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Some aspects of strong gravity effects on the electromagnetic field of a Radio pulsar magnetosphere: solving the Maxwell's equations

by

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Abstract

The general relativistic (GR) effects of a neutron star play a substantial role on the physics at the stellar surface. These neutron stars also host a very strong magnetic field and spin with periods of a few seconds to as high as milliseconds. In order to account for the motion of charged particles in the magnetosphere immediately outside the stellar surface, it is essential to include the GR effects in the Maxwell's equations. To account for the frame dragging effects due to the stellar spin, we have, in this dissertation, considered a 3+1 decomposition of the spacetime and applied them to find the solutions to Maxwell's equations of an isolated neutron star in a vacuum, for different cases. In order to derive our solutions we made use of the vector spherical harmonics in a curved spacetime. We first considered an aligned dipole magnetic field from which we formed a general formalism for the magnetic and electric fields for higher orders. We then considered an orthogonal dipole magnetic field for which we solved only for the non-rotating case. In a realistic scenario for a radio pulsar, the radio beams which originate from the pole caps of the magnetic field, have a finite angle with the spin axis and hence it is necessary to find a model for an oblique rotator. This study will be helpful in the future for the understanding of the charged particle interaction at the pulsar pole caps and hence for the emission mechanism of a radio pulsar.

Declaration

I, Kathleen Sellick (214504945), declare that this thesis titled, 'Some aspects of strong gravity effects on the electromagnetic field of a Radio pulsar magnetosphere: solving the Maxwell's equations' and the work presented in it are my own. I confirm that:

- The research reported in this dissertation, except where otherwise indicated, is my original research.
- This dissertation has not been submitted for any degree or examination at any other university.
- This dissertation does not contain other persons' data, pictures, graphs or other information, unless specifically acknowledged as being sourced from other persons.
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Finally, I wish to thank my mother for her endless support and encouragement.

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Physical Constants

Speed of Light	c	$=$	$2.997\ 924\ 58 \times 10^{8}$ ms^{-1}
Gravitational constant	G	$=$	$6.674\ 08 \times 10^{-11}$ kg^{-1} m^{3} s^{-2}
Permittivity of free space	ε_0	$=$	$8.854\ 187\ 817 \times 10^{-12}$ kg^{-1} m^{-3} s^{4} A^{2}
Permeability of free space	μ_0	$=$	$1.256\ 637\ 061 \times 10^{-6}$ kg m s^{-2} A^{-2}
Solar mass	$M_\odot$	$=$	2×10^{30} kg
kiloparsec	kpc	$=$	3.0857×10^{19} m

Symbols

M	mass of neutron star
R	radius of neutron star
R_s	Schwarzschild radius
r_L	light cylinder radius
a	spin
J	angular momentum
I	moment of inertia
ω	angular frequency
Ω	rotation rate
$\tilde{\omega}$	relative rotation

Chapter 1

Introduction

The theory of the existence of radio pulsars first came about in the early 1930s[1, 2] long before the first observation was made in 1967 by Jocelyn Bell and Antony Hewish[3]. Over 90% of neutron stars are observed as radio pulsars, which are isolated neutron stars emitting beams of electromagnetic radiation in the radio frequency from the magnetic poles. The remaining observed neutron stars are observed as X-ray pulsars. These neutron stars occur in binary systems, with the partner usually being a main sequence star. The neutron star in the binary system accretes matter from its partner star and emits X-rays (stronger than it emits radio waves) and hence we observe them as X-ray pulsars. The pulses from both the radio and X-ray pulsars have an optical depth of only a few kpc ($3-5$ kpc) and hence we can observe only a small fraction of them found in our galaxy. Neutron stars are extremely dense objects with masses ranging between $1.4-2.16$ $M_{\odot}$ [4] with an average estimated radius of $10-15$ km thus giving a mean density of around $10^{15} g.cm^{-3}$. The stars have rotation periods of a few seconds to as high as milliseconds which primarily occur due to conservation of angular momentum during core collapse. Isolated radio pulsars are often younger stars with slower spins than the binary system pulsars, with their spin being thought to be close to their birth spin[5, 6]. These young pulsars have a high spin-down rate due to their electromagnetic wave radiation and outflow of relativistic particles which decreases their rotational kinetic energy. Similarly, the stars have superstrong magnetic fields of order 10^{12} G or more which are thought to occur from the conservation of magnetic flux[7]. Neutron stars with magnetic field strengths of $10^{13}-10^{15}$ G are known as magnetars[8]. The compactness of the star also gives rise to a strong gravitational field close to the star surface. Because of this, it is important, when studying the physics near the stellar surface, to take general relativistic (GR) effects into consideration. It is not fully understood how the radio beams from the pulsars are created, however, it is thought that the superstrong magnetic field of the neutron star funnels jets of particles out along the two magnetic poles. These accelerated particles then create the intense beams which rotate with the magnetic field around the rotation axis of the star, provided the magnetic poles do not coincide with the rotational axis.

There are many theories which exist on the different possible emission mechanisms of the radio pulsars. A comprehensive review on the emission mechanisms can be found in Ginzburg and Zheleznyakov (1975)[9] and Melrose and Yuen (2016)[10]. A widely accepted theory is that primary particles near the neutron star surface are accelerated along the curved magnetic field lines (e.g. Goldreich and Julian (1969)[11]) which then emit gamma rays. These photons propagate in the curved magnetic field and reach the particle generation threshold where they create electron-positron pairs. Secondary particles then emit synchrotron photons which are accelerated and emit new gamma rays and so the process is repeated[12–14]. This model was then improved upon by Ruderman and Sutherland (1975)[15] who introduced discharge and drifting subpulses to the model, which requires that the polar caps be a source of relativistic particles. There also exists alternative models with an electrosphere[16] as apposed to a polar-cap model, and an ion proton plasma[17] instead of a pair plasma. The reason for such variation in opinion of the emission mechanisms of pulsars is that we are only able to observe the period P and its rate of increase $\dot{P}$. Things like the source region of the radio emission, plasma parameters and properties of the radio-emitting particles thus cannot be clearly identified from observations. Hence, no agreement has been reached on the most plausible mechanism. It is also required that the radio pulsar emission have a 'coherent' emission mechanism which again brings about difference in opinion on how this occurs.

When a radio pulsar is observed, it is due to the magnetic axis being at an angle with the rotation axis. It is highly unlikely that one finds a neutron star whose magnetic axis is either aligned or perpendicular with its rotation axis. As a matter of fact, a magnetic axis aligned with the rotation axis will not be observed as it will not create any pulse. It is, therefore, useful to consider an oblique rotator model when solving for the electromagnetic equations on the surface of a neutron star. The first general solution in a flat spacetime for an oblique rotator of a magnetised star in a vacuum was found by Deutsch (1955)[18]. Deutsch found that stars, in this model, with high rotation periods and strong magnetic fields induce electric fields so strong that they can accelerate particles to relativistic speeds. He thus hypothesised that this could be the source of cosmic rays. This theory is still valid in today's research. Pacini (1967, 1968)[19, 20] then proposed the first model for an electromagnetically active neutron star and applied the vacuum dipole model to the neutron star contemporary to the observations of the radio pulsars[3]. Goldreich and Julian (1969)[11] investigated the case of the aligned rotating dipole in a flat spacetime and identified some key characteristics regarding the properties of the region surrounding rotating neutron stars including the Goldreich-Julian charge density ρ_{GJ}. They argued that it is impossible to have a practical scenario of a neutron star in a vacuum and hence it needs to be surrounded by a plasma of electrons and positrons. The arguments in Goldreich and Julian (1969), however, were not generally accepted to correspond directly to the oblique rotator where the electromagnetic field around the neutron star changed considerably. This was studied and discussed in detail by Mestel (1971)[21].

We know that the physics of the magnetospheres of neutron stars are sufficiently influenced by GR effects at the stellar surface and this is particularly true at the polar caps. An exact analytical solution for the static magnetic dipole in the Schwarzschild spacetime is given by Ginzburg and Ozernoy (1964)[22]. Muslimov and Tsygan (1992)[23] confirmed the importance of the GR effects at the polar caps as they found some interesting effects on the electric field of an isolated, rotating, magnetic neutron star due to the GR effects, particularly at the polar caps. This can be further seen by Sakai and Shibata (2013)[24] who studied the motion of charged particles in a pulsar magnetosphere at the polar caps and found the GR effects to be particularly significant. The GR effects also influence frame-dragging effects on the star and the generation of particles in the vicinity of the radio pulsars, especially in the free particle escape models.[7].

While studying the electromagnetic field of a neutron star under GR considerations, one cannot use the standard Maxwell's equations in a flat spacetime. The Maxwell's equations have to thus be altered according to the curved spacetime with the relevant spacetime metric and coordinate system. Rezzolla, Ahmedov and Miller (2001)[25] gave analytical solutions of Maxwell's equations in a curved spacetime for both the internal and external fields of a slowly rotating, isolated neutron star in a vacuum with dipole magnetic field not aligned with the rotation axis.

There are many different approaches which can be used when solving equations in a curved spacetime. One very useful approach which will be used in this paper is the $3+1$ splitting of spacetime, developed by Ehlers (1961)[26] and Ellis (1971)[27], where the spacetime is foliated such that the spacetime metric is split into three parts space and one part time. This will be explained in detail in chapter (2). The electrodynamic equations of curved spacetime were first expressed using a $3+1$ formalism by Thorne and MacDonald (1982)[28] and has been greatly studied and developed over the years. For example, we follow the methods outlined by Alcubierre (2006)[29] and Komissarov (2011)[30] for our derivation of the $3+1$ equations. See also Yu (2007)[31] who uses the $3+1$ approach to describe general relativistic force-free electrodynamics of a stationary, axisymmetric black hole magnetosphere, re-deriving the Grad-Shafranov equations and defining the Maxwell equations in the same way as seen in Petri (2013)[32].

