On the Integrity of Domination in Graphs

by

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To my parents and grandmother

PREFACE

The research on which this thesis is based was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, from January 1992 to January 1993, under the supervision of Professor Henda C. Swart.

Unless specifically indicated to the contrary in the thesis, this thesis represents the author's own work and has not been submitted in any form to another university.

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My gratitude to my parents is very great. Throughout my studies, my parents have been earnestly and unfailingly supportive, both in word and in deed, and much of what I have accomplished is due to their positive influence. With much love, I dedicate this thesis to them, and to my late grandmother, who was enthusiastically in my corner from the day I was born.

Thanks are also extended to my friends, who have shown interest and provided much-appreciated encouragement from the start to the end of this endeavour.

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ABSTRACT

This thesis deals with an investigation of the integrity of domination in a graph, i.e., the extent to which domination properties of a graph are preserved if the graph is altered by the deletion of vertices or edges or by the insertion of new edges.

A brief historical introduction and motivation are provided in Chapter 1. Chapter 2 deals with k-edge-(domination-)critical graphs, i.e., graphs G such that $\gamma(G) = k$ and $\gamma(G+e) < k$ for all $e \in E(\bar{G})$. We explore fundamental properties of such graphs and their characterization for small values of k. Particular attention is devoted to 3-edge-critical graphs.

In Chapter 3, the changes in domination number brought about by vertex removal are investigated. Parameters $\gamma^{+\prime}(G)$ (and $\gamma^{-}(G)$), denoting the smallest number of vertices of G in a set S such that $\gamma(G-S) > \gamma(G)$ ($\gamma(G-S) < \gamma(G)$, respectively), are investigated, as are k-vertex-critical graphs G (with $\gamma(G) = k$ and $\gamma(G-v) < k$ for all $v \in V(G)$). The existence of smallest domination-forcing sets of vertices of graphs is considered.

The bondage number $\gamma^{+\prime}(G)$, i.e., the smallest number of edges of a graph G in a set F such that $\gamma(G-F) > \gamma(G)$, is investigated in Chapter 4, as are associated extremal graphs. Graphs with dominating sets or domination numbers that are insensitive to the removal of an arbitrary edge are considered, with particular reference to such graphs of minimum size.

Finally, in Chapter 5, we discuss n-dominating sets D of a graph G (such that each vertex in G-D is adjacent to at least n vertices in D) and associated parameters. All chapters but the first and fourth contain a listing of unsolved problems and conjectures.

CONTENTS

Chapter 1

INTRODUCTION AND DEFINITIONS

1.1 DEFINITIONS

Unless otherwise specified, the quantities dealt with in this thesis are positive integers. Where convenient, a rectangle in the figures will denote a complete graph.

In this thesis, all our graphs will be finite, undirected and without loops or multiple edges. A set S of vertices is said to *dominate* a graph G (or to be a *dominating set* of G) if, for each v $\in$ V(G) - S, there is a vertex u $\in$ S with u adjacent to v. The smallest cardinality of any such dominating set is called the *domination number* of G and is denoted by $\gamma(G)$. Let G be a graph, and let A and B be subsets of V(G), and H a subgraph of G. If A dominates B (or H), i.e., every vertex in B (or H) either belongs to A or is adjacent to a vertex of A, then we write $A \rightarrowtail B$ (or $A \rightarrowtail H$, respectively). We denote the neighbourhood of a vertex v in a graph G by $N_G(v)$, or by N(v) if no ambiguity is possible. If G is a graph, then $\overline{G}$ denotes the complement of G. Let G be graph. Then, for D $\subseteq$ V(G), the *neigbourhood N(D) of D* is defined to be the set {v $\in$ V(G); v is adjacent to at least one vertex of D}. If G is a graph and S, T $\subseteq$ V(G), then by $[S, T]_G$ we mean the set {uv $\in$ E(G); u $\in$ S and v $\in$ T}; if no ambiguity is possible, $[S, T]_G$ may be denoted by [S, T]. For any graph G with vertex set $\{v_1, v_2, \ldots, v_p\}$, we define G^+ to be the graph obtained from G by adding p new vertices u_1, u_2, ..., u_p and the edges u_iv_i, $i = 1, 2, \ldots, p$, to G. Let $P: x_1, x_2, \ldots, x_n$ be a path in a graph G. We will write $P^{\rightarrow}$ if the order of the vertices in P is to be considered from x_1 to x_n, or $P^{\leftarrow}$ if the order is in the opposite direction. For $x_i, x_j \in V(P)$, $i < j$, we write $x_iP^{\rightarrow}x_j$ to indicate the segment on P originating at x_i and terminating at x_j, and we write $x_jP^{\leftarrow}x_i$ to denote the same segment in the opposite direction. For a vertex x on P, we denote by ${}^Px^+$ the vertex on $P^{\rightarrow}$ that immediately follows x, and denote by ${}^Px^-$ the vertex on $P^{\rightarrow}$ that immediately precedes x on $P^{\rightarrow}$. If no ambiguity is possible, we denote ${}^Px^+$ by x^+ and ${}^Px^-$ by x^-. If the vertices of a cycle (path, respectively) form a dominating set, then we will call this cycle (path, respectively) a *dominating cycle* (*dominating path*, respectively). The *wheel* W_p of order $p \geq 4$ is the graph obtained from the join of a graph isomorphic to C_{p-1} and a graph isomorphic to K_1. For a graph G, we define $\varepsilon(G)$ to be the maximum possible number of end-edges in a spanning forest of G, where an *end-edge* is an edge incident with an end-vertex. For a graph G, a $\gamma^+(G)$*-set* (respectively, a $\gamma^-(G)$*-set*) denotes any smallest subset S of V(G) whose removal from G creates a graph with greater (respectively, lower) domination number than $\gamma(G)$. Given any set A, an *n-subset* B of A is any subset of A of cardinality n. A *clique* W of a graph G is a complete subgraph of G; W may or may not be maximal with respect to the property of being complete. The *double star S(m,n)* is the graph obtained from the (disjoint) union of two stars $K_{1,m}$ and $K_{1,n}$ $(m, n \geq 2)$ by joining the two central vertices by an edge. For a graph G, we define *i(G)*, the *independent dominating number* of G, to be the cardinality of a smallest independent dominating set (or, alternatively, a smallest maximally independent set) of G. For any graph G, and $k \in \{0, 1, \ldots, p(G) - 1\}$, let $S_k(G)$ be defined to be the set of vertices of G of degree at most k, and let $s_k(G)$ denote $|S_k(G)|$. Let G be a graph, and suppose $E(G) = \{e_1, e_2, \ldots, e_{q(G)}\}$. Then, the *subdivision graph S(G)* of G is defined to be graph H satisfying $V(H) = V(G) \cup \{x_1, x_2, \ldots, x_{q(G)}\}$, and $E(H) = \bigcup_{i=1}^{q(G)} \{ux_i, x_iv;\ e_i = uv\}$. Let G be a graph, and let $n \in \mathbb{N}$. The *intersection graph* $I(\mathscr{F})$ of a family $\mathscr{F}$ of sets $A_1, A_2, \ldots, A_n$ is a graph with $V(I(\mathscr{F})) = \mathscr{F}$ and $E(I(\mathscr{F})) = \{A_iA_j;\ A_i \cap A_j \neq \varnothing\}$. Given disjoint graphs G and H, and vertices $x \in V(G)$ and $y \in V(H)$, the *(x,y)-coalescence of G and H*, denoted by $(G,x)\bullet(H,y)$, is the graph obtained from G and H by identifying the vertices x and y. We denote by $u_{(G,x)\bullet(H,y)}$ the vertex of $(G,x)\bullet(H,y)$ that is the result of the identification of x and y. If the identified vertices x and y of G and H, respectively, are understood, we write $G\bullet H$ instead of $(G,x)\bullet(H,y)$. We refer to the graph $K_{1,n}$ $(n \in \mathbb{N})$ as a *star graph*, or, more simply, a *star*. For a graph G, a set $F \subseteq E(G)$ is said to be an *edge-cover* of G if $\langle F\rangle_G$ is a spanning subgraph of G, i.e., if every vertex of G is incident with at least one element of F. In the set theoretical sense, we shall use the symbol $\subseteq$ to indicate inclusion and the symbol $\subset$ to indicate strict inclusion.

For concepts and notation not defined above, but occurring in the thesis, we refer to [CL1].

1.2 INTRODUCTION

The roots of domination theory may be traced back to the nineteenth century, when the notion of dominating sets of queens on a chessboard was first considered [D1]. Domination theory was formally initiated by Ore in 1962 [O1] and Berge in 1973 [B2], and soon thereafter, many related concepts were introduced, such as total domination [CDH1], independent domination [AL1], connected domination [SW1], k-domination ([CGS1], [CR1], [F2]), and others. (See the survey [C1] and the comprehensive collection of papers in [HL1].)

Domination theory is applicable to diverse fields, such as communication theory, political science, social network theory, experimental sciences, coding theory and computer science. As a simple example, let the vertices of G represent entities that may or may not be in direct communication with each other, where two vertices of G are adjacent if a direct communication link exists between the corresponding entities. For instance, the vertices may represent intersections in a street grid of a city, where adjacent vertices represent intersections that are exactly one city block apart; or centres in a transmission network where adjacent vertices represent centres that are within receiving range of each other. Computers in a microprocessor network may be represented by vertices which are adjacent if transferral of information between the corresponding computers can be accomplished in a single unit of time. Members of a human, animal or bacteriological population may be represented by vertices that are adjacent if, for example, the corresponding members can communicate directly or are adjacent in a food network or differ from each other within some prescribed limits. A minimum dominating set then represents a smallest set D of entities such that each entity not contained in D is able to communicate directly with a member of D. For instance, the vertices in D may represent intersections in a street grid where facilities (fire hydrants, telephones, police posts, etc.) may be placed such that every inhabitant of the city is within a city block of such a facility. The vertices in D may denote a smallest subset of centres from which radio signals can be transmitted to reach all centres in the relevant network, or smallest sets of computers from which stored data can be communicated within unit time to all computers in a network. A minimum dominating set may represent a smallest subgroup of a human population that can inform or influence all members of the population directly or, in a biological population, a minimum dominating set may correspond to a smallest representative subset of the population.

We shall investigate the extent to which domination properties of a graph, especially its domination number, are retained or altered if vertices or edges are removed from the graph or if additional edges are inserted.

It is interesting to speculate on applications for γ-insensitive graphs, i.e., graphs such that $\gamma(G-e) = \gamma(G)$ for all $e \in E(G)$. For a graph G with domination number γ, one can, for example, imagine that G represents a communication network having p stations and the property that γ of them can transmit a message to the remaining $p - \gamma$ stations with no message traversing more than one communication link in succession. For networks corresponding to γ-insensitive graphs, this property is preserved whenever a single communication link fails; furthermore, we shall deal with the corresponding problem in which γ transmitters suffice if a prescribed number of stations or links fail.

A graph is vertex-domination-critical if $\gamma(G-v) < \gamma(G)$ for all $v \in V(G)$. If the graph G on which a microprocessor network is modelled is vertex-domination-critical, then the network has the characteristics that (1) the failure of any processor leaves a network which requires one fewer "dominating" processor and (as a consequence of (1)) that (2) any processor can be included in a minimum set of these dominating processors (to see why (2) is true, consider the fact that, for any $v \in V(G)$, $G-v$ has a dominating set D of cardinality $\gamma(G) - 1$ and $D \cup \{v\}$ is a dominating set of G with $|D \cup \{v\}| = \gamma(G)$).

A graph G is edge-domination-critical if $\gamma(G+e) < \gamma(G)$ for all $e \in E(\bar{G})$. If, for instance, the facilities location problem is modelled on an edge-domination-critical graph G, it may perhaps be advisable to introduce a new thoroughfare if that can be accomplished for less than the installation and maintenance cost of a facility. Such edge-domination-critical graphs may also be used to model political or social populations which allow for maximum communication while guarding against the existence of a small power block (i.e., a set of influential individuals of cardinality smaller than an acceptable value of γ).

In an experiment in which a representative set of the population under consideration is to be kept as small as possible, the basis on which "closeness" of members of the population is decided may well be adjusted to yield a graph G with $\gamma(G)$ acceptably small, but such that $\gamma(G-e) > \gamma(G)$ for every $e \in E(G)$ (i.e., G is what we shall call $\gamma(G)$-$\gamma^{+\prime}$-critical). In this case, the imposition of more stringent "closeness" requirements would yield a representative set which is too large for the purposes of the experiment.

Chapter 2

DECREASING DOMINATION NUMBER BY ADDITION OF ANY EDGE

2.1 INTRODUCTION AND BASIC PROPERTIES OF EDGE-DOMINATION-CRITICAL GRAPHS

In this chapter, we will investigate those graphs G that have the property that the domination number of the graph obtained from G by the addition of any edge from $\overline{G}$ is less than $\gamma(G)$. We obtain characterizations of certain classes, and investigate the hamiltonian properties, of these graphs.

Unless stated to the contrary below, all results in sections 2.1 to 2.4 appear in [SB1] and [S1], except for Theorems 2.1.10 and 2.1.11, which are from [BCD2], those in sections 2.8 and 2.9 are from [S1] alone, and, finally, those in sections 2.5 to 2.7 are from [W1]. We have supplied the statement of Theorem 2.2.6, the statement and proof of Proposition 2.1.2, 2.2.8, 2.2.9, 2.2.18, 2.2.19, 2.2.26, 2.2.27, Lemma 2.2.20, Theorem 2.2.5, as well as the proof and most of the statement of Theorem 2.8.1, and Remark 2.2.10. We have slightly extended the statement of Theorem 2.2.24; we have slightly extended the statement of, and supplied a proof for, Theorem 2.2.25 and 2.8.1. We have expanded the proof of Theorem 2.4.5, 2.4.6, 2.4.11, 2.6.2, and 2.7.1 (considerably) and of Lemma 2.2.15, 2.4.3 (considerably), and that of Lemma 2.2.16 (slightly) and 2.6.1. A large portion of the proof of Theorem 2.2.24 has been newly supplied. We have slightly generalized Definition 2.2.7, introduced Definition 2.2.12, and slightly altered Definition 2.2.14. We have supplied the proof of Proposition 2.2.1, 2.2.3, 2.2.9, 2.2.11, 2.2.17, 2.4.8, 2.5.2, 2.5.4 and of Theorem 2.1.6, 2.2.17, 2.3.3, 2.8.1, 2.8.2 and Lemma 2.1.5, 2.2.21, 2.2.22, 2.2.23, 2.4.10, as well as the proof of Case 1 in the proof of Proposition 2.2.28 (of which the statement has been extended to include disconnected graphs). The statement and proof of Proposition 2.1.5 have been generalized. We have slightly modified the proof of Theorem 2.1.7. We have modified the statement of Theorem 2.4.6 and 2.4.11 by supplying a lower bound on the order of the graphs for which the theorems are valid. We have supplied the example in Remark 2.2.4, and we have clarified Remark 2.4.9.

2.1.1 Definition: A graph G is defined to be *edge-domination-critical* if $\gamma(G+e) < \gamma(G)$ for each $e \in E(\overline{G})$. For $k \in \mathbb{N}$, an edge-domination-critical graph G will be called *k-edge-critical* if $\gamma(G) = k$.

We note that every complete graph is 1-edge-critical, and that every empty graph of order p is p-edge-critical.

2.1.2 Proposition: If G is a non-complete edge-domination-critical graph, then $\gamma(G+e) = \gamma(G) - 1$ for each $e \in E(\overline{G})$.

Proof: Suppose, to the contrary, that there exists an edge-domination-critical graph G, an edge $e = uv \in E(\overline{G})$, and a minimum dominating set S of G+e with $|S| < \gamma(G) - 1$. Now, if $|\{u, v\} \cap S| \in \{0, 2\}$, then $S \mapsto G$ and $\gamma(G) \leq |S| < \gamma(G)$, which is not possible; if $|\{u, v\} \cap S| = 1$, then $S \cup \{u, v\} \mapsto G$ and $\gamma(G) \leq |\{u, v\} \cup S| = |S| + 1 < \gamma(G)$, which, again, is not possible. These contradictions show that no such edge-domination-critical graph G exists, and the proposition follows. □

The following result is one that we shall use often in the first part of this chapter.

2.1.3 Proposition: For any graph G, $\Delta(G) \leq p(G) - \gamma(G)$.

Proof: Let G be any graph, and let $w \in V(G)$ with $\deg_G w = \Delta(G)$. Clearly, $V(G) - N_G(w) \rightarrowtail G$, and so

$$\gamma(G) \leq |V(G) - N(w)| = p(G) - |N_G(w)| = p(G) - \Delta(G). \qquad \square$$

A sufficient condition for a graph to be edge-domination-critical is given next.

2.1.4 Proposition: If G is a regular graph that is not complete, and $p(G) = \Delta(G) + \gamma(G)$, then G is edge-domination-critical.

Proof: Let G be a graph satisfying the hypothesis of the proposition. Note that, since G is not complete, $E(\bar{G}) \neq \emptyset$. Let $e \in E(\bar{G})$. Then, by Proposition 2.1.3,

$$p(G) = p(G+e) \geq \Delta(G+e) + \gamma(G+e) = \Delta(G) + 1 + \gamma(G+e),$$

i.e.,

$$\gamma(G+e) \leq p(G) - \Delta(G) - 1 < \gamma(G).$$

Thus, G is edge-domination-critical. $\square$

The next result, a basic property of edge-domination-critical graphs G with $\gamma(G) \geq 3$, is one that we shall use often.

2.1.5 Lemma: If G is a k-edge-critical graph for $k \geq 3$, then no two end-vertices of G have a common neighbour.

Proof: Suppose, to the contrary, that there exists a k-edge-critical graph G, $k \geq 3$, with end-vertices a, b of G having a common neighbour, v say, in G. Then, since $ab \notin E(G)$, there exists $S \subseteq V(G) - \{a, b\}$ with $|S| = k - 2$ such that $S \cup \{a\} \rightarrowtail G-b$ or $S \cup \{b\} \rightarrowtail G-a$. Clearly, $v \notin S$ since $S \not\rightarrowtail \{b\}$. However, then $S \cup \{v\} \rightarrowtail G$, whence $\gamma(G) \leq |S| + 1 < k = \gamma(G)$, which is impossible. So, no such k-edge-critical graph G exists. $\square$

In general, the diameter of a connected graph having domination number k can be as large as $3k - 1$ (for example, $\gamma(P_{3k}) = k$, diam $P_{3k} = 3k - 1$, for $k \in \mathbb{N}$). For k-edge-critical graphs, the situation is more restrictive.

2.1.6 Theorem: For $k \geq 2$, the diameter of a (connected) k-edge-critical graph is at most $3k - 4$.

Proof: Suppose, to the contrary, that there exist $k \geq 2$ and a k-edge-critical G such that diam $G \geq 3k - 3$. Let a, b be two vertices on a diametrical path of G such that $d(a,b) = 3k - 3$. Let $P:(a=)v_0,v_1,\ldots,v_{3k-3}$ be a shortest a-b path in G. Since $v_0v_{3k-3} \in E(\overline{G})$, there exists $S \subseteq V(G) - \{v_0, v_{3k-3}\}$ such that $|S| = k - 2$ and $S \cup \{v_0\} \mapsto G+v_0v_{3k-3}$ or $S \cup \{v_{3k-3}\} \mapsto G+v_0v_{3k-3}$; suppose the former holds. Now, each vertex in S dominates at most three vertices of P, and v_0 dominates at most three vertices of P (in $G+v_0v_{3k-3}$); so,

$$\begin{aligned} p(P) &= \Big|\bigcup_{s\in S}(N[s] \cap V(P)) \cup \big(N[v_0] \cap V(P)\big)\Big| \\ &\leq \Big|\bigcup_{s\in S}(N[s] \cap V(P))\Big| + |N[v_0] \cap V(P)| \\ &\leq \sum_{s\in S} |N[s] \cap V(P)| + |N[v_0] \cap V(P)| \\ &\leq 3\cdot|S| + 3 \\ &= 3k - 3 < p(P), \end{aligned}$$

which is absurd. So, no such k-edge-critical graph exists, and the theorem follows. □

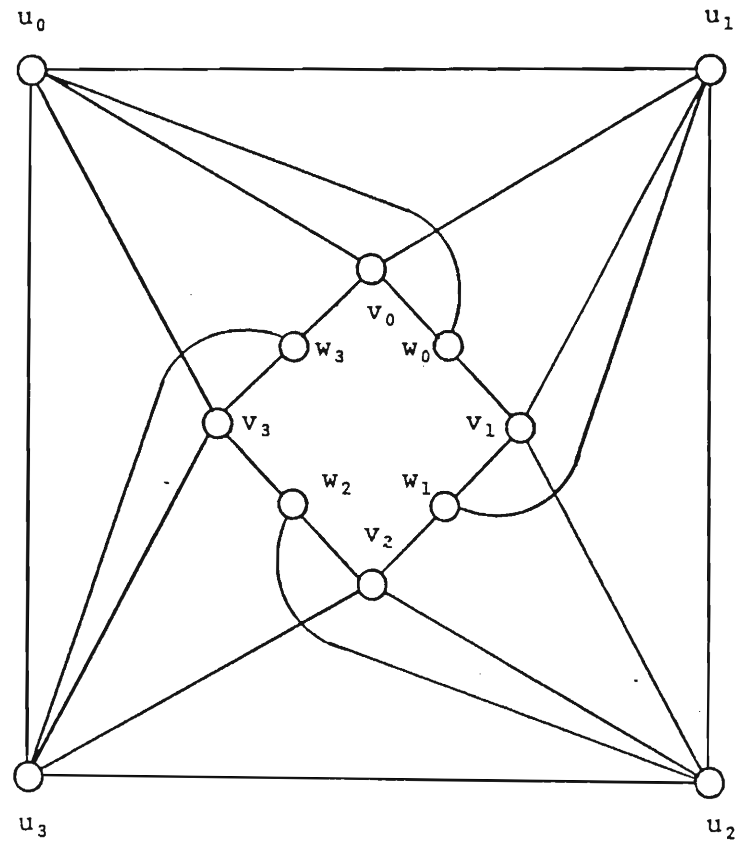
We mention in passing that the claim in [S1] that the proof of 2.1.6 appears in [SB1] is erroneous. That Theorem 2.1.6 is not best possible for $k = 3$ is demonstrated by the following theorem.

2.1.7 Theorem: The diameter of a connected 3-edge-critical graph is at most three.

Proof: Suppose, to the contrary, that there exists a connected 3-edge-critical graph with diam $G \geq 4$. Let $a, b \in V(G)$ with $d(a,b) =$ diam G. Let $A = N(a)$, $B = N(b)$, and $C = V(G) - (N[a] \cup N[b])$. Since $d(a,b) \geq 4$, $C \neq \varnothing$.

We show first that $\langle A\rangle_G$ and $\langle B\rangle_G$ cannot both be non-complete. Suppose, to the contrary, that there exist $x, x' \in A$ and $y, y' \in B$ such that $xx', yy' \notin E(G)$. Clearly, $xy \notin E(G)$ (else, $d(a,b) \leq 3$), so we assume, without loss of generality, that there exists $w \in V(G)$ such that $\{x, w\} \mapsto G-y$. However, then $x'w, wb \in E(G)$, implying $d(a,b) \leq 3$, a contradiction. So, at least one of $\langle A\rangle_G$ and $\langle B\rangle_G$ is complete; suppose $\langle A\rangle_G$ is complete.

Now, let $r \in A$ and consider any $t \in B$. Since $rt \notin E(G)$, there exists $y \in V(G)$ with $\{t, y\} \mapsto G-r$ or $\{r, y\} \mapsto G-t$. If $\{t, y\} \mapsto G-r$, then, since $\langle N[a]\rangle_G$ is complete, $y \notin N[a]$; but


u0
u1
v0
w3
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w2
w1
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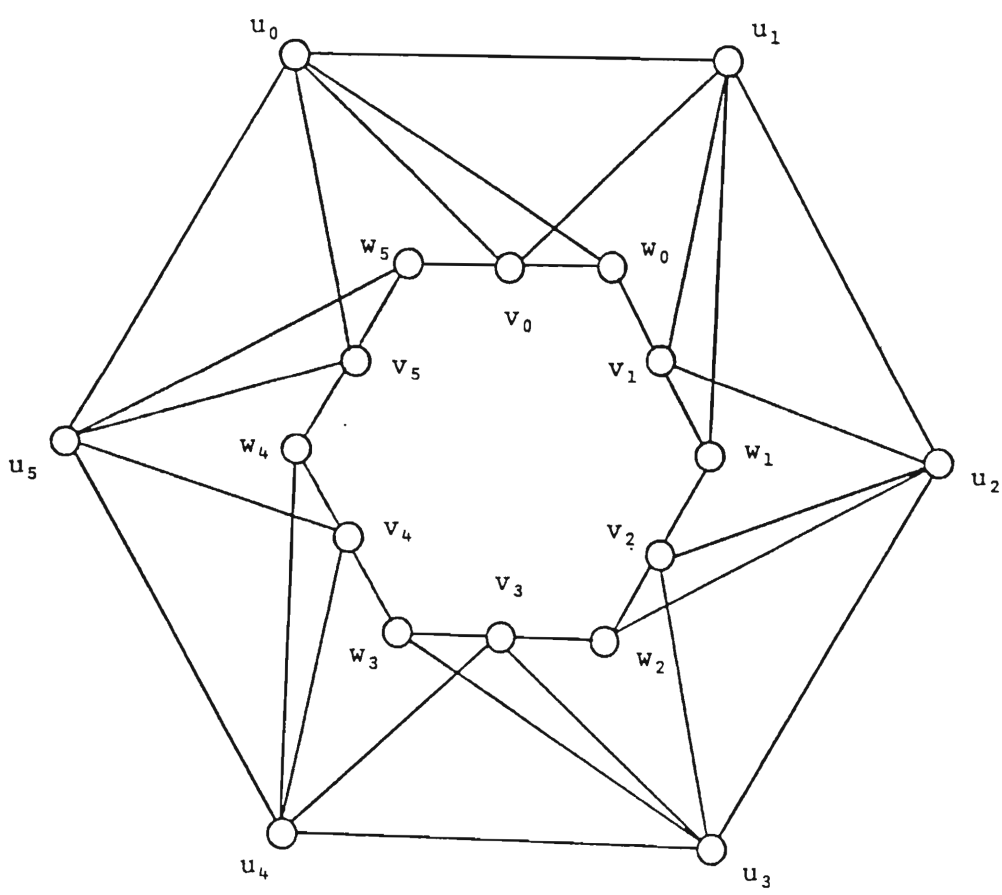
u2

Fig. 2.1.1


u0
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then a is not dominated by $\{t, y\}$. Thus, it must be the case that, for each $t \in B$, there exists a vertex t^* with $\{r, t^*\} \mapsto G-t$. Also, since $tt^* \notin E(G)$, $t^* \neq b$; but $t^* \in N[b]$ (since $\{t^*\} \mapsto \{b\}$), and hence $t^* \in B$. Thus, t^* dominates all of $B - \{t\}$. Furthermore, for any $t \in B$, t^* is unique and $(t^*)^* = t$, as we now show. Let $t \in B$. By what we have just proved, there exists $t^* \in B$ such that $\{t^*\} \mapsto B - \{t\}$. In particular, there exists $t^{**} = (t^*)^*$ such that $\{t^{**}\} \mapsto B - \{t^*\}$. But, t is the only vertex in B non-adjacent to t^*; so, $t^{**} = t$, and the desired uniqueness follows. Furthermore, t is adjacent to every vertex in $B - \{t^*\}$. So, since t^* is adjacent to every vertex of $B - \{t\}$, we see that B can be partitioned into non-adjacent pairs $\{t, t^*\}$.

Since G is connected, there exist $r \in A$ and $c \in C$ with $rc \in E(G)$. If $C \subseteq N(r)$, then $\{r, b\} \mapsto G$, contrary to $\gamma(G) = 3$; so, there exists $c' \in C$ with $rc' \notin E(G)$. Now, since $ac' \notin E(G)$, there exists $x \in V(G)$ with $\{a, x\} \mapsto G-c'$ or $\{c', x\} \mapsto G-a$. Suppose $\{a, x\} \mapsto G-c'$. Then, in order for $\{a, x\}$ to dominate b, we must have $x \in N[b]$. But, x must dominate c; so, $x \in B$. However, then $\{a, x\} \not\mapsto \{x^*\}$. On the other hand, if $\{c', x\} \mapsto G-a$, then x must dominate both r and b, whence $d(a,b) \leq 3$, a contradiction. So, our original assumption is false, and every connected 3-edge-critical graph has diameter at most three. □

This result is best possible, since diam $H(K_3^+) = 3$. Next, we give an example of a class of edge-domination-critical graphs.

2.1.8 Proposition: For $k \geq 3$, define the graph Q_k as follows: $V(Q_k) = \{u_i, v_i, w_i;\ 0 \leq i \leq k-1\}$ and $E(Q_k) = \{u_iu_{i-1}, u_iu_{i+1}, u_iv_{i-1}, u_iv_i, u_iw_i, v_iw_{i-1}, v_iw_i;\ 0 \leq i \leq k-1\}$, where the subscript arithmetic is interpreted modulo k. Then, Q_k is k-edge-critical. (See Figs 2.1.1 and 2.1.2 for Q_4 and Q_6, respectively.)

Proof: Let $k \geq 3$, and let the graph Q_k be defined as above. That $\gamma(Q_k) \leq k$ follows from the observation that, for instance, Q_k is dominated by $\{u_i;\ 0 \leq i \leq k-1\}$. By inspection, one may conclude that no set of cardinality less than k dominates Q_k. So, $\gamma(Q_k) = k$. Showing that $\gamma(Q_k+e) = k - 1$ for every $e \in E(\overline{Q_k})$ involves a lengthy case-study which we omit. □

The following result appears in [V1].

2.1.9 Lemma: For any graph G,

$$\gamma(G) \leq p(G) + 1 - \sqrt{2q(G)+1}.$$

2.1.10 Theorem: If G is a graph such that

$$\gamma(G) > p(G) + 1 - \sqrt{2q(G)+3},$$

then G is edge-domination-critical.

Proof: Suppose, to the contrary, that there exists a graph G satisfying the hypothesis of the theorem, but for which there exists $e \in E(\bar{G})$ with $\gamma(G+e) = \gamma(G)$. By Lemma 2.1.9,

$$\gamma(G+e) \leq p(G+e) + 1 - \sqrt{2q(G+e)+1},$$

i.e.,

$$\gamma(G) \leq p(G) + 1 - \sqrt{2q(G)+3},$$

so that, by our assumption,

$$p(G) + 1 - \sqrt{2q(G) + 3} < \gamma(G) \leq p(G) + 1 - \sqrt{2q(G)+3}.$$

This is not possible, and the desired result follows. □

2.1.11 Theorem: If G is a graph such that

$$\gamma(G) = p(G) + 1 - \sqrt{2q(G)+1},$$

then G is edge-domination-critical.

Proof: Let G be a graph satisfying the hypothesis of the theorem. Suppose $E(\bar{G}) \neq \varnothing$; let $e \in E(\bar{G})$. Then, by Lemma 2.1.9,

$$\gamma(G+e) \leq p(G+e) + 1 - \sqrt{2q(G+e)+1} = p(G) + 1 - \sqrt{2q(G) + 3},$$

and thus, by our assumption,

$$\gamma(G+e) \leq p(G+e) + 1 - \sqrt{2q(G)+3} < p(G) + 1 - \sqrt{2q(G) + 1} = \gamma(G).$$

So, G is edge-domination-critical. □

2.2 TOWARDS A CHARACTERIZATION OF EDGE-DOMINATION-CRITICAL GRAPHS

2.2.1 Proposition: A graph G is 1-edge-critical if and only if $G \cong K_n$ ($n \in \mathbb{N}$).

Proof: If G is a complete graph, then G is 1-edge-critical. If G is a graph with $\gamma(G) = 1$ and $\gamma(G+e) = 0$ for all $e \in E(\overline{G})$, then, since every graph has positive domination number, we must have $E(\overline{G}) = \varnothing$, i.e., G is complete. □

2.2.2 Theorem: A graph G is 2-edge-critical if and only if

$$\overline{G} \cong \bigcup_{i=1}^{n} K_{1,n_i},$$

$n, n_i \in \mathbb{N}$ for $i = 1, 2, \ldots, n$.

Proof: Let G be a 2-edge-critical graph. Then, for any edge $e \in E(\overline{G})$, say $e = uv$, we have $\gamma(G+e) = 1$. Thus, without loss of generality, we may assume that $\{v\} \mapsto G+e$, and so v is an isolated vertex of $\overline{G}-e$, which implies that v has degree 1 in $\overline{G}$. Hence, every edge of $\overline{G}$ is incident with an end-vertex of $\overline{G}$; so $\overline{G}$ is a disjoint union of stars.

Conversely, suppose that G is a graph whose complement is a disjoint union of stars. Since no vertex in $\overline{G}$ is isolated, no vertex of G has degree $p(G) - 1$, and so $\gamma(G) \geq 2$. On the other hand, the central vertex and any non-central vertex of any star of $\overline{G}$ form a dominating set for G. So, $\gamma(G) = 2$. Finally, since $\overline{G}$-e has an isolated vertex for any $e \in E(\overline{G})$, G+e has a vertex of degree $p(G) - 1$. So, G is edge-domination-critical, and the result follows. □

The characterization of k-edge-critical graphs with $k \geq 3$ is more complicated. Our chief interest will be 3-edge-critical graphs. We begin by characterizing disconnected 3-edge-critical graphs.

2.2.3 Proposition: If G is a disconnected 3-edge-critical graph, then

(1) $G = K_{p(G)-2} \cup 2K_1$, or

(2) $G = H \cup K_n$ ($n \in \mathbb{N}$), where H is a connected 2-edge-critical graph.

Proof: Let G be a disconnected 3-edge-critical graph. If $k(G) > 3$, then $\gamma(G) \geq 4$, so $k(G) \in \{2, 3\}$. Let $H_1, H_2, \ldots, H_{k(G)}$ be the components of G. Suppose first that $k(G) = 3$. If there exists $H \in \{H_1, H_2, H_3\}$ with $p(H) \geq 3$ and H not complete, then, for all $e \in E(\overline{H})$, $k(G+e) = 3$, so that $\gamma(G+e) \geq 3$, which contradicts the 3-edge-criticality of G. So, each of H_1, H_2, H_3 is complete. Furthermore, if more than one component is non-trivial, then, for any edge f in $\overline{G}$ with one end in one non-trivial component and the other end in another non-trivial component, we clearly have $\gamma(G+f) = 3$, a contradiction. So, at least two of H_1, H_2, H_3 are trivial, and $G \cong K_{p(G)-2} \cup 2K_1$.

Suppose now that $k(G) = 2$. Since $\gamma(G) = \gamma(H_1) + \gamma(H_2)$, $\gamma(H_1) = 1$ and $\gamma(H_2) = 2$, or $\gamma(H_1) = 2$ and $\gamma(H_2) = 1$; suppose the former is true. Now, if H_1 is not complete, then $E(\overline{H}) \neq \varnothing$ and, for $e \in E(\overline{H}_1)$, $\gamma(G+e) = 3$, a contradiction. So, H_1 is complete. For all $e \in E(\overline{H}_2) \subseteq E(\overline{G})$, $\gamma(G+e) = \gamma(H_1) + \gamma(H_2+e) = 1 + \gamma(H_2+e)$, and so, since G is 3-edge-critical, $\gamma(H_2+e) = 1$; i.e., H_2 is 2-edge-critical. Hence, G is the disjoint union of a connected 2-edge-critical graph and a complete graph. □

2.2.4 Remark: Note that the converse of Proposition 2.2.3 is false. For example, for any $n \geq 2$, the graph $H = (\overline{F}_1 + \overline{F}_2) \cup F_3$, where $F_1 \cong K_{1,1} = K_2$, $F_2 \cong K_{1,2} = P_3$, and $F_3 \cong K_n$, is *not* 3-edge-critical since, for instance, $\gamma(H+uv) = \gamma(H) = 3$ for $u \in V(F_3)$ and $v \in V(F_2)$ with $\deg_H v = 2$. However, $H = K_{2,2,\ldots,2} \cup K_n$ $(n \in \mathbb{N})$ *is* a 3-edge-critical graph that is the union of a 2-edge-critical graph and a complete graph. In fact, we have

2.2.5 Theorem: Let G be a graph of the form $G = H \cup K_n$ $(n \in \mathbb{N})$, where H is 2-edge-critical, and $p = p(G) \geq 4$. Then, G is 3-edge-critical if and only if $n = 1$ or $\overline{H} \cong mK_2$, for $m \geq 2$.

Proof: Let G be a graph of the form $H \cup F$, where H is a 2-edge-critical graph and $F \cong K_n$ $(n \in \mathbb{N})$. Suppose that $p = p(G) \geq 4$, and that G is 3-edge-critical. We shall assume that $n \geq 2$, and show that $\overline{H} = mK_2$ for $m \geq 2$. We consider two cases.

Case 1: Suppose that H is connected. Then, by Theorem 2.2.2,

$$\overline{H} = \bigcup_{i=1}^{n} K_{1,m_i}$$

for $m \geq 2$, $m_i \in \mathbb{N}$ for $i = 1, \ldots, m$. We consider two subcases.

Subcase 1.1: Suppose $m_i = 1$ for each $i = 1, \ldots, m$. Then, $\overline{H} \cong mK_2$ $(m \geq 2)$.

Subcase 1.2: Suppose $m_i \geq 2$ for at least one $i \in \{1, \ldots, m\}$. Let S be a star component of $\overline{H}$ with order at least 3, with central vertex x, say, and let u be any non-central vertex of S. Furthermore, let v be any vertex of F. Since G is 3-edge-critical, there is a minimum dominating set D of $G+uv$ of cardinality 2. If $v \notin D$, then $v' \in D$ for some $v' \in V(F) - \{v\}$ and u must dominate H, which is not possible, since $\gamma(H) = 2$. Thus, v belongs to every minimum dominating set of $G+uv$. So, if $D = \{v, d\}$, then $\{d\} \mapsto V(H) - \{u\}$.

Now, since $|N_{\overline{H}}(x)| = p(S) - 1 \geq 2$, $\{x\} \not\mapsto H-u$; so, $d \neq x$. Certainly, $d \notin V(S) - \{x\}$ since, otherwise, $\{d\} \not\mapsto \{x\} \subseteq V(H) - \{u\}$. So, d must belong to a star component S^* in $\overline{H}$ distinct from S. Suppose y is a central vertex of S^* (note, possibly, $S^* \cong K_2$). Then, if $d = y$, $\{d\} \not\mapsto V(S^*) - \{d\}$ $(\neq \varnothing) \subseteq V(H) - \{u\}$, and if d is a non-central vertex of S^*, $\{d\} \not\mapsto \{y\} \subseteq V(H) - \{u\}$, both situations being contrary to the fact that $\{d\} \mapsto V(H) - \{u\}$. So, Subcase 1.2 does not occur.

Case 2: Suppose that H is disconnected, so that $\overline{H} \cong K_{1,m}$ for some $m \in \mathbb{N}$. We claim that $m = 1$. Suppose, to the contrary, that $m \geq 2$. Then, $G \cong K_1 \cup G_1 \cup G_2$, where $G_1 \cong K_m$ and $G_2 \cong K_n$ (where, we recall, $n \geq 2$). Then, clearly, for any vertex $v \in V(G_2)$ and any vertex $u \in V(G_1)$, $\gamma(G+uv) = 3$, which is a contradiction. So, $m = 1$, whence $\overline{H} \cong K_2$.

Conversely, suppose G is a graph of the form $H \cup F$, where $F \cong K_n$ $(n \in \mathbb{N})$ and H is a 2-edge-critical graph. Now, if $n = 1$ or $\overline{H} \cong mK_2$ for some $m \in \mathbb{N}$, then clearly $\gamma(G) = 3$.

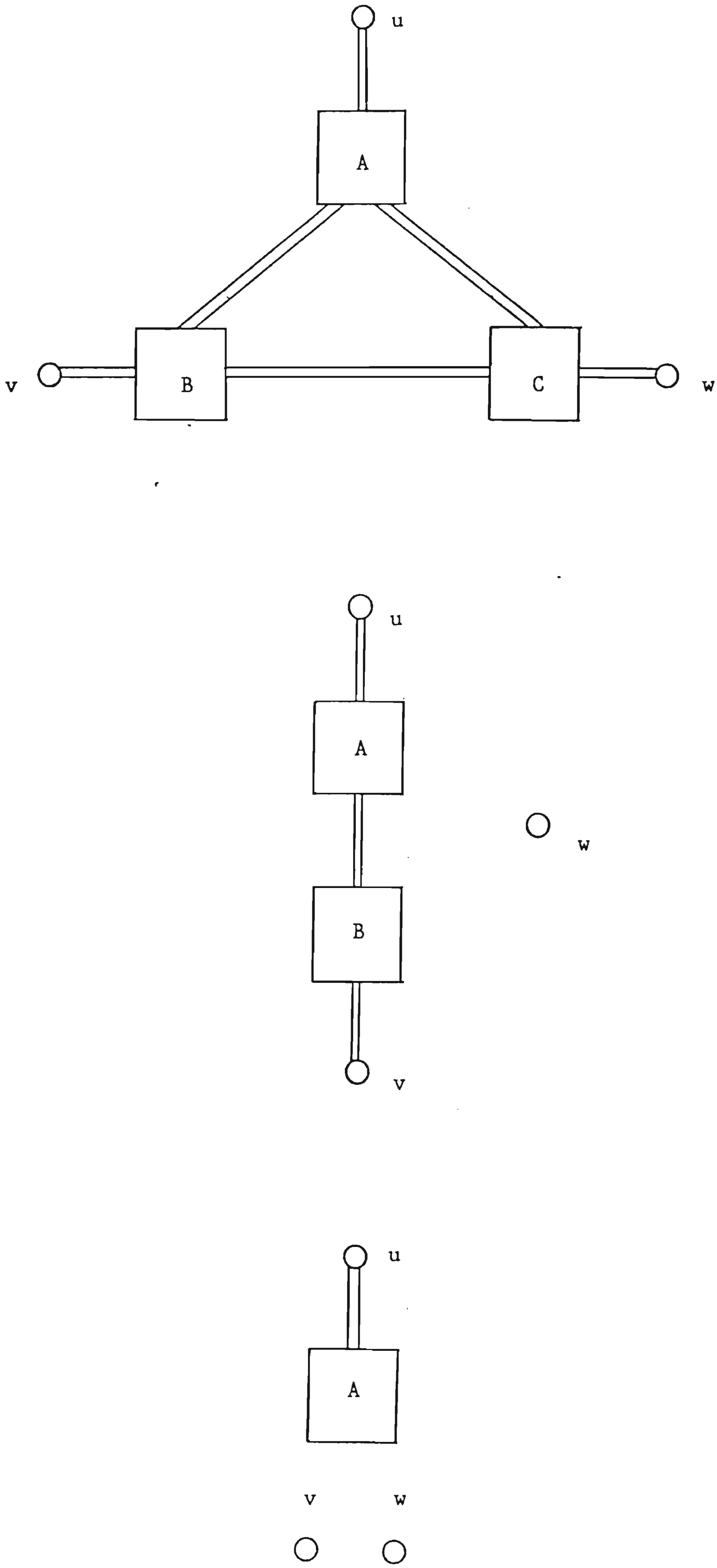


Fig. 2.2.1

Case 3: Suppose $\overline{H} \cong mK_2$ for some $m \in \mathbb{N}$. If $m = 1$, then $G \cong 2K_1 \cup K_{p-2}$, which is clearly 3-edge-critical; so, suppose $m \geq 2$. If $uu' \in E(\overline{H})$, then $\{u, v\} \rightarrowtail G + uu'$, where $v \in V(F)$. If $uv \in E(\overline{G})$ where $u \in V(H)$ and $v \in V(F)$, then $\{u', v\} \rightarrowtail G+uv$, where u' is the (unique) vertex in H satisfying $uu' \in E(\overline{H})$. Thus, $\gamma(G+e) \leq 2 < 3$ for each $e \in E(\overline{G})$, and G is 3-edge-critical in this case, also.

Case 4: Suppose that $n = 1$. If $\overline{H}$ is a (single) star, then, again, $G \cong 2K_1 \cup K_{p-2}$; so, suppose

$$\overline{H} \cong \bigcup_{i=1}^{m} K_{1,m_i},$$

for $m \geq 2$, $m_i \in \mathbb{N}$, for $i = 1, 2, \ldots, m$. Let $e \in E(\overline{G})$. If $e \in E(\overline{H})$, then $\gamma(G+e) = 2$, so suppose that $e = uv \in E(\overline{G}) - E(\overline{H})$; assume that $\langle\{v\}\rangle$ is the trivial component of G. If u is the central vertex of a star in $\overline{H}$, then it is easily seen that $\{u, w\} \rightarrowtail G+uv$, where w is any vertex in $V(H) - \{u\}$. If, on the other hand, u is a non-central vertex of a star in $\overline{H}$, with y, say, as central vertex, then $\{u, y\} \rightarrowtail G+uv$. So, in this case, too, G is 3-edge-critical. □

Combining these last results, we obtain our characterization of disconnected 3-edge-critical graphs.

2.2.6 Theorem: A graph G is a disconnected 3-edge-critical graph if and only if either $G \cong K_n \cup 2K_1$ ($n \in \mathbb{N}$) or $G \cong H \cup K_n$ ($n \in \mathbb{N}$) where H is a connected 2-edge-critical graph and, furthermore, $n = 1$ or $\overline{H} \cong mK_2$ for $m \geq 2$.

As we shall see, the following definition provides a general class of 3-edge-critical graphs.

2.2.7 Definition: For any $p \in \mathbb{N}$, $p \geq 5$, let non-negative integers a, b, and c satisfy $a + b + c = p - 3$. Let G be a complete graph on $p - 3$ vertices and let $A, B, C \subseteq V(G)$ satisfy $A \cup B \cup C = V(G)$ with $A \cap B = A \cap C = B \cap C = \emptyset$, and $|A| = a$, $|B| = b$, and $|C| = c$. Form the graph *H(a,b,c)* by adding to G the new vertices u, v, and w with $N_{H(a,b,c)}(u) = A$, $N_{H(a,b,c)}(v) = B$, and $N_{H(a,b,c)}(w) = C$. The graph H(a,b,c) is depicted in Fig. 2.2.1.

2.2.8 Proposition: Let a, b, c be non-negative integers. Then, the size of H(a,b,c) is

$$\binom{p(H(a,b,c)) - 2}{2} = \binom{a+b+c+1}{2}.$$

Proof: Let a, b, c be non-negative integers, and let p = p(H(a,b,c)) = a + b + c + 3. Then,

$$q(H(a,b,c)) = \binom{p-3}{2} + (p-3) = \frac{(p-3)(p-4)}{2} + p - 3 = \frac{p^2-5p+6}{2} = \binom{p-2}{2}. \qquad \square$$

2.2.9 Proposition: The graph H(a,b,c) in Definition 2.2.7 has domination number 3.

Proof: Let a, b, c be non-negative integers, and let $H \cong H(a,b,c)$. Certainly, $\gamma(H) \leq 3$ since $\{u, v, w\} \mapsto H$. Let D be a minimum dominating set of H. Since $N_H[u] = A \cup \{u\}$, $N_H[v] = B \cup \{v\}$, and $N_H[w] = C \cup \{w\}$, we note that the three sets $N_H[u]$, $N_H[v]$, and $N_H[w]$ are pairwise disjoint. Furthermore, as $D \mapsto \{u, v, w\}$, D contains at least one vertex from each of these three closed neighbourhoods, and so $|D| \geq 3$. Hence, $|D| = 3$ and $\gamma(H) = 3$. $\square$

2.2.10 Remark: It is easy to see that the graph H(a,b,c) defined in 2.2.7 is not 3-edge-critical if exactly one of a, b, c is zero: if, for example, c = 0, a, b $\in \mathbb{N}$ and $H \cong H(a,b,c)$, then $\gamma(H+uv) = \gamma(H) = 3$. Hence, we have the following.

2.2.11 Proposition: For non-negative integers a, b, and c, the graph H(a,b,c) defined in 2.2.7 is 3-edge-critical if all of a, b, c are non-zero, or exactly two of a, b, c are zero.

Proof: Let a, b, and c be non-negative integers, and let $H \cong H(a,b,c)$. Let $e \in E(\overline{H})$. We consider two cases.

Case 1: Suppose a, b, and c are all non-zero. Without loss of generality, assume that e = uy, where $y \in B \cup C \cup \{v, w\}$. If $y \in C$, then $\{v, y\} \mapsto H+uy$; if $y \in B$, then $\{w, y\} \mapsto G+uy$. If y = v, then $\{u, c\} \mapsto H+uy$, for any $c \in C$; if y = w, then $\{u, b\} \mapsto H+uy$, for any $b \in B$. So, $\gamma(H+uy) = 2$.

Case 2: Suppose a > 0 and b, c = 0. If e = rv (or e = rw) for some $r \in A \cup \{u\}$, then $\{r, v\} \mapsto G+e$ (or $\{r, w\} \mapsto G+e$, respectively). If e = vw, then $\{u, v\} \mapsto G+e$. So, $\gamma(G+e) = 2$.

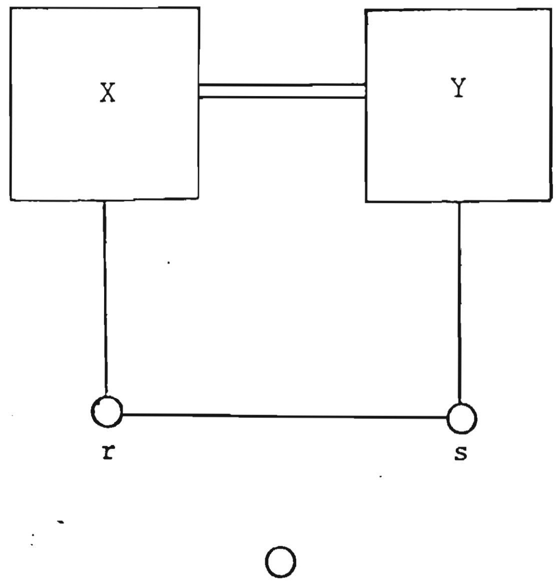
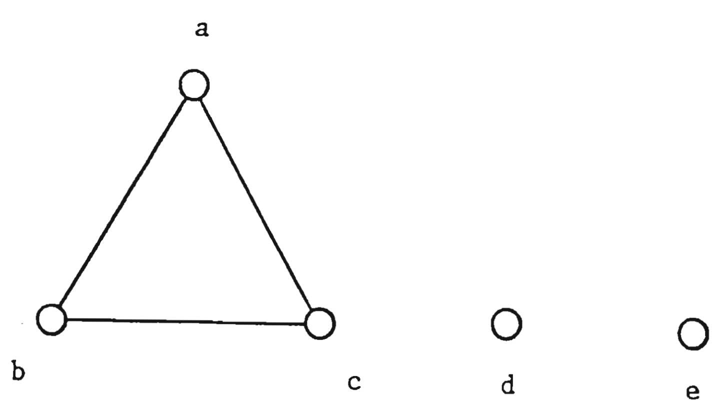
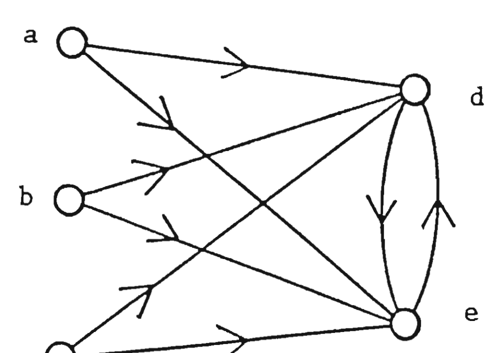


Fig. 2.2.2





Hence, G is 3-edge-critical. □

2.2.12 Definition: For positive integers x and y, we define a graph *H(x,y)* of order $p \geq 5$ to be a graph that consists of an isolated vertex together with a complete graph of order $p - 3$ whose vertex set is partitioned into two sets X and Y, with two additional vertices r and s satisfying $N_{H(x,y)}(r) = X \cup \{s\}$ and $N_{H(x,y)}(s) = Y \cup \{r\}$, where $|X| = x$, $|Y| = y$. The graph H(x,y) is depicted in Fig. 2.2.2.

2.2.13 Remark: Suppose that G is a 3-edge-critical graph. Then, if u, v are any two distinct non-adjacent vertices of G, then $\gamma(G+uv) = 2$ and so (as the proof of Proposition 2.1.2 shows) there exists a vertex x with $\{u, x\} \mapsto G-v$ or $\{v, x\} \mapsto G-u$. Thus, there is a natural orientation induced on the edges of $\overline{G}$, as we indicate in the following definition:

2.2.14 Definition: Let G be a 3-edge-critical graph. The *digraph obtained (from $\overline{G}$) by domination ordering on G* is the digraph D with $V(D) = V(G)$ such that, for $u, v \in V(D)$, $(u, v) \in E(D)$ if and only if $uv \in E(\overline{G})$ and there exists $x \in V(G)$ with $\{u, x\} \mapsto G-v$. We note that D is not necessarily asymmetric, as the example in Fig. 2.2.3 shows.

The next two lemmas, particularly the first, will be used often in this chapter.

2.2.15 Lemma: Let G be a 3-edge-critical graph and S an independent set of vertices of G. If $n = |S| \geq 4$, then the vertices of S may be ordered as $a_1, a_2, \ldots, a_n$ in such a way that there exists a path $x_1, x_2, \ldots, x_{n-1}$ in $G-S$ with $\{a_i, x_i\} \mapsto G-a_{i+1}$ for $i = 1, 2, \ldots, n - 1$.

Proof: Let G be a 3-edge-critical graph, and let S be an independent set of vertices of G with $n = |S| \geq 4$. Since S is independent in G, $\langle S \rangle_{\overline{G}}$ is complete and hence the domination ordering on G induces on S a complete digraph consisting of a tournament with possibly a few extra arcs. Thus, since every tournament has a hamiltonian path, we may label the vertices of S as $a_1, a_2, \ldots, a_n$ so that (a_i, a_{i+1}) is an arc of D for each $i = 1, 2, \ldots, n - 1$. Hence, for each $i = 1, 2, \ldots, n - 1$, there exists $x_i \in V(G)$ such that $\{a_i, x_i\} \mapsto G-a_{i+1}$.

Now, since $|S| \geq 4$, $x_i \notin S$, for each $i = 1, 2, \ldots, n - 1$. To see this, suppose that $x_i \in S$ for some $i \in \{1, 2, \ldots, n - 1\}$. Since $\{a_i, x_i\} \mapsto G-a_{i+1}$, it follows that every vertex in $V(G) - \{a_i, x_i, a_{i+1}\}$ is dominated by a_i or x_i. Therefore, since a_i is not adjacent to any other vertex in S (since S is independent), it follows that x_i must be adjacent to every vertex in

$S - \{a_i, x_i, a_{i+1}\}$, where $|S - \{a_i, x_i, a_{i+1}\}| \geq |S| - 3 \geq 4 - 3 = 1$. However, this is impossible since $x_i \in S$ and S is independent.

Let $i, j \in \{1, 2, \ldots, n - 1\}$ with $i \neq j$; assume, without loss of generality, that $j < i$. Then, $\{a_j, x_j\} \mapsto G - a_{j+1}$ (with $a_{j+1} \neq a_{i+1}$), so $\{a_j, x_j\} \mapsto \{a_{i+1}\}$. Hence, $\{a_j\} \mapsto \{a_{i+1}\}$ or $\{x_j\} \mapsto \{a_{i+1}\}$. If $\{a_j\} \mapsto \{a_{i+1}\}$, then (as $a_ja_{i+1} \notin E(G)$, since S is independent), it follows that $a_j = a_{i+1}$, contradicting the assumption that $j < i$. So, $\{x_j\} \mapsto \{a_{i+1}\}$; but $x_j \neq a_{i+1}$ (as $x_j \notin S$). Hence, $x_ja_{i+1} \in E(G)$. This, combined with the fact that $x_ia_{i+1} \notin E(G)$, yields that $x_j \neq x_i$.

Finally, since for $i = 2, 3, \ldots, n - 1$, we have $\{a_i, x_i\} \mapsto G - a_{i+1}$ and a_i non-adjacent to x_{i-1}, we have that x_i is adjacent to x_{i-1}. Thus, $x_1, x_2, \ldots, x_{n-1}$ is the required path. □

2.2.16 <u>Lemma</u>: If S is an independent set of vertices of a connected 3-edge-critical graph with $|S| \geq n$ for some $n \in \mathbb{N}$, then there exists $x \in S$ with $\deg x \geq n - 2$.

<u>Proof</u>: Let G be a connected 3-edge-critical graph, let S be an independent set of vertices of G, and suppose $|S| \geq n$ for some $n \in \mathbb{N}$. For $n = 1$ or $n = 2$, certainly $\deg x \geq n - 2$ for any $x \in V(G)$; if $n = 3$, then, since G is connected, $\deg x \geq 1 = n - 2$ for every $x \in V(G)$. So, we assume that $n \geq 4$ and let $S = \{a_1, a_2, \ldots, a_n\}$ be ordered as in Lemma 2.2.15. Now, in the course of the proof of Lemma 2.2.15, it was shown that for $i, j \in \{1, 2, \ldots, n - 1\}$ with $j < i$, x_j is adjacent to a_{i+1}, so, in particular, x_j is adjacent to a_n, and thus $\{x_1, x_2, \ldots, x_{n-2}\} \subseteq N(a_n)$, giving $\deg a_n \geq n - 2$. □

Note that the connectedness of G was used in the case $n = 3$ only, in order to guarantee that some vertex in S is not isolated. In fact, this lemma holds if we require that G have at least four vertices and we dispense with the demand that G is connected.

2.2.17 <u>Proposition</u>: If S is an independent set of vertices of a 3-edge-critical graph, of order at least 4, with $|S| = n$ for some $n \in \mathbb{N}$, then there exists $x \in S$ with $\deg x \geq n - 2$.

<u>Proof</u>: Let G be a 3-edge-critical graph of order at least 4, and let $S \subseteq V(G)$ be independent in G. Let $n = |S|$. If $n \leq 2$, the result is trivial. Suppose $n = 3$ and that every vertex in S has degree 0 in G. Then, clearly, G has at least four components, which implies $\gamma(G) \geq 4$, a contradiction. So, there is indeed a vertex in S with degree at least $n - 2$. The case for $n \geq 4$ is proved as in Lemma 2.2.16. □

2.2.18 Proposition: A 3-edge-critical graph G has order 3 if and only if $G \cong \bar{K}_3$.

2.2.19 Proposition: A 3-edge-critical graph G has order 4 if and only if $G \cong 2K_1 \cup K_2$.

Proof: Let G be a 3-edge-critical graph of order 4. Then (by Proposition 2.1.3), $\Delta(G) \leq p(G) - \gamma(G) = 1$; however, $G \not\cong \bar{K}_4$, so $\Delta(G) = 1$, and G is isomorphic to an element of $\{2K_2, K_2 \cup 2K_1\}$. Clearly, $K_2 \cup 2K_1$ is 3-edge-critical and $\gamma(2K_2) = 2$; hence, $G \cong K_2 \cup 2K_1$. □

2.2.20 Lemma: No connected 3-edge-critical graph of order 5 exists.

Proof: Suppose that there exists a connected 3-edge-critical graph G of order 5 that contains no triangle. If $C_5 \subset G$, then $\gamma(G) \leq 2$; so, $C_5 \not\subset G$, and G is bipartite, with partite sets V and W, say. One of these partite sets, say V, has cardinality 1 or 2; hence, since G is connected, $V \mapsto G$, and $\gamma(G) \leq 2$, contrary to $\gamma(G) = 3$. So, if a connected 3-edge-critical graph G of order 5 exists, then G contains a triangle, H (say), but, since $\Delta(G) \leq 5 - 3 = 2$, H is a component of G and G is disconnected. Hence, no 3-edge-critical graph of order 5 exists. □

2.2.21 Lemma: Every connected 3-edge-critical graph of order 6 contains a triangle.

Proof: Suppose that there exists a connected 3-edge-critical graph G of order 6 that does not contain a triangle. Suppose $C_5 \subset G$, say $\langle\{v_0, v_1, v_2, v_3, v_4\}\rangle_G \cong C_5$. Since G is connected, the vertex $u \in V(G) - \{v_0, v_1, v_2, v_3, v_4\}$ is adjacent to v_i for some $i \in \{0, 1, 2, 3, 4\}$; however, then $\{v_i, v_{i+2}\} \mapsto G$ (where the subscripts are taken modulo 5). So, neither C_3 nor C_5 is a subgraph of G, and G is bipartite, with partite sets V and W, say. If either of V and W has fewer than 3 vertices, then (as before) $\gamma(G) \leq 2$. So, $|V|, |W| \geq 3$, and (since $|V| + |W| = 6$) we have $|V| = |W| = 3$. Let $V = \{v_1, v_2, v_3\}$, $W = \{w_1, w_2, w_3\}$. Since $\gamma(G) > 2$, at most one of V and W contains a vertex of degree 3. Note that $\Delta(G) \leq 3$. If $\Delta(G) \leq 2$, then since G is connected, $G \cong C_6$ or $G \cong P_6$, which is not possible, since $\gamma(P_6) = \gamma(C_6) = 2 < \gamma(G)$. So, $\Delta(G) = 3$ and there exists $x \in V(G)$ with deg $x = 3$; say, $x = v_1$. By Lemma 2.1.5, at most one of w_1, w_2, w_3 has degree 1.

Case 1: Suppose w_1 is an end-vertex of G. Then, $w_1v_2, w_1v_3 \notin E(G)$. Since G is connected and $\Delta(G - v_1) \leq 2$, w_2 and w_3 must be adjacent to distinct elements of $\{v_2, v_3\}$; suppose $w_2v_2, w_3v_3 \in E(G)$. Then, $w_2v_3, w_3v_2 \notin E(G)$. (Thus, G is isomorphic to the graph obtained from $K_{1,3}$ by sub-dividing two edges.) However, $\gamma(G+w_2w_3) = \gamma(G)$, contrary to the edge-domination-criticality of G. So, this case does not occur.

Case 2: Suppose that none of w_1, w_2, w_3 are end-vertices of G. Since $\Delta(G-v_1) \leq 2$, exactly two of w_1, w_2, w_3 are adjacent to one of v_2, v_3; assume w_1v_2, w_2v_2, $w_3v_3 \in E(G)$ (it follows that v_3 is an end-vertex of G). However, we now have that $\{v_2, w_3\} \mapsto G$, contrary to the fact that $\gamma(G) = 3$. So, this case, too, does not occur.

So, if a connected 3-edge-critical graph G of order 6 exists,then G contains a triangle. □

2.2.22 Lemma: Every connected 3-edge-critical graph of order 7 contains a triangle.

Proof: Suppose that there exists a connected 3-edge-critical graph G of order 7 that contains no triangle. We shall show that neither C_5 nor C_7 is a subgraph of G, whence it will follow, as above, that G is bipartite; finally, we shall show that $\gamma(G) = 2$, which contradiction will establish the desired result.

Suppose, to the contrary, that G contains a 5-cycle $H:v_1,v_2,v_3,v_4,v_5,v_1$. Note that, since $C_3 \not\subset G$, H has no diagonals in G. Let $\{a_1, a_2\} = V(G) - V(H)$. As $\gamma(G) = 3$, no vertex v_i ($i \in \{1, 2, 3, 4, 5\}$) is adjacent to both a_1 and a_2. Since $v_1v_3 \notin E(G)$, there exists a vertex $x \in V(G)$ such that $\{v_1, x\} \mapsto G-v_3$ or $\{v_3, x\} \mapsto G-v_1$. We shall assume, without loss of generality, that $\{v_1, x\} \mapsto G-v_3$. Since $\{x\} \not\mapsto \{v_3\}$, we have $x \in \{v_5, a_1, a_2\}$.

Case 1: Suppose $x = v_5$. Then, say, a_1v_1, $a_2v_5 \in E(G)$. Now, a_1v_2, a_1v_5, a_2v_1, $a_2v_4 \notin E(G)$ (otherwise, $K_3 \subset G$). Also, a_1v_3, $a_2v_3 \notin E(G)$ (otherwise, $\gamma(G) = 2$). Furthermore, not both a_1v_4 and a_2v_2 belong to E(G) (otherwise, $\gamma(G) = 2$). So, suppose, without loss of generality, that $a_2v_2 \notin E(G)$. Then, the graph I with $V(I) = V(G)$ and $E(I) = E(H) \cup \{a_1v_1, a_2v_5, a_1a_2, a_1v_4\}$ is a supergraph of G. However, $\gamma(I+v_1v_4) = 3$, whence $\gamma(G+v_1v_4) \geq \gamma(I+v_1v_4) = 3$, a contradiction. So, this case does not occur.

Case 2: Suppose $x \in \{a_1, a_2\}$. Without loss of generality, we assume $x = a_1$. Since $v_1v_4 \notin E(G)$, we have $a_1v_4 \in E(G)$. (Then, $a_2v_4 \notin E(G)$.) Further, a_2v_1, $a_2v_2 \notin E(G)$ (otherwise, $\gamma(G) = 2$). Since $a_2v_1 \notin E(G)$ and $\{a_1, v_1\} \mapsto G-v_3$, we have $a_1a_2 \in E(G)$. Since $K_3 \not\subset G$, we have a_1v_3, $a_1v_5 \notin E(G)$. Then, the graph J with $V(J) = V(G)$ and $E(J) = E(H) \cup \{a_1v_4, a_1a_2, a_2v_3, a_2v_5\}$ is a supergraph of G; however, $\gamma(J+v_3v_5) = 3$.

Thus, $\gamma(G+v_3v_5) \geq \gamma(J+v_3v_5) = 3$, contrary to the 3-edge-criticality of G. So, this case does not occur, either.

Hence, $C_5 \not\subset G$. We show now that $C_7 \not\subset G$. Assume, to the contrary, that G contains a 7-cycle $H{:}v_0,v_1,v_2,v_3,v_4,v_5,v_6,v_0$. We note first that $G \neq H$, since, otherwise, $\gamma(G+v_1v_4) = \gamma(G)$, contrary to the assumption that G is 3-edge-critical. So, since $K_3 \not\subset G$, E(G) contains at least one element v_iv_{i+3} of $\{v_jv_{j+3};\ 0 \leq j \leq 6\}$, where the subscripts are taken modulo 7. However, then $C{:}v_i,v_{i+3},v_{i+4},v_{i+5},v_{i+6},v_i$ (where the subscripts are taken modulo 7) is a 5-cycle in G, contrary to the result established above. Thus, G contains no 7-cycle.

We have established that G contains no odd cycles, so G is bipartite, with partite sets V and W, say. Suppose $|V| \leq |W|$; thus, $|V| \leq 3$. Note that, since G is connected, $V \mapsto G$. Hence, $|V| \geq 3$, and so $|V| = 3$. Let $V = \{v_1, v_2, v_3\}$, $W = \{w_1, w_2, w_3, w_4\}$.

Since $w_1w_2 \notin E(G)$, there exists a vertex $x \in V(G)$ such that $\{w_1, x\} \mapsto G-w_2$ or $\{w_2, x\} \mapsto G-w_1$. Assume,without loss of generality, that $\{w_1, x\} \mapsto G-w_2$. If $x \in W$, then, in $G+w_1w_2$, $|N[\{w_1, x\}] \cap W| \leq 3 < |W|$, so that at least one element of W is undominated; thus, $x \in V$. Without loss of generality, suppose $x = v_1$. Since $[\{w_1\}, \{w_3, w_4\}] = \varnothing$ in $G+w_1w_2$, we have $v_1w_3, v_1w_4 \in E(G)$, and since $[\{v_1\}, \{v_2, v_3\}] = \varnothing$ in $G+w_1w_2$, we have w_1v_2, $w_1v_3 \in E(G)$. Further, $v_1w_2 \notin E(G)$ (otherwise, $\{v_1, w_1\} \mapsto G$).

Since $w_1w_3 \notin E(G)$, there exists a vertex $y \in V(G)$ such that $\{w_1, y\} \mapsto G-w_3$ or $\{w_3, y\} \mapsto G-w_1$. By the same reasoning used above, $y \in V$; also, $yw_2, yw_4 \in E(G)$ since $[\{w_1, w_3\}, \{w_2, w_4\}] = \varnothing$ in $G+w_1w_3$. Since $yw_2 \in E(G)$, $y \neq v_1$; $y = v_2$, say. Now, $\{w_3, v_2\} \mapsto G-w_1$ is not possible, since $v_2w_1 \in E(G)$; so, $\{w_1, v_2\} \mapsto G-w_3$. Of course, then, $v_1w_1 \in E(G)$, and $v_2w_3 \notin E(G)$.

Since $w_1w_4 \notin E(G)$, there exists a vertex $z \in V(G)$ such that $\{w_1, z\} \mapsto G-w_4$ or $\{w_4, z\} \mapsto G-w_1$. Since $\{z\} \mapsto \{w_2, w_3\}$, $z \notin \{v_1, v_2\}$. Hence, $z = v_3$. Since $v_3w_1 \in E(G)$, $\{w_4,v_3\} \mapsto G-w_1$ is not possible; so, we have $\{w_1,v_3\} \mapsto G-w_4$. Thus, $v_3w_2, v_3w_3 \in E(G)$ and $v_3w_4 \notin E(G)$. However, now, $\{v_1, w_2\} \mapsto G$, a contradiction.

Thus, every connected 3-edge-critical graph of order 7 contains a triangle. □

2.2.23 <u>Lemma</u>: Every connected 3-edge-critical graph of order 8 contains a triangle.

Proof: Suppose that there exists a connected 3-edge-critical graph of order 8 that does not contain a triangle. We begin by noting that $\beta(G) \leq 3$ (otherwise, if there exist four independent vertices a_1, a_2, a_3, a_4 in G, then (by Proposition 2.2.15), there exist $x_1, x_2, x_3 \in V(G) - \{a_1, a_2, a_3, a_4\}$ such that $x_2x_3 \in E(G)$ and $\{a_4\} \mapsto \{x_1, x_2\}$, whence $\langle\{a_4, x_1, x_2\}\rangle \cong K_3$, a contradiction). Then, G is not bipartite, since, otherwise, at least one partite set of G would contain at least four (independent) vertices, contradicting $\beta(G) \leq 3$. So, G contains an odd cycle. However, G does not contain an *induced* subgraph isomorphic to C_7, as we now show. Suppose that $V(G) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, a\}$ and that $\langle\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}\rangle$ is an induced 7-cycle $C:v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_1$ in G; since G is connected, $[\{a\}, V(G)] \neq \emptyset$ - say $av_1 \in E(G)$. Hence, $av_2, av_7 \notin E(G)$ and there exists $x \in V(G)$ such that $\{x, a\} \mapsto G-v_2$ or $\{x, v_2\} \mapsto G-a$. If $\{x, a\} \mapsto G-v_2$, then $\{x\} \mapsto \{v_7\}$ and since $\{x\} \not\mapsto \{v_2\}$, $x \in \{v_6, v_7\}$. Thus, $\{x\} \not\mapsto \{v_3\}, \{v_4\}$, and so $av_3, av_4, v_3v_4 \in E(G)$, contradicting the assumption that G contains no 3-cycle. Hence, $\{x, v_2\} \mapsto G-a \cong C_7$, which is impossible as $\gamma(C_7) = 3$.

Thus, either G contains no subgraph isomorphic to C_7, and hence (since G is not bipartite) contains a 5-cycle, or G contains a subgraph which is isomorphic to C_7 but which has a diagonal, which diagonal forms one edge of a 5-cycle (since $K_3 \not\subset G$). In either case, G contains an induced 5-cycle, $C:v_1,v_2,v_3,v_4,v_5,v_1$, say, and three other vertices, a_1, a_2, a_3, which do not induce a K_3; suppose, without loss of generality, that $a_1a_2 \notin E(G)$. Then, there exists x such that (without loss of generality) $\{a_1, x\} \mapsto G-a_2$, where $x \in V(G) - \{a_1, a_2\}$. We observe that, since $K_3 \not\subset G$, a_i is adjacent to at most two vertices in $\{v_1, v_2, v_3, v_4, v_5\}$, for each $i \in \{1, 2, 3\}$. So, if $x = a_3$, then $|N[\{a_1, x\}]| \leq 6 < |V(G) - \{a_2\}|$, a contradiction. Consequently, $x \in \{v_1, v_2, v_3, v_4, v_5\}$; assume, without loss of generality, that $x = v_1$. From $\{v_1, a_1\} \mapsto G-a_2$, it follows (since $v_1v_3, v_1v_4 \notin E(G)$) that a_1v_3, a_1v_4 and (of course) v_3v_4 are edges of G, whence $K_3 \subset G$, a contradiction. □

2.2.24 Theorem: Every connected 3-edge-critical graph has order at least 6 and contains a triangle (i.e, $\omega(G) \geq 3$ for every 3-edge-critical graph G).

Proof: By Propositions 2.2.18, 2.2.19 and 2.2.20, any connected 3-edge-critical graph has order at least 6. By Lemmas 2.2.21, 2.2.22, and 2.2.23 any connected 3-edge-critical graph of order 6, 7, or 8 contains a triangle.

Now, let G be a connected 3-edge-critical graph of order at least 9, and assume, to the contrary, that $K_3 \not\subset G$. Then, since the Ramsey number $r(3,4) = 9$, G must contain an independent set S of cardinality at least 4. Let $S = \{a_1, a_2, \ldots, a_n\}$ be ordered as in Lemma 2.2.15 and let $x_1, x_2, \ldots, x_{n-1}$ be the associated path in $G - S$. Then, $\langle\{x_1, x_2, a_4\}\rangle$ is a triangle in G, a contradiction. □

That $\omega(G) \leq p - 2$ for every 3-edge-critical graph is obvious, $G \cong K_{p-2} \cup 2K_1$ being the only 3-edge-critical graph of order p with $\omega(G) = p - 2$. We next identify all 3-edge-critical graphs G of order p with $\omega(G) = p - 3$.

2.2.25 <u>Theorem</u>: Let G be a 3-edge-critical graph on p (≥ 5) vertices having maximum clique size $\omega(G) = p - 3$. If $p = 5$, then $G \cong H(1,1)$. If $p \geq 6$, then $G \cong H(a,b,c)$ for positive integers a, b, c satisfying $a + b + c = p - 3$, or $G \cong H(x,y)$ for some positive integers x and y satisfying $x + y = p - 3$.

<u>Proof</u>: Let G be a 3-edge-critical graph on p (≥ 5) vertices having maximum clique size $\omega(G) = p - 3$. Let $W \subseteq V(G)$ satisfy $\langle W\rangle_G \cong K_{p-3}$. Let $V(G) = W \cup \{u, v, w\}$. Let $A = N_G(u)$, $B = N_G(v)$, $C = N_G(w)$. Note that at most one of A, B, and C is empty, as the only 3-edge-critical graph with 2 isolated vertices is $K_{p-2} \cup 2K_1$ with clique number $p - 2$.

<u>Case 1</u>: Suppose that $A \cup B \cup C \subseteq W$. (This implies, of course, that $\{u, v, w\} \cap (A \cup B \cup C) = \varnothing$.) If A, B, and C are not pairwise disjoint, then assume, without loss of generality, that $A \cap B \neq \varnothing$. Then, for any $y \in A \cap B$, $\{y, w\} \rightarrowtail G$. This contradicts the fact that $\gamma(G) = 3$. So, A, B, and C are pairwise disjoint.

Next, we show that $A \cup B \cup C = W$. Suppose, to the contrary, that there exists $x \in W - (A \cup B \cup C)$. Now, since u is non-adjacent to x, there exists $y \in V(G)$ such that $\{u, y\} \rightarrowtail G-x$, or $\{x, y\} \rightarrowtail G-u$. Suppose $\{u, y\} \rightarrowtail G-x$. Since y is non-adjacent to x, $y \notin W$. So, $y \in \{v, w\}$; however, then neither u nor y dominates $\{v, w\} - \{y\}$, which is not possible. So, we assume $\{x, y\} \rightarrowtail G-u$. Thus, $y \notin A$. However, then one of v and w is not dominated by $\{x, y\}$. So, our assumption is false and $A \cup B \cup C = W$.

Finally, we observe that, since A, B, C are subsets of W, where $\{u, v, w\} \cap W = \varnothing$, we have that $\langle\{u, v, w\}\rangle \cong \overline{K}_3$. So, $G \cong H(a,b,c)$, with either all of $a = |A|$, $b = |B|$, $c = |C|$ non-zero, or (by our earlier remark) exactly one of a, b, c equal to zero. By Remark 2.2.10, we must have $a, b, c > 0$, as required.

Case 2: Suppose that $A \cup B \cup C \not\subset W$. Let $A' = W \cap A$, $B' = W \cap B$, $B' = W \cap B$. The argument we used in Case 1 to show that A, B, and C are mutually disjoint shows that A', B', and C' are also mutually disjoint.

We observe first that P_3 is not a subgraph of $F = \langle\{u, v, w\}\rangle$, since otherwise $\gamma(G) = 2$ (if, for example, $\deg_F u = 2$, then $\{u, y\} \mapsto G$ for any $y \in W$). So, $F \cong K_1 \cup K_2$ or $F \cong \bar{K}_3$. But, $A \cup B \cup C \not\subset W$, so $F \cong K_1 \cup K_2$. Assume, without loss of generality, that $\deg_F w = 0$.

Next, we show that $A' \cup B' \cup C' = W$. Suppose, to the contrary, that there exists $x \in W - (A' \cup B' \cup C')$. Since $ux \notin E(G)$, there exists $y \in V(G)$ such that $\{u, y\} \mapsto G-x$, or $\{x, y\} \mapsto G-u$. Suppose $\{x, y\} \mapsto G-u$. Since $\{y\} \not\mapsto \{u\}$, $y \in V(G) - N_G[u]$. However, then at least one of v and w is not dominated. So, $\{u, y\} \mapsto G-x$. Since $yx \notin E(G)$, $y \notin W$. If $y = v$, then $\{u, y\} \not\mapsto \{w\}$. So, $y = w$. Further, $B' = \emptyset$, since, otherwise, $\{u, y\} \not\mapsto B'$, contrary to $\{u, y\} \mapsto G-x$. However, for $z \in N_G(w) = C'$, $\{u, z\} \mapsto G$, which is not possible, since $\gamma(G) = 3$. So, our assumption is false, and $A' \cup B' \cup C' = W$. Finally, we notice that $C' = \emptyset$ (otherwise, $\{z, u\} \mapsto G$, for any $z \in C'$). So, $G \cong H(|A'|,|B'|)$. If $p = 5$, then, by the definition of $H(x,y)$ (see 2.2.12), we must have $|A'| = |B'| = 1$, and $G \cong H(1,1)$, as required. □

We now characterize the 3-edge-critical graphs of order 5 and 6.

2.2.26 Proposition: A 3-edge-critical graph G has order 5 if and only if G is isomorphic to an element of $\{2K_1 \cup K_3, C_4 \cup K_1\}$.

Proof: Let F be the set of all 3-edge-critical graphs of order 5. It is a simple matter to verify that $2K_1 \cup K_3, C_4 \cup K_1 \in F$. Now, let G be a 3-edge-critical on 5 vertices, and observe that (by Proposition 2.1.3) $\Delta(G) \leq 2$, and, by Proposition 2.2.20, G is disconnected. Now, since $\gamma(G) = 3$, it follows that $\omega(G) \in \{2, 3\}$. If $\omega(G) = 2 = 5 - 3$, then, by Theorem 2.2.25, $G \cong H(1,1)$, i.e. $G \cong C_4 \cup K_1$. Suppose now $\omega(G) = 3$; assume $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, where $G_1 = \langle\{v_1, v_2, v_3\}\rangle \cong K_3$. If G_1 is not a component of G, or $G \cong K_3 \cup K_2$, then $\gamma(G) \leq 2$, a contradiction. So, $G \cong K_3 \cup 2K_1$, as required. □

2.2.27 Proposition: A 3-edge-critical graph G has order 6 if and only if G is isomorphic to an element of $\{K_3^+ (= H(1,1,1)), K_4 \cup 2K_1, H(1,2), C_4 \cup K_2\}$.

Proof: Suppose that G is disconnected. Then, by Proposition 2.2.3, $G \cong K_4 \cup 2K_1$ or $G \cong H \cup K_n$ ($n \in \mathbb{N}$), where H is a connected, 2-edge-critical graph. Suppose G is the latter graph. Clearly, $n \in \{1, 2, 3, 4\}$. If $n = 4$, then $H \cong 2K_1$ (since $\gamma(G) = 3$), which is impossible since H is connected. If $n = 3$, then, since $\overline{H}$ contains no isolated vertices, $\overline{H} \cong K_{1,2}$, whence $H \cong K_1 \cup K_2$; however, then H is not connected. If $n = 2$, then, since $\gamma(H) = 2$, we have $H \cong C_4$ or $H \cong P_4$; since $P_4 \cup K_2$ is not 3-edge-critical, $G \cong C_4 \cup K_2$. Suppose now $n = 1$. By Theorem 2.2.2,

$$\overline{H} \cong \bigcup_{i=1}^{m} K_{1,m_i},$$

for $m, m_i \in \mathbb{N}$, $i = 1, 2, \ldots, m$; so, since $\Sigma_{i=1}^{m}(m_i + 1) = 5$, it follows that $\overline{H} \cong K_2 \cup K_{1,2}$ or $\overline{H} \cong K_{1,4}$. However, if $\overline{H} \cong K_{1,4}$, then H is disconnected. So, $G \cong (\overline{K}_2 + \overline{K}_{1,2}) \cup K_1 \cong H(1,2)$.

Suppose now that G is connected. Since $K_3 \subset G$ (by Theorem 2.2.24) and $\Delta(G) \leq p(G) - \gamma(G) = 3$ (by Proposition 2.1.3), obviously, $\omega(G) \in \{3, 4\}$. However, if $\omega(G) = 4 = p(G) - 2$, then $G \cong K_4 \cup 2K_1$, which is not connected. So, $\omega(G) = 3$ and, by Theorem 2.2.25, $G \cong K_3^+$.

Conversely, H(1,1,1) is 3-edge-critical by Proposition 2.2.11, $K_4 \cup 2K_1$ and $C_4 \cup K_2$ are 3-edge-critical by Theorem 2.2.6, and H(1,2) is seen to be 3-edge-critical by inspection. □

2.2.28 Proposition: G is a 3-edge-critical graph with exactly two end-vertices if and only if G is isomorphic to an element of $\{K_2 \cup 2K_1, K_2 \cup K_{2,\ldots,2}, H(1,1,p-5)$, where $p \geq 6\}$.

