best price we are also looking for the best equivalent martingale measure.

We shall look at some of the measures proposed, that gives an optimal price when the market is incomplete, in the next Chapter. In this dissertation we are mostly concerned with the completion of incomplete market models, as complete market model have a unique martingale measure and hence, a unique price.

Chapter 3

Pricing of Contingent Claims and the Black-Scholes Formula

The core of financial mathematics is the pricing of financial derivatives such as options and futures. Almost all, if not all, books and literature of the subject-matter of financial mathematics are concerned with finding an optimal pricing formula for the value of a contingent claim. After four decades of its discovery, the Black-Scholes formula is the universally accepted formula for option pricing even though the formula fails when a market is incomplete. This is due to its bias assumptions of constant volatility and no transactional costs.

3.1 Itô Formula

3.1.1 Basic One Dimensional Itô Formula

Suppose we have a stochastic differential equation given by:

$$dX(t) = \mu(t, X)dt + \beta(t, X)dB(t), \tag{3.1}$$

for some random variable $X(t)$. Where $\mu(t, X)$ is the drift term, $\beta(t, X)$ is the volatility and $B(t)$ is the Brownian motion or the Wiener process. If we are given a twice continuously differentiable function $f(t, X)$, then we have, by the Itô formula that

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2, \tag{3.2}$$

where $(dX)^2 = (dX)(dX)$ is calculated according to the rules

$$dt.dt = dt.dB = dB.dt = 0, \qquad dB.dB = dt. \tag{3.3}$$

Equation (3.2) is known as the one dimensional Itô formula.

3.1.2 The Multi-dimensional Itô Formula

Let $B(t,\omega) = (B_1(t,\omega), \ldots, B_m(t,\omega))$ denote an m-dimensional Brownian motion and $X(t) = (X_1(t), \ldots, X_n(t))$ be an n dimensional Itô process satisfying:

$$dX_i(t) = \mu_i(t,X)dt + \beta_i(t,X)dX_i(t), \tag{3.4}$$

for each i such that $1 \leqslant i \leqslant n$. Now suppose we have a twice-differentiable function $f(t, X_1(t), \ldots, X_n(t))$. Then f is a Itô process, with an Itô formula given by:

$$df = \frac{\partial f}{\partial t}dt + \sum_{i=1}^{n} \frac{\partial f}{\partial X_{(i)}} dX_{(i)} + \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial X_{(i)} \partial X_{(j)}} dX_{(i)} dX_{(j)}, \tag{3.5}$$

where $dB_{(i)}dB_{(j)} = \delta_{(ij)}dt$. The above equation (3.5) is known as the multi-dimensional Itô formula.

3.2 The Black-Scholes Model for Pricing Stock Options

There are some assumptions underlining the behavior of stock prices over time. These assumptions may be the difference between pricing in a complete or incomplete market. Like any other model, the Nobel prize winning Black-Scholes model also known as Black-Scholes-Marton model has its own assumptions about how stock prices evolve over time.

3.2.1 Assumptions Underlying The Black-Scholes Model

- Stock price behavior follows a geometric Brownian motion given by:

  $$dS(t) = S(t)(\mu dt + \sigma dB(t) \tag{3.6}$$

  for some constant μ and σ, where as before $B(t)$ is a standard Brownian motion.

- There are no transactional costs or tax. All securities are perfectly divisible

- There are no dividends on the stock during the life of the option.

- There are no riskless arbitrage opportunities.

- Security trading is continuous.
- Investors can borrow or lend at the same risk-free rate of interest.
- The short-term risk-free rate of interest, r, is also constant.

3.2.2 Derivation of the Black-Scholes Equation

Suppose we have a European call option $f(S,t)$ with the behavior of stock prices $S(t)$ given above by the Black-Scholes assumptions, where the payoff is given by $f(S,T) = \Phi(S)$. Now given a portfolio consisting of one option and $-\Delta$ unit of the underlying asset, the value of this portfolio is

$$\Pi = f - \Delta S. \tag{3.7}$$

The change of the portfolio is thus given by

$$d\Pi = df - \Delta dS. \tag{3.8}$$

Then by Itô's formula

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2, \tag{3.9}$$

with

$$(dS)^2 = \sigma^2 S^2 dt,$$

it follows that

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}(\mu S dt + \sigma S dB) + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(\sigma^2 S^2 dt). \tag{3.10}$$

After substituting we obtain

$$\begin{aligned} d\Pi &= \left(\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt + \sigma S\frac{\partial f}{\partial S}dB - \Delta(\mu S dt + \sigma S dB) \\ &= \left[\frac{\partial f}{\partial t} + \mu S(\frac{\partial f}{\partial S} - \Delta) + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right]dt + \sigma S\left(\frac{\partial f}{\partial S} - \Delta\right)dB. \end{aligned}$$

By choosing $\Delta = \frac{\partial f}{\partial S}$ we are able to remove the randomness dB and Π will consequently become risk-free, which means

$$d\Pi = \left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}\right)dt, \tag{3.11}$$

but by the principal of no-arbitrage, Π must therefore instantaneously earn the risk-free interest rate, hence:

$$\begin{aligned} d\Pi &= r\Pi dt \\ &= (rf - r\Delta S)dt. \end{aligned}$$

Which then imply that

$$\begin{aligned} rf - r\Delta S &= \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} \qquad (3.12) \\ rf - rS\frac{\partial f}{\partial S} &= \frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}. \end{aligned}$$

Which gives the nondividend Black-Scholes partial differential equation (PDE) for a European option:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf. \qquad (3.13)$$

One can solve the above equation to get the price of a European option, by either the two transformation methods given in Zhu-Wu-Chern [41] where they reduce the above Black-Scholes (PDE) to a heat equation. Because Green's function of the heat equation has analytic expression, they obtain an analytic expression of Green's function for the Black-Scholes equation using the inverse transformation. Based on these analytic expressions, a European option price can be derived. One may also use numerical methods or the Feynman-Kac formulae (3.29) to solve the above (PDE) which gives the famous Black-Scholes option pricing formula.

3.2.3 The Black-Scholes Option Pricing Formula

The derivation of the Black-Scholes option pricing formula Black and Scholes [9] is considered as one of the most important developments in the field of mathematics of finance. Based on Black-Scholes given assumptions of the evolution of stock prices, the solution of the Black-Scholes (PDE) (3.13) for a European call option is given by

$$C = SN(d_1) - Ke^{-r(T-t)}N(d_2), \qquad (3.14)$$

which is known as the Black-Scholes option pricing formula.
Where $d_1 = \frac{\log(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.
Also note that for a call option $f(S,T) = [S-K,0]^+$. The standard normal distribution function is given by

$$N(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{-\frac{x^2}{2}} dx. \qquad (3.15)$$

By arbitrage arguments, it can be shown that the relationship between the European call option and put option is given by

$$C = P + S - Ke^{-r(T-t)}, \tag{3.16}$$

which is called the put-call parity. By using the put-call parity we can derive the put price, which is given by

$$P = Ke^{-r(T-t)}N(-d_2) - SN(-d_1). \tag{3.17}$$

3.2.4 Robustness Property of The Black-Scholes Hedging Procedure

The result of this section can be found in Fouque, Papanicolaou and Sircar [22], which aims at finding an error or contrast between a chosen model for the stock price process and the actual stock price process for a European contingent claim $f(t, S(t))$.

Suppose the stock price process in reality is not modeled by the Black-Scholes price process given in equation (3.6), but modeled by a stock price process given by

$$dS(t) = \alpha(t)S(t)dt + \beta(t)dB(t). \tag{3.18}$$

Where the drift term α and the volatility term β are general $\mathcal{F}_t$-adapted processes, which could also depend on $S(t)$. Now suppose a trader believes that the price process for stock prices is actually given by the Black-Scholes model with some specific volatility σ. Then the trader will automatically hedge according to the Black-Scholes dynamics, by holding $\Delta(t) = \frac{\partial f}{\partial S}$ units of stocks and placing the remaining amount which is given by $(V(t) - \Delta(t)S(t))$ into the bank account. Where $f(t, S(t))$ is the solution of the Black-Scholes PDE given by (3.13).

The change in the value of our portfolio is thus given by

$$dV(t) = \Delta(t)dS(t) + (V(t) - \Delta(t)S(t))rdt \tag{3.19}$$

For a self-financing portfolio, where $V(0) = f(0, S(0))$ if we write the option at the Black-Scholes price.

Define the process $Y(t) = V(t) - f(t, S(t))$, then $dY(t) = dV(t) - df(t, S(t))$

Itô lemma thus gives

$$df = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}dS + \frac{1}{2}\beta^2 S^2 \frac{\partial^2 f}{\partial S^2}dt \tag{3.20}$$

So that

$$\begin{aligned} dY &= \Delta dS + (V - \Delta S)\Delta rdt - df \\ &= \frac{\partial f}{\partial S}dS + (V - \frac{\partial f}{\partial S})rdt - \frac{\partial f}{\partial t} - \frac{\partial f}{\partial S}dS - \frac{1}{2}\beta^2 S^2 \frac{\partial^2 f}{\partial S^2}dt \end{aligned}$$

Now substituting the Black-Scholes PDF, equation (3.13), into $\frac{\partial f}{\partial t}$ we get

$$\begin{aligned}
dY &= (V-f)rdt + \frac{1}{2}S^2\frac{\partial^2 f}{\partial S^2}(\sigma^2-\beta^2)dt \\
&= Yrdt + \frac{1}{2}S^2\frac{\partial^2 f}{\partial S^2}(\sigma^2-\beta^2)dt
\end{aligned}$$

Now using an integrating factor on Y we see that

$$\begin{aligned}
de^{-rt}Y &= -re-rtYdt + e^{-rt}dY \\
&= \frac{1}{2}e^{-rt}S^2\frac{\partial^2 f}{\partial S^2}(\sigma^2-\beta^2)dt
\end{aligned}$$

So that

$$\begin{aligned}
e^{-rt}Y &= \frac{1}{2}\int_0^T e^{-rt}S^2\frac{\partial^2 f}{\partial S^2}(\sigma^2-\beta^2)dt \\
\Rightarrow Y(T) &= \frac{1}{2}\int_0^T e^{r(T-t)}S^2\frac{\partial^2 f}{\partial S^2}(\sigma^2-\beta^2)dt
\end{aligned}$$

Where $Y(0)=0$.
The gamma of a call option or any option with a convex pay-off is positive for any log-type model. For the proof of this argument see Joshi [31] Chapter 15. A function is said to be convex if the chord between any two point on the graph lies above the graph, that is, a function is convex if the second derivative is non-negative. We therefore have that $\frac{\partial^2 f}{\partial S^2} > 0$ for a put and call option, hence $Y(T) \geq 0$
Thus our hedging strategy makes a profit with probability one as long as $\sigma^2 \geq \beta^2$. This shows that successful hedging is entirely a matter of good volatility estimation. We consistently make a profit if the Black-Scholes volatility σ dominates the true volatility β regardless of other details of the price dynamistic. In fact the difference between the true volatility (β) and the Black-Scholes volatility (σ) is known as the hedging error of the Black-Scholes model.

3.2.5 Time Dependent Parameters

Suppose we allow the parameters of the Black-Scholes model to be deterministic function of time. That is, we assume that $r(t)$ and $\sigma = \beta(t)$ vary as in equation (3.18) according to some prescribed rule. Then the market given by

$$\begin{aligned}
dR(t) &= r(t)R(t)dt, \\
dS(t) &= S(t)\mu(t)dt + \sigma(t)S(t)dB(t),
\end{aligned}$$

is a complete market model with a unique equivalent martingale measure Q with a Black-Scholes PDE for a European option given by

$$\frac{\partial f}{\partial t} + r(t)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 f}{\partial S^2} = r(t)f. \tag{3.21}$$

And a unique price for a European call option given by

$$f = SN(\widetilde{d}_1) - Ke^{-\int_0^T r(t)dt}N(\widetilde{d}_2)$$

Where

$$\widetilde{d}_1 = \frac{ln(\frac{S}{K}) + \int_0^T \Big(r(t) + \frac{1}{2}\sigma^2(t)\Big)dt}{\sqrt{\int_0^T \sigma^2(t)dt}}$$

And

$$\widetilde{d}_2 = \widetilde{d}_1 - \sqrt{\int_0^T \sigma^2(t)dt}$$

3.2.6 Derivation Of Black-Scholes Option Pricing Formula

We will make use of the Feynman-Kac formulae to derive the Black-Scholes Option Pricing Formula, which does not depend on the expected future price nor investors attitudes towards risk.

The Generation of an Itô Diffusion

Definition 3.1. Let $X(t)$ be an Itô diffusion, then the generator A of $X(t)$ is

$$Af(X) = lim_{t\to 0^-}\frac{E_x[f(X_t)] - f(X)}{t} \qquad \text{for all } x \in \mathbb{R} \tag{3.22}$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists at x is denoted by $\mathcal{D}_A(x)$ while $\mathcal{D}_A$ denote the set of functions for which the limit exists for all $x \in \mathbb{R}^n$

Theorem 3.1. *Let $X(t)$ be an Itô diffusion of the form*

$$dX(t) = b(X(t))dt + \sigma(X(t))dB(t) \tag{3.23}$$

if $f \in C^2(\mathbb{R})^n$ and f has compact support, then $f \in \mathcal{D}_A$ and

$$Af(x) = \Sigma_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\sigma(x)\frac{\partial^2 f}{\partial x_i \partial x_j} \tag{3.24}$$

Proof see Øksendal [37], Chapter 7, page 118

Example 3.1

Let $B(t)$ denote a 1-dimensional Brownian motion and let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be the solution of the stochastic differential equation given by

$$dX_1(t) = dt, \qquad X_1(0) = t_0$$
$$dX_2(t) = dB(t), \qquad X_2(0) = x_0$$

that is

$$dX(t) = \mu dt + \sigma dB(t) \qquad X(0) = \begin{pmatrix} t_0 \\ x_0 \end{pmatrix} \tag{3.25}$$

where $\mu = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\sigma = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Then the generator A of X is given by

$$Af = \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \qquad f(t,x) \in C^2(\mathbb{R})^n \tag{3.26}$$

The Dynkin Formula

Let $f \in C^2(\mathbb{R}^n)$. Suppose t is a stopping time. $E_x[t] < \infty$. Then

$$E_x[f(X(t))] = f(x) + E_x\left[\int_0^t Af(X(s))ds\right] \tag{3.27}$$

Kolmogorov's Backward Equation

Let $f \in C^2(\mathbb{R}^n)$ and define

$$u(t,x) = E_x[f(X(t))] \tag{3.28}$$

Then $u(t,x) \in \mathcal{D}_A$ for each t and

$$\begin{aligned} &\frac{\partial u}{\partial t} = Au, \qquad t > o \quad x \in \mathbb{R}^n \\ &u(0,x) = f(x) \qquad x \in \mathbb{R}^n \end{aligned}$$

The Feynman-Kac Formula

Let $f \in C^2(\mathbb{R}^n)$ and $q \in C(\mathbb{R}^n)$. Assume that q is lower bounded.
Put

$$v(t,x) = E_x[e^{(-\int_0^t q(X(s))ds)} f(X(t))] \tag{3.29}$$

Then

$$\begin{aligned} &\frac{\partial v}{\partial t} = Av - qv, \qquad t > 0 \quad x \in \mathbb{R}^n \\ &v(0,x) = f(x) \qquad x \in \mathbb{R}^n \end{aligned}$$

The Feynman-Kac Formula establishes a relationship between the martingale approach (which was pioneered by Harrison and Kreps [24] and Harrison and Pliska [25]) to option pricing and the original Black-Scholes approach which leads to solving PDE's under boundary constraints. We can now use the above Feynman-Kac Formula to solve the Black-Scholes option price for a call option $f(S(t)) = [S(t) - K]^+$

Example 3.2
Given the Black-Scholes PDE with initial condition $f(0, S) = [S - K]^+$ for a European call option, we can use Feynman-Kac Formula to show that

$$f(x,t) = \frac{e^{-rt}}{\sqrt{2\pi t}} \int_{\mathbb{R}} (x.exp[(r - \frac{1}{2}\sigma^2)t + \sigma y] - K)^+ e^{-\frac{y^2}{2t}} dy \tag{3.30}$$

Proof
In connection with the deduction of the Black-Scholes formula for the price of a European option, we showed that its PDE (3.13) is given by

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf \tag{3.31}$$

with $f(0, S) = (S - K)^+$ and $S(0) = x$. Then by the Feynman-Kac formula we see that

$$Af - qf = -rf + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}, \tag{3.32}$$

so that

$$Af = rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}, \tag{3.33}$$

when $q = r$. If we apply the generator of the Itô diffusion we see that

$$dS(t) = rS(t)dt + \sigma S(t)dB(t). \tag{3.34}$$

Integrating the above stochastic differential equation yields

$$S(t) = x\exp[(r - \frac{1}{2}\sigma^2)t + \sigma B(t)] \qquad S(0) = x. \tag{3.35}$$

Also note that Girsanov's theorem gives the same value for the discounted stock price $S(t)$.
The Feynman-Kac formula thus gives

$$f(t,x) = E_x[e^{(-\int_0^t q(X(s))ds)} f(X(t))], \tag{3.36}$$

as a solution of the above PDE, so substituting the obtained value for $f(S(t))$ and $q(X(s))$ we get

$$\begin{aligned} f(t,x) &= E_x[e^{(-\int_0^t rds)} f(x\exp[(r - \frac{1}{2}\sigma^2)t + \sigma B(t)])]] \\ &= E_x[e^{(-\int_0^t rds)} (x\exp[(r - \frac{1}{2}\sigma^2)t + \sigma B(t)] - K)^+]] \\ &= \frac{e^{-rt}}{\sqrt{2\pi t}} \int_{\mathbb{R}} (x\exp[(r - \frac{1}{2}\sigma^2)t + \sigma y] - K)^+ . e^{(\frac{-y^2}{2t})} dy \end{aligned} \tag{3.37}$$

as required. We can further solve the above claim $f(t, x)$ to derive the Black-Scholes option pricing formula for a European call option by noting that $y =$

$B(t) \sim N(0,t)$. We also know that if $f(x) = [x - K]^+$ for a European call option, then $x > K$ otherwise $f(x) = 0$. So

$$x \exp[(r - \frac{1}{2}\sigma^2)t + \sigma B(t)] > K \qquad \text{for all } t \in [0,T]. \tag{3.38}$$

Then

$$-B(t) < \frac{log(\frac{x}{K}) + (r - \frac{1}{2}\sigma^2)t}{\sigma} = d. \tag{3.39}$$

Let $y = -B(t)$ so that

$$\begin{aligned} f(t,x) &= \frac{e^{-rt}}{\sqrt{2\pi t}} \int_{-\infty}^{d} (x \exp[(r - \frac{1}{2}\sigma^2)t - \sigma y] - K).e^{(\frac{-y^2}{2t})} dy \\ &= x\frac{e^{-rt}}{\sqrt{2\pi t}} e^{(r-\frac{1}{2}\sigma^2)t} \int_{-\infty}^{d} \exp -\frac{1}{2t}(y^2 + 2\sigma t y) dy - Ke^{-rt} P[Y \leq d] \\ &= x\frac{e^{-\frac{1}{2}\sigma^2 t}}{\sqrt{2\pi t}} \int_{-\infty}^{d} \exp -\frac{1}{2t}(y + \sigma t)^2 + \frac{\sigma^2 t}{2} dy - Ke^{-rt} N(d_2) \\ &= x\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{d} \exp -\frac{1}{2t}(y + \sigma t)^2 dy - Ke^{-rt} N(d_2). \end{aligned}$$

Where $d_2 = \frac{d}{\sqrt{t}} = \frac{log(\frac{x}{K})+(r-\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$ since for the standardized normal distribution we have $\frac{y}{\sqrt{t}} \sim N(0,1)$. Now let $z = y + t\sigma$ so that $dz = dy$.
This gives

$$\begin{aligned} f(t,S) &= x\frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{d_*} \exp -\frac{1}{2t}(z)^2 dz - Ke^{-rt} N(d_2) \\ &= xP[Z < d_*] - Ke^{-rt} N(d_2) \\ &= xN(d_1) - e^{-rt} K N(d_2) \\ &= SN(d_1) - e^{-rt} K N(d_2), \end{aligned}$$

where $d_* = d + \sigma t$ and $d_1 = \frac{d_*}{\sqrt{t}} = \frac{log(\frac{x}{K})+(r+\frac{1}{2}\sigma^2)t}{\sigma\sqrt{t}}$.
Which is the Black-Scholes option pricing formula for a European call option. Similarly one can derive the Black-Scholes price for a put option by setting $f(x) = (k - x)^+$ or by just using the put call parity.

3.3 Pricing in Incomplete Markets

In this section we will try to answer the question of pricing non attainable contingent claims. We assume that the market is free of arbitrage, that is there exist equivalent martingale measures which are not necessarily unique. Due to the fact that the EMM's are not unique, we must have claims that cannot be hedged by a self-financing portfolio in this market. We impose a method of pricing the non-attainable contingent claims by selecting out of the pool of EMM's an equivalent martingale measure that minimizes the risk. Further approaches to the problems of pricing and hedging contingent claims in incomplete markets are treated in $\{[1],[10],[11],[20],[32],[34]\}$. With [33] covering the general theory of incomplete markets. This section is based on the work by Bingham and Kiesel [8].

3.3.1 A General Option Pricing Formula

Utility Function

Definition 3.2. A continuous function $U : (0,\infty) \to \mathbb{R}$ which is strictly increasing, strictly concave and continuously differentiable with $\lim_{x\to\infty} U'(x) = 0$ and $\lim_{x\to 0} U'(x) = \infty$ is called a utility function.

The function $U(x)$, where x denotes a cash payoff, is used to measure the satisfaction of the investor attitude towards risk.
One of the most commonly used utility functions is the exponential utility function given by:

$$U(X) = 1 - e^{-cX} \qquad (c > 0) \tag{3.40}$$

Where c is the risk-aversion constant, X is the amount of money, and $U(X)$ measures the satisfaction of the investor.
An investor with such utility function U and initial endowment x trading only in the underlying asset $S = \{S_0, \dots, S_d\}$ forms a dynamic portfolio φ, whose value at time t is given by $V_{\varphi,x}(t)$. His objective is to maximize his expected utility under the original probability measure of his final wealth at time T given that he is allowed to choose his trading strategy φ from a suitable subset Φ_a of the set of self-financing trading strategies. We write

$$\widetilde{U}(x) = \sup_{\varphi\in\Phi_a} E[U(V_{\varphi,x}(T))] \tag{3.41}$$

for the maximal utility. Now suppose that a contingent claim X is made available for trading with current purchase price p. The question in hand is if the maximal utility given above could be increased. To find a fair price $\widehat{p}$ for a contingent claim we follow a martingale rate of substitution argument' quite commonly used in pricing: $\widehat{p}$ is a fair price for the contingent claim if diverting a little of funds into it at time zero has a neutral effect on the investor's achievable utility. More precisely, if we set

$$W(\delta, x, p) = \sup_{\phi\in\Phi_a} \{E\left[U\left(V_{\varphi,x-\delta}(T) + \frac{\delta}{p}X\right)\right], \tag{3.42}$$

we can then state,

Definition 3.3. Suppose that for each fixed (x, p) the function $W(\delta, x, p)$ is differentiable as a function of δ for $\delta = 0$, that there is a unique solution $\widehat{p}(x)$ of the equation

$$\frac{\partial W}{\partial \delta}(0, p, x) = 0,$$

then $\widehat{p}(x)$ is the fair option price at time $t = 0$.