In several articles Petri has explored the physics of pulsars and their different features, such as their electromagnetic field solutions, striped winds and emission mechanisms. Petri (2012)[33] attempts to find accurate solutions to the almost stationary force-free pulsar magnetosphere and determine its link to the pulsar striped wind for different spin periods and arbitrary inclination angle. He solves for the time-dependent force-free Maxwell's equations using a vector spherical harmonic (VSH) expansion of the electromagnetic field in spherical coordinates. The general relativistic effects are found to be of importance particularly at the polar caps. Petri (2013)[32] again uses a VSH expansion of the fields along with a $3+1$ split of the spacetime metric to solve for the fields of an aligned and orthogonal rotator and find a general formalism of the fields in vacuum space. His hope was that his analysis could be used to test numerical codes of magnetospheres. He extends this work in Petri (2014)[34] for the time-dependent Maxwell's equations. The effects of multipoles are explored in Petri (2015, 2017)[35, 36] and the pulsar emissions are discusses in depth in Petri (2018)[37].

In this study we began by looking into the radio pulsar magnetosphere, with the intent of finding a plausible model for the emission mechanism of the pulse beams of radio pulsars. It became apparent that this topic was too broad to tackle without studying, more thoroughly, the physical foundations behind the neutron star. After further study, it was decided that the electrodynamics of the magnetosphere needed to be better understood and solving for Maxwell's equations in a strong magnetic field would be a good foundation. We follow closely the work of Petri (2013)[32], who uses the $3+1$ formalism and a vector spherical harmonics expansion of the electromagnetic field to solve Maxwell's equations for both an aligned and orthogonal dipole, with the help of other similar works, such as Rezzolla (2001)[25]. In chapter (2), we lay the foundations of what is needed to solve for the electromagnetic field equations in a curved spacetime using a $3+1$ splitting. The relevant spacetime metric and coordinate system is set out and the Maxwell's equations are adjusted according to the the $3+1$ formalism. We also give the force-free conditions and explain the VSH in a curved spacetime with some useful identities which will be used to aid in solving the Maxwell's equations. Chapter (3) sets out to solve for the dipole magnetic field of an aligned non-rotating neutron star in a vacuum where we also confirm whether the far field limit matches the aligned flat spacetime dipole magnetic field. In chapter (4) we introduce rotation to the system and solve for the electric field of the aligned rotating dipole. We solve for the electric field in two steps, first by neglecting the effects of frame-dragging to simplify the equations and once that electric field solution is obtained, we introduce frame-dragging to the equations and find the final solution for the electric field. We go on, in chapter (5), to derive a general formalism of the fields for the aligned rotating dipole such that the electromagnetic fields can be determined up to any order in the spin for ones desired degree of accuracy. Chapter (6) introduces the extreme case of the orthogonal dipole magnetic field in a vacuum where we solve only for the magnetic field without rotation and again confirm whether the far field limit matches the orthogonal flat spacetime dipole magnetic field. In chapter (7) we review the results and provide a summary of the work.

Chapter 2

Electromagnetic Equations in Strong Gravity

In this chapter we shall lay the foundations of what is needed to solve for the electromagnetic field equations for our system. We will first explain the 3+1 formalism used in this work. We will then show Maxwell's equations and how they change in the curved spacetime with the 3+1 formalism followed by the force-free conditions. We will then briefly derive and explain the VSH in a curved spacetime which will aid in solving the field equations.

2.1 The $3+1$ Decomposition

In order to predict the evolution in time of a system, one must formulate an initial value problem, or Cauchy problem, with adequate boundary conditions. The problem, when working with the Einstein Field Equations (EFE), is that they are covariant and written in such a way that there is no clear distinction between space and time. It is, therefore, difficult to predict the evolution of the gravitational field in time. Hence, in order to rewrite the EFE as a Cauchy problem, we must first split the space and time components so that they are no longer dependent on each other and can be analysed separately. To do this, we use a 3+1 formalism[38] where we split the spacetime metric into three parts space and one part time. The idea behind this is that the spacetime is foliated, or 'sliced' into three-dimensional pieces, so that each 'slice' is spacelike, assuming that the spacetime is globally hyperbolic (i.e. the spacetime has no closed timelike curves and, therefore, does not allow time travel to the past). We can then parametrise the foliation with the parameter t which can be considered as the universal time. These spacelike hypersurfaces are then regarded as "absolute space" at different instances of time t. Now consider a specific foliation, such as that in figure (2.2). Consider two hypersurfaces adjacent to each other, Σ_1 and Σ_2 or rather Σ_t and Σ_{t+dt}, where we now wish to determine the geometry of the region between these two hypersurfaces.

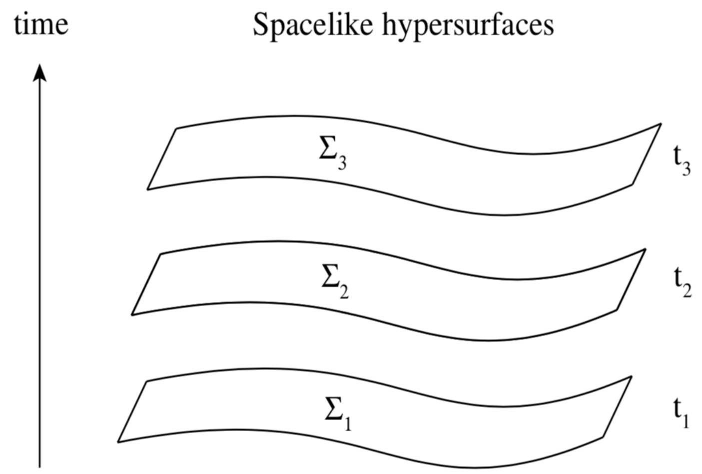

FIGURE 2.1: The spacetime foliation for a 3+1 formalism. (Alcubierre, 2006)[29]

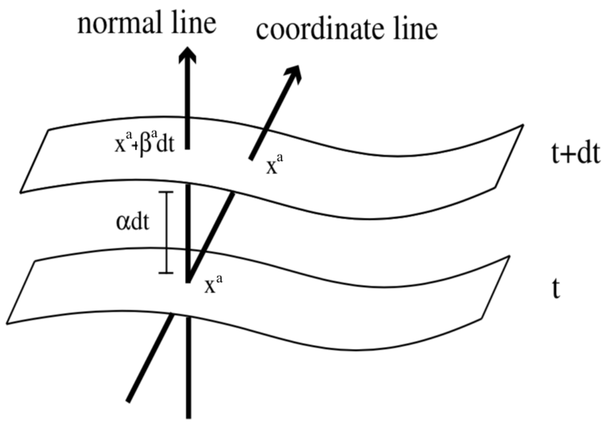

FIGURE 2.2: Two adjacent hypersurfaces showing the definitions of the lapse function and shift vector. (Alcubierre, 2006)[29]

To do this, we will need three new variables. In our notation, we use latin letters to represent the vector indices. Letters running from a to h are used to represent the 3-vectors in absolute space, which run in $\{1, 2, 3\}$, and letters running from i to z are used to represent 4-vectors and tensors, which run in $\{0, 1, 2, 3\}$. Now, the first is the three-dimensional space metric γ_{ab} which measures the proper distances within the hypersurface such that

$$dl^2 = \gamma_{ab} dx^a dx^b. \tag{2.1}$$

Second is the lapse of the proper time between both hypersurfaces, which is measured from the observer moving along the normal to the hypersurfaces (Eulerian observer), given by

$$d\tau = \alpha(t, x^a) dt \tag{2.2}$$

where τ is the proper time and α is known as the lapse function.

The third variable we need to define is the relative velocity β^a between the normal line (Eulerian observers) and the coordinate line, which corresponds to constant spatial coordinates. We have, for Eulerian observers

$$x^a_{t+dt} = x^a_t + \beta^a(t, x^b) c dt. \tag{2.3}$$

The relative velocity β^a is known as the shift vector.

It is clear that both the way in which the spacetime is foliated and the way in which the spatial coordinates propagate between the hypersurfaces are not unique. Hence both α and β^a can be freely specified and these functions govern our choice of coordinate system and are known as gauge functions. We choose to use the metric signature $(+, -, -, -)$. Taking $\beta_a = \gamma_{ab}\beta^b$ and $\beta^2 = \beta^a\beta_a$, we determine the spacetime metric to be

$$\begin{aligned}
ds^2 &= g_{ij}dx^i dx^j = c^2 d\tau^2 - dl^2 \\
&= \alpha^2 c^2 dt^2 - \gamma_{ab}dx^a dx^b \\
&= \alpha^2 c^2 dt^2 - \gamma_{ab}\left(dx^a + \beta^a cdt\right)\left(dx^b + \beta^b cdt\right) \\
&= \alpha^2 c^2 dt^2 - \gamma_{ab}\left[dx^a dx^b + \beta^a cdtdx^b + \beta^b cdtdx^a \beta^a \beta^b c^2 dt^2\right] \\
&= \left(\alpha^2 - \beta^2\right) c^2 dt^2 - 2\beta_a cdtdx^a - \gamma_{ab}dx^a dx^b
\end{aligned} \tag{2.4}$$

where $x^i = (ct, x^a)$. We can see that g_{ij} can be written in matrix form as

$$g_{ij} = \begin{pmatrix} \alpha^2 - \beta^2 & -\beta_b \\ -\beta_a & -\gamma_{ab} \end{pmatrix} = \begin{pmatrix} \alpha^2 - \beta^2 & -\beta_1 & -\beta_2 & -\beta_3 \\ -\beta_1 & -\gamma_{11} & -\gamma_{12} & -\gamma_{13} \\ -\beta_2 & -\gamma_{21} & -\gamma_{22} & -\gamma_{23} \\ -\beta_3 & -\gamma_{31} & -\gamma_{32} & -\gamma_{33} \end{pmatrix} \tag{2.5}$$

with its inverse being

$$g^{ij} = \begin{pmatrix} 1/\alpha^2 & -\beta^b/\alpha^2 \\ -\beta^a/\alpha^2 & -\gamma^{ab} + \beta^a\beta^b/\alpha^2 \end{pmatrix} \tag{2.6}$$

$$= \begin{pmatrix} 1/\alpha^2 & -\beta^1/\alpha^2 & -\beta^2/\alpha^2 & -\beta^3/\alpha^2 \\ -\beta^1/\alpha^2 & -\gamma^{11} + \beta^1\beta^1/\alpha^2 & -\gamma^{12} + \beta^1\beta^2/\alpha^2 & -\gamma^{13} + \beta^1\beta^3/\alpha^2 \\ -\beta^2/\alpha^2 & -\gamma^{21} + \beta^2\beta^1/\alpha^2 & -\gamma^{22} + \beta^2\beta^2/\alpha^2 & -\gamma^{23} + \beta^2\beta^3/\alpha^2 \\ -\beta^3/\alpha^2 & -\gamma^{31} + \beta^3\beta^1/\alpha^2 & -\gamma^{32} + \beta^3\beta^2/\alpha^2 & -\gamma^{33} + \beta^3\beta^3/\alpha^2 \end{pmatrix} \tag{2.7}$$

.

