Asymptotic Analysis of Singularly Perturbed Dynamical Systems

by

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As the candidate's supervisor I have approved this thesis for submission.

Signed: .... Name: J. BANASIAK .... Date: 08/11/2011

Preface

The work described in this thesis was carried out in the School of Mathematical Sciences, University of KwaZulu-Natal, Durban, under the supervision of Professor Jacek Banasiak.

These studies represent original work by the author and have not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others it is duly acknowledged in the text.

Abstract

According to the needs, real systems can be modeled at various level of resolution. It can be detailed interactions at the individual level (or at microscopic level) or a sample of the system (or at mesoscopic level) and also by averaging over mesoscopic (structural) states; that is, at the level of interactions between subsystems of the original system (or at macroscopic level).

With the microscopic study one can get a detailed information of the interaction but at a cost of heavy computational work. Also sometimes such a detailed information is redundant. On the other hand, macroscopic analysis, computationally less involved and easy to verify by experiments. But the results obtained may be too crude for some applications.

Thus, the mesoscopic level of analysis has been quite popular in recent years for studying real systems. Here we will focus on structured population models where we can observe various level of organization such as individual, a group of population, or a community. Due to fast movement of the individual compare of the other demographic processes (like death and birth), the problem is multiple-scale.

There are various methods to handle multiple-scale problem. In this work we will follow asymptotic analysis (or more precisely compressed Chapman–Enskog method) to approximate the microscopic model by the averaged one at a given level of accuracy.

We also generalize our model by introducing reducible migration structure. Along with this, considering age dependency of the migration rates and the mortality rates, the thesis offers improvement of the existing literature.

DECLARATION 1 - PLAGIARISM

I, AMARTYA GOSWAMI declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.
2. This thesis has not been submitted for any degree or examination at any other university.
3. This thesis does not contain other person's data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.
4. This thesis does not contain other person's writing, unless specifically acknowledged as being sourced from other researchers. Where other written sources have been quoted, then:
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DECLARATION 2 - PUBLICATIONS

Paper published

1. Banasiak, J., Goswami, A., Shindin, S., Aggregation in age and space structured population models: An asymptotic approach, *Journal of Evolution Equations*, **11**(1) (2011), 121–154.

2. Banasiak, J., Goswami, A., Shindin, S., Asymptotic analysis of structured population models, *Numerical Analysis and Applied Mathematics:* AIP Conference Proceedings, **1048** (2008), 5–8.

Paper submitted

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Papers in preparation

1. J. Banasiak, A. Goswami, Aggregation in structured population models - the case of a reducible migration matrix.

Signed: ....

I dedicate this dissertation to

Lord Ganesh

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Contents

CHAPTER 1

Introduction

The history of mathematical studies of population modeling is quite long. In 1798, Malthus [41] proposed a model of population dynamics, and a more realistic model of population growth was proposed by Verhulst [61] in 1838. The models of Malthus and Verhulst are examples of continuous or deterministic population models. For many populations, consideration of the age distribution within the population leads to a more realistic and useful mathematical model. One of the crucial steps in this line is by Sharpe and Lotka [57] in 1911 and McKendrick [43] in 1926. With time many more models came and things were getting more and more challenging due to the complicated structures of the models. It is therefore important to investigate systematic simplification methods for this class of models. One would like to elucidate under which conditions the general problem can be simplified in such a way that the essential information one would like to obtain from the model is not lost.

Here, we are interested in three natural phenomena related to population. Namely, birth, death and migration of the population from one place to another. The model we are going to investigate was proposed in [5], where the model was considered for fish population. The special feature about this model is that it is a *singularly perturbed* (which reflects different time scales) initial boundary valued problem of *resonance type*.

The idea of the time scale argument is that when one time-scale is very fast, as compared to the other time-scale inherent in the system, one assumes that the process on the fast time-scale is actually in equilibrium at all time (however, this equilibrium changes slowly as the slower process change in time). This then leads to a system of differential equations of lower dimension, capturing the essentials of the original bigger system. The equilibrium assumption is usually called the quasi-steady-state hypothesis. The mathematical counterpart of this intuitive notion starts by scaling the original system of differential equations in such a way that it can be rewritten as a singular perturbation problem.

In order to start our discussion, we are going to set up an adequate work frame. We consider two equations:

$$\text{equation } A_0 : L_0 u = f_0$$

$$\text{equation } A_\epsilon : L_0 u + \epsilon L_1 u = f_0 + \epsilon f_1$$

Here L_0 and L_1 are given operators, f_0 and f_1 are known functions, ϵ is a small scalar parameter (we will consider $\epsilon > 0$), u is the unknown function of the independent variable t. Equation A_0 might be a simplified model of some process and then equation A_ϵ corresponds to the extended model. The terms $\epsilon L_1 u$ and ϵf_1 represent perturbations. If A_0 and A_ϵ are differential equations, we must add necessary initial (and/or boundary) conditions. These conditions might also contain a small parameter ϵ. We denote the solution of A_0 by $u_0(t)$ and the solution of A_ϵ by $u_\epsilon(t)$ for $t \in D$, where D is some domain. The main question of perturbation theory might be posed as follows:
does the difference $u_\epsilon(t) - u_0(t)$ approach zero (in some norm space) as $\epsilon \to 0$? The answer to this question depends also on the choice of the norm.

The problem A_ϵ is called *regularly perturbed* in a domain D if there exists a solution $u_0(t)$ of the problem A_0 such that

$$\sup_D \|u_\epsilon(t) - u_0(t)\| \to 0 \quad \text{when} \quad \epsilon \to 0.$$

Otherwise, A_ϵ is said to be *singularly perturbed* with respect to the same norm. It follows from the definition that for a singularly perturbed problem A_ϵ, $u_0(t)$ will not be close to u_ϵ for all small ϵ at least in some part of domain D.

Consider a function $U_\epsilon(t)$ defined in a sub domain D_1 of D. The function $U_\epsilon(t)$ is called an *asymptotic approximation* of the solution u_ϵ with respect to the parameter ϵ in the sub domain D_1 if

$$\|u_\epsilon(t) - U_\epsilon(t)\| \longrightarrow 0 \quad \text{as} \quad \epsilon \to 0.$$

Moreover, if $\|u_\epsilon(t) - U_\epsilon(t)\| = O(\epsilon^k)$ for all $t \in D_1$ then we say that $U_\epsilon(t)$ is the asymptotic approximation of $u_\epsilon(t)$ in D_1 to within accuracy of the order ϵ^k.

On domain D, the function U_ϵ may fail to satisfy the condition $\|u_\epsilon(t) - U_\epsilon(t)\| = O(\epsilon^k)$. The reason is the existence of the initial layer in the vicinity of $t = 0$ and/or boundary layer for a boundary value problem, extending over the interval of the order ϵ. To handle this, we introduce a new time variable $\tau = t/\epsilon$ and we do series expansion similar as before. For linear problems, the new solution becomes the sum of the original approximate solution and this initial layer. For an initial boundary value problem, we also add the corresponding boundary layer correction. Among different methods, in our work we follow the compressed method which is a modified Chapman–Enskog method. For details about this method, we refer to [11].

1.1 Outline of the Thesis

The aim of this thesis is to do a rigorous error analysis needed due to approximation of a mesoscopic age and space structured population model to the corresponding aggregated one using asymptotic method. We also consider a model with reducible mixing structure which is an improvement, from analysis point of view, of the existing literature.

In Chapter 2, we present a brief summary of the necessary mathematical tools that we use. Starting from vector spaces and matrices with non-negative elements, we give an outline of calculus on Banach spaces and the necessary properties of C_0 type semigroups of linear operators. We also include a short section on the well-posedness problem of evolution type equations.

Chapter 3 is a short overview of deterministic population models. Starting with continuous-time model of Sharpe and Lotka, we talk next about its limitations and development of McKendrick–Von Foerster Model. Here we find the integral solution of this model using the method of characteristics. Specifically we look on the properties of the characteristic equation. For long time behaviour of this model, several methods have been used. Among these, we refer to [65] for semigroup theoretic approach. We finish this chapter with the concept of the multiregional demography, which is one of our main subsequent matter of study.

We introduce our main model in Chapter 4, it is basically a singularly perturbed initial boundary valued vector equation of resonance type. Using Perron–Frobenius Theorem, we show that the irreducible migration matrix $\mathbf{C}(a)$ with Kolmogorov type has zero as a dominant eigenvalue and rest of the eigenvalues have negative real parts. Following [28], next we shall prove the well-posedness of the model, i.e., the existence of a C_0-semigroup $\{\mathbf{T}(t)\}_{t\geq 0}$ generated by $\mathfrak{S} - \mathbf{M}(a) + \frac{1}{\epsilon}\mathbf{C}(a)$ using Hille-Yosida theorem. We also analyze further spectral properties of this C_0- semigroup. Along with this, we give an estimate of the norm of $\{\mathbf{T}(t)\}_{t\geq 0}$.

In Chapter 5, we first approximate our perturbed model and then we derive formulae for the asymptotic expansion. We project the system of equations into the subspace generated by the eigenvector corresponding to the zero eigenvalue of $\mathbf{C}(a)$, usually called the 'hydrodynamic space' and to the complementary subspace, called the 'kinetic space'. We first do bulk part approximation which however is not sufficient to handle initial condition and hence we proceed towards initial layer correction by blowing the time parameter. This introduces some problematic terms on the boundary. To get rid of this problematic terms, we similarly do boundary layer corrections. Both of these necessitate corner layer corrections. We postpone the detail error analysis until Chapter 7, where we do it with integral formulation.

We generalize the perturbed model in different directions in Chapter 6. First we take full mortality and birth matrix and show that we obtain a similar system to that found at the beginning of Chapter 5. Later, we consider a reducible migration matrix along with general mortality and birth matrices. We construct proper bases for defining projection operators. After stating necessary hypotheses, we prove necessary lifting theorems required for the error analysis. Next we do formal asymptotic expansions, similar to Chapter 5. We point out the limitations of the differential equation presentations of the error equations and do integral formulation of the problem in Chapter 7.

Chapter 7 deals with the integral formulation of the perturbed model and its asymptotic analysis in integral form. This way we need less assumptions on the model and also overcome some of the technical difficulties.

With an illustrative example, in Chapter 8, we demonstrate the application of our compressed Enskog–Chapman method. We take a reducible migration matrix with some simple numerical values and construct bases for the null space and the adjoint null space of the migration matrix. We do formal asymptotic expansion and work out the error analysis including all the layer corrections.

In Chapter 9 we summarize the results obtained in Chapters 5, 6 and 7. We discuss further possibilities of generalizations of the model and compare our work with the existing literature.

CHAPTER 2

Mathematical Framework

In this chapter we discuss the standard properties of linear algebra, normed vector spaces and operator semigroups. In the text, we give precisely those results which are necessary for our subsequent work. To incorporate all the theory in detail in the text would be extremely oppressive and would obscure the principal lines of thought inherent in the basic aspects of the subject. We refer to [45], [34], [29], [35], [20] and [21] for detailed study.

2.1 Eigenvalues and Eigenvectors

For an $n \times n$ matrix A, scalars λ and vectors $x \neq 0$ satisfying $Ax = \lambda x$ are called *eigenvalues* and *right eigenvectors* of A, respectively. The set of distinct eigenvalues, denoted by $\sigma(A)$, is called the *spectrum* of A.

- $\lambda \in \sigma(A)$ if and only if $A - \lambda I$ is singular if and only if $\det(A - \lambda I) = 0$.
- $\{x \neq 0 : x \in N(A - \lambda I)\}$ is the set of all eigenvectors associated with λ, where $N(A - \lambda I) := \{x : (A - \lambda I)x = 0\}$. $N(A - \lambda I)$ is called an *eigenspace* for A.
- Non-zero row vectors y^* such that $y^*(A - \lambda I) = 0$ are called *left eigenvectors* for A.
- $\det(A - \lambda I)$ is called the *characteristic polynomial* for A and $\det(A - \lambda I) = 0$ is the corresponding *characteristic equation* for A.

- Let us consider a vector v with n components, namely $(v_1, \dots, v_n)$. Then $v \geq 0$ means each of $v_1, \dots, v_n$ is non-negative and at least one of these is non-zero.

Let $\lambda \in \sigma(A)$. The *algebraic multiplicity* of λ is the number of times it is repeated as a root of the characteristic polynomial. When $algmult_A(\lambda) = 1, \lambda$ is called a *simple eigenvalue*. The *geometric multiplicity* of λ is $\dim N(A - \lambda I)$. Eigenvalues such that $algmult_A(\lambda) = geomult_A(\lambda)$ are called *semisimple eigenvalues* of A.

For every singular matrix $A_{n \times n}$, there exists a positive integer k such that range of A^k, $R(A^k)$ and $N(A^k)$ are complementary subspaces; that is,

$$\mathbb{R}^n = R(A^k) \oplus N(A^k). \tag{2.1}$$

The smallest positive k for which (2.1) holds is called the index of A. For nonsingular matrices we define $index(A) = 0$.

Jordan Form

Following [45], for every $A \in \mathbb{C}^{n \times n}$ with distinct eigenvalues $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$, there is a non-singular matrix P such that

$$P^{-1}AP := J := \begin{bmatrix} J(\lambda_1) & 0 & \cdots & 0 \\ 0 & J(\lambda_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J(\lambda_s) \end{bmatrix}$$

J has one Jordan segment $J(\lambda_j)$ for each eigenvalue $\lambda_j \in \sigma(A)$. Each segment $J(\lambda_j)$ is made up of $t_j = \dim N(A - \lambda_j I)$ Jordan blocks $J_*(\lambda_j)$ as described below.

$$J(\lambda_j) := \begin{bmatrix} J_1(\lambda_j) & 0 & \cdots & 0 \\ 0 & J_2(\lambda_j) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{t_j}(\lambda_j) \end{bmatrix} \quad \text{with} \quad J_*(\lambda_j) := \begin{bmatrix} \lambda_j & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_j \end{bmatrix}.$$

The largest Jordan block in $J(\lambda_j)$ is $k_j \times k_j$, where $k_j = index(\lambda_j)$. The number of $i \times i$ Jordan blocks in $J(\lambda_j)$ is given by

$$\nu_j(\lambda_j) = r_{i-1}(\lambda_j) - 2r_i(\lambda_j) + r_{i+1}(\lambda_j) \text{ with } r_i(\lambda_j) = rank((A - \lambda_j I)^i).$$

The matrix J is called the *Jordan form* for A. The structure of this form is unique in the sense that the number of Jordan segments in J as well as the number of sizes of the Jordan blocks in each segment is uniquely determined by the entries in A.

2.2 Matrices with Non-negative Elements

Definition 2.1 [24, p. 50], *We will call a matrix A with real elements*

$$A = [a_{ik}] \qquad (1 \leq i \leq m;\ 1 \leq k \leq n)$$

non-negative $A \geq 0$ or positive $A > 0$, if all the elements of the matrix A are non-negative (respectively positive): $a_{ik} \geq 0$ (respectively > 0).

Definition 2.2 [24, p. 50] *The matrix $A = [a_{ik}]_{1 \leq i,k \leq n}$ is called reducible, if there is a permutation of the indices which reduces it to the form*

$$\tilde{A} := \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B and D are square matrices. Otherwise the matrix A is called irreducible.

Theorem 2.3 (Frobenius) *An irreducible non-negative matrix $A = [a_{ik}]_{1 \leq i,k \leq n}$ always has a positive simple eigenvalue λ (also known as dominant eigenvalue, Perron root or Frobenius root). The moduli of all the other eigenvalues are at most λ. There is an eigenvector x with positive coordinates that corresponds to λ.*

Proof. See [24, p. 53]. □

Corollary 2.4 *The irreducible matrix $A \geq 0$ cannot have two linearly independent non-negative eigenvectors corresponding to the dominant eigenvalue λ.*

Proof. See [24, p. 63]. □

The spectral properties of irreducible non-negative matrices described in previous theorem are not valid for reducible matrices. For an arbitrary non-negative matrix $A = [a_{ik}]_{1 \leq i,k \leq n}$ we have the following theorem:

Theorem 2.5 *A non-negative matrix $A = [a_{ik}]_{1 \leq i,k \leq n}$ always has a non-negative eigenvalue λ such that no eigenvalue of the matrix A has modulus exceeding λ. To this "dominant" eigenvalue λ there corresponds a non-negative eigenvector y:*

$$Ay = \lambda y \qquad (y \geq 0,\ y \neq 0).$$

Proof. See [24, p. 66]. □

2.3 Normed Vector Spaces

A *vector space* X is a set of elements, called *vectors*, $u, v, \ldots$, for which linear operations (addition $u + v$ of two vectors and multiplication αu of a vector u by a scalar α) are defined and obey the usual rules of such operations. For the rest of the work, scalars are assumed to be the set of complex numbers $\mathbb{C}$ unless otherwise stated. Vectors $u_1, u_2, \ldots, u_n$ are said to be *linearly independent* if their linear combination $\alpha_1 u_1 + \ldots + \alpha_n u_n = 0$ if and only if $\alpha_1 = \ldots = \alpha_n = 0$; otherwise they are *linearly dependent*. The dimension of X, denoted by $\dim X$, is the largest number of linearly vectors that exist in X. A subset M of X is a *subspace* if M is itself a vector space under the same linear operations as in X. Let X be an n-dimensional vector space and let $x_1, \ldots, x_n$ be a family of n linearly independent vectors. Then their span coincides with X and each $u \in X$ can be expanded in the form

$$u = \sum_{i=1}^{n} \alpha_i x_i$$

in a unique way. The family $\{x_i\}$ is called a *basis* of X.

A *normed vector space* (over $\mathbb{C}$) is a vector space X together with a function on X denoted by $x \longmapsto \|x\|$ (real valued) such that:

1. We have $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$.
2. If $\alpha \in \mathbb{C}$ and $x \in X$, then $\|\alpha x\| = |\alpha| \, \|x\|$.
3. If $x, y \in X$, then $\left\|x + y\right\| \leq \|x\| + \left\|y\right\|$.

Let $\{x_n\}$ be a sequence in a normed vector space X. This sequence is said to be *Cauchy* if given any ϵ (always assumed > 0) there exists N such that for all $m, n \geq N$ we have

$$\|x_m - x_n\| < \epsilon.$$

This sequence is said to *converge* to an element x if given ϵ, there exists N such that for all $n > N$ we have

$$\|x - x_n\| < \epsilon.$$

Let S be a set. A map $f : S \to F$ of S into a normed vector space F said to be *bounded* if there exists a number $C > 0$ such that $\|f(x)\| \leq C$ for all $x \in S$. If f is bounded, define

$$\left\|f\right\|_S := \sup_{x \in S} \left\|f(x)\right\|,$$

sup meaning the least upper bound.

Let E be the space of continuous X-valued functions on $[0, 1]$. For $f \in E$ define

$$\left\|f\right\|_{L^1} := \int_0^1 \left\|f(x)\right\| \, dx.$$

Then $\|\cdot\|_{L^1}$ is a norm on E, called the L^1-norm. This norm will be one of the major objects in our subsequent work. A normed vector space X is said to be *complete* (or *Banach space*) if every Cauchy sequence converges, i.e. has a limit in X. A continuous (bounded) linear map between Banach spaces is called an *operator*. From now on by a space we will mean a Banach space unless otherwise stated.

A complex-valued function $f(u)$ defined on a space X is called a *anti-linear* form if

$$f(\alpha u + \beta v) = \bar{\alpha} f(u) + \bar{\beta} f(v)$$

where $\bar{\alpha}$ denotes the complex conjugate of α. The set of all semilinear forms on X becomes a vector space, called the *adjoint space* of X and denoted by X^*. It is convenient to treat X^* on the same level as X. To this end we write $f(u) = (f, u)$.

Let $\{x_j\}$ be a basis of X (N-dimensional space). As in the case of linear forms, for each scalar α_k there is an $f \in X^*$ such that $(f, x_k) = \alpha_k$. In particular, it follows that for each j, there exists a unique $e_j \in X^*$ such that

$$(e_j, x_k) = \delta_{jk}\ ,\quad j,k = 1,\ldots,N.$$

Each $f \in X^*$ can be expressed in a unique way as a linear combination of the e_j, according to

$$f = \sum_j \alpha_j e_j\ ,\ \text{ where }\ \alpha_j = (f, x_j).$$

Thus the N vectors e_j form a basis of X^*, called the basis *adjoint* to the basis $\{x_j\}$ of X. For each $u \in X$ we have

$$u = \sum_j \xi_j x_j \ \text{ where }\ \xi_j = \overline{(e_j, u)}.$$

It follows that

$$(f, u) = \sum_j \alpha_j \bar{\xi}_j = \sum (f, x_j)\,(e_j, u).$$

2.4 Projections

Let M, N be two *complementary* linear subspaces of X, i.e.,

$$X = M \oplus N.$$

Thus each $u \in X$ can be uniquely expressed in the form $u = u' + u''$ with $u' \in M$ and $u'' \in N$. If we set $u' =: Pu$, it follows that P is a linear operator on X. P is called the *projection operator* (or simply the *projection*) *on* M *along* N. $(I - P)$ is the projection on N along M. The following result will be used in our later chapters.

Theorem 2.6 [29, Problem 3.20] *Let $v \neq 0$ and $v \in X$ (where X is finite dimensional) and $f \in X^*$ be given. Then operator P defined by $Pu := (u, f)\, v$ for all $u \in X$ is a projection if and only if $(v, f) = 1$. In this case PX is the one-dimensional subspace $[v]$ spanned by v, and $(I - P)$ is the closed linear subspace of X consisting of all u with $(u, f) = 0$.*

2.5 Calculus in a Banach Space

2.5.1 Integration in One Variable

Following [34], let $[a, b]$ be a closed interval and E a Banach space. By a *step map* $f : [a, b] \to E$ we mean a map for which there exists a partition

$$P : a = a_0 \leq a_1 \leq \ldots \leq a_n = b$$

and elements $v_1, \ldots, v_n \in E$ such that if $a_i < t < a_{i+1}$, then $f(t) = v_i$. We then say that f is a step map with respect to P. The step maps form a subspace of the space of all bounded maps, and we deal with the L^1-norm on this space.

We define the *integral of a step map f with respect to a partition P* by

$$I_P(f) := \sum_{i=1}^{n} (a_i - a_{i-1}) v_i.$$

This is in fact independent of P and we write simply $I(f)$. It is then easily seen that I is linear and that $|I(f)| \leq (b-a)\|f\|$, so I is continuous, with bound $b-a$. We can therefore extend I to the closure of the space of step maps by the linear extension theorem. If f lies in this closure, we denote $I(f)$ by

$$\int_a^b f$$

and call it the *integral*.

2.5.2 The Derivative as a Linear Map

Let U be an open set in a Banach space E and let $x \in U$. Let $f : U \to F$ be a map to a Banach space F. We shall say f is *differentiable* at x if there exists a continuous linear map $\lambda : E \to F$ and a map ψ defined for all sufficiently small h in E, with values in F, such that

$$\lim_{h \to 0} \psi(h) = 0,$$

and such that

$$f(x+h) = f(x) + \lambda(h) + |h|\psi(h).$$

If f is differentiable at every point x of U then we say that f is *differentiable on U*. In that case, the derivative f' is the map

$$Df := f' : U \to L(E, F)$$

from U into the space of continuous linear maps $L(E, F)$ assigning to each $x \in U$, the linear map $f'(x) = \lambda \in L(E, F)$. If f' is continuous, we say that f is of class C^1. Since f' maps U into the Banach space $L(E, F)$, we can define inductively f to be of class C^p if all derivatives $D^k f$ exist and are continuous for $1 \leq k \leq p$.

2.5.3 Properties of the Derivative

Theorem 2.7 *Let E, F be Banach spaces and let U be open in E. Let $f, g : U \to F$ be maps which are differentiable at $x \in U$. Then $f + g$ is differentiable at x and*

$$(f+g)'(x) = f'(x) + g'(x).$$

If c is a number, then

$$(cf)'(x) = cf'(x).$$

Proof. See [34, p. 101]. □

Theorem 2.8 *Let F_1, F_2, G be Banach spaces and let $F_1 \times F_2 \to G$ be a continuous bilinear map. Let U be an open set in E and let $f : U \to F_1$ and $g : U \to F_2$ be maps differentiable at $x \in U$. Then the product map fg is differentiable at x and*

$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

Proof. See [34, p. 102]. □

Theorem 2.9 *Let E, F be Banach spaces. Let U be open in E and let V be open in F. Let $f : U \to V$ and $g : V \to G$ be maps. Let $x \in U$. Assume that f is differentiable at x and g is differentiable at $f(x)$. Then $g \circ f$ is differentiable at x and*

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x).$$

Proof. See [34, p. 103]. □

Theorem 2.10 (Map with coordinates) *Let $E, F_1, \ldots, F_m$ be Banach spaces. Let U be open in E, let*

$$f : U \to F_1 \times \ldots \times F_m,$$

and let $f = (f_1, \ldots, f_m)$ be its expression in terms of coordinate maps. Then f is differentiable at x if and only if f_i is differentiable at x and if this is the case, then

$$f'(x) = (f_1'(x), \ldots, f_m'(x)).$$

Proof. See [34, p. 104]. □

Theorem 2.11 (Fundamental Theorem of Calculus) *Let f be regulated (the closure of the space of step maps will be called the space of regulated maps) on an interval $[a, b]$ and assume that f is continuous at a point c of $[a, b]$. Then the map*

$$t \mapsto \phi(t) := \int_a^t f$$

is differentiable at c and its derivative is $f(c)$.

Proof. See [34, p. 105]. □

Theorem 2.12 (Taylor's formula) *Let E, F be Banach spaces. Let U be open E and let $f : U \to F$ be of class C^p. let $x \in U$ and let $y \in E$ be such that the segment $x + ty$, $0 \le t \le 1$, is contained in U. Denote by $y^{(k)}$ the k-tuple $(y, \dots, y)$. Then*

$$f(x+y) = f(x) + \frac{Df(x)y}{1!} + \dots + \frac{D^{p-1}f(x)y^{p-1}}{(p-1)!} + R_p,$$

where

$$R_p = \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x+ty)y^{(p)}\, dt.$$

Proof. See [34, p. 115]. □

2.6 Operator Semigroups

A *dynamical system* may be defined as a family $\{T(t)\}_{t\ge 0}$ of mappings on a Banach space X satisfying

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \ge 0, \\ T(0) = id, \end{cases} \tag{2.2}$$

where *id* is the identity operator. If $T(t)$ is a linear operator on X then $\{T(t)\}_{t\ge 0}$ is called a (one-parameter) *operator semigroup*.

The standard situation in which such operator semigroups naturally appear are so-called *Abstract Cauchy Problems*

$$\begin{cases} \dot{u}(t) = Au(t) & \text{for } t \ge 0, \\ u(0) = x, \end{cases}$$

where A is a linear operator on a Banach space X.

A family $\{T(t)\}_{t\ge 0}$ of bounded operators on a Banach space X is called a *strongly continuous* (one-parameter) semigroup (or C_0*-semigroup*) if it satisfies the functional equation (2.2) and is strongly continuous in the following sense.
For every $x \in X$ the orbit maps

$$\xi_x : t \mapsto \xi_x(t) := T(t)x$$

are continuous from $\mathbb{R}_+$ into X for every $x \in X$. For the sake of simplicity we use the notation $T(t)$ for $\{T(t)\}_{t\ge 0}$.

Theorem 2.13 *For a semigroup $T(t)$ on a Banach space X, the following assertions are equivalent.*

1. *$T(t)$ is strongly continuous.*

2. $\lim_{t\downarrow 0} T(t)x = x$ *for all* $x \in X$.

3. *There exist* $\delta > 0$, $M \geq 1$, *and a dense subset* $D \subset X$ *such that*
   (a) $\|T(t)\| \leq M$ *for all* $t \in [0, \delta]$,
   (b) $\lim_{t\downarrow 0} T(t)x = x$ *for all* $x \in D$.

Proof. See [20, p. 4]. □

Theorem 2.14 *For every C_0-semigroup $T(t)$, there exist constants $w \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{wt}$$

for all $t \geq 0$.

Proof. See [20, p. 5]. □

For a C_0-semigroup $T(t)$, we define its *growth bound* as

$$w_0 := inf\{w \in \mathbb{R} : \exists M_w \text{ such that } \|T(t)\| \leq M_w e^{wt} \text{ for all } t \geq 0\}.$$

A semigroup is called *bounded* if we can take $w = 0$, *quasi-contractive* if $M_w = 1$ and *contractive* if $w = 0$ and $M_w = 1$ is possible.
The *generator* $A : D(A) \subseteq X \to X$ of a C_0-semigroup $T(t)$ on a Banach space X is the operator

$$Ax := \lim_{h\downarrow 0} \frac{T(h)x - x}{h}$$

defined for every x in its *domain*

$$D(A) := \{x \in X : \lim_{h\downarrow 0} \frac{T(h)x - x}{h} \text{ exists}\}.$$

Theorem 2.15 *For the generator $A, D(A)$ of a C_0-semigroup $T(t)$, the following properties hold.*

1. *$A : D(A) \subseteq X \to X$ is a linear operator.*

2. *If $x \in D(A)$, then $T(t)x \in D(A)$ and*

$$\frac{d}{dt}T(t)x = T(t)Ax = AT(t)x \text{ for all } t \geq 0.$$

3. *For every $t \geq 0$ and $x \in X$, one has*

$$\int_o^t T(s)x\, ds \in D(A).$$

4. *For every $t \geq 0$, one has*

$$T(t)x - x = \begin{cases} A \displaystyle\int_0^t T(s)x\,ds, & \text{if } x \in X; \\ \displaystyle\int_0^t T(s)Ax\,ds, & \text{if } x \in D(A). \end{cases}$$

Proof. See [20, p. 37]. □

Theorem 2.16 *The generator of a C_0-semigroup is a closed and densely defined linear operator that determines the semigroup uniquely.*

Proof. See [20, p. 38]. □

Let $(A, D(A))$ be a closed operator on a Banach space X. Then we define

1. *spectrum* $\sigma(A) := \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not bijective}\}$,
2. *resolvent set* $\rho(A) := \mathbb{C} \setminus \sigma(A)$,
3. *resolvent* $R(\lambda, A) := (\lambda I - A)^{-1}$ at $\lambda \in \rho(A)$,
4. *spectral bound* $s(A) := sup\,\{Re\lambda : \lambda \in \sigma(A)\}$,
5. *spectral radius* $Sp(A) := sup\,\{|\lambda| : \lambda \in \sigma(A)\}$.

Theorem 2.17 *For the spectral bound $s(A)$ of a generator A and for the growth bound w_0 of the generated semigroup $T(t)$, one has*

$$-\infty \leq s(A) \leq w_0.$$

Proof. See [20, p. 168]. □

Theorem 2.18 (Hille–Yosida) *Let $w \in \mathbb{R}$. For a linear operator $(A, D(A)$ on a Banach space X the following conditions are equivalent.*

1. *$(A, D(A))$ generates a C_0-semigroup $T(t)$ satisfying*

$$||T(t)|| \leq e^{wt} \quad for\ t \geq 0.$$

2. *$(A, D(A))$ is closed, densely defined and for each $\lambda \in \mathbb{C}$ with $Re\lambda > w$ one has $\lambda \in \sigma(A)$ and*

$$||R(\lambda, A)|| \leq \frac{1}{Re\lambda - w}.$$

Proof. See [20, p. 68]. □

Theorem 2.19 (Bounded Perturbation Theorem) *Let $(A, D(A))$ be the generator of a C_0-semigroup $T(t)$ on a Banach space X satisfying*

$$\|T(t)\| \leq Me^{wt} \text{ for all } t \geq 0$$

and some $w \in \mathbb{R}, M \geq 1$. If B is a bounded operator on X, then

$$C := A + B \text{ with } D(C) := D(A)$$

generates a C_0-semigroup $S(t)$ satisfying

$$\|S(t)\| \leq Me^{(w+M\|B\|)t} \textit{ for all } t \geq 0.$$

Proof. See [20, p. 117]. □

Let us consider a Banach-space-valued linear initial value problems of the form

$$\begin{cases} \dot{u}(t) = Au(t), & (t > 0) \\ u(0) = x, \end{cases} \tag{2.3}$$

where the independent variable t represents time, $u(\cdot)$ is a function with values in a Banach space X, $A : D(A) \subset X \to X$ a linear operator and $x \in X$ the initial value. The initial value problem (2.3) is called the *abstract Cauchy problem* associated with $(A, D(A))$ and the initial value x. A function $u : \mathbb{R}_+ \to X$ is called a *classical solution* of (2.3) if u is continuously differentiable, $u(t) \in D(A)$ for all $t \geq 0$ and (2.3) holds.

Theorem 2.20 *Let $(A, D(A))$ be the generator of the C_0-semigroup $T(t)$. Then, for every $x \in D(A)$, the function*

$$u : t \mapsto u(t) := T(t)x$$

is the unique classical solution of (2.3).

Proof. See [20, p. 110]. □

The important point is that classical solutions exist if and only if the initial value x belongs to $D(A)$. A continuous function $u : \mathbb{R}_+ \to X$ is called a *mild solution* of (2.3) if $\int_0^t u(s)\,ds \in D(A)$ for all $t \geq 0$ and

$$u(t) = A \int_0^t u(s)\,ds + x.$$

Theorem 2.21 *Let $(A, D(A))$ be the generator of the C_0-semigroup $T(t)$. then for every $x \in X$, the orbit map*

$$u : t \mapsto u(t) := T(t)x$$

is the unique mild solution of a associated abstract Cauchy problem (2.3).

Proof. See [20, p. 111]. □

Theorem 2.22 *Let $A : D(A) \subset X \to X$ be a closed operator. Then for the associated abstract Cauchy problem* (2.3)*, the following properties are equivalent.*

1. *A generates a C_0-semigroup.*
2. *A satisfies the following condition.*
   For every $x \in D(A)$, there exists a unique solution $u(\cdot, x)$ of (2.3) *and $\rho(A) \neq 0$.*
3. *A satisfies existence and uniqueness condition 2, has dense domain and for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset D(A)$ satisfying $\lim_{n\to\infty} x_n = 0$, one has $\lim_{n\to\infty} u(t, x_n) = 0$ uniformly in compact intervals* $[0, t_0]$.

Proof. See [20, p. 112]. □

Condition 3 of the Theorem (2.22) expresses what we expect from a “well-posed" problem and its solutions:

$$\textit{existence} + \textit{uniqueness} + \textit{continuous dependence on the data}.$$

CHAPTER 3

Deterministic Population Models - An Overview

3.1 Introduction

Malthus can possibly be credited with formulating the first mathematical population model [41] in 1798. If we denote the population size by u, time by t and the rate of increase of the population by r, Malthus's model is represented by the following first-order differential equation

$$\frac{du}{dt} = ru. \tag{3.1}$$

The mathematical model of Lotka [37] was also based on (3.1). In fact many of the population models are such that asymptotically the total population grows exponentially, and they obey equation (3.1).

In practice, a population cannot grow exponentially for ever. If the growth rate r is negative, it will shrink. If the growth rate r is positive, the population will become too large for the environment to support. Population mathematicians have therefore modified equation (3.1) to obtain the differential equation for the so-called *logistic population*:

$$\frac{du}{dt} = ru\left(1 - \frac{u}{U}\right).$$

This type of population grows exponentially while it is small, but the growth rate tapers off as the population size increases, and the population cannot exceed a certain maximum size U (unless of course it had been made greater than U to begin with artificially, in which case it will decrease to size U). However for human populations, the logistic law of growth has not proved very satisfactory.

3.2 The Continuous-time Model of Sharpe and Lotka

Although Malthus [41] and Lotka [37] proposed elementary mathematical models for human populations, the paper which might be considered to be the beginning of the subject of population dynamics is that of Sharpe and Lotka [57] in 1911. Demographers usually find it more convenient to apply one-sex model to the female sex, and describe the model in terms of females.

In order to derive Lotka's model, following [7], let $B(t)\Delta t$ is approximately the number of females births that occur in the time interval $[t, t+\Delta t)$. Let $n(a,t)$ is the density of females of age a at time t; that is, $n(a,t)\Delta a$ is the number of females of age $[a, a+\Delta a)$. Next let $l(a)$ denotes the fraction of newborn females surviving to age a. Here we assume that $l(a)$ is continuous and piecewise differentiable. Also, l must be non-increasing and there is some maximum age of survivorship w. Lastly, assume $m(a)\Delta a$ represents the number of females born, on average, to a female of age between a and $a+\Delta a$. Here m is assumed to be continuous and piecewise smooth and there is a minimum age of reproduction α (menarche) and a maximum age of reproduction β (menopause). Now the births can be divided into two classes: one class attributed to females born between time 0 and t and the other due to females which were alive at time 0. Females that are of age a at time t were born at time t were born at time $t-a$. The number of females born around $t-a$ is given by $B(t-a)\Delta t$. The number of them that survive till the age a (that is, till time t) is $l(a)B(t-a)\Delta t$ and thus the number of births by females of around age a is $l(a)B(t-a)\Delta t m(a)\Delta a$. Summing up, we obtain

$$\Delta t \int_0^t B(t-a)l(a)m(a)\,da.$$

To find the contribution of the females who were present at time $t=0$ we begin with taking the number of females of around age a present at $t=0$; that is, $n(a,0)\Delta a$. Now, these females must live till the age $t+a$, that is, we must take survival rate till $t+a$, $l(t+a)$ conditioned upon the females having survived till a. Since

$$l(a+t) = l(a)\cdot\{\text{fraction of age}\,a\,\text{females surviving till}\,t+a\},$$

we see that $n(a,0)l(a+t)/l(a)\Delta a$ females survived till time t. These gave birth to $m(a+t)n(a,0)l(a+t)/l(a)\Delta a$ new females. TO find the number of all births due to females older than t we again integrate over all ages. However, no individual survives beyond w so that the integration terminates at $w-t$ (no female older than $w-t$ at time $t=0$ will survive till t. Combining these two formula and dropping Δt we obtain the renewal equation

$$B(t) = \int_0^t B(t-a)l(a)m(a)\,da + G(t), \tag{3.2}$$

where

$$G(t) := \int_0^{w-t} m(a+t)n(a,0)\frac{l(a+t)}{l(a)}\,da,$$

is a known function. Equation (3.2) is known as the basic Lotka one-sex deterministic population model.

3.3 The McKendrick–Von Foerster Model

Apparently, the first formulation of a PDE for the age distribution of a population is due to McKendrick [43]. Much later, Von Foerster [64] independently derived a similar equation and applied it to the dynamics of blood cell population.

Following [31], if $u(a,t)$ is the population (say of females) at age a and time t, the female population of age $a+\Delta a$ at time $t+\Delta t$ is $u(a+\Delta a, t+\Delta t)$. If $\Delta a = \Delta t$, the latter includes the same individuals as were counted in $u(a,t)$, only subject to deductions for mortality. The equation of change involving mortality $\mu(a)$, a function of age but not time, is

$$u(a+\Delta a, t+\Delta t) = u(a,t) - \mu(a)u(a,t)\Delta t.$$

Expanding $u(a+\Delta a, t+\Delta t)$ by Taylor's theorem for two independent variables and canceling $u(a,t)$ and higher order terms from both sides leaves

$$\frac{\partial u(a,t)}{\partial t}\Delta t + \frac{\partial u(a,t)}{\partial a}\Delta a = -\mu(a)u(a,t)\Delta t.$$

Dividing by Δa, which is equal to Δt, we have

$$\frac{\partial u(a,t)}{\partial t} + \frac{\partial u(a,t)}{\partial a} = -\mu(a)u(a,t). \tag{3.3}$$

Births enter as a boundary condition at age zero:

$$u(0,t) = \int_0^\infty \beta(s)u(s,t)\,ds, \tag{3.4}$$

where $\beta(a)$ is the age-specific birth rate, supposed to be invariant with respect to time. The initial condition

$$u(a,0) = \phi(a), \tag{3.5}$$

says that the population at time $t=0$ has a given age distribution $\phi(a)$.

3.3.1 Solution of the McKendrick–Von Foerster Model

We can solve the system (3.3)–(3.5) by characteristics which are given by

$$\frac{da}{dt} = 1, \tag{3.6}$$

on which

$$\frac{du}{dt} = -\mu u. \tag{3.7}$$

The characteristics are the straight lines

$$a = \begin{cases} t + a_0, & a > t; \\ t - t_0, & a < t. \end{cases} \tag{3.8}$$

Here a_0, t_0 are respectively the initial age of an individual at time $t = 0$ in the original population and the time of birth of an individual. Equation (3.7) which holds along each characteristic, has a different solution according to whether $a > t$ or $a < t$, that is, one for the population that was present at $t = 0$, namely, $a > t$, and the other for those born after $t = 0$, that is $a < t$. On integrating equation (3.7), using $da/dt = 1$ and (3.8), the solutions are

$$u(a,t) = \begin{cases} \phi(a-t)\, e^{-\int_{a-t}^{a} \mu(s)\, ds}, & a > t; \\ \\ u(0, t-a)\, e^{-\int_0^a \mu(s)\, ds}, & a < t. \end{cases} \tag{3.9}$$

From (3.9) we see that if the birth rate $u(0,t)$ can be determined as a function of t, then the density function u becomes known. Equations (3.4) and (3.9) can be used to arrive at single equation for the birth rate $u(0,t)$. Substitute (3.9) into (3.4) to obtain

$$\begin{aligned} u(0,t) = &\int_0^t \beta(s) u(0, t-s) e^{-\int_0^s \mu(v)\, dv}\, ds \\ &+ \int_t^\infty \beta(s)\phi(s-t)\, e^{-\int_{s-t}^{s} \mu(v)\, dv}\, ds. \end{aligned}$$

3.3.2 Characteristic Equation

The ingredients of this problem are similar to those for Lotka's integral equation (3.2). For Lotka's integral equation, the population asymptotically approached exponential growth and a stable distribution. Taking that as our clue, we may try a solution of the form

$$u(a,t) = e^{\lambda t} r(a). \tag{3.10}$$

That is, the age distribution is simply changed by a factor which either grows or decays with time according to whether $\lambda > 0$ or $\lambda < 0$. Substitution of (3.10) into (3.3) gives

$$\frac{dr}{da} = -\left[\mu(a) + \lambda\right] r,$$

and so

$$r(a) = r(0)\, e^{\left[-\lambda a - \int_0^a \mu(s)\, ds\right]}. \tag{3.11}$$

With this $r(a)$ in (3.10), the resulting $u(a,t)$, when inserted into the boundary condition (3.4), gives

$$e^{\lambda t} r(0) = \int_0^\infty \beta(a)\, e^{\lambda t} r(0)\, e^{\left[-\lambda a - \int_0^a \mu(s)\, ds\right]}\, da,$$

and hence, on canceling $e^{\lambda t}\, r(0)$,

$$1 = \int_0^\infty \beta(a)\, e^{\left[-\lambda a - \int_0^a \mu(s)\, ds\right]}\, da, \tag{3.12}$$

where $r(0) \neq 0$. Equation (3.12), which is known as the ***characteristic equation***, was discovered by Lotka in his model in 1922. The expression $\psi(a) := \beta(a) e^{-\int_0^a \mu(s)\, ds}$ in the characteristic equation (3.12) is called the ***net maternity function***. This characteristic equation has some special spectral properties which we are going to discuss in the following theorems. To prove these results, we will assume $\beta(a)$ and $e^{-\int_0^a \mu(s)\, ds}$ are continuous and defined for all $a \geq 0$.

Theorem 3.1 *The characteristic equation $\int_0^\infty e^{-\lambda a} \psi(a)\, da - 1 = 0$ admits exactly one real root for λ.*

Proof. Define $\Psi(a) := \int_0^\infty e^{-\lambda a} \psi(a)\, da$. The net maternity function $\psi(a)$ can not be negative and also $e^{-\lambda a}$ is real and positive. For non-trivial case $\psi(a) > 0$ on some open subinterval of $[0, \infty)$, $\Psi(\lambda) > 0$ for all $\lambda \in \mathbb{R}$. Also we notice that

$$\lim_{\lambda \to -\infty} \Psi(\lambda) = \infty, \quad \lim_{\lambda \to +\infty} \Psi(\lambda) = 0.$$

Now $\Psi'(\lambda) = -\int_0^\infty \lambda\, e^{-\lambda a} \psi(a)\, da < 0$ for all $\lambda \in \mathbb{R}$. Thus $\Psi(a)$ is a positive, monotonically decreasing function of λ which implies there is exactly one real λ for which $\Psi(\lambda) = 1$ and this completes the proof. □

Theorem 3.2 *All complex roots of the characteristic equation occur in conjugate pairs.*

Proof. Let $\lambda = \alpha + i\beta$ be a complex root of the characteristic equation.
Then,

$$\int_0^\infty e^{-(\alpha + i\beta)a} \psi(a)\, da = 1.$$

By expansion of the left-hand side, and identification of the real and imaginary parts, we find

$$\int_0^\infty e^{-\alpha a} \cos(\beta a) \psi(a)\, da = 1,$$

and

$$\int_0^\infty e^{-\alpha a} \sin(\beta a) \psi(a)\, da = 0.$$

Substitution of $\bar{\lambda} = \alpha - i\beta$ into the characteristic equation leads to the same result. This shows that $\lambda = \alpha - i\beta$ is also a root of the characteristic equation. □

Theorem 3.3 *If r is the real root and α is the real part of any other complex root, then $r > \alpha$.*

Proof. Let $\lambda = \alpha + i\beta$ denote any complex root of the characteristic equation. Substituting λ in the characteristic equation we have

$$\int_0^\infty e^{-\alpha a} cos(\beta a)\, \psi(a)\, da - i \int_0^\infty e^{-\alpha a} \sin(\beta a)\, \psi(a)\, da = 1.$$

Comparing real and imaginary parts from the both sides of the above equation we have,

$$\int_0^\infty e^{-\alpha a} \cos(\beta a)\, \psi(a)\, da = 1, \tag{3.13}$$

and

$$\int_0^\infty e^{-\alpha a} \sin(\beta a)\, \psi(a)\, da = 0.$$

Because $\cos(\beta a) < 1$ for some βa in the range of integration for equation (3.13), so we must have

$$\int_0^\infty e^{-\alpha a} \psi(a)\, da > 1.$$

But

$$\int_0^\infty e^{-r a} \psi(a)\, da = 1,$$

giving $e^{-r a} < e^{-\alpha a}$ which implying $r > \alpha$ as required. □

3.3.3 Remark

In this model we are generally interested in the long term behaviour of the population. If r denotes the above mentioned unique real root of the characteristic equation then under appropriate assumptions on β, μ and the initial age distribution ϕ, it can be shown [65] that every solution $u(a,t)$ of the PDE has the property that, uniformly in finite intervals of a,

$$\lim_{t\to\infty} \left| e^{-rt} u(a,t) - C(\phi) \exp\left[-ra - \int_0^a \mu(b)\, db \right] \right| = 0,$$

where $C(\phi)$ is a value which depends on ϕ. This result is originally due to Lotka [38], [39] and was proved rigorously by Feller [23]. A semigroup theoretic proof of the above result is available in [65].