Proof: Let G be a 3-edge-critical graph with exactly two end-vertices. Then (by Proposition 2.1.5), $p(G) \geq 4$. We consider two cases.

Case 1: Suppose G is disconnected. Then, by Proposition 2.2.3, $G \cong K_{p-2} \cup 2K_1$, or $G \cong H \cup K_n$ ($n \in \mathbb{N}$) where H is a connected 2-edge-critical graph. If G is the former, then (since G has 2 end-vertices) K_{p-2} must be isomorphic to K_2 and $G \cong K_2 \cup 2K_1$; if G is the latter graph, then, by Theorem 2.2.5, $n = 1$ or $\overline{H} \cong mK_2$, $m \in \mathbb{N}$.

Subcase 1.1: Suppose $\overline{H} = mK_2$, $m \in \mathbb{N}$.

Subcase 1.1.1: Suppose $n \geq 3$ (then the end-vertices of G belong to H). If $m = 1$, then G has no end-vertices, so $m \geq 2$. However, then every vertex of H has degree $2m - 2 > 1$, so, again, G has no end-vertices. This contradiction shows that this subcase does not occur.

Subcase 1.1.2: Suppose $n = 1$. Then, the end-vertices of G must belong to H. This implies, since $\overline{H}$ is a union of stars, that $\overline{H}$ must be isomorphic to $K_{1,2}$. This, however, is contrary to our assumption about $\overline{H}$. So, this subcase also does not occur.

Subcase 1.1.3: Suppose $n = 2$. Then, the two vertices in the complete component of G are the two end-vertices of G, and we must have $\delta(\overline{H}) \geq 2$. So, $\overline{H}$ cannot be a single star (as $\delta(H) \neq 0$), which implies $m \geq 2$, and $G \cong K_2 \cup K_{2,\ldots,2}$.

Subcase 1.2: Suppose $n = 1$ and

$$\overline{H} \cong \bigcup_{i=1}^{m} K_{1,m_i},$$

for $m \in \mathbb{N}$, where $m_i > 1$ for at least one $i \in \{1, 2, \ldots, m\}$. Since $n = 1$, H must have exactly two end-vertices. As in Subcase 1.1.2, this implies that $\overline{H} \cong K_{1,2}$. So, $G \cong K_2 \cup 2K_1$.

Case 2: Suppose that G is connected. By Propositions 2.2.18, 2.2.19, and 2.2.26, no edge-critical graph on fewer than 6 vertices has two end-vertices, so $p(G) \geq 6$. Let a and b be end-vertices of G, and let $N(a) = \{a_1\}$, $N(b) = \{b_1\}$. We show first that $G-\{a, b, a_1, b_1\}$ is complete. Suppose, to the contrary, that there exist $u, v \in V(G) - \{a, b, a_1, b_1\}$ such that $uv \notin E(G)$. Then, there exists a vertex $x \in V(G)$ which we assume, without loss of generality, satisfies $\{u, x\} \mapsto G-v$. However (since $u \notin \{a, b, a_1, b_1\}$), x must dominate a and b, which is not possible, by Lemma 2.1.5. Hence, $G-\{a, b, a_1, b_1\}$ is complete.

Since $\gamma(G) = 3$, there exists a vertex $w \in V(G) - N_G[\{a_1, b_1\}]$; in particular, $w \notin \{a, b, a_1, b_1\}$. We now show that $G-\{a, b, w\}$ is complete. Suppose, to the contrary, that there exist vertices u, v $\in$ V(G) - {a, b, w} satisfying uv $\notin$ E(G). We may assume, without loss of generality, that there exists a vertex x $\in$ V(G) such that {u, x} $\mapsto$ G-v. Since (by Lemma 2.1.5) x does not dominate both a and b, we may suppose that {u} $\mapsto$ {a}. Since u $\in$ V(G) - {a, b, w}, we have u = a_1. Thus, {x} $\mapsto$ {b}, so that x $\in$ {b, b_1}. However, this produces a contradiction, since {u, x} $\mapsto$ G-v implies that {u} $\mapsto$ {w} (i.e., {a_1} $\mapsto$ {w}), or {x} $\mapsto$ {w} (i.e., {b} $\mapsto$ {w} or {b_1} $\mapsto$ {w}), which is not possible because w $\notin$ N_G[{a_1, b_1}] = N_G[{a_1, b, b_1}]. Hence, G-{a, b, w} is complete.

It now follows from the definition of w and the fact that G-{a, b, a_1, b_1} and G-{a, b, w} are complete that G $\cong$ H(1,1,p-5).

Conversely, we show that if G is isomorphic to an element of {$K_2 \cup 2K_1$, $K_2 \cup K_{2,\ldots,2}$, H(1,1,p-5), where p $\geq$ 6}, then G is 3-edge-critical with two end-vertices. That each of these graphs has two end-vertices is obvious. By Proposition 2.2.3, $K_2 \cup 2K_1$ is 3-edge-critical; by Theorem 2.2.5, $K_2 \cup K_{2,\ldots,2}$ is 3-edge-critical; by Proposition 2.2.11, H(1,1,p-5) is 3-edge-critical. □

2.3 MATCHINGS IN 3-EDGE-CRITICAL GRAPHS

Any graph with a 1-factor (or perfect matching) must, of necessity, have even order. We show next that, for 3-edge-critical graphs, this obvious condition is also sufficient. We need the following lemma.

2.3.1 Lemma: Let G be a connected 3-edge-critical graph. For any T $\subset$ V(G), G-T has at most |T| + 1 components.

Proof: Suppose, to the contrary, that there exists a connected, 3-edge-critical graph with a non-empty proper subset T of V(G) such that k(G-T) $\geq$ |T| + 2. Assume first that T = {v} and let A, B, and C be three distinct components of G-T. By Lemma 2.1.5, at most one of A, B, and C is trivial; we shall assume that |A|, |B| $\geq$ 2. Now, let a, b $\in$ N_G(v) with a $\in$ A and b $\in$ B. Since ab $\notin$ E(G), we may assume, without loss of generality, that {a, x} $\mapsto$ G-b for some x $\in$ V(G). Since xb $\notin$ E(G), we have x $\neq$ v. Furthermore, x must belong to C since {a, x} $\mapsto$ C , N(a) $\cap$ C = $\varnothing$, and x $\neq$ v. However, then {a, x} $\not\mapsto$ B - {b}($\neq$ $\varnothing$), a contradiction. Thus, we must have n = |T| $\geq$ 2; let A_1, A_2, ..., A_{n+1}, A_{n+2} be n + 2 distinct components of $G-T$. For each $i \in \{1, 2, \ldots, n+2\}$, let $a_i \in A_i$. Then, $S = \{a_1, a_2, \ldots, a_{n+2}\}$ is independent in G with $|S| = n + 2 \geq 4$. We assume then that S is ordered as in Lemma 2.2.15, and let $x_1, x_2, \ldots, x_{n+1}$ be a path in $G-S$ with $\{a_i, x_i\} \mapsto G-a_{i+1}$, for each $i = 1, 2, \ldots, n+1$. Then, for each $i \in \{1, 2, \ldots, n+1\}$, x_i is adjacent to vertices in at least $n + 2 - |\{i, i+1\}| = n \geq 2$ of the components A_i, and hence each x_i ($i = 1, 2, \ldots, n+1$) must belong to T. However, then $|T| \geq n + 1 > n = |T|$, which is absurd. Thus, no such 3-edge-critical graph G exists, and the lemma follows. □

The well-known theorem of Tutte concerning the existence of a 1-factor in a (general) graph states that a graph G has a 1-factor if and only if it does not contain a set S of vertices such that $G-S$ has more than $|S|$ odd components. We may also phrase Tutte's theorem as follows.

2.3.2 Theorem: A connected graph G of even order has a 1-factor if and only if G does not contain a set S of vertices such that $G-S$ has at least $|S| + 2$ odd components.

Proof: Let G be a connected graph of even order. By Tutte's theorem, G has a 1-factor if and only if G does not contain a set S of vertices such that $G-S$ has at least $|S| + 1$ odd components. Suppose that G contains a set S of vertices such that $G-S$ has at least $|S| + 1$ odd components. Let $A = \bigcup\{V(C);\ C$ is an odd component of $G-S\}$, and $B = \bigcup\{V(C);\ C$ is an even component of $G-S\}$. Now, if $|S|$ is even (odd), then (since $p(G) = |A| + |B| + |S|$ is even), $G-S$ must have an even (odd) number of odd components. So, the statement, concerning a graph G of even order, that *G has a set S of vertices such that $G-S$ has at least $|S| + 1$ odd components* is equivalent to the statement that *G has a set S of vertices such that $G-S$ has at least $|S| + 2$ odd components*. □

The main result of this section now follows from Lemma 2.3.1 and the above theorem.

2.3.3 Theorem: If G is a connected 3-edge-critical graph of even order, then G contains a 1-factor.

Proof: Let G be a connected, 3-edge-critical graph of even order. By Lemma 2.3.1, G has no set T of vertices such that $G-T$ has at least $|T| + 2$ components; in particular, there is no set $T \subset V(G)$ such that $G-T$ has at least $|T| + 2$ *odd* components. So, by Theorem 2.3.2 (since G has even order), G has a 1-factor. □

n = 6

1 1 1 3 3 3

n = 7

1 1 2 3 3 4 4
2 2 2 3 3 3 3

n = 8

1 1 3 5 5 5 5 5
1 2 2 5 5 5 5 5
1 2 3 3 4 5 5 5
1 3 3 3 4 4 5 5
2 2 3 3 4 4 4 4
2 2 3 3 4 4 5 5
2 2 3 4 4 5 5 5
2 3 3 3 3 3 4 5
3 3 3 3 3 3 3 3
3 3 3 3 4 4 4 4

n = 9

1 1 4 6 6 6 6 6 6
1 2 3 6 6 6 6 6 6
1 2 4 4 5 6 6 6 6
1 3 3 4 5 6 6 6 6
1 3 4 4 5 5 6 6 6
1 4 4 4 4 5 6 6 6
2 2 2 6 6 6 6 6 6
2 2 3 3 5 5 6 6 6
2 2 3 4 5 5 5 6 6
2 2 4 4 4 4 4 5 5
2 2 4 4 4 5 5 5 5
2 2 4 5 5 5 5 6 6
2 2 4 5 5 6 6 6 6
2 3 3 3 3 3 5 5 5
2 3 3 4 4 4 4 4 6
2 3 3 4 4 4 4 5 5
2 3 3 4 4 4 4 6 6
2 3 3 4 4 5 5 5 5
2 3 3 4 5 5 6 6 6
2 3 3 5 5 6 6 6 6
2 3 4 4 4 4 4 5 6
2 3 4 4 4 4 5 6 6
2 3 4 4 4 5 5 5 6
2 3 4 4 5 5 5 6 6
2 4 4 4 4 4 4 4 4
2 4 4 4 4 4 4 5 5
2 4 4 4 4 4 6 6 6
2 4 4 4 5 5 5 5 6
3 3 3 3 4 4 4 4 4
3 3 3 4 4 4 4 4 5
3 3 3 4 4 4 4 5 6
3 3 3 4 4 4 5 5 5
3 3 3 4 4 5 5 5 6
3 3 3 4 5 5 5 5 5
3 3 4 4 4 4 4 5 5
3 3 4 4 4 5 5 5 5
3 4 4 4 4 4 4 4 5
4 4 4 4 4 4 4 4 4

Fig. 2.4.1

2.4 DEGREE SEQUENCES/SETS OF 3-EDGE-CRITICAL GRAPHS

A graph G has a vertex of degree $p(G) - 1$ if and only if $\gamma(G) = 1$. If a and b are non-adjacent vertices in a 2-edge-critical graph H, then, in H+ab, either a or b must have degree $p(H) - 1$, implying that $deg_H a = p(H) - 2$, or $deg_H b = p(H) - 2$. With these kinds of considerations in mind, one would expect there to be restrictions on the degrees of the vertices of 3-edge-critical graphs. In fact, the following theorems demonstrate that such restrictions do hold.

Fig. 2.4.1 shows degree sequences which are known to be degree sequences of 3-edge-critical graphs of order at most 9, as listed in [S1]. As it is a very simple matter to list the degrees of the vertices of any graph, whereas the characterization of k-edge-critical graphs has thus far proved to be difficult, it would be desirable to characterize degree sequences of k-edge-critical graphs completely. Even establishing properties of such sequences would be of use in exploring the structure of the graphs concerned. A reasonable conjecture relating to degree sequences of 3-edge-critical graphs is given in [S1]:

2.4.1 Conjecture: If $d_1, d_2, \ldots, d_p$, with $d_1 \leq d_2 \leq \ldots \leq d_p$, is the degree sequence of a 3-edge-critical graph G, then, for each $i = 0, 2, \ldots, \lfloor {}^n/_2 \rfloor$, we have $d_{i+1} + d_{n-i} \geq p - 3$.

Recall the following defintion.

2.4.2 Definition: For any graph G, and $k \in \{0, 1, \ldots, p(G) - 1\}$, $S_k(G)$ is defined to be the set of vertices of G of degree at most k, and $s_k(G)$ denotes $|S_k(G)|$.

2.4.3 Lemma: Let G be a connected 3-edge-critical graph with $s_k(G) \geq 3k + 1$ for some $k \geq 2$. Then, there do not exist vertices $x, y, z \in S_k(G)$ with $\{x, y\} \mapsto G-z$.

Proof: Let G be a connected, 3-edge-critical graph of order p. For $k \in \{0, 1, \ldots, p - 1\}$, let $S_k = S_k(G)$ and $s_k = s_k(G)$. Suppose that some $k \in \{3, 4, \ldots, p(G) - 1\}$ satisfies $s_k \geq 3k + 1$ and assume that there exist $x, y, z \in S_k(G)$ with $\{x, y\} \mapsto G-z$. Then,

$$\begin{aligned} 3k + 1 &\leq |S_k| \\ &\leq p = |N[x] \cup N[y] \cup \{z\}| \\ &= |N(x) \cup N(y) \cup \{x, y, z\}| \\ &\leq \deg x + \deg y + 3 \\ &\leq 2k + 3, \end{aligned}$$

and so $k \leq 2$, a contradiction. So, there are no $x, y, z \in S_k$ with $\{x, y\} \mapsto G-z$, if $k \geq 3$.

Suppose now that $s_2 \geq 3.2 + 1 = 7$. Suppose that there exist vertices $x, y, z \in S_2$ with $\{x, y\} \mapsto G-z$. Clearly, $V(G) = N(x) \cup N(y) \cup \{x, y, z\}$, so $p = |V(G)| \leq |N(x)| + |N(y)| + 3 \leq 7$. However, $p \geq |S_2| = s_2 \geq 7$. So, $|S_2| = 7$ and $V(G) = S_2$ and every vertex of G has degree 1 or 2.

Now, $|N[x] \cup N[y]| = 6$, so we must have $\deg x = \deg y = 2$ and $N[x] \cap N[y] = \varnothing$. Let $N(x) = \{x_1, x_2\}$, $N(y) = \{y_1, y_2\}$. Then, since $\Delta(G) \leq 2$, we have $xy_1, xy, xy_2, xz, yx_1, yx_2, yz \notin E(G)$. We consider two cases.

Case 1: Suppose $\deg z = 1$. Then, z is adjacent to (at least) one of $\{x_1, x_2, y_1, y_2\}$ - say, $zx_1 \in E(G)$. Then, $zx_2, zy_1, zy_2 \notin E(G)$ (since $\deg z = 1$) and $x_1y_1, x_1y_2, x_1x_2 \notin E(G)$ (since $\deg x_1 = 2$). Now, since G is connected, (exactly) one of x_2y_2 and x_2y_1 must belong to E(G). Say, $x_2y_1 \in E(G)$. We have now shown that the 3-edge-critical graph G is isomorphic to P_7. However, by Theorem 2.2.23, this produces a contradiction, since P_7 contains no triangle. So, Case 1 does not occur.

Case 2: Suppose $\deg z = 2$. If $N(z) = N(x)$ or $N(z) = N(y)$, then C_4 is a subgraph of G which must, in fact, be a component of G (since $\Delta(G) \leq 2$), which is impossible since G is connected. So, suppose, without loss of generality, that $N(z) = \{x_1, y_1\}$. Then, $zx_2, zy_2, y_1y_2, y_1x_2, x_1x_2, x_1y_2, x_1y_1 \notin E(G)$. If $x_2y_2 \notin E(G)$, then $G \cong P_7$, which is not possible, so $x_2y_2 \in E(G)$, and $G \cong C_7$. However, then, again by Theorem 2.2.23, G is not 3-edge-critical. So, Case 2 also does not occur.

Hence, no such vertices $x, y, z \in S_2$ with $\{x, y\} \mapsto G-z$ exist. □

2.4.4 Lemma: Let G be a connected 3-edge-critical graph with $s_k(G) \geq 3k + 1$ for some $k \geq 2$. Then, $\beta(\langle S_k \rangle) \leq k + 1$.

Proof: Suppose, to the contrary, that there exists a connected, 3-edge-critical graph G of order p and $k \in \{0, 1, \ldots, p - 1\}$ such that $s_k(G) \geq 3k + 1$ but $\beta(\langle S_k \rangle) \geq k + 2$. Let $S_k = S_k(G)$, and let $A = \{a_1, a_2, \ldots, a_{k+2}\}$ be an independent set of vertices in $\langle S_k \rangle$ ordered as in Lemma 2.2.15. There is an associated path $x_1, x_2, \ldots, x_{k+1}$ in $G-A$ satisfying $\{x_i, a_i\} \mapsto G-a_{i+1}$, for $i = 1, 2, \ldots, k + 1$. Also, by Lemma 2.4.3, $x_i \notin S_k$ for each $i \in \{1, 2, \ldots, k + 1\}$. Let $b \in S_k - A$ ($3k + 1 > k + 2$ implies $S_k - A \neq \emptyset$). Then, for each $i = 1, 2, \ldots, k + 1$, b is adjacent to one of x_i or a_i, which implies (since $x_i \notin S_k$ and hence $b \neq x_i$, for each $i \in \{1, 2, \ldots, k + 1\}$) that $|N(b)| \geq k + 1$. However, this is impossible since $b \in S_k$ implies $\deg b \leq k$. □

For a 3-edge-critical graph G, and for all $k \in \{1, 2, \ldots, p(G) - 1\}$, we now derive an upper bound on $s_k(G)$.

2.4.5 Theorem: Let G be a connected 3-edge-critical graph. Then, for each $k \in \{0, 1, \ldots, p(G) - 1\}$, $s_k(G) \leq 3k$.

Proof: Let G be a connected 3-edge-critical graph of order p. For each $k \in \{1, 2, \ldots, p - 1\}$, let S_k denote $S_k(G)$ and s_k denote $s_k(G)$. We claim that S_1 is independent. Suppose, to the contrary, that there exist $u, v \in S_1$ such that $uv \in E(G)$. Then, $\langle \{u, v\} \rangle \cong K_2$ is a component of G (by the definition of S_1), which implies, since G is connected, that $G \cong K_2$; however, this is impossible, since $\gamma(K_2) = 1 \neq 3 = \gamma(G)$. So, S_1 is indeed independent. Thus, by Lemma 2.2.16, there exists $x \in S_1$ with $1 \geq \deg x \geq |S_1| - 2 = s_1 - 2$, whence $s_1 \leq 3$, as required.

Now, suppose there is a $k \in \{2, 3, \ldots, p(G) - 1\}$ with $s_k \geq 3k + 1$. Let $H = \langle S_k \rangle_{\bar{G}}$. Now, $\deg_{\bar{H}} v \leq \deg_G v \leq k$ for any $v \in V(H)$, and so each vertex $v \in V(H)$ satisfies $\deg_H v = p(H) - 1 - \deg_{\bar{H}} v \geq 3k + 1 - 1 - k = 2k$. We show now that, if we consider the subdigraph $D^* = \langle V(H) \rangle_D$ of the digraph D obtained from $\bar{G}$ by domination ordering on G, there must be a vertex $v \in V(H)$ with $od_{D^*}(v) \geq k$: We observe first that, since $\deg_H u \geq 2k$ for each $u \in V(H)$, we have $2q(H) = \Sigma_{u \in V(H)} \deg_H u \geq 2k.p(H)$, i.e., $q(H) \geq k.p(H)$. Now, if every vertex u of H satisfies $od_{D^*}(u) \leq k - 1$, then $q(H) \leq q(D^*) = \Sigma_{u \in V(H)} od_{D^*}(u) \leq (k - 1).p(H) < k.p(H)$, a contradiction. Hence, there exists $v \in V(H)$ with $od_{D^*}(v) \geq k$.

Next, let $A = \{a_1, a_2, \ldots, a_k\}$ be a k-subset of the out-neighbourhood of v in D^*. Thus, for each $i \in \{1, 2, \ldots, k\}$, there exists $x_i \in V(G)$ such that $\{v, x_i\} \mapsto G-a_i$. That the vertices $x_1, x_2, \ldots, x_k$ are distinct may be seen as follows: Suppose there exist $i, j \in \{1, 2, \ldots, k\}$, $i \neq j$, such that $x_i = x_j$. Now, $\{v, x_i\} \mapsto G-a_i$ implies va_i, $x_ia_i \notin E(G)$, and $\{v, x_i\} \mapsto a_j$; on the other hand, $\{v, x_j\} \mapsto G-a_j$ implies $\{v, x_j\} = \{v, x_i\} \not\mapsto \{a_j\}$. This contradiction establishes our claim. Note that, by Lemma 2.4.3, $x_i \notin S_k$, for each $i = 1, 2, \ldots, k$. Now, let B be any set of k vertices in $S_k - (A \cup \{v\})$ which are non-adjacent to v in G. (We know $\deg_H v \geq 2k$ (where $H \subset \bar{G}$), so at least 2k vertices of $S_k - \{v\}$ are non-adjacent to v in G; the set A accounts for k of these at least 2k vertices, so such a set B exists.) Let $b \in B$ and $i \in \{1, 2, \ldots, k\}$. Now, $\{v, x_i\} \mapsto G-a_i$, $b \neq a_i$ (since $b \notin A$), $vb \notin E(G)$, and $b \neq x_i$ (since $B \subseteq S_k$ and $x_i \notin S_k$), so we must have $x_ib \in E(G)$; i.e., each element of B is adjacent in G to each of the k distinct vertices x_i ($i \in \{1, 2, \ldots, k\}$). So, since each element b of B lies in S_k and hence has degree at most k, we must have $N(b) \cap S_k = \emptyset$, and thus, in particular, $[B, \{v, a_i\}] = \emptyset$, for every $i \in \{1, 2, \ldots, k\}$. Thus, for any $i \in \{1, 2, \ldots, k\}$, $B \cup \{v,a_i\}$ is an independent set of $k + 2$ vertices in $\langle S_k \rangle_G$ $(=\bar{H})$. However, this contradicts Lemma 2.4.4. Hence, no $k \geq 2$ with $s_k \geq 3k + 1$ exists. □

We note that, for $k = 1$, K_3^+ is a 3-edge-critical graph for which $s_k = 3k$ and so the above bound is best possible for $k = 1$. For larger values of k, no example has been found for which $s_k = 3k$ and, for large values of p (relative to k), the bound can certainly be improved.

2.4.6 Theorem: If G is a connected 3-edge-critical graph of order p and $k \in \{1, 2, \ldots, p - 1\}$ is such that $p > 6k^2 + 3k$, then $s_k(G) \leq k + 1$.

Proof: Let G be a connected, 3-edge-critical graph of order p, and let $k \in \{1, 2, \ldots, p - 1\}$ be such that $p > 6k^2 + 3k$. Denote $S_k(G)$ and $s_k(G)$ by S_k and s_k, respectively. Let $W = \{v \in V(G); N(v) \cap S_k = \emptyset\}$, and $M = V(G) - (S_k \cup W)$. So, M consists of all vertices of degree at least $k + 1$ which are adjacent to at least one vertex of S_k.

By Theorem 2.4.5, $s_k \leq 3k$. We will now show that $W \neq \emptyset$. Suppose, to the contrary, that W is empty. This means that every vertex $x \in V(G)$ has at least one neighbour x' belonging to S_k. So, $|[S_k, V(G) - S_k]| \geq |V(G) - S_k| \geq p - 3k$. Since

$$|[S_k,V(G)-S_k]| \leq |[S_k,V(G)]| = \sum_{u \in S_k} \deg_G u \leq k.3k = 3k^2,$$

we have $p - 3k \leq 3k^2$; so, $p \leq 3k^2 + 3k < 6k^2 + 3k$, contrary to assumption. So, W is indeed non-empty.

Since G is connected, there must exist $a \in S_k$ such that a is adjacent to some vertex not in S_k. Now, if $|N(a) \cap S_k| \geq s_k - 2$, then, since (by definition) a is adjacent to at least one vertex not in S_k, we see that $k \geq \deg a \geq |N(a) \cap S_k| + 1 \geq (s_k - 2) + 1 = s_k - 1$, whence we have $s_k \leq k + 1$, as required. So, we assume now that $|N(a) \cap S_k| \leq s_k - 3$. Therefore, a is non-adjacent to at least two vertices of S_k other than itself. Let $r = |N(a) \cap S_k|$, let $T = S_k - N[a]$, and $t = |T|$. Then, $r + t = s_k - 1$ and $t \geq 2$.

Now, for each $b \in T$ and $x \in W$, $xb \notin E(G)$ and so, for some $y \in V(G)$, $\{b, y\} \mapsto G-x$ or $\{x, y\} \mapsto G-b$. Let $C = \{x \in V(G);\ x \in W$ and for some $b, y \in V(G)$ with $b \in T$, $\{b,y\} \mapsto G-x\}$. We claim that $W - C \neq \varnothing$, i.e., we claim that there exists $x \in W$ such that, for every $b \in T$ and $y \in V(G)$, $\{b, y\} \not\mapsto G-x$ (and so $\{x, y\} \mapsto G-b$). Let $x \in C$. Then, for some $b, y \in V(G)$ with $b \in T$, we have $\{b, y\} \mapsto G-x$. Now, by definition of T, b is not adjacent to a, so it follows that $\{y\} \mapsto \{a\}$. Thus, $y \notin W$. Also, because p(G) is assumed to be large, we have $W \neq \{x\}$: Suppose, to the contrary, that $|W| = 1$. Then (by the definition of W), every vertex w of $V(G) - \{x\}$ has at least one neighbour w' belonging to S_k. So, $|[S_k, V(G) - (S_k \cup \{x\})]| \geq |V(G) - (S_k \cup \{x\})| \geq p - 3k - 1$. Since

$$|[S_k,V(G)-(S_k \cup \{x\}]| \leq |[S_k,V(G)]| = \sum_{u \in S_k} \deg_G u \leq 3k^2,$$

we have $p - 3k - 1 \leq 3k^2$; so $p \leq 3k^2 + 3k + 1 < 6k^2 + 3k$, contrary to assumption. So, $W \neq \{x\}$; say, $(x \neq) z \in W$. Since $\{b, y\} \mapsto G-x$, we have, in particular, that $\{b, y\} \mapsto \{z\}$. Since $z \in W$ and $b \in S_k$, $bz \notin E(G)$, so $yz \in E(G)$, which implies $y \notin S_k$. Hence, $y \in M$, and $\{y\} \mapsto W - \{x\}$. Thus, for a fixed $b \in T$ and any $x \in C$, there exists $y_x \in M$ with $N(y_x) \cap W = W - \{x\}$. Furthermore, if $x \neq x'$, then $y_x \neq y_{x'}$. (For our fixed b, if $\{b, y_x\} \mapsto G-x$, then we must have $y_x x' \in E(G)$, while we have $y_{x'}x' \notin E(G)$ from $\{b, y_{x'}\} \mapsto G-x'$.) So, $|C| \leq |M|$. Now, by the definition of W, every vertex in M is adjacent to a vertex of S_k. So, since $|N(S_k)| \leq |S_k|.k = s_k.k \leq 3k^2$, we have $|C| \leq |M| \leq 3k^2$. We claim that $|W| > 3k^2$: Suppose, to the contrary, that $|W| \leq 3k^2$. Every vertex s of $V(G) - W$

($\neq \emptyset$, since $p - |W| \geq 3k^2 + 3k > 0$) has at least one neighbour s' belonging to S_k. So, $|[S_k, V(G) - (S_k \cup W)]| \geq |V(G) - (S_k \cup W)| \geq p - (3k + 3k^2)$. Since

$$|[S_k, V(G) - (S_k \cup W)]| \leq |[S_k, V(G)]| = \sum_{u \in S_k} \deg_G u \leq 3k^2,$$

we have $p - 3k - 3k^2 \leq 3k^2$; so $p \leq 6k^2 + 3k$, contrary to assumption. Hence, $|W| > 3k^2 \geq |C|$, and we have $C \neq W$.

Now, let $z \in W - C$. For each $b \in T$, there exists y_b with $\{z, y_b\} \mapsto G-b$ (since $z \notin C$). Thus (since $[\{z\}, S_k] = \emptyset$), $\{y_b\} \mapsto S_k - \{b\}$. Recall that a is non-adjacent to at least two other elements of S_k. Hence, $y_b \neq a$ for every $b \in T$. So, since $\{z, y_b\} \mapsto \{a\}$, it must be that $y_b \in N(a) \cap S_k \subseteq S - T$, or $y_b \in N(a) \cap M$. Let $R = \{y_b; y_b \in M\}$, and $U = \{y_b; y_b \in S_k\}$. (So, if $Y = \{y_b; \{z, y_b\} - G-b$ for some $b \in T\}$, then $Y = R \cup U$.)

We first observe that if $b, b' \in T$ with $b \neq b'$, then $y_b \neq y_{b'}$ (since y_b is adjacent to b' while $y_{b'}$ is not adjacent to b'). Now, let $b \in T$ such that $y_b \in U$; then z is not adjacent to y_b and so there exists $w_b \in V(G)$ such that $\{z, w_b\} \mapsto G-y_b$, or $\{y_b, w_b\} \mapsto G-z$.

Suppose $\{y_b, w_b\} \mapsto G-z$. Using the fact that $p > 6k^2 + 3k$, we showed that $|W| \geq 2$; say, $(z\neq)x \in W$. Since $\{y_b, w_b\} \mapsto G-z$, $\{y_b, w_b\} \mapsto \{x\}$. Since $x \in W$ and $y_b \in U \subseteq S_k$, $y_bx \notin E(G)$; so, $w_bx \in E(G)$, which implies $w_b \notin S_k$. But now, since $\{z, y_b\} \mapsto G-b$, while w_b is not adjacent to z, and since $b \neq w_b$ ($w_b \notin S_k$), we must have w_b adjacent to y_b. Further, if $|N(y_b) \cap S_k| \geq s_k - 2$, then, since $y_bw_b \in E(G)$ and $w_b \notin S_k$, we have (as before) that $s_k \leq k + 1$, as required; so we assume now that $|N(y_b) \cap S_k| \leq s_k - 3$. However, $N(y_b) \cap S_k = S_k - \{b, y_b\}$ (see above), which is a contradiction since $|S_k - \{b, y_b\}| = s_k - 2$. So, the case $\{y_b, w_b\} \mapsto G-z$ does not occur.

Therefore, for each $y_b \in U$, there exists $w_b \in V(G)$ with $\{z, w_b\} \mapsto G-y_b$. However, if $w_b \in S_k$, then, since $w_by_b \notin E(G)$ and $\{z, y_b\} \mapsto G-b$, it follows that $w_b = b$ (if $w_b \neq b$, then $y_bw_b \notin E(G)$ implies (by $\{z, y_b\} \mapsto G-b$) that $zw_b \in E(G)$; however, this is impossible since $z \in W$ and $w_b \in S_k$). However, $w_b = b$ is not possible, since $\{z, b\} \not\mapsto \{a\}$ (while $\{z, w_b\} \mapsto G-y_b$). Thus, it follows that $w_b \notin S_k$. Let $L = \{w_b; y_b \in U\}$. Notice that if b and b' are distinct elements of T with $y_b, y_{b'} \in U$ (and so $y_b \neq y_{b'}$), then $w_b \neq w_{b'}$ since w_b is not adjacent to y_b ($\{z, w_b\} \mapsto G-y_b$, after all), while $w_{b'}$ is adjacent to y_b. (So, $|L| = |U|$.) Also, if $w_b \in L$, then, since w_b is adjacent to every element of T (this is so because $[\{z\}, S_k] = \emptyset$ and $\{z, w_b\} \mapsto G-y_b$),

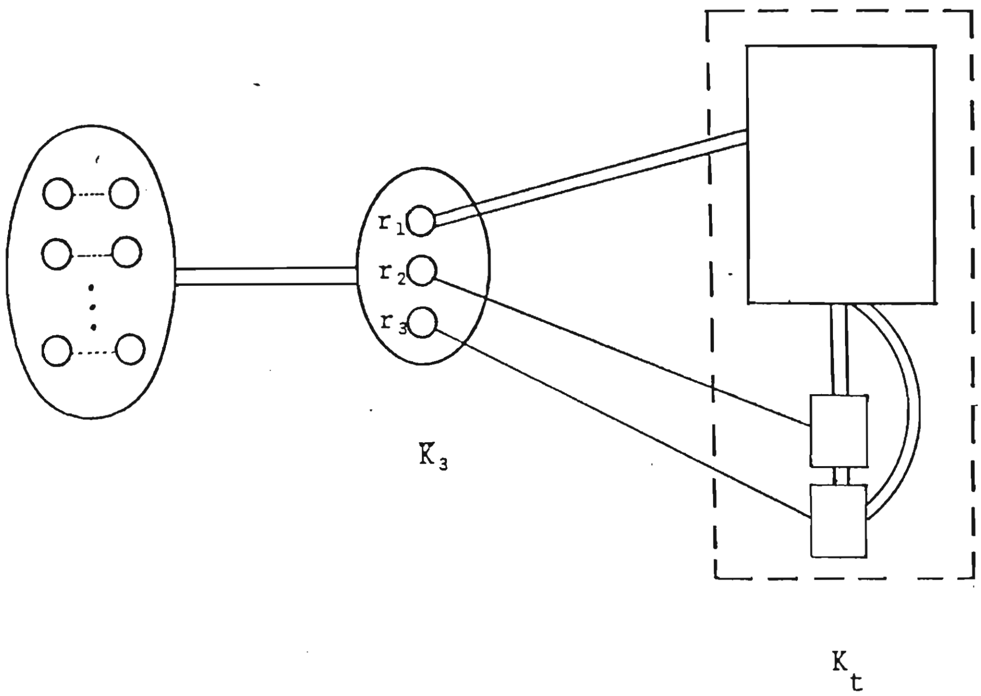


Fig. 2.4.2

we have $w_b \notin R$. (If $w_b \in R$, then by definition of R), w_b satisfies $\{z, w_b\} \mapsto G-b$, so that, in particular, $w_b b \notin E(G)$, which is a contradiction since $b \in T$.) So, $L \cap R = \varnothing$. Also, $R \cap S_k = \varnothing$ since $R \subseteq M = V(G) - (S_k \cup W)$. Finally, $L \cap S_k = \varnothing$ since $L = \{w_b; y_b \in U\}$ and every w_b with $y_b \in U$ satisfies $w_b \notin S_k$ (see earlier in this paragraph). So, L, R, and S_k are pairwise disjoint.

Now, let B_1 be the set of vertices b in T that give rise to a vertex $y_b \in M$, i.e., $y_b \in R$. Recall that $b \neq b'$ implies $y_b \neq y_{b'}$; so $|B_1| = |R|$. Also, the vertices b in $B_2 = T - B_1$ give rise to (distinct) vertices y_b in U, so $|U| = |B_2|$, and, of course, $|B_1| + |B_2| = |T|$, so that $|R| + |L| = |R| + |U| = |T|$.

Next, we recall that, for every $y_b \in R$, $\{z, y_b\} \mapsto G-b$, whence $ay_b \in E(G)$, and for every $w_b \in L$, $\{z, w_b\} \mapsto G-y_b$, whence $aw_b \in E(G)$. So, a is adjacent to every element of $L \cup R$. Hence,

$$\begin{aligned} k &\geq \deg a \\ &\geq |(N(a) \cap S_k) \cup R \cup L| \\ &= |N(a) \cap S_k| + |R| + |L| \\ &= |N(a) \cap S_k| + |T| \\ &= |S_k| - |\{a\}| \\ &= s_k - 1, \end{aligned}$$

whence we obtain $s_k \leq k + 1$, as desired. □

We consider now whether the result of Theorem 2.4.6 is best possible or not. It is easy to see that s_1 can be two for arbitrarily large graphs (see, for example, Proposition 2.2.28). We begin by considering the following class of 3-edge-critical graphs.

2.4.7 <u>Definition</u>: Let $k, t, a, b, c \in \mathbb{N}$ be given, where $a + b + c = t$, and let $G_1 \cong K_{2k}$, $G_2 \cong K_t$, F a 1-factor of G_1, and $S \cong \overline{K}_3$, with $V(S) = \{r_1, r_2, r_3\}$. Let G(k,t,a,b,c) be the graph obtained from $((G_1\text{-}F) + S) \cup G_2$ by partitioning $V(G_2)$ into subsets A, B, and C with $|A| = a$, $|B| = b$, and $|C| = c$, and joining r_1, r_2, and r_3 to all vertices in A, B, and C, respectively (see Fig. 2.4.2).

2.4.8 Proposition: The graph $G(k,t,a,b,c)$ defined in 2.4.7 is 3-edge-critical.

Proof: Let k, t, a, b, $c \in \mathbb{N}$ be given, where $a + b + c = t$, and let G_1, G_2, F, S, and $G(k,t,a,b,c)$ be as defined above. Denote $G(k,t,a,b,c)$ by G. Certainly, $\gamma(G) \leq 3$ since $\{r_1, r_2, r_3\} \mapsto G$. No vertex of G has degree $p(G) - 1$, so $\gamma(G) \geq 2$. Let D be a minimum dominating set of G, and suppose $|D| = 2$. Since $N[r_1] \cap V(G_2)$, $N[r_2] \cap V(G_2)$, and $N[r_3] \cap V(G_2)$ are pairwise disjoint, we must have $D \cap V(G_1) \neq \emptyset$ (since $D \mapsto S$). If $|D \cap V(G_1)| = 1$ - say $D \cap V(G_1) = \{y\}$ - then some $r \in V(S)$ belongs to D, so that the vertex of G_1 not dominated by y is dominated by D. However, then $D \not\mapsto V(G_2) - N(r)$. If $|D \cap V(G_1)| = 2$, then $D \not\mapsto V(G_2)$. So, we must have $\gamma(G) \geq 3$. Hence, $|D| = 3$ and $\gamma(G) = 3$.

Let $uv \in E(\overline{G})$. If $u, v \in V(G_1)$, then $\{u, y\} \mapsto G$ for any $y \in V(G_2)$. If $u \in V(G_1)$ and $v \in V(G_2)$, then $\{u', v\} \mapsto G$, where u' is the unique vertex of G_1 distinct from u that is non-adjacent to u. If $uv \in E(\overline{S})$, then $\{u, y\} \mapsto G$ for any $y \in N(S - \{u, v\}) \cap V(G_2)$. If $u \in V(S)$ and $v \in V(G_2) - N(u)$, say $v \in N(w)$, where $w \in V(S) - \{u\}$, then $\{v, y\} \mapsto G$ where $y \in V(S) - \{u, w\}$. So, $G(k,t,a,b,c)$ is indeed 3-edge-critical. □

2.4.9 Remark: Consider the 3-edge-critical graphs $G \cong G(m,t,1,1,t-2)$, where $m \in \mathbb{N}$ and $t \geq 2m + 2$. Then, the $2m + 2$ vertices in $V(G_1) \cup V(S) - \{r_3\}$ have degree $2m + 1$, $\deg r_1 = \deg r_2 = 2m + 1$ and the t vertices in $V(G_2)$ have degree $t \geq 2m + 2$. Consequently, for $k = 2m + 1$, $s_k = k + 1$. So, for every odd value of $k \geq 3$, the bound in Theorem 2.4.6 is attained by an infinite class of 3-edge-critical graphs. That the bound in Theorem 2.4.6 is not best possible for $k = 2$ is shown in Theorem 2.4.11. First, however, we need the following lemma.

(Note that it is stated erroneously in [W1] that a proof of the following lemma appears in [S1].)

2.4.10 Lemma: If v is a cut-vertex of a connected 3-edge-critical graph, then v is adjacent to an end-vertex of G.

Proof: Assume, to the contrary, that there exists a 3-edge-critical graph G with a cut-vertex v such that v is not adjacent to an end-vertex of G. Then, $k(G-v) = 2$ (by Lemma 2.3.1); say, $G-v = G_1 \cup G_2$, with $p(G_1)$, $p(G_2) \geq 2$. Notice that, if there exist $x \in V(G_1)$, $y \in V(G_2)$ with vx, $vy \in E(\overline{G})$, then (by Theorem 2.1.7) $d(x,y) \geq 4 > 3 \geq \operatorname{diam} G$, which is impossible. So, v is adjacent to every vertex in G_1 or every vertex in G_2; say, $V(G_2) \subseteq N(v)$. Since (by Proposition 2.1.3) deg $v \leq \Delta(G) \leq p(G) - 3$, we have that v is non-adjacent to at least two vertices, a_1 and a_2, in G_1.

We show next that G_2 is complete. Suppose there exist vertices $b_1, b_2 \in V(G_2)$ with $b_1b_2 \in E(\overline{G})$. Then, without loss of generality, we may assume that there exists a vertex $x \in V(G) - \{b_1, b_2\}$ such that $\{b_1, x\} \mapsto G-b_2$. Now, since v dominates b_2 (see previous paragraph), $x \neq v$. Furthermore, we clearly have $\{x\} \mapsto V(G_1)$. However, then $\{x, v\} \mapsto G$, which is impossible. So, G_2 is indeed complete.

Now, let $c \in N(v) \cap V(G_1)$, and $u \in V(G_2)$. Since $uc \in E(\overline{G})$, there exists $z \in V(G) - \{c, u\}$ such that $\{c, z\} \mapsto G-u$ or $\{u, z\} \mapsto G-c$. If $\{u, z\} \mapsto G-c$, then $z \neq v$ (since $\{v\} \mapsto \{c\}$) and $\{z\} \mapsto V(G_1) - \{c\}$. However, then $\{z, v\} \mapsto G$, a contradiction. So, $\{c, z\} \mapsto G-u$. Again, $z \neq v$ (since $\{v\} \mapsto \{u\}$). Now, $\{c, z\} \mapsto G-u$, $V(G_2) - \{u\} \neq \varnothing$ (since $p(G_2) \geq 2$), and $(V(G_2) - \{u\}) \cap N[c] = \varnothing$; hence, $\{z\} \mapsto V(G_2)$ and, in particular, $z \in V(G_2)$. However, this implies (since G_2 is complete) that $\{z\} \mapsto \{u\}$, a contradiction. Hence, no such 3-edge-critical graph with cut-vertex v exists, and the lemma follows. □

2.4.11 <u>Theorem</u>: If G is a connected 3-edge-critical graph of order $p > 30$, then $s_2 \leq 2$.

<u>Proof</u>: Suppose, to the contrary, that there exists a 3-edge-critical graph G of order $p \geq 31$ with $s_2 \geq 3$. Since $p > 30 = 6.2^2 + 3.2$, we have, by Theorem 2.4.6, that $s_2 \leq 3$. Suppose $S_2 = \{a, b, c\}$. Let $M = N(S_2)$ and $W = V(G) - (M \cup S_2)$.

We show first that $P_3 \not\subset \langle S_2 \rangle_G$. Suppose, to the contrary, that (say) $ab, cb \in E(G)$. Since G is connected, there exists $x \in M - S_2$ such that $ax \in E(G)$ or $cx \in E(G)$; assume, without loss of generality, that $ax \in E(G)$. Suppose $cx \in E(G)$; then deg a = deg b = deg c = 2, and x is a cut-vertex of G. Then, by Lemma 2.4.10, there exists a vertex, w say, of degree 1 with $xw \in E(G)$. By the definition of S_2, $w \in \{a, b, c\}$. However, this is not possible (since deg a = deg b = deg c = 2). So, $cx \notin E(G)$. Thus, there exists $y \in V(G)$ such that $\{c, y\} \mapsto G-x$ or $\{x, y\} \mapsto G-c$. Suppose $\{c, y\} \mapsto G-x$. Then, since $a \notin N[c]$, we have $y \in N[a] = \{a, b, x\}$. Since $\{y\} \not\mapsto \{x\}$, we have $y = b$. However, $|N[\{c, b\}]| \leq 4 < p - 1$, contrary to $\{c, b\} \mapsto G-x$. So, $\{x, y\} \mapsto G-c$. Since $b \notin N[x]$, we have $y = a$. However, then $\{b, x\} \mapsto G$, contrary to $\gamma(G) = 3$. Hence, our assumption that $P_3 \subset \langle S_2 \rangle_G$ is false. We shall show next that S_2 is independent.

Suppose, to the contrary, that $ab \in E(G)$ (and, hence $ac, cb \notin E(G)$). Since G is connected, we assume, without loss of generality, that deg $a = 2$; let $N(a) = \{x, b\}$. Suppose $xb \in E(G)$. Then, x is a cut-vertex of G and (by Lemma 2.4.10) c is an end-vertex of G adjacent to x. Hence, there is $r \in V(G) - \{b, c\}$ with $\{b, r\} \mapsto G-c$ or $\{c, r\} \mapsto G-b$; in either case, $r \neq x$. However, $N[b] \cup N[c] \subset N[x]$ and $\{x\} \mapsto \{b, c\}$; so, $\{r, x\} \mapsto G$, which is not possible. So, $xb \notin E(G)$.

We show next that deg $b = 2$. Suppose, to the contrary, that b is an end-vertex of G. Then, x is a cut-vertex of G and, by Lemma 2.3.1, $G-x$ has exactly two components, one of which is trivial. However, the components of $G-x$ in this case are $\langle\{a, b\}\rangle$ and $G-\{a, b, x\}$, neither of which is trivial. So, deg $b = 2$; say $N(b) = \{a, y\}$. If $y = x$, then (as above) a contradiction arises; so, $y \neq x$ and $ay \notin E(G)$.

Let $W = V(G) - N[\{a, b, c\}]$; then $W \neq \varnothing$. Let $w \in V(G) - W$. Since $cw \notin E(G)$, there exists $s \in V(G) - \{c, w\}$ with $\{c, s\} \mapsto G-w$ or $\{w, s\} \mapsto G-c$. In either case, $\{s\} \mapsto \{a, b\}$, which implies $s \in \{a, b\}$. However, then $\{c, s\} \not\mapsto G-w$ (because $|N[\{a, b, c\}]| \leq 7 < p - 1$); so, $\{w, s\} \mapsto G-c$. (Hence (since w is an arbitrary vertex in W), $\langle W\rangle$ is complete.) Furthermore, if $s = a$ ($s = b$), then $\{w, s\} \mapsto G-c$ implies $\{a\} \not\mapsto \{y\}$ ($\{b\} \not\mapsto \{x\}$), whence $\{w\} \mapsto \{y\} \subset N(b)$ ($\{w\} \mapsto \{x\} \subset N(a)$); so, every vertex in W is adjacent to a neighbour of a or a neighbour of b. Suppose, without loss of generality, that $s = a$, i.e., $\{w, a\} \mapsto G-c$. Then, $yw \in E(G)$. Also, $aw \notin E(G)$, so there exists $t \in V(G) - \{a, w\}$ such that $\{a, t\} \mapsto G-w$ or $\{w, t\} \mapsto G-a$. If $\{a, t\} \mapsto G-w$, then $\{t\} \mapsto \{c\}$ and $t \notin W \cup \{y\}$ (since $\{t\} \not\mapsto \{w\}$); however, $\{t\} \mapsto W - \{w\}$. If $t = x$, then $\{x\} \mapsto G-\{b, w\}$ and so $\{x, y\} \mapsto G$, which is not possible. So, $t \neq x$. Furthermore, $t \notin \{b, c\}$, since $|N[\{a, b, c\}]| \leq 7 < p - 1$. But (by $\{a, w\} \mapsto G-c$), $\{w\} \mapsto G-(N[a] \cup \{c\})$; so, in particular, $\{w\} \mapsto \{t\}$, which is a contradiction. Hence, $\{a, t\} \not\mapsto G-w$ and we must have $\{w, t\} \mapsto G-a$. Clearly, $\{t\} \mapsto \{b, c\}$, whence $t = y$, and thus $yc \in E(G)$. So (by the properties of the elements of W proved above), $\{x, y\} \mapsto V(G) - (N(c) - \{y\})$ (where, we recall, $|N(c)| \leq 2$). Since $\gamma(G) = 3$, $V(G) - (N(c) - \{y\}) \neq V(G)$, i.e., c has (exactly) one neighbour z ($\{z\} = N(c) - \{y\}$)which is not dominated by $\{x, y\}$. In particular, $xc \notin E(G)$ (otherwise, $\{x, y\} \mapsto G$), so there exists $f \in V(G) - \{x, c\}$ such that $\{x, f\} \mapsto G-c$ or $\{c, f\} \mapsto G-x$. If $\{x, f\} \mapsto G-c$, then $\{f\} \mapsto \{b, z\}$, which is not possible since $za, zy \in E(\bar{G})$; so, $\{c, f\} \mapsto G-x$. Then, since $N[c] = \{c, y, z\}$, we have $\{f\} \mapsto \{a, b\}$; but, $\{f\} \not\mapsto \{x\}$. Thus, $f = b$, and $\{b, c\} \mapsto G-x$. However, this is not possible since $|N[\{b, c\}]| = 5 < p - 1$. So, S_2 is indeed independent, as claimed.

Let $C_a = \{x \in W;$ for some $y \in V(G), \{a, y\} \mapsto G-x\}$. Note that $|W| = p - |M| - |S_2| \geq 31 - 6 - 3 = 22$. We claim that $W - C_a \neq \varnothing$. Let $x \in W$; then, there exists $y \in V(G)$ such that $\{a, y\} \mapsto G-x$ or $\{x, y\} \mapsto G-a$. Suppose $\{a, y\} \mapsto G-x$. Since $[\{a\}, S_2] = \varnothing$, yb, yc $\in E(G)$ and so $y \in M$. Also, we note that y is adjacent to all vertices in $W - \{x\}$. Thus, for any $x \in C_a$, there exists $y_x \in M$ with $N(y_x) \cap W = W - \{x\}$. So, for $x, x' \in W$, $x \neq x'$ implies $y_x \neq y_{x'}$. Thus, $|C_a| \leq |M| \leq 6 < |W|$, and $W - C_a \neq \varnothing$. Hence, there exists $x \in W - C_a \subseteq W$ such that, for some $y \in M$, $\{x, y\} \mapsto G-a$. Similarly, we may assume that there is some $r \in W$ and $s \in M$ such that $\{r, s\} \mapsto G-b$, and some $u \in W$ and $v \in M$ such that $\{v, u\} \mapsto G-c$. But, then, we have $\{s, v\} \subseteq N(a)$, $\{v, y\} \subseteq N(b)$, $\{y, s\} \subseteq N(c)$. However, s, v, and y are all distinct. (To see this, note that, for example, by $\{r, s\} \mapsto G-b$, we have sa, sc $\in E(G)$ and sb $\notin E(G)$, and by $\{v, u\} \mapsto G-c$, we have va, vb $\in E(G)$ and vc $\notin E(G)$, so that v and s cannot be the same vertex.) Hence, $N(a) = \{s, v\}$, $N(b) = \{v, y\}$, and $N(c) = \{y, s\}$.

But now, since ab $\notin E(G)$, we may assume that there exists $w \in V(G)$ such that (without loss of generality) $\{a, w\} \mapsto G-b$. So, $w \in N(c)$; by the previous paragraph, this implies $w = y$ or $w = s$. If $w = y$, then yb $\in E(G)$, contrary to $\{a, w\} \mapsto G-b$; so, $w = s$. However, then $\{s, v\} \mapsto G$, which is impossible. Hence, no such 3-edge-critical graph G exists, and the theorem follows. □

So, although the bound $s_k \leq k + 1$ has been shown to be best possible for odd values of k (see 2.4.9), the question remains as to whether the bound is best possible for even values of $k \geq 4$.

2.5 END-VERTICES OF 3-EDGE-CRITICAL GRAPHS

2.5.1 Remark: We begin by recalling from Theorem 2.4.5 that, for a 3-edge-critical graph G, $s_1(G)$ (the number of end-vertices of G) satisfies $s_1(G) \leq 3$. By Remark 2.4.9, we know that this result is best possible, while Proposition 2.5.2 will show that this upper bound is attained for exactly one 3-edge-critical graph, namely, one of order 6, whence it follows that all 3-edge-critical graphs of order more than 6 have at most two vertices of degree one.

The following result is quoted in [W1] where the reader is referred to [S1] for a proof; however, no such proof is provided in [S1].

2.5.2 Proposition: If a 3-edge-critical graph G has three end-vertices, then $G \cong H(1,1,1)$.

Proof: Let G be a 3-edge-critical graph with three end-vertices, a, b, and c. Let $\{a_1\} = N(a)$, $\{b_1\} = N(b)$, and $\{c_1\} = N(c)$. By Lemma 2.1.5, the vertices a_1, b_1, and c_1 are distinct. So, $p(G) \geq 6$. We claim that $\langle\{a_1, b_1, c_1\}\rangle_G \cong K_3$. Suppose, to the contrary, that (say) $a_1b_1 \in E(\overline{G})$. Then, we may assume, without loss of generality, that there exists $x \in V(G) - \{a_1, b_1\}$ such that $\{a_1, x\} \mapsto G-b_1$. As $\{a_1\} \not\mapsto \{b\}, \{c\}$, we have $\{x\} \mapsto \{b, c\}$; however, this is impossible by Lemma 2.1.5. So, $\langle\{a_1, b_1, c_1\}\rangle_G$ is complete.

We show now that $p(G) = 6$, whence the desired result will follow. Suppose there exists $y \in V(G) - \{a, b, c, a_1, b_1, c_1\}$. Then, $ya \in E(\overline{G})$, and there exists $z \in V(G) - \{a, y\}$ such that $\{a, z\} \mapsto G-y$ or $\{y, z\} \mapsto G-a$. Suppose $\{a, z\} \mapsto G-y$. Since $N[a] = \{a, a_1\}$, we have $\{z\} \mapsto V(G) - \{a, a_1, y\}$; in particular, $\{z\} \mapsto \{b, c\}$. However, this is not possible (by Lemma 2.1.5, again). So, $\{y, z\} \mapsto G-a$. Since $\{y, z\} \mapsto \{b, c\}$ and b, c are end-vertices of G, it follows that (say) $y \in \{b, b_1\}$ and $z \in \{c, c_1\}$. However, this is contrary to our choice of y. So, $p(G) = 6$, and $G \cong H(1,1,1)$, as desired. □

We shall use the following result in Theorem 2.6.2.

2.5.3 Theorem: Let G be a connected 3-edge-critical graph and let A be the set of all end-vertices of G. Then, G-A is 2-connected.

Proof: Assume, to the contrary, that there exists a 3-edge-critical graph G such that $\kappa(G-A) = 1$, where $A = \{v \in V(G); \deg v = 1\}$. Let x be a cut-vertex of G-A. We show first that x is a cut-vertex of G. Suppose, to the contrary, that G-x is connected. Let Y and W be distinct components of G-A-x, and let $y \in Y$ and $w \in W$. Since, by assumption, G-x is connected, there is a y-w path P in G-x. However, since $[Y, W]_{G-A-x} = \varnothing$, it follows that there exists $z \in A$ such that z is an (internal) vertex of P. However, then $\deg_G z \geq 2$, which contradicts the definition of A. So, x is indeed a cut-vertex of G.

By Lemma 2.3.1, G-x has (exactly) two components, say G_1 and G_2. By Lemma 2.4.10, one of these components, say G_2, consists of a single vertex, z say, which is thus an end-vertex of G. However, then (G-A)-x is connected (no set of end-vertices of a graph is a vertex cutset of the graph), which contradicts the fact that x is a cut-vertex of G-A. Hence, no such connected 3-edge-critical graph G exists, and the theorem follows. □

2.5.4 Proposition: Every connected 3-edge-critical graph with exactly two end-vertices has a hamiltonian path and, furthermore, has a cycle that contains all vertices of degree at least 2.

Proof: Let G be a connected 3-edge-critical graph of order p with exactly two end-vertices. By Proposition 2.2.28, $G \cong H(1,1,p\text{-}5)$, where (by Remark 2.5.1), $p \geq 7$. Let u and v be the end-vertices of G, and let w be the vertex of G of degree $p - 5$. Let $V(G) = \{u, v, w, u_1, u_2, \ldots, u_{p\text{-}5}, u_{p\text{-}4}, u_{p\text{-}3}\}$, and let $N(w) = \{u_2, u_3, \ldots, u_{p\text{-}4}\}$, $N(u) = \{u_1\}$, and $N(v) = \{u_{p\text{-}3}\}$. Then, $P{:}u,u_1,u_2,w,u_3,u_4,\ldots,u_{p\text{-}4},u_{p\text{-}3},v$ is a hamiltonian path of G. Moreover, $C{:}u_1,u_2,w,u_3,u_4,\ldots,u_{p\text{-}4},u_{p\text{-}3},u_1$ is a cycle of G that contains all non-end-vertices of G. □

2.6 DOMINATING CYCLES IN 3-EDGE-CRITICAL GRAPHS

We next show that every connected 3-edge-critical graph has a dominating cycle. We shall need the following lemma.

2.6.1 Lemma: Every 3-edge-critical graph of order at least 7 contains three vertices u, v, x of degree at least two such that $\{u, x\} \rightarrowtail G-v$.

Proof: Let G be a connected 3-edge-critical graph of order $p \geq 7$. Since $p(G) \geq 7$, it follows (by Remark 2.5.1) that G has fewer than three end-vertices. Suppose $\delta(G) \geq 2$. Since G is 3-edge-critical, G is not complete; so, there exist vertices u and v in G with $uv \notin E(G)$. Therefore, there exists $x \in V(G)$ such that $\{u, x\} \rightarrowtail G-v$ or $\{v, x\} \rightarrowtail G-u$. Since G has no end-vertices, we have deg u, deg v, deg $x \geq 2$, and the lemma follows.

Suppose now that G has two end-vertices. Then, by Proposition 2.2.28, $G \cong H(1,1,p\text{-}5)$. Let $a, b, c \in V(G)$ with deg a = deg b = 1, deg $c = p - 5$, and let $\{a_1\} = N(a)$, $\{b_1\} = N(b)$. Clearly, $\{a_1, b_1\} \rightarrowtail G-c$, and deg a_1 = deg $b_1 = p - 3 \geq 4$, deg $c = p - 5 \geq 2$. So, $u = a_1$, $x = b_1$, and $v = c$ satisfy the statement of the lemma.

We assume now that G has exactly one end-vertex, say w. Let u be the neighbour of w. Obviously, deg $u \geq 2$. Since $\gamma(G) = 3$, there exists a vertex $v \in V(G)$ such that $uv \notin E(G)$; clearly, then, $v \neq w$, so deg $v \geq 2$. Then, there exists a vertex $x \in V(G)$ such that either $\{u, x\} \rightarrowtail G-v$ or $\{v, x\} \rightarrowtail G-u$. Suppose $x = w$. Since u is adjacent to $w(=x)$, $\{v, w\} \rightarrowtail G-u$

does not hold, so we must have $\{u, w\} \rightarrowtail G-v$. However, then it follows that $\{u, v\} \rightarrowtail G$, contrary to $\gamma(G) = 3$. So, $x \neq w$, whence $\deg x \geq 2$, and the lemma follows. $\square$

2.6.2 Theorem: If G is a connected 3-edge-critical graph, then G has a dominating cycle.

Proof: Let G be a connected 3-edge-critical graph of order p. If $p = 6$, then, by Proposition 2.2.28, $G \cong K_3^+$; the vertices of the 3-cycle of G clearly form a dominating set of G. Suppose, now, that $p \geq 7$, and let u, v, x be three distinct vertices of G whose existence is guaranteed by Lemma 2.6.1, i.e., each of u, v, x is of degree at least two and $\{u, x\} \rightarrowtail G-v$. So, if A is the set of all end-vertices of G, then $u, v, x \in V(G) - A$, where (by Theorem 2.5.3), $G-A$ is 2-connected. We recall that Whitney's Theorem states that a non-trivial graph F is n-connected ($n \in \mathbb{N}$) if and only if, for each pair r, s of distinct vertices of F, there are at least n internally disjoint r-s paths in F. Hence, since $\kappa(G-A) \geq 2$, it follows that $G-A$ has as subgraph a cycle C containing u and x. If v is adjacent to a vertex, w say, on this cycle, then the theorem follows, since $u, x, w \in V(C)$ and $\{u, x\} \rightarrowtail V(G) - \{v\}$; hence, $\{u, x, w\} \subseteq V(C) \rightarrowtail G$, and C is a dominating cycle of G. So, suppose now that v is not adjacent to any vertex on C.

Now, let a′, b′ be distinct vertices on C. Recall that if F is an n-connected graph ($n \in \mathbb{N}$) and y, $y_1, y_2, \ldots, y_n$ are $n + 1$ distinct vertices of F, then for $i = 1, 2, \ldots, n$, there exist internally disjoint y-y_i paths (cf. Theorem 5.7 in [CL1]). Consequently, $G-A$ has two internally disjoint v-a′, v-b′ paths P_1' and P_2', respectively. Let a be the first vertex of P_1' that belongs to C; b the first vertex of P_2' that belongs to C; and let P_1 and P_2 be the v-a, v-b subpaths of P_1' and P_2', respectively.

Suppose $C{:}v_0,v_1,v_2,\ldots,v_n,v_0$, for some $n \geq 3$; assume $u = v_0$, and let $i, j, k \in \{0, 1, \ldots, n-1, n\}$ such that $x = v_i$, $a = v_j$, and $b = v_k$, where $i \neq 0$ and where, possibly, $|\{0, i, j, k\}| \in \{2, 3\}$. Note, though, that $j \neq k$ (since P_1' and P_2' are internally disjoint).

Now, u and x divide C into two segments, namely, paths $Q_1{:}v_0,v_1,v_2,\ldots,v_i$ and $Q_2{:}v_i,v_{i+1},v_{i+1},\ldots,v_{n-1},v_n,v_0$.

Case 1: Suppose that $a(=v_j)$ and $b(=v_k)$ both lie on Q_1, or both lie on Q_2. Assume, without loss of generality, that $a, b \in V(Q_1)$ and that $k \geq j$. Then, $C^*{:}vP_1^{\rightarrow}a(=v_j)C^{\leftarrow}(v_k=)bP_2^{\leftarrow}v$ is a dominating cycle of $G-A$ (and hence of G) since $u, x, v \in V(C^*)$ and $\{u, x\} \rightarrowtail G-v$.

Case 2: Suppose that $a \in V(Q_r)$, $b \in V(Q_s)$ where $r, s \in \{1, 2\}$, $r \neq s$. Assume, without loss of generality, that $a \in V(Q_1)$, $b \in V(Q_2)$ and that $0 < j < i$ and $i < k \leq n$ (since instances where $j, k \in \{0, i\}$ have been dealt with in Case 1). Let t be the vertex on P_2 that follows v (by our assumption, $t \neq b$). Since $\{u, x\} \mapsto G-v$, either $\{u\} \mapsto \{t\}$ or $\{x\} \mapsto \{t\}$. Suppose $\{x\} \mapsto \{t\}$ (then $xt \in E(G)$). In this instance, $C^*:(x=)v_i,v_{i-1},\ldots,v_j,v_{j-1},\ldots,v_0(=u),v_n,v_{n-1},\ldots,v_{k-1},(v_k=)b\overleftarrow{P_2}t,v_i(=x)$ is a dominating cycle of G. If $\{u\} \mapsto \{t\}$, then $ut \in E(G)$, and $C^*:(b=)v_k,v_{k-1},\ldots,v_i(=x),v_{i-1},\ldots,v_{j+1},(v_j=)a\overrightarrow{P_1}v,t,v_0(=u),v_nv_{n-1},\ldots,v_{k+1},v_k(=b)$ is a dominating cycle of G. □

2.6.3 Corollary: If G is a connected 3-edge-critical graph, then G has a dominating path.

2.7 HAMILTONIAN PATHS IN 3-EDGE-CRITICAL GRAPHS

In [S1], Sumner conjectured that every 3-edge-critical graph of order exceeding 6 has a hamiltonian path. The conjecture was valid for a large collection of computer generated graphs studied by Sumner and was proved by Wojcicka in [W1]. The proof of the conjecture given below is an elaboration on that given in [W1], and is presented as a series of lemmata. Notation and definitions introduced will be retained without repetition throughout the proof.

2.7.1 Theorem: Every connected 3-edge-critical graph on more than 6 vertices has a hamiltonian path.

Proof: Suppose, to the contrary, that there exists a connected 3-edge-critical graph G of order at least 7 that contains no hamiltonian path. By Corollary 2.6.3, G contains a dominating path; let $P:(a=)x_1,x_2,\ldots,x_n(=b)$ $(n \geq 3)$ be a longest such dominating path. By our assumption, $V(P) \subset V(G)$. We shall establish the theorem by deriving a contradiction. We recall that, for any vertex $x \in V(P) - \{a, b\}$, x^+ and x^- will denote the successor and predecessor (respectively) of x on P; i.e., if $x = x_i$, then $x^- = x_{i-1}$ and $x^+ = x_{i+1}$. There exists $y \in V(G) - V(P)$. Let $Y = N(y) \cap V(P) = \{y_1, y_2, \ldots, y_k\}$ $(k \in \mathbb{N})$, ordered so that, if $i, j \in \{1, 2, \ldots, k\}$, $i < j$, then y_i precedes y_j on $a\overrightarrow{P}b$ ($Y \neq \emptyset$, as P is a dominating path). Now, suppose that $k > 1$ and that there exists $i \in \{1, \ldots, k-1\}$ with $y_i^+ = y_{i+1}$; suppose $y_i = x_j$. Then, $x_1,x_2,\ldots,x_j(=y_i),y,(y_{i+1}=)x_{j+1},x_{j+2},\ldots,x_n$ would be a path Q with V(P) a proper subset of V(Q), i.e., Q is a dominating path of G, where the length of Q is strictly greater than the length of P. This contradicts the fact that P is a longest dominating path in G. Hence, if $k > 1$, then, for all $i \in \{1, \ldots, k-1\}$, we have $y_i^+ \neq y_{i+1}$.

In order to complete the proof of the theorem, we next prove a few lemmas, using the terminology introduced above.

2.7.2 Lemma: If there exists a vertex in G of degree at least 2 that does not lie on P, then there exists a vertex $z \in V(G) - V(P)$ such that $|N(z) \cap V(P)| \geq 2$.

Proof: Suppose there exists $z \in V(G) - V(P)$ with $\deg z \geq 2$, and assume, to the contrary, that every vertex in $V(G) - V(P)$ is adjacent to only one vertex in P. In particular, z is adjacent to only one vertex of P, say $N(z) \cap V(P) = \{x_i\}$ for some $i \in \{1, 2, \ldots, n\}$. By the maximality of P, we have $x_i \notin \{a, b\}$.

Suppose that x_i separates z from P, i.e., that $G-x_i$ is a disconnected graph in which z lies in a component distinct from every component containing vertices of P. Now, $|\{x_i\}| = 1$, so (by Lemma 2.3.1), $G-x_i$ has exactly two components (one containing z and the other containing the vertices of $P-x_i$); by Lemma 2.4.10, one of these components, say F, must contain a single vertex. If $V(P-x_i) \subseteq V(F)$, then $P-x_i = F$ and $x_i \in \{a, b\}$.; however, this contradicts our earlier remarks. So, $F = \langle\{z\}\rangle$, which implies $N_G(z) = \{x_i\}$, contradicting the fact that $\deg z \geq 2$. So, x_i does not separate z from P. Thus, there exists a path P_z:$z,x,w_1,w_2,\ldots,w_m(=x_k)$ from z to a vertex x_k, for some $k \in \{1, 2, \ldots, n\}$, $k \neq i$. We will assume further that the vertex z and its associated path P_z have been chosen so that P_z is as short as possible. Since we have assumed $|N(z) \cap V(P)| = 1$, $P_z \not\cong K_2$. So, x does not lie on P; but, since P is a dominating path, x must be adjacent to some vertex in P. If x is adjacent to x_i and x_i is the only vertex on P to which x is adjacent, then x together with the path $x,w_1,\ldots,w_m$ is a path Q from a vertex x not on P that has degree at least two in G and which satisfies $|N(x) \cap V(P)| = 1$, where Q is a path that is shorter than P_z; this contradicts our optimal choice of z and P_z. Thus, we may assume that x is adjacent to some vertex x_j in P, where $j \in \{1, 2, \ldots, n\}$, $j \neq i$. It follows, then, that P_z can be chosen to be the path z,x,x_j.

Note that the following hold: x_j is distinct from both x_i^+ and x_i^- (since, otherwise, $aP^{\rightarrow}x_i,z,x,x_i^+(=x_j)P^{\rightarrow}b$ or $aP^{\rightarrow}x_i^-(=x_j),x,z,x_iP^{\rightarrow}b$, respectively, is a longer dominating path in G than P); x is not adjacent to either of x_i^+ or x_i^- (since, otherwise, $aP^{\rightarrow}x_i,z,x,x_i^+P^{\rightarrow}b$ or $aP^{\rightarrow}x_i^-,x,z,x_iP^{\rightarrow}b$ is a longer dominating path in G); x is not adjacent to either of a or b (since otherwise $x,aP^{\rightarrow}b$ or

$aP^{\rightarrow}b,x$ is a longer dominating path in G). Also, $x \notin \{a, b\}$ since $x \notin V(P)$. We will assume, without loss of generality, that x_i precedes x_j on P; we note that this ensures $x_j^+ \neq b$.

Note that $ax_i^+ \notin E(G)$ (since, otherwise, $bP^{\leftarrow}x_i^+,aP^{\rightarrow}x_i,z$ is a longer dominating path than P), so there exists a vertex $w \in V(G) - \{a, x_i^+\}$ such that $\{w, a\} \mapsto G-x_i^+$ or $\{w, x_i^+\} \mapsto G-a$. Suppose $\{w, a\} \mapsto G-x_i^+$. Since, by our previous comments, $x \notin N[a]$, we have $\{w\} \mapsto \{x\}$. Furthermore, $a \neq z$ and $za \notin E(G)$ (since, otherwise, $z,aP^{\rightarrow}b$ is a longer dominating path in G than P), so, $\{w\} \mapsto \{z\}$. Suppose $\{w, x_i^+\} \mapsto G-a$. Again, by our previous comments, $x \notin N[x_i^+]$, so $\{w\} \mapsto \{x\}$, and $x_i^+z \notin E(G)$ (since $|N(z) \cap V(P)| = 1$), so $\{w\} \mapsto \{z\}$. So, in either case, $\{w\} \mapsto \{z, x\}$. Furthermore, $ab \notin E(G)$ (otherwise, $x_i^+P^{\rightarrow}b,aP^{\rightarrow}x_i,z$ would be a longer dominating path than P) and $x_i^+b \notin E(G)$ (otherwise, $aP^{\rightarrow}x_i,z,x_j,x_jP^{\rightarrow}b,x_i^+P^{\rightarrow}x_j^-$ would be a longer dominating path than P). Hence (by $\{w, a\} \mapsto G-x_i^+$ or $\{w, x_i^+\} \mapsto G-a$), $\{w\} \mapsto \{b\}$. So, w dominates $\{z, x, b\}$. However, if $w = x$, then $xb \in E(G)$, which (we showed) is impossible, and if $w = z$ or $w = b$, then $zb \in E(G)$, which is also contrary to what we have proved before. So, $w \notin \{z, x, b\}$. In particular, $wz \in E(G)$, whence $w \in V(P)$ (otherwise, $aP^{\rightarrow}b,w,y$ is a longer dominating path in G than P). But, $N(z) \cap V(P) = \{x_i\}$; so, $w = x_i$. However, then, since w is adjacent to x, we have that x is a vertex in G not lying on P which satisfies $|N(x) \cap V(P)| \geq |\{x_j, x_i(=w)\}| = 2$, which contradicts our original assumption. □

2.7.3 <u>Lemma</u>: Suppose that $|Y| = k \geq 2$. Then,

(1) For all $i \in \{1, \ldots, k\}$, (a) $ay_i^+ \notin E(G)$, (b) $by_i^- \notin E(G)$.

(2) For $i,j \in \{1, \ldots, k\}$, $i \neq j$ (a) $y_i^+y_j^+ \notin E(G)$ (b) $y_i^-y_j^- \notin E(G)$ (for $k \geq 2$).

(3) (a) For all $i \in \{2, \ldots, k\}$, $ay_i^- \notin E(G)$ (for $k \geq 2$).

(b) For all $i \in \{1, \ldots, k-1\}$, $by_i^+ \notin E(G)$ (for $k \geq 2$).

(4) If $\{y_i\} \mapsto \{a\}$ or $\{y_i\} \mapsto \{b\}$, then $y_i^+y_i^- \notin E(G)$.

<u>Proof</u>: The lemma follows from the maximality of P:

(1a) If $i \in \{1, \ldots, k\}$ satisfies $ay_i^+ \in E(G)$, then $y,y_iP^{\leftarrow}a,y_i^+P^{\rightarrow}b$ is a longer dominating path than P.

(b) If $i \in \{1, \ldots, k\}$ satisfies $by_i^- \in E(G)$, then $y,y_iP^{\rightarrow}b,y_i^-P^{\leftarrow}a$ is a longer dominating path than P.

(2a) If there exist $i, j \in \{1, \ldots, k\}$, $i < j$, such that $y_i^+y_j^+ \in E(G)$, then $aP^{\rightarrow}y_i,y,y_jP^{\leftarrow}y_i^+,y_j^+P^{\rightarrow}b$ is a longer dominating path than P. Similarly, (2b) holds.

(3a) If there is an $i \in \{2, \ldots, k\}$, such that $ay_i^- \in E(G)$, then $y_1^+P^{\rightarrow}y_i^-,aP^{\rightarrow}y_1,y,y_iP^{\rightarrow}b$ is a longer dominating path than P.

(b) If there is an $i \in \{1, \ldots, k-1\}$ such that $by_i^+ \in E(G)$, then $aP^{\rightarrow}y_i, y, y_k P^{\rightarrow} b, y_i^+ P^{\rightarrow} y_K^-$ is a longer dominating path in G than P.

(4) Suppose that there exists $i \in \{1, \ldots, k\}$ such that $by_i \in E(G)$, but for which $y_i^+ y_i^- \in E(G)$. Then, $aP^{\rightarrow}y_i^- y_i^+ P^{\rightarrow} b, y_i, y$ is a longer dominating path than P. A similar argument holds if there exists $i \in \{1, \ldots, k\}$ such that $ay_i \in E(G)$, but for which $y_i^+ y_i^- \in E(G)$. □

For the next few lemmas, we need the following definitions. Let $A = \{w \in Y;\ w^+w^- \notin E(G)\}$. We define a directed graph G^* as follows: $V(G^*) = A$, and (v, w) is an arc in G^* if and only if $\{v^+, w\} \rightarrowtail G - v^-$ or $\{v^-, w\} \rightarrowtail G - v^+$.

2.7.4 Lemma: If $|Y| \geq 2$, then $A \neq \varnothing$.

Proof: Let $i \in \{1, \ldots, k-1\}$. By Lemma 2.7.3(1), $ay_i^+ \notin E(G)$, so there exists $w \in V(G) - \{a, y_i^+\}$ such that $\{y_i^+, w\} \rightarrowtail G - a$ or $\{a, w\} \rightarrowtail G - y_i^+$.

Suppose $\{y_i^+, w\} \rightarrowtail G - a$. We recall from our introductory remarks that if $k > 1$, then, for all $j \in \{1, \ldots, k-1\}$, we have $y_j^+ \neq y_{j+1}$. So, certainly, $y_i^+ y \notin E(G)$ and $y_i^+ \neq y$ (since y is not on P), so $\{w\} \rightarrowtail \{y\}$. Also, by Lemma 2.7.3(3b), $by_i^+ \notin E(G)$, and $y_i^+ \neq b$ (since $i < k$), so we have that $\{w\} \rightarrowtail \{b\}$.

Suppose $\{a, w\} \rightarrowtail G - y_i^+$. By the maximality of P, $ay \notin E(G)$, so w must dominate y. Also, $ab \notin E(G)$ (otherwise, $y_i^+ P^{\rightarrow} b, aP^{\rightarrow} y_i, y$ would be a longer dominating path than P), so w must dominate b.

Hence, in either case, w dominates y and b. Now, $w \notin \{y, b\}$ (since y is non-adjacent to b); so, $yw, yb \in E(G)$. If $w \notin Y$, then $aP^{\rightarrow}b, w$ is a longer dominating path than P; so, we must have $w \in Y$. Furthermore, by Lemma 2.7.3(4), $w^+w^- \notin E(G)$. Thus, $w \in A$, and the lemma is proved. □

2.7.5 Lemma: Suppose $|Y| \geq 2$. By Lemma 2.7.4, $A \neq \varnothing$; let $r \in A$. Then, there exists $w \in A - \{r\}$ such that

(1) $(r, w) \in E(G^*)$, and

(2) w is adjacent to one of the end-vertices of P.

Proof: Suppose $|Y| \geq 2$. Let $r \in A$. Since $r^+r^- \notin E(G)$, there exists $w \in V(G)$ such that $\{r^+, w\} \mapsto G-r^-$ or $\{r^-, w\} \mapsto G-r^+$. Thus, w is non-adjacent to at least one of r^+ and r^-, and so $w \neq r$. By Lemma 2.7.3(1a), $r^+a \notin E(G)$, and by Lemma 2.7.3(1b), $r^-b \notin E(G)$; so, neither r^+ nor r^- can dominate both of the end-vertices of P. Also, $w \neq y$ since w dominates a and b. Hence, since r^+y, $r^-y \in E(\bar{G})$ and $|N(r^+) \cap \{a, b\}|$, $|N(r^-) \cap \{a, b\}| \leq 1$, we have that $\{r^+, w\} \mapsto G-r^-$ or $\{r^-, w\} \mapsto G-r^+$ implies that w dominates both y and at least one of the end-vertices of P. Thus, w must lie on P (otherwise, $w,aP^{\rightarrow}b$ or $aP^{\rightarrow}b,w$ is a longer dominating path than P), and, since w dominates y, $w \in Y$. Therefore, by Lemma 2.7.3(4), $w^+w^- \notin E(G)$. So, $w \in A$, and the lemma follows. □

2.7.6 Lemma: Suppose that $|Y| \geq 2$. Let $r \in A$ and suppose that r is adjacent to one of the end-vertices of P. Then, if (s, w) and (r, w) are arcs in G^*, then $r = s$.

Proof: Suppose, to the contrary, that there exist r, s, $w \in A$ such that (s, w), $(r, w) \in E(G^*)$, and $r \neq s$. There are four possibilities to consider.

Case 1: Suppose that $\{r^+, w\} \mapsto G-r^-$ and $\{s^+, w\} \mapsto G-s^-$. Without loss of generality, assume that r follows s on P. Then, since $ws^- \notin E(G)$ and $\{r^+, w\} \mapsto G-r^-$, it follows that $r^+s^- \in E(G)$. Now, if $rb \in E(G)$, then $r^-P^{\leftarrow}s,y,r,bP^{\leftarrow}r^+,s^-P^{\leftarrow}a$ will be a longer dominating path than P. If $ra \in E(G)$, then $r^-P^{\leftarrow}s,y,r,aP^{\rightarrow}s^-,r^+P^{\rightarrow}b$ will be a longer dominating path than P. So, r does not dominate an end-vertex of P, a contradiction.

Case 2: Suppose that $\{r^-, w\} \mapsto G-r^+$ and $\{s^-, w\} \mapsto G-s^+$. This case is analogous to Case 1.

Case 3: Suppose $\{r^-, w\} \mapsto G-r^+$ and $\{s^+, w\} \mapsto G-s^-$. Without loss of generality, assume that r precedes s on P. Since $wr^+ \notin E(G)$ and $\{s^+, w\} \mapsto G-s^-$, it follows that s^+ must dominate r^+. However, this contradicts the conclusion of Lemma 2.7.3(2) (since we assume $r \neq s$).

Case 4: Suppose $\{r^+, w\} \mapsto G-r^-$ and $\{s^-, w\} \mapsto G-s^+$. This case is analogous to Case 3.

Hence, our assumption is false, and the lemma follows. □

2.7.7 <u>Lemma</u>: Suppose that $|Y| \geq 2$. Then, for each $w \in A$, there exists $v \in A - \{w\}$ such that (v, w) is an arc in G^* (consequently, $|A| \geq 2$).

<u>Proof</u>: Suppose $|Y| \geq 2$, and let $w \in A$. Let $v_0 = w$. Then, by Lemma 2.7.5, there exists $v_1 \in A - \{w\}$ such that (w, v_1) is an arc in G^*, and v_1 dominates one of the end-vertices of P. Again, using Lemma 2.7.5, we can find $v_2 \in A - \{v_1\}$ such that (v_1,v_2) is an arc of G^*, and $v_2a \in E(G)$ or $v_2b \in E(G)$. Now, if $v_2 = w$, then we are done ($v = v_1$ satisfies the lemma). So, suppose v_2 is distinct from w. Using Lemma 2.7.5 again, we can find $v_3 \in A - \{v_2\}$ such that v_2v_3 is an arc in G^*, and $v_3a \in E(G)$ or $v_3b \in E(G)$. Notice that v_3 is distinct from v_1 since, otherwise, we would have (w, v_1) and (v_2, v_1) as distinct arcs in G^*, which would contradict Lemma 2.7.6 (given that v_2 is adjacent to an end-vertex of P and our assumption that $v_2 \neq w$).

Let $Q:(w=v_0),v_1,v_2,\ldots,v_t$ be a longest path in G^* such that for all $i \in \{1, \ldots, t\}$, the v_i are distinct (and each of them dominates one of the end-vertices of P). Note that, by the previous paragraph, we need only consider the case where $(w=)v_0$ is distinct from each of $v_1, v_2, \ldots, v_t$; otherwise, as shown earlier, the proof is complete. Since $v_t \in A$, there exists $v_{t+1} \in A - \{v\}$ such that (v_t, v_{t+1}) is an arc of G^*, and $v_{t+1}a \in E(G)$ or $v_{t+1}b \in E(G)$. Since Q is a longest path in G^* starting at w, with the described properties, it must be the case that $v_{t+1} \in V(Q)$. If $v_{t+1} = v_i$ for some $i \in \{1, \ldots, t-1\}$, then (v_t, v_{t+1}) and (v_{i-1}, v_{t+1}) would be distinct arcs in G^*. By Lemma 2.7.6, this implies $v_t = v_{i-1}$, which is contrary to assumption. Consequently, $v_{t+1} = w$. Thus, $v = v_t$ is the required vertex. □

2.7.8 <u>Lemma</u>: Suppose that $|Y| \geq 2$, and let $w \in A$. Then, (y_i, w) is an arc of G^* for some $i \in \{1, \ldots, k\}$, and we have the following:

(1) Suppose $\{y_i^+, w\} \mapsto G-y_i^-$. Then,

(a) If $1 \leq i \leq k-1$, then w is adjacent to a and b.