We write γ_{ab} in matrix form as

$$\gamma_{ab} = \begin{pmatrix} \alpha^{-2} & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \tag{2.8}$$

.

We then have a relationship between the determinants of the spacetime metric, g, and spatial metric, γ, as

$$\sqrt{-g} = \alpha\sqrt{\gamma}. \tag{2.9}$$

It is also easy to show that the components of the normal vector to the spatial hypersurfaces, or the 4-velocity of the Fiducial Observer (FIDO), n^i are

$$n^i = \frac{dx^i}{d\tau} = \frac{c}{\alpha}(1, -\vec{\beta}) \tag{2.10a}$$

$$n_i = (\alpha c, \vec{0}). \tag{2.10b}$$

Following the Schwarzschild solution, we define α, for a slowly rotating neutron star, as

$$\alpha = \sqrt{1 - \frac{R_s}{r}} \tag{2.11}$$

where $R_s = 2GM/c^2$ is the Schwarzschild radius and G is the gravitational constant and M the mass of the star. The shift vector $\vec{\beta}$ is defined by

$$c\vec{\beta} = -\omega r \sin\theta \hat{\phi}. \tag{2.12}$$

Note: for our calculations we use a spherical coordinate system (r, θ, ϕ) with the orthonormal spatial basis $(\hat{r}, \hat{\theta}, \hat{\phi})$. Here the angular frequency ω is defined as

$$\omega = \frac{R_s ac}{r^3} \tag{2.13}$$

where the spin, a, is

$$\frac{a}{R_s} = \frac{2}{5}\frac{R}{R_s}\frac{R}{r_L} \tag{2.14}$$

and R is the radius of the neutron star. The light cylinder radius r_L is defined by

$$r_L = \frac{c}{\Omega} \tag{2.15}$$

and Ω is the rotation rate of the neutron star. The light cylinder is the distance from the pulsar where the linear velocity of the rotating pulse beam reaches the speed of light. We can relate the spin a to the angular momentum J by

$$J = Mac. \tag{2.16}$$

We can see that a has units of length and should satisfy $a \leq R_s/2$. We can also write the angular momentum in terms of the moment of inertia I as

$$J = I\Omega \tag{2.17}$$

where, when we have the special case of a homogeneous and uniform neutron star with spherical symmetry, the moment of inertia is

$$I = \frac{2}{5}MR^2. \tag{2.18}$$

It is useful to introduce the relative rotation of the neutron star as

$$\tilde{\omega} = \Omega - \omega. \tag{2.19}$$

2.2 Maxwell's Equations in the 3+1 formalism

We know the standard Maxwell's equations in a flat spacetime to be

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.20a}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} \tag{2.20b}$$

$$\vec{\nabla} \cdot \vec{D} = \rho \tag{2.20c}$$

$$\vec{\nabla} \times \vec{H} = \frac{\partial}{\partial t}\vec{D} + \vec{J}. \tag{2.20d}$$

where ρ is the electric charge density and $\vec{J}$ the electric current density. These equations are no longer sufficient in a curved spacetime and we now use the electromagnetic field tensors to describe the fields and subsequently make corrections to the Maxwell equations (Landau and Lifshitz, 1987)[39].

The contravariant form of the electromagnetic field tensor, known as the Maxwell tensor, is expressed as

$$\mathcal{F}^{ij} = \frac{1}{\sqrt{-g}} \begin{pmatrix} 0 & -\sqrt{\gamma}D^1/\varepsilon_0 c & -\sqrt{\gamma}D^2/\varepsilon_0 c & -\sqrt{\gamma}D^3/\varepsilon_0 c \\ \sqrt{\gamma}D^1/\varepsilon_0 c & 0 & -\mu_0 H_3 & \mu_0 H_2 \\ \sqrt{\gamma}D^2/\varepsilon_0 c & \mu_0 H_3 & 0 & -\mu_0 H_1 \\ \sqrt{\gamma}D^3/\varepsilon_0 c & -\mu_0 H_2 & \mu_0 H_1 & 0 \end{pmatrix} \tag{2.21}$$

where we have used the $\vec{D}$ and $\vec{H}$ vector fields. The dual of equation (2.21), known as the Faraday tensor, using the $\vec{E}$ and $\vec{B}$ vector fields, is

$$^*\mathcal{F}^{ij} = \frac{1}{\sqrt{-g}} \begin{pmatrix} 0 & -\sqrt{\gamma}B^1 & -\sqrt{\gamma}B^2 & -\sqrt{\gamma}B^3 \\ \sqrt{\gamma}B^1 & 0 & E_3/c & -E_2/c \\ \sqrt{\gamma}B^2 & -E_3/c & 0 & E_1/c \\ \sqrt{\gamma}B^3 & E_2/c & -E_1/c & 0 \end{pmatrix} \tag{2.22}$$

The electromagnetic field tensor expressed in its covariant form, using the vector fields $\vec{E}$ and $\vec{B}$, is

$$\mathcal{F}_{ij} = \begin{pmatrix} 0 & E_1/c & E_2/c & E_3/c \\ -E_1/c & 0 & -\sqrt{\gamma}B^3 & \sqrt{\gamma}B^2 \\ -E_2/c & \sqrt{\gamma}B^3 & 0 & -\sqrt{\gamma}B^1 \\ -E_3/c & -\sqrt{\gamma}B^2 & \sqrt{\gamma}B^1 & 0 \end{pmatrix} \tag{2.23}$$

and its dual is

$$^*\mathcal{F}_{ij} = \begin{pmatrix} 0 & \mu_0 H_1 & \mu_0 H_2 & \mu_0 H_3 \\ -\mu_0 H_1 & 0 & \sqrt{\gamma}D^3/\varepsilon_0 c & -\sqrt{\gamma}D^2/\varepsilon_0 c \\ -\mu_0 H_2 & -\sqrt{\gamma}D^3/\varepsilon_0 c & 0 & \sqrt{\gamma}D^1/\varepsilon_0 c \\ -\mu_0 H_3 & \sqrt{\gamma}D^2/\varepsilon_0 c & -\sqrt{\gamma}D^1/\varepsilon_0 c & 0 \end{pmatrix} \tag{2.24}$$

expressed using the $\vec{D}$ and $\vec{H}$ vector fields. The covariant Maxwell equations are [40]

$$\nabla_j {}^*\mathcal{F}^{ij} = 0 \tag{2.25a}$$

$$\nabla_j \mathcal{F}^{ij} = I^i \tag{2.25b}$$

where I^i is the 4-current density defined by the charge density ρ and the current density $\vec{J}$ as

$$\rho c \equiv \alpha I^0 \tag{2.26a}$$

$$J^a \equiv \alpha I^a. \tag{2.26b}$$

In order to $3+1$ split the covariant Maxwell equations, we can write them in components and, introducing the spatial vector fields, $\vec{B}$, $\vec{E}$, $\vec{D}$ and $\vec{H}$, we can write them in a form similar that in equation (2.20). For example, equation (2.25a) can be split into two parts, one time and the other space, such that the time part reads

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^a}\left(\alpha\sqrt{\gamma}^*\mathcal{F}^{0a}\right) = 0 \tag{2.27}$$

and the space part reads

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\alpha\sqrt{\gamma}^{*}\mathcal{F}^{b0}\right)+\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^{a}}\left(\alpha\sqrt{\gamma}^{*}\mathcal{F}^{ba}\right)=0. \tag{2.28}$$

Now, from the contravariant field tensors, we can express the spatial vector fields, $\vec{B}$, $\vec{E}$, $\vec{D}$ and $\vec{H}$ as

$$B^{a} = \alpha^{*}\mathcal{F}^{a0} \tag{2.29a}$$
$$E_{a} = \frac{\alpha}{2}e_{abc}c^{*}\mathcal{F}^{bc} \tag{2.29b}$$
$$D^{a} = \varepsilon_{0}c\alpha\mathcal{F}^{a0} \tag{2.29c}$$
$$H_{a} = -\frac{\alpha}{2\mu_{0}}e_{abc}\mathcal{F}^{bc} \tag{2.29d}$$

where ε_0 is the permittivity of free space, μ_0 is the permeability of free space and

$$e_{abc} = \sqrt{\gamma}\varepsilon_{abc} \tag{2.30a}$$
$$e^{abc} = \frac{\varepsilon^{abc}}{\sqrt{\gamma}} \tag{2.30b}$$

is the Levi-Civita tensor of absolute space with ε_{abc} being the three-dimensional Levi-Civita symbol. Equations (2.29a) to (2.29d) can be inverted such that the field tensor is the subject of the equation as follows

$$^{*}\mathcal{F}^{a0} = \frac{B^{a}}{\alpha} \tag{2.31a}$$
$$^{*}\mathcal{F}^{ab} = \frac{1}{c\alpha}e^{abc}E_{c} = \frac{1}{c\sqrt{-g}}\varepsilon^{abc}E_{c} \tag{2.31b}$$
$$\mathcal{F}^{a0} = \frac{D^{a}}{\varepsilon_{0}c\alpha} \tag{2.31c}$$
$$\mathcal{F}^{ab} = -\frac{\mu_{0}}{\alpha}e^{abc}H_{c} = -\frac{\mu_{0}}{\sqrt{-g}}\varepsilon^{abc}H_{c}. \tag{2.31d}$$