3.4 Multiregional Demography

Formal demography is concerned with the mathematical description of human (or nonhuman) populations, particularly with their structure with regard to age and sex, and the components of change, such as births and deaths, which occur over time to alter that structure. Accordingly, demographers have focused their attention on population *stocks* and on population *events*. Our work is based on a model, which deals with the extension of that focus to include the *flows* that interconnect and weld several regional populations into a multiregional population system. One of the first major contribution along this line is by Rogers [52].

Following [52] and [53], "formal multiregional demography, therefore, is concerned with the mathematical description of the evolution of populations over time and space. The trifold focus of such descriptions is on the *stocks* of population groups at different points in time and locations in space, the vital *events* that occur among these populations, and the *flows* of members of such populations across the spatial borders that delineate the constituent regions of the mulitiregional population system. A biological population may experience multiple states in two ways:

First, it may visit different states in the course of time, the whole population experiencing the same (possibly age-specific) vital rates at any one time. For example, a troop of baboons moves from one area to another of its range, with associated changes in food supply and risks of predation [2]. A human population experiences fluctuating crop yields from one year to the next, with associated effects on childbearing and survival. These are *serial* changes of state of a *homogeneous* population.

Second, the population may be subdivided into *inhomogeneous* subpopulation that experience different states in *parallel*. Individuals may migrate from one state to another in the course of time. The states may corresponds to geographical regions, work status, marital status, health status, or other classifications [54]. Individuals within a given state at a given time are assumed to be homogeneous with respect to their vital rates. In general multistate stable population theory is used for two reasons:

1. To explore the relation between individual life histories and population characteristics.
2. To explore the consequences of current life histories and variations in life histories. Stable populations are a population that experiences particular demographic regimes (transitions) over a long period. The theory is used to magnify the effects of a current demographic regime and to assess the consequences of small changes in demographic behaviour (microscopic view)."

For more detailed study of multidimensional demography, we refer to [33], [53].

CHAPTER 4

Singularly Perturbed Model

4.1 The Model

We are going to consider a closed (with no immigration and emigration), one-sex population model which is linear and continuously depends on age. The whole population is additionally subdivided into several groups. We assume the population is distributed into n patches or states which may be interpreted as geographical areas or any other classifications with proper meaning. The basic assumptions regarding this population model are as follows:

1. Migrations between patches occur in a discrete manner.
2. The migrations occur on a much faster time scale than the demographic events (ageing, births and deaths).
3. The migration rates between the states, death rates and the birth rates may be age dependent.
4. The movement process between the patches is conservative with respect to the life dynamics of the population.

To describe the model, let $u_i(a,t)$, for $1 \le i \le n$, be the density of a population at time t of individuals residing in patch i and being of age a so that the number of inhabitants in the age interval of $a + da$ is approximately equal to $u_i(a,t)da$ and precisely

$$\int_a^{a+da} u_i(s,t)\, ds.$$

Let $\mathbf{M}(a) := \text{diag}[\mu_i(a)]_{1\le i\le n}$ denote the mortality matrix in which $\mu_i(a)$ is the mortality rate at age a in the ith patch. Let the matrix $\mathbf{C}(a) := [c_{ij}(a)]_{1\le i,j\le n}$ describe the transfer of individuals between patches, that is, $c_{ij}(a)$, for $i \neq j$, denote the migration rates at age a from state j to state i and the diagonal elements $c_{ii}(a)$, defined as $c_{ii}(a) := -\sum_{i=1,j\neq i}^{n} c_{ji}(a)$ indicate the net loss of population in ith state due to migration to other states. Let $\mathbf{B}(a) := \text{diag}[\beta_i(a)]_{1\le i\le n}$ denote the fertility matrix in which $\beta_i(a)$ is the average number of offspring of state i per unit time produced by an individual at age a. Let $\boldsymbol{\phi}(a)$ be the initial population vector. Migration between the patches occur at a much faster time scale than the demographic processes such as aging, and this is incorporated by introducing a large parameter $1/\epsilon$ multiplying the migration matrix $\mathbf{C}(a)$. For all our future work, we will assume that $\epsilon > 0$ is an arbitrary small number.

Denoting the population vector as $\mathbf{u}(a,t) := (u_1(a,t),\ldots,u_n(a,t))$ we have the following system of equations (in vector form) with proper initial and boundary conditions

$$\mathbf{u}_t = -\mathbf{u}_a - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon}\mathbf{C}\mathbf{u}, \tag{4.1}$$

$$[\gamma\mathbf{u}](t) := \mathbf{u}(0,t) = [\mathfrak{B}\mathbf{u}](t) := \int_0^\infty \mathbf{B}(s)\mathbf{u}(s,t)\,ds, \tag{4.2}$$

$$\mathbf{u}(a,0) = \boldsymbol{\phi}(a), \tag{4.3}$$

where subscripts t and a, respectively denote differentiation with respect to the variables t and a. To avoid technical difficulties, we have taken the upper limit of the integral in boundary condition (4.2) as infinity. From a practical point of view, one can always assume a finite upper limit of age (say, w) and the rest of the range of integral (i.e., from w to infinity) as zero. We consider (4.1)–(4.3) in the space $\mathbf{X} = L_1(\mathbb{R}_+,\mathbb{R}^n)$, where the norm of a non-negative element gives the size of the total population. Here we assume $\mathbb{R}^n$ is equipped with the l^1-type norm $\|x\| := \sum_{i=1}^n |x_i|$.

4.1.1 The Irreducible Migration Matrix

Definition 4.1 [26] *A matrix* $\mathbf{C} := [c_{ij}]_{1\le i,j\le n}$ *will be called a* ***Kolmogorov matrix*** *if* $c_{ij} \ge 0$ *for* $i \neq j$ *and* $c_{ii} := -\sum_{\substack{j=1\\ j\neq i}}^{n} c_{ji}$.

In terms of the above definition, our migration matrix $\mathbf{C}$ is a Kolmorogov matrix. We assume that $\mathbf{C}$ is irreducible. Also we assume $a \to \mathbf{C}(a) \in C_b^2(\mathbb{R}_+,\mathbb{R}^{n^2})$. With this irreducibility assumption and Kolmogorov matrix structure, the migration matrix $\mathbf{C}$ (a) does have special spectral properties (Theorem 4.4). Results (Theorem 4.2) are available only for irreducible non-negative matrices. Since our migration matrix $\mathbf{C}$ has non-negative off-diagonal elements, we need specific results for this type of matrices which can be obtained using Theorem 4.2 and we prove them in Corollary 4.3. Here for the sake of completeness, Theorem 2.3 is rewritten as Theorem 4.2 below.

Theorem 4.2 (**Perron–Frobenius**) *An irreducible non-negative matrix always has a positive eigenvalue λ that is simple root of the characteristic equation. The moduli of all the other eigenvalues do not exceed λ. To the eigenvalue λ there corresponds an eigenvector with positive coordinates.*

Proof. See [24, p. 53]. □

Corollary 4.3 [58] *A matrix $\mathbf{A}$ with non-negative off-diagonal entries has the following properties:*
1. The spectral bound $s(\mathbf{A})$ is in $\sigma(\mathbf{A})$ and there is a vector $\mathbf{v} > 0$ such that $\mathbf{Av} = s(\mathbf{A})\mathbf{v}$.
2. $Re\lambda < s(\mathbf{A})$ for all $\lambda \in \sigma(\mathbf{A}) \setminus \{s(\mathbf{A})\}$.
3. If, in addition, $\mathbf{A}$ is irreducible, then $s(\mathbf{A})$ has algebraic multiplicity one.

Proof. $\mathbf{A} + c\mathbf{I} \geq 0$ for all large c since $\mathbf{A}$ has non-negative off-diagonal entries. Therefore Theorem 4.2 applies to $\mathbf{A} + c\mathbf{I}$ for such c. In particular, $Sp\,(\mathbf{A} + c\mathbf{I})$ is positive and it is an eigenvalue of $\mathbf{A} + c\mathbf{I}$ for all large c. Moreover there exists a corresponding non-negative eigenvector. Since adding $c\mathbf{I}$ to $\mathbf{A}$ results in $\sigma(\mathbf{A} + c\mathbf{I}) = c + \sigma(\mathbf{A})$, it follows from Theorem 4.2, that $s(\mathbf{A} + c\mathbf{I}) = Sp\,(\mathbf{A} + c\mathbf{I}) = s(\mathbf{A}) + c$ and therefore $s(\mathbf{A})$ is an eigenvalue of $\mathbf{A}$. Also, adding $\mathbf{cI}$ to $\mathbf{A}$ preserves the property of irreducibility in the sense that $\mathbf{A}$ is irreducible if and only if $\mathbf{A}$+$\mathbf{cI}$ is irreducible. Since $s(\mathbf{A})$ satisfies the condition of Theorem4.2, it follows that $s(\mathbf{A})$ has algebraic multiplicity one. □

Using Corollary 4.3, we can now prove the following spectral properties of our migration matrix $\mathbf{C}$.

Theorem 4.4 $\mathbf{C}$ *has zero as a simple, dominant eigenvalue and the rest of the eigenvalues have negative real parts.* $\mathbf{1}$ *is the left eigenvector corresponding to the zero eigenvalue of* $\mathbf{C}$*.*

Proof. The fact that $\mathbf{C}$ has zero as an eigenvalue follows from the fact that the sum of the entries of its column is zero. Let $\mathbf{k}$ be the corresponding right eigenvector of the zero eigenvalue. Now let us prove that zero is a dominant eigenvalue. Following Corollary 4.3, we have

$$\mathbf{Ck} = s(\mathbf{C})\,\mathbf{k}.$$

Summing over this system of equations we obtain

$$0 = s(\mathbf{C})(k_1 + \ldots + k_n),$$

where $\mathbf{k} := (k_1, \cdots, k_n)$. The reason of getting zero on the left hand side is that the sum of the entries of each column of $\mathbf{C}$ is zero. Since from Corollary 4.3, $\mathbf{k} > 0$, we must have $s(\mathbf{C}) = 0$. Since $s(\mathbf{C})$ is defined as the largest number among the real parts of the eigenvalues of $\mathbf{C}$, zero is a dominant eigenvalue and since $\mathbf{C}$ is irreducible, zero is also a simple eigenvalue. Since the sum of the elements of a column of $\mathbf{C}$ is zero, it follows that $\mathbf{1}$ is a left eigenvector corresponding to the zero eigenvalue of $\mathbf{C}$ and this completes the proof. □

4.2 Well-Posedness of the Model

To avoid technical difficulties, in this section we assume that there is a finite upper limit of age and denote it as w. In order to write the system (4.1)–(4.3) as an abstract Cauchy problem we introduce the operator $\mathbf{A}$ as

$$(\mathbf{A}\boldsymbol{\phi})(a) := -\boldsymbol{\phi}'(a) + \mathbf{Q}(a)\boldsymbol{\phi}(a),$$

where $\mathbf{Q}(a) = [q_{ij}(a)]_{1\le i,j\le n} := -\mathbf{M}(a) + \frac{1}{\epsilon}\mathbf{C}(a)$, $\boldsymbol{\phi} \in \mathfrak{D}(\mathbf{A})$ and the domain is defined by
$$\mathfrak{D}(\mathbf{A}) = \left\{\boldsymbol{\phi} \in L^1([0,w],\mathbb{R}^n) : \boldsymbol{\phi} \text{ is absolutely continuous, } \boldsymbol{\phi}(0) = \int_0^w \mathbf{B}(s)\boldsymbol{\phi}(s)\,ds\right\}.$$

For this section, we adopt the following assumptions:

A1: The state space of age distribution functions $\mathbf{u}(\cdot,t)$ is $L^1([0,w],\mathbb{R}^n)$, where w is a real number.

A2: $\mu_i(a)$ and $q_{ij}(a)$, where $i \neq j$, $1 \le i,j \le n$ are nonnegative continuous functions and define $\underline{\mu} := \inf\limits_{i,a} \mu_i(a)$.

A3: $\beta_{ij}(a) \in L^\infty_+(0,w;\mathbb{R})$ for $1 \le i,j \le n$ and $\beta_{ij}(a) = 0$ for $a \ge w, 1 \le i,j \le n$. Also we define $\bar{\beta} := \operatorname*{ess\,sup}\limits_{0\le a\le w} \|\,\mathbf{B}(a)\| < +\infty$.

With this set up, we prove the following result.

Theorem 4.5 $\mathbf{A}$ *generates a* C_0*-semigroup* $\{\mathbf{T}(t)\}_{t\ge 0}$ *such that* $\|\mathbf{T}(t)\| \le e^{(\bar{\beta}-\underline{\mu})t}$.

We use the Hille–Yosida Theorem 2.18 to prove Theorem 4.5.

Before proceeding with the proof, let us introduce the survival matrix $\mathbf{L}(a)$, which is the multistate analog of the survival function in a single-state population. $\mathbf{L}(a)$ is defined as the solution of the matrix differential equation:

$$\frac{d}{da}\mathbf{L}(\mathrm{a}) = \mathbf{Q}(\mathrm{a})\mathbf{L}(\mathrm{a}), \qquad \mathbf{L}(0) = \mathbf{I}, \tag{4.4}$$

where $\mathbf{I}$ denotes the $n \times n$ identity matrix. From the theory of ordinary differential equations, we know that the survival matrix $\mathbf{L}(a)$ is uniquely determined by the system (4.4), and the following holds [15]:

$$\det \mathbf{L}(a) = \exp\left(\int_0^a \sum_{i=1}^n q_{ii}(s)\,ds\right). \tag{4.5}$$

From (4.5), we know that $\det \mathbf{L}(a) \neq 0$ for all $a \in [0,w]$, and the inverse matrix $\mathbf{L}^{-1}(a)$ always exists for all $a \in [0,w]$. The (i,j)th element of $\mathbf{L}(a)$ denotes the probability a person born in the jth state will survive and be in the ith state at age a. Furthermore, we define $\mathbf{L}(b,a)$ as $\mathbf{L}(b)\mathbf{L}^{-1}(a)$. The (i,j)th element $l_{ij}(b,a)$ of $\mathbf{L}(b,a)$ denotes the probability that a person in the jth state at age a will survive to age b in the ith state. We can prove the following.

Lemma 4.6 *For $\mathbf{L}(a,b)$, the following holds:*

1. $\mathbf{L}(b,a)$ is nonnegative,

2. $\|\mathbf{L}(b,a)\| \le e^{-\underline{\mu}(b-a)}$,
where the operator norm is related to the l^1-type norm in $\mathbb{R}^n$.

Proof. 1. Clearly, $\mathbf{L}(b,a)$ satisfies

$$\frac{d}{db}\mathbf{L}(b,a) = \mathbf{Q}(b)\mathbf{L}(b,a), \qquad \mathbf{L}(a,a) = \mathbf{I}.$$

Let $\eta := \sup_{i,a} |q_{ii}(a)|$. Then $\mathbf{Q}(a) + \eta\mathbf{I}$ is a nonnegative matrix, and we have

$$\frac{d}{db}\left\{\mathbf{L}(b,a)e^{\eta(b-a)}\right\} = \left[\mathbf{Q}(b) + \eta\mathbf{I}\right]\mathbf{L}(b,a)e^{\eta(b-a)}.$$

By Picard's iteration method, we have the following representation

$$\begin{aligned}\mathbf{L}(b,a)e^{\eta(b-a)} = \mathbf{I} &+ \int_a^b \left[\mathbf{Q}(\rho) + \eta\mathbf{I}\right] d\rho \\ &+ \int_a^b \left[\mathbf{Q}(\rho_1) + \eta\mathbf{I}\right] \int_a^{\rho_1} \left[\mathbf{Q}(\rho_2) + \eta\mathbf{I}\right] d\rho_2\, d\rho_1 + \cdots .\end{aligned}$$

Since the right-hand side is nonnegative, we can conclude that $\mathbf{L}(b,a)$ is nonnegative.

2. Next, let $l_{ij}(b,a)$ denote the (i,j)th element of $\mathbf{L}(b,a)$. Then

$$\frac{d}{db}l_{ij}(b,a) = \sum_{k=1}^{n} q_{ik}(b)l_{kj}(b,a), \qquad \text{where } l_{ij}(a,a) = \delta_{ij},$$

and summing over index i, we have

$$\begin{aligned}\frac{d}{db}\sum_{i=1}^{n} l_{ij}(b,a) = \sum_{k=1}^{n}\sum_{i=1}^{n} q_{ik}(b)l_{kj}(b,a) &= \sum_{k=1}^{n} \left(-\mu_k(b)\right) l_{kj}(b,a) \\ &\le (-\underline{\mu}) \sum_{i=1}^{n} l_{ij}(b,a).\end{aligned}$$

Thus we obtain

$$\sum_{i=1}^{n} l_{ij}(b,a) \le e^{-\underline{\mu}(b-a)}.$$

This shows that $\|\mathbf{L}(b,a)\| \leq e^{-\underline{\mu}(b-a)}$. This completes the proof. □

In the following series of lemmas, we are going to prove the second condition of Theorem 2.18, and that will subsequently prove our main Theorem 4.5.

Lemma 4.7 **A** *is a closed linear operator in* $L^1(0,w;\mathbb{R}^n)$.

Proof. Linearity of **A** follows immediately from linear properties of differentiation and matrix multiplication.

Next, in order to show that **A** is a closed operator, we prove that if $\boldsymbol{\phi}_n \in \mathfrak{D}(\mathbf{A})$, $\lim\limits_{n\to\infty} \boldsymbol{\phi}_n =: \boldsymbol{\phi} \in L^1([0,w],\mathbb{R}^n)$ and $\lim\limits_{n\to\infty} \mathbf{A}\boldsymbol{\phi}_n =: \mathbf{v} \in L^1([0,w],\mathbb{R}^n)$, then $\boldsymbol{\phi} \in \mathfrak{D}(\mathbf{A})$ and $\mathbf{A}\boldsymbol{\phi} = \mathbf{v}$.

Since

$$\left(\mathbf{A}\boldsymbol{\phi}_n\right)(a) = -\frac{d}{da}\boldsymbol{\phi}_n(a) + \mathbf{Q}(a)\boldsymbol{\phi}_n(a),$$

it follows that

$$\begin{aligned}
\int_0^a \left(\mathbf{A}\boldsymbol{\phi}_n\right)(\sigma)\,d\sigma &= -\int_0^a \frac{d}{d\sigma}\boldsymbol{\phi}_n(\sigma)\,d\sigma + \int_0^a \mathbf{Q}(\sigma)\boldsymbol{\phi}_n(\sigma)\,d\sigma \\
&= -\boldsymbol{\phi}_n(a) + \boldsymbol{\phi}_n(0)\mathbf{L}(a) + \int_0^a \mathbf{Q}(\sigma)\boldsymbol{\phi}_n(\sigma)\,d\sigma \\
\text{or,} \qquad \boldsymbol{\phi}_n(0) &= \boldsymbol{\phi}_n(a) + \int_0^a \left(\mathbf{A}\boldsymbol{\phi}_n\right)(\sigma)\,d\sigma + \int_0^a \mathbf{Q}(a)\boldsymbol{\phi}_n(\sigma)\,d\sigma.
\end{aligned}$$

Then we find that

$$\left\|\boldsymbol{\phi}_n(0) - \boldsymbol{\phi}_m(0)\right\|_{\mathbb{R}^n} \|\mathbf{L}\|_{L^1} \leq \left\|\boldsymbol{\phi}_n - \boldsymbol{\phi}_m\right\|_{L^1} + w\left\|\mathbf{A}\boldsymbol{\phi}_n - \mathbf{A}\boldsymbol{\phi}_m\right\|_{L^1} + w\left\|\mathbf{Q}\boldsymbol{\phi}_n - \mathbf{Q}\boldsymbol{\phi}_m\right\|_{L^1}.$$

Hence $\{\boldsymbol{\phi}_n(0)\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $\lim\limits_{n\to\infty} \boldsymbol{\phi}_n(0) = \boldsymbol{\alpha}$. If we define

$$\boldsymbol{\omega}(a) := \boldsymbol{\alpha}\mathbf{L}(a) - \int_0^a \mathbf{v}(\sigma)\mathbf{L}(a,\sigma)\,d\sigma,$$

then we can see that $\boldsymbol{\omega}$ is absolutely continuous, because the right hand side is differentiable almost everywhere and its derivative is integrable on $[0,w]$ (we assumed that $\mathbf{Q}(a)$ is bounded in $[0,w]$). Also we have

$$\boldsymbol{\phi}_n(a) - \boldsymbol{\omega}(a) = \boldsymbol{\phi}_n(0)\mathbf{L}(a) - \int_0^a \left(\mathbf{A}\boldsymbol{\phi}_n\right)(\sigma)\mathbf{L}(a,\sigma)\,d\sigma - \boldsymbol{\alpha}\mathbf{L}(a) + \int_0^a \mathbf{v}(\sigma)\mathbf{L}(a,\sigma)\,d\sigma,$$

which shows that $\lim_{n\to\infty} \left\|\boldsymbol{\phi}_n - \boldsymbol{\omega}\right\|_{L^1} = 0$, so $\boldsymbol{\phi} = \boldsymbol{\omega}$ almost everywhere. Moreover, from $\boldsymbol{\phi}_n(0) = \int_0^w \mathbf{B}(s)\boldsymbol{\phi}_n(s)\,ds$ we have $\boldsymbol{\alpha} = \int_0^w \mathbf{B}(s)\boldsymbol{\phi}(s)\,ds$ when $n \to \infty$. Since $\boldsymbol{\phi} = \boldsymbol{\omega}$ almost everywhere, we obtain

$$\boldsymbol{\alpha} = \boldsymbol{\omega}(0) = \int_0^w \mathbf{B}(s)\boldsymbol{\omega}(s)\,ds.$$

Since we can identify $\boldsymbol{\phi}$ as $\boldsymbol{\omega}$ (as an element of L^1) by modifying the values on the null set, so we can say that $\boldsymbol{\phi} \in \mathfrak{D}(\mathbf{A})$ and $\mathbf{A}\boldsymbol{\phi} = \mathbf{v}$. This completes the proof. □

Lemma 4.8 $\rho(\mathbf{A}) \supset \{\lambda : Re\lambda > \bar{\beta} - \underline{\mu}\}$.

Proof. [28] Define the characteristic matrix $\tilde{\Gamma} : \mathbb{C} \to \mathbb{B}(\mathbb{R}^n, \mathbb{R}^n)$ by

$$\tilde{\Gamma}(\lambda) := \int_0^w \Gamma(a)e^{-\lambda a}da,$$

with

$$\Gamma(a) := \mathbf{B}(a)\mathbf{L}(a).$$

Let us first prove the following result which we are going to use in this lemma.

$$\sigma(\mathbf{A}) = \left\{\lambda : \lambda \in \mathbb{C},\ \det\left(\mathbf{I} - \tilde{\Gamma}(\lambda)\right) = 0\right\}. \tag{4.6}$$

To prove (4.6), let $\lambda \in \mathbb{C}$ be such that $\det\left(\mathbf{I} - \tilde{\Gamma}(\lambda)\right) = 0$. Then there must exist $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$ such that $\left(\mathbf{I} - \tilde{\Gamma}(\lambda)\right)\mathbf{x} = \mathbf{0}$. Define

$$\boldsymbol{\phi}(0) := \mathbf{x}, \tag{4.7}$$
$$\boldsymbol{\phi}(a) := e^{-\lambda a}\mathbf{L}(a)\boldsymbol{\phi}(0),\ a \geq 0. \tag{4.8}$$

Then $\boldsymbol{\phi} \neq \mathbf{0}$ and using $\left(\mathbf{I} - \tilde{\Gamma}(\lambda)\right)\mathbf{x} = \mathbf{0}$ we get

$$\begin{aligned}\boldsymbol{\phi}(0) &= \int_0^w e^{-\lambda a}\mathbf{B}(a)\mathbf{L}(a)\boldsymbol{\phi}(0)\,da \\ &= \int_0^w \mathbf{B}(a)\boldsymbol{\phi}(a)\,da,\end{aligned}$$

i.e., $\boldsymbol{\phi} \in \mathfrak{D}(\mathbf{A})$. Differentiating (4.8) with respect to a, we get

$$\begin{aligned}\boldsymbol{\phi}'(a) &= -\lambda e^{-\lambda a}\mathbf{L}(a)\boldsymbol{\phi}(0) + e^{-\lambda a}\frac{d}{da}\mathbf{L}(a)\boldsymbol{\phi}(0) \\ &= -\lambda e^{-\lambda a}\mathbf{L}(a)\boldsymbol{\phi}(0) + e^{-\lambda a}\mathbf{Q}(a)\mathbf{L}(a)\boldsymbol{\phi}(0) && \text{(by (4.4))} \\ &= -\lambda\boldsymbol{\phi}(a) + \mathbf{Q}(a)\boldsymbol{\phi}(a) && \text{(by (4.8)} \\ \text{or, } \lambda\boldsymbol{\phi}(a) &= -\boldsymbol{\phi}'(a) + \mathbf{Q}(a)\boldsymbol{\phi}(a) = \mathbf{A}\boldsymbol{\phi}(a),\end{aligned}$$

and this implies $\lambda \in \sigma(\mathbf{A})$. Now to prove our lemma, let $F(\lambda)$, $\lambda \in \mathbb{R}$ denote the Frobenius root of the characteristic matrix $\tilde{\Gamma}(\lambda)$, let $\tilde{\Gamma}_{ij}(\lambda)$ be the (i,j)th element of $\tilde{\Gamma}(\lambda)$. Let $Sp(\mathbf{A})$ denote the spectral radius of the operator $\mathbf{A}$. Then it can be shown that $Sp(\tilde{\Gamma}(\lambda)) \leq F(\lambda)$ ([24], $p.$ 57). From

$$\begin{aligned}
\left\|\tilde{\Gamma}(Re\lambda)\right\| &\leq \int_0^w \|\Gamma(v)\| \, e^{-vRe\lambda} dv \\
&\leq \int_0^w \|\mathbf{B}(v)\| \, \|\mathbf{L}(v)\| \, e^{-vRe\lambda} dv \\
&\leq \bar{\beta} \int_0^w e^{-\left(\underline{\mu}+\mathrm{Re}\lambda\right)v} \, dv \qquad \text{(by Lemma 4.6 and assumption } (\mathbf{A3})) \\
&\leq \frac{\bar{\beta}}{\mathrm{Re}\lambda + \underline{\mu}} \left[1 - e^{-\left(Re\lambda+\underline{\mu}\right)w}\right],
\end{aligned}$$

we obtain

$$\begin{aligned}
F(\mathrm{Re}\lambda) \leq \max_{\mathrm{j}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \tilde{\Gamma}_{\mathrm{ij}}(\mathrm{Re}\lambda) &= \left\|\tilde{\Gamma}(Re\lambda)\right\| \qquad (4.9) \\
&\leq \frac{\bar{\beta}}{\mathrm{Re}\lambda + \underline{\mu}} \left[1 - e^{-(Re\lambda+\underline{\mu})w}\right],
\end{aligned}$$

where for the proof of the inequality in (4.9), we refer to [24, p.63]. Therefore if $\mathrm{Re}\lambda > \bar{\beta} - \underline{\mu}$ then $Sp(\tilde{\Gamma}(\lambda)) \leq F(\mathrm{Re}\lambda) < 1$. Thus $\det(\mathbf{I} - \tilde{\Gamma}(\lambda)) \neq 0$ for $\mathrm{Re}\lambda > \bar{\beta} - \underline{\mu}$. Hence using (4.6), we get $\rho(\mathbf{A}) \supset \{\lambda : Re\lambda > \bar{\beta} - \underline{\mu}\}$. □

Lemma 4.9 *If $\lambda \in \rho(\mathbf{A})$ with $\lambda > \bar{\beta} - \underline{\mu}$, then $R(\lambda, \mathbf{A})$ is given by*

$$\begin{aligned}
R(\lambda, \mathbf{A})\boldsymbol{\psi}(a) = e^{-\lambda a}\, \mathbf{L}(a) &\left(\mathbf{I} - \int_0^w \mathbf{B}(b)\mathbf{L}(b)\, e^{-\lambda b} db\right)^{-1} \\
&\times \int_0^w \mathbf{B}(a)\mathbf{L}(a)\, e^{-\lambda a} \int_0^a e^{\lambda b}\mathbf{L}^{-1}(b)\boldsymbol{\psi}(b)\, db\, da + e^{-\lambda a}\mathbf{L}(a) \int_0^a e^{\lambda b}\, \mathbf{L}^{-1}(b)\boldsymbol{\psi}(b)\, db.
\end{aligned}$$

Proof. Following [65], define

$$\Delta(\lambda)\mathbf{x} := \mathbf{x} - \int_0^w e^{-\lambda a}\, \mathbf{B}(a)\mathbf{L}(a)\, \mathbf{x}\, da,$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathrm{Re}\lambda > \bar{\beta} - \underline{\mu}$. Let $\boldsymbol{\psi} \in \mathbf{L}^1$. There exists $\boldsymbol{\phi} \in L^1$ satisfying $(\lambda\mathbf{I} - \mathbf{A})\boldsymbol{\phi} = \boldsymbol{\psi}$ if and only if

$$\lambda\boldsymbol{\phi}(a) + \boldsymbol{\phi}'(a) - \mathbf{Q}(a)\boldsymbol{\phi}(a) = \boldsymbol{\psi}(a), \qquad (4.10)$$

$$\boldsymbol{\phi}(0) = \int_0^w \mathbf{B}(a)\boldsymbol{\phi}(a)da. \qquad (4.11)$$

The solution of (4.10) is given by (see [42, Proposition 5.2, p. 242])

$$\boldsymbol{\phi}(a) = e^{-\lambda a}\,\mathbf{L}(a)\boldsymbol{\phi}(0) + \int_0^a e^{-\lambda(a-b)}\,\mathbf{L}(a,b)\boldsymbol{\psi}(b)db. \tag{4.12}$$

Substituting this formula for $\boldsymbol{\phi}$ into the boundary equation (4.11), we obtain

$$\boldsymbol{\phi}(0) = \int_0^w e^{-\lambda a}\,\mathbf{B}(a)\mathbf{L}(a)\boldsymbol{\phi}(0)\,da + \int_0^w \mathbf{B}(a)\left[\int_0^a e^{-\lambda(a-b)}\,\mathbf{L}(a,b)\boldsymbol{\psi}(b)\,db\right] da.$$

Since $\lambda > \overline{\beta} - \underline{\mu}$, $\lambda \notin \sigma(\mathbf{A})$, so that $\Delta(\lambda)^{-1}$ exists and thus we get

$$\boldsymbol{\phi}(0) = \left(\mathbf{I} - \int_0^w e^{-\lambda a}\,\mathbf{B}(a)\mathbf{L}(a)\,da\right)^{-1} \int_0^w \mathbf{B}(a)\left[\int_0^a e^{-\lambda(a-b)}\,\mathbf{L}(a)\mathbf{L}^{-1}(b)\boldsymbol{\psi}(b)\,db\right] da. \tag{4.13}$$

Hence, putting the expression of $\boldsymbol{\phi}(0)$ from (4.13) into the equation (4.12), we get

$$\begin{aligned}\boldsymbol{\phi}(a) = R(\lambda,\mathbf{A})\boldsymbol{\psi}(a) &= e^{-\lambda a}\,\mathbf{L}(a)\left(\mathbf{I} - \int_0^w \mathbf{B}(b)\mathbf{L}(b)\,e^{-\lambda b}db\right)^{-1}\\ &\times \int_0^w \mathbf{B}(a)\mathbf{L}(a)\,e^{-\lambda a}\int_0^a e^{\lambda b}\,\mathbf{L}^{-1}(b)\boldsymbol{\psi}(b)\,db\,da + e^{-\lambda a}\,\mathbf{L}(a)\int_0^a e^{\lambda b}\,\mathbf{L}^{-1}(b)\boldsymbol{\psi}(b)\,db.\end{aligned}$$

□

Lemma 4.10 $\overline{\mathfrak{D}(A)} = \boldsymbol{L}^1(0,w;\,\mathbb{C}^n)$.

Proof. To prove the result, we follow [28] but we will go through the detailed calculations. If $\lambda > \bar{\beta} - \mu$, we can define $\boldsymbol{\phi}_\lambda = \lambda(\lambda\mathbf{I} - \mathbf{A})^{-1}\boldsymbol{\psi}$ for all $\boldsymbol{\psi} \in \mathbf{L}^1(0,w;\,\mathbb{C}^n)$. Since $\boldsymbol{\phi}_\lambda \in \mathfrak{D}(\mathbf{A})$, it is sufficient to show that $\boldsymbol{\phi}_\lambda \to \boldsymbol{\psi}$ as $\lambda \to \infty$ in $\mathbf{L}^1(0,w;\mathbb{C}^n)$. Now using the expression of the resolvent from Lemma 4.9 we have

$$\begin{aligned}\boldsymbol{\phi}_\lambda(a) &= \lambda e^{-\lambda a}\,\mathbf{L}(a)\left(\mathbf{I} - \int_0^w \mathbf{B}(b)\mathbf{L}(b)\,e^{-\lambda b}db\right)^{-1}\\ &\times \int_0^w \mathbf{B}(u)\mathbf{L}(u)\,e^{-\lambda u}\int_0^u e^{\lambda v}\,\mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\,dv\,du + \lambda e^{-\lambda a}\,\mathbf{L}(a)\int_0^a e^{\lambda v}\,\mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\,dv.\end{aligned}$$

We can write $\left\|\boldsymbol{\phi}_\lambda - \boldsymbol{\psi}\right\|_{L_1} \le J_1 + J_2$, where

$$\begin{aligned}J_1 = \lambda \int_0^w \bigg\| e^{-\lambda a}\,\mathbf{L}(a)&\left(\mathbf{I} - \int_0^w \mathbf{B}(b)\mathbf{L}(b)\,e^{-\lambda b}\,db\right)^{-1}\\ &\times \int_0^w \mathbf{B}(u)\mathbf{L}(u)\,e^{-\lambda u}\int_0^u e^{\lambda v}\,\mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\,dv\,du\bigg\|\,da,\end{aligned}$$

and

$$J_2 = \int_0^w \left\| \lambda e^{-\lambda a}\, \mathbf{L}(a) \int_0^a e^{\lambda v}\, \mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\, dv - \boldsymbol{\psi}(a) \right\| da. \tag{4.14}$$

Now

$$\begin{aligned}
\left\| \mathbf{I} - \int_0^w e^{-\lambda a}\, \mathbf{B}(a)\mathbf{L}(a)\, da \right\| &\geq \left| 1 - \left\| \int_0^w e^{-\lambda a}\, \mathbf{L}(a)\mathbf{B}(a)\, da \right\| \right| \\
&\geq 1 - \overline{\beta} \int_0^w e^{-(\lambda+\underline{\mu})a}\, da \quad \left(\because \overline{\beta} \int_0^w e^{-(\lambda+\underline{\mu})a}\, da < \frac{\bar{\beta}}{\lambda + \underline{\mu}} < 1 \right)
\end{aligned}$$

$$\text{or,} \quad \left\| \left(I - \int_0^w e^{-\lambda a} \mathbf{B}(a)\mathbf{L}(a)\, da \right)^{-1} \right\| \leq \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu}) + \overline{\beta} e^{-(\lambda+\underline{\mu})w}} \leq \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})}.$$

So

$$\begin{aligned}
J_1 &\leq \lambda \int_0^w \Bigg[e^{-(\lambda+\underline{\mu})a} \left\| \left(\mathbf{I} - \int_0^w e^{-\lambda a}\, \mathbf{B}(a)\mathbf{L}(a)\, da \right)^{-1} \right\| \\
&\qquad \int_0^w \left\| \mathbf{B}(u)\mathbf{L}(u)\, e^{-\lambda u} \right\| \times \int_0^u e^{\lambda v} \left\| L^{-1}(v)\boldsymbol{\psi}(v) \right\| \, dv\, du \Bigg] da \\
&\leq \lambda\overline{\beta} \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})} \int_0^w \left[e^{-(\lambda+\underline{\mu})a} \int_0^w e^{-(\lambda+\underline{\mu})u} \int_0^u e^{(\lambda+\underline{\mu})v} \left\| \boldsymbol{\psi}(v) \right\| \, dv\, du \right] da \\
&= \lambda\overline{\beta} \frac{\lambda + \underline{\mu}}{\lambda - (\overline{\beta} - \underline{\mu})} \int_0^w \left[e^{-(\lambda+\underline{\mu})a} \int_0^w e^{(\lambda+\underline{\mu})v} \left\| \boldsymbol{\psi}(v) \right\| \int_v^w e^{-(\lambda+\underline{\mu})u}\, du\, dv \right] da \\
&\leq \lambda\overline{\beta} \frac{1}{\lambda - (\overline{\beta} - \underline{\mu})} \int_0^w \left[e^{-(\lambda+\underline{\mu})a} \int_0^w e^{-(\lambda+\underline{\mu})v} . e^{(\lambda+\underline{\mu})v} \left\| \boldsymbol{\psi}(v) \right\| \, dv \right] da \\
&= \lambda\overline{\beta} \frac{1}{\lambda - (\overline{\beta} - \underline{\mu})} . \frac{\left(1 - e^{-(\lambda+\underline{\mu})w} \right)}{\lambda + \underline{\mu}} . \left\| \boldsymbol{\psi} \right\|_{L_1}
\end{aligned}$$

$$\text{or} \quad J_1 \leq \lambda \frac{1 - e^{-(\lambda+\underline{\mu})w}}{\lambda - (\bar{\beta} - \underline{\mu})} . \frac{\bar{\beta}}{\lambda + \underline{\mu}} \left\| \boldsymbol{\psi} \right\|_{L_1} , \tag{4.15}$$

and hence $\lim\limits_{\lambda\to\infty} J_1 = 0$.

To show $\lim_{\lambda\to\infty} J_2 = 0$, we notice that the expression J_2 in (4.14) can be written as

$$\begin{aligned}
J_2 &= \int_0^w \left\| \lambda \mathbf{L}(a) \int_0^a e^{-\lambda(a-v)}\, \mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\,dv - \boldsymbol{\psi}(a) \right\| da, \\
&\le \int_0^w \|\mathbf{L}(a)\| \left\| \int_0^a \lambda e^{-\lambda(a-v)} \mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\,dv - \mathbf{L}^{-1}(a)\boldsymbol{\psi}(a) \right\| da \\
&\le \int_0^w \left\| \lambda \int_0^a \lambda e^{-\underline{\mu} a} e^{-\lambda(a-v)} \mathbf{L}^{-1}(v)\boldsymbol{\psi}(v)\,dv - e^{-\underline{\mu} a}\mathbf{L}^{-1}(a)\boldsymbol{\psi}(a) \right\| da \qquad \left(\because \|\mathbf{L}(a)\| \le e^{-\underline{\mu} a}\right) \\
&= \int_0^w \left\| \int_0^a \lambda e^{-\underline{\mu} t} e^{-\lambda t} e^{-\underline{\mu}(a-t)} \mathbf{L}^{-1}(a-t)\boldsymbol{\psi}(a-t)\,dt - e^{-\underline{\mu} a}\mathbf{L}^{-1}(a)\boldsymbol{\psi}(a) \right\| da \quad (\text{where } t := a-v) \\
&= \int_0^w \left\| \int_0^a \lambda e^{-\lambda t} e^{-\underline{\mu} t} \bar{\boldsymbol{\psi}}(a-t)\,dt - \bar{\boldsymbol{\psi}}(a) \right\| da, \qquad (4.16)
\end{aligned}$$

where $\bar{\boldsymbol{\psi}}(a) := e^{-\underline{\mu} a}\mathbf{L}^{-1}(a)\boldsymbol{\psi}(a)$ and since

$$\left\|\bar{\boldsymbol{\psi}}(a)\right\| = e^{-\underline{\mu} a} \left\|\mathbf{L}^{-1}(a)\right\| \left\|\boldsymbol{\psi}(a)\right\| \le e^{-\underline{\mu} a} e^{\underline{\mu} a} \left\|\boldsymbol{\psi}(a)\right\| = \left\|\boldsymbol{\psi}(a)\right\|,$$

we have $\bar{\boldsymbol{\psi}} \in L^1$. Now, following [65], to prove that the expression (4.16) tends to zero as $\lambda \to \infty$, let $t > 0$ and define $\boldsymbol{\psi}^t \in L^1$ by

$$\boldsymbol{\psi}^t(a) := \begin{cases} \mathbf{0}, & a < t; \\ e^{-\underline{\mu} t}\bar{\boldsymbol{\psi}}(a-t), & a > t. \end{cases}$$

Since

$$\begin{aligned}
\left\|\boldsymbol{\psi}^t - \bar{\boldsymbol{\psi}}\right\|_{L^1} &= \int_0^\infty \left\|\boldsymbol{\psi}^t(a) - \bar{\boldsymbol{\psi}}(a)\right\| da \\
&= \int_0^t \left\|\bar{\boldsymbol{\psi}}(a)\right\| da + \int_t^\infty \left\| e^{-\underline{\mu} t}\bar{\boldsymbol{\psi}}(a-t) - \bar{\boldsymbol{\psi}}(a)\right\| da \\
&\le \int_0^t \left\|\bar{\boldsymbol{\psi}}(a)\right\| da + \int_t^\infty \left\| \left(e^{-\underline{\mu} t} - 1\right)\bar{\boldsymbol{\psi}}(a-t)\right\| da + \int_t^\infty \left\|\bar{\boldsymbol{\psi}}(a-t) - \bar{\boldsymbol{\psi}}(a)\right\| da \\
&= \int_0^t \left\|\bar{\boldsymbol{\psi}}(a)\right\| da + \int_0^\infty \left\| \left(e^{-\underline{\mu} t} - 1\right)\bar{\boldsymbol{\psi}}(z)\right\| dz + \int_0^\infty \left\|\bar{\boldsymbol{\psi}}(z) - \bar{\boldsymbol{\psi}}(t+z)\right\| dz,
\end{aligned} \tag{4.17}$$

we claim that

$$\lim_{t\to 0} \left\|\boldsymbol{\psi}^t - \bar{\boldsymbol{\psi}}\right\|_{L^1} = 0. \tag{4.18}$$

To prove (4.18), the last term from (4.17) needs some attention and we show that

$$\lim_{t\to 0} \int_0^\infty \left\|\bar{\boldsymbol{\psi}}(t+z) - \bar{\boldsymbol{\psi}}(z)\right\| dz = 0$$

$$\text{or,} \qquad \lim_{t\to 0} \left\|\bar{\boldsymbol{\psi}}_t - \bar{\boldsymbol{\psi}}\right\|_{L^1} = 0, \tag{4.19}$$

where $\bar{\boldsymbol{\psi}}_t(z) := \bar{\boldsymbol{\psi}}(t+z)$.