Theorem 3.2. *(Davis)*
Suppose that $\widetilde{U}$ is differentiable at each $x \in \mathbb{R}^+$ and that $\widetilde{U}'(x) > 0$. Then the fair price $\widehat{p}(x)$ of Definition [3.2] *is given by*

$$\widehat{p} = \frac{E\left[U'\left(V_{\varphi^*,x}(T)\right)X\right]}{\widetilde{U}'(x)} \tag{3.43}$$

Proof
see Bingham and Kiesel [8]

3.3.2 The Esscher Measure

Let $S(t)$ denote the price at time t of a non-dividend paying stock. Assume that there is a stochastic process $\{X(t)\}$ with independent and stationary increments such that

$$S(t) = S(0)e^{X(t)} \qquad t \geq 0 \tag{3.44}$$

Assume that the moment-generating function of $X(t)$, given by

$$M(h, t) \quad = \quad E\left[e^{hX(t)}\right]$$

exists, then

$$M(h, t) \quad = \quad [M(h, 1)]^t$$

the process

$$\left\{e^{hX(t)} M(h, 1)^{-t}\right\}_{t \geq 0}$$

is a positive martingale and can be used to define a change of probability measure, i.e., it can be used to define the Radon-Nikodym derivative $\frac{dQ}{dP}$ of a new probability measure Q with respect to the original probability measure P. Q is called the Esscher measure of parameter h. Gerber and Shiu [23] introduced the risk-neutral Esscher measure: the Esscher measure of parameter $h = h^*$ such that the discounted price process

$$\{e^{-rt}S(t)\}_{t \geq 0},$$

is a martingale. The condition that

$$E[e^{-rt}S(t);h^*] = S(0),$$

yields

$$\begin{aligned} e^{rt} &= \mathbb{E}\left[e^{X(t)};h^*\right] \\ &= \mathbb{E}\left[\frac{e^{X(t)+h*X(t)}}{M(h^*,1)^t}\right] \\ &= \left[\frac{M(1+h^*,1)}{M(h^*,1)}\right]^t \end{aligned}$$

or $e^r = \frac{M(1+h^*,1)}{M(h^*,1)}$.
This equation then uniquely determines the parameter h^*, because for $t \geq 0$

$$\begin{aligned} e^{hX(t)}M(h,1)^{-t} &= \frac{e^{hX(t)}}{\mathbb{E}[e^{hX(t)}]} \\ &= \frac{S(t)^h}{\mathbb{E}[S(t)^h]}. \end{aligned}$$

We thus have the following lemma.

Lemma 3.1. *(Factorization Formula)*

For g a measurable function and h, k and t real numbers, with $t \geq 0$

$$E[S(t)^k g(S(t));h] = E[S(t)^k;h]E[g(S(t));k+h] \tag{3.45}$$

Proof
see Bingham and Kiesel [8]. The above Esscher measure offers us an attractive way of finding an equivalent martingale measure in an incomplete market model.

Example 3.3
Consider the following incomplete market model :

$$dS(t) = S(t)[\mu dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t)] \tag{3.46}$$

integrating the above (S.D.E) gives

$$\ln(S(t)) - \ln(S(0)) = \mu t + \sigma_1 B_1(t) + \sigma_2 B_2(t),$$

so that

$$S(t) = S(o)e^{[\mu t+\sigma_1 B_1(t)+\sigma_2 B_2(t)]}. \tag{3.47}$$

But by the Itô formula applied to two dimensional Brownian motion we have

$$dS(t) = S(t)\mu dt + S(t)[\sigma_1 dB_1(t) + \sigma_2 dB_2(t)] + \frac{1}{2}(\sigma_1^2 + \sigma_2^2)dt. \tag{3.48}$$

Hence, we must have

$$S(t) = S(0)e^{[(\mu - \frac{1}{2}(\sigma_1^2+\sigma_2^2))t+\sigma_1 B_1(t)+\sigma_2 B_2(t)]}. \tag{3.49}$$

Let

$$X(t) = [(\mu - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))t + \sigma_1 B_1(t) + \sigma_2 B_2(t)],$$

so that

$$S(t) = S(0)e^{X(t)}.$$

We can thus apply the Esscher measure technique to evaluate the price of a European call option with maturity T and strike price K on the underlying stock, with price dynamics $S(t)$. By the risk-neutral valuation principle, we have to calculate

$$\begin{aligned} & E[e^{-rT}(S(T)-K)^+;h^*] \\ = \; & E[e^{-rT}(S(T)-K)1_{\{S(T)>K\}};h^*] \\ = \; & e^{-rT}\left[E[S(T)1_{\{S(T)>K\}};h^*] - KE[1_{\{S(T)>K\}};h^*]\right]. \end{aligned}$$

To evaluate the first term we apply the factorization formula with $k = 1, h = h^*$ and $g(x) = 1_{\{x>K\}}$ and get

$$\begin{aligned} E[S(T)1_{\{S(T)>K\}};h^*] &= E[S(T);h^*]E[1_{\{S(T)>K\}};h^*+1] \\ &= E[e^{-rT}S(T);h^*]e^{rT}P[S(T)>K;h^*+1] \\ &= S(0)e^{rT}P[S(T)>K;h^*+1], \end{aligned}$$

where we have used the martingale property of the discounted stock price process $e^{-rt}S(t)$ under the risk-neutral Esscher measure for the last step. Thus the pricing formula for a European call option becomes

$$S(0)P[S(T)>K;h^*+1] - e^{-rT}KP[S(T)>K;h^*], \tag{3.50}$$

when $\sigma_2 = 0$, that is when there is only one source of randomness, the above formula recovers the Black-Scholes pricing formula.

Chapter 4

Completion of a Market that is Incomplete Due to More Randomness than Tradable Assets

In a complete market every contingent claim can be replicated by trading in the underlying assets. That is, every contingent claim is attainable. While in an incomplete market we have claims that cannot be hedged by a self financing portfolio. In this chapter we look at a market with one tradable risky asset and two sources of risk. We show that such a market is incomplete, i.e, has more randomness than tradable assets. We then suggest a method of completing such a market, and show that such a market can be completed by trading in the stock and its quadratic variation.

4.1 Incompleteness Due to More Randomness than Tradable Assets

Suppose one has an incomplete market given by

$$\begin{aligned} dX_0 &= qX_0dt, \\ dX_1 &= \alpha dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t). \end{aligned} \tag{4.1}$$

We have shown in Chapter 2 (example 2.4) that this market is incomplete, and that any claim given by $F\{\omega\} = g(B_2(t))$ in this market cannot be hedged. We will make use of the Girsanov's Theorem to change the probability measure P to a new risk neutral measure Q, and enlarge the market using quadratic variation assets to show that such a market can be completed in a similar fashion as Corcuera and Nualart [12] did in their paper on incomplete markets due to jumps. We will then complete this market using two independent options, where we will use the second option to hedge away the risk associated with the second Brownian motion.

4.2 Quadratic Variation Assets

By Girsanov's Theorem, applied to the above market (4.1), we have

$$\begin{pmatrix} \sigma_1 & \sigma_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \alpha - qX_1(t).$$

It follows that

$$\sigma_1 u_1 + \sigma_2 u_2 = \alpha - qX_1(t).$$

Therefore

$$\begin{aligned} u_1 &= \frac{\alpha - qX_1(t) - \sigma_2\lambda}{\sigma_1}, \\ u_2 &= \lambda, \end{aligned}$$

for some constant $\lambda \in \mathbb{R}$.
One can immediately see from the above market price of risk u, that the equivalent risk neutral measure Q is not unique and depends on the chosen value of λ. Hence, the market is incomplete. Also $\widetilde{B}(t)$ given by $d\widetilde{B}(t) = u(t)dt + dB(t)$ is by Girsanov's Theorem a Q Brownian motion where

$$\frac{dQ(\omega)}{dP(\omega)} = M(t),$$

with

$$M(t) = exp\{-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds\}, \quad 0 \le t \le T.$$

Thus

$$\begin{aligned} d\widetilde{B}_1(t) &= \frac{\alpha - qX_1(t) - \sigma_2\lambda}{\sigma_1}dt + dB_1(t), \\ d\widetilde{B}_2(t) &= \lambda dt + dB_2(t), \end{aligned}$$

our market in (4.1) then becomes,

$$\begin{aligned} dX_1(t) &= \alpha d(t) + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) \\ &= \alpha dt + \sigma_1[d\widetilde{B}_1(t) - \frac{\alpha - qX_1(t) - \sigma_2\lambda}{\sigma_1}dt] + \sigma_2[d\widetilde{B}_2(t) - \lambda dt] \\ &= qX_1(t)dt + \sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t). \end{aligned} \tag{4.2}$$

Which is the equivalent martingale measure Q under which our market in (1.1) remains the same. It is well known that a classical friction free model containing a risky stock and a bank account admits no arbitrage if and only if there exists a probability measure Q on the model under which the stock price discounted by the interest rate on the bank account is a martingale. Thus the discounted stock price process given by $Y(t) = Y_1(t) = \frac{X_1(t)}{X_0(t)}$ remains a Q martingale. We will enlarge our market with what we will call the ith-variation process.

Let $Z_1(t) = Y_1^2(t) - \langle Y_1 \rangle_t$.
We know from the Doob-Meyer decomposition that the process $Z_1(t)$ is martingale. Also let $W_1(t) = e^{qt} Z_1(t)$ where we assume the riskless rate of interest q to be a constant and $X_0(t) = e^{qt}$ is the riskfree bond or bank account at time t

Note that the discounted process $W_1(t)$ is a Q martingale since

$$\begin{aligned} & E[e^{-qt} W_1(t) | \mathcal{F}_s] \\ = \; & E[Z_1(t) | \mathcal{F}_s] \\ = \; & Z_1(s) \qquad 0 \leq s \leq t. \end{aligned}$$

Also note that

$$dZ_1(t) = 2Y_1(t) dY_1(t) \tag{4.3}$$

where

$$\begin{aligned} \frac{dY_1(t)}{Y_1(t)} &= \frac{dX_1(t)}{X_1(t)} - \frac{dX_0(t)}{X_0(t)} - \frac{dX_1(t)}{X_1(t)} \frac{dX_0(t)}{X_0(t)} - (\frac{dX_0(t)}{X_0(t)})^2 \\ &= qdt + \frac{1}{X_1(t)} [\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)] - qdt \\ &= \frac{1}{X_1(t)} [\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)] \end{aligned} \tag{4.4}$$

hence

$$dY_1(t) = X_0^{-1} [\sigma_1 d\widetilde{B}_1 + \sigma_2 d\widetilde{B}_2(t)] \tag{4.5}$$

and

$$dZ_1(t) = 2Y_1(t) e^{-qt} [\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)] \tag{4.6}$$

so that

$$\begin{aligned} dW_1(t) &= qe^{qt} Z_1(t) dt + e^{qt} dZ_1(t) \\ &= qe^{qt} Z_1(t) dt + 2Y_1(t) [\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)]. \end{aligned}$$

Now let $Y_2(t) = e^{-qt} W_1(t)$, which we have shown to be a martingale under the risk neutral measure Q equivalent to the real world measure P. We thus have again that

$$Z_2(t) = Y_2^2(t) - \langle Y_2 \rangle_t,$$

is again a Q martingale, so if we let

$$W_2(t) = e^{qt} Z_2(t),$$

we know that its discounted version given by

$$Y_3(t) = e^{-qt} W_2(t),$$

is a martingale with respect to Q and its differential form is given by

$$dW_2(t) = qe^{qt} Z_2(t)dt + e^{qt} dZ_2(t). \tag{4.7}$$

Where

$$\begin{aligned} dZ_2(t) &= 2Y_2(t)dY_2(t) + dY_2(t)dY_2(t) - d\langle Y_2\rangle_t \\ &= 2Y_2(t)dY_2(t) \\ &= 4Y_1(t)Y_2(t)X_0^{-1}[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)]. \end{aligned}$$

Thus

$$\begin{aligned} dW_2(t) &= qe^{qt} Z_2(t)dt + e^{qt} dZ_2(t) \\ &= qe^{qt} Z_2(t)dt + 4e^{qt} Y_1(t)Y_2(t)X_0^{-1}[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)] \end{aligned}$$

which justifies the fact that the process $Z(t)$ is indeed a martingale with respect to Q as it does not contain the drift part (dt) and the fact that it is driven by the two dimensional Q Brownian motion $\widetilde{B}(t) = \{\widetilde{B}_1(t), \widetilde{B}_2(t)\}$. Now if we let

$$Z_3(t) = Y_3^2(t) - \langle Y_3\rangle_t,$$

the process $Z_3(t)$ is again a Q martingale, so we have

$$W_3(t) = e^{qt} Z_3(t),$$

where

$$dW_3(t) = qe^{qt} Z_3(t)dt + e^{qt} dZ_3(t) \tag{4.8}$$

and

$$\begin{aligned} dZ_3(t) &= 2Y_3(t)dY_3(t) + dY_3(t)dY_3(t) - d\langle Y_3\rangle_t \\ &= 2Y_3(t)dY_3(t) \\ &= 8Y_1(t)Y_2(t)Y_3(t)X_0^{-1}[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)], \end{aligned}$$

so that

$$\begin{aligned} dW_3(t) &= qe^{qt} Z_3(t)dt + e^{qt} dZ_3(t) \\ &= qe^{qt} Z_3(t)dt + e^{qt} 8Y_1(t)Y_2(t)Y_3(t)X_0^{-1}[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)]. \end{aligned}$$

The above mentioned process gives us three additional tradable asset with price process $W_i(t) = \{W_1(t), W_2(t), W_3(t)\}$ driven by the following stochastic differential equations

$$\begin{aligned}
dW_1(t) &= qe^{qt}Z_1(t)dt + 2Y_1(t)[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)], \\
dW_2(t) &= qe^{qt}Z_2(t)dt + 4e^{qt}Y_1(t)Y_2(t)X_0^{-1}[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)], \\
dW_3(t) &= qe^{qt}Z_3(t)dt + e^{qt}8Y_1(t)Y_2(t)Y_3(t)X_0^{-1}[\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)].
\end{aligned}$$

We call our processes $W_i = \{W_i(t), t \geq 0\}$ quadratic variation processes.

4.3 Completeness and Martingale Representation

Theorem 4.1. *(The Martingale Representation Theorem)]*

Let $B(t) = (B_1(t), B_2(t), ..., B_n(t))$ be n-dimensional. Suppose M_t is an $\mathcal{F}_t^{(n)}$-martingale (w.r.t P) for all t. Then there exists a unique stochastic process $\Gamma(u) = \{\Gamma_1(u), \Gamma_2(u), ..., \Gamma_n(t)\}$ for every $t > 0$ such that

$$M_t(u) = E[M_0] + \int_0^t \Gamma(u)dB(u) \qquad \textit{a.s.} \quad \textit{for all} \quad t \geq 0. \tag{4.9}$$

Proof
see Øksendal [37]

We will make use of martingale representative property (MRP) which says that any square-integrable Q-martingale M_t can be represented as follows:

$$M_t = M_0 + \int_0^t \Gamma(u)d\widetilde{B}(u) \qquad 0 < t < T \tag{4.10}$$

such that

$$E[\int_0^t |\Gamma(u)|^2 du] < \infty.$$

Now consider an incomplete market driven by an SDE given below

$$\begin{aligned}
dX_0(t) &= qX_0(t)dt, \\
dX_1(t) &= \alpha dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t) + ... + \sigma_p dB_p(t).
\end{aligned}$$

Then by Girsanov's Theorem we have

$$\begin{pmatrix} \sigma_1 & \sigma_2 & \dots & \sigma_p \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix} = \alpha - qX_1(t).$$

Which in the same way as our previous example (4.1) gives

$$dX_1(t) = qX_1(t)dt + \sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t) + ... + \sigma_p d\widetilde{B}_P(t), \quad (4.11)$$

which is an equivalent martingale measure Q under which our market remains the same and the discounted price process $Y_1(t) = \frac{\widehat{X}_1(t)}{X_0(t)}$ remains a Q martingale. We can easily derive the set of quadratic variation processes using the method described in section 4.2, which we will use to enlarge our market model. These quadratic variation processes are given by

$$W(t) = \{W_1(t); W_2(t); ...; W_N(t)\}.$$

Proposition 4.1. *An incomplete model given above enlarged using the quadratic variation process, can be completed, in the sense that any square-integrable contingent claim $X(t)$ can be replicated.*

Proof

Consider a square-integrable contingent claim $X(t)$ with maturity T.
Let $M_t = E_Q[e^{-qT}X|\mathcal{F}_t]$
By the (MRP) given by Theorem 4.1, define:

$$M_t^N = M_0 + \int_0^t \omega_1(u)dY(u) + \sum_{i=2}^{N}\int_0^t \omega_i(u)dZ_i(u),$$

where $\omega_i(u)$,$\{i = 1, 2, \ldots, N\}$ are predictable processes, such that

$$E[\int_0^t |\omega_i(u)|^2 du] < \infty.$$

Define the sequence of portfolios

$$\theta^N = \{\theta_t^N = (\alpha_t^N; \beta_1(t); \beta_2(t); ...; \beta_N(t)) \quad t \geq 0\} \quad (4.12)$$

by

$$\begin{aligned}
\alpha_t^N &= M_t^N - \beta_1(t)X_1(t)e^{-qt} - e^{-qt}\sum_{i=2}^{N}\beta_i(t)W_i(t),\\
\beta_1(t) &= e^{qt}\omega(t)X_0^{-1}(t),\\
\beta_i(t) &= \omega_i(t) \qquad i \geq 2.
\end{aligned}$$

The portfolio $\{\theta^N, N \geq 2\}$ is the sequence of self-financing portfolios which replicates $X(t)$. In fact the value of θ^N at time t is given by

$$\begin{aligned}
V_t^N &= \alpha_t^N e^{qt} + \beta_1(t)X_1(t) + \sum_{i=2}^{N}\beta_i(t)W_i(t)\\
&= e^{qt}M_t^N,
\end{aligned}$$

so the sequence of portfolios $\{\theta^N, N > 2\}$ is replicating the claim. Denote by

$$G_u^N = q\int_0^u \alpha_t^N e^{qt}dt + \int_0^u \beta_1(t)dX_1(t) + \sum_{i=2}^N \int_0^u \beta_i(t)dW_i(t), \tag{4.13}$$

the gain process.
We want to show that $G_u^N + M_0 = M_u^N e^{qu}$. Which implies that the portfolio is self-financing.
We have

$$\begin{aligned} G_u^N &= q\int_0^u M_t^N e^{qt}dt - q\int_0^u \omega_t e^{qt}dt - q\sum_{i=2}^N \int_0^u \omega_i(t)W_i(t)dt \\ &\quad + \int_0^u \omega(t)e^{qt}X_0^{-1}(t)dX_1(t) + \sum_{i=2}^N \int_0^u \omega_i(t)dW_i(t). \end{aligned} \tag{4.14}$$

Now integrating by part gives us

$$\begin{aligned} &q\int_0^u M_t^N e^{qt}dt \\ &= e^{qu}M_u^N - M_0 - \int_0^u \omega_1(t)e^{qt}dY(t) - \sum_{i=2}^N \int_0^u \omega_i(t)e^{qt}dZ_i(t) \end{aligned} \tag{4.15}$$

since $dM_t^N = \omega_1(t)dY(t) + \sum_{i=2}^N \omega_i(t)dZ_i(t)$.
Now substituting (4.15) into (4.14) yields

$$\begin{aligned} G_u^N &= e^{qu}M_u^N - M_0 - \int_o^t \omega_1(t)e^{qt}dY(t) - \sum_{i=2}^N \int_0^u \omega_i(t)e^{qt}dZ_i(t) \\ &\quad -q\int_0^u \omega_1(t)Y(t)e^{qt}dt - q\sum_{i=2}^N \int_0^u \omega_i(t)W_i(t)dt \\ &\quad + \int_0^u \omega(t)e^{qt}X_0^{-1}dX_1(t) + \sum_{i=2}^N \int_0^u \omega_i(t)dW_i(t) \\ &= e^{qu}M_u^N - M_0 - q\int_0^u \omega_1(t)Y(t)e^{qt}dt - \int_0^u \omega_1(t)e^{qt}dY(t) \\ &\quad + \int_0^u \omega(t)e^{qt}X_0^{-1}(t)dX_1(t) \\ &= e^{qt}M_u^N - M_0. \end{aligned}$$

Which completes our market model as required.
Thus such an incomplete market model can be completed as shown in this project irrespective of the number of diffusion terms, since such terms can be compensated by increasing the quadratic variation process shown in this paper to account for the extra diffusion terms.