The spatial field vectors can also be written using the covariant field tensors such that

$$B^a = -\frac{1}{2}e^{abc}\mathcal{F}_{bc} \tag{2.32a}$$

$$E_a = c\mathcal{F}_{0a} \tag{2.32b}$$

$$D^a = \frac{\varepsilon_0 c}{2}e^{abc*}\mathcal{F}_{bc} \tag{2.32c}$$

$$H_a = \frac{{}^*\mathcal{F}_{0a}}{\mu_0} \tag{2.32d}$$

and again inverting them so that the field tensor is the subject of the equation, we have

$$\mathcal{F}_{ab} = -e_{abc}B^c = -\sqrt{\gamma}\varepsilon_{abc}B^c \tag{2.33a}$$

$$\mathcal{F}_{0a} = \frac{E_a}{c} \tag{2.33b}$$

$${}^*\mathcal{F}_{ab} = \frac{e_{abc}}{\varepsilon_0 c}D^c = \frac{\sqrt{\gamma}}{\varepsilon_0 c} \tag{2.33c}$$

$${}^*\mathcal{F}_{0a} = \mu_0 H_a \varepsilon_{abc} D^c. \tag{2.33d}$$

Now in a curved three-dimensional spacetime, the covariant Maxwell equations become

$$\vec{\nabla}\cdot\vec{B} = 0 \tag{2.34a}$$

$$\vec{\nabla}\times\vec{E} = -\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\vec{B}) \tag{2.34b}$$

$$\vec{\nabla}\cdot\vec{D} = \rho \tag{2.34c}$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}(\sqrt{\gamma}\vec{D}). \tag{2.34d}$$

When we have $\frac{\partial}{\partial t}\gamma = 0$ we obtain the classical form of Maxwell's equations[30] as in equation (2.20). The vector fields are now not independent of each other as they would be in a flat spacetime. They are connected by the constitutive relations given below as

$$\varepsilon_0 \vec{E} = \alpha \vec{D} + \varepsilon_0 c \vec{\beta} \times \vec{B} \tag{2.35a}$$

$$\mu_0 \vec{H} = \alpha \vec{B} - \frac{\vec{\beta} \times \vec{D}}{\varepsilon_0 c}. \tag{2.35b}$$

These constitutive relations take into account the curvature of the absolute space by the lapse function α, which occurs in the first term on the RHS of the two relations. The frame-dragging effect is taken into account by the cross product of $\vec{\beta}$ in the second term on the RHS of the two relations. In a flat spacetime, we would have $\alpha = 1$ and $\vec{\beta} = 0$ and hence we can retrieve the known form of $\varepsilon_0 \vec{E} = \vec{D}$ and $\mu_0 \vec{H} = \vec{B}$.

We note that we are able to express each of the spatial vector fields using a spacetime vector where the time component vanishes and the spatial component is that of the spatial vector field. We can thus write the $\vec{B}$ and $\vec{D}$ fields as

$$cB^a = {}^*\mathcal{F}^{aj} n_j \tag{2.36a}$$

$$\frac{D^a}{\varepsilon_0} = \mathcal{F}^{aj} n_j. \tag{2.36b}$$

Hence $\vec{B}$ and $\vec{D}$ are the magnetic and electric fields respectively measured by the FIDO. We can also then write that

$$\rho c^2 = I^j n_j \tag{2.37}$$

and hence ρ is the electric charge density as measured by the FIDO. However, $\vec{J}$ is not the electric current density measured by the FIDO. If we denote the electric current density as measured by the FIDO as $\vec{j}$, then this is the component of the 4-current density I^i that is normal to n^i. We use the projection tensor

$$p_i^j = \delta_i^j - \frac{n_i n^j}{c^2} \tag{2.38}$$

and find the electric current density $\vec{j}$ to be

$$\alpha\vec{j} = \vec{J} + \rho c\vec{\beta}. \tag{2.39}$$

2.3 Force Free Conditions

We wish to express the current in the limit of a force-free plasma as we are considering a pulsar magnetosphere, where we neglect inertia and pressure since the electromagnetic field is so strong. The force-free condition enables us to write the electric current in terms of the electromagnetic field and its spatial derivatives in an alternative form of Ohm's law. The derivation to follow is similar to that of Gruzinov (1999)[41] except we shall consider the effects of general relativity. The force-free condition in covariant form is

$$\mathcal{F}_{ij}I^j = 0. \tag{2.40}$$

where, in the $3+1$ formulation, it can be split into

$$\vec{J} \cdot \vec{E} = 0 \tag{2.41}$$

and

$$\rho\vec{E} + \vec{J} \times \vec{B} = \vec{0}. \tag{2.42}$$

Equation (2.42) implies that

$$\vec{E} \cdot \vec{B} = 0 \tag{2.43}$$

which, combined with the constitutive relation equation (2.35a), implies that

$$\vec{D} \cdot \vec{B} = 0. \tag{2.44}$$

We find the component of the electric current density that is normal to the magnetic field by cross-multiplying equation (2.42) by $\vec{B}$ and find

$$\vec{J}_{\perp} = \rho \frac{\vec{E} \times \vec{B}}{B^2}. \tag{2.45}$$

For the parallel component of the electric current density, we first note that equation (2.44) also implies that

$$\frac{\partial}{\partial t} \left(\sqrt{\gamma} \vec{D} \cdot \vec{B} \right) = 0 \tag{2.46}$$

which we use with equations (2.34b), (2.34d) and (2.44) to obtain

$$\left(\vec{\nabla} \times \vec{H} - \vec{J} \right) \cdot \vec{B} - \left(\vec{\nabla} \times \vec{E} \right) \cdot \vec{D} = 0. \tag{2.47}$$

This now has no time derivative of γ and solving for the parallel component of $\vec{J}$ we have

$$\vec{J}_{\parallel} = \frac{\vec{B} \cdot \left(\vec{\nabla} \times \vec{H} \right) - \vec{D} \cdot \left(\vec{\nabla} \times \vec{E} \right)}{B^2} \vec{B}. \tag{2.48}$$

Combining the perpendicular and parallel components of the electric current density, we find it to be the same as that in the special relativistic case such that

$$\vec{J} = \rho \frac{\vec{E} \times \vec{B}}{B^2} + \frac{\vec{B} \cdot \vec{\nabla} \times \vec{H} - \vec{D} \cdot \vec{\nabla} \times \vec{E}}{B^2} \vec{B}. \tag{2.49}$$

We now have the background system needed to solve any prescribed metric using Maxwell's equations ((2.34a) to (2.34d)), the constitutive relations (equations (2.35a) and (2.35b)) and the prescription for the source terms. Next we will introduce and define the VSH in a curved spacetime, which will be used to help solve this system of equations, where we will focus only on the vacuum solutions.

2.4 Vector Spherical Harmonics

We make use of VSH to help solve for the electric and magnetic fields. Below is a brief derivation for the VSH in a flat spacetime which are easily converted to the curved spacetime as the only difference is that $d/dr = \alpha d/dr$. The VSH are just an extension of the scalar spherical harmonics (SSH) which are special functions defined on the surface of a sphere and are found by solving Laplace's equation in spherical coordinates, denoted as $Y_{l,m}(\theta, \phi)$ where θ is the polar coordinate and ϕ the azimuthal coordinate.

The Laplacian can be written in spherical coordinates as

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \theta}(\sin\theta\frac{\partial}{\partial \theta}) + \frac{1}{r^2 \sin^2\theta}\partial^2_\phi. \tag{2.50}$$

Laplace's equation can then be solved using separation of variables

$$\vec{\nabla}^2 \mathcal{A}(r, \theta, \phi) = 0 \tag{2.51}$$

where

$$\mathcal{A}(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi). \tag{2.52}$$

We shall skip over the full derivation which can be found at [42]. The spherical harmonics $Y_{l,m}(\theta, \phi)$ are found to be

$$Y_{l,m}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta)e^{im\phi}, \tag{2.53}$$

Now, in order to construct the VSH, we will need the orthogonality condition and the closure relation as given in Barrera et al., 1985 [43]. The completeness or closure condition is a vital property of the SSH, which states that any arbitrary function of θ, ϕ, say $g(\theta, \phi)$, can be expanded onto the SSH such that

$$g(\theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-1}^{l} \mathcal{G}_{l,m}Y_{l,m}(\theta, \phi). \tag{2.54}$$

If the function, g, is a function of other variables along with θ and ϕ, such as r, then the expansion coefficients, $\mathcal{G}_{l,m}$, are also functions of the additional variables, such that

$$g(r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-1}^{l} \mathcal{G}_{l,m}(r)Y_{l,m}(\theta,\phi). \tag{2.55}$$

In order to determine the expansion coefficients, one will now need the orthogonality condition

$$\oint d\Omega Y^*_{l,m}(\theta,\phi)Y_{l',m'}(\theta,\phi) = \delta_{l,l}\cdot\delta_{m,m'} \tag{2.56}$$

where $d\Omega \equiv \sin\theta d\theta d\phi$. The coefficients are then given by

$$\mathcal{G}_{l,m} = \int d\Omega Y^*_{l,m}(\theta,\phi)g(\theta,\phi). \tag{2.57}$$

We will now go on to define the VSH. We use equation (2.55) as our scalar field and take the gradient to attain a vector field

$$\begin{aligned}\vec{\nabla}g &= \sum_{l=0}^{\infty}\sum_{m=-1}^{l}(Y_{l,m}\vec{\nabla}\mathcal{G}_{l,m} + \mathcal{G}_{l,m}\vec{\nabla}Y_{l,m}) \\ &= \sum_{l=0}^{\infty}\sum_{m=-1}^{l}\left(\frac{d}{dr}\mathcal{G}_{l,m}(r)Y_{l,m}\hat{r} + \mathcal{G}_{l,m}\vec{\nabla}Y_{l,m}\right)\end{aligned} \tag{2.58}$$

which in the curved spacetime would be

$$\vec{\nabla}g = \sum_{l=0}^{\infty}\sum_{m=-1}^{l}\left(\alpha\frac{d}{dr}\mathcal{G}_{l,m}(r)Y_{l,m}\hat{r} + \mathcal{G}_{l,m}\vec{\nabla}Y_{l,m}\right), \tag{2.59}$$

For simplicity, it is required that the gradient of a spherical harmonic expansion also be a spherical harmonic expansion. Thus equation (2.58) can be considered as a generalised expansion of the VSH. Here it can be seen that the radial part of the vector $\vec{\nabla}g$ is simply expanded with $Y_{l,m}$, while the angular parts, $\hat{\theta}$ and $\hat{\phi}$, are expanded in terms of $\vec{\nabla}Y_{l,m}$. This encourages us to introduce the following vector

$$\vec{\Psi}_{l,m}(\theta,\phi) \equiv r\vec{\nabla}Y_{l,m}(\theta,\phi) \tag{2.60}$$

where the r factor helps in making equation (2.60) dimensionless, like $Y_{l,m}$. We now consider another vector equation

$$\vec{\mathcal{C}} = \hat{r} \times \vec{\Psi}_{l,m}. \tag{2.61}$$

This vector can neither be expanded in terms of $Y_{l,m}\hat{r}$, nor can it be expanded in terms of $\vec{\Psi}_{l,m}$, since $\vec{\mathcal{C}}$ is orthogonal to $\hat{r}$. We thus conclude that $\hat{r} \times \vec{\Psi}_{l,m}$ is a new type of vector which we will need in the expansion and define it as

$$\vec{\Phi}_{l,m} \equiv \hat{r} \times \vec{\Psi}_{l,m} = \vec{r} \times \vec{\nabla} Y_{l,m}. \tag{2.62}$$

We shall also define another symbol for the radial part seen in equation (2.58) such that

$$\vec{Y}_{l,m} \equiv Y_{l,m}\hat{r}. \tag{2.63}$$

Thus we have our three vector spherical harmonics $(\vec{Y}_{l,m}, \vec{\Psi}_{l,m}, \vec{\Phi}_{l,m})$.