(b) If $i = k$, then w is adjacent to a.

(2) Suppose $\{y_i^-, w\} \mapsto G-y_i^+$.

(a) If $2 \leq i \leq k$, then w is adjacent to a and b.

(b) If $i = 1$, then w is adjacent to b.

<u>Proof</u>: Let $w \in A$. By Lemma 2.7.7, there exists $i \in \{1, \ldots, k\}$ such that $(y_i, w) \in E(G^*)$.

(1a) Suppose $\{y_i^+, w\} \mapsto G-y_i^-$ and $1 \leq i \leq k-1$. By Lemma 2.7.3(1a), $ay_i^+ \notin E(G)$, and hence w is adjacent to a. By Lemma 2.7.3(3b), $by_i^+ \notin E(G)$, and thus w is adjacent to b.

(b) Suppose $\{y_i^+, w\} \mapsto G-y_i^-$ and $i = k$. From Lemma 2.7.3(1a), $ay_k^+ \notin E(G)$, so w is adjacent to a.

(2a) Suppose $\{y_i^-, w\} \mapsto G-y_i^+$ and $2 \leq i \leq k$. By Lemma 2.7.3(3a), $ay_i^- \notin E(G)$, and hence w is adjacent to a. By Lemma 2.7.3(1b), $by_i^- \notin E(G)$, and thus w is adjacent to b.

(b) Suppose $\{y_i^-, w\} \mapsto G-y_i^+$ and $i = 1$. From Lemma 2.7.3(1b), $by_i^- \notin E(G)$, so w is adjacent to b. □

2.7.9 <u>Lemma</u>: Suppose that $|Y| \geq 2$, and let $w \in A$. Then,

(1) w is adjacent to one of the end-vertices of P; and

(2) if $wb \notin E(G)$, then $\{y_k^+, w\} \mapsto G-y_k^-$; and

(3) if $wa \notin E(G)$, then $\{y_1^-, w\} \mapsto G-y_1^+$.

<u>Proof</u>: (1) follows directly from Lemma 2.7.8.

(2) Let $w \in A$, and suppose $wb \in E(\bar{G})$. By Lemma 2.7.7, $(y_i, w) \in E(G^*)$ for some $i \in \{1, ..., k\}$. If the conditions of (1a), (2a), or (2b) of Lemma 2.7.8 hold, then $wb \in E(G)$, a contradiction. So, the conditions of Lemma 2.7.3(1b) hold, and $\{y_k^+, w\} \mapsto G-y_k^-$.

(3) is proved in a similar manner to (2). □

We are now ready to complete the proof of Theorem 2.7.1.

<u>Case 1</u>: Suppose that there exists a vertex in $V(G) - V(P)$, that has degree at least 2. Then, by Lemma 2.7.2, we may choose $y \in V(G) - V(P)$ such that $|N(y) \cap V(P)| > 1$. Let $Y = N(y) \cap V(P) = \{y_1, y_2, ..., y_\ell\}$ (so, $\ell = |Y| \geq 2$). Since $ab \notin E(G)$, there exists $r \in V(G)$ such that (without loss of generality) $\{a, r\} \mapsto G-b$. Now, a does not dominate y, $a \neq y_1^+$, and (by Lemma 2.7.3(1a)) $ay_1^+ \notin E(G)$. Thus, r must dominate y and y_1^+. Since $yy_1^+ \notin E(G)$, r is distinct from both y and y_1^+. If r is not on P, then $aP^{\rightarrow}y_1,y,r,y_1^+P^{\rightarrow}b$ would be a longer dominating path in G than P. So, $r \in V(P)$, and, since $ry \in E(G)$, $r \in Y$.

Now, if $r \in Y - A$, then $r^+r^- \in E(G)$. Suppose that $y_2^+ = b$. Then, $Y = \{y_1, y_2\}$ (i.e., $\ell = 2$), and y_2 dominates an end-vertex of P (namely, b). Hence, by Lemma 2.7.3(4), $y_2^-y_2^+ \in E(\bar{G})$, and so $A = \{y_2\}$ or $A = \{y_1, y_2\}$. But, $r \in Y - A$, so we must have $A = \{y_2\}$. However, this contradicts Lemma 2.7.7. So, $y_2^+ \neq b$ (and $\ell \geq 3$). If $r = y_1$, then $y_1y_2^+ \in E(G)$ (since $\{a, r\} \mapsto G-b$ and $ay_2^+ \notin E(G)$ (Lemma 2.7.3(1a)) and $y_2^+ \neq b$); however, then $aP^{\rightarrow}y_1^-,y_1^+P^{\rightarrow}y_2,y,y_1,y_2^+P^{\rightarrow}b$ would be a longer dominating path in G than P.

So, $r \neq y_1$; but then, since $ry_1^+ \in E(G)$, we have $aP^{\to}y_1,y,r,y_1^+P^{\to}r^-,r^+P^{\to}b$ being a longer dominating path than P. Therefore, it must be true that $r \in A$. Since $rb \notin E(G)$ (by $\{a, r\} \mapsto G-b$), it follows from Lemma 2.7.9 (since $r \in A$) that $\{y_k^+, r\} \mapsto G-y_k^-$, which implies that $ry_k^- \notin E(G)$. But, since $\{a, r\} \mapsto G-b$, a must dominate y_k^-; however, this contradicts Lemma 2.7.3(3a). Hence, Case 1 does not occur.

<u>Case 2</u>: Suppose that no vertex in $V(G) - V(P)$ has degree at least two, i.e., (since G is connected) every vertex in $V(G) - V(P)$ is adjacent to a vertex of P and has degree 1. Since G has order more than 6, it follows from Remark 2.5.1 that G has at most two end-vertices. If G has exactly two end-vertices, then (by Proposition 2.2.8) $G \cong H(1,1,p\text{-}5)$; however, then (by Proposition 2.5.4), G contains a hamiltonian path, which is contrary to our original assumption. So, we may assume that G has exactly one end-vertex, say y. Then, $V(G) - V(P) = \{y\}$, and $\deg_G a, \deg_G b > 1$. Let $\{y_1\} = N(y) \cap V(P)$. Since $ab \notin E(G)$, there exists $w \in V(G)$ such that, without loss of generality, $\{a, w\} \mapsto G-b$. Now, $ay \in E(\overline{G})$, $a \neq y_1^+$, and (by Lemma 2.7.3(1a)), $ay_1^+ \in E(\overline{G})$, whence w must dominate y_1^+ and y. Since $w \neq y$ ($w = y$ implies $N(y) \supseteq \{y_1^+, y_1\}$, i.e., $\deg y > 1$), $wy \in E(G)$ and so $w = y_1$.

We show next that $y_1^+y_1^- \in E(G)$. If $y_1^+y_1^- \notin E(G)$, then there exists $v \in V(G)$ such that $\{y_1^+, v\} \mapsto G-y_1^-$ or $\{y_1^-, v\} \mapsto G-y_1^+$. Clearly, in either case, $\{v\} \mapsto \{y\}$. If $y_1^+ = b$, then $aP^{\to}y_1,y$ is a path in G that contains all vertices of G except $b = y_1^+$ (which is adjacent to y_1), i.e., $Q{:}aP^{\to}y_1,y$ is a longest dominating path in G, where the vertex $b \in V(G) - V(Q)$ has degree at least 2. However, as we showed in Case 1, this situation is impossible. So, $y_1^+ \neq b$. Similarly, $y_1^- \neq a$. Suppose $\{y_1^+, v\} \mapsto G-y_1^-$. Since $y_1^- \neq a$ and $y_1^+a \notin E(G)$ (by Lemma 2.7.3(1a)), we have $va \in E(G)$ ($v \neq a$ since $\{v\} \mapsto \{y\}$). If $\{y_1^-, v\} \mapsto G-y_1^+$, then, since $y_1^+ \neq b$ and $y_1^-b \notin E(G)$ (by Lemma 2.7.3(1b)), we have $vb \in E(G)$. Thus, in either case, we may conclude that v dominates y and one of the end-vertices of P. Therefore, (since $ya, yb \notin E(G)$), it follows that v must be y_1. However, this is impossible, since $vy_1^- \notin E(G)$ or $vy_1^+ \notin E(G)$. Hence, $y_1^+y_1^- \in E(G)$.

Since $\deg b > 1$, there exists $v \in V(P)$ distinct from b^- such that $vb \in E(G)$. We will show that $v^+ \in V(G) - \{b\}$ is not dominated by $\{a, y_1\}$, which will provide us with our desired contradiction, since $\{a, w(=y_1)\} \mapsto G-b$. Obviously, $v^+ \neq a$. Further, note that $v \neq y_1$ since, otherwise, $\{a, v(=y_1)\} \mapsto G-b$ implies $vb \notin E(G)$ while, by definition, $vb \in E(G)$. So, we consider the following two subcases.

Subcase 2.1: Suppose that v follows y_1 on P. Then, certainly, $v^+ \neq y_1$. If $y_1v^+ \in E(G)$, then $y,y_1,v^+P^{\rightarrow}b,v,P^{\leftarrow}y_1^+,y_1^-P^{\leftarrow}a$ is a longer dominating path in G than P, which is impossible. If $av^+ \in E(G)$, then $y,y_1P^{\leftarrow}a,v^+P^{\rightarrow}b,vP^{\leftarrow}y_1^+$ is a longer dominating path than P. So, this subcase does not occur.

Subcase 2.2: Suppose that v precedes y_1 on P. If $v^+ = y_1$, then $aP^{\rightarrow}v,bP^{\leftarrow}v^+,y$ would be a longer dominating path than P. If $y_1v^+ \in E(G)$, then $aP^{\rightarrow}v,bP^{\leftarrow}y_1^+$, $y_1^-P^{\leftarrow}v^+,y_1,y$ (if $y_1^- \neq v^+$) or $aP^{\rightarrow}v,bP^{\leftarrow}y_1^+,y_1^-,y_1,y$ (if $y_1^- = v^+$) is a longer dominating path in G than P. If $av^+ \in E(G)$, then $y,y_1P^{\rightarrow}b,vP^{\leftarrow}a,v^+P^{\rightarrow}y_1^-$ is a longer dominating path than P.

The above two subcases show that $v^+ \in V(G) - \{b\}$ is not dominated by $\{a, y_1\}$. This contradicts $\{a, w(=y_1)\} \rightarrowtail G-b$. Hence, Case 2 does not occur either, and the theorem is proved. □

2.8 INDEPENDENT SETS IN 3-EDGE-CRITICAL GRAPHS

The following theorem gives exact values for β and Δ for disconnected 3-edge-critical graphs and shows that the independence number of a connected 3-edge-critical graph G is bounded above by $\Delta(G)$.

2.8.1 Theorem: Let G be a 3-edge-critical graph of order p.

(1) If G is connected, then $\beta(G) \leq \Delta(G)$.

(2) If G is disconnected, then $\beta(G) = 3$ and

(a) if $G \cong K_{p-2} \cup 2K_1$, then $\Delta(G) = \begin{cases} 0, & p = 3 \\ p-1, & p \geq 4 \end{cases}$;

(b) if $G \cong H \cup K_n$, where H is a connected 2-edge-critical graph and $n \in \mathbb{N}$, then $\Delta(G) = p - n - 1$ if $n \leq \frac{1}{2}p$ and $\Delta(G) = n - 1$, otherwise.

(c) if $G \cong H \cup K_1$, where H is a 2-edge-critical graph, then $\Delta(G) = \begin{cases} p-2, & \text{if H is connected} \\ p-3, & \text{if H is disconnected}. \end{cases}$

Proof: (1) Suppose, to the contrary, that there exists a connected 3-edge-critical graph G such that $\beta(G) \geq \Delta(G) + 1$. Let $\beta = \beta(G)$ and $\Delta = \Delta(G)$. We consider two cases.

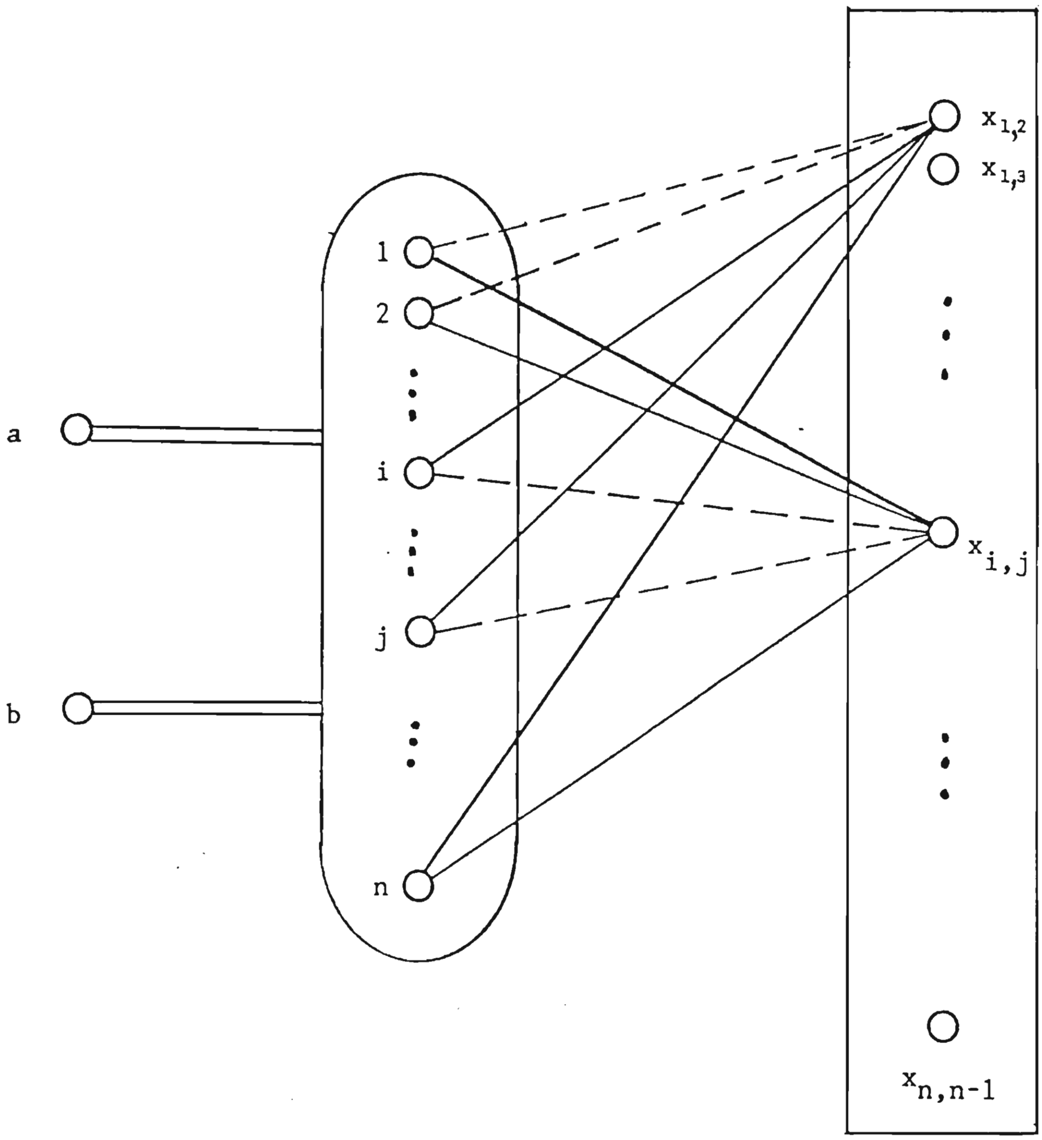


Fig. 2.8.1

Case 1: Suppose $\Delta \geq 3$. Then, $\beta \geq 4$; let S be a maximum independent set in G. Then, by Lemma 2.2.15, there exists a path $x_1, x_2, \ldots, x_{\beta-1}$ in G and an ordering $a_1, a_2, \ldots, a_\beta$ of the vertices in S in such a way that $\{a_i, x_i\} \mapsto G - a_{i+1}$ for $i = 1, 2, \ldots, \beta - 1$. From the proof of Lemma 2.2.15, we see that $N(x_2) \supseteq \{x_1, x_2\} \cup (S - \{a_2, a_3\})$, whence $\deg x_2 \geq |S| = \beta > \Delta$, which is absurd.

Case 2: Suppose $\Delta \leq 2$. Then, G is a path or a cycle of order at least 7 ($p(G) \geq 7$ since $\gamma(G) = 3$) that contains no triangle. This contradicts Theorem 2.2.24.

(2a) That $\beta(G) = 3$ and $\Delta(G) = \begin{cases} 0; & p = 3 \\ p-1; & p \geq 4 \end{cases}$ for $G \cong K_{p-2} \cup 2K_1$ is obvious.

(2b) Let G be a graph of the form $H \cup K_n$, where H is a connected 2-edge-critical graph. By Theorem 2.2.2, $\overline{H}$ is the union of one or more star graphs. So, $\beta(H) = \omega(\overline{H}) = 2$, whence $\beta(G) = 3$ follows, and $\Delta(G) = \max\{n - 1, p(H) - \delta(\overline{H})\} = \max\{n - 1, p - n - 1\}$; so, $\Delta(G) = n - 1$ if $2n \geq p$ and $\Delta(G) = p - n - 1$, otherwise.

(2c) Let G be a graph of the form $H \cup K_1$, where H is a 2-edge-critical graph. As in (2b), $\beta(G) = 3$. If H is isomorphic to $K_{2,\ldots,2}$, or to any other connected 2-edge-critical graph, then, by (2b), $\Delta(G) = p - 1 - 1 = p - 2$. If H is disconnected, then $\overline{H}$ is a single star, so that $\Delta(G) = \Delta(H) = p(H) - 1 - \delta(\overline{H}) = (p - 1) - 1 - 1 = p - 3$. □

That there is no upper bound on the cardinality of independent sets in edge-domination-critical graphs is shown by Theorem 2.8.2.

2.8.2 Theorem: For every integer $n \geq 3$, there exists a 3-edge-critical graph G with $\beta(G) = n$.

Proof: Let $n \geq 3$, and define a graph G_n as follows. Let $V(G_n) = \{a, b\} \cup S \cup T$, where the unions are disjoint, $T = \{x_{i,j}, x_{j,i};\ i, j \in \{1, 2, \ldots, n\}, i \neq j\}$, $\langle T \rangle$ is complete, and $S = \{1, 2, \ldots, n\}$ is independent. Let $N(a) = N(b) = S$, and for each $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $N(x_{i,j}) = S - \{i, j\}$. (See Fig. 2.8.1.) Clearly, $\beta(G_n) = n$. We show now that G_n is 3-edge-critical.

First, we establish $\gamma(G_n) = 3$. Let D be a minimum dominating set of G_n. Since $\Delta(G_n) < p(G_n) - 1$, we have $|D| \geq 2$. Suppose $|D| = 2$. If $a \in D$, then (since $ab \notin E(G)$), either $b \in D$ (in which case, $D \not\mapsto T$), or $i \in D$ for some $i \in \{1, 2, \ldots, n\}$ (in which case,

$D \not\mapsto \{x_{i,k}, x_{k,i};\ 1 \leq k \leq n, k \neq i\}$). So, $a \notin D$; similarly, $b \notin D$. But, $D \mapsto \{a, b\}$; so, there is $i \in \{1, 2, ..., n\}$ with $i \in D$. If $D \subseteq S$, then (since S is independent) $D \not\mapsto D - S(\neq \emptyset)$. So, there exist $k, \ell \in \{1, 2, ..., n\}$, $k \neq \ell$, with $x_{k,\ell} \in D$. However, then $D \not\mapsto \{k, \ell\} - \{i\}$. Hence, it follows that our assumption is false, and $|D| \geq 3$. Since $\{a, b, x\} \mapsto G_n$ for any $x \in T$, it follows that $\gamma(G_n) = 3$, as required.

We show now that G_n is edge-domination-critical. Let $u, v \in V(G_n)$ be distinct, non-adjacent vertices. We consider four cases.

Case 2.1: Suppose $u = a$ (or $u = b$) and $v = x_{i,j}$ for some $i, j \in \{1, 2, ..., n\}$, $i \neq j$. Then, $\{b, x_{i,j}\} \mapsto G+ux_{i,j}$ (or $\{a, x_{i,j}\} \mapsto G+ux_{i,j}$).

Case 2.2: Suppose $u = i$ for some $i \in \{1, 2, ..., n\}$, and $v = v_{i,k}$ (or $v = x_{k,i}$) for some $k \in \{1, 2, ..., n\}$ with $k \neq i$. Then, $\{v_{i,k}, k\} \mapsto G+ix_{i,k}$ (or $\{x_{k,i}, k\} \mapsto G+ix_{k,i}$).

Case 2.3: Suppose $\{u, v\} = \{a, b\}$. Then, $\{a, x\} \mapsto G+ab$ for any $x \in T$.

Case 2.4: Suppose $u = i$, $v = j$ for some $i, j \in \{1, 2, ..., n\}$ with $i \neq j$. Then, $\{i, x_{i,j}\} \mapsto G+ij$.

So, G_n is 3-edge-critical. □

2.9 CONJECTURES AND UNSOLVED PROBLEMS

In [V1], Vizing provided an upper bound on the number q of edges in a graph G of order p and having domination number γ, namely,

$$q \leq \frac{(p-\gamma)(p-\gamma+2)}{2}.$$

If G has domination number 3, then this bound becomes

$$q \leq \frac{(p-1)(p-3)}{2}.$$

Clearly, then, the following result, established by Blitch [B1] for connected 3-edge-critical graphs, is an improvement on Vizing's bound.

p	minimum number of edges in a connected 3-edge-critical graph of order p
6	6
7	10
8	12
9	16
10	21
11	26
12	31
13	37
14	44

Table 1

The minimum number of edges in a connected 3-edge-critical graph

2.9.1 Theorem: If G is a connected 3-edge-critical graph on p vertices, then

$$q(G) \le \binom{p-2}{2}.$$

By Proposition 2.2.8, Theorem 2.9.1 is best possible.

A far more difficult problem is the computation of the minimum number of edges in a connected k-edge-critical graph. For 3-edge-critical graphs, Sumner has conjectured in [S1] that the graphs G(m,t,1,1,t-2) ($m \in \mathbb{N}$, $t \ge 2m + 2$, see 2.4.7 and 2.4.9) have as few edges as possible, namely, $q(G(m,t,1,1,t-2) = \binom{2m}{2} + m + t + \binom{t}{2}$.

2.9.2 Conjecture: If G is a connected 3-edge-critical graph of order p, then

$$q(G) \le \min\left\{ \binom{p-k}{2} + \binom{k}{2} + \frac{k-4}{2} \right\},$$

where the minimum is taken over all even k, $2 \le k \le p$. For $p \ge 10$, this minimum is achieved at essentially $k = ½p$ ([S1]).

Table 1 shows the conjectured minimum number of edges for 3-edge-critical graphs of order at most 14. This table has been verified by Sumner ([S1]) using computer search for $p \le 9$. For values of $p > 9$, 3-edge-critical graphs of order p may be generated using an algorithm given in [S1]. A good deal of heuristic evidence lead D. P. Sumner to the following conjecture.

2.9.3 Conjecture: If G is a connected 3-edge-critical graph with diam G = 2 and $\delta(G) \ge 3$, then $\gamma(G) = i(G)$.

The following conjecture appeared in [SB1].

2.9.4 Conjecture: For every k-edge-critical graph G, $\gamma(G) = i(G)$.

Suppose that a graph G models, for example, a street network and that a smallest dominating set represents a set of intersections at which facilities are to be located. Then (assuming that the cost of building a street is less than the cost of the installation of a facility), it is economically advantageous to construct a new street if, as a result, the number of facilities can be reduced by one . The cost of the new thoroughfare is likely to be relatively small if it links two intersections which are not far apart in the original street network. Hence, it is of interest to study a modified version of k-edge-critical graphs, namely, (k,d)-edge-critical graphs, introduced and defined in [HOS1] as follows: For $k, d \in \mathbb{N}$, $d \geq 2$, a graph G is (k,d)-edge-critical if $\gamma(G) = k$ and $\gamma(G+uv) < k$ for every $uv \in E(\bar{G})$ such that $d_G(u,v) \leq d$. In [HOS1], (k,d)-edge-critical graphs are investigated. It is shown that a graph G is (2,2)-edge-critical if and only if $\bar{G}$ is a double star or a union of disjoint stars. It is shown that the diameter of a (3,2)-edge-critical is at most 4 and that the only (3,2)-edge-critical graphs of diameter 4 are contained in the set H consisting of the graphs G defined as follows.: Let $H_1 \cong K_r$ $(r \geq 2)$, $H_2 \cong K_s$ $(s \geq 1)$ and let H_3 be obtained from a complete graph K_{2m} $(m \geq 2)$ by removing the edges in a 1-factor. Let $u \in V(H_1)$ and let $v \in V(H_3)$. Let G be obtained from the disjoint union of H_1, H_2, and H_3 by joining every vertex of H_2 to every vertex of $H_1 \cup H_3$ distinct from u and v. Whereas each 3-edge-critical graph of diameter 3 is also (3,2)-edge-critical, it is shown that the graph G defined below is (3,2)-edge-critical but not 3-edge-critical. Let $H_1 \cong K_m$ $(m \geq 2)$, $H_2 \cong H_3 \cong K_n$ and $H_4 \cong K_1$. Let v be a vertex of H_1 and suppose $V(H_4) = \{u\}$. Let G be obtained from $H_1 \cup H_2 \cup H_3 \cup H_4$ by first joining every vertex of $H_1 - v$ to every vertex of H_2. Next, join u to every vertex of H_2. Finally, if $V(H_2) = \{v_1, v_2, \ldots, v_n\}$ and $V(H_3) = \{u_1, u_2, \ldots, u_n\}$, then join the vertex v_i to the vertex u_j for each pair $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. The following problem remains open:

<u>Problem</u>: Characterize the (3,2)-edge-critical graphs of diameter 3 that are not 3-edge-critical, or, failing this, obtain properties of such graphs. For instance, does such a graph necessarily contain a dominating path?

It is also shown in [HOS1] that the diameter of a (4,2)-edge-critical graph is at most 6 and that, for $k \geq 2$, each (k,2)-edge-critical graph G satisfies $\gamma(G-v) \leq k$ for each $v \in V(G)$, though G is not necessarily k-vertex-critical. The investigation of further properties of (k,2)-edge-critical graphs merits attention; for instance, the following question posed by the authors is as yet unanswered: If G is a (k,2)-edge-critical graph $(k \geq 3)$, is it true that $i(G) = \gamma(G)$?

The characterization of 3-edge-critical graphs that are minimal with respect to the property of being edge-domination-critical has not yet been investigated. It is known, however (see [S1]), that a 3-edge-critical graph G that has the smallest possible order satisfies $2\beta(G) \leq p(G) \leq 3\beta(G)$.

Chapter 3

DOMINATION NUMBER ALTERATION BY REMOVAL OF VERTICES

3.1 INTRODUCTION

Whereas, in Chapter 2, we considered the situation where a graph H is produced from a graph G by the *insertion* of *edges* so that $\gamma(H) < \gamma(G)$, we shall consider in this chapter the changes in domination numbers of graphs brought about by the *removal* of *vertices*.

All results in sections 3.1 and 3.2 are from [BHNS1], with the exçeption of Proposition 3.2.13 and 3.2.16, which come from [BCD2], and Theorem 3.2.36, which comes from [SB1], and all those in sections 3.3 to 3.9 are from [BCD1] and [BCD2]. All examples have been generalized, apart from that in Fig. 3.2.1. In addition, the examples in Figs 3.2.2, 3.2.5, 3.2.8 and 3.8.1 are new. We have generalized Lemmas 3.2.15 and 3.2.17 and Theorem 3.4.5. We have supplied Corollary 3.4.8 and 3.6.4, and the statement and proof of Proposition 3.2.10 and 3.2.11, and Theorem 3.4.5. We have made the statement of Proposition 3.2.38 more precise (than in [SB1]) and supplied a proof; we have made the statement of Theorem 3.2.6 slightly more informative. We have supplied Remark 3.2.5, as well as a proof for Proposition 3.2.6, 3.2.22, 3.2.28, 3.2.31, 3.4.4, 3.3.2.6.1,

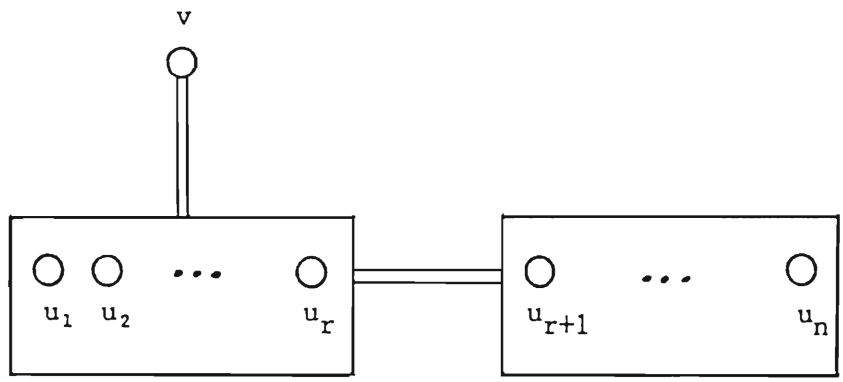


Fig. 3.1.1

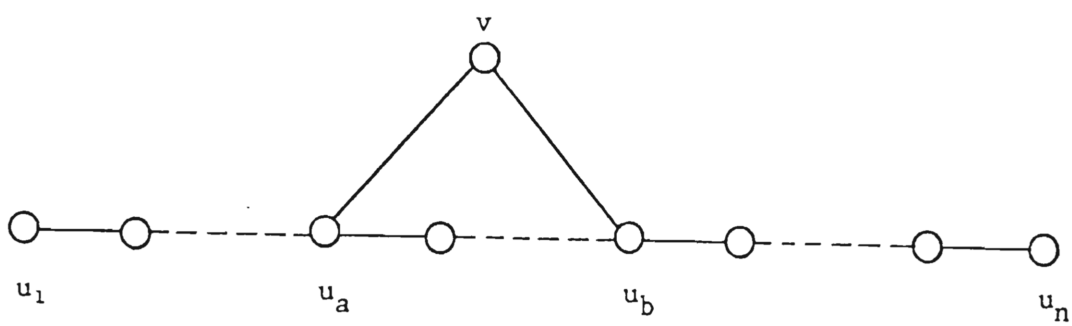


Fig. 3.1.2

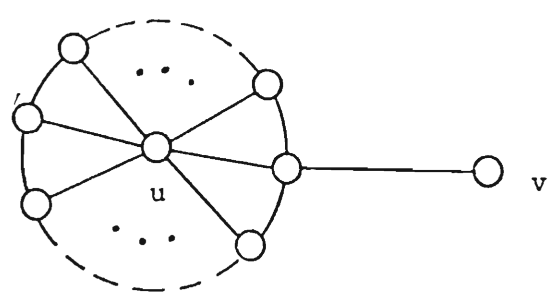


Fig. 3.1.3

3.3.2.7, 3.3.2.8.1, 3.3.2.9.2, 3.5.8, 3.5.10, 3.6.9, 3.8.8, 3.8.9, 3.8.11, Theorem 3.2.18, 3.2.20, 3.2.24, (most of) 3.6.7, 3.6.8, 3.8.6, 3.9.6, Corollary 3.2.14, 3.3.2.4, 3.4.6, 3.5.4, 3.8.12, 3.8.16, Lemma 3.6.5, 3.8.15, and most of 3.6.3, as well as of Remark 3.2.30 and Example 3.3.2.3. We have expanded Remark 3.2.27, 3.5.2, and 3.8.7, as well as the proof of Theorem 3.2.23, 3.2.26 (substantially), 3.2.32, 3.2.33 and 3.2.34 (considerably), 3.4.7, 3.5.12, 3.6.6, 3.8.13, 3.9.4, Lemma 3.4.9, 3.8.3, 3.8.4, 3.8.5 and Corollary 3.8.14 (slightly). Also, we have considerably clarified and expanded Theorem 3.9.2. Finally, we remark that section 3.10 constitutes original work done jointly with P. J. Slater and H. C. Swart.

3.1.1 <u>Definition</u> [BHNS1]: Let G be a graph and $\mu = \mu(G)$ be an arbitrary parameter of G. The *μ-stability* of G is the minimum number of vertices in a set $S \subset V(G)$ such that $\mu(G-S) \neq \mu(G)$, if such a set S exists.

3.1.2 <u>Remark</u>: Some parameters, such as the clique number, $\omega(G)$, the chromatic number, $\chi(G)$, the independence number, $\beta(G)$, and the vertex arboricity, a(G), of a graph G, have the property that removal of any subset S of V(G) does not result in a graph for which the parameter is greater than the value of the parameter for G. For other graphical parameters, there exist graphs G and subsets S_1 and S_2 of V(G) such that $\mu(G-S_1) > \mu(G)$ and $\mu(G-S_2) < \mu(G)$. One example of such a parameter is the connectivity κ. Consider the graph G shown in Fig. 3.1.1 with $V(G) = \{u_1, u_2, \ldots, u_r, \ldots, u_n, v\}$ such that $\langle\{u_1,u_2,\ldots,u_r,\ldots,u_n\}\rangle$ is complete and v is adjacent to $u_1, u_2, \ldots, u_r$, where $n, r \in \mathbb{N}$ and $n \geq r + 2$. Here, $\kappa(G-u_1) = r - 1 < \kappa(G) = r < \kappa(G-v) = n - 1$. Another example is the diameter of a graph. The graph G obtained from the path $u_1,u_2,\ldots,u_a,\ldots,u_b,\ldots,u_n$ where $3 < a + 2 < b < n$, by the addition of a vertex v, adjacent to u_a and u_b (see Fig. 3.1.2), is such that diam $G-u_1 <$ diam $G <$ diam $G-v$. Such parameters μ are known as *exceptional*.

3.1.3 <u>Definition</u>: For an exceptional parameter μ and a graph G, we define $\mu^+(G)$ to be the minimum number of vertices of G whose removal from G produces a graph H such that $\mu(H) > \mu(G)$; the minimum number of vertices whose removal from G results in a graph H with $\mu(H) < \mu(G)$ is defined to be $\mu^-(G)$.

3.1.4 <u>Remark</u>: In the graph G of Fig. 3.1.3, obtained from a wheel on r + 1 vertices, with centre u, by the addition of a vertex v, made adjacent to one peripheral vertex of the wheel, where $r \geq 7$, we see that $\gamma(G) = 2$, $\gamma(G-v) = 1$, and

$$\gamma(G-u) = 1 + \gamma(P_{r-3}) = 1 + \left\lceil \frac{r-3}{3} \right\rceil = \left\lceil \frac{r}{3} \right\rceil \geq 3.$$

Thus, $\gamma^+(G) = \gamma^-(G) = 1$. We shall now concentrate on the *domination number* and *domination alteration sets*, i.e., sets of vertices of a graph G whose removal results in a graph with domination number different from $\gamma(G)$.

3.2 STABILITY OF γ

We note first that, for some graphs, namely, graphs G containing one or more vertices of degree $p(G) - 1$, $\gamma^-(G)$ is not defined, as $\gamma(G) = 1 \leq \gamma(G-S)$ for all $S \subset V(G)$.

3.2.1 Definition: If, for a graph G, there exists no proper subset S of V(G) such that $\gamma(G-S) < \gamma(G)$, then we will *define* $\gamma^-(G)$ to be p(G).

3.2.2 Remark: In [BHNS1], the concept of the discrete, or null, graph (a graph with order 0) is used in order to avoid the above definition, where the domination number of the null graph is 0. We shall, however, not deal with null graphs.

It is also true that, for some graphs G, $\gamma^+(G)$ is not defined, for example, P_4 and K_n ($n \in \mathbb{N}$).

3.2.3 Definition: If G is a graph for which there exists no subset S of V(G) such that $\gamma(G-S) > \gamma(G)$, then we *define* $\gamma^+(G)$ to be p(G).

The following definitions will be useful.

3.2.4 Definition: Let G be a graph, A a minimum dominating set of G, and $v \in A$. We define $A^*(v)$ by

$$A^*(v) = \{u \in V(G) - A;\ N_G(u) \cap A = \{v\}\}.$$

In addition, let

$$m(G) = \min \{\ |A^*(v)|\ ;\ v \in A \text{ and } A \text{ is a minimum dominating set of graph } G\}.$$

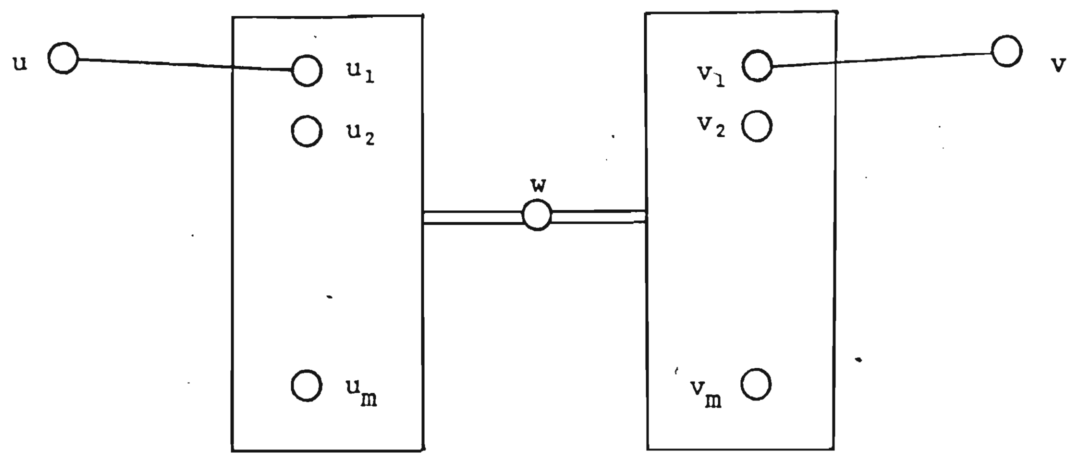


Fig. 3.2.1

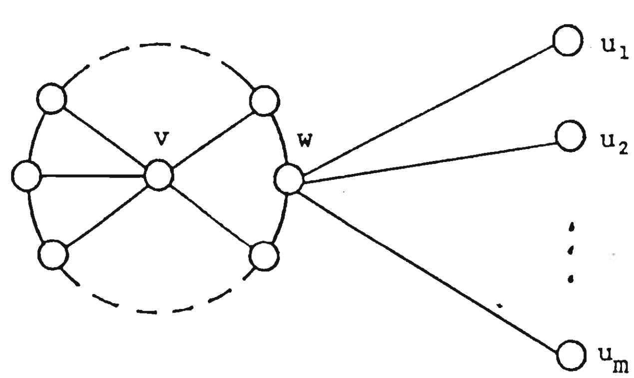


Fig. 3.2.2

3.2.5 Remark: Note that, for a graph G, $m(G) = 0$ if and only if there exists $v \in V(G)$ and a minimum dominating set A for G such that $A^*(v) = \emptyset$, i.e., if and only if there exists $v \in V(G)$ and a minimum dominating set A of G such that $A - \{v\} \mapsto G-v$ and (since A is a *minimum* dominating set) $[A, \{v\}] = \emptyset$ (i.e., the only vertex of G *not* dominated by $A - \{v\}$ is v). Observe also that $p(G) - \gamma(G)$ is an upper bound for $m(G)$, and is, in fact, attained by, for example, the graphs $K_{1,n}$ ($n \in \mathbb{N}$).

We now present an upper bound for $\gamma^-(G)$.

3.2.6 Proposition: For any graph G, $\gamma^-(G) \le m(G) + 1 \le p(G) - \gamma(G) + 1$.

Proof: Let G be a graph, and let A be a minimum dominating set of G with v a vertex in A such that $m(G) = |A^*(v)|$. Then, $A - \{v\} \mapsto G-A^*(v)-v$, i.e.,

$$\gamma(G-A^*(v)-v) \le |A - \{v\}| = \gamma(G) - 1 < \gamma(G).$$

So, $\gamma^-(G) \le |A^*(v) \cup \{v\}| = m(G) + 1$, which, with the observation that $m(G) \le p(G) - \gamma(G)$, completes the proof of the proposition. □

3.2.7 Remark: We show now that equality does not hold in general. Let $m \ge 2$, and let $G_1 \cong G_2 \cong K_m$ with $V(G_1) = \{u_1, u_2, \ldots, u_m\}$, and $V(G_2) = \{v_1, v_2, \ldots, v_m\}$ and let G be obtained from $G_1 \cup G_2$ by the addition of three vertices, u, v, and w, as well as the edges uu_1, vv_1, wu_i, wv_i ($i = 1, \ldots, m$) (see Fig. 3.2.1). Then, G has a (unique) minimum dominating set $A = \{u_1, v_1\}$ with $A^*(u_1) = \{u, u_2, u_3, \ldots, u_m\}$, $A^*(v_1) = \{v, v_2, v_3, \ldots, v_m\}$. So, we have $\gamma(G) = 2$, $m(G) = m \ge 2$, implying $m(G) + 1 \ge 3$, while $\gamma(G-\{u, v\}) = 1$, implying $\gamma^-(G) \le 2$.

To see that the bound given in Proposition 3.2.6 is sharp, consider the graph G shown in Fig. 3.2.2, where G is obtained from a wheel on $r + 1$ vertices, with central vertex v, by the addition of m new vertices, all adjacent only to a single peripheral vertex, w say, of the wheel, where $m \ge 2$ and $r \ge m + 4$, and, finally, by removing the edge vw. We see that G has a unique minimum dominating set, namely $\{v, w\}$, $m(G) = \min \{r - 3, m\} = m$, and $\gamma^-(G) = m + 1 = m(G) + 1$.

3.2.8 Corollary: For any graph G, $\gamma^-(G) = 1$ if and only if $m(G) = 0$.

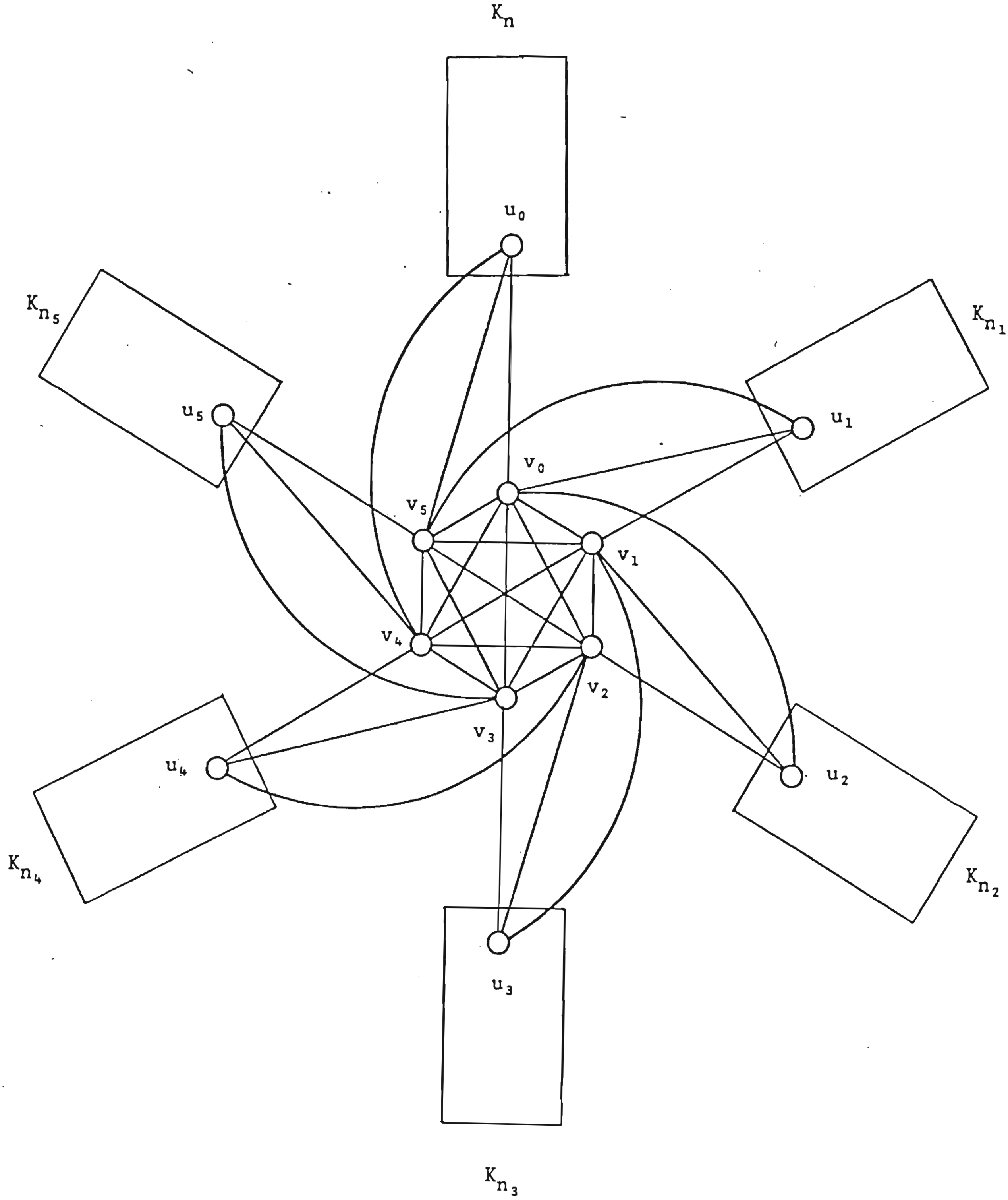


Fig. 3.2.3

Proof: Let G be a graph. If $m(G) = 0$, then $\gamma^-(G) \le 1$ by Proposition 3.2.6. However, $\gamma^-(G) \ge 1$; so, $\gamma^-(G) = 1$. Conversely, suppose that $\gamma^-(G) = 1$; let $v \in V(G)$ with $\gamma(G-v) < \gamma(G)$, and let B be a minimum dominating set for $G-v$. Then, $A = B \cup \{v\}$ is clearly a dominating set for G, and it is a *minimum* dominating set (since, otherwise, $\gamma(G) \le |A| - 1 = |B| = \gamma(G-v)$, contrary to the fact that $\gamma(G-v) < \gamma(G)$). Thus, $A - \{v\} \not\mapsto G$; in particular, $A - \{v\} \not\mapsto \{v\}$; also, $A - \{v\} = B \mapsto G-v$. So, by Remark 3.2.5, $A^*(v) = \varnothing$, whence $m(G) = 0$. □

3.2.9 Remark: Now, for any given $k \in \mathbb{N}$, there exists a graph G and a $\gamma^+(G)$-set S of G such that $\gamma(G-S) - \gamma(G) = k$, namely the graph $k_{1,k+1}$ (where S is the singleton containing the central vertex of $K_{1,k+1}$). We also have

3.2.10 Proposition: Given $m, k, n \in \mathbb{N}$ with $m \le k$ and $n \ge 2$, there exists a graph G with $\gamma(G) = k$, and $\gamma^+(G) = m$ and $\gamma^-(G) = n$.

Proof: Let m, k, and n satisfy the hypothesis of the proposition. Let $n_0 = n$, and, for $i = 1, \ldots, k-1$, let $n_i \ge n$ be any integer, and define

$$G_i \cong K_{n_i};$$

for each $i = 0, 1, \ldots, k-1$, let $u_i \in V(G_i)$. Also, let $G \cong K_k$, and suppose that $V(G) = \{v_0, v_1, \ldots, v_{k-1}\}$. Define a new graph H by

$$V(H) = V(G) \cup \bigcup_{i=0}^{k-1} V(G_i),$$

and

$$E(H) = \bigcup_{i=0}^{k-1} E(G_i) \cup E(G) \cup \bigcup_{i=0}^{k-1} \{v_i u_j \mid i \le j \le i+m-1\},$$

where addition is taken modulo k. (See Fig. 3.2.3 for the case where $k = 6$, $m = 3$.) Then,

$$\gamma(H) = k,\ \gamma^+(H) = |\{u_0, u_1, \ldots, u_{m-1}\}| = m, \text{ and } \gamma^-(H) = n. \qquad \square$$

However, we may generalize further:

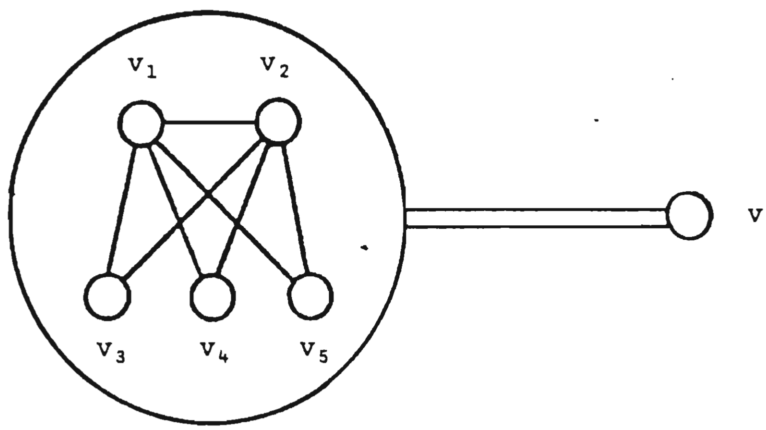


Fig. 3.2.4

3.2.11 Proposition: Given m, k $\in \mathbb{N}$, there exists a graph G with $\gamma^+(G) = m$ such that, for some $S \subset V(G)$ with $|S| = \gamma^+(G) = m$, we have $\gamma(G-S) - \gamma(G) = k$.

Proof: Let $G^* = K_{1,k+m}$ and suppose $V(G^*) = \{v, v_1, v_2, \ldots, v_{k+m}\}$, where v is the central vertex of G^*. If m = 1, then G^* has the desired property. Now, suppose m $\geq$ 2 and define a graph G by $V(G) = V(G^*)$, and

$$E(G) = E(G^*) \cup \bigcup_{i=1}^{m-1} \{v_iv_j \mid j \in \{1, 2, \ldots, k+m\}, i \neq j\};$$

so, $G \cong K_1 + (K_{m-1} + \overline{K}_{k+1})$. (See Fig. 3.2.4 for the case where k = 2 and m = 3.) Then, $\gamma(G) = 1$ (since $\{v\} \mapsto G$). Also, for $A_t = \{v, v_1, \ldots, v_t\}$, $\Delta(G-A_t) = p(G-A_t) - 1$, for t = 0, 1, 2, ..., m - 2, so that $\gamma(G-A_t) = 1$ for each t = 0, 1, ..., m - 2. However, $\Delta(G-S) = 0$, where $S = \{v, v_1, v_2, \ldots, v_{m-1}\}$, so that

$$\gamma(G-S) = p(G-S) = p(G) - m = k + m + 1 - m = k + 1.$$

Hence,

$$\gamma(G-S) - \gamma(G) = (k + 1) - 1 = k. \qquad \square$$

That the difference $\gamma(G-D) - \gamma(G)$, where D is a $\gamma^-(G)$-set, cannot be made arbitrarily large is shown by the next theorem, which shows that if $\gamma(G-T) < \gamma(G)$ for a graph G and $T \subset V(G)$, then, in fact, $\gamma(G-T) = \gamma(G) - 1$. First, however, we introduce the following definition.

3.2.12 Definition: Let G be a graph and v a vertex of G. Then, v is a *critical* vertex of G, or a G-*critical* vertex, if and only if $\gamma(G-v) < \gamma(G)$. If no ambiguity is possible, we simply write that v is critical.

3.2.13 Lemma: If G is a graph with a critical vertex v, then $\gamma(G-v) = \gamma(G) - 1$.

Proof: Let G be a graph with a critical vertex v, and let D^* be a minimum dominating set of G-v. Clearly, $D^* \cup \{v\} \mapsto G$, whence $|D^*| + 1 \geq \gamma(G)$, i.e., $|D^*| \geq \gamma(G) - 1$. Since v is critical, $|D^*| \leq \gamma(G) - 1$. Thus, $\gamma(G-v) = |D^*| = \gamma(G) - 1$. $\square$

The following corollary is a result that we shall use often.

3.2.14 Corollary: For any graph G, $\gamma(G-v) \geq \gamma(G) - 1$ for all $v \in V(G)$ (i.e., the removal of a single vertex from a graph can decrease the domination number by at most one).

Proof: Let G be any graph, and v any vertex of G. If v is not critical, then $\gamma(G-v) \geq \gamma(G)$; if v is critical, then, by Lemma 3.2.13, $\gamma(G-v) = \gamma(G) - 1$. □

A result with a proof similar to that of Lemma 3.2.13 is the following.

3.2.15 Proposition: For any graph G and any $v \in V(G)$,

$$\gamma(G-S) \geq \gamma(G) - 1$$

for any subset S of $N_G[v], v \in S$.

Proof: Suppose, to the contrary, that there exists a graph G with $v \in V(G)$ such that $\gamma(G-S) < \gamma(G) - 1$ for some $S \subseteq N_G[v]$, $v \in S$. Clearly, if D^* is a minimum dominating set of G-S, then $D^* \cup \{v\} \mapsto G$, whence $\gamma(G) \leq |D^* \cup \{v\}| < \gamma(G)$, which is not possible. Thus, no such graph G and vertex v exist. □

The following lemma is another result that we will use repeatedly.

3.2.16 Lemma: If G is a vertex-domination-critical graph, then, for every $v \in V(G)$, no vertex in a minimum dominating set of G-v is adjacent to v in G.

Proof: If G is a vertex-domination-critical graph with a vertex v and a minimum dominating set D of G-v satisfying $N_G(v) \cap D \neq \varnothing$, then $D \mapsto G$, whence $\gamma(G) \leq |D| = \gamma(G) - 1$, which is impossible. □

3.2.17 Proposition: If G is any graph and v is a G-critical vertex, then

$$\gamma(G-S) = \gamma(G) - 1,$$

for any $S \subseteq N_G[v]$, $v \in S$.

Proof: Let G be any graph with a critical vertex v, and let D^* be a minimum dominating set of G-v. Then, by Lemma 3.2.13, $|D^*| = \gamma(G) - 1$. Furthermore, $N[v] \cap D^* = \varnothing$ (by

Lemma 3.2.16). Thus, D^* is a subset of $V(G) - N[v]$ and hence of $V(G) - S$. So, since $D^* \mapsto G-v$, we certainly have $D^* \mapsto G-S$. Hence, $\gamma(G-S) \leq |D^*| \leq \gamma(G) - 1$. By Proposition 3.2.15, the desired result follows. □

3.2.18 <u>Theorem</u>: Let $W = \{u_1, u_2, \ldots, u_n\}$ be a minimal set of vertices of a graph G such that $\gamma(G-W) < \gamma(G)$. Then,

$$\gamma(G-W) = \gamma(G) - 1$$

and

$$\gamma(G-Y) = \gamma(G),$$

for any subset Y of W with cardinality $n - 1$.

<u>Proof</u>: Let G be a graph and let $W = \{u_1, u_2, \ldots, u_n\}$ be a minimal set of vertices of G such that $\gamma(G-W) < \gamma(G)$. Let Y be any $(n - 1)$-subset of W; suppose, without loss of generality, that $Y = \{u_1, u_2, \ldots, u_{n-1}\}$. Since Y is a proper subset of W, we have, by the minimality of W, that $\gamma(G-Y) \geq \gamma(G)$. By definition of $W = Y \cup \{u_n\}$, we have

$$\gamma((G-Y) - u_n) \leq \gamma(G) - 1. \tag{i}$$

By Corollary 3.2.14, $\gamma((G-Y) - u_n) \geq \gamma(G-Y) - 1$, whence we obtain $\gamma(G-Y) - 1 \leq \gamma(G) - 1$, i.e.,

$$\gamma(G-Y) \leq \gamma(G).$$

Hence (by the reverse inequality established earlier),

$$\gamma(G-Y) = \gamma(G),$$

and so

$$\gamma((G-Y) - u_n) \geq \gamma(G) - 1.$$

Combined with (i), this gives

$$\gamma(G-W) = \gamma((G-Y) - u_n) = \gamma(G) - 1,$$

as required. □

3.2.19 <u>Remark</u>: We note that if Y is a minimal set of vertices of a graph G such that $\gamma(G-Y) < \gamma(G)$ and Y′ is a proper subset of Y, it is possible for $\gamma(G-Y')$ to exceed $\gamma(G)$. In

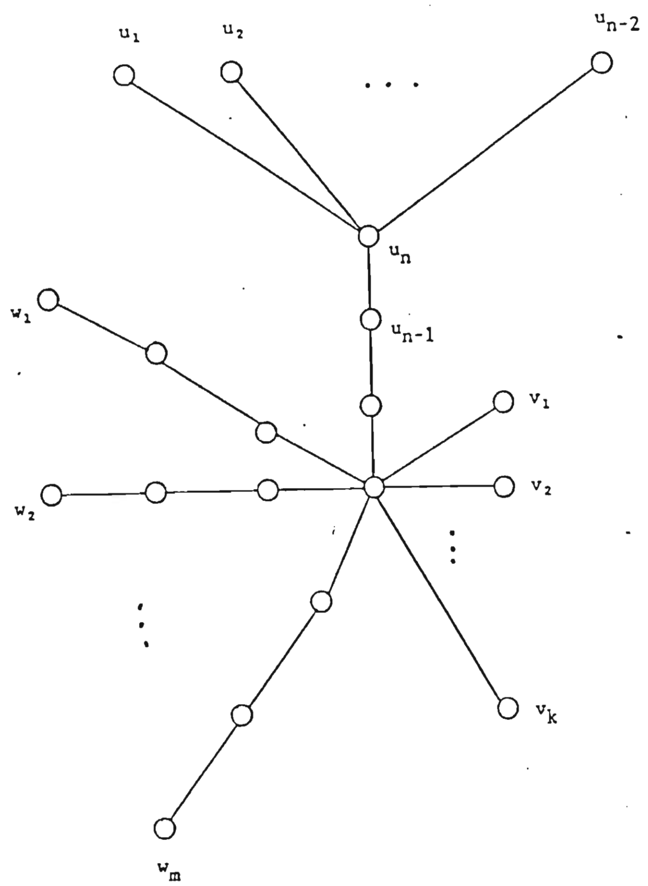


Fig. 3.2.5

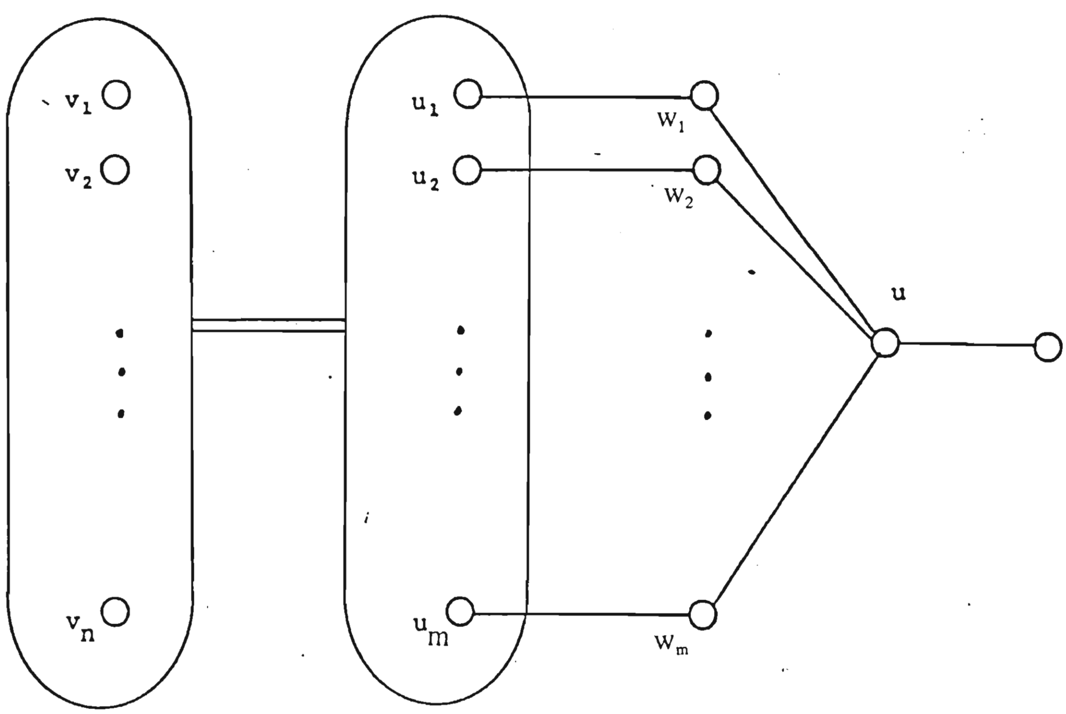


Fig. 3.2.5, where $3 \le n \le k$ and $m \in \mathbb{N}$, we have $\gamma(G) = m + 2$, and $S = \{u_1, u_2, \ldots, u_n\}$ is a minimal set whose removal decreases $\gamma(G)$ ($\gamma(G-S) = m + 1$). Then, $S' = \{u_n\} \subset S$ satisfies

$$\gamma(G-S') = m + 1 + n - 1 = m + n > m + 2 = \gamma G).$$

It is also possible that a minimal set S of vertices of a graph G that satisfies $\gamma(G-S) > \gamma(G)$ may properly contain a subset S' of vertices such that $\gamma(G-S') < \gamma(G)$: The graph G shown in Fig. 3.2.6 is obtained from the union of $K_{n,m}$ (with partite sets $\{v_1, v_2, \ldots, v_n\}$ and $\{u_1, u_2, \ldots, u_m\}$) and $K_{1,m+1}$ (with end-vertices $w_1, w_2, \ldots, w_{m+1}$) by the insertion of the edges u_iw_i ($i = 1, 2, \ldots, m$), $m > 2$, $n \ge 2$, and has $\{u_1, v_1, u\}$ as a minimum dominating set. Clearly, $S = \{v_1, v_2, \ldots, v_n\}$ is a minimal set for which $\gamma(G-S) = m + 1 > \gamma(G) = 3$, while $\gamma(G-\{v_1, v_2, \ldots, v_n\}) = 2 < \gamma(G)$.

Next, we prove a result which characterizes single vertices whose removal from a graph G produces a graph with domination number greater than $\gamma(G)$.

3.2.20 Theorem: A vertex v of a graph G is such that $\gamma(G-v) > \gamma(G)$ if and only if

(i) v is not isolated and is in every minimum dominating set for G, and

(ii) there is no dominating set for $G-N[v]$ having $\gamma(G)$ vertices which also dominates $N(v)$.

Proof: Let G be a graph. Suppose, first, that there exists $v \in V(G)$ such that $\gamma(G-v) > \gamma(G)$. If v is isolated, then, for any dominating set D of G, $v \in D$ and $D - \{v\}$ dominates $G-v$, i.e., $\gamma(G-v) < \gamma(G)$, a contradiction. So, v is not isolated. If D is a minimum dominating set of G that does not contain v, then, clearly, D is a dominating set of $G-v$ and $\gamma(G-v) \le \gamma(G)$, again a contradiction. So, v belongs to every minimum dominating set of G, i.e., v satisfies condition (i). If there exists $S \subseteq V(G) - N[v]$ such that $S \mapsto V(G) - N[v]$, S has $\gamma(G)$ vertices, and $S \mapsto N(v)$, then $S \mapsto G-v$, whence $\gamma(G-v) \le |S| = \gamma(G)$, which, again, produces a contradiction. So, v satisfies condition (ii), also.

Conversely, suppose that v is a vertex of G for which conditions (i) and (ii) hold. We shall prove that $\gamma(G-v) > \gamma(G)$. Suppose, to the contrary, that $\gamma(G-v) \le \gamma(G)$. Let S be a dominating set of $G-v$ with $|S| = \gamma(G)$. From (i), it follows that $N(v) \neq \varnothing$. Now, S clearly dominates $N(v)$, so, by condition (ii), S cannot be a subset of $V(G) - N[v]$. Hence, S must contain at least one vertex from $N(v)$, so that, in fact, $S \mapsto G$. Thus, S is a minimum dominating set of G which does not contain v, which is contrary to condition (i). Hence, it follows that $\gamma(G-v) > \gamma(G)$, as required. □

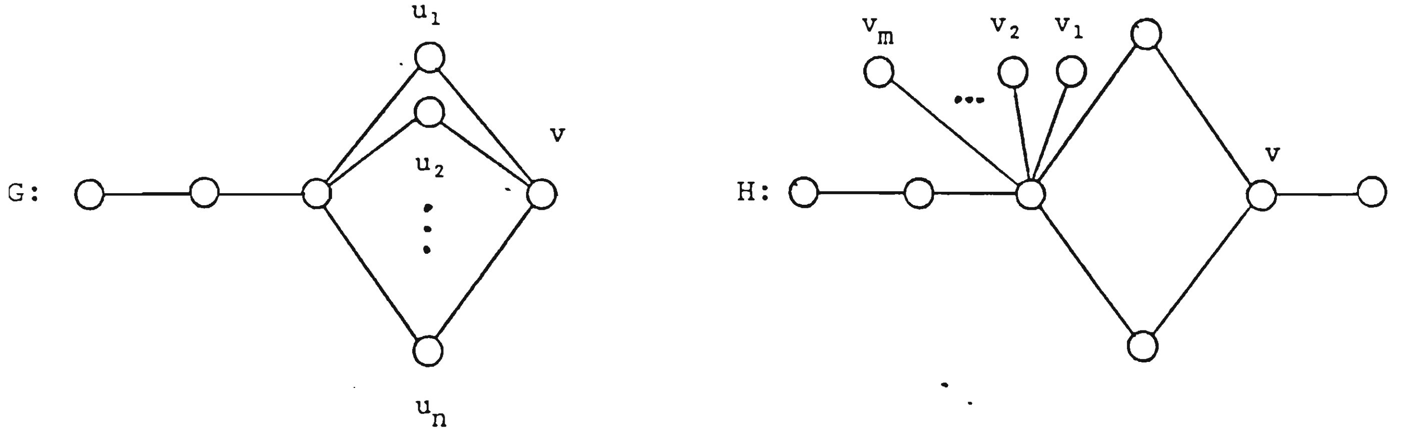


Fig. 3.2.7

3.2.21 Remark: The graphs G and H in Fig. 3.2.7 with m, n ≥ 2 show that neither of the above conditions is sufficient. Clearly, v is in every minimum dominating set for G, yet $\gamma(G-v) = \gamma(G) = 2$ (i.e., condition (i) is not sufficient). It is also easy to see that there is no two-vertex dominating set for $H-N_H[v]$ which dominates $N_H(v)$; however, $\gamma(H-v) = \gamma(H) = 3$ (so condition (ii) is not sufficient). However, condition (i) is sufficient if G is a tree, as we shall show in Theorem 3.2.23. We note first the following.

3.2.22 Proposition: If a vertex v is in every minimum dominating set of a tree T, then v is not an end-vertex of T.

Proof: The result obviously holds for a trivial tree, so suppose that there exist a non-trivial tree T and $v \in V(T)$ such that v belongs to every minimum dominating set of T but such that v is an end-vertex of T; suppose that w is the neighbour of v in T. Let D be a minimum dominating set of T. Since $v \in D$, w does not belong to D (otherwise, $D - \{v\}$ would be a smaller dominating set of T). Now, v dominates the vertices v and w, and no others. Since w dominates v and w, $(D - v) \cup \{w\}$ is a minimum dominating set of T that does not contain v, which is a contradiction. So, no such non-trivial tree T and vertex v exist, and the proposition follows. □

3.2.23 Theorem: For any tree T of order at least 3, and any $v \in V(T)$, $\gamma(T-v) > \gamma(T)$ if and only if v is in every minimum dominating set of T.

Proof: Let T be any tree of order at least three. If $v \in V(T)$ such that $\gamma(T-v) > \gamma(T)$, then, by Theorem 3.2.20, v is in every minimum dominating set of T. Conversely, suppose that $v \in V(T)$ such that v is in every minimum dominating set of T. Suppose $\gamma(T-v) < \gamma(T)$. Let D be a minimum dominating set for T−v. If $N(v) \cap D \neq \varnothing$, then $D \mapsto T$, which is impossible, since $|D| = \gamma(T-v) < \gamma(T)$. So, D contains no neighbour of v. Since T is connected and $p(T) > 1$, $N(v) \neq \varnothing$. Then, for any $w \in N(v)$, $D \cup \{w\}$ is a dominating set for T that does not contain v, where $|D \cup \{w\}| \leq \gamma(T)$. However, this contradicts the assumption that v belongs to every minimum dominating set for T. So, $\gamma(T-v) \geq \gamma(T)$.

Let $N(v) = \{v_1, v_2, \ldots, v_m\}$ and, for each $i \in \{1, 2, \ldots, m\}$, let T_i be the component of T−v containing v_i. (Note that, since T is acyclic, $i, j \in \{1, 2, \ldots, m\}$ with $i \neq j$ implies $T_i \neq T_j$.) Suppose that $\gamma(T-v) = \gamma(T)$, and suppose that there exists $i \in \{1, 2, \ldots, m\}$ such that v_i belongs to some minimum dominating set D_i for T_i. For each $j \in \{1, 2, \ldots, m\}$, $j \neq i$, let D_j be a minimum dominating set for T_j. Since $\gamma(T-v)$ is the sum of the domination numbers of the components of T−v, we have that $D = \bigcup_{k=1}^{m} D_k$ is a minimum dominating set for T−v, with $|D| = \gamma(T-v) = \gamma(T)$. Since D contains a neighbour (namely, v_i) of v, we see that D is a dominating set for T that does not contain v and which has cardinality $\gamma(T)$. However, this again contradicts the fact that v belongs to every minimum dominating set for T. So, for each $i \in \{1, 2, \ldots, m\}$, v_i is in no minimum dominating set of T_i.

Now, for each $i \in \{1, 2, \ldots, m\}$, the graph

$$T_i' = T - \bigcup_{j \in S} V(T_j)$$

where $S = \{1, 2, \ldots, m\} - \{i\}$, consists of the component T_i, together with the vertex v joined to v_i. Let $i \in \{1, 2, \ldots, m\}$. Since v is an end-vertex of T_i', it follows from Proposition 3.2.22 that there exists a minimum dominating set D_i' for T_i' such that $v \notin D_i'$. So, in order that v might be dominated by D_i' in T_i', we must have $v_i \in D_i'$. So, D_i' is a dominating set for T_i that is not a minimum dominating set (by the result established at the end of the previous paragraph). Thus, $\gamma(T_i) < |D_i'| = \gamma(T_i')$. Hence, $\gamma(T_i') \geq \gamma(T_i) + 1$, for each $i \in \{1, 2, \ldots, m\}$.

Now, suppose that there exists a dominating set D of T and $i \in \{1, 2, \ldots, m\}$ such that $|D \cap V(T_i)| < \gamma(T_i)$. Then, since $D \cap V(T_i) \mapsto T_i - v_i$, we have $(D \cap V(T_i)) \cup \{v\} \mapsto T_i'$, so that

$$\gamma(T_i') \leq |(D \cap V(T_i) \cup \{v\}| < \gamma(T_i) + 1,$$

which contradicts the result established in the previous paragraph. So, for any dominating set D of T, $|D \cap V(T_i)| \geq \gamma(T_i)$, for each $i \in \{1, 2, \ldots, m\}$.

Finally, let D be a minimum dominating set for T. Then, $v \in D$ (by assumption). Since, for each $i \in \{1, 2, \ldots, m\}$, $|D \cap V(T_i)| \geq \gamma(T_i)$, we have

$$\gamma(T) = 1 + \sum_{i=1}^{m} |D \cap V(T_i)| \geq 1 + \sum_{i=1}^{m} \gamma(T_i) = 1 + \gamma(T-v) = 1 + \gamma(T),$$

which is impossible. Hence, our assumption that $\gamma(T-v) = \gamma(T)$ is false, and we have $\gamma(T-v) > \gamma(T)$, as desired. □

We now investigate the situation for graphs in general.

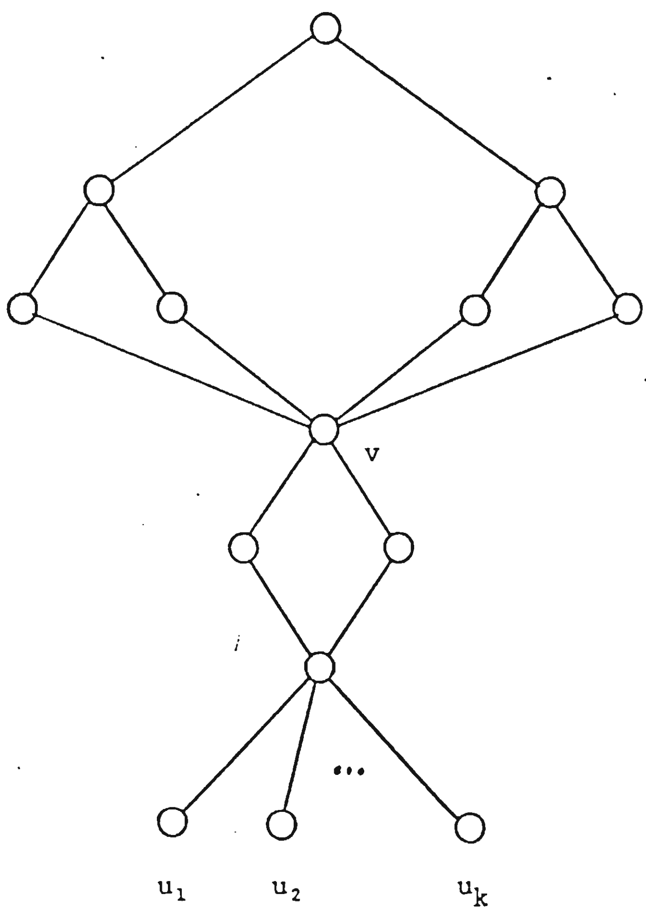


Fig. 3.2.8

3.2.24 Theorem: If a cut-vertex v of a graph G is in every minimum dominating set for G, then $\gamma(G-v) \geq \gamma(G)$ (i.e., v is a non-critical vertex of G).

Proof: Suppose, to the contrary, that there exists a graph G with a cut-vertex v that belongs to every minimum dominating set for G, but for which $\gamma(G-v) < \gamma(G)$. Then, v is a critical vertex of G. Let D be a minimum dominating set for G-v. By Lemma 3.2.16, $N(v) \cap D = \emptyset$. Since v is a cut-vertex of G, i.e. $k(G-v) > k(G)$, we have $N(v) \neq \emptyset$. Clearly, for any $w \in N(v)$, $D \cup \{w\}$ is a dominating set for G that does not contain v, where, by Lemma 3.2.13, $|D \cup \{w\}| = \gamma(G)$. This contradicts the fact that v belongs to every minimum dominating set for G. So, $\gamma(G-v) \geq \gamma(G)$. □

3.2.25 Remark: That equality can hold in the above result is illustrated by the graph G shown in Fig. 3.2.8. It can be easily verified that v belongs to every minimum dominating set of G, and that $\gamma(G) = 3$. However, $\gamma(G-v) = 2 + 1 = 3$. We note that the statement that strict inequality in Theorem 3.2.24 holds for all graphs G with a cut-vertex v, which is made in [BHNS1], is false.

In the next theorem, we provide an extension of Theorem 3.2.23 by describing the properties of those trees T for which $\gamma^+(T) = 2$ (for instance, the tree P_5).

3.2.26 Theorem: Let T be a tree. Then, $\gamma^+(T) = 2$ if and only if there are vertices u and v in T such that

(i) every minimum dominating set of T contains u or v,

(ii) v is in every minimum dominating set of T-u, and u is in every minimum dominating set of T-v, and

(iii) no vertex is in every minimum dominating set for T.

Proof: Let T be a tree. Suppose first that $\gamma^+(T) = 2$. (Then, $p(T) \geq 3$.) Then, there exist distinct vertices u and v of T such that $\gamma(T-\{u, v\}) > \gamma(T)$, and, for each $w \in V(T)$, $\gamma(T-w) \leq \gamma(T)$. If there exists a minimum dominating set D for T that contains neither u nor v, then D is a dominating set for $T-\{u, v\}$, i.e., $\gamma(T-\{u, v\}) \leq \gamma(T)$, which contradicts our choice of u and v. So, condition (i) is satisfied by u and v. Suppose that condition (ii) is not satisfied; assume, without loss of generality, that there exists a minimum dominating set D for T-u that does not contain v. Then, D is a dominating set for (T-u)-v, whence $\gamma(T-\{u, v\}) \leq |D| =$

$\gamma(T-u) \leq \gamma(T)$, again a contradiction. So, condition (ii) holds. Finally, condition (iii) holds, by Theorem 3.2.23 and the fact that $\gamma^+(G) = 2$.

Conversely, let u and v be distinct vertices of T that satisfy conditions (i) to (iii). We observe first that, by condition (iii) and Theorem 3.2.23, $\gamma^+(T) \geq 2$. So, $p(T) \geq 4$.

We show first that $\gamma(T-v) = \gamma(T)$. Since $\gamma^+(T) \geq 2$, $\gamma(T-v) \leq \gamma(T)$. Suppose that $\gamma(T-v) < \gamma(T)$. By conditions (i) and (iii), there exists a minimum dominating set S for T which contains v but not u. Let $v_1, v_2, \ldots, v_m$ be the vertices adjacent to v, and, for each $i \in \{1, 2, \ldots, m\}$, let T_i be the component of $T-v$ that contains v_i. Then, for $S_i = V(T_i) \cap S$, $S = \bigcup_{i=1}^{m} S_i \cup \{v\}$, where S_i is the smallest subset of $V(T_i)$ (not necessarily of $V(T_i) - \{v_i\}$) that dominates $T_i - v_i$ ($i \in \{1, 2, \ldots, m\}$). Then, certainly, $\gamma(T_i) \geq |S_i| \geq \gamma(T_i) - 1$ for all $i = 1, 2, \ldots, m$. We now consider two cases.

Case 1: Suppose that at least one $i \in \{1, 2, \ldots, m\}$ satisfies $\gamma(T_i) = |S_i| + 1$. Then,

$$\gamma(T-v) = \sum_{i=1}^{m} \gamma(T_i) \geq \left(\sum_{i=1}^{m} |S_i| \right) + 1 = |S| = \gamma(T),$$

which is contrary to our assumption that $\gamma(T-v) < \gamma(T)$. So, this case does not occur.

Case 2: Suppose $\gamma(T_i) = |S_i|$ for all $i \in \{1, \ldots, m\}$. Assume $u \in V(T_1)$, let $S_1' = S_1$, let $i \in \{2, \ldots, m\}$. If $S_i \mapsto T_i$, let $S_i' = S_i$; if $S_i \not\mapsto T_i$ (i.e., if $S_i \not\mapsto \{v_i\}$), then we let S_i' be any minimum dominating set of T_i. Clearly, $\bigcup_{i=1}^{m} S_i' \mapsto \bigcup_{i=1}^{m} T_i$ and, since $\gamma(T_i) = |S_i|$, $|\bigcup_{i=1}^{m} S_i'| = |S - \{v\}|$, so $\bigcup_{i=1}^{m} S_i' \not\mapsto \{v\}$ (otherwise, $\gamma(T) \leq |\bigcup_{i=1}^{m} S_i'| = \gamma(T) - 1$). Since $S_i' \mapsto T_i$ for each $i \in \{1, 2, \ldots, m\}$ and $S_1 \cup \{v_1\} \mapsto \langle V(T_1) \cup \{v\} \rangle$, while $u \notin S_1 \subset S$, it follows that $S' = S_1 \cup \{v_1\} \cup S_2' \cup \ldots \cup S_m'$ is a dominating set of T of cardinality $|S'| = |S| = \gamma(T)$ which contains neither u nor v (notice that $u \neq v_1$: Suppose, to the contrary, that $u = v_1$; then $uv \in E(G)$; if D is a minimum dominating set of $T-v$, then (by condition (ii)), $u \in D$, and so $D \mapsto T$. Hence, $\gamma(T) \leq \gamma(T-v)$. However, this contradicts our assumption that $\gamma(T-v) < \gamma(T)$). These properties of S' provide a contradiction to condition (ii).

Thus, neither Case 1 nor Case 2 occurs, and our assumption that $\gamma(T-v) < \gamma(T)$ is false. Hence, $\gamma(T-v) = \gamma(T)$. Let $u \in V(T_j)$ for some $j \in \{1, \ldots, m\}$. Now, by condition (ii), u is in every minimum dominating set of $T-v$, hence of T_j; so, $T_j \not\cong P_2$. If $T_j \cong P_1$, then $T-u$ is a tree and (as above) $\gamma(T-u) = \gamma(T)$; since v is in every minimum dominating set of $T-u$ (by condition (ii)) and

$p(T-u) \geq 3$, it follows from Theorem 3.2.23, applied to $T-u$, that $\gamma(T-\{u, v\}) = \gamma((T-u)-v) > \gamma(T-u) = \gamma(T)$. Otherwise, $p(T_j) \geq 3$ and application of Theorem 3.2.23 to T_j yields

$$\gamma(T-\{u,v\}) = \gamma((T-v)-u) = \sum_{\substack{i=1 \\ i \neq j}}^{m} \gamma(T_i) + \gamma(T_j - u) > \sum_{i=1}^{m} \gamma(T_i) = \gamma(T-v) = \gamma(T).$$

Hence, $\gamma^+(T) \leq 2$. Combined with the reverse inequality which we derived earlier, we have $\gamma^+(T) = 2$, as required. □

3.2.27 Remark: For graphs in general, γ, γ^+, and γ^- can be made as large as we wish. For example, let G be the graph constructed by joining a vertex v to one vertex in each of $m \geq 2$ distinct copies $G_1, G_2, \ldots, G_m$ of K_m; let $N(v) \cap V(G_i) = \{v_i\}$, for each $i \in \{1, 2, \ldots, m\}$. Then, $S = \{v_1, v_2, \ldots, v_m\}$ is a dominating set for G, with no smaller set dominating G; so $\gamma(G) = m$. Further, S satisfies $\gamma(G-S) = m + 1 > \gamma(G)$; since the removal of no smaller set of vertices from G produces a graph with domination number greater than $\gamma(G)$, we have $\gamma^+(G) = |S| = m$. Finally, $\gamma(G-V(G_1)) = m - 1 < \gamma(G)$, so $\gamma^-(G) = |V(G_1)| = m$. However, graphs with large values for γ^+ and γ^- have a large minimum degree δ.