Now as the continuous functions with compact support are dense in L^1 ([59, Theorem 2.4]), the proof of the fact (4.19) is a simple consequence of the approximation of integrable functions by continuous functions with compact support. In fact for any $\epsilon > 0$, we can find such a function $\mathbf{g}$ such that $\|\bar{\boldsymbol{\psi}} - \mathbf{g}\|_{L^1} < \epsilon$. Now

$$\bar{\boldsymbol{\psi}}_t - \bar{\boldsymbol{\psi}} = (\mathbf{g}_t - \mathbf{g}) + \left(\bar{\boldsymbol{\psi}}_t - \mathbf{g}_t\right) - (\bar{\boldsymbol{\psi}} - \mathbf{g}),$$

where $\mathbf{g}_t(x) := \mathbf{g}(t+x)$. However, $\|\bar{\boldsymbol{\psi}}_t - \mathbf{g}_t\|_{L^1} = \|\bar{\boldsymbol{\psi}} - \mathbf{g}\|_{L^1} < \epsilon$, while since $\mathbf{g}$ is continuous and has compact support, we have

$$\left\|\mathbf{g}_t - \mathbf{g}\right\|_{L^1} = \int_0^\infty \left\|\mathbf{g}(t+z) - \mathbf{g}(z)\right\| dz \to 0 \text{ as } t \to 0.$$

Therefore, if $|t| < \delta$, where δ is sufficiently small, then $\|\mathbf{g}_t - \mathbf{g}\|_{L^1} < \epsilon$, and as a result $\|\bar{\boldsymbol{\psi}}_t - \bar{\boldsymbol{\psi}}\|_{L^1} < 3\epsilon$, whenever $|t| < \delta$. This proves (4.19) and subsequently we have (4.18). Now from (4.16), we have

$$\begin{aligned}
J_2 &\le \int_0^\infty \left\| \int_0^a \lambda e^{-\lambda t} e^{-\underline{\mu} t} \bar{\boldsymbol{\psi}}(a-t)\, dt - \bar{\boldsymbol{\psi}}(a) \right\| da \\
&= \int_0^\infty \left\| \int_0^a \lambda e^{-\lambda t} e^{-\underline{\mu} t} \bar{\boldsymbol{\psi}}(a-t)\, dt - \int_0^\infty \lambda e^{-\lambda t} \bar{\boldsymbol{\psi}}(a)\, dt \right\| da \qquad \left(\because \int_0^\infty \lambda e^{-\lambda t}\, dt = 1\right) \\
&= \int_0^\infty \left\| \int_0^\infty \lambda e^{-\lambda t} \left[\boldsymbol{\psi}^t(a) - \bar{\boldsymbol{\psi}}(a)\right] dt \right\| da \\
&\le \int_0^\infty \left\{ \lambda e^{-\lambda t} \int_0^\infty \left\|\boldsymbol{\psi}^t(a) - \bar{\boldsymbol{\psi}}(a)\right\| da \right\} dt \\
&\le \sup_{0 \le t \le \epsilon} \left\|\boldsymbol{\psi}^t - \bar{\boldsymbol{\psi}}\right\|_{L^1} \int_0^\epsilon \lambda e^{-\lambda t}\, dt + 2\left\|\bar{\boldsymbol{\psi}}\right\|_{L^1} \int_\epsilon^\infty \lambda e^{-\lambda t}\, dt \\
&\le \sup_{0 \le t \le \epsilon} \left\|\boldsymbol{\psi}^t - \bar{\boldsymbol{\psi}}\right\|_{L^1} + 2\left\|\bar{\boldsymbol{\psi}}\right\|_{L^1} e^{-\lambda \epsilon}.
\end{aligned}$$

Therefore, using (4.18), we find $\lim_{\lambda \to \infty} J_2 = 0$. Hence we can conclude that

$$\lim_{\lambda \to \infty} \left\|\boldsymbol{\phi}_\lambda - \boldsymbol{\psi}\right\|_{L^1} = 0,$$

and this completes the proof. □

Lemma 4.11 $\|(\lambda \mathbf{I} - \mathbf{A})^{-1}\| \le \dfrac{1}{\lambda - (\bar{\beta} - \underline{\mu})}$.

Proof. From the proof of Lemma (4.10) we have

$$\left\|(\lambda \mathbf{I} - \mathbf{A})^{-1} \boldsymbol{\psi}\right\| \le \frac{1}{\lambda} J_1 + \int_0^w e^{-\lambda a} \int_0^a e^{\lambda v} \|\mathbf{L}(a,v)\| \left\|\boldsymbol{\psi}(v)\right\| dv\, da,$$

where, from (4.15), we have

$$J_1 \le \frac{\lambda\bar{\beta}}{\lambda+\underline{\mu}}.\frac{1}{\lambda-(\bar{\beta}-\underline{\mu})}.\left\|\psi\right\|_{L^1},$$

and

$$\int_0^w e^{-\lambda a}\int_0^a e^{\lambda v}\,\|\mathbf{L}(a,v)\|\left\|\psi(v)\right\|\,dv\,da \le \frac{1}{\lambda+\underline{\mu}}.\left\|\psi\right\|_{L^1}.$$

So, we have

$$\left\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\right\| \le \frac{1}{\lambda+\underline{\mu}}\left[\frac{\bar{\beta}}{\lambda-(\bar{\beta}-\underline{\mu})}+1\right] = \frac{1}{\lambda-(\bar{\beta}-\underline{\mu})}. \qquad \square$$

Proof of Theorem 4.5. Considering above lemmas, we can see that $(\mathbf{A},\mathfrak{D}(\mathbf{A}))$ satisfies condition (2) of Hille–Yosida Theorem 2.18. Hence, the proof of Theorem 4.5 follows immediately due to equivalence conditions (1) and (2) in theorem 2.18. □

The specific spectral properties of the migration matrix what has been discussed in this chapter is going to be used in the asymptotic analysis of the model in next chapter. Also the existence of the C_0-semigroup (Theorem 4.5) is one of the crucial result needed for all of our further analysis.

CHAPTER 5

Asymptotic Analysis of the Perturbed Model

5.1 Aggregated Model

Biological heuristics suggests that no geographical structure should persist for very large interstate migration rates; that is, for $\epsilon \to 0$. Here we also note that both biological and mathematical analysis rely on $\lambda = 0$ being the dominant simple eigenvalue of $\mathbf{C}(a)$ for each $a \in \mathbb{R}_+$ with the corresponding positive eigenvector, denoted by $\mathbf{k}(a)$, and the left eigenvector $\mathbf{1} = (1, 1, \ldots, 1)$. Vector $\mathbf{k}(a)$ is normalized to satisfy $\mathbf{1} \cdot \mathbf{k} = 1$ and $\mathbf{k}(a) = (k_1(a), \ldots, k_n(a))$ is the so-called stable patch structure; that is, the asymptotic (as $t \to \infty$ and disregarding demographic processes) distribution of the population among the patches for a given age a. Thus, in population theory, the components of $\mathbf{k}$ are approximated as $k_i \approx u_i/u$ for $i = 1, \ldots, n$, where

$$u := \mathbf{u} \cdot \mathbf{1} = \sum_{i=1}^{n} u_i.$$

Adding together equations in (4.1) and using the above approximation we obtain

$$u_t \approx -u_a - \mu^* u, \tag{5.1}$$

where $\mu^* := \mathbf{1} \cdot \mathbf{Mk}$ is the 'aggregated' mortality. This model, supplemented with boundary condition

$$u(0, t) \approx \int_0^\infty \beta^*(a) u(a, t)\, da, \tag{5.2}$$

where $\beta^* := \mathbf{1} \cdot \mathbf{Bk}$ is the 'aggregated fertility', is expected to provide an approximate description of the averaged population. Thus, (5.1) is the macroscopic and (4.1) the mesoscopic description of the population.

The main goal of this chapter is a rigorous validation of the above heuristics; that is, that the true total population u can be approximated by the solution $\bar{u}$ of the aggregated problem (5.1)–(5.2) (where '$\approx$' is replaced by '$=$') with an ϵ-order error. The analysis is involved due to the initial and boundary conditions which are not consistent with those of the aggregated model. This makes the problem singularly perturbed and thus necessitates a careful analysis of the boundary, corner and initial layer phenomena.

5.2 Formal Asymptotic Expansion

In order to achieve our goal, we derive formulae for the asymptotic expansion, which are formal in the sense that they are valid if all terms are smooth enough to allow for applications of necessary operations. But it is not always so and a full justification of the validity of the expansion requires using integral formulation of the problem which is much more involved and is referred to Chapter 7. However, the results given here serve as a guideline for the proper analysis and, once validated, are easier to use. To start our asymptotic analysis, using the spectral properties of the migration matrix $\mathbf{C}$, first we define projection operators on the space $\mathbb{R}^n$ and subsequently we apply these projection operators on the perturbed system. The validity of these projections follows from Theorem 2.6.

5.2.1 Spectral Projections

The assumptions on $\mathbf{C}(a)$ ensure that for each $a \in \mathbb{R}_+$, 0 is the simple dominant eigenvalue of $\mathbf{C}(a)$ with positive eigenvector $\mathbf{k}(a)$. The null-space of the adjoint matrix is spanned by $\mathbf{1} = (1, \ldots, 1)$ and we will normalize $\mathbf{k}$ to satisfy $\mathbf{1} \cdot \mathbf{k} = 1$. The spectral projections $\mathbf{P}$, $\mathbf{Q}$ (depending on a) are defined as follows

$$\mathbf{Pu} := (\mathbf{u} \cdot \mathbf{1})\mathbf{k} = \mathbf{k} \sum_{i=1}^{n} u_i \, ,$$

while the complementary projection is given by $\mathbf{Qu} := \mathbf{u} - \mathbf{Pu}$. The 'eigenspace' corresponding to $\lambda = 0$ (also known as *hydrodynamic space*) is a-dependent and is given as $V(a) := Span\{\mathbf{k}(a)\} =: [\mathbf{k}(a)]$. However, the complementary space to $V(a)$ (also known as *kinetic space*) is independent of a and it is given by $S := \mathrm{Im}\mathbf{Q} := \{\mathbf{x} : \mathbf{1} \cdot \mathbf{x} = 0\}$. Hence any element $\mathbf{u} \in \mathbb{R}^n$ can be decomposed as

$$\mathbf{u} = \mathbf{Pu} + \mathbf{Qu} =: \mathbf{v} + \mathbf{w} =: u\mathbf{k} + \mathbf{w},$$

where u is a scalar defined as $\sum_{i=1}^{n} u_i$. For each $a \in \mathbb{R}_+$ the decomposition $\mathbb{R}^n = V(a) \oplus S$ reduces $\mathbf{C}(a)$. Its part in $V(a)$ is zero whereas for $\mathbf{C}_S := \mathbf{QCQ} = \mathbf{C}|_S$ we have

$$s\left(\mathbf{C}_S(a)\right) := \max\left\{\mathrm{Re}\lambda(a) : \lambda(a) \in \sigma(\mathbf{C}_S(a))\right\} < 0.$$

For the asymptotic analysis of the perturbed system we need

$$\sup_{a \in \mathbb{R}_+} s(\mathbf{C}_S(a)) =: \zeta < 0.$$

Further properties of $\mathbf{C}$ are mentioned in the following lemma.

Lemma 5.1 *Under the above assumptions, $\mathbf{C}_S^{-1} \in C_b^2(\mathbb{R}_+, \mathbb{R}^{n^2})$ and $\mathbf{k} \in C_b^1(\mathbb{R}_+, \mathbb{R}^n)$.*

Proof: The first statement is obvious since the determinant of $\mathbf{C}_S(a)$ is twice differentiable and bounded away from zero by uniform invertibility of $\mathbf{C}_S(a)$.
To prove the second statement, we note that the spectral projection onto the eigenspace associated with $\lambda = 0$ is defined by

$$\mathbf{P}(a) := \frac{1}{2\pi i} \int_\Gamma \left(\lambda \mathbf{I} - \mathbf{C}_S(a)\right)^{-1} \, d\lambda, \tag{5.3}$$

where Γ is the circle surrounding the eigenvalue 0 of, say radius $\rho = -\zeta/2$. Then Γ is contained in the intersection of resolvent sets of each $\mathbf{C}_S(a)$. Thus we can apply [29] to claim that $\mathbf{P}(a)$ is as smooth as $\mathbf{C}_S$. But $\mathbf{k}$ can be expressed as $\mathbf{k}(a) = \mathbf{P}(a)\mathbf{x}/(\mathbf{x} \cdot \mathbf{1})$ for a fixed vector $\mathbf{x}$, so $\mathbf{k}$ is smooth as $\mathbf{P}$. Since $\lambda \in \Gamma$ which is at least $-\zeta/2$ away from any eigenvalue of $\mathbf{C}_S(a), a \in \mathbb{R}_+$ it is clear that differentiation of (5.3) will produce bounded derivatives and hence the required derivatives of $\mathbf{k}$ are bounded. □

5.2.2 Projected System of Equations

To apply projections on (4.1)–(4.3), we will use the following lemma.

Lemma 5.2 *For sufficiently regular function $a \to \mathbf{u}(a)$ we have*

$$\mathbf{P}(u_a\mathbf{k}) = u_a\mathbf{k}, \qquad \mathbf{Q}(u_a\mathbf{k}) = \mathbf{0}, \qquad \mathbf{Q}(u\mathbf{k}_a) = u\mathbf{k}_a,$$

$$\mathbf{P}(u\mathbf{k}_a) = 0, \qquad \mathbf{PCu} = 0, \qquad \mathbf{QCPu} = 0.$$

Proof. Now $\mathbf{P}(u_a\mathbf{k}) = u_a(\mathbf{1} \cdot \mathbf{k})\mathbf{k} = u_a\mathbf{k}$. Next using this we have

$$\mathbf{Q}(u_a\mathbf{k}) = u_a\mathbf{k} - \mathbf{P}(u_a\mathbf{k}) = u_a\mathbf{k} - u_a\mathbf{k} = \mathbf{0}.$$

Also

$$\mathbf{P}(u\mathbf{k}_a) = u(\mathbf{1} \cdot \mathbf{k}_a)\mathbf{k} = 0.$$

Since $\mathbf{1} \cdot \mathbf{k} = 1$, we get $\mathbf{1} \cdot \mathbf{k}_a = 0$ and thus

$$\mathbf{Q}(u\mathbf{k}_a) = u\mathbf{k}_a - \mathbf{P}(u\mathbf{k}_a) = u\mathbf{k}_a - u\,(\mathbf{1} \cdot \mathbf{k}_a)\,\mathbf{k} = u\mathbf{k}_a - 0 = u\mathbf{k}_a,$$

and

$$\mathbf{PCu} = (\mathbf{1} \cdot \mathbf{Cu})\,\mathbf{k} = \left(\sum_{j=1}^{n} \sum_{i=1}^{n} c_{ij} u_j \right) \mathbf{k} = 0,$$

since the sum of the entries of a column of $\mathbf{C}$ is zero. Also

$$\mathbf{QCPu} = \mathbf{CPu} - \mathbf{PCPu} = \mathbf{Cv} - \mathbf{PCv} = u\mathbf{Ck} - u(\mathbf{1} \cdot \mathbf{Ck})\mathbf{k} = 0,$$

as $\mathbf{Ck} = 0$. We note that

$$\mathbf{1} \cdot \mathbf{w}_a = \mathbf{1} \cdot \mathbf{Qu}_a = \mathbf{1} \cdot (\mathbf{u}_a - \mathbf{Pu}_a) = \mathbf{1} \cdot \mathbf{u}_a - \mathbf{1} \cdot \mathbf{Pu}_a = u_a - \mathbf{1} \cdot (\mathbf{k}u_a) = u_a - u_a = 0,$$

since $\mathbf{1} \cdot \mathbf{k} = 0$. This shows that $\mathbf{w}_a \in S$. □

Now, using Lemma 5.2 and operating formally with $\mathbf{P}$ and $\mathbf{Q}$ on both sides of (4.1) we get

$$\mathbf{v}_t = -\mathbf{v}_a - \mathbf{PMv} - \mathbf{PMw}, \tag{5.4}$$

and

$$\epsilon\mathbf{w}_t = -\epsilon u\mathbf{k}_a - \epsilon\mathbf{w}_a - \epsilon\mathbf{QMv} - \epsilon\mathbf{QMw} + \mathbf{C}_S\mathbf{w}, \tag{5.5}$$

where we have used the following fact

$$\mathbf{u}_a = \mathbf{v}_a + \mathbf{w}_a = u\mathbf{k}_a + u_a\mathbf{k} + \mathbf{w}_a\,.$$

Similarly applying projections $\mathbf{P}$, $\mathbf{Q}$ on initial condition (4.3) and boundary condition (4.2) we get, respectively,

$$\mathbf{v}(a,0) =: \mathbf{v}_0(a), \qquad \mathbf{w}(a,0) =: \mathbf{w}_0(a), \tag{5.6}$$

and

$$\mathbf{v}(0,t) = \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{v}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{w}(s,t))\,ds, \tag{5.7}$$

$$\mathbf{w}(0,t) = \int_0^\infty \mathbf{B}(s)\mathbf{v}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\mathbf{w}(s,t)\,ds - \mathbf{v}(0,t), \tag{5.8}$$

where, for (5.8), we have used the following relation:

$$\mathbf{u}(0,t) = \mathbf{v}(0,t) + \mathbf{w}(0,t) = \int_0^\infty \mathbf{B}(s)\mathbf{v}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\mathbf{w}(s,t)\,ds.$$

5.2.3 Bulk Approximation

Following the idea of the Chapman–Enskog asymptotic method as described in [11], we put $\mathbf{w} = \mathbf{w}_0 + \epsilon\, \mathbf{w}_1 + \cdots$ leaving, however, $\mathbf{v}$ unexpanded. Thus

$$(\mathbf{v}(a,t), \mathbf{w}(a,t)) = (\mathbf{v}(a,t), \mathbf{w}_0(a,t) + \epsilon \mathbf{w}_1(a,t) + \cdots).$$

Inserting these into (5.4)–(5.8) we have

$$\begin{aligned}
\mathbf{v}_t &= -\mathbf{v}_a - \mathbf{PMv} - \mathbf{PM}(\mathbf{w}_0 + \epsilon\, \mathbf{w}_1 + \cdots), &(5.9)\\
\epsilon \mathbf{w}_{0,t} + \epsilon^2 \mathbf{w}_{1,t} + \cdots &= -\epsilon \mathbf{k}_a u - \epsilon \mathbf{w}_{0,a} - \epsilon^2 \mathbf{w}_{1,a} - \cdots - \epsilon \mathbf{QMv}\\
&\quad - \epsilon \mathbf{QM}(\mathbf{w}_0 + \epsilon \mathbf{w}_1 + \cdots) + \mathbf{C}_S(\mathbf{w}_0 + \epsilon\, \mathbf{w}_1 + \cdots), &(5.10)\\
\mathbf{v}(0,t) &= \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{v}(s,t))\, ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)(\mathbf{w}_0 + \epsilon\, \mathbf{w}_1 + \cdots))\, ds, &(5.11)\\
\mathbf{w}_0(0,t) + \epsilon\, \mathbf{w}_1(0,t) + \cdots &= \int_0^\infty \mathbf{B}(s)\mathbf{v}(s,t)\, ds + \int_0^\infty \mathbf{B}(s)(\mathbf{w}_0 + \epsilon\, \mathbf{w}_1 + \cdots)\, ds - \mathbf{v}(0,t),\\
\mathbf{v}(a,0) &= \mathbf{v}_0(a), &(5.12)\\
\mathbf{w}_0(a,0) + \epsilon\, \mathbf{w}_1(a,0) + \cdots &= \mathbf{w}_0(a).
\end{aligned}$$

Let $\bar{\mathbf{v}}$ and $\bar{\mathbf{w}} := \bar{\mathbf{w}}_0 + \epsilon \bar{\mathbf{w}}_1$ denote the solutions (called *bulk solutions*) to the truncated equations. Comparing ϵ^0 order term of (5.9), we have

$$\bar{\mathbf{v}}_t = -\bar{\mathbf{v}}_a - \mathbf{PM}\bar{\mathbf{v}} - \mathbf{PM}\bar{\mathbf{w}}_0. \tag{5.13}$$

Comparing ϵ^0 and ϵ order terms of (5.10), we get respectively

$$\mathbf{C}_S \bar{\mathbf{w}}_0 = \mathbf{0}, \tag{5.14}$$

and

$$\bar{\mathbf{w}}_{0,t} = -\,\mathbf{k}_a \bar{u} - \bar{\mathbf{w}}_{0,a} - \mathbf{QM}\bar{\mathbf{v}} - \mathbf{QM}\bar{\mathbf{w}}_0 + \mathbf{C}_S \bar{\mathbf{w}}_1, \tag{5.15}$$

where $\bar{\mathbf{v}} = \bar{u}\mathbf{k}$.

Now $\bar{\mathbf{w}}_0 \in \mathbf{S}$. Restriction of $\mathbf{Q}$ to the subspace $\mathbf{S}$ becomes an identity map and all the eigenvalues of $\mathbf{C}$ have negative real parts in $\mathbf{S}$. Therefore $\mathbf{C}_S^{-1}$ exists. Hence, from (5.14), we have

$$\bar{\mathbf{w}}_0 = \mathbf{0},$$

while (5.15) gives

$$\mathbf{C}_S \bar{\mathbf{w}}_1 = \mathbf{k}_a \bar{u} + \mathbf{QM}\bar{\mathbf{v}}. \tag{5.16}$$

Comparing ϵ^0 order terms of (5.11) we have

$$\bar{\mathbf{v}}(0,t) = \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\bar{\mathbf{v}}(s,t))\, ds. \tag{5.17}$$

Using the definition of $\bar{\mathbf{v}} := \mathbf{k}\bar{u}$ in equations of (5.13), (5.17) and (4.3), respectively, we get

$$\begin{aligned} \bar{u}_t &= -\bar{u}_a - \mu^*\bar{u}, \\ \bar{u}(0,t) &= \int_0^\infty \beta^*(s)\bar{u}(s,t)\,ds, \\ \bar{u}(a,0) &= \phi(a), \end{aligned} \tag{5.18}$$

where we have defined $\phi(a) := \sum_{i=1}^n \phi_i(a)$. The system (5.18) is precisely the 'aggregated model' (5.1)–(5.2).

The error due to the approximation we made only using the bulk part is

$$\begin{aligned} \mathbf{E}(a,t) &:= (\mathbf{e}(a,t), \mathbf{f}(a,t)) = (e\mathbf{k}, \mathbf{f}) \\ &:= (\mathbf{v}(t) - \overline{\mathbf{v}}(t), \mathbf{w}(t) - \overline{\mathbf{w}}(t)) \\ &= (\mathbf{v}(t) - \overline{\mathbf{v}}(t), \mathbf{w}(t) - \epsilon\overline{\mathbf{w}}_1(t)). \end{aligned} \tag{5.19}$$

From this definition we have the following system of error equations

$$\begin{aligned} \mathbf{e}_t &= \mathbf{v}_t - \bar{\mathbf{v}}_t \\ &= -\mathbf{v}_a - \mathbf{PMv} - \mathbf{PMw} - \bar{\mathbf{v}}_t \\ &= -\mathbf{e}_a - \mathbf{PMe} - \mathbf{PMf} - \bar{\mathbf{v}}_t - \bar{\mathbf{v}}_a - \mathbf{PM}\bar{\mathbf{v}} + \epsilon\mathbf{PM}\bar{\mathbf{w}}_1 \\ &= -\mathbf{e}_a - \mathbf{PMe} - \mathbf{PMf} - \epsilon\mathbf{PM}\bar{\mathbf{w}}_1, \end{aligned}$$

where we have used (5.13). Next

$$\begin{aligned} \mathbf{f}_t &= \mathbf{w}_t - \epsilon\bar{\mathbf{w}}_{1,t} \\ &= -\mathbf{k}_a u - \mathbf{w}_a - \mathbf{QMv} - \mathbf{QMw} + \frac{1}{\epsilon}\mathbf{C}_S\mathbf{w} - \epsilon\bar{\mathbf{w}}_{1,t} \\ &= -\mathbf{k}_a u - \mathbf{f}_a - \epsilon\bar{\mathbf{w}}_{1,a} - \mathbf{QMe} - \mathbf{QM}\bar{\mathbf{v}} - \mathbf{QMf} - \epsilon\mathbf{QM}\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\mathbf{f} + \mathbf{C}_S\bar{\mathbf{w}}_1 - \epsilon\bar{\mathbf{w}}_{1,t} \\ &= -e\mathbf{k}_a - \mathbf{f}_a - \epsilon\bar{\mathbf{w}}_{1,a} - \mathbf{QMe} - \mathbf{QMf} - \epsilon\mathbf{QM}\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\mathbf{f} - \epsilon\bar{\mathbf{w}}_{1,t}\,, \end{aligned}$$

where we have used (5.16). The initial conditions are

$$\mathbf{e}(a,0) = \mathbf{v}(a,0) - \bar{\mathbf{v}}(a,0) = \mathbf{0},$$

and

$$\begin{aligned} \mathbf{f}(a,0) &= \mathbf{w}(a,0) - \epsilon\bar{\mathbf{w}}_1(a,0) \\ &= \mathbf{w}_0(a) - \epsilon\mathbf{C}_S^{-1}(a)\left(\mathbf{k}_a\bar{u}(a,0) + \mathbf{Q}(a)\mathbf{M}(a)\bar{\mathbf{v}}(a,0)\right), \end{aligned}$$

while the first boundary conditions is

$$
\begin{aligned}
\mathbf{e}(0,t) &= \mathbf{v}(0,t) - \bar{\mathbf{v}}(0,t) \\
&= \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{v}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{w}(s,t))\,ds - \bar{\mathbf{v}}(0,t) \\
&= \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{e}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{f}(s,t))\,ds \\
&\quad + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\bar{\mathbf{v}}(s,t))\,ds + \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\bar{\mathbf{w}}_1(s,t))\,ds - \bar{\mathbf{v}}(0,t) \\
&= \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{e}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\mathbf{f}(s,t))\,ds + \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\bar{\mathbf{w}}_1(s,t))\,ds,
\end{aligned}
$$

where we used (5.17). Further,

$$
\begin{aligned}
\mathbf{f}(0,t) &= \mathbf{w}(0,t) - \epsilon\,\bar{\mathbf{w}}_1(0,t) \\
&= \int_0^\infty \mathbf{B}(s)\mathbf{v}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\mathbf{w}(s,t)\,ds - \mathbf{v}(0,t) - \epsilon\bar{\mathbf{w}}_1(0,t) \\
&= \int_0^\infty \mathbf{B}(s)\mathbf{e}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\mathbf{f}(s,t)\,ds \\
&\quad + \epsilon \int_0^\infty \mathbf{B}(s)\bar{\mathbf{w}}_1(s,t)\,ds - \mathbf{e}(0,t) - \bar{\mathbf{v}}(0,t) - \epsilon\bar{\mathbf{w}}_1(0,t).
\end{aligned}
$$

From the above set of error equations, we observe that in the initial and boundary error equations we do have some terms which are not of order ϵ. Therefore we cannot hope for (5.19) to be an $O(\epsilon)$ approximation of $\bar{u}$. To remedy the situation we have to introduce corrections which will take care of the transient phenomena occurring close to $t = 0$ and to the boundary $a = 0$. They should not 'spoil' the approximation away from spatial and temporal boundaries and thus should rapidly decrease to zero with increasing distance from both boundaries.

5.2.4 Initial Layer

In order to construct the initial layer corrector we blow up the neighbourhood of $t = 0$ by introducing the 'fast' time $\tau := t/\epsilon$ and the initial layer corrections by $\tilde{\mathbf{u}}(a,\tau) = (\tilde{\mathbf{v}}(a,\tau), \tilde{\mathbf{w}}(a,\tau))$. Thanks to the linearity of the problem, we approximate the solution $\mathbf{u}$ as the sum of the bulk part obtained above and the initial layer which we construct below. We insert the formal expansion

$$
\tilde{\mathbf{v}}(a,\tau) = \tilde{\mathbf{v}}_0(a,\tau) + \epsilon\tilde{\mathbf{v}}_1(a,\tau) + \cdots, \qquad \tilde{\mathbf{w}}(a,\tau) = \tilde{\mathbf{w}}_0(a,\tau) + \epsilon\tilde{\mathbf{w}}_1(a,\tau) + \cdots
$$

into (5.4) and (5.5), getting,

$$
\tilde{\mathbf{v}}_{0,\tau} + \epsilon\tilde{\mathbf{v}}_{1,\tau} + \cdots = -\epsilon\left(\tilde{\mathbf{v}}_{0,a} + \epsilon\tilde{\mathbf{v}}_{1,a} + \cdots\right) - \epsilon\,\mathbf{PM}(\tilde{\mathbf{v}}_0 + \epsilon\tilde{\mathbf{v}}_1 + \cdots) - \epsilon\mathbf{PM}(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots), \tag{5.20}
$$

and

$$\tilde{\mathbf{w}}_{0,\tau} + \epsilon\tilde{\mathbf{w}}_{1,\tau} + \cdots = -\epsilon\mathbf{k}_a\tilde{u} - \left(\epsilon\tilde{\mathbf{w}}_{0,a} + \epsilon^2\tilde{\mathbf{w}}_{1,a} + \cdots\right) - \epsilon\mathbf{QM}(\tilde{\mathbf{v}}_0 + \epsilon\tilde{\mathbf{v}}_1 + \cdots) \\ - \epsilon\mathbf{QM}(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots) + \mathbf{C}_S(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots), \tag{5.21}$$

where $\tilde{u} := \mathbf{1}\cdot\tilde{\mathbf{u}}$. Comparing ϵ^0 order terms from (5.20) and (5.21) we get respectively

$$\tilde{\mathbf{v}}_{0,\tau} = \mathbf{0}, \tag{5.22}$$

$$\tilde{\mathbf{w}}_{0,\tau} = \mathbf{C}_S\tilde{\mathbf{w}}_0. \tag{5.23}$$

Since $\tilde{\mathbf{v}}_0$ is a layer term, so it should be important only in a small vicinity of the initial point $t = 0$. Since from (5.22), $\tilde{\mathbf{v}}_0$ = constant, we must take

$$\tilde{\mathbf{v}}_0 = \mathbf{0}.$$

Now (5.23) gives

$$\tilde{\mathbf{w}}_0(a,\tau) = e^{\tau\mathbf{Q}(a)\mathbf{C}(a)}\tilde{\mathbf{w}}_0(a,0).$$

We define $\tilde{\mathbf{w}}_0(a,0) := \mathbf{w}_0(a)$ in order to get rid of $\mathbf{w}_0$ from the error equation of the kinetic part, namely $\mathbf{f}(a,0)$. Due to the assumption that $\lambda = 0$ is the dominant eigenvalue of $\mathbf{C}$ uniformly in a, the type of $\{\exp(\tau\mathbf{C}_S)\}_{\tau\geq 0}$ in S is negative uniformly in a and thus $\exp(\tau\mathbf{C}_S)$ decays to zero exponentially fast. We also note that the initial layer is fully determined by the initial condition $\mathbf{w}_0$ and thus no corrections to the boundary conditions can be made at this level; on the contrary, as we shall see, the initial layer introduces an additional error on the boundary.

We modify the approximation made by the bulk part and take into account the initial layer:

$$(\mathbf{v}(a,t),\mathbf{w}(a,t) \approx (\bar{\mathbf{v}}(a,t),\epsilon\bar{\mathbf{w}}_1(a,t) + \tilde{\mathbf{w}}_0(a,t/\epsilon)).$$

Then the new error of the approximation is given by

$$\begin{aligned}\tilde{\mathbf{E}}(a,t) &:= (\tilde{\mathbf{e}}(a,t),\tilde{\mathbf{f}}(a,t)) = (\tilde{e}(a,t)\mathbf{k},\tilde{\mathbf{f}}(a,t)) \\ &= (\mathbf{v}(a,t) - \bar{\mathbf{v}}(a,t),\mathbf{w}(a,t) - \epsilon\bar{\mathbf{w}}_1(a,t) - \tilde{\mathbf{w}}_0(a,t/\epsilon)) \\ &= (\mathbf{e}(a,t),\mathbf{f}(a,t) - \tilde{\mathbf{w}}_0(a,t/\epsilon)).\end{aligned}$$

Again, assuming that all the terms are sufficiently smooth and using linearity of the problem, we have the following set of error equations

$$\tilde{\mathbf{e}}_t = -\tilde{\mathbf{e}}_a - \mathbf{PM}\tilde{\mathbf{e}} - \mathbf{PM}\tilde{\mathbf{f}} - \mathbf{PM}\tilde{\mathbf{w}}_0 - \epsilon\mathbf{PM}\bar{\mathbf{w}}_1,$$

and

$$\begin{aligned}\tilde{\mathbf{f}}_t &= \mathbf{f}_t - \frac{1}{\epsilon}\tilde{\mathbf{w}}_{0,\tau} \\ &= -\tilde{\mathbf{f}}_a - \tilde{\mathbf{w}}_{0,a} - \epsilon\bar{\mathbf{w}}_{1,a} - \mathbf{QM}\tilde{\mathbf{e}} - \mathbf{QM}\tilde{\mathbf{f}} - \tilde{e}\mathbf{k}_a - \mathbf{QM}\tilde{\mathbf{w}}_0 - \epsilon\mathbf{QM}\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\tilde{\mathbf{f}} - \epsilon\bar{\mathbf{w}}_{1,t}\,,\end{aligned} \tag{5.24}$$

where we used (5.23) in (5.24). The initial conditions are as follows,

$$\tilde{\mathbf{e}}(a,0) = \mathbf{0},$$

and

$$\tilde{\mathbf{f}}(a,0) = \mathbf{f}(a,0) - \tilde{\mathbf{w}}(a,0) = -\epsilon\bar{\mathbf{w}}_1(a,0),$$

while for the boundary conditions we have

$$\begin{aligned}\tilde{\mathbf{e}}(0,t) = \mathbf{e}(0,t) &= \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\tilde{\mathbf{e}}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\tilde{\mathbf{f}}(s,t))\,ds \\ &\quad + \epsilon\int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\bar{\mathbf{w}}_1(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau))\,ds, \qquad (5.25)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{f}}(0,t) &= \mathbf{f}(0,t) - \tilde{\mathbf{w}}(0,\tau) \\ &= \int_0^\infty \mathbf{B}(s)\tilde{\mathbf{e}}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\tilde{\mathbf{f}}(s,t)\,ds + \int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds \\ &\quad + \int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds + \epsilon\int_0^\infty \mathbf{B}(s)\bar{\mathbf{w}}_1(s,t)\,ds - \tilde{\mathbf{e}}(0,t) \\ &\quad - \bar{\mathbf{v}}(0,t) - \epsilon\bar{\mathbf{w}}_1(0,t) - \tilde{\mathbf{w}}_0(0,t/\epsilon). \qquad (5.26)\end{aligned}$$

As expected, the troublesome $O(1)$ term $\int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds - \bar{\mathbf{v}}(0,t)$ in the boundary condition (5.26) has been unaffected by the initial layer. Also the initial layer has introduced a new short range error at $a = 0$. This necessitates introduction of the boundary layer.

5.2.5 Boundary Layer

To eliminate $O(1)$ term $\int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds - \bar{\mathbf{v}}(0,t)$ from (5.26) we introduce boundary layer corrections. We cannot eliminate the $O(1)$ terms $\int_0^\infty \mathbf{P}(0)\,(\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau))\,ds$ from (5.25) and $\int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds$ and $\tilde{\mathbf{w}}_0(0,\tau)$ from (5.26) at this level as these terms are on a different time scale.

The boundary layer is constructed by blowing up the state variable a according to $\alpha := a/\epsilon$ and defining

$$\hat{\mathbf{u}}(\alpha,t) = (\hat{\mathbf{v}}(\alpha,t), \hat{\mathbf{w}}(\alpha,t)).$$

Again, the linearity allows us to approximate the solution $\mathbf{u}$ by the sum of the bulk and initial layer parts, obtained above, and the boundary layer. Thus, inserting the expansions

$$\hat{\mathbf{v}}(\alpha,t) = \hat{\mathbf{v}}_0(\alpha,t) + \epsilon\hat{\mathbf{v}}_1(\alpha,t) + \cdots \quad \text{and} \quad \hat{\mathbf{w}}(\alpha,t) = \hat{\mathbf{w}}_0(\alpha,t) + \epsilon\hat{\mathbf{w}}_1(\alpha,t) + \cdots,$$

as before into the projected equations (5.4), (5.5) we get respectively

$$\begin{aligned}\epsilon\hat{\mathbf{v}}_{0,t} + \epsilon^2\hat{\mathbf{v}}_{1,t} + \cdots = &-(\hat{\mathbf{v}}_{0,\alpha} + \epsilon\hat{\mathbf{v}}_{1,\alpha} + \cdots)\\ &-\epsilon(\mathbf{P}(0)\mathbf{M}(0) + a\mathbf{P}'(0)\mathbf{M}(0) + a\mathbf{P}(0)\mathbf{M}'(0) + \cdots)(\bar{\mathbf{v}}_0 + \epsilon\bar{\mathbf{v}}_1 + \cdots)\\ &-\epsilon(\mathbf{P}(0)\mathbf{M}(0) + a\mathbf{P}'(0)\mathbf{M}(0) + a\mathbf{P}(0)\mathbf{M}'(0) + \cdots)(\bar{\mathbf{w}}_0 + \epsilon\bar{\mathbf{w}}_1 + \cdots),\end{aligned} \tag{5.27}$$

and

$$\begin{aligned}\epsilon\hat{\mathbf{w}}_{0,t} + \epsilon^2\hat{\mathbf{w}}_{1,t} + \cdots = &-\epsilon\mathbf{k}_a\hat{\mathbf{u}} - (\hat{\mathbf{w}}_{0,\alpha} + \epsilon\hat{\mathbf{w}}_{1,\alpha} + \cdots)\\ &-\epsilon(\mathbf{Q}(0)\mathbf{M}(0) + a\mathbf{Q}'(0)\mathbf{M}(0) + a\mathbf{Q}(0)\mathbf{M}'(0) + \cdots)(\hat{\mathbf{v}}_0 + \epsilon\hat{\mathbf{v}}_1 + \cdots)\\ &-\epsilon(\mathbf{Q}(0)\mathbf{M}(0) + a\mathbf{Q}'(0)\mathbf{M}(0) + a\mathbf{Q}(0)\mathbf{M}'(0) + \cdots)(\hat{\mathbf{w}}_{0,\epsilon} + \epsilon\hat{\mathbf{w}}_1 + \cdots)\\ &+(\mathbf{C}_S(0) + a\mathbf{C}_S'(0) + \ldots)(\hat{\mathbf{w}}_0 + \epsilon\hat{\mathbf{w}}_1 + \ldots).\end{aligned} \tag{5.28}$$

At the ϵ^0 order of the hydrodynamic part (5.27) we have

$$\hat{\mathbf{v}}_{0,\alpha} = \mathbf{0}. \tag{5.29}$$

Since $\hat{\mathbf{v}}_0$ is a layer term, it should be important only in a small vicinity of the initial point $a = 0$. Since (5.29) gives $\hat{\mathbf{v}}_0$ = constant, we must take $\hat{\mathbf{v}}_0 = \mathbf{0}$.

Now at the ϵ^0 order in the kinetic part (5.28) we have

$$\hat{\mathbf{w}}_{0,\alpha} = \mathbf{C}_S(0)\hat{\mathbf{w}}_0\,,$$

and this gives

$$\hat{\mathbf{w}}_0(\alpha, t) = e^{\alpha\mathbf{C}_S(0)}\hat{\mathbf{w}}_0(0, t). \tag{5.30}$$

At this stage we are free to choose the boundary conditions which will help to eliminate the $O(1)$ term $\int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds - \bar{\mathbf{v}}(0,t)$ and, for this, we define

$$\hat{\mathbf{w}}_0(0,t) := \int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds - \bar{\mathbf{v}}(0,t). \tag{5.31}$$

Let us define the new error

$$\begin{aligned}\hat{\mathbf{E}}(a,t) &:= (\hat{e}(a,t)\mathbf{k}, \hat{\mathbf{f}}(a,t))\\ &:= (\tilde{e}\mathbf{k}(a,t), \tilde{\mathbf{f}}(a,t) - \hat{\mathbf{w}}_0(a/\epsilon, t)),\end{aligned}$$

where $\hat{\mathbf{w}}_0$ is given by (5.30) and (5.31). The new error equations are as follows:

$$\hat{\mathbf{e}}_t = -\hat{\mathbf{e}}_a - \mathbf{PM}\hat{\mathbf{e}} - \mathbf{PM}\hat{\mathbf{f}} - \mathbf{PM}\hat{\mathbf{w}}_0 - \epsilon\mathbf{PM}\bar{\mathbf{w}}_1 - \mathbf{PM}\tilde{\mathbf{w}}_0,$$

and

$$\begin{aligned}\hat{\mathbf{f}}_t &= \tilde{\mathbf{f}}_t - \hat{\mathbf{w}}_{0,t}\\ &= -\hat{\mathbf{f}}_a - \tilde{\mathbf{w}}_{0,a} - \epsilon\bar{\mathbf{w}}_{1,a} - \mathbf{QM}\hat{\mathbf{e}} - \mathbf{QM}\hat{\mathbf{f}} - \mathbf{QM}\hat{\mathbf{w}}_0 - \mathbf{QM}\tilde{\mathbf{w}}_0 - \epsilon\mathbf{QM}\bar{\mathbf{w}}_1 - \hat{e}\mathbf{k}_a\\ &\quad + \frac{1}{\epsilon}\left(\mathbf{C}_S(a) - \mathbf{C}_S(0)\right)\hat{\mathbf{w}}_0 + \frac{1}{\epsilon}\mathbf{C}_S\hat{\mathbf{f}} - \epsilon\bar{\mathbf{w}}_{1,t} - \hat{\mathbf{w}}_{0,t}.\end{aligned}$$

For the boundary conditions we obtain

$$\begin{aligned}\hat{\mathbf{e}}(0,t) = &\int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\hat{\mathbf{e}}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\hat{\mathbf{f}}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau))\,ds \\ &+ \epsilon \int_0^\infty \mathbf{P}(a)\left(\mathbf{B}(s)\bar{\mathbf{w}}_1(s,t)\right)\,ds + \epsilon \int_0^\infty \mathbf{P}(0)\left(\mathbf{B}(\epsilon\alpha)\hat{\mathbf{w}}_0(\alpha,t)\right)\,d\alpha, \end{aligned} \tag{5.32}$$

and

$$\begin{aligned}\hat{\mathbf{f}}(0,t) &= \tilde{\mathbf{f}}(0,t) - \hat{\mathbf{w}}(0,t) \\ &= \int_0^\infty \mathbf{B}(s)\hat{\mathbf{e}}(s,t)\,ds + \epsilon \int_0^\infty \mathbf{B}(\epsilon\alpha)\hat{\mathbf{w}}_0(\alpha,t)\,d\alpha - \hat{\mathbf{e}}(0,t) + \int_0^\infty \mathbf{B}(s)\hat{\mathbf{f}}(s,t)\,ds \\ &\quad + \int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds - \tilde{\mathbf{w}}_0(0,\tau) + \epsilon\int_0^\infty \mathbf{B}(s)\bar{\mathbf{w}}_1(s,t)\,ds - \epsilon\bar{\mathbf{w}}_1(0,t), \end{aligned} \tag{5.33}$$

where we have used (5.31).

The initial conditions take the following form:

$$\hat{\mathbf{e}}(a,0) = \mathbf{0}, \quad \text{and} \quad \hat{\mathbf{f}}(a,0) = -\hat{\mathbf{w}}_0(a/\epsilon,0) - \epsilon\bar{\mathbf{w}}_1(a,0).$$

We note that, even with the boundary layer, we still have terms depending on t/ϵ, namely $\int_0^\infty \mathbf{P}(0)\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds$ in (5.32) and $\int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds$, $\tilde{\mathbf{w}}_0(0,\tau)$ in (5.33). This necessitates introduction of the corner layer.