There have been other definitions of the VSH, such as Hill (1954)[44]. In Hill's definition of the VSH, there are a set of vectors $(\vec{V}_{l,m}, \vec{W}_{l,m}, \vec{X}_{l,m})$ which are related to the vectors $(\vec{Y}_{l,m}, \vec{\Psi}_{l,m}, \vec{\Phi}_{l,m})$ as follows

$$\vec{V}_{l,m} = \sqrt{\frac{l+1}{2l+1}}\vec{Y}_{l,m} + \frac{1}{\sqrt{(l+1)(2l+1)}}\vec{\Psi}_{l,m} \tag{2.64a}$$

$$\vec{W}_{l,m} = \sqrt{\frac{l}{2l+1}}\vec{Y}_{l,m} + \frac{1}{\sqrt{l(2l+1)}}\vec{\Psi}_{l,m} \tag{2.64b}$$

$$\vec{X}_{l,m} = \frac{-i}{\sqrt{l(l+1)}}\vec{\Phi}_{l,m}. \tag{2.64c}$$

For the purposes of this work, the VSH in a curved spacetime are defined by

$$\vec{Y}_{l,m} = Y_{l,m}\hat{r} \tag{2.65}$$

where $Y_{l,m}$ is given by equation (2.53), and

$$\vec{\Psi}_{l,m} = \frac{r}{\sqrt{l(l+1)}}\vec{\nabla}Y_{l,m}, \tag{2.66}$$

$$\vec{\Phi}_{l,m} = \frac{\vec{r}}{\sqrt{l(l+1)}} \times \vec{\nabla}Y_{l,m}. \tag{2.67}$$

In this case, we can relate our VSH to that of Hill's by

$$\vec{X}_{l,m} = \frac{-i}{\sqrt{l(l+1)}}\vec{r} \times \vec{\nabla}Y_{l,m} = -i\vec{\Phi}_{l,m}. \tag{2.68}$$

We can take note of some useful properties in a curved space that will be used to solve for our vector fields.

The gradient and Laplacian of any scalar field, $f(r, \theta, \phi)$, are, respectively

$$\vec{\nabla}f = \alpha\frac{\partial}{\partial r}f\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}f\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}f\hat{\phi} \tag{2.69}$$

and

$$\vec{\Delta}f = \frac{\alpha}{r^2}\frac{\partial}{\partial r}(\alpha r^2\frac{\partial}{\partial r}f) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f. \tag{2.70}$$

The divergence and the curl for any vector field, $\vec{V}(r, \theta, \phi)$, are, respectively

$$\vec{\nabla}\cdot\vec{V} = \frac{\alpha}{r^2}\frac{\partial}{\partial r}(r^2V^{\hat{r}}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta V^{\hat{\theta}}) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V^{\hat{\phi}} \tag{2.71}$$

and

$$(\vec{\nabla}\times\vec{V})^{\hat{r}} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta V^{\hat{\phi}}) - \frac{\partial}{\partial\phi}V^{\hat{\theta}}\right] \tag{2.72a}$$

$$(\vec{\nabla}\times\vec{V})^{\hat{\theta}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V^{\hat{r}} - \frac{\alpha}{r}\frac{\partial}{\partial r}(rV^{\hat{\phi}}) \tag{2.72b}$$

$$(\vec{\nabla}\times\vec{V})^{\hat{\phi}} = \frac{\alpha}{r}\frac{\partial}{\partial r}(rV^{\hat{\theta}}) - \frac{1}{r}\frac{\partial}{\partial\theta}V^{\hat{r}} \tag{2.72c}$$

where the physical components of the vector fields are represented by hatted indices. The only difference between equations (2.71) and (2.72) and the divergence and curl equations for a flat space is that here we have $\alpha\frac{\partial}{\partial r}$ instead of just $\frac{\partial}{\partial r}$.

It is also useful that we have the divergence and curl of our scalar f as a function only of r with each of the VSH. For the divergence we have

$$\vec{\nabla}\cdot\left(f(r)\vec{Y}_{l,m}\right) = \frac{\alpha}{r^2}\frac{\partial}{\partial r}\left(r^2 f\right)Y_{l,m} \tag{2.73a}$$

$$\vec{\nabla}\cdot\left(f(r)\vec{\Psi}_{l,m}\right) = -\frac{\sqrt{l(l+1)}}{r}fY_{l,m} \tag{2.73b}$$

$$\vec{\nabla}\cdot\left(f(r)\vec{\Phi}_{l,m}\right) = 0 \tag{2.73c}$$

and for the curl

$$\vec{\nabla}\times\left(f(r)\vec{Y}_{l,m}\right) = -\frac{\sqrt{l(l+1)}}{r}f\vec{\Phi}_{l,m} \tag{2.74a}$$

$$\vec{\nabla}\times\left(f(r)\vec{\Psi}_{l,m}\right) = \frac{\alpha}{r}\frac{\partial}{\partial r}(rf)\vec{\Phi}_{l,m} \tag{2.74b}$$

$$\vec{\nabla}\times\left(f(r)\vec{\Phi}_{l,m}\right) = -\frac{\sqrt{l(l+1)}}{r}f\vec{Y}_{l,m} - \frac{\alpha}{r}\frac{\partial}{\partial r}(rf)\vec{\Psi}_{l,m}. \tag{2.74c}$$

It can then be shown, using these properties, that

$$\vec{\nabla}\times\left(\alpha\vec{\nabla}\times\left(f(r)\vec{\Phi}_{l,m}\right)\right) = -\alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}(rf)\right) - \frac{l(l+1)}{r^2}f\right]\vec{\Phi}_{l,m}. \tag{2.75}$$

We now introduce some useful properties for frame dragging. The frame-dragging effects are included in the constitutive relations in equations (2.35a) and (2.35b) from the cross product of two divergenceless vector fields, $\vec{\beta}$ with $\vec{B}$ and $\vec{\beta}$ with $\vec{D}$ respectively. From our definition of the VSH and the shift vector $\vec{\beta}$ we find

$$\vec{\beta}\times(f\vec{\Phi}_{l,m}) = -i\frac{m}{\sqrt{l(l+1)}}\frac{\omega r}{c}f\vec{Y}_{l,m} \tag{2.76}$$

and hence the curl of equation (2.76) is

$$\vec{\nabla} \times (\vec{\beta} \times (f\vec{\Phi}_{l,m})) = im\frac{\omega}{c}f\vec{\Phi}_{l,m}. \tag{2.77}$$

Another useful property which we find from equations (2.12) and (2.74c) is

$$\vec{\beta} \times \vec{\nabla} \times (f\vec{\Phi}_{l,m}) = \frac{\omega}{c}\sin\theta\left[\sqrt{l(l+1)}fY_{l,m}\hat{\theta} - \frac{\alpha}{\sqrt{l(l+1)}}\frac{\partial}{\partial r}(rf)\frac{\partial}{\partial\theta}Y_{l,m}\hat{r}\right]. \tag{2.78}$$

The curl of equation (2.78), using the VSH definitions and the SSH eigenfunction properties and applying straightforward algebra[32], is then

$$\begin{aligned}\vec{\nabla} \times (\vec{\beta} \times \vec{\nabla} \times (f\vec{\Phi}_{l,m})) &= im\frac{\omega r^3}{c}\vec{\nabla} \times \left(\frac{f}{r^3}\vec{\Phi}_{l,m}\right) \\ &+ 3\alpha\frac{\omega}{cr}f\left[\sqrt{l(l+2)}J_{l+1,m}\vec{\Phi}_{l+1,m} - \sqrt{(l+1)(l-1)}J_{l,m}\vec{\Phi}_{l-1,m}\right]\end{aligned} \tag{2.79}$$

where

$$J_{l,m} = \sqrt{\frac{l^2 - m^2}{4\,l^2 - 1}}. \tag{2.80}$$

We have now laid out all the foundations needed to solve for our field equations and shall put them to use in the following chapters.

Chapter 3

Electromagnetic field of an aligned dipole without rotation

We begin with a very simple case of an aligned dipole magnetic field in a vacuum without rotation. Here we are only solving for the magnetic field as, without rotation, there will be no induced electric field. Following this we will introduce rotation to the aligned dipole and solve for the electric field, from which we will be able to determine a general formalism for higher order fields. We will then solve separately for the perpendicular magnetic field.