3.2.28 Proposition: For all graphs G,

$$\min\{\gamma^+(G), \gamma^-(G)\} \leq \delta(G) + 1.$$

Proof: Let G be a graph, and let v be a vertex of minimum degree in G. If $G \cong K_{p(G)}$, then (by 3.2.1, 3.2.3) $\gamma^-(G) = \gamma^+(G) = p(G) = \delta(G) + 1$. If $G \not\cong K_{p(G)}$, then $N[v] \neq V(G)$. Then, if $\gamma(G-N[v]) > \gamma(G)$, we have $\gamma^+(G) \leq 1 + \delta(G)$, and if $\gamma(G-N[v]) < \gamma(G)$, we have $\gamma^-(G) \leq 1 + \delta(G)$. In either case, we have

$$\min\{\gamma^+(G), \gamma^-(G)\} \leq \delta(G) + 1.$$

If, on the other hand, $\gamma(G-N[v]) = \gamma(G)$, then

$$\gamma(G-N(v)) = \gamma(\langle\{v\}\rangle \cup (G-N[v])) = 1 + \gamma(G-N[v]) > \gamma(G);$$

so, $\gamma^+(G) \leq \delta(G)$ and

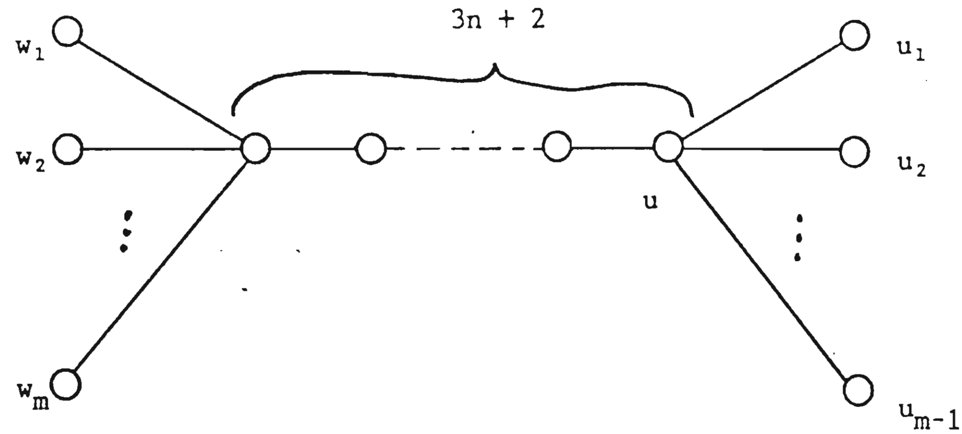
$$\min\{\gamma^+(G), \gamma^-(G)\} \leq \delta(G) + 1$$

certainly holds. □

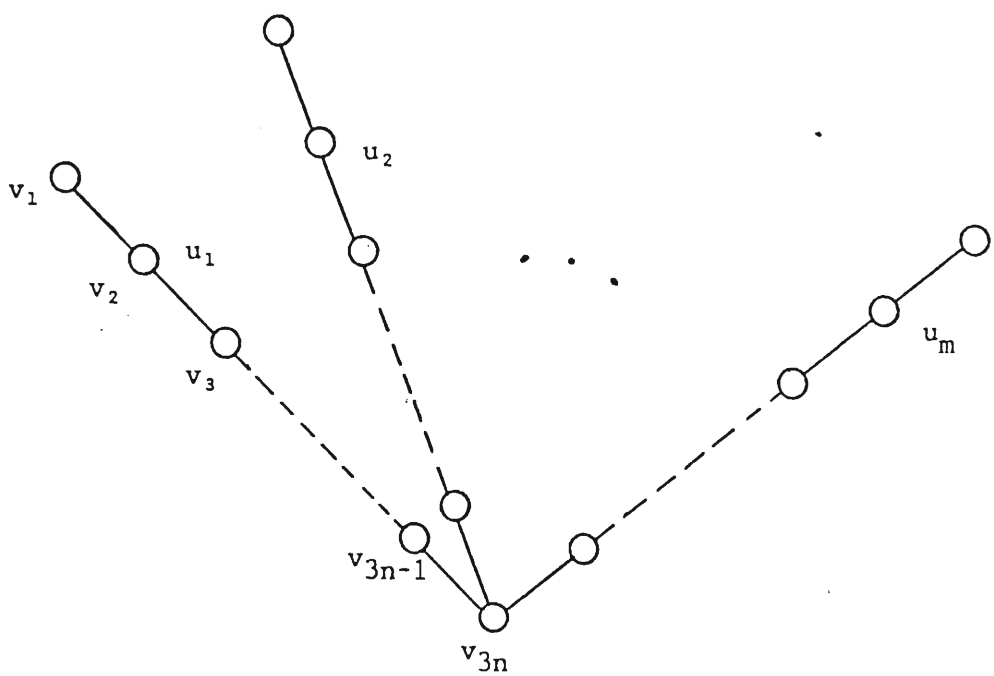
Notice that the bound in the previous proposition is best possible, since, for the graph G described in 3.2.27, $\gamma^+(G) = \gamma^-(G) = m = \delta(G) + 1$.

3.2.29 Proposition: If G is a graph with an end-vertex, then

$$\gamma^+(G) \geq 2 \text{ implies } \gamma^-(G) \leq 2.$$



(a)



(b)

Fig.3.2.9

Proof: Let G be a graph with an end-vertex u; let v be the neighbour of u in G. If $\gamma(G-v) < \gamma(G)$ (and, hence $\gamma^-(G) = 1$), then the proof is complete, since $\gamma^+(G) \geq 2$ implies $\gamma^-(G) \leq 2$ is true, regardless of the value of $\gamma^+(G)$.

Suppose now that $\gamma(G-v) \geq \gamma(G)$, and assume that $\gamma^+(G) \geq 2$. Then, $\gamma(G-v) = \gamma(G)$. Now,

$$\gamma(G-v) = \gamma((G-\{u, v\}) \cup \langle\{u\}\rangle) = 1 + \gamma(G-\{u, v\}),$$

so that

$$\gamma(G-\{u, v\}) < \gamma(G-v) = \gamma(G).$$

Thus, $\gamma^-(G) \leq 2$, as required. □

Note that, since every non-trivial tree has at least two end-vertices, $\gamma^+(T) \geq 2$ implies $\gamma^-(T) \leq 2$ for each non-trivial tree T.

3.2.30 Remark: The examples in Fig. 3.2.9 serve to show that the only restriction on γ^+ and γ^- for trees is given in the above proposition.

(1) For the graph T depicted in Fig. 3.2.9(a), with $n \geq 2$, $m \geq 3$,

$$\gamma(T) = \gamma(P_{3n+2-4}) + 2 = \left\lceil \frac{3n-2}{3} \right\rceil + 2 = n + 2.$$

Furthermore,

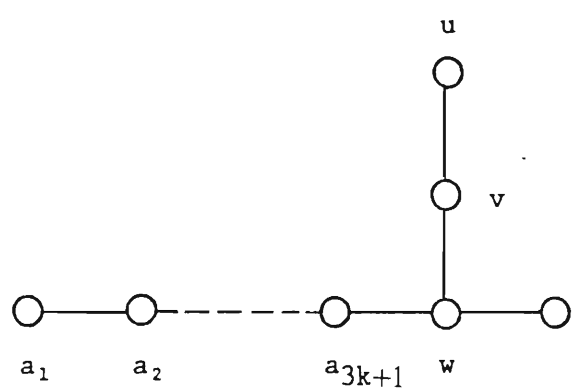
$$\gamma(T-u) = (m-1) + 1 + \gamma(P_{3n+2-2-1}) = m + n > \gamma(T)$$

(since $m > 2$); so, $\gamma^+(T) = 1$. Finally,

$$\gamma(T-\{u, u_1, u_2, \ldots, u_{m-1}\}) = \gamma(P_{3n+2-2-1}) + 1 = n + 1 < \gamma(T);$$

since the removal of no smaller set of vertices from T produces a graph with domination number less than $\gamma(T)$, it follows that $\gamma^-(T) = m$. So, this example shows that, if $\gamma^+(T)$ is very small, i.e., $\gamma^+(T) = 1$, then $\gamma^-(T)$ can be arbitrarily large.

(2) For the tree T depicted in Fig. 3.2.9(b), with $n, m \geq 2$, $\gamma(T) = nm$, and



(c)

Fig 3.2.9

$$\gamma(T-\{u_1, \ldots, u_m\}) = 1 + m\gamma(P_{3n-4}) + m = 1 + (n-1)m + m = mn + 1 > \gamma(T);$$

since the removal of no smaller set of vertices of T produces a graph with domination number greater than $\gamma(T)$, we have $\gamma^+(T) = m$ (≥ 2). Further, if S' is a minimum dominating set for $T - \{v_i; 1 \leq i \leq 3n-1\}$ and $S = \{v_4, v_7, \ldots, v_{3n-2}\} \cup S'$, then

$$\gamma(T-\{v_1, v_2\}) = |S| = (m-1)n + (n-1) = mn - 1 < \gamma(T).$$

The removal of no subset of T smaller than S results in a graph with domination number less than $\gamma(T)$, so $\gamma^-(T) = 2$. So, Fig. 3.2.9(b) illustrates the fact that γ^+ can be made arbitrarily large (with γ^- remaining in the set $\{1, 2\}$, as allowed by Proposition 3.2.29).

(3) For the tree T depicted in Fig. 3.2.9(c), we have $\gamma(T) = \gamma(P_{3k}) + 2 = k + 2$. Since $\gamma(T-u) = k + 1$, we have $\gamma^-(T) = 1$, and $\gamma^+(T) = 1$ since $\gamma(T-w) = \gamma(P_{3k+1}) + 2 = k + 3$. This shows that both γ^- and γ^+ can be very small.

We show now that every tree contains a vertex the removal of which creates a forest with domination number equal to that of the original tree. First, we prove

3.2.31 Lemma: If T is a tree of order at least three such that every vertex of T is adjacent to at most one end-vertex of T, then T contains a vertex of degree 2 which is adjacent to an end-vertex of T.

Proof: Let T be a tree of order $p \geq 3$ with the property that every vertex of T is adjacent to at most one end-vertex of T. Let a and b be two end-vertices of T such that $d_T(a,b)$ is as great as possible. Let P be the a-b path in T, and let w be the neighbour of a in T. We claim that $\deg_T w = 2$. Suppose, to the contrary, that $\deg w > 2$. Then, w must have a neighbour, z say, that does not lie on P. By our assumption, z is not an end-vertex of T; however, there is an end-vertex y of T that is joined to w by a (unique) path of length at least two which contains z. Hence, $d_T(y,b) > d_T(a,b)$. This is contrary to our choice of a and b. Thus, w is indeed a vertex with the desired properties. □

3.2.32 Theorem: For every non-trivial tree T, there exists a vertex $v \in V(T)$ such that $\gamma(T-v) = \gamma(T)$.

Proof: Let T be a non-trivial tree. If $T \cong K_2$, then either vertex of T has the desired property, so we may assume $p(T) \geq 3$. Suppose that there exists a vertex $v \in V(T)$ which is adjacent to two or more end-vertices of T; let Y be the set of neighbours of v that are end-vertices. Suppose that there exists a minimum dominating set S for T that does not contain v. Then, we must have $Y \subseteq S$, where $|Y| \geq 2$. However, then $(S - Y) \cup \{v\}$ is a dominating set of T with $|(S - Y) \cup \{v\}| \leq |S| - 1 = \gamma(T) - 1$, which is impossible. So, v belongs to every minimum dominating set for T, whence $\gamma(T-y) = \gamma(T)$ for any $y \in Y$.

So, suppose now that every vertex of T is adjacent to at most one end-vertex of T. Then, by Lemma 3.2.31, T contains a vertex w of degree two which is adjacent to an end-vertex u of T. Obviously, if v is an end-vertex of a graph G, then $\gamma(G-v) \leq \gamma(G)$. Hence, since $\deg_{T-u} w = 1$, we have

$$\gamma(T-u-w) \leq \gamma(T-u) \leq \gamma(T). \tag{i}$$

Now, since, for any minimum dominating set S for T-u-w, the set $S \cup \{w\}$ dominates T, we have

$$\gamma(T-u-w) + 1 = |S \cup \{w\}| \geq \gamma(T).$$

Thus, $\gamma(T-u-w) \geq \gamma(T) - 1$, and, by (i), we have

$$\gamma(T-u-w) \in \{\gamma(T), \gamma(T) - 1\}.$$

If $\gamma(T-u-w) = \gamma(T)$, then (by (i)) $\gamma(T-u-w) = \gamma(T-u) = \gamma(T)$, so that u is the vertex whose existence we wish to prove. Suppose $\gamma(T-u-w) = \gamma(T) - 1$. Then, if S is a minimum dominating set for T-u-w, we have that $S^* = S \cup \{u\}$ is a dominating set for $(T-w-u) \cup \langle\{u\}\rangle = T-w$, and (by the definition of S), no smaller set dominates T-w. Hence, $\gamma(T-w) = |S \cup \{u\}| = \gamma(T)$, and w is the vertex we seek. □

We shall show next that, for sufficiently large n, the quantity $\gamma^+ + \gamma^-$ is a constant for paths P_n $(n \in \mathbb{N})$ and cycles C_n $(n \geq 3)$. First note that

$$\gamma(P_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil, \qquad \text{if } n \geq 3.$$

3.2.33 Theorem: For $n \geq 7$, $\gamma^+(P_n) + \gamma^-(P_n) = 4$, where

$$\text{for } n \equiv 0 \pmod 3,\ \gamma^+(P_n) = 1 \text{ and } \gamma^-(P_n) = 3,$$
$$\text{for } n \equiv 1 \pmod 3,\ \gamma^+(P_n) = 3 \text{ and } \gamma^-(P_n) = 1,$$
$$\text{for } n \equiv 2 \pmod 3,\ \gamma^+(P_n) = 2 \text{ and } \gamma^-(P_n) = 2.$$

Proof: Let $n \in \mathbb{N}$, with $n \geq 7$. Consider the path P_n: $v_1, v_2, \ldots, v_n$. We shall show that $\gamma^+(P_n) + \gamma^-(P_n) = 4$ by proving this result separately for $n \equiv 0$, 1, and 2 (mod 3).

Case 1: Suppose that $n \equiv 0 \pmod 3$. Clearly, v_2 belongs to every minimum dominating set (since $D = \{v_1, v_4, v_7, \ldots, v_{n-2}, v_n\}$ is the smallest dominating set of P_n not containing v_2, and $|\{v_1, v_4, v_7, \ldots, v_{n-2}, v_n\}| = {}^n/_3 + 1 > \gamma(P_n)$). Hence, by Theorem 3.2.23, $\gamma^+(P_n) = 1$.

To see that $\gamma^-(P_n) = 3$, first note that $\gamma(P_{n-3}) = \gamma(P_n) - 1$, whence we obtain $\gamma^-(P_n) \leq 3$. Since $\gamma(P_{n-1}) = \gamma(P_{n-2}) = \gamma(P_n)$, the only way to lower the domination number of P_n by removing one or two vertices is to remove a vertex cutset S of P_n, with $|S| = 1$ or 2. If P_n-S has two components, A and B, containing a and b vertices, respectively, then,

$$\gamma(A) + \gamma(B) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil \geq \frac{a}{3} + \frac{b}{3} \geq \frac{1}{3}(n-2) = \gamma(P_n) - \frac{2}{3}$$

and so $\gamma(A) + \gamma(B) \geq \gamma(P_n)$. So, if $\gamma(P_n\text{-}S) < \gamma(P_n)$ for some $S \subseteq V(P_n)$ with $|S| < 3$, then P_n-S must have three components, A, B, C, and $|S| = 2$. Let $p(A) = a$, $p(B) = b$, $p(C) = c$. Then,

$$\gamma(A) + \gamma(B) + \gamma(C) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil + \left\lceil \frac{c}{3} \right\rceil \geq \frac{a}{3} + \frac{b}{3} + \frac{c}{3} = \frac{1}{3}(n-2) = \gamma(P_n) - \frac{2}{3}$$

whence $\gamma(A) + \gamma(B) + \gamma(C) \geq \gamma(P_n)$. Hence, $\gamma^-(P_n) \geq 3$ must hold, and the proposition follows in this case.

Case 2: Suppose that $n \equiv 1 \pmod 3$. Now,

$$\gamma(P_{n-1}) = \left\lceil \frac{n-1}{3} \right\rceil = \frac{n-1}{3} = \frac{n}{3} + \frac{2}{3} - 1 = \frac{n+2}{3} - 1 = \left\lceil \frac{n}{3} \right\rceil - 1 = \gamma(P_n) - 1,$$

and hence $\gamma^-(P_n) = 1$. Note that $P_n - \{v_2, v_4, v_6\} \cong 3K_1 \cup P_{n-6}$. Since

$$\gamma(P_n) - 2 = \left\lceil \frac{n}{3} \right\rceil - 2 = \frac{n-1}{3} + 1 - 2 = \frac{n-4}{3}$$

and

$$\gamma(P_{n-6}) = \left\lceil \frac{n-6}{3} \right\rceil = \left\lceil \frac{n-4}{3} - \frac{2}{3} \right\rceil = \frac{n-4}{3},$$

i.e., $\gamma(P_{n-6}) = \gamma(P_n) - 2$, and hence $\gamma(P_n - \{v_2, v_4, v_6\}) = 3 + \gamma(P_n) - 2 > \gamma(P_n)$, we conclude that $\gamma^+(P_n) \leq 3$. Now note that no vertex of P_n is in every minimum dominating set for P_n, which implies, by Theorem 3.2.23, that $\gamma^+(P_n) \geq 2$, and that condition (iii) of Theorem 3.2.26 is satisfied. However, no pair of vertices of P_n satisfy conditions (i) and (ii) of Theorem 3.2.26: the only pairs of vertices satisfying condition (i) are $\{v_1, v_2\}$ and $\{v_{n-1}, v_n\}$. However,neither pair satisfies condition (ii). Hence, by Theorem 3.2.26, $\gamma^+(P_n) = 3$, and the proposition follows in this case, also.

<u>Case 3</u>: Suppose that $n \equiv 2 \pmod 3$. It is easily verified that v_2 and v_{n-1} satisfy conditions (i) and (ii) of Theorem 3.2.26 and that no vertex belongs to every minimum dominating set for P_n; thus, $\gamma^+(P_n) = 2$. Now, by Proposition 3.2.29, $\gamma^-(P_n) \leq 2$. We show now that $\gamma^-(P_n) \neq 1$. Since $\gamma(P_{n-1}) = \gamma(P_n)$, the only way to lower the domination number of P_n by the removal of a single vertex is to disconnect P_n. Suppose there exists $v \in V(P_n)$ such that $P_n - v$ has two components, A and B, containing a and b vertices, respectively, then

$$\gamma(A) + \gamma(B) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil \geq \frac{a}{3} + \frac{b}{3} \geq \frac{1}{3}(n-1) = \frac{n+1}{3} - \frac{2}{3} = \gamma(P_n) - \frac{2}{3},$$

and so $\gamma(A) + \gamma(B) \geq \gamma(P_n)$. Hence, $\gamma^-(P_n) = 2$ and $\gamma^+(P_n) + \gamma^-(P_n) = 4$, as desired. □

3.2.34 <u>Theorem</u>: For $n \geq 8$, $\gamma^+(C_n) + \gamma^-(C_n) = 6$, where

for $n \equiv 0 \pmod 3$, $\gamma^+(C_n) = 3 = \gamma^-(C_n)$,
for $n \equiv 1 \pmod 3$, $\gamma^+(C_n) = 5$ and $\gamma^-(C_n) = 1$,
for $n \equiv 2 \pmod 3$, $\gamma^+(C_n) = 4$ and $\gamma^-(C_n) = 2$.

Proof: Let $n \in \mathbb{N}$ such that $n \geq 8$, and suppose that $C_n: v_0, v_1, \ldots, v_n = v_0$. We consider three cases.

Case 1: Suppose that $n \equiv 0 \pmod 3$. Since C_n is regular of degree 2, $m(C_n) \leq 2$, so that, by Proposition 3.2.6, $\gamma^-(C_n) \leq 3$. Since

$$\gamma(P_{n-1}) = \left\lceil \frac{n}{3} - \frac{1}{3} \right\rceil = \left\lceil \frac{n}{3} - \frac{2}{3} \right\rceil = \gamma(P_{n-2}) = \gamma(C_n),$$

it follows that, if there exists a set $S \subset V(C_n)$ such that $\gamma(C_n - S) < \gamma(C_n)$ and $|S| = 2$, then $C_n - S$ is disconnected with components (say) A and B, containing a and b vertices, respectively. Then,

$$\gamma(A) + \gamma(B) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil \geq \frac{a}{3} + \frac{b}{3} = \frac{1}{3}(n-2) = \gamma(C_n) - \frac{2}{3},$$

so that $\gamma(A) + \gamma(B) \geq \gamma(C_n)$. Hence, $\gamma^-(C_n) = 3$.

We note that $C_n - \{v_0, v_2, v_4\} \cong 2K_1 \cup P_{n-5}$. Since

$$\gamma(P_{n-5} \cup 2K_1) = 2 + \gamma(P_{n-5}) = 2 + \left\lceil \frac{n-3}{3} - \frac{2}{3} \right\rceil = 2 + \frac{n-3}{3} = \frac{n}{3} + 1 = \gamma(C_n) + 1,$$

we have $\gamma^+(C_n) \leq 3$. Now, note that $\gamma(P_{n-1}) = \lceil ⅓(n-1) \rceil = \lceil {}^n/_3 \rceil = \gamma(C_n)$. So, at least two vertices must be removed from C_n to produce a graph with domination number greater than $\gamma(C_n)$; so, $\gamma^+(C_n) \geq 2$. Since $\gamma(P_{n-2}) = \gamma(C_n)$ (see above), if $\gamma(C_n - S) > \gamma(C_n)$ for some $S \subset V(G)$ with $|S| = 2$, then $C_n - S$ must be disconnected, with two path-compoments, A and B, say, containing a and b vertices, respectively. Clearly, $n - 2 \equiv 1 \pmod 3$.

Subcase 1.1: Suppose $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$. Then,

$$\begin{aligned}\gamma(A) + \gamma(B) &= \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil = \left\lceil \frac{a-1}{3} + \frac{1}{3} \right\rceil + \frac{b}{3} = \frac{a+b-1}{3} + 1 \\ &= \frac{a+b+2}{3} = \frac{n}{3} = \gamma(C_n)\end{aligned}$$

Subcase 1.2: Suppose $a \equiv b \equiv 2 \pmod 3$. Then,

$$\begin{aligned}\gamma(A) + \gamma(B) &= \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil = \left\lceil \frac{a-2}{3} + \frac{2}{3} \right\rceil + \left\lceil \frac{b-2}{3} + \frac{2}{3} \right\rceil \\ &= \frac{a+b-4}{3} + 2 = \frac{a+b+2}{3} = \gamma(C_n)\end{aligned}$$

Hence, $\gamma^+(C_n) > 2$, so that $\gamma^+(C_n) = 3$.

Case 2: Suppose that $n \equiv 1 \pmod 3$. (Then, $n \geq 10$.) Note that $C_n - \{v_0, v_2, v_4, v_6, v_8\} \cong 4K_1 \cup P_{n-9}$. However,

$$\gamma(P_{n-9}) = \left\lceil \frac{n-9}{3} \right\rceil = \frac{n-7}{3} = \frac{n+2}{3} - 3 = \gamma(C_n) - 3$$

and thus

$$\gamma(C_n - \{v_0, v_2, v_4, v_6, v_8\}) = 4 + \gamma(C_n) - 3 > \gamma(C_n).$$

So, $\gamma^+(C_n) \leq 5$. Since

$$\gamma(P_{n-1}) = \frac{n-1}{3} = \left\lceil \frac{n}{3} \right\rceil - 1 = \gamma(C_n) - 1,$$

it is clear that $\gamma^-(C_n) = 1$ and $\gamma^+(C_n) \geq 2$.

Since $\gamma(P_{n-2}) = \frac{1}{3}(n - 1) = \gamma(C_n) - 1$, it follows that removing two adjacent vertices of C_n does not produce a graph with domination number greater than $\gamma(C_n)$. So, if $\gamma(C_n - S) > \gamma(C_n)$ for $S \subset V(C_n)$ with $|S| = 2$, then $C_n - S$ must be disconnected. Suppose $C_n - S$ has two components A and B with $p(A) = a$, $p(B) = b$. Clearly, $n - 2 \equiv 2 \pmod 3$. However:

Subcase 2.1: Suppose $a \equiv b \equiv 1 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil = \left\lceil \frac{a+2}{3} - \frac{2}{3} \right\rceil + \left\lceil \frac{b+2}{3} - \frac{2}{3} \right\rceil$$
$$= \frac{a+b+4}{3} = \frac{n+2}{3} = \gamma(C_n).$$

Subcase 2.2: Suppose $a \equiv 0 \pmod 3$, $b \equiv 2 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil = \frac{a}{3} + \left\lceil \frac{b+1}{3} \right\rceil$$
$$= \frac{a+b+1}{3} = \frac{n-1}{3} = \gamma(C_n) - 1.$$

Hence, it follows that $\gamma^+(C_n) \geq 3$.

Since

$$\gamma(P_{n-3}) = \left\lceil \frac{n-1}{3} - \frac{2}{3} \right\rceil = \frac{n-1}{3} = \gamma(C_n) - 1,$$

it follows that, in order for the $\gamma(C_n - S)$ to exceed $\gamma(C_n)$, where $S \subseteq V(G)$ and $|S| = 3$, $C_n - S$ must, as before, be disconnected. We suppose first that $C_n - S$ has two components A and B, with $p(A) = a$, $p(B) = b$. Then, $n - 3 \equiv 1 \pmod 3$.

<u>Subcase 2.3</u>: Suppose $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \left\lceil \frac{a+2}{3} - \frac{1}{3} \right\rceil + \frac{b}{3} = \frac{a+b+2}{3} = \frac{n-1}{3} = \gamma(C_n) - 1.$$

<u>Subcase 2.4</u>: Suppose $a \equiv b \equiv 2 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \frac{a-2}{3} + 1 + \frac{b-2}{3} + 1 = \frac{a+b+2}{3} = \gamma(C_n) - 1.$$

So, suppose now that $|S| = 3$ and $C_n - S$ has three components A_i, $1 \leq i \leq 3$, containing a_i vertices, respectively. Suppose that $\Sigma_{i=1}^{3} \gamma(A_i) \geq \gamma(C_n) + 1$. Let $i \in \{1, 2, 3\}$. If a_i is of the form $3m$, then $\gamma(A_i) = \frac{1}{3} a_i$; if a_i is of the form $3m+1$ or $3m+2$, then $\gamma(A_i) = \frac{1}{3}(a_i + 2)$ or $\gamma(A_i) = \frac{1}{3}(a_i + 1)$. In any case, $a_i \geq 3\gamma(A_i) - 2$. So,

$$\Sigma_{i=1}^{3} a_i \geq 3\, \Sigma_{i=1}^{3} \gamma(A_i) - 6 \geq 3\,(\gamma(C_n) + 1) - 6 = 3\,\gamma(C_n) - 3.$$

However,

$$\Sigma_{i=1}^{3} a_i = n - 3 = (3\,\gamma(C_n) - 2) - 3 = 3\,\gamma(C_n) - 5,$$

which is a contradiction. So, $\gamma^+(C_n) \geq 4$.

Since $\gamma(P_{n-4}) = \frac{1}{3}(n-4) = \gamma(C_n) - 2$, it follows that, if $\gamma(C_n - S) > \gamma(C_n)$ for $S \subset V(G)$ with $|S| = 4$, then $C_n - S$ must be disconnected. Suppose that $C_n - S$ has two components A and B with $p(A) = a$, $p(B) = b$, respectively. Clearly, $n - 4 \equiv 0 \pmod 3$.

<u>Subcase 2.5</u>: Suppose $a \equiv b \equiv 0 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \frac{a+b}{3} = \frac{n-4}{3} = \gamma(C_n) - 2.$$

Subcase 2.6: Suppose a $\equiv$ 1 (mod 3) and b $\equiv$ 2 (mod 3), or a $\equiv$ 2 (mod 3) and b $\equiv$ 1 (mod 3). Then,

$$\gamma(A) + \gamma(B) = \frac{a+b+3}{3} = \frac{n-1}{3} = \gamma(C_n) - 1.$$

Suppose then that C_n-S has three components A_i, $1 \leq i \leq 3$, containing a_i vertices, respectively. Suppose that $\Sigma_{i=1}^{3} \gamma(A_i) \geq \gamma(C_n) + 1$. As above, $\Sigma_{i=1}^{3} a_i \geq 3\,\gamma(C_n) - 3$, which contradicts

$$\Sigma_{i=1}^{4} a_i = n - 4 = (3\,\gamma(C_n) - 2) - 4 = 3\,\gamma(C_n) - 6.$$

Suppose now that C_n-S has four components A_i, $1 \leq i \leq 4$, containing a_i vertices, respectively. Suppose that $\Sigma_{i=1}^{4} \gamma(A_i) \geq \gamma(C_n) + 1$. As above,

$$\Sigma_{i=1}^{4} a_i \geq 3\,[\Sigma\,\gamma(A_i)] - 8 \geq 3\,(\,\gamma(C_n) + 1) - 8 > 3\,\gamma(C_n) - 5.$$

However, $\Sigma_{i=1}^{4} a_i = n - 4 = (3\ \gamma(C_n) - 2) - 4 = 3\ \gamma(C_n) - 6$, which produces a contradiction. Thus, $\gamma^+(C_n) > 4$. Hence, by the inequality $\gamma^+(C_n) \leq 5$, we have $\gamma^+(C_n) = 5$, as desired.

Case 3: Suppose that n $\equiv$ 2 (mod 3). Since $\gamma(P_{n-1}) = \frac{1}{3}(n+1) = \gamma(C_n)$, we have $\gamma^-(C_n) \geq 2$, and $\gamma^+(C_n) \geq 2$. Since $C_n-\{v_0, v_4\} \cong P_3 \cup P_{n-5}$ and

$$\gamma(P_3 \cup P_{n-5}) = 1 + \frac{n-5}{3} = \frac{n+1}{3} - 1 = \gamma(C_n) - 1,$$

we have $\gamma^-(C_n) \leq 2$. So, $\gamma^-(C_n) = 2$. We now show that $\gamma^+(G) = 4$. As $C_n-\{v_0, v_2, v_4, v_6\} \cong 3K_1 \cup P_{n-7}$ and

$$3 + \gamma(P_{n-7}) = 3 + \left\lceil\frac{n-7}{3}\right\rceil = 3 + \frac{n-5}{3} = \frac{n+1}{3} + 1 = \gamma(C_n) + 1,$$

we have $\gamma^+(C_n) \leq 4$.

Since $\gamma(P_{n-2}) = \gamma(C_n) - 1$, it follows that we cannot produce from C_n a graph with domination number greater than $\gamma(C_n)$ by the removal of two adjacent vertices. So, if the domination number of $\gamma(C_n-S) > \gamma(C_n)$ for some $S \subseteq V(G)$ with $|S| = 2$, then C_n-S

must be disconnected. Suppose that C_n-S has two components A and B, with $p(A) = a$, $p(B) = b$. Clearly, $n - 2 \equiv 0 \pmod 3$.

Subcase 3.1: Suppose $a \equiv b \equiv 0 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \frac{a+b}{3} = \frac{n-2}{3} = \gamma(C_n) - 1.$$

Subcase 3.2: Suppose $a \equiv 1 \pmod 3$ and $b \equiv 2 \pmod 3$, or $a \equiv 2 \pmod 3$ and $b \equiv 1 \pmod 3$. Then,

$$\gamma(A)+\gamma(B) = \frac{a+b+3}{3} = \frac{n+1}{3} = \gamma(C_n).$$

Hence, it follows that $\gamma^+(C_n) \geq 3$. Since $\gamma(P_{n-3}) = \frac{1}{3}(n - 2) = \gamma(C_n) - 1$, it follows that, if $\gamma(C_n-S) > \gamma(C_n)$ for some $S \subset V(G)$ with $|S| = 3$, then C_n-S must be disconnected. Suppose C_n-S has two components A and B with $p(A) = a$, $p(B) = b$. Then, $n - 3 \equiv 2 \pmod 3$.

Subcase 3.3: Suppose $a \equiv b \equiv 1 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \frac{a-1}{3} + \frac{b-1}{3} + 2 = \frac{n+1}{3} = \gamma(C_n).$$

Subcase 3.4: Suppose $a \equiv 0 \pmod 3$, $b \equiv 2 \pmod 3$. Then,

$$\gamma(A) + \gamma(B) = \frac{a}{3} + \frac{b-2}{3} + 1 = \frac{n-2}{3} = \gamma(C_n) - 1.$$

Suppose now that C_n-S has three components A_i, $1 \leq i \leq 3$, containing a_i vertices, respectively. If $\Sigma_{i=1}^{3} \gamma(A_i) \geq \gamma(C_n) + 1$, then, as above, $\Sigma_{i=1}^{3} a_i > 3\gamma(C_n) - 3$, which is contrary to

$$\Sigma_{i=1}^{3} a_i = n - 3 = (3\gamma(C_n) - 1) - 3 = 3\gamma(C_n) - 4.$$

Thus, $\gamma^+(C_n) > 3$, whence, $\gamma^+(C_n) = 4$, as desired. □

3.2.35 Remark: It is obvious that, in general, the removal of a single vertex from a graph G can result in a graph with domination number much greater than that of $\gamma(G)$. The next theorem, however, shows that this is not the case for the edge-domination-critical graphs of Chapter 2.

3.2.36 Theorem: If G is a non-trivial k-edge-critical graph ($k \in \mathbb{N}$), then, for every vertex $v \in V(G)$, $\gamma(G-v) \in \{k - 1, k\}$.

Proof: Let G be a non-trivial k-edge-critical graph, and let $v \in V(G)$. That $\gamma(G-v) \geq k - 1$ follows from Corollary 3.2.14. We show now that $\gamma(G-v) \leq k$. If $k = 1$, then G is complete (by Proposition 2.2.1), so that, trivially, $\gamma(G-v) = k$ for each $v \in V(G)$. So, we assume henceforth that $k \geq 2$. Let $v \in V(G)$, $A = N(v)$, and $B = V(G) - N[v]$. We consider two cases.

Case 1: Suppose that $\langle A \rangle$ is not complete. Then, there exist $a, b \in A$ with $ab \notin E(G)$, and we may assume, without loss of generality, that there exists a set $S \subset V(G)$ with $|S| = k - 2$ such that $S \cup \{a\} \mapsto G-b$. Since no element of S is adjacent to b, $v \notin S$. Thus, $S \cup \{a, b\} \mapsto G-v$, whence $\gamma(G-v) \leq k$.

Case 2: Suppose that $\langle A \rangle$ is complete, and let $w \in B$. Then, since $vw \notin E(G)$, there exists a set $S \subset V(G) - \{v, w\}$ with $|S| = k - 2$ such that $S \cup \{v\} \mapsto G-w$ or $S \cup \{w\} \mapsto G-v$. If $S \cup \{w\} \mapsto G-v$, then $\gamma(G-v) \leq k - 1$. On the other hand, if $S \cup \{v\} \mapsto G-w$, then $S \mapsto B - \{w\}$, so that, for any $a \in A$, we have $S \cup \{a, w\} \mapsto G-v$ (since $\langle A \rangle$ is complete), whence $\gamma(G-v) \leq k$. □

3.2.37 Remark: The 3-edge-critical graph $G = C_4 \cup K_1$ (see Proposition 2.2.26) is an example that illustrates the fact that it may not always be possible to find a vertex v of a k-edge-critical graph H such that $\gamma(H-v) = k$: For each $v \in V(G)$, $\gamma(G-v) = 2$. However, we do have the following result.

3.2.38 Proposition: For every vertex v of a k-edge-critical graph G with $k \geq 2$, at least one vertex in $N_{\bar{G}}[v]$ is G-critical.

Proof: Let G be a k-edge-critical graph with $k \geq 2$, and let v be a vertex of G of degree at most $p(G) - 2$. Let $uv \in E(\bar{G})$. By the edge-domination-criticality of G, there is a set $S \subset V(G)$ such that $S \mapsto G+uv$ and $|S| = \gamma(G) - 1$. Now, $|S \cap \{u, v\}| = 1$ since, otherwise, $S \mapsto G$ and $\gamma(G) \leq |S| < \gamma(G)$, which is impossible. If $u \in S$, then $S \mapsto (G+uv)-v = G-v$ so that $\gamma(G-v) \leq |S| < \gamma(G)$, and u is the vertex we seek; otherwise, v is a critical vertex of G. □

3.3 INTRODUCTION TO VERTEX-DOMINATION-CRITICAL GRAPHS

We begin with the following definition.

3.3.1 Definition [BCD1]: A graph G is called *vertex-domination-critical* if the domination number of the graph produced by the removal of any single vertex of G is less than $\gamma(G)$, i.e., if every vertex of G is G-critical. For $k \geq 2$, a graph G is said to be k-vertex-critical if $\gamma(G) = k$ and (by Lemma 3.2.13) $\gamma(G-v) = k - 1$ for every $v \in V(G)$. (We define k-vertex-criticality for $k \geq 2$ only, since, obviously, if H is a graph with $\gamma(H) = 1$, then H has no proper (vertex-)induced subgraphs with domination number less than $\gamma(H)$.)

3.3.2 Examples: Illustrative examples of families of vertex-domination-critical graphs are considered next.

3.3.2.1 Example: For $p \geq 2$, $G = \overline{K}_p$ is vertex-domination-critical since $\gamma(G) = p$ while $\gamma(G-v) = p - 1$ for any vertex $v \in V(G)$.

3.3.2.2 Example: For $n \in \mathbb{N}$, the graph $G \cong C_{3n+1}$ is vertex-domination-critical since $\gamma(G) = \lceil ⅓(3n + 1)\rceil = n + 1$, and, for any $v \in V(G)$, $G-v \cong P_{3n}$, where $\gamma(P_{3n}) = n$. However, neither C_{3n+2} nor C_{3n} is vertex-domination-critical since $\gamma(C_{3n+2}) = n + 1 = \gamma(P_{3n+1})$ and $\gamma(C_{3n}) = n = \gamma(P_{3n-1})$.

3.3.2.3 Example: A graph G is 2-vertex-critical if and only if $G \cong K_{2n}-F$, where $n \in \mathbb{N}$ and F is the edge set of a 1-factor of G.

Proof: Let G be a 2-vertex-critical graph of order p. Let $v \in V(G)$. Since $\gamma(G-v) = 1$, there exists a vertex u, say, in $G-v$ that has degree $p - 2$ in $G-v$. Clearly, u must have degree $p - 2$ in G also, since, otherwise, $\gamma(G) = 1$. So, v is the unique vertex distinct from u that is non-adjacent to u in G. Similarly, in the graph $G-u$, there is a vertex, w, say, with $\deg_{G-u}w = \deg_G w = p - 2$. So, u is the unique vertex distinct from w that is non-adjacent to w in G. So, $w = v$, and V(G) can be partitioned into pairs $\{u, v\}$ where

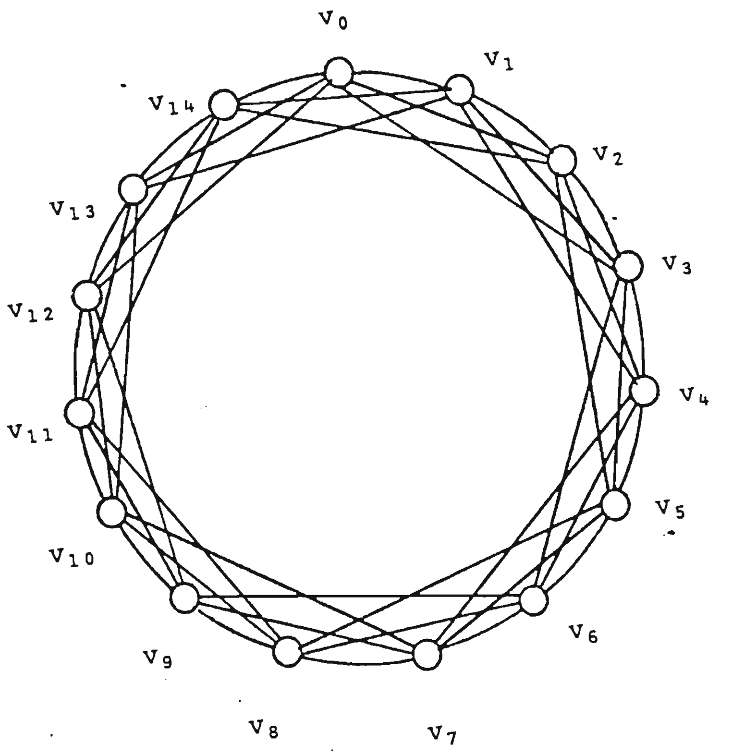


Fig. 3.3.1

$N(u) = N(v) = V(G) - \{u, v\}$, i.e., $G \cong K_{2n}-F$, where n is some positive integer and F is the edge set of a 1-factor of G.

Conversely, let $G \cong K_{2n}-F$ for some $n \in \mathbb{N}$ and F the edge set of some 1-factor of G. Since G is $(p-2)$-regular, $\gamma(G) > 1$. If $x \in V(G)$ and x' is the unique vertex of G satisfying $xx' \notin E(G)$, then $\{x, x'\} \mapsto G$, so that $\gamma(G) \leq 2$; hence, $\gamma(G) = 2$. Since $\{x'\} \mapsto G-x$ (where x was chosen arbitrarily), it follows that G is 2-vertex-critical. □

3.3.2.4 <u>Corollary</u>: Every 2-vertex-critical graph is 2-edge-critical.

<u>Proof</u>: Let G be a 2-vertex-critical graph; then (by Example 3.3.2.3), G is isomorphic to the graph obtained from a complete graph of even order by the removal of a perfect matching. Thus, if $uv \in E(\bar{G})$, then $\deg_{G+uv}u = \deg_{G+uv}v = p(G) - 1$, and, consequently, $\gamma(G+uv) = 1$, whence the edge-domination-criticality of G follows. □

3.3.2.5 <u>Example</u>: For integers m, $n \geq 2$, where m is even, the circulant graph $[C_{1+(n-1)(m+1)}]^{\frac{1}{2}m}$ is n-vertex-critical, as we see below.

3.3.2.6.1 <u>Proposition</u>: For integers m, $n \geq 2$, define the graph $G_{m,n}$ as follows: $V(G_{m,n}) = \{v_0, v_1, \ldots, v_{(n-1)(m+1)}\}$ and $E(G_{m,n}) = \{v_iv_j;\ 1 \leq i - j \pmod{[(n-1)(m+1)+1]} \leq {}^m/_2\}$. Then, if m is even, $G_{m,n}$ is n-vertex-critical. (See Fig. 3.3.1 for $G_{6,3}$.)

<u>Proof</u>: Let m, $n \geq 2$ be integers with m even. Let $G_{m,n}$ be as described above. Clearly, $G_{m,n}$ is m-regular. Hence, since every vertex in $G_{m,n}$ dominates exactly $m + 1$ vertices,

$$\gamma(G_{m,n}) \geq \left\lceil \frac{p(G_{m,n})}{m+1} \right\rceil = \left\lceil \frac{(n-1)(m+1)+1}{m+1} \right\rceil = (n-1)+1 = n.$$

Let

$$D_i = \{v_i, v_{i+(m+1)}, v_{i+2(m+1)}, \ldots, v_{i+(n-2)(m+1)}, v_{i+(n-2)(m+1)+\frac{m}{2}+1}\},$$

where the subscripts are taken modulo $(n-1)(m+1)+1$, for $i \in \{0, 1, 2, \ldots, (n-1)(m+1)\}$. Then, $D_i \mapsto G_{m,n}$ and $|D_i| = n$; hence, D_i is a minimum dominating set of $G_{m,n}$ and $\gamma(G_{m,n}) = n$. Furthermore, if $j \in \{0, 1, 2, \ldots, (n-1)(m+1)\}$, then,

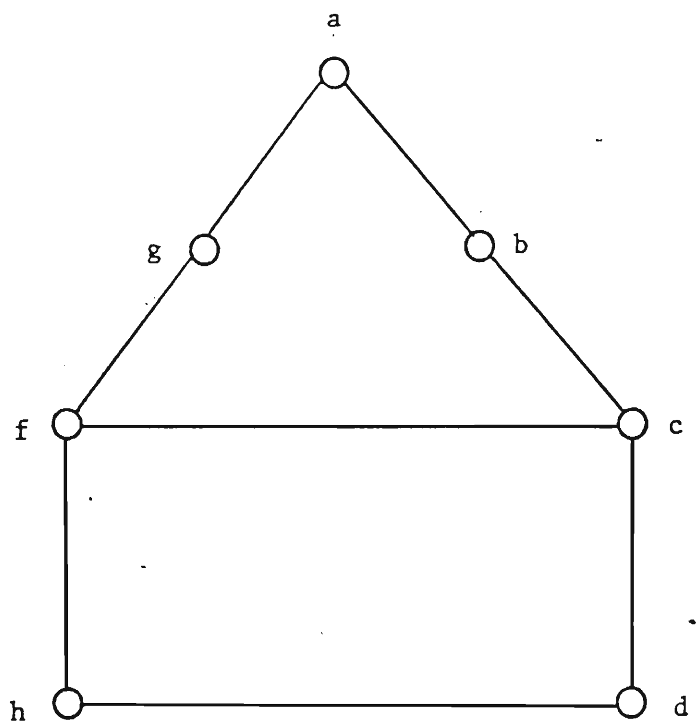
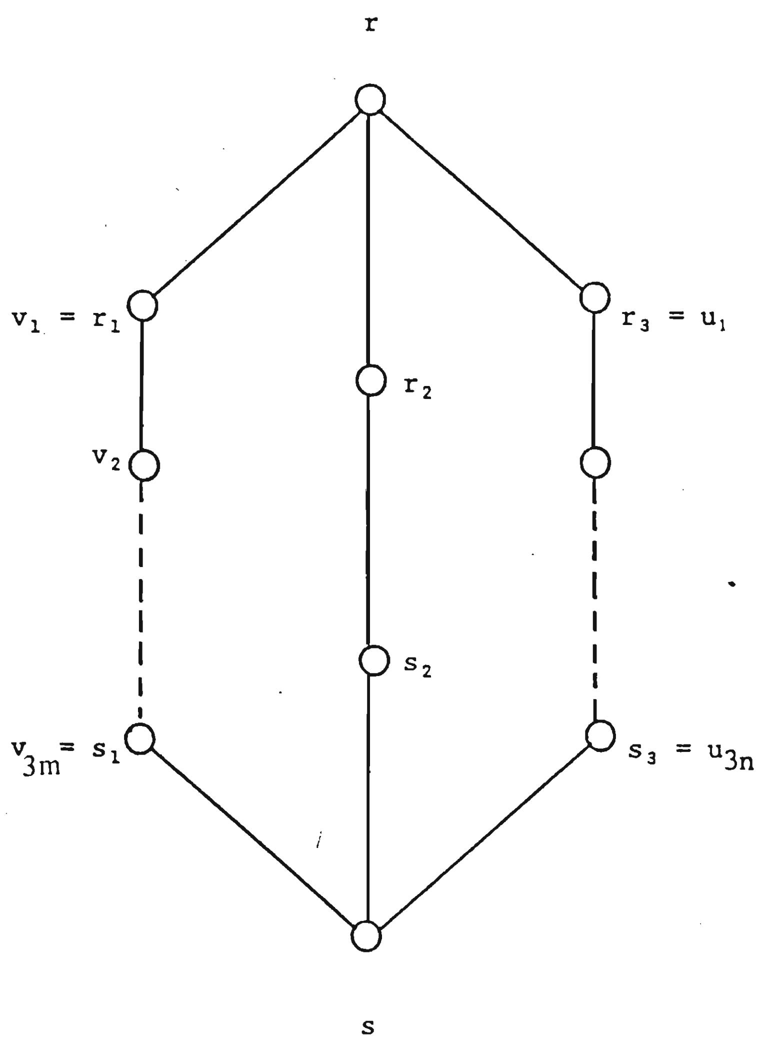


Fig. 3.3.2



$$D_i - \{v_j\} \mapsto G_{m,n}-v_j,$$

where $i + (n - 2)(m + 1) + \frac{1}{2}m + 1 = j$, i.e.,

$$i = j - n(m+1) + \frac{3}{2}m + 1,$$

whence $\gamma(G_{m,n}-v_j) \leq n - 1$. By Corollary 3.2.14, $\gamma(G_{m,n}-v_j) \geq n - 1$, and so $\gamma(G_{m,n}-v_j) = n - 1$. Thus, $G_{m,n}$ is indeed n-vertex-critical. □

3.3.2.6.2 Remark: We observe that, while $\gamma(G_{m,n}) = n$ if m is odd, we have $\gamma(G_{m,n}-v) = n$ for any $v \in V(G_{m,n})$ for such an m.

3.3.2.6.3 Remark: For $n \in \mathbb{N}$, $C_{3n+1} \cong G_{2,n+1}$, and a 2-vertex-critical graph H of order 2n satisfies $H \cong G_{2n-2,2}$.

3.3.2.7 Proposition: The graph G in Fig. 3.3.2, obtained by joining two diametrical vertices in a graph isomorphic to C_7, is 3-vertex-critical.

Proof: Let G be the graph in Fig. 3.3.2. By inspection, it may be seen that no 2-subset of V(G) dominates G; so $\gamma(G) \geq 3$. However, $\{a, d, f\} \mapsto G$, whence $\gamma(G) = 3$. Now, let $v \in V(G)$. If $v = h$, then $\{a, c\} \mapsto G-v$. If $v = f$, then $G-v \cong P_6$, and so $\gamma(G-v) = 2$. If $v = g$, then $\{b, h\} \mapsto G-v$. If $v = a$, then $\{c, f\} \mapsto G-v$. By symmetry, $\gamma(G-v) = 2$ for each $v \in \{b, c, d\}$. So, $\gamma(G-v) = 2$ for each $v \in V(G)$, and G is indeed 3-vertex-critical. □

3.3.2.8.1 Proposition: The graph of Fig. 3.3.3, obtained by connecting vertices r and s by three internally disjoint paths, of lengths $3m + 1$, 3, and $3n + 1$, respectively ($m, n \in \mathbb{N}$), is vertex-domination-critical.

Proof: Let G be the graph in Fig. 3.3.3, and let $U_1 = \{v_1, v_2, \ldots, v_{3m}\}$, $U_2 = \{r, s, r_2, s_2\}$, and $U_3 = \{u_1, u_2, \ldots, u_{3n}\}$. We show first that $\gamma(G) = m + n + 2$. Let D be a minimum dominating set of G. We consider four cases.

Case 1: Suppose that $r, s \in D$. Now, $\{r, s\} \mapsto A = \{r_i, s_i;\ 1 \leq i \leq 3\} \cup \{r, s\}$, so $D - \{r, s\}$ must dominate $G-A$. Since $G-A \cong P_{3m-2} \cup P_{3n-2}$, it follows, by the minimality of D, that $|D - \{r, s\}| = \gamma(P_{3m-2} \cup P_{3n-2}) = \left\lceil \frac{3m-2}{3} \right\rceil + \left\lceil \frac{3n-2}{3} \right\rceil = m + n$, whence $|D| = m + n + 2$.

Case 2: Suppose $r \in D$, $s \notin D$. Then, as $D - \{r\} \mapsto s_2$, there exists $t \in \{r_2, s_2\}$ such that $t \in D$. So, $D' = (D - \{t\}) \cup \{s\}$ is a minimum dominating set of G for which r, $s \in D'$, and, as in Case 1, $\gamma(G) = |D'| = m + n + 2$.

Case 3: Suppose $r \notin D$, $s \in D$. This case is similar to Case 2, and we obtain $|D| = m + n + 2$ as above.

Case 4: Suppose r, $s \notin D$. Assume, without loss of generality, that $r_2 \in D$; then $D - \{r_2\} \mapsto G-\{r, r_2, s_2\} \cong P_{3m+3n+1}$ and so $|D - \{r_2\}| \geq m + n + 1$. However, by the minimality of D, $D - \{r_2\}$ is a minimum dominating set of $G-\{r, r_2, s_2\}$, and so $\gamma(G) = |D| = 1 + \gamma(P_{3m+3n+1}) = m + n + 2$.

Next, we show that $\gamma(G-v) = m + n + 1$ for any vertex v of G. Let $v \in V(G)$; then (by Corollary 3.2.14), $\gamma(G-v) \geq m + n + 1$. We consider five cases.

Case 5: Suppose that $v = r$. If L is a minimum dominating set of $G-(U_2 \cup U_3)$ ($\cong P_{3m}$), and R is a minimum dominating set of $G-(U_1 \cup U_2)$ ($\cong P_{3n}$), then $L \cup R \cup \{s_2\} \mapsto G-r$, whence $\gamma(G-v) = |L| + |R| + 1 = m + n + 1$.

Case 6: Suppose that $v = s$. This case is similar to Case 5, and $\gamma(G-v) = m + n + 1$.

Case 7: Suppose $v = r_2$. Then, $G-v$ is a $(3m + 3n + 2)$-cycle with a pendant edge incident with s. Thus, if D is a minimum dominating set of $G-\{r_2, s_2\}$ that contains s, we have $D \mapsto G-v$ and $|D| = m + n + 1$.

Case 8: Suppose $v = s_2$. This case is similar to Case 7.

Case 9: Suppose $v \notin \{r, s, s_2, r_2\}$. Without loss of generality, suppose $v \in U_1$.

Subcase 9.1: Suppose $\langle U_1 - \{v\}\rangle$ is connected; then, $v = r_1$ or $v = s_1$ - say $v = s_1$. Clearly, then, $H = \langle (U_1 \cup \{r\}) - \{s_1\}\rangle \cong P_{3m}$. If D_1 is a minimum dominating set of H and D_2 is a minimum dominating set of $\langle U_3\rangle \cong P_{3n}$, then

$D_1 \cup D_2 \mapsto U_1 \cup U_3 \cup \{r\}$ so that $D^* = D_1 \cup D_2 \cup \{s_2\} \mapsto G-v$, with $|D^*| = m + n + 1$.

Subcase 9.2: Suppose $\langle U_1 - \{v\}\rangle$ is disconnected. Let $H = \langle (U_1 \cup \{r\}) - \{v\}\rangle$; then $H \cong F_1 \cup F_2$ with $F_1 \cong P_a$, $F_2 \cong P_b$, $a + b = 3m$, $r \in V(F_1)$.

Subcase 9.2.1: Suppose $a \equiv 0 \pmod 3$. If D_1 is a minimum dominating set of $\langle U_3\rangle$, D_2 is a minimum dominating set of F_1, where $F_1 \cong P_{3k}$ for some $k \in \mathbb{N}$, and D_3 is a minimum dominating set of F_2 $(\cong P_{3(m-k)})$, then $D_1 \cup D_2 \cup D_3 \mapsto U_1 \cup U_3 \cup \{r\}$, so that $D^* = D_1 \cup D_2 \cup D_3 \cup \{s_2\} \mapsto G-v$ with

$$|D^*| = \left\lceil \frac{3n}{3} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3(m-k)}{3} \right\rceil + 1 = m + n + 1.$$

Subcase 9.2.2: Suppose $a \equiv 2 \pmod 3$. If D_1 is a minimum dominating set of $\langle U_3 - \{r_3, s_3\}\rangle \cong P_{3n-2}$, D_2 is a minimum dominating set of $\langle V(F_1) - \{r, r_1\}\rangle \cong P_{3k}$, where $3k = a - 2$ for some $k \in \mathbb{N}$, and D_3 is a minimum dominating set of $\langle V(F_2) - \{s_1\}\rangle \cong P_{3(m-1-k)}$, then $D_1 \cup D_2 \cup D_3 \mapsto G-(N[r] \cup N[s])$. Thus, $D^* = D_1 \cup D_2 \cup D_3 \cup \{r, s\} \mapsto G-v$, where $|D^*| = n + k + m - 1 - k + 2 = n + m + 1$.

Subcase 9.2.3: Suppose $a \equiv 1 \pmod 3$. Let D_1 be a minimum dominating set of $\langle U_3\rangle$ $(\cong P_{3n})$, D_2 a minimum dominating set of $\langle V(F_1) - \{r\}\rangle \cong P_{3k}$, where $3k = a - 1$, and D_3 a minimum dominating set of $\langle V(F_2) \cup \{s\}\rangle$ $(\cong P_{3(m-k)})$. Then, $D_1 \cup D_2 \cup D_3 \mapsto G-N[r_2]$. So, $D^* = D_1 \cup D_2 \cup D_3 \cup \{r_2\} \mapsto G-v$, where

$$|D^*| = \left\lceil \frac{3n}{3} \right\rceil + \left\lceil \frac{3k}{3} \right\rceil + \left\lceil \frac{3(m-k)}{3} \right\rceil + 1 = m + n + 1.$$

So, G is indeed vertex-domination-critical. □

3.3.2.8.2 <u>Remark</u>: The examples of vertex-domination-critical graphs given in Examples 3.3.2.2, 3.3.2.3 (for $n \geq 2$), 3.3.2.5, and 3.3.2.7 are hamiltonian. That a vertex-domination-critical graph need *not* be hamiltonian is illustrated by Example 3.3.2.1 and the above proposition.

3.3.2.9.1 <u>Remark</u>: The family of graphs that we describe in the next proposition is one that we encountered in Chapter 2; in 2.1.8, we indicated that the graphs belonging to this family are edge-domination-critical. By Proposition 3.3.2.9.2, we know that these graphs are also vertex-domination-critical.

3.3.2.9.2 <u>Proposition</u>: For an integer $n \geq 3$, define the graph Q_n as follows: $V(Q_n) = \{u_i, v_i, w_i; 0 \leq i \leq n-1\}$ and $E(Q_n) = \{u_iu_{i-1}, u_iu_{i+1}, u_iv_{i-1}, u_iv_i, u_iw_i, v_iw_{i-1}, v_iw_i;\ i = 1, 2, \ldots, n\}$, where the subscript arithmetic is interpreted modulo n. Then, Q_n is n-vertex-critical.

<u>Proof</u>: Let n be an integer with $n \geq 3$, and let the graph Q_n be defined as above. By Proposition 2.1.8, $\gamma(Q_n) = n$. Let $y \in V(Q_n)$; then there exists $j \in \{0, 1, \ldots, n-1\}$ such that $y = u_j$, v_j, or w_j. We consider each case separately.

<u>Case 1</u>: Suppose $y = u_j$. Since $N(w_i) = \{u_i, v_i, v_{i+1}\}$, we have that $\{w_0, w_1, \ldots, w_{j-1}, w_{j+1}, \ldots, w_{n-1}\} \succ Q_n - u_j$, and $\gamma(Q_n - u_j) = n - 1$.

<u>Case 2</u>: Suppose $y = v_j$. Observe that $N(v_i) = \{u_i, u_{i+1}, w_{i-1}, w_i\}$, and so $\{v_0, v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_{n-1}\} \succ Q_n - v_j$, and $\gamma(Q_n - v_j) = n - 1$.

<u>Case 3</u>: Suppose $y = w_j$. Since $N(u_i) = \{u_{i-1}, u_{i+1}, v_{i-1}, v_i, w_i\}$, it follows that $\{u_0, u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_{n-1}\} \succ Q_n - w_j$, whence $\gamma(Q_n - w_j) = n - 1$.

Cases 1, 2, and 3 thus show that $\gamma(Q_n - y) = n - 1$ for each $y \in V(Q_n)$, and the desired result follows. □

3.3.2.9.3 <u>Remark</u>: Observe that, for $n \geq 4$ and any $i \in \{0, 1, \ldots, n-1\}$, Q_{n-1} can be formed from Q_n by deleting the vertices u_i, v_i, w_i and adding edges $w_{i-1}v_{i+1}$, $v_{i-1}u_{i+1}$, and $u_{i-1}u_{i+1}$, where the subscripts are taken modulo n.

3.3.3 <u>Remark</u>: The concepts of edge- and vertex-domination-criticality are independent. As we mentioned in Remark 3.3.2.9.1, the family Q_n of graphs (defined in 3.3.2.9.2) are both edge- and vertex-domination-critical, as is the cycle C_4 (by Theorem 2.2.2 and Example 3.3.2.2), while the cycle C_7 is vertex-domination-critical (again, see Example 3.3.2.2), but not edge-domination-critical (since $\gamma(G+e) = 3 = \gamma(G)$ for any edge $e \in E(\overline{C_7})$ that joins two diametrical vertices in C_7). On the other hand, any graph G that is obtained from a complete graph of order at least four by the subdivision of one edge is 2-edge-critical but not 2-vertex-critical ($\gamma(G-v) = 2$ for any vertex v of G not incident with the subdivided edge). However, any vertex-domination-critical graph can be extended to a graph that is both vertex- and edge-domination-critical, as the next theorem shows.

3.3.4 Theorem: For every k-vertex-critical graph G ($k \geq 2$), there exists a graph H such that

(1) G is a spanning subgraph of H, and

(2) H is k-vertex-critical and k-edge-critical.

Proof: Let G be a vertex-domination-critical graph with $\gamma(G) \geq 2$. There exists a finite sequence $(G=)G_0, G_1, \ldots, G_n$ of graphs where, for each $i \in \{1, \ldots, n\}$, $G_i = G_{i-1} + u_iv_i$ where u_iv_i is an element of $E(\overline{G}_{i-1})$ with $\gamma(G_i) = \gamma(G_{i-1}) = k$, and where $\gamma(G_n+e) < \gamma(G_n)$ for each $e \in E(\overline{G}_n)$. Then, $H = G_n$ is a k-edge-critical graph and has G as a spanning subgraph. So, for any $v \in V(H) = V(G)$, a subset D of V(G) of cardinality $k - 1$ that dominates $G-v$ by the vertex-domination-criticality of G also dominates $H-v$, whence $\gamma(H-v) = \gamma(H) - 1$ for each $v \in V(H)$. So, (1) and (2) hold, and the theorem follows. □

3.4 BASIC PROPERTIES OF VERTEX-DOMINATION-CRITICAL GRAPHS

3.4.1 Proposition: If G is a vertex-domination-critical graph, then, for every vertex $v \in V(G)$, there exists a minimum dominating set D of G such that

(1) $v \in D$, and

(2) $D \cap N(v) = \varnothing$.

Proof: Let G be a vertex-domination-critical graph, and let $v \in V(G)$. Then, for any minimum dominating D^* of $G-v$, $D = D^* \cup \{v\} \mapsto G$ with $|D| = \gamma(G)$, $v \in D$, and, by Lemma 3.2.16, $D \cap N(v) = \varnothing$. □

3.4.2 Proposition: Let G be any vertex-domination-critical graph, and u, v any two distinct vertices of G. If D_1 is any minimum dominating set of $G-u$ and D_2 is any minimum dominating set of $G-v$, then $D_1 \neq D_2$.

Proof: Suppose, to the contrary, that there exists a vertex-domination-critical graph G with distinct vertices u and v such that G contains a minimum dominating set D_1 of $G-u$ and a minimum dominating set D_2 of $G-v$ satisfying $D_1 = D_2$. Since $u \notin D_1 = D_2$, some vertex $w \in D_2$ must be adjacent to u (in $G-v$). But, $D_1 = D_2$; so, $w \in D_1$, which implies $N_G(u) \cap D_1 \neq \varnothing$. This contradicts Lemma 3.2.16. Thus, no such vertex-domination-critical graph exists. □

3.4.3 Corollary: If G is a vertex-domination-critical graph and u, v are distinct vertices of G, then $N_G[u] \not\subseteq N_G[v]$.

Proof: Let G be a vertex-domination-critical graph. Let $u, v \in V(G)$ with $u \neq v$, and let D^* be a minimum dominating set of $G-v$. Clearly, in $G-v$, u is dominated by some $w \in D^*$, so $w \in N_G[u]$. By Lemma 3.2.16, w is not adjacent to v; so, $w \notin N_G[v]$. Hence, $N_G[u] \not\subseteq N_G[v]$. □

3.4.4 Proposition: If G is a vertex-domination-critical graph and $N_G(u) \subseteq N_G(v)$ for some vertices u and v of G, then $uv \notin E(G)$ and u belongs to every minimum dominating set of $G-v$.

Proof: Let G be a vertex-domination-critical graph, and suppose that $N_G(u) \subseteq N_G(v)$ for some distinct $u, v \in V(G)$. Then, u and v must be non-adjacent in G; otherwise, $v \in N_G(u)$, whence $v \in N_G(v)$, which is impossible. Let D^* be a minimum dominating set of $G-v$. In $G-v$, u is dominated by some vertex $w \in D^*$. If $w \neq u$, then $w \in N_G(u)$. However, $N_G(u) \subseteq N_G(v)$; so $w \in N_G(v)$, which contradicts Lemma 3.2.16. So, $w = u$, and $u \in D^*$. Since D^* is an arbitrary minimum dominating set of $G-v$, it follows that u belongs to every minimum dominating set of $G-v$. □

3.4.5 Theorem: If G is any graph with a critical vertex v and a non-critical vertex w, then

$$\gamma(G-S-w) \geq \gamma(G-S)$$

for any $S \subseteq N_G[v]$, $v \in S$.

Proof: Let G be any graph with a critical vertex v and a non-critical vertex w, and let $S \subseteq N_G[v]$, $v \in S$. If $w \in S$, the result holds trivially, so suppose that $w \notin S$, and assume, to the contrary, that $\gamma(G-S-w) < \gamma(G-S)$. Then, by Lemma 3.2.13 and Proposition 3.2.17, we obtain

$$\gamma(G-S-w) = \gamma(G-S) - 1 = [\gamma(G) - 1] - 1 = \gamma(G) - 2.$$

If $w \notin N_G(v)$, then $N_{G-w}(v) = N_G(v)$ and $S \subseteq N_{G-w}(v)$. If $w \in N_G(v)$, then $N_{G-w}(v) = N_G(v) - \{w\}$; however, $w \notin S$, so we have $S \subseteq N_{G-w}(v)$ in this case, too. Thus, we can apply Proposition 3.2.15 to the graph G-w, and obtain

$$\gamma(G-w-S) \geq \gamma(G-w) - 1.$$

Thus,

$$\gamma(G-w) \leq \gamma(G-w-S) + 1 = \gamma(G) - 2 + 1 = \gamma(G) - 1 < \gamma(G).$$

This contradicts our assumption that w is a non-critical vertex of G. □

3.4.6 Corollary: If G is any graph with $s < p(G)$ non-critical vertices and v is G-critical, then G-v has at least s non-critical vertices.

Proof: The result follows from Theorem 3.4.5 if we observe that setting $S = \{v\}$ shows that every non-critical vertex of G is also a non-critical vertex of G-v. □

3.4.7 Theorem: If G is a vertex-domination-critical graph, then each non-isolated vertex of G is contained in at least two cliques of G.

Proof: Suppose, to the contrary, that there exists a vertex-domination-critical graph G with a vertex u such that $\deg_G u \geq 1$ and u belongs to exactly one clique, H say. Since $\deg_G u \geq 1$, $p(H) \geq 2$; let $v \in V(H) - \{u\}$. Now, if u is adjacent to a vertex $x \notin V(H)$, then u belongs to the complete subgraph $\langle\{ux\}\rangle$, where $ux \notin E(H)$. This implies that u belongs to at least two cliques of G, namely H and some clique that contains the edge ux, which is contrary to assumption. So, $N_G[u] = V(H)$. Since $V(H) \subseteq N_G[v]$, it follows that $N_G[u] \subseteq N_G[v]$. This contradicts Corollary 3.4.3. Thus, no such vertex-domination-critical graph exists. □

3.4.8 Corollary: No vertex-domination-critical graph has an end-vertex.

Another result concerning complete subgraphs is the following.

3.4.9 Lemma: If a graph G has a non-isolated vertex v such that $\langle N_G(v)\rangle$ is complete, then G is not vertex-domination-critical.

Proof: Suppose, to the contrary, that there exists a vertex-domination-critical graph G with a non-isolated vertex v such that $\langle N_G(v)\rangle \cong K_{\deg\, v}$. Let $u \in N_G(v)$. Clearly, any minimum dominating set D for G-u must contain at least one vertex x from $N_G[v] - \{u\}$. Since G is vertex-domination-critical, $|D| = \gamma(G) - 1$. Now, since $N_G[v]$ is complete, we have $xu \in E(G)$, i.e., $D \mapsto u$ in G. So, $D \mapsto G$ and $\gamma(G) \leq |D| < \gamma(G)$. This absurdity establishes the lemma. □

3.4.10 Remark: Notice that Corollary 3.4.8 is also a corollary of Lemma 3.4.9.

3.5 RESULTS INVOLVING OTHER PARAMETERS OF VERTEX-DOMINATION-CRITICAL GRAPHS

3.5.1 Proposition: The minimum number of vertices which must be removed from a non-complete graph G to produce a vertex-domination-critical graph is at most $p(G) - \beta(G)$.

Proof: Let G be any non-complete graph, and let S be a maximum independent set in G. Then, the graph $H = G-(V(G) - S) = \langle S\rangle$ has $q(H) = 0$ and $p(H) \geq 2$; so, since a non-trivial, empty graph is vertex-domination-critical (by Example 3.3.2.1), the result follows. □

3.5.2 Remark: The bound in Proposition 3.5.1 is sharp.

Proof: Let $c_m(G)$ denote the minimum number of vertices that must be removed from G to produce a graph that is vertex-domination-critical. Let $p \geq 2$. Then, any one of the following three observations provides a proof of the remark.

1. $0 \leq c_m(\overline{K}_p) \leq p - \beta(\overline{K}_p) = p - p = 0$, so that $c_m(\overline{K}_p) = p - \beta(\overline{K}_p)$.

2. Recall from Corollary 3.4.8 that a vertex-domination-critical graph has no end-vertices. Thus, P_p is itself not vertex-domination-critical, and the only induced subgraph of the path P_p that is vertex-domination-critical is an induced subgraph that is empty. Thus, a largest, induced, vertex-

domination-critical subgraph of G (i.e., an induced vertex-domination-critical subgraph of G that is obtained by the removal of the smallest number of vertices) has order $\beta(G)$ and so $c_m(P_p) = p - \beta(G)$.

3. We have seen before (Example 3.3.2.2) that the cycles C_p are not vertex-domination-critical for $p \equiv 0, 2 \pmod 3$. Suppose that $p \equiv 0$ or $2 \pmod 3$. Then, as above, any induced subgraph of C_p that is a vertex-domination-critical graph must be empty; the largest such graph has order $\beta(C_p)$ and $c_m(C_p) = p - \beta(C_p)$. □

3.5.3 Remark: From Theorem 2.2.5 of [LW1], we know that the order p and maximum degree Δ of any graph are related to its domination number γ by $p \leq (\Delta + 1)\gamma$ (see also Corollary 5.2.4). Since $(a + 1)(b - 1) + 1 < (a + 1)b$ for $a, b \in \mathbb{N}$, the bound presented in the theorem below for graphs having critical vertices is an improvement on the bound $p \leq (\Delta + 1)\gamma$, provided G is non-empty.

3.5.4 Theorem: If G is a graph with at least one critical vertex, then

$$p(G) \leq [\Delta(G) + 1][\gamma(G) - 1] + 1.$$

Proof: Let G be a graph of order p, domination number γ and maximum degree Δ, for which there exists $v \in V(G)$ with $\gamma(G-v) < \gamma(G)$. By Lemma 3.2.13, $\gamma(G-v) = \gamma - 1$. Applying the result from [LW1] mentioned in Remark 3.5.3, we thus have

$$p(G-v) \leq [\Delta(G-v) + 1]\, \gamma(G-v),$$

i.e.,

$$p - 1 \leq [\Delta(G-v) + 1]\, [\gamma - 1];$$

since $\Delta(G-v) \leq \Delta(G)$, we have finally that $p \leq (\Delta + 1)(\gamma - 1) + 1$. □

3.5.5 Corollary: For every vertex-domination-critical graph G,

$$p(G) \leq [\Delta(G) + 1]\, [\gamma(G) - 1] + 1.$$

3.5.6 Remark: Both the bounds $p(G) \leq (\Delta(G) + 1)\gamma(G)$ and $p(G) \leq [\Delta(G) + 1][\gamma(G) - 1] + 1$ for a graph G are sharp, since they are attained for empty graphs. Another class of graphs that shows that the bound of Theorem 3.5.4 is best possible is the infinite class of n-vertex-critical graphs $G_{m,n}$ defined in Proposition 3.3.2.6.1, since $p(G_{m,n}) = (n - 1)(m + 1) + 1 = [\Delta(G_{m,n}) + 1]\,[\gamma(G_{m,n}) - 1] + 1$.

The following theorem extends the result of Corollary 3.5.5.

3.5.7 Theorem: If G is a vertex-domination-critical graph, then

$$p(G) \leq [\Delta(G) + 1]\,[\gamma(G) - k(G)] + k(G).$$

Proof: Let G be a vertex-domination-critical graph, with $p = p(G)$, $\gamma = \gamma(G)$, $\Delta = \Delta(G)$ and $k = k(G)$. Let the components of G be $G_1, G_2, \ldots, G_k$ and let $p_i = p(G_i)$, $\Delta_i = \Delta(G_i)$ and $\gamma_i = \gamma(G_i)$ for $i \in \{1, 2, \ldots, k\}$. Clearly, each component G_i $(i \leq i \leq k)$ of G is vertex-domination-critical. By Corollary 3.5.5, $p_i \leq (\Delta_i + 1)(\gamma_i - 1) + 1$ for $i = 1, 2, \ldots, k$. Thus,

$$\begin{aligned} p = \sum_{i=1}^{k} p_i &\leq \sum_{i=1}^{k} \left[(\Delta_i + 1)(\gamma_i - 1) + 1\right] \\ &\leq \sum_{i=1}^{k} \left[(\Delta + 1)(\gamma_i - 1)\right] + k \\ &\leq (\Delta + 1)\sum_{i=1}^{k}(\gamma_i - 1) + k \\ &= (\Delta + 1)(\gamma - k) + k. \qquad \Box \end{aligned}$$

3.5.8 Proposition: If G is a regular graph that is not complete, and $p(G) = \Delta(G) + \gamma(G)$, then G is vertex-domination-critical.

Proof: Let G be a graph satisfying the hypothesis of the proposition. By Proposition 2.1.3, for any $v \in V(G)$,

$$p(G-v) \geq \Delta(G-v) + \gamma(G-v),$$

i.e.,

$$p(G) - 1 \geq \Delta(G) + \gamma(G-v),$$

i.e.,

$$\gamma(G-v) \leq p(G) - \Delta(G) - 1 = \gamma(G) - 1 < \gamma(G).$$

So, G is vertex-domination-critical. $\Box$

3.5.9 Remark: Notice that the above proposition, combined with Proposition 2.1.4, shows that, for a graph G, the property of being regular and non-complete and having $p(G) = \Delta(G) + \gamma(G)$ is sufficient to ensure both vertex- and edge-domination-criticality.

3.5.10 Proposition: Let G be a non-empty graph of order $p > 2$. Then, G is regular, non-complete and satisfies $p = \Delta(G) + \gamma(G)$ if and only if p is even and G is isomorphic to the graph obtained from K_p by the removal of the edges in a 1-factor.

Proof: Let $n \in \mathbb{N}$, and let $G = H-F$, where $H \cong K_{2n}$ and F is the edge set of a 1-factor of H. That G is non-complete and $(p - 2)$-regular follows immediately. By Example 3.3.2.3, G is 2-vertex-critical, so $\gamma(G) = 2$ and $\Delta(G) + \gamma(G) = p - 2 + 2 = p$.

For the converse, suppose that G is a non-empty, Δ-regular, non-complete graph such that $p = \Delta(G) + \gamma(G) = \Delta + \gamma(G)$. By Proposition 3.5.8, G is k-vertex-critical for some $k \geq 2$. We shall show that $k = 2$, whence the proposition will follow. Let $v \in V(G)$ and suppose, to the contrary, that $k > 2$. Then, $\Delta = p - k \leq p - 3$, so that $|V(G) - N[v]| \geq 2$. Since G is non-empty, $|N(v)| = \Delta \geq 1$. We consider two cases.

Case 1: Suppose that $G-N[v]$ is empty. Then (since G is Δ-regular), it must follow that every vertex in $V(G) - N[v]$ is adjacent to every vertex in $N(v)$. So, for any $v_1 \in N(v)$, $\{v, v_1\} \mapsto G$, and $k = \gamma(G) \leq 2$, a contradiction.

Case 2: Suppose that $G-N[v]$ is non-empty; let $ab \in E(\langle V(G) - N[v]\rangle)$. Then, $V(G) - (N(v) \cup \{a\}) \mapsto G$, so that $\gamma(G) \leq p - \Delta - 1$. Thus, $\Delta(G) + \gamma(G) \leq p - 1 < p$, a contradiction. □

3.5.11 Remark: In the following theorem, we establish another upper bound on the order of a vertex-domination-critical graph.

3.5.12 Theorem: If G is a vertex-domination-critical graph with $p = p(G)$, $q = q(G)$, $\gamma = \gamma(G)$, and $\Delta = \Delta(G)$, then $p \leq \frac{1}{3}(2q + 3\gamma - \Delta)$.

Proof: Let G be a γ-vertex-critical graph, $\gamma \geq 2$, and let $\Delta = \Delta(G)$, $p = p(G)$, and $q = q(G)$. Suppose first that G has no isolated vertices. By Corollary 3.4.8, every vertex of G has at least two neighbours. Let v be a vertex of G of maximum degree Δ and let $D = D' \cup \{v\}$, where D'

is a minimum dominating set of $G-v$. Since G is vertex-domination-critical, $\bar{D}$ is a minimum dominating set of G.

By Lemma 3.2.16, no vertex of $N(v)$ belongs to D', so each vertex x of $N(v)$ has (at least) 2 neighbours in D, namely v and some vertex in D'. Furthermore, each vertex y of $V(G) - (D \cup N(v))$ is adjacent to at least one vertex y' of D' (by the definition of D'), and so a further $p - \gamma - \Delta$ edges of G are thus accounted for. Also, each of these $p - \gamma - \Delta$ vertices y in $V(G) - (D \cup N(v))$ has a neighbour y'' distinct from y', and thus the vertices of $V(G) - (D \cup N(v))$ contribute at least another $\lceil \frac{1}{2}(p - \gamma - \Delta) \rceil$ more edges to G. Therefore,

$$q \geq 2|N(v)| + (p - \gamma - \Delta) + \left\lceil \frac{p-\gamma-\Delta}{2} \right\rceil = p - \gamma + \Delta + \left\lceil \frac{p-\gamma-\Delta}{2} \right\rceil .$$

We consider two cases, dependent on the parity of $p - \gamma - \Delta$.

Case 1: If $p - \gamma - \Delta$ is even, then

$$q \geq \frac{2p - 2\gamma + 2\Delta + p - \gamma - \Delta}{2} = \frac{3p - 3\gamma + \Delta}{2},$$

whence $p \leq \frac{1}{3}(2q + 3\gamma - \Delta)$.

Case 2: Suppose $p - \gamma - \Delta$ is odd. Then

$$q \geq \frac{2p - 2\gamma + 2\Delta}{2} + \frac{p - \gamma + \Delta + 1}{2} = \frac{3p - 3\gamma + \Delta + 1}{2},$$

whence $p \leq \frac{1}{3}(2q + 3\gamma - \Delta - 1) < \frac{1}{3}(2q + 3\gamma - \Delta)$.

Thus, Cases 1 and 2 show that the result holds for vertex-domination-critical graphs without isolated vertices. Suppose now that $G = \overline{K}_t \cup G^*$ where $p = p(G) = p(G^*) + t$ and G^* has no isolated vertices. Clearly, $p(G^*) = p - t$, $\Delta(G^*) = \Delta$, $q(G^*) = q$, and $\gamma(G^*) = \gamma$. Applying our result to the graph G^*, we obtain

$$p - t \leq \tfrac{1}{3}[2q + 3(\gamma - t) - \Delta] = \tfrac{1}{3}(2q + 3\gamma - \Delta) - t,$$

and the desired result follows for graphs with isolated vertices also. □

3.5.13 Remark: The bound in the above theorem is best possible since, for example, for $k \in \mathbb{N}$, C_{3k+1} is a vertex-domination-critical graph (see Example 3.3.2.2) and

$$\tfrac{1}{3}[2q(C_{3k+1}) + 3\gamma(C_{3k+1}) - \Delta(C_{3k+1})] = \tfrac{1}{3}[2(3k+1) + 3(k+1) - 2] = 3k+1 = p(C_{3k+1}).$$

3.5.14 Remark: We recall the following result of Vizing [V1] (also given in [C1]). Extensions of, and results based on, Lemma 2.1.9 are given in the three succeeding theorems.

2.1.9 Lemma: For any graph G,

$$\gamma(G) \le p(G) + 1 - \sqrt{2q(G)+1}.$$

3.5.15 Theorem: If G is a graph such that

$$\gamma(G) > p(G) + 1 - \sqrt{2q(G)+2},$$

then G is vertex-domination-critical.

Proof: Suppose, to the contrary, that there exists a graph G, with order p, size q, and domination number γ, that satisfies the hypothesis of the theorem, but for which there exists $v \in V(G)$ with $\gamma(G-v) \ge \gamma(G)$. By Lemma 2.1.9, we know that

$$\begin{aligned}\gamma(G-v) &\le p(G-v) + 1 - \sqrt{2q(G-v)+1}\\ &= p - 1 + 1 - \sqrt{2(q-\deg_G v)+1}\\ &= p - \sqrt{2q - 2\deg_G v + 1}.\end{aligned}$$

Thus, we have

$$p - \sqrt{2q - 2\deg_G v + 1} \ge \gamma(G-v) \ge \gamma > p + 1 - \sqrt{2q+2},$$

i.e.,

$$\sqrt{2q+2} - 1 > \sqrt{2q - 2\deg_G v + 1},$$

and squaring gives

$$2q + 2 - 2\sqrt{2q+2} + 1 > 2q - 2\deg_G v + 1.$$

whence

$$2\deg_G v + 2 > 2\sqrt{2q + 2}.$$

So, since, by our assumption,

$$\sqrt{2q+2} > p - \gamma + 1,$$

we have

$$2 \deg_G v + 2 > 2 (p - \gamma + 1),$$

i.e.,

$$\deg_G v > p - \gamma.$$

Hence, $\Delta(G) \geq \deg_G v > p - \gamma \geq \Delta(G)$, which is impossible. So, the desired result follows.□

3.5.16 <u>Theorem</u>: If G is a graph such that

$$\gamma(G) = p(G) + 1 - \sqrt{2q(G)+1},$$

then G is vertex-domination-critical.