5.2.6 Corner Layer

To eliminate $O(1)$ terms $\int_0^\infty \mathbf{P}(0)\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds$ from (5.32) and $\int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds$, $\tilde{\mathbf{w}}_0(0,\tau)$ from (5.33), we need corner layer corrections. Here we enlarge both time and age variables as $\alpha := a/\epsilon$ and $\tau := t/\epsilon$. As before we use linearity and seek the corner layer independently by inserting the formal expansions

$$\check{\mathbf{v}}(\alpha,\tau) = \check{\mathbf{v}}_0(\alpha,\tau) + \epsilon\check{\mathbf{v}}_1(\alpha,\tau) + \cdots, \quad \text{and} \quad \check{\mathbf{w}}(\alpha,\tau) = \check{\mathbf{w}}_0(\alpha,\tau) + \epsilon\check{\mathbf{w}}_1(\alpha,\tau) + \cdots$$

into the projected system (5.4), (5.5) we get respectively

$$\begin{aligned}\frac{1}{\epsilon}\check{\mathbf{v}}_{0,\tau} + \check{\mathbf{v}}_{1,\tau} + \cdots = &-\left(\frac{1}{\epsilon}\check{\mathbf{v}}_{0,\alpha} + \check{\mathbf{v}}_{1,\alpha} + \cdots\right) \\ &- (\mathbf{P}(0)\mathbf{M}(0) + a\mathbf{P}'(0)\mathbf{M}(0) + +a\mathbf{P}(0)\mathbf{M}'(0)\cdots)(\check{\mathbf{v}}_0 + \epsilon\check{\mathbf{v}}_1 + \cdots) \\ &- (\mathbf{P}(0)\mathbf{M}(0) + a\mathbf{P}'(0)\mathbf{M}(0) + a\mathbf{P}(0)\mathbf{M}'(0) + \cdots)(\check{\mathbf{w}}_0 + \epsilon\check{\mathbf{w}}_1 + \cdots),\end{aligned}$$

and

$$\begin{aligned}\check{\mathbf{w}}_{0,\tau} + \epsilon\check{\mathbf{w}}_{1,\tau} + \cdots = &-\epsilon\mathbf{k}_a(0)u - (\check{\mathbf{w}}_{0,\alpha} + \epsilon\check{\mathbf{w}}_{1,\alpha} + \cdots) \\ &- \epsilon(\mathbf{Q}(0)\mathbf{M}(0) + a\mathbf{Q}'(0)\mathbf{M}(0) + a\mathbf{Q}(0)\mathbf{M}'(0) + \cdots)(\check{\mathbf{v}}_0 + \epsilon\check{\mathbf{v}}_1 + \cdots) \\ &- \epsilon(\mathbf{Q}(0)\mathbf{M}(0) + a\mathbf{Q}'(0)\mathbf{M}(0) + a\mathbf{Q}(0)\mathbf{M}'(0) + \cdots)(\check{\mathbf{w}}_0 + \epsilon\check{\mathbf{w}}_1 + \cdots) \\ &+ (\mathbf{C}_S(0) + a\mathbf{C}_S'(0) + \cdots)(\check{\mathbf{w}}_0 + \epsilon\check{\mathbf{w}}_1 + \cdots).\end{aligned}$$

Comparing coefficients at the ϵ^0 level we have

$$\check{\mathbf{v}}_{0,\tau} + \check{\mathbf{v}}_{0,\alpha} = \mathbf{0}, \tag{5.34}$$
$$\check{\mathbf{w}}_{0,\tau} = -\check{\mathbf{w}}_{0,\alpha} + \mathbf{C}_S(0)\check{\mathbf{w}}_0. \tag{5.35}$$

Here we have freedom of choosing both the boundary and initial conditions (in (α, τ)-variables) which will help to eliminate the problematic terms on the boundary. Therefore we define boundary conditions

$$\check{\mathbf{v}}_0(0,\tau) := \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau))\,ds, \tag{5.36}$$

and

$$\check{\mathbf{w}}_0(0,\tau) := \int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds - \tilde{\mathbf{w}}_0(0,\tau) - \check{\mathbf{v}}_0(0,\tau). \tag{5.37}$$

Define the new approximation

$$\begin{aligned} \mathbf{u}(a,t) = (\mathbf{v}(a,t), \mathbf{w}(a,t)) \approx (&\bar{\mathbf{v}}(a,t) + \check{\mathbf{v}}(a/\epsilon, t/\epsilon), \\ &\epsilon\bar{\mathbf{w}}_1(a,t) + \tilde{\mathbf{w}}_0(a,t/\epsilon) + \hat{\mathbf{w}}(a/\epsilon,t) + \check{\mathbf{w}}(a/\epsilon,t/\epsilon)), \end{aligned}$$

with the error of this approximation given by

$$\begin{aligned} \check{\mathbf{E}} &:= (\check{\mathbf{e}}(a,t), \check{\mathbf{f}}(a,t)) \\ &:= (\hat{\mathbf{e}}(a,t) - \check{\mathbf{v}}_0(\alpha,\tau), \hat{\mathbf{f}}(a,t) - \check{\mathbf{w}}_0(\alpha,\tau)). \end{aligned}$$

Taking all layers into account, we find that the final error formally satisfies

$$\begin{aligned} \check{\mathbf{e}}_t &= \hat{\mathbf{e}}_t - \frac{1}{\epsilon}\check{\mathbf{v}}_{0,\tau} \\ &= -\hat{\mathbf{e}}_a - \mathbf{PM}\hat{\mathbf{e}} - \mathbf{PM}\hat{\mathbf{f}} - \mathbf{PM}\hat{\mathbf{w}}_0 - \mathbf{PM}\tilde{\mathbf{w}}_0 - \epsilon\mathbf{PM}\bar{\mathbf{w}}_1 - \frac{1}{\epsilon}\check{\mathbf{v}}_{0,\tau} \\ &= -\check{\mathbf{e}}_a - \mathbf{PM}\check{\mathbf{e}} - \mathbf{PM}\check{\mathbf{f}} - \mathbf{PM}\check{\mathbf{v}}_0 - \mathbf{PM}\check{\mathbf{w}}_0 - \mathbf{PM}\hat{\mathbf{w}}_0 - \epsilon\mathbf{PM}\bar{\mathbf{w}}_1 - \mathbf{PM}\tilde{\mathbf{w}}_0\,, \end{aligned} \tag{5.38}$$

where we used (5.34). Further

$$\begin{aligned} \check{\mathbf{f}}_t &= \hat{\mathbf{f}}_t - \frac{1}{\epsilon}\check{\mathbf{w}}_{0,\tau} \\ &= -\check{\mathbf{f}}_a - \tilde{\mathbf{w}}_{0,a} - \epsilon\bar{\mathbf{w}}_{1,a} - \check{u}_0\mathbf{k}_a - \mathbf{QM}\check{\mathbf{e}} - \mathbf{QM}\check{\mathbf{v}}_0 - \mathbf{QM}\check{\mathbf{f}} - \mathbf{QM}\check{\mathbf{w}}_0 - \mathbf{QM}\hat{\mathbf{w}}_0 - \mathbf{QM}\tilde{\mathbf{w}}_0 \\ &\quad - \epsilon\mathbf{QM}\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}(\mathbf{C}_S(a) - \mathbf{C}_S(0))\hat{\mathbf{w}}_0 + \frac{1}{\epsilon}\mathbf{C}_S\check{\mathbf{f}} + \frac{1}{\epsilon}(\mathbf{C}_S(a) - \mathbf{C}_S(0))\check{\mathbf{w}}_0 - \epsilon\bar{\mathbf{w}}_{1,t} - \hat{\mathbf{w}}_{0,t}\,, \end{aligned} \tag{5.39}$$

where we used (5.35). The boundary condition for the hydrodynamic part is

$$\begin{aligned}
\check{\mathbf{e}}(0,t) &= \hat{\mathbf{e}}(0,t) - \check{\mathbf{v}}(0,\tau) \\
&= \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\check{\mathbf{e}}(s,t))\,ds + \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\check{\mathbf{f}}(s,t))\,ds \\
&\quad + \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(\epsilon\alpha)\check{\mathbf{v}}_0(\alpha,\tau))\,d\alpha + \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(\epsilon\alpha)\check{\mathbf{w}}_0(\alpha,\tau))\,d\alpha \\
&\quad + \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(s)\bar{\mathbf{w}}_1(s,\tau))\,ds + \epsilon \int_0^\infty \mathbf{P}(0)(\mathbf{B}(\epsilon\alpha)\hat{\mathbf{w}}_0(\alpha,t))\,d\alpha,
\end{aligned} \tag{5.40}$$

where we have used (5.36). Further,

$$\begin{aligned}
\check{\mathbf{f}}(0,t) &= \hat{\mathbf{f}}(0,t) - \check{\mathbf{w}}_0(0,\tau) \\
&= \int_0^\infty \mathbf{B}(s)\check{\mathbf{e}}(s,t)\,ds + \epsilon \int_0^\infty \mathbf{B}(s)\bar{\mathbf{w}}_1(s,t)\,ds \\
&\quad + \int_0^\infty \mathbf{B}(s)\check{\mathbf{f}}(s,t)\,ds + \epsilon \int_0^\infty \mathbf{B}(\epsilon\alpha)\check{\mathbf{w}}_0(\alpha,\tau)\,d\alpha - \epsilon\bar{\mathbf{w}}_1(0,t) \\
&\quad + \epsilon \int_0^\infty \mathbf{B}(\epsilon\alpha)\check{\mathbf{v}}_0(\alpha,\tau)\,d\alpha + \epsilon \int_0^\infty \mathbf{B}(\epsilon\alpha)\hat{\mathbf{w}}_0(\alpha,t)\,d\alpha,
\end{aligned} \tag{5.41}$$

where we have used (5.37). We introduce homogeneous initial conditions

$$\check{\mathbf{v}}_0(a,0) = \mathbf{0},\ \check{\mathbf{w}}_0(a,0) = \mathbf{0}. \tag{5.42}$$

The error equations (5.38)–(5.42) can be written in a compact form as

$$\begin{aligned}
\check{\mathbf{E}}_t &= -\check{\mathbf{E}}_a - \mathbf{M}\check{\mathbf{E}} + \frac{1}{\epsilon}\mathbf{C}\check{\mathbf{E}} + \mathbf{Y}(\cdot), \\
\check{\mathbf{E}}(0,t) &= \int_0^\infty \mathbf{B}(s)\check{\mathbf{E}}(s,t)\,ds + \mathbf{Z}(\cdot), \\
\check{\mathbf{E}}(a,0) &= \mathbf{0}.
\end{aligned}$$

We postpone a detailed study of the L_1-norm estimates of the error $\check{\mathbf{E}}$ to Chapter 7, where we deal with a more general model and provide rigorous error estimates with the integral formulation.

We close this chapter by showing all the layer related terms appear in $\mathbf{Y}(\cdot)$ and $\mathbf{Z}(\cdot)$ are either of ϵ order or decay to zero as $\tau \to \infty$ or $\alpha \to \infty$. To do this we start with the hydrodynamic terms of the corner layer and for this we have the following set of equations:

$$\begin{aligned}
\check{\mathbf{v}}_{0,\tau} + \check{\mathbf{v}}_{0,\alpha} &= \mathbf{0}, \\
\check{\mathbf{v}}_0(\alpha,0) &= \mathbf{0}, \\
\check{\mathbf{v}}_0(0,\tau) &= \int_0^\infty \mathbf{P}(0)\mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\,ds.
\end{aligned}$$

The solution of the above system can be represented as

$$\check{\mathbf{v}}_0(\alpha,\tau) = \begin{cases} \mathbf{k}(0)\displaystyle\int_0^\infty \mathbf{1}\cdot\left[\mathbf{B}(s)\odot e^{(\tau-\alpha)\mathbf{C}_S(s)}\mathbf{w}_0(s)\right] ds, & \tau > \alpha; \\ \mathbf{0}, & \tau < \alpha, \end{cases}$$

where $\odot$ denotes the component-wise multiplications of two vectors. Now, by assumption $\sup_{a\in\mathbf{R}_+} \mathrm{Re}\lambda(a) = \zeta < 0$, whenever $\lambda(a) \in \sigma(\mathbf{C}_S(a))$.
Thus

$$\begin{aligned} \left\|\check{\mathbf{v}}_0\left(\cdot,\frac{t}{\epsilon}\right)\right\|_{L^1} &\le \int_0^t \left\|\mathbf{k}(0)\int_0^\infty \mathbf{B}(s)\odot e^{(t/\epsilon-\alpha)\mathbf{C}_S(s)}\mathbf{w}_0(s)\,ds\right\| da \\ &\le \epsilon D\int_0^{t/\epsilon} e^{(t/\epsilon-\alpha)\zeta}\,d\alpha = \epsilon D\frac{1}{\zeta}\left(e^{\zeta t/\epsilon}-1\right) = O(\epsilon), \end{aligned}$$

where D is a positive constant. Next from (5.30) we have

$$\begin{aligned} \|\hat{\mathbf{w}}_0(\cdot,t)\|_{L^1} &= \int_0^t \left\|e^{\alpha\mathbf{C}_S(0)}\left(\int_0^\infty \mathbf{B}(s)\bar{\mathbf{v}}(s,t)\,ds - \bar{\mathbf{v}}(0,t)\right)\right\| da \\ &\le \epsilon\int_0^{t/\epsilon} e^{\alpha\zeta'}\left(\int_0^\infty \|\mathbf{B}(s)\bar{\mathbf{v}}(s,t)\|\,ds + \|\bar{\mathbf{v}}(0,t)\|\right) d\alpha \\ &\le \epsilon F\int_0^{t/\epsilon} e^{\alpha\zeta'}\,d\alpha = O(\epsilon), \end{aligned}$$

where F is a positive constant. Also,

$$\|\hat{\mathbf{w}}_{0,t}(\cdot,t)\|_{L^1} \le \epsilon G\int_0^{t/\epsilon} e^{\alpha\zeta'}\,d\alpha = O(\epsilon),$$

where G is a positive constant. For the kinetic part of the initial layer term, we have

$$\begin{aligned} \left\|\tilde{\mathbf{w}}_0\left(\cdot,\frac{t}{\epsilon}\right)\right\|_{L^1} &= \int_0^{t/\epsilon} \left\|e^{t/\epsilon\mathbf{C}_S(s)}\mathbf{w}_0(s)\right\| ds \\ &\le e^{\zeta t/\epsilon}\int_0^{t/\epsilon} \|\mathbf{w}_0(s)\|\,ds, \end{aligned}$$

and

$$\begin{aligned} \left\|\tilde{\mathbf{w}}_{0,a}\left(\cdot,\frac{t}{\epsilon}\right)\right\|_{L^1} &= \int_0^{t/\epsilon} \left\|\mathbf{Q}(s)\mathbf{C}(s)e^{t/\epsilon\mathbf{C}_S(s)}\mathbf{w}_0(s) + e^{t/\epsilon\mathbf{C}_S(s)}\mathbf{w}_{0,a}(s)\right\| ds \\ &\le \zeta G_1 e^{\zeta t/\epsilon}\int_0^{t/\epsilon} \|\mathbf{w}_0(s)\|\,ds + e^{\zeta t/\epsilon}\int_0^{t/\epsilon} \left\|\mathbf{w}_{0,a}(s)\right\| ds, \end{aligned}$$

where G_1 is a positive constant. Next we solve the following set of corner layer equations:

$$\begin{aligned}
\breve{\mathbf{w}}_{0,\tau} &= -\breve{\mathbf{w}}_{0,\alpha} + \mathbf{C}_S(0)\breve{\mathbf{w}}_0, \\
\breve{\mathbf{w}}_0(\alpha, 0) &= \mathbf{0}, \\
\breve{\mathbf{w}}_0(0,\tau) &= \int_0^\infty \mathbf{B}(s)\tilde{\mathbf{w}}_0(s,\tau)\, ds - \tilde{\mathbf{w}}_0(0,\tau) - \check{\mathbf{v}}_0(0,\tau).
\end{aligned}$$

The solution of the above system can be represented as

$$\breve{\mathbf{w}}_0(\alpha,\tau) = \begin{cases} \mathbf{V}(\alpha)\, \breve{\mathbf{w}}_0(0, \tau-\alpha), & \tau > \alpha; \\ \mathbf{0}, & \tau < \alpha, \end{cases}$$

where $\mathbf{V}(\alpha)$ is the fundamental matrix solution of

$$\frac{d}{d\alpha}\mathbf{V}(\alpha) = \mathbf{C}_S(0)\mathbf{V}(\alpha) \quad \text{and} \quad \mathbf{V}(0) = \mathbf{I}.$$

Thus,

$$\begin{aligned}
\left\| \breve{\mathbf{w}}_0\left(\cdot, \frac{t}{\epsilon}\right) \right\|_{L^1} &= \int_0^t \|\mathbf{V}(\alpha)\breve{\mathbf{w}}_0(0, t/\epsilon - \alpha)\|\, da \\
&\leq \int_0^t \|\mathbf{V}(\alpha) \int_0^\infty \mathbf{B}(s)\breve{\mathbf{w}}_0(s, t/\epsilon - \alpha)\, ds - \breve{\mathbf{w}}_{0,\epsilon}(0, t/\epsilon - \alpha)\|\, da \\
&\leq \epsilon \int_0^{t/\epsilon} \|\mathbf{V}(\alpha) \int_0^\infty \mathbf{B}(s)\breve{\mathbf{w}}_0(s, t/\epsilon - \alpha)\, ds - \tilde{\mathbf{w}}_{0,\epsilon}(0, t/\epsilon - \alpha) - \check{\mathbf{v}}_0(0, t/\epsilon - \alpha)\|\, d\alpha \\
&\leq \epsilon \int_0^{t/\epsilon} \|\mathbf{V}(\alpha) \int_0^\infty \mathbf{B}(s) e^{(t/\epsilon - \alpha)\mathbf{C}_S(s)} \mathbf{w}_0(s)\, ds\|\, d\alpha \\
&\qquad + \int_0^{t/\epsilon} \|\mathbf{V}(\alpha)\, e^{(t/\epsilon-\alpha)\mathbf{C}_S(0)}\mathbf{w}_0(0)\|\, d\alpha + \int_0^{t/\epsilon} \|\mathbf{V}(\alpha)\check{\mathbf{v}}_0(0, t/\epsilon - \alpha)\|\, d\alpha \\
&\leq \epsilon \|\mathbf{V}(\cdot)\| \left[\|\mathbf{B}(s)\|\, \|\mathbf{w}_0(s)\| \int_0^{t/\epsilon} e^{(t/\epsilon - \alpha)\zeta}\, d\alpha \right. \\
&\qquad \left. + \|\mathbf{w}_0(0)\| \int_0^{t/\epsilon} e^{(t/\epsilon-\alpha)\zeta}\, d\alpha + \int_0^{t/\epsilon} \check{\mathbf{v}}_0(0, t/\epsilon - \alpha)\, d\alpha \right] \\
&\leq \epsilon H \int_0^{t/\epsilon} e^{(t/\epsilon - \alpha)\zeta}\, d\alpha = O(\epsilon),
\end{aligned}$$

where H is a positive constant.

CHAPTER 6

Generalizations of the Perturbed Model

The perturbed model considered in Chapters 4 and 5 deals with irreducible migration matrix which essentially indicates that any geographical patch is accessible from any other. This assumption often proves too restrictive. In many cases, it is found that some patches are either isolated or only admitting emigration (or only immigration). As typical examples, it is found that due to economic situation, a population may leave a region to inhibit several other regions or we can consider fishes leaving rivers, move to sea whereas sea-fishes remain confined in sea. These type of phenomenons can be modeled with reducible migration matrix what we are going to discuss in this chapter. Along with this, in this chapter, we also consider full birth and death matrices instead of diagonal structures.

6.1 Model with General Mortality and Birth Matrices

The first attempt towards generalization is to consider general mortality and birth matrices instead of the diagonal ones. We will use the same notation, to denote these matrices. Thus, with these changes, our new perturbed model becomes

$$\begin{aligned} \mathbf{u}_t &= -\mathbf{u}_a - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon}\mathbf{C}\mathbf{u}, \\ \mathbf{u}(0,t) &= \int_0^\infty \mathbf{B}(s)\mathbf{u}(s,t)\,ds, \\ \mathbf{u}(a,0) &= \boldsymbol{\phi}(a), \end{aligned} \tag{6.1}$$

where

$$\mathbf{M}(a) = \left[\mu_{ij}(a)\right]_{1 \le i,j \le n} \quad \text{and} \quad \mathbf{B}(a) = \left[\beta_{ij}(a)\right]_{1 \le i,j \le n}.$$

Here we assume that the matrix **M** has positive off-diagonal elements. The reason for this assumption has been explained in Section 6.3.1. Also other properties of **M** will be specified in Hypothesis **H3**, in Section 6.3.1. So, essentially **M** not only reflects the mortality but some migration as well occurring, however, in a much slower scale compared to the migration phenomena modeled by **C**. With an abuse of terminology, we will still call it a mortality matrix. In matrix **B**, a non zero entry β_{ij} for $i \neq j$, gives a contribution of patch j to births in patch i.

Summing the system of equations (6.1), we get

$$\begin{aligned}
\sum_{i=1}^{n} u_{i,t} &= -\sum_{i=1}^{n} u_{i,a} - \left(\sum_{i=1}^{n} \mu_{1i} u_i + \cdots + \sum_{i=1}^{n} \mu_{ni} u_i \right), \\
\sum_{i=1}^{n} u_i(0,t) &= \int_0^{\infty} \left(\sum_{i=1}^{n} \beta_{1i}(s) u_i(s,t) + \cdots + \sum_{i=1}^{n} \beta_{ni}(s) u_i(s,t) \right) ds, \\
\sum_{i=1}^{n} u_i(a,0) &= \sum_{i=1}^{n} \phi_i(a).
\end{aligned}$$

Since the right Perron vector of **C** represents the stable patch structure, we approximate the original system exactly as we did in Chapter 5, i.e.,

$$k_i \approx \frac{u_i}{\sum\limits_{i=1}^{n} u_i}, \qquad \text{where} \quad i = 1, \cdots, n.$$

Defining as before, $u(a,t) := \sum\limits_{i=1}^{n} u_i(a,t)$, heuristically we have the following approximations:

$$\sum_{i=1}^{n} \mu_{1i} u_i \approx \left(\sum_{i=1}^{n} \mu_{1i} k_i \right) u =: \mu_1^*(a) u,$$

$$\vdots$$

$$\sum_{i=1}^{n} \mu_{ni} u_i \approx \left(\sum_{i=1}^{n} \mu_{ni} k_i \right) u =: \mu_n^*(a) u,$$

and on the boundary,

$$\sum_{i=1}^{n} \beta_{1i} u_i \approx \left(\sum_{i=1}^{n} \beta_{1i} k_i \right) u =: \beta_1^*(a) u,$$

$$\vdots$$

$$\sum_{i=1}^{n} \beta_{ni} u_i \approx \left(\sum_{i=1}^{n} \beta_{ni} k_i \right) u =: \beta_n^*(a) u.$$

With these approximations, we have the following aggregated system:

$$\begin{aligned} u_t &\approx -u_a - \mu^{**}(a) u, \\ u(0,t) &\approx \int_0^\infty \beta^{**}(s) u(s,t)\, ds, \\ u(a,0) &\approx \sum_{i=1}^{n} \phi_i(a), \end{aligned}$$

where, $\mu^{**}(a) := \sum_{i=1}^{n} \mu_i^*(a)$ and $\beta^{**}(a) := \sum_{i=1}^{n} \beta_i^*(a)$. So, essentially we found same 'approximated system' as we got in Chapter 5.

Along with these general mortality and birth matrices, our next attempt is to weaken the irreducibility assumption on the migration matrix **C** and to analyze the behaviour with reducible **C**. For the rest of the work, we will again use the same notation **C** to denote reducible migration matrix.

6.2 The Reducible Migration Matrix

In this section we are going to state the spectral properties of our reducible migration matrix. Specifically, we focus on reducible Kolmogorov matrices. Theorem 6.1 below is valid for arbitrary non-negative matrices and it is different from Theorem 4.2 in the sense that the eigenvector corresponding to the dominant eigenvalue is not necessarily a positive vector. Since our reducible migration matrix **C** has non-negative off-diagonal elements, we need a specific result (Theorem 6.3) for this type of matrices, which can be obtained using Theorem 6.1 and is cited in Corollary 6.2. Since the proofs of the Corollary 6.2 and the Theorem 6.3 are essentially same as the corresponding results proved in Chapter 4, we are not repeating them here.

Theorem 6.1 *The spectral radius of a non-negative matrix* **A** *is a dominant eigenvalue of* **A** *and there is a corresponding eigenvector* $\mathbf{k} \geq 0$*.*

Proof. We refer to [24, p. 66]. □

Corollary 6.2 *The spectral bound of a non-negative off-diagonal matrix* **A** *is an eigenvalue of* **A** *and there is a corresponding eigenvector* $\mathbf{k} \geq 0$*.*

Using Corollary 6.2, we can now prove the following spectral properties of a reducible Kolmogorov matrix.

Theorem 6.3 *A reducible Kolmogorov matrix has zero as the dominant eigenvalue and* **1** *is a left eigenvector corresponding to this dominant eigenvalue.*

Due to the reducibility property, the left and right Perron vectors have a special structures as described in the following theorems.

Theorem 6.4 *An arbitrary reducible, non-negative matrix cannot have both left and right positive eigenvectors corresponding to the dominant eigenvalue unless the matrix is block diagonal.*

Proof. See [24, p. 78]

Theorem 6.5 *For a reducible Kolmogorov matrix, the right eigenvector (Perron vector) must contain some zero elements unless the matrix is block diagonal.*

Proof. The proof immediately follows from Theorems 6.3 since **1** is a left eigenvector corresponding to the zero eigenvalue. □

With respect to migration states, reducibility of the migration matrix **C** produces different behaviour compared to Chapter 4. For our migration matrix **C**, we divide the states into two sets: the states $\mathscr{T}$ from which systems can make transit to any other states are called ***transient states*** and the set of states $\mathscr{C}$ called ***closed states***, where, once the system is in, cannot transit to other states no mater how long we iterate. Using the definition, a reducible matrix **C** can be represented as

$$\begin{bmatrix} \mathbf{C}' & \mathbf{A} \\ \mathbf{0} & \mathbf{T} \end{bmatrix}, \tag{6.2}$$

where $\mathbf{C}', \mathbf{T}$ are irreducible, square matrices. $\mathbf{C}'$ corresponds the set of closed states $\mathscr{C}$ with the property that $\mathbf{C}'$ has zero as dominant eigenvalue and **T** corresponds the transient states $\mathscr{T}$ with $\sigma(\mathbf{T}) := \{\lambda \in \mathbb{C} : Re\lambda < 0\}$. If one of the matrices $\mathbf{C}'$ or **T** is reducible then it can also be represented in a form similar to (6.2) and we can carry on the above decomposition. Finally with suitable permutations, **C** can be represented in the following normal form

$$\begin{bmatrix} \mathbf{C}_{n_1} & \dots & \mathbf{0} & \mathbf{A}_{n_1,n_{m+1}} & \dots & \mathbf{A}_{n_1,n_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{C}_{n_m} & \mathbf{A}_{n_m,n_{m+1}} & \dots & \mathbf{A}_{n_m,n_n} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{T}_{n_{m+1},n_{m+1}} & \dots & \mathbf{T}_{n_{m+1},n_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{T}_{n_n,n_n} \end{bmatrix}. \tag{6.3}$$

Define $\mathfrak{C} := \begin{bmatrix} \mathbf{C}_{n_1} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{C}_{n_m} \end{bmatrix}$, $\mathbb{A} := \begin{bmatrix} \mathbf{A}_{n_1,n_{m+1}} & \dots & \mathbf{A}_{n_1,n_n} \\ \vdots & \vdots & \vdots \\ \mathbf{A}_{n_m,n_{m+1}} & \dots & \mathbf{A}_{n_m,n_n} \end{bmatrix}$, $\mathbb{O} := \begin{bmatrix} \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{0} \end{bmatrix}$,
$\mathbb{T} := \begin{bmatrix} \mathbf{T}_{n_{m+1},n_{m+1}} & \dots & \mathbf{T}_{n_{m+1},n_n} \\ \vdots & \vdots & \vdots \\ \mathbf{0} & \dots & \mathbf{T}_{n_n,n_n} \end{bmatrix}$.

Then the normal form of $\mathbf{C}$ can be represented as $\begin{bmatrix} \mathfrak{C} & \mathbb{A} \\ \mathbb{O} & \mathbb{T} \end{bmatrix}$, where $\mathfrak{C}$ corresponds all the closed states and $\mathbb{T}$ corresponds all the transient states.

There is an interesting connection between the transient states, closed states and the right Perron vector of a Kolmogorov matrix as described in the following result.

Theorem 6.6 *For a Kolmogorov matrix in its normal form, the closed states correspond to the non-zero elements of the right Perron vector and the transient states correspond to the zero elements of the right Perron vector.*

Proof. From the property of the normal form, all $\mathbf{C}_{n_i}$, where $1 \le i \le m$, have zero as the dominant eigenvalue and all $\mathbf{T}_{j,j}$, where $n_{m+1} \le j \le n_n$, have negative real parts. Let all square matrices $\mathbf{C}_{n_i}$ have orders, respectively, n_i, where $1 \le i \le m$ and all $\mathbf{T}_{j,j}$ have orders, respectively n_j, where $n_{m+1} \le j \le n_n$. Now $\mathbf{C}$ has $\mathbf{1}$ as a left Perron vector and, by Theorem 6.5, a right Perron vector of $\mathbf{C}$ must have some zero components. Let $\mathbf{k} := (\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n)$ be a right Perron vector of $\mathbf{C}$, where $\mathbf{k}_1 := (k_1, \dots, k_{n_1}), \dots,$ $\mathbf{k}_m := (k_{n_{m-1}+1}, \dots, k_{n_m}), \mathbf{k}_{m+1} := (k_{n_m+1}, \dots, k_{n_{m+1}}), \dots, \mathbf{k}_n := (k_{n_{n-1}+1}, \dots, k_{n_n})$. Corresponding to the zero eigenvalue, we have the following set of equations

$$\begin{aligned} \mathbf{C}_{n_1}\mathbf{k}_1 + \dots + \mathbf{0} + \mathbf{A}_{n_1,n_{m+1}}\mathbf{k}_{m+1} + \dots + \mathbf{A}_{n_1,n_n}\mathbf{k}_n &= \mathbf{0}, \\ &\vdots \\ \mathbf{0} + \dots + \mathbf{C}_{n_m}\mathbf{k}_m + \mathbf{A}_{n_m,n_{m+1}}\mathbf{k}_{m+1} + \dots + \mathbf{A}_{n_m,n_n}\mathbf{k}_n &= \mathbf{0}, \\ \mathbf{0} + \dots + \mathbf{0} + \mathbf{T}_{n_{m+1},n_{m+1}}\mathbf{k}_{m+1} + \dots + \mathbf{T}_{n_{m+1},n_n}\mathbf{k}_n &= \mathbf{0}, \\ &\vdots \\ \mathbf{0} + \dots + \mathbf{0} + \mathbf{0} + \dots + \mathbf{T}_{n_n,n_n}\mathbf{k}_n &= \mathbf{0}. \end{aligned}$$

Since each $\mathbf{T}_{j,j}$ ($n_{m+1} \le j \le n_n$) is non-singular, it is invertible and hence $\mathbf{k}_{m+1} = \dots = \mathbf{k}_n = \mathbf{0}$. Since each $\mathbf{C}_{n_i}$ is irreducible, it has positive Perron vector corresponding to the dominant eigenvalue zero and hence each $\mathbf{k}_1, \dots, \mathbf{k}_m$ is non-zero. This completes the proof. □

Remarks:

1. Since each $\mathbf{C}_{n_i}$, where $i = 1, \dots, m$, is an irreducible Kolmogorov matrix, by Theorem 4.4, we find that zero is a simple dominant eigenvalue for each $\mathbf{C}_{n_i}$,

where $i = 1, \ldots, m$, and hence zero is a semi-simple eigenvalue of $\mathbf{C}$. Each $\mathbf{T}_{j,j}$, where $n_{m+1} \leq j \leq n_n$, is nonsingular.

2. The matrix $\mathbf{C}$ has zero as a dominant eigenvalue with multiplicity m and rest of the eigenvalues have negative real parts.

6.2.1 Spectral Projections

In this section we shall prepare the ground in order to define spectral projection operators which are going to be applied to the perturbed system for asymptotic analysis. We define a basis set for the null space of the migration operator $\mathbf{C}$ and also construct the basis system for the adjoint null space of $\mathbf{C}$. In rest of the work, we shall assume that our migration matrix $\mathbf{C}$ depends on age but that its normal form remains unchanged as a varies. By Remarks 1 and 2, the eigenspace of $\mathbf{C}$ corresponding to $\lambda = 0$ is m-dimensional. We denote its basis by

$$\mathbf{e}_1 := \left(\frac{k_1}{\sum_{i=1}^{n_1} k_i}, \ldots, \frac{k_{n_1}}{\sum_{i=1}^{n_1} k_i}, 0, \ldots, 0 \right),$$

$$\vdots$$

$$\mathbf{e}_m := \left(0, \ldots, 0, \frac{k_{n_{m-1}+1}}{\sum_{i=n_{(m-1)}}^{n_m} k_i}, \ldots, \frac{k_{n_m}}{\sum_{i=n_{(m-1)}}^{n_m} k_i}, 0, \ldots, 0 \right).$$

Here for defining the basis vectors $\mathbf{e}_1, \ldots, \mathbf{e}_m$, we have used Theorem 6.6 to assign 0 as the value of all transient state components of the right Perron vector.

Since $\mathbf{1}$ is a left eigenvector corresponding to the zero eigenvalue of $\mathbf{C}$, we can construct a basis for the adjoint null space of $\mathbf{C}$ as follows:
Let $(\mathbf{y}_1, \ldots, \mathbf{y}_n)$ be a left eigenvector corresponding to the zero eigenvalue of $\mathbf{C}$, where $\mathbf{y}_1$ has n_1 components, $\ldots, \mathbf{y}_m$ has n_m components. If our block $\mathbb{T}$ has only non-zero diagonal elements, then we have the following explicit expressions of the corresponding left basis. Using the normal form of $\mathbf{C}$ we have the following set of equations:

$$\begin{aligned}
\mathbf{y}_1 \mathbf{C}_{n_1} &= \mathbf{0}, \\
&\vdots \\
\mathbf{y}_m \mathbf{C}_{n_m} &= \mathbf{0}, \\
\mathbf{y}_1 \mathbf{A}_{n_1, n_{m+1}} + \cdots + \mathbf{y}_m \mathbf{A}_{n_m, n_{m+1}} + \mathbf{y}_{m+1} \mathbf{T}_{n_{m+1}, n_{m+1}} &= \mathbf{0}, \\
&\vdots \\
\mathbf{y}_1 \mathbf{A}_{n_1, n_n} + \cdots + \mathbf{y}_m \mathbf{A}_{n_m, n_n} + \mathbf{y}_n \mathbf{T}_{n_n, n_n} &= \mathbf{0}.
\end{aligned}$$

Since all $\mathbf{T}_{n_{m+1},n_{m+1}}, \ldots, \mathbf{T}_{n_n,n_n}$ are invertible, we get

$$\begin{aligned}\mathbf{y}_{m+1} &= -\left(\mathbf{y}_1\mathbf{A}_{n_1,n_{m+1}} + \cdots + \mathbf{y}_m\mathbf{A}_{n_m,n_{m+1}}\right)\mathbf{T}^{-1}_{n_{m+1},n_{m+1}},\\ &\vdots\\ \mathbf{y}_n &= -\left(\mathbf{y}_1\mathbf{A}_{n_1,n_n} + \cdots + \mathbf{y}_m\mathbf{A}_{n_m,n_n}\right)\mathbf{T}^{-1}_{n_n,n_n}.\end{aligned}$$

Now, for the basis vector $\mathbf{x}_1$, choose

$$\begin{aligned}\mathbf{y}_{m+1} &= -\,\mathbf{y}_1\mathbf{A}_{n_1,n_{m+1}}\mathbf{T}^{-1}_{n_{m+1},n_{m+1}},\\ &\vdots\\ \mathbf{y}_n &= -\mathbf{y}_1\mathbf{A}_{n_1,n_n}\mathbf{T}^{-1}_{n_n,n_n},\end{aligned}$$

where $\mathbf{y}_1 = (\underbrace{1,\ldots,1}_{n_1\ times})$.

Following the same method, we get the full set of a left basis as follows:

$$\begin{aligned}\mathbf{x}_1 &= \left(\mathbf{y}_1, \mathbf{0}, \ldots, \mathbf{0}, -\,\mathbf{y}_1\mathbf{A}_{n_1,\,n_{m+1}}\mathbf{T}^{-1}_{n_{m+1},n_{m+1}}, \ldots, -\mathbf{y}_1\mathbf{A}_{n_1,n_n}\mathbf{T}^{-1}_{n_n,n_n}\right),\\ &\vdots\\ \mathbf{x}_m &= \left(\mathbf{0}, \ldots, \mathbf{0}, \mathbf{y}_m, -\,\mathbf{y}_m\mathbf{A}_{n_m,n_{m+1}}\mathbf{T}^{-1}_{n_{m+1},n_{m+1}}, \ldots, -\mathbf{y}_m\mathbf{A}_{n_m,n_n}\mathbf{T}^{-1}_{n_n,n_n}\right),\end{aligned}$$

where, $\mathbf{y}_1 = (\underbrace{1,\ldots,1}_{n_1\ times}), \ldots, \mathbf{y}_m = (\underbrace{1,\ldots,1}_{n_m\ times})$.

For a general upper triangular form of $\mathbb{T}$, the components of this basis vectors will be denoted by the following short-hand notation:

$$\begin{aligned}\mathbf{x}_1 &:= \left(\mathbf{y}_1, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{f}_{n_1,n_{m+1}}, \ldots, \mathbf{f}_{n_1,n_n}\right),\\ &\vdots\\ \mathbf{x}_m &:= \left(\mathbf{0}, \ldots, \mathbf{0}, \mathbf{y}_m, \mathbf{f}_{n_m,n_{m+1}}, \ldots, \mathbf{f}_{n_m,n_n}\right).\end{aligned}$$

Since $\mathbf{T}_{n_i,n_j}$, for $m+1 \le i,j \le n$ are of negative type and $\mathbf{A}_{n_i,n_j}$, for $1 \le i \le m$ and $m+1 \le j \le n$ are of positive type, $\mathbf{f}_{ij} \ge 0$. Also we note that $\mathbf{1} = \mathbf{x}_1 + \cdots + \mathbf{x}_m$.

Also, for example,

$$\begin{aligned}\sum_{i=1}^{m}\mathbf{f}_{n_i,n_{m+1}} &= \mathbf{y}_1\mathbf{A}_{n_1,n_{m+1}}\mathbf{T}^{-1}_{n_{m+1},n_{m+1}} + \cdots + \mathbf{y}_m\mathbf{A}_{n_m,n_{m+1}}\mathbf{T}^{-1}_{n_{m+1},n_{m+1}}\\ &= \left(\mathbf{y}_1\mathbf{A}_{n_1,n_{m+1}} + \cdots + \mathbf{y}_m\mathbf{A}_{n_m,n_{m+1}}\right)\mathbf{T}^{-1}_{n_{m+1},n_{m+1}}\\ &= \mathbf{T}_{n_{m+1},n_{m+1}}\mathbf{T}^{-1}_{n_{m+1},n_{m+1}} = \mathbf{1},\end{aligned}$$

since $\mathbf{C}$ is a Kolmogorov matrix, the sum of the elements of a particular column is zero. Therefore, it follows that

$$\sum_{i=1}^{m} \mathbf{f}_{n_i,\, n_{m+1}} = \cdots = \sum_{i=1}^{m} \mathbf{f}_{n_i, n_n} = \mathbf{1}. \tag{6.4}$$

Now we can split the space as follows:

$$\mathbb{R}^n = [\mathbf{e}_1] \oplus \ldots \oplus [\mathbf{e}_m] \oplus S,$$

where $S := \left\{ \mathbf{x} \in \mathbb{R}^n : \mathbf{P}_j \mathbf{x} = 0, \text{ for } 1 \leq j \leq m \right\}$, where $\mathbf{P}_j$ for $1 \leq j \leq m$ are projections defined below.

Using the above basis vectors, let us now define the set of projection operators $\mathbf{P}_1, \ldots, \mathbf{P}_m, \mathbf{Q} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ as follows:

$$\mathbf{P}_1 \mathbf{u} =: \mathbf{v}_1 := (\mathbf{x}_1 \cdot \mathbf{u})\ \mathbf{e}_1 = \left(\sum_{i=1}^{n_1} u_i + \sum_{i=n_{m+1}}^{n_n} \mathbf{f}_{n_1, n_i} \cdot \mathbf{u}_{n_i} \right) \mathbf{e}_1 =: u^{n_1} \mathbf{e}_1,$$

$$\vdots$$

$$\mathbf{P}_m \mathbf{u} =: \mathbf{v}_m := (\mathbf{x}_m \cdot \mathbf{u})\, \mathbf{e}_m = \left(\sum_{i=n_{m-1}+1}^{n_m} u_i + \sum_{i=n_{m+1}}^{n_n} \mathbf{f}_{n_m, n_i} \cdot \mathbf{u}_{n_i} \right) \mathbf{e}_m =: u^{n_m} \mathbf{e}_m\,,$$

and

$$\mathbf{Q}\, \mathbf{u} := \mathbf{w} := [\, \mathbf{I} - (\mathbf{P}_1 + \ldots + \mathbf{P}_m)]\, \mathbf{u}.$$

6.3 Model with General M, B and Reducible C

Along with the reducible migration matrix $\mathbf{C}$ and considering general mortality $\mathbf{M}$ and birth $\mathbf{B}$ matrices, the perturbed system becomes,

$$\mathbf{u}_t = -\mathbf{u}_a - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon} \mathbf{C}\, \mathbf{u}, \tag{6.5}$$

$$\mathbf{u}(0,t) = \int_0^\infty \mathbf{B}(s) \mathbf{u}(s,t)\, ds, \tag{6.6}$$

$$\mathbf{u}(a,0) = \boldsymbol{\phi}(a). \tag{6.7}$$

For the rest of the work, we use the following notation:

$$\mathbf{x}^m(a,t) := \sum_{i=1}^{m} x^{n_i}(a,t)\, \mathbf{e}_i(a)\ ,\ \mathbf{x}_0^m(a,t) := \sum_{i=1}^{m} x_0^{n_i}(a,t)\, \mathbf{e}_i(a),\ [\mathfrak{B}\, \mathbf{x}](t) := \int_0^\infty \mathbf{B}(s)\, \mathbf{x}(s,t)\, ds$$

and for any matrix $\mathbf{A}$, $\mathbf{A}_S := \mathbf{Q}\mathbf{A}$.

6.3.1 Hypotheses & Properties

We are going to work in the Banach space $\mathbf{X} := L^1(\mathbb{R}^+, \mathbb{R}^n)$. The necessary assumptions to be made on $\mathbf{C}(a)$, $\mathbf{M}(a) := [\mu_{ij}(a)]_{1 \le i,j \le n}$ and $\mathbf{B}(a) := [\beta_{ij}(a)]_{1 \le i,j \le n}$ are summarized in the next hypotheses.

H1: $\beta_{ij}(a) \ge 0$, a.e. in $\mathbb{R}^+$;

H2: $\beta_{ij} \ge 0$ and $\mu_{ij}, \beta_{ij} \in L^\infty(\mathbb{R}^+)$, that is, there exist $\underline{\mu}, \bar{\beta} \in \mathbb{R}^+$ such that $\mu_{ij}(a) \le \underline{\mu}$, $\beta_{ij}(a) \le \bar{\beta}$, all most everywhere in $\mathbb{R}^+$;

H3: $-\mathbf{M}(a)$ is a sub-Kolmogorov matrix; that is, $\mu_{ij} \le 0$ for $i \ne j$, and satisfies $-\sum_{j=1}^{n} \mu_{ji}(a) \le 0$ for any $1 \le i \le n$ and $a \in \mathbb{R}_+$. Also $a \mapsto \mathbf{M}(a) \in C_b^1(\mathbb{R}_+, \mathbb{R}^{n^2})$;

H4: The migration matrix $\mathbf{C}$ is a reducible Kolmogorov matrix and $\mathbf{C}_S := \mathbf{C}|_S$ has all eigenvalues with negative real parts and let

$$\max_{\lambda \in \sigma(\mathbf{C}_S(a))} Re\,\lambda =: \zeta < 0. \tag{6.8}$$

Lemma 6.7 *Under the above assumptions, $\mathbf{C}_S^{-1} \in C_b^2(\mathbb{R}_+, \mathbb{R}^{n^2})$ and $\mathbf{e}_j \in C_b^1(\mathbb{R}_+, \mathbb{R}^n)$ for $1 \le j \le m$.*

Proof. The first statement is obvious since the determinant of $\mathbf{C}_S(a)$ is twice differentiable and bounded away from zero by uniform invertibility of $\mathbf{C}_S(a)$. To prove the second statement, we note that $\mathbf{e}_j := (0, \ldots, \mathbf{e}_{n_j}, \ldots, 0)$ for $1 \le j \le m$, where $\mathbf{e}_{n_j}$ are the Perron vectors corresponding to the irreducible matrices $\mathbf{C}_{n_j}$ and hence differentiability of $\mathbf{e}_{n_j}$ follows from Lemma 5.1. □

Regarding solvability of (6.5)–(6.7), it follows, [65, Proposition 3.2], that $\mathfrak{S} - \mathbf{M} + \frac{1}{\epsilon}\mathbf{C}$ (where $\mathfrak{S}\mathbf{u} := -\mathbf{u}_a$) on the domain $\mathfrak{D}(\mathfrak{S}) := \{\mathbf{u} \in \mathbf{X} : \mathbf{u}(0) = \mathfrak{B}\mathbf{u}\}$ generates a C_0-semigroup, say $(\mathbf{G}_\epsilon(t))_{t \ge 0}$, of type $(1, w)$ where $w \le \|\mathfrak{B}\| + \| - \mathbf{M} + \frac{1}{\epsilon}\mathbf{C}\|$. This estimate is not satisfactory as it depends on ϵ. However, $-\mathbf{M} + \frac{1}{\epsilon}\mathbf{C}$ is also positive off-diagonal and hence it generates a positive semigroup of contractions. Thus the assumptions of the Trotter formula, [20, Corollary III 5.8], are satisfied and therefore the type of $(\mathbf{G}_\epsilon(t))_{t \ge 0}$ is the same as of the semigroup generated by $(\mathfrak{S}, \mathfrak{D}(\mathfrak{S}))$. Hence $w \le \|\mathfrak{B}\|$, independently of ϵ.

6.3.2 Lifting Theorem

While the semigroup theory, via the Duhamel formula, provides satisfactory estimates for the problem (6.5)–(6.7), with the inhomogeneity in (6.5), it is insufficient to handle inhomogeneous boundary conditions $\mathbf{u}(0,t) = \mathfrak{B}\mathbf{u} + \mathbf{g}$ where $\mathbf{g}$ is a vector, possibly depending on time. There are various versions of trace theorems which can lift the inhomogeneity from the boundary to the interior but here the problem is complicated due to the presence of a small parameter. We provide one which gives estimates uniform in ϵ.

Lemma 6.8 *There is a bounded solution operator $\mathbf{L}_{\epsilon,\lambda} : \mathbb{R}^N \to \mathbf{X}$ of the problem*

$$\lambda\mathbf{u} = \mathfrak{S}\mathbf{u} - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon}\mathbf{C}\mathbf{u}, \;\; \mathbf{u}(0,t) = \mathbf{g}, \tag{6.9}$$

which satisfies $\mathbf{L}_{\epsilon,\lambda}\mathbf{g} \in \mathfrak{D}(\mathfrak{S})$ and $\|\mathbf{L}_{\epsilon,\lambda}\| \to 0$ as $\lambda \to \infty$ uniformly in $\epsilon \in (0,\epsilon_0)$ for some $\epsilon_0 > 0$.