Starting with our aligned magnetic dipole without any rotation, we set the spin to zero ($a = 0$), which in turn sets the shift vector to zero, $\vec{\beta} = 0$, and simplifies the constitutive relations. The electric field is zero, $\vec{E} = \vec{D} = 0$, and for the magnetic field we have $\mu_0 \vec{H} = \alpha \vec{B}$. The static part $\partial / \partial t = 0$ is thus satisfied and Maxwell's equations simplify as:

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{3.1a}$$

$$\vec{\nabla} \times \left(\alpha \vec{B} \right) = 0 \tag{3.1b}$$

Far from the neutron star, as $\lim_{r \to \infty}$, we expect that the magnetic field should be the same as the aligned dipole flat spacetime magnetic field[45]. This is given by

$$\begin{aligned}\vec{B} &= \frac{\mu_0}{4\pi r^3}\left[\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2} - \vec{\mu}\right] = \frac{\mu_0}{4\pi r^3}\left[3(\vec{\mu}\cdot\hat{r})\hat{r} - \vec{\mu}\right] \\ &= \frac{\mu_0 \mu}{4\pi r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)\end{aligned} \tag{3.2}$$

where the vector $\vec{\mu}$ can be rewritten as

$$\vec{\mu} = \mu(\cos\theta\hat{r} - \sin\theta\hat{\theta}). \tag{3.3}$$

Now, we define our aligned dipole magnetic field in a curved spacetime as being expressed only with the first vector spherical harmonics $\vec{\Phi}_{1,0}$ which corresponds to $(l, m) = (1, 0)$ since $l = 1$ corresponds to a dipole field and $m = 0$ corresponds to symmetry around the z-axis in accordance with our field being aligned. We can thus express the magnetic field according to its divergenceless state and write $\vec{B}$ as

$$\vec{B} = \mathrm{Re}\left[\vec{\nabla} \times (f^B_{1,0}(r)\vec{\Phi}_{1,0})\right], \tag{3.4}$$

since the divergence of the curl of a vector is zero. Here we have the scalar function $f^B_{1,0}(r)$ which is dependent only on r and is the unique unknown in the equation. Now in order to satisfy the far field limit in equation (3.2) we take $\lim_{r\to\infty}$ of equation (3.4) and equate it to equation (3.2) to solve for $\lim_{r\to\infty} f^B_{1,0}(r)$. First we calculate the value of $\vec{\Phi}_{1,0}$. From equation (2.65) we have

$$\vec{Y}_{1,0} = \sqrt{\frac{3}{4\pi}}\cos\theta\hat{r}. \tag{3.5}$$

Using equation (2.69) to take the gradient of equation (3.5) we get

$$\vec{\nabla}\vec{Y}_{1,0} = -\frac{1}{r}\sqrt{\frac{3}{4\pi}}\sin\theta\hat{\theta}. \tag{3.6}$$

Thus from equation (2.67) we obtain

$$\vec{\Phi}_{1,0} = \frac{r\hat{r}}{\sqrt{2}} \times -\frac{1}{r}\sqrt{\frac{3}{4\pi}}\sin\theta\hat{\theta}$$
$$= -\sqrt{\frac{3}{8\pi}}\sin\theta\hat{\phi}. \tag{3.7}$$

Now taking the limit of equation (3.4) and substituting in equation (3.7)

$$\vec{B} = -\sqrt{\frac{3}{8\pi}}\vec{\nabla} \times (\lim_{r\to\infty} f^B_{1,0}(r)\sin\theta\hat{\phi})$$
$$= -\sqrt{\frac{3}{8\pi}}\left[\frac{2}{r}\lim_{r\to\infty} f^B_{1,0}(r)\cos\theta\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\lim_{r\to\infty} f^B_{1,0}(r)\right)\sin\theta\hat{\theta}\right] \tag{3.8}$$

Equating the $\hat{r}$ components of equations (3.2) and (3.8) we find $\lim_{r\to\infty} f^B_{1,0}(r)$ to be

$$-\sqrt{\frac{3}{8\pi}}\frac{2}{r}\lim_{r\to\infty} f^B_{1,0}(r)\cos\theta = \frac{\mu_0\mu}{4\pi r^3}2\cos\theta$$
$$\Rightarrow \lim_{r\to\infty} f^B_{1,0}(r) = -\sqrt{\frac{8\pi}{3}}\frac{\mu_0\mu}{4\pi r^2} \tag{3.9}$$

and hence

$$\vec{B} = \frac{\mu_0}{4\pi r^3}\left[\frac{3(\vec{\mu}\cdot\vec{r})\vec{r}}{r^2} - \vec{\mu}\right] = -\sqrt{\frac{8\pi}{3}}\frac{\mu_0\mu}{4\pi}\mathrm{Re}\left[\vec{\nabla} \times \frac{\vec{\Phi}_{1,0}}{r^2}\right] \tag{3.10}$$

We now wish to find a separable solution for equation (3.4) using the boundary condition in equation (3.9). We know that equation (3.1a) is automatically satisfied by our construction of $\vec{B}$ in equation (3.4). Now, in order to satisfy equation (3.1b), we insert equation (3.4) into equation (3.1b) as follows

$$\vec{\nabla} \times \left(\alpha\vec{B}\right) = \vec{\nabla} \times \left(\alpha\mathrm{Re}\left[\vec{\nabla} \times (f^B_{1,0}(r)\vec{\Phi}_{1,0})\right]\right) = 0 \tag{3.11}$$

From equation (3.7) and using equation (2.72) we obtain the curl of $f^B_{1,0}(r)\vec{\Phi}_{1,0}$ as

$$\begin{aligned}\vec{B} &= \vec{\nabla} \times (f^B_{1,0}(r)\vec{\Phi}_{1,0}) \\ &= -\frac{1}{r}\sqrt{\frac{3}{2\pi}} f^B_{1,0}(r)\cos\theta\hat{r} + \frac{\alpha}{r}\sqrt{\frac{3}{8\pi}}\frac{\partial}{\partial r}(r f^B_{1,0}(r))\sin\theta\hat{\theta}\end{aligned} \tag{3.12}$$

which can also be obtained from equation (2.74c). Now solving for equation (3.11) we find

$$\begin{aligned}&\left[\frac{\alpha}{r}\frac{\partial}{\partial r}(r\frac{\alpha^2}{r}\sqrt{\frac{3}{8\pi}}\frac{\partial}{\partial r}(r f^B_{1,0}(r))\sin\theta) + \frac{1}{r}\frac{\partial}{\partial\theta}(\frac{\alpha}{r}\sqrt{\frac{3}{2\pi}} f^B_{1,0}(r)\cos\theta)\right]\hat{\phi} = 0 \\ \Rightarrow&\frac{\sin\theta}{2}\frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(r f^B_{1,0}(r))) - \frac{1}{r}\sin\theta f^B_{1,0}(r) = 0 \\ \Rightarrow&\frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(r f^B_{1,0}(r))) - \frac{2}{r} f^B_{1,0}(r) = 0\end{aligned} \tag{3.13}$$

and hence we have a second-order linear ordinary differential equation for the scalar $f^B_{1,0}(r)$ as

$$\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}(r f^B_{1,0}(r))\right) - \frac{2}{r} f^B_{1,0}(r) = 0. \tag{3.14}$$

Now we need only solve for the scalar $f^B_{1,0}(r)$. This is done as follows:

First we substitute $f^B_{1,0}(r)/r = q_1(r)$ into equation (3.14) and substituting for α, we can rewrite equation (3.14) as

$$\frac{d}{dr}\left[\left(1 - \frac{R_s}{r}\right)\frac{d}{dr}(r^2 q_1)\right] - 2q_1 = 0. \tag{3.15}$$

Now let $x = 1 - 2r/R_s$. We then have

$$\frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)\frac{d}{dx}\left[(1-x)^2 q_1\right]\right\} + 2q_1 = 0 \tag{3.16}$$

Let us rewrite this as

$$\frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)\frac{d}{dx}\left[(1-x)^2 q_l\right]\right\} + l(l+1)q_l = 0 \tag{3.17}$$

where we are now looking for the solution to the equation (3.17) for the function q_l. Note, equation (3.17) reduces to equation (3.16) when $l = 1$. We can rewrite equation (3.17) as follows

$$\begin{aligned}
&\frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)\frac{d}{dx}\left[(1-x)^2 q_l\right]\right\} + l(l+1)q_l = 0 \\
\Rightarrow\ &\frac{d}{dx}\left\{-2(1+x)q_l + (1-x^2)\frac{d}{dx}q_l\right\} + l(l+1)q_l = 0 \\
\Rightarrow\ &-2q_l - 2(1+x)\frac{d}{dx}q_l - 2x\frac{d}{dx}q_l + (1-x^2)\frac{d^2}{dx^2}q_l + l(l+1)q_l = 0 \\
\Rightarrow\ &(1-x^2)\frac{d^2}{dx^2}q_l - 2(1+2x)\frac{d}{dx}q_l + [l(l+1)-2]q_l = 0.
\end{aligned} \tag{3.18}$$

We then have a solution of the form

$$q_l = \frac{d}{dx}\left[(1+\mathrm{x})\frac{d}{dx}Q_l\right] \tag{3.19}$$

where Q_l is the associated Legendre function of second kind. As proof that our solution is of the form of the associated Legendre function, we can show that equation (3.18) reduces to the associated Legendre differential equation. The first two derivatives of equation (3.19) are

$$q_l = \frac{d^2}{dx^2}Q_l + x\frac{d^2}{dx^2}Q_l + \frac{d}{dx}Q_l \tag{3.20a}$$

$$\frac{d}{dx}q_l = \frac{d^3}{dx^3}Q_l + x\frac{d^3}{dx^3}Q_l + 2\frac{d^2}{dx^2}Q_l \tag{3.20b}$$

$$\frac{d^2}{dx^2}q_l = \frac{d^4}{dx^4}Q_l + x\frac{d^4}{dx^4}Q_l + 3\frac{d^3}{dx^3}Q_l \tag{3.20c}$$

We substitute equation (3.20) into equation (3.18) as follows

$$
\begin{aligned}
& (1-x^2)\frac{d^2}{dx^2}q_l - 2(1+2x)\frac{d}{dx}q_l + [l(l+1)-2]q_l = 0 \\
\Rightarrow \quad & (1-x^2)\left(\frac{d^4}{dx^4}Q_l + x\frac{d^4}{dx^4}Q_l + 3\frac{d^3}{dx^3}Q_l\right) \\
& \quad - 2(1+2x)\left(\frac{d^3}{dx^3}Q_l + x\frac{d^3}{dx^3}Q_l + 2\frac{d^2}{dx^2}Q_l\right) \\
& \quad\quad + [l(l+1)-2]\left(\frac{d^2}{dx^2}Q_l + x\frac{d^2}{dx^2}Q_l + \frac{d}{dx}Q_l\right) = 0 \\
\Rightarrow \quad & (1-x^2)(1+x)\frac{d^4}{dx^4}Q_l + \left[3(1-x^2) - 2(1+2x)(1+x)\right]\frac{d^3}{dx^3}Q_l \\
& \quad + (1+x)[l(l+1)-2]\frac{d^2}{dx^2}Q_l \\
& \quad\quad - 4(1+2x)\frac{d^2}{dx^2}Q_l + [l(l+1)-2]\frac{d}{dx}Q_l = 0 \\
\Rightarrow \quad & (1+x)\left[(1-x^2)\frac{d^4}{dx^4}Q_l + (1-7x)\frac{d^3}{dx^3}Q_l + [l(l+1)-2]\frac{d^2}{dx^2}Q_l\right] \\
& \quad\quad - 4(1+2x)\frac{d^2}{dx^2}Q_l + [l(l+1)-2]\frac{d}{dx}Q_l = 0. \qquad (3.21)
\end{aligned}
$$