<u>Proof</u>: The result follows from Theorem 3.5.15 but can also be proved independently, as follows.

Suppose, to the contrary, that there exists a graph G that satisfies the hypothesis of the theorem, but which contains a non-trivial vertex v. Let $p = p(G)$, $q = q(G)$, and $\gamma = \gamma(G)$. By Lemma 2.1.9,

$$\gamma(G-v) \leq p(G-v) + 1 - \sqrt{2q(G-v) + 1} = p - \sqrt{2q-2\deg_G v + 1},$$

and so, since v is non-G-critical,

$$p - \sqrt{2q-2\deg_G v+1} \geq \gamma(G-v) \geq \gamma = p + 1 - \sqrt{2q+1}.$$

Then,

$$\sqrt{2q + 1} \geq 1 + \sqrt{2q - 2\deg_G v + 1},$$

and squaring gives

$$2q + 1 \geq 1 + 2\sqrt{2q - 2\deg_G v + 1} + 2q - 2\deg_G v + 1,$$

i.e.,

$$2\deg_G v - 1 \geq 2\sqrt{2q - 2\deg_G v + 1}.$$

By squaring again, we obtain

$$4(\deg_G v)^2 - 4\deg_G v + 1 \geq 4(2q - 2\deg_G v + 1),$$

i.e.,

$$4(\deg_G v)^2 + 4\deg_G v + 1 \geq 4(2q + 1),$$

i.e.,

$$(2\deg_G v + 1)^2 \geq 4(2q + 1),$$

i.e.,

$$2\deg_G v + 1 \geq 2\sqrt{2q + 1},$$

whence

$$2\deg_G v \geq 2\sqrt{2q + 1} - 1. \qquad \text{..........(1)}$$

Furthermore, we have

$$p + 1 - \sqrt{2q+1} = \gamma \leq p - \Delta,$$

i.e.,

$$\sqrt{2q + 1} \geq \Delta + 1. \qquad \text{..........(2)}$$

Hence,

$$\begin{aligned} 2\Delta &\geq 2\deg_G v \geq 2\sqrt{2q + 1} - 1 && \text{(from (1))} \\ &\geq 2(\Delta + 1) - 1 && \text{(from (2))} \\ &= 2\Delta + 1, \end{aligned}$$

which is absurd. Thus, no such graph G exists, and the result follows. □

3.5.17 Remark: By Theorem 2.1.11 and the preceding theorem, it follows that, for any graph G, the condition

$$\gamma(G) = p(G) + 1 - \sqrt{2q(G)+1},$$

is sufficient to ensure that G is both vertex- and edge-domination-critical.

3.5.18 Theorem: If G is a graph such that

$$\gamma(G) \geq p(G) - \sqrt{2q(G) - 2\Delta(G)},$$

then G is vertex-domination-critical.

Proof: Suppose, to the contrary, that there exists a (p,q) graph G, having maximum degree Δ and domination number γ, that satisfies the hypothesis of the theorem but that contains a non-critical vertex v. By Lemma 2.1.9,

$$\gamma(G-v) \leq p(G-v) + 1 - \sqrt{2q(G-v) + 1} = p - \sqrt{2q - 2\deg_G v + 1}.$$

Thus,

$$p - \sqrt{2q - 2\deg_G v + 1} \geq \gamma(G-v) \geq \gamma(G) \geq p - \sqrt{2q - 2\Delta},$$

i.e.,

$$\sqrt{2q - 2\Delta} \geq \sqrt{2q - 2\deg_G v + 1},$$

i.e.,

$$2q - 2\Delta \geq 2q - 2\deg_G v + 1,$$

so that

$$2\deg_G v \geq 2\Delta + 1.$$

However, this is impossible since $\Delta \geq \deg_G v$. Hence, no such graph G exists and the theorem follows. □

3.6 CONSTRUCTING VERTEX-DOMINATION-CRITICAL GRAPHS

In this section, we show how new vertex-domination-critical graphs can be generated from smaller vertex-domination-critical graphs. Recall first the following definition.

3.6.1 Definition: Given disjoint graphs G and H, and vertices $x \in V(G)$ and $y \in V(H)$, the *(u,v)-coalescence* of G and H, denoted by $(G,x)\bullet(H,y)$, is the graph obtained from G and H by identifying the vertices x and y. We denote by $u_{(G,x)\bullet(H,y)}$ the vertex of $(G,x)\bullet(H,y)$ that is the result of the identification of x and y. If the identified vertices x and y of G and H, respectively, are understood, we write $G\bullet H$ instead of $(G,x)\bullet(H,y)$.

3.6.2 Remark: It is immediately obvious that a graph G is vertex-domination-critical if and only if every component of G is vertex-domination-critical.

3.6.3 Lemma: Let G and H be any non-trivial graphs, and consider any coalescence $G \bullet H$ of G and H. Then,

(1) $\gamma(G) + \gamma(H) - 1 \leq \gamma(G \bullet H) \leq \gamma(G) + \gamma(H)$, and

(2) if both G and H are vertex-domination-critical, or if $G \bullet H$ is vertex-domination-critical, then $\gamma(G \bullet H) = \gamma(G) + \gamma(H) - 1$.

Proof: Let G and H be any two non-trivial graphs and let $(G,u_G) \bullet (H,u_H)$ be any coalescence of G and H. Let

$$u = u_{(G,u_G)\cdot(H,u_H)}$$

Then, $V(G \bullet H) = V_H \cup V_G$, where $V_G = (V(G) - \{u_G\}) \cup \{u\}$, and $V_H = (V(H) - \{u_H\}) \cup \{u\}$.

The upper bound is easy to establish. If D_1 and D_2 are minimum dominating sets of $\langle V_G \rangle$ and $\langle V_H \rangle$, respectively, then $|D_1| = \gamma(G)$, $|D_2| = \gamma(H)$, and $D_1 \cup D_2 \mapsto G \bullet H$; so

$$\gamma(G \bullet H) \leq |D_1 \cup D_2| \leq |D_1| + |D_2| = \gamma(G) + \gamma(H).$$

Now suppose there exists a subset D of $V(G \bullet H)$ with $|D| = \gamma(G) + \gamma(H) - 2$ such that $D \mapsto G \bullet H$. Let $D_H = D \cap V_H$ and $D_G = D \cap V_G$. We consider two cases.

Case 1: Suppose $u \notin D$. Then, since $D \mapsto \{u\}$, at least one of $N_{G \bullet H}(u) \cap D_G$, $N_{G \bullet H}(u) \cap D_H \neq \varnothing$; suppose $N_{G \bullet H}(u) \cap D_G \neq \varnothing$. So, $D_G \mapsto \langle V_G \rangle \cong G$, whence $|D_G| \geq \gamma(G)$, and $D_H \mapsto \langle V_H - \{u\} \rangle \cong H - u_H$, whence $|D_H| \geq \gamma(H - u_H)$. Thus, we have

$$|D_H| = |D| - |D_G| \leq [\gamma(G) + \gamma(H) - 2] - \gamma(G) = \gamma(H) - 2.$$

However, then $\gamma(H - u_H) \leq |D_H| \leq \gamma(H) - 2$, which contradicts Corollary 3.2.14. This contradiction shows that Case 1 does not occur.

Case 2: Suppose $u \in D$. Then, $D_G \mapsto \langle V_G \rangle \cong G$ and $D_H \mapsto \langle V_H \rangle \cong H$, so that $|D_G| \geq \gamma(G)$, and $|D_G \cap D_H| = 1$. Thus,

$$|D_H| = |D| - |D_G| + 1 \leq [\gamma(G) + \gamma(H) - 2] - \gamma(G) + 1 = \gamma(H) - 1,$$

which implies that D_H that does not dominate $\langle V_H \rangle \cong H$. So, Case 2 does not occur, either.

Thus, we must have $\gamma(G \bullet H) \geq \gamma(G) + \gamma(H) - 1$, and statement (1) of the lemma follows.

Suppose now that both G and H are vertex-domination-critical. Let D_1 and D_2 be minimum dominating sets for $G - u_G$ and $H - u_H$, respectively. Since G and H are vertex-domination-critical, we know that $|D_1| = \gamma(G) - 1$, $|D_2| = \gamma(H) - 1$. Clearly, $D_1 \cup D_2 \cup \{u\} \rightarrowtail G \bullet H$, whence

$$\gamma(G \bullet H) \leq |D_1| + |D_2| + 1 = [\gamma(G) - 1] + [\gamma(H) - 1] + 1 = \gamma(G) + \gamma(H) - 1.$$

Since $\gamma(G \bullet H) \in \{\gamma(G) + \gamma(H) - 1, \gamma(G) + \gamma(H)\}$, the desired result follows.

Finally, suppose that $G \bullet H$ is vertex-domination-critical. Suppose, to the contrary, that $\gamma(G \bullet H) = \gamma(G) + \gamma(H)$. Since $G \bullet H$ is vertex-domination-critical,

$$\gamma((G \bullet H) - u) = \gamma(G \bullet H) - 1 = \gamma(G) + \gamma(H) - 1.$$

Now, $(G \bullet H)$-u consists of two components, namely $H - u_H$ and $G - u_G$. We have proved above that if u_G and u_H are critical vertices of G and H, respectively, then $\gamma(G \bullet H) = \gamma(G) + \gamma(H) - 1$. So, suppose now, without loss of generality, that $\gamma(H - u) \geq \gamma(H)$. Then, from

$$\gamma(G) + \gamma(H) - 1 = \gamma((G \bullet H) - u) = \gamma(G - u) + \gamma(H - u) \geq \gamma(G - u) + \gamma(H),$$

we have $\gamma(G - u) \leq \gamma(G) - 1$. So, by Corollary 3.2.14, $\gamma(G - u) = \gamma(G) - 1$. However, then, if D_1 and D_2 are minimum dominating set of $G - u$ and H, respectively, we have that $D^* = D_1 \cup D_2$ is a dominating set of $G \bullet H$, whence

$$\gamma(G \bullet H) \leq |D^*| = |D_1| + |D_2| = \gamma(G) + \gamma(H) - 1.$$

This contradicts our assumption. So, $\gamma(G \bullet H)$ is indeed $\gamma(G) + \gamma(H) - 1$. □

In the course of the above proof, the following corollary has been shown to hold:

3.6.4 <u>Corollary</u>: If G and H are graphs with critical vertices x and y, respectively, then

$$\gamma((G,x)\bullet(H,y)) = \gamma(G) + \gamma(H) - 1.$$

3.6.5 Lemma: For any graphs G and H, any coalescence G•H is vertex-domination-critical if and only if both G and H are vertex-domination-critical.

Proof: Let G and H be any two graphs; consider any coalescence $(G,u_G)\bullet(H,u_H)$ of G and H. Let

$$u = u_{(G,u_G)\cdot(H,u_H)},$$

$V_G = (V(G) - \{u_G\}) \cup \{u\}$, and $V_H = (V(H) - \{u_H\}) \cup \{u\}$.

We suppose first that both G and H are vertex-domination-critical. By Lemma 3.6.3(2), this implies that $\gamma(G\bullet H) = \gamma(G) + \gamma(H) - 1$. Let $v \in V(G\bullet H)$. We consider three cases.

Case 1: Suppose $v = u$. Then, if D_G and D_H are minimum dominating sets of $G-u_G$ and $H-u_H$, respectively, $|D_G| = \gamma(G) - 1$ and $|D_H| = \gamma(H) - 1$ (by the vertex-domination-criticality of G and H), and, furthermore, $D = D_G \cup D_H \mapsto (G\bullet H)-u$, where $|D| = \gamma(G) + \gamma(H) - 2$. Thus, by Corollary 3.2.14, $\gamma(G\bullet H-u) = \gamma(G\bullet H) - 1$.

Case 2: Suppose $v \in V(G) - \{u_G\}$. Let D_G be a minimum dominating set of $G-v$. Then, $|D_G| = \gamma(G) - 1$; clearly, $D_G \mapsto \{u\}$. Let D_H be a minimum dominating set of $H-u_H$; then $|D_H| = \gamma(H) - 1$. Since $D = D_G \cup D_H \mapsto (V_G - \{v\}) \cup V_H = V(G\bullet H) - \{v\}$, we have

$$\gamma((G\bullet H)-v) \leq |D| = \gamma(G) + \gamma(H) - 2,$$

and it follows that

$$\gamma((G\bullet H)-v) = \gamma(G\bullet H) - 1.$$

Case 3: Suppose $v \in V(H) - \{u_H\}$. This case proceeds analogously to Case 2.

Cases 1 to 3 show that G•H is vertex-domination-critical.

To prove the converse, suppose that G•H is vertex-domination-critical. As before, this implies, by Lemma 3.6.3(2), that $\gamma(G\bullet H) = \gamma(G) + \gamma(H) - 1$. We shall show that G is vertex-domination-critical; that H is vertex-domination-critical also is shown in a similar way.

Let $v \in V(G)$, and let D be a minimum dominating set of $(G \bullet H)-v$. Then, $|D| = \gamma(G) + \gamma(H) - 2$. We consider two cases.

<u>Case 1</u>: Suppose $v = u_G$. Let $D_G = D \cap V_G$, $D_H = D \cap V_H$. Clearly, $D_G \mapsto G-u_G$ and $D_H \mapsto H-u_H$. If $|D_G| \geq \gamma(G)$, then

$$|D_H| = |D| - |D_G| \leq (\gamma(G) + \gamma(H) - 2) - \gamma(G) = \gamma(H) - 2.$$

However, this contradicts the fact that $D_H \mapsto H-u_H$, since $\gamma(H-u_H) \geq \gamma(H) - 1$. So, $|D_G| \leq \gamma(G) - 1$. But, $D_G \mapsto G-u_G$ implies $|D_G| \geq \gamma(G-u_G) \geq \gamma(G) - 1$. Thus, $\gamma(G-v) = \gamma(G-u_G) = |D_G| = \gamma(G) - 1$.

<u>Case 2</u>: Suppose $v \in V(G) - \{u_G\}$ and assume that $\gamma(G-v) \geq \gamma(G)$; so, $\gamma(G-\{u_G, v\}) \geq \gamma(G) - 1$. Let $D_H = D \cap V_H$ and $D_G = D \cap V_G$.

<u>Subcase 2.1</u>: Suppose $u \in D$. Then, $D_H \mapsto \langle V_H \rangle \cong H$, whence $|D_H| \geq \gamma(H)$ and $D_G \mapsto \langle V_G - \{v\} \rangle \cong G-v$, whence $|D_G| \geq \gamma(G) - 1$. Now, $\gamma(G) + \gamma(H) - 2 = |D_G| + |D_H| - 1$; so, we must have $|D_H| = \gamma(H)$ and $|D_G| = \gamma(G) - 1$. Thus, $\gamma(G) - 1 \leq \gamma(G-v) \leq |D_G|$ gives $\gamma(G-v) = \gamma(G) - 1$, contradicting our assumption that $\gamma(G-v) \geq \gamma(G)$.

<u>Subcase 2.2</u>: Suppose $u \notin D$. Then, $D_H \mapsto H-u_H$ and $D_G \mapsto \langle V_G - \{v\} \rangle \cong G-v$, whence $|D_H| \geq \gamma(H) - 1$ and $|D_G| \geq \gamma(G) - 1$, or $D_H \mapsto H$ and $D_G \mapsto \langle V_G - \{u, v\} \rangle \cong G-\{u_G, v\}$, whence $|D_H| \geq \gamma(H)$ and $|D_G| \geq \gamma(G) - 1$. However, the latter possibility does not hold, since, otherwise, $|D| = |D_G| + |D_H| \geq \gamma(G) + \gamma(H) - 1 > |D|$. So, the first possiblity holds, and, in fact, $|D| = \gamma(G) + \gamma(H) - 2 = |D_G| + |D_H|$ implies that $|D_H| = \gamma(H) - 1$ and $|D_G| = \gamma(G) - 1$. Thus, as in the previous case, $\gamma(G-v) = \gamma(G) - 1$, and a contradiction to our assumption that $\gamma(G-v) \geq \gamma(G)$ is produced.

Thus, Case 2 shows that $\gamma(G-v) < \gamma(G)$ for all $v \in V(G) - \{u_G\}$. Combined with Case 1, this proves that G is vertex-domination-critical. As mentioned above, the vertex-domination-criticality of H follows similarly. □

The following theorem provides a method of constructing large classes of vertex-domination-critical graphs.

3.6.6 <u>Theorem</u>: (1) A graph G is vertex-domination-critical if and only if every block of G is vertex-domination-critical.

(2) If a graph G is

$$\gamma(G) = \sum_{i=1}^{n} \gamma(G_i) - (n-1).$$

Proof: We will proceed by induction on n, the number of blocks in G. If $n = 1$, then result (1) is immediate. Also, $\Sigma_{i=i}^{n} \gamma(G_i) - (n - 1) = \gamma(G_1) = \gamma(G)$, so (2) holds in this case also. Assume that both results hold for $n = k$, where $k \geq 1$. Suppose, now, that G is a graph with $k + 1$ blocks $G_1, G_2, \ldots, G_{k+1}$. Since $k + 1 \geq 2$, G has a cut-vertex, and we assume that these $k + 1$ blocks have been labelled so that G_{k+1} is an end-block of G, i.e., G_{k+1} contains only one cut-vertex, v say, of G. Then, if $H = \langle [V(G) - V(G_{k+1})] \cup \{v\} \rangle_G$, i.e., if $H = \langle \bigcup_{i=1}^{k} V(G_i) \rangle_G$, then $G = H \bullet G_{k+1}$, where it is the vertex v of G_{k+1} and the vertex v of H that have been identified in the coalescence.

Suppose first that every block of G is vertex-domination-critical. This means, by the inductive hypothesis, that H, comprising the k blocks $G_1, G_2, \ldots, G_k$, is vertex-domination-critical. Since G_{k+1} is, by assumption, also vertex-domination-critical, it follows, by Lemma 3.6.5, that $G = H \bullet G_{k+1}$ is vertex-domination-critical. Conversely, if we assume that $G = H \bullet G_{k+1}$ is vertex-domination-critical, then, again by Lemma 3.6.5, H and G_{k+1} are both vertex-domination-critical. By the inductive hypothesis, H being vertex-domination-critical implies that each of its blocks $G_1, G_2, \ldots, G_k$ is vertex-domination-critical. Thus, result (1) follows.

We now consider the second result of the theorem. By the inductive hypothesis, if H is vertex-domination-critical, then

$$\gamma(H) = \sum_{i=1}^{k} \gamma(G_i) - (k-1).$$

Suppose now that G is vertex-domination-critical. Then, by Lemma 3.6.3(2), we have

$$\begin{aligned}
\gamma(G) &= \gamma(H \cdot G_{k+1}) = \gamma(H) + \gamma(G_{k+1}) - 1 \\
&= \left[\sum_{i=1}^{k} \gamma(G_i) - (k-1)\right] + \gamma(G_{k+1}) - 1 \\
&= \sum_{i=1}^{k+1} \gamma(G_i) - k \\
&= \sum_{i=1}^{k+1} \gamma(G_i) - [(k+1) - 1].
\end{aligned}$$

The theorem now follows, by the Principle of Mathematical Induction. □

The following theorem illustrates another method of constructing infinite classes of vertex-domination-critical graphs.

3.6.7 Theorem: Let G be any graph of order p and size q. Let $E(G) = \{e_1, e_2, \ldots, e_q\}$. For $i \in \{1, 2, \ldots, q\}$, let H_i be a vertex-domination-critical graph having two vertices u_i and v_i with the following properties:

(i) u_i and v_i are both in some minimum dominating set of H_i,

(ii) u_i belongs to some minimum dominating set of $H-v_i$, and

(iii) v_i belongs to some minimum dominating set of $H-u_i$.

Construct G^* from G by replacing, for each $i \in \{1, 2, \ldots, q\}$, the edge e_i by H_i where the end-vertices of e_i in G are identified in G^* with vertices u_i and v_i of H_i. Then,

(1) $\gamma(G) = p + \Sigma_{i=1}^{q} [\gamma(H_i) - 2]$, and

(2) G^* is a vertex-domination-critical graph.

Proof: Let G be any graph. Let $E(G) = \{e_1, e_2, \ldots, e_q\}$, and let H_i, u_i, and v_i be as defined in the theorem hypothesis for each $i \in \{1, 2, \ldots, q\}$. Also, let $S = \{u_i, v_i;\ 1 \le i \le q\}$. Denote by V_1 the set of isolates of G. Clearly, $|S| = |V(G) - V_1| = |V(G)| - |V_1|$. We shall refer to vertices of G^* as G-vertices or non-G-vertices according to whether they correspond to vertices which were originally in G or not.

Now, for each $i \in \{1, 2, \ldots, q\}$, let D_i denote a minimum dominating set of H_i that contains the vertices u_i and v_i (by (i), such a minimum dominating set of H_i exists). Then, clearly, $D^* = \bigcup_{i=1}^{q} D_i \mapsto \bigcup_{i=1}^{q} V(H_i)$ and $D^* \cup V_1 \mapsto G^*$. That $D^* \cup V_1$ is, in fact, a minimum dominating set for G^* follows from the observation that every dominating set D' of G must contain V_1, as well as a dominating set D_i' of H_i for every $i \in \{1, 2, \ldots, q\}$ and that $|D'|$ is minimized if D_i' is a minimum dominating set of H_i for $i = 1, 2, \ldots, q$ and if $D_i' \cap D_j' \neq \emptyset$ for as many pairs i, j (with $i \neq j$) as possible, i.e., if D_i' contains u_i and v_i for every $i \in \{1, 2, \ldots, q\}$. Thus,

$$\begin{aligned}\gamma(G^*) &= |D^* \cup V_1| = |D^*| + |V_1| \\ &= |\{u_i, v_i;\ 1 \le i \le q\}| + \left|\bigcup_{i=1}^{q} (D_i - \{u_i, v_i\})\right| + |V_1| \\ &= |S| + \sum_{i=1}^{q} (\gamma(H_i) - 2) + |V_1| \\ &= p + \sum_{i=1}^{q} (\gamma(H_i) - 2).\end{aligned}$$

We show next that G^* is vertex-domination-critical. Let $x^* \in V(G^*)$; we consider two cases.

Case 1: Suppose that x^* is a G-vertex, i.e., $x^* = u_i$ or v_i for some $i \in \{1, 2, ..., q\}$; suppose, without loss of generality, that $x = u_i$. Let U be the family of vertices $u_k = x^*$, indexed by the set $I_U = \{ k; 1 \le k \le q$ and $u_k = x^*\}$; and let W be the family of vertices $v_k = x^*$, indexed by the set $I_W = \{ k; 1 \le k \le q$ and $v_k = x^*\}$; then, if x is the vertex of G to which x^* corresponds, we have $\deg_G x = |I_U| + |I_W|$. Let $J = I_U \cup I_W$; then, $|J| = \deg_G x$. For each $j \in J$, let D_j^* be a minimum dominating set of $H_j - u_j$ (if $u_j \in U$) that contains v_j (by (iii), such a set D_j^* exists), or of $H_j - v_j$ (if $v_j \in W$) that contains u_j (by (ii), such a set D_j^* exists). Since, by assumption, H_k is vertex-domination-critical for each $k \in \{1, 2, ..., q\}$, we have $|D_j^*| = \gamma(H_j) - 1$ for each $j \in J$. For each $j \in \{1, 2, ..., q\} - J$, let D_j^* be a minimum dominating set of H_j containing both u_j and v_j (such a set D_j^* exists by (i)). Then,

$$D = \bigcup_{k \in J} D_k^* \cup \bigcup_{\substack{k=1 \\ k \notin J}}^{q} D_k^* \cup V_1 \mapsto G^* - x^*$$

whence

$$\begin{aligned}
\gamma(G^*-x^*) &\le \sum_{k \in J}(\gamma(H_k) - 1) + \sum_{\substack{k=1 \\ k \notin J}}^{q} |D_k^* - \{u_k, v_k\}| + \left|\bigcup_{\substack{k=1 \\ k \notin J}}^{q} (D_k^* \cap \{u_k, v_k\})\right| + |V_1| \\
&= \sum_{k \in J}(\gamma(H_k) - 2) + |J| + \sum_{\substack{k=1 \\ k \notin J}}^{q}(\gamma(H_k) - 2) + |S - \{u_j, v_j; j \in J\}| + |V_1| \\
&= \sum_{k=1}^{q}(\gamma(H_k) - 2) + (|J| + |V(G)| - |V_1| - (\deg_G x + 1)) + |V_1| \\
&= \sum_{k=1}^{q}(\gamma(H_k) - 2) + |J| + |V(G)| - |V_1| - |J| - 1 + |V_1| \\
&= p - 1 + \sum_{k=1}^{k}(\gamma(H_k) - 2) \\
&= \gamma(G^*) - 1.
\end{aligned}$$

Thus, every G-vertex of G^* is a G^*-critical vertex.

Case 2: Suppose that x^* is a non-G-vertex, say $x^* \in V(H_i) - \{u_i, v_i\}$ for some $i \in \{1, 2, ..., q\}$. Let J denote the set of all indices j for which e_j is adjacent to e_i in G and let w_j be the vertex common to e_i and e_j ($j \in J$). A dominating set D of G^* may be constructed as the union of the following sets:

(i) a minimum dominating set D_i of $H_i - x^*$ with $|D_i| = \gamma(H_i) - 1$ (where D_i may contain both, one, or neither of u_i and v_i, but D_i obviously dominates both u_i and v_i),

(ii) for each $j \in J$, a minimum dominating set D_j of $H_j - w_j$, containing $\{u_j, v_j\} - \{w_j\}$, with $|D_j| = \gamma(H_j) - 1$, where $w_j \in \{u_i, v_i\}$ (and so w_j is dominated by D_i);

(iii) for each $k \in \{1, 2, \ldots, q\} - (J \cup \{i\})$, a minimum dominating set D_k of H_k, containing u_k and v_k, of cardinality $\gamma(H_k)$; and

(iv) V_1.

Let $V_i' = V(G) - \{u_i, v_i\}$; then, $V_i' \subseteq D$. We note that, for $j \in J$ and $k \in \{1, 2, \ldots, q\} - (J \cup i)$, the numbers of vertices which are contained in $D_j - V_i'$ and in $D_k - V_i'$ are, respectively, $|D_j| - 1 = \gamma(H_j) - 2$ and $|D_k| - 2 = \gamma(H_k) - 2$. Hence,

$$\begin{aligned}
\gamma(G^*-x^*) \le |D| &= |V_i'| + |D_i| + \sum_{j\in J}\left(|D_j| - 1\right) + \sum_{k\in\{1,2,\ldots,q\}-(J\cup\{i\})}\left(|D_k| - 2\right) \\
&= p - 2 + \gamma(H_i) - 1 + \sum_{\substack{l=1\\ j\neq i}}^{q}\left(\gamma(H_l) - 2\right) \\
&= p + \sum_{l=1}^{q}\left(\gamma(H_l) - 2\right) - 1 \\
&= \gamma(G^*) - 1.
\end{aligned}$$

Therefore, $\gamma(G^*-x^*) < \gamma(G^*)$, as required. □

3.6.8 Remark: Examples of graphs from which the graphs H_i of Theorem 3.6.7 may be chosen include any 2-vertex-critical graph, as the following proposition shows.

3.6.9 Proposition: If G is a 2-vertex-critical graph, then there exist $u, v \in V(G)$ such that

(1) there exists a minimum dominating set of G containing both u and v;

(2) there exists a minimum dominating set of $G-u$ containing v; and

(3) there exists a minimum dominating set of $G-v$ containing u.

Proof: Suppose that $G \cong H-F$ where $H \cong K_{2n}$, for some $n \in \mathbb{N}$, and F is the edge set of a 1-factor of H. Let u be any vertex of G, and let v be the unique vertex of G such that $uv \in E(\overline{G})$.

Clearly, $\{u,v\}$ is a minimum dominating set of G; so (1) holds. Since the only vertex of $G-u$ of degree $p(G-u) - 1$ is v, (2) holds; similarly, (3) holds. □

Later on (in Theorem 3.9.6), we shall show that every graph is an induced subgraph of some vertex-domination-critical graph. At present, we can derive the following result.

3.6.10 Theorem: Given any graph G, there exists a vertex-domination-critical graph G^* that has an induced subgraph isomorphic to the subdivision graph S(G) of G.

Proof: Let G be a graph of size q with $E(G) = \{u_iv_i;\ i = 1, 2, \ldots, q\}$. Now, for each $i \in \{1, 2, \ldots, q\}$, let $H_i \cong C_4$, where $V(H_i) = \{u_i, v_i, w_i, y_i\}$ and $E(H_i) = \{u_iy_i, y_iv_i, v_iw_i, w_iu_i\}$, and form the vertex-domination-critical graph G^* from $\{H_i;\ 1 \leq i \leq q(G)\}$ as described in Theorem 3.6.7. Clearly,

$$\langle\{u_i, v_i, y_i;\ 1 \leq i \leq q\}\rangle_G.$$

is an induced subgraph of G^* isomorphic to S(G). □

3.7 VERTEX-DOMINATION-CRITICAL GRAPHS, γ^+ AND γ^-

Recall from Definition 3.1.3 that, in general, the $\gamma^+(\gamma^-)$-stability of a graph G is the minimum number of vertices whose removal from G results in a graph H with $\gamma(H) > \gamma(G)$ ($\gamma(H) < \gamma(G)$). Now, the definition of vertex-domination-criticality prompts another question: Do there exist graphs G for which

$$\gamma(G-v) > \gamma(G) \text{ for every } v \in V(G)\ ?$$

Any such graph, of course, would be a special case of a graph H with $\gamma^+(H) = 1$. The following proposition answers this question in the negative.

3.7.1 Proposition: There does not exist any graph G such that $\gamma(G-v) > \gamma(G)$ for each $v \in V(G)$.

Proof: Let G be a graph of order p. If $\gamma(G) = p$, then $G \cong \overline{K}_p$ and $\gamma(G-v) = \gamma(G) - 1$ for each $v \in V(G)$. So, suppose now that $\gamma(G) < p(G)$. Let D be a minimum dominating set of G.

Then, $\gamma(G) < p(G)$ implies that $V(G) - D \neq \varnothing$. Clearly, for any $v \in V(G) - D$, $D \mapsto G-v$, whence $\gamma(G-v) \leq \gamma(G)$. Thus, no graph G satisfies $\gamma(G-v) > \gamma(G)$ for every $v \in V(G)$. □

3.7.2 Remark: There do exist graphs G such that $\gamma(G-v) = \gamma(G)$ for all $v \in V(G)$, i.e., $\gamma^+(G) \neq 1$ and $\gamma^-(G) \neq 1$; for instance, for $n \geq 3$ and $n \equiv 0, 2 \pmod 3$, if $G \cong C_n$, then $\gamma(G-v) = \gamma(P_{n-1}) = \lceil \frac{1}{3}(n-1) \rceil = \gamma(G)$ for any vertex $v \in V(G)$. In fact, there exist graphs with $\gamma^+(G) = m$ and $\gamma^-(G) = n$ for any prescribed $m, n \in \mathbb{N}$ (see Proposition 3.2.10).

3.8 BOUNDS ON THE DOMINATION NUMBERS OF A GRAPH AND ITS COMPLEMENT

In this section, we investigate relationships between the domination number of a graph and the domination number of its complement. Not all results in this section relate to vertex-domination-critical graphs.

From the observation that, for every vertex v in a graph G of order p, $\deg_G v + \deg_{\bar{G}} v = p - 1$, the following lemma is immediately apparent.

3.8.1 Lemma: For any graph G, $p(G) - \Delta(G) = \delta(\bar{G}) + 1$.

3.8.2 Theorem: For any graph G, $\gamma(G) + \gamma(\bar{G}) \leq \delta(G) + \delta(\bar{G}) + 2$.

Proof: Let G be any graph. By Proposition 2.1.3 and Lemma 3.8.1, we have

$$\gamma(G) \leq p(G) - \Delta(G) = \delta(\bar{G}) + 1.$$

Similarly, $\gamma(\bar{G}) \leq \delta(G) + 1$, and the result follows. □

3.8.3 Lemma: For any graph G, $\gamma(G) \leq \kappa(\bar{G}) + 2$.

Proof: Let G be any graph of order p. If $\bar{G}$ is complete, then $\gamma(G) = p$, $\kappa(\bar{G}) + 2 = p - 1 + 2 = p + 1$, and the result holds. Suppose now that $\bar{G}$ is not complete, and let V^* be a minimum vertex cutset of $\bar{G}$. Let $G_1, G_2, \ldots, G_n$ be the components of $\bar{G}-V^*$; let $u \in V(G_1)$ and $v \in V(G_2)$. Since $[V(G_1), V(G_2) \cup V(G_3) \cup \ldots \cup V(G_n)]_{\bar{G}} = \varnothing$, it follows that

$V(G_2) \cup V(G_3) \cup \ldots \cup V(G_n)$ is contained in $N_G(u)$ (so, $uv \in E(G)$). Similarly, $V(G_1)$ is contained in $N_G(v)$. So, $\{u, v\} \mapsto V(G_1) \cup \ldots \cup V(G_n) = V(G) - V^*$. Thus, $V^* \cup \{u, v\} \mapsto G$, and $\gamma(G) \leq \kappa(\overline{G}) + 2$. □

3.8.4 Lemma: For any non-empty, non-complete graph G, the number of isolated vertices of G cannot exceed $\kappa(\overline{G})$.

Proof: Suppose, to the contrary, that there exists a graph G that is neither empty nor complete, but such that the set J of isolated vertices of G satisfies $|J| > \kappa(\overline{G})$. Then, in $\overline{G}$, the vertices of J are mutually adjacent, and, furthermore, every vertex of J is adjacent to every vertex of $V(G) - J$. Let S be any subset of V(G) with $|S| = \kappa(\overline{G})$. Clearly, by our assumption, $J - S \neq \varnothing$; so, for any $v \in J - S$, $\overline{G}-S$ is dominated by v and is connected; hence, $\kappa(\overline{G}) > |S| = \kappa(\overline{G})$, a contradiction. □

3.8.5 Lemma: For a graph G, $\gamma(G) = \kappa(\overline{G}) + 2$ if and only if

(i) G has $\kappa(\overline{G})$ isolated vertices, and

(ii) $\Delta(G) \leq p - \kappa(\overline{G}) - 2$.

Proof: Let G be any graph. Suppose first that $\gamma(G) = \kappa(\overline{G}) + 2$. If G is complete, then $\kappa(\overline{G}) + 2 = 2$ and $\gamma(G) = 1 \neq 2$, and if $\overline{G}$ is complete, then (as we saw in the proof of Lemma 3.8.3) $\kappa(\overline{G})+ 2 \neq \gamma(G)$; so, neither G nor $\overline{G}$ is complete. Let $J = \{v \in V(G); \deg_G v = 0\}$. By Lemma 3.8.4, $|J| \leq \kappa(\overline{G})$. If $\kappa(\overline{G}) = 0$, then $|J| = 0$, and (i) holds. Suppose now that $\kappa(\overline{G}) > 0$. Let V^* be a minimum vertex cutset of $\overline{G}$, and let u and v be vertices of $\overline{G}-V^*$ that lie in distinct components of $\overline{G}-V^*$. Then, as in the proof of Lemma 3.8.3, $\{u, v\} \mapsto G- V^*$ and $V^* \cup \{u, v\} \mapsto G$, and so

$$\kappa(\overline{G}) + 2 = |V^* \cup \{u, v\}| \geq \gamma(G).$$

Since we have assumed $\gamma(G) = \kappa(\overline{G}) + 2$, it follows that $V^* \cup \{u, v\}$ is a minimum dominating set of G. Notice, first, that $\langle V^*\rangle_G$ must be empty, since if $xy \in E(\langle V^*\rangle_G)$, then $(V^* - \{x\}) \cup \{u, v\}$ is a dominating set of G of cardinality $\gamma(G) - 1$, which is not possible. Furthermore, every vertex of V^* must be non-adjacent in G to u, since if $x \in V^*$ with $xu \in E(G)$, then, again, $(V^* - \{x\}) \cup \{u, v\} \mapsto G$. However, u is an arbitrary element of $V(G) - V^*$; so, every vertex of V^* is non-adjacent in G to every vertex of $G-V^*$. Thus, V^* consists of isolated vertices of G and we have $|J| \geq |V^*| = \kappa(\overline{G})$. Combining this with the reverse inequality above, we have $|J| = \kappa(\overline{G})$, and (i) follows. Finally, let $w \in V(G)$. If $w \in V^*$, $\deg_G w = 0$. Suppose now that

$w \in V(G) - V^*$. If every vertex of $V(G) - V^*$ is adjacent to w, then $\{w\} \rightarrowtail V(G) - V^*$, whence $\gamma(G) \leq |V^* \cup \{w\}| < \gamma(G)$. So, there exists at least one vertex y, say, of $V(G) - V^*$ such that $yw \notin E(G)$. Then, $N_G(w) \subseteq V(G) - (V^* \cup \{w, y\})$, and so $\deg_G w \leq p(G) - \kappa(\overline{G}) - 2$. Since w is arbitrary, $\Delta(G) \leq p(G) - \kappa(\overline{G}) - 2$, and (ii) holds.

For the converse, suppose now that (i) and (ii) hold. If $\overline{G}$ is complete, then G contains $p(G) \neq p(G) - 1 = \kappa(\overline{G})$ isolated vertices (which contradicts (i)), and if G is complete, then $\Delta(G) = p(G) - 1 > p(G) - \kappa(\overline{G}) - 2 = p(G) - 2$ (which contradicts (ii)); so, again, neither G nor $\overline{G}$ is complete. Now, let D be any minimum dominating set of G. Clearly, the set J of the $\kappa(\overline{G})$ isolated vertices of G is contained in D, and at least one vertex x from $V(G) - J$ must belong to D, in order that $V(G) - J$ is dominated. However, by (ii), $\deg_G x \leq p(G) - \kappa(\overline{G}) - 2 = |V(G) - J| - 2$; so x is adjacent to at most $|V(G) - J| - 2$ vertices of $V(G) - J$. Thus, at least 2 distinct vertices of $V(G) - J$ belong to D; so, $\gamma(G) = |D| \geq |J| + 2 = \kappa(\overline{G}) + 2$. Since, by Lemma 3.8.3, $\gamma(G) \leq \kappa(\overline{G}) + 2$, the desired result follows. □

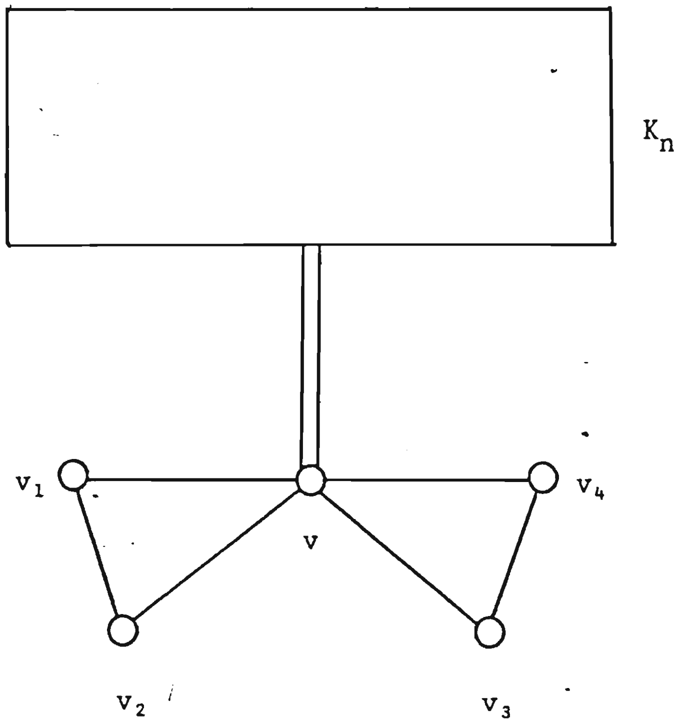
3.8.6 <u>Theorem</u>: For any graph G, $\gamma(G) + \gamma(\overline{G}) \leq \kappa(G) + \kappa(\overline{G}) + 3$.

<u>Proof</u>: Let G be any graph. By Lemma 3.8.3, $\gamma(G) \leq \kappa(\overline{G}) + 2$ and $\gamma(\overline{G}) \leq \kappa(G) + 2$.

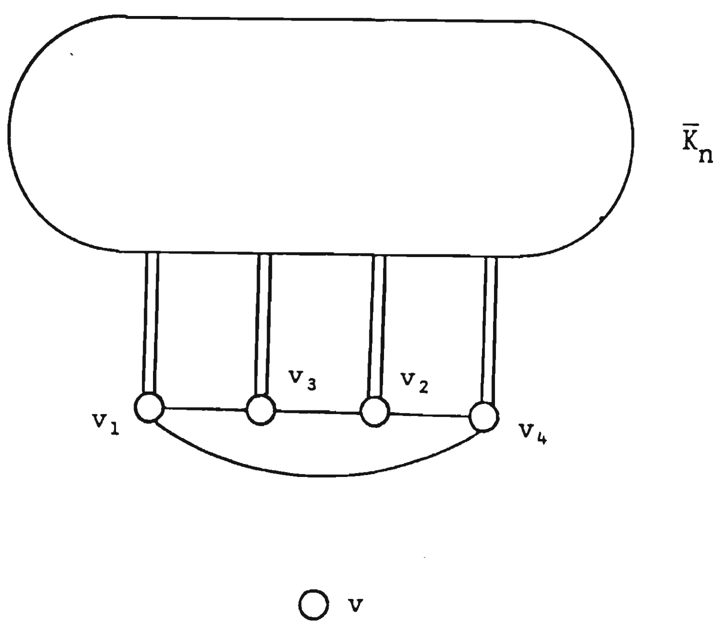
<u>Case 1</u>: Suppose $\gamma(G) \neq \kappa(\overline{G}) + 2$ or $\gamma(\overline{G}) \neq \kappa(G) + 2$; then, $\gamma(G) + \gamma(\overline{G}) \leq \kappa(G) + \kappa(\overline{G}) + 3$ follows immediately.

<u>Case 2</u>: Suppose that both $\gamma(G) = \kappa(\overline{G}) + 2$ and $\gamma(\overline{G}) = \kappa(G) + 2$ hold. Since, for any graph F, at least one of F and $\overline{F}$ is connected, we have that at least one of $\kappa(G)$, $\kappa(\overline{G})$ is positive; suppose, without loss of generality, that $\kappa(\overline{G}) > 0$. Now, since $\overline{G}$ is connected, $\overline{G}$ has no isolated vertices. Since $\gamma(\overline{G}) = \kappa(G) + 2$, it follows by Lemma 3.8.5 that $\overline{G}$ has $\kappa(G)$ isolated vertices. So, $\kappa(G) = 0$ and G is disconnected. Furthermore, since $\kappa(\overline{G}) > 0$ and $\gamma(G) = \kappa(\overline{G}) + 2$, we have, again by Lemma 3.8.5, that G has at least one isolated vertex, x say. Then, $\deg_{\overline{G}} x = p(\overline{G}) - 1$, i.e., $\gamma(\overline{G}) = 1$. However, then $\gamma(\overline{G}) = 1 \neq \kappa(G) + 2 = 2$, which contradicts our assumption. So, Case 2 cannot occur, and the theorem follows. □

3.8.7 <u>Remark</u>: In some instances, the bound given in Theorem 3.8.2 is better than that provided by Theorem 3.8.6; for example, if G is a complete graph, then the bound in Theorem 3.8.2 is $p(G) + 1$, whereas that in Theorem 3.8.6 has the value $p(G) + 2$ (this example also shows, in



(a)



(b)

Fig. 3.8.1

fact, that the bound of Theorem 3.8.2 is sharp). On the other hand, if G is a complete bipartite graph $K_{m,n}$, with $1 < n \leq m$, then the bound provided by Theorem 3.8.6 is almost twice as good as that given in Theorem 3.8.2, since

$$\delta(G) + \delta(\overline{G}) + 2 = n + (n - 1) + 2 = 2n + 1$$

and

$$\kappa(G) + \kappa(\overline{G}) + 3 = n + 0 + 3 = n + 3 < 2n + 2$$

(where, of course, $\overline{G} = K_m \cup K_n$). Consider the graph G (shown in Fig. 3.8.1(a)) which is obtained from a complete graph $G_1 \cong K_{n+1}$ ($n \in \mathbb{N}$) and two 3-cycles G_2 and G_3, where $u \in V(G_1)$, $v \in V(G_2)$, and $w \in V(G_3)$, by identifying the vertices u, v and w. (The graph $\overline{G}$ is shown in Fig. 3.8.1(b).) This graph shows that the bound in Theorem 3.8.6 is sharp, since $\gamma(G) + \gamma(\overline{G}) = 4 = \kappa(G) + \kappa(\overline{G}) + 3$.

We consider next some sufficient conditions on $\overline{G}$ under which $\gamma(G) \in \{1, 2, 3\}$, for a graph G.

3.8.8 Proposition: If G is a graph. Then,

$$\gamma(G) = \begin{cases} 1, & \text{if } \delta(\overline{G}) = 0 \\ 2, & \text{if } \delta(\overline{G}) \geq 1 \text{ and either } \kappa(\overline{G}) = 0 \text{ or } \kappa(\overline{G}) = 1 \\ 3, & \text{if } \delta(\overline{G}) \geq 2,\ \kappa(\overline{G}) = 1, \text{ and } \Delta(\overline{G}) = p(G) - 1 \end{cases} .$$

Proof: Let G be a graph.

Case 1: Suppose $\delta(\overline{G}) = 0$. Then, G contains a vertex of degree $p(G) - 1$, and hence $\gamma(G) = 1$.

Case 2: Suppose $\delta(\overline{G}) \geq 1$. Then, no vertex of $\overline{G}$ is isolated, and so no vertex of G is adjacent to every other vertex of G. Thus, $\gamma(G) \geq 2$.

Subcase 2.1: Suppose $\kappa(\overline{G}) = 0$. Then, $\overline{G}$ is disconnected, with components G_1, G_2, ..., $G_{k(\overline{G})}$, $k(\overline{G}) \geq 2$. Let $u \in V(G_1)$, $v \in V(G_2)$. Then, as in the proof of Lemma 3.8.3,

$$\left[\{u\}, \bigcup_{i=2}^{k(\bar{G})} V(G_i)\right]_{\bar{G}} \subseteq \left[V(G_1), \bigcup_{i=2}^{k(\bar{G})} V(G_i)\right]_{\bar{G}} = \emptyset,$$

which implies that, in G, $\{u\} \mapsto \bigcup_{i=2}^{k(\bar{G})} V(G_i)$; similarly, $\{v\} \mapsto V(G_1)$. So, $\{u, v\} \mapsto G$ and $\gamma(G) \leq 2$. Combined with our earlier observation, this gives $\gamma(G) = 2$.

Subcase 2.2: Suppose $\kappa(\bar{G}) = 1$. Let v be a cut-vertex of $\bar{G}$, and let $G_1, G_2, \ldots, G_n$ be the components of $\bar{G}-v$ $(n \geq 2)$.

Subcase 2.2.1: Suppose $N_{\bar{G}}(v) \neq V(G) - \{v\}$. Let $x \in V(G) - N_{\bar{G}}[v]$, and let $i \in \{1, 2, \ldots, n\}$ be such that $x \in V(G_i)$. Then, in G,

$$\{x\} \mapsto \{v\} \cup \bigcup_{\substack{k=1 \\ k \neq i}}^{n} V(G_k),$$

and, for any $j \in \{1, 2, \ldots, n\}$, $j \neq i$, and any $y \in V(G_j)$, we have $\{y\} \mapsto V(G_i)$ in G. So, $\{x, y\} \mapsto G$, and $\gamma(G) = 2$.

Subcase 2.2.2: Suppose $N_{\bar{G}}[v] = V(G)$. Then, v is an isolated vertex in G so that $\kappa(G) = 0$, and $\gamma(\bar{G}) = 1$. Then, since $\gamma(\bar{G}) = 1 \neq \kappa(G) + 2 = 2$, we have, by Lemma 3.8.5 and Lemma 3.8.4, that $\bar{G}$ has fewer than $\kappa(G) = 0$ isolated vertices, which is impossible, or $\Delta(G) > p(G) - \kappa(G) - 2 = p(G) - 2$. So, $\Delta(G) \geq p(G) - 1$ (and, hence, $\Delta(G) = p(G) - 1$), which implies $\delta(\bar{G}) = 0$, which is contrary to assumption.

Case 3: Suppose $\delta(\bar{G}) \geq 2$, $\kappa(\bar{G}) = 1$, and $\Delta(\bar{G}) = p(G) - 1$. By Lemma 3.8.3, $\gamma(G) \leq \kappa(\bar{G}) + 2 = 3$. Since $\Delta(\bar{G}) = p(G) - 1$, G contains an isolated vertex, and, since $\delta(\bar{G}) \geq 2$, $p(G) \geq 3$. Thus, $\gamma(G) \geq 2$. So, $\gamma(G) \in \{2, 3\}$. Suppose that $\gamma(G) = 2$. Then, clearly, the set J of isolated vertices of G satisfies $|J| \leq 2$. If $|J| = \gamma(G) = 2$, then $p(G) = 2 < 3$, a contradiction. So, G has at most one isolated vertex. Since $\Delta(\bar{G}) = p(G) - 1$, we have $\delta(G) = 0$, i.e., G has at least one isolated vertex, say v, and v must belong to every minimum dominating set of G. This implies, since $\gamma(G) = 2$, that

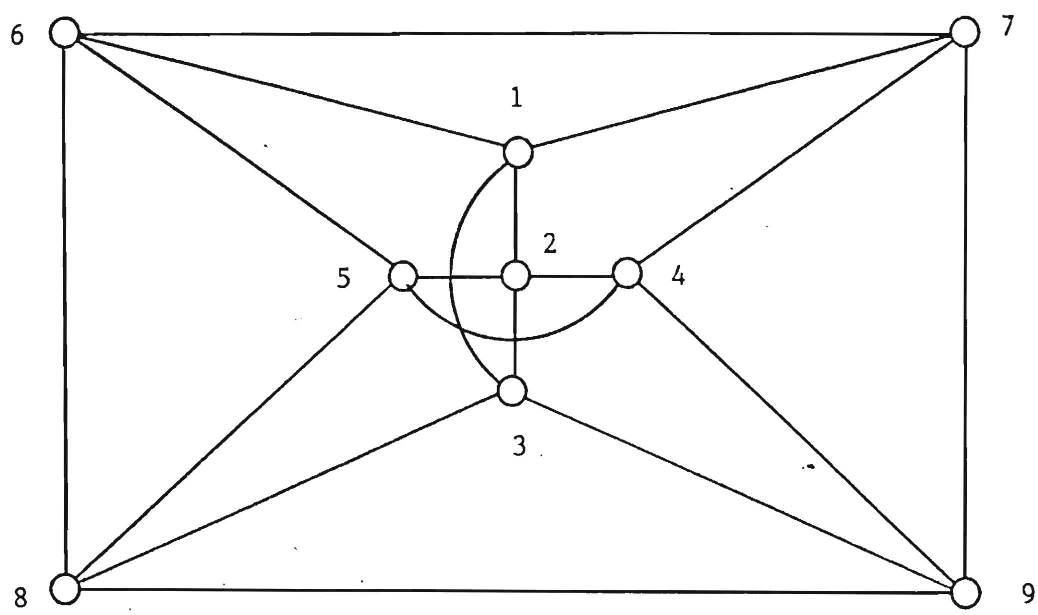


Fig. 3.8.2

there exists a vertex $w \in V(G) - \{v\}$ with $\deg_G w = p(G) - 2$. However, no such vertex exists, since $\delta(\overline{G}) \geq 2$ implies $\Delta(G) = [p(G) - 1] - \delta(G) \leq p(G) - 3$. This contradiction establishes the fact that $\gamma(G) = 3$. □

We consider next graphs G for which both G and $\overline{G}$ are vertex-domination-critical. Such graphs do exist, as the following proposition proves.

3.8.9 Proposition: There is at least one graph G such that both G and its complement are vertex-domination-critical.

Proof: Consider the graph G depicted in Fig. 3.8.2; G is obtained from the coalescence $(G_1,2)\bullet(G_2,2)$ of two 3-cycles G_1 and G_2, where $V(G_1) = \{1, 2, 3\}$ and $V(G_2) = \{2, 4, 5\}$, by the addition of four new vertices 6, 7, 8, and 9, and the insertion of the edges 61, 65, 71, 74, 93, 94, 83, 85, 67, 79, 98 and 86. The mapping $\Theta: V(G) = \{1, 2, ..., 9\} \mapsto V(G)$ defined by $\Theta(1) = 7$, $\Theta(2) = 2$, $\Theta(3) = 8$, $\Theta(4) = 9$, $\Theta(5) = 6$, $\Theta(6) = 3$, $\Theta(7) = 5$, $\Theta(8) = 4$, $\Theta(9) = 1$ is an isomorphism between G and $\overline{G}$; so G is self-complementary. The desired result will follow once we have shown that G is 3-vertex-critical.

By inspection, it is easy to see that $\{2, 6, 9\} \mapsto G$, and that no smaller subset of V(G) dominates G. So, $\gamma(G) = \gamma(\overline{G}) = 3$. Now, let $i \in \{1, 2, ..., 9\}$. We consider three cases.

Case 1: Suppose $i \in \{6, 7, 8, 9\}$; without loss of generality, assume $i = 6$. Then, $\{2, 9\} \mapsto G-i$.

Case 2: Suppose $i \in \{1, 3, 4, 5\}$; without loss of generality, assume $i = 1$. Then, $\{4, 8\} \mapsto G-i$.

Case 3: Suppose $i = 2$. Then, $\{6, 9\} \mapsto G-i$.

So, G is 3-vertex-critical. □

3.8.10 Remark: We make the observation, as an aside, that it is not only the domination number of the graph G in Fig. 3.8.2 that has value 3; the independent domination number, i(G), and the total domination number, $\gamma^t(G)$, also have value 3 (consider the vertex subsets $\{2, 6, 9\}$ and $\{1, 2, 3\}$).

3.8.11 Proposition: The complement of any 2-vertex-critical graph is not vertex-domination-critical.

Proof: Let G be a 2-vertex-critical graph. Then, by Example 3.3.2.3, $G \cong H-F$, where $H \cong K_{2n}$ ($n \in \mathbb{N}$) and F is the edge set of a 1-factor of H. Clearly, then, $\overline{G} \cong nK_2$, so $\gamma(\overline{G}) = n$ and $\gamma(\overline{G}-v) = n$ for any $v \in V(G)$, i.e., G is not k-vertex-critical for any $k \in \mathbb{N}$. □

3.8.12 Corollary: If G is a graph for which both G and $\overline{G}$ are vertex-domination-critical, then $\gamma(G) \geq 3$ and $\gamma(\overline{G}) \geq 3$.

Proof: Let G be a graph satisfying the hypothesis of the proposition; then, $\gamma(G) \geq 2$. If $\gamma(G) = 2$, then (by Proposition 3.8.11), $\overline{G}$ is not vertex-domination-critical, a contradiction. Similarly, $\gamma(\overline{G}) \geq 3$. □

3.8.13 Theorem: If G is a graph for which both G and $\overline{G}$ are vertex-domination-critical, then both G and $\overline{G}$ are blocks.

Proof: Let G be a graph such that both G and $\overline{G}$ are vertex-domination-critical. Then, $p(G) \geq \gamma(G) \geq 3$ (by Corollary 3.8.12). Suppose, to the contrary, that $\overline{G}$ is not a block. Then, either $\overline{G}$ is disconnected or $\kappa(\overline{G}) = 1$.

Case 1: Suppose $\kappa(\overline{G}) = 1$. Then, $\overline{G}$ is connected with $p(G) > 1$, so G has no isolated vertices, whence $\delta(\overline{G}) \geq 1$. Our assumption that $\kappa(\overline{G}) = 1$ implies further, by Proposition 3.8.8, that $\gamma(G) = 2$, contradicting Corollary 3.8.12.

Case 2: Suppose $\overline{G}$ is disconnected. Then, $\kappa(\overline{G}) = 0$. Furthermore, $\delta(\overline{G}) \geq 1$ (otherwise, if $\delta(\overline{G}) = 0$, then G contains a vertex of degree $p(G) - 1$, and $\gamma(G) = 1$, whence G is not a vertex-domination-critical graph). So, again by Proposition 3.8.8, $\gamma(G) = 2$, and a contradiction results as above.

That G is a block follows from the observation that G is the complement of $\overline{G}$. □

3.8.14 Corollary: If G is a graph such that both G and $\overline{G}$ are vertex-domination-critical, then

$$\gamma(G) + \gamma(\bar{G}) \leq \kappa(G) + \kappa(\bar{G}) + 2.$$

Proof: Let G be a graph satisfying the hypothesis of the corollary. By Theorem 3.8.13, both G and $\bar{G}$ are blocks and are therefore both connected, i.e., $\kappa(G) > 0$ and $\kappa(\bar{G}) > 0$. By Lemma 3.8.3, we know that $\gamma(G) \leq \kappa(\bar{G}) + 2$ and $\gamma(\bar{G}) \leq \kappa(G) + 2$. If, say, $\gamma(G) = \kappa(\bar{G}) + 2$, then, by Lemma 3.8.5, it follows that G has at least one isolated vertex; however, this contradicts the fact that G is connected. We similarly obtain a contradiction if we assume $\gamma(\bar{G}) = \kappa(G) + 2$. So, $\gamma(G) \leq \kappa(\bar{G}) + 1$ and $\gamma(\bar{G}) \leq \kappa(G) + 1$, whence the desired result follows immediately. □

For the remainder of this section, we return to graphs in general.

3.8.15 Lemma: For any graph G and any subset X of V(G), where $|X| \leq \gamma(G) - 1$, there exists an independent set $W \subseteq V(G) - X$ such that

(1) $|W| \geq \gamma(G) - |X|$, and

(2) $W \cup \{x\}$ is independent for all $x \in X$.

Proof: Let G be any graph and let X be any subset of V(G) with $|X| \leq \gamma(G) - 1$. Certainly, then, X is not a dominating set of G. Let W′ be the set of vertices of V(G) − X not dominated by X. If W′ is independent, then we let W = W′. If W′ is not independent, then we consider any maximal independent set W of $\langle W' \rangle_G$; of course, then $W \mapsto \langle W' \rangle_G$. In either case, W is an independent subset of V(G) − X and is such that $X \cup W \mapsto G$, whence $\gamma(G) \leq |X \cup W| = |X| + |W|$, i.e., $|W| \geq \gamma(G) - |X|$. Finally, that $W \cup \{x\}$ is independent for all $x \in X$ follows immediately from the definition of W′. □

3.8.16 Corollary: For any graph G,

(1) every pair of distinct vertices has a set S of at least $\gamma(\bar{G}) - 2$ common neighbours in G, with $\langle S \rangle_G$ complete, and,

(2) if $\gamma(\bar{G}) \geq 3$, then diam (G) ≤ 2 and $\gamma(G) \leq \kappa(G)$.

Proof: Let G be a graph, and let u and v be distinct vertices of G. If $\gamma(\bar{G}) = 1$ or 2, then (trivially) $|N_G(u) \cap N_G(v)| \geq 0 \geq \gamma(\bar{G}) - 2$. So, we assume now that $\gamma(\bar{G}) \geq 3$. If we let $X = \{u, v\}$, then $|X| \leq \gamma(\bar{G}) - 1$ so that, by Lemma 3.8.15, there exists a subset W of V(G) − X such that $|W| \geq \gamma(\bar{G}) - |X|$ and $\langle W \cup \{u\} \rangle_G$ and $\langle W \cup \{v\} \rangle_G$ are cliques. Clearly, then, W is contained in both $N_G(u)$ and $N_G(v)$, where $|W| \geq \gamma(\bar{G}) - |X| = \gamma(\bar{G}) - 2$. So, the first result follows.

Now suppose that $\gamma(\bar{G}) \geq 3$. Then, by (1), for every two vertices u, v $\in$ V(G), $|N_G(u) \cap N_G(v)| \geq \gamma(\bar{G}) - 2 \geq 1$, i.e., there is a path of length two joining u and v in G. So, diam (G) ≤ 2. If $G \cong K_n$, for some $n \in \mathbb{N}$, then $n = \gamma(\bar{G}) \geq 3$ and diam(G) $= 1 \leq 2$ and $\gamma(G) = 1 < \kappa(G)$. Suppose now that G is not complete, and let D be a minimum vertex cut-set of G; say, $G_1, G_2, \ldots, G_n$, where $n = k(G-D) \geq 2$, are the components of G-D. We will show that $D \mapsto G$, whence it will follow that $\gamma(G) \leq \kappa(G)$. Suppose, to the contrary, that there is some $i \in \{1, 2, \ldots, n\}$ and $u \in V(G_i)$ such that $D \not\mapsto \{u\}$. Let $v \in V(G_j)$ for any $j \in \{1, 2, \ldots, n\}$, $j \neq i$. While u and v are not connected in G-D, there is, by what we proved above, at least one vertex, w say, such that uv, vw $\in$ E(G), and w must belong to D. However, then $D \mapsto \{u\}$. This contradiction establishes the desired result. □

3.8.17 Remark: We note that it follows directly from the proof above that if G is a graph and diam (G) ≤ 2, then $\gamma(G) \leq \kappa(G)$. In Theorem 3.8.18, we obtain a Nordhaus-Gaddum-type upper bound on the sum $\gamma(G) + \gamma(\bar{G})$ for arbitrary graphs G satisfying $\gamma(\bar{G}) \geq 3$. This result may be compared with the well-known Nordhaus-Gaddum-type results obtained by Jaegar and Payan [JP1], namely, (a) $\gamma(G)\,\gamma(\bar{G}) \leq p(G)$ and (b) $\gamma(G) + \gamma(\bar{G}) \leq p(G) + 1$, for any graph G. For graphs G with $\kappa(G) \leq p(G) - 3$, Theorem 3.8.18 is an improvement on (b).

3.8.18 Theorem: For any graph G with $\gamma(\bar{G}) \geq 3$, $\gamma(G) + \gamma(\bar{G}) \leq \kappa(G) + 3$.

Proof: Let G be a graph of order p. If $G \cong K_p$, then $\kappa(G) + 3 = (p - 1) + 3 = p + 2 > p + 1 = \gamma(G) + \gamma(\bar{G})$. We suppose now that G is not complete. Let Z be a minimum vertex cutset of G, and let u and v be vertices in distinct components of G-Z. By Corollary 3.8.16(1), we know that there exists a set S of $\gamma(\bar{G}) - 2$ vertices such that S is contained in $N_G(u)$ and in $N_G(v)$ and $\langle S\rangle_G$ is complete. Since u and v are not connected in G-Z, it is clear that $S \subseteq Z$. Let $D = Z - (S - \{x\}) = (Z - S) \cup \{x\}$, where x is any vertex of S. Clearly, all vertices of Z are dominated by this set D, since $Z - S \mapsto Z - S$ and $\{x\} \mapsto S$.

We now show that $D \mapsto V(G) - Z$. Let w and z be vertices in distinct components of G-Z. Then, by Corollary 3.8.16, there exists a subset T of V(G) with $|T| \geq \gamma(\bar{G}) - 2$ such that w is adjacent to every vertex of T, and z is adjacent to every vertex of T. Since w and z belong to distinct components of G-Z, T must be contained in Z. Since $|T| > |S - \{x\}| = \gamma(\bar{G}) - 3$, T must have a non-empty intersection with D. Since each vertex of $T \mapsto \{w\}$, it follows that $D \mapsto \{w\}$. Since $w \in V(G) - Z$ is arbitrary, it follows that $D \mapsto V(G) - Z$. Hence, $D \mapsto G$, and

$$\begin{aligned}\gamma(G) &\leq |D| = |(Z - S) \cup \{x\}| = |Z - S| + 1 = \kappa(G) - [\gamma(\bar{G}) - 2] + 1 \\ &= \kappa(G) - \gamma(\bar{G}) + 3.\end{aligned}$$ □

By Proposition 3.8.12, the following corollary applies to vertex-domination-critical graphs G for which $\bar{G}$ is also vertex-domination-critical.

3.8.19 Corollary: For any graph with $\gamma(G) \geq 3$ and $\gamma(\bar{G}) \geq 3$,

$$\gamma(G) + \gamma(\bar{G}) \leq \min\{\kappa(G), \kappa(\bar{G})\} + 3.$$

3.9 CHARACTERIZATION OF VERTEX-DOMINATION-CRITICAL GRAPHS

Finding a characterization of vertex-domination-critical graphs appears to be a difficult problem. However, it is possible to characterize those vertex-domination-critical graphs G that have the smallest order among graphs with maximum degree $\Delta(G)$ and domination number $\gamma(G)$, i.e., $p(G) = \Delta(G) + \gamma(G)$ (recall that $p(H) \geq \Delta(H) + \gamma(H)$ for every graph H). Before we prove this result, we state the following definitions.

3.9.1 Definition: Let G be a graph and U a subset of V(G), and consider a partition $U_1, U_2, \ldots, U_n$ ($n \in \mathbb{N}$) of U. Then,

(a) a *system of distinct representatives* of $U_1, U_2, \ldots, U_n$ is a set $\{u_1, u_2, \ldots, u_n\}$ of n (distinct) vertices where $u_i \in U_i$ for $1 \leq i \leq n$;

(b) a *sub-U domination* is a non-empty subcollection

$$S = \{U_{i_1}, U_{i_2}, \ldots, U_{i_k}\}$$

of the elements of the partition of U such that S possesses a system of distinct representatives which forms a dominating set of the subgraph of G induced by

$$U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_k}.$$

3.9.2 Theorem: Let $\gamma, \Delta \in \mathbb{N}$ be given, and let G be a graph. Then, G is a vertex-domination-critical graph having $\gamma(G) = \gamma$, $\Delta(G) = \Delta$, and $p(G) = \Delta + \gamma$ if and only if $V(G) = \{v\} \cup U \cup$

$\bar{W}$, where v is a vertex of G of degree Δ, $U = N(v) = \{u_1, u_2, \ldots, u_\Delta\}$, $W = V(G) - N[v] = \{w_1, w_2, \ldots, w_{\gamma-1}\}$ and

(i) the set W is independent in G,

(ii) every vertex of U is adjacent to exactly one vertex of W (this property of G results in a partition $U_1, U_2, \ldots, U_{\gamma-1}$ of U being induced, where the vertices of U_i are all adjacent to w_i),

(iii) the partition of (ii) has no sub-U domination,

(iv) for any $i \in \{1, 2, \ldots, \gamma - 1\}$ and any $u \in U_i$, define a partition P_i of $U - \{u\}$ which has all the members of the partition of (ii) except that U_i is replaced by $U_i - u$; then, there is a sub-$(U - \{u\})$ domination which includes $U_i - \{u\}$ such that the representative of $U_i - \{u\}$ is not adjacent to u.

Proof: Let $\gamma, \Delta \in \mathbb{N}$ be given, and let G be a graph.

Suppose, first, that $V(G) = \{v\} \cup U \cup W$, where v, U and W are as described above, and satisfy conditions (i) - (iv). We show that G is a vertex-domination-critical graph with $\gamma(G) = \gamma$, $\Delta(G) = \Delta$ and, hence, $p(G) = \Delta(G) + \gamma(G)$.

Obviously, $\{v\} \cup W \mapsto G$; hence, $\gamma(G) \leq \gamma$. Let D be a minimum dominating set of G. It follows from conditions (i) and (ii) that, for each $i \in \{1, \ldots, \gamma - 1\}$, in order that w_i is dominated by D, D must contain w_i or some element of U_i. Hence, D contains at least $\gamma - 1$ elements of $U \cup W$ and certainly $\gamma(G) = |D| \geq \gamma - 1$.

If $|D| = \gamma - 1$, then $v \notin D$ and D must be W, as may be seen as follows: Suppose that $|D| = \gamma - 1$ and that $D \neq W$; say (without loss of generality) that $D = \{u_1, \ldots, u_m, w_{m+1}, \ldots, w_{\gamma-1}\}$, where $u_i \in U_i$ for $1 \leq i \leq m$. Since no vertex in $U_1 \cup \ldots \cup U_m$ is dominated by $\{w_{m+1}, \ldots, w_{\gamma-1}\}$, it follows that $\{u_1, \ldots, u_m\} \mapsto U_1 \cup \ldots \cup U_m$ and therefore the partition of U into subsets $U_1, \ldots, U_\Delta$ has a sub-U domination (namely, $U_1, U_2, \ldots, U_m$), contrary to condition (iii). So, if $|D| = \gamma - 1$, then $D = W$; however, then $D \not\mapsto \{v\}$, contrary to the fact that D is a dominating set of G. Consequently, it follows that $\gamma(G) = |D| = \gamma$.

Clearly, $\Delta(G) \geq \deg v = \Delta$; if $\Delta(G) > \Delta$, then (by conditions (i) and (ii)) there exists $u \in U$ (say $u = u_1 \in U_1$) such that $N[u_1] = \{v\} \cup U \cup \{w_1\}$ and so $\{u_1\} \cup (W - \{w_1\}) \mapsto G$, i.e., $\gamma(G) \leq \gamma - 1$, contradicting the result that $\gamma(G) = \gamma$ (established earlier). Hence, $\Delta(G) = \Delta$ and $p(G) = 1 + \Delta + (\gamma - 1) = \Delta(G) + \gamma(G)$.

To show that $\gamma(G-x) \leq \gamma - 1$ for every $x \in V(G)$, we consider three cases: If $x = v$, then $W \mapsto G-x$; if $x = w_i \in W$, then $\{v\} \cup (W - \{w_i\}) \mapsto G-x$; so $\gamma(G-x) \leq \gamma - 1$ in both cases. If $x \in U_1$, there exists a sub-$(U - \{x\})$ domination of the partition P of $U - \{x\}$ guaranteed by (iv) consisting of (without loss of generality) the sets $U_1 - \{x\}, U_2, \ldots, U_k$, possessing distinct representatives $x_1, x_2, \ldots, x_k$ (respectively) such that $xx_1 \notin E(G)$ and $\{x_1, \ldots, x_k\} \mapsto (U_1 - \{x\}) \cup U_2 \cup \ldots \cup U_k \cup \{w_1, \ldots, w_k\} \cup \{v\}$. So, $\{x_1, \ldots, x_k, w_{k+1}, \ldots, w_{\gamma-1}\} \mapsto G-x$ and $\gamma(G-x) \leq \gamma - 1$. It follows similarly that, for all $x \in U - U_1$, $\gamma(G-x) \leq \gamma - 1$ and the vertex-domination-criticality of G follows.

Conversely, suppose that G is a vertex-domination-critical graph with $V(G) = \gamma$, $\Delta(G) = \Delta$, and $p(G) = \gamma + \Delta$. Let $v \in V(G)$ with $\deg v = \Delta$ and define $U = N(v) = \{u_1, \ldots, u_\Delta\}$, $W = V(G) - N[v] = \{w_1, w_2, \ldots, w_t\}$, where $t = p - \Delta - 1 = \gamma - 1$. Then, (i) W is independent, since, if $w_iw_j \in E(G)$ for some $w_i, w_j \in W$, then $\{v\} \cup (W - \{w_j\}) \mapsto G$ and so $\gamma(G) \leq \gamma - 1$, contrary to assumption. So, (i) holds. We now prove (ii). Suppose that some $u \in U$ is adjacent to at least two vertices, w_i and w_j, of W; then $\{v, u\} \cup (W - \{w_i, w_j\}) \mapsto G$ and so $\gamma(G) \leq \gamma - 1$, again a contradiction. So, each vertex in U is adjacent to at most one vertex of W. If a vertex u of U is adjacent to no vertex of W, let D′ be a minimum dominating set of $G-v$; then $|D'| = \gamma - 1$ (as G is γ-vertex-critical) and D′ contains a vertex y that dominates u; so, $y \in V(G) - W = N[v]$ and hence $D' \mapsto G$, contrary to our assumption that $\gamma(G) = \gamma$. Thus, (ii) is true and U may be partitioned into subsets $U_1, \ldots, U_{\gamma-1}$, as given above. That condition (iii) holds follows from the observation that, if the partition $U_1, \ldots, U_{\gamma-1}$ has a sub-U domination S, with (say) $S = \{U_1, \ldots, U_m\}$, with distinct representatives $x_1, \ldots, x_m$, respectively ($m \geq 1$), then $\{x_1, \ldots, x_m, w_{m+1}, \ldots, w_{\gamma-1}\} \mapsto G$ and, again, $\gamma(G) \leq \gamma - 1$, a contradiction.

Finally, to prove that condition (iv) is satisfied, we select $u \in U_i$ for some $i \in \{1, \ldots, \gamma - 1\}$ and let D″ be a minimum dominating set of $G-u$ (so, $|D''| = \gamma - 1$). Since $D'' \not\mapsto \{u\}$, neither v nor w_i is an element of D″. Now, D″ has the following properties: (a) D″ contains an element of $U_j \cup \{w_j\}$ for each $j \in \{1, \ldots, \gamma - 1\}$, $j \neq i$ (because $D'' \mapsto \{w_j\}$), and (b) D″ contains an element x of $U_i - \{u\}$. (Note that $U_i - \{u\} \neq \emptyset$ since $x \in (U_i - \{u\}) \cap D''$ is required to dominate w_i as $w_i \notin D''$). Furthermore, $xu \notin E(G)$, otherwise $\gamma(G) < \gamma$. Let $P = \{U_1', U_2', \ldots, U_\Delta'\}$ be the partition of $U - \{x\}$ obtained from the partition $\{U_1, U_2, \ldots, U_\Delta\}$ as described in (iv) (where U_n' is the member of P obtained from U_n, $1 \leq n \leq \Delta$). Now, by (a) and (b),

$$\gamma - 1 = |D''| \geq \sum_{\substack{j=1\\ j\neq i}}^{\gamma-1} \left[D'' \cap (U_j \cup \{w_j\})\right] + |\{x\}| \geq (\gamma - 2) + 1 = \gamma - 1,$$

and so, in fact, D'' contains *exactly* one element, t_j (say), of $U_j \cup \{w_j\}$ for each $j \in \{1, 2, \ldots, \gamma - 1\}$, $j \neq i$; clearly, if $t_j \in U_j$ ($j \in \{1, 2, \ldots, \gamma - 1\}$, $j \neq i$), then $\{t_j\} \mapsto U_j$. Let

$$U_{i_1}', U_{i_2}', \ldots, U_{i_k}'$$

be the elements of P for which

$$t_{i_r} \in U_{i_r}$$

($1 \leq r \leq k$). Then, $\{t_j;\ t_j \in U_j,\ 1 \leq j \leq \gamma - 1,\ j \neq i\}$ is a system of representatives of

$$U_{i_1}', U_{i_2}', \ldots, U_{i_k}'$$

such that

$$\{t_1, t_2, \ldots, t_k\} \mapsto \bigcup_{r=1}^{k} U_{i_r}'.$$

Since $x \in D'' \cap (U_i - \{u\}) = D'' \cap U_i$, it follows that $\{x\} \mapsto U_i - \{u\}$ and if we let $t_{k+1} = x$ and

$$U_{i_{k+1}}' = U_i',$$

then

$$S = \{U_{i_1}', U_{i_2}', \ldots, U_{i_{k+1}}'\}$$

is a sub-$(U - \{x\})$ domination satisfying (iv). □

3.9.3 <u>Remark</u>: In the above theorem, it was not required that G should be connected. It is indeed possible that some of the vertices in W may be isolated, although each vertex in U must be adjacent to a vertex in W; for instance, the graph G to which we refer in Theorem 3.9.2 may be $C_4 \cup \overline{K}_{\gamma-1}$. However, the theorem is easily refined to apply to connected graphs.

3.9.4 <u>Theorem</u>: Let $\gamma, \Delta \in \mathbb{N}$ be given, and let G be a graph. Then, G is a connected, vertex-domination-critical graph with $\gamma(G) = \gamma$, $\Delta(G) = \Delta$ and $p(G) = \Delta + \gamma$, if and only if $V(G) = \{v\} \cup U \cup W$, where v is a vertex of G of degree Δ, $U = N(v) = \{u_1, \ldots, u_\Delta\}$, $W = V(G) -$

$\bar{N}[v] = \{w_1, \ldots, w_{\gamma-1}\}$ and the conditions (i) to (iv) of Theorem 3.9.2 are satisfied, as well as (v) every vertex in W is adjacent to at least one vertex in U.

Proof: The statement follows from the proof of Theorem 3.9.2 together with the following observations: In part (a) of the proof, condition (v) immediately implies connectedness of G, whereas, in part (b) of the proof, once condition (i) has been established, the requirement that G be connected, together with the independence of W, implies validity of (v). □

In the rest of this section, we shall show that it is not possible to characterize vertex-domination-critical graphs in terms of forbidden subgraphs. We consider first

3.9.5 Theorem: For any graph G^*, there exists a vertex-domination-critical graph H such that G^* is an induced subgraph of H.

Proof: Let G^* be a graph of order p. If $\gamma(G^*) \leq 2$, let G be the graph G^* together with two isolated vertices; otherwise, let $G = G^*$. In either case, $\gamma(G) \geq 3$ and $p(G) = p \geq 3$. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Define a graph H by $V(H) = V(G) \cup W \cup X$, where the unions are disjoint and $W = \{w_1, w_2, \ldots, w_p\}$, $X = \{x_1, x_2, \ldots, x_p\}$ and $E(H) = E(G) \cup \{w_ix_j, w_iv_j, x_iv_j; 1 \leq i, j \leq p, i \neq j\}$.

Since $E(\langle\{v_1, v_2, \ldots, v_p\}\rangle_H) = E(G)$, G is an induced subgraph of H (if $G \neq G^*$, then, since G^* is an induced subgraph of G, G^* is an induced subgraph of H). Clearly, $\{v_i, x_i, w_i\} \mapsto H$ for any $i \in \{1, 2, \ldots, p\}$, and so $\gamma(H) \leq 3$. To show that $\gamma(H) \geq 3$, we show that no two-element subset of V(H) dominates H. Suppose $S \subseteq V(G)$ with $|S| = 2$ and $S \mapsto H$. Firstly, S is not contained in V(G), since, otherwise, $\gamma(G) \leq 2$. If S is a subset of W, then $S \not\mapsto W - S$. Similarly, S is not contained in X. If $S = \{w_i, x_j\}$ for some $i, j \in \{1,2,\ldots,p\}$, then $S \not\mapsto \{w_j\}$, if $i \neq j$, or $S \not\mapsto \{v_i\}$ if $i = j$; similarly, if $S = \{w_i, v_j\}$, or if $S = \{v_i, x_j\}$, $S \not\mapsto H$. So, $\gamma(H) = 3$.

Now, let $u \in V(H)$. Then, $u = v_i$, x_i, or w_i, for some $i \in \{1, 2, \ldots, p\}$. Since $\{x_i, v_i, w_i\} - \{u\} \mapsto H-u$, we have $\gamma(H-u) \leq 2$; since $\gamma(H-u) \geq \gamma(H) - 1 = 2$, it follows that $\gamma(H-u) = 2$ for every $u \in V(H)$. Thus, H is vertex-domination-critical, and contains G^* as an induced subgraph. □

3.9.6 Theorem: It is not possible to characterize vertex-domination-critical graphs in terms of forbidden subgraphs.

Proof: Suppose, to the contrary, that there exists a non-empty family $\mathscr{F}$ of graphs such that G is vertex-domination-critical if and only if G contains no member of $\mathscr{F}$ as an induced subgraph. Let H^* be any member of $\mathscr{F}$. Then, by Theorem 3.9.5, there exists a vertex-domination-critical graph H such that H^* is an induced subgraph of H. Now, by our assumption, the vertex-domination-criticality of H implies that no member of $\mathscr{F}$ is an induced subgraph of H; in particular, H^* is not an induced subgraph of H. This contradiction establishes the theorem. □

3.9.7 Theorem: Any graph G with $\gamma(G) \geq 3$ can be embedded as an induced subgraph in a vertex-domination-critical graph G^* where $\gamma(G^*) = \gamma(G)$.

Proof: Let G be any graph with $\gamma(G) \geq 3$, and let H be the vertex-domination-critical graph, containing G as an induced subgraph, that is constructed in the proof of Theorem 3.9.5. Recall that $\gamma(H) = 3$. If $\gamma(G) = 3$, then H is a graph G^* with the required properties. Suppose now that $\gamma(G) \geq 4$. Clearly, any coalescence $H \bullet C_n$, where $n = 3[\gamma(G) - 3] + 1$, contains G as an induced subgraph. Since $n \equiv 1 \pmod 3$, we have (by Example 3.3.2.2) that C_n is vertex-domination-critical graph, and so, by Lemma 3.6.3,

$$\gamma(H \cdot C_n) = \gamma(H) + \gamma(C_n) - 1 = 3 + \left\lceil \frac{3(\gamma(G) - 3) + 1}{3} \right\rceil - 1$$

$$= 3 + (\gamma(G) - 2) - 1 = \gamma(G).$$

In this case, then, $G^* = H \bullet C_n$ is a graph with the desired properties. □

3.10 DOMINATION-FORCING SETS OF GRAPHS

P. J. Slater recently proposed, in a private communication, the investigation of smallest subsets S of vertices of a graph G which cannot be dominated by subsets of V(G) containing fewer than $\gamma(G)$ vertices.

3.10.1 Definitions: Let G be a graph and T a non-empty set of vertices of G. A *T-dominating set in G* is a set $D \subseteq V(G)$ such that $D \rightarrowtail T$. (Note that it is not required that $D \subseteq T$; hence, a T-dominating set in G is not necessarily a dominating set of $\langle T \rangle_G$.) A T-dominating set in G of minimum cardinality is called a *minimum T-dominating set in G* and its cardinality, denoted by $\gamma(T,G)$, is called the *T-domination number in G.*

3.10.2 <u>Examples</u>: (a) If $G \cong K_p$ and $T \subseteq V(G)$, $T \neq \emptyset$, then $\gamma(T,G) = 1 = \gamma(G)$, any singleton subset of $V(G)$ being a T-dominating set in G.

(b) If $G \cong K_{m,n}$, $2 \leq m \leq n$, with partite sets V_1 and V_2, then, for $T \subseteq V(G)$ such that $|T \cap V_i| \geq 2$ for $i \in \{1, 2\}$, we have $\gamma(T,G) = 2 = \gamma(G)$, whereas $\gamma(T,G) = 1$ if $|T \cap V_i| \leq 1$ for some $i \in \{1, 2\}$.

(c) If $G \cong \overline{K}_p$ and $T \subseteq V(G)$, $T \neq \emptyset$, then $\gamma(T,G) = |T|$, T being the only T-dominating set in G.

Hence, we note that there exist graphs G having proper subsets T of V(G) for which $\gamma(T,G) = \gamma(G)$.