Proof. Since $\mathfrak{S}$ is the diagonal differentiation with respect to a, (6.9) is just the Cauchy problem for the system of linear non-autonomous equations $\mathbf{u}_a = Q_\epsilon(a)\mathbf{u}$, where $Q_\epsilon(a) := -\lambda\mathbf{I} - \mathbf{M}(a) + \frac{1}{\epsilon}\mathbf{C}(a)$. Since $Q_\epsilon(a) := [q_{ij}(a)]_{1\le i,j\le n}$ is positive off-diagonal, the solution $\mathbf{u}$ is non-negative. Indeed let us denote by $\mathbf{L}_{\epsilon,\lambda}(a) := \{l_{\epsilon,ij}(a)\}_{1\le i,j\le n}$ the fundamental matrix of (6.9) corresponding to the unit vectors of $\mathbb{R}^n$, $\mathbf{p}_i = \{\delta_{i,j}\}_{1\le j\le n}$, where $i = 1,\cdots,n$. Let us first prove that, $\mathbf{L}_{\epsilon,\lambda}(a)$ is a non-negative matrix. To do this, we note that $L_{\epsilon,\lambda}(a)$ satisfies

$$\frac{d}{da}\mathbf{L}_{\epsilon,\lambda}(a) = Q_\epsilon(a)\mathbf{L}_{\epsilon,\lambda}(a). \tag{6.10}$$

Let $\eta := \sup_{i,a} |q_{ii}(a)|$. Then $Q_\epsilon(a) + \eta\mathbf{I}$ is a non-negative matrix, and we have

$$\begin{aligned}\frac{d}{da}\left\{e^{\eta a}\mathbf{L}_{\epsilon,\lambda}(a)\right\} &= e^{\eta a}\frac{d}{da}\mathbf{L}_{\epsilon,\lambda}(a) + \eta e^{\eta a}\mathbf{L}_{\epsilon,\lambda}(a)\\ &= e^{\eta a}Q_\epsilon(a)\mathbf{L}_{\epsilon,\lambda}(a) + \eta e^{\eta a}\mathbf{L}_{\epsilon,\lambda}(a) \qquad \text{(by (6.10))}\\ &= \left[Q_\epsilon(a) + \eta\mathbf{I}\right]e^{\eta a}\mathbf{L}_{\epsilon,\lambda}(a).\end{aligned}$$

Accordingly, we have the following representation

$$e^{\eta a}\mathbf{L}_{\epsilon,\lambda}(a) \;=\; \mathbf{I} + \int_0^a \left[Q_\epsilon(s) + \eta\mathbf{I}\right] ds + \int_0^a \left[Q_\epsilon(s_1) + \eta\mathbf{I}\right]\int_0^{s_1}\left[Q_\epsilon(s_2) + \eta\mathbf{I}\right] ds_2\, ds_1 + \cdots. \tag{6.11}$$

Since the right-hand side of (6.11) is non-negative, then we know that $\mathbf{L}_{\epsilon,\lambda}(a)$ is non-negative.
Now, considered for each a as the operator in $\mathbb{R}^n$, l^1 norm of $\mathbf{L}_{\epsilon,\lambda}(a)$ is $\left\|\mathbf{L}_{\epsilon,\lambda}(a)\right\|_{\mathbb{R}^n,1} = \max_{1\le j\le n}\sum_{i=1}^n l_{\epsilon,ij}(a)$. Further, for any $1 \le j \le n$,

$$\frac{d}{da}\sum_{i=1}^n l_{\epsilon,ij}(a) = \sum_{k=1}^n\sum_{i=1}^n q_{ik}(a)l_{\epsilon,kj}(a) \le -\lambda\sum_{i=1}^n l_{\epsilon,ij}(a),$$

since $-\mathbf{M}(a)$ is a sub-Kolmogorov matrix for each a. So $\sum_{i=1}^{n} l_{\epsilon,ij}(a) \leq e^{-\lambda a}$, which implies that $\|\mathbf{L}_{\epsilon,\lambda}\|_{\mathbf{X}} \leq \lambda^{-1}$ where the latter norm is the operator norm from $\mathbb{R}^n$ into $\mathbf{X}$. □

Lemma 6.9 *Let $\mathfrak{B}$ be a bounded operator between $\mathbf{X}$ into $\mathbb{R}^n$. For sufficiently large λ there is a solution operator $\mathcal{H}_{\epsilon,\lambda} : \mathbb{R}^n \to \mathbf{X}$ of the problem*

$$\lambda \mathbf{u} = -\mathbf{u}_a - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon}\mathbf{C}\mathbf{u}, \quad \mathbf{u}(0,t) = \mathfrak{B}\mathbf{u} + \mathbf{f}, \tag{6.12}$$

with $\|\mathcal{H}_{\epsilon,\lambda}\|$ bounded independently of ϵ.

Proof. Consider $\mathbf{L}_{\epsilon,\lambda}\mathbf{g}$ for an unspecified, for a moment, vector $\mathbf{g}$. Then our problem will be solved if we can find $\mathbf{g}$ satisfying $\mathbf{g} = \mathfrak{B}\mathbf{L}_{\epsilon,\lambda}\mathbf{g} + \mathbf{f}$. Now,

$$\left\|\mathfrak{B}\mathbf{L}_{\epsilon,\lambda}\mathbf{g}\right\|_{\mathbb{R}^n} \leq \|\mathfrak{B}\| \left\|\mathbf{L}_{\epsilon,\lambda}\mathbf{g}\right\|_{\mathbb{R}^n} \leq \lambda^{-1} \|\mathfrak{B}\| \left\|\mathbf{g}\right\|_{\mathbb{R}^n},$$

hence $q := \|\mathfrak{B}\| \|\mathbf{L}_{\epsilon,\lambda}\| < 1$ provided λ is large enough. Clearly, λ and q can be chosen independently of ϵ. Then $\mathbf{g} = (\mathbf{I} - \mathfrak{B}\mathbf{L}_{\epsilon,\lambda})^{-1}\mathbf{f}$ and, by the Neumann expansion, $\left\|(\mathbf{I} - \mathfrak{B}\mathbf{L}_{\epsilon,\lambda})^{-1}\right\| \leq (1-q)^{-1}$. Hence, the solution $\mathbf{u}$ to (6.9) is given by

$$\mathbf{u} = \mathcal{H}_{\epsilon,\lambda}\mathbf{f} = \mathbf{L}_{\epsilon,\lambda}\mathbf{g} = \mathbf{L}_{\epsilon,\lambda}\left(\mathbf{I} - \mathfrak{B}\mathbf{L}_{\epsilon,\lambda}\right)^{-1}\mathbf{f},$$

with $\left\|\mathcal{H}_{\epsilon,\lambda}\right\|_{\mathbf{X}} \leq \frac{1}{\lambda(1-q)}$. □

Remark:

In further applications, the boundary data $\mathbf{f}$ depends on t. Since the construction above does not depend on t, $\mathbf{u}$ has the same regularity in t as $\mathbf{f}$ with bounds on derivatives independent of ϵ. Furthermore, the operation $\left(\mathbf{I} - \mathfrak{B}\mathbf{L}_{\epsilon,\lambda}\right)^{-1}$ acts between $\mathbb{R}^n$ and $\mathbb{R}^n$ and thus is a-independent. Since, $\mathbf{u}$ is a solution of a Cauchy problem for a differential equation in a, it is differentiable with respect to a.

The main application of $\mathcal{H}_{\epsilon,\lambda}$ is to reduce the inhomogeneous boundary problem

$$\mathbf{u}_t = -\mathbf{u}_a - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon}\mathbf{C}\mathbf{u} + \mathbf{h}, \quad \mathbf{u}(0,t) = \mathfrak{B}\mathbf{u} + \mathbf{f}, \quad \mathbf{u}(a,0) = \boldsymbol{\phi},$$

where $\mathbf{f}$ is an $\mathbb{R}^n$-valued function differentiable with respect to t, to a problem which is homogeneous on the boundary. By introducing $\mathbf{U} := \mathbf{u} - \mathcal{H}_{\epsilon,\lambda}\mathbf{f}$, we obtain

$$\begin{aligned}
\mathbf{U}_t &= \mathbf{u}_t - \mathcal{H}_{\epsilon,\lambda}\mathbf{f}_t = -\mathbf{M}\mathbf{U} - \mathbf{U}_a + \frac{1}{\epsilon}\mathbf{C}\mathbf{U} + \lambda\mathcal{H}_{\epsilon,\lambda}\mathbf{f} - \mathcal{H}_{\epsilon,\lambda}\mathbf{f}_t + \mathbf{h}, \\
\mathbf{U}(0,t) &= \mathbf{u}(0,t) - \mathcal{H}_{\epsilon,\lambda}\mathbf{f}(0,t) = \mathfrak{B}\mathbf{U} + \mathfrak{B}\mathcal{H}_{\epsilon,\lambda}\mathbf{f} + \mathbf{f} - \mathcal{H}_{\epsilon,\lambda}\mathbf{f}(0,t), \\
\mathbf{U}(a,0) &= \boldsymbol{\phi} - \mathcal{H}_{\epsilon,\lambda}\mathbf{f}(a,0).
\end{aligned} \tag{6.13}$$

We note that in this approach the lifting of $\mathbf{f}$ produces its time derivative on the right hand side of the equation which creates some problems in the asymptotic analysis. This necessitates a refinement of this method which will be discussed later when we consider an integral formulation of (6.5)–(6.7).

6.4 Formal Asymptotic Expansion

To apply the projection operators on the perturbed model (6.5)–(6.7), we are going to use the following series of lemmas.

Lemma 6.10 *For all projection operators $\mathbf{P}_j$, where $1 \le j \le m$, we have $\mathbf{P}_j\,\mathbf{Cu} = 0$.*

Proof. As an example consider, $\mathbf{P}_1\mathbf{Cu} := (\mathbf{x}_1 \cdot \mathbf{Cu})\,\mathbf{e}_1$. Using the above normal form of $\mathbf{C}$, $\mathbf{Cu}$ gives

$$\begin{bmatrix} \mathbf{C}_{n_1}\mathbf{u}_1 + \cdots + \mathbf{0} + \mathbf{A}_{n_1,n_{m+1}}\mathbf{u}_{n_{m+1}} + \cdots + \mathbf{A}_{n_1,n_n}\mathbf{u}_{n_n} \\ \vdots \\ \mathbf{0} + \cdots + \mathbf{C}_{n_m}\mathbf{u}_{n_m} + \mathbf{A}_{n_m,n_{m+1}}\mathbf{u}_{n_{m+1}} + \cdots + \mathbf{A}_{n_m,n_n}\mathbf{u}_{n_n} \\ \mathbf{0} + \cdots + \mathbf{0} + \mathbf{T}_{n_{m+1},n_{m+1}}\mathbf{u}_{n_{m+1}} + \cdots + \mathbf{T}_{n_{m+1},n_n}\mathbf{u}_{n_n} \\ \vdots \\ \mathbf{0} + \cdots + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{T}_{n_n,n_n}\mathbf{u}_{n_n} \end{bmatrix}, \tag{6.14}$$

where $\mathbf{u}_{n_1} := (u_1, \dots, u_{n_1}), \dots, \mathbf{u}_{n_m} := (u_{n_{m-1}+1}, \dots, u_{n_m}), \mathbf{u}_{n_{m+1}} := (u_{n_m+1}, \dots, u_{n_{m+1}}), \dots,$ $\mathbf{u}_{n_n} := (u_{n_{n-1}+1}, \dots, u_n)$. Again using the normal form of $\mathbf{C}$ and considering the fact that $\mathbf{C}_{n_1}$ is a Kolmogorov matrix, we get $\mathbf{x}_1 \cdot \mathbf{C}_{n_1} = 0$. Since $\mathbf{x}_1$ is an eigenvector of $\mathbf{C}$, multiplication of any particular column, say $(\mathbf{A}_{n_1,n_{m+1}}, \dots, \mathbf{A}_{n_m,n_{m+1}}, \mathbf{T}_{n_{m+1},n_{m+1}})$, by $\mathbf{x}_1$, gives zero. Hence $\mathbf{P}_1\mathbf{C}\,\mathbf{u} = 0$. So, in general, for $1 \le j \le m$, we have $\mathbf{P}_j\mathbf{Cu} = 0$ and this completes the proof. □

Lemma 6.11 *For $1 \le j \le m$,*

$$\mathbf{P}_j\mathbf{Mu} = \left(\sum_{i=1}^{m} u^{n_i}\mu^*_{ji}\right)\mathbf{e}_j + \left(\mathbf{x}_j \cdot \mathbf{Mw}\right)\mathbf{e}_j \qquad \textit{and} \qquad \mathbf{P}_j\mathbf{Bu} = \left(\sum_{i=1}^{m} u^{n_i}\beta^*_{ij}\right)\mathbf{e}_j + \left(\mathbf{x}_j \cdot \mathbf{Bw}\right)\mathbf{e}_j\,,$$

where $u^{n_j} := \left(\sum^{n_j} u_i + \sum^{n} l_{ji}\,u_i\right)$, $\mu^*_{ji} := \mathbf{x}_j \cdot \mathbf{Me}_i$ and $\beta^*_{ji} := \mathbf{x}_j \cdot \mathbf{Be}_i$.

Proof.

$$\begin{aligned} \mathbf{P}_j\mathbf{Mu} &= \mathbf{P}_j\mathbf{M}\sum_{k=1}^{m}\mathbf{P}_k\mathbf{u} + \mathbf{P}_j\mathbf{MQu} \\ &= \left[\left(\sum_{i=1}^{n_1} u_i + \sum_{i=n_m+1}^{n} l_{1i}u_i\right)\left(\mathbf{x}_j \cdot \mathbf{Me}_1\right) + \right. \\ &\qquad \left. \cdots + \left(\sum_{i=n_{m-1}+1}^{n_m} u_i + \sum_{i=n_m+1}^{n} l_{mi}u_i\right)\left(\mathbf{x}_j \cdot \mathbf{Me}_m\right)\right]\mathbf{e}_j + \left(\mathbf{x}_j \cdot \mathbf{Mw}\right)\mathbf{e}_j \\ &=: \left[u^{n_1}\mu^*_{1j} + \cdots + u^{n_m}\mu^*_{mj}\right]\mathbf{e}_j + \left(\mathbf{x}_j \cdot \mathbf{Mw}\right)\mathbf{e}_j \\ &=: \left(\sum_{i=1}^{m} u^{n_i}\mu^*_{ji}\right)\mathbf{e}_j + \left(\mathbf{x}_j \cdot \mathbf{Mw}\right)\mathbf{e}_j\,. \end{aligned}$$

Similarly, the expressions for $\mathbf{P}_j\mathbf{B}\mathbf{u}$ are obtained as follows:

$$
\begin{aligned}
\mathbf{P}_j\mathbf{B}\mathbf{u} &= \mathbf{P}_j\mathbf{B}\sum_{k=1}^{m}\mathbf{P}_k\mathbf{u} + \mathbf{P}_j\mathbf{B}\mathbf{Q}\mathbf{u} \\
&= \left[\left(\sum_{i=1}^{n_1} u_i + \sum_{i=n_m+1}^{n} l_{1i}u_i\right)\left(\mathbf{x}_j\cdot\mathbf{B}\mathbf{e}_1\right) + \right. \\
&\qquad \left. \cdots + \left(\sum_{i=n_{m-1}+1}^{n_m} u_i + \sum_{i=n_m+1}^{n} l_{mi}u_i\right)\left(\mathbf{x}_j\cdot\mathbf{B}\mathbf{e}_m\right)\right]\mathbf{e}_j + \left(\mathbf{x}_j\cdot\mathbf{B}\mathbf{w}\right)\mathbf{e}_j \\
&=: \left[u^{n_1}\beta^*_{1j} + \cdots + u^{n_m}\beta^*_{mj}\right]\mathbf{e}_j + \left(\mathbf{x}_j\cdot\mathbf{B}\mathbf{w}\right)\mathbf{e}_j \\
&=: \left(\sum_{i=1}^{m} u^{n_i}\beta^*_{ij}\right)\mathbf{e}_j + \left(\mathbf{x}_j\cdot\mathbf{B}\mathbf{w}\right)\mathbf{e}_j\,. \qquad \square
\end{aligned}
$$

Lemma 6.12 $-[\mu^*_{ji}]_{1\le i,j\le m}$ *is positive off-diagonal and satisfies* $-\sum_{j=1}^{m}\mu^*_{ji} \le 0$ *for any* $1 \le i \le m$.

Proof. μ^*_{ji} defined as

$$\mu^*_{ji} := \mathbf{x}_j\cdot\mathbf{M}\mathbf{e}_i.$$

Each co-ordinate of the vectors $\mathbf{x}_j$ and $\mathbf{e}_i$ is non-negative for $1 \le i, j \le m$ and thus the sign of the terms $\mathbf{x}_j\cdot\mathbf{M}\mathbf{e}_i$ solely depends on the structure of $\mathbf{M}$. For $i \ne j$, $\mathbf{x}_j\cdot\mathbf{M}\mathbf{e}_i$ does not contain any diagonal elements of $\mathbf{M}$ can be shown as follows:
For example let us prove it for $j = 1, i = m$.

Now

$$\mathbf{x}_1 := (\mathbf{1}, \mathbf{0}, \ldots, \mathbf{0}, \mathbf{f}_{n_1,n_{m+1}}, \ldots, \mathbf{f}_{n_1,n_n})\,,$$

where

$$\mathbf{1} := \underbrace{(1, \ldots, 1)}_{n_1\ times}, \mathbf{f}_{n_1,n_{m+1}} := (l_{n_1,n_m+1}, \ldots, l_{n_1,n_{m+1}})\,, \ldots, \mathbf{f}_{n_1,n_n} := (l_{n_1,n_{n-1}+1}, \ldots, l_{n_1,n})\,.$$

and

$$\mathbf{e}_m := \left(0, \ldots, 0, e^m_{n_{m-1}+1}, \ldots, e^m_{n_m}, 0, \ldots, 0\right).$$

Therefore

$$
\begin{aligned}
\mathbf{x}_1\cdot\mathbf{M}\mathbf{e}_m := &\left(\mu_{1,n_{m-1}+1}\,e^m_{n_{m-1}+1} + \cdots + \mu_{1,n_m}e^m_{n_m}\right) + \cdots \\
&+ \left(\mu_{n_1,n_{m-1}+1}\,e^m_{n_{m-1}+1} + \cdots + \mu_{n_1,n_m}\,e^m_{n_m}\right) \\
&+ \left(\mu_{n_m+1,n_{m-1}+1}\,e^m_{n_{m-1}+1} + \cdots + \mu_{n_m+1,n_m}\,e^m_{n_m}\right) l_{n_1,n_m+1} + \cdots \\
&+ \left(\mu_{n_{m+1},n_{m-1}+1}\,e^m_{n_{m-1}+1} + \cdots + \mu_{n_{m+1},n_m}\,e^m_{n_m}\right) l_{n_1,n_{m+1}} + \cdots \\
&+ \left(\mu_{n_{m-1}+1,n_{m-1}+1}\,e^m_{n_{m-1}+1} + \cdots + \mu_{n_n,n_m}\,e^m_{n_m}\right) l_{n_1,n_{n-1}+1} + \cdots \\
&+ \left(\mu_{n_n,n_{m-1}+1}\,e^m_{n_{m-1}+1} + \cdots + \mu_{n_n,n_m}\,e^m_{n_m}\right) l_{n_1,n}\,,
\end{aligned}
$$

which is positive and in general since by hypothesis **H3**, $-\mathbf{M}$ is positive off-diagonal, $-[\mu^*_{ji}]_{1\le i,j\le m}$ is positive off-diagonal.

Next consider

$$\sum_{i=1}^{m} \mu^*_{ij}(a) = \sum_{i=1}^{m} \mathbf{x}_i \cdot \mathbf{M}(a)\mathbf{e}_j\,, \qquad \text{for } 1 \le j \le m.$$

For example, for $j = 1$, we have $\sum\limits_{i=1}^{m} \mu^*_{i1} = \sum\limits_{i=1}^{m} \mathbf{x}_i \cdot \mathbf{M}\mathbf{e}_1$ and terms of this sum are as follows:

$$\begin{aligned}
\mathbf{x}_1 \cdot \mathbf{M}\mathbf{e}_1 = {} & \left(\mu_{11} e^m_1 + \cdots + \mu_{1,n_1} e^m_{n_1}\right) + \cdots + \left(\mu_{n_1,1}\, e^m_1 + \cdots + \mu_{n_1,n_1}\, e^m_{n_1}\right) + \cdots \\
& + \left(\mu_{n_m+1,1}\, e^m_1 + \cdots + \mu_{n_m+1,n_1}\, e^m_{n_1}\right) l_{n_1,n_m+1} + \cdots \\
& + \left(\mu_{n_{m+1},1}\, e^m_1 + \cdots + \mu_{n_{m+1},n_1}\, e^m_{n_1}\right) l_{n_1,n_{m+1}} + \cdots \\
& + \left(\mu_{n_{n-1}+1,1}\, e^m_1 + \cdots + \mu_{n_{n-1}+1,n_1}\, e^m_{n_1}\right) l_{n_1,n_{n-1}+1} + \cdots \\
& + \left(\mu_{n1}\, e^m_1 + \cdots + \mu_{n,n_1}\, e^m_{n_1}\right) l_{n_1,n}\,, \\
& \vdots \\
\mathbf{x}_m \cdot \mathbf{M}\mathbf{e}_1 = {} & \left(\mu_{n_{m-1}+1,1} e^m_1 + \cdots + \mu_{n_{m-1}+1,n_1} e^m_{n_1}\right) + \cdots + \left(\mu_{n_m,1}\, e^m_1 + \cdots + \mu_{n_m,n_1}\, e^m_{n_1}\right) \\
& + \cdots + \left(\mu_{n_m+1,1}\, e^m_1 + \cdots + \mu_{n_m+1,n_1}\, e^m_{n_1}\right) l_{n_m,n_m+1} + \cdots \\
& + \left(\mu_{n_{m+1},1}\, e^m_1 + \cdots + \mu_{n_{m+1},n_1}\, e^m_{n_1}\right) l_{n_m,n_{m+1}} + \cdots \\
& + \left(\mu_{n_{n-1}+1,1}\, e^m_1 + \cdots + \mu_{n_{n-1}+1,n_1}\, e^m_{n_1}\right) l_{n_m,n_{n-1}+1} + \cdots \\
& + \left(\mu_{n1}\, e^m_1 + \cdots + \mu_{n,n_1}\, e^m_{n_1}\right) l_{n_m,n}.
\end{aligned}$$

If we look, for example, on the elements of the first column of $\mathbf{M}$ we have

$$\begin{aligned}
\left(\mu_{11} + \cdots + \mu_{n_m,1}\right) e^m_1 + \cdots + \mu_{n_m+1,1}\, e^m_1 \left(l_{n_1,n_m+1} + \cdots + l_{n_m,n_m+1}\right) + \cdots \\
+ \mu_{n1} e^m_1 \left(l_{n_1,n} + \cdots + l_{n_m,n}\right) \\
= \left(\sum_{j=1}^{n} \mu_{j1}\right) e^m_1,
\end{aligned}$$

where we have used (6.4). Since $-\sum\limits_{j=1}^{n} \mu_{ji} \le 0$, for $1 \le i \le m$, the required result immediately follows. $\square$

Lemma 6.13 *For $1 \le i, j \le m$, we have the following results*

1. $\mathbf{P}_i\mathbf{P}_j\mathbf{u} = 0$,

2. $\mathbf{P}_i\mathbf{u}_a = u_a^{n_i}\mathbf{e}_i$ and $\displaystyle\sum_{j=1}^{m} u^{n_j}\mathbf{e}_{j,a} \in S$,

3. $\mathbf{P}_j\mathbf{w}_a = -\left(\mathbf{f}'_{n_j} \cdot \mathbf{w}\right)\mathbf{e}_j\,,$

4. $\mathbf{Q}\mathbf{w}_a = \mathbf{w}_a + \displaystyle\sum_{i=n_m+1}^{n} w_i \left(\sum_{j=1}^{m} l'_{n_j,i}\mathbf{e}_j\right).$

Proof. 1. For $i \ne j$, we have

$$\mathbf{P}_i\mathbf{P}_j\mathbf{u} = \mathbf{P}_i(\mathbf{x}_j \cdot \mathbf{u})\mathbf{e}_j = (\mathbf{x}_j \cdot \mathbf{u})(\mathbf{x}_i \cdot \mathbf{e}_j)\mathbf{e}_i = 0,$$

since $\mathbf{x}_i \cdot \mathbf{e}_j = 0$ for $i \ne j$.

2. Using spectral projections $\mathbf{P}_j, \mathbf{Q}$, where $1 \le j \le m$, we can represent $\mathbf{u}$ as follows:

$$\mathbf{u} = \sum_{j=1}^{m} \mathbf{P}_j\mathbf{u} + \mathbf{Q}\mathbf{u} = \sum_{j=1}^{m} u^{n_j}\mathbf{e}_j + \mathbf{w}.$$

Differentiating with respect to a, we get

$$\mathbf{u}_a = \sum_{j=1}^{m} u_a^{n_j}\mathbf{e}_j + \sum_{j=1}^{m} u^{n_j}\mathbf{e}_{j,a} + \mathbf{w}_a.$$

Applying $\mathbf{P}_i$, where $1 \le i \le m$, we have

$$\mathbf{P}_i\mathbf{u}_a = \mathbf{P}_i(u_a^{n_i}\mathbf{e}_i + u^{n_i}\mathbf{e}_{i,a}) = u_a^{n_i}\mathbf{e}_i\,,$$

where we have used the following fact:

$$\mathbf{P}_i(u^{n_i}\mathbf{e}_{i,a}) = u^{n_i}(\mathbf{x}_i \cdot \mathbf{e}_{i,a})\mathbf{e}_i = u^{n_i}(\mathbf{x}_i \cdot \mathbf{e}_i)'\mathbf{e}_i = 0 \;\; [\text{as } \mathbf{x}_i \cdot \mathbf{e}_i = 1 \implies (\mathbf{x}_i \cdot \mathbf{e}_i)' = 0]\,.$$

Here prime denotes the derivative with respect to a. Using the definition of the subspace S and $\mathbf{P}_i(u^{n_i}\mathbf{e}_{i,a}) = 0$, we conclude that $\sum_{j=1}^{m} u^{n_j}\mathbf{e}_{j,a}$ is in S.

3. From the definition of S, if $\mathbf{w} \in S$, then $\mathbf{x}_j \cdot \mathbf{w} = 0$. Differentiating with respect to a, we get $(\mathbf{x}_j)' \cdot \mathbf{w} + \mathbf{x}_j \cdot \mathbf{w}_a = 0$. Now using the definition of $\mathbf{P}_j$, where $1 \le j \le m$, we have

$$\mathbf{P}_j\mathbf{w}_a = (\mathbf{x}_j \cdot \mathbf{w}_a)\mathbf{e}_j = -\left((\mathbf{x}_j)' \cdot \mathbf{w}\right)\mathbf{e}_j = -\left(\sum_{i=n_m+1}^{n} l'_{n_j,i}\, w_i\right)\mathbf{e}_j = -\left(\mathbf{f}'_{n_j} \cdot \mathbf{w}\right)\mathbf{e}_j\,, \tag{6.15}$$

where $\mathbf{f}_{n_j} := \Big(\mathbf{0}, l_{n_j,n_j+1}, \ldots, l_{n_j,n}\Big)$.

4. Using the expression of $\mathbf{P}_j\mathbf{w}_a$ from (6.15) we get

$$\begin{aligned}
\mathbf{Q}\mathbf{w}_a = \mathbf{w}_a - \sum_{j=1}^{m} \mathbf{P}_j\mathbf{w}_a &= \mathbf{w}_a - \sum_{j=1}^{m} \Big(\mathbf{x}_j \cdot \mathbf{w}_a\Big)\,\mathbf{e}_j \\
&= \mathbf{w}_a + \sum_{j=1}^{m} \Big(\big(\mathbf{x}_j\big)' \cdot \mathbf{w}\Big)\,\mathbf{e}_j \\
&= \mathbf{w}_a + \sum_{j=1}^{m} \left(\sum_{i=n_m+1}^{n} l'_{n_j,i} w_i \right) \mathbf{e}_j \\
&= \mathbf{w}_a + \sum_{i=n_m+1}^{n} w_i \left(\sum_{j=1}^{m} l'_{n_j,i} \mathbf{e}_j \right).
\end{aligned}$$

□

Now, using the above lemmas and applying projections $\mathbf{P}_j$, where $1 \le j \le m$, and $\mathbf{Q}$ on (6.5)–(6.7) we get respectively,

$$u_t^{n_j} = -u_a^{n_j} - \sum_{i=1}^{m} \mu_{ji}^{*} u^{n_i} + \mathbf{f}'_{n_j} \cdot \mathbf{w} - \mathbf{x}_j \cdot \mathbf{M}\mathbf{w}, \tag{6.16}$$

and

$$\epsilon \mathbf{w}_t = -\epsilon \mathbf{w}_a - \epsilon \sum_{j=1}^{m} \Big(\mathbf{f}'_{n_j} \cdot \mathbf{w}\Big)\,\mathbf{e}_j - \epsilon\, u^m \mathbf{e}_a - \epsilon \mathbf{M}_S \mathbf{w} - \epsilon \mathbf{M}_S \mathbf{u}^m + \mathbf{C}_S \mathbf{w}, \tag{6.17}$$

where

$$u^m \mathbf{e}_a := \sum_{i=1}^{m} u^{n_i} \mathbf{e}_{i,a} \qquad \text{and} \qquad \mathbf{u}^m := \sum_{j=1}^{m} u^{n_j} \mathbf{e}_j.$$

For the boundary, we have

$$u^{n_j}(0,t) = \int_0^{\infty} \left(\sum_{i=1}^{m} \beta_{ji}^{*}(s)\, u^{n_i}(s,t) \right) ds + \int_0^{\infty} \mathbf{x}_j \cdot \mathbf{B}(s)\mathbf{w}(s,t)\, ds, \tag{6.18}$$

and

$$\mathbf{w}(0,t) = \mathfrak{B}\mathbf{u}^m(\cdot,t) + \mathfrak{B}\mathbf{w}(\cdot,t) - \mathbf{u}^m(0,t). \tag{6.19}$$

The initial conditions are

$$\mathbf{P}_1\, \mathbf{u}(a,0) =: \mathbf{v}_1^0(a), \ldots, \mathbf{P}_m\, \mathbf{u}(a,0) =: \mathbf{v}_m^0(a)\,; \;\; \mathbf{Q}\, \mathbf{u}(a,0) =: \mathbf{w}_0(a). \tag{6.20}$$

6.4.1 Bulk Approximation

Following Section 5.2.3, we put $\mathbf{w}$ as $\mathbf{w} = \mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots$, leaving however, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ unexpanded. Thus

$$\mathbf{u} = (\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}) = (\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots).$$

Inserting these into the projected system (6.16)–(6.20) we get respectively:

$$u_t^{n_j} = -u_a^{n_j} - \sum_{i=1}^{m} \mu_{ji}^* u^{n_i} + \mathbf{f}'_{n_j} \cdot (\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots) - \mathbf{x}_j \cdot \mathbf{M}(\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots), \tag{6.21}$$

and

$$\begin{aligned}\epsilon\mathbf{w}_{0,t} + \epsilon^2\mathbf{w}_{1,t} + \cdots = -\epsilon\mathbf{w}_{0,a} - \epsilon^2\mathbf{w}_{1,a} + \cdots - \epsilon \sum_{j=1}^{m} \Big(\mathbf{f}'_{n_j} \cdot (\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots)\Big)\mathbf{e}_j - \epsilon\mathbf{M}_S\mathbf{u}^m \\ - \epsilon\, u^m\mathbf{e}_a - \epsilon\mathbf{M}_S(\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots) + \mathbf{C}_S(\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots),\end{aligned} \tag{6.22}$$

with boundary conditions as

$$u^{n_j}(0,t) = \int_0^\infty \Big(\sum_{i=1}^{m} \beta_{ij}^* u^{n_i}\Big) + \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}(\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots), \tag{6.23}$$

and

$$\mathbf{w}_0(0,t) + \epsilon\mathbf{w}_1(0,t) + \cdots = \mathfrak{B}\mathbf{u}^m + \mathfrak{B}(\mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots) - \mathbf{u}^m(0,t).$$

For the initial conditions, we get

$$\begin{aligned} u^{n_j}(a,0) &= \mathbf{x}_j \cdot \boldsymbol{\phi}(a), \qquad (6.24)\\ \mathbf{w}_0(a,0) + \epsilon\mathbf{w}_1(a,0) + \cdots &= \mathbf{w}_0(a), \end{aligned}$$

where $1 \leq j \leq m$.

Let $\bar{u}^{n_j}$ and $\bar{\mathbf{w}} := \bar{\mathbf{w}}_0 + \epsilon\bar{\mathbf{w}}_1$ denote the solutions to the truncated equations. Comparing coefficients of ϵ^0 order terms from (6.21) and (6.22), we have respectively,

$$\bar{u}_t^{n_j} = -\bar{u}_a^{n_j} - \sum_{i=1}^{m} \mu_{ji}^* \bar{u}^{n_i}, \tag{6.25}$$

$$\mathbf{C}_S\,\bar{\mathbf{w}}_0 = \mathbf{0}. \tag{6.26}$$

Again, from Section 5.2.3, we know that $\mathbf{C}_S$ is invertible and thus the last equation (6.26) gives $\bar{\mathbf{w}}_0 = \mathbf{0}$.

Comparing ϵ order terms from (6.22), we have

$$\bar{\mathbf{w}}_{0,t} = -\bar{\mathbf{w}}_{0,a} - \mathbf{M}_S\bar{\mathbf{u}}^m - \bar{u}^m\mathbf{e}_a - \mathbf{M}_S\bar{\mathbf{w}}_0 + \mathbf{C}_S\bar{\mathbf{w}}_1\,,$$

and from this we get

$$\bar{\mathbf{w}}_1 = \mathbf{C}_S^{-1}\,[\,\bar{u}^m \mathbf{e}_a + \mathbf{M}_S \bar{\mathbf{u}}^m\,]\,, \tag{6.27}$$

where we have used $\bar{\mathbf{w}}_0 = 0$.

On the boundary part, comparing ϵ^0 order terms on the hydrodynamic subspace (i.e., from equation (6.23)) we have

$$\bar{u}^{n_j}(0,t) = \int_0^\infty \left(\sum_{i=1}^m \beta_{ij}^*(s)\,\bar{u}^{n_i}(s,t) \right) ds. \tag{6.28}$$

Collecting equations (6.25), (6.28) and (6.24), we arrive at the closed system for $\bar{u}^{n_j}$:

$$\begin{aligned} \bar{u}_t^{n_j} &= -\bar{u}_a^{n_j} - \sum_{i=1}^m \mu_{ji}^*\,\bar{u}^{n_i}, \\ \bar{u}^{n_j}(0,t) &= \int_0^\infty \left(\sum_{i=1}^m \beta_{ij}^*(s)\,\bar{u}^{n_i}(s,t) \right) ds, \\ \bar{u}^{n_j}(a,0) &= \mathbf{x}_j \cdot \boldsymbol{\phi}(a). \end{aligned} \tag{6.29}$$

Now, in order to solve this system, let $\bar{\mathbf{v}} := (\bar{u}^{n_1}, \ldots, \bar{u}^{n_m})$, $\mathbf{M}_v(a) := [\mu_{ji}^*(a)]_{1 \le i,j \le m}$, $\boldsymbol{\phi}^m(a) := (\mathbf{x}_1 \cdot \boldsymbol{\phi}(a), \ldots, \mathbf{x}_m \cdot \boldsymbol{\phi}(a))$ and $\mathbf{B}_v(a) := [\beta_{ji}^*(a)]_{1 \le i,j \le m}$. With this notation, the closed system (6.29) for $\bar{u}^{n_j}$ is equivalent to

$$\begin{aligned} \bar{\mathbf{v}}_t + \bar{\mathbf{v}}_a &= -\mathbf{M}_v(a)\bar{\mathbf{v}}, \\ \bar{\mathbf{v}}(a,0) &= \boldsymbol{\phi}^m(a), \\ \bar{\mathbf{v}}(0,t) &= \int_0^\infty \mathbf{B}_v(s)\bar{\mathbf{v}}(s,t)\,ds. \end{aligned} \tag{6.30}$$

Let $\mathbf{L}_v(a)$ be the fundamental solution to

$$\mathbf{L}_v'(a) = -\mathbf{M}_v(a)\mathbf{L}_v(a), \qquad \mathbf{L}_v(0) = \mathbf{I}.$$

We define $\mathbf{L}_v(b,a)$ as

$$\mathbf{L}_v(b,a) := \mathbf{L}_v(b)\mathbf{L}_v^{-1}(a).$$

Then solution of (6.30) is given by

$$\bar{\mathbf{v}}(a,t) = \begin{cases} \mathbf{L}_v(a)\bar{\mathbf{v}}(0,t-a), & a < t; \\ \mathbf{L}_v(a,a-t)\boldsymbol{\phi}^m(a-t), & a > t, \end{cases}$$

where

$$\boldsymbol{\psi}(t) := \bar{\mathbf{v}}(0,t) = \int_0^t \mathbf{B}_v(a)\mathbf{L}_v(a)\boldsymbol{\psi}(t-a)\,da + \int_t^\infty \mathbf{B}_v(a)\mathbf{L}_v(a)\mathbf{L}_v^{-1}(a-t)\boldsymbol{\phi}^m(a-t)\,da. \tag{6.31}$$

In order to write the system (6.30) as an abstract Cauchy problem we introduce the operator $\mathcal{A}$ as

$$\mathcal{A}\,\mathbf{x}(a) := -\mathbf{x}'(a) + \mathbf{M}_v(a)\mathbf{x}(a),$$

on the domain

$$\mathcal{D}(\mathcal{A}) := \left\{ \mathbf{x} \in L_1(0, w; \mathbb{C}^n) : \mathbf{x} \text{ is absolutely continuous}, \ \mathbf{x}(0) = \int_0^\infty \mathbf{B}_v(s)\mathbf{x}(s)\, ds \right\}.$$

Using the semigroup method as we did in Chapter 4, we can prove the following result.

Theorem 6.14 *$\mathcal{A}$ generates a strongly continuous semigroup $\{\mathcal{T}(t)\}_{t\geq 0}$ such that*

$$\|\mathcal{T}(t)\| \leq e^{(\bar{b}-\underline{m})t},$$

where $\bar{b} := \operatorname*{ess\,sup}_{0\leq a\leq w} \|\boldsymbol{B}_v(a)\| < +\infty$, $\underline{m} := \inf_{j,a} \mu_j^*(a)$ and $\mu_j^*(a) := \mu_{jj}^*(a)$.

Here we also note that

$$\begin{aligned}
\sum_{j=1}^{m} u^{n_j} &= \left(\sum_{i=1}^{n_1} u_i + \cdots + \sum_{i=n_m+1}^{n_m} u_i \right) + \left(\sum_{i=n_m+1}^{n} l_{1i}u_i + \cdots + \sum_{i=n_m+1}^{n} l_{mi}u_i \right) \\
&= \sum_{i=1}^{n_m} u_i + \sum_{i=n_m+1}^{n} u_i \sum_{k=1}^{m} l_{ki} \\
&= \sum_{i=1}^{n_m} u_i + \sum_{i=n_m+1}^{n} u_i \qquad [\text{by}(6.4)] \\
&= \sum_{i=1}^{n} u_i\,,
\end{aligned}$$

which implies that the sum of the scalars of the hydrodynamic space gives the total population.
The error due to the approximation we made in the bulk part is

$$\bar{\mathbf{E}} := (\bar{\mathbf{g}}_j, \bar{\mathbf{h}}) := (\bar{g}_j\, \mathbf{e}_j, \bar{\mathbf{h}}) := (u^{n_j}\, \mathbf{e}_j - \bar{u}^{n_j}\, \mathbf{e}_j, \mathbf{w} - \epsilon\, \bar{\mathbf{w}}_1), \tag{6.32}$$

where $1 \leq j \leq m$. The error satisfies:

$$\begin{aligned}
\bar{g}_{j,t} = u_t^{n_j} - \bar{u}_t^{n_j} &= -u^{n_j} + \mathbf{f}'_{n_j} \cdot \mathbf{w} - \sum_{i=1}^{m} \mu_{ji}^* u^{n_i} - \mathbf{x}_j \cdot \mathbf{M}\mathbf{w} + \bar{u}_a^{n_j} + \sum_{i=1}^{m} \mu_{ji}^* \bar{u}^{n_i} \\
&= -\bar{g}_{j,a} + \mathbf{f}'_{n_j} \cdot \bar{\mathbf{h}} + \epsilon \mathbf{f}'_{n_j} \cdot \bar{\mathbf{w}}_1 - \sum_{i=1}^{m} \mu_{ji}^* \bar{g}_i - \mathbf{x}_j \cdot \mathbf{M}\bar{\mathbf{h}} - \epsilon\, \mathbf{x}_j \cdot \mathbf{M}\bar{\mathbf{w}}_1,
\end{aligned}$$

and

$$
\begin{aligned}
\bar{\mathbf{h}}_t &= \mathbf{w}_t - \epsilon\bar{\mathbf{w}}_{1,t} \\
&= -\mathbf{w}_a - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\mathbf{w}\right)\mathbf{e}_j - \mathbf{M}_S\mathbf{u}^m - u^m\mathbf{e}_a - \mathbf{M}_S\mathbf{w} + \frac{1}{\epsilon}\mathbf{C}_S\mathbf{w} - \epsilon\,\bar{\mathbf{w}}_{1,t} \\
&= -\bar{\mathbf{h}}_a - \epsilon\bar{\mathbf{w}}_{1,a} - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{h}}\right)\mathbf{e}_j - \epsilon\sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1\right)\mathbf{e}_j - \mathbf{M}_S\sum_{i=1}^{m}\bar{\mathbf{g}}_i \\
&\quad - \mathbf{M}_S\bar{\mathbf{u}}^m - \bar{u}^m\mathbf{e}_a - \bar{g}^m\mathbf{e}_a - \mathbf{M}_S\bar{\mathbf{h}} - \epsilon\mathbf{M}_S\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\bar{\mathbf{h}} + \mathbf{C}_S\bar{\mathbf{w}}_1 - \epsilon\bar{\mathbf{w}}_{1,t} \\
&= -\bar{\mathbf{h}}_a - \epsilon\bar{\mathbf{w}}_{1,a} - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{h}}\right)\mathbf{e}_j - \epsilon\sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1\right)\mathbf{e}_j - \mathbf{M}_S\sum_{i=1}^{m}\bar{\mathbf{g}}_i \\
&\quad - \bar{g}^m\mathbf{e}_a - \mathbf{M}_S\bar{\mathbf{h}} - \epsilon\mathbf{M}_S\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\bar{\mathbf{h}} - \epsilon\bar{\mathbf{w}}_{1,t}\,,
\end{aligned}
$$

with initial conditions

$$\bar{g}_j(a,0) = 0,$$

and

$$
\begin{aligned}
\bar{\mathbf{f}}(a,0) &= \mathbf{w}(a,0) - \epsilon\bar{\mathbf{w}}_1(a,0) \\
&= \mathbf{w}_0(a) - \epsilon\mathbf{C}_S^{-1}(a)\left[\bar{u}^m(a,0)\mathbf{e}_a(a) + \mathbf{M}_S(a)\bar{\mathbf{u}}^m(a,0)\right]. \qquad (6.33)
\end{aligned}
$$

The boundary conditions for the error are as follows.

$$
\begin{aligned}
\bar{g}_j(0,t) &= u^{n_j}(0,t) - \bar{u}^{n_j}(0,t) \\
&= \int_0^\infty\left(\sum_{i=1}^{m}\beta^*_{ji}\,\bar{g}_i\right) + \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}\bar{\mathbf{h}} + \epsilon\int_0^\infty \mathbf{x}_j\cdot\mathbf{B}\bar{\mathbf{w}}_1\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\bar{\mathbf{h}}(0,t) &= \mathbf{w}(0,t) - \epsilon\mathbf{w}_1(0,t) \\
&= \mathfrak{B}\mathbf{u}^m + \mathfrak{B}\mathbf{w} - \mathbf{u}^m(0,t) - \epsilon\mathbf{w}_1(0,t) \\
&= \mathfrak{B}\bar{\mathbf{u}}^m + \mathfrak{B}\sum_{i=1}^{m}\bar{\mathbf{g}}_i^m + \mathfrak{B}\bar{\mathbf{h}} + \epsilon\mathfrak{B}\bar{\mathbf{w}}_1 - \sum_{i=1}^{m}\bar{\mathbf{g}}^m(0,t) - \bar{\mathbf{u}}^m(0,t) - \epsilon\bar{\mathbf{w}}_1(0,t).
\end{aligned}
$$

From the above set of error equations, we observe that in the initial and boundary equations do have some terms which are not of order ϵ. To remedy the situation we need to introduce corrections which will take care of the transient phenomena occurring close to $t = 0$ and to the boundary $a = 0$. They should not 'spoil' the approximation away from spatial and temporal boundaries and thus should rapidly decrease to zero with increasing distance from both boundaries.

6.4.2 Initial Layer

To construct the initial layer corrector we elongate the time parameter t as $\tau := t/\epsilon$ and do microscopic analysis near $t = 0$. The initial layer corrections are denoted by $\tilde{\mathbf{u}} := (\tilde{u}^{n_j}\mathbf{e}_j, \tilde{\mathbf{w}})$. Thanks to the linearity of the problem, we approximate the solution $\mathbf{u}$ as the sum of the bulk part obtained above and the initial layer we construct below. Inserting the formal expansion

$$\tilde{u}^{n_j}(a,\tau) = \tilde{u}_0^{n_j}(a,\tau) + \epsilon\tilde{u}_1^{n_j}(a,\tau) + \cdots, \quad \text{and} \quad \tilde{\mathbf{w}}(a,\tau) = \tilde{\mathbf{w}}_0(a,\tau) + \epsilon\tilde{\mathbf{w}}_1(a,\tau) + \cdots$$

into the (6.16)–(6.20) we get respectively:

$$\begin{aligned}
\epsilon^{-1}(\tilde{u}_{0,\tau}^{n_j} + \epsilon\tilde{u}_{1,\tau}^{n_j} + \cdots) &= -(\tilde{u}_{0,a}^{n_j} + \epsilon\tilde{u}_{1,a}^{n_j} + \cdots) + \mathbf{f}'_{n_j}\cdot(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots) \\
&\quad - \sum_{i=1}^{m}\mu_{ji}^{*}(\tilde{u}_0^{n_i} + \epsilon\tilde{u}_1^{n_i} + \cdots) - \mathbf{x}_j\cdot\mathbf{M}(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots), \quad (6.34) \\
\epsilon^{-1}(\tilde{\mathbf{w}}_{0,\tau} + \epsilon\tilde{\mathbf{w}}_{1,\tau} + \cdots) &= -(\tilde{\mathbf{w}}_{0,a} + \epsilon\tilde{\mathbf{w}}_{1,a} + \cdots) - \sum_{j=1}^{m}\mathbf{f}'_{n_j}\cdot(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots)\mathbf{e}_j \\
&\quad - (\tilde{u}_0^{m} + \epsilon\tilde{u}_1^{m} + \cdots)\mathbf{e}_a - \mathbf{M}_S(\tilde{\mathbf{u}}_0^{m} + \epsilon\tilde{\mathbf{u}}_1^{m} + \cdots) \\
&\quad + \frac{1}{\epsilon}\mathbf{C}_S(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots), \quad (6.35) \\
\tilde{u}_0^{n_j}(0,t) + \epsilon\tilde{u}_1^{n_j}(0,t) + \cdots &= \int_0^\infty \left(\sum_{i=1}^{m}\beta_{ij}^{*}(s,t)\,(\tilde{u}_0^{n_i}(s,t) + \epsilon\tilde{u}_1^{n_i}(s,t) + \cdots)\right) ds \\
&\quad + \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}(s)(\tilde{\mathbf{w}}_0(s,t) + \epsilon\tilde{\mathbf{w}}_1(s,t) + \cdots)\,ds, \\
\tilde{\mathbf{w}}_0(0,t) + \epsilon\tilde{\mathbf{w}}_1(0,t) + \cdots &= \mathfrak{B}(\tilde{\mathbf{u}}_0^{m} + \epsilon\tilde{\mathbf{u}}_1^{m} + \cdots) + \mathfrak{B}(\tilde{\mathbf{w}}_0 + \epsilon\tilde{\mathbf{w}}_1 + \cdots) \\
&\quad - \left(\tilde{\mathbf{u}}_0^{m}(0,t) + \epsilon\tilde{\mathbf{u}}_1^{m}(0,t) + \cdots\right), \\
\tilde{u}^{n_j}(a,0) &= 0, \\
\tilde{\mathbf{w}}_0(a,0) + \epsilon\tilde{\mathbf{w}}_1(a,0) + \cdots &= \mathbf{w}_0\,,
\end{aligned}$$

where, in the initial condition, we have taken into account that the exact initial condition for the hydrodynamic part is already satisfied by the bulk hydrodynamic approximation but the bulk kinetic part cannot satisfy the exact initial condition. Comparing coefficients of ϵ^0, from equation (6.34) we obtain $\tilde{u}_{0,\tau}^{n_j} = 0$ and with the same argument as done in Section 5.2.4, we can take $\tilde{u}_0^{n_j} = 0$.