Now returning to the incomplete market model given in (4.1) we have the following proposition

Proposition 4.2. *An incomplete model enlarged using the quadratic variation process as shown is complete, in the sense that any square-integrable contingent claim $X(t)$ can be replicated.*

Proof
Consider a square-integrable contingent claim $X(t)$ with maturity T.
Let $M_t = E_Q[e^{-qT}X|\mathcal{F}_t]$.
By the (MRP) given in theorem (4.1), define:

$$M_t^3 = M_0 + \int_0^t \omega_1(u)dY(u) + \sum_{i=2}^{3}\int_0^t \omega_i(u)dZ_i(u).$$

Where $\omega_i(u)$,$\{i = 1, 2, \dots, N\}$ are predictable processes, such that

$$E[\int_0^t |\omega_i(u)|^2 du] < \infty.$$

Define the sequence of portfolios

$$\theta^3 = \{\theta_t^3 = (\alpha_t^3; \beta_1(t); \beta_2(t); \beta_3(t)) \quad t \geq 0\}, \tag{4.16}$$

by

$$\begin{aligned}
\alpha_t^3 &= M_t^3 - \beta_1(t)X_1(t)e^{-qt} - e^{-qt}\sum_{i=2}^{3}\beta_i(t)W_i(t),\\
\beta_1(t) &= e^{qt}\omega(t)X_0^{-1}(t),\\
\beta_i(t) &= \omega_i(t) \qquad i = 2, 3.
\end{aligned}$$

Where α_t^3 is the amount of money in the bank account at time t, $\beta_1(t)$ is the number of stocks at time t and $\beta_i(t) \quad i = 2, 3$ is the number of quadratic variation assets $\{W_i(t)\}$, that one need to hold at time t.
The portfolio $\{\theta^3\}$ is the sequence of self-financing portfolios which replicates $X(t)$. In fact the value of θ^3 at time t is given by

$$\begin{aligned}
V_t^3 &= \alpha_t^3 e^{qt} + \beta_1(t)X_1(t) + \sum_{i=2}^{3}\beta_i(t)W_i(t)\\
&= e^{qt}M_t^3
\end{aligned}$$

so the sequence of portfolios $\{\theta^3\}$ is replicating the claim. Denote by

$$G_u^3 = q\int_0^u \alpha_t^3 e^{qt}dt + \int_0^u \beta_1(t)dX_1(t) + \sum_{i=2}^{3}\int_0^u \beta_i(t)dW_i(t) \tag{4.17}$$

the gain process. We want to show that $G_u^3 + M_0 = M_u^3 e^{qu}$. Which implies that our portfolio given above is self-financing hedging portfolio that replicates any contingent claim $X(T)$.
We have

$$G_u^3 = q\int_0^u M_t^3 e^{qt}dt - q\int_0^u \omega_1(t)Y(t)e^{qt}dt - q\sum_{i=2}^3\int_0^u \omega_i(t)W_i(t)dt$$
$$+\int_0^u \omega_1(t)e^{qt}X_0^{-1}(t)dX_1(t) + \sum_{i=2}^3\int_0^u \omega_i(t)dW_i(t). \tag{4.18}$$

Now integrating by part we get

$$\begin{aligned}
& q\int_0^u M_t^3 e^{qt}dt \\
= \;& e^{qu}M_u^3 - M_0 - \int_0^u e^{qt}dM_t^3 \\
= \;& e^{qu}M_u^3 - M_0 - \int_0^u \omega_1(t)e^{qt}dY(t) - \sum_{i=2}^3\int_0^u \omega_i(t)e^{qt}dZ_i(t), \qquad (4.19)
\end{aligned}$$

since $dM_t^3 = \omega_1(t)dY(t) + \sum_{i=2}^3 \omega_i(t)dZ_i(t)$.
Now substituting (4.19) into (4.18) yields

$$\begin{aligned}
G_u^3 \;=\; & e^{qu}M_u^3 - M_0 - \int_o^t \omega_1(t)e^{qt}dY(t) - \sum_{i=2}^3\int_0^u \omega_i(t)e^{qt}dZ_i(t) \\
& -q\int_0^u \omega_1(t)Y(t)e^{qt}dt - q\sum_{i=2}^3\int_0^u \omega_i(t)W_i(t)dt \\
& +\int_0^u \omega_1(t)e^{qt}X_0^{-1}dX_1(t) + \sum_{i=2}^3\int_0^u \omega_i(t)dW_i(t) \qquad (4.20) \\
=\; & e^{qu}M_u^3 - M_0 - q\int_0^u \omega_1(t)Y(t)e^{qt}dt - \int_0^u \omega_1(t)e^{qt}dY(t) \\
& +\int_0^u \omega_1(t)e^{qt}X_0^{-1}(t)dX_1(t) \qquad (4.21) \\
=\; & e^{qt}M_u^3 - M_0 \qquad (4.22)
\end{aligned}$$

which complete our market model as required. We have thus obtained a self-financing portfolio which hedges any contingent claim $F(\omega) = X(t))$ in this market
Note that the equality sign from the first line above (4.20) to the second line (4.21) comes from the fact that $Z_i(t) = e^{-qt}W_i(t)$ which then implies that

$$dZ_i(t) = -qe^{-qt}W_i(t)dt + e^{-qt}dW_i(t), \tag{4.23}$$

so we must have

$$\begin{aligned}
& \sum_{i=2}^{3} \int_0^u \omega_i(t) e^{qt} dZ_i(t) \\
= & \sum_{i=2}^{3} \int_0^u \omega_i(t) e^{qt} (-q e^{-qt} W_i(t) dt + e^{-qt} dW_i(t)) \\
= & -q \sum_{i=2}^{3} \int_0^u \omega_i(t) W_i(t) dt + \sum_{i=2}^{3} \int_0^u \omega_i(t) dW_i(t).
\end{aligned}$$

Which after substitution gives the required result. The equality from the second line (4.21) above to the last line above (4.22), comes from the fact that

$$\begin{aligned}
dY(t) &= X_0^{-1}(t)(\sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t)) \\
dX_1(t) &= qX_1(t)dt + \sigma_1 d\widetilde{B}_1(t) + \sigma_2 d\widetilde{B}_2(t).
\end{aligned}$$

Which after equating the two equations above gives

$$dX_1(t) = qX_1(t)dt + X_0(t)dY(t). \tag{4.24}$$

So we have

$$\begin{aligned}
& \int_0^u \omega_1(t) e^{qt} X_0^{-1}(t) dX_1(t) \\
= & \int_0^u \omega_1(t) e^{qt} X_0^{-1}(t)(qX_1(t)dt + X_0(t)dY(t)) \\
= & q \int_0^u \omega_1(t) e^{qt} X_0^{-1}(t) X_1(t) dt + \int_0^u \omega_1(t) e^{qt} X_0^{-1}(t) X_0(t) dY(t) \\
= & q \int_0^u \omega_1(t) e^{qt} Y(t) dt + \int_0^u \omega_1(t) e^{qt} dY(t).
\end{aligned}$$

Where as before $Y(t) = X_0^{-1}(t)X_1(t)$, thus the result follows.

4.4 Construction of The Hedging Portfolio

In this section we show how to construct the self financing portfolio used to hedge contingent claims in the enlarged market, which consist of bonds (or bank account), quadratic variation asset and stocks.
Consider a market given by

$$\begin{aligned}
dX_1 &= qX_1 dt + \sigma_1 dB_1 + \sigma_2 dB_2, \\
dZ_2 &= \phi_2(t)dt + \xi_2(t)[\sigma_1 dB_1 + \sigma_2 dB_2], \\
dZ_3 &= \phi_3(t)dt + \xi_3(t)[\sigma_1 dB_1 + \sigma_2 dB_2],
\end{aligned}$$

where $X_1(t)$ is our stock price process while $Z_2(t)$ and $Z_3(t)$ is our second and third quadratic variation price process respectively as defined in section (4.2). A contingent claim $X(t) = f(S(T))$ at time t is given by

$$F(t,S) = \exp\left(-q(T-t)\right)E_Q[f(S(T))\ \mathcal{F}_t]$$

We call $F(t,x)$ the price of the contingent claim X
A portfolio consisting of $-\Delta$ units of the underlying is given by

$$\Pi = f - \Delta X_1$$

And the change in the value of this portfolio is given by

$$d\Pi = df - \Delta dX_1$$

When we apply the Itô lemma on f, we get

$$df = \tfrac{\partial f}{\partial t}dt + \tfrac{\partial f}{\partial X_1}dX_1 + \tfrac{1}{2}\textstyle\sum_{(i,j)=1}^{3}\tfrac{\partial^2 f}{\partial Z_i \partial Z_j}\xi_i\xi_j(\sigma_1^2+\sigma_2^2)dt + \textstyle\sum_{i=2}^{3}\tfrac{\partial f}{\partial Z_i}dZ_i$$

Where we have, without loss of generality,
let $X_1 = Z_1$, $\phi_1(t) = qX_1(t)$ and $\xi_1(t) = 1$. So we must have

$$\begin{aligned}
d\Pi &= [\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{(i,j)=1}^{3}\frac{\partial^2 f}{\partial Z_i\partial Z_j}\xi_i\xi_j(\sigma_1^2+\sigma_2^2)]dt \\
&\quad +(\frac{\partial f}{\partial X_1} - \Delta)dX_1 + \sum_{i=2}^{3}\frac{\partial f}{\partial Z_i}dZ_i
\end{aligned}$$

Now setting $\Delta = \frac{\partial f}{\partial X_1}$ and using the no-arbitrage argument used in the Black-Scholes model of Chapter (3) we have

$$\begin{aligned}
d\Pi &= [\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{(i,j)=1}^{3}\frac{\partial^2 f}{\partial Z_i\partial Z_j}\xi_i\xi_j(\sigma_1^2+\sigma_2^2)]dt + \sum_{i=2}^{3}\frac{\partial f}{\partial Z_i}dZ_i \\
&= q(f - \Delta X_1)dt \\
&= q(f - X_1\frac{\partial f}{\partial X_1})dt
\end{aligned}$$

Now comparing the dt terms we see that

$$\frac{\partial f}{\partial t} + \frac{1}{2}\sum_{(i,j)=1}^{3}\frac{\partial^2 f}{\partial Z_i\partial Z_j}\xi_i\xi_j(\sigma_1^2+\sigma_2^2) = q(f - X_1\frac{\partial f}{\partial X_1}). \tag{4.25}$$

Hence

$$\frac{1}{2}\sum_{(i,j)=1}^{3}\frac{\partial^2 f}{\partial Z_i\partial Z_j}\xi_i\xi_j(\sigma_1^2+\sigma_2^2) = \frac{\partial f}{\partial t} + qf - qX_1\frac{\partial f}{\partial X_1}. \tag{4.26}$$

We can now calculate the sequence of self-financing portfolios that replicates the contingent claim $X(t)$.

Proposition 4.3. *The sequence of self-financing portfolios replicating a contingent claim X with a payoff only depending on the stock price value equation* (4.14) *at maturity and the price function $F(t,x) \in \mathbb{C}^2$ is given at time t by*

$$\Phi_t^N = \{\Phi_t^N = (\alpha_t^N, \beta_1(t), \ldots, \beta_N(t)), t \geq 0\} \tag{4.27}$$

$$\textit{number of bonds} = \alpha_t^N = X_0^{-1}(f - X_1 \frac{\partial f}{\partial X_1}) - X_0^{-1} \sum_2^N X_0^{-1} \frac{\partial f}{\partial Z_i(t)} W_i(t)$$

$$\textit{number of stocks} = \beta_1(t) = \frac{\partial f}{\partial X_1(t)}$$

$$\textit{number of quadratic variation assets} = \beta_i(t) = X_0^{-1} \frac{\partial f}{\partial Z_i(t)} \quad i = 2, 3, \ldots, N.$$

Proof

Applying Itô lemma to $f(t, X_1(t))$ for the price process given in equation (4.14), gives us

$$\begin{aligned}
f(t, X_1) - f(0, x) &= \int_0^t \frac{\partial f}{\partial t} ds + \int_0^t \frac{1}{2} \sum_{(i,j)=1}^N \frac{\partial^2 f}{\partial Z_i \partial Z_j} \xi_i \xi_j (\sigma_1^2 + \sigma_2^2) ds + \int_0^t \frac{\partial f}{\partial X_1} dX_1 \\
&+ \int_0^t \sum_2^N \frac{\partial f}{\partial Z_i(t)} dZ_i \\
&= \int_0^t \frac{\frac{\partial f}{\partial t} + \int_0^t \frac{1}{2} \sum_{(i,j)=1}^N \frac{\partial^2 f}{\partial Z_i \partial Z_j} \xi_i \xi_j (\sigma_1^2 + \sigma_2^2)}{q X_0} dX_0 \\
&+ \int_0^t \frac{\partial f}{\partial X_1} dX_1 + \int_0^t \sum_2^N \frac{\partial f}{\partial Z_i(t)} dZ_i \\
&= \int_0^t \frac{\frac{\partial f}{\partial t} + \int_0^t \frac{1}{2} \sum_{(i,j)=1}^N \frac{\partial^2 f}{\partial Z_i \partial Z_j} \xi_i \xi_j (\sigma_1^2 + \sigma_2^2) - q \sum_2^N Z_i \frac{\partial f}{\partial Z_i}}{q X_0} dX_0 \\
&+ \int_0^t \frac{\partial f}{\partial X_1} dX_1 + \int_0^t \sum_2^N X_0^{-1} \frac{\partial f}{\partial Z_i(t)} dW_i(t) \\
&= \int_0^t X_0^{-1} \left[f + X_1 \frac{\partial f}{\partial X_1} - \sum_2^N X_0^{-1} \frac{\partial f}{\partial Z_i} W_i(t) \right] dX_0 \\
&+ \int_0^t \frac{\partial f}{\partial X_1} dX_1 + \int_0^t \sum_2^N X_0^{-1} \frac{\partial f}{\partial Z_i(t)} dW_i(t).
\end{aligned}$$

Where we have used the PDE given by

$$\frac{1}{2}\sum_{(i,j)=1}^{N}\frac{\partial^2 f}{\partial Z_i \partial Z_j}\xi_i\xi_j(\sigma_1^2+\sigma_2^2) = \frac{\partial f}{\partial t} + qf - qX_1\frac{\partial f}{\partial X_1}, \tag{4.28}$$

for the market consisting of N quadratic variation price processes and the fact that $Z_i(t) = e^{(-qt)}W_i(t)$ (for $i \geq 2$) or equivalently we have

$$\begin{aligned} dZ_i(t) &= -qe^{(-qt)}W_i(t)dt + e^{(-qt)}dW_i(t) \\ &= -qZ_i(t)dt + X_0^{-1}dW_i(t). \end{aligned} \tag{4.29}$$

For the market given in equation (4.27) for $N = 3$ we have the following proposition.

Proposition 4.4. *The sequence of self-financing portfolios replicating a contingent claim X with a payoff only depending on the stock price value equation* (4.3) *at maturity and the price function $F(t,x) \in \mathbb{C}^2$ is given at time t by*

$$\Phi_t^3 = \{\Phi_t^3 = (\alpha_t^3, \beta_1(t), \beta_2(t), \beta_3(t)), t \geq 0\} \tag{4.30}$$

$$\textit{number of bonds} = \alpha_t^3 = X_0^{-1}(f - X_1\frac{\partial f}{\partial X_1}) - X_0^{-1}\sum_2^3 X_0^{-1}\frac{\partial f}{\partial Z_i(t)}W_i(t)$$

$$\textit{number of stocks} = \beta_1(t) = \frac{\partial f}{\partial X_1(t)}$$

$$\textit{number of quadratic variation assets} = \beta_i(t) = X_0^{-1}\frac{\partial f}{\partial Z_i(t)} \quad i = 2,3.$$

4.5 Pricing in an Incomplete Market with more Randomness than Tradable Assets

Consider again the incomplete market given by

$$\begin{aligned} dX_0 &= qX_0dt, \\ dX_1 &= \alpha dt + \sigma_1 dB_1(t) + \sigma_2 dB_2(t). \end{aligned}$$

Now consider a portfolio consisting of the option we wish to use to price the underlying stock f as well as another option, independent of our first option f_1, to hedge the risk associated with the second randomness. Then the value of our portfolio is given by

$$\begin{aligned} &\Pi = f - \Delta S - \Delta_1 f_1 \\ &d\Pi = df - \Delta dS - \Delta_1 df_1. \end{aligned}$$

By the Itô Lemma on f we have that

$$\begin{aligned}
df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial X_0}dX_0 + \frac{\partial f}{\partial S}dS + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 \\
&= \frac{\partial f}{\partial t}dt + qX_0\frac{\partial f}{\partial X_0}dt + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}dt + \\
&\quad \alpha\frac{\partial f}{\partial S}dt + \frac{\partial f}{\partial s}(\sigma_1 dB_1(t) + \sigma_2 dB_2(t)) \\
&= \left[\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \alpha\frac{\partial f}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}\right]dt + \\
&\quad \sigma_1\frac{\partial f}{\partial S}dB_1(t) + \sigma_2\frac{\partial f}{\partial S}dB_2(t)
\end{aligned}$$

We can produce a similar process for f_1. Substituting both f and f_1 into our portfolio equation (4.29) we get

$$\begin{aligned}
d\Pi &= \left[\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \alpha\frac{\partial f}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}\right]dt \\
&\quad -\Delta_1\left[\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \alpha\frac{\partial f_1}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2}\right]dt + \\
&\quad (\sigma_1\frac{\partial f}{\partial S} - \Delta_1\sigma_1\frac{\partial f_1}{\partial S})dB_1(t) + (\sigma_2\frac{\partial f}{\partial S} - \Delta_1\sigma_2\frac{\partial f_1}{\partial S})dB_2(t) \\
&\quad -\alpha\Delta dt - \Delta\sigma_1 dB_1(t) - \Delta\sigma_2 dB_2(t)
\end{aligned}$$

Which then implies that

$$\begin{aligned}
d\Pi &= \left[\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \alpha\frac{\partial f}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2} - \alpha\Delta\right]dt \\
&\quad -\Delta_1\left[\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \alpha\frac{\partial f_1}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2}\right]dt + \\
&\quad (\sigma_1\frac{\partial f}{\partial S} - \Delta_1\sigma_1\frac{\partial f_1}{\partial S} - \Delta\sigma_1)dB_1(t) + (\sigma_2\frac{\partial f}{\partial S} - \Delta_1\sigma_2\frac{\partial f_1}{\partial S} - \Delta\sigma_2)dB_2(t)
\end{aligned}$$

To remove the randomness from our portfolio, we must choose Δ and Δ_1 in such a way that

$$\sigma_1\frac{\partial f}{\partial S} - \Delta_1\sigma_1\frac{\partial f_1}{\partial S} - \Delta\sigma_1 = 0 \tag{4.31}$$

and

$$\sigma_2\frac{\partial f}{\partial S} - \Delta_1\sigma_2\frac{\partial f_1}{\partial S} - \Delta\sigma_2 = 0. \tag{4.32}$$

This change of our portfolio then gives us a portfolio consisting of only the dt term which we call the return of the portfolio. Keeping in mind that we are working in a no-arbitrage environment and thus our portfolio cannot earn more than the risk-free rate. That is

$$d\Pi = q\Pi dt \tag{4.33}$$

for no-arbitrage opportunities to exist, which then gives

$$\begin{aligned} d\Pi &= \left[\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \alpha\frac{\partial f}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2} - \alpha\Delta\right]dt \\ &\quad -\Delta_1\left[\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \alpha\frac{\partial f_1}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2}\right]dt \\ &= q\Pi dt \\ &= q(f - \Delta S - \Delta_1 f_1)dt. \end{aligned}$$

This then implies that

$$\begin{aligned} &\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \alpha\frac{\partial f}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2} - \alpha\Delta \\ &-\Delta_1\left[\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \alpha\frac{\partial f_1}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2}\right] \\ = \;& qf - \Delta qS - \Delta_1 f_1. \end{aligned}$$

But $\Delta = \frac{\partial f}{\partial S} - \Delta_1\frac{\partial f_1}{\partial S}$, so we must have

$$\begin{aligned} &\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \alpha\frac{\partial f}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2} - \alpha\frac{\partial f}{\partial S} + \alpha\Delta_1\frac{\partial f_1}{\partial S} - rf \\ = \;& \Delta_1\left[\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \alpha\frac{\partial f_1}{\partial S} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2}\right] \\ &-qS\frac{\partial f}{\partial S} + \Delta_1 qS\frac{\partial f_1}{\partial S} - \Delta_1 qf_1. \end{aligned}$$

Which then implies that

$$\begin{aligned} &\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2} + qS\frac{\partial f}{\partial S} - rf \\ = \;& \Delta_1\left[\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2} + qS\frac{\partial f_1}{\partial S} - qf_1\right]. \end{aligned}$$

Now by setting $\Delta_1 = \frac{\partial f}{\partial S}/\frac{\partial f_1}{\partial S}$ as the value of the number of second options f_1 required in other to hedge our portfolio, our equation above then becomes

$$\begin{aligned} &\frac{\frac{\partial f}{\partial t} + qX_0\frac{\partial f}{\partial X_0} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2} + qS\frac{\partial f}{\partial S} - qf}{\frac{\partial f}{\partial s}} \\ = \;& \frac{\frac{\partial f_1}{\partial t} + qX_0\frac{\partial f_1}{\partial X_0} + \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2} + qS\frac{\partial f_1}{\partial S} - qf_1}{\frac{\partial f_1}{\partial S}}. \end{aligned}$$

From this equation one can see that the left-hand side of the equation is a function of f which is independent of f_1, similarly the right-hand side is a function of f_1 which is also independent of f.
Since both f and f_1 will generally have different payoffs, time to maturities and

strike price for an option, the only way the above equation will hold is if both sides are independent of contract type. Since each side is independent of our chosen derivatives f and f_1, we can put both sides equal to some arbitrary function $\lambda(t,S)$,
which gives us

$$\frac{\frac{\partial f}{\partial t}+qX_0\frac{\partial f}{\partial X_0}+\frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}+qS\frac{\partial f}{\partial S}-qf}{\frac{\partial f}{\partial s}}=\lambda(t,S). \tag{4.34}$$

Hence

$$\frac{\partial f}{\partial t}+qX_0\frac{\partial f}{\partial X_0}+\frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}+(qS-\lambda)\frac{\partial f}{\partial S}-qf=0. \tag{4.35}$$

In fact

$$\frac{\partial f}{\partial t}+\frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}+(qS-\lambda)\frac{\partial f}{\partial S}-qf=0. \tag{4.36}$$

Similarly for f_1

$$\frac{\partial f_1}{\partial t}+\frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f_1}{\partial S^2}+(qS-\lambda)\frac{\partial f_1}{\partial S}-qf_1=0, \tag{4.37}$$

the function $\lambda(t,S)$ is called the market price of risk.