(d) Let G be any graph that contains an induced subgraph isomorphic to P_3 (for example, if G is connected and non-complete), and let x,y,z be an induced path in G. Then, $T = \{x, z\}$ is such that $\gamma(\langle T\rangle_G) = 2 \neq 1 = \gamma(T,G)$.

3.10.3 <u>Definition</u>: Let G be a graph. A set $S \subseteq V(G)$ for which $\gamma(S,G) = \gamma(G)$ is called a *domination-forcing set* of G or (briefly) a *γ-forcing set* of G. (Clearly, such a set exists for every graph G as $\gamma(V(G),G) = \gamma(G)$.) A γ-forcing set of G of minimum cardinality is known as a *$\theta(G)$-set* and its cardinality, denoted by $\theta(G)$, is called the *γ-forcing number* of G.

3.10.4 <u>Examples</u>: (a) If $G \cong K_p$, then any singleton subset of V(G) is a $\theta(G)$-set and $\theta(G) = 1$.

(b) If $G \cong K_{m,n}$ with $2 \leq m \leq n$, then any 4-set of vertices containing two vertices from each of the partite sets of G is a γ-forcing set of G (and hence a $\theta(G)$-set, as a set $S \subseteq V(G)$ containing at most one vertex from some partite set of G has $\gamma(S,G) = 1 < 2 = \gamma(G)$); so, $\theta(G) = 4$.

(c) If $G \cong \overline{K}_p$, then V(G) is the only γ-forcing set of G and so $\theta(G) = p$.

(d) $G \cong P_3$ and $H \cong K_1 \cup K_2$ are the non-complete graphs of smallest order for which the order exceeds the γ-forcing number. Any $S \subseteq V(G)$ with $S \neq \emptyset$ is a γ-forcing set of G (so $\theta(G) = 1$) and the subsets of V(H) containing at least one vertex from each component of H are γ-forcing sets of H (so $\theta(H) = 2$).

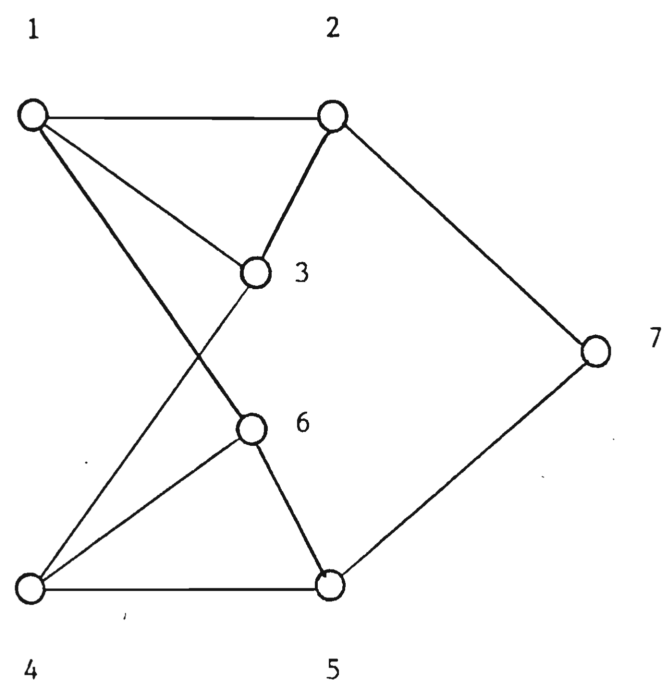


Fig. 3.10.1

(e) If $G \cong S_{m,n}$ ($2 \leq m \leq n$) with central vertices u and v, adjacent to the end-vertices u_1, u_2, ..., u_m and v_1, v_2, ..., v_n, respectively, and $S = \{u_1, v_1\}$, then $\gamma(S,G) = 2 = \gamma(G)$ and S is (obviously) a $\theta(G)$-set.

3.10.5 <u>Remark</u>: It is immediately obvious that, for any graph G and $S \subseteq V(G)$, $\gamma(S,G) \leq \min\{\gamma(G), \gamma(\langle S\rangle_G)\}$. The examples in 3.10.4 all have the property that, for any $\theta(G)$-set S, $\gamma(\langle S\rangle_G) = \gamma(S,G)$ ($= \gamma(G)$). That this is not true for every graph G is shown by the following example, in which is exhibited a graph G and a $\theta(G)$-set S for which $\gamma(\langle S\rangle_G) > \gamma(S,G)$ ($= \gamma(G)$).

3.10.6 <u>Example</u>: The graph G shown in Fig. 3.10.1 has domination number 2 and {2, 5} is a minimum dominating set of G. Since the vertices in every pair of distinct, non-adjacent vertices in G have a common neighbour, $\gamma(T,G) = 1$ if $T \subseteq V(G)$ and $1 \leq |T| \leq 2$; hence, $\theta(G) \geq 3$. As the set $S = \{1, 4, 7\}$ satisfies $\gamma(S,G) = |\{2, 4\}| = 2 = \gamma(G)$ and $|S| = 3$, it follows that $\theta(G) = 3$ and that S is a $\theta(G)$-set; furthermore, since S is independent, $\gamma(\langle S\rangle_G) = 3 > \gamma(S,G) = \gamma(G) = 2$.

3.10.7 <u>Remark</u>: We next investigate the relationship between $\theta(G)$ and $\gamma(G)$ for a graph G. A dominating set D of a graph is said to be *efficient* if $\Sigma_{v\in D} (1 + \deg_G v) = p(G)$, i.e., if every vertex of G is dominated by a unique vertex of D. Now, let G be a graph with an efficient dominating set D; then, no two vertices of D are adjacent or have a common neighbour in G. Hence, each vertex in any D-dominating set in G dominates at most one vertex of D, so that, if D′ is a minimum D-dominating set in G, we have $\gamma(D,G) = |D'| \geq |D|$. Since $D \mapsto D$, we have $\gamma(D,G) \leq |D|$, whence it follows that $\gamma(D,G) = |D|$. Consequently, since $\gamma(D,G) \leq \gamma(G) \leq |D|$ (by Remark 3.10.5), D is a *minimum* dominating set of D.

3.10.8 <u>Proposition</u>: For any graph G,

(1) $\gamma(G) \leq \theta(G)$, and

(2) $\gamma(G) = \theta(G)$ if G has an efficient dominating set.

<u>Proof</u>: Let G be any graph.

(1) If $S \subseteq V(G)$ and $|S| < \gamma(G)$, then $\gamma(S,G) \leq \gamma(\langle S\rangle_G) \leq |S| < \gamma(G)$ and S is not a $\theta(G)$-set. Hence, for any $\theta(G)$-set S, $\theta(G) = |S| \geq \gamma(G)$.

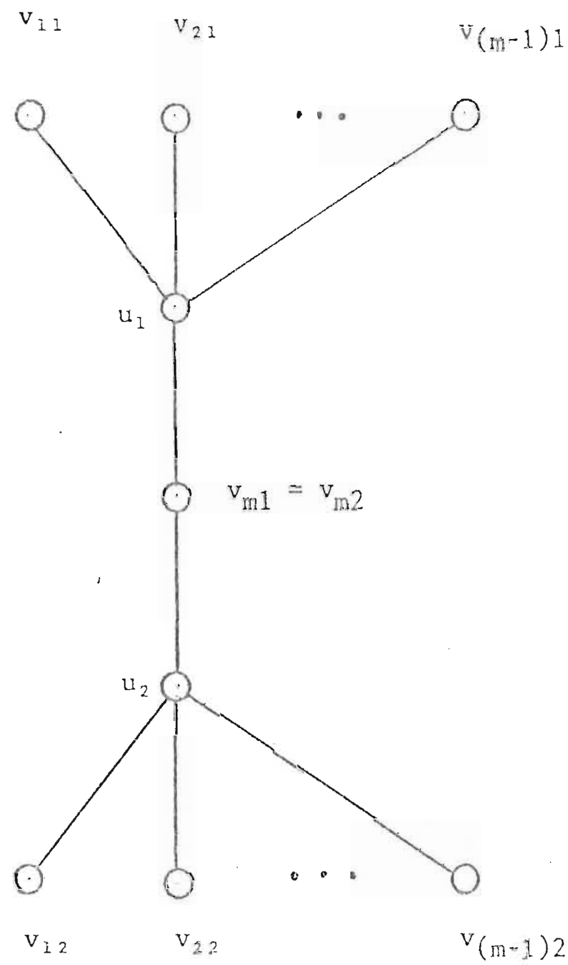


Fig. 3.10.2

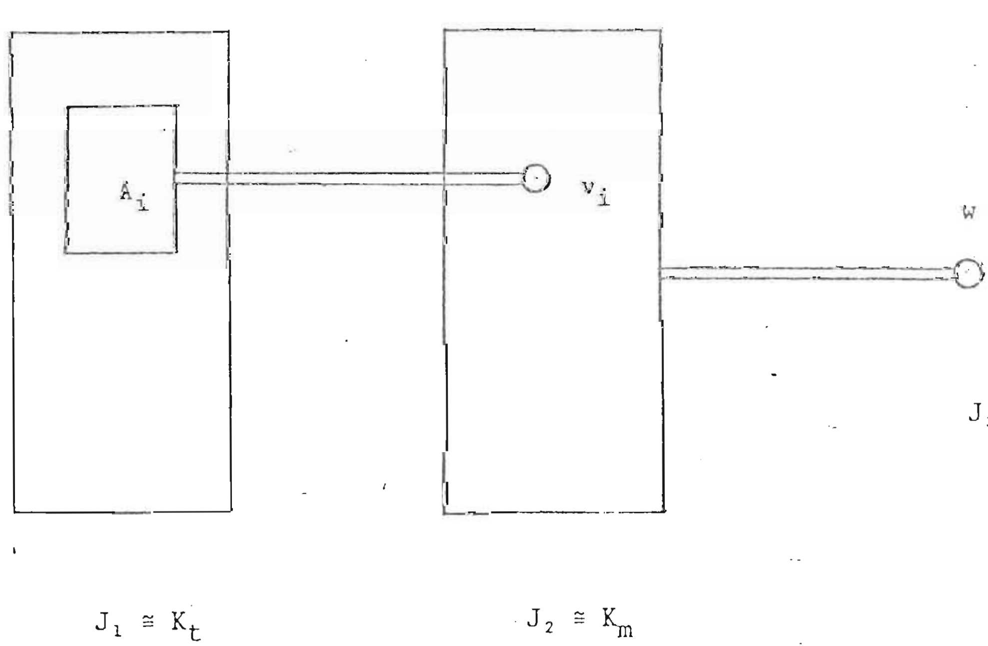


Fig. 3.10.3

(2) If G has an efficient dominating set D, then $\gamma(D,G) = |D| = \gamma(G)$ (by Remark 3.10.7). Hence, D is a γ-forcing set of G and $\theta(G) \leq |D| = \gamma(G)$, which, with (1), yields $\theta(G) = \gamma(G)$. □

3.10.9 Remark: That the (sufficient) condition given in Proposition 3.10.8(2) is not necessary to ensure that $\theta(G) = \gamma(G)$ may be seen by consideration of the graph G in Fig. 3.10.2, obtained from $G_1 \cup G_2$ with $G_1, G_2 \cong K_{1,m}$, where G_i has centre u_i and end-vertices $v_{1i}, v_{2i}, \ldots, v_{mi}$, by identifying v_{m1} and v_{m2} ($m \geq 3$). The only minimum dominating set of G is $D = \{u_1, u_2\}$ and $S = \{v_{11}, v_{12}\}$ satisfies $\gamma(S,G) = 2 = \gamma(G) = |S|$, whence S is a $\theta(G)$-set and $\theta(G) = 2 = \gamma(G)$. Certainly, D is not an efficient dominating set of G (since $d(u_1,u_2) = 2$), and so (by Remark 3.10.7), *no* dominating set of G is efficient.

We shall show next that, for any given positive integers j, t with $j < t$, there exists a graph G for which $\gamma(G) = 2$, $\theta(G) - \gamma(G) = j$ and $p(G) - \theta(G) \geq 2t + 1$.

3.10.10 Definition: For $j, t \in \mathbb{N}$ with $t \geq j + 1$, let $m = \binom{t}{j}$ and define the graph $J_{t,j}$ as follows: Let $J_1 \cong K_t$, $J_2 \cong K_m$ and $J_3 \cong K_1$, with $V(J_1) = \{u_1, u_2, \ldots, u_t\}$, $V(J_2) = \{v_1, \ldots, v_m\}$ and $V(J_3) = \{w\}$, and let $A_1, A_2, \ldots, A_m$ be the m distinct subsets of $V(J_1)$ that have cardinality j. Let $V(J_{t,j}) = V(J_1) \cup V(J_2) \cup V(J_3)$ and $E(J_{t,j}) = E(J_1) \cup E(J_2) \cup \{wv_i;\ i = 1, 2, \ldots, m\} \cup F$, where $F = \bigcup_{i=1}^{m} \{v_iu_k;\ u_k \in A_i\}$. (See Fig. 3.10.3.)

3.10.11 Proposition: For $t, j \in \mathbb{N}$, $t \geq j + 1$ and $J_{t,j}$ defined as in 3.10.10 above,

(1) $\gamma(J_{t,j}) = 2$, and

(2) $\theta(J_{t,j}) = j + 2 = \gamma(J_{t,j}) + j$, and

(3) $p(J_{t,j}) = t + \binom{t}{j} + 1 \geq 2t + 1 \geq 2\theta(J_{t,j}) - 1$.

Proof: Let t, j satisfy the hypothesis of the proposition.

(1) Since $\Delta(J_{t,j}) < p(J_{t,j}) - 1$, it follows that $\gamma(J_{t,j}) \geq 2$; hence, as $\{u_1, w\} \rightarrowtail J_{t,j}$, $\gamma(J_{t,j}) = 2$.

(2) Let $B \subseteq V(J_{t,j})$ such that $|B \cap V(J_1)| \leq j$. Then, there exists $k \in \{1, 2, \ldots, m\}$ such that $B \cap V(J_1) \subseteq A_k$; consequently, $\{v_k\} \rightarrowtail B$ and $\gamma(B,J_{t,j}) = 1$. Hence, it follows that, if S is a $\theta(G)$-set (so $\gamma(S,J_{t,j}) = 2$), then $|S \cap V(J_1)| \geq j + 1$. Furthermore, $S \not\subseteq V(J_1)$ (since, otherwise, $\{u_1\} \rightarrowtail S$ and $\gamma(S,J_{t,j}) = 1$), so $S - V(J_1) \neq \emptyset$ and $\theta(G) = |S| \geq (j + 1) + |S - V(J_1)| \geq j + 2$. To show that $\theta(G) \leq j + 2$, let $T = \{u_1, u_2, \ldots, u_{j+1}, w\}$. Then, $\gamma(T,J_{t,j}) \geq 2$ since,

otherwise, if there exists $y \in V(J_{t,j})$ with $\{y\} \mapsto T$, then $y \notin V(J_2) \cup V(J_3)$ (as no vertex in $V(J_2) \cup V(J_3)$ is adjacent to $j + 1$ vertices in $V(J_1)$) and so $y \in V(J_1)$, whence $\{y\} \not\mapsto \{w\}$, contradicting $\{y\} \mapsto T$. So, by (1) and Remark 3.10.5, we have $\gamma(J_{t,j}) = 2 \leq \gamma(T,J_{t,j}) \leq \gamma(J_{t,j})$, i.e., $\gamma(T,J_{t,j}) = \gamma(J_{t,j})$ and T is a γ-forcing set of G, whence $\theta(G) \leq |T| = j + 2$. Hence, $\theta(G) = j + 2$.

(3)

$$\begin{aligned}
p(J_{t,j}) &= t + \binom{t}{j} + 1 \\
&= t + t\,\frac{(t-1)(t-2)\cdots(t-j+1)}{j(j-1)\cdots 2.1} + 1 \\
&\geq t + t + 1 = 2t + 1 \;\geq\; 2j + 3 \;=\; 2\theta(J_{t,j}) - 1.
\end{aligned}$$

3.10.12 <u>Remark</u>: For $t = 2$, $j = 1$, we obtain a graph $J_{t,j}$ $(= J_{2,1})$ of smallest possible order (namely, $p(J_{2,1}) = 5$), and we have $\theta(J_{2,1}) = 3$ and $\gamma(J_{2,1}) = 2$. In this case,

$$\frac{\theta(J_{2,1})}{p(J_{2,1})} = \frac{3}{5} > \frac{1}{2}.$$

In general, if $t = j + 1$, then

$$\frac{\theta(J_{j+1,j})}{p(J_{j+1,j})} = \frac{j+2}{2j+3} = \frac{1}{2} + \frac{1}{4j+6} \in \left(\frac{1}{2}, \frac{3}{5}\right]$$

and

$$\lim_{j\to\infty} \frac{\theta(J_{j+1,j})}{p(J_{j+1,j})} = \frac{1}{2}.$$

If $t = j + 2$, then $p(J_{t,j}) = j + \binom{j+2}{j} + 1$ and

$$\lim_{j\to\infty} \frac{\theta(J_{j+2,j})}{p(J_{j+2,j})} = 0;$$

furthermore, for any fixed $j \in \mathbb{N}$, we see from Proposition 3.10.11 (2) and (3) that

$$\lim_{t\to\infty} \frac{\theta(J_{t,j})}{p(J_{t,j})} = 0.$$

In the above example, $\gamma(J_{t,j}) = 2$. We shall show that, for prescribed $n \geq 2$, M and N, there exists a graph G for which $\gamma(G) = n$, $\theta(G) - \gamma(G) \geq M$ and $p(G) - \theta(G) \geq N$.

3.10.13 Example: For $t, j \in \mathbb{N}$ with $n \geq 2$, $t \geq (n-1)j + 1$, $m = \binom{t}{j}$, let $G_1, G_2, \ldots, G_{n-1} \cong J_{t,j}$ (see Definition 3.10.10) and, in G_i, let V_{1i}, V_{2i}, V_{3i}, u_{1i}, u_{2i}, ..., u_{ti}, v_{1i}, v_{2i}, ..., v_{mi} and w_i correspond to $V(J_1)$, $V(J_2)$, $V(J_3)$, u_1, u_2, ..., u_t, v_1, v_2, ..., v_m, and w, respectively, for $i = 1, \ldots, n-1$. Let $J_{t,j,n}$ be the graph obtained from $G_1 \cup G_2 \cup \ldots \cup G_{n-1}$ by identifying the vertices v_{i1}, v_{i2}, ..., $v_{i(n-1)}$ to form a new vertex v_i corresponding to the vertex $v_i \in V(J_2)$ in $J_{t,j}$, for $i = 1, 2, \ldots, m$. Denote the resulting set $\{v_1, v_2, \ldots, v_m\}$ by V_2, and the subset of V_{1i} corresponding to A_k by A_{ki} ($i \in \{1, \ldots, n-1\}$, $k \in \{1, \ldots, m\}$). (Note that $J_{t,j,2} = J_{t,j}$.)

3.10.14 Proposition: For $t, j \in \mathbb{N}$ with $t \geq (n-1)j + 1$, $n \geq 2$, and $G = J_{t,j,n}$ (as in Definition 3.10.13), we have

(1) $\gamma(G) = n$,

(2) $\theta(G) = (n-1)j + 2 = \gamma(G) + (n-1)(j-1) + 1$, and

(3) $p(G) = (n-1)(t+1) + \binom{t}{j} = \theta(G) + (n-1)(t+1-j) + \binom{t}{j} - 2$.

Proof: Let t, j satisfy the hypothesis of the proposition.

(1) That $\gamma(G) \leq n$ follows from the observation that $\{v_1, u_{11}, u_{12}, \ldots, u_{1(n-1)}\} \mapsto G$. If there exists a dominating set D of G with $|D| \leq n-1$, then $D \not\subset \bigcup_{i=1}^{n-1} V_{1i}$ (otherwise, $D \not\mapsto \{w_1, \ldots, w_{n-1}\}$); hence, $D \cap V_{1i} = \varnothing$ for at least one value of $i \in \{1, 2, \ldots, n-1\}$. So, V_{1i} is dominated by (at most $n-1$) vertices in $D \cap V_2$; however,

$$|N_{G_j}(D \cap V_2)| \leq |D \cap V_2|.j \leq (n-1).j < t = |V_{1j}|,$$

so that $D \cap V_2 \not\mapsto V_{1j}$, a contradiction. So, any dominating set of G has cardinality at least n. So, $\gamma(G) = n$.

(2) Let S be $\theta(G)$-set. We note that, if $U \subseteq V(J_1) \subseteq V(J_{t,j})$ is the set of all vertices in $V(J_{t,j})$ corresponding to at least one vertex in $S \cap (\bigcup_{i=1}^{n-1} V_{1i})$, then $|U| \geq (n-1)j + 1$ (otherwise, U is the union of at most $n-1$ subsets A_k from $\{A_1, \ldots, A_m\}$ and at most $n-1$ corresponding vertices v_k from $V_2 \subseteq V(G)$ serve to form an S-dominating set in G, contradicting $\gamma(S,G) = n$). Furthermore, $S \not\subset \bigcup_{i=1}^{n-1} V_{1i}$, since, otherwise, $\{u_{11}, u_{12}, \ldots, u_{1(n-1)}\}$ is an S-dominating set in G. So, $\theta(G) = |S| \geq (n-1)j + 2$. Now, let $U_1 = \{u_1, \ldots, u_j\}$, $U_2 = \{u_{1+j}, \ldots, u_{2j}\}$, ..., $U_{n-2} = \{u_{1+(n-3)j}, \ldots, u_{(n-2)j}\}$ and $U_{n-1} = \{u_{1+(n-2)j}, \ldots, u_{(n-1)j}, u_{(n-1)j+1}\}$ (so that $U = \bigcup_{i=1}^{n-1} U_i$ satisfies

$|\bar{U}| = (n-1)j+1$) and denote by U_i' the subset of V_{1i}, corresponding to U_i ($i \in \{1, 2, ..., n-1\}$). The only $(n-1)$-set of vertices that dominates $U' = \bigcup_{i=1}^{n-1} U_i'$ in G is contained in $\bigcup_{i=1}^{n-1} V_{1i}$ and so $S = U' \cup \{w_1\}$ (say) satisfies $|S| = (n-1)j+2$ and $\gamma(S,G) = n = \gamma(G)$, whence $\theta(G) \leq (n-1)j+2$. Hence, $\theta(G) = (n-1)j+2$, as required, which completes the proof of (2), from which (3) follows immediately. □

3.11 CONJECTURES AND UNSOLVED PROBLEMS

The following are four open questions posed by Brigham, Chinn, Dutton in [BCD1] and [BCD2].

(1) For a vertex-domination-critical graph G, is

$$p(G) \geq [\delta(G)+1]\,[\gamma(G)-1]+1\ ?$$

This is trivially true when p(G) attains the upper bound $p(G) = [\Delta(G)+1]\,[\gamma(G)-1]+1$, the maximum possible value of p(G), given in Theorem 3.5.4, and also holds when $p(G) = \gamma(G) + \Delta(G)$, the minimum possible value of p(G) [BCD2].

(2) Is it true that $i(G) = \gamma(G)$ for every vertex-domination-critical graph G? As in (1), the statement is true when the order of the vertex-domination-critical graph is the minimum or maximum value it can attain [BCD2]. Recall that a similar conjecture for edge-domination-critical graphs is made in 2.9.4 (cf. [SB1]).

(3) For a vertex-domination-critical graph G, is

$$\operatorname{diam} G \leq 2[\gamma(G)-1]\ ?$$

The relation holds when $p(G) = \gamma(G) + \Delta(G)$ or $\gamma(G) \leq 5$ [BCD2].

(4) If G is a vertex-domination-critical graph and $v \in V(G)$, does there exist a vertex u and a minimum dominating set D_u of G-u such that $v \in D_u$? It has been shown that, if G is a vertex-domination-critical graph and $u, v \in V(G)$ such that $u \neq v$ and $\gamma(G-\{u, v\}) \neq \gamma - 1$, then $v \in D_u$ for some minimum dominating set D_u of G-u (cf. [BCD2]).

Chapter 4

DOMINATION NUMBER ALTERATION BY REMOVAL OF EDGES

4.1 INTRODUCTION

Whereas in Chapter 2 we studied graphs G that have the property that adding any single edge of $\bar{G}$ to G produces a graph with domination number lower than $\gamma(G)$, and whereas in Chapter 3 we investigated graphs H that possess sets of vertices the removal of which results in graphs with domination number different from $\gamma(H)$, in the present chapter we consider the effect of the removal of a set of *edges* on the domination numbers of graphs. In particular, for a graph I, we investigate the minimum cardinality of a set of edges the removal of which yields a graph with domination number greater than $\gamma(I)$, and subsequently consider extremal graphs with dominating sets or domination numbers that are impervious to the removal of arbitrary edges.

All results in sections 4.2 to 4.5 are from [BHNS1], with the exception of Theorem 4.2.8 and Corollary 4.5.5 which come from [AW1], Theorem 4.2.7 which comes from [FJ1], and Remark 4.5.6 which comes from [S1], and all results in sections 4.6 to 4.8 are from [BD1]. In addition, we have expanded Remark 4.4.10, as well as the proof of Theorem 4.2.8 (slightly), 4.4.1, 4.4.9

(slightly), 4.6.8, Theorem 4.6.10 (considerably), 4.7.3(1), 4.7.4 (considerably), 4.7.6 (very considerably), 4.8.4, 4.8.5 (slightly), 4.8.8, and Proposition 4.5.4, 4.6.5, 4.8.6 (considerably). We have slightly modified the proof of Theorem 4.3.6, and slightly rearranged the proof of Theorem 4.3.2. We have clarified and expanded Theorem 4.3.4, 4.4.11, and 4.7.19. We have supplied Remark 4.2.2, 4.2.4, 4.4.3 and 4.4.6. We have provided the statement and proof of Proposition 4.2.3, 4.5.2, Corollary 4.2.6, and Theorem 4.7.3(2), and 4.7.5. We have modified Definition 4.4.4, which originally appeared in [BHNS1]. We have supplied Corollary 4.5.3, as well as the proof of Theorem 4.2.7, 4.4.5, Proposition 4.2.5, 4.3.1, 4.4.13, 4.6.6, 4.7.17, 4.7.18, 4.8.3, 4.8.7, 4.8.9, Corollary 4.3.3, and Lemma 4.7.15. Finally, we have modified the statement of, and provided a proof for, Corollary 4.6.9.

4.2 INTRODUCTION TO γ-EDGE-STABILITY NUMBER (BONDAGE NUMBER) OF A GRAPH

4.2.1 Definition [BHNS1]: For a graphical parameter μ, the *μ-edge-stability number* of a graph G is defined to be the minimum number of edges in any set $F \subseteq E(G)$ such that $\mu(G-F) \neq \mu(G)$, provided that such a set F exists. In particular, $\mu^{+\prime}(G)$ (or $\mu^{-\prime}(G)$) denotes the minimum number of edges in $F \subseteq E(G)$ for which $\mu(G-F) > \mu(G)$ (or $\mu(G-F) < \mu(G)$), if such a set F exists.

For instance, if $G \cong K_p$ $(p \geq 3)$ and $e \in E(G)$, then $\beta(G-e) = \beta(G) + 1$ and $\kappa(G-e) = \kappa(G) - 1$; hence, $\beta^{+\prime}(G) = 1 = \kappa^{-\prime}(G)$.

4.2.2 Remark: Obviously, if G is any graph and $F \subseteq E(G)$, then $\gamma(G-F) \geq \gamma(G)$; so, the parameter $\gamma^{-\prime}$ is *not* defined for any graph G. That $\gamma^{+\prime}(G)$ (known as the *bondage number* of G) is well-defined for every non-empty graph G is shown by the following proposition, since $\gamma(G-E(G)) = \gamma(\overline{K}_{p(G)}) > \gamma(G)$ (see Proposition 4.2.3) and therefore a smallest subset F of E(G) exists for which $\gamma(G-F) > \gamma(G)$.

4.2.3 Proposition: For every non-empty graph G, $\gamma(G) < p(G)$.

Proof: If G is a non-empty graph and $uv \in E(G)$, then $\{v\} \mapsto \{u\}$ and $V(G) - \{u\} \mapsto V(G) - \{u\}$, whence $V(G) - \{u\} \mapsto G$ and $\gamma(G) \leq p(G) - 1$. □

4.2.4 Remark: It should be stressed that, if G is a graph, $F \subseteq E(G)$ such that $\gamma(G-F) > \gamma(G)$ and $|F| = \gamma^{+\prime}(G)$, then F is a smallest set of edges for which *no* minimum dominating set of G also dominates $G-F$. There may well exist a minimum dominating set D of G and a set $F' \subseteq E(G)$ with $|F'| < |F|$ such that $D \not\mapsto G-F'$, but, in this case, some minimum dominating set D' ($\neq D$) of G will exist such that $D' \mapsto G-F'$. In the following results, culminating in Theorem 4.2.7, we shall prove that, for any non-trivial graph G and any minimum dominating set D of G, there exists a set $F' \subseteq E(G)$ with $|F'| \leq 2$ such that $D \not\mapsto G-F'$, even if $\gamma^{+\prime} \geq 3$.

So, the bondage number $\gamma^{+\prime}$ may be regarded as a measure of the integrity of the domination *number* (as opposed to the dominating property of a *particular* minimum dominating set) of a graph with respect to edge removal.

We shall need the following result of Ore [O1].

4.2.5 Proposition: For any non-empty graph G with no isolated vertices, there exists a minimum dominating set D of G such that, for each $v \in D$, there exists $u \in V(G) - D$ such that $N(u) \cap D = \{v\}$. (We shall call u a *private neighbour* of v.)

Proof: Let G be a non-empty graph with no isolated vertices. If $\gamma(G) = 1$, the result follows immediately; so suppose $\gamma(G) \geq 2$. For any minimum dominating set D of G and, for each $d \in D$, we know that at least one of the following is true:

(1) there exists $v \in V(G) - D$ such that $\{d\} = N(v) \cap D$;

(2) $N(d) \cap D = \varnothing$.

We shall prove that G must contain some minimum dominating set D' for which each $d \in D'$ satisfies (1).

Suppose, to the contrary, that no minimum dominating set of G is such that each of its vertices satisfies (1). Let D be a minimum dominating set of G such that $q(\langle D\rangle)$ is a maximum, and let d be a vertex of D that does not satisfy (1). Then, (2) holds (so that $[\{d\}, V(G)]$ contributes no edges to $E(\langle D\rangle)$ and, furthermore, any vertex $w \in V(G) - D$ that is adjacent to d satisfies $|N(w) \cap D| \geq 2$. Since d is not an isolate of G, d has a neighbour v which, by condition (2), lies in $V(G) - D$. By our earlier comment, then, there exists $d' \in D - \{d\}$ ($\neq \varnothing$) such that $vd' \in E(G)$. Then, since $\{v\} \mapsto \{d\}$ and $N(w) \cap (D - \{d\}) \neq \varnothing$ for each $w \in N(d)$, we have that $D^* = (D - \{d\}) \cup \{v\}$ is a minimum dominating set of G and $E(\langle D^*\rangle) \supseteq E(\langle D\rangle) \cup \{vd'\}$, whence $q(\langle D^{*}\rangle) \geq q(\langle D\rangle) + 1$, which contradicts our choice of D. Hence, any minimum dominating set D of G for which $q(\langle D\rangle)$ is a maximum satisfies (1) for each of its vertices. $\square$

4.2.6 <u>Corollary</u>: For any non-empty graph G with no isolated vertices, there exists a minimum dominating set D for which there is an edge e in G such that D is not a dominating set in G−e.

<u>Proof</u>: Let G be a non-empty graph with no trivial components, and let D′ be a dominating set whose existence is guaranteed by the above theorem. Then, if v is any vertex of D, and u is a vertex of $V(G) - D$ satisfying $N(u) \cup D = \{v\}$, then D is clearly not a dominating set of G−uv. $\square$

4.2.7 <u>Theorem</u>: For any non-empty G and minimum dominating set D of G, there exists a vertex $u \in V(G) - D$ such that $|N(u) \cap D| \leq 2$.

<u>Proof</u>: Suppose, to the contrary, that there exists a non-empty graph G and a minimum dominating set D of G such that $|N(x) \cap D| \geq 3$ for each $x \in V(G) - D$. Let $u \in V(G) - D$, and suppose that v and w are two of the (at least three) vertices in D that dominate u. Then, $N(x) \cap (D - \{v, w\}) \neq \varnothing$ for every $x \in V(G) - D$ and so $(D - \{v, w\}) \cup \{u\}$ is a dominating set of G of cardinality less than $\gamma(G)$, which is not possible. The desired result follows. $\square$

This concludes the discussion of Remark 4.2.4.

4.2.8 <u>Theorem</u>: Let G be a graph and $e = uv \in E(G)$. Then, $\gamma(G\text{-}uv) > \gamma(G)$ if and only if, for every minimum dominating set D of G, the following two conditions hold:

(i) $e \in [D, V(G) - D]$; say, $u \in D$ and $v \in V(G) - D$;

(ii) $N_G(v) \cap D = \{u\}$.

<u>Proof</u>: We prove the necessity first. Assume, to the contrary, that there exists a graph G and an edge $uv \in E(G)$ such that $\gamma(G-uv) > \gamma(G)$ but for which there is a minimum dominating set D such that (i) or (ii) is not satisfied. If (i) is false, i.e., if $u, v \in D$ or $u, v \in V(G) - D$, then, clearly, $D \rightarrowtail G-uv$ and $\gamma(G-uv) \leq \gamma(G)$, contrary to our choice of u and v. So, (i) holds and (ii) is false, whence it follows that $|N_G(v) \cap D| \geq 2$, which, in turn, implies that $D \rightarrowtail \{v\}$ in G−uv. So, $D \rightarrowtail G-uv$ and $\gamma(G-uv) \leq \gamma(G)$, again a contradiction. So, no such graph G exists, and the necessity follows.

Conversely, let G be a graph and let $uv \in E(G)$ such that, for every minimum dominating set D of G, both (i) and (ii) hold. Let D be a minimum dominating set of G. By (i), it follows that $D \cup \{v\} \mapsto G-uv$, whence $\gamma(G-uv) \leq \gamma(G) + 1$. We claim $\gamma(G-uv) = \gamma(G) + 1$. Suppose this equality does not hold. Then, since $\gamma(G-e) \geq \gamma(G)$ for all $e \in E(G)$, we have $\gamma(G-uv) = \gamma(G)$. Let D_0 be a minimum dominating set of G-uv; clearly, D_0 is a minimum dominating set of G, since D_0 dominates G-uv, a spanning subgraph of G, and $\gamma(G) = \gamma(G-uv)$. Since D_0 dominates a graph (namely, G-uv) in which u and v are non-adjacent, we have (a) $u, v \in D_0$ or (b) $u, v \in V(G) - D_0$ or (c) $u \in D_0$, $v \in V(G) - D_0$ and $|N_{G-uv}(v) \cap D_0| \geq 1$, i.e., $|N_G(v) \cap D_0| \geq 2$ or (d) $v \in D_0$, $u \in V(G) - D_0$ and $|N_{G-uv}(u) \cap D_0| \geq 1$. However, by our choice of u and v, i.e., because (i) and (ii) hold, none of the four afore-mentioned possibilities can occur. So, we have $\gamma(G-uv) = \gamma(G) + 1 > \gamma(G)$, as claimed, and the theorem follows. □

4.3 EXAMPLES OF BONDAGE NUMBERS OF GRAPHS

In this section, we investigate the value of $\gamma^{+\prime}$ for several classes of graphs, namely, complete graphs, cycles, paths, complete t-partite graphs ($t \geq 2$) and trees.

4.3.1 Proposition: For $n \geq 2$, $\gamma^{+\prime}(K_n) = \lceil n/2 \rceil$.

Proof: Let $n \geq 2$, let $G_n \cong K_n$, and let $F \subseteq E(G_n)$ such that, for $G = G_n - F$, $\gamma(G) > \gamma(G_n) = 1$. Then, $\Delta(G) \leq n - 2$, and so F is an edge cover of G_n, whence $|F| \geq \lceil n/2 \rceil$. In particular, then, $\gamma^{+\prime}(K_n) = \gamma^{+\prime}(G_n) \geq \lceil n/2 \rceil$. If n is even, G_n possesses a 1-factor with edge set F' such that $|F'| = n/2 = \lceil n/2 \rceil$ and $\gamma(G_n - F') = 2$. If n is odd (and $n \geq 3$), let $v \in V(G_n)$; then the set F' consisting of the edges in a 1-factor of $G_n - v$, together with any edge incident with v, contains $\frac{1}{2}(n - 1) + 1 = \frac{1}{2}(n + 1) = \lceil n/2 \rceil$ edges and $\gamma(G_n - F') = 2$. So, $\gamma^{+\prime}(G_n) \leq \lceil n/2 \rceil$. Hence, $\gamma^{+\prime}(K_n) = \gamma^{+\prime}(G_n) = \lceil n/2 \rceil$, as required. □

4.3.2 Theorem: For $n \geq 3$,

$$\gamma^{+\prime}(C_n) = \begin{cases} 3, & \text{if } n \equiv 1 \pmod 3 \\ 2, & \text{otherwise} \end{cases}.$$

Proof: Let $n \geq 3$. Since $C_n - e \cong P_n$ for $e \in E(C_n)$ and $\gamma(C_n) = \gamma(P_n)$, we have $\gamma^{+\prime}(C_n) \geq 2$. We consider two cases.

Case 1: Suppose that $n \equiv 1 \pmod 3$. Then, the removal of two non-adjacent edges from C_n leaves a graph H which is the (disjoint) union of two paths A and B, with $p(A) = a$, $p(B) = b$, where $a + b = n$.

Subcase 1.1: Suppose $a \equiv 0 \pmod 3$ and $b \equiv 1 \pmod 3$ (or, $a \equiv 1 \pmod 3$ and $b \equiv 0 \pmod 3$). Then,

$$\begin{aligned}\gamma(H) &= \gamma(A) + \gamma(B) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil = \frac{a}{3} + \frac{b+2}{3} = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil \\ &= \gamma(C_n).\end{aligned}$$

Subcase 1.2: Suppose $a \equiv b \equiv 2 \pmod 3$. Then,

$$\gamma(H) = \left\lceil \frac{a}{3} \right\rceil + \left\lceil \frac{b}{3} \right\rceil = \frac{a+1}{3} + \frac{b+1}{3} = \frac{n+2}{3} = \left\lceil \frac{n}{3} \right\rceil = \gamma(C_n).$$

The removal of two adjacent edges from C_n yields a graph $H \cong P_{n-1} \cup K_1$, where $\gamma(H) = 1 + \lceil ⅓(n - 1) \rceil = \lceil n/3 \rceil = \gamma(C_n)$. Thus, it follows that $\gamma^{+\prime}(C_n) \geq 3$. To show the reverse inequality holds, we consider the graph I resulting from the deletion of three consecutive edges of C_n. Since $I \cong 2K_1 \cup P_{n-2}$, we have

$$\gamma(I) = 2 + \left\lceil \frac{n-2}{3} \right\rceil = 1 + \left\lceil \frac{n+1}{3} \right\rceil = 1 + \left\lceil \frac{n}{3} \right\rceil = 1 + \gamma(C_n),$$

whence $\gamma^{+\prime}(C_n) \leq 3$, and so $\gamma^{+\prime}(C_n) = 3$.

Case 2: Suppose $n \equiv 0$ or $n \equiv 2 \pmod 3$. Let H be a graph obtained by the removal of two adjacent edges from C_n; then, $H \cong K_1 \cup P_{n-1}$. So,

$$\gamma(H) = 1 + \left\lceil \frac{n-1}{3} \right\rceil = 1 + \left\lceil \frac{n}{3} \right\rceil = 1 + \gamma(C_n),$$

so that $\gamma^{+\prime}(C_n) \leq 2$. The required result now follows by the already established reverse inequality. □

4.3.3 Corollary: For $n \geq 2$,

$$\gamma^{+'}(P_n) = \begin{cases} 2, & \text{if } n \equiv 1 \pmod 3 \\ 1, & \text{otherwise} \end{cases}.$$

Proof: For any $n \geq 3$, since $C_n - e \cong P_n$ for any edge $e \in E(C_n)$, it follows that $\gamma^{+'}(C_n) = \gamma^{+'}(P_n) + 1$, and the desired result follows immediately from Theorem 4.3.2. That $\gamma^{+'}(P_2) = 1$ follows from Proposition 4.3.1. □

We next consider the bondage numbers of complete t-partite graphs ($t \geq 2$).

4.3.4 Theorem: If

$$G \cong K_{n_1, n_2, \ldots, n_t},$$

where $n_1, n_2, \ldots, n_t$ satisfy $1 \leq n_1 \leq n_2 \leq \ldots \leq n_t$ and $t \geq 2$, then

$$\gamma^{+'}(G) = \begin{cases} \left\lceil \frac{m}{2} \right\rceil, & \text{if } n_m = 1 \text{ and } n_{m+1} \geq 2 \text{ for some } m \in \{1, \ldots, t-1\}, \text{ or } n_t = 1 \\ 2t-1, & \text{if } n_1 = n_2 = \ldots = n_t = 2 \\ \sum_{i=1}^{t-1} n_i, & \text{otherwise} \end{cases}$$

Proof: Let $t \geq 2$ and let G be a t-partite graph with partite sets V_i, where $|V_i| = n_i$, for $i = 1, 2, \ldots, t$, and $1 \leq n_1 \leq n_2 \ldots \leq n_t$. We consider four cases.

Case 1: Suppose that $n_1 = n_2 = \ldots = n_m = 1$ and $n_{m+1} \geq 2$ for some $m \in \{1, 2, \ldots, t-1\}$. Then, $\gamma(G) = 1$ and the set of vertices of degree $p(G) - 1$ is $W = \bigcup_{i=1}^{m} V_i$, where (obviously) $\langle W \rangle \cong K_m$. Now, $\gamma^{+'}(G)$ is the smallest number of edges of G in a set F whose removal from G reduces the degree of each vertex in W; if $m = 1$, then the set F consisting of a single edge from $[V_1, V(G) - V_1]$ is such a smallest set and, if $m \geq 2$, $F \subseteq E(\langle W \rangle)$ with $\Delta(\langle W \rangle - F) \leq m - 2$ and $|F| = \gamma^{+'}(K_m)$ is such a smallest set, and so, by Proposition 4.3.1, $\gamma^{+'}(G) = \lceil m/2 \rceil$.

Case 2: Suppose $n_t = 1$. Then, $G \cong K_t$ and $\gamma^{+'}(G) = \lceil t/2 \rceil$ by Proposition 4.3.1.

Case 3: Suppose that $n_1 = n_2 = \ldots = n_t = 2$, and note that $\deg v = 2t - 2$ for all $v \in V(G)$ and $\gamma(G) = 2$. We show first that $\gamma^{+\prime}(G) \geq 2t - 1$. Assume, to the contrary, that there is a set F of edges of G such that $|F| = 2t - 2$ and $\gamma(G-F) > \gamma(G)$. Observe that $\delta(G-F) > 0$ since any (spanning) subgraph of G that has an isolated vertex and size $q(G) - 2(t - 1)$ is necessarily isomorphic to $K_1 \cup K_{1,2,2,\ldots,2}$ (where the latter graph is t-partite) and has domination number 2 (= $\gamma(G)$), contrary to our assumption that F satisfies $\gamma(G-F) > \gamma(G)$. Also, if $\Delta(G-F) = 2t - 2$, then $\gamma(G-F) = 2$, contrary to assumption. Thus, $1 \leq \deg_{G-F}u \leq 2t - 3$ for each vertex $u \in V(G)$.

We show now that there exists a vertex x_1 with $\deg_{G-F}x_1 = 2t - 3$. We observe first that $q(G) = 2t^2 - 2t$, and that $q(G-F) = q(F) - |F| = 2t^2 - 2t - (2t - 2) = 2t^2 - 4t + 2$. Now, suppose, to the contrary, that $\deg_{G-F}u \leq 2t - 4$ for each $u \in V(G-F)$. Then, $4t^2 - 8t + 4 = 2\, q(G-F) = \Sigma_{u \in V(G)} \deg_{G-F}u \leq 2t(2t - 4) = 4t^2 - 8t$, which is impossible. So, at least one vertex x_1 of G-F has degree $2t - 3$ in G-F. Let x_2 be the other vertex of G that belongs to the same partite set as x_1, and let y_1 be the unique vertex distinct from x_2 that is not adjacent to x_1 in G-F. Now, if $y_1x_2 \in E(G-F)$, then $\{x_1, x_2\} \mapsto G-F$ since $\{x_1\} \mapsto \{x_1\} \cup [V(G) - \{x_2, y_1\}]$ and $\{x_2\} \mapsto \{x_2, y_1\}$, so that $\gamma(G-F) \leq 2$. This is contrary to our assumption about F. So, $y_1x_2 \in F$. Let y_2 be the other member of the partite set in G that contains y_1. If there exists a vertex $u \in V(G) - \{x_1, x_2, y_1, y_2\}$ that is adjacent to both x_2 and y_1, then $\{x_1, u\} \mapsto G-F$, which, again, contradicts $\gamma(G-F) > 2$. So, each vertex of V(G) different from x_1, x_2, y_1, y_2 must be non-adjacent with at least one of x_2 and y_1 in G-F. Hence, since $|V(G) - \{x_1, x_2, y_1, y_2\}| = 2t - 4$, it follows that F contains at least $(2t - 4) + |\{x_1y_1, x_2y_1\}| = 2t - 2$ edges. But, F has exactly $2t - 2$ elements. So, we have fully described F: F consists of the set $\{y_1x_1, y_1x_2\}$ and exactly one edge from $[\{u\}, \{x_2,y_1\}]$ for each $u \in V(G) - \{x_1, x_2, y_1, y_2\}$. As none of these edges in F is incident with y_2, we see that y_2 has degree $2t - 2$ in G-F, contrary to the result obtained above. Hence, our assumption that $\gamma^{+\prime}(G) \leq 2t - 2$ is false, and we have $\gamma^{+\prime}(G) \geq 2t - 1$, as required.

To obtain the reverse inequality, we consider the following. If $\{x_1, x_2\}$ is any partite set of G and H is the graph obtained by removing from G the $2t - 2$ edges incident with x_1 and one edge incident with x_2, then $\gamma(H) = 3$.

Case 4: Suppose that $n_1 \geq 2$ and $n_t \geq 3$; then $\gamma(G) = 2$. Let $s = \Sigma_{i=1}^{t-1} n_i = p(G) - n_t$. Assume, to the contrary, that there is a set $F \subseteq E(G)$ such that $|F| < s$ and $\gamma(G-F) >$

$\gamma(G)$. We show first that each vertex of G is incident with at least one member of F. Suppose, to the contrary, that there exists a vertex $v \in V_i$ (say) ($i \in \{1, 2, ..., t\}$) that is not incident with a member of F. Then, $|V_i| = n_i \leq n_t$, so that $|V(G) - V_i| \geq p(G) - n_t = (s + n_t) - n_t = s$. Each vertex x in $V(G) - V_i$ must be non-adjacent in G−F to at least one member of V_i (otherwise, $\{v, x\} \mapsto G-F$). However, this implies that $|F| \geq s$, contrary to our assumption. Thus, each vertex of G is incident with at least one edge in F. Further, if every vertex of G is incident with two or more edges of F, then $|F| \geq ½(2p(G)) = s + n_t > s$, a contradiction; so there must be a vertex x_1 incident with exactly one edge, say e, in F. Let $e = x_1y_1$, and let $x_1 \in V_k$, $y_1 \in V_j$ (say), where $V_k = \{x_1, x_2, ..., x_n\}$, $n \geq 2$. Since x_1 is adjacent in G−F to every vertex in $V(G) - (\{y_1\} \cup V_k)$ and since $\gamma(G-F) > 2$, it follows that each vertex u in $V(G) - (V_j \cup V_k)$ must be non-adjacent to at least one of the vertices $y_1, x_2, ..., x_n$ in G−F (otherwise, $\{x_1, u\} \mapsto G-F$, a contradiction). Hence, F contains a subset F_1 such that $F_1 \subseteq [V(G) - (V_j \cup V_k), \{y_1, x_2, ..., x_n\}]_G$ and $|F_1| \geq |V(G) - (V_j \cup V_k)|$. Furthermore, since each vertex in V(G), and hence in $V_j - \{y_1\}$, is incident with an edge of F, F contains a subset F_2 such that $F_1 \cap F_2 = \emptyset$, $F_2 \subseteq [V_j - \{y_1\}, V(G) - V_j]$ and $|F_2| \geq |V_j - \{y_1\}|$. So, $|F| \geq |F_1| + |F_2| + |\{e\}| \geq |V(G) - (V_j \cup V_k)| + |V_j| = |V(G)| - |V_k| \geq (s + n_t) - n_t = s$, a contradiction. So, $\gamma^{+\prime}(G) \geq s$.

Finally, consider the graph H obtained by removing the s edges incident with a vertex v in V_t. Clearly, any dominating set of H must contain v. Since $|V_t| = n_t \geq 3$, no single vertex of $V_t - \{v\}$ can dominate G−v, and since every other partite set of G has at least two elements, no single vertex of any other partite set of G can dominate G−v. So, $\gamma(H) \geq 3$. However, for any vertex $v' \in V_t - \{v\}$ and any vertex $u \in V(G) - V_t$, we have $\{v, v', u\} \mapsto H$, i.e., $\gamma(H) = 3 > \gamma(G)$. Hence, we conclude that $\gamma^{+\prime}(G) = s$, as desired. □

We consider now the value of $\gamma^{+\prime}$ for trees.

4.3.6 <u>Theorem</u>: If T is a non-trivial tree, then $\gamma^{+\prime}(T) \leq 2$.

<u>Proof</u>: Let T be a non-trivial tree. If T contains a vertex v that is adjacent to at least two end-vertices, then v is in every minimum dominating set for T. However, if u is an end-vertex of T adjacent to v, then both u and either v or another end-vertex adjacent to v will be in every

dominating set for T-uv. So, $\gamma(T-uv) = \gamma(T) + 1$. Thus, $\gamma^{+\prime}(T) \leq 1$. Since $\gamma^{+\prime}(T) \geq 1$ always, we have $\gamma^{+\prime}(T) = 1$ in this case.

If no vertex of T is adjacent to two or more end-vertices, then either $T \cong K_2$, in which case $\gamma^{+\prime}(T) = 1$ follows immediately, or $p(T) \geq 3$ and (by Lemma 3.2.31) T has an end-vertex u that is adjacent to a vertex w of degree 2. In the latter case, let $\{y\} = N(w) - \{u\}$, and let D be a minimum dominating set for $T-\{wu, wy\}$. Then, both u and w belong to D and $D - \{u\}$ is a dominating set for T. Hence,

$$\gamma(T) \leq \gamma(T-\{wu, wy\}) - 1 < \gamma(T-\{wu, wy\})$$

and $\gamma^{+\prime}(G) \leq |\{wu, wy\}| = 2$. □

In the course of the above proof, the following result has been established.

4.3.7 Corollary: If any vertex of a tree T is adjacent with two or more end-vertices, then $\gamma^{+\prime}(T) = 1$.

4.3.8 Remark: That the converse of Corollary 4.3.7 does not hold is illustrated by the fact that $\gamma^{+\prime}(P_m) = 1$ for $m \equiv 0, 2 \pmod 3$ (see Corollary 4.3.3) and $\gamma^{+\prime}(S(K_{1,n})) = 1$ $(n \geq 1)$, while neither P_m ($m \equiv 0, 2 \pmod 3$) nor $S(K_{1,n})$ $(n \geq 1)$ have a vertex adjacent to more than one end-vertex. The problem of characterizing the class of trees T with $\gamma^{+\prime}(T) = 1$ is as yet unsolved. The following theorem shows that such trees cannot be characterized in terms of forbidden subgraphs.

4.3.9 Theorem: If F is a forest, then F is isomorphic to an induced subgraph of a tree S with $\gamma^{+\prime}(S) = 1$, and a tree T with $\gamma^{+\prime}(T) = 2$.

Proof: Let F be any forest, and let $S_0 \cong P_3$ with u as the central vertex of S_0. Let S be obtained from $S_0 \cup F$ by selecting from each component of F one vertex and inserting an edge from that vertex to u. The resulting tree S contains F as an induced subgraph and has a vertex, namely u, adjacent to two end-vertices. By Corollary 4.3.7, $\gamma^{+\prime}(S) = 1$.

To prove the existence of a tree T with $\gamma^{+\prime}(T) = 2$ that contains an induced subgraph isomorphic to F, we use induction on the order p of F. If $p = 2$, then F is isomorphic to an induced subgraph

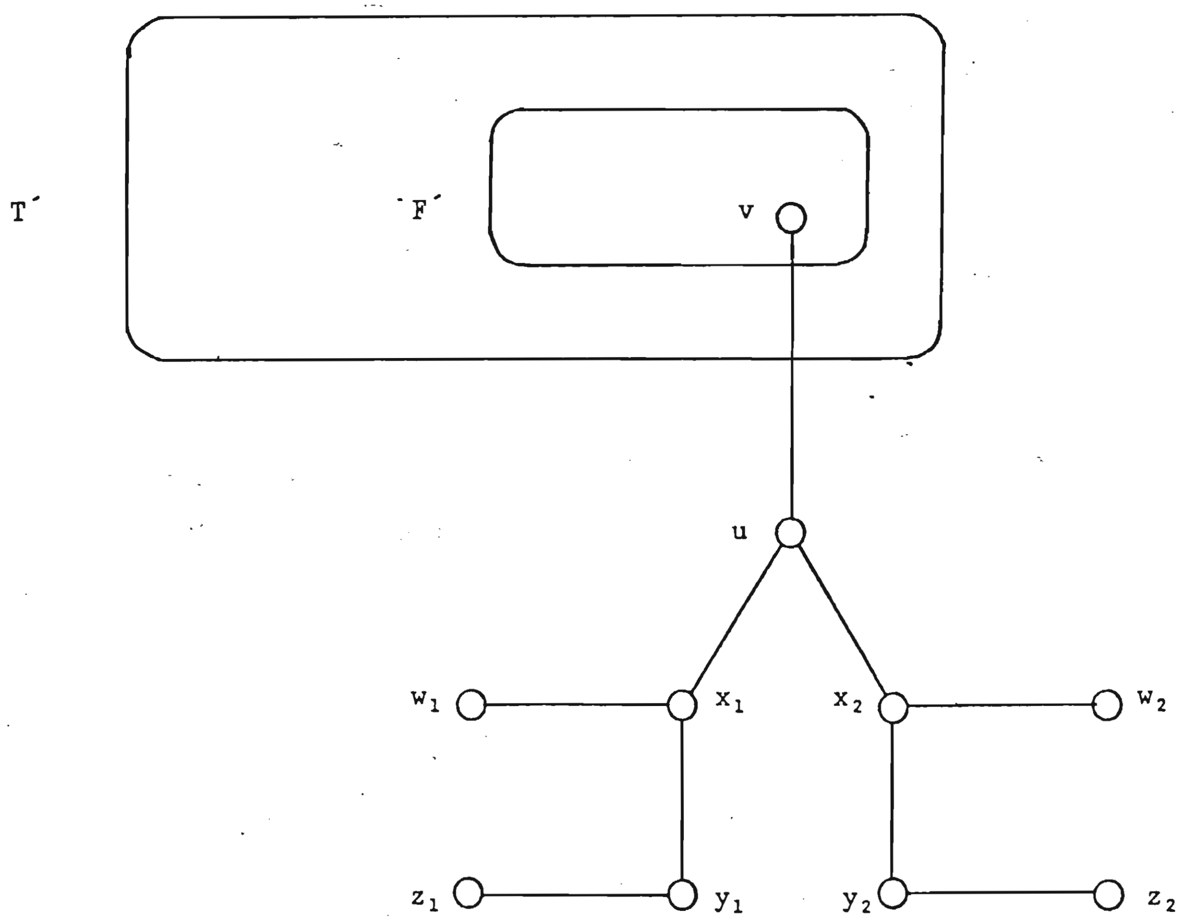


Fig. 4.3.1

of the tree T_2 obtained by sub-dividing two edges of $K_{1,3}$; clearly, $\gamma^{+\prime}(T_2) = 2$. Assume now that the claim is true for every forest of order p (≥ 2), and let F be a forest of order p + 1. If F is empty, let $T \cong P_n$ with $n \equiv 1 \pmod 3$ and $n \geq 2p + 1$. Then, T contains an independent set of p + 1 vertices which induces a subgraph isomorphic to F in T, and, by Corollary 4.3.3, has $\gamma^{+\prime}(T) = 2$. Suppose now that F is non-empty. Let u be an end-vertex of F, and let $uv \in E(F)$. By the inductive hypothesis, the forest $F' = F-u$ of order p is an induced subgraph of a tree T' with $\gamma^{+\prime}(T') = 2$. Let $H \cong 2P_4$, and let the components of H be the paths w_i,x_i,y_i,z_i (i = 1, 2). Let T be the tree obtained from $H \cup T'$ by adding the vertex u together with edges uv, ux_1, and ux_2 (see Fig. 4.3.1). Clearly, F is an induced subgraph of T. Also, from each pair of vertices $\{w_1, x_1\}$, $\{y_1, z_1\}$, $\{w_2, x_2\}$, $\{y_2, z_2\}$, exactly one must be in every minimum dominating set for T, and none of these vertices dominates a vertex of T'; thus, $\gamma(T) \geq \gamma(T') + 4$. If D' is a dominating set for T', then $D = D' \cup \{x_1, y_1, x_2, y_2\}$ is a dominating set for T of cardinality $\gamma(T') + 4$. Thus, $\gamma(T) = \gamma(T') + 4$. Finally, we show that $\gamma^{+\prime}(T) = 2$. Since $\gamma^{+\prime}(T') = 2$, we have $\gamma(T-e) = \gamma(T)$ for any edge e of T'. Furthermore, if e belongs to the subgraph $J = \langle V(T) - (V(T') - \{v\})\rangle = \langle\{u, v, w_1, x_1, y_1, z_1, w_2, x_2, y_2, z_2\}\rangle$, then $\gamma(J) = 5$, and it is easily verified that $\gamma^{+\prime}(J) = 2$; so $\gamma(J-e) = \gamma(J)$, and since a minimum dominating set of T'$-$v and a minimum dominating set of J combine to give a minimum dominating set of T, we may conclude that $\gamma(T-e) = \gamma(T)$. So, $\gamma^{+\prime}(T) \geq 2$, which, by Theorem 4.3.6, implies $\gamma^{+\prime}(T) = 2$. □

4.4 UPPER BOUNDS ON THE BONDAGE NUMBERS OF GRAPHS

In this section, we shall establish upper bounds on the bondage number $\gamma^{+\prime}(G)$ of a graph G in terms of other parameters of G.

4.4.1 Theorem: If G is a connected graph of order $p \geq 2$, then $\gamma^{+\prime}(G) \leq p - 1$.

Proof: Let G be a connected graph of order $p \geq 2$. Let u and v be adjacent vertices of G with $\deg_G u \leq \deg_G v$. If $\gamma^{+\prime}(G) \leq \deg_G u$, then, obviously, $\gamma^{+\prime}(G) \leq p - 1$; so we suppose that $\gamma^{+\prime}(G) > \deg_G u$. Let $E_u = [\{u\}, V(G)]$. Then, $\gamma(G - E_u) = \gamma(G)$, and so, since $\gamma(G - E_u) = \gamma((G-u) \cup \langle\{u\}\rangle) = \gamma(G-u) + 1$, we have

$$\gamma(G-u) = \gamma(G-E_u) - 1 = \gamma(G) - 1.$$

Let D denote the union of all minimum dominating sets for $G-u$. If there is $w \in D$ such that $uw \in E(G)$, then there exists a minimum dominating set D' of $G-u$ containing w and $D' \mapsto G$, whence $\gamma(G) \leq \gamma(G-u) = \gamma(G) - 1$, which is absurd. So, u is adjacent in G to no vertex of D. Thus, $|E_u| = \deg_G u \leq (p - 1) - |D|$ and $v \notin D$. Now, let $F_v = [\{v\}, D]_G$; we claim that $\gamma(G-u-F_v) > \gamma(G-u)$. Suppose, to the contrary, that $\gamma(G-u-F_v) = \gamma(G-u)$; let D^* be a minimum dominating set for $G-u-F_v$. In particular, $D^* \mapsto \{v\}$, which implies that there exists $d \in D^* \subseteq D$ such that $d = v$ or $dv \in E(G-u-F_v)$. Now, if D^* is a minimum dominating set for $G-u-F_v$, then it is a minimum dominating set for $G-u$; so, $D^* \subseteq D$. Since $v \notin D$, we must have $dv \in E(G-u-F_v)$ and so $dv \in E(G-u)$, where $d \in D$. However, then the latter fact implies $dv \in [\{v\}, D] = F_v$, while the former implies $dv \notin F_v$. This contradiction proves that

$$\gamma(G-u-F_v) > \gamma(G-u).$$

Equivalently,

$$\gamma(G-u-F_v) > \gamma(G) - 1.$$

Thus, $\gamma(G-(E_u \cup F_v)) = \gamma(G-u-F_v) + \gamma(\langle\{u\}\rangle) > \gamma(G)$, and we see that

$$\gamma^{+\prime}(G) \leq |E_u \cup F_v| \leq (p - 1 - |D|) + |D| = p - 1.$$

This completes the proof. $\square$

4.4.2 Remark: That the bound in Theorem 4.4.1 is attainable is shown by the following: If $G \cong K_{2,2,\ldots,2}$, then, by Theorem 4.3.4, $\gamma^{+\prime}(G) = p(G) - 1$. However, for many classes of graphs, the bound of Theorem 4.4.1 is poor: For instance, $\gamma^{+\prime}(P_n) = 2$ if $n \equiv 1 \pmod 3$ and $\gamma^{+\prime}(P_n) = 1$ if $n \equiv 0, 2 \pmod 3$; so, the bound of Theorem 4.4.1 is accurate for paths $G \cong P_2$ and is poor if $G \cong P_n$ with n very large, since

$$[p(P_n) - 1] - \gamma^{+\prime}(P_n) = n - 1 - \gamma^{+\prime}(P_n) \geq n - 3.$$

Similarly, $\gamma^{+\prime}(C_n) = 3$ if $n \equiv 1 \pmod 3$ and $\gamma^{+\prime}(C_n) = 2$ if $n \equiv 0, 2 \pmod 3$, whence it follows that the bound of Theorem 4.4.1 is attained by cycles C_n if $n = 3$ or $n = 4$, but is a bad bound for large n, since

$$[p(C_n) - 1] - \gamma^{+\prime}(C_n) = (n - 1) - \gamma^{+\prime}(C_n) \geq n - 4.$$

We note that, for trees T of order $p \geq 4$, the bound is poor, since, by Theorem 4.3.6, $\gamma^{+\prime}(T) \leq 2$, and hence $[p - 1] - \gamma^{+\prime}(T) \geq p - 3$. As for complete graphs K_p, the bound in Theorem 4.4.1 is exact for $p = 2$ and $p = 3$ since

$$\gamma^{+\prime}(K_2) = \lceil {}^2/_2 \rceil = 1 = p(K_2) - 1 \text{ and } \gamma^{+\prime}(K_3) = \lceil {}^3/_2 \rceil = 2 = p(K_3) - 1;$$

however, the bound is poor for complete graphs of order much greater that 3 since

$$[p(K_n) - 1] - \gamma^{+\prime}(K_n) = (n - 1) - \lceil {}^n/_2 \rceil \geq \tfrac{1}{2}(n - 3).$$

4.4.3 Remark: The proof of Theorem 4.4.1 also suffices to prove that, if G is a graph with at least one non-trivial component, then $\gamma^{+\prime}(G) \leq p(G) - 1$. However, if G is disconnected, with $G_1, G_2, \ldots, G_m$ as its non-trivial components, then $\gamma^{+\prime}(G) = \min\{\gamma^{+\prime}(G_i);\ 1 \leq i \leq m\}$, so that $\gamma^{+\prime}(G) \leq \min\{p(G_i) - 1;\ 1 \leq i \leq m\} < p(G) - 1$. Hence, the bound in Theorem 4.4.1 is not attained by any disconnected graph.

4.4.4 Definition: For a non-empty graph G, define the *degree of an edge uv* of G, $deg'_G(uv)$, to be $|[\{u, v\}, V - \{u, v\}]|$; i.e.,

$$\deg_G u + \deg_G v - 2,$$

and set

$$\delta'(G) = \delta(L(G)) = \min\{\deg'_G(uv);\ uv \in E(G)\}.$$

(So, $\deg'_G uv = \deg_{L(G)} uv$.)

4.4.5 Theorem: For any graph G, $\gamma^{+\prime}(G) \leq \delta'(G) + 1$.

Proof: Let G be a graph and let uv be an edge of G that satisfies $\delta'(G) = \deg_G(uv)$. Let $F = [\{u, v\}, V(G)]$. Then, $|F| = \delta'(G) + 1$. For any minimum dominating set S for G-u-v, we have that $S \cup \{u\}$ is a dominating set for G, and so $\gamma(G) \leq \gamma(G-u-v) + 1$; i.e., $\gamma(G-u-v) \geq \gamma(G) - 1$. So,

$$\gamma(G-F) = \gamma(G-u-v) + \gamma(\langle\{u\}\rangle) + \gamma(\langle\{v\}\rangle) \geq (\gamma(G) - 1) + 2 > \gamma(G),$$

whence $\gamma^{+\prime}(G) \leq |F| = \delta'(G) + 1$. □

4.4.6 Remark: While there are graphs for which strict inequality holds in Theorem 4.4.5 (for example, $\gamma^{+\prime}(K_n) = \lceil n/2 \rceil < 2n - 3 = \delta'(K_n) + 1$, for $n \geq 3$, and $\gamma^{+\prime}(S(m,n)) = 1 < \min\{m, n\} = \delta'(S(m,n))$ for the double star $S(m,n)$ with $m, n \geq 2$, equality does hold for others. For example, $\gamma^{+\prime}(K_2) = 1 = \delta'(K_2) + 1$, as well as $\gamma^{+\prime}(C_n) = 3 = \delta'(C_n) + 1$ and $\gamma^{+\prime}(P_n) = 2 = \delta'(P_n) + 1$ for $n \equiv 1 \pmod 3$. The fact that the bound in Theorem 4.4.5 is exact for cycles C_n and paths P_n for $n \geq 4$ shows that the bound of Theorem 4.4.5 is better than that of Theorem 4.4.1 in these instances. For $G \cong K_{2,2,\ldots,2}$, we have $\gamma^{+\prime}(G) = p(G) - 1$ and $\delta'(G) + 1 = 2p(G) - 5 > \gamma^{+\prime}(G)$ for $p(G) \geq 5$; so, the bound in Theorem 4.4.1 is attained and is better than that in Theorem 4.4.5 in this instance.

As a corollary to Theorem 4.4.5, we have the following easily computed bound.

4.4.7 Corollary: If G is a graph for which $\delta(G) > 0$, then $\gamma^{+\prime}(G) \leq \Delta(G) + \delta(G) + 1$.

Proof: Let G be a graph with no isolated vertices, let u be a vertex of degree $\delta(G)$, and let $v \in N_G(u)$. Then, by Theorem 4.4.5,

$$\gamma^{+\prime}(G) \leq \deg_G u + \deg_G v - 1 = \delta(G) + \deg_G v - 1 \leq \delta(G) + \Delta(G) - 1. \qquad \square$$

4.4.8 Remark: The next theorem provides a further bound on $\gamma^{+\prime}$ that involves the maximum degree Δ of a graph. Notice that Theorem 4.4.9 gives a relationship between $\gamma^{+\prime}$ and γ.

4.4.9 Theorem: If G is a non-empty graph with $\gamma(G) \geq 2$, then $\gamma^{+\prime}(G) \leq (\gamma(G) - 1)\Delta(G) + 1$.

Proof: We use induction on the domination number $\gamma(G)$.

Let G be a non-empty graph of order p with $\gamma(G) = 2$, let $\Delta = \Delta(G)$, and assume, to the contrary, that $\gamma^{+\prime}(G) \geq \Delta + 2$. Note that, as $\gamma(G) > 1$, $\Delta \leq p - 2$. Let $u \in V(G)$ with $\deg_G u = \Delta$; then, since $|[\{u\}, V(G)]| = \deg_G u = \Delta < \gamma^{+\prime}(G)$, we have $\gamma(G) = \gamma(G - [\{u\}, V(G)]) = \gamma(G-u) + 1$, i.e., $\gamma(G-u) = \gamma(G) - 1 = 1$, and so, since $|E(G) - E(G-u)| = \Delta$ and $\gamma^{+\prime}(G) \geq \Delta + 2$, we have $\gamma^{+\prime}(G-u) \geq 2$. Since $\gamma(G) = 2$ while $\gamma(G-u) = 1$, there must exist $v \in V(G)$ with $N[v] = V(G) - \{u\}$; so $\deg_G v = p - 2$, which implies $\Delta = p - 2$ and hence that $N[u] = V(G) - \{v\}$. Since $\gamma^{+\prime}(G-u) \geq 2$ and $\gamma(G-u) = 1$, for any edge $e = vy$ incident with v, we have $\gamma(G-u-e) = 1$. Thus, for each $y \in V(G) - \{u, v\}$, there exists $w_y \in V(G) - \{u, v, y\}$ such that w_y is adjacent to every vertex in $V(G) - \{u\}$. But, since v is the only vertex of G that is not adjacent with u, any such vertex w_y must be adjacent in G with u. However, then $\deg_G w_y = p(G) - 1$, whence $\gamma(G) = 1$, a contradiction. Thus, $\gamma^{+\prime}(G) \leq \Delta + 1$ if $\gamma(G) = 2$.

Now, assume validity of the statement for all non-empty graphs G with $\gamma(G) = k$ (where $k \geq 2$), let G be a non-empty graph with $\gamma(G) = k + 1$, and assume, to the contrary, that $\gamma^{+\prime}(G) > k \cdot \Delta(G) + 1$. Let u be any vertex of G. Since $\deg_G u \leq \Delta < \gamma^{+\prime}(G)$,

$$\gamma(G) = \gamma(G-[\{u\}, V(G)]) = \gamma(G-u) + 1,$$

i.e., $\gamma(G-u) = \gamma(G) - 1 = (k + 1) - 1 = k$. Now, clearly, if S is a smallest subset of $E(G-u)$ such that $\gamma((G-u)-S) > \gamma(G-u)$, then $S' = S \cup [\{u\}, V(G)]$ satisfies $\gamma(G-S') = \gamma(G-[\{u\}, V(G)]-S) = \gamma(\langle\{u\}\rangle) + \gamma((G-u)-S) > 1 + \gamma(G-u) = \gamma(G)$, whence we obtain $\gamma^{+\prime}(G) \leq |S'| = \gamma^{+\prime}(G-u) + \deg_G u$. Since $\gamma(G-u) = k$, we have (by the inductive hypothesis applied to $G-u$) that

$$\gamma^{+\prime}(G) \leq [(k - 1) \cdot \Delta(G-u) + 1] + \deg_G u \leq (k - 1) \cdot \Delta(G) + 1 + \Delta(G),$$

i.e.,

$$\gamma^{+\prime}(G) \leq k \cdot \Delta(G) + 1,$$

contrary to our assumption that $\gamma^{+\prime}(G) > k \cdot \Delta(G) + 1$. Thus, $\gamma^{+\prime}(G) \leq k \cdot \Delta + 1$, and, by the principle of mathematical induction, the proof is complete. □

4.4.10 Remark: If G is the graph consisting of a 3-cycle with a pendant path of length 2, then G is a graph for which inequality holds in Theorem 4.4.9, since $\gamma(G) = 2$, $\gamma^{+\prime}(G) = 2$ and $\Delta(G) = 3$. However, by considering graphs such as the complete graph K_2, the cycle C_4, the path P_3, and the complete t-partite graph $K_{2,2,\ldots,2}$, we see that Theorem 4.4.9 provides a sharp bound on $\gamma^{+\prime}$. This last graph can also be used to demonstrate that the bound given in Theorem 4.4.11 is sharp.

4.4.11 Theorem: If G is a connected graph of order $p \geq 2$, then $\gamma^{+\prime}(G) \leq p - \gamma(G) + 1$.

Proof: If G is a connected graph of order at least 2 for which $\gamma(G) \leq 2$, then the desired inequality follows from Theorem 4.4.1. Thus, we assume that there exists a connected graph G of order p with $\gamma = \gamma(G) \geq 3$ and $\gamma^{+\prime}(G) \geq p - \gamma(G) + 2$. Let $x \in V(G)$, and let $E_x = [\{x\}, V(G)]$. Since $V(G) - N_G(x)$ is a dominating set for G, we have $\gamma(G) \leq p - \deg_G x$, i.e.,

$$\deg_G x \leq p - \gamma(G) < \gamma^{+\prime}(G),$$

so, $\gamma(G-E_x) = \gamma(G)$, but $\gamma(G-E_x) = \gamma(G-x) + 1$, and thus $\gamma(G-x) = \gamma(G) - 1$. Furthermore, as in the proof of Theorem 4.4.1, if D denotes the union of all minimum dominating sets for $G-x$, we have that $N_G(x) \cap D = \varnothing$; thus, $|E_x| \leq p - 1 - |D|$. Now, let $z \in V(G) - D - \{x\}$, and let $F_z = [\{z\}, D]_G$. Then, since $z \notin D$, we have that $\gamma(G-x-F_z) > \gamma(G-x)$. To see this, suppose $\gamma(G-x-F_z) = \gamma(G-x)$, and let D' be a minimum dominating set of $G-x-F_z$. Then, D' is a minimum dominating set of $G-x$ and so $D' \subseteq D$, which is impossible, as no vertex of D dominates z in $G-x-F_z$. So, $\gamma(G-x-F_z) > \gamma(G-x)$. Since $\gamma(G-x) = \gamma(G) - 1$, we have

$$\gamma(G-E_x-F_z) = \gamma(G-x-F_z) + \gamma(\langle\{x\}\rangle) > \gamma(G).$$

From this last inequality, we obtain

$$|E_x| + |F_z| \geq \gamma^{+\prime}(G) \geq p - \gamma(G) + 2.$$

Hence,

$$|F_z| \geq (p - \gamma(G) + 2) - (p - 1 - |D|)$$

i.e.,

$$|F_z| \geq |D| - \gamma(G) + 3. \tag{i}$$

Now, let J be a minimum dominating set for $G-x$ (so $|J| = \gamma(G) - 1$). Suppose that z is adjacent to exactly one vertex w of J (of course, $|[\{z\}, J]| \geq 1$, since $J \mapsto G-x$). Then, $[\{z\}, D - J] = F_z - \{zw\}$, so that, from (i),

$$|D| = |J| + |D - J| \geq |J| + (|F_z| - 1) \geq \gamma(G) - 1 + |D| - \gamma(G) + 2 = |D| + 1,$$

which is absurd. So, we must have that z is adjacent to at least two vertices of J.

We shall assume now that $z \in V(G) - D - \{x\}$ is fixed and that $z \in N_G(x)$ (note that this is a valid assumption since we have shown that $N_G(x) \cap D = \varnothing$). Let J_1 be the set of vertices of J that are adjacent to z, and let J_2 denote the set of vertices in $D - J$ that are not adjacent to z. Then,

$$\begin{aligned}|F_z| &= |D - J - J_2| + |J_1| \\ &= |D| - |J| - |J_2| + |J_1| \\ &= |D| - (\gamma(G) - 1) - |J_2| + |J_1|,\end{aligned}$$

so that (from (i))

$$|D| - (\gamma(G) - 1) - |J_2| + |J_1| \geq |D| - \gamma(G) + 3,$$

i.e.,

$$|J_1| \geq |J_2| + 2.$$

Now, let J_2^* be the set of vertices in J_2 each of which is adjacent to exactly one vertex of J_1. Since $|J_1| > |J_2| \geq |J_2^*|$, there must be a vertex $v \in J_1$ that is adjacent to no vertex of J_2^*. We show now that the set $K = (J - \{v\}) \cup \{z\}$ is a dominating set of $G-x$. Certainly, $J - \{v\} \mapsto (G-x) - N_G[v]$ and $\{z\} \mapsto \{v\}$. Suppose there exists $a \in V(G) - J - \{x\}$ such that $K \not\mapsto \{a\}$; then $a \notin J_1 \cup (D - J - J_2) \subseteq N_G(z)$ and $a \notin J_2^*$ (by the definition of v). Furthermore, $a \notin J_2 - J_2^*$, since (by definition of J_2^*) $|N_G(w) \cap J_1| \geq 2 > |\{v\}|$ for all $w \in J_2 - J_2^*$, from which it follows that $J - \{v\} \mapsto J_2 - J_2^*$. So, we must have $a \in V(G) - D - \{x\}$. Since $K \not\mapsto \{a\}$, certainly $J - \{v\} \not\mapsto \{a\}$, from which it follows that $|N_G(a) \cap J| \leq 1$. However, this now produces a contradiction, since we proved above that every vertex belonging to the set $V(G) - D - \{x\}$ is adjacent to at least *two* vertices of J. So, K is indeed a minimum dominating set for G-x, and $K \subseteq D$, a contradiction since $z \notin D$. Thus, $\gamma^{+\prime}(G) \leq p - \gamma(G) + 1$. □

The following conjecture appeared in [FJKR1].

4.4.12 Conjecture: If G is a non-empty graph, then $\gamma^{+\prime}(G) \leq \Delta(G) + 1$.

This conjecture is supported by the following proposition of [BHNS1].

4.4.13 Proposition: If G is a graph with at least one non-critical vertex, then $\gamma^{+\prime}(G) \leq \Delta(G)$.

Proof: Let G be a graph containing a vertex v such that $\gamma(G-v) \geq \gamma(G)$. Then, if $F = [\{v\}, V(G)]$, we have

$$\gamma(G-F) = \gamma((G-v) \cup \langle\{v\}\rangle) = \gamma(G-v) + 1 \geq \gamma(G) + 1 > \gamma(G),$$

i.e., $\gamma^{+\prime}(G) \leq |F| = \deg_G v \leq \Delta(G)$. □

4.4.14 Remark: To see that the hypothesis in the above proposition is required, consider the vertex-domination-critical cycle C_{3n+1} ($n \in \mathbb{N}$); by Theorem 4.3.2, $\gamma^{+\prime}(G) = 3$, while $\Delta(G) = 2 < 3$.

4.5 CHARACTERIZATION OF k-$\gamma^{+\prime}$-CRITICAL GRAPHS

In this section, we regard a concept which is dual to k-vertex-criticality.

4.5.1 Definition [BHNS1]: We define a graph G to be *k-$\gamma^{+\prime}$-critical* if $\gamma(G) = k$, and, for each edge $e \in E(G)$, $\gamma(G-e) > k$.

These graphs can be characterized as follows.

4.5.2 Proposition: If G is a graph and $e \in E(G)$ such that $\gamma(G-e) > \gamma(G)$, then $\gamma(G-e) = \gamma(G) + 1$.

Proof: Let G be a graph with an edge $e = uv$ satisfying $\gamma(G-e) \geq \gamma(G) + 1$. Let D be a minimum dominating set of G. If $u, v \in D$, or if $u, v \notin D$, then $D \mapsto G-e$, whence $\gamma(G-e) \leq \gamma(G)$, a contradiction. So, $|\{u, v\} \cap D| = 1$; then $D \cup \{u, v\} \mapsto G-e$, whence $\gamma(G-e) \leq |D \cup \{u, v\}| = \gamma(G) + 1$. Combined with our first inequality, this yields the desired result. □

4.5.3 Corollary: If G is a k-$\gamma^{+\prime}$-critical graph, then $\gamma(G-e) = k + 1$ for each $e \in E(G)$.

4.5.4 Proposition: A graph G is k-γ^{+}-critical if and only if $k(G) = k$ and each non-trivial component of G is a star.

Proof: The sufficiency is clear. Suppose now that G is a k-$\gamma^{+\prime}$-critical graph. Let D be a minimum dominating set for G. Suppose that there exists a vertex $v \in V(G)$ of degree at least two such that $v \notin D$. Then, every neighbour of v is dominated by D, and some neighbour y of v belongs to D. Hence, if $x \in N(v) - \{y\}$, then D is a dominating set for $G-vx$, i.e., $\gamma(G-vx) \leq \gamma(G)$. However, this contradicts the k-$\gamma^{+\prime}$-criticality of G. So, every vertex of degree at least 2 must belong to D. However, no two vertices in D are adjacent (otherwise, if $e \in E(\langle D \rangle)$, then $\gamma(G-e) \leq \gamma(G)$, which, again, contradicts the k-$\gamma^{+\prime}$-criticality of G). Hence, every vertex of degree 2 or more is adjacent only to end-vertices of G. Thus, G has k components, each of which is K_1 or K_2 or a star of order at least 3. □

4.5.5 Corollary: If G is a connected k-$\gamma^{+\prime}$-critical graph for some $k \in \mathbb{N}$, then $k = 1$.

4.5.6 Remark: By comparing Proposition 4.5.4 with Theorem 2.2.2, one may immediately observe that the k-$\gamma^{+\prime}$-critical graphs are precisely the complements of the 2-edge-critical graphs. Hence, Proposition 4.5.4 provides an alternative characterization of 2-edge-critical graphs.

4.6 INTRODUCTION TO EDGE-DOMINATION-INSENSITIVE GRAPHS

In the next three sections, we shall consider graphs G for which $\gamma^{+\prime}(G) \geq 2$ and, in particular, such graphs of given order p and minimum size.

4.6.1 Definition [BD1]: The graph G will be called *edge-domination-insensitive* if $\gamma(G) = \gamma(G-e)$ for every edge e of G, i.e., if $\gamma^{+\prime}(G) \geq 2$. For brevity, we shall say that G is *domination-insensitive*, or, even more simply, *γ-insensitive* when $\gamma(G)$ is known to be γ.

4.6.2 Remark: Within this general framework, three sub-problems will be discussed in this and the following two sections. Here, we shall consider the simplest of the three problems, namely, to determine the minimum number of edges in a graph G with p vertices, domination number γ and having the property that some minimum dominating set of G exists which also dominates every edge-deleted subgraph $G-e$ of G. We shall denote by $q_F(p,\gamma)$ the minimum number of edges of an insensitive graph G of order p with $\gamma(G) = \gamma$ in this case. In section 4.7, we demand no restrictions other than the connectedness of the γ-insensitive graph G, and we will denote the minimum number of edges for such graphs by $q(p,\gamma)$. Finally, in section 4.8, we consider the case where we make the demand that the graph remain connected after any edge is removed; here, G is 2-edge-connected and $q_c(p,\gamma)$ will represent the minimum number of edges.

We will assume throughout this section and sections 4.7 and 4.8 that the graphs G under consideration are connected, so $\gamma(G) \leq {}^{p(G)}/_2$ for these graphs (see [OR1]). This assumption of connectedness implies no loss of generality since, if each component of a disconnected graph is domination-insensitive, then so is the graph. This point will be discussed in slightly more detail in the sections concerned.