Next, comparing coefficients at like powers of ϵ^0, from equation (6.35) we have

$$\tilde{\mathbf{w}}_{0,\tau} = \mathbf{C}_S\tilde{\mathbf{w}}_0\,, \tag{6.36}$$

and solving (6.36), we obtain

$$\tilde{\mathbf{w}}_0(a,\tau) = e^{\tau\mathbf{C}_S(a)}\mathbf{w}_0(a),$$

where we defined $\tilde{\mathbf{w}}_0(a,0) := \mathbf{w}_0(a)$ in order to get rid of $\mathbf{w}_0(a)$ from the initial error equation of the bulk kinetic part (6.33). 0 is a dominant eigenvalue of $\mathbf{C}$. Thus in $\mathbf{S}$, all the eigenvalues of $\mathbf{C}_S$ have negative real parts and hence $\tilde{\mathbf{w}}_0$ decays to $\mathbf{0}$ exponentially fast as expected from a layer term. At this level, we cannot do any corrections to the boundary condition. We will also notice there is additional error on the boundary due to initial layer.
Modifying the approximation (6.32) due to initial layer, the new error defined as:

$$\begin{aligned}\tilde{\mathbf{E}}(a,t) &:= (\tilde{\mathbf{g}}_j(a,t), \tilde{\mathbf{h}}(a,t)) = (\,\tilde{g}_j(a,t)\,\mathbf{e}_j,\, \tilde{\mathbf{h}})\\ &= \left(u^{n_j}\mathbf{e}_j - \bar{u}^{n_j}\mathbf{e}_j, \mathbf{w}(a,t) - \epsilon\bar{\mathbf{w}}_1(a,t) - \tilde{\mathbf{w}}_0(a,t/\epsilon)\right)\\ &= \left(\bar{\mathbf{g}}_j(a,t), \bar{\mathbf{h}}(a,t) - \tilde{\mathbf{w}}_0(a,t/\epsilon)\right). \end{aligned} \tag{6.37}$$

Thus we have the following set of error equations:

$$\begin{aligned}\tilde{g}_{j,t} &= -\tilde{g}_{j,a} + \mathbf{f}'_{n_j}\cdot\tilde{\mathbf{h}} + \mathbf{f}'_{n_j}\cdot\tilde{\mathbf{w}}_0 + \epsilon\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1\\ &\quad - \sum_{i=1}^{m}\mu^*_{ji}\,\tilde{g}_i - \mathbf{x}_j\cdot\mathbf{M}\tilde{\mathbf{h}} - \mathbf{x}_j\cdot\mathbf{M}\tilde{\mathbf{w}}_0 - \epsilon\,\mathbf{x}_j\cdot\mathbf{M}\bar{\mathbf{w}}_1\,,\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathbf{h}}_t &= -\bar{\mathbf{h}}_a - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{h}}\right)\mathbf{e}_j - \epsilon\sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1\right)\mathbf{e}_j - \epsilon\bar{\mathbf{w}}_{1,a} - \mathbf{M}_S\sum_{i=1}^{m}\bar{\mathbf{g}}_i\\ &\quad - \bar{g}^m\mathbf{e}_a - \mathbf{M}_S\bar{\mathbf{h}} - \epsilon\mathbf{M}_S\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\bar{\mathbf{h}} - \epsilon\bar{\mathbf{w}}_{1,t} - \frac{1}{\epsilon}\tilde{\mathbf{w}}_{0,\tau}\\ &= -\tilde{\mathbf{h}}_a - \tilde{\mathbf{w}}_{0,a} - \epsilon\bar{\mathbf{w}}_{1,a} - \mathbf{M}_S\sum_{i=1}^{m}\tilde{\mathbf{g}}_i - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\tilde{\mathbf{h}}\right)\mathbf{e}_j - \epsilon\sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1\right)\mathbf{e}_j\\ &\quad - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\tilde{\mathbf{w}}_0\right)\mathbf{e}_j - \tilde{g}^m\mathbf{e}_a - \mathbf{M}_S\tilde{\mathbf{h}} - \mathbf{M}_S\tilde{\mathbf{w}}_0 - \epsilon\mathbf{M}_S\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\tilde{\mathbf{h}} - \epsilon\bar{\mathbf{w}}_{1,t},\end{aligned}$$

with initial conditions as

$$\begin{aligned}&\tilde{g}_j(a,0) = 0,\\ &\tilde{\mathbf{h}}(a,0) = -\epsilon\,\mathbf{C}_S^{-1}(a)\,[\bar{u}^m(a,0)\mathbf{e}_a(a) + \mathbf{M}_S(a)\bar{\mathbf{u}}^m(a,0)]\,,\end{aligned}$$

and for boundary we have

$$\tilde{g}_j(0,t) = \int_0^\infty\left(\sum_{i=1}^{m}\beta^*_{ji}\,\tilde{g}_i\right) + \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}\tilde{\mathbf{h}} + \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}\tilde{\mathbf{w}}_0 + \epsilon\int_0^\infty \mathbf{x}_j\cdot\mathbf{B}\bar{\mathbf{w}}_1$$

and

$$\begin{aligned}\tilde{\mathbf{h}}(0,t) &= \mathfrak{B}\bar{\mathbf{u}}^m + \mathfrak{B}\sum_{i=1}^{m}\tilde{\mathbf{g}}_i^m + \mathfrak{B}\tilde{\mathbf{h}} + \mathfrak{B}\tilde{\mathbf{w}}_0 + \epsilon\mathfrak{B}\bar{\mathbf{w}}_1\\ &\quad - \sum_{i=1}^{m}\tilde{\mathbf{g}}_i^m(0,t) - \bar{\mathbf{u}}^m(0,t) - \epsilon\bar{\mathbf{w}}_1(0,t) - \tilde{\mathbf{w}}_0(0,\tau).\end{aligned}$$

We can see that $O(1)$ terms remain in boundary error equation as expected and also a new error term introduced in boundary due to initial layer. This suggests we need boundary layer.

6.4.3 Boundary Layer

The boundary layer is constructed by blowing up the state variable a according to $\alpha := a/\epsilon$ and defining

$$\hat{\mathbf{u}}(a,t) := (\hat{\mathbf{v}}_j(\alpha,t), \hat{\mathbf{w}}(\alpha,t)).$$

Expanding $\hat{\mathbf{v}}_j(\alpha,t)$, $\hat{\mathbf{w}}(\alpha,t)$, inserting the expressions into (6.16), (6.17) and comparing the coefficients of ϵ^0 order we get

$$\hat{u}^{n_j}_{0,\alpha} = 0,$$

and

$$\hat{\mathbf{w}}_{0,\alpha} = \mathbf{C}_S(0)\hat{\mathbf{w}}_0.$$

Now, the new error is

$$\begin{aligned}\hat{\mathbf{E}}(a,t) &:= (\hat{g}_j(a,t)\mathbf{e}_j, \hat{\mathbf{h}}(a,t)) \\ &= \left(\tilde{g}_j(a,t)\mathbf{e}_j, \tilde{\mathbf{h}}(a,t) - \hat{\mathbf{w}}_0(a/\epsilon,t)\right).\end{aligned}$$

With this new error, we have the following set of error equations:

$$\begin{aligned}\hat{g}_{j,t} &= \tilde{g}_{j,t} - \hat{u}^{n_j}_{0,t} \\ &= -\hat{g}_{j,a} + \mathbf{f}'_{n_j}\cdot\hat{\mathbf{h}} + \mathbf{f}'_{n_j}\cdot\hat{\mathbf{w}}_0 + \mathbf{f}'_{n_j}\cdot\tilde{\mathbf{w}}_0 + \epsilon\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1 - \sum_{i=1}^{m}\mu^*_{ji}\,\hat{g}_i \\ &\quad - \mathbf{x}_j\cdot\mathbf{M}\hat{\mathbf{h}} - \mathbf{x}_j\cdot\mathbf{M}\hat{\mathbf{w}}_0 - \mathbf{x}_j\cdot\mathbf{M}\tilde{\mathbf{w}}_0 - \epsilon\mathbf{x}_j\cdot\mathbf{M}\bar{\mathbf{w}}_1\,,\end{aligned}$$

and

$$\begin{aligned}\hat{\mathbf{h}}_t &= \tilde{\mathbf{h}}_t - \hat{\mathbf{w}}_{0,t} \\ &= -\hat{\mathbf{h}}_a - \tilde{\mathbf{w}}_{0,a} - \epsilon\bar{\mathbf{w}}_{1,a} - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\hat{\mathbf{h}}\right)\mathbf{e}_j - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\hat{\mathbf{w}}_0\right)\mathbf{e}_j - \sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\tilde{\mathbf{w}}_0\right)\mathbf{e}_j \\ &\quad - \epsilon\sum_{j=1}^{m}\left(\mathbf{f}'_{n_j}\cdot\bar{\mathbf{w}}_1\right)\mathbf{e}_j - \mathbf{M}_S\sum_{i=1}^{m}\hat{\mathbf{g}}_i - \tilde{g}^m\mathbf{e}_a - \mathbf{M}_S\hat{\mathbf{h}} - \mathbf{M}_S\hat{\mathbf{w}}_0 - \mathbf{M}_S\tilde{\mathbf{w}}_0 \\ &\quad - \epsilon\mathbf{M}_S\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}\mathbf{C}_S\hat{\mathbf{h}} - \epsilon\bar{\mathbf{w}}_{1,t} + \frac{1}{\epsilon}\left(\mathbf{C}_S - \mathbf{C}_S(0)\right)\hat{\mathbf{w}}_0 - \hat{\mathbf{w}}_{0,t}\,,\end{aligned}$$

and for the boundary conditions we obtain

$$\begin{aligned}\hat{g}_j(0,t) = & \int_0^\infty \left(\sum_{i=1}^m \beta^*_{ij} \hat{g}_i \right) + \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\hat{\mathbf{h}} + \epsilon \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\hat{\mathbf{w}}_0 \\ & + \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\tilde{\mathbf{w}}_0 + \epsilon \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\bar{\mathbf{w}}_1 \,,\end{aligned}$$

and

$$\begin{aligned}\hat{\mathbf{h}}(0,t) = & \mathfrak{B}\bar{\mathbf{u}}^m + \mathfrak{B} \sum_{i=1}^m \hat{\mathbf{g}}_i^m + \mathfrak{B}\hat{\mathbf{h}} + \epsilon \mathfrak{B}\hat{\mathbf{w}} + \mathfrak{B}\tilde{\mathbf{w}}_0 + \epsilon \mathfrak{B}\bar{\mathbf{w}}_1 \\ & - \sum_{i=1}^m \hat{\mathbf{g}}_i^m(0,t) - \bar{\mathbf{u}}^m(0,t) - \epsilon \bar{\mathbf{w}}_1(0,t) - \tilde{\mathbf{w}}_0(0,\tau) - \hat{\mathbf{w}}_0(0,t).\end{aligned}$$

Thus to eliminate the bulk term on the boundary, the boundary layer should be the solution to

$$\hat{\mathbf{w}}_{0,\alpha} = \mathbf{C}_S(0)\hat{\mathbf{w}}_0, \qquad \hat{\mathbf{w}}_0(0,t) := \mathfrak{B}\bar{\mathbf{u}}^m - \bar{\mathbf{u}}^m(0,t).$$

The initial conditions for the system of above error equations take the following form:

$$\hat{g}_j(a,0) = 0,$$

and

$$\hat{\mathbf{h}}(a,0) = -\hat{\mathbf{w}}_0(a/\epsilon, 0) - \epsilon \bar{\mathbf{w}}_1(a,0).$$

We note that, even with the boundary layer, we still have terms depending on t/ϵ and this necessitates introduction of the corner layer.

6.4.4 Corner Layer

To eliminate the initial layer contribution on the boundary, we need to introduce the corner layer by simultaneously rescaling time and space: $\tau := t/\epsilon$ and $\alpha := a/\epsilon$. As before, inserting the series expansions of $\breve{\mathbf{v}}(\alpha,\tau)$ and $\breve{\mathbf{w}}(\alpha,\tau)$ into the system (6.16), (6.17) and comparing the coefficients of ϵ^0 order we get respectively

$$\breve{u}^{n_j}_{0,\tau} = -\breve{u}^{n_j}_{0,\alpha}, \tag{6.38}$$

$$\breve{\mathbf{w}}_{0,\tau} = -\breve{\mathbf{w}}_{0,\alpha} + \mathbf{C}_S(0)\breve{\mathbf{w}}_0 \,, \tag{6.39}$$

which is the unperturbed original equation in (α,τ)-variable with coefficient frozen at $a = 0$. Hence, here we do have freedom of choosing both the boundary and initial conditions (in (α,τ)-variables) which will help to eliminate the problematic terms on the boundary. To find the proper side conditions, let us assume that we have a solution to the above equations with, for the moment, unspecified boundary condition and define the new error approximation

$$\begin{aligned}\mathbf{u}(a,t) &= (\mathbf{v}(a,t), \mathbf{w}(a,t)) \\ &\approx (\bar{\mathbf{v}}(a,t) + \breve{\mathbf{v}}(a/\epsilon, t/\epsilon), \epsilon\bar{\mathbf{w}}_1(a,t) + \tilde{\mathbf{w}}_0(a,t/\epsilon) + \hat{\mathbf{w}}(a/\epsilon, t) + \breve{\mathbf{w}}(a/\epsilon, t/\epsilon))\,,\end{aligned}$$

with the error of this approximation given by

$$\begin{aligned}\breve{\mathbf{E}}(a,t) &:= (\breve{g}_j(a,t)\mathbf{e}_j, \breve{\mathbf{h}}(a,t)) \\ &= (\hat{g}_j(a,t)\mathbf{e}_j - \breve{u}_0^{n_j}(a/\epsilon, t/\epsilon)\mathbf{e}_j, \hat{\mathbf{h}}(a,t) - \breve{\mathbf{w}}_0(a/\epsilon, t/\epsilon)).\end{aligned}$$

Following the procedure described for the boundary layer, we find that to eliminate the $O(1)$ entries in the equation for the error on the boundary we have to impose the following boundary conditions for (6.38) and (6.39) respectively.

$$\breve{u}_0^{n_j}(0,\tau) = \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\tilde{\mathbf{w}}_0\,,$$

and

$$\breve{\mathbf{w}}_0(0,\tau) = \mathfrak{B}\tilde{\mathbf{w}}_0 - \tilde{\mathbf{w}}_0(0,\tau) - \breve{\mathbf{u}}_0^m(0,\tau).$$

We complement the problem for the corner layer by the homogeneous initial conditions: $\breve{u}_0^{n_j}(a,0) = 0$, $\breve{\mathbf{w}}_0(a,0) = \mathbf{0}$.

Taking all layers into account, we find that the final error satisfies

$$\begin{aligned}\breve{g}_{j,t} &= -\breve{g}_{j,a} + \mathbf{f}'_{n_j} \cdot \breve{\mathbf{h}} + \mathbf{f}'_{n_j} \cdot \breve{\mathbf{w}}_0 + \mathbf{f}'_{n_j} \cdot \tilde{\mathbf{w}}_0 + \mathbf{f}'_{n_j} \cdot \hat{\mathbf{w}}_0 + \epsilon\mathbf{f}'_{n_j} \cdot \bar{\mathbf{w}}_1 - \sum_{i=1}^m \mu^*_{ji}\,\breve{g}_i \\ &\quad - \sum_{i=1}^m \mu^*_{ji}\,\breve{u}_0^{n_j} - \mathbf{x}_j \cdot \mathbf{M}\breve{\mathbf{h}} - \mathbf{x}_j \cdot \mathbf{M}\hat{\mathbf{w}}_0 - \mathbf{x}_j \cdot \mathbf{M}\breve{\mathbf{w}}_0 - \mathbf{x}_j \cdot \mathbf{M}\tilde{\mathbf{w}}_0 - \epsilon\mathbf{x}_j \cdot \mathbf{M}\bar{\mathbf{w}}_1, \qquad (6.40)\end{aligned}$$

and

$$\begin{aligned}\breve{\mathbf{h}}_t &= -\breve{\mathbf{h}}_a - \tilde{\mathbf{w}}_{0,a} - \epsilon\bar{\mathbf{w}}_{1,a} - \sum_{j=1}^m \left(\mathbf{f}'_{n_j} \cdot \breve{\mathbf{h}}\right)\mathbf{e}_j - \sum_{j=1}^m \left(\mathbf{f}'_{n_j} . \breve{\mathbf{w}}_0\right)\mathbf{e}_j - \sum_{j=1}^m \left(\mathbf{f}'_{n_j} \cdot \hat{\mathbf{w}}_0\right)\mathbf{e}_j \\ &\quad - \sum_{j=1}^m \left(\mathbf{f}'_{n_j} \cdot \tilde{\mathbf{w}}_0\right)\mathbf{e}_j - \epsilon\sum_{j=1}^m \left(\mathbf{f}'_{n_j} \cdot \bar{\mathbf{w}}_1\right)\mathbf{e}_j - \mathbf{M}_S \sum_{i=1}^m \breve{\mathbf{g}}_i - \breve{g}^m \mathbf{e}_a - \breve{u}_0^m \mathbf{e}_a - \mathbf{M}_S \breve{\mathbf{u}}_0^m \\ &\quad - \mathbf{M}_S\breve{\mathbf{h}} - \mathbf{M}_S\breve{\mathbf{w}}_0 - \mathbf{M}_S\hat{\mathbf{w}}_0 - \mathbf{M}_S\tilde{\mathbf{w}}_0 - \epsilon\mathbf{M}_S\bar{\mathbf{w}}_1 + \frac{1}{\epsilon}(\mathbf{C}_S - \mathbf{C}_S(0))\hat{\mathbf{w}}_0 \\ &\quad + \frac{1}{\epsilon}\mathbf{C}_S\breve{\mathbf{h}} - \epsilon\bar{\mathbf{w}}_{1,t} - \hat{\mathbf{w}}_{0,t} + \frac{1}{\epsilon}(\mathbf{C}_S - \mathbf{C}_S(0))\breve{\mathbf{w}}_0, \qquad (6.41)\end{aligned}$$

with boundary conditions,

$$\begin{aligned}\breve{g}_j(0,t) &= \int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}\,\breve{g}_i\right) + \int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}\,\breve{u}_0^{n_i}\right) + \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\breve{\mathbf{h}} \\ &\quad + \epsilon\int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\breve{\mathbf{w}}_0 + \epsilon\int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\hat{\mathbf{w}}_0 + \epsilon\int_0^\infty \mathbf{x}_j \cdot \mathbf{B}\bar{\mathbf{w}}_1 \text{ and}\end{aligned}$$

$$\breve{\mathbf{h}}(0,t) = \mathfrak{B}\sum_{i=1}^m \breve{\mathbf{g}}_i^m + \mathfrak{B}\breve{\mathbf{u}}_0^m + \mathfrak{B}\breve{\mathbf{h}} + \epsilon\mathfrak{B}\breve{\mathbf{w}}_0 + \epsilon\mathfrak{B}\hat{\mathbf{w}}_0 + \epsilon\mathfrak{B}\bar{\mathbf{w}}_1 - \breve{\mathbf{g}}^m(0,t) - \epsilon\bar{\mathbf{w}}_1(0,t).$$

For the initial conditions, we have

$$\breve{g}_j(a,0) = 0, \qquad \text{and} \qquad \breve{\mathbf{h}}(a,0) = -\hat{\mathbf{w}}_0(a/\epsilon,0) - \epsilon\bar{\mathbf{w}}_1(a,0).$$

For (6.40), (6.41) to be valid, the solution $\mathbf{u}$ and all terms of the asymptotic expansion must be strongly differentiable with respect to t and belong to the domain of the generator which implies $\mathbf{u} \in W_1^1(\mathbb{R}_+, \mathbb{R}^n)$ and $\mathbf{u}(0) = \mathfrak{B}\mathbf{u}$, where W_1^1 denotes the standard Sobolev space. This is not always easy to achieve. In fact, in general an initial condition $\boldsymbol{\phi}$ which satisfies $\boldsymbol{\phi}(0) = \mathfrak{B}\boldsymbol{\phi}$, will not satisfy the condition $\mathbf{x}_j \cdot \boldsymbol{\phi} = \int_0^\infty \left(\sum_{i=1}^m \beta_{ji}^*(s)\phi_i(s)\right) ds$, required for differentiability of the solution of the aggregated problem.

CHAPTER 7

Integral Formulation

Considering the problem mentioned in Chapter 6, it turns out that we have to work with mild solutions of the equations. To set the stage, let us consider our population model (6.5)–(6.7) in a more compact form:

$$\begin{aligned} \mathbf{u}_t(a,t) &= -\mathbf{u}_a(a,t) + \mathcal{K}[\mathbf{u}(\cdot,t)], \\ \mathbf{u}(a,0) &= \boldsymbol{\phi}(a), \\ \mathbf{u}(0,t) &= \mathfrak{B}[\mathbf{u}(\cdot,t)]. \end{aligned} \tag{7.1}$$

The operators $\mathcal{K} : \mathbf{X} \to \mathbf{X}$ and $\mathfrak{B} : \mathbf{X} \to \mathbb{R}^n$ are linear and bounded.

The system (7.1) can be reduced to an integral equation by integration along characteristics. It turns out that the solution of this integral equation defines the semigroup generated by the operator $\mathcal{A}\mathbf{u} := -\mathbf{u}_a + \mathcal{K}\mathbf{u}$ on the domain $\mathfrak{D}(\mathcal{A}) := \{\mathbf{u} \in W_1^1(\mathbb{R}_+, \mathbb{R}^n) : \mathbf{u}(0) = \mathfrak{B}\mathbf{u}\}$.

Let us consider the integral equation obtained by integrating (7.1) along the characteristics $a - t = \text{constant}$:

$$\mathbf{u}(a,t) = \begin{cases} \boldsymbol{\phi}(a-t) + \displaystyle\int_0^t \mathcal{K}[\mathbf{u}(\cdot,s)](s+a-t)\,ds, & a > t; \\ \mathfrak{B}[\mathbf{u}(\cdot,t-a)] + \displaystyle\int_{t-a}^t \mathcal{K}[\mathbf{u}(\cdot,s)](s+a-t)\,ds, & a < t, \end{cases} \tag{7.2}$$

where here and afterwards the notation $a < t$ and $a > t$ is understood as the respective inequality almost everywhere. Then the family of operators defined as

$[\mathcal{G}(t)\boldsymbol{\phi}](a) := \mathbf{u}(a,t)$, where $\mathbf{u}(a,t)$ is the solution of (7.2) with $\boldsymbol{\phi} \in \mathbf{X}$ is the semigroup on $\mathbf{X}$ generated by $(\mathcal{A}, \mathfrak{D}(\mathcal{A}))$, see [65]. In the error estimates we shall need mild solutions of the inhomogeneous problem associated with (7.1):

$$\mathbf{u}_t(a,t) = -\mathbf{u}_a(a,t) + \mathcal{K}[\mathbf{u}(\cdot,t)] + \mathbf{f}(a,t), \tag{7.3}$$

with the same initial and boundary conditions as in (7.1), where $t \to \mathbf{f}(t)$ is a function from $(0,\infty)$ to $\mathbf{X}$. However, (7.3) does not make sense if $\mathbf{u}$ is not differentiable which, in turn, cannot be achieved unless $\boldsymbol{\phi} \in \mathfrak{D}(\mathcal{A})$ and $\mathbf{f}$ is an $\mathbf{X}$-differentiable, or a $\mathfrak{D}(\mathcal{A})$-continuous function. In general, we have to work with mild solutions of (7.3) defined by

$$\mathbf{u}(t) = \mathcal{G}(t)\boldsymbol{\phi} + \int_0^t \mathcal{G}(t-s)\,\mathbf{f}(s)\,ds. \tag{7.4}$$

This definition is not very helpful as it views $(\mathcal{G}(t))_{t\geq 0}$ somewhat globally without noticing the structure visible in (7.2). However, we can prove the following result:

Proposition 7.1 *A function $\mathbf{u} \in C(\mathbb{R}_+, \mathbf{X})$ is a mild solution of* (7.3) *if and only if*

$$\mathbf{u}(a,t) = \begin{cases} \boldsymbol{\phi}(a-t) + \displaystyle\int_0^t \mathcal{K}[\mathbf{u}(\cdot,\sigma)](\sigma+a-t)\,d\sigma + \int_0^t \mathbf{f}(\sigma+a-t,\sigma)\,d\sigma, & a > t; \\ \mathfrak{B}[\mathbf{u}(\cdot,t-a)] + \displaystyle\int_{t-a}^t \mathcal{K}[\mathbf{u}(\cdot,\sigma)](\sigma+a-t)\,d\sigma + \int_{t-a}^t \mathbf{f}(\sigma+a-t,\sigma)\,d\sigma, & a < t. \end{cases} \tag{7.5}$$

Proof. First, to shorten notation, we denote, for arbitrary numbers a, σ, t,

$$\sigma_{a,t} := (\sigma + a - t, \sigma),$$

and, for any a-dependent operation A and function $(a,t) \to u(a,t)$ we denote $A[u(\cdot,t)](a) = [Au](a,t)$ (or$[Au](t)$ if the output is a-independent).
It can be proved, [8], that a function $\mathbf{u} \in C(\mathbb{R}_+, \mathbf{X})$ is a mild solution to (7.3) with $\mathbf{f} \in L_1(\mathbb{R}_+, \mathbf{X})$ if and only if $\int_0^t \mathbf{u}(s)\,ds \in \mathfrak{D}(\mathcal{A})$ and

$$\mathbf{u}(t) = \boldsymbol{\phi} + \mathcal{A}\int_0^t \mathbf{u}(s)\,ds + \int_0^t \mathbf{f}(s)\,ds, \quad t \geq 0. \tag{7.6}$$

Hence, $\mathbf{u}$ is a mild solution to (7.3) if and only if $\mathbf{v}(t) := \int_0^t \mathbf{u}(s)\,ds \in \mathfrak{D}(\mathcal{A})$ is the classical solution to

$$\mathbf{v}_t(a,t) = \boldsymbol{\phi}(a) - \mathbf{v}_a(a,t) + \mathcal{K}[\mathbf{v}(\cdot,t)](a) + \mathbf{F}(a,t), \tag{7.7}$$

with $\mathbf{v}(a,0) = \mathbf{0}$ and $\mathbf{v}(0,t) = \mathfrak{B}[\mathbf{v}(\cdot,t)]$, where $\mathbf{F}(a,t) = \int_0^t \mathbf{f}(a,s)\,ds$. Equation (7.7) is satisfied pointwise and thus we can integrate them along characteristics

$$\mathbf{v}(a,t) = \begin{cases} \displaystyle\int_0^t \mathcal{K}[\mathbf{v}(\cdot,s)](s+a-t)\,ds + \int_0^t \mathbf{F}(s_{a,t})\,ds + \int_0^t \boldsymbol{\phi}(s+a-t)\,ds, & a > t; \\ \displaystyle\mathfrak{B}[\mathbf{v}(\cdot,t-a)] + \int_{t-a}^t \mathcal{K}[\mathbf{v}(\cdot,s)](s+a-t)\,ds + \int_{t-a}^t \mathbf{F}(s_{a,t})\,ds & \\ \displaystyle\qquad + \int_{t-a}^t \boldsymbol{\phi}(s+a-t)\,ds, & a < t. \end{cases} \tag{7.8}$$

Then, by changing the order of integration and changing variables in respective terms, we find

$$\mathbf{v}(a,t) = \begin{cases} \displaystyle\int_0^t \boldsymbol{\phi}(a-\sigma)\,d\sigma + \int_0^t \left(\int_0^s \mathcal{K}[\mathbf{u}(\cdot,\sigma)](\sigma+a-s)\,d\sigma\right) ds & \\ \displaystyle\qquad + \int_0^t \left(\int_0^s \mathbf{f}(\sigma_{a,s})\,d\sigma\right) ds, & a > t; \\ \displaystyle\int_0^t \mathfrak{B}[\mathbf{u}(\cdot,\sigma-a)]\,d\sigma + \int_0^t \left(\int_{s-a}^s \mathcal{K}[\mathbf{u}(\cdot,\sigma)](\sigma+a-s)\,d\sigma\right) ds & \\ \displaystyle\quad + \int_0^t \left(\int_{s-a}^t \mathbf{f}(\sigma_{a,s})\,d\sigma\right) ds + \int_0^a \boldsymbol{\phi}(z)\,dz & \\ \displaystyle\quad + \int_0^a \left(\int_0^s \mathbf{f}(s,\sigma)\,d\sigma\right) ds + \int_0^a \left(\int_0^s \mathcal{K}[\mathbf{u}(\cdot,\sigma)](s)\,d\sigma\right) ds, & a < t, \end{cases} \tag{7.9}$$

and, using $\mathbf{v}(a,t) = \int_0^t \mathbf{u}(a,\sigma)\,d\sigma$, upon differentiation we arrive at (7.5). □

Various terms of the asymptotic expansion appear in a direct form which is incompatible with (7.2) and must be re-written to allow for accommodation into the integral formulation. First, from Lemma 6.8 we know that for sufficiently large λ there is a classical solution of the stationary problem

$$\lambda\mathbf{w} = -\mathbf{w}_a + \mathcal{K}\mathbf{w}, \quad \mathbf{w}(0,t) = \mathfrak{B}\mathbf{w} + \mathbf{g}, \tag{7.10}$$

where $\mathbf{g}$ may depend on $t > 0$. Moreover, $\mathbf{w}$ is differentiable with respect to a (as a solution of a system of ODEs) and with respect to t, provided $\mathbf{g}(t)$ is differentiable. Since equation (7.10) is satisfied point-wise, we can integrate it along characteristics to obtain

$$\begin{aligned} \lambda \int_0^t \mathbf{w}(\sigma_{a,t})\,d\sigma &= -\int_0^t \mathbf{w}_{,1}(\sigma_{a,t})\,d\sigma + \int_0^t [\mathcal{K}\mathbf{w}](\sigma_{a,t})\,d\sigma, \qquad a > t; \\ \lambda \int_{t-a}^t \mathbf{w}(\sigma_{a,t})\,d\sigma &= -\int_{t-a}^t \mathbf{w}_{,1}(\sigma_{a,t})\,d\sigma + \int_{t-a}^t [\mathcal{K}\mathbf{w}]\,(\sigma_{a,t})\,d\sigma, \qquad a < t, \end{aligned} \tag{7.11}$$

where $\mathbf{w}_{,i}$ denotes the partial derivative with respect to the i-th variable. Now

$$\frac{\partial}{\partial\sigma}\mathbf{w}(\sigma_{a,t}) = \mathbf{w}_{,1}(\sigma_{a,t}) + \mathbf{w}_{,2}(\sigma_{a,t}),$$

and therefore, integrating with respect to σ from 0 to t we obtain

$$\begin{aligned}
\mathbf{w}(a,t) - \mathbf{w}(a-t,0) &= \int_0^t \mathbf{w}_{,1}\left(\sigma_{a,t}\right) d\sigma + \int_0^t \mathbf{w}_{,2}\left(\sigma_{a,t}\right) d\sigma, && a > t;\\
\mathbf{w}(a,t) - \mathbf{w}(0,t-a) &= \int_{t-a}^t \mathbf{w}_{,1}\left(\sigma_{a,t}\right) d\sigma + \int_{t-a}^t \mathbf{w}_{,2}\left(\sigma_{a,t}\right) d\sigma, && a < t.
\end{aligned}$$

Combining these with (7.11) we obtain

$$\mathbf{w}(a,t) = \begin{cases} \mathbf{w}(a-t,0) + \int_0^t [\mathcal{K}\mathbf{w}]\left(\sigma_{a,t}\right) d\sigma + \int_0^t \mathbf{w}_{,2}\left(\sigma_{a,t}\right) d\sigma - \lambda \int_0^t \mathbf{w}\left(\sigma_{a,t}\right) d\sigma, & a > t;\\ [\mathfrak{B}\mathbf{w}](t-a) + \mathbf{g}(t-a) + \int_{t-a}^t [\mathcal{K}\mathbf{w}]\left(\sigma_{a,t}\right) d\sigma + \int_{t-a}^t \mathbf{w}_{,2}\left(\sigma_{a,t}\right) d\sigma & \\ \qquad -\lambda \int_{t-a}^t \mathbf{w}\left(\sigma_{a,t}\right) d\sigma, & a < t. \end{cases} \tag{7.12}$$

It turns out that the inhomogeneous boundary data are better treated separately. By linearity, we can consider the case with $\boldsymbol{\phi} = \mathbf{0}$ and $\mathbf{f}(t) = \mathbf{0}$.

Denote by $\mathcal{V}_{\mathcal{K}}$ the fundamental solution matrix of the equation $\mathbf{z}'_a(a) = \mathcal{K}(a)\mathbf{z}(a)$; that is $\mathbf{z}(a) = \mathcal{V}_{\mathcal{K}}(a)\mathbf{z}_0$ satisfies the above equation with $\mathbf{z}(0) = \mathbf{z}_0$. We note that here $\mathcal{K} := \mathbf{L}_{\epsilon,0} = -\mathbf{M} + \frac{1}{\epsilon}\mathbf{C}$ and using same l^1 norm as in Lemma 6.8 with $\lambda = 0$ we have

$$\sup_{a\in\mathbb{R}_+} \|\mathcal{V}_{\mathcal{K}}(a)\|_{\mathbb{R}^n,1} \leq 1. \tag{7.13}$$

Lemma 7.2 *Assume that, in addition to assumption of this section, $\mathcal{K}$ satisfies* (7.13) *and let* $\mathbf{g} \in C([0,\infty), \mathbb{R}^n)$. *Then* $\mathbf{u}$ *is a continuous solution to*

$$\mathbf{u}(a,t) = \begin{cases} \int_0^t \mathcal{K}[\mathbf{u}(\cdot,s)](s+a-t)\, ds, & a > t;\\ \\ \mathfrak{B}[\mathbf{u}(\cdot,t-a)] + \mathbf{g}(t-a) + \int_{t-a}^t \mathcal{K}[\mathbf{u}(\cdot,s)](s+a-t)\, ds, & a < t, \end{cases} \tag{7.14}$$

if and only if

$$\mathbf{u}(a,t) = \mathcal{V}_{\mathcal{K}}(a)\boldsymbol{\omega}(a,t), \tag{7.15}$$

where

$$\boldsymbol{\omega}(a,t) = \begin{cases} \mathbf{0}, & a > t;\\ \left((\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}\mathbf{g}\right)(t-a), & a < t. \end{cases}$$

Proof. For regular $\mathbf{g}$ we can re-write the problem as a differential equation (satisfied in each triangle $t < a$ and $t > a$) and, using invertibility of $\mathcal{V}_{\mathcal{K}}$, we see that $\boldsymbol{\omega}$ defined by (7.15) satisfies

$$\boldsymbol{\omega}(a,t) = \begin{cases} \mathbf{0}, & a > t; \\ \mathfrak{B}\mathcal{V}_{\mathcal{K}}[\boldsymbol{\omega}(\cdot, t-a)] + \mathbf{g}(t-a), & a < t. \end{cases}$$

The solution $\boldsymbol{\omega}$ of this problem is given by the solution of the simple problem

$$\boldsymbol{\omega}(a,t) = \begin{cases} \mathbf{0}, & a > t; \\ \boldsymbol{\psi}(t-a), & a < t, \end{cases} \tag{7.16}$$

provided $\boldsymbol{\psi}(t) = \mathfrak{B}\mathcal{V}_{\mathcal{K}}[\boldsymbol{\psi}(t-\cdot)] + \mathbf{g}(t)$. This is Volterra a equation which, considered in $C([0,T],\mathbb{R}^n)$ for any fixed $T < +\infty$, can be solved by using standard the Picard iterations yielding a unique solution

$$\boldsymbol{\psi}(t) = \left[(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}\mathbf{g}\right](t),$$

with $\left\|(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}\right\|_{C([0,T],\mathbb{R}^n)} \leq e^{mT}$, where

$$m = \sup_{s\in[0,T]} \|\mathfrak{B}(s)\mathcal{V}_{\mathcal{K}}(s)\|_{\mathbb{R}^n}.$$

Let us take a sequence of W_1^1 function $\mathbf{g}_n$ converging uniformly on $[0,T]$ to a continuous function $\mathbf{g}$. Then $\boldsymbol{\psi}_n = [(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}\mathbf{g}_n]$ converges uniformly on $[0,T]$ to

$$\boldsymbol{\psi} = [(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}\mathbf{g}], \tag{7.17}$$

as $(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}$ is a continuous operator on $C([0,T],\mathbb{R}^n)$ (in fact, on $L_\infty([0,T],\mathbb{R}^n)$). Thus,

$$\boldsymbol{\omega}_n(a,t) = \begin{cases} \mathbf{0}, & t < a < T; \\ \boldsymbol{\psi}_n(t-a), & 0 < a < t, \end{cases}$$

converges uniformly on $[0,T]\times[0,T]$ to $\boldsymbol{\omega}$ given by (7.16) and hence $\mathcal{V}_{\mathcal{K}}(a)\boldsymbol{\omega}_n(a,t)$ uniformly converges to a continuous function on $[0,t]\times[0,T]$ and to zero on $(t,\infty)\times[0,T]$; we denote the limit by $\bar{\mathbf{u}}(a,t)$. Clearly $\bar{\mathbf{u}}(a,t)$ is a solution of (7.14) as all operators in (7.14) are bounded. Moreover $\bar{\mathbf{u}}(a,t)$ treated as a function $t \to \bar{\mathbf{u}}(\cdot,t)$ is in $C([0,T],L_1(\mathbb{R}_+))$ by

$$\int_0^\infty \|\bar{\mathbf{u}}(a,t+h) - \bar{\mathbf{u}}(a,t)\|_{\mathbb{R}^n}\,da = \int_0^t \|\bar{\mathbf{u}}(a,t+h) - \bar{\mathbf{u}}(a,t)\|_{\mathbb{R}^n}\,da + \int_t^{t+h} \|\bar{\mathbf{u}}(a,t+h)\|_{\mathbb{R}^n}\,da,$$

and the uniform continuity of $\bar{\mathbf{u}}(a,t)$ as a function of two variables in the triangle $[0,t]\times[0,T]$. But the difference of two solutions to (7.14) satisfies its homogeneous version (with $\mathbf{g} = \mathbf{0}$) for which we can use the semigroup theory which ensures the uniqueness. Hence, the only solution to (7.14) with continuous $\mathbf{g}$ is given by (7.15). the converse statement follows similarly by applying $\mathcal{V}_{\mathcal{K}}$ to the equation satisfied by $\boldsymbol{\omega}$. □

7.1 Error Analysis

As we noticed, in general, it is impossible to have differentiable solutions of all problems involved in the construction of the asymptotic expansion. Thus we have to re-write the error system (6.40), (6.41) in the form of integrated equation (7.5). For $1 \le j \le m$, the mild solution of (6.5) in the projected form satisfy

$$u^{n_j}(a,t) = \begin{cases} \phi^j(a-t) - \int_0^t \left[\sum_{i=1}^m \mu^*_{ji} u^{n_i}\right](\sigma_{a,t})\, d\sigma - \int_0^t \left[\mathbf{x}_j \cdot \mathbf{M}\mathbf{w}\right](\sigma_{a,t})\, d\sigma \\ \quad + \int_0^t \left[\mathbf{f}'_{n_j} \cdot \mathbf{w}\right](\sigma_{a,t})\, d\sigma, & a > t; \\ \int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}(s) u^{n_i}(s, t-a)\right) ds + \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}(s)\mathbf{w}(s, t-a)\, ds \\ \quad - \int_{t-a}^t \left[\sum_{i=1}^m \mu^*_{ji} u^{n_i}\right](\sigma_{a,t})\, d\sigma - \int_{t-a}^t \left[\mathbf{x}_j \cdot \mathbf{M}\mathbf{w}\right](\sigma_{a,t})\, d\sigma \\ \quad + \int_{t-a}^t \left[\mathbf{f}'_{n_j} \cdot \mathbf{w}\right](\sigma_{a,t})\, d\sigma, & a < t, \end{cases} \tag{7.18}$$

and

$$\mathbf{w}(a,t) = \begin{cases} \mathbf{w}_0(a-t) - \int_0^t \left[\mathbf{M}_S \mathbf{u}^m + u^m \mathbf{e}_a\right](\sigma_{a,t})\, d\sigma - \int_0^t [\mathbf{M}_S \mathbf{w}](\sigma_{a,t})\, d\sigma \\ \quad + \int_0^t \sum_{j=1}^m \left(\mathbf{f}'_{n_j} \cdot \mathbf{w}\right) \mathbf{e}_j(\sigma_{a,t})\, d\sigma + \frac{1}{\epsilon} \int_0^t [\mathbf{C}_S \mathbf{w}](\sigma_{a,t})\, d\sigma, & a > t; \\ [\mathfrak{B}\mathbf{u}^m](t-a) + [\mathfrak{B}\mathbf{w}](t-a) - \mathbf{u}^m(0, t-a) \\ \quad - \int_{t-a}^t [\mathbf{M}_S \mathbf{u}^m + u^m \mathbf{e}_a](\sigma_{a,t})\, d\sigma - \int_{t-a}^t [\mathbf{M}_S \mathbf{w}](\sigma_{a,t})\, d\sigma \\ \quad + \int_{t-a}^t \sum_{j=1}^m \left(\mathbf{f}'_{n_j} \cdot \mathbf{w}\right) \mathbf{e}_j(\sigma_{a,t})\, d\sigma + \frac{1}{\epsilon} \int_{t-a}^t [\mathbf{C}_S \mathbf{w}](\sigma_{a,t})\, d\sigma, & a < t. \end{cases} \tag{7.19}$$

In the same manner, the solution of the aggregated equations (6.29) satisfies

$$\bar{u}^{n_j}(a,t) = \begin{cases} \phi^j(a-t) - \int_0^t \left[\sum_{i=1}^m \mu^*_{ji} \bar{u}^{n_i}\right](\sigma_{a,t})\, d\sigma, & a > t; \\ \int_0^\infty \left[\beta^*_{ji}(s) \sum_{i=1}^m \bar{u}^{n_i}(s, t-a)\right] ds - \int_{t-a}^t \left[\sum_{i=1}^m \mu^*_{ji} \bar{u}^{n_i}\right](\sigma_{a,t})\, d\sigma, & a < t. \end{cases} \tag{7.20}$$

The cohort functions $\sigma \to \bar{u}^{n_j}(\eta,\sigma), \eta = a - t$ are continuously differentiable with respect to σ for all $\eta < 0$ and almost all $\eta > 0$, with

$$\frac{d}{d\sigma}\bar{u}^{n_j}(\sigma+\eta,\sigma) = -\mu(\sigma+\eta)\,\bar{u}^{n_j}(\sigma+\eta,\sigma). \tag{7.21}$$

In the next step we write $\bar{\mathbf{w}}_1 = \mathbf{C}_S^{-1}[\mathbf{M}_S(a)\bar{\mathbf{u}}^m + \bar{u}^m\mathbf{e}_a]$ in the integrated form. Using the time derivative of the cohort function, we have

$$\bar{\mathbf{w}}_1(a,t) - \bar{\mathbf{w}}_1(a-t,0) = \int_0^t \frac{d}{d\sigma}\bar{\mathbf{w}}_1(\sigma_{a,t})\,d\sigma, \quad a > t;$$
$$\bar{\mathbf{w}}_1(a,t) - \bar{\mathbf{w}}_1(0,t-a) = \int_{t-a}^t \frac{d}{d\sigma}\bar{\mathbf{w}}_1(\sigma_{a,t})\,d\sigma, \quad a < t.$$

But

$$\begin{aligned}\frac{d}{d\sigma}\bar{\mathbf{w}}_1(\sigma_{a,t}) &= \frac{d}{d\sigma}\mathbf{C}_S^{-1}\left[\mathbf{M}_S(\sigma+a-t)\bar{\mathbf{u}}^m(\sigma_{a,t}) + u^m(\sigma_{a,t})\mathbf{e}_a(\sigma+a-t)\right]\\ &= \mathfrak{L}(\sigma+a-t)\bar{\mathbf{u}}^m(\sigma_{a,t}),\end{aligned} \tag{7.22}$$

where the function $\mathfrak{L}$ is bounded, again by Lemma 6.7 and assumptions on $\mathbf{M}$.
Hence

$$\bar{\mathbf{w}}_1(a,t) = \begin{cases}\mathbf{C}_S^{-1}\left[\mathbf{M}_S\boldsymbol{\phi}_m + \boldsymbol{\phi}_m\mathbf{e}_a\right](a-t) + \displaystyle\int_0^t \mathfrak{L}(\sigma+a-t)\bar{\mathbf{u}}^m(\sigma_{a,t})\,d\sigma, & a > t;\\ \displaystyle\sum_{i=1}^m\left(\int_0^\infty\left(\sum_{k=1}^m \beta_{ki}^*(s)\,\bar{u}^{n_k}(s,t-a)\right)ds\right)\mathbf{e}_i & \\ \qquad\qquad + \displaystyle\int_{t-a}^t \mathfrak{L}(\sigma+a-t)\bar{\mathbf{u}}^m(\sigma_{a,t})\,d\sigma, & a < t,\end{cases} \tag{7.23}$$

where $\boldsymbol{\phi}_m := \sum_{i=1}^m \phi_i\mathbf{e}_i$.