Proposition 4.5. *The above options f and f_1 are the options that complete our market model* 4.1 *in the sense that the claims $F\omega = g(B_2(t))$, which we could not hedge before, can now be hedged by the options f and f_1.*

Proof

Consider the PDE obtained above for the price of the option f given by

$$\frac{\partial f}{\partial t}+\frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}+(qS-\lambda)\frac{\partial f}{\partial S}-qf=0. \tag{4.38}$$

Applying the Feynman-Kac formula to the above PDE we get that

$$\begin{aligned} Af-qf &= -qf+\frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}+(qS-\lambda)\frac{\partial f}{\partial S} \\ \Rightarrow Af &= \frac{1}{2}(\sigma_1^2+\sigma_2^2)\frac{\partial^2 f}{\partial S^2}+(qS-\lambda)\frac{\partial f}{\partial S}. \end{aligned}$$

If we apply the generator of the Itô diffusion we get that

$$dX_1(t)=(qX_1(t)-\lambda(t,X_1))dt+\sigma_* dW(t). \tag{4.39}$$

Where $X_1=S$, $\sigma_*^2=(\sigma_1^2+\sigma_2^2)$ and $W(t)$ is a Brownian motion, which complete our market model as required.
If for example one wanted to price a contingent claim for f given by a European

call option $f(t, X_1) = (X_1 - K)^+$ in the above market. By the Feynman-Kac formula we have that

$$\begin{aligned} f(t,x) &= E_x[e^{(-\int_0^t q(X(s))ds)} f(X(t))] \\ &= E_x[e^{qt} f(X_1(t))]. \end{aligned}$$

Where $X_1(t)$ is given by

$$X_1(t) = X_1(0) + \int_0^t \delta(t, X_1)dt + \sigma_* W(t). \tag{4.40}$$

With the mean and variance of X_1 given by

$$\begin{aligned} E[X_1] &= \zeta(X_1) \\ Var[X_1] &= E[X_1^2] + (E[X_1])^2 \\ &= \zeta^2 + \int_0^t \sigma_*^2 dt - \zeta^2 \\ &= \sigma_*^2 t. \end{aligned} \tag{4.41}$$

For easy notation we have made $\delta(t, X_1) = (qX_1(t) - \lambda(t, X_1))$ and $\zeta(X_1) = X_1(0) + \int_0^t \delta(t, X_1)dt$. Now returning to our price for a European call option we have

$$f(t, X_1) = \frac{e^{-qt}}{\sqrt{2\pi\sigma_*}} \int_{\mathbb{R}} (X_1 - K)^+ e^{-\frac{1}{2\sigma_*}(X_1 - \zeta)^2} dX_1. \tag{4.42}$$

If we use the fact that $-W(t) \sim N(0, t)$ and that $X_1(t) > K$ for a European call option, otherwise the option is not exercised, we have that

$$\begin{aligned} & X_1(0) + \int_0^t \delta(t, X_1)dt + \sigma_* W(t) > K, \\ & -W(t) < \frac{-K + X_1(0) + \int_0^t \delta(t, X_1)dt}{\sigma_*} = d, \end{aligned}$$

so that

$$\begin{aligned} f(t, X_1) &= \frac{e^{-qt}}{\sqrt{2\pi t}} \int_{-\infty}^{d} (\zeta - \sigma_* y - K) e^{-\frac{1}{2t}y^2} dy \\ &= \Psi + \frac{e^{-qt}}{\sqrt{2\pi t}} \sigma_* t \int_{-\infty}^{d_1} e^u du - K \frac{e^{-qt}}{\sqrt{2\pi t}} \int_{-\infty}^{d} e^{-\frac{1}{2t}y^2} dy \\ &= \Psi + \frac{e^{(d_1 - qt)}}{\sqrt{2\pi t}} \sigma_* t - K e^{-qt} N(d_2). \end{aligned}$$

Where $d_1 = -\frac{d^2}{2t}$,

$$d_2 = \frac{-K + X_1(0) + \int_0^t \delta(t, X_1)dt}{\sigma_* \sqrt{t}} \tag{4.43}$$

and

$$\Psi = \frac{e^{-qt}}{\sqrt{2\pi t}} \int_{-\infty}^{d} \zeta e^{-\frac{1}{2t}y^2} dy. \tag{4.44}$$

Using the same argument we can also obtain a price for the contingency claim $f_1(t, X_1)$. When we compare our price with the one obtained in the Black-Scholes option price for a call option, we see that the only difference lies in the dynamics of the actual underlying stock price process. The Black-Scholes model has stocks being modeled by a geometric Brownian motion, where else our completed market had stocks being modeled by a process with a deterministic rate of return.

Chapter 5

Market Paying Transactional Costs

Transactional costs are costs of buying and selling stocks and bonds in the market. The process of buying and selling assets is not free, there are costs incurred when one buys and sells assets. We will make use of a continuous market model considered in the paper by Cvitanić and Karatzas [8], where they find an optimal hedging portfolio under transactional costs. Such markets are incomplete due to the cost incurred when buying and selling stocks and bonds. Hedging is expensive, buyer's and seller's prices do not agree to the same value. Rather than finding an optimal portfolio for the market, we attempt to complete such a market model.

5.1 Transactional Costs Model

Consider a financial market consisting of a bond or bank account (riskless asset) and one stock (risky asset) driven by the stochastic equation

$$\begin{aligned} dS_0(t) &= S_0(t)r(t)dt, \qquad S_0(0) = 1, & (5.1) \\ dS(t) &= S(t)[b(t)dt + \sigma(t)dB(t)], \qquad S(0) = x, \quad x\epsilon(0, \infty), & (5.2) \end{aligned}$$

where as before, $t \in [0, T]$ and $B(t)$ is a one-dimensional Brownian motion. Now, a trading strategy is a pair (L, M) of $\mathcal{F}$-adapted processes on $[0, T]$ with left continuous-nondecreasing paths and $L(0) = M(0) = 0$.
$L(t)$ (respectively, $M(t)$) represent the total amount of funds transferred from bank account to stock (respectively from stock to bank account) by time t. Given proportional transactional costs $0 < \lambda, \mu < 1$ for such transfers, and initial holdings x and y in bank and stock respectively. The portfolio holding $X(t)$ and $Y(t)$ corresponding to a given trading strategy (M, L),

evolve according to the equations

$$\begin{aligned} X(t) &= x-(1+\lambda)L(t)+(1-\mu)M(t)+\int_0^t X(u)r(u)du, \qquad 0\leq t\leq T, \\ Y(t) &= y+L(t)-M(t)+\int_0^t Y(u)[b(u)du+\sigma(u)dB(u)], \end{aligned}$$

is an incomplete market model. Hence, there are claims in this market which cannot be hedged by a self financing portfolio.

5.2 Claims that Cannot be Hedged in a Market Paying Transactional Costs

If one holds θ_1 funds in the bank account and θ_2 funds in stocks then the value of such a portfolio holding is given by

$$V(t)=\theta_1(t)X(t)+\theta_2(t)Y(t) \qquad t\epsilon[0,T] \tag{5.3}$$

For such a self financing portfolio we have

$$\begin{aligned} dV(t) &= \theta_1(t)dX(t)+\theta_2(t)dY(t) \qquad (5.4)\\ &= \theta_1(t)X(t)r(t)dt+\theta_1(t)(1-\mu)dM(t)-\theta_1(t)(1+\lambda)dL(t)+\\ &\quad \theta_2(t)Y(t)b(t)dt+\theta_2(t)Y(t)\sigma(t)dB(t)+\theta_2(t)[dL(t)-dM(t)]\\ &= r(t)[V(t)-\theta_2(t)Y(t)]dt+\theta_2(t)b(t)Y(t)dt+\theta_1(t)(1-\mu)dM(t)-\\ &\quad \theta_1(t)(1+\lambda)dL(t)+\theta_2(t)\sigma(t)Y(t)dB(t)+\theta_2(t)[dL(t)-dM(t)]\\ &= r(t)V(t)dt+\Big[\frac{b(t)-r(t)}{\sigma(t)}\Big]\theta_2(t)Y(t)\sigma(t)dt+\theta_2(t)\sigma(t)Y(t)dB(t)+\\ &\quad \theta_1(t)(1-\mu)dM(t)-\theta_1(t)(1+\lambda)dL(t)+\theta_2(t)[dL(t)-dM(t)]\\ &= r(t)V(t)dt+\theta_2(t)Y(t)\sigma(t)d\widetilde{B}(t)+\theta_1(t)(1-\mu)dM(t)\\ &\quad -\theta_1(t)(1+\lambda)dL(t)+\theta_2(t)[dL(t)-dM(t)]\\ &= r(t)V(t)dt+\theta_2(t)Y(t)\sigma(t)d\widetilde{B}(t)+[\theta_1(t)(1-\mu)-\theta_2(t)]dM(t)\\ &\quad +[\theta_2(t)-\theta_1(t)(1+\lambda)]dL(t). \end{aligned}$$

After integration gives the value at time t of our portfolio to be given by

$$\begin{aligned} V(t) &= V(0)+\int_0^t r(s)V(s)ds+\int_0^t \theta_2(s)Y(s)\sigma(s)d\widetilde{B}(s)\\ &\quad +[\theta_1(t)(1-\mu)-\theta_2(t)]M(t)+[\theta_2(t)-\theta_1(t)(1+\lambda)]L(t), \qquad (5.5) \end{aligned}$$

where $\theta=(\theta_1,\theta_2)$ are self-financing trading strategies.
Now if our portfolio $\theta=(\theta_1,\theta_2)$ hedges a contingent T-claim $F(\omega)$ we have

$$\begin{aligned} F(\omega) &= V(0)+\int_0^T r(s)V(s)ds+\int_0^T \theta_2(s)Y(s)\sigma(s)d\widetilde{B}(s)+\\ &\quad [\theta_1(T)(1-\mu)-\theta_2(T)]M(T)+[\theta_2(T)-\theta_1(T)(1+\lambda)]L(T). \end{aligned}$$

If we choose the claim to be given by any claim $F(\omega) = g(\widetilde{B}(T))$, then by the Itô representation theorem applied to the one dimensional Brownian motion $\widetilde{B}(T)$ there is a unique $\phi(t,\omega)$ such that

$$g(\widetilde{B}(T) = E_Q[g(\widetilde{B}(T)] + \int_0^T \phi(t,\omega)d\widetilde{B}(t). \tag{5.6}$$

For this to be true we must have $\phi(t) = \sigma(t)\theta_2(t)Y(t)$ and $E_Q[g(\widetilde{B}(T)] = V(0)$, i.e. we must have

$$[\theta_1(t)(1-\mu) - \theta_2(t)]M(t) + [\theta_2(t) - \theta_1(t)(1+\lambda)]L(t) = 0.$$

Therefore the market is incomplete due to nonzero transactional costs incurred when we transfer funds from bank to stock. Holding such a portfolio will be very expensive and risky since we are not sure of how large the amount $\varphi(M,L) = [\theta_1(t)(1-\mu) - \theta_2(t)]M(t) + [\theta_2(t) - \theta_1(t)(1+\lambda)]L(t)$ will be at some future time T when we continuously transfer funds from bank to stock and vice versa. Any claim of the amount $\varphi(M,L)$ cannot be hedged by a self financing portfolio if they are non-zero. Hence this market is an incomplete market.

5.3 Completion of the Market Model

To complete a market model paying transactional costs with a trading strategy given by

$$\begin{aligned} dX(t) &= X(t)r(t)dt + (1-\mu)dM(t) - (1+\lambda)dL(t), && (5.7)\\ dY(t) &= Y(t)b(t)dt + Y(t)\sigma(t)dB(t) + dL(t) - dM(t), && (5.8)\end{aligned}$$

we first consider the case when $dM(t) = M(t)b(t)dt$ and $dL(t) = L(t)r(t)dt$ then we have

$$\begin{aligned} dY(t) &= Y(t)b(t)dt + Y(t)\sigma(t)dB(t) + L(t)r(t)dt - M(t)b(t)dt \\ &= [L(t)r(t) + Y(t)b(t) - M(t)b(t)]dt + Y(t)\sigma(t)dB(t) \\ &= L(t)r(t)dt + Y(t)\sigma(t)b(t)\left[\frac{Y(t)-M(t)}{Y(t)\sigma(t)}dt + dB(t)\right] \\ &= L(t)r(t)dt + Y(t)\sigma(t)b(t)d\widetilde{B}(t),\end{aligned}$$

and

$$\begin{aligned} dX(t) &= X(t)r(t)dt + \widetilde{\mu}M(t)b(t)dt - \widetilde{\lambda}L(t)r(t)dt \\ &= (X(t)r(t) - \widetilde{\lambda}L(t)r(t) + \widetilde{\mu}M(t)b(t))dt.\end{aligned}$$

We therefore have under the risk-neutral measure Q, which is equivalent to the real world measure P, a market consisting of

$$\begin{aligned} dX(t) &= X^*(L,M,r,t)dt \qquad \text{funds in bank account,}\\ dY(t) &= L(t)r(t)dt + Y(t)b(t)\sigma(t)d\widetilde{B}(t) \qquad \text{funds in stock.}\end{aligned}$$

We thus easily see that this is a complete market model with only one source of randomness. Even though this market is complete it is very expensive to hedge due to continuous payment of transactional costs when we continuously transfer funds from stock to bank account and vice-versa. Now consider the case when $dM(t) = M(t)[b(t)dt + \sigma(t)dB(t)]$ and $dL(t) = r(t)L(t)dt$. The reason we consider such a case is the fact that the amount of funds taken out of the bank gains the same interest rate $r(t)$ as any other funds in the bank account, while the amount of funds in stock has the same risk $B(t)$, the same interest rate $r(t)$ and volatility $\sigma(t)$. Our bank and stock account satisfy:

$$dX(t) = Xrdt + (1-\mu)M[bdt + \sigma dB(t)] - (1+\lambda)rLdt \tag{5.9}$$

and

$$\begin{aligned}
dY(t) &= Y(t)b(t)dt + Y(t)\sigma(t)dB(t) + r(t)L(t)dt - M(t)[b(t)dt + \sigma(t)dB(t)] \\
&= [Y(t)b(t) + r(t)L(t) - M(t)b(t)]dt + [Y(t) - M(t)]\sigma(t)dB(t) \\
&= \Big[r(t)L(t) + \frac{Y(t)-M(t)}{\sigma(t)}\sigma(t)b(t)\Big]dt + \frac{Y(t)-M(t)}{\sigma(t)}\sigma^2(t)dB(t) \\
&= [r(t)L(t) + \widetilde{M}(t)\sigma(t)b(t)]dt + \widetilde{M}(t)\sigma^2(t)dB(t) \\
&= r(t)L(t)dt + \sigma(t)d\widetilde{B}(t).
\end{aligned}$$

Where $d\widetilde{B}(t) = \widetilde{M}(t)b(t)dt + \widetilde{M}(t)\sigma(t)dB(t)$ and $\widetilde{M}(t) = \frac{Y(t)-M(t)}{\sigma(t)}$.
Letting $\widetilde{\mu} = (1-\mu)$ and $\widetilde{\lambda} = (1+\lambda)$ we get

$$\begin{aligned}
dX(t) &= X(t)r(t)dt + \widetilde{\mu}(Y(t) - \sigma(t)\widetilde{M}(t))[b(t)dt + \sigma(t)dB(t)] - \widetilde{\lambda}r(t)L(t)dt \\
&= X(t)r(t)dt - \widetilde{\lambda}r(t)L(t)dt + \widetilde{\mu}Y(t)b(t)dt + \widetilde{\mu}Y(t)\sigma(t)dB(t) \\
&\quad -\widetilde{\mu}\sigma(t)\widetilde{M}(t)b(t)dt - \widetilde{\mu}\sigma^2(t)\widetilde{M}(t)dB(t) \\
&= X(t)r(t)dt - \widetilde{\lambda}r(t)L(t)dt + \widetilde{\mu}Y(t)b(t)dt + \widetilde{\mu}Y(t)\sigma(t)dB(t) \\
&\quad -\widetilde{\mu}\sigma(t)d\widetilde{B}(t) \\
&= X(t)r(t)dt - \widetilde{\lambda}r(t)L(t)dt + \widetilde{\mu}Y(t)b(t)dt + \\
&\quad \widetilde{\mu}Y(t)\Big[\frac{d\widetilde{B}(t) - \widetilde{M}(t)b(t)dt}{\widetilde{M}(t)}\Big] - \widetilde{\mu}\sigma(t)d\widetilde{B}(t) \\
&= X(t)r(t)dt - \widetilde{\lambda}r(t)L(t)dt + \widetilde{\mu}Y(t)b(t)dt - \\
&\quad \widetilde{\mu}Y(t)b(t)dt + \widetilde{\mu}Y(t)\widetilde{M}^{-1}(t)d\widetilde{B}(t) - \widetilde{\mu}\sigma(t)d\widetilde{B}(t) \\
&= [X(t) - \widetilde{\lambda}L(t)]r(t)dt + \Big[\frac{Y(t)}{Y(t)-M(t)} - 1\Big]\widetilde{\mu}\sigma(t)d\widetilde{B}(t) \\
&= [X(t) - \widetilde{\lambda}L(t)]r(t)dt + \frac{M(t)}{Y(t)-M(t)}\widetilde{\mu}\sigma(t)d\widetilde{B}(t) \\
&= \widehat{X}(t)r(t)dt + M^*(t)\widetilde{\mu}\sigma(t)d\widetilde{B}(t),
\end{aligned}$$

which gives us a market, under an equivalent measure Q, driven by the following stochastic differential equation

$$dX(t) = \widehat{X}(t)r(t)dt + M^*(t)\widetilde{\mu}\sigma(t)d\widetilde{B}(t), \tag{5.10}$$

$$dY(t) = r(t)L(t)dt + \sigma(t)d\widetilde{B}(t). \quad (5.11)$$

Proposition 5.1. *A market model driven by the following SDE*

$$\begin{aligned} dX(t) &= \widehat{X}(t)r(t)dt + M^*(t)\widetilde{\mu}\sigma(t)d\widetilde{B}(t), & (5.12)\\ dY(t) &= r(t)L(t)dt + \sigma(t)d\widetilde{B}(t), & (5.13)\end{aligned}$$

is a complete market model.

Proof
Using Girsanov's Theorem we have

$$\begin{pmatrix} M^*(t)\widetilde{\mu}\sigma(t) \\ \sigma(t) \end{pmatrix} u = \begin{pmatrix} \widehat{X}(t)r(t) \\ r(t)L(t) \end{pmatrix}$$

$$\begin{pmatrix} 0 & \frac{1}{\sigma(t)} \end{pmatrix}\begin{pmatrix} M^*(t)\widetilde{\mu}\sigma(t) \\ \sigma(t) \end{pmatrix} u = \begin{pmatrix} 0 & \frac{1}{\sigma(t)} \end{pmatrix}\begin{pmatrix} \widehat{X}(t)r(t) \\ r(t)L(t) \end{pmatrix}$$

$\therefore\ u = \frac{r(t)L(t)}{\sigma(t)}$, thus $\Lambda = \begin{pmatrix} 0 & \frac{1}{\sigma(t)} \end{pmatrix}$ is a left inverse of $\underline{\sigma} = \begin{pmatrix} M^*(t)\widetilde{\mu}\sigma(t) \\ \sigma(t) \end{pmatrix}$ hence by theorem [2.5] the market is complete. Hence, a market paying continuous transactional costs is in fact a complete market model, if we consider the funds transferred from stocks to the bank account to be stochastic and driven by the same Brownian motion as our stocks.
Note that even though our market is complete, we are still not sure of how large our transactional costs will be when we continuously transfer funds from stock to bank account, thus holding such a portfolio will be very expensive.
Now suppose $dM(t) = M(t)b(t)dt + M(t)\sigma_2(t)dB_2(t)$ and as before $dL(t) = L(t)r(t)dt$, where $B_2(t)$ is a Brownian motion which has zero correlation with $B(t)$, that is $dB(t)dB_2(t) = 0$. Then

$$\begin{aligned} dY(t) &= Ybdt + Y\sigma dB(t) + Lrdt - Mbdt - M\sigma_2 dB_2(t) \\ &= [Yb + Lr - Mb]dt + Y\sigma dB(t) - M\sigma_2 dB_2(t),\end{aligned}$$

and

$$\begin{aligned} dX(t) &= Xrdt + \widetilde{\mu}Mbdt + \widetilde{\mu}M\sigma_2 dB_2(t) - \widetilde{\lambda}Lrdt \\ &= [Xr + \widetilde{\mu}Mb - \widetilde{\lambda}Lr]dt + \widetilde{\mu}M\sigma_2 dB_2(t).\end{aligned}$$

It is easy to see that the market given by

$$\begin{aligned} dX(t) &= [X(t)r(t) + \widetilde{\mu}M(t)b(t) - \widetilde{\lambda}L(t)r(t)]dt + \widetilde{\mu}M(t)\sigma_2(t)dB_2(t), \\ dY(t) &= [Y(t)b(t) + L(t)r(t) - M(t)b(t)]dt + Y(t)\sigma(t)dB(t) - M(t)\sigma_2(t)dB_2(t),\end{aligned}$$

is a complete market model, and from Corollary 2.1 we have that,

$$\begin{pmatrix} Y\sigma & -M\sigma_2 \\ 0 & \widetilde{\mu}M\sigma_2 \end{pmatrix}\begin{pmatrix} \frac{1}{Y\sigma} & \frac{1}{\widetilde{\mu}Y\sigma} \\ 0 & \frac{1}{\widetilde{\mu}M\sigma_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence $\overline{\sigma}(t) = \begin{pmatrix} Y\sigma & -M\sigma_2 \\ 0 & \widetilde{\mu}M\sigma_2 \end{pmatrix}$ is invertible, therefore the market is complete.

Now consider the case when

$$\begin{aligned} dM(t) &= M(t)b(t)dt + M(t)\sigma_1(t)dB_1(t), \\ dL(t) &= L(t)dt + L(t)\sigma_2(t)dB_2(t). \end{aligned}$$

Where $(B(t), B_1(t), B_2(t))$ are Brownian motions with zero correlation between them. We therefore have

$$\begin{aligned} dY(t) &= Y(t)b(t)dt + Y(t)\sigma(t)dB(t) + L(t)r(t)dt + L(t)\sigma_2(t)dB_2(t) \\ &\quad -M(t)b(t)dt - M(t)\sigma_1(t)dB_1(t) \\ &= [Y(t)b(t) + L(t)r(t) - M(t)b(t)]dt + Y(t)\sigma(t)dB(t) \\ &\quad -M(t)\sigma_1(t)dB_1(t) + L(t)\sigma_2(t)dB_2(t), \end{aligned}$$

and

$$\begin{aligned} dX(t) &= [X(t)r(t) + \widetilde{\mu}M(t)b(t) - \widetilde{\lambda}L(t)r(t)]dt \\ &\quad +M(t)\sigma_1(t)dB_1(t) - \widetilde{\lambda}L(t)\sigma_2(t)dB_2(t). \end{aligned}$$

We thus get a market consisting of

$$\begin{aligned} dX(t) &= X^*(L, M, t)dt + M(t)\sigma_1(t)dB_1(t) - \widetilde{\lambda}L(t)\sigma_2(t)dB_2(t), \\ dY(t) &= Y^*(L, M, t)dt + Y(t)\sigma(t)dB(t) - M(t)\sigma_1(t)dB_1(t) + L(t)\sigma_2(t)dB_2(t), \end{aligned}$$

which is an incomplete market model, since we have three sources of randomness and only two tradable assets. We know that there exist an equivalent martingale measure Q for the above market. Also note that transactional costs can never create arbitrage, in other words, if a price cannot be arbitrage in a world free of transactional costs, it cannot be arbitrage in world with them either.
Joshi [31], Chapter 4 uses the following argument for the above proof.