We can rewrite equation (3.21) as

$$
\begin{aligned}
& (1+x)\left[(1-x^2)\frac{d^4}{dx^4}Q_l + (1-x)\frac{d^3}{dx^3}Q_l \right. \\
& \quad \left. -6x\frac{d^3}{dx^3}Q_l + [l(l+1)-6]\frac{d^2}{dx^2}Q_l + 4\frac{d^2}{dx^2}Q_l\right] \\
& \quad\quad - 4(1+2x)\frac{d^2}{dx^2}Q_l + [l(l+1)-2]\frac{d}{dx}Q_l = 0 \\
\Rightarrow \quad & (1+x)\left[(1-x^2)\frac{d^4}{dx^4}Q_l - 6x\frac{d^3}{dx^3}Q_l + [l(l+1)-6]\frac{d^2}{dx^2}Q_l\right] \\
& \quad + (1-x^2)\frac{d^3}{dx^3}Q_l - 4x\frac{d^2}{dx^2}Q_l + [l(l+1)-2]\frac{d}{dx}Q_l = 0. \qquad (3.22)
\end{aligned}
$$

Now, we know the Legendre differential equation with $m = 0$ is

$$
(1-x^2)\frac{d^2}{dx^2}Q_l - 2x\frac{d}{dx}Q_l + l(l+1)Q_l = 0. \qquad (3.23)
$$

Taking the first two derivatives of Legendre's equation, we get

$$(1-x^2)\frac{d^3}{dx^3}Q_l - 4x\frac{d^2}{dx^2}Q_l + [l(l+1)-2]\frac{d}{dx}Q_l = 0 \tag{3.24}$$

and

$$(1-x^2)\frac{d^4}{dx^4}Q_l - 6x\frac{d^3}{dx^3}Q_l + [l(l+1)-6]\frac{d^2}{dx^2}Q_l = 0. \tag{3.25}$$

We can now see that the content in the first square bracket of equation (3.22) is just the second derivative of the Legendre equation, and the remaining terms outside the bracket make up the first derivative of the Legendre equation. Thus equation (3.22) is identically satisfied.

Now, for $l = 1$, the associated Legendre function of second kind is

$$Q_1 = \frac{x}{2}\ln\left|\frac{1+x}{1-x}\right| - 1. \tag{3.26}$$

Substituting in equation (3.26) into equation (3.19) and finding the first two derivatives we obtain

$$q_1 = \frac{1}{2}\ln\left|\frac{1+x}{1-x}\right| + \frac{1}{1-x} + \frac{1}{(1-x)^2} \tag{3.27a}$$

$$\frac{d}{dx}q_1 = \frac{1}{1-x^2} + \frac{1}{(1-x)^2} + \frac{2}{(1-x)^3} \tag{3.27b}$$

$$\frac{d^2}{dx^2}q_1 = \frac{2x}{(1-x^2)^2} + \frac{2}{(1-x)^3} + \frac{6}{(1-x)^4}. \tag{3.27c}$$

Substituting equation (3.27) into equation (3.18) we verify the solution to be zero. Thus equation (3.27a) is a solution of equation (3.16). Now in order to solve for $f^B_{1,0}(r)$ we have

$$f_{l,m} = F_l r q_l \tag{3.28}$$

where F_l is a constant obtained from the boundary conditions. Thus, substituting for x, we have

$$
\begin{aligned}
f^B_{1,0}(r) &= F_1 r\left[\frac{1}{2}\ln\left|\frac{1+1-\frac{2r}{R_s}}{1-1+\frac{2r}{R_s}}\right| + \frac{1}{1-1+\frac{2r}{R_s}} + \frac{1}{\left(1-1+\frac{2r}{R_s}\right)^2}\right] \\
&= F_1 r\left[\frac{1}{2}\ln\left|\frac{R_s}{r}-1\right| + \frac{R_s}{2r} + \frac{R_s^2}{4r^2}\right] \\
&= F_1\frac{r}{2}\left[\ln\left|-\alpha^2\right| + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right] \\
&= F_1\frac{r}{2}\left[\ln\alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right] \\
&= F_1' r\left[\ln\alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]
\end{aligned}
\tag{3.29}
$$

where $F_1' = F_1/2$. Note that $r > R_s$ for the exterior of the star and hence $0 < 1 - R_s/R < \alpha^2 < 1$ and thus $-1 < \alpha^2 < R_s/R - 1 < 0$, where R is the radius of the star. Solving for F_1' using the boundary condition in equation (3.9) and using the limit

$$
\lim_{r\to\infty}\ln\left(1-\frac{R_s}{r}\right) = -\frac{R_s}{r} - \frac{R_s^2}{2r^2} - \frac{R_s^3}{3r^3} - h.o. \tag{3.30}
$$

we have

$$
\begin{aligned}
\lim_{r\to\infty} f^B_{1,0}(r) &= \lim_{r\to\infty} F_1' r q_1 \\
\Rightarrow -\frac{\mu_0\mu}{4\pi r^2}\sqrt{\frac{8\pi}{3}} &= \lim_{r\to\infty} F_1' r\left[-\frac{R_s}{r} - \frac{R_s^2}{2r^2} - \frac{R_s^3}{3r^3} - \frac{R_s^4}{4r^4} - h.o. + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right] \\
&= \lim_{r\to\infty} -F_1' r\left[\frac{R_s^3}{3r^3} + \frac{R_s^4}{4r^4} + h.o.\right] \\
&= \lim_{r\to\infty} -F_1' r\frac{R_s^3}{3r^3}\left[1 + \frac{3R_s}{4r} + \frac{3R_s^2}{5r^2} + h.o.\right] \\
&= -F_1'\frac{R_s^3}{3r^2}.
\end{aligned}
\tag{3.31}
$$

Therefore

$$
F_1' = \frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3}{R_s^3} \tag{3.32}
$$

and

$$f_{1,0}^{B(\text{dip})} = \frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3r}{R_s^3}\left[\ln\alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]. \tag{3.33}$$

In order to solve for the magnetic field components, we now substitute equation (3.33) into equation (3.12). We will need $\frac{\partial}{\partial r}(rf_{1,0}^{B(\text{dip})}(r))$ which is

$$\begin{aligned}
\frac{\partial}{\partial r}(rf_{1,0}^{B(\text{dip})}(r)) &= \frac{\partial}{\partial r}\left(\frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3}{R_s^3}\left[r^2\ln\alpha^2 + rR_s + \frac{R_s^2}{2}\right]\right)\\
&= \frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3}{R_s^3}\left[2r\ln\alpha^2 + \frac{R_s}{\alpha^2} + R_s\right]\\
&= \frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3r}{\alpha R_s^3}\left[2\alpha\ln\alpha^2 + \frac{R_s}{\alpha r} + \frac{\alpha R_s}{r}\right]\\
&= \frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3r}{\alpha R_s^3}\left[2\alpha\ln\alpha^2 + \frac{R_s}{r}\frac{2r-R_s}{\alpha r}\right]\\
&= \frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3r}{\alpha R_s^3}\left[2\sqrt{1-\frac{R_s}{r}}\ln\left(1-\frac{R_s}{r}\right) + \frac{R_s}{r}\frac{2r-R_s}{\sqrt{r(r-R_s)}}\right]
\end{aligned} \tag{3.34}$$

We now find the magnetic field components from equation (3.12) to be

$$\begin{aligned}
B^{\hat{r}} &= -\frac{1}{r}\sqrt{\frac{3}{2\pi}}f_{1,0}^{B}(r)\cos\theta\\
&= -6\frac{\mu_0\mu}{4\pi R_s^3}\left[\ln\left(1-\frac{R_s}{r}\right) + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]\cos\theta
\end{aligned} \tag{3.35a}$$

$$\begin{aligned}
B^{\hat{\theta}} &= \frac{\alpha}{r}\sqrt{\frac{3}{8\pi}}\frac{\partial}{\partial r}(rf_{1,0}^{B}(r))\sin\theta\\
&= 3\frac{\mu_0\mu}{4\pi R_s^3}\left[2\sqrt{1-\frac{R_s}{r}}\ln\left(1-\frac{R_s}{r}\right) + \frac{R_s}{r}\frac{2r-R_s}{\sqrt{r(r-R_s)}}\right]\sin\theta
\end{aligned} \tag{3.35b}$$

$$B^{\hat{\phi}} = 0. \tag{3.35c}$$

Using equation (3.30) and the following limits

$$\lim_{r\to\infty}\sqrt{1-\frac{R_s}{r}} = 1 - \frac{R_s}{2r} - \frac{R_s^2}{8r^2} - \frac{R_s^3}{16r^3} - h.o. \tag{3.36a}$$

$$\lim_{r\to\infty}\left(1-\frac{R_s}{r}\right)^{-1} = 1 + \frac{R_s}{r} + \frac{R_s^2}{r^2} + \frac{R_s^3}{r^3} + h.o. \tag{3.36b}$$

we take $\lim_{r\to\infty}$ on the set of equations (3.35a) to (3.35c) to compare to the flat spacetime magnetic field in equation (3.2) and find