Following the same strategy, we write the initial layer term $\tilde{\mathbf{w}}_0(a,\tau) = e^{\tau\mathbf{C}_S}\mathbf{w}_0(a)$ into the integrated form. For this we note that (6.36) is of the same form as (7.10) if we introduce $\tilde{\mathbf{w}}_\epsilon(a,t) := \tilde{\mathbf{w}}_0(a,\tau)$ and put $\lambda = 0$, $\mathcal{K} = (1/\epsilon)\mathbf{C}_S$ and $\mathfrak{B} = 0$ (and with t and a variables interchanged); that is,

$$\tilde{\mathbf{w}}_\epsilon(a,t) = \begin{cases}\mathbf{w}_0(a-t) + \displaystyle\int_0^t \tilde{\mathbf{w}}_{\epsilon,1}(\sigma_{a,t})\,d\sigma + \frac{1}{\epsilon}\int_0^t [\mathbf{C}_S\tilde{\mathbf{w}}_\epsilon](\sigma_{a,t})\,d\sigma, & a > t;\\ \exp\left(\dfrac{t-a}{\epsilon}\mathbf{C}_S(0)\right)\mathbf{w}_0(0) + \displaystyle\int_{t-a}^t \tilde{\mathbf{w}}_{\epsilon,1}(\sigma_{a,t})\,d\sigma + \frac{1}{\epsilon}\int_{t-a}^t [\mathbf{C}_S\tilde{\mathbf{w}}_\epsilon](\sigma_{a,t})\,d\sigma, & a < t.\end{cases} \tag{7.24}$$

In the same way for the boundary layer $\hat{\mathbf{w}}_\epsilon(a,t) := \hat{\mathbf{w}}_0(\alpha,t)$ we obtain the representation

$$\hat{\mathbf{w}}_\epsilon(a,t) = \begin{cases} \exp\left(\frac{a-t}{\epsilon}\mathbf{C}_S(0)\right)[\mathfrak{B}\bar{\mathbf{u}}^m(\cdot,0) - \bar{\mathbf{u}}^m(0,0)] \\ \qquad +\frac{1}{\epsilon}\int_0^t [\mathbf{C}_S(0)\hat{\mathbf{w}}_\epsilon](\sigma_{a,t})\,d\sigma + \int_0^t \hat{\mathbf{w}}_{\epsilon,1}(\sigma_{a,t})\,d\sigma, & a > t; \\ \\ [\mathfrak{B}\bar{\mathbf{u}}^m](t-a) - \bar{\mathbf{u}}^m(0,t-a) \\ \qquad +\frac{1}{\epsilon}\int_{t-a}^t [\mathbf{C}_S(0)\hat{\mathbf{w}}_\epsilon](\sigma_{a,t})\,d\sigma + \int_{t-a}^t \hat{\mathbf{w}}_{\epsilon,2}(\sigma_{a,t})\,d\sigma, & a < t. \end{cases} \tag{7.25}$$

where since $\boldsymbol{\phi} \in W_1^1(\mathbb{R}_+,\mathbb{R}^n)$, the values $\phi_i(0)$ are well defined and so is $\bar{\mathbf{u}}^m(0,0)$. Also $\mathfrak{B}\bar{\mathbf{u}}^m(\cdot,0)$ is well defined, indeed

$$\mathfrak{B}(\bar{\mathbf{u}}^m) \to \mathfrak{B}\left(\sum_{j=1}^m \bar{u}^{n_j}(a,t)\mathbf{e}_j\right),$$

for $t \to 0^+$ as $\mathfrak{B}$ is bounded and $\bar{\mathbf{u}}^m = \sum_{j=1}^m \bar{u}^{n_j}(a,t)\mathbf{e}_j$, $\sum_{j=1}^m \bar{u}^{n_j}(a,t)$ being continuous in t, X-valued solution to (7.18).

Finally, we find the integral representation of the corner layer. The corner layer solves the equation of the same type as the original equation so there is no need to perform any additional transformations. However, it is clear that the boundary conditions of the corner layer correction are not compatible at $\alpha = \tau = 0$ with the homogeneous initial conditions and thus the problem must be considered in the integrated form.

First let us note that the equations in (6.38) and (6.39) are decoupled. The problem for $\breve{u}_0^{n_j}$ is of the form

$$\breve{u}_{0,\tau}^{n_j}(\alpha,\tau) = -\breve{u}_{0,\alpha}^{n_j}(\alpha,\tau),\ \ \breve{u}_0^{n_j}(\alpha,0) = 0,\ \ \breve{u}_0^{n_j}(0,\tau) =: \mathcal{F}(\tau), \tag{7.26}$$

where $\mathcal{F}(\tau) := \int_0^\infty \left[\mathbf{x}_j \cdot \mathbf{B}(s)e^{\tau\mathbf{C}_S(s)}\mathbf{w}_0(s)\right]ds$.
Hence,

$$\breve{u}_0^{n_j}(\alpha,\tau) = \begin{cases} 0, & \alpha > \tau; \\ \\ \int_0^\infty \left[\mathbf{x}_j \cdot \mathbf{B}(s)\,e^{(\tau-\alpha)\mathbf{C}_S(s)}\mathbf{w}_0(s)\right]ds, & \alpha < \tau. \end{cases} \tag{7.27}$$

The kinetic part of the corner layer, $\breve{\mathbf{w}}_0$, satisfies

$$\breve{\mathbf{w}}_0(\alpha,\tau) = \begin{cases} \int_0^\tau [\mathbf{C}_S(0)\breve{\mathbf{w}}_0](\sigma_{\alpha,\tau})\,d\sigma, & \alpha > \tau; \\ \\ \mathbf{H}(\tau-\alpha) + \int_{\tau-\alpha}^\tau [\mathbf{C}_S(0)\breve{\mathbf{w}}_0]\,(\sigma_{\alpha,\tau})\,d\sigma, & \alpha < \tau, \end{cases} \tag{7.28}$$

where $\sigma_{\alpha,\tau} := (\sigma + \alpha - \tau, \sigma)$ and

$$\mathbf{H}(\tau) := \mathfrak{B}\tilde{\mathbf{w}}_0 - \tilde{\mathbf{w}}_0(0,\tau) - \breve{\mathbf{u}}_0^m(0,\tau). \tag{7.29}$$

We note that (7.28) can be simplified as in Lemma 7.2. In this case the fundamental solution matrix of the equation $\mathbf{z}'_a(a) = \mathbf{C}_S(0)\mathbf{z}(a)$ is simply the matrix exponential: $\mathbf{z}(a) = e^{a\mathbf{C}_S(0)}$. Using the fact that the initial value is 0 and $\mathfrak{B} = \mathbf{0}$, we immediately obtain

$$\breve{\mathbf{w}}_0(\alpha,\tau) = \begin{cases} \mathbf{0}, & \alpha > \tau; \\ e^{\alpha\mathbf{C}_S(0)}\,\mathbf{H}(\tau-\alpha), & \alpha < \tau. \end{cases} \tag{7.30}$$

To simplify notation, let

$$\tilde{\mathbf{w}}_{0,\epsilon}(a,t) := \tilde{\mathbf{w}}_0(a,\tau), \;\; \hat{\mathbf{w}}_{0,\epsilon}(a,t) := \hat{\mathbf{w}}_0(\alpha,t),$$

and

$$\breve{\mathbf{w}}_{0,\epsilon}(a,t) := \breve{\mathbf{w}}_0(\alpha,\tau), \;\; \breve{u}_{0,\epsilon}^{n_j}(a,t) := \breve{u}_0^{n_j}(\alpha,\tau).$$

Combining the above, we arrive at the following equations of the error in the integral form:

(i) for the aggregated ('hydrodynamic') part and $a > t$:

$$\begin{aligned} \breve{g}_j(a,t) = &-\int_0^t \Big[\sum_{i=1}^m \mu^*_{ji}\breve{g}_i\Big](\sigma_{a,t})\,d\sigma + \int_0^t [\mathbf{f}'_{n_j}\cdot\breve{\mathbf{h}}](\sigma_{a,t})\,d\sigma \\ &+ \int_0^t [\mathbf{f}'_{n_j}\cdot(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})](\sigma_{a,t})\,d\sigma \\ &- \int_0^t [\mathbf{x}_j.\mathbf{M}\breve{\mathbf{h}}](\sigma_{a,t})\,d\sigma - \int_0^t [\mathbf{x}_j\cdot\mathbf{M}(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})](\sigma_{a,t})\,d\sigma, \end{aligned} \tag{7.31}$$

where we have used the fact that $\breve{u}_0^{n_j} = 0$ for $a > t$;

(ii) for the aggregated part and $a < t$:

$$\begin{aligned} \breve{g}_j(a,t) = &\int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}(s)\,\breve{g}_i(s,t-a)\right) ds + \int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}(s)\,\breve{u}_{0,\epsilon}^{n_i}(s,t-a)\right) ds \\ &+ \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}(s)(\epsilon\bar{\mathbf{w}}_1(s,t-a) + \hat{\mathbf{w}}_{0,\epsilon}(s,t-a) + \breve{\mathbf{w}}_{0,\epsilon}(s,t-a))\,ds \\ &+ \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}(s)\,\breve{\mathbf{h}}(s,t-a)\,ds - \int_{t-a}^t \left[\sum_{i=1}^m \mu^*_{ji}\,\breve{g}_i\right](\sigma_{a,t})\,d\sigma \\ &- \int_{t-a}^t \left[\mathbf{x}_j\cdot\mathbf{M}\breve{\mathbf{h}}\right](\sigma_{a,t})\,d\sigma + \int_{t-a}^t \left[\mathbf{f}'_{n_j}\cdot\breve{\mathbf{h}}\right](\sigma_{a,t})\,d\sigma - \int_{t-a}^t \left[\mu^*_{ji}\,\breve{u}_{0,\epsilon}^{n_i}\right](\sigma_{a,t})\,d\sigma \\ &+ \int_{t-a}^t \left[\mathbf{f}'_{n_j}\cdot(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})\right](\sigma_{a,t})\,d\sigma \\ &- \int_{t-a}^t \left[\mathbf{x}_j\cdot\mathbf{M}(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})\right](\sigma_{a,t})\,d\sigma, \end{aligned} \tag{7.32}$$

(iii) for the complementary ('kinetic') part and $a > t$:

$$\begin{aligned}
\breve{\mathbf{h}}(a,t) &= -\epsilon \mathbf{C}_S^{-1}[\mathbf{M}_S\boldsymbol{\phi}_m + \boldsymbol{\phi}_m\mathbf{e}_a](a-t) - \exp\left(\frac{a-t}{\epsilon}\mathbf{C}_S(0)\right)[\mathfrak{B}\bar{\mathbf{u}}^m(\cdot,0) - \bar{\mathbf{u}}^m(0,0)] \\
&\quad - \int_0^t [\mathbf{M}_S\sum_{i=1}^m \breve{\mathbf{g}}_i + \breve{g}^m\mathbf{e}_a](\sigma_{a,t})\,d\sigma - \int_0^t [\mathbf{M}_S\breve{\mathbf{h}}](\sigma_{a,t})\,d\sigma + \int_0^t \sum_{j=1}^m [\mathbf{f}'_{n_j}\cdot\breve{\mathbf{h}}]\mathbf{e}_j(\sigma_{a,t})\,d\sigma \\
&\quad + \frac{1}{\epsilon}\int_0^t [\mathbf{C}_S\breve{\mathbf{h}}](\sigma_{a,t})\,d\sigma - \int_0^t [\mathbf{M}_S\breve{\mathbf{u}}_0^m + \breve{u}_0^m\mathbf{e}_a](\sigma_{a,t})\,d\sigma \\
&\quad + \frac{1}{\epsilon}\int_0^\infty [(\mathbf{C}_s - \mathbf{C}_S(0))\hat{\mathbf{w}}_{0,\epsilon}](\sigma_{a,t})\,d\sigma + \frac{1}{\epsilon}\int_0^\infty [(\mathbf{C}_s - \mathbf{C}_S(0))\breve{\mathbf{w}}_{0,\epsilon}](\sigma_{a,t})\,d\sigma \\
&\quad - \int_0^t [\mathbf{M}_S(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})](\sigma_{a,t})\,d\sigma \\
&\quad - \int_0^t \sum_{j=1}^m [\mathbf{f}'_{n_j}\cdot(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})]\mathbf{e}_j(\sigma_{a,t})\,d\sigma \\
&\quad - \epsilon\int_0^t \mathfrak{L}(\sigma + a - t)\bar{\mathbf{u}}^m(\sigma_{a,t})\,d\sigma - \int_0^t \tilde{\mathbf{w}}_{0,\epsilon,1}(\sigma_{a,t})\,d\sigma - \int_0^t \hat{\mathbf{w}}_{\epsilon,1}(\sigma_{a,t})\,d\sigma, \qquad (7.33)
\end{aligned}$$

(iv) for the complementary ('kinetic') part and $t > a$:

$$\begin{aligned}
\breve{\mathbf{h}}(a,t) &= \left[\mathfrak{B}\sum_{i=1}^n \breve{\mathbf{g}}_i^m\right](t-a) - \sum_{i=1}^n \breve{\mathbf{g}}_i^m(0,t-a) + [\mathfrak{B}\breve{\mathbf{h}}](t-a) + [\mathfrak{B}\breve{\mathbf{u}}_{0,\epsilon}^m](t-a) \\
&\quad + [\mathfrak{B}(\epsilon\bar{\mathbf{w}}_1 + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})](t-a) - \epsilon\sum_{i=1}^m\left(\int_0^\infty\left(\sum_{i=1}^m \beta_{ki}^*(s)\bar{u}^{n_k}(s,t)\right)ds\right)\mathbf{e}_i \\
&\quad - \int_{t-a}^t \sum_{j=1}^m \left(\mathbf{f}'_{n_j}\cdot\breve{\mathbf{h}}\right)\mathbf{e}_j - \int_{t-a}^t \sum_{j=1}^m \left[\mathbf{f}'_{n_j}\cdot(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})\right]\mathbf{e}_j(\sigma_{a,t})\,d\sigma \\
&\quad - \int_{t-a}^t \left[\mathbf{M}_S\sum_{i=1}^n \breve{\mathbf{g}}_i + \breve{g}^m\mathbf{e}_a\right](\sigma_{a,t})\,d\sigma - \int_{t-a}^t [\mathbf{M}_S\breve{\mathbf{h}}](\sigma_{a,t})\,d\sigma + \frac{1}{\epsilon}\int_{t-a}^t [\mathbf{C}_S\breve{\mathbf{h}}](\sigma_{a,t})\,d\sigma \\
&\quad - \int_{t-a}^t [\mathbf{M}_S\breve{\mathbf{u}}_{0,\epsilon}^m + \breve{u}_{0,\epsilon}^m\mathbf{e}_a](\sigma_{a,t})\,d\sigma - \int_{t-a}^t [\mathbf{M}_S(\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_{0,\epsilon} + \hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon})](\sigma_{a,t})\,d\sigma \\
&\quad - \epsilon\int_{t-a}^t \mathfrak{L}(\sigma + a - t)\bar{\mathbf{u}}^m(\sigma_{a,t})\,d\sigma + \frac{1}{\epsilon}\int_{t-a}^t [(\mathbf{C}_S - \mathbf{C}_S(0))(\hat{\mathbf{w}}_{0,\epsilon} + \breve{\mathbf{w}}_{0,\epsilon}](\sigma_{a,t})\,d\sigma \\
&\quad - \int_{t-a}^t \tilde{\mathbf{w}}_{0,\epsilon,1}(\sigma_{a,t})\,d\sigma - \int_{t-a}^t \hat{\mathbf{w}}_{0,\epsilon,1}(\sigma_{a,t})\,d\sigma. \qquad (7.34)
\end{aligned}$$

The initial value of the error $\breve{\mathbf{E}}(a,0) = (\breve{\mathbf{g}}(a,0), \breve{\mathbf{h}}(a,0))$ is thus

$$\breve{\mathbf{E}}(a,0) = \begin{bmatrix} \mathbf{0} \\ -\epsilon\mathbf{C}_S^{-1}\left[\mathbf{M}_S\boldsymbol{\phi}_m + \boldsymbol{\phi}_m\mathbf{e}_a\right](a) - e^{\frac{a}{\epsilon}\mathbf{C}_S}[\mathfrak{B}\bar{\mathbf{u}}^m(\cdot,0) - \bar{\mathbf{u}}^m(0,0)] \end{bmatrix}, \qquad (7.35)$$

the inhomogeneity in the equation is given by

$$
\begin{aligned}
\breve{\mathbf{F}}(a,t) = &- \begin{bmatrix} \mathbf{x}_j \cdot \mathbf{M}(a)\tilde{\mathbf{w}}_0\left(a,\frac{t}{\epsilon}\right) - \mathbf{f}'_{n_j}(a)\cdot \tilde{\mathbf{w}}_0\left(a,\frac{t}{\epsilon}\right) \\ \tilde{\mathbf{w}}_{0,a}\left(a,\frac{t}{\epsilon}\right) + \mathbf{M}_S(a)\tilde{\mathbf{w}}_0\left(a,\frac{t}{\epsilon}\right) + \sum_{j=1}^{m}\left(\mathbf{f}(a)'_{n_j}\cdot \tilde{\mathbf{w}}_0\left(a,\frac{t}{\epsilon}\right)\right)\mathbf{e}_j \end{bmatrix} \\
&- \epsilon \begin{bmatrix} \mathbf{x}_j \cdot \mathbf{M}(a)\bar{\mathbf{w}}_1(a,t) - \mathbf{f}'_{n_j}(a)\cdot \bar{\mathbf{w}}_1(a,t) \\ \mathfrak{L}(a)\bar{\mathbf{u}}^m(a,t) + \sum_{j=1}^{m}\left(\mathbf{f}(a)'_{n_j}\cdot \bar{\mathbf{w}}_1(a,t)\right)\mathbf{e}_j(a) + \mathbf{M}_S(a)\bar{\mathbf{w}}_1(a,t) \end{bmatrix} \\
&- \begin{bmatrix} \mathbf{x}_j \cdot \mathbf{M}(a)\hat{\mathbf{w}}_0\left(\frac{a}{\epsilon},t\right) - \mathbf{f}'_{n_j}(a)\cdot \hat{\mathbf{w}}_0\left(\frac{a}{\epsilon},t\right) \\ \sum_{j=1}^{m}\left(\mathbf{f}(a)'_{n_j}\cdot \hat{\mathbf{w}}_0\left(\frac{a}{\epsilon},t\right)\right)\mathbf{e}_j + \mathbf{M}_S(a)\hat{\mathbf{w}}_0\left(\frac{a}{\epsilon},t\right) \\ +\hat{\mathbf{w}}_{0,t}\left(\frac{a}{\epsilon},t\right) - \epsilon^{-1}(\mathbf{C}_S(a)-\mathbf{C}_S(0))\hat{\mathbf{w}}_0\left(\frac{a}{\epsilon},t\right) \end{bmatrix} \\
&- \begin{bmatrix} \sum_{i=1}^{m}\mu^*_{ji}(a)\breve{u}_0^{n_j}\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right) + \mathbf{x}_j\cdot\mathbf{M}(a)\breve{\mathbf{w}}_0\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right) - \mathbf{f}'_{n_j}(a)\cdot\breve{\mathbf{w}}_0\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right) \\ \sum_{j=1}^{m}\left(\mathbf{f}(a)'_{n_j}\cdot\breve{\mathbf{w}}_0\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right)\right)\mathbf{e}_j(a) + \mathbf{M}_S(a)\breve{\mathbf{u}}_0^m\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right) + \mathbf{M}_S(a)\breve{\mathbf{w}}_0\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right) \\ +\breve{u}_0^m\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right)\mathbf{e}_a(a) - \frac{1}{\epsilon}(\mathbf{C}_S(a)-\mathbf{C}_S(0))\breve{\mathbf{w}}_0\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right) \end{bmatrix}
\end{aligned}
$$

$$
=: \mathbf{F}_1\left(a,\frac{t}{\epsilon}\right) + \mathbf{F}_2(a,t) + \mathbf{F}_3\left(\frac{a}{\epsilon},t\right) + \mathbf{F}_4\left(\frac{a}{\epsilon},\frac{t}{\epsilon}\right), \tag{7.36}
$$

which is similar to (6.40) and (6.41) but for $\bar{\mathbf{w}}_{1,t} + \bar{\mathbf{w}}_{1,a}$ which has been replaced, thanks to (7.22), by the term $\mathfrak{L}\bar{\mathbf{u}}^m$ which requires lower regularity from the data. Finally, the inhomogeneity on the boundary is given by

$$\mathbf{H}(t) = \begin{bmatrix} \int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}(s)\breve{u}^{n_i}_{0,\epsilon}\right) ds + \int_0^\infty \mathbf{x}_j \cdot \mathbf{B}(s)(\epsilon\bar{\mathbf{w}}_1(s,t) + \hat{\mathbf{w}}_{0,\epsilon}(s,t) + \breve{\mathbf{w}}_{0,\epsilon}(s,t))\, ds \\ \mathfrak{B}\breve{\mathbf{u}}^m_0 + [\mathfrak{B}(\epsilon\bar{\mathbf{w}}_1(\cdot,t) + \hat{\mathbf{w}}_{0,\epsilon}(\cdot,t) + \breve{\mathbf{w}}_{0,\epsilon}(\cdot,t))] \\ -\epsilon\mathbf{C}_S^{-1}\left[\mathbf{M}_S(0)\sum_{i=1}^m \left(\int_0^\infty \left(\sum_{k=1}^m \beta^*_{ki}(s)\bar{u}^{n_k}(s,t)\right) ds\right)\mathbf{e}_i \right. \\ \left. + \sum_{i=1}^m \left(\int_0^\infty \left(\sum_{k=1}^m \beta^*_{ki}(s)\bar{u}^{n_k}(s,t)\right) ds\right)\mathbf{e}_{a,i}\right] \end{bmatrix}. \tag{7.37}$$

Theorem 7.3 *Let us assume that* $\mathbf{C}, \mathbf{B}$ *and* $\mathbf{M}$ *satisfy assumptions introduced in Subsection 6.3.1 and* $\mathbf{u}_\epsilon(a,t) := [\mathcal{G}_\epsilon(t)\boldsymbol{\phi}](a) = \sum_{i=1}^m u^{n_i}_\epsilon(a,t)\mathbf{e}_i(a) + \mathbf{w}_\epsilon(a,t)$ *be a solution to (6.5). Then, for each* $T < \infty$ *there exists a constant* $C(T, \mathbf{M}, \mathbf{B}, \mathbf{C})$ *such that for any* $\boldsymbol{\phi} \in W^1_1(\mathbb{R}_+, \mathbb{R}^n)$ *and uniformly on* $[0, T]$ *we have*

$$\left\|u^{n_j}_\epsilon(\cdot,t) - \bar{u}^{n_j}(\cdot,t)\right\|_{L_1(\mathbb{R}_+)} \leq \epsilon\, C(\mathrm{T}, \mathbf{M}, \mathbf{B}, \mathbf{C}) \left\|\boldsymbol{\phi}\right\|_{W^1_1(\mathbb{R}_+,\mathbb{R}^n)}, \tag{7.38}$$

and

$$\left\|\mathbf{w}_\epsilon(\cdot,t) - e^{\frac{t}{\epsilon}\mathbf{C}(\cdot)}\mathbf{w}_0(\cdot)\right\|_{L_1(\mathbb{R}_+,\mathbb{R}^n)} \leq \epsilon\, C(\mathrm{T}, \mathbf{M}, \mathbf{B}, \mathbf{C}) \left\|\boldsymbol{\phi}\right\|_{W^1_1(\mathbb{R}_+,\mathbb{R}^n)}. \tag{7.39}$$

Proof. We use linearity and first estimate the part of the error, denoted by $\breve{\mathbf{E}}_1$, coming from $\breve{\mathbf{F}}$ and the initial condition (7.35) with $\mathbf{g} = 0$ using the semigroup formula (7.4) and then we let the initial conditions and $\breve{\mathbf{F}}$ equal to zero and use (7.15) to estimate the part of the error $\breve{\mathbf{E}}_2$ due to the non-zero $\mathbf{g}$.

Let us recall that the semigroup $(\mathcal{G}_\epsilon(t))_{t\geq 0}$ generated by the system (6.5)–(6.7) is equibounded in ϵ: $\|\mathcal{G}_\epsilon(t)\| \leq e^{wt}$ with w independent of ϵ. By [65] and (7.31)–(7.34), $\breve{\mathbf{E}}_1$ satisfies

$$\breve{\mathbf{E}}_1(t) = \mathcal{G}_\epsilon(t)\breve{\mathbf{E}}(\cdot,0) + \int_0^t \mathcal{G}_\epsilon(t-s)\breve{\mathbf{F}}(s)\, ds.$$

Let us fix $0 < T < \infty$. Then, for any $t \in [0, T]$,

$$\left\|\breve{\mathbf{E}}_1(t)\right\|_{\mathbf{X}} \leq e^{\omega T}\left(\left\|\breve{\mathbf{E}}(\cdot,0)\right\|_{\mathbf{X}} + \int_0^t \left\|\breve{\mathbf{F}}(s)\right\|_{\mathbf{X}} ds\right).$$

In what follows, constants c_i depend only on the coefficients of the problem and T but not on the initial data. Due (6.27) and assumptions on $\mathbf{M}$ we have

$$\left\|\epsilon\, \mathbf{C}_S^{-1}\left[\mathbf{M}_S\boldsymbol{\phi}_m + \boldsymbol{\phi}_m\mathbf{e}_a\right]\right\|_{\mathbf{X}} \leq \epsilon c_1 \left\|\boldsymbol{\phi}_m\right\|_{W^1_1(\mathbb{R}_+,\mathbb{R}^n)}.$$

Similarly, due to (6.8) and the assumptions on $\mathbf{B}$, for some $0 < \sigma < -\zeta$, we have

$$\begin{aligned}\left\|\exp\left(\frac{a}{\epsilon}\mathbf{C}_S(0)\right)[\mathfrak{B}\bar{\mathbf{u}}^m(\cdot,0) - \bar{\mathbf{u}}^m(0,0)]\right\|_{\mathbf{X}} &\leq c_2 \left\|\boldsymbol{\phi}_m\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)} \int_0^\infty e^{-\sigma\frac{s}{\epsilon}} ds\\ &\leq \epsilon c_2 \sigma^{-1} \left\|\boldsymbol{\phi}_m\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}.\end{aligned}$$

Next let us consider $\mathbf{F}_1(a, t/\epsilon)$. First we observe that the term $\tilde{\mathbf{w}}_{0,a}(a, t/\epsilon)$ is well defined due to the assumption that $\sum_{i=1}^m \phi_i \mathbf{e}_i \in W_1^1(\mathbb{R}_+, \mathbb{R}^n)$, Lemma 6.8 (as $\mathbf{w}_0 = \boldsymbol{\phi} - \sum_{i=1}^m \phi_i \mathbf{e}_i$). Thus, the error estimates involving $\mathbf{F}_1$ are all of the form $\int_0^t e^{-\sigma\frac{t}{\epsilon}} dt \leq \epsilon/\sigma$, where σ is as above. Hence

$$\left\|\int_0^t \mathcal{G}_\epsilon(t-s)\mathbf{F}_1(s)\, ds\right\|_{\mathbf{X}} \leq \epsilon c_3(T) \left\|\boldsymbol{\phi}\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}. \tag{7.40}$$

Let us consider the contribution of $\mathbf{F}_2$ to the error. By (6.27) we immediately find that

$$\left\|\int_0^t \mathcal{G}_\epsilon(t-s)\mathbf{F}_2(s)\, ds\right\|_{\mathbf{X}} \leq \epsilon c_4(T) \left\|\boldsymbol{\phi}\right\|_{\mathbf{X}}. \tag{7.41}$$

Estimates related to $\mathbf{F}_3$ and some other terms of the error are more involved. Before we go on, let us mention some additional properties of the operator in (7.17). First as in [27, Theorem 4.3], $(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}$ can be extended to a continuous operator on $L_1([0,T], \mathbb{R}^n)$ with

$$\left\|(\mathbf{I} - \mathfrak{B}\mathcal{V}_{\mathcal{K}})^{-1}\,\mathbf{g}\right\|_{L_1([0,T],\mathbb{R}^n)} \leq e^{mT} \left\|\mathbf{g}\right\|_{L_1([0,T],\mathbb{R}^n)}. \tag{7.42}$$

Next, we need estimates of the derivative of $\bar{\mathbf{v}}(0,\cdot) = \{\bar{u}^{n_j}(0,\cdot)\}_{1\leq j\leq m}$. The fact that $\bar{\mathbf{v}}(0,\cdot) \in W^1_{1,loc}(\mathbb{R}_+, \mathbb{R}^n)$ follows from e.g., [27, Theorem 4.1]. Denoting $\boldsymbol{\psi}(t) = \bar{\mathbf{v}}(0,t)$ and using (6.31), we find that $\boldsymbol{\psi}$ is determined from the equation

$$\boldsymbol{\psi}(t) := \int_0^t \mathbf{B}_v(a)\mathbf{L}_v(a)\boldsymbol{\psi}(t-a)\, da + \int_t^\infty \mathbf{B}_v(a)\mathbf{L}_v(a)\mathbf{L}_v^{-1}(a-t)\boldsymbol{\phi}^m(a-t)\, da. \tag{7.43}$$

If $\boldsymbol{\psi}$ is differentiable, then using the results on differentiability of convolutions (e.g., [4, Proposition 1.3.6]), we get

$$\boldsymbol{\psi}'(t) = \int_0^t \mathbf{B}_v(a)\mathbf{L}_v(a)\boldsymbol{\psi}'(t-a)\, da + \mathbf{q}(t), \tag{7.44}$$

where

$$\begin{aligned}\mathbf{q}(t) = \mathbf{B}_v(t)\boldsymbol{\psi}(0) - \mathbf{B}_v(t)\mathbf{L}_v(t)\boldsymbol{\phi}^m(0) - \int_t^\infty \mathbf{B}_v(a)\mathbf{L}_v(a)(\mathbf{L}_v^{-1})'(a-t)\boldsymbol{\phi}^m(a-t)\, da\\ - \int_t^\infty \mathbf{B}_v(a)\mathbf{L}_v(a)\mathbf{L}_v^{-1}(a-t)(\boldsymbol{\phi}^m)'(a-t)\, da.\end{aligned} \tag{7.45}$$

By (7.17),

$$\operatorname*{ess\,sup}_{t\in[0,T]} \left\|\psi'(t)\right\| \leq C \operatorname*{ess\,sup}_{t\in[0,T]} \left\|\mathbf{q}(t)\right\|,$$

and thus, by (7.43),

$$\begin{aligned}\left\|\psi(0)\right\| \leq \operatorname*{ess\,sup}_{t\in[0,T]} \left\|\psi(t)\right\| &\leq C_1 \operatorname*{ess\,sup}_{t\in[0,T]} \int_t^\infty \left\|\mathbf{B}_v(a)\mathbf{L}_v(a)\mathbf{L}_v^{-1}(a-t)\phi(a-t)\right\| da \\ &\leq C_2 \left\|\phi^m\right\|_{\mathbf{X}}. \end{aligned} \tag{7.46}$$

Therefore

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in[0,T]} \left\|\psi'(t)\right\| &\leq C_3 \left(\left\|\phi^m\right\|_{\mathbf{X}} + \left\|\phi^m(0)\right\|_{\mathbb{R}^n} + \left\|\phi^m\right\|_{\mathbf{X}} + \left\|(\phi^m)'\right\|_{\mathbf{X}}\right) \\ &\leq C_4 \left\|\phi^m\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}. \end{aligned} \tag{7.47}$$

To estimate $\mathbf{F}_3$, first consider

$$\begin{aligned}\frac{1}{\epsilon}\int_0^t \left\|(\mathbf{C}_S(\cdot)-\mathbf{C}_S(0))\,\hat{\mathbf{w}}_0\left(\frac{\cdot}{\epsilon},s\right)\right\|_{\mathbf{X}} ds &= \frac{1}{\epsilon}\int_0^t \left(\int_0^\infty \left\|(\mathbf{C}_S(a)-\mathbf{C}_S(0))e^{\frac{a}{\epsilon}\mathbf{C}_S}\hat{\mathbf{w}}_0(0,s)\right\|_{\mathbb{R}^n} da\right) ds \\ &\leq c_4' \int_0^t \left(\int_0^\infty \frac{a}{\epsilon}e^{-\sigma\frac{a}{\epsilon}}\, da\right) \|\hat{\mathbf{w}}_0(0,s)\|_{\mathbb{R}^n}\, ds \\ &\leq \epsilon c_4'' \int_0^t \|\hat{\mathbf{w}}_0(0,s)\|_{\mathbb{R}^n}\, ds \\ &= \epsilon c_4'' \int_0^t \|\mathfrak{B}\bar{\mathbf{u}}^m(\cdot,s) - \bar{\mathbf{u}}^m(0,s)\|_{\mathbb{R}^n}\, ds \\ &\leq \epsilon c_4''' \left(\int_0^t \|\bar{\mathbf{u}}^m(\cdot,s)\|_{\mathbf{X}}\, ds \right.\\ &\qquad \left. + \int_0^t \|\bar{\mathbf{u}}^m(0,s)\|_{\mathbf{X}}\, ds\right) \\ &\leq \epsilon c_4^{iv} \|\phi_m\|_{\mathbf{X}}. \end{aligned} \tag{7.48}$$

In the last inequality we have used (7.42). The next term which requires some reflection is $\hat{\mathbf{w}}_{0,t}(a/\epsilon,t)$. The differentiability of $t \to \int_0^\infty \mathbf{B}(s)\bar{\mathbf{u}}^m(s,t)\,ds$ follows by writing it as in (7.43) and arguing in the same manner. Hence

$$\begin{aligned}\left\|\hat{\mathbf{w}}_{0,t}\left(\frac{a}{\epsilon},t\right)\right\|_{\mathbf{X}} \leq \int_0^\infty \left\|\hat{\mathbf{w}}_{0,t}\left(\frac{a}{\epsilon},t\right)\right\|_{\mathbb{R}^n} da &= \int_0^\infty \left\|e^{\frac{a}{\epsilon}\mathbf{C}_S}(a)\hat{\mathbf{w}}_{0,t}\right\|_{\mathbb{R}^n} da \\ &\leq \epsilon\left(\int_0^\infty e^{-\sigma\alpha}\, d\alpha\right)\left\|\hat{\mathbf{w}}_{0,t}(0,t)\right\|_{\mathbb{R}^n} \\ &\leq \epsilon c_5' \left\|\mathfrak{B}\bar{\mathbf{u}}_t^m(a,t) - \bar{\mathbf{u}}_t^m(0,t)\right\|_{\mathbb{R}^n} \\ &\leq \epsilon c_5'' \left\|\phi_m\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}. \end{aligned} \tag{7.49}$$

The other two terms in $\mathbf{F}_3$ can be easily estimated by $c\epsilon \left\|\boldsymbol{\phi}_m\right\|_{\mathbf{X}}$. Consequently

$$\left\|\int_0^t \mathcal{G}_\epsilon(t-s)\mathbf{F}_3(s)\,ds\right\|_{\mathbf{X}} \le \epsilon c_5(T)\left\|\boldsymbol{\phi}_m\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}.$$

Next let us move to $\mathbf{F}_4$. Using (7.27) and (7.30), we find

$$\begin{aligned}\left\|\int_0^t \mathcal{G}_\epsilon(t-s)\mathbf{F}_4\left(\cdot,\frac{s}{\epsilon}\right)ds\right\|_{\mathbf{X}} &\le e^{\omega T}\int_0^t\left(\int_0^s\left\|\mathbf{F}_4\left(\frac{a}{\epsilon},\frac{s}{\epsilon}\right)\right\|_{\mathbb{R}^n}da\right)ds\\ &= \epsilon e^{\omega T}\int_0^t\left(\int_0^{s/\epsilon}\left\|\mathbf{F}_4\left(\alpha,\frac{s}{\epsilon}\right)\right\|_{\mathbb{R}^n}d\alpha\right)ds. \qquad (7.50)\end{aligned}$$

By (7.36) estimates of $\check{\mathbf{F}}_4$ involve four type of expressions. First, by (7.27), the terms involving $\breve{u}_0^{n_j}$ satisfy

$$\begin{aligned}&\int_0^t\left(\int_0^{s/\epsilon}\left(\int_0^\infty\left\|e^{(s/\epsilon-\alpha)\mathbf{C}_S(a)}\mathbf{w}_0(a)\right\|_{\mathbb{R}^n}da\right)d\alpha\right)ds\\ &\qquad\le \int_0^t\left(\int_0^{s/\epsilon}\left(\int_0^\infty e^{-\sigma(s/\epsilon-\alpha)}\|\mathbf{w}_0(a)\|_{\mathbb{R}^n}\,da\right)d\alpha\right)ds\\ &\qquad\le \epsilon\|\mathbf{w}_0\|_{\mathbf{X}}\int_0^{t/\epsilon}\int_0^\eta e^{-\sigma(\eta-\alpha)}\,d\alpha\,d\eta \;\le\; t\sigma^{-1}\|\mathbf{w}_0\|_{\mathbf{X}}. \qquad (7.51)\end{aligned}$$

Second, we have the terms involving $\check{\mathbf{w}}_0$. By (7.29), the first and last terms contain $\tilde{\mathbf{w}}_0(a,\tau) = e^{\tau\mathbf{C}_S(a)}\mathbf{w}_0(a)$ and, similarly to the above, they can be estimated as

$$\begin{aligned}&\int_0^t\int_0^{s/\epsilon}\left\|e^{\alpha\mathbf{C}_S(0)}\left(\int_0^\infty e^{(\frac{s}{\epsilon}-\alpha)\mathbf{C}_S(a)}\mathbf{w}_0(a)\,da\right)\right\|_{\mathbb{R}^n}d\alpha\,ds\\ &\qquad\le \int_0^t\int_0^{s/\epsilon}e^{\zeta\alpha}\int_0^\infty e^{-\sigma(\frac{s}{\epsilon}-\alpha)}\left\|\mathbf{w}_0(a)\right\|_{\mathbb{R}^n}da\,d\alpha\,ds\\ &\qquad\le \epsilon\|\mathbf{w}_0\|_{\mathbf{X}}\int_0^{t/\epsilon}e^{-\sigma\eta}\eta\,d\eta \;\le \epsilon\,\sigma^{-2}\|\mathbf{w}_0\|_{\mathbf{X}}. \qquad (7.52)\end{aligned}$$

The second term in (7.29) is $\tilde{\mathbf{w}}_0(0,\tau) = e^{\tau\mathbf{C}_S(0)}\mathbf{w}_0(0)$ which is well defined under the assumption $\mathbf{w}_0 \in W_1^1(\mathbb{R}_+,\mathbb{R}^n)$. Estimates related to this case are as follows

$$\begin{aligned}&\int_0^t\int_0^{s/\epsilon}\left\|e^{\alpha\mathbf{C}_S(0)}e^{(\frac{s}{\epsilon}-\alpha)\mathbf{C}_S(0)}\mathbf{w}_0(0)\right\|_{\mathbb{R}^n}d\alpha\,ds\\ &\qquad\le \int_0^t\int_0^{s/\epsilon}e^{-\sigma\alpha}e^{-\sigma(\frac{s}{\epsilon}-\alpha)}\,\|\mathbf{w}_0(0)\|_{\mathbb{R}^n}\,d\alpha\,ds \;\le \epsilon\sigma^{-2}\,\|\mathbf{w}_0\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}. \qquad (7.53)\end{aligned}$$

The last term requiring our attention is $\epsilon^{-1}(\mathbf{C}_S(a) - \mathbf{C}_S(0))\breve{\mathbf{w}}_0(a/\epsilon, t/\epsilon)$. As above we have two cases depending on the terms in (7.29).
The first two can be estimated by the following terms

$$\frac{1}{\epsilon}\int_0^t\left(\int_0^{s/\epsilon}\left\|(\mathbf{C}_S(a) - \mathbf{C}_S(0))e^{\alpha\mathbf{C}_S(0)}\int_0^\infty e^{(\frac{s}{\epsilon}-\alpha)\mathbf{C}_S(a)}\mathbf{w}_0(a)\,da\right\|_{\mathbb{R}^n} d\alpha\right)ds$$
$$\leq c'\int_0^{t/\epsilon}\left(\int_0^\eta \alpha e^{-\sigma\alpha}\int_0^\infty e^{-\sigma(\eta-\alpha)}\|\mathbf{w}_0\|_{\mathbb{R}^n}\,da\,d\alpha\right)d\eta$$
$$\leq \frac{1}{2}c'\|\mathbf{w}_0\|_{\mathbf{X}}\int_0^\infty \eta^2 e^{-\sigma\eta}\,d\eta = c'\|\mathbf{w}_0\|_{\mathbf{X}}. \quad (7.54)$$

The estimate of the term containing $\tilde{\mathbf{w}}_0(0,\tau) = e^{\tau\mathbf{C}_S(0)}\mathbf{w}_0(0)$ is follows:

$$\epsilon e^{\varpi T}\frac{1}{\epsilon}\int_0^t\left(\int_0^{s/\epsilon}\left\|e^{\alpha\mathbf{C}_S(0)}(\mathbf{C}_S(a) - \mathbf{C}_S(0))e^{\left(\frac{s}{\epsilon}-\alpha\right)\mathbf{C}_S(0)}\mathbf{w}_0(0)\right\|_{\mathbb{R}^n} d\alpha\right)ds$$
$$\leq \epsilon c'(T)\int_0^t\left(\int_0^{s/\epsilon}\left\|\alpha e^{-\sigma\frac{s}{\epsilon}\mathbf{C}_S(0)}\mathbf{w}_0(0)\right\|_{\mathbb{R}^n} d\alpha\right)ds$$
$$\leq \frac{1}{\epsilon}c'(T)\,\|\mathbf{w}_0\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}\int_0^t e^{-\sigma\frac{s}{\epsilon}}s^2\,ds \leq \epsilon^2 c'(T)\,\|\mathbf{w}_0\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}.$$

Inserting the above estimates into (7.50) we obtain

$$\left\|\int_0^t \mathcal{G}_\epsilon(t-s)\mathbf{F}_4\left(\cdot,\frac{s}{\epsilon}\right)ds\right\|_{\mathbf{X}} \leq \epsilon c_6(T)\left\|\boldsymbol{\phi}\right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)}. \quad (7.55)$$

It remains to estimates the contribution of the boundary terms. For this we use equation (7.15) in which $\mathcal{K}(a) := -\mathbf{M}(a) + \epsilon^{-1}\mathbf{C}$. By Lemma 7.2, $\mathcal{V}_\mathcal{K}$ is bounded on $\mathbb{R}_+$ and thus

$$\left\|\breve{\mathbf{E}}_2(t)\right\|_{\mathbf{X}} \leq \|\boldsymbol{\omega}(\cdot,t)\|_{\mathbf{X}} = \int_0^t\left\|\left[(I - \mathfrak{B}\mathcal{V}_\mathcal{K})^{-1}\mathbf{g}\right](t-a)\right\|_{\mathbb{R}^n} da.$$

Therefore, by (7.42),

$$\left\|\breve{\mathbf{E}}_2(t)\right\|_{\mathbf{X}} \leq \|\boldsymbol{\omega}(\cdot,t)\|_{\mathbf{X}} \leq c_7(T)\left\|\mathbf{g}\right\|_{L_1([0,T],\mathbb{R}^n)}.$$

Since $t \to \bar{u}^{n_j}(\cdot,t)$ is a mild solution to (6.29), it is strongly continuous and thus the L_1 norms of the terms

$$\epsilon\int_0^\infty \mathbf{x}_j\cdot\mathbf{B}(s)\,\bar{\mathbf{w}}_1(s,t)\,ds, \quad \epsilon\mathfrak{B}\bar{\mathbf{w}}_1(\cdot,t),$$

and

$$\epsilon\mathbf{C}_S^{-1}\left[\mathbf{M}_S(0)\sum_{i=1}^m\left(\int_0^\infty\left(\sum_{k=1}^m \beta_{ki}^*(s)\bar{u}^{n_k}(s,t)\right)ds\right)\mathbf{e}_i + \sum_{i=1}^m\left(\int_0^\infty\left(\sum_{k=1}^m \beta_{ki}^*(s)\bar{u}^{n_k}(s,t)\right)ds\right)\mathbf{e}_{a,i}\right]$$

are bounded by $\epsilon c_8(T)\|\phi_m\|_X$, where $c_8(T)$ is related to the type of the solution $\bar{u}^{n_j}$. Next, consider the corner layer terms:

$$\int_0^\infty \left(\sum_{i=1}^m \beta^*_{ji}(s)\,\breve{u}^{n_j}_{0,\epsilon}(s,t)\right) ds,\quad \int_0^\infty \mathbf{x}_j\cdot\mathbf{B}(s)\breve{\mathbf{w}}_{0,\epsilon}(s,t)\,ds,\quad \mathfrak{B}\breve{\mathbf{u}}^m_0,\quad \mathfrak{B}\breve{\mathbf{w}}_{0,\epsilon}(\cdot,t),$$

and using (7.27), (7.29) and (7.30), we see that all the terms in these expressions have the generic form

$$\int_0^t \mathbf{B}_1(s)\left(\int_0^\infty \mathbf{B}_2(\sigma)\,e^{\frac{t-s}{\epsilon}\mathbf{C}_S(\sigma)}\mathbf{x}(\sigma)\,d\sigma\right) ds \quad \text{or} \quad \int_0^t \mathbf{B}_1(s)\,e^{\frac{t-s}{\epsilon}\mathbf{C}_S(0)}\mathbf{x}(0)\,ds,$$

where $\mathbf{x}\in W^1_1(\mathbb{R}_+,\mathbb{R}^k)$, $\mathbf{B}_i\in L_\infty(\mathbb{R}_+,\mathcal{B}_{\mathbb{R}^k\times\mathbb{R}^l})$, $i=1,2$, with k,l equal to either 1 or n. Hence, the estimates of the L_1 norm of them are of the same type as (7.50) combined with (7.51)–(7.53). Finally, the estimates of the boundary layer terms

$$\int_0^\infty \mathbf{x}_j\cdot\mathbf{B}(s)\hat{\mathbf{w}}_{0,\epsilon}(s,t)\,ds \qquad \text{and} \qquad \mathfrak{B}\hat{\mathbf{w}}_{0,\epsilon}(\cdot,t)$$

follow from (7.48) due to the boundedness of the coefficients of $\mathbf{B}$.