Suppose a price is arbitrageurs in the world with transactional costs. Then we can set up a portfolio taking into account transactional costs at zero or negative costs today, which will be of non-negative and possible positive value in future. If we neglect to take into account transactional costs then the initial set up cost of the portfolio will be even lower and thus still negative or zero. The final value of the portfolio will be at least as high as there will be no cash drain from any transactional costs during the portfolio's life. We therefore conclude that the portfolio is also an arbitrage portfolio in a world free of transactional costs. Thus the existence of arbitrage in the world with transactional costs implies arbitrage in the world free of them. Under the equivalent martingale measure Q we have

$$\begin{aligned} d\widetilde{X}(t) &= M(t)\sigma_1(t)d\widetilde{B}_1(t) - \widetilde{\lambda}L(t)\sigma_2(t)d\widetilde{B}_2(t), \\ d\widetilde{Y}(t) &= Y(t)\sigma(t)d\widetilde{B}(t) - M(t)\sigma_1(t)d\widetilde{B}_1(t) + L(t)\sigma_2(t)d\widetilde{B}_2(t). \end{aligned}$$

Following proposition (4.1), Chapter 4, we can construct a self financing portfolio consisting of the bonds, stocks and quadratic variation asset (Z_1, Z_2, Z_3, Z_4) given by $\phi^3 = \phi_t^3 = (\alpha_t^3, \beta_1(t), \beta_2(t), \beta_3(t))$ to complete the incomplete market given above as the above market is incomplete due to more sources of randomness than tradable asset.

Consider another market paying transactional costs given by

$$\begin{aligned} dS(t) &= bS(t)dt + \sigma S(t)dB(t), \\ dY(t) &= dL(t) - dM(t), \\ dX(t) &= rX(t)dt - (1+\lambda)dL(t) + (1-\mu)dM(t). \end{aligned}$$

Where as before X and Y representing funds held in bank and stocks respectively, with transactional costs given by $0 < \lambda, \mu < 1$ and the funds transferred from bank to stock, respectively stock to bank given by $L(t)$ and $M(t)$. If we again let

$$\begin{aligned} dL(t) &= L(t)rdt, \\ dM(t) &= bM(t)dt + \sigma M(t)dB(t), \end{aligned}$$

so that

$$\begin{aligned} dY(t) &= L(t)rdt - bM(t)dt - \sigma M(t)dB(t), \\ dX(t) &= rX(t)dt - (1+\lambda)L(t)rdt + (1-\mu)[bM(t)dt + \sigma M(t)dB(t)]. \end{aligned}$$

Thus

$$\begin{aligned} dY(t) &= L(t)rdt - \sigma M(t)\left[\frac{b}{\sigma}dt + dB(t)\right] \\ &= L(t)rdt - \sigma M(t)d\widetilde{B}(t), \end{aligned}$$

where $d\widetilde{B}(t) = \frac{b}{\sigma}dt + dB(t)$.
The change in our bank account is then given by

$$\begin{aligned} dX(t) &= rX(t)dt - \widetilde{\lambda}L(t)rdt + \widetilde{\mu}bM(t)dt + \widetilde{\mu}\sigma M(t)dB(t) \\ &= rX(t)dt - \widetilde{\lambda}L(t)rdt + \widetilde{\mu}bM(t)dt + \widetilde{\mu}\sigma M(t)\left[d\widetilde{B}(t) - \frac{b}{\sigma}dt\right] \\ &= rX(t)dt - \widetilde{\lambda}L(t)rdt + \widetilde{\mu}\sigma M(t)d\widetilde{B}(t) \\ &= X^*(t)rdt + \widetilde{\mu}\sigma M(t)d\widetilde{B}(t), \end{aligned}$$

where $X^*(t) = X(t) - \widetilde{\lambda}L(t)$. We therefore have under the risk neutral measure Q, a market model given by

$$\begin{aligned} dX(t) &= X^*(t)rdt + \widetilde{\mu}\sigma M(t)d\widetilde{B}(t), \\ dY(t) &= L(t)rdt - \sigma M(t)d\widetilde{B}(t). \end{aligned}$$

We can thus easily see that the above market is a complete market model. If we normalize the above market using $X(t)$ as a numeraire, that is, if we let $Z(t) = \frac{Y(t)}{X(t)}$ for the normalized process we get

$$\begin{aligned} \frac{dZ}{Z} &= \frac{1}{Y}[Lrdt - \sigma M d\widetilde{B}] - \frac{\widetilde{\mu}\sigma^2 M^2}{XY}dt + \frac{\widetilde{\mu}^2\sigma^2 M^2}{X^2}dt - \frac{1}{X}[X^*rdt + \widetilde{\mu}\sigma M d\widetilde{B}] \\ &= \frac{1}{XY}\left[[XLr - YX^*r + \left(\frac{\widetilde{\mu}Y}{X} - 1\right)\widetilde{\mu}\sigma^2 M^2]dt - (X + \widetilde{\mu}Y)\sigma M d\widetilde{B}\right] \\ &= \frac{1}{XY}\left[[XLr - YX^*r]dt - (X + \widetilde{\mu}Y)\sigma M d\widehat{B}(t)\right], \end{aligned}$$

where

$$\begin{aligned} d\widehat{B}(t) &= \frac{\frac{\widetilde{\mu}Y}{X} - 1}{X + \widetilde{\mu}Y}\widetilde{\mu}\sigma M dt + d\widetilde{B} \\ &= \phi(X, Y, M, \sigma, t)dt + d\widetilde{B} \end{aligned}$$

is a Brownian motion for some measure $Q^* \sim Q$.
One can thus compare the above price process with the time dependent parameter model of Chapter 3 to show that the price of a European call option for a market paying transactional costs is given by

$$f = ZN(\widetilde{d}_1) - Ke^{-\int_0^T (\frac{1}{XY}(XLr - YX^*r))dt}N(\widetilde{d}_2)$$

Where

$$\widetilde{d}_1 = \frac{ln(\frac{Z}{K}) + \int_0^T (\frac{1}{XY}(XLr - YX^*r) + \frac{1}{2}(\frac{1}{XY}(X + \widetilde{\mu}Y)\sigma M)^2)dt}{\sqrt{\int_0^T ((\frac{1}{XY}(X + \widetilde{\mu}Y)\sigma M)^2)dt}}$$

And

$$\widetilde{d}_2 = \widetilde{d}_1 - \sqrt{\int_0^T ((\frac{1}{XY}(X + \widetilde{\mu}Y)\sigma M)^2)dt}.$$

Chapter 6

Stochastic Volatility

There is evidence that volatility is not constant but stochastic, that is, volatility changes randomly according to some stochastic differential equation. There is thus an immense volume of research furthering the field of stochastic volatility. It is known that models with stochastic volatility are incomplete market models due to the extra source of risk introduced by the volatility coefficient. We shall look at a few models with stochastic volatility, and attempt to complete one of the models. We suppose that the stock price process $S(t)$ and its volatility process $\sigma(t)$ evolves according to the following SDE:

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)dB_1(t), \qquad (6.1)\\ d\sigma(t) &= \alpha(t,S,\sigma)dt + \beta(t,S,\sigma)dB_2(t). \end{aligned}$$

Where $B_1(t)$ and $B_2(t)$ are correlated Brownian motions. It is a popular opinion to choose the function α and β to have mean-reverting volatility process, where volatility strives to reach a certain level in the long run.

6.1 Cox-Ingersoll-Ross (CIR) Model

The Cox-Ingersoll-Ross (CIR) model is a mean-reverting model with a stochastic differential equation given by

$$dS(t) = \mu S(t)dt + \sigma(t)S(t)dB_1(t), \qquad (6.2)$$

$$dv(t) = (a + bv(t))dt + c\sqrt{v(t)}dB_2(t). \qquad (6.3)$$

Where a, b and c are constant. The volatility is given by $v(t) = \sigma^2(t)$. The process $B_1(t)$ and $B_2(t)$ are correlated according to

$$dB_2(t) = \rho dB_1(t) + \sqrt{1-\rho^2}dB_3(t). \qquad (6.4)$$

Where $B_1(t)$ and $B_3(t)$ are correlated. The process $v(t)$ is mean-reverting if $a > 0$ and $b < 0$. In the CIR model $v(t)$ is a non-central chi-square distribution with mean of

$$E[v(t)\ v(0) = y] = -\frac{a}{b} + (y + \frac{a}{b})e^{-bt}, \qquad (6.5)$$

and variance of

$$Var[\upsilon(t)\ \upsilon(0) = y] = \frac{ac^2}{2b^2} - \frac{c^2}{b}(y + \frac{a}{b})e^{-bt} + \frac{c^2}{b}(y + \frac{a}{2b})e^{-2bt}. \quad (6.6)$$

Using the above equation we can find the limiting distribution of $\upsilon(t)$ which is a gamma distribution with mean of $\frac{-a}{b}$ and the variance of $\frac{ac^2}{2b^2}$.

6.2 The Heston Model

The model proposed by Heston [Heston [42]] extends the Black and Scholes model for stock prices. The model is derived from the the CIR model of Cox, Ingersoll and Ross, which also cites the Feller[17] paper. Heston's model take into account non-lognormal distribution of assets returns, leverage effect and the mean-reverting property of stochastic volatility. It is not possible to build a replicating portfolio if we formulate the statement that the volatility of the asset varies stochastically due to the fact that volatility is not a tradable security. This implies that the model is an incomplete market model which is driven by the following stochastic differential equation:

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sqrt{\sigma(t)}S(t)dB_1(t), \quad (6.7)\\ d\sigma(t) &= \delta(\theta - \sigma(t))dt + \kappa\sqrt{\sigma(t)}dB_2(t). \end{aligned}$$

Where δ is the speed of σ's reversion to long-run mean θ and the correlation between the two Brownian motions dB_1 and dB_2 is ρ. This model is one of the few stochastic volatility models with a tractable closed form solution as well as a non-zero correlation between stock prices and volatility.

6.3 The Hull and White Model

The stochastic volatility model presented in (Hull and White [22] model) is a two-factor model in which the variance follows a lognormal stochastic process. In their model, Hull and White considered a derivative asset f with a price that depends on some security price, $S(t)$, and instantaneous variance, $\upsilon(t) = \sigma^2(t)$, which obey the following stochastic processes:

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma(t)S(t)dB_1(t), \quad (6.8)\\ d\upsilon(t) &= \mu\upsilon(t)dt + \kappa\upsilon(t)dB_2(t). \end{aligned}$$

Where μ and κ may depend on σ and t, but they do not depend on S. The two Brownian motions have correlation ρ. Hull and White [22] analyzed the model for the case when $\rho = 0$ and $\rho \neq 0$.
When $\rho = 0$, the price of a call option can be shown to be given by

$$C(t, S, y) = E[C_{BS}(t, S, K, T, \sqrt{\overline{\sigma}^2})|\upsilon(t) = y] \quad (6.9)$$

where $\overline{\sigma}^2 = \frac{1}{T-t}\int_t^T f(\upsilon(x))^2 dx$ and $\upsilon(t)$ is a Markov process with two states. C_{BS} denotes the standard Black-Scholes formula.

6.4 The Stein and Stein Model

Stein and Stein [38] studied stock price processes with stochastically varying volatility parameter. They assumed that the volatility is governed by an arithmetic Ornstein-Uhlenbeck process, where the volatility tends to a long-run mean, and where the Brownian motions describing the randomness of the stock price and volatility are independent. Assuming volatility is uncorrelated with the asset price, an exact closed-form solution for the stock price distribution was derived. They also used analytic techniques to develop an approximation to the distribution. Then they used their results to develop closed form option pricing formulas, and to sketch some links between stochastic volatility and the nature of fat tails in stock price distributions.
The Stein and Stein model is described by the following processes:

$$\begin{aligned} dS(t) &= \mu S(t)dt + \sigma(t)S(t)dB_1(t), \\ d\sigma(t) &= \delta(\sigma(t) - \theta)dt + \kappa dB_2(t). \end{aligned} \tag{6.10}$$

Where $S(t)$ is the stock price, $\sigma(t)$ is the volatility of the stock, κ, μ, δ and θ are fixed constants. The two Brownian motions dB_1 and dB_2 are independent.

6.5 Market Completeness

For the stochastic volatility market model given by:

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)dB_1(t) \\ d\sigma(t) &= \alpha(t,S,\sigma)dt + \beta(t,S,\sigma)dB_2(t) \end{aligned} \tag{6.11}$$

with $dB_1(t)dB_2(t) = \rho dt$. We can write $B_2(t) = \rho B_1(t) + \rho^{'} B_1^{'}(t)$ where $B_1^{'}(t)$ is a Brownian motion independent of $B_1(t)$ and $\rho^{'} = \sqrt{1-\rho^2}$. The risk neutral measures Q thus has a density of the form

$$\begin{aligned} \frac{dQ}{dP} &= \exp\left(\int_0^T \Phi(s)dB_1(s) - \frac{1}{2}\int_0^T \Phi^2(s)ds \right. \\ &\quad \left. + \int_0^T \Psi(s)dB_1^{'}(s) - \frac{1}{2}\int_0^T \Psi^2(s)ds \right) \end{aligned}$$

for some integrands Φ and Ψ. Taking $\Phi = \frac{r-\mu}{\sigma}$ and $\Psi = \Psi(S,\sigma)$ we find that under the risk neutral measure Q our stochastic volatility model becomes

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma(t)d\widetilde{B}_1(t) \\ d\sigma(t) &= \widetilde{\alpha}(t,S,\sigma)dt + \beta(t,S,\sigma)d\widetilde{B}_2(t) \end{aligned} \tag{6.12}$$

where $\widetilde{B}_1$ and $\widetilde{B}_2$ are Q-Brownian motions with $d\widetilde{B}_1 d\widetilde{B}_2 = \rho dt$ and $\widetilde{\alpha}(t,S,\sigma) = \alpha + \beta\rho\Phi + \beta\rho^{'}\Psi$. Then $S(t)$ has the riskless growth rate r, but σ is not a traded asset, so arbitrage conditions do not determine the drift of σ, leaving Ψ as an

arbitrary choice.
Now letting $Z(t) = \frac{S(t)}{\sigma(t)}$ we get

$$\begin{aligned}\frac{dZ(t)}{Z(t)} &= \frac{dS(t)}{S(t)} - \frac{d\sigma(t)}{\sigma(t)} - (\frac{dS(t)}{S(t)})(\frac{d\sigma(t)}{\sigma(t)}) + (\frac{d\sigma(t)}{\sigma(t)})^2 \\ &= rdt + \sigma(t)d\widetilde{B}_1(t) - \sigma^{-1}(t)\widetilde{\alpha}dt - \\ &\quad \sigma^{-1}(t)\beta d\widetilde{B}_2(t) - \rho\beta dt + \beta^2 dt \\ &= [r(t) - \sigma^{-1}(t)\widetilde{\alpha} - \rho\beta + \beta^2]dt + \\ &\quad \sigma(t)d\widetilde{B}_1(t) - \sigma^{-1}(t)\beta d\widetilde{B}_2(t).\end{aligned}$$

The market given by $Z(t)$ above is an incomplete market model (incomplete due more randomness than tradable assets). We have shown in Chapter 2, example 2.4, that for such a market, any claim given by $F\omega = g(\widetilde{B}_2(t))$ cannot be hedged by a self-financing portfolio.

6.6 Completion of a Market with Stochastic Volatility

For the market given by

$$\begin{aligned} dS(t) &= r(t)S(t)dt + \sigma(t)dB_1(t), && (6.13)\\ d\sigma(t) &= \widetilde{\alpha}(t,S,\sigma)dt + \beta(t,S,\sigma)dB_2(t), && (6.14)\end{aligned}$$

setting $Z(t) = \frac{S(t)}{\sigma(t)}$ gives:

$$\begin{aligned}\frac{dZ(t)}{Z(t)} &= \frac{dS(t)}{S(t)} - \frac{d\sigma(t)}{\sigma(t)} - (\frac{dS(t)}{S(t)})(\frac{d\sigma(t)}{\sigma(t)}) + (\frac{d\sigma(t)}{\sigma(t)})^2 \\ &= \frac{dS(t)}{S(t)} - \frac{d\sigma(t)}{\sigma(t)} - \beta(t,S,\sigma)\rho dt + \beta^2(t,S,\sigma)dt,\end{aligned}$$

but by Back [4], $Z(t)$ is a normalized price process with the volatility process as its numeraire, hence must be a martingale. Thus the drift of the process $Z(t)$ must be zero. Therefore the drift of $\frac{d\sigma(t)}{\sigma(t)}$ must be $(r(t) + \beta^2(t,S,\sigma))$ and the drift of $\frac{dS(t)}{S(t)}$ must be $(r(t) + \rho\beta^(t,S,\sigma) - \sigma^{-1}(t)\widetilde{\alpha}(t,S,\sigma))$. This implies that

$$\begin{aligned}\frac{dS(t)}{S(t)} &= [r(t) + \rho\beta^(t,S,\sigma) - \sigma^{-1}(t)\widetilde{\alpha}(t,S,\sigma)]dt + \sigma(t)d\widehat{B}_1(t) \\ \frac{d\sigma(t)}{\sigma(t)} &= [r(t) + \beta^2(t,S,\sigma)]dt - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t),\end{aligned}$$

where $\widehat{B}_1$ and $\widehat{B}_2$ are correlated Brownian motions. This implies that our martingale process $Z(t)$ is given by:

$$\frac{dZ(t)}{Z(t)} = \sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t). \tag{6.15}$$

6.6.1 Enlarging the Stochastic Volatility Model with Quadratic Variation Assets

Without loss of generality consider the normalized market given by

$$dY(t) = Y(t)[\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)]. \tag{6.16}$$

Using the same method employed in Chapter 4, section 4.2, we construct quadratic variation assets for the normalized stochastic volatility given by (6.16) as follows:

Let $Z_1(t) = Y_1^2(t) - \langle Y_1 \rangle_t$.
Then by the Doob-Meyer decomposition $Z_1(t)$ is a martingale. The differential form of $Z_1(t)$ is then given by

$$dZ_1(t) = 2Y_1(t)dY_1(t).$$

Now if we let $W_1(t) = e^{\int_0^t r(s)ds} Z_1(t)$, the discounted process of $W_1(t)$ is again a martingale, where $r(t)$ is the deterministic riskless rate of interest for our bank account since

$$\begin{aligned} & E[e^{-\int_0^t r(s)ds} W_1(t)|\mathcal{F}_s] \\ = \; & E[Z_1(t)|\mathcal{F}_s] \\ = \; & Z_1(s) \qquad 0 \le s \le t. \end{aligned}$$

Thus

$$\begin{aligned} dW_1(t) &= r(t)e^{-\int_0^t r(s)ds} Z_1(t)dt + e^{-\int_0^t r(s)ds} dZ_1(t) \\ &= r(t)e^{-\int_0^t r(s)ds} Z_1(t)dt + 2e^{-\int_0^t r(s)ds} Y_1(t)dY_1(t) \\ &= r(t)e^{-\int_0^t r(s)ds} Z_1(t)dt + \\ & \quad\; 2e^{\int_0^t r(s)ds} Y_1^2(t)[\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)]. \end{aligned}$$

Let $Y_2(t) = e^{-qt}W_1(t)$ which we have shown to be a Q martingale, then
$Z_2(t) = Y_2^2(t) - \langle Y_2 \rangle_t$ is again a Q martingale with its differential form given by

$$dZ_2(t) = 2Y_2(t)dY_2(t)$$

and $W_2(t) = e^{-\int_0^t r(s)ds} Z_2(t)$, which in its differential form is given by

$$\begin{aligned} dW_2(t) &= r(t)e^{\int_0^t r(s)ds} Z_2(t)dt + e^{\int_0^t r(s)ds} dZ_2(t) \\ &= r(t)e^{\int_0^t r(s)ds} Z_2(t)dt + 4e^{\int_0^t r(s)ds} Y_2(t)Y_1(t)dY_1(t) \\ &= r(t)e^{\int_0^t r(s)ds} Z_2 dt + 4e^{\int_0^t r(s)ds} Y_2 Y_1^2[\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)]. \end{aligned}$$

The process $Y_3(t) = e^{-\int_0^t r(s)ds} W_2(t)$ is again a Q martingale, so that

$$Z_3(t) = Y_3^2(t) - \langle Y_3 \rangle_t,$$

according to the Doob-Meyer decomposition is a Q martingale with a differential form given by

$$dZ_3(t) = 2Y_3(t)dY_3(t)$$

and $W_3(t) = e^{\int_0^t r(s)ds} Z_3(t)$ with

$$\begin{aligned} dW_3(t) &= r(t)e^{\int_0^t r(s)ds} Z_3(t)dt + e^{\int_0^t r(s)ds} dZ_3(t) \\ &= r(t)e^{\int_0^t r(s)ds} Z_3(t)dt + 2e^{\int_0^t r(s)ds} Y_3(t)dY_3(t) \\ &= re^{\int_0^t r(s)ds} Z_3 dt + 8e^{\int_0^t r(s)ds} Y_3 Y_2 Y_1^2 [\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)] \end{aligned}$$

We thus have additional tradable assets, called the quadratic variation assets, to hedge away the risk associated with the second randomness given by the non-tradable volatility coefficient.

6.6.2 Hedging with Quadratic Variation Assets in the Presence of Stochastic Volatility

The quadratic variation assets, $W(t) = W_1(t), W_2(t), W_3(t)$ driven by the following stochastic differential equation:

$$\begin{aligned} dW_1 &= re^{-\int_0^t r(s)ds} Z_1 dt + 2e^{\int_0^t r(s)ds} Y_1^2 [\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)], \\ dW_2 &= r(t)e^{\int_0^t r(s)ds} Z_2 dt + 4e^{\int_0^t r(s)ds} Y_2 Y_1^2 [\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)], \\ dW_3 &= re^{\int_0^t r(s)ds} Z_3 dt + 8e^{\int_0^t r(s)ds} Y_3 Y_2 Y_1^2 [\sigma(t)d\widehat{B}_1(t) - \sigma^{-1}(t)\beta(t,S,\sigma)d\widehat{B}_2(t)], \end{aligned}$$

will be used to enlarge our market model, (6.16), to a new market model consisting of quadratic variation assets, stocks and bonds. We can easily verify by using Proposition 4.1 or Proposition 4.2 with $N = 3$ of Chapter 4 that these quadratic variation assets complete a markets model with stochastic volatility given by equation (6.13) and equation (6.14). The portfolio that hedges the market model is given by

$$\phi^3 = \phi_t^3 = (\alpha_t^3, \beta_1(t), \beta_2(t), \beta_3(t))$$

where

$$\begin{aligned} \alpha_t^3 &= M_t^3 - \beta_1(t)S(t)e^{-\int_0^t r(s)ds} - e^{\int_0^t r(s)ds} \sum_{i=2}^{3} \beta_i(t)W_i(t) \\ \beta_1(t) &= e^{\int_0^t r(s)ds} \omega(t)S^{-1}(t) \\ \beta_i(t) &= \omega_i(t) \qquad i = 2, 3 \end{aligned}$$

with α_t^3 corresponding to the number of bonds (or funds in bank account) at time t, $\beta_1(t)$ is the number of stocks at time t and $\beta_i(t)$ is the number of quadratic variation assets $W_i(t)$ that one needs to hold at time t in order to replicate any contingent claim $F(S)$.