4.6.3 Definition: For $p \geq 2$ and $\gamma \geq 1$, let $G_F(p,\gamma)$ denote the set of all connected, non-trivial graphs G of order p with $\gamma(G) = \gamma$, having the property that G contains a minimum dominating set V^* which satisfies $V^* \mapsto G-e$ for each edge e of G, and, furthermore, the property that G has the minimum number $q_F(p,\gamma)$ of edges over all such graphs. For brevity, we shall use the notation $G \in G_F(p,\gamma;V_1)$ to indicate that $G \in G_F(p,\gamma)$ and that V_1 is a minimum dominating set of G such that V_1 dominates each of the edge-deleted subgraphs $G-e$, $e \in E(G)$, of G.

That extremal graphs of the kind in $G_F(p,\gamma)$ do exist is illustrated by the fact that $K_{2,2} \in G_F(4,2;V_1)$, where V_1 is a partite set of $K_{2,2}$.

4.6.4 Remark: Note that, if G is a disconnected graph with components $G_1, G_2, .., G_k$, and each component G_i of G belongs to $G_F(p_i,\gamma_i)$, for some positive integers p_i and γ_i $(1 \leq i \leq k)$, then, for $p = \Sigma_{i=1}^k p_i$ and $\gamma = \Sigma_{i=1}^k \gamma_i$, G is a graph of order p and minimum size such that $\gamma(G) = \gamma(G-e)$ for all $e \in E(G)$. So, no loss of generality ensues from the restriction of our investigation to connected graphs.

4.6.5 Proposition: If $p \geq 2$, $\gamma \in \mathbb{N}$, and $G_F(p,\gamma) \neq \varnothing$, then $\gamma \geq 2$ and so $p \geq 4$.

Proof: If G is a connected, non-trivial graph which is dominated by a single vertex v and if e is an edge of G incident with v, then $\{v\} \not\mapsto G-e$. Hence, for $\gamma = 1$, $G_F(p,\gamma) = \varnothing$. So, $G_F(p,\gamma) \neq \varnothing$ implies that $\gamma \geq 2$ and, since $G \in G_F(p,\gamma)$ is connected, $p \geq 2\gamma \geq 4$. □

We shall characterize the graphs in $G_F(p,\gamma)$ for $p, \gamma \geq 2$ (and hence $p \geq 4$).

4.6.6 Proposition: If $\gamma \geq 2$, $p \geq 4$ are such that $G_F(p,\gamma) \neq \varnothing$ and if $G \in G_F(p,\gamma;V_1)$, then G is bipartite with partite sets V_1 and $V_2 = V(G) - V_1$. Moreover, each vertex in V_2 has degree 2.

Proof: Let G be a graph satisfying the hypothesis of the proposition, and define $V_2 = V(G) - V_1$.

We show first that V_1 and V_2 are independent. Suppose that, for some $i \in \{1, 2\}$, there exist u, $v \in V_i$ with $uv \in E(G)$, and let $G^* = G - uv$. Then, since $G \in G_F(p,\gamma,V_1)$, it follows that $V_1 \mapsto G^*$. Now, for any $e \in E(G^*)$, $V_1 \mapsto G-e$, and so, since $uv \notin [V_1, V_2]$, $V_1 \mapsto (G-e)-uv = G^*-e$. Hence, $V_1 \mapsto G^*$ and $V_1 \mapsto G^*-e$ for every $e \in E(G^*)$. This produces a contradiction, since $G \in G_F(p,\gamma)$ and $q(G^*) < q(G)$. We conclude that V_1 and V_2 are independent sets and that G is bipartite with V_1 and V_2 as partite sets.

Finally, we prove that each vertex of V_2 has degree 2. Certainly, since $V_1 \mapsto G-e$ for all $e \in E(G)$, each vertex of V_2 has degree at least two. Suppose that some vertex $v \in V_2$ has at least three neighbours $v_1, v_2, \ldots, v_{\deg v}$ ($\deg v \geq 3$). Let $G^* = G - vv_{\deg v}$. Then, $\deg_{G^*} u \geq 2$ for every $u \in V_2$ and V_1 is a minimum dominating set for G^* (since $G \in \mathcal{G}_F(p,\gamma,V_1)$). Furthermore, if $e \in E(G^*)$, then $\deg_{G^*-e} u \geq 1$ for every $u \in V_2$, which implies that $[\{u\}, V_1]_{G^*-e} \neq \emptyset$ for each $u \in V_2$, i.e., $V_1 \mapsto G^*-e$ for all $e \in E(G^*)$. This, together with the fact that $q(G^*) < q(G)$, contradicts our choice of G from $\mathcal{G}_F(p,\gamma)$; hence, $\deg_G v = 2$ for every $v \in V_2$. □

4.6.7 <u>Corollary</u>: If, for some $p \geq 4$ and $\gamma \geq 2$, $\mathcal{G}_F(p,\gamma) \neq \emptyset$, then $q_F(p,\gamma) = 2p - 2\gamma$.

4.6.8 <u>Theorem</u>: Let $p \geq 2$ and $\gamma \geq 1$ be such that $\mathcal{G}_F(p,\gamma) \neq \emptyset$. (Then, $\gamma \geq 2$ and $p \geq 4$.) Let $G \in \mathcal{G}_F(p,\gamma;V_1)$ with $V_1 = \{a_1, a_2, \ldots, a_\gamma\}$ and $A_i = N_G(a_i)$ for $i \in \{1, 2, \ldots, \gamma\}$. Then,

(1) $|A_i \cap A_j| \neq 1$, where $i, j \in \{1, 2, \ldots, \gamma\}$, $i \neq j$;

(2) the intersection graph I of $A_1, A_2, \ldots, A_\gamma$ is connected, whence it follows that I has at least $\gamma - 1$ edges;

(3) G has at least $4\gamma - 4$ edges; and

(4) $p \geq 3\gamma - 2$.

<u>Proof</u>: Let $p \geq 4$, $\gamma \geq 2$ and let G be a graph satisfying the hypothesis of the theorem. Then, G is bipartite with V_1 and $V_2 = V(G) - V_1$ as partite sets.

(1) Suppose, to the contrary, that there exist distinct $i, j \in \{1, 2, \ldots, \gamma\}$ with $A_i \cap A_j = \{v\}$. Then, $V^* = \{a_k;\ k \in \{1, 2, \ldots, \gamma\} - \{i, j\}\} \cup \{v\}$ is a dominating set for G, as we now show. Let $x \in V_2 - \{v\}$. Let a and b be the two vertices of V_1 that are adjacent to x (see Lemma 4.6.6). If $\{a, b\} \cap \{a_i, a_j\} = \emptyset$, then clearly V^* (which contains $\{a, b\}$) dominates x. On the other hand, if, say, $a \in \{a_i, a_j\}$, then, since $A_i \cap A_j = \{v\} \neq \{x\}$, we have $b \notin \{a_i, a_j\}$, i.e., $b \in V^*$, so that, again, $V^* \mapsto \{x\}$. Thus, $V^* \mapsto V^* \cup (V_2 - \{v\})$; since $\{v\} \mapsto \{v\} \cup \{a_i, a_j\}$, it follows that $V^* \mapsto G$. However, then, $\gamma(G) = \gamma \leq |V^*| = \gamma - 1$, which is impossible. So, (1) does indeed hold.

(2) Let $i, j \in \{1, 2, \ldots, \gamma\}$ with $i \neq j$. We will show that I contains an A_i-A_j path. Since G is connected, G contains an a_i-a_j path. Since G is bipartite, with partite sets V_1 and V_2, any such a_i-a_j path consists of an alternating sequence of elements of V_1 and V_2. Suppose

$$P: (a_i =)\, a_{i_1}, x_1, a_{i_2}, x_2, \ldots, x_{n-1}, a_{i_n} (= a_j),$$

is one such a_i-a_j path in G. Clearly,

$$x_k \in A_{i_k} \cap A_{i_{(k+1)}},$$

so

$$A_{i_k}A_{i_{(k+1)}} \in E(I)$$

for $k = 1, 2, \ldots, n - 1$. Hence,

$$(A_i=)A_{i_1}, A_{i_2}, \ldots, A_{i_n}(=A_j)$$

is an A_i-A_j path in I. So, I is connected.

(3) For each pair i, j for which A_iA_j is an edge of I, let $S_{i,j} = [A_i \cap A_j, V_1]$. Let $i, j \in \{1, 2, \ldots, \gamma\}$ with $A_iA_j \in E(I)$. We observe first that, since each vertex in V_2 has degree 2 and $|A_i \cap A_j| \geq 2$, we have $|S_{i,j}| = 2\ |A_i \cap A_j| \geq 4$. Now, let $k, \ell \in \{1, 2, \ldots, \gamma\}$ so that $|\{k, \ell\} \cap \{i, j\}| \leq 1$ and $A_kA_\ell \in E(I)$; we claim that $S_{i,j}$ and $S_{k,\ell}$ are disjoint. Suppose, to the contrary, that there exists $e = uv \in S_{i,j} \cap S_{k,\ell}$, where $u \in V_1$ and $v \in (A_i \cap A_j) \cap (A_k \cap A_\ell)$. Then, $a_iv, a_jv, a_kv, a_\ell v \in E(G)$, where $3 \leq |\{i, j, k, \ell\}| \leq 4$, i.e., $\deg_G v \geq 3$. However, by Lemma 4.6.6, this is impossible. So, $S_{i,j} \cap S_{k,\ell} = \varnothing$, as required. Hence, under the mapping $A_iA_j \mapsto S_{i,j}$ for $A_iA_j \in E(I)$, the q(I) edges of I are associated with q(I) disjoint subsets of edges of E(G), each containing at least 4 edges of E(G). So, $q(G) \geq 4\, q(I) \geq 4(\gamma - 1)$, as required.

(4) Since, by Corollary 4.6.7, $q(G) = 2p - 2\gamma$, it follows from (3) that $2p - 2\gamma \geq 4\gamma - 4$, i.e., $p \geq 3\gamma - 2$. □

Obviously, if $G_F(p,\gamma) \neq \varnothing$, then $\gamma \leq {}^p/_2$ and so $p \geq 2\gamma$. However, the condition $p \geq 2\gamma$ does not guarantee that $G_F(p,\gamma) \neq \varnothing$, as the next result shows.

4.6.9 <u>Corollary</u>: For every $\gamma \in \mathbb{N} - \{2, 3\}$, there exists $p_\gamma \geq 2\gamma$ such that $G_F(p_\gamma,\gamma) = \varnothing$.

<u>Proof</u>: Let $\gamma \in \mathbb{N}$. If $\gamma = 1$, then (as seen in the proof of Proposition 4.6.5), $G_F(p,\gamma) = \varnothing$ for every $p \geq 2$. For $\gamma \geq 4$, let p_γ be any element of $\{2\gamma, 2\gamma + 1, \ldots, 3\gamma - 3\}$; then, by Theorem 4.6.8(4), $G_F(p_\gamma,\gamma) = \varnothing$. □

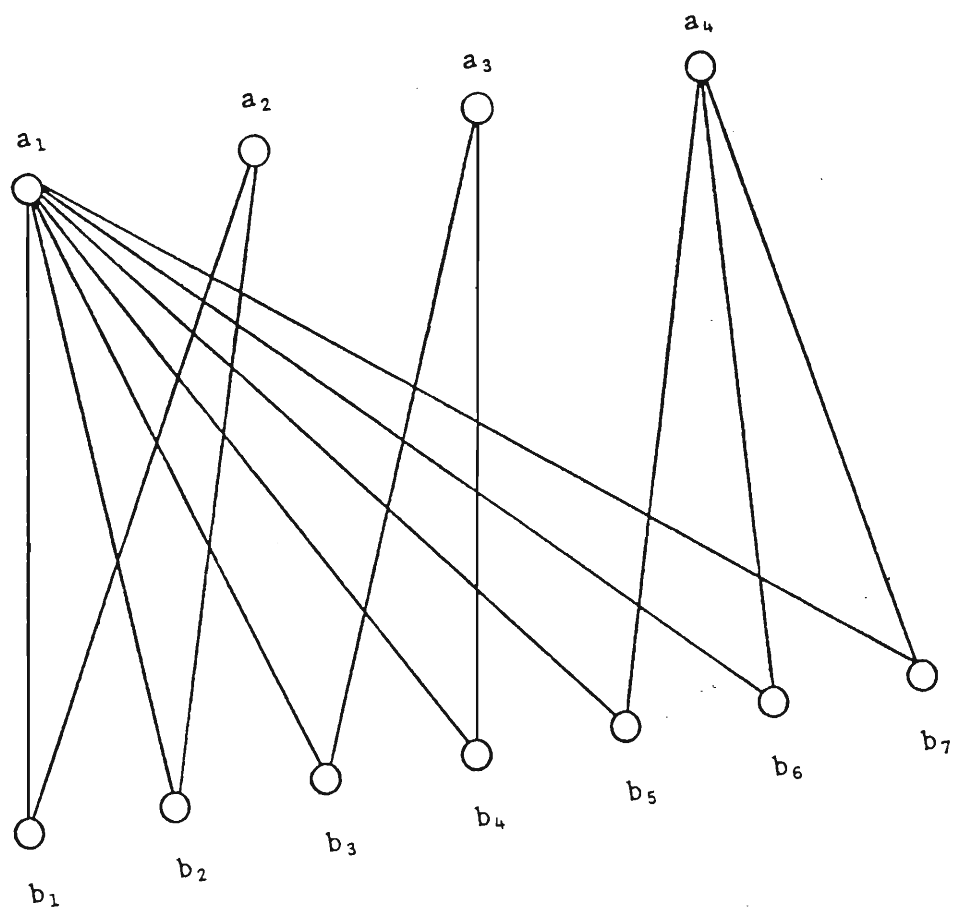


Fig. 4.6.1

4.6.10 Theorem: For any $\gamma \geq 2$ and $p \geq 3\gamma - 2$, $G_F(p,\gamma) \neq \varnothing$.

Proof: Let γ be an integer with $\gamma \geq 2$, and let p be an integer with $p \geq 3\gamma-2$. We construct (what we will show is) an element G of $G_F(p,\gamma)$ by defining $V(G) = V_1 \cup V_2$, where $V_1 = \{a_1, a_2, \ldots, a_\gamma\}$, $V_2 = \{b_1, b_2, \ldots, b_{p-\gamma}\}$, and

$$E(G) = \{a_1b_i; 1 = 1, 2, \ldots, p - \gamma\} \cup \{a_ib_{2i-3}, a_ib_{2i-2}; i = 2, 3, \ldots, \gamma - 1\} \cup \{a_\gamma b_i; i = 2\gamma - 3, 2\gamma - 2, \ldots, p - \gamma\}.$$

(Fig. 4.6.1 illustrates the case when $\gamma = 4$ and $p = 11$. Since $p \geq 3\gamma - 2$, $\deg a_i \geq 2$ for $1 \leq i \leq \gamma$ and, obviously, $\deg b_i = 2$ for $1 \leq i \leq p - \gamma$.

Since $N(a_1) = V_2$, $V_1 = \{a_1, a_2, \ldots, a_\gamma\}$ is clearly a dominating set for G. Moreover, $V_1 \mapsto G-e$, for each $e \in E(G)$, since G is bipartite with partite sets V_1 and V_2 and $\deg b_i = 2$ for each $b_i \in V_2$.

We show next that $\gamma(G) = \gamma$. Suppose, to the contrary, that $\gamma(G) < \gamma$. We remark that, given any $a_i \in V_1$, it is possible, by the definition of E(G), to determine $b_j \in V_2$ with $a_ib_j \in E(G)$, and given any $b_j \in V_2$, it is possible to determine $a_i \in V_1$ with $a_ib_j \in E(G)$. Let D be a minimum dominating set for G. We consider two cases.

Case 1: Suppose that $a_1 \in D$ (then $D \supseteq \{a_1\} \mapsto V_2$). If $V_1 \subseteq D$, then $|D| \geq \gamma$ (contrary to assumption), so let

$$A = \{a_{i_1}, a_{i_2}, \ldots, a_{i_n}\}$$

be the set of vertices of V_1 not contained in D, where $1 \leq n \leq \gamma - 1$. Now, by the definition of E(G), no vertex of V_2 is adjacent to more than one vertex of $V_1 - \{a_1\}$. So, for each $k \in \{1, 2, \ldots, n\}$, $a_{i_k} \in D$ or $b \in D$ where b is one of the two vertices in V_2 adjacent to a_{i_k}, i.e., a set B of at least n elements of $V_2 \cup A$ must be contained in D in order that A is dominated by D. However, then

$$|D| \geq |(V_1 - A) \cup B| = |V_1 - A| + |B| \geq (\gamma - n) + n = \gamma,$$

a contradiction. So, this case does not occur.

Case 2: Suppose that $a_1 \notin D$. Assume first that $V_1 - \{a_1\} \subseteq D$. Then, since $V_1 - \{a_1\} \not\mapsto \{a_1\}$, and $a_1 \notin D$, at least one element of V_2 must belong to D (since we know $D \mapsto \{a_1\}$). However, then $|D| \geq |V_1 - \{a_1\}| + 1 = |V_1| = \gamma$, a contradiction. So, if A is defined by

$$A = \{a_{i_1}, a_{i_2}, ..., a_{i_n}\} = V_1 - D$$

where $1 = i_1 < i_2 < \ldots < i_n$, then $2 \leq n \leq \gamma$.

Now, for each $k \in \{2, \ldots, n - 1\}$, if we define $B_k = N(a_{i_k})$, then $B_k = \{b_{r_k}, b_{t_k}\}$, where $r_k = 2i_k - 3$ and $t_k = 2i_k - 2$, and since

$$N(b_{s_k}) = N(b_{t_k}) = \{a_1, a_{i_k}\},$$

we must have that $B_k \subseteq D$. If $a_\gamma \in D$ (i.e., if $i_n < \gamma$), then we define $B_n = \{b_{r_n}, b_{t_n}\}$, where $r_n = 2i_n - 3$, $t_n = 2i_n - 2$. Suppose that $a_\gamma \notin D$ (i.e., $i_n = \gamma$); define B_n by $B_n = \{b_k; 2\gamma - 3 \leq k \leq p - \gamma\}$. Then, since $N(B_k) = \{a_1, a_\gamma\}$ for each $k \in \{2\gamma - 3, 2\gamma - 2, \ldots, p - \gamma\}$, it follows that B_n must be contained in D. In either case, then, $|D| \geq |V_1 - A| + |\bigcup_{k=1}^{n} B_k|$.

Thus, if $a_\gamma \in D$, then

$$|D| \geq (\gamma - n) + 2(n - 1) = \gamma + n - 2 \geq \gamma,$$

and if $a_\gamma \notin D$, then

$$\begin{aligned} |D| &\geq (\gamma - n) + 2(n - 1) + [p - \gamma - (2\gamma - 3) + 1] \\ &= n + 2 - 2\gamma + p \\ &\geq n + 2 - 2\gamma + 3\gamma - 2 \\ &= \gamma + n \geq \gamma + 1. \end{aligned}$$

In either case, a contradiction to our assumption that $\gamma(G) < \gamma$ is produced.

So, Case 2 does not occur, and it follows that $\gamma(G) = \gamma$, as required.

Finally, we observe that

$$q(G) = (p-\gamma) + (\gamma-2).2 + (p + 4 - 3\gamma) = 2p - 2\gamma = q_F(p,\gamma). \qquad \square$$

The results of this section are summarized in the following theorem.

4.6.11 Theorem: $G_F(p,\gamma) \neq \emptyset$ and $q_F(p,\gamma) = 2p - 2\gamma$ if $\gamma \geq 2$ and $p \geq 3\gamma - 2$, and $G_F(p,\gamma) = \emptyset$, otherwise.

4.7 EDGE-DOMINATION-INSENSITIVE GRAPHS OF MINIMUM SIZE

In this section, we shall investigate the existence and minimum size of graphs G for which $\gamma(G-e) = \gamma(G)$ for each $e \in E(G)$.

4.7.1 Definition: For $p \geq 2$ and $\gamma \in \mathbb{N}$, let *$G(p,\gamma)$* denote the set of all connected graphs G of order p with $\gamma(G) = \gamma$, having the property that $\gamma(G-e) = \gamma(G) = \gamma$ for each edge e of G, and, furthermore, the property that G has the minimum number *$q(p,\gamma)$* of edges over all such graphs.

As shown later (in Theorem 4.7.3), for any $p \geq 3$, any graph of the form $K_3 + \overline{K}_{p-3} = K_{1,1,1,p-3}$ belongs to $G(p,1)$. So, extremal graphs of the kind in $G(p,\gamma)$ do exist.

4.7.2 Remark: Observe that, if G is a disconnected graph with components $G_1, G_2, .., G_k$, and each component G_i of G belongs to $G(p_i,\gamma_i)$, for some positive integers p_i and γ_i $(1 \leq i \leq k)$, then, for $p = \Sigma_{i=1}^k p_i$ and $\gamma = \Sigma_{i=1}^k \gamma_i$, we have that G is a graph of order p and minimum size such that $\gamma(G-e) = \gamma(G)$ for each $e \in E(G)$. Thus, as in the previous section, no loss of generality is incurred if we restrict our investigation to connected graphs. It follows, then, that $q(p,\gamma) \geq p - 1$. We first treat the special case of $\gamma = 1$.

4.7.3 Theorem: For any $p \geq 3$, we have

(1) $q(p,1) = 3p - 6$ for $p \geq 3$; and

(2) $G(p,1) = \{K_3 + \overline{K}_{p-3}\}$.

Proof: Let $p \geq 3$.

(1) If $G \in G(p,1)$, then $\Delta(G) = p - 1$ and G must have at least three vertices of degree $p - 1$ (since a graph H having precisely two vertices u, v of degree $p(H) - 1$ satisfies $\gamma(H-uv) \geq 2 >$

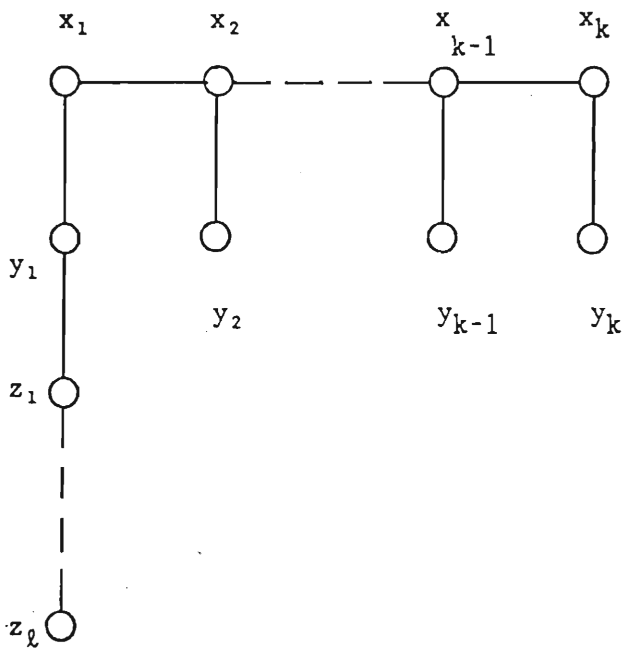


Fig. 4.7.1

$\bar{\gamma}(H)$, as does a graph H with exactly one vertex u of degree $p(H) - 1$, where $v \in V(H) - \{u\})$. Thus, $q(p,1) \geq (p - 1) + (p - 2) + (p - 3) = 3p - 6$.

To prove the reverse inequality, let G be a graph of order p, with exactly three vertices u, v, and y of degree $p - 1$, and every other vertex having degree 3 (i.e., no edges are present in G other than those incident with u, v, or y). Then, clearly, $\gamma(G-e) = 1 = \gamma(G)$ for each $e \in E(G)$. Since $q(G) = 3p - 6$, we have $q(p,1) \leq 3p - 6$, as required.

(2) By our comments in (1), it is clear that $K_3 + \bar{K}_{p-3}$ is a spanning subgraph of any element of $G(p,1)$. However, the size of the graph $K_3 + \bar{K}_{p-3}$ is $3p - 6$, the minimum number of edges of any such element , i.e., $K_3 + \bar{K}_{p-3}$ belongs to $G(p,1)$ and no proper supergraph of $K_3 + \bar{K}_{p-3}$ belongs to $G(p,1)$. □

We now assume $\gamma \geq 2$ and consider three cases: $p \leq 3\gamma - 2$, $p = 3\gamma - 1$, and $p \geq 3\gamma$.

4.7.4 <u>Theorem</u>: If $\gamma \geq 2$ and $2\gamma \leq p \leq 3\gamma - 2$, then $G(p,\gamma) \neq \emptyset$ and $q(p,\gamma) = p - 1$.

<u>Proof</u>: Let γ, p be integers with $\gamma \geq 2$ and $2\gamma \leq p \leq 3\gamma - 2$. We will show that a tree T of order $2k + \ell$ constructed from a path of length $k + \ell$, viz., $z_\ell, z_{\ell-1}, \ldots, z_1, y_1, x_1, \ldots, x_k$, by joining a new vertex y_i to x_i for $i = 2, 3, \ldots, k$ with $k = 3\gamma - p$ (≥ 2 since $p \leq 3\gamma - 2$) and $\ell = 3p - 6\gamma$ (≥ 0 since $p \geq 2\gamma$) has order p, domination number γ, and the property that $\gamma(T-e) = \gamma(T) = \gamma$ for each $e \in E(T)$ (see Fig. 4.7.1). This will prove that $q(p,\gamma) \leq q(T) = p - 1$, whence $q(p,\gamma) = p - 1$ will follow by our earlier observation that $q(p,\gamma) \geq p - 1$.

Let $H = \langle\{x_i, y_i; i = 1, 2, \ldots, k\}\rangle \cong P_k^+$, and let $Q_{i,j} = \langle\{z_i, z_{i+1}, \ldots, z_j\}\rangle$ for $i, j \in \{1, 2, \ldots, m\}$, $i < j$. We note first that $p(T) = 2k + \ell = 2(3\gamma - p) + (3p - 6\gamma) = p$. Since $\gamma(P_n^+) = n$, $\gamma(H) = k = 3\gamma - p$.

Let $D_1 = \{y_1, y_2, \ldots, y_k\}$; then D_1 is a minimum dominating set of H and D_1 also dominates z_1. Since $\gamma(P_n) = \lceil n/3 \rceil$, we have, for $Q_{2,\ell} = \langle\{z_2, z_3, \ldots, z_\ell\}\rangle$, that

$$\gamma(Q_{2,\ell}) = \lceil ⅓(\ell - 1)\rceil = \lceil ⅓(3p - 6\gamma - 1)\rceil = p - 2\gamma.$$

It is easy to see that, if D' is a minimum dominating set for $Q_{2,\ell}$, then $D_1 \cup D'$ is a minimum dominating set for T, so $\gamma(T) = (3\gamma - p) + (p - 2\gamma) = \gamma$.

We now show that T is γ-insensitive. Let $e \in E(T)$. We consider three cases.

Case 1: Suppose $e = y_1z_1$. Then,

$$\gamma(T-e) = \gamma(H) + \gamma(P_\ell) = (3\gamma - p) + \lceil ⅓(3p - 6\gamma)\rceil = 3\gamma - p + p - 2\gamma = \gamma.$$

Case 2: Suppose that $e = x_jy_j$ (where $j \in \{1, 2, ..., k\}$). Since $\gamma(T-e) \geq \gamma(T)$ for any $e \in E(T)$, it will suffice to exhibit the existence of a dominating set D for T-e with $|D| = \gamma$. We know that $D^* = \{y_1, y_2, ..., y_k\} \cup D'$, where D' is a minimum dominating set for the path $Q_{2,\ell}$ of order $\ell - 1$, is a minimum dominating set for T. If we now define a new set D by

$$D = \begin{cases} (D^* - \{y_{j+1}\}) \cup \{x_{j+1}\}, & \text{if } 1 \leq j \leq k-1 \\ (D^* - \{y_{j-1}\}) \cup \{x_{j-1}\}, & \text{if } j = k \end{cases},$$

then it is obvious that $D \mapsto T-e$ and $|D| = |D^*| = \gamma$.

Case 3: Suppose $e = z_mz_{m+1}$, where $1 \leq m \leq \ell - 1$. Then, $Q_{2,\ell}-e \cong P_a \cup P_b$, where $a + b = \ell - 1$. We need simply show that, for any $a \in \{1, 2, ..., \ell - 2\}$, we have

$$\gamma(P_{\ell-1-a} \cup P_a) = \gamma(P_{\ell-1}),$$

where $\gamma(P_{\ell-1}) = \lceil ⅓(3p - 6\gamma - 1)\rceil = \lceil ⅓(3p - 6\gamma)\rceil = p - 2\gamma$ (since then $\gamma(T-e) = (3\gamma - p) + (p - 2\gamma) = \gamma$, as in Case 1). Since $\ell = 3p - 6\gamma \equiv 0 \pmod 3$ and $a + b = \ell - 1 \equiv 2 \pmod 3$, we have, by symmetry, that no loss of generality is incurred if we consider only the cases where $a \equiv 0 \pmod 3$ and $a \equiv 1 \pmod 3$.

Subcase 3.1: Suppose $a \equiv 0 \pmod 3$. Since $a = 3n$ for some $n \in \mathbb{N}$, we have

$$\begin{aligned} \gamma(P_b \cup P_a) &= \lceil ⅓(\ell - 1 - a)\rceil + \lceil ^a/_3\rceil \\ &= \lceil ⅓(3p - 6\gamma - 1 - 3n)\rceil + \lceil ^{3n}/_3\rceil \\ &= (p - 2\gamma - n) + n = p - 2\gamma. \end{aligned}$$

Subcase 3.2: Suppose $a \equiv 1 \pmod 3$ (so $b \equiv 1 \pmod 3$); then $a = 3n + 1$, for some $n \in \mathbb{N}$. We assume that $\ell - m = a$ (i.e., $Q_{m+1,\ell} = \langle\{z_{m+1}, z_{m+2}, ...,$

$z_l\}\rangle \cong P_a)$. Observe that, if D_1 is a minimum dominating set for $Q_{1,m}$ that contains z_1 (D_1 always exists since $m = \ell - a \equiv (3p - 6\gamma) - (3n + 1) \equiv -1 \equiv 2 \pmod 3$)), D_2 is a minimum dominating set for $Q_{m+1,\ell}$ and $D_3 = \{x_2, x_3, \ldots, x_k\}$, then $D = D_1 \cup D_2 \cup D_3$ is a dominating set for $T-e$ that has cardinality

$$\begin{aligned} |D| &= |D_1| + |D_2| + |D_3| = \gamma(P_{\ell-a}) + \gamma(P_a) + (k-1) \\ &= \lceil ⅓(3p - 6\gamma - 3n - 1)\rceil + \lceil ⅓(3n+1)\rceil + (3\gamma - p - 1) \\ &= (p - 2\gamma - n) + (n + 1) + (3\gamma - p - 1) = \gamma. \end{aligned}$$

Cases 1 to 3, then, show that T is γ-insensitive. □

The following theorem will be used in the proof of Theorem 4.7.6, but is also interesting in its own right.

4.7.5 Theorem: If $p \geq 4$ and $\gamma \geq 2$ are such that $\mathcal{G}(p,\gamma) \neq \emptyset$, and if $G \in \mathcal{G}(p,\gamma)$, then G has at least two dominating sets of cardinality γ.

Proof: Let $p \geq 4$ and $\gamma \geq 2$ with $\mathcal{G}(p,\gamma) \neq \emptyset$, and let $G \in \mathcal{G}(p,\gamma)$. Since G is non-trivial and connected (so that G has no isolated vertices and is non-empty), we have, by Proposition 4.2.5, that G contains a minimum dominating set D which has the property that, for each $d \in D$, there exists $v_d \in V(G) - D$ such that $N_G(v_d) \cap D = \{d\}$. Clearly, then, for any $d \in D$, D is not a dominating set of $G-dv_d$. Hence, since $\gamma(G-dv_d) = \gamma(G)$, there exists a dominating set D′ of $G-dv_d$ (and hence of G) with $|D'| = \gamma(G)$ and $D \neq D'$. □

The following theorem requires a lengthy and complicated proof which will be presented as a sequence of lemmata. Notation and definitions introduced will be retained without repetition throughout the proof.

4.7.6 Theorem: If p and γ satisfy $p \geq 3\gamma \geq 6$, then $q(p,\gamma) = 2p - 3\gamma$.

Proof: Let $p, \gamma \in \mathbb{N}$ with $p \geq 3\gamma \geq 6$ and consider a graph G in $\mathcal{G}(p,\gamma)$. Let the minimum dominating sets of G be denoted by $D_0, D_1, \ldots, D_n$ (by Theorem 4.7.5, $n \geq 1$), where we shall assume that D_0 has been selected so that D_0 is a minimum dominating set whose existence is guaranteed by Proposition 4.2.5, whence it follows that the set A_0, which we define to be the set of all vertices in $V(G) - D_0$ which have a unique neighbour in D_0, is non-empty. (Notice that, in the notation of Definition 3.2.4, we have $A_0 = \bigcup \{D_0^*(v);\ v \in D_0\}$.) We next define subsets of D_0 and A_0 as follows:

$$X_i = (D_0 \cap D_1 \cap \ldots \cap D_{i-1}) - D_i, \qquad \text{for } 1 \leq i \leq n,$$
$$X_{n+1} = D_0 \cap D_1 \cap \ldots\ D_n,$$
$$A_i = \{a \in A_0;\ N(a) \cap D_0 \subseteq X_i\}, \qquad \text{for } 1 \leq i \leq n+1.$$

It is possible that some of the above sets may be empty; that the non-empty sets partition D_0 and A_0, respectively, is shown next.

4.7.7 Lemma:

(1) For distinct $i, j \in \{1, 2, \ldots, n+1\}$,

(a) $X_i \cap X_j = \varnothing$; (b) $A_i \cap A_j = \varnothing$.

(2) (a) $\bigcup_{i=1}^{n+1} X_i = D_0$; (b) $\bigcup_{i=1}^{n+1} A_i = A_0$.

Proof: (1a) Suppose, to the contrary, that there exist distinct $i, j \in \{1, 2, \ldots, n\}$ with $X_i \cap X_j \neq \varnothing$. Let $x \in X_i \cap X_j$ and assume, without loss of generality, that $i < j$. Now, $X_j = (D_0 \cap D_1 \cap \ldots \cap D_{j-1}) - D_j$, so we must have $x \in D_i$ (and $x \notin D_j$); however, $X_i = (D_0 \cap D_1 \cap \ldots \cap D_{i-1}) - D_i$ implies $x \notin D_i$. This contradiction establishes that $X_1, X_2, \ldots, X_{n+1}$ are disjoint.

(1b) Again, suppose that there are $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ and $A_i \cap A_j \neq \varnothing$. Let $a \in A_i \cap A_j$. By the definition of A_i and A_j, a satisfies $a \in A_0$ and $N(a) \cap D_o \subseteq X_i \cap X_j$, which, with (1a), yields $N(a) \cap D_0 = \varnothing$, a contradiction, since $|N(a) \cap D_0| = 1$. So, $A_1, A_2, \ldots, A_{n+1}$ are indeed disjoint.

(2a) By the definition of X_i ($1 \leq i \leq n+1$), $X_i \subseteq D_0$, and so $\bigcup_{i=1}^{n+1} X_i \subseteq D_0$. Let $v \in D_0$ and let k be the largest integer such that $v \in D_0 \cap D_1 \cap \ldots \cap D_{k-1}$; clearly, k exists, $1 \leq k \leq n+1$, and $v \in X_k$ (since the definition of k implies $v \notin D_k$). Hence, $D_0 \subseteq \bigcup_{i=1}^{n+1} X_i$. So, (2a) follows.

(2b) That $\bigcup_{i=1}^{n+1} A_i \subseteq A_0$ follows immediately from the definition of A_i ($1 \leq i \leq n+1$). Let $v \in A_0$. By the definition of A_0, there exists $d \in D_0$ such that $N(v) \cap D_0 = \{d\}$. Since $\bigcup_{i=1}^{n+1} X_i = D_0$, there exists $j \in \{1, 2, \ldots, n+1\}$ with $d \in X_j$. This implies, by the definition of $A_1, A_2, \ldots, A_{n+1}$, that $v \in A_j$. Hence, $A_0 \subseteq \bigcup_{i=1}^{n+1} A_i$, and the desired result follows. □

Another collection of sets Z_i ($1 \leq i \leq n$) is defined as follows:

(1) $Z_i \subseteq D_i \cap \{a \in A_0; |N(a) \cap (D_0 \cap D_1 \cap \ldots \cap D_{i-1})| = 1\}$;

(2) $D_i - Z_i \mapsto G - (X_i \cup A_i)$;

(3) Z_i is maximal with respect to properties (1) and (2).

4.7.8 Lemma: For $i \in \{1, 2, \ldots, n\}$,

(1) $Z_i \subseteq D_i \cap A_0$ and, for any $z \in Z_i$, $N(z) \cap D_0 = \{z^*\} \subseteq D_0 \cap D_1 \cap \ldots \cap D_{i-1}$;

(2) each vertex z in Z_i is in A_i or dominates at least one vertex in A_i which is dominated by no other vertex of D_i.

Proof: The results listed in (1) follow immediately from the definition of Z_i and A_0.

To prove (2), we suppose that $z \in Z_i - A_i$. Then, by the definition of Z_i, $D_i - Z_i \mapsto V(G) - (X_i \cup A_i)$ whereas, by the minimality of D_i, $D_i - \{z\} \not\mapsto V(G)$; consequently, there exists a vertex z' (say) in $V(G) - (D_i - \{z\})$ which is dominated by z and by no other vertex in D_i. If $z' \notin X_i \cup A_i$, then $D_i - \{z\} \supseteq D_i - Z_i \mapsto G-(X_i \cup A_i) \ni z'$, contrary to the property of z'. So, $z' \in X_i \cup A_i$. Furthermore, since $z \in A_0 - A_i$, the (unique) vertex z^* in D_0 which is adjacent to z is not contained in X_i. Hence, $z' \in A_i$, as required. □

4.7.9 Remark: In view of Lemma 4.7.8, for each $i \in \{1, 2, \ldots, n\}$, we now define an injective function $f_i:Z_i \mapsto A_i$ where, for $z \in Z_i$, $f_i(z) = z$ if $z \in A_i$ and, if $z \in Z_i - A_i$, $f_i(z) = z'$, where z' is any vertex of A_i which is dominated by z and by no other vertex of D_i, arbitrarily selected and then fixed as $f_i(z)$. Denoting $f_i(Z_i-A_i)$ by A_i', we define

$$B_i = f_i(Z_i) = (Z_i \cap A_i) \cup A_i',$$

for $1 \leq i \leq n$.

4.7.10 Lemma: For $i \in \{1, 2, \ldots, n\}$, $|B_i| = |Z_i| \leq |X_i|$.

Proof: Let $i \in \{1, 2, \ldots, n\}$. We claim first that $(Z_i \cap A_i) \cap A_i' = \varnothing$; if $y \in Z_i - A_i$ ($\subseteq D_i$) and $y' \in A_i' \cap (Z_i \cap A_i) \subseteq D_i$, then $N[y'] \cap D_i \supseteq \{y, y'\}$, contrary to the fact that y is the only vertex of D_i that dominates y'). This and the fact that $|Z_i - A_i| = |A_i'|$, proves $|B_i| = |Z_i|$.

To show that $|Z_i| \leq |X_i|$, we note that $X_i \mapsto X_i \cup A_i$ and $D_i - Z_i \mapsto V(G) - (X_i \cup A_i)$; so $(D_i - Z_i) \cup X_i \mapsto V(G)$, whence $|(D_i - Z_i) \cup X_i| \geq \gamma(G)$. As $X_i \cap D_i = \varnothing$ and $D_i - Z_i \subseteq D_i$, we obtain $|D_i| = \gamma \leq |D_i| - |Z_i| + |X_i|$, whence $|Z_i| \leq |X_i|$. □

4.7.11 Lemma: For each $i \in \{1, 2, \ldots, n\}$, A_i covers at least $2|A_i| - |X_i|$ edges of G.

Proof: Let $i \in \{1, 2, .., n\}$. For each $a \in A_i$, denote the unique neighbour of a in X_i by a^* and let E_i be the set of $|A_i|$ edges aa^* obtained thus. We shall show that every vertex a in $A_i - B_i$ is incident with a specified edge $av_a \neq aa^*$ and denote by F_i the set of such edges av_a. Let $a \in A_i - B_i$; we consider two cases.

Case 1: Suppose that $a \in D_j$ for some $j < i$; then $j \in \{1, 2, \ldots, i - 1\}$ as $a \notin D_0$.

Subcase 1.1: Suppose there is $k \in \{1, 2, \ldots, i - 1\}$ with $a \in Z_k$. Then, since $a \in A_i$ and $A_i \cap A_k = \varnothing$, it follows that $a \in Z_k - A_k$. Consequently, by Remark 4.7.9, a is adjacent to a vertex $v_a = f_k(a) \in A_k' \subseteq B_k \subseteq A_k$. Since $A_k \cap X_i = \varnothing$, we have $v_a \neq a^*$.

Subcase 1.2: Suppose $a \notin Z_\ell$ for all $\ell \in \{1, 2, \ldots, i - 1\}$. Let k denote the smallest ℓ for which $a \in D_\ell$ (clearly, $k \leq j$). We know $D_k - Z_k \mapsto V(G) - (A_k \cup X_k)$; also, by the maximality of Z_k, we have $D_k - (Z_k \cup \{a\}) \not\mapsto V(G) - (A_k \cup X_k)$. Hence, there exists a vertex $v_a \in V(G) - (A_k \cup X_k)$ which is dominated by a and by no other vertex of D_k. It is shown later that $v_a \neq a^*$.

Case 2: Suppose that $a \notin D_j$ for all $j < i$. We again consider two subcases.

Subcase 2.1: Suppose that $a \notin D_i$. Then, there exists $v_a \in D_i$ such that a is dominated by (i.e., is adjacent to) v_a. (Note that, since D_i and X_i are disjoint, $v_a \neq a^*$.)

Subcase 2.2: Suppose that $a \in D_i$. Since $a \in A_i - B_i$, we have $a \notin Z_i$. Hence (as in Subcase 1.2), $D_i - Z_i \mapsto V(G) - (A_i \cup X_i)$ while, by the maximality of Z_i, $D_i - (Z_i \cup \{a\}) \not\mapsto V(G) - (A_i \cup X_i)$, from which it follows that there exists

$v_a \in V(G) - (A_i \cup X_i)$ which is dominated by a and by no other vertex of D_i. (As above, $v_a \neq a^*$.)

Thus, in each of the above cases, a unique vertex v_a has been chosen in N(a) and we let $F_i = \{av_a \in E(G);\ a \in A_i - B_i\}$. In Subcases 1.1, 2.1, and 2.2, it is clear that $v_a \notin X_i$. In Subcase 1.2, since $a \in A_i$, a has a unique neighbour a^* in $X_i = (D_0 \cap D_1 \cap \ldots \cap D_{i-1}) - D_i$; so, if $v_a = a^*$, it follows (since $k < i$) that $v_a \in D_k$ and so v_a is dominated by at least two vertices, a and v_a, in D_k, contrary to the definition of v_a in this case. Consequently, $F_i \cap E_i = \emptyset$.

To show that F_i consists of $|A_i - B_i|$ distinct edges, we show that, for every pair of distinct elements a, a′ of $A_i - B_i$, if $v_a = a'$, then $v_{a'} \neq a$. In Subcases 1.1 and 2.2, $v_a \notin A_i$ for any $a \in A_i - B_i$, so $v_a \neq a'$ for any $a' \in A_i - B_i - \{a\}$. Next, we consider the vertex v_a selected in Subcase 2.1. Suppose that there is a vertex $a \in A_i - B_i$ such that the vertex v_a selected in Subcase 2.1 belongs to $A_i - B_i$; say, $v_a = a'$. By the choice of v_a in Subcase 2.1, we have $a' = v_a \in D_i$. So, the vertex $a' \in A_i - B_i$ satisfies the conditions of Subcase 2.2, and $v_{a'}$ is chosen to be an element of $V(G) - (A_i \cup X_i)$; in particular, $v_{a'} \notin A_i$, from which it follows that $v_{a'} \neq a$.

Finally, consider the vertex v_a selected in Subcase 1.2. Suppose that there is a vertex $a \in A_i - B_i$ such that the vertex v_a selected in Subcase 1.2 belongs to $A_i - B_i$. By the choice of v_a in Subcase 1.2, we have that $v_a \in V(G) - (A_k \cup X_k)$ and v_a is adjacent to a and to no other vertex of D_k. Now, by the definition of A_i, there is a (unique) vertex y in X_i with $v_a y \in E(G)$. But, $X_i \subseteq D_k$ (since $k < i$), and so y is a vertex of D_k adjacent to v_a. Finally, we observe that $y \in X_i$ implies $y \neq a$ ($\in A_i$), so that v_a is adjacent to two distinct vertices of D_k, contrary to our choice of v_a.

Therefore, F_i contains $|A_i - B_i| = |A_i| - |B_i|$ edges and $|E_i \cup F_i| = 2|A_i| - |B_i|$, which, with Lemma 4.7.10, completes the proof of the lemma. □

The following notation will be used to simplify the exposition in the remaining lemmas: For $1 \leq i \leq n$ and $a \in A_i - B_i$, the specified edge av_a of F_i selected in Lemma 4.7.11 will also be denoted by $g_i(a)$ and $F_i(1.1)$, $F_i(1.2)$, $F_i(2.1)$, and $F_i(2.2)$ will denote the set of all edges of the form $g_i(a)$ where a satisfies the condition listed in the Subcases 1.1, 1.2, 2.1, and 2.2, respectively.

4.7.12 <u>Lemma</u>: For $1 \leq i < m \leq n$, $(E_i \cup F_i) \cap (E_m \cup F_m) = \emptyset$.

Proof: Let $i \in \{1, 2, \ldots, m-1\}$. That $E_i \cap (E_m \cup F_m) = \emptyset$ follows immediately from the observation that, if $e \in E_i$, then $e \in [A_i, X_i]$ and so neither end of e is contained in A_m, whereas each edge in $E_m \cup F_m$ contains a vertex of A_m. This argument also proves $E_m \cap (E_i \cup F_i) = \emptyset$. We show now that $F_i \cap F_m = \emptyset$.

If $av_a = g_i(a) \in F_i(1.1)$, then $v_a \in A_k$ for some $k < i < m$; consequently, neither end of av_a is in A_m and so

$$F_i(1.1) \cap F_m = \emptyset.$$

If $av_a = g_m(a) \in F_m(1.1)$, then $a \in A_m - B_m$ and $v_a \in B_{j'} \subseteq A_{j'}$ for some $j' < m$. Suppose that $av_a \in F_i \cup E_i$; then $v_a \in A_i - B_i$, so that $j' = i$ as $A_{j'} \cap A_i = \emptyset$ for $i \neq j'$. However, this yields $v_a \in B_i$ and $v_a \in A_i - B_i$, a contradiction. So, $av_a \notin F_i$ and

$$F_m(1.1) \cap F_i = \emptyset.$$

We show now that $F_i(1.2) \cap F_m = \emptyset$. Let $av = g_i(a) \in F_i(1.2)$; then $a \in A_i - B_i$ and $a \notin Z_\ell$ for $1 \leq \ell < i$, but $a \in D_k$ for some $k < i$, where k is chosen to be as small as possible. We consider two cases.

Case 1: Suppose $av \in F_t$ for some $t > k$, with $t \neq i$. Then, $av = g_t(v)$ and $v \in A_t$; hence, v is dominated by some vertex $v^* \in X_t = (D_0 \cap D_1 \cap \ldots \cap D_k \cap \ldots \cap D_{t-1}) - D_t$ and $v^* \neq a$ ($v^* \in D_0$, and $a \in A_i$ implies $a \notin D_0$). So, v is dominated by distinct vertices v^*, $a \in D_k$, contrary to the choice of $v = v_a$ in Subcase 1.2. Thus, $av \notin F_t$ for all $t \in \{k+1, k+2, \ldots, m, \ldots, n\} - \{i\}$, i.e.,

$$F_i(1.2) \cap F_t = \emptyset, \qquad \text{for } t \in \{k+1, k+2, \ldots, n\} - \{i\}.$$

Case 2: Suppose $av \in F_t$ for some $t \leq k$ $(< i)$. Then, $av = g_t(v) \in A_t - B_t$. Now, we proved above that $F_i(1.1) \cap (E_m \cup F_m) = \emptyset$, where i and m are arbitrary, distinct elements of $\{1, 2, \ldots, n\}$ with $i < m$. So, since $t < i$, we have $F_t(1.1) \cap (E_i \cup F_i) = \emptyset$; in particular, this implies (since $av \in F_i(1.2) \subseteq F_i$) that $av \notin F_t(1.1)$. So, $av \in F_t$ implies $av = g_t(v) \in F_t(1.2) \cup F_t(2.1) \cup F_t(2.2)$.

Subcase 2.1: Suppose $av = g_t(v) \in F_t(1.2)$. Let $k' < t$ be the smallest index for which $v \in D_{k'}$. Then, a is dominated by $v \in D_{k'}$ and by no other vertexof $D_{k'}$;

however, a is dominated by $a^* \in X_i = (D_0 \cap \ldots \cap D_{k'} \cap \ldots \cap D_t \cap \ldots \cap D_{i-1}) - D_i$ and, as $a^* \in D_0$ and $v \notin D_0$, a is dominated by $a^* \in D_\ell - \{v\}$, a contradiction. Hence, $av \notin F_t(1.2)$. Thus, $av \notin F_t(1.2)$ for all $t \in \{1, 2, \ldots, k\}$; so

$$F_i(1.2) \cap F_t(1.2) = \varnothing, \qquad \text{for } t \in \{1, 2, \ldots, k\}.$$

Subcase 2.2: Suppose $av = g_t(v) \in F_t(2.1)$. Then, $a \in D_t - X_t$, so that $k \leq t$ (by the choice of k). Now (by our assumption in Case 2), $t \leq k$; so $t = k$. Then, since $av = g_i(a) \in F_i(1.2)$, we have $v \in V(G) - (A_k \cup X_k)$, so $v \in A_k$; but $av = g_t(v) \in F_t(2.1)$ and $k = t$ implies that $av = g_k(v) \in F_k(2.1)$ and $v \in A_k - B_k$, contradicting $v \notin A_k$. Hence, $av \notin F_t(2.1)$ and

$$F_i(1.2) \cap F_t(2.1) = \varnothing, \qquad \text{for } t \in \{1, 2, \ldots, k\}.$$

Subcase 2.3: Suppose $av = g_t(v) \in F_t(2.2)$. Then, $v \in D_t$ and (by the definition of $F_t(2.2)$) v is the only vertex in D_t which dominates a; however, as $a \in A_i$, a is dominated by some vertex $a^* \in X_i = (D_0 \cap \ldots \cap D_t \cap \ldots \cap D_{i-1}) - D_i$ and $a^* \neq v$ (since $a^* \in D_0$ and $v \notin D_0$), so a^* and v are distinct vertices in D_t which dominate a. This produces a contradiction. Thus, $av \notin F_t(2.2)$ for all $t \in \{1, 2, \ldots, k\}$. So,

$$F_i(1.2) \cap F_t(2.2) = \varnothing, \qquad \text{for } t \in \{1, 2, \ldots, k\}.$$

Cases 1 and 2 thus show that $F_i(1.2) \cap F_t = \varnothing$ if $t \neq i$, and so $F_i(1.2) \cap F_m = \varnothing$ and $F_m(1.2) \cap F_i = \varnothing$.

We show now that $F_i(2.1) \cap F_m = \varnothing$. If $av = g_i(a) \in F_i(2.1)$, then a $\notin D_j$ for $1 \leq j \leq i$, and $v \in D_i$. We consider two cases.

Case 3: Suppose $av = g_m(v) \in F_m(2.1)$. Then, $v \notin D_\ell$ for all $\ell \in \{0, 1, \ldots, m\}$, which provides a contradiction as $av = g_i(a)$ implies $v \in D_i$, where $i < m$. Hence, $av \notin F_m(2.1)$ and

$$F_i(2.1) \cap F_m(2.1) = \varnothing.$$

Case 4: Suppose that $F_i(2.1) \cap F_m(2.2) \neq \emptyset$. Let $av = g_i(a) = g_m(v) \in F_i(2.1) \cap F_m(2.2)$. Then, since $av = g_i(a) \in F_i(2.1)$, it follows that $v \in D_i - X_i$; however, since $av = g_m(v) \in F_i(2.2)$, $v \notin D_\ell$ for all $\ell < m$ and so $v \notin D_i$, a contradiction. Hence,

$$F_i(2.1) \cap F_m(2.2) = \emptyset.$$

Thus, $F_i(2.1) \cap F_m = \emptyset$.

To complete the proof, we have only to show that $F_i(2.2) \cap [F_m(2.1) \cup F_m(2.2)] = \emptyset$.

Case 5: Suppose that $F_i(2.2) \cap [F_m(2.1) \cup F_m(2.2)] \neq \emptyset$. Let $av = g_i(a) = g_m(v) \in F_i(2.2) \cap [F_m(2.1) \cup F_m(2.2)]$. Then, as $av = g_i(a) \in F_i(2.2)$, v is dominated by a and by no other vertex in D_i; however, $v \in A_m$ and so v is dominated by some $v^* \in X_m = (D_0 \cap \ldots \cap D_i \cap \ldots \cap D_{m-1}) - D_m$. We note that $v^* \neq a$ as $v^* \in D_0$ and $a \notin D_0$. So, v is dominated by $v^* \in D_i - \{a\}$, a contradiction. So,

$$F_i(2.2) \cap [F_m(2.1) \cup F_m(2.2)] = \emptyset. \qquad \square$$

4.7.13 Lemma: If $G \in G(p,\gamma)$ with $\gamma \geq 2$, then A_{n+1} covers at least $2|A_{n+1}|$ edges which are not contained in $\bigcup_{i=1}^{n} (E_i \cup F_i)$.

Proof: If $A_{n+1} = \emptyset$, the statement holds trivially, so now assume that $A_{n+1} \neq \emptyset$ and let $a \in A_{n+1}$. Then, a is adjacent to a vertex a^* in $X_{n+1} = D_0 \cap D_1 \cap \ldots \cap D_n$. We consider two cases.

Case 1: Suppose $a \in D_j$ for some $j \in \{1, 2, \ldots, n\}$. Then, as in Lemma 4.7.11, two subcases arise.

Subcase 1.1: Suppose there is $k \in \{1, 2, \ldots, n\}$ with $a \in Z_k$. Then, since $a \in A_{n+1}$ and $A_{n+1} \cap A_k = \emptyset$, it follows that $a \in Z_k - A_k$. Consequently, by Remark 4.7.9, a is adjacent to a vertex $v_a = f_k(a) \in B_k \subseteq A_k$.

Subcase 1.2: Suppose $a \notin Z_\ell$ for all $\ell \in \{1, 2, \ldots, n\}$. Choose $k \in \{1, 2, \ldots, n\}$ to be a minimum subject to $a \in D_k$. We know $D_k - Z_k \mapsto G-(A_k \cup X_k)$; by the maximality of Z_k, we have $D_k - (Z_k \cup \{a\}) \not\mapsto V(G) - (A_k \cup X_k)$. Hence, there exists a vertex $v_a \in V(G) - (A_k \cup X_k)$ which is dominated by a and by no other vertex of D_k.

Case 2: Suppose $a \notin D_1 \cup D_2 \ldots \cup D_n$. Then, since $G \in G(p,\gamma)$, $\gamma(G-aa^*) = \gamma$, and there is some value of $j \in \{1, 2, \ldots, n\}$ such that $D_j \mapsto G-aa^*$. Hence, a is dominated by a vertex v_a(say) of D_j with $v_a \neq a^*$.

Cases 1 and 2 thus show that every vertex a of A_{n+1} is incident with an edge $av \neq aa^*$.

We will denote by $F_{n+1}(1.1)$, $F_{n+1}(1.2)$, and $F_{n+1}(2)$ the set of all edges $av = g_{n+1}(a)$ obtained in Subcases 1.1 and 1.2, and Case 2, respectively, and we shall let $F_{n+1} = F_{n+1}(1.1) \cup F_{n+1}(1.2) \cup F_{n+1}(2)$ and $E_{n+1} = \{aa^*;\ a \in A_{n+1}\}$. We show now that $F_{n+1}(1.1)$, $F_{n+1}(1.2)$, and $F_{n+1}(2)$ are mutually disjoint.

Suppose $av = g_{n+1}(a) \in F_{n+1}(1.1) \cap [F_{n+1}(1.2) \cup F_{n+1}(2)]$. Then, $av = g_{n+1}(a) \in F_{n+1}(1.1)$ implies that $a \in Z_k \subseteq D_k$ for some $k \in \{1, 2, \ldots, n\}$ and $v \in B_k \subseteq A_k$. Now, since $a \in Z_k \subseteq D_k$ and since $av \in F_{n+1}(1.2) \cup F_{n+1}(2)$, we cannot have $av = g_{n+1}(a) \in F_{n+1}(1.2) \cup F_{n+1}(2)$. So, $av = g_{n+1}(v) \in F_{n+1}(1.2) \cup F_{n+1}(2)$, and $v \in A_{n+1}$. However, this contradicts the fact that $A_k \cap A_{n+1} = \varnothing$. Hence,

$$F_{n+1}(1.1) \cap F_{n+1}(1.2) = F_{n+1}(1.1) \cap F_{n+1}(2) = \varnothing.$$

Suppose $F_{n+1}(1.2) \cap F_{n+1}(2) \neq \varnothing$. Let $av \in F_{n+1}(1.2) \cap F_{n+1}(2)$. Suppose $av = g_{n+1}(a) \in F_{n+1}(1.2)$. Then, $a \in D_j$ for some $j \in \{1, 2, \ldots, n\}$, so that $av = g_{n+1}(a) \notin F_{n+1}(2)$. So, $av = g_{n+1}(v) \in F_{n+1}(2)$. Since $av = g_{n+1}(a) \in F_{n+1}(1.2)$, $a \in A_{n+1} \cap D_k$ for some $k \in \{1, 2, \ldots, n\}$ (chosen to be as small as possible) and v is dominated by a and by no other vertex in D_k. However, $av = g_{n+1}(v) \in F_{n+1}(2)$ implies that $v \in A_{n+1}$ and so v is dominated by some vertex $v^* \in X_{n+1} = \bigcap_{i=1}^{n} D_i \subseteq D_k$ and $v^* \neq a$ (as $a \notin X_{n+1}$), which provides a contradiction. So,

$$F_{n+1}(1.2) \cap F_{n+1}(2) = \varnothing.$$

We now show that, if $av = g_{n+1}(v) \in F_{n+1}(1.1)$, $F_{n+1}(1.2)$, or $F_{n+1}(2)$, then either $v \notin A_{n+1}$ or $g_{n+1}(v) \neq g_{n+1}(a)$; by what we have proved above, we need only show that $v \notin A_{n+1}$ or $g_{n+1}(v) \notin F_{n+1}(1.1)$, $F_{n+1}(1.2)$, or $F_{n+1}(2)$, respectively.

Certainly, if $av = g_{n+1}(a) \in F_{n+1}(1.2)$, where k is the smallest index for which $a \in D_k$, then $v \notin A_{n+1}$ (otherwise, v is dominated by some $v^* \in X_{n+1} \subseteq D_k$, and $v^* \neq a$ (since $v^* \in D_0$ and $a \notin D_0$), which contradicts the choice of v as a vertex dominated by a and by no other vertex of D_k). Hence, if $av = g_{n+1}(a) \in F_{n+1}(1.2)$, $v \notin A_{n+1}$.

If $av = g_{n+1}(a) \in F_{n+1}(1.1)$, where $j \in \{1, 2, \ldots, n\}$ satisfies $a \in D_j$, then there is $k \in \{1, 2, \ldots, n\}$ with $a \in Z_k$, and $v = f_k(a) \in B_k \subseteq A_k$. But, $A_k \cap A_{n+1} = \varnothing$; hence, $v \notin A_{n+1}$.

Finally, if $av = g_{n+1}(a) \in F_{n+1}(2)$, then, for some $j \in \{1, 2, \ldots, n\}$, $v \in D_j$ and v is adjacent to a; if $av = g_{n+1}(v) \in F_{n+1}(2)$, then it follows that $a \in D_{j'}$ for some $j' \in \{1, 2, \ldots, n\}$, contrary to the fact that $av = g_{n+1}(a) \in F_{n+1}(2)$.

Hence, for distinct vertices $a, b \in A_{n+1}$, the edges $g_{n+1}(a)$ and $g_{n+1}(b)$ are distinct.

We show next that $E_{n+1} \cap F_{n+1} = \varnothing$. Suppose, to the contrary, that there exists $av = g_{n+1}(a) \in E_{n+1} \cap F_{n+1}$. Since $av = g_{n+1}(a) \in E_{n+1}$, we have $v = a^* \in X_{n+1} = \bigcap_{i=1}^{n} D_i$. By the conditions of Case 2, $av = g_{n+1}(a) \notin F_{n+1}(2)$. Furthermore, if $av = g_{n+1}(v) \in F_{n+1}(2)$, then $v \notin D_1, D_2, \ldots, D_n$, whence $v \notin X_{n+1}$, a contradiction. So, $av \in F_{n+1}(1.1) \cup F_{n+1}(1.2)$. If $av = g_{n+1}(a) \in F_{n+1}(1.1)$, then $v \in A_k$ for some $k \in \{1, 2, \ldots, n\}$; however, this produces a contradiction since $A_k \cap X_{n+1} = \varnothing$. If $av = g_{n+1}(v) \in F_{n+1}(1.1) \cup F_{n+1}(1.2)$, then $v \in A_{n+1}$, which contradicts $v \in X_{n+1}$ as $X_{n+1} \cap A_{n+1} = \varnothing$. If $av = g_{n+1}(a) \in F_{n+1}(1.2)$, then v is dominated by a and by no other vertex in D_k. However, $v \in X_{n+1} = D_0 \cap D_1 \cap \ldots \cap D_k \cap \ldots \cap D_n$ implies that $v \in D_k$, so that v is dominated by at least *two* vertices, namely a and v, of D_k. Hence, E_{n+1} and F_{n+1} are indeed disjoint.

Thus, we have proved that $E_{n+1} \cup F_{n+1}$ contains $2|A_{n+1}|$ distinct edges. It remains only to show that $(E_{n+1} \cup F_{n+1}) \cap (E_i \cup F_i) = \varnothing$ for all $i \in \{1, 2, \ldots, n\}$. This may be accomplished by the use of techniques developed in proving Lemma 4.7.12. The proof is lengthy and so similar to that of Lemma 4.7.12 that we shall omit it here. □

So, the number of edges of G found so far that are distinct (see Lemmas 4.7.12 and 4.7.13) is (see Lemmas 4.7.11 and 4.7.13) at least

$$\sum_{i=1}^{n}(2|A_i| - |X_i|) + 2|A_{n+1}| = 2\left(\sum_{i=1}^{n+1}|A_i|\right) - \left(\sum_{i=1}^{n+1}|X_i|\right) + |X_{n+1}|$$

$$= 2|A_0| - \gamma(G) + |X_{n+1}| \geq 2|A_0| - \gamma(G) \qquad (*)$$

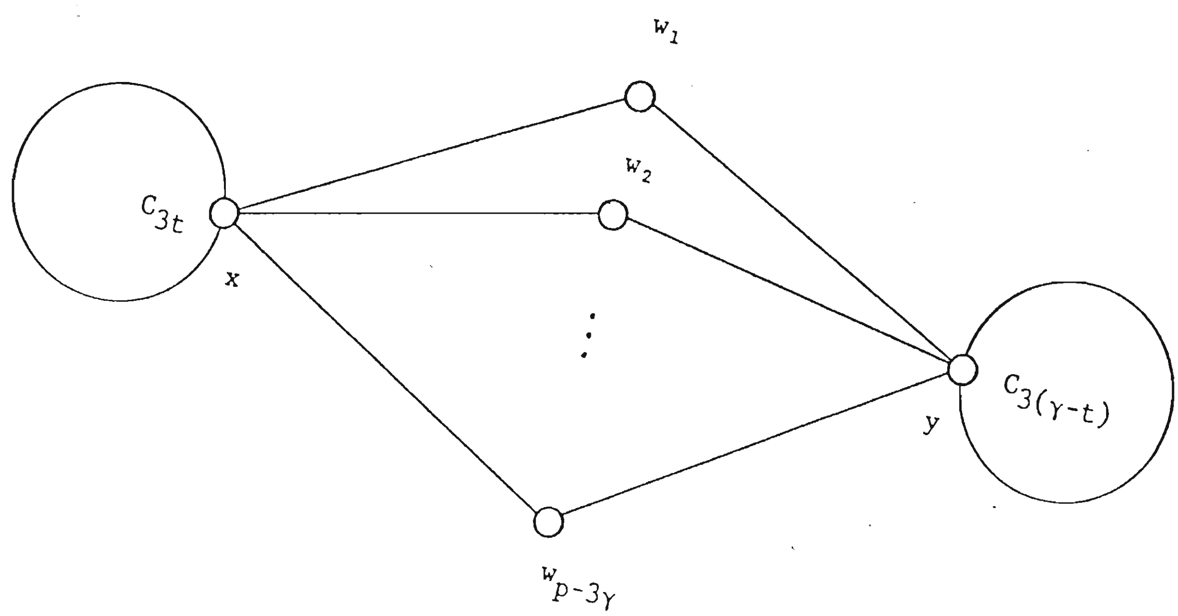


Fig. 4.7.2

(since $\bigcup_{i=1}^{n+1} X_i = D_0$, where $|D_0| = \gamma(G)$); i.e., any γ-insensitive graph with $\gamma \geq 2$ must have at least $2|A_0| - \gamma$ edges with an end in A_0. By definition of A_0, each vertex of $G-(D_0 \cup A_0)$ is incident with at least two edges with ends in D_0. Since

$$\{u, v; uv \in E(G), u \in V(G) - (D_0 \cup A_0), v \in D_0\} \cap A_0 \neq \emptyset,$$

we have

$$q(p,\gamma) \geq (2|A_0| - \gamma) + 2|V(G) - (D_0 \cup A_0)| = 2|A_0| - \gamma + 2(p - \gamma - |A_0|) = 2p - 3\gamma.$$

To complete the proof of Theorem 4.7.6, we shall show that, when $p \geq 3\gamma$, $2p - 3\gamma$ is an upper bound for $q(p,\gamma)$ by constructing connected γ-insensitive graphs of order p which have $2p - 3\gamma$ edges. If $p = 3\gamma$, then (by Theorem 4.3.2) the cycle $C_{3\gamma}$ is such a graph. So, we assume now that $p > 3\gamma$ and let t be any positive integer less than γ. Consider two disjoint graphs F_1 and F_2 with $F_1 \cong C_{3t}$ and $F_2 \cong C_{3(\gamma-t)}$, and suppose $x \in V(F_1)$, $y \in V(F_2)$. For each $i \in \{1, 2, ..., p - 3\gamma\}$, add to the graph $F_1 \cup F_2$ a vertex w_i and edges w_ix, w_iy. Call the resulting (connected) graph $I_{p,\gamma}$ (see Fig. 4.7.2). Then,

$$p(I_{p,\gamma}) = 3t + 3(\gamma - t) + p - 3\gamma = p$$

and

$$q(I_{p,\gamma}) = 3t + 3(\gamma - t) + 2(p - 3\gamma) = 2p - 3\gamma.$$

Furthermore, $I_{p,\gamma}$ is γ-insensitive: Let D_1 be a minimum dominating set of F_1 containing x, and let D_2 be a minimum dominating set of F_2 containing y. Then, $D_1 \cup D_2$ is a dominating set of $I_{p,\gamma}$ that is easily seen to be minimum. Thus,

$$\gamma(I_{p,\gamma}) = |D_1| + |D_2| = \lceil 3t/3 \rceil + \lceil ⅓(3(\gamma - t)) \rceil = \gamma.$$

Clearly, for any edge e of the form xw_i or yw_i ($i \in \{1, 2, ..., p - 3\gamma\}$), $D_1 \cup D_2 \mapsto I_{p,\gamma}-e$; so, $\gamma(G-e) = \gamma$. If $e \in E(F_1)$, then, if D is a minimum dominating set of the path F_1-e of order 3t, we have $|D| = t$, $D \mapsto F_1-e$, and $D_2 \mapsto \langle\{w_1, w_2, ..., w_{p-3\gamma}\}\rangle \cup F_2$, so that, again, $\gamma(I_{p,\gamma}-e) = \gamma$. If $e \in E(F_2)$, then $D^* \mapsto F_2-e$ and $D_1 \mapsto \langle\{w_1, w_2,...,w_{p-3\gamma}\}\rangle \cup F_1$, where D^* is a minimum dominating set of the path F_2-e of order $3(\gamma - t)$; in this case, $|D^*| = \gamma - t$ so that $\gamma(I_{p,\gamma}-e) =$

$t + (\gamma - t) = \gamma$. Thus, $I_{p,\gamma}$ is γ-insensitive and $I_{p,\gamma} \in G(p,\gamma)$. Hence, $q(p,\gamma) = 2p - 3\gamma$ for $p \geq 3\gamma \geq 6$, as desired. □

4.7.14 Remark: At this stage, $q(p,\gamma)$ is completely determined except for the case when $p = 3\gamma - 1$. Since the cycle $C_{3\gamma-1}$ is a connected γ-insensitive graph (see Theorem 4.3.2), we know that, if G is a connected γ-insensitive graph with $q(3\gamma-1,\gamma)$ edges, then

$$p(G) - 1 \leq q(3\gamma-1,\gamma) \leq p(G).$$

We show that, in fact, $q(3\gamma-1,\gamma) = p(G)$, i.e., $q(3\gamma-1,\gamma) = 3\gamma - 1$. The following lemmas will be useful.

4.7.15 Lemma: If G is a γ-insensitive graph for some $\gamma \geq 2$, then each vertex of G is adjacent to at most one end-vertex of G.

Proof: Suppose, to the contrary, that there exists an integer $\gamma \geq 2$ and a graph G that is γ-insensitive for which there is $v \in V(G)$ with v adjacent to at least two end-vertices u_1, u_2 of G. Then, clearly, any minimum dominating set D for G must contain the vertex v but neither of the vertices u_1 and u_2. Now, the graph $G-u_1v$ consists of the components $G-u_1$ and $\langle\{u_1\}\rangle$. Any minimum dominating set of $G-u_1$ must contain v or u_2, and so $\gamma(G-u_1) = \gamma(G)$. Hence,

$$\gamma(G-u_1v) = \gamma(G-u_1) + \gamma(\langle\{u_1\}\rangle) = \gamma(G) + 1 > \gamma(G),$$

which contradicts the γ-insensitivity of G. So, no such graph G and $\gamma \geq 2$ exist, and the proposition follows. □

The following proposition is a direct consequence of Lemma 4.7.15 and the proof of Lemma 3.2.31.

4.7.16 Proposition: If G is a γ-insensitive tree for some integer $\gamma \geq 2$, the end-vertices of any maximum length path are vertices of degree 1 and both are adjacent to a vertex of degree 2.

4.7.17 Proposition: If G is a graph with $\gamma(G) = k \geq 3$ which contains distinct vertices x, y, z that satisfy $\deg z = 1$, $\deg y = \deg x = 2$, $N(y) = \{x, z\}$, then $\gamma(G-\{x, y, z\}) = k - 1$.

Proof: Let G be a graph satisfying the above hypothesis. Let D be a minimum dominating set of G, and let $N(x) = \{y, t\}$. If $z \in D$, then $D' = (D - \{z\}) \cup \{y\} \mapsto G$ and $|D'| = |D| = k$; hence, we shall assume that $y \in D$ and $z \notin D$. Furthermore, if $x \in D$, then $D'' = (D - \{x\}) \cup \{t\}$ dominates G and $|D''| = |D| = k$, so we shall assume that $x \notin D$.

Now, y dominates only x, y, and z, and so $D - \{y\} \mapsto G-\{x, y, z\}$. Hence, $\gamma(G-\{x, y, z\}) \leq k - 1$. However, if $\gamma(G-\{x, y, z\}) < k - 1$ and D' is a minimum dominating set of $G-\{x, y, z\}$, then $D' \cup \{y\} \mapsto G$ and $|D' \cup \{y\}| < k = \gamma(G)$, a contradiction. Hence, $\gamma(G-\{x, y, z\}) = k - 1$. □

4.7.18 Proposition: If G is a graph with $\gamma(G) = k \geq 3$ which contains distinct vertices x, y, z, v, and s, where $N(x) \supseteq \{y, v\}$, $\deg y = \deg v = 2$, $\deg z = \deg s = 1$, $N(y) = \{x, z\}$ and $N(v) = \{x, s\}$. Then, $\gamma(G-\{y, z\}) = k - 1$.

Proof: Let G be a graph satisfying the above hypothesis. Let D be a minimum dominating set of G. Then, $|D| = k$ and D contains either y or z and either v or s. Since $D' = (D - \{z, s\}) \cup \{y, v\}$ is also a minimum dominating set of G, we shall assume that $y, v \in D$. Then, $D - \{y\}$ is a dominating set of $G-\{y, z\}\}$ and so $\gamma(G-\{y, z\}) \leq |D - \{y\}| = k - 1$. If $\gamma(G-\{y, z\}) < k - 1$, then, for any minimum dominating set D' for $G-\{y, z\}$, $D' \cup \{y\} \mapsto G$, and $|D' \cup \{y\}| < k = \gamma(G)$, a contradiction. Hence, $\gamma(G-\{y, z\}) = k - 1$. □

4.7.19 Theorem: For $k \geq 2$, $q(3k-1,k) = 3k - 1$.

Proof: Suppose, to the contrary, that, for some $k \geq 2$, there exists a graph $G \in G(3k-1,k)$ such that $q(G) \neq 3k - 1$. By Remark 4.7.14, $q(G) \leq p(G) = 3k - 1$ and, since G is connected, $q(G) \geq 3k - 2$. Hence, $q(G) = 3k - 2$ and G is a tree. Let k be the smallest integer such that $k \geq 2$ and $G(3k-1,k)$ contains a tree G.

Let $P: x_1, x_2, \ldots, x_n$ be a longest path in G. Since $\gamma(G) \geq 2$, G is not a star and $n \geq 4$. That $k \geq 3$ may be seen as follows. Suppose that $k = 2$; then $G \in G(5,2)$ and $p(P) = n \geq 4$; furthermore, by Proposition 4.7.16, $\deg x_2 = \deg x_{n-1} = 2$, so $n > 4$ and so $G \cong P_5: x_1, x_2, x_3, x_4, x_5$. However, $\gamma(P_5 - x_1x_2) = 3 > \gamma(P_5)$, contrary to the 2-insensitivity of the elements of $G(5,2)$. So, we conclude that $k \geq 3$.

It follows from Proposition 4.7.16 that $\deg x_2 = 2$. Suppose $\deg x_3 = 2$. Then, G, k, $x = x_3$, $y = x_2$, and $z = x_1$ satisfy the conditions of Proposition 4.7.17 and we have, for $H = G-\{x_1, x_2,$

$x_3\}$, that H is connected and $\gamma(H) = k - 1$. Now, if $\gamma(H-e) \leq k - 2$ for some $e \in E(H)$, then $\gamma(G-e) \leq k - 1 < \gamma(G)$, which contradicts the k-insensitivity of G. So, $\gamma(H-e) = k - 1 = \gamma(H)$ for each $e \in E(H)$. So, for $\ell = k - 1$, H is a connected, ℓ-insensitive graph with $q(H) = 3\ell - 2 = p(H) - 1$. Since no connected graph of order $3\ell - 1$ has fewer than $3\ell - 2$ edges, no connected ℓ-insensitive graph of order $3\ell - 1$ has fewer than $3\ell - 2$ edges, and so $H \in G(3\ell-1,\ell)$. However, then $2 \leq \ell < k$ contradicts our choice of k. So, deg $x_3 \geq 3$. By Proposition 4.7.15, x_3 is adjacent to at most one end-vertex; so (by definition of P) the paths emanating from x_3 in $G-\{x_2, x_4\}$ are, with one possible exception, of length 2. For ease of reading, we shall let $x = x_3$, $y = x_2$, and $z = x_1$.

We shall examine several cases dependent upon the degree of x. In each case, two or three vertices will be removed from G so that the remaining subgraph F has domination number $k - 1$, is a tree, and is $(k - 1)$-insensitive. If two vertices are removed, the remaining graph has order $3(k - 1)$, and Theorem 4.7.6 indicates that this graph cannot be a tree. To see this, observe that, since $p(F) = 3(k - 1)$, $\gamma(F) = k - 1$, and $k \geq 2$, we have $p(F) \geq 3\,\gamma(F) \geq 6$, and we can apply Theorem 4.7.6 to obtain $q(3(k-1),k-1) = 2.3(k - 1) - 3(k - 1) = 3(k - 1)$. Thus, since F is $(k-1)$-insensitive with order $3(k - 1)$, we must have $q(F) \geq q(3(k-1),k-1) = 3(k - 1) = p(F) > p(F) - 1$.

Thus, F cannot be a tree. If three vertices are removed, the remaining graph has order $3(k - 1) - 1$ and cannot be a tree because G represents a smallest tree T for which $p(T) = 3\gamma(T) - 1$ and $\gamma(T-e) = \gamma(T)$ for each $e \in E(G)$. These contradictions will prove the theorem.

Case 1: Suppose that deg $x \geq 4$. Then, in $G - E(P)$, at least two non-trivial paths emanate from x, say $Q{:}x,v,s$ and $R{:}x,w,t$ or $R{:}x,w$. By Proposition 4.7.18, $\gamma(G-\{y, z\}) = k - 1$. Let $e \in E(G-\{y, z\})$ and let D be a minimum dominating set of $G-e$. If $e \neq xv$, then (as before) we may choose D to contain y and v (if $e \neq vs$) or y and x (if $e = vs$). Thus, since $|N_{G-\{y,z\}}[x] \cap D| \geq 2$, $D - \{y\}$ is a dominating set of $(G-\{y, z\})-e$, whence $\gamma((G-\{y, z\})-e) \leq |D| - 1 = k - 1$. If $e = xv$, then D may be chosen to contain y and w (if R is x,w,t) or y and x (if R is x,w), so that, in this case also, $|N_{G-\{y,z\}}[x] \cap D| \geq 2$ and $\gamma((G-\{y, z\})-e) \leq k - 1$. Thus $\gamma(G-\{y, z\}) = k - 1$ for each $e \in E(G-\{y, z\})$ and the tree $G-\{y, z\}$ is the subgraph F we seek.

Case 2: Suppose that deg $x = 3$ with x adjacent to a vertex w of degree 1. By arguments similar to those used above, it may easily be seen that there exists a minimum dominating set D of G that contains x and y. Clearly, then, $D - \{y\} \mapsto G-\{w, y, z\}$, so that $\gamma(G-\{w, y, z\}) \leq k - 1$. If $\gamma(G-\{w, y, z\}) < k - 2$, then, if D^* is a minimum dominating set of $G-\{w, y, z\}$, then $D^* \cup \{w, z\} \mapsto G$ with $|D^* \cup \{w, z\}| < k$, a

Suppose first that $\gamma(G-\{w, y, z\}) = k - 1$ and let $e \in E(G-\{w, y, z\})$. There is a minimum dominating D set of $G-e$ that contains both x and y, and clearly, $D - \{y\} \mapsto (G-\{w, y, z\})-e$, so $G-\{w, y, z\}$ is $(k - 1)$-insensitive and is a subgraph F with the desired properties.

Suppose now that $\gamma(G-\{w, y, z\}) = k - 2$. Then, no minimum dominating set D of $G-\{w, y, z\}$ can contain x since otherwise $D \cup \{y\}$ would be a $(k - 1)$-element dominating set of G. If $\gamma(G-\{y, z\}) < k - 1$, then $\gamma(G) < k$; so, $\gamma(G-\{y, z\}) \geq k - 1$. If D is a minimum dominating set of G, then either y or z belongs to D, and $D - \{y, z\} \mapsto G-\{y, z\}$, so that $\gamma(G-\{y, z\}) \leq |D - \{y, z\}| = |D| - 1 = k - 1$. Thus, we conclude that $\gamma(G-\{y, z\}) = k - 1$. Now, let $e \in E(G-\{y, z\})$. If $e = xw$, let D be a minimum dominating set for $G-\{w, y, z\}$; then $|D| = k - 2$ and $D \cup \{w\} \mapsto G-\{w, y, z\} \cup \langle\{w\}\rangle = G-\{y, z\} - xw$, whence $\gamma((G-\{y, z\}-e) \leq (k - 2) + 1 = k - 1$. If $e \in E(G-\{w, y, z\})$, then, for a smallest set D' that dominates $G-e$ and contains both x and y, we have $D' - \{y\} \mapsto (G-\{y, z\})-e$, so that, again, $\gamma((G-\{y,z\})-e) \leq k - 1$. The $(k - 1)$-insensitivity of $G-\{y, z\}$ follows, and $G-\{y, z\}$ is the subgraph F we seek.

Case 3: Suppose $\deg x = 3$ and in $G - E(P)$ a path $Q{:}x,v,s$ emanates from x. Consider $F = G-\{y, z\}$ and note that, as in Case 1, if $e \neq vx$, then $\gamma(F-e) = k - 1$. Furthermore, if $e = vx$ and some minimum dominating set D of $G-vx$ contains a vertex of $N_G[x] - \{y, v\}$, then $\gamma(F-e) = \gamma(F)$ as in Case 1. So, we consider now the case that arises if $e = vx$ and no minimum dominating set of $G-vx$ (which, of course, has cardinality k) contains a vertex from $N_G[x] - \{y, v\}$. Let D be a minimum dominating set of $G-e$; we may assume that $v, y \in D$. Now, let $H = G-\{x, y, v, s\}$. Then, if D' is a minimum dominating set of H (in particular, there is $w \in D'$ with $\{w\} = N_G(x) - \{v, y\} \mapsto \{x\}$), then we have $|D'| \leq |(D - \{v, y\}) \cup \{x\}| = k - 1$, but $|D'| \geq k - 1$ since, otherwise, if $|D'| \leq k - 2$, then $D' \cup \{v, y\}$ is a dominating set of $G-e$ of cardinality at most k that contains an element (namely, w) of $N_G[x] - \{v, y\}$, contrary to assumption. Hence, $\gamma(H) = |D'| = k - 1$.