Now, using the spectral decomposition of $\mathbf{u}_\epsilon$, we have

$$\begin{aligned}
\mathbf{u}_\epsilon &= \sum_{j=1}^n \mathbf{u}^{n_j}_\epsilon \mathbf{e}_j + \mathbf{w}_\epsilon \\
&= \sum_{j=1}^n \left(\mathbf{u}^{n_j}_\epsilon - \bar{u}^{n_j} - \breve{u}^{n_j}_0\right)\mathbf{e}_j + (\mathbf{w}_\epsilon - \epsilon\bar{\mathbf{w}}_1 - \tilde{\mathbf{w}}_0 - \hat{\mathbf{w}}_0 - \breve{\mathbf{w}}_0) \\
&\quad + \sum_{j=1}^n \left(\bar{u}^{n_j} + \breve{u}^{n_j}_0\right)\mathbf{e}_j + (\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_0 + \hat{\mathbf{w}}_0 + \breve{\mathbf{w}}_0),
\end{aligned}$$

and thus, using all estimates done above, we have proved that for any $T<\infty$ there is a constant $C = C(T,\mathbf{M},\mathbf{B},\mathbf{C})$ such that

$$\begin{aligned}
&\left\|\mathbf{u}_\epsilon - \sum_{j=1}^n \left(\bar{u}^{n_j} + \breve{u}^{n_j}_0\right)\mathbf{e}_j - (\epsilon\bar{\mathbf{w}}_1 + \tilde{\mathbf{w}}_0 + \hat{\mathbf{w}}_0 + \breve{\mathbf{w}}_0)\right\|_X \\
&\qquad = \left\|\sum_{j=1}^n \left(\mathbf{u}^{n_j}_\epsilon - \bar{u}^{n_j} - \breve{u}^{n_j}_0\right)\mathbf{e}_j + (\mathbf{w}_\epsilon - \epsilon\bar{\mathbf{w}}_1 - \tilde{\mathbf{w}}_0 - \hat{\mathbf{w}}_0 - \breve{\mathbf{w}}_0)\right\|_X \\
&\qquad \le \epsilon C(T,\mathbf{M},\mathbf{B},\mathbf{C})\left\|\phi\right\|_{W^1_1(\mathbb{R}_+,\mathbb{R}^n)}. \qquad (7.56)
\end{aligned}$$

However

$$\begin{aligned}
\left\|\breve{u}^{n_j}_0\left(\cdot,\frac{t}{\epsilon}\right)\right\|_{L_1(\mathbb{R}_+)} &\le \int_0^t \left\|\left(\int_0^\infty \left(\mathbf{x}_j\cdot\mathbf{B}(s)e^{(\frac{t-a}{\epsilon})\mathbf{C}_S(s)}\mathbf{w}_0(s)\right)ds\right)\right\|_{\mathbb{R}^n} da \\
&\le \epsilon c_6\left(\int_0^{\frac{t}{\epsilon}} e^{-\sigma(\frac{t-a}{\epsilon})}\right) d\alpha\ \|\mathbf{w}_0\|_X \le \frac{\epsilon c_6\|\mathbf{w}_0\|_X}{\sigma},
\end{aligned}$$

and

$$\left\| \breve{\mathbf{w}}_0\left(\cdot, \frac{t}{\epsilon}\right) \right\|_{\mathbf{X}} \leq c_7 \left\| \mathbf{w}_0 \right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)} \int_0^t e^{-\sigma \frac{a}{\epsilon}}\, da$$
$$\leq \epsilon c_7 \left\| \mathbf{w}_0 \right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)} \left(\max_{z \in \mathbb{R}_+} z e^{-\sigma z} \right) \leq \frac{\epsilon c_7}{\sigma e} \left\| \mathbf{w}_0 \right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)} .$$

Also, we note that

$$\left\| \epsilon \bar{\mathbf{w}}_1(\cdot, t) \right\|_{\mathbf{X}} \leq \epsilon c_8(T) \left\| \phi \right\|_{\mathbf{X}} .$$

Combining the above estimates, we can move $\epsilon \bar{\mathbf{w}}_1$ as well as the boundary and corner layer terms to the right-hand side and re-write (7.56) as

$$\left\| \mathbf{u}_\epsilon - \sum_{j=1}^{n} \bar{u}^{n_j} \mathbf{e}_j - \tilde{\mathbf{w}}_0 \right\|_{\mathbf{X}} \leq \epsilon\, C_1(T, \mathbf{M}, \mathbf{B}, \mathbf{C}) \left\| \phi \right\|_{W_1^1(\mathbb{R}_+,\mathbb{R}^n)} \tag{7.57}$$

uniformly in $t \in [0, T]$.

CHAPTER 8

An Illustrative Example

In this chapter we are going to apply the asymptotic analysis method to the McKendrick models with 5 patches with migrations between them described by a reducible, Kolmogorov matrix. To simplify calculations, we assume that the migration matrix **C**, mortality matrix **M** and the birth matrix **B** are all age independent and also take simple numerical values of **C**. Let the reducible migration matrix be

$$\mathbf{C} := \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & -2 \end{bmatrix}.$$

This matrix **C** has zero as the dominant eigenvalue with multiplicity two and the null space is also two dimensional. The rest of the eigenvalues have negative real parts. Let $\mathbf{k} := (k_1, k_2, k_3, k_4, k_5)$ be a right eigenvector corresponding to the zero eigenvalue of **C**. Then **k** has the following structure:

$$k_1 = k_2 = \alpha \text{ (say)}, \qquad k_3 = k_4 = \alpha' \text{ (say)} \qquad \text{and} \qquad k_5 = 0.$$

We can choose the corresponding right basis (normalized) vectors for the two dimensional null space of **C** as follows:

$$\mathbf{e}_1 := \left(\frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \qquad \text{and} \qquad \mathbf{e}_2 := \left(0, 0, \frac{1}{2}, \frac{1}{2}, 0\right).$$

Let $\mathbf{x} := (x_1, x_2, x_3, x_4, x_5)$ be a left eigenvector corresponding to the zero eigenvalue of $\mathbf{C}$. Then $\mathbf{x}$ has the following structure:

$$x_1 = x_2 = p \text{ (say)}, \quad x_3 = x_4 = p' \text{ (say)} \quad \text{and} \qquad x_5 = \frac{1}{2}(p + p').$$

The adjoint basis vectors of the two dimensional adjoint null space of $\mathbf{C}$ can be represented as follows:

$$\mathbf{y}_1 := \left(1, 1, 0, 0, \frac{1}{2}\right) \qquad \text{and} \qquad \mathbf{y}_2 := \left(0, 0, 1, 1, \frac{1}{2}\right).$$

Let the population vector be denoted by $\mathbf{u} := (u_1, u_2, u_3, u_4, u_5)$. Following the same method as in Chapter 6, the projection operators $\mathbf{P}_j$, where $1 \le j \le 2$, are defined as follows:

$$\mathbf{P}_j\mathbf{u} := \left(\mathbf{y}_j \cdot \mathbf{u}\right)\mathbf{e}_j = u^j\mathbf{e}_j\,,$$

where

$$u^1 := u_1 + u_2 + \frac{1}{2}u_5 \qquad \text{and} \qquad u^2 := u_3 + u_4 + \frac{1}{2}u_5.$$

The complementary projection is given by $\mathbf{Qu} := \mathbf{u} - \mathbf{Pu}$ and the corresponding subspace of image of $\mathbf{Q}$ is given by

$$\mathbf{S} := \Im\mathbf{Q} := \left\{\mathbf{u} \in \mathbb{R}^5 : \mathbf{P}_1\mathbf{u} = 0, \mathbf{P}_2\mathbf{u} = 0\right\}.$$

Therefore whole space $\mathbb{R}^5$ can be split as

$$\mathbb{R}^5 = [\mathbf{e}_1] \oplus [\mathbf{e}_2] \oplus \mathbf{S}.$$

The subspace $\mathbf{S}$ is 3 dimensional and the corresponding basis vectors can be chosen as

$$\mathbf{f}_1 := (1, -1, 0, 0, 0)\,, \quad \mathbf{f}_2 := (0, 0, 1, -1, 0)\,, \quad \text{and} \quad \mathbf{f}_3 := \left(0, -\frac{1}{2}, 0, -\frac{1}{2}, 1\right).$$

So, in terms of the basis vectors, $\mathbf{Qu}$ can be represented as follows:

$$\mathbf{Qu} := \mathbf{w} = u^{q1}\mathbf{f}_1 + u^{q2}\mathbf{f}_2 + u^{q3}\mathbf{f}_3\,,$$

where

$$u^{q1} := \frac{1}{2}u_1 - \frac{1}{2}u_2 - \frac{1}{4}u_5, \quad u^{q2} := \frac{1}{2}u_3 - \frac{1}{2}u_4 - \frac{1}{4}u_5, \quad \text{and} \quad u^{q3} := u_5.$$

We observe that the fractional contributions of hydrodynamic and kinetic parts in terms of the original variables are as follows:

$$
\begin{aligned}
u_1 &= \frac{1}{2}u^1 + u^{q1},\\
u_2 &= \frac{1}{2}u^1 - u^{q1} - \frac{1}{2}u^{q3},\\
u_3 &= \frac{1}{2}u^2 + u^{q2},\\
u_4 &= \frac{1}{2}u^2 - u^{q2} - \frac{1}{2}u^{q3},\\
u_5 &= u^{q3},
\end{aligned}
$$

and adding these equations we get

$$u_1 + u_2 + u_3 + u_4 + u_5 = u^1 + u^2.$$

Now, using the basis vectors, $\mathbf{u}$ can be represented as

$$\mathbf{u} = \mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u} + \mathbf{Q}\mathbf{u} = u^1\mathbf{e}_1 + u^2\mathbf{e}_2 + u^{q1}\mathbf{f}_1 + u^{q2}\mathbf{f}_2 + u^{q3}\mathbf{f}_3.$$

Define the mortality matrix $\mathbf{M}$ and the birth matrix $\mathbf{B}$ as

$$\mathbf{M} := [\mu_{ij}]_{1\le i,j\le 5} \quad \text{and} \quad \mathbf{B} := [\beta_{ij}]_{1\le i,j\le 5}.$$

Therefore, using above structures of $\mathbf{u}$, $\mathbf{C}$, $\mathbf{M}$ and $\mathbf{B}$, our perturbed model becomes

$$
\begin{aligned}
\mathbf{u}_t &= \mathbf{u}_a - \mathbf{M}\mathbf{u} + \frac{1}{\epsilon}\mathbf{C}\mathbf{u}, && (8.1)\\
\mathbf{u}(0,t) &= \int_0^w \mathbf{B}(s)\mathbf{u}(s,t)\,ds, && (8.2)\\
\mathbf{u}(a,0) &= \boldsymbol{\phi}(a), && (8.3)
\end{aligned}
$$

where $\boldsymbol{\phi}(a) := \big(\phi_1(a), \phi_2(a), \phi_3(a), \phi_4(a), \phi_5(a)\big)$, describes the initial population distribution.

Applying projections $\mathbf{P}_j$, where $1 \le j \le 2$, and $\mathbf{Q}$ on (8.1), we get respectively

$$u_t^1\mathbf{e}_1 = -u_a^1\mathbf{e}_1 - \mathbf{P}_1\mathbf{M}\left(\mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u}\right) - \mathbf{P}_1\mathbf{M}\mathbf{Q}\mathbf{u}, \tag{8.4}$$

$$u_t^2\mathbf{e}_2 = -u_a^2\mathbf{e}_2 - \mathbf{P}_2\mathbf{M}\left(\mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u}\right) - \mathbf{P}_2\mathbf{M}\mathbf{Q}\mathbf{u}, \tag{8.5}$$

and

$$\mathbf{w}_t = -\mathbf{w}_a - \mathbf{Q}\mathbf{M}\left(\mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u}\right) - \mathbf{Q}\mathbf{M}\mathbf{Q}\mathbf{u} + \frac{1}{\epsilon}\mathbf{Q}\mathbf{C}\mathbf{Q}\mathbf{u}, \tag{8.6}$$

where we have used Lemma 5.2 and Lemma 6.13. Now, let us find the explicit expressions of the terms involved in the projected equations (8.4), (8.5) and (8.6). First consider

$$\mathbf{P}_1\mathbf{M}\left(\mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u}\right) = \left(\mathbf{y}_1 \cdot \mathbf{M}\left(u^1\mathbf{e}_1 + u^2\mathbf{e}_2\right)\right)\mathbf{e}_1 =: \left(\mu_1^* u^1 + \mu_2^* u^2\right)\mathbf{e}_1, \tag{8.7}$$

where

$$\mu_1^* := \frac{1}{2}\left(\mu_{11} + \mu_{12} + \mu_{21} + \mu_{22} + \frac{1}{2}\mu_{51} + \frac{1}{2}\mu_{52}\right),$$

and

$$\mu_2^* := \frac{1}{2}\left(\mu_{13} + \mu_{14} + \mu_{23} + \mu_{24} + \frac{1}{2}\mu_{53} + \frac{1}{2}\mu_{54}\right).$$

Similarly

$$\mathbf{P}_2\mathbf{M}(\mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u}) = (\mu_3^* u^1 + \mu_4^* u^2)\mathbf{e}_2, \tag{8.8}$$

where

$$\mu_3^* := \frac{1}{2}\left(\mu_{31} + \mu_{32} + \mu_{41} + \mu_{42} + \frac{1}{2}\mu_{51} + \frac{1}{2}\mu_{52}\right),$$

and

$$\mu_4^* := \frac{1}{2}\left(\mu_{33} + \mu_{34} + \mu_{43} + \mu_{44} + \frac{1}{2}\mu_{53} + \frac{1}{2}\mu_{54}\right).$$

Next, we take

$$\mathbf{P}_1\mathbf{M}\mathbf{Q}\mathbf{u} = \left(\mathbf{y}_1 \cdot \mathbf{M}\mathbf{Q}\mathbf{u}\right)\mathbf{e}_1 = \left(\mu_{1q}^* u^{q1} + \mu_{2q}^* u^{q2} + \mu_{3q}^* u^{q3}\right)\mathbf{e}_1\,, \tag{8.9}$$

where

$$\mu_{1q}^* := \left(\mu_{11} - \mu_{12} + \mu_{21} - \mu_{22} + \frac{1}{2}\mu_{51} - \frac{1}{2}\mu_{52}\right),$$

$$\mu_{2q}^* := \left(\mu_{13} - \mu_{14} + \mu_{23} - \mu_{24} + \frac{1}{2}\mu_{53} - \frac{1}{2}\mu_{54}\right),$$

and

$$\mu_{3q}^* := \left(-\frac{1}{2}\mu_{12} - \frac{1}{2}\mu_{14} + \mu_{15} - \frac{1}{2}\mu_{22} - \frac{1}{2}\mu_{24} + \mu_{25} - \frac{1}{4}\mu_{52} - \frac{1}{4}\mu_{54} + \frac{1}{2}\mu_{55}\right).$$

Similarly, we have

$$\mathbf{P}_2\mathbf{M}\mathbf{Q}\mathbf{u} = \left(\mu_{4q}^* u^{q1} + \mu_{5q}^* u^{q2} + \mu_{6q}^* u^{q3}\right)\mathbf{e}_2\,, \tag{8.10}$$

where

$$\mu_{4q}^* := \left(\mu_{31} - \mu_{32} + \mu_{41} - \mu_{42} + \frac{1}{2}\mu_{51} - \frac{1}{2}\mu_{52}\right),$$

$$\mu_{5q}^* := \left(\mu_{33} - \mu_{34} + \mu_{43} - \mu_{44} + \frac{1}{2}\mu_{53} - \frac{1}{2}\mu_{54}\right),$$

and

$$\mu_{6q}^* := \left(-\frac{1}{2}\mu_{32} - \frac{1}{2}\mu_{34} + \mu_{35} - \frac{1}{2}\mu_{42} - \frac{1}{2}\mu_{44} + \mu_{45} - \frac{1}{4}\mu_{52} - \frac{1}{4}\mu_{54} + \frac{1}{2}\mu_{55}\right).$$

Next, considering terms from equation (8.6), we have

$$\mathbf{QM}(\mathbf{P}_1\mathbf{u} + \mathbf{P}_2\mathbf{u}) = u^{11}\mathbf{f}_1 + u^{12}\mathbf{f}_2 + u^{13}\mathbf{f}_3\,, \tag{8.11}$$

where

$$u^{11} := \left(\frac{1}{2}(\mu_{11} + \mu_{12}) - \frac{1}{2}\mu_1^*\right)u^1 + \left(\frac{1}{2}(\mu_{13} + \mu_{14}) - \frac{1}{2}\mu_2^*\right)u^2,$$

$$u^{12} := \left(\frac{1}{2}(\mu_{31} + \mu_{32}) - \frac{1}{2}\mu_3^*\right)u^1 + \left(\frac{1}{2}(\mu_{33} + \mu_{34}) - \frac{1}{2}\mu_4^*\right)u^2,$$

and

$$u^{13} := \left(\frac{1}{2}(\mu_{51} + \mu_{52})\right)u^1 + \left(\frac{1}{2}(\mu_{53} + \mu_{54})\right)u^2.$$

Similarly

$$\mathbf{QMQu} = \mathbf{QMw} = u^{q11}\mathbf{f}_1 + u^{q22}\mathbf{f}_2 + u^{q33}\mathbf{f}_3\,, \tag{8.12}$$

where

$$u^{q11} := \left(\mu_{11} - \mu_{12} - \frac{1}{2}\mu_{1q}^*\right)u^{q1} + \left(\mu_{13} - \mu_{14} - \frac{1}{2}\mu_{2q}^*\right)u^{q2} + \left(-\frac{1}{2}\mu_{12} - \frac{1}{2}\mu_{14} + \mu_{15} - \frac{1}{2}\mu_{3q}^*\right)u^{q3},$$

$$u^{q22} := \left(\mu_{31} - \mu_{32} - \frac{1}{2}\mu_{4q}^*\right)u^{q1} + \left(\mu_{33} - \mu_{34} - \frac{1}{2}\mu_{5q}^*\right)u^{q2} + \left(-\frac{1}{2}\mu_{32} - \frac{1}{2}\mu_{34} + \mu_{35} - \frac{1}{2}\mu_{6q}^*\right)u^{q3},$$

and

$$u^{q33} := (\mu_{51} - \mu_{52})\,u^{q1} + (\mu_{53} - \mu_{54})\,u^{q2} + \left(-\frac{1}{2}\mu_{52} - \frac{1}{2}\mu_{54}\right)u^{q3}.$$

Further

$$\mathbf{QCw} = \left(-2u^{q1} - \frac{1}{2}u^{q3}\right)\mathbf{f}_1 + \left(-2u^{q2} - \frac{1}{2}u^{q3}\right)\mathbf{f}_2 - (2u^{q3})\mathbf{f}_3. \tag{8.13}$$

Now, using the expressions from (8.7) and (8.9), the projected equation (8.4) on the hydrodynamic space can be written as follows:

$$u_t^1 = -u_a^1 - (\mu_1^* u^1 + \mu_2^* u^2) - (\mu_{1q}^* u^{q1} + \mu_{2q}^* u^{q2} + \mu_{3q}^* u^{3q}). \tag{8.14}$$

Similarly, with the use of expressions (8.8) and (8.10), the equation (8.5) becomes

$$u_t^2 = u_a^2 - (\mu_3^* u^1 + \mu_4^* u^2) - (\mu_{4q}^* u^{q1} + \mu_{5q}^* u^{q2} + \mu_{6q}^* u^{3q}). \tag{8.15}$$

For the projected equation (8.6) on the kinetic part, the expressions (8.11), (8.12) and (8.13) will be used.

Using projections $\mathbf{P}_j$, where $1 \leq j \leq 2$, and $\mathbf{Q}$, we can derive the corresponding expressions of the boundary equations on the hydrodynamic space as

$$u^1(0,t) := \gamma u^1 = \int_0^\infty \left(\beta_1^* u^1 + \beta_2^* u^2\right) + \int_0^\infty \left(\beta_{1q}^* u^{q1} + \beta_{2q}^* u^{q2} + \beta_{3q}^* u^{3q}\right),$$

and

$$u^3(0,t) := \gamma u^2 = \int_0^\infty \left(\beta_3^* u^1 + \beta_4^* u^2\right) + \int_0^\infty \left(\beta_{4q}^* u^{q1} + \beta_{5q}^* u^{q2} + \beta_{6q}^* u^{3q}\right).$$

The projected initial conditions on the hydrodynamic space are as follows:

$$u^1(a,0) =: \phi_1^*(a),$$

$$u^2(a,0) =: \phi_2^*(a).$$

On the kinetic space, the respective boundary and the initial conditions are

$$\mathbf{w}(0,t) = \int_0^\infty \mathbf{QB}(s)\left(\mathbf{P}_1\mathbf{u}(s,t) + \mathbf{P}_2\mathbf{u}(s,t)\right)\, ds + \int_0^\infty \mathbf{QB}(s)\mathbf{w}(s,t)\, ds,$$

and

$$\mathbf{w}(a,0) =: \mathbf{w}_0.$$

Bulk approximation

Following Section 6.4.1, first we want to find the expression of the term $\bar{\mathbf{w}}_1$. To do this, let

$$\bar{\mathbf{w}}_1 := \bar{u}_1^{q1}\mathbf{f}_1 + \bar{u}_1^{q2}\mathbf{f}_2 + \bar{u}_1^{q3}\mathbf{f}_3\,.$$

Now, inserting the series expansion $\mathbf{w} = \mathbf{w}_0 + \epsilon\mathbf{w}_1 + \cdots$, keeping the hydrodynamic part as it is, into the projected system (8.6) and comparing terms at the ϵ order level, we get

$$\mathbf{QC}\bar{\mathbf{w}}_1 = \mathbf{QM}\left(\bar{u}^1\mathbf{e}_1 + \bar{u}^2\mathbf{e}_2\right)$$

$$\text{or,}\quad \left(-2\bar{u}_1^{q1} - \frac{1}{2}\bar{u}_1^{q3}\right)\mathbf{f}_1 + \left(-2\bar{u}_1^{q2} - \frac{1}{2}\bar{u}_1^{q3}\right)\mathbf{f}_2 + \left(-2\bar{u}_1^{q3}\right)\mathbf{f}_3 = \bar{u}^{11}\mathbf{f}_1 + \bar{u}^{12}\mathbf{f}_2 + \bar{u}^{13}\mathbf{f}_3\,,$$

where

$$\bar{u}^{11} := \left(\frac{1}{2}\left(\mu_{11} + \mu_{12}\right) - \frac{1}{2}\mu_1^*\right)\bar{u}^1 + \left(\frac{1}{2}\left(\mu_{13} + \mu_{14}\right) - \frac{1}{2}\mu_2^*\right)\bar{u}^2,$$

$$\bar{u}^{12} := \left(\frac{1}{2}\left(\mu_{31} + \mu_{32}\right) - \frac{1}{2}\mu_3^*\right)\bar{u}^1 + \left(\frac{1}{2}\left(\mu_{33} + \mu_{34}\right) - \frac{1}{2}\mu_4^*\right)\bar{u}^2,$$

and

$$\bar{u}^{13} := \left(\frac{1}{2}\left(\mu_{51} + \mu_{52}\right)\right)\bar{u}^1 + \left(\frac{1}{2}\left(\mu_{53} + \mu_{54}\right)\right)\bar{u}^2.$$

Solving for $\bar{u}_1^{q1}$, $\bar{u}_1^{q2}$, $\bar{u}_1^{q3}$ we get

$$\bar{u}_1^{q1} = \frac{1}{8}\bar{u}^{13} - \frac{1}{2}\bar{u}^{11}, \quad \bar{u}_1^{q2} = \frac{1}{8}\bar{u}^{13} - \frac{1}{2}\bar{u}^{12}, \quad \text{and} \quad \bar{u}_1^{q3} = -\frac{1}{2}\bar{u}^{13}.$$

Therefore

$$\bar{\mathbf{w}}_1 = \left(\frac{1}{8}\bar{u}^{13} - \frac{1}{2}\bar{u}^{11}\right)\mathbf{f}_1 + \left(\frac{1}{8}\bar{u}^{13} - \frac{1}{2}\bar{u}^{12}\right)\mathbf{f}_2 + \left(-\frac{1}{2}\bar{u}^{13}\right)\mathbf{f}_3.$$

Similarly to the system (6.29), we have the following set of equations for $\bar{u}^j$, where $1 \leq j \leq 2$.

$$\begin{bmatrix} \bar{u}^1_t \\ \\ \bar{u}^2_t \end{bmatrix} = -\begin{bmatrix} \bar{u}^1_a \\ \\ \bar{u}^2_a \end{bmatrix} - \begin{bmatrix} \mu^*_1 & \mu^*_2 \\ \\ \mu^*_3 & \mu^*_4 \end{bmatrix} \begin{bmatrix} \bar{u}^1 \\ \\ \bar{u}^2 \end{bmatrix}$$

$$\begin{bmatrix} \bar{u}^1(0,t) \\ \\ \bar{u}^2(0,t) \end{bmatrix} = \int_0^\infty \begin{bmatrix} \beta^*_1 & \beta^*_2 \\ \\ \beta^*_3 & \beta^*_4 \end{bmatrix} \begin{bmatrix} \bar{u}^1 \\ \\ \bar{u}^2 \end{bmatrix}, \qquad \begin{bmatrix} \bar{u}^1(a,0) \\ \\ \bar{u}^2(a,0) \end{bmatrix} = \begin{bmatrix} \phi^*_1(a) \\ \\ \phi^*_2(a) \end{bmatrix}.$$

This system can be written in matrix form as follows:

$$\begin{aligned} \bar{\mathbf{u}}_t &= -\bar{\mathbf{u}}_a - \boldsymbol{\mu}\bar{\mathbf{u}}, \\ \bar{\mathbf{u}}(0,t) &= \int_0^\infty \boldsymbol{\beta}\bar{\mathbf{u}}, \\ \bar{\mathbf{u}}(a,0) &= \boldsymbol{\phi}^*(a). \end{aligned}$$

Let $\mathbf{L}(a)$ be the fundamental solution to

$$\frac{d}{da}\mathbf{L}(a) = -\boldsymbol{\mu}\mathbf{L}(a), \qquad \mathbf{L}(0) = \mathbf{I}.$$

Define

$$\mathbf{L}(a,b) := \mathbf{L}(a)\mathbf{L}^{-1}(b).$$

Then the solution of the above system can be represented as

$$\bar{\mathbf{u}}(a,t) = \begin{cases} \mathbf{L}(a)\bar{\mathbf{u}}(0,t-a), & a < t; \\ \mathbf{L}(a,a-t)\boldsymbol{\phi}(a-t), & a > t, \end{cases}$$

where

$$\boldsymbol{\psi}(t) := \bar{\mathbf{u}}(0,t) = \int_0^\infty \boldsymbol{\beta}(a)\mathbf{L}(a)\boldsymbol{\psi}(t-a)\,da + \int_t^\infty \boldsymbol{\beta}(a)\mathbf{L}(a)\mathbf{L}^{-1}(a-t)\boldsymbol{\phi}^*(a-t)\,da.$$

Initial Layer

For the initial layer, considering the ϵ^0 order kinetic equation (6.36), as done in Chapter 6, we have

$$\tilde{\mathbf{w}}_{0,\tau} = \mathbf{QC}\tilde{\mathbf{w}}_0\,,$$

and in component form,

$$\tilde{u}^{q1}_{0,\tau}\mathbf{f}_1 + \tilde{u}^{q2}_{0,\tau}\mathbf{f}_2 + \tilde{u}^{q3}_{0,\tau}\mathbf{f}_3 = \left(-2\tilde{u}^{q1}_0 - \frac{1}{2}\tilde{u}^{q3}_0\right)\mathbf{f}_1 + \left(-2\tilde{u}^{q2}_0 - \frac{1}{2}\tilde{u}^{q3}_0\right)\mathbf{f}_2 + (-2\tilde{\mathbf{u}}^{q3}_0)\mathbf{f}_3\,.$$

Comparing components and solving, we have

$$\tilde{u}^{q1}_0 = e^{-2\tau}w^{q1}_0, \quad \tilde{u}^{q2}_0 = e^{-2\tau}w^{q2}_0, \quad \text{and} \quad \tilde{u}^{q3}_0 = e^{-2\tau}w^{q3}_0,$$

where

$$\mathbf{Qu}(a,0) := w_0^{q1}\mathbf{f}_1 + w_0^{q2}\mathbf{f}_2 + w_0^{q3}\mathbf{f}_3.$$

Therefore

$$\tilde{\mathbf{w}}_0(a,\tau) = e^{-2\tau}w_0^{q1}(a)\mathbf{f}_1 + e^{-2\tau}w_0^{q2}(a)\mathbf{f}_2 + e^{-2\tau}w_0^{q3}(a)\mathbf{f}_3.$$

Boundary Layer

Similarly, following the ϵ^0 kinetic equation for the boundary layer we have

$$\hat{\mathbf{w}}_{0,\alpha} = \mathbf{QC}\hat{\mathbf{w}}_0\,,$$

which yields

$$\hat{u}_0^{q1} = e^{-2\alpha}\hat{w}_0^1(0,t), \quad \hat{u}_0^{q2} = e^{-2\alpha}\hat{w}_0^2(0,t), \quad \text{and} \quad \hat{u}_0^{q3} = e^{-2\alpha}\hat{w}_0^3(0,t).$$

where

$$\hat{\mathbf{w}}_0(0,t) := \left(\hat{w}_0^1(0,t), \hat{w}_0^2(0,t), \hat{w}_0^3(0,t)\right),$$

and

$$\hat{\mathbf{w}}_0(0,t) := \int_0^\infty \mathbf{B}(s)\sum_{i=1}^{2}\bar{u}^i(s,t)\mathbf{e}_i\,ds - \sum_{i=1}^{2}\bar{u}^i(0,t)\mathbf{e}_i.$$

Therefore

$$\hat{\mathbf{w}}_0(\alpha,t) = e^{-2\alpha}\hat{w}_0^1(0,t)\mathbf{f}_1 + e^{-2\alpha}\hat{w}_0^2(0,t)\mathbf{f}_2 + e^{-2\alpha}\hat{w}_0^3(0,t)\mathbf{f}_3.$$

Corner Layer

For the corner layer we have the following system of equations as found in Chapter 6:

$$\begin{aligned}
\breve{u}^1_{0,\tau} + \breve{u}^1_{0,\alpha} &= 0,\\
\breve{u}^1_0(\alpha,0) &= 0,\\
\breve{u}^1_0(0,\tau) &= \int_0^\infty (\mathbf{y}_1\cdot\mathbf{B}\tilde{\mathbf{w}}_0)\\
&= e^{-2\tau}\int_0^\infty\left(\beta^*_{1q}w_0^{q1} + \beta^*_{2q}w_0^{q2} + \beta^*_{3q}w_0^{q3}\right),
\end{aligned}$$

which upon solving, gives

$$\breve{u}^1_0(\alpha,\tau) = \begin{cases} 0, & \tau<\alpha;\\ e^{-2(\tau-\alpha)}\displaystyle\int_0^\infty\left(\beta^*_{1q}w_0^{q1} + \beta^*_{2q}w_0^{q2} + \beta^*_{3q}w_0^{q3}\right), & \tau>\alpha.\end{cases}$$

Similarly, for $\breve{u}^2_0$, the solution can be represented as

$$\breve{u}^2_0(\alpha,\tau) = \begin{cases} 0, & \tau<\alpha;\\ e^{-2(\tau-\alpha)}\displaystyle\int_0^\infty\left(\beta^*_{4q}w_0^{q1} + \beta^*_{5q}w_0^{q2} + \beta^*_{6q}w_0^{q3}\right), & \tau>\alpha.\end{cases}$$

For the kinetic part we have the following system of equations:

$$\begin{aligned}\breve{\mathbf{w}}_{0,\tau} &= -\breve{\mathbf{w}}_{0,\alpha} + \mathbf{QC}\breve{\mathbf{w}}_0,\\ \breve{\mathbf{w}}_0(\alpha,0) &= \mathbf{0},\\ \breve{\mathbf{w}}_0(0,\tau) &= \int_0^\infty \mathbf{B}\tilde{\mathbf{w}}_0 - \tilde{\mathbf{w}}_0(0,\tau) - \sum_{i=1}^{2} \breve{u}_0^i(0,\tau)\mathbf{e}_i\,,\end{aligned}$$

which upon solving, yields

$$\breve{\mathbf{w}}_0(\alpha,\tau) = \begin{cases}\mathbf{0}, & \tau < \alpha;\\ e^{-2\alpha}\breve{\mathbf{w}}_0(0,\tau-\alpha), & \tau > \alpha.\end{cases}$$

where

$$\breve{\mathbf{w}}_0(0,\tau) = \breve{w}_0^1(0,\tau)\mathbf{f}_1 + \breve{w}_0^1(0,\tau)\mathbf{f}_1 + \breve{w}_0^1(0,\tau)\mathbf{f}_1,$$

with

$$\breve{w}_0^1(0,\tau) = e^{-2\tau}\left(w_0^{q1}(a) - w_0^{q1}(0)\right) - \frac{1}{2}e^{-2\tau}\int_0^\infty \left(\beta_{1q}^* w_0^{q1} + \beta_{2q}^* w_0^{q2} + \beta_{3q}^* w_0^{q3}\right),$$

$$\breve{w}_0^2(0,\tau) = e^{-2\tau}\left(w_0^{q2}(a) - w_0^{q2}(0)\right) - \frac{1}{2}e^{-2\tau}\int_0^\infty \left(\beta_{4q}^* w_0^{q1} + \beta_{5q}^* w_0^{q2} + \beta_{6q}^* w_0^{q3}\right),$$

and

$$\breve{w}_0^3(0,\tau) = e^{-2\tau}\left(w_0^{q3}(a) - w_0^{q3}(0)\right).$$

Error estimates

Let us work out the error estimates of the layer terms which subsequently will help us to use the final form (7.57) of the error as done in Chapter 7.

1.

$$\begin{aligned}\|\tilde{\mathbf{w}}_0(\cdot,\tau)\|_{\mathbf{X}} &\le \int_0^t \left\|e^{-2\tau}w_0^{q1}(s)\mathbf{f}_1 + e^{-2\tau}w_0^{q2}(s)\mathbf{f}_2 + e^{-2\tau}w_0^{q3}(s)\mathbf{f}_3\right\| ds\\ &\le e^{-2\tau}\int_0^t \|\mathbf{w}_0(s)\|\, ds.\end{aligned}$$

2.

$$\begin{aligned}\|\hat{\mathbf{w}}_0(\cdot,t)\|_{\mathbf{X}} &\le \epsilon\int_0^\tau e^{-2\alpha}\left\|\mathbf{B}(s)\sum_{i=1}^{2}\bar{u}^i(s,t)\mathbf{e}_i\, ds - \sum_{i=1}^{2}\bar{u}^i(0,t)\mathbf{e}_i\right\| d\alpha\\ &\le \epsilon\, G\int_0^\tau e^{-2\alpha}\, d\alpha = \frac{1}{2}\epsilon\, G\left(1 - e^{-2\tau}\right),\end{aligned}$$

where, G is a positive constant.

3.

$$
\begin{aligned}
\left\|\breve{u}_0^1(\cdot,\tau)\right\|_{L^1(\mathbb{R}_+)} &\leq \int_0^t \left\|e^{-2(\tau-\alpha)}\int_0^\infty \left(\beta_{1q}^* w_0^{q1} + \beta_{2q}^* w_0^{q2} + \beta_{3q}^* w_0^{q3}\right)\right\| da \\
\leq \epsilon \int_0^\tau \left(\int_0^\infty \left(\left\|\beta_{1q}^*(s)\right\|\left\|w_0^{q1}(s)\right\| + \left\|\beta_{2q}^*(s)\right\|\left\|w_0^{q2}(s)\right\| + \left\|\beta_{3q}^*(s)\right\|\left\|w_0^{q3}(s)\right\|\right) ds\right) e^{-2(\tau-\alpha)}\, d\alpha \\
&\leq \epsilon D\left(\int_0^\tau e^{-2(\tau-\alpha)}\, d\alpha\right)\|\mathbf{w}_0\| = \frac{\epsilon}{2} D\left(1-e^{-2\tau}\right)\|\mathbf{w}_0\|.
\end{aligned}
$$

where, D is a positive constant.

4. Similarly

$$
\left\|\breve{u}_0^2(\cdot,\tau)\right\|_{L^1(\mathbb{R}_+)} \leq \frac{\epsilon}{2} E\left(1-e^{-2\tau}\right)\|\mathbf{w}_0\|.
$$

where, E is a positive constant.

5.

$$
\begin{aligned}
\|\breve{\mathbf{w}}_0(\cdot,\tau)\|_{\mathbf{X}} &\leq \int_0^t \left\|e^{-2\alpha}\, \breve{\mathbf{w}}_0(0,\tau-\alpha)\right\| da \\
&\leq \epsilon \int_0^\tau \left\|e^{-2\alpha}\, \breve{\mathbf{w}}_0(0,\tau-\alpha)\right\| d\alpha \\
&\leq \epsilon F\left(\int_0^\tau e^{-2\alpha}. e^{-2(\tau-\alpha)}\, d\alpha\right)\|\mathbf{w}_0\| \\
&= \epsilon F\left(\int_0^\tau e^{-2\tau}\, d\alpha\right)\|\mathbf{w}_0\| \\
&\leq \epsilon F\, \|\mathbf{w}_0\|\left(\max_{z\in\mathbf{R}_+} z e^{-2z}\right) \leq \frac{\epsilon F}{2e}\|\mathbf{w}_0\|.
\end{aligned}
$$

where, F is a positive constant.

Finally in order to apply (7.57) for the error estimates, we write $\mathbf{u}_\epsilon - \sum_{j=1}^2 \bar{u}^{n_j}\mathbf{e}_j - \tilde{\mathbf{w}}_0$ into components as

$$
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} - \bar{u}^1 \begin{bmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \bar{u}^2 \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \\ 0 \end{bmatrix} - e^{-2\tau} w_0^{q1} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - e^{-2\tau} w_0^{q2} \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - e^{-2\tau} w_0^{q3} \begin{bmatrix} 0 \\ -1/2 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}.
$$

Now, using (7.57), we have the following set of error estimates:

$$\begin{aligned}
\left|u_1 - \frac{1}{2}\bar{u}_1 - e^{-2\tau} w_0^{q1}\right| &\leq O(\epsilon), \\
\left|u_2 - \frac{1}{2}\bar{u}_1 + e^{-2\tau} w_0^{q1} + \frac{1}{2} e^{-2\tau} w_0^{q3}\right| &\leq O(\epsilon), \\
\left|u_3 - \frac{1}{2}\bar{u}_2 - e^{-2\tau} w_0^{q2}\right| &\leq O(\epsilon), \\
\left|u_4 - \frac{1}{2}\bar{u}_2 + e^{-2\tau} w_0^{q2} + \frac{1}{2} e^{-2\tau} w_0^{q3}\right| &\leq O(\epsilon), \\
\left|u_5 - e^{-2\tau} w_0^{q3}\right| &\leq O(\epsilon).
\end{aligned}$$

This completes our error analysis with the illustrative example.

CHAPTER 9

Conclusion

The main aim of this work was to extend the exiting results on multidimensional singularly perturbed age-structured population model and this has been achieved in various ways.

After reviewing the necessary mathematical framework in Chapter 2, we gave an overview of the deterministic population models in Chapter 3. For long time behaviour of the classical McKendrick-Von Foerster model, we referred to [65]. We included an introduction to the multiregional demography which was an essential part of our perturbed model.

In Chapter 4, we introduced our singularly perturbed population model. We discussed in details the spectral properties of our irreducible migration matrix. Following [28] and using Hille–Yosida theorem, we proved the well-posedness of the perturbed problem.

We showed how application of classical techniques of asymptotic analysis and, in particular, of the Chapman–Enskog procedure, can yield the aggregation of variables in more systematic way and deliver a simpler approximation formula than the ad hoc method of [5], [16]. We explicitly calculated layer correction terms namely, initial, boundary and corner layers. The solution of the perturbed problem and all terms of the asymptotic expansion must be strongly differentiable with respect to t and belong to the domain of the generator which, as mentioned, equals $\{\mathbf{u} \in W_1^1(\mathbb{R}_+, \mathbb{R}^n) : \mathbf{u}(0) = \mathfrak{B}\mathbf{u}\}$. This is not always easy to achieve in practice. We deferred the work with mild solution to Chapter 7, where we did rigorous error analysis using the integral formulation.

In literature, most of the works on multipatch population models have been done with the assumption of irreducibility of the migration matrix. The advantage with this assumptions is the existence of unique (up to scalar multiple) positive eigenvector corresponding to the dominant eigenvalue of the migration matrix. But in practice, it is quite reasonable to consider reducible migration structure. Chapter 6 dealt with formal asymptotic calculations with reducible migration matrix. We also note that typically in linear models the matrix $\mathbf{M}$ is diagonal which reflects the fact that death is an intra-patch phenomenon. However, linear models with general matrix $\mathbf{M}$, with off-diagonal positive entries, not only reflect the mortality but also migration occurring on a slower time scale. On the other hand, births in a particular patch can easily depend on the population density in other patches (e.g. females could move to a safer patch just to give birth) and thus considering full matrix $\mathbf{B}$ is perfectly reasonable. This made our analysis more general than that in [5, 16, 36], where only diagonal matrices $\mathbf{M}$ and $\mathbf{B}$ are considered.

Aggregation for our singularly perturbed model has been studied quite extensively in [5, 6, 16] and in [36]. The results of the former are similar to our final error estimates described in Theorem 7.3. However, to get estimates valid up to $t = 0$, the authors used the solution of the full problem restricted to the subspace complementary to right Perron vector $\mathbf{k}(a)$ so that in practice finding the approximation presents difficulties comparable to solving the original problem. In our approach the asymptotic analysis provides the necessary correction in a systematic way as an explicit solution of a linear autonomous system of differential equations so that using this approximation is computationally viable. Moreover, there are some gaps in the argument of [5], one of them being that the projected boundary conditions in [5] are correct only if $\mathbf{k}$ is independent of age. Moreover, classical solutions of the perturbed system and the aggregated system exist only with initial data satisfying nonlocal compatibility conditions and, unless additional necessary constraints are imposed on the initial data, both problems should be considered in their mild form, as discussed in Chapter 7. This approach, though computationally more involved, allows to remove several technical assumptions imposed in [5].

The essential difference between the reducible and irreducible structure of our migration matrix is that the reducible structure yields multidimensional null space of the migration matrix and the basis vector corresponding to the adjoint null space is age dependent, which contributes a substantial changes in asymptotic analysis in Chapter 6 and Chapter 7 compare to Chapter 5. Physical significance of multidimensional hydrodynamic space occurred in reducible case is still not well understood.

It may seem strange that the constructed elaborate hierarchy of layers is only used in intermediate steps of the analysis but, apart from the initial layer, does not appear in the final approximation. In our opinion this is one of the advantage of the method which, while providing all potentially significant terms of the expansion, allows for discarding all these which are not absolutely necessary. In our case the absence of the boundary and the corner layers in the final approximation is due to the choice of the state space $L_1(\mathbb{R}_+, \mathbb{R}^n)$. The norm of $L_1(\mathbb{R}_+, \mathbb{R}^n)$ averages the terms of layers which decay exponentially fast in a/ϵ and thus makes them negligible. It can be shown [10] that these terms would be essential to get uniform approximation if the $L_\infty(\mathbb{R}_+, \mathbb{R}^n)$ norm was used.

From realistic point of view this model still may be oversimplified. Various generalizations are possible. For instance, instead of linear model one can consider nonlinearity and analyze the situation. The migration matrix can be considered irreducible for up to certain age limit and after that it can be reduced into groups which is also a quite natural assumption.

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