6.7 Pricing In A Market With Stochastic Volatility

Consider again the stochastic volatility model given by

$$\begin{aligned} dS(t) &= \mu(t)S(t)dt + \sigma(t)dB_1(t), \qquad (6.17)\\ d\sigma(t) &= \alpha(t,S,\sigma)dt + \beta(t,S,\sigma)dB_2(t). \end{aligned}$$

Now consider a portfolio consisting of an option with value $f(S,\sigma,t)$ and another option with value $f_1(S,\sigma,t)$, both dependent on the same stochastic volatility and stock price, but with different strike price and different expiration date given by

$$\begin{aligned} \Pi &= f - \Delta S - \Delta_1 f_1,\\ d\Pi &= df - \Delta dS - \Delta_1 df_1. \qquad (6.18) \end{aligned}$$

From Itô Lemma we have

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial \sigma}d\sigma + \frac{\partial f}{\partial S}dS + \frac{\partial^2 f}{\partial\sigma\partial S}dSd\sigma + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \frac{1}{2}\frac{\partial^2 f}{\partial \sigma^2}(d\sigma)^2\\ &= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\beta^2\frac{\partial^2 f}{\partial\sigma^2} + \rho\sigma\beta S\frac{\partial^2 f}{\partial\sigma\partial S}\right]dt\\ &\quad + \frac{\partial f}{\partial\sigma}d\sigma + \frac{\partial f}{\partial S}dS. \end{aligned}$$

We can produce a similar process for f_1. Now substituting f and f_1 into our portfolio (6.18) we get

$$\begin{aligned} d\Pi &= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\beta^2\frac{\partial^2 f}{\partial\sigma^2} + \rho\sigma\beta S\frac{\partial^2 f}{\partial\sigma\partial S}\right]dt\\ &\quad -\Delta_1\left[\frac{\partial f_1}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f_1}{\partial S^2} + \frac{1}{2}\beta^2\frac{\partial^2 f_1}{\partial\sigma^2} + \rho\sigma\beta S\frac{\partial^2 f_1}{\partial\sigma\partial S}\right]dt\\ &\quad +\left(\frac{\partial f}{\partial S} - \Delta_1\frac{\partial f_1}{\partial S} - \Delta\right)dS + \left(\frac{\partial f}{\partial\sigma} - \Delta_1\frac{\partial f_1}{\partial\sigma}\right)d\sigma. \end{aligned}$$

We then choose Δ and Δ_1 in such a way that the randomness is removed from our portfolio in order to have a riskless hedge. All the risk lies in the dS and $d\sigma$ terms, so we need to make the coefficient of these terms zero by choosing Δ and Δ_1 to satisfy

$$\begin{aligned} &\left(\frac{\partial f}{\partial S} - \Delta_1\frac{\partial f_1}{\partial S} - \Delta\right) = 0,\\ &\left(\frac{\partial f}{\partial\sigma} - \Delta_1\frac{\partial f_1}{\partial\sigma}\right) = 0, \end{aligned}$$

which gives the hedge ratios

$$\begin{aligned} \Delta &= \frac{\partial f}{\partial S} - \Delta_1\frac{\partial f_1}{\partial S},\\ \Delta_1 &= \frac{\partial f}{\partial\sigma}\Big/\frac{\partial f_1}{\partial\sigma}. \end{aligned}$$

With these hedges in place, the change in the value of our hedged portfolio is then given by

$$\begin{aligned}
d\Pi &= \left[\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\beta^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho\sigma\beta S \frac{\partial^2 f}{\partial \sigma \partial S}\right] dt \\
&\quad -\Delta_1 \left[\frac{\partial f_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_1}{\partial S^2} + \frac{1}{2}\beta^2 \frac{\partial^2 f_1}{\partial \sigma^2} + \rho\sigma\beta S \frac{\partial^2 f_1}{\partial \sigma \partial S}\right] dt \\
&= r\Pi dt \\
&= r(f - \Delta S - \Delta_1 f_1)dt.
\end{aligned}$$

Where we have used the no-arbitrage condition as before (Chapter 3 and 4). Now rearranging and substituting for Δ and Δ_1 we get

$$\begin{aligned}
&\frac{\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\beta^2 \frac{\partial^2 f}{\partial \sigma^2} + \rho\sigma\beta S \frac{\partial^2 f}{\partial \sigma \partial S} + rs\frac{\partial f}{\partial S} - rf}{\frac{\partial f}{\partial \sigma}} \\
= &\frac{\frac{\partial f_1}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f_1}{\partial S^2} + \frac{1}{2}\beta^2 \frac{\partial^2 f_1}{\partial \sigma^2} + \rho\sigma\beta S \frac{\partial^2 f_1}{\partial \sigma \partial S} + rs\frac{\partial f_1}{\partial S} - rf_1}{\frac{\partial f_1}{\partial \sigma}}.
\end{aligned}$$

As before, the left-hand side is therefore a function of f which is independent of f_1, similarly the right-hand side is a function of f_1 independent of f. We can thus as before (Chapter 4) put both sides equal to some arbitrary function $-\lambda(t, S, \sigma)$ which is called the market price of volatility risk. We therefore have

$$\begin{aligned}
&\frac{\partial f}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{1}{2}\beta^2 \frac{\partial^2 f}{\partial \sigma^2} \\
&+\rho\sigma\beta S \frac{\partial^2 f}{\partial \sigma \partial S} + rs\frac{\partial f}{\partial S} + \lambda \frac{\partial f}{\partial \sigma} - rf = 0.
\end{aligned}$$

Which gives the partial differential equation for the value of as option with stochastic volatility.

Chapter 7

Lévy Processes

Lévy market models, except of course for the geometric Brownian model which we have in the previous chapters (Chapter 3 to be precise) shown to be a complete market model, are incomplete market models. Every claim in the market with its price process following a geometric Brownian motion can be hedged and the unique price is given by the Black-Scholes formula. The geometric Poissonian Lévy market model is also a complete market model with a unique equivalent martingale measure (EMM). General geometric Lévy market models, defined below, on the other hand are incomplete and have many equivalent martingale measures. In this chapter we will follow the work by Corécuera and Nualart [12] which shows that general Lévy processes market model can be completed by their so called power-jump processes. We then find an equivalent martingale measure Q with minimal relative entropy for the market completed with power jump assets.

Definition 7.1. A stochastic process $X = X(t), t \geq 0$ is said to be a Lévy process if

- Each $X(0) = 0 \quad (a.s.)$
- X has independent and stationary increments
- X is stochastically continuous, i.e. for all $a \neq 0$ and for all $s \geq 0$

$$\lim_{t\to s} P(\mid X(t) - X(s) \mid > a) = 0.$$

One can thus use the above definition to determine whether a particular process is a Lévy process or not. It is very easy to check that processes like Brownian motion, Gaussian, Poisson, Compound Poisson and Interlacing processes are indeed Lévy processes as they all satisfy the condition of definition 7.1

The model that we will consider for the behavior of stock prices in the market will be the geometric Lévy market model. Under the real world measure P, the dynamics of stock prices $S = \{S(t), t \geq 0\}$ are modeled by a stochastic differential equation (SDE) driven by a general Lévy process $Z = \{Z(t), t \geq 0\}$ satisfying the following conditions:

$$\begin{aligned} dR(t) &= rR(t)dt \qquad R(0) = 1, \\ dS(t) &= S(t_-)[bdt + dZ(t)]. \end{aligned} \tag{7.1}$$

7.1 The General Lévy Process

General Lévy processes take into account jumps, which is the cause of incompleteness. They have many equivalent martingale measures and thus contingent claims cannot be hedged by a self-financing portfolio. Before we can look at the completion of such models, we will need to understand the representation of jump processes and their integrals. Such a representation is made possible by the Lévy-Kintchine formula. Given a stochastic process $Z = \{Z(t), t \geq 0\}$ is a Lévy process with characteristic function $\phi(z)$.
The function $\psi(z) = \log \phi(z) = \log E[e^{(izZ_1)}]$ is called the characteristic exponent and it satisfies the following Lévy-khintchine formula

$$\psi(z) = i\alpha z - \frac{c^2}{2}z^2 + \int_{-\infty}^{+\infty} e^{(izx)} - 1 - izx1_{\{|x|<1\}}\nu(dx), \tag{7.2}$$

where $\alpha \in \mathbb{R}$,$c \geq 0$ and ν is a σ-finite measure on $\mathbb{R} - \{0\}$ satisfying

$$\nu\{0\} = 0 \text{ and } \int_{-\infty} +\infty(1 \wedge x^2)\nu(dx).$$

The infinitely divisible distribution is said to have a triplet of Lévy characteristics $[\alpha, c^2, \nu(dx)]$. The measure $\nu(dx)$ is called the Lévy measure of Z and indicates how the jumps occur. Jumps of sizes in the set A occur according to a Poisson process with parameter $\int_A \nu(dx)$.
From the Lévy-Khintchine formula, we can deduce that Z must be a linear combination of a standard Brownian motion $W = \{W(t), t \geq 0\}$ and a pure jump process $X = \{X(t), t \geq 0\}$ such that

$$Z(t) = \sigma W(t) + X(t),$$

and W is independent of X. Moreover,

$$X(t) = \mu t + \int_{\{|x|<1\}} x\widetilde{N}(t, dx) + \int_{\{|x|\geq 1\}} xN(t, dx), \tag{7.3}$$

where $N(t, A)$ is a Poisson process, with intensity parameter $\nu(A)$, that count the number of jumps up to time t, and the Borel subset A represents the range of possible jump sizes. The compensated Poisson measure, which is a martingale-valued measure, is given by the process $\{\widetilde{N}(t, A), t \geq 0\}$ defined by,

$$\widetilde{N}(t, A) = N(t, A) - t\nu(A).$$

A stochastic process $Z = \{Z(t), t \geq 0\}$ is said to be a semimartingale if it is an adapted process that admits representation as

$$Z(t) = Z(0) + M(t) + C(t), \tag{7.4}$$

where $M = \{M(t), t \geq)\}$ is a martingale and $C = \{C(t), t \geq)\}$ is an adapted process.
Since the compensated Poisson process $\widetilde{N}(t, A)$ and the Brownian motion $W(t)$ are both martingale, if we define

$M^*(t) = \sigma W(t) + \int_{\{|x|<1\}} x\widetilde{N}(t, dx)$ and $C(t) = \mu t + \int_{\{|x|\geq 1\}} xN(t, dx)$

we see that the Lévy process $Z(t)$ is indeed a semimartingale.
For the purpose of our model we require the process $Z(t)$ to satisfy certain conditions. We will suppose that the Lévy measure satisfies for some $\varepsilon > 0$, and $\lambda > 0$

$$\int_{(-\varepsilon,\varepsilon)^c} \exp(\lambda \mid x \mid)\nu(dx) < \infty \tag{7.5}$$

This implies that

$$\int_{-\infty}^{\infty} \mid x \mid^i \nu(dx) < \infty \qquad i \geq 1,$$

and that

$$E[\exp(-\theta Z_1)] < \infty \text{ for all } \theta \in (-\theta_1, \theta_2),$$

where $0 < \theta_1, \theta_2 \leq \infty$. Such a requirement will then imply that $Z(t)$ has finite moment of all order.
With this requirement imposed on the Lévy process $\{Z(t), t \geq 0\}$ we can thus write the Lévy-Itó decomposition as

$$Z(t) = at + \sigma W(t) + \int_{\mathbb{R}} x\widetilde{N}(t, dx), \tag{7.6}$$

where $a = E(X_1)$, $\{W(t), t \geq 0\}$ is the standard Brownian motion and $\{M(t) = \int_{\mathbb{R}} x\widetilde{N}(t, dx)\}$ is the process responsible for all the jumps which is independent of the Brownian motion. From (7.3) we see that

$$\begin{aligned} E[X(t)] &= E\left[\int_{\{|x|<1\}} x\widetilde{N}(t, dx)\right] + E\left[\int_{\{|x|\geq 1\}} xN(t, dx) + \mu t\right] \\ &= E\left[\int_{\{|x|\geq 1\}} xN(t, dx)\right] + \mu t, \end{aligned} \tag{7.7}$$

the compensated Poisson process is a martingale, and thus has zero expectation. Equation (7.7) then gives

$$\mu = E[X_1] - \int_{\{|x|\geq 1\}} xN(1, dx)$$

Since every Lévy process is a semimartingale, we can write $X(t)$ as a sum of a martingale and a predictable process of finite variation as follows:

$$X(t) = M(t) + at \tag{7.8}$$

where $M = \{M(t), t \geq 0\}$ is a martingale and $E[L_1] = a$.
We can thus write the SDE for stock prices (7.1) as follows

$$dS(t) = S(t_-)[(a + b)dt + \sigma dW(t) + dM(t), \tag{7.9}$$

with $Z(t)$ as defined in (7.6).
We will need the following theorem to solve the above SDE for stock prices.

Theorem 7.1. *(Itô Lemma for Lévy processes)*
Let $X = \{X(t), t \geq 0\}$ be a càdlàg semimartingale stochastic process, then for each $f \in \mathbf{C}^{1,2}(\mathbb{R}^+, \mathbb{R})$ we have with probability one that

$$\begin{aligned} f(t, X_t) - f(0, X_0) = &\int_0^t \frac{\partial f}{\partial s}(s, X_{s-})ds + \int_0^t \frac{\partial f}{\partial x}(s, X_{s-})dX_s \\ &+\frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, X_{s-})d\langle X^c\rangle_s \\ &+ \sum_{0\leq s\leq t} [f(s, X_s) - f(s, X_{s-}) - \frac{\partial f}{\partial x}(s, X_{s-})\Delta X_s]. \end{aligned} \tag{7.10}$$

Proof
see Applebaum [3]

Where the Lévy process can be broken up into continuous and discontinuous part such that

$$(X(t) = X^c(t) + X^d(t),$$

where $X^c(t)$ represent the continuous part and $X^d(t)$ represent the discontinuous part of the Lévy process. We can thus split up the quadratic variation of the Lévy process in the similar manner as follows

$$\langle X\rangle_t = \langle X^c\rangle_t + \sum_{0\leq s\leq t} (\Delta X s^d)^2, \tag{7.11}$$

and hence

$$\langle X\rangle_t = t + \int_{\{|x| \ \leq 1\}} x^2 N(t, dx) + \int_{\{|x|\geq 1\}} x^2 N(t, dx). \tag{7.12}$$

Returning to the SDE for stock prices (7.9), we can find a solution to this equation by applying the above Itô lemma for Lévy processes to the function $\ln S(t)$, noting that $\Delta S(t) = S(s_-)\Delta Z(t) = S(s_-)\Delta M(s)$ and $\langle Z^c\rangle_t = \langle W\rangle_t = t$ which then gives us

$$\begin{aligned}
\ln S(t) &= \ln S(0) + \int_0^t \frac{1}{S(u_-)}dS(u) + \frac{1}{2}\int_0^t \left(\frac{-1}{S^2(u_-)}\right)d\langle S^c\rangle_u \\
&\quad + \sum_{0\le u\le t} [\ln S(u) - \ln S(u_-) - \frac{1}{S(u_-)}\Delta S(u)] \\
&= \ln S(0) + \int_0^t \Big((a+b)du + \sigma dW(u) + dM(u)\Big) - \frac{1}{2}\int_0^t \sigma^2 du \\
&\quad + \sum_{0\le u\le t} [\ln \frac{S(u)}{S(u_-)} - \frac{1}{S(u_-)}\Delta S(u)] \\
&= \ln S(0) + \int_0^t (a+b-\frac{1}{2}\sigma^2)du + \int_0^t \Big(\sigma dW(u) + dM(u)\Big) \\
&\quad + \sum_{0\le u\le t} [\ln(1+\Delta M(u)) - \Delta M(u)] \\
&= \ln S(0) + (a+b-\frac{1}{2}\sigma^2)t + \sigma W(t) + M(t) \\
&\quad + \sum_{0\le u\le t} [\ln(1+\Delta M(u)) - \Delta M(u)].
\end{aligned}$$

We finally get

$$\begin{aligned}
S(t) &= S(0)\exp\left(\sigma W(t) + M(t) + (a+b-\frac{1}{2}\sigma^2)t\right) \times \\
&\prod_{0\le s\le t} (1+\Delta M(s))\exp\big(-\Delta M(s)\big). \qquad (7.13)
\end{aligned}$$

Since the stock price process has to be non-negative for all time t in order to guarantee that $1+\Delta M(t) \ge 0$ for every $t \ge 0$ $a.s.$, we should put the constraint that $\Delta\ M(t) \ge -1$. We thus need that the Lévy measure ν to be supported on the subset of $[-1, +\infty]$. The riskless rate of interest is assumed to be a constant r.

7.2 Power-Jump Processes

Corécuera and Nualart [12] considers the following transformation of the Lévy process
$Z = \{Z(t), t \ge 0\}$ which will play an important role in our analysis. Where they set

$$Z^{(i)}(t) = \sum_{0<s\le t} (\Delta Z(s))^i \qquad i \ge 2. \qquad (7.14)$$

The jump process $\Delta Z(t) = \{\Delta Z(t), t \ge 0\}$ is defined by

$$\Delta Z(t) = Z(t) - Z(t_-). \qquad (7.15)$$

Where $Z(t_-)$ is the left limit of Z at time point t (i.e., $Z(t_-) = \lim_{s\uparrow t} Z(s)$).
For convenience we set $Z^{(1)}(t) = Z(t)$. We should also note that
$Z(t) = \sum_{0<s\leq t} \Delta Z(s)$ is not necessarily true, it is only true in the bounded variation case. If we define $X^{(i)}(t)$ in the same way, we have that $X^{(i)}(t) = Z^{(i)}(t), i \geq 2$ and clearly the quadratic variation $\langle X \rangle_t = X^{(2)}(t)$.
The processes $X^{(i)} = \{X^{(i)}(t), t \geq 0\}, i \geq 2$, are again Lévy processes and are called the ith-power-jump processes (or the power jump processes of order i). They jump at the same point as the original Lévy process, but the jumps sizes are the ith power of the jump size of the original Lévy process.
We have $E[X(t)] = E[X^{(1)}] = ta = tm_1 < \infty$ and

$$E[X^{(i)}(t)] = E[\sum_{0<s\leq t} (\Delta X(s))^i] = t\int_{-\infty}^{\infty} x^i \nu(dx) = m_i t < \infty, \quad i \geq 2. \tag{7.16}$$

We denote by

$$Y^{(i)}(t) = Z^{(i)}(t) - [Z^{(i)}(t)] = Z^{(i)}(t) - m_i t, \qquad i \geq 1 \tag{7.17}$$

the compensated processes. We then orthonormalize the sequence of martingales $\{Y^{(i)}, i \geq 1\}$ (see Corécuera and Nualart [12]) and take a suitable linear combination of the $Y^{(i)}$ to obtain a set of pairwise strongly orthonormal martingales $\{T^{(i)}, i \geq 1\}$. Each $T^{(i)}$ is a linear combination of the $Y^{(j)}, j = 1, 2, \ldots, i$ such that

$$T^{(i)} = c_{i,i}Y^{(i)} + c_{i,i-1}Y^{(i-1)} + \ldots + c_{i,1}Y^{(1)} \quad i \geq 1. \tag{7.18}$$

The constants can be calculated as described in Nualart and Schoutens [35], they correspond to the coefficients of the orthonormalization of the polynomials $\{x^n, n \geq 0\}$ with respect to the measure $\mu(dx) = x^2\nu(dx) + c^2\delta_0(dx)$. The resulting processes $T^{(i)} = \{T^{(i)}(t), t \geq 0\}$ are called the orthormalized ith-power-jump processes.

7.3 Equivalent Martingale Measure for Lévy Processes

We are interested in finding an equivalent probability measure Q, for which the discounted price process of (7.1) is a martingale under the probability measure Q. If at least one of such measures exist, then we know that our discounted price process does not allow arbitrage. If on the other hand the equivalent martingale measure Q is unique, then our market is complete and every claim in the market is attainable. The unique price of a contingent claim for the complete market will be given by the expectation of the discounted payoff at maturity under the equivalent martingale measure Q. In order to find such a Q we need to consider the following stochastic differential equation

$$\begin{aligned} dL(t) &= L(t_-)\Big[h(t)dW(t) + \int_{\{|x|<1\}} (H(t,x)-1)\widetilde{N}(dt,dx) \\ &\quad + \int_{\{|x|\geq 1\}} (F(t,x)-1)\widetilde{N}(dt,dx)\Big] \end{aligned} \tag{7.19}$$

Where the integrands are predictable processes, with $H : \mathbb{R} \to (0, \infty)$ a Borel function satisfying

$\int_{-\infty}^{+\infty} \left(1 - \sqrt{H(x)}\right)^2 \nu(dx) < \infty$

If we let

$$\begin{aligned} Y(t) &= \int_0^t h(t)dW(t) + \int_0^t \int_{\{|x|<1\}} (H(t,x) - 1)\widetilde{N}(dt, dx) \\ &\quad + \int_0^t \int_{\{|x|\geq 1\}} (F(t,x) - 1)\widetilde{N}(dt, dx) \end{aligned} \tag{7.20}$$

then the process $Y = \{Y(t), t \geq 0\}$ is a Lévy process. We can thus rewrite (7.19) as

$$dL(t) = L(t_-)dY(t), \tag{7.21}$$

which has a solution given by the stochastic exponential process (also known as the Doléans-Dade exponential) defined (see R.J. Elliot [17]) by

$$L(t) = \exp\left(Y(t) - \frac{1}{2}\langle Y^c \rangle_t \prod_{0 \leq s \leq t} [1 + \Delta Y(s)]e^{-\Delta Y(s)}\right), \tag{7.22}$$

where $L(t)$ is strictly positive for all time t. The following Girsanov's Theorem for Lévy processes, will be used to change the standard Brownian motion under the real world measure P to a standard Browian motion under the equivalent martingale measure Q.