Now, let $f \in E(H)$ and let D'' be a minimum dominating set of $G-f$; then $|D''| = k$ (since G is k-insensitive) and we may assume that D'' is chosen to contain v and y. Furthermore, $(N_G[x] - \{y, v\}) \cap D'' = \varnothing$ (i.e., $x \notin D''$), otherwise D'' is a minimum dominating set of $G-vx$ that contains a vertex from $N_G[x] - \{v, y\}$, contrary to assumption. But $(D'' - \{v, y\}) \cup \{x\} \mapsto H-f$ and so $\gamma(H-f) \leq |D''| - 1 = k - 1$; obviously H is a tree. This shows, as before, that H is a tree in $G_c(3k-4,k-1)$, contrary to assumption, which completes the proof. □

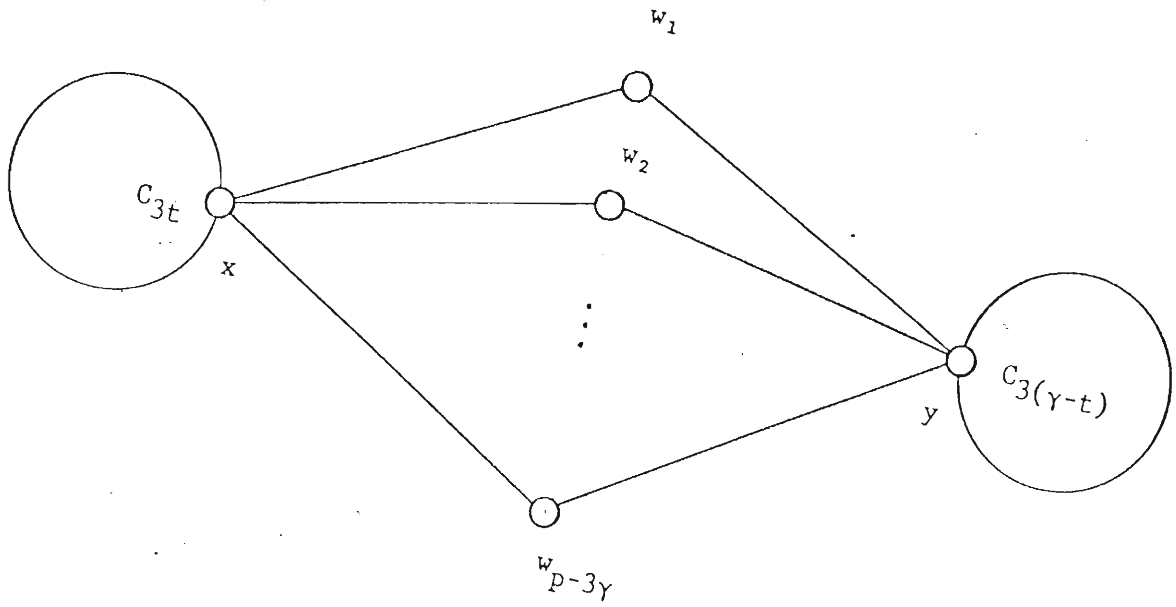


Fig. 4.7.2

We summarize the results of this section in the following theorem.

4.7.20 Theorem: For p, $\gamma \in \mathbb{N}$,

$$q(p,\gamma) = \begin{cases} 3p - 6 \text{ (and } |G(p,\gamma)| = 1), & \text{if } \gamma = 1 \text{ and } p \geq 3 \\ p - 1, & \text{if } \gamma \geq 2 \text{ and } 2\gamma \leq p \leq 3\gamma - 2 \\ p, & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma - 1 \\ 2p - 3\gamma, & \text{if } \gamma \geq 2 \text{ and } p \geq 3\gamma \end{cases}$$

$G(p,\gamma) = \varnothing$, otherwise.

4.8 GRAPHS WHOSE DOMINATION NUMBER AND NUMBER OF COMPONENTS ARE PRESERVED UPON THE REMOVAL OF A SINGLE EDGE

In this section, we shall investigate the existence of graphs G for which the $\gamma(G-e) = \gamma(G)$ and $k(G-e) = k(G)$ for each edge $e \in E(G)$.

4.8.1 Definition: For $p \geq 2$ and $\gamma \geq 1$, define $G_c(p,\gamma) = \{G;$ G is a graph, $\gamma(G) = \gamma$, and, for each $e \in E(G)$, $G-e$ is connected and $\gamma(G-e) = \gamma\}$; so $G_c(p,\gamma) = \{G \in G(p,\gamma);$ G is 2-edge-connected$\}$.

4.8.2 Remark: Observe that the graphs in $G_c(p,\gamma)$ are connected and that, if G is disconnected with components $G_1, G_2, \ldots, G_k$ and each component G_i belongs to $G_c(p_i,\gamma_i)$, for some $p_i, \gamma_i \in \mathbb{N}$ $(1 \leq i \leq k)$, then, for $p = \Sigma_{i=1}^{n+1} p_i$ and $\gamma = \Sigma_{i=1}^{n+1} \gamma_i$, G is a graph of order p and minimum size for which $\gamma(G-e) = \gamma(G) = \gamma$ and $k(G-e) = k(G)$ for all $e \in E(G)$. So, no loss of generality results from the demand that the graphs in $G_c(p,\gamma)$ are connected.

That extremal graphs of the kind in $G_F(p,\gamma)$ do exist is illustrated by the fact that the graph G in Fig. 4.7.2 (which has the property that $\gamma(G-e) = \gamma(G) = \gamma$ for each $e \in E(G)$) is such that $G-e$ is connected for each $e \in E(G)$ when $p - 3\gamma \geq 2$.

4.8.3 Proposition: If p and γ are such that $G_c(p,\gamma) \neq \varnothing$ and $G \in G_c(p,\gamma)$, then

(1) $q(G) \geq p$ and so $q_c(p,\gamma) \geq p$;

(2) $G(p,\gamma) \neq \emptyset$ and $q_c(p,\gamma) \geq q(p,\gamma)$.

Proof: The result in (a) is a direct consequence of the observation that no tree remains connected upon the deletion of one of its edges, while (b) is immediately obvious. □

4.8.4 Theorem: Let $p \geq 2$. Then,

(1) $G_c(p,1)$, $G(p,1) \neq \emptyset$ and $q_c(p,1) = q(p,1) = 3p - 6$, for $p \geq 3$;

(2) $G_c(p,\gamma)$, $G(p,\gamma) \neq \emptyset$ and $q_c(p,\gamma) = q(p,\gamma)$, for $\gamma \geq 2$, $p \geq 3\gamma - 1$ and $p \neq 3\gamma + 1$.

Proof: Let $p \geq 2$, $\gamma \geq 1$.

(1) From Theorem 4.7.3, we know $G(p,1) = \{K_3 + \overline{K}_{p-3}\}$. We claim now that $G(p,1) \subseteq G_c(p,\gamma)$. Let $G \cong K_3 + \overline{K}_{p-3}$ $(p \geq 3)$, and let $e \in E(G)$. Then, e is contained in a cycle of G and is not a bridge. So, $G-e$ is connected. Thus, since G is 1-insensitive, we have $G \in G_c(p,1)$; so, $G_c(p,1) \neq \emptyset$ and $q_c(p,1) \leq q(G) = q(p,1)$. So, by Proposition 4.8.3(2), $q_c(p,1) = q(p,1)$.

(2) Since the cycle $C_{3\gamma-1}$ has order $3\gamma - 1$ (≥ 5), and size $3\gamma - 1 = p$, is connected, satisfies $k(C_{3\gamma-1}-e) = 1$ and $\gamma(C_{3\gamma-1}-e) = \gamma(P_{3\gamma-1}) = \gamma(C_{3\gamma-1})$, for each $e \in E(C_{3\gamma-1})$, and since $q_c(p,\gamma) \geq p$, we have $C_{3\gamma-1} \in G_c(3\gamma-1,\gamma)$ and

$$q_c(3\gamma-1,\gamma) = q(C_{3\gamma-1}) = q(3\gamma-1,\gamma).$$

Applying the argument above to the graph $C_{3\gamma}$, we may show $q_c(p,\gamma) = q(p,\gamma)$ for $p = 3\gamma$.

For $p \geq 3\gamma + 2$, the graphs in $G(p,\gamma)$ shown in Fig. 4.7.2 (introduced in Theorem 4.7.6) contain no bridges, and so $q_c(p,\gamma) \leq q(I_{p,\gamma}) = q(p,\gamma)$. By Proposition 4.8.3(2), the theorem follows. (Note that, for $p = 3\gamma + 1$ (i.e., $p - 3\gamma = 1$), both the edges w_1x, w_1y in Fig. 4.7.2 are bridges of $I_{p,\gamma}$.) □

4.8.5 Theorem: If $\gamma \geq 2$ and $p = 3\gamma - 2$, then

$$G_c(p,\gamma) \neq \emptyset \text{ and } q_c(p,\gamma) = 3\gamma - 2 = q(p,\gamma) + 1.$$

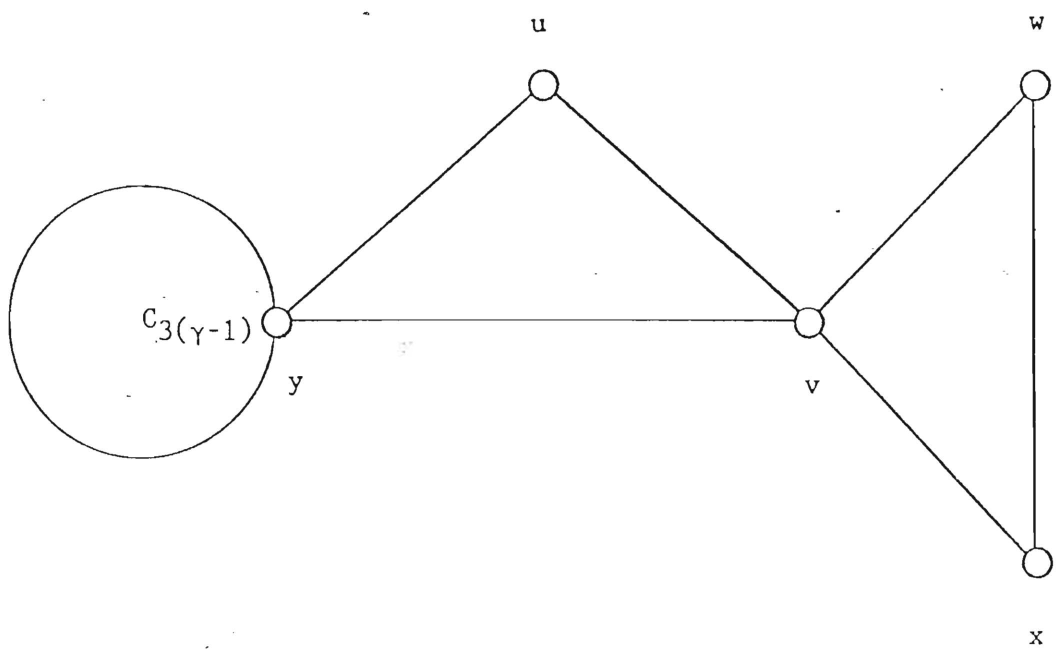


Fig. 4.8.1

Proof: Let $\gamma \geq 2$ and $p = 3\gamma - 2 \geq 4$. Let $G \cong C_{3\gamma-2}$. Now, $\gamma(G) = \lceil \frac{1}{3}(3\gamma - 2) \rceil = \gamma$ and $\gamma(P_{3\gamma-2}) = \gamma$; so G is γ-insensitive. Furthermore, G-e is connected for each $e \in E(G)$, and $q(G) = 3\gamma - 2 = p(G)$. So, $q_c(3\gamma-2,\gamma) \leq q(G) = 3\gamma - 2 = p$. By Proposition 4.8.3, $q_c(p,\gamma) \geq p$; so, $q_c(3\gamma-2,\gamma) = q(G) = 3\gamma - 2$ and $G \in G_c(p,\gamma)$. By Theorem 4.7.20 (since $\gamma \geq 2$ implies $2\gamma \leq p$), we have $q(3\gamma-2,\gamma) = (3\gamma - 2) - 1 = 3\gamma - 3$, and the theorem follows. □

4.8.6 Proposition: Let $p, \gamma \geq 2$ with $p < 3\gamma - 2$. If $G_c(p,\gamma) \neq \varnothing$, then $q_c(p,\gamma) \geq p + 1$.

Proof: Suppose, to the contrary, that there exists $p, \gamma \geq 2$ with $p < 3\gamma - 2$ such that $G_c(p,\gamma) \neq \varnothing$ and $q_c(p,\gamma) \leq p$. Then, by Proposition 4.8.3(1), $q_c(p,\gamma) = p$. Let $G \in G(p,\gamma)$. Since G-e is connected for any $e \in E(G)$, G is not a tree, and contains at least one cycle; however, $q(G) = p$. So, G is unicyclic; let $C{:}u_1,u_2,\ldots,u_n,u_1$ be the cycle in G. We claim that $n = p$. Assume, to the contrary, that C is not a hamiltonian cycle of G. Let $w \in V(G) - V(C)$ and let e be an edge incident with w. Since $k(G-e) = 1$, e is not a bridge of G and so e is contained in a cycle of G. However, this is impossible, as w does not lie on the unique cycle C of G. So, C is indeed a hamiltonian cycle of G. Since $q(C) = p$ and $q(G) = p$, it follows that $G \cong C_p$. However, $p \leq 3\gamma - 3$ implies that $\gamma(G) = \gamma(C_p) = \lceil p/3 \rceil < \gamma$, which is a contradiction. Hence, $q_c(p,\gamma) = q(G) \geq p + 1$, as required. □

4.8.7 Proposition: Let $\gamma \geq 2$ and let G be the graph of order $p = 3\gamma + 1$ shown in Fig. 4.8.1 obtained from the union of the cycles $H = C_{3(\gamma-1)}{:}u_1,u_2,\ldots,u_{3(\gamma-1)},u_1$, $C_3'{:}y,v,u,y$ and $C_3''{:}x,w,z,x$ by identifying u_1 with y and v with z. Then,

(1) G is a γ-insensitive graph of order p for which G-e is connected for each $e \in E(G)$, and

(2) $q(G) = 3\gamma + 3$.

Proof: Let G be the connected graph defined above. If D is a minimum dominating set for H that contains y, then it is not difficult to see that the set $D \cup \{v\}$ dominates G and is a smallest such dominating set. So, $\gamma(G) = \lceil \frac{1}{3}(3(\gamma - 1)) \rceil + 1 = \gamma$. Further, since each edge of G lies on a cycle, it is clear that G-e is connected for each $e \in E(G)$.

Next, we show that G is γ-insensitive. Let $e \in E(G)$. If $e \in \{uy, uv, xw, vy\}$, then $D \cup \{v\} \mapsto G-e$. If $e \in \{vx, vw\}$, then $D \cup \{x\} \mapsto G-e$. If $e \in E(H)$, then $D' \cup \{v\} \mapsto G-e$ where D' is a minimum dominating set of the path H-e of order $3(\gamma - 1)$. Finally,

$$q(G) = q(C_{3(\gamma-1)}) + 6 = 3\gamma + 3. \qquad \square$$

4.8.8 Theorem: Let $\gamma \geq 2$. Then, for $p = 3\gamma + 1 \geq 7$,

$$G_c(p,\gamma) \neq \emptyset \text{ and } q_c(p,\gamma) = q(p,\gamma) + 1 = p + 2.$$

Proof: Let $\gamma \geq 2$ and $p = 3\gamma + 1$. By Proposition 4.8.7(1), the graph G of Fig. 4.8.1 is a connected graph of order p satisfying $k(G-e) = k(G) = 1$ and $\gamma(G-e) = \gamma(G) = \gamma$ for each $e \in E(G)$, and, by Proposition 4.8.7(2), $q(G) = 3\gamma + 3$. Hence, $G_c(p,\gamma) \neq \emptyset$ and $q_c(p,\gamma) \leq 3\gamma + 3$. Since, by Theorem 4.7.20, $q(3\gamma+1,\gamma) = 2(3\gamma + 1) - 3\gamma = 3\gamma + 2$, we have $q_c(p,\gamma) \leq q(p,\gamma) + 1 = p + 2$. Suppose that $q_c(p,\gamma) \neq p + 2$, i.e., since $q_c(p,\gamma) \geq q(p,\gamma) = p + 1$, we assume $q_c(p,\gamma) = p + 1 = q(p,\gamma)$.

Let $G \in G_c(p,\gamma)$; for $i = 1, 2, \ldots, p - 1$, let k_i denote the number of vertices of degree i in G. Then (by the First Theorem of Graph Theory),

$$2q_c(p,\gamma) = 2q(G) = 2p + 2 = \Sigma_{i=1}^{p-1} ik_i.$$

Since G is 2-edge-connected, $k_1 = 0$. Clearly, $k_2 = p - (k_3 + k_4 + \ldots + k_{p-1})$. Thus,

$$\begin{aligned} 2p + 2 &= 2[p - (k_3 + k_4 + \ldots k_{p-1})] + \sum_{i=3}^{p-1} ik_i \\ &= 2p + \sum_{i=3}^{p-1} (i - 2)k_i. \end{aligned}$$

Therefore, $2 = \Sigma_{i=3}^{p-1} (i - 2)k_i$, which implies immediately that $k_i = 0$ for $i \geq 5$, and either $k_3 = 2$ and $k_4 = 0$, or $k_3 = 0$ and $k_4 = 1$. We consider the following two cases.

Case 1: Suppose $k_3 = 0$ and $k_4 = 1$ (then G has $p - 1$ vertices of degree 2 and one vertex of degree 4). Observe that if there is a minimum dominating set D of G that does not contain the vertex, t say, of degree 4, then such a minimum dominating set contains only vertices of degree 2 and could thus dominate at most $3\gamma < p$ vertices. So, every minimum dominating set of G must contain the vertex of degree 4. Since $p \geq 3\gamma$, from the proof of Theorem 4.7.6 (see (*)) it follows that, if $H \in G(p,\gamma)$ and if $X_{n+1} = X_{n+1}(H)$ is the set of vertices of H that appear in every minimum dominating set of H, then $q(p,\gamma) = q(H) \geq 2p - 3\gamma + |X_{n+1}|$. By the properties of the elements of $G(p,\gamma)$ and

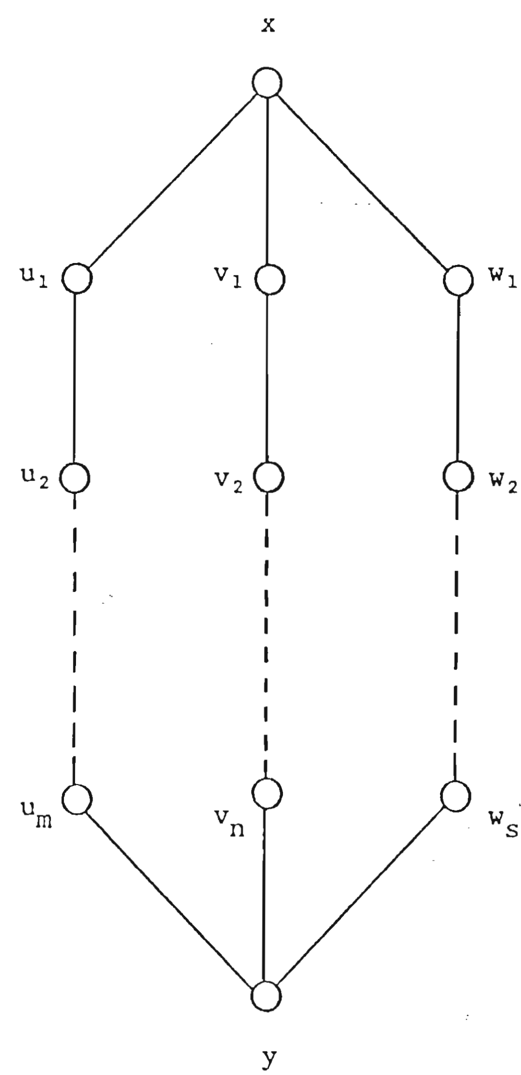


Fig. 4.8.2

$G_c(p,\gamma)$, and the fact that $q_c(p,\gamma) = q(p,\gamma)$, it follows that $G_c(p,\gamma) \subseteq G(p,\gamma)$. In particular, $G \in G(p,\gamma)$, and (from our comments above), we have

$$q(G) \geq 2p - 3\gamma + |X_{n+1}(G)|.$$

So,

$$q_c(p,\gamma) = q(G) = q(p,\gamma) \geq 2p - 3\gamma + |\{t\}| = 2p - 3 \cdot \tfrac{1}{3}(p - 1) + 1 = p + 2,$$

which contradicts our assumption that $q_c(p,\gamma) = p + 1$.

Case 2: Suppose $k_3 = 2$ and $k_4 = 0$. Then, by the properties possessed by G, it follows that G is as shown in Fig. 4.8.2, where G is obtained by joining vertices x and y by three internally disjoint paths of lengths $m + 1$, $n + 1$, and $s + 1$, respectively, where $m \leq n \leq s$ and $m + n + s = p - 2 = 3\gamma - 1 \geq 5$.

We begin by noting that, for any edge e incident with x in G, the graph $G-e$ may be described as consisting of a cycle and a path, where the cycle and path have exactly one vertex in common. Suppose that v is a vertex adjacent to x; without loss of generality, suppose that $v = u_1$. Let $G' = G-vx$. Since G is γ-insensitive, $\gamma(G') = \gamma(G) = \gamma$ and G' consists of a cycle C on $n + s + 2 = 3\gamma + 1 - m$ vertices, with the attached path P having length m. We discuss three cases dependent upon the value of m.

Subcase 2.1: Suppose $m \equiv 0 \pmod 3$; say, $m = 3k$ for some $k \in \mathbb{N}$. Then, $p(C) \equiv 3\gamma - 2 - m \equiv 1 \pmod 3$. Let D be a minimum dominating set of G'. If $y \in D$, then $D = D_1 \cup D_2 \cup \{y\}$ where D_1 is a minimum dominating set of the path $\langle V(P) - N[y]\rangle_G \cong P_{m-1}$, and D_2 is a minimum dominating set of the path $\langle V(C) - N[y]\rangle_G \cong P_{n+s-1}$; if $y \notin D$, then $D = D_3 \cup D_4$ where D_3 is the minimum dominating set for $\langle V(P) - \{y\}\rangle_G \cong P_m$ and D_4 is a minimum dominating set for C (that does not contain y), or $D = D_5 \cup D_6$, where D_5 is a smallest dominating set of $\langle V(P) - \{y\}\rangle_G$ that contains the neighbour u_m of y on P, and D_6 is the minimum dominating set of $\langle V(C) - \{y\}\rangle_G$. In the first instance, $|D|$ is calculated as

$$|D| = \lceil ⅓(3k - 1)\rceil + \lceil ⅓(3\gamma + 1 - 3k - 3)\rceil + 1$$
$$= k + (\gamma - k) + 1 = \gamma + 1,$$

and, in the second, as

$$|D| = \lceil {}^{3k}/_{3}\rceil + \lceil ⅓(3\gamma + 1 - m)\rceil = k + (\gamma - k + 1) = \gamma + 1,$$

or

$$|D| = |D_5| + |D_6| = (\lceil {}^{3k}/_{3}\rceil + 1) + \lceil ⅓(3\gamma + 1 - 3k - 1)\rceil$$
$$= (k + 1) + (\gamma - k) = \gamma + 1,$$

respectively.

So, $\gamma(G') = \gamma + 1$, i.e., the edge $e = vx$ satisfies $\gamma(G-e) = \gamma(G) + 1$. However, this contradicts the γ-insensitivity of G; so Subcase 2.1 does not occur.

Subcase 2.2: Suppose $m \equiv 2 \pmod 3$; then, $m = 3k + 2$ for some $k \in \mathbb{N}$. Let D be a minimum dominating set of G′. Then, it is easy to see that $D = D_1 \cup D_2$ where D_1 is a minimum dominating set of P that contains u_m and D_2 is a minimum dominating set of $\langle V(C) - \{y\}\rangle_G \cong P_{n+s+1}$, in which case

$$|D| = \lceil ⅓(3k + 2)\rceil + \lceil ⅓(n + s + 1)\rceil$$
$$= k + 1 + \lceil ⅓(3\gamma - 3k - 2)\rceil$$
$$= (k + 1) + (\gamma - k) = \gamma + 1,$$

or $D = D_3 \cup D_4$ where D_3 is a minimum dominating set of $\langle V(P) - \{y\}\rangle \cong P_{3k+1}$ and D_4 is a minimum dominating set of C, in which case

$$|D| = \lceil ⅓(3k + 1)\rceil + \lceil ⅓(3\gamma + 1 - 3k - 2)\rceil$$
$$= (k + 1) + (\gamma - k) = \gamma + 1.$$

So, $\gamma(G-vx) = \gamma(G) + 1$. Since this contradicts the γ-insensitivity of G, it follows that Subcase 2.2 does not occur either.

So, $m \equiv 1 \pmod 3$. Now, since we chose $u_1 = v \in N_G(x)$ arbitrarily, Subcases 2.1 and 2.2 apply equally well to n and s, i.e., Subcases 2.1 and 2.2 show that $n \equiv m \equiv s \equiv 1$

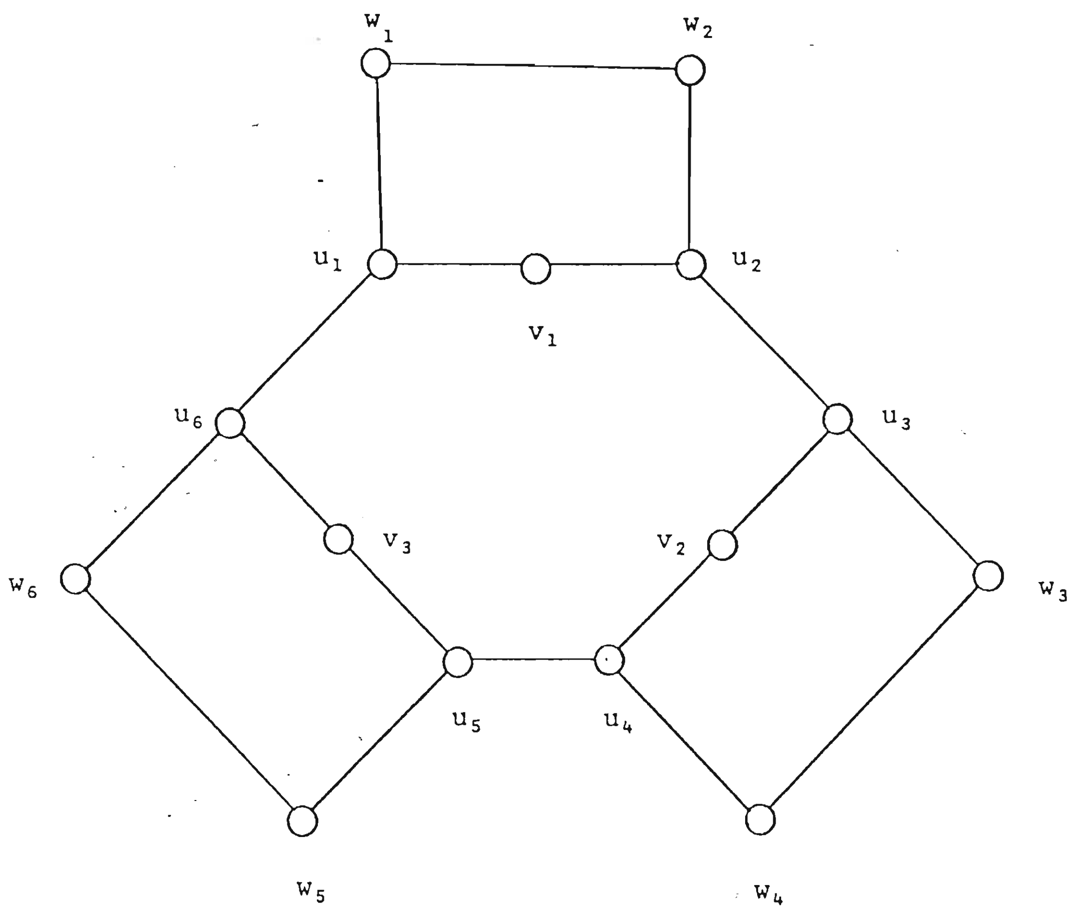


Fig. 4.8.3

(mod 3). Hence, there must exist $\ell_1, \ell_2, \ell_3 \in \mathbb{N}$ with $m = 3\ell_1 + 1$, $n = 3\ell_2 + 1$, $s = 3\ell_3 + 1$. However, then

$$p = m + n + s + 2 = 3(\ell_1 + \ell_2 + \ell_3 + 1) + 2 \neq 3\gamma + 1 = p,$$

which is absurd. So, our original assumption that $q_c(p,\gamma) = p + 1$ is incorrect, and $p + 1 \leq q_c(p,\gamma) \leq p + 2$ implies $q_c(p,\gamma) = p + 2$, as required. $\square$

We summarize the results of this section in the following theorem.

4.8.9 Theorem: Let $p \geq 2$ and $\gamma \geq 1$. Then,

$$q_c(p,\gamma) = \begin{cases} q(p,\gamma) = 3p - 6, & \text{if } \gamma = 1 \text{ and } p \geq 3 \\ q(p,\gamma) + 1 = p, & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma - 2 \\ q(p,\gamma) = p, & \text{if } \gamma \geq 2, \text{ and } p = 3\gamma - 1 \text{ or } p = 3\gamma \\ q(p,\gamma) + 1 = p + 2 & \text{if } \gamma \geq 2 \text{ and } p = 3\gamma + 1 \\ q(p,\gamma) = 2p - 3\gamma, & \text{if } \gamma \geq 2 \text{ and } p \geq 3\gamma + 2 \end{cases}.$$

The situation when $\gamma \geq 2$ and $(4 \leq)\ p \leq 3\gamma - 3$ is unknown. In [BD1], it is stated that the authors have been able to show that $G_c(p,\gamma) = \varnothing$ when $p \leq 3\gamma - 3$ and $\gamma = 2$, 3, or 4, but it is pointed out that, for example, the graph in Fig. 4.8.3 is a 6-insensitive graph with $p = 3\gamma - 3$ which remains connected when any edge is removed (as the following proposition shows), i.e., $G(15,6) \neq \varnothing$.

4.8.10 Proposition: The graph in Fig. 4.8.3, obtained from the union of three (disjoint) cycles, $u_i, v_i, u_{i+1}, w_{i+1}, w_i, u_i$ ($i \in \{1, 3\ 5\}$) by the insertion of the edges u_2u_3, u_4u_5, and u_6u_1, y, is a 6-insensitive graph with $p = 3\gamma - 3$ which remains connected when any edge is removed.

Proof: Let G be the graph defined above. We note first that $D = \{u_1, u_3, u_5, w_1, w_3, w_5\}$ is a dominating set of G, and it is not difficult to see, by inspection, that no smaller subset of V(G) dominates G. So, $\gamma = \gamma(G) = 6$; since $p = p(G) = 15$, we do indeed have $p = 3\gamma - 3$. Let $e \in E(G)$. We consider the following four cases.

Case 1: If $e = w_iw_{i+1}$ for some $i \in \{1, 2, ..., 5\}$, then $D^* = (D - \{w_i\}) \cup \{w_{i+1}\} \mapsto G-e$.

Case 2: If $e = u_1v_1$, u_3v_2, or u_5v_3, then $D^* = (D - \{u_1, w_5\}) \cup \{u_2, w_6\}$, or $D^* = (D - \{u_3, w_1\}) \cup \{u_4, w_2\}$, or $D^* = (D - \{u_5, w_3\}) \cup \{u_6, w_4\}$, respectively, dominates $G-e$.

Case 3: If $e = u_iw_i$ for some $i \in \{1, 2, ..., 6\}$, or $e = v_iu_{2i}$ for some $i \in \{1, 2, 3\}$, or $e = u_iw_i$ for some $i \in \{2, 4, 6\}$, then $D^* = D \mapsto G-e$.

Case 4: If $e \in \{u_1u_6, u_2u_3, u_4u_5\}$, then $D^* = (D - \{w_5\}) \cup \{w_6\}$, or $D^* = (D - \{w_1\}) \cup \{w_2\}$, or $D^* = (D - \{w_3\}) \cup \{w_4\}$, respectively, satisfies $D^* \mapsto G-e$.

Since, in each case above, $|D^*| = |D| = 6 = \gamma(G)$, we have that G is 6-insensitive. Finally, since every edge of G lies on a cycle, $k(G-e) = 1$ for each $e \in E(G)$. □

Chapter 5

n-DOMINATION

5.1 INTRODUCTION

In this chapter, we shall consider dominating sets of high integrity which retain the property of domination if at most a given number of vertices or edges are removed from the graphs. We shall generalize the classical notion of domination in graphs to include a prescribed degree of redundance in domination. Most of the results in sections 5.1 to 5.3 appear in the seminal paper [FJ1], except for Theorem 5.3.5 and Corollary 5.3.6, which appear in [F1], while most of the results in section 5.4 appear in [FJ2]. The following are the exceptions. We have supplied the proof of Proposition 5.2.1, Corollary 5.2.9, 5.3.6, 5.3.9, 5.4.11, and Theorem 5.4.4. We have expanded Remark 5.4.6, as well as the proof of Theorem 5.2.2 (slightly), 5.3.4, 5.3.5, 5.3.7, and 5.3.14 (slightly). We have slightly modified the proof of Corollary 5.3.12, as well as the statement of Theorem 5.2.6. We have clarified and slightly modified the proof of Theorem 5.4.9, 5.4.12, and 5.4.13. Finally, we have provided the second example given prior to Theorem 5.3.7.

We recall the following result of Chapter 4.

4.2.7 Theorem: For any non-empty graph G and minimum dominating set D of G, there exists a vertex $u \in V(G) - D$ such that $|N(u) \cap D| \leq 2$.

This theorem illustrates the fact that the dominating property of any minimum dominating set of a graph can be destroyed by the removal of at most two edges or vertices from the graph; for example, if G is a non-empty graph, D is a minimum dominating set of G, u is a vertex of $V(G) - D$ adjacent to at most two vertices of D and $U = N(u) \cap D$, then D is not a dominating set of $G-[\{u\}, U]$, nor a dominating set of $G-U$. We have already encountered many examples of a graph H with minimum dominating set D and a vertex v_0 or an edge e_0 such that $D \not\succ H-v_0$ or $D \not\succ H-e_0$ (see, for example, Corollary 4.2.6). This brings us to the following definitions.

5.1.1 Definition: Let G be a graph, and let $n \in \mathbb{N}$. If $D \subseteq V(G)$ and $u \in V(G) - D$ such that u is adjacent to at least n members of D, we say that u is *n-dominated* by D. If every vertex in $V(G) - D$ is n-dominated by D, then D is called an *n-dominating set* of G. If D has a smallest cardinality among all n-dominating sets of the graph G, then D is a *minimum n-dominating set* of G and its cardinality is the *n-domination number* $\gamma_n(G)$ of G.

We observe immediately that every n-dominating set ($n \in \mathbb{N}$) of a graph G is a dominating set of G in the usual sense; thus, we have

5.1.2 Proposition: For every graph G and each $n \in \mathbb{N}$, $\gamma(G) \leq \gamma_n(G)$.

5.1.3 Remark: In particular, a minimum 1-dominating set is a minimum dominating set and $\gamma(G) = \gamma_1(G)$. More generally, for m and n satisfying $m \leq n$, every n-dominating set in G is also an m-dominating set and thus $\gamma_m(G) \leq \gamma_n(G)$. It is our purpose in this chapter to obtain both bounds and exact values for the parameter γ_n, as well as an understanding of the behaviour of γ_n.

5.2 PROPERTIES OF γ_n

We now set about establishing a more accurate relationship between γ and γ_n than that given in 5.1.2. We begin with the following stronger version of Proposition 5.1.2.

5.2.1 Proposition: If G is a graph with $\Delta(G) \geq 3$, then $\gamma_n(G) > \gamma(G)$ for all $n \geq 3$.

Proof: Suppose, to the contrary, that there is a graph G for which there exists $n \geq 3$ with $\gamma_n(G) = \gamma(G)$, i.e., for which there exists $n \geq 3$ and a minimum dominating set D of G such that D is an n-dominating set of G. Then, if $u \in V(G) - D$ is a vertex whose existence is guaranteed by Theorem 4.2.7, we have $3 \leq n \leq |N_G(u) \cap D| \leq 2$, an absurdity. So, the proposition holds. □

We note that, for n as small as 3, the difference $\gamma_n(G) - \gamma(G)$ can be made arbitrarily large for a suitable graph G: Consider $G \cong W_p$ ($p \geq 3$); here, $\gamma(G) = 1$ and $\gamma_3(G) = \lceil \frac{1}{2}(p-1) \rceil + 1$, whence $\gamma_3(G) - \gamma(G) = \lceil \frac{1}{2}(p-1) \rceil$. The observation that, for $p > 3$, $\lceil \frac{1}{2}(p-1) \rceil > 1$ shows that the following theorem, which gives a bound on $\gamma_n(G)$ (for $n \geq 2$), is not best possible. Theorem 5.2.2 yields more information than Proposition 5.2.1 and, in fact, produces Proposition 5.2.1 as a corollary.

5.2.2 Theorem: If G is a graph with $\Delta(G) \geq n \geq 2$, then $\gamma_n(G) \geq \gamma(G) + n - 2$.

Proof: Let $n \geq 2$, let G be a graph satisfying $\Delta(G) \geq n$, and let D be a minimum n-dominating set of G. If $V(G) - D = \varnothing$ and $w \in V(G)$ with $\deg_G w = \Delta(G)$, then, since $|N_G(w) \cap (D - \{w\})| = |N_G(w) \cap D| = \Delta(G) \geq n$, it follows that $D - \{w\}$ is an n-dominating set of G of cardinality less than $\gamma_n(G)$, which is impossible. So, $V(G) - D \neq \varnothing$. Let $u \in V(G) - D$, and let $v_1, v_2, \ldots, v_n$ be distinct members of D that dominate u. Since D is an n-dominating set of G, each vertex in $V(G) - D$ is adjacent to at least one member of $D - \{v_2, v_3, \ldots, v_n\}$. Therefore, since $\{v_2, v_3, \ldots, v_n\} \subseteq N_G(u)$, the set

$$D^* = (D - \{v_2, v_3, \ldots, v_n\}) \cup \{u\}$$

is a dominating set in G. Hence,

$$\gamma(G) \leq |D^*| = \gamma_n(G) - (n-1) + 1,$$

so that

$$\gamma_n(G) \geq \gamma(G) + n - 2.$$ □

While Theorem 5.2.2 yields a lower bound on the quantity $\gamma_n(G) - \gamma(G)$ (for $n \geq 2$), in many cases it does not provide a lower bound on $\gamma_n(G)$ that is easily determined, since the calculation of $\gamma(G)$ is often difficult. In contrast, the next two theorems provide lower bounds on γ_n that depend on easily computed parameters.

5.2.3 Theorem: If G is a graph of order p and maximum degree Δ, then

$$\gamma_n(G) \geq \frac{np}{\Delta + n}.$$

Proof: Let G be a graph, and let $p = p(G)$, $\Delta = \Delta(G)$. Let D be a minimum n-dominating set in G, let $S = V(G) - D$, and let $t = |[D, S]|$. Then,

$$t = \sum_{v \in D} |N(v) \cap S| \leq \sum_{v \in D} \deg v \leq \Delta \sum_{v \in D} 1 = \Delta \cdot |D| = \Delta \cdot \gamma_n(G).$$

Furthermore, each vertex in S is adjacent to at least n members of D, so

$$t \geq n \cdot |S| = n \cdot [p - \gamma_n(G)].$$

The two inequalities now yield

$$n \cdot [p - \gamma_n(G)] \leq \Delta \cdot \gamma_n(G),$$

and thus

$$\gamma_n(G) \geq \frac{np}{\Delta + n}. \qquad \square$$

The bound on γ_n provided by Theorem 5.2.3 is best possible since the bound is attained by the graph $K_{n,n}$.

5.2.4 Corollary: If G is a graph of order p and maximum degree Δ, then $\gamma(G) \geq p/(1 + \Delta)$.

Before we present the next theorem, we introduce the following definition.

5.2.5 Definition: A bipartite graph G is said to be an *n-semiregular bipartite graph* if V(G) can be bipartitioned in such a way that every vertex in one of the partite sets has degree n; the partite set each of whose vertices has degree n is called the *n-regular partite set* of G.

5.2.6 Theorem: If G is a (p,q) graph, then, for each $n \in \mathbb{N}$,

(1) $\gamma_n(G) \geq p - {}^q/_n$, and

(2) $\gamma_n(G) = p - {}^q/_n$ if and only if G is an n-semiregular bipartite graph.

Proof: Let G be a (p,q) graph, let $n \in \mathbb{N}$, let D be a minimum n-dominating set in G, and let $S = V(G) - D$. Each vertex in S is adjacent to n or more vertices of D, so

$$q = |E(G)| \geq |[S, D]| \geq n \cdot |S| = n \cdot [p - \gamma_n(G)], \tag{i}$$

whence

$$\gamma_n(G) \geq p - {}^q/_n. \tag{ii}$$

If $\gamma_n(G) = p - {}^q/_n$, then (i) becomes $q \geq |[S, D]| \geq n.|S| = q$, from which it follows that $E(G) = [S, D]$, i.e., each edge of G joins a vertex in D and a vertex in S. Thus, D and S are independent sets and (since $q = |[S, D]| = n.|S|$), the degree of each vertex in S is exactly n; i.e., G is an n-semiregular bipartite graph.

Conversely, suppose that G is an n-semiregular bipartite graph and that the n-regular partite set, N say, of G has cardinality a. Clearly, $V(G) - N$ is an n-dominating set for G, so $\gamma_n(G) \leq p - a$. By (ii), $\gamma_n(G) \geq p - {}^{(an)}/_n = p - a$. Hence, $\gamma_n(G) = p - a = p - {}^q/_n$. □

5.2.7 Corollary: For any tree T of order p, $\gamma_2(T) \geq \frac{1}{2}(p + 1)$.

The simple observation that a graph is a 2-semiregular bipartite graph if and only if it is the subdivision graph of some multigraph leads us to the following two corollaries of Theorem 5.2.6.

5.2.8 Corollary: If G is a non-empty (p,q) graph, then $\gamma_2(G) = p - {}^q/_2$ if and only if G is the subdivision graph of some multigraph.

5.2.9 Corollary: If T is a tree of order $p \geq 2$, then $\gamma_2(T) = \frac{1}{2}(p + 1)$ if and only if T is the subdivision graph of another tree.

Proof: The sufficiency follows from Corollary 5.2.8. The necessity follows from Corollary 5.2.7 and the simple fact that, if a tree T is the subdivision graph of a (multi)graph G, then G must itself be a tree. □

The following theorem provides an exact value for $\gamma_n(G)$ for any non-empty graph G.

5.2.10 Theorem: If G is a graph with $\Delta(G) \geq n$, then $\gamma_n(G) = \min\{\gamma_n(H)\}$ where this minimum is taken over all spanning n-semiregular bipartite subgraphs H of G.

Proof: Let $n \in \mathbb{N}$ and let G be a graph with $\Delta(G) \geq n$. If H is any spanning subgraph of G, then $\gamma_n(G) \leq \gamma_n(H)$; so, $\gamma_n(G) \leq \min\{\gamma_n(H);$ H is a spanning subgraph of G$\} \leq \min\{\gamma_n(H);$ H is a spanning n-semiregular bipartite subgraph of G$\}$.

To obtain the reverse inequality, let D be a minimum n-dominating set in G, and let $S = V(G) - D$. Since each vertex in S is adjacent to at least n vertices in D, we may construct a spanning subgraph H of G as follows: Let $V(H) = V(G)$, and for each vertex v in S, select exactly n edges of G that join v to vertices in D. Then, H is a spanning n-semiregular bipartite subgraph of G, and D is an n-dominating set in H. Thus, $\gamma_n(H) \leq |D| = \gamma_n(G)$. So, $\gamma_n(G) \geq \min\{\gamma_n(H);$ H is a spanning n-semiregular bipartite subgraph of G$\}$, and the desired result follows. □

For the next two results, we need the following definition.

5.2.11 Definition: For a graph G and $n \in \mathbb{N}$, we define $\eta_n(G)$ to be $\eta_n(G) = \max\{|S|;$ S is an n-regular partite set of H$\}$, where the maximum is taken over all spanning n-semiregular bipartite subgraphs H of G.

The observation that $\gamma_n(G) = p(G) - |N|$ for any n-semiregular bipartite graph G with n-regular partite set N leads immediately to the following corollary of Theorem 5.2.10.

5.2.12 Corollary: If G is a graph of order $p \geq 2$, then $\gamma_n(G) = p - \eta_n(G)$.

Theorem 5.2.10 and Corollary 5.2.12 now yield the following theorem of Nieminen [N1] concerning the usual domination number. (Recall that $\varepsilon(G)$ denotes the maximum possible number of end-edges in a spanning forest of a graph G.)

5.2.13 Theorem: For any graph G, $\gamma(G) + \varepsilon(G) = p(G)$.

Proof: Let G be a graph. We will show first that $\eta_1(G) = \varepsilon(G)$. Let F be a spanning forest of G having $\varepsilon(G)$ end-edges and let H be a spanning 1-semiregular bipartite subgraph of G having a largest 1-regular partite set S, i.e., $|S| = \eta_1(G)$. Every component of H is a star or an isolated vertex, so H is a 1-semiregular bipartite spanning forest of G having $\eta_1(G)$ vertices in its 1-regular partite set and hence having $\eta_1(G)$ edges, each of which is an end-edge. Thus, $\eta_1(G) \leq \varepsilon(G)$.

Let F' be the subgraph of F obtained by deleting from F all non-end-edges. Then, F' is the union of stars (where the edges of the stars are precisely the ($\varepsilon(G)$) end-edges of F) and (possibly) isolated vertices. Hence, F' is a spanning 1-semiregular bipartite subgraph of G that contains $\varepsilon(G)$ edges, so that its 1-regular partite set contains $\varepsilon(G)$ vertices; thus, $\eta_1(G) \geq \varepsilon(G)$.

The above two paragraphs thus give us $\varepsilon(G) = \eta_1(G)$. Hence, by Corollary 5.2.11, we have $\gamma(G) + \varepsilon(G) = p(G)$. □

5.3 n-DOMINATION AND n-DEPENDENCE OF GRAPHS

The following well-known theorem of Ore [O1] depends on the fact that every maximal independent set of vertices in a graph G is a dominating set of the graph:

5.3.1 Theorem: For every graph G, $\gamma(G) \leq \beta(G)$.

Our main efforts in this section will concern an investigation of similar relationships between n-domination and the generalized notion of independence which we give next.

5.3.2 Definition: Let G be a graph. Then, $S \subseteq V(G)$ is called an *n-dependent set* of G if and only if $\Delta(\langle S \rangle) \leq n$. An n-dependent set of largest possible cardinality in G is a *maximum n-dependent set* of G, and its cardinality, denoted $\beta_n(G)$, is called the *n-dependence number* of G.

5.3.3 Remark: Obviously, for any graph G, if k and m satisfy $k \leq m$, then any k-dependent set of G is also an m-dependent set of G and so $\beta_k(G) \leq \beta_m(G)$. Also, an independent set of vertices of G is precisely a 0-dependent set of G, and so $\beta(G) = \beta_0(G)$ for every graph G. Therefore, Theorem 5.3.1 expresses the relationship $\gamma_1(G) \leq \beta_0(G)$.

That a similar relationship holds between γ_2 and β_1 for every graph is shown next.

5.3.4 Theorem: For every graph G, $\gamma_2(G) \leq \beta_1(G)$.

Proof: Let G be any graph. If $\Delta(G) \leq 1$, then V(G) is the only 2-dominating set of G, as well as a maximum 1-dependent set of vertices of G, so that $\gamma_2(G) = p(G) = \beta_1(G)$. So, we assume now that $\Delta(G) \geq 2$. Let D be a maximum 1-dependent set of G such that $q(\langle D\rangle) \leq q(\langle D^*\rangle)$ for every maximum 1-dependent set D^* of G. We show now that D is a 2-dominating set of G.

Suppose, to the contrary, that there is a vertex $u \in V(G) - D$ that is not 2-dominated by D, i.e., $|N(u) \cap D| \leq 1$. Since D is a maximum 1-dependent set, $D \cup \{u\}$ cannot be a 1-dependent set, and so there must exist two vertices v and w in D such that $\Delta(\langle\{u, v, w\}\rangle) \geq 2$, i.e., uv, uw $\in E(G)$, or vu, vw $\in E(G)$, or wv, wu $\in E(G)$. However, since $|N(u) \cap D| < 2$, the first of these three situations does not arise, so we may assume, without loss of generality, that v is adjacent to both u and w. However, since v is the only vertex in D that is adjacent to u, it follows that $\deg_{\langle D'\rangle} u = 0$, where $D' = (D - \{v\}) \cup \{u\}$. So, D′ is a 1-dependent set of G of cardinality $\beta_1(G)$ and $E(\langle D'\rangle) = (E(\langle D\rangle) - [\{v\}, D]) \cup [\{u\}, D'] = E(\langle D\rangle) - \{vw\}$, i.e., $q(\langle D\rangle) = q(\langle D'\rangle) + 1$. This contradicts our choice of D. So, D is indeed a 2-dominating set of G, and $\gamma_2(G) \leq |D| = \beta_1(G)$, as required. □

The following theorem of Favaron [F1] provides a corollary that settles in the affirmative a conjecture made by Fink and Jacobson in [FJ1].

5.3.5 Theorem: For any graph G and $n \in \mathbb{N}$, if D is an $(n - 1)$-dependent set of G such that $n|D| - q(\langle D\rangle)$ is a maximum, then D is an n-dominating set of G.

Proof: Let G be a graph, let $n \in \mathbb{N}$, and let D be an $(n - 1)$-dependent set of G with $n|D| - q(\langle D\rangle) = \max \{ n|D'| - q(\langle D'\rangle);\ D' \text{ is an } (n-1)\text{-dependent set of } G\}$. Suppose, to the contrary, that D is not an n-dominating set of G. Then, there exists $v \in V(G) - D$ such that v is not n-dominated by D; let $B = N_G(v) \cap D$ (so, $0 \leq |B| < n$), let

$$A = \{a \in B;\ |N(a) \cap D| = n - 1\},$$

and let S be a maximal independent set of A. (So, $S \subseteq A \subseteq B \subseteq D$.)

Now, let $C = (D - S) \cup \{v\}$. Then, C is $(n - 1)$-dependent since

(i) $\deg_{\langle C\rangle} v = |N_G(v) \cap C| = |N_G(v) \cap (D - S)| \leq |N_G(v) \cap D| = |B| \leq n - 1$;

(ii) for any $x \in D - B$,

$$\deg_{\langle C\rangle} x = |N_G(x) \cap C| = |N_G(x) \cap (D - S)| \leq |N_G(x) \cap D| = \deg_{\langle D\rangle} x \leq n - 1$$

(by the definition of D);

(iii) for any $b \in B - A$,

$$\begin{aligned} \deg_{\langle C\rangle} b &= |N_G(b) \cap [(D - S) \cup \{v\}]| = |\{v\} \cup (N_{\langle D\rangle}(b) - S)| \\ &\leq 1 + |N_{\langle D\rangle}(b)| \leq 1 + (n - 2) = n - 1 \end{aligned}$$

(by the definition of A); and,

(iv) for any $a \in A - S$,

$$\deg_{\langle C\rangle} a = |\{v\} \cup (N_{\langle D\rangle}(a) - S)| \leq 1 + |N_{\langle D\rangle}(a)| - 1 = n - 1$$

(since S being a maximal independent set in A implies that every vertex in $A - S$ is adjacent to at least one vertex of S).

Now, $S \subseteq A = \{a \in N_G(v) \cap D;\ |N_G(a) \cap D| = n - 1\}$, and S is independent, so $E(\langle S\rangle) = \varnothing$ and $[\{s\}, D - \{s\}] = [\{s\}, D - S]$, for each $s \in S$, so that

$$[S, D] = [S, D - S] = \bigcup_{s\in S} [\{s\}, D - \{s\}] = (n - 1)|S|.$$

Hence,

$$E(\langle C\rangle) = (E(\langle D\rangle) - [S, D]) \cup [\{v\}, D - S]$$

so that

$$\begin{aligned} q(\langle C\rangle) &= q(\langle D\rangle) - (n - 1)\,|S| + |N_G(v) \cap (D - S)| \\ &= q(\langle D\rangle) - n|S| + |S| + |B| - |S| \\ &= q(\langle D\rangle) - n|S| + |B|. \end{aligned}$$

Thus, since $|C| = |D| - |S| + 1$, we have

$$\begin{aligned} n|C| - q(\langle C\rangle) &= n|D| - n|S| + n - q(\langle D\rangle) + n|S| - |B| \\ &= n|D| + n - q(\langle D\rangle) - |B| \\ &> n|D| - q(\langle D\rangle) \text{ (since } |B| < n). \end{aligned}$$

However, this contradicts our choice of D. So, D is indeed an n-dominating set of G. □

5.3.6 <u>Corollary</u>: For any graph G and $n \in \mathbb{N}$, $\gamma_{n+1}(G) \leq \beta_n(G)$.

<u>Proof</u>: Let G be a graph and $n \in \mathbb{N}$. If D is an n-dependent set of G such that $(n + 1).|D| - q(\langle D\rangle)$ is a maximum among all n-dependent sets D of G, then, by the above theorem,

$$\gamma_{n+1}(G) \leq |D| \leq \beta_n(G).$$

□

As an example of a graph G for which $\gamma_{n+1}(G) = \beta_n(G)$, let $G \cong K_p$, where $n \leq p - 1$; then any $(n + 1)$-subset of $V(G)$ is both a minimum $(n + 1)$-dominating set and a maximum n-dependent set of G. If $G \cong K_{1,n+1}$ $(n \geq 2)$ and S is the set of end-vertices of G, then S is both a minimum n-dominating set and a minimum $(n + 1)$-dominating set in G; furthermore, S is a maximum $(n - 1)$-dependent set of G. So, G is a graph for which $\gamma_{n+1}(G) = \gamma_n(G)$.

5.3.7 Theorem: If G is a graph with $\gamma_n(G) = \gamma_{n+1}(G)$ and D is a minimum $(n + 1)$-dominating set in G, then D is a maximal $(n - 1)$-dependent set of G.

Proof: Let $n \in \mathbb{N}$, let G be a graph for which $\gamma_n(G) = \gamma_{n+1}(G)$, and let D be a minimum $(n + 1)$-dominating set in G. If $u \in V(G) - D$, then, since each vertex of $V(G) - D$ is $(n + 1)$-dominated by D, the star graph $K_{1,n+1}$ is a subgraph of $\langle D \cup \{u\}\rangle$, whence $\Delta(\langle D \cup \{u\}\rangle) \geq n + 1$; i.e., $D \cup \{u\}$ does not possess the property of being $(n - 1)$-dependent for any $u \in V(G) - D$. Hence, it will be sufficient to show that D is an $(n - 1)$-dependent set of G, for then the maximality condition on the $(n - 1)$-dependence of D will follow.

Suppose there exists $u \in D$ such that $|N(u) \cap D| \geq n$ (so, u is n-dominated by $D - \{u\}$). Furthermore, for any $v \in V(G) - D$, since $|N(v) \cap D| \geq n + 1$, it follows that $|N(v) \cap (D - \{u\})| \geq n$. Hence, $D - \{u\}$ n-dominates $\{u\} \cup (V(G) - D)$ (i.e., $D - \{u\}$ is an n-dominating set of G) and so $\gamma_n(G) < |D| = \gamma_{n+1}(G)$; however, this contradicts the hypothesis that $\gamma_n(G) = \gamma_{n+1}(G)$. So, D is indeed a maximal $(n - 1)$-dependent set of G. □

5.3.8 Corollary: If G is a graph for which $\gamma_n(G) = \gamma_{n+1}(G)$, then $\gamma_{n+1}(G) \leq \beta_{n-1}(G)$.

Since every maximal independent set of vertices is a dominating set, Theorem 5.3.7 has a further corollary relating to the independent dominating number i(G).

5.3.9 Corollary: If G is a graph for which $\gamma(G) = \gamma_2(G)$, then every minimum 2-dominating set in G is a maximal independent set and $\gamma(G) = i(G) = \gamma_2(G)$.

Proof: Let G be a graph satisfying $\gamma(G) = \gamma_2(G)$. That every minimum 2-dominating set in G is a maximal independent set of G follows immediately from Theorem 5.3.7. This implies at once (by the definition of i(G)), that $i(G) \leq \gamma_2(G)$. Since a (smallest) maximal independent set is a dominating set, we have $\gamma(G) \leq i(G)$. Then, $\gamma(G) \leq i(G) \leq \gamma_2(G) = \gamma(G)$ yields the desired result. □

The next theorem provides a further relationship between the generalized domination and independence parameters.

5.3.10 Theorem: If G is a graph of order p and maximum degree $\Delta \geq n$, then $\gamma_n(G) \geq p - \beta_{\Delta-n}(G)$.

Proof: Let G be a graph of order p and maximum degree Δ, let D be a minimum n-dominating set of G, and let $S = V(G) - D$. Then, $\Delta(\langle S \rangle) \leq \Delta - n$, i.e., S is a $(\Delta - n)$-dependent set. So, $|S| \leq \beta_{\Delta-n}(G)$, and since $\gamma_n(G) = |D| = p - |S|$, the desired result follows. □

Using Theorem 5.3.10, we may now establish some results concerning n-domination and n-dependence of regular graphs. As an aside, we mention that our first result of this kind, given in Theorem 5.3.11, shows that the lower bound on γ_n provided by Theorem 5.3.10 is, in fact, best possible.

5.3.11 Theorem: If G is an r-regular graph of order p, and $n \leq r$, then $\gamma_n(G) = p - \beta_{r-n}(G)$.

Proof: Let G be an r-regular graph of order p, and let n satisfy $n \leq r$. Let S be a maximum $(r - n)$-dependent set in G, and let $D = V(G) - S$. Then, each vertex u in S is adjacent to $\deg_G u - \deg_{\langle S \rangle} u \geq r - (r - n) = n$ vertices of D. Thus, D is an n-dominating set in G and $\gamma_n(G) \leq |D| = p - \beta_{r-n}(G)$. Since, by Theorem 5.3.10, $\gamma_n(G) \geq p - \beta_{r-n}(G)$, the desired result follows. □

By introducing the chromatic number χ, we may present, in Corollary 5.3.12, another upper bound on γ_n, this time for regular graphs.

5.3.12 Corollary: If G is an r-regular graph of order p and $n \leq r$, then

$$\gamma_n(G) \leq p\left(1 - \frac{1}{\chi(G)}\right).$$

Proof: Let G be an r-regular graph of order p with $n \leq r$. By Remark 5.1.3 and Theorem 5.3.11, we have $\gamma_n(G) \leq \gamma_r(G) = p - \beta G)$. Since $\beta(G) \geq {}^{p}/_{\chi(G)}$, we have

$$\gamma_n(G) \leq \gamma_r(G) = p - \beta(G) \leq p - \frac{p}{\chi(G)},$$

whence the desired inequality follows. □

The chromatic number is not an easily determined quantity for many graphs, and so the (weaker) upper bound provided by the next result is probably more useful in practice.

5.3.13 Corollary: If G is an r-regular graph of order p such that no component of G is a complete graph or an odd cycle and n satisfies $n \leq r$, then $\gamma_n(G) \leq p - {}^p/_r$.

Proof: Let G be an r-regular graph of order p satisfying the hypothesis of the corollary. Since no component of G is a complete graph or an odd cycle, Brooks' Theorem gives $\chi(G) \leq \Delta(G) = r$, and so, by Corollary 5.3.12,

$$\gamma_n(G) \leq p\left(1 - \frac{1}{\chi(G)}\right) \leq p\left(1 - \frac{1}{r}\right) = p - \frac{p}{r}.$$ □

Using Theorem 5.3.11 and Corollary 5.3.12, we can obtain an exact value for γ_n for the large class $K_m \times K_{n+2-m}$ of n-regular graphs ($m \leq \frac{1}{2}(n+2)$).

5.3.14 Theorem: If $m \leq \frac{1}{2}(n + 2)$ and G is the Cartesian product $K_m \times K_{n+2-m}$, then $\gamma_n(G) = p(G) - m$.

Proof: Let $m, n \in \mathbb{N}$ with $m \leq \frac{1}{2}(n + 2)$, and let $G \cong K_m \times K_{n+2-m}$. Clearly, for each $u \in V(G)$, $\deg u = (m - 1) + (n + 2 - m - 1) = n$, i.e., G is n-regular. From a result of Behzad and Mahmoodian [BM1], we have $\chi(G) = n + 2 - m = {}^p/_m$; so, by Corollary 5.3.12,

$$\gamma_n(G) \leq p - p({}^m/_p) = p - m. \qquad \text{(i)}$$

Since V(G) can be partitioned into m subsets $S_1, S_2, \ldots, S_m$, each of which induces a subgraph isomorphic to K_{n+2-m} where each vertex in $\langle S_i \rangle$ is non-adjacent to all but one vertex in $\langle S_j \rangle$ if $i \neq j$ ($i, j \in \{1, 2, \ldots, m\}$), and since $m \leq \frac{1}{2}(n + 2)$ implies $m \leq n + 2 - m$, we see that $\beta(G) = \min\{m, n + 2 - m\} = m$. So, by Theorem 5.3.11,

$$\gamma_n(G) = p - \beta_0(G) = p - \beta(G) \geq p - m. \tag{ii}$$

The theorem now follows from (i) and (ii). □

5.3.15 Remark: We conclude this section by remarking that, since $\chi(G) = {}^{p}/_{m}$ for the graph $K_m \times K_{n+2-m}$ if $m \leq \frac{1}{2}(n + 2)$, Theorem 5.3.14 shows that the bound on $\gamma_n(G)$ given in Corollary 5.3.12 is sharp. In particular, if $m = 2$, then $K_m \times K_{n+2-m} = K_2 \times K_n$ and if $G \cong K_2 \times K_n$ and $p = p(G) = 2n$, then $\gamma_n(G) = p - 2 = p - {}^{p}/_{n}$, which shows that the bound given in Corollary 5.3.13 is best possible.

The following is an open problem concerning the rate at which the n-domination number increases as n increases.

Problem: Find a sharp bounding function $f(n)$ such that, if G is a graph with $\delta(G) \geq n$ and $m \geq f(n)$, then $\gamma_n(G) < \gamma_m(G)$.

5.4 CLAW-FREE GRAPHS AND GENERALIZED INDEPENDENT DOMINATION NUMBERS

The relationship between the domination number γ and other graphical parameters has been the subject of a fair amount of investigation. The independent domination number i, in particular, has been studied in relation to γ. By the definition of i, we have $\gamma(G) \leq i(G)$ for every graph G; what has received a lot of attention (see, for example, [AL1]) is the question of which graphs G satisfy $\gamma(G) = i(G)$. Such graphs are significant since the task of finding a minimum dominating set for such graphs is reduced to the task of finding a smallest maximally independent set in the graphs. In [AL1], Allan and Laskar presented a "forbidden subgraph" condition on a graph G that is sufficient to ensure $\gamma(G) = i(G)$:

5.4.1 Theorem: If G is a graph which does not have an induced subgraph isomorphic to $K_{1,3}$, then $\gamma(G) = i(G)$.

We omit the proof since the theorem is proved in a more general context in Theorem 5.4.9.

In the spirit of this result, we shall, in this section, investigate the relationship between $\gamma_n(G)$ and $\beta_k(G)$ for graphs which fail to contain certain induced subgraphs; such a forbidden subgraph that will receive the most consideration is the star $K_{1,3}$. In particular, we shall present a generalization of Theorem 5.4.1 in terms of n-domination and n-dependence.

The following definition will prove useful.

5.4.2 Definition: Let G, H_1, H_2, ..., H_m be graphs. We say that G is *$(H_1,H_2,\ldots,H_m)$-free* whenever G contains no induced subgraph isomorphic to any of H_1, H_2, ..., or H_m

In [FJ3], the following conjecture is made by Fink and Jacobson.

5.4.3 Conjecture: If G is a graph with $\delta(G) \geq n$, then $\gamma_n(G) < \gamma_{2n+1}(G)$.

That the conjecture is, in fact, false was shown by Dick Schelp (personal communication), who constructed a graph G with $\delta(G) \geq n$ and $\gamma_n(G) = \gamma_{\frac{1}{4}(n^2+n)}(G)$ (from which, of course, it follows that $\gamma_n(G) = \ldots = \gamma_{2n+1}(G) = \ldots. = \gamma_{\frac{1}{4}(n^2+n)}(G)$, if $n \geq 8$). However, the following theorem shows that the conclusion of the conjecture does hold if we demand that G is $K_{1,3}$-free.

5.4.4 Theorem: If G is a $K_{1,3}$-free graph with $\Delta(G) \geq n$, then $\gamma_n(G) < \gamma_{2n}(G)$.

Proof: Let G be a graph of order p satisfying the hypothesis of the theorem. We consider three cases.

Case 1: Suppose that $\Delta(G) \leq 2n - 1$. Then, obviously, no proper subset of V(G) is a 2n-dominating set of G. Furthermore, $\Delta(G) \geq n$ implies that a smallest n-dominating set of G is a proper subset of V(G) (since, if u is any vertex of G with $\deg_G u = \Delta(G)$, then $V(G) - \{u\}$ is an n-dominating set of G). Hence,

$$\gamma_n(G) < p = \gamma_{2n}(G).$$

Case 2: Suppose $\Delta(G) \geq 2n$. Let D be a 2n-dominating set. Then, D is an n-dominating set which we show is not minimal. As above, we can show $\gamma_{2n}(G) < p$, so $V(G) - D \neq \emptyset$. Let $x \in V(G) - D$ and consider the graph $H_x = \langle N(x) \cap D \rangle$, which has order at least 2n. If H_x contains a set A of three independent vertices, then

$\langle\{x\} \cup A\rangle \cong K_{1,3}$, which contradicts our assumption about G. Hence, any independent set of vertices of H_x contains at most two vertices.

<u>Subcase 2.1</u>: Suppose that $\Delta(\langle D\rangle) \leq n - 1$. Now, if $v \in V(H_x)$, then, since H_x is a subgraph of $\langle D\rangle$,

$$\deg_{H_x} v \leq \deg_{\langle D\rangle} v \leq \Delta(\langle D\rangle) \leq n - 1,$$

so that

$$\deg_{\overline{H_x}} v = p(H_x) - 1 - \deg_{H_x} v \geq 2n - 1 - (n - 1) = n,$$

i.e., there are at least n vertices in H_x non-adjacent in H_x to v. Furthermore, no pair y, z of these at least n vertices can be non-adjacent, since, otherwise, $\{v, y, z\}$ is an independent subset of H_x containing more than two vertices, which we have shown is not possible. So, $\langle N_{H_x}(v)\rangle$ is complete and $\langle N_{H_x}[v]\rangle \cong K_m \subset H_x$ where $m \geq n + 1$, whence $\Delta(\langle D\rangle) \geq \Delta(H_x) \geq n$, a contradiction.

<u>Subcase 2.2</u>: Suppose $\Delta(\langle D\rangle) \geq n$. Then, for any vertex u of D with $\deg_{\langle D\rangle} u \geq n$, $D - \{u\}$ is an n-dominating set of G (since any vertex of $V(G) - D$ that is adjacent to u is adjacent to n other vertices of D, and u had n neighbours in $D - \{u\}$). So, as in the previous case, D is not a minimal n-dominating set.

So, since D is n-dominating set of G that is not *minimal*, D is not a *minimum* n-dominating set of G, and so $\gamma_n(G) < |D| = \gamma_{2n}(G)$. □

The following result is an immediate consequence of Theorem 5.4.4 and Remark 5.1.3.

5.4.5 <u>Corollary</u>: If G is a $K_{1,3}$-free graph with $\Delta(G) \geq n$, then $\gamma_n(G) < \gamma_{2n+1}(G)$.

5.4.6 <u>Remark</u>: The result of Theorem 5.4.4 is best possible in the sense that $\gamma_n(G) < \gamma_{2n-1}(G)$ is not true for all claw-free graphs G with $\Delta(G) \geq n$. For example, let $G \cong K_{n+1} \bullet K_n$; then, G is $K_{1,3}$-free with $\Delta(G) \geq n$ and, if u is the coalesced vertex of G, $V(G) - \{u\}$ is the only n-dominating set and the only $(2n - 1)$-dominating set of G, so that $\gamma_n(G) = \gamma_{2n-1}(G) = p - 1 = 2n - 1$. On the other hand, we note that there are graphs which are *not* $K_{1,3}$-free and which have $\gamma_n(G) = \gamma_{2n}(G)$; for instance, let $G \cong K_{2n,2n}$, $n \geq 2$. Then, $\Delta(G) = 2n \geq n$ and, for any 2n-subset $A \cup B$ of V(G), where A and B lie in distinct partite sets of G and either $|A| = |B| =$

n, or $|A| = 2n$ and $B = \varnothing$, $A \cup B$ is a smallest subset of V(G) that n-dominates G, and for any partite set C (which has cardinality 2n), C is a smallest subset of V(G) that 2n-dominates G. So, $\gamma_n(G) = 2n = \gamma_{2n}(G)$.

The following result is reminiscent of a result we used in Proposition 4.1.2.

5.4.7 Proposition: For a graph G and $n \in \mathbb{N}$, an n-dominating set D of G is minimal if and only if, for each $v \in D$

(i) $|N(v) \cap D| < n$, or

(ii) there exists $x \in V(G) - D$ such that $|N(x) \cap D| = n$ and $v \in N(x)$.

Proof: Suppose, to the contrary, that there exist a graph G, $n \in \mathbb{N}$, a minimal n-dominating set D of G, and $v \in D$ such that $|N(v) \cap D| \geq n$ and for every $x \in V(G) - D$, either $|N(x) \cap D| > n$ or $v \notin N(x)$. Then, v is adjacent to at least n vertices of $D - \{v\}$ and each vertex $u \in V(G) - D$ is such that $|N(u) \cap (D - \{v\})| = |N(u) \cap D| - 1 \geq n$ (if $v \in N(u)$) or $|N(u) \cap (D - \{v\})| = |N(u) \cap D| \geq n$ (if $v \notin N(u)$); so, $D - \{v\}$ is an n-dominating set of G. However, this contradicts the minimality of D. Thus, (i) and (ii) follow if D is a minimal n-dominating set of G.

Conversely, let G be a graph, let $n \in \mathbb{N}$, and suppose that D is an n-dominating set such that, for each $v \in D$, $|N(v) \cap D| < n$ or there exists $x \in V(G) - D$ such that $|N(x) \cap D| = n$ and $v \in N(x)$. Let $v \in D$. If $|N(v) \cap D| < n$, then $D - \{v\}$ is not n-dominating in G; if there is $x \in V(G) - D$ such that $|N(x) \cap D| = n$ and $v \in N(x)$, then x is adjacent to exactly $n - 1$ vertices of $D - \{v\}$. So, in neither case is $D - \{v\}$ an n-dominating set of G. Hence, since v is an arbitrary element of D, it follows that D is a minimal n-dominating set of G. □

The rest of this section is devoted to the presentation of results relating the concepts of n-domination and m-dependence. We introduce the following definition.

5.4.8 Definition: For a non-negative integer j, positive integer n and graph G, we define the j-dependent-n-domination number i(j,n;G) to be the cardinality of the smallest j-dependent, n-dominating set of G, provided that such a set exists.

5.4.9 Theorem: If a graph G is $K_{1,3}$-free, then $i(2n-2,n;G) = \gamma_n(G)$ (i.e., G has a smallest n-dominating set that is also $(2n - 2)$-dependent).

Proof: Let G be a graph that is $K_{1,3}$-free, and let $n \in \mathbb{N}$. We establish the desired result by proving that G possesses a $(2n-2)$-dependent, n-dominating set of G, which shows that $i(2n-2,n;G)$ exists and that $\gamma_n(G) \leq i(2n-2,n;G)$, and, secondly, by showing that $i(2n-2,n;G) \leq \gamma_n(G)$. We achieve both these tasks by showing that G contains minimum n-dominating sets that are $(2n-2)$-dependent.

Let D be a minimum n-dominating set of G such that $\langle D\rangle$ has as few edges as possible among all minimum n-dominating sets of G. For $z \in V(G)$, let N_z denote the set $N(z) \cap D$. If D is $(2n-2)$-dependent, then the proof is complete, so we will suppose that there exists $v \in D$ such that $|N(v) \cap D| \geq 2n-1$. Since D is a minimal n-dominating set, it follows, by Proposition 5.4.7 (and the fact that $|N(v) \cap D| \geq n$), that there exists $y \in V(G) - D$ such that $|N(y) \cap D| = n$ and $y \in N(v)$. So, the set $T \subseteq V(G) - D$ defined by $T = \{x \in V(G) - D;\ |N(x) \cap D| = n \text{ and } x \in N(v)\}$ is non-empty. We show now that $\langle T\rangle$ is complete. If $|T| = 1$, then, trivially, $\langle T\rangle$ is complete; so suppose that $|T| \geq 2$, and let x, y be distinct elements of T. Now, $|N_x| = |N_y| = n$ and $v \in N_x \cap N_y$ and $|N_x \cap N_y| \leq 2n-1$. So, $|N_v \cap (N_x \cap N_y)| \leq 2n-2$; however, $|N_v| \geq 2n-1$. Consequently, there exists $z \in N_v - (N_x \cup N_y)$. So, xz, $yz \in E(\overline{G})$ which implies, since $\langle\{v, x, y, z\}\rangle \not\cong K_{1,3}$, that $xy \in E(G)$. The vertices x and y are arbitrary elements of T, so $\langle T\rangle$ is complete.

Now, let x be any vertex in T, and let $D' = (D - \{v\}) \cup \{x\}$. Then,

$$E(\langle D'\rangle) = (E(\langle D\rangle) - [\{v\}, D]) \cup [\{x\}, D - \{v\}],$$

so that

$$q(\langle D'\rangle) = q(\langle D\rangle) - |N_v| + |N_x - \{v\}| \leq q(\langle D\rangle) - (2n-1) + (n-1) = q(\langle D\rangle) - n < q(\langle D\rangle),$$

i.e., $\langle D'\rangle$ contains fewer edges than $\langle D\rangle$. Thus, by our choice of D and the fact that $|D'| = |D|$, it follows that D′ is not an n-dominating set of G, i.e., there must exist a vertex $p \in V(G) - D'$ that is adjacent to fewer than n vertices of D′. By the definition of D and T, either $p = v$ or p belongs to $T - \{x\}$. However, v is adjacent to $|N_v| + |\{x\}| \geq 2n$ vertices of D′, so $p \neq v$. Furthermore, for any $w \in T - \{x\}$, $|N_G(w) \cap D'| = |N_w - \{v\}) \cup \{x\}| = (n-1) + 1 = n$ (since D is n-dominating and $\langle T\rangle$ is complete), which means $p \notin T - \{x\}$. This is a contradiction. Thus, our assumption that D is not $(2n-2)$-dependent is false, and so D is a $(2n-2)$-dependent, n-dominating set of G, and (by our introductory comments) $i(2n-2,n;G) = \gamma_n(G)$. □

5.4.10 Remark: Notice that Theorem 5.4.1 follows as a corollary from Theorem 5.4.9. Furthermore, as in [AL1], related results pertaining to L(G), the line graph of G, follow as corollaries to Theorem 5.4.9.

5.4.11 Corollary: For any graph G and $n \in \mathbb{N}$, $\gamma_n(L(G)) = i(2n-2,n;L(G))$.

Proof: The corollary follows immediately from Theorem 5.4.9 and the result which states: A graph H is a line graph if and only if (a) $K_{1,3}$ is not an induced subgraph of H, and (b) if $K_{1,1,2}$ is an induced subgraph of H, then at least one of its two triangles is even (cf. [CL1]). □

5.4.12 Theorem: Let $n \in \mathbb{N}$, $H \cong K_{1,3}$ and let $e \in E(\overline{H})$. Then, if G is (H,H+e)-free, then $i(n-1,n;G) = \gamma_n(G)$.

Proof: Let G and H be graphs satisfying the hypothesis of the theorem, and let $n \in \mathbb{N}$. As in the proof of Theorem 5.4.9, we shall establish the desired result by proving that G contains an n-dominating, (n - 1)-dependent set, from which it will follow that $i(n-1,n;G)$ exists and that $\gamma_n(G) \leq i(n-1,n;G)$, and, finally, by showing that $i(n-1,n;G) \leq \gamma_n(G)$. We achieve both these objectives by showing that G contains a minimum n-dominating set that is (n - 1)-dependent.

Select a minimum n-dominating set D of G such that $q(\langle D\rangle) \leq q(\langle D^*\rangle)$ for all minimum n-dominating sets D^* of G. For any $z \in V(G)$, let N_z denote the set $N(z) \cap D$. If D is (n - 1)-dependent, the theorem is proved, so we assume now that D is not (n - 1)-dependent. Let $v \in D$ such that $|N(v) \cap D| \geq n$. It follows, then, by Proposition 5.4.7, that (since D is a minimal n-dominating set) there exists $x \in V(G) - D$ such that $|N(x) \cap D| = n$ and $v \in N(x)$. So (as in the proof of Theorem 5.4.9), the set T defined by $T = \{x \in V(G) - D;\ |N(x) \cap D| = n$ and $x \in N(v)\}$ is non-empty. If $|T| = 1$, then (trivially) $\langle T\rangle$ is complete; suppose now that $|T| \geq 2$, and let x, y be distinct elements of T. Now, $|N_x| = n$ and $v \in N_x$, so $|N_x \cap N_y| \leq |N_x| - 1 = n - 1$, while $|N_y| \geq n$. Hence, there must exist $z \in N_y - N_x$. Clearly, $zx \notin E(G)$. Then, since $\langle\{x, y, z, v\}\rangle$ is not isomorphic to H ($\cong K_{1,3}$) or to H+e, we have $xy \in E(G)$. Since, x, y are arbitrary elements of T, it follows that $\langle T\rangle$ is complete.

Let $D' = (D - \{v\}) \cup \{x\}$. As in the proof of Theorem 5.4.9, we have that $q(\langle D'\rangle) < q(\langle D\rangle)$, which proves, by our choice of D, that D′ is not n-dominating. So, there exists $p \in V(G) - D'$ such that $|N(p) \cap D'| < n$. By the definition of D and T, either $p = v$ or $p \in T - \{x\}$. However, $|N(v) \cap D'| = |N_v \cup \{x\}| = n + 1 > n$, i.e., $p \neq v$, and, for any $w \in T - \{x\}$,

$|N(w) \cap D'| = |(N_w - \{v\}) \cup \{x\}| = n$, i.e., $p \notin T - \{x\}$. This contradiction establishes the theorem. □

Another result in a similar vein is the following.

5.4.13 Theorem: Let $G_1, G_2 \cong K_3$, let $B_1 = G_1 \bullet G_2$, and let $B_2 = B_1 + v_1v_2$, where $v_1v_2 \in E(\overline{B})$ with $v_i \in V(G_i)$, $i = 1, 2$. Then, if G is a $(K_{1,3}, B_1, B_2)$-free graph and $n \in \mathbb{N}$, then $i(n-1,n;G) = \gamma_n(G)$.

Proof: Let G, B_1, and B_2 be graphs satisfying the hypothesis of the theorem, and let $n \in \mathbb{N}$. Using the same technique employed in the proof of Theorems 5.4.9 and 5.4.12, we prove $i(n-1,n;G) = \gamma_n(G)$ by finding minimum n-dominating sets that are $(n-1)$-dependent. Let D be a minimum n-dominating set of G such that $\langle D \rangle$ has the smallest possible size. If D is $(n - 1)$-dependent, the result follows, so we suppose now that D is *not* $(n - 1)$-dependent. Then, there exists $v \in D$ such that $|N(v) \cap D| \geq n$. By Proposition 5.4.7, there must exist $x \in V(G) - D$ such that $|N(x) \cap D| = n$ and $v \in N(x)$. Again, let T be the set of all such vertices x. If $|T| = 1$, then $\langle T \rangle$ is complete, so suppose $|T| \geq 2$, and consider any two distinct elements x, y of T. We show that $xy \in E(G)$. By the same reasoning used in the proof of Theorem 5.4.12, there exists $x_1 \in N(v) \cap D$ such that $xx_1 \notin E(G)$. Similarly, there exists $y_1 \in N(v) \cap D$ such that $yy_1 \notin E(G)$. If $x_1 = y_1$, then, since $\langle \{v, x, y, x_1\} \rangle \not\cong K_{1,3}$, we have $xy \in E(G)$. On the other hand, suppose that $x_1 \neq y_1$; assume, to the contrary that $xy \notin E(G)$. By the definitions of x_1 and y_1, we have $\langle \{x_1, y_1, x, y, v\} \rangle \cong K_3 \bullet K_3$ $(\cong B_1)$ or $\langle \{x_1, y_1, x, y, v\} \rangle \cong B_2$, a contradiction. So, $xy \in E(G)$. Since x and y are arbitrary, distinct elements of T, it follows that $\langle T \rangle$ is complete. Now, by considering the set $(D - \{v\}) \cup \{x\}$, we may derive a contradiction exactly as we did in the proof of Theorem 5.4.12. □

5.5 CONJECTURES AND UNSOLVED PROBLEMS

In [FJ1], a number of interesting problems related to the parameters γ_n and β_n, as well as $i(j,n;G)$ for a graph G are listed. For example:

Question 1: Are there other classes of graphs G, perhaps also characterized by forbidden induced subgraphs, for which relationships between $\gamma_n(G)$, $i(k,n;G)$ and $\beta_m(G)$ exist?

Question 2: Are there other parameters which naturally relate to $\gamma_n(G)$, i(k,n;G) or $\beta_m(G)$?

Question 3: It was shown in [JP2] that the problems of deciding whether (for a given graph G and integers k and n) $\gamma_k(G) \leq n$ or $\beta_k(G) \leq n$ are NP-complete. In [JP2], linear-time algorithms (in the number of vertices) are developed to determine $\gamma_k(G)$ and $\beta_k(G)$ for the cases occurring if G is a tree or a series-parallel graph. It remains an open problem to find linear-time or polynomially bounded algorithms to determine these parameters for other classes of graphs.

Related to this problem is the establishment of upper and lower bounds on $\gamma_k(G)$ for classes of graphs for which no such algorithms can be found. It is known that, for a graph G,

$$\gamma_k(G) \leq \frac{k\,p(G)}{\Delta(G) + k}$$

[FJ1] and, if $\delta(G) \geq k$, then

$$\gamma_k(G) \geq \frac{k\,p(G)}{k+1}$$

[CGS1]. The latter result was slightly extended in [CR1]. It may be worthwhile to seek to improve these bounds for selected classes of graphs.

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