Theorem 7.2. *Girsanov's Theorem*
Let $\{W(t), t \geq 0\}$ be a standard Brownian motion under the measure P. Then if the stochastic exponential defined by

$$L(t) = \exp\left(Y(t) - \frac{1}{2}\langle Y^c \rangle_t \prod_{0 \leq s \leq t} [1 + \Delta Y(s)]e^{-\Delta Y(s)}\right), \tag{7.23}$$

is a martingale under the measure P for which $E_P[L(t)] = 1$ where the process $Y = \{Y(t), t \geq 0\}$ is a Lévy process containing the Brownian motion, then a new process $\widetilde{W} = \{\widetilde{W}(t), t \geq 0\}$ defined by

$$\widetilde{W}(t) = W(t) - \int_0^t h(s)ds, \tag{7.24}$$

has a standard Brownian motion under the measure Q defined by the following Radon-Nikodym derivative linking P to Q

$$\frac{dQ}{dP} = L(T). \tag{7.25}$$

Proof
see Elliott [17]

We can also represent the jump processes of the Lévy process as a martingale under the measure Q by considering the process $M = \{M(t), t \geq 0\}$ under the measure P which is defined by

$$M(t) = \int_0^t \int_{|x|<1} F(s,x)\widetilde{N}(ds,dx), \tag{7.26}$$

where

$$E_P\left[\int_0^T \int_{|x|<1} F^2(s,x)\nu(dx)ds\right] < \infty. \tag{7.27}$$

If we define the change of measure P to an equivalent martingale measure Q by

$$\frac{dQ}{dP} = L(T),$$

and define a new process $\widetilde{M} = \{\widetilde{M}(t), t \geq 0\}$ by

$$\begin{aligned}\widetilde{M}(t) &= M(t) - \int_0^t \int_{|x|<1} F(s,x)(H(s,x)-1)\nu(dx)ds \\ &= \int_0^t F(s,x)[N(ds,dx) - H(s,x)\nu(dx)ds] \\ &= \int_0^t F(s,x)\widetilde{N}_Q(ds,dx),\end{aligned} \tag{7.28}$$

where $\widetilde{N}_Q(ds,dx) = N(ds,dx) - H(s,x)\nu(dx)ds$. It can thus be shown that the process $\widetilde{M} = \{\widetilde{M}(t), t \geq 0\}$ is a Q martingale, see Jacod and Shriyaev [30]. In general martingales of the form

$$J(t) = \int_0^t \int_{|x|<1} K(s,x)\widetilde{N}(ds,dx), \tag{7.29}$$

under the measure P have a representation as a martingale under the measure Q given by

$$\widetilde{J}(t) = J(t) - \int_0^t \int_{|x|<1} K(s,x)(F(s,x)-1)\nu(dx)ds. \tag{7.30}$$

Now returning to the stock price process (7.1), if we let $\widetilde{S}(t) = \frac{S(t)}{R(t)}$ be our discounted price process with the bank account $R(t)$ as our pricing numeraire, we have that

$$
\begin{aligned}
\frac{d\widetilde{S}(t)}{\widetilde{S}(t)} &= \frac{dS(t)}{S(t)} - \frac{dR(t)}{R(t)} - \frac{dS(t)}{S(t)}\frac{dR(t)}{R(t)} + \left(\frac{dR(t)}{R(t)}\right) \\
&= \frac{S(t_-)}{S(t)}\Big[bdt + dZ(t)\Big] - rdt \\
\Rightarrow \qquad & d\widetilde{S}(t) = \frac{S(t_-)}{R(t)}\Big[bdt + dZ(t)\Big] - \frac{S(t)}{R(t)}rdt,
\end{aligned}
$$

so that

$$
\begin{aligned}
\widetilde{S}(t) &= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)(b-r)du + \int_0^t \widetilde{S}(u_-)dZ(u) \\
&= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)(b-r)du + \int_0^t \widetilde{S}(u_-)d[au + \sigma W(u) + M(u)] \\
&= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)(a+b-r)du \\
&\quad + \int_0^t \widetilde{S}(u_-)\sigma dW(u) + \int_0^t \widetilde{S}(u_-)dM(u).
\end{aligned}
$$

Now substituting (7.24) of Girsanov's Theorem gives

$$
\begin{aligned}
\widetilde{S}(t) &= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)(a+b-r)du \\
&\quad + \int_0^t \widetilde{S}(u_-)\sigma\Big[d\widetilde{W}(u) + h(u)du\Big] + \int_0^t \widetilde{S}(u_-)dM(u) \\
&= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)\Big[a+b+h(u)-r\Big]du \\
&\quad + \int_0^t \widetilde{S}(u_-)\sigma d\widetilde{W} + \int_0^t \widetilde{S}(u_-)dM(u).
\end{aligned}
$$

Using (7.30) to change the measure for the jump processes $M(t)$ we get

$$
\begin{aligned}
\widetilde{S}(t) &= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)\Big[a+b+h(u)-r\Big]du \\
&\quad + \int_0^t \widetilde{S}(u_-)\sigma d\widetilde{W} + \int_0^t \widetilde{S}(u_-)(d\widetilde{M}(u) + \int_{|x|<1} x(H(s,x)-1)\nu(dx)du \\
&= \widetilde{S}(0) + \int_0^t \widetilde{S}(u_-)\Big[a+b+h(u)-r + \int_{|x|<1} x(H(s,x)-1)\nu(dx)\Big]du \\
&\quad + \int_0^t \widetilde{S}(u_-)\sigma d\widetilde{W} + \int_0^t \widetilde{S}(u_-)d\widetilde{M}(u). \qquad (7.31)
\end{aligned}
$$

But the discounted stock price process is a martingale under the equivalent measure Q. The du term of (7.31) must be zero, hence

$$
a + b + h(u) - r + \int_{|x|<1} x(H(s,x)-1)\nu(dx) = 0, \qquad (7.32)
$$

so that the discounted stock price price process under the measure Q equivalent to P is given by

$$\widetilde{S}(t) = \widetilde{S}(0) + \int_0^t \widetilde{S}(u)\sigma d\widetilde{W}(u) + \int_0^t \widetilde{S}(u)d\widetilde{M}(u). \tag{7.33}$$

Which in it differential form is given by

$$d\widetilde{S}(t) = \widetilde{S}(t)\sigma d\widetilde{W}(t) + \widetilde{S}(t)d\widetilde{M}(t), \tag{7.34}$$

where

$$E_Q\left[\int_0^T \widetilde{S}^2(u)\sigma^2 du\right] < \infty, \tag{7.35}$$

and

$$E_Q\left[\int_0^T \int_{\mathbb{R}} \widetilde{S}^2(u_-)x^2 H^2(u,x)\nu(dx)du\right] < \infty, \tag{7.36}$$

is a Q martingale. Since (7.32) does not uniquely determine the function $h(t)$ and $H(t,x)$, the resulting equivalent martingale measure is defined by the Radon-Nikodym derivative linking P to Q given by

$$\frac{dQ}{dP} = L(T),$$

is not unique martingale measure. We thus have infinitely many equivalent martingale measure's, one for each choice of the function $h(t)$ and $H(t,x)$. Hence our market given by (7.1) is an incomplete market model with infinitely many prices, one for each choice of Q. In order to complete such a market model Corécuera and Nualart [12] proposed an enlargement of the market model, with the so called power-jump assets defined in section 7.2 and prove that such an enlargement complete the Lévy market model given by (7.1)

7.4 Enlarging the Lévy Market Model

Corécuera and Nualart [12] enlarge the market with the orthonormalized ith-power-jump assets introduced in section 7.1. More precisely they allow trade in assets with price process $\overline{H}^{(i)} = \{\overline{H}^{(i)}(t), t \geq 0\}$, where

$$\overline{H}^{(i)}(t) = \exp(rt)T^{(i)}(t), i = 2, 3, \ldots$$

Corécuera and Nualart [12] also considers an enlargement based on assets with price process $H^{(i)} = \{H^{(i)}(t), t \geq 0\}$, where

$$H^{(i)}(t) = \exp(rt)Y^{(i)}(t) \qquad i = 2, 3, \ldots$$

Also note that the discounted version of the assets $H^{(i)}$ (and $\overline{H}^{(i)}$) are the (orthonormalized) power-jump assets, and hence martingale

$$E_Q[\exp(-rt)\overline{H}^{(i)}(t)|\mathcal{F}_s] = E_Q[T^{(i)}(t)|\mathcal{F}_s] = T^{(i)}(s), \quad 0 \leq s \leq t,$$

and

$$E_Q[\exp(-rt)H^{(i)}(t)|\mathcal{F}_s] = E_Q[Y^{(i)}(t)|\mathcal{F}_s] = Y^{(i)}(s), \quad 0 \leq s \leq t.$$

Hence the market allowing trade in the bank account, stocks and (orthonormalized) power-jump assets remains arbitrage free.

"Corécuera and Nualart [12] motivate trading in the power jump assets by the fact that one can trade in volatility, which is a trading strategy designed to speculate on changes in the volatility of the market rather than the direction of the market". The power-jump assets are also trading strategies based on the volatility of stock prices. If we consider the 2nd power-jump asset for instance which is just the quadratic variation assets defined in Chapter 4, section 4.2, for Lévy type processes. This quadratic variation process measures the volatility of the stock, since it accounts for the square of the jumps. If one believes that in the future there will be a more volatile environment than the current market's anticipate, trading in the quadratic variation asset can be of interest. Also if one would like to cover against periods of high (or low) volatility, they can be useful: Buying 2nd quadratic variation assets can thus cover the possible losses due to such unfavorable periods. The same strategy holds for higher order variation assets. Where the 3rd-power-jump assets is measuring asymmetry (or skewness) while the 4th power-jump asset measures extremal movements (or kurtosis). Trade in these assets can be useful if one likes to bet on the realized skewness or realized kurtosis of the stock. If one believes that the market is not counting in asymmetry and possible extremal moves rightly. An insurance against a possible crash can be easily built from the 4th-power-jump (or ith-power jump, $i \geq 4$) assets.

7.5 Completion of the Market with Jumps

The method of completion makes use of the martingale representative property (MRP) derived in Nualart and Schoutens [35], which says that any square-integrable Q martingale $M(t)$ can be represented as follows:

$$M(t) = M(0) + \int_0^t h(s)d\widetilde{Z}(s) + \sum_{i=2}^{\infty}\int_0^t h^{(i)}(s)dT^{(i)}(s), \tag{7.37}$$

where $h(s)$ and $h^{(i)}(s)$, $i \geq 2$ are predictable processes such that

$$E\left[\int_0^t \mid h(s) \mid^2 ds\right] < \infty,$$

and

$$E\left[\int_0^t \sum_{i=2}^{\infty} \mid h^{(i)}(s) \mid^2 ds\right] < \infty.$$

Theorem 7.3. *The Lévy market model enlarged with the ith-power-jump assets is complete, in the sense that any square-integrable contingent claim X can be replicated.*

Proof

Consider a square-integrable contingent claim X with maturity T. Let

$$M(t) = E_Q[\exp(-rT)X|\mathcal{F}_t].$$

We apply to this martingale the MRP given by (7.37). We then look for a self-financing strategy

$$\phi = \{\phi(t) = (\alpha(t), \beta(t), \beta^{(2)}(t), \beta^{(3)}(t), \ldots), t \geq 0\}, \tag{7.38}$$

to replicate the contingent claim X. We claim that the self-financing strategy replicating the claim is given by

$$\begin{aligned}
&\alpha(t) = M(t_-) - \beta(t)S(t_-)R^{-1}(t) - \sum_{i=2}^{\infty} \beta^{(i)}(t)\overline{H}^{(i)}(t_-)R^{-1}(t) \\
&\beta(t) = h(t)R(t)S^{-1}(t) \\
&\beta^{(i)}(t) = h^{(i)}(t) \qquad i = 2, 3, \ldots
\end{aligned}$$

where $\alpha(t)$ corresponds to the number of bonds at time t; $\beta(t)$ is the number of stocks at that time and $\beta^{(i)}(t)$ is the number of assets $\overline{H}(i), i = 2, 3, \ldots,$ that one needs to hold at time t. We claim that $\phi(t)$ is a sequence of self-financing portfolios which replicates X. In fact, the value $V(t)$ of ϕ at time t is given by

$$V(t) = \alpha(t)R(t) + \beta(t)S(t) + \sum_{i=2}^{\infty} \beta^{(i)}(t)\overline{H}^{(i)}(t) = R(t)M(t),$$

which is the price of the claim at time t. So the sequence of portfolio ϕ is replicating the claim. To show that the sequence of portfolio ϕ is self-financing, we let

$$G(u) = \int_0^u \alpha(t)dR(t) + \int_0^u \beta(t)dS(t) + \sum_{i=2}^{\infty} \int_0^u \beta^{(i)}(t)d\overline{H}^{(i)}(t),$$

denote the gain process, i.e., the gains or losses obtained up to time u by following ϕ. We will show that

$$G(u) + M(0) = M(u)R(u)$$

which implies that the portfolio ϕ is self-financing.
We have

$$\begin{aligned} G(u) &= \int_0^u M_{t-}dR(t) - \int_0^u h(t)dR(t) - \sum_{i=2}^{\infty}\int_0^u h_t^{(i)}\overline{H}_t^{(i)}R^{-1}(t)dR(t) \\ &\quad + \int_0^u h(t)R(t)S^{-1}(t_-)dS(t) + \sum_{i=2}^{\infty}\int_0^u h^{(i)}(t)d\overline{H}^{(i)}(t_-). \end{aligned}$$

Now

$$\begin{aligned} &\int_0^u M(t)dR(t) \\ &= \int_0^u \left(M(0) + \int_0^{t-} h(s)d\widetilde{Z}(s) + \sum_{i=2}^{\infty}\int_0^{t-} h^{(i)}(s)dT^{(i)}(s)\right)dR(t) \\ &= M(0)(R(u)-R(0)) + \int_0^u\int_0^{t-} h(s)d\widetilde{Z}(s)dR(t) \\ &\quad + \sum_{i=2}^{\infty}\int_0^u\int_0^{t-} h^{(i)}(s)dT^{(i)}(s)dR(t) \\ &= M(0)(R(u)-R(0)) + \int_0^u\int_{s_+}^u h(s)dR(t)d\widetilde{Z}(s) \\ &\quad + \sum_{i=2}^{\infty}\int_0^u\int_{s_+}^u h^{(i)}(s)dR(t)dT^{(i)}(s) \\ &= M(0)(R(u)-R(0)) + \int_0^u h(s)(R(u)-R(s))d\widetilde{Z}(s) \\ &\quad + \sum_{i=2}^{\infty}\int_0^u h^{(i)}(s)(R(u)-R(s))dT^{(i)}(s) \\ &= M(0)(R(u)-R(0)) + R(u)\int_0^u h(s)d\widetilde{Z}(s) - \int_0^u h(s)R(s)d\widetilde{Z}(s) \\ &\quad + R(u)\sum_{i=2}^{\infty}\int_0^u h^{(i)}(s)dT^{(i)}(s) - \sum_{i=2}^{\infty}\int_0^u h^{(i)}(s)R(s)dT^{(i)}(s) \\ &= M(0)(R(u)-R(0)) + R(u)\int_0^u dM(t) - \int_0^u h(s)R(s)d\widetilde{Z}(s) \\ &\quad - \sum_{i=2}^{\infty}\int_0^u h^{(i)}(s)R(s)dT^{(i)}(s) \\ &= M(0)(R(u)-R(0)) + R(u)(M(u)-M(0)) - \int_0^u h(s)R(s)d\widetilde{Z}(s) \\ &\quad - \sum_{i=2}^{\infty}\int_0^u h^{(i)}(s)R(s)dT^{(i)}(s). \end{aligned}$$

Where

$$dM(t) = h(t)d\widetilde{Z}(t) + \sum_{i=2}^{\infty} h^{(i)}(t)dT^{(i)}(t).$$

Hence

$$\begin{aligned}
& G(u) \\
= \quad & M(u)R(u) - M(0) - \int_0^u h(t)dR(t) - \sum_{i=2}^{\infty}\int_0^u h^{(i)}(t)\overline{H}^{(i)}(t_-)R^{-1}(t)dR(t) \\
& - \int_0^u h(s)R(s)d\widetilde{Z}(s) + \int_0^u h(t)R(t)S^{-1}(t_-)dS(t) \\
& + \sum_{i=2}^{\infty}\int_0^u h^{(i)}(t)d\overline{H}^{(i)}(t_-) - \sum_{i=2}^{\infty}\int_0^u h^{(i)}(s)R(s)dT^{(i)}(s) \\
= \quad & M(u)R(u) - M(0) - \int_0^u h(t)dR(t) - \sum_{i=2}^{\infty}\int_0^u h^{(i)}(t)\overline{H}^{(i)}(t_-)R^{-1}(t)dR(t) - \\
& \int_0^u h(s)R(s)d\widetilde{Z}(s) + \int_0^u h(t)R(t)S^{-1}(t_-)dS(t) + \sum_{i=2}^{\infty}\int_0^u h^{(i)}(t)T^{(i)}(t_-)dR(t) \\
= \quad & M(u)R(u) - M(0) - \int_0^u h(t)dR(t) - \\
& \int_0^u h(t)R(t)S^{-1}(t_-)dS(t) + \int_0^u h(t)dR(t) + \int_0^u h(t)R(t)S^{-1}(t_-)dS(t) \\
& M(u)R(u) - M(0).
\end{aligned}$$

Where $\overline{H}^{(i)}(t) = R(t)T^{(i)}(t)$ implies that $d\overline{H}^{(i)}(t) = R(t)dT^{(i)}(t) + T^{(i)}(t)dR(t)$ and the derivative of the discounted stock price (7.33) gives $dS(t) = S(t)dZ(t)$

Which completes the proof.

7.6 Hedging Portfolio for Lévy Processes

The value of the contingent claim $X = F(t, S(t))$ at time t is given by

$$F(t, S(t)) = E_Q[\exp(-rt)X|\mathcal{F}_t).$$

We call $F(t, x)$ the price function of the contingent claim X. It is clear that the discounted contingent claim $e^{-rt}F(t, S(t))$ is a Q martingale. From the Itô lemma for Lévy processes (7.10) we can write our contingent claim $F(t, S(t))$ as

$$F(t, S(t)) - F(0, S(0)) = \int_0^t \frac{\partial F}{\partial u}(u, S(u_-))du + \int_0^t \frac{\partial F}{\partial S}(u, S(u_-))dS(u)$$

$$
\begin{aligned}
&+\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial S^2}(u,S(u_-))d\langle S^c\rangle_u \\
&+\sum_{0\le u\le t}[F(u,S(u))-F(u,S(u_-))-\frac{\partial F}{\partial S}(u,S(u_-))\Delta S(u)]. \qquad (7.39)
\end{aligned}
$$

We can define the random finite sum of the jumps as

$$
\sum_{0\le u\le t} F(u,\Delta X(u))\mathbf{1}_{(\Delta X(u)\in A)} = \int_0^t\int_A F(s,x)N(ds,dx), \qquad (7.40)
$$

for some predictable process $F(s,x)$ with

$$
E_P\left[\int_0^t\int_A \mid F(s,x)\mid^2 \nu(dx)ds\right] < \infty. \qquad (7.41)
$$

Keeping in mind that $\Delta S(t) = S(t_-)\Delta M(t)$ where $\Delta S(t) = S(t) - S(t_-)$, we can write (7.39) as

$$
\begin{aligned}
&F(t,S(t)) - F(0,S(0)) = \int_0^t \frac{\partial F}{\partial u}(u,S(u_-))du + \int_0^t \frac{\partial F}{\partial S}(u,S(u_-))dS(u) \\
&+\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial S^2}(u,S(u_-))d\langle S^c\rangle_u \\
&+\int_0^t\int_{\mathbb{R}}[F(u,S(u_-)(1+y))-F(u,S(u_-))-S(u_-)y\frac{\partial F}{\partial S}(u,S(u_-))]N(du,dx).
\end{aligned}
$$

Differentiating both sides gives

$$
\begin{aligned}
&dF(t,S(t)) = D_1F(t,S(t))dt + D_2F(t,S(t_-))dS(t) + \frac{1}{2}D2^2F(t,S(t_-))d\langle S^c\rangle_t \\
&+\int_{\mathbb{R}}[F(t,S(t_-)(1+y))-F(t,S(t_-)) \\
&-S(t_-)yD_2F(t,S(t_-))]N(du,dx). \qquad (7.42)
\end{aligned}
$$

Where D_1 is the differential operation with respect to the time variable and D_2 is the differential operator with respect to stock price $S(t)$. Applying Itô lemma to the discounted contingent claim $e^{-rt}F(t,S(t))$ yields

$$
d\Big(e^{-rt}F(t,S(t))\Big) = -re^{-rt}F(t,S(t))dt + e^{-rt}dF(t,S(t)). \qquad (7.43)
$$

Substituting (7.42) into (7.43) gives

$$
\begin{aligned}
&d\Big(e^{-rt}F(t,S(t))\Big) \\
= \;& -re^{-rt}F(t,S_t)dt + e^{-rt}\Big[D_1F(t,S_t)dt + D_2F(t,S_{t-})dS(t) + \frac{1}{2}D2^2F(t,S_{t-})d\langle S^c\rangle_t \\
&+\int_{\mathbb{R}}[F(t,S(t_-)(1+y))-F(t,S(t_-))-S(t_-)yD_2F(t,S(t_-))]N(du,dx)\Big]. \qquad (7.44)
\end{aligned}
$$

From (7.34) we can write the stocks price process under the risk neutral measure Q as

$$dS(t) = S(t)[rdt + \sigma d\widetilde{W}(t) + d\widetilde{M}(t)], \tag{7.45}$$

which we can substitute into (7.44) to get

$$\begin{aligned} & d\Big(e^{-rt}F(t,S(t))\Big) \\ = \;& -re^{-rt}F(t,S_t)dt + e^{-rt}\Big[D_1F(t,S_t)dt + \sigma S_{t-}D_2F(t,S_{t-})d\widetilde{W}(t) + \\ & S_{t-}D_2F(t,S_{t-})d\widetilde{M}(t) + rS_{t-}D_2F(t,S_{t-})dt + \frac{1}{2}\sigma^2 S_{t-}^2 D_2^2 F(t,S_{t-})dt \\ & + \int_{\mathbb{R}} [F(t,S(t_-)(1+y)) - F(t,S(t_-)) - S(t_-)yD_2F(t,S(t_-))]\widetilde{\nu}(dy)\Big]. \end{aligned}$$

Where $\widetilde{\nu}(dy)$ is the compensator of the Poisson process $N(dt,dy)$ under the risk neutral measure Q. Since the discounted contingent claim is a martingale under Q, the dt terms must be zero. So we must have

$$D_1F(t,x) + rxD_2F(t,x) + \frac{1}{2}\sigma^2x^2D_2^2F(t,x) + \mathcal{D}F(t,x) = rF(t,x), \tag{7.46}$$

where

$$\mathcal{D}F(t,x) = \int_{\mathbb{R}}[F(t,x(1+y)) - F(t,x) - xyD_2F(t,x)]\widetilde{\nu}(dy).$$

Equation (7.46) is the partial differential equation for the value of an option when the stocks price process is driven by Lévy processes with jumps. We will need the following lemma to derive the hedging portfolio for our enlarged market.

Lemma 7.1. *Consider a real function $h(s,x,y) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ which is infinitely differentiable in the y variable. Set*

$$a_i(s,x) = \frac{1}{i!}\frac{\partial^i h}{\partial y^i}(s,x,0),$$

and assume that

$$\sup_{x<K, s\le s_0} \sum_{i=2}^{\infty} \mid a_i(s,x) \mid R^i < \infty \tag{7.47}$$

for all $K, R > 0, s_0 > 0$.

Then we have

$$\sum_{t<s\le T} h(s,S_{s-},\Delta X_s) = \sum_{i=2}^{\infty}\int_t^T \frac{1}{i!}\frac{\partial^i}{\partial y^i}h(s,S_{s-},0)dY_s^{(i)} + \int_t^T\int_{-\infty}^{\infty} h(s,S_{s-},y)\widetilde{\nu}(dy)ds.$$

Proof
see Corécuera and Nualart [12].

We now calculate the sequence of self-financing portfolios that replicates the contingent claim X

Theorem 7.4. *The sequence of self-financing portfolios replicating a contingent claim X with a payoff only depending on the stock price value at maturity and a price function $F(t,x) \in \mathbf{C}^{1,\infty}$ which satisfies*

$$\sup_{x<K, t\le t_0} \sum_{n=2}^{\infty} |D_2^n F(t,x)| R^n < \infty \tag{7.48}$$

for all $K, R > 0, t_0 > 0$,

is given at time t by

number of bonds

$$\alpha(t) = R^{-1}(t)(F(t,S_{t-}) - S_{t-}D_2F(t,S_{t-}))$$
$$-R^{-1}(t)\sum_{i=2}^{\infty}\frac{S_{t-}^i D_2^i F(t,S_{t-})}{i!R(t)} H_{t-}^{(i)},$$

number of stocks

$$\beta(t) = D_2F(t,S_{t-}),$$

number of ith-power-jump assets

$$\beta^{(i)}(t) = \frac{S_{t-}^i D_2^i F(t,S_{t-})}{i!R(t)} \quad i = 2,3,\dots$$

Proof
An application of the Itô lemma to $F(t,S(t))$ gave us (7.39) which is

$$F(t,S(t)) - F(0,S(0)) = \int_0^t \frac{\partial F}{\partial u}(u,S(u_-))du + \int_0^t \frac{\partial F}{\partial S}(u,S(u_-))dS(u)$$
$$+\frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial S^2}(u,S(u_-))d\langle S^c\rangle_u$$
$$+\sum_{0\le u\le t}[F(u,S(u)) - F(u,S(u_-)) - \frac{\partial F}{\partial S}(u,S(u_-))\Delta S(u)],$$

with $\Delta S(t) = S(t_-)\Delta M(t)$ so that

$$F(t,S(t)) - F(0,S(0)) = \int_0^t D_1F(u,S(u_-))du + \int_0^t D_2(u,S(u_-))dS(u)$$
$$+\frac{1}{2}\int_0^t \sigma^2 S^2(u_-)D_2^2F(u,S(u_-))du$$
$$+\sum_{0\le u\le t}[F(u,S(u_-)(1+\Delta M(u))) - F(u,S(u_-)) - \Delta M(u)S(u_-)D_2F(u,S(u_-))].$$

If we let $h(t,x,y) = F(u,x(1+y)) - F(u,x) - xyD_2F(u,x)$. It is clear that

$$
\begin{aligned}
&h(t,x,0) = 0 \\
&\frac{\partial}{\partial y}h(t,x,0) = 0 \\
&\frac{\partial^n}{\partial y^n}h(t,x,0) = x^n D_2^n F(t,x) \qquad n = 2,3,\dots
\end{aligned}
$$

Since $F(t,x) \in \mathbf{C}^{1,\infty}$ satisfies conditions of (7.48), then h satisfies conditions of Lemma 7.1. Application of Lemma 7.1 gives

$$
\begin{aligned}
&F(t,S(t)) - F(0,S(0)) \\
= \;& \int_0^t D_1F(u,S(u_-))du + \frac{1}{2}\int_0^t \sigma^2 S^2(u_-)D_2^2F(u,S(u_-))du \\
&+ \int_0^t D_2(u,S(u_-))dS(u) + \sum_{i=2}^{\infty}\int_0^t \frac{S_{u-}^i D_2^i F(u,S_{u-})}{i!}dY^{(i)}(u) \\
&+ \int_0^t \int_{-\infty}^{\infty} F(u,x(1+y)) - F(u,x) - xyD_2F(u,x)\widetilde{\nu}(dy)du \\
= \;& \int_0^t \frac{D_1F(u,S_{u-}) + \frac{1}{2}\sigma^2 S_{u-}^2 D_2^2F(u,S_{u-}) + \mathcal{D}F(u,x)}{rR(u)}dR(u) \\
&+ \int_0^t D_2(u,S(u_-))dS(u) + \sum_{i=2}^{\infty}\int_0^t \frac{S_{u-}^i D_2^i F(u,S_{u-})}{i!R(u)}dH^{(i)}(u) \\
&- \sum_{i=2}^{\infty}\int_0^t \frac{S_{u-}^i D_2^i F(u,S_{u-})}{i!R(u)}Y^{(i)}(u_-)dR(u).
\end{aligned}
$$

Now using the partial differential equation for the price (7.46), we get

$$
\begin{aligned}
&F(t,S(t)) - F(0,S(0)) \\
= \;& \int_0^t \frac{F(u,S_{u-}) - S_{u-}D_2F(u,S_{u-}) - \sum_{i=2}^{\infty}\int_0^t \frac{S_{u-}^i D_2^i F(u,S_{u-})}{i!R(u)}H^{(i)}(u_-)}{R(u)}dR(u) \\
&+ \int_0^t D_2(u,S(u_-))dS(u) + \sum_{i=2}^{\infty}\int_0^t \frac{S_{u-}^i D_2^i F(u,S_{u-})}{i!R(u)}dH^{(i)}(u).
\end{aligned}
$$

Which gives us the required hedging portfolio for the number of bonds, stocks and ith-power-jump assets.

For the Black-Scholes model covered in Chapter 3, with the stock price process $S = \{S(t), t \geq 0\}$ given by

$$S(t) = S(0)\exp((r - \frac{1}{2}\sigma^2)t + \sigma W(t)),$$

where $W = \{W(t), t \geq 0\}$ is a standard Brownian motion, all the power-jump processes $H^{(i)}, i = 2,3,\dots$ are equal to zero. The market is already complete and an enlargement is not necessary. The hedging portfolio is given by

$$\frac{F(u,S(u)) - S(u)D_2F(u,S(u))}{R(u)}$$

number of bonds and

$$D_2F(u,S(u))$$

number of stocks.

7.7 Pricing Formula for Lêvy Processes

Consider the value at time t of a contingent claim X with a payoff function $f(S(T)) = F(T,S(T))$ only depending on the stock price at maturity:

$$\begin{aligned} F(t,S(t)) &= \exp(-r(T-t))E_Q[X|\mathcal{F}_t] \\ &= \exp(-r(T-t))E_Q[f(S(T))|\mathcal{F}_t] \end{aligned} \tag{7.49}$$

Where we have $r = a + b$ under the risk-neutral measure. Now using (7.13) we get

$$\begin{aligned} & F(t,S(t)) \\ = \;& \exp(-r(T-t))E_Q\bigg[f\bigg(S_t \exp\Big(\sigma(W_T - W_t) + (M_T - M_t) + (r - \frac{1}{2}\sigma^2)(T-t)\Big) \\ & \prod_{0\le s\le t}\Big(1+\Delta M(s))\exp\big(-\Delta M(s)\big)\Big)\bigg|\mathcal{F}_t\bigg] \\ = \;& \exp(-r(T-t))E_Q\bigg[f\bigg(x \exp\Big(\sigma(W_{T-t}) + (M_{T-t}) + (r - \frac{1}{2}\sigma^2)(T-t)\Big) \\ & \prod_{0\le s\le t}\Big(1+\Delta M(s))\exp\big(-\Delta M(s)\big)\Big)\bigg]. \end{aligned}$$

Introducing the Black-Scholes option price

$$F_{BS}(t,x) = \exp(-r(T-t))E_Q\bigg[f\bigg(x\exp\Big(\sigma(W_{T-t}) + (r-\tfrac{1}{2}\sigma^2)(T-t)\Big)\bigg)\bigg],$$

gives us

$$F(t,x) = E_Q\bigg[F_{BS}\bigg(t, xe^{M_{T-t}}\prod_{0\le s\le t}\Big(1+\Delta M(s))\exp\big(-\Delta M(s)\big)\Big)\bigg]. \tag{7.50}$$

The n^{th} derivative with respect to x, which is needed in the formula for the number of n^{th} power-jump assets in the replicating portfolio is given by

$$\begin{aligned} D_2^n F(t,x) &= E_Q\bigg[e^{nM_{T-t}\prod_{0\le s\le T-t}(1+\Delta M_s)^n e^{-n\Delta M_s}} \times \\ & \qquad D_2^n F_{BS}\bigg(t, xe^{M_{T-t}}\prod_{0\le s\le T-t}\Big((1+\Delta M(s))\exp(-\Delta M(s))\Big)\bigg], \end{aligned}$$

for the European call option, the $D_2^n F_{BS}$ are very simple. They are given in terms of cumulative probability distribution function $N(x)$ and the density function $n(x)$ of a Standard Normal random variable by

$$D_2^1 F_{BS}(t,x) = N(d_1) = N\left(\frac{\log(\frac{x}{T-t}) + (r + \frac{1}{2}\sigma^2)}{\sigma\sqrt{T-t}}\right),$$

and

$$D_2^2 F_{BS} = \frac{n(d_1)}{x\sigma\sqrt{T-t}},$$

which are also known as the delta and gamma of the option.

We can use the Esscher transformation described in Chapter 3, section 3.3, to find an equivalent martingale measure Q with minimal relative entropy since the stock price process $S(t)$ is given by

$$\begin{aligned}
S(t) &= S(0)\exp\left(\sigma W_t + M_t + (r - \frac{1}{2}\sigma^2)t\right)\prod_{0\le s\le t}(1+\Delta M(s))\exp(-\Delta M(s)) \\
&= S(0)\exp\left(\sigma W_t + M_t + (r - \frac{1}{2}\sigma^2)t\right)\prod_{0\le s\le t}\exp\left(\ln(1+\Delta M(s)) - \Delta M(s)\right) \\
&= S(0)\exp\left(\sigma W_t + M_t + (r - \frac{1}{2}\sigma^2)t\right)\sum_{0\le s\le t}\left(\ln(1+\Delta M(s)) - \Delta M(s)\right) \\
&= S(0)\exp\left(\sigma W_t + (r - \frac{1}{2}\sigma^2)t + \sum_{0\le s\le t}\ln(1+\Delta M(s))\right).
\end{aligned}$$

If we let $U = \Delta M(s)$ and assume that the intensity process of the jumps modelled by the variable $(1+U)$ follows a log-normal distribution with mean δ and variance β^2, that is $\ln(1+U) \sim N(\delta, \beta^2)$ so that

$$\begin{aligned}
S(t) &= S(0)\exp\left(\sigma W_t + (r - \frac{1}{2}\sigma^2)t + \sum_{i=1}^{N(t)}\ln(U+1)\right) \qquad (7.51)\\
&= S(0)e^{X^*(t)}.
\end{aligned}$$

Since the stock price process $S(t)$ is of the above form, we can indeed use the Esscher transform of parameter θ to find a minimal martingale measure Q for the above process. Chan [11] proves that $\theta(t) = h(t) = -\lambda(t)\sigma$ produces a minimal change of measure P to an equivalent measure Q.

The Esscher transformation of parameter θ is given by

$$L(t) = \frac{e^{\theta\widetilde{X}(t)}}{E\left[e^{\theta\widetilde{X}(t)}\right]} \qquad \theta \in \mathbb{R},$$

where

$$\widetilde{X}(t) = \sigma W_t + \sum_{i=1}^{N(t)} \ln(U+1).$$

With

$$\begin{aligned}
& E\Big[e^{\theta \widetilde{X}(t)}\Big] \\
= \;& E\Big[e^{\theta\sigma W(t)}\Big] E\Big[e^{\theta \sum_{i=1}^{N(t)} \ln(U+1)}\Big] \\
= \;& e^{\frac{1}{2}\sigma^2\theta^2 t} E\Big[E\Big[\prod_{i=1}^{N(t)} e^{\theta \ln(U+1)} | \; N(t)\Big]\Big] \\
= \;& e^{\frac{1}{2}\sigma^2\theta^2 t} E\Big[\prod_{i=1}^{N(t)} e^{\theta\delta + \frac{1}{2}\beta^2\theta^2}\Big] \\
= \;& \exp\left\{\frac{1}{2}\sigma^2\theta^2 t + \lambda t (e^{\theta\delta + \frac{1}{2}\beta^2\theta^2} - 1)\right\},
\end{aligned}$$

so that

$$L(t) = \exp\left\{\sigma\theta W(t) + \sum_{i=1}^{N(t)} \ln(U+1) - \frac{1}{2}\sigma^2\theta^2 t - \lambda t(e^{\theta\delta + \frac{1}{2}\beta^2\theta^2} - 1)\right\}. \tag{7.52}$$

The Radon-Nikodým derivative linking P to Q is thus given by

$$\frac{dQ}{dP} = L(T), \tag{7.53}$$

where $L(T)$ is given in equation (7.52). Girsanov's theorem states that there exists a process $\widetilde{W} = \{\widetilde{W}(t), t \geq 0\}$, such that

$$\widetilde{W}(t) = W(t) - \theta t, \tag{7.54}$$

is a Brownian motion under Q, so that (7.51) becomes

$$S(t) = S(0)\exp\left(\sigma\widetilde{W}_t - \sigma\theta t + (r - \frac{1}{2}\sigma^2)t + \sum_{i=1}^{N(t)} \ln(U+1)\right).$$

Then by (7.49) we get

$$\begin{aligned}
& F(t, S_t) \\
= \;& \exp(-r\tau) E_Q\left[f\left(S_t \exp\left(\sigma\widetilde{W}_\tau + (r - \sigma\theta - \frac{1}{2}\sigma^2)\tau + \sum_{i=1}^{N(\tau)} \ln(U+1)\right)\right); \theta \middle| \mathcal{F}_t\right],
\end{aligned}$$

where $\tau = T - t$. For $S(t) = x$ we have

$$
\begin{aligned}
& F(t,x) \\
= \;& \exp(-r\tau)E_Q\Big[f\Big(x\exp\Big(\sigma\widetilde{W}_\tau + (r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + \sum_{i=1}^{N(\tau)}\ln(U+1)\Big)\Big);\theta\Big] \\
= \;& \exp(-r\tau)\sum_{n=0}^{\infty}E_Q\Big[f\Big(x\exp\Big(\sigma\widetilde{W}_\tau + (r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + \sum_{i=1}^{N(\tau)}\ln(U+1)\Big)\Big); \\
& \theta\Big|N(\tau)=n\Big] \times Q(N(\tau)=n) \\
= \;& \exp(-r\tau)\sum_{n=0}^{\infty}E_Q\Big[f\Big(x\exp\Big(\sigma\widetilde{W}_\tau + (r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + \sum_{i=1}^{N(\tau)}\ln(U+1)\Big)\Big); \\
& \theta\Big|N(\tau)=n\Big] \times \frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!},
\end{aligned}
$$

where

$$\sigma\widetilde{W}_\tau + (r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + \sum_{i=1}^{N(\tau)}\ln(U+1) \sim N\Big((r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + n\delta, \sigma^2\tau + n\beta^2\Big).$$

But

$$\frac{\sigma\widetilde{W}_\tau + \sum_{i=1}^{N(\tau)}\ln(U+1) - n\delta}{\sqrt{\sigma^2\tau + n\beta^2}} \sim N(0,1),$$

and

$$\frac{\widetilde{W}(\tau)}{\sqrt{\tau}} \sim N(0,1).$$

We have, by uniqueness of probability density function, that

$$\frac{\sigma\widetilde{W}_\tau + \sum_{i=1}^{N(\tau)}\ln(U+1) - n\delta}{\sqrt{\sigma^2\tau + n\beta^2}} = \frac{\widetilde{W}(\tau)}{\sqrt{\tau}},$$

so that

$$\sigma\widetilde{W}_\tau + \sum_{i=1}^{N(\tau)}\ln(U+1) = \sqrt{\frac{\sigma^2\tau + n\beta^2}{\tau}}\widetilde{W}(\tau) + n\delta.$$

Hence

$$(r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + \sqrt{\frac{\sigma^2\tau + n\beta^2}{\tau}}\widetilde{W}(\tau) + n\delta \sim N\Big((r-\sigma\theta-\frac{1}{2}\sigma^2)\tau + n\delta, \sigma^2\tau + n\beta^2\Big).$$

Therefore (7.49) becomes

$$\begin{aligned} F(t,x) &= \exp(-r\tau)\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\times \\ & E_Q\left[f\left(x\exp\left((r-\sigma\theta-\frac{1}{2}\sigma^2)\tau+\sqrt{\frac{\sigma^2\tau+n\beta^2}{\tau}}\widetilde{W}(\tau)+n\delta\right)\right)\right]. \end{aligned}$$

By adding and subtracting $\frac{n\beta^2}{2\tau}$ into the exponential function we get

$$\begin{aligned} F(t,x) &= \exp(-r\tau)\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\times \\ & E_Q\left[f\left(x\exp\left((r-\sigma\theta-\frac{1}{2}\sigma^2+\frac{n\beta^2}{2\tau}-\frac{n\beta^2}{2\tau})\tau+\sqrt{\frac{\sigma^2\tau+n\beta^2}{\tau}}\widetilde{W}(\tau)+n\delta\right)\right)\right] \\ F(t,x) &= \exp(-r\tau)\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\times \\ & E_Q\left[f\left(x\exp\left((r-\sigma\theta-\frac{1}{2}(\sigma^2+\frac{n\beta^2}{\tau})+\frac{n\beta^2}{2\tau})\tau+\sqrt{\frac{\sigma^2\tau+n\beta^2}{\tau}}\widetilde{W}(\tau)+n\delta\right)\right)\right] \\ F(t,x) &= \exp(-r\tau)\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\times \\ & E_Q\left[f\left(x\exp\left((r-\sigma\theta-\frac{1}{2}\sigma_n^2+\frac{n\beta^2}{2\tau})\tau+\sigma_n\widetilde{W}(\tau)+n\delta\right)\right)\right] \\ F(t,x) &= \exp(-r\tau)\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\times \\ & E_Q\left[f\left(x\exp\left(\left(n\delta-\sigma\theta\tau+\frac{n\beta^2}{2}\right)\right)\times\exp\left((r-\frac{1}{2}\sigma_n^2)\tau+\sigma_n\widetilde{W}(\tau)\right)\right)\right], \end{aligned}$$

where $\sigma_n^2=\sigma^2+\frac{n\beta^2}{\tau}$. If we let $S_n=x\exp\left(\left(n\delta-\sigma\theta\tau+\frac{n\beta^2}{\tau}\right)\right)$ we get

$$\begin{aligned} F(t,S(t)) &= \sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\exp(-r\tau)E_Q\left[f\left(S_n\exp\left((r-\frac{1}{2}\sigma_n^2)\tau+\sigma_n\widetilde{W}(\tau)\right)\right)\right] \\ &= \sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}F_{BS}(\tau,S_n;c_n), \end{aligned} \tag{7.55}$$

which represents the weighted average of Black-Scholes price in terms of n number of jumps. We thus have that

$$D_2^1F(t,x)=\sum_{n=0}^{\infty}\frac{(\widetilde{\lambda}\tau)^n e^{-\widetilde{\lambda}\tau}}{n!}\left[\exp\left(\left(n\delta-\sigma\theta\tau+\frac{n\beta^2}{2}\right)\right)\times D_2^1F_{BS}(\tau,S_n;c_n)\right]. \tag{7.56}$$

Chapter 8

Conclusion

We have seen the imperfections of the Black and Scholes [9] option pricing theory, in Chapter 3. One of them being its insufficiency for modelling stock prices that jump discontinuously and the volatility of stock prices being stochastic. With all of its drawbacks its still serves as a benchmark by which other option pricing models are judged.

Features such as jumps and stochastic volatility of market assets prices may cause incompleteness, depending on the available trading opportunities. For example, in the Stein and Stein [38] (Chapter 6) model of stocks with stochastic volatility, the market is incomplete because it is impossible to hedge the risk factor associated with stochastic volatility. However, if our so called quadratic variation assets on stocks (Chapter 4) were also to be marketed, both the risk factors could be hedged by trading in the stocks and the quadratic variation assets, and the market would be complete as shown in Chapter 6.

Jumps tend to cause incompleteness except in very simple or unusual models (see for example Dritschel and Protter [15]). It is not that easy to hedge against potential jumps of various sizes because their values are nonlinear. To complete a market in which jumps of all sizes are possible might require many more marketed securities, for example, vanilla European options of all strikes and maturities or a larger number of power-jump assets as introduced by Corécuera and Nualart [12]

Other phenomena causing incompleteness in the market which we covered in this dissertation were market frictions such as transactional costs. For the case of transactional costs we considered a market model paying continuous transactional costs, like the one presented by Cvitanić and Karatzas [13]. We showed that the market is already completed and there is no need to enlarge the market if transactional costs are constant or deterministic. For the case when transactional costs are stochastic, the market is incomplete, and an enlargement with what we called quadratic variation assets to hedge away the risk associated with the extra source of randomness was required. Even though we were able to complete the market paying transactional costs, trading is not optimal as the hedging price for contingent claims in the completed market will be too expensive for practitioners as it contains an unknown amount of transactional costs which could be infinitely large.

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