$$
\begin{aligned}
\lim_{r\to\infty} B^{\hat{r}} &= \lim_{r\to\infty} -6\frac{\mu_0\mu}{4\pi R_s^3}\left[-\frac{R_s}{r} - \frac{R_s^2}{2r^2} - \frac{R_s^3}{3r^3} - \frac{R_s^4}{4r^4} - h.o. + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]\cos\theta \\
&= \lim_{r\to\infty} -6\frac{\mu_0\mu}{4\pi R_s^3}\left[-\frac{R_s^3}{3r^3} - \frac{R_s^4}{4r^4} - h.o.\right]\cos\theta \\
&= \lim_{r\to\infty} \frac{\mu_0\mu}{4\pi r^3} 2\cos\theta\left[1 + \frac{3}{4}\frac{R_s}{r} + h.o.\right] \\
&= \frac{\mu_0\mu}{4\pi r^3} 2\cos\theta \qquad (3.37a) \\
\lim_{r\to\infty} B^{\hat{\theta}} &= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[2\sqrt{1-\frac{R_s}{r}}\ln\left(1-\frac{R_s}{r}\right) + \frac{R_s}{r}\frac{2r-R_s}{\sqrt{r(r-R_s)}}\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[\sqrt{1-\frac{R_s}{r}}\left(2\ln\left(1-\frac{R_s}{r}\right) + \frac{R_s}{r}\frac{2r-R_s}{r-R_s}\right)\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[\sqrt{1-\frac{R_s}{r}}\left(2\ln\left(1-\frac{R_s}{r}\right) + \frac{R_s}{r} + \frac{R_s}{r-R_s}\right)\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[\sqrt{1-\frac{R_s}{r}}\left(2\ln\left(1-\frac{R_s}{r}\right)\right.\right. \\
&\qquad\qquad \left.\left. + \frac{R_s}{r}\left[1 + \left(1-\frac{R_s}{r}\right)^{-1}\right]\right)\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[\left(1 - \frac{R_s}{2r} - \frac{R_s^2}{8r^2} - \frac{R_s^3}{16r^3} - h.o.\right)\right. \\
&\qquad \times\left(2\left(-\frac{R_s}{r} - \frac{R_s^2}{2r^2} - \frac{R_s^3}{3r^3} - \frac{R_s^4}{4r^4} - h.o.\right)\right. \\
&\qquad \left.\left. + \frac{R_s}{r}\left[1 + \left(1 + \frac{R_s}{r} + \frac{R_s^2}{r^2} + \frac{R_s^3}{r^3} + h.o.\right)\right]\right)\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[\left(1 - \frac{R_s}{2r} - \frac{R_s^2}{8r^2} - \frac{R_s^3}{16r^3} - h.o.\right)\left(-\frac{2R_s}{r} - \frac{R_s^2}{r^2} - \frac{2R_s^3}{3r^3}\right.\right. \\
&\qquad \left.\left. - \frac{R_s^4}{2r^4} - h.o. + \frac{2R_s}{r} + \frac{R_s^2}{r^2} + \frac{R_s^3}{r^3} + \frac{R_s^4}{r^4} + h.o.\right)\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[-\frac{2R_s^3}{3r^3} - \frac{R_s^4}{2r^4} + \frac{R_s^3}{r^3} + \frac{R_s^4}{r^4} + \frac{R_s^4}{3r^4} - \frac{R_s^4}{2r^4} + h.o.\right]\sin\theta \\
&= \lim_{r\to\infty} 3\frac{\mu_0\mu}{4\pi R_s^3}\left[\frac{R_s^3}{3r^3} + \frac{R_s^4}{3r^4} + h.o.\right]\sin\theta \\
&= \lim_{r\to\infty} \frac{\mu_0\mu}{4\pi r^3}\left[1 + \frac{R_s}{r} + h.o.\right]\sin\theta \\
&= \frac{\mu_0\mu}{4\pi r^3}\sin\theta \qquad (3.37b) \\
B^{\hat{\phi}} &= 0 \qquad (3.37c)
\end{aligned}
$$

as expected.

In this chapter we have solved for the magnetic field of an aligned dipole without rotation in a curved space. We have taken the far field limit as $r \to \infty$ and proven that it is equal to the flat spacetime magnetic field, as it should be. We will now go on to introduce rotation to the system and solve for the electric field also.

Chapter 4

Electromagnetic field of an aligned dipole with rotation

We shall now take our aligned dipole magnetic field and introduce rotation to the system which will induce an electric field. We know through the constitutive relations, equations (2.35a) and (2.35b), that the the electric and magnetic fields mix from frame-dragging effects. To make things simpler, we will first exclude the frame-dragging effects by setting $\beta = 0$, as we did in chapter (3), and find the electric field without any frame-dragging and then we will go on to include the frame-dragging effects.

4.1 Without frame-dragging ($\vec{\beta} = 0$)

We are now introducing rotation to our system, however, we are still setting the shift vector to zero, $\vec{\beta} = 0$, as with the previous chapter, so as to neglect the frame-dragging effects at this stage. This situation can arise when we consider the case of very slow rotation. Maxwell's equations then become

$$\vec{\nabla} \cdot \vec{D} = 0 \tag{4.1a}$$

$$\vec{\nabla} \times (\alpha \vec{D}) = 0 \tag{4.1b}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4.1c}$$

$$\vec{\nabla} \times (\alpha \vec{B}) = 0 \tag{4.1d}$$

The electric field is taken to be quadrupolar (from the flat spacetime solution), hence $l = 2$. The fields are then written as

$$\vec{D} = \mathrm{Re}\left[\vec{\nabla} \times (f^D_{2,0} \vec{\Phi}_{2,0})\right] \tag{4.2a}$$

$$\vec{B} = \mathrm{Re}\left[\vec{\nabla} \times (f^B_{1,0} \vec{\Phi}_{1,0})\right]. \tag{4.2b}$$

Both fields are automatically divergenceless. The magnetic field remains the same as before and we use similar methods to solve for the electric field where we now have the unique unknown scalar function $f^D_{2,0}(r)$. We now insert equation (4.2a) into equation (4.1b) as follows

$$\vec{\nabla} \times \left(\alpha \vec{D}\right) = \vec{\nabla} \times \left(\alpha \mathrm{Re}\left[\vec{\nabla} \times (f^D_{2,0}(r) \vec{\Phi}_{2,0})\right]\right) = 0 \tag{4.3}$$

First we need to find $\vec{\Phi}_{2,0}$ from equation (2.67). From equation (2.65) we have

$$\vec{Y}_{2,0} = \frac{1}{2}\sqrt{\frac{5}{4\pi}}(3\cos^2\theta - 1)\hat{r}. \tag{4.4}$$

Using equation (2.69) to take the gradient of equation (4.4) we get

$$\vec{\nabla}\vec{Y}_{2,0} = -\frac{3}{r}\sqrt{\frac{5}{4\pi}}\cos\theta\sin\theta\hat{\theta}. \tag{4.5}$$

Thus we have

$$
\begin{aligned}
\vec{\Phi}_{2,0} &= \frac{r\hat{r}}{\sqrt{6}} \times -\frac{3}{r}\sqrt{\frac{5}{4\pi}}\cos\theta\sin\theta\hat{\theta} \\
&= -\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\hat{\phi}.
\end{aligned}
\tag{4.6}
$$

Equation (4.2a) then becomes

$$
\begin{aligned}
\vec{D} &= \vec{\nabla} \times (f^D_{2,0}(r)\vec{\Phi}_{2,0}) \\
&= -\frac{1}{r}\sqrt{\frac{15}{8\pi}} f^D_{2,0}(r)(3\cos^2\theta - 1)\hat{r} + \frac{\alpha}{r}\sqrt{\frac{15}{8\pi}}\frac{\partial}{\partial r}(r f^D_{2,0}(r))\cos\theta\sin\theta\hat{\theta}
\end{aligned}
\tag{4.7}
$$

which can also be obtained from equation (2.74c). Now solving for equation (4.3) we find

$$
\begin{aligned}
&\left[\frac{\alpha}{r}\frac{\partial}{\partial r}(r\frac{\alpha^2}{r}\sqrt{\frac{15}{8\pi}}\frac{\partial}{\partial r}(r f^D_{2,0}(r))\cos\theta\sin\theta) + \frac{1}{r}\frac{\partial}{\partial\theta}(\frac{\alpha}{r}\sqrt{\frac{15}{8\pi}}f^D_{2,0}(r)(3\cos^2\theta - 1))\right]\hat{\phi} = 0 \\
\Rightarrow \quad & \cos\theta\sin\theta\frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(r f^D_{2,0}(r))) - \frac{1}{r}6\cos\theta\sin\theta f^D_{2,0}(r) = 0 \\
\Rightarrow \quad & \frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(r f^D_{2,0}(r))) - \frac{6}{r}f^D_{2,0}(r) = 0.
\end{aligned}
\tag{4.8}
$$

and hence we have another second-order linear ordinary differential equation, now for our scalar $f^D_{2,0}(r)$ as

$$
\frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(r f^D_{2,0}(r))) - \frac{6}{r}f^D_{2,0}(r) = 0.
\tag{4.9}
$$

Now we wish to solve for the scalar $f^D_{2,0}(r)$ using similar methods as were used in chapter (3) to solve for $f^B_{1,0}(r)$. First we substitute $l = 2$ into equation (3.17) to obtain

$$
\frac{d}{dx}\left\{\left(\frac{1+x}{1-x}\right)\frac{d}{dx}\left[(1-x)^2 q_2\right]\right\} + 6q_2 = 0
\tag{4.10}
$$

For $l = 2$ the associated Legendre function of second kind is

$$Q_2 = \frac{3x^2-1}{4}\ln\left|\frac{1+x}{1-x}\right| - \frac{3x}{2} \tag{4.11}$$

and thus, after substituting equation (4.11) into equation (3.19) and finding the first two derivatives, we find

$$q_2 = \frac{3}{2}(2x+1)\ln\left|\frac{1+x}{1-x}\right| - \frac{6x^2-9x+2}{(x-1)^2} \tag{4.12a}$$

$$\frac{d}{dx}q_2 = 3\ln\left|\frac{1+x}{1-x}\right| - \frac{2(3x^3-6x^2+x+4)}{(x-1)^3(x+1)} \tag{4.12b}$$

$$\frac{d^2}{dx^2}q_2 = \frac{24}{(x-1)^4(x+1)^2}. \tag{4.12c}$$

We substitute equation (4.12) into equation (3.18) and verify it to be zero. Hence equation (4.12a) is a solution to equation (4.10). Now, substituting for x, we can solve for $f^D_{2,0}(r)$ as

$$\begin{aligned}
f^D_{2,0}(r) &= F_2 r q_2 \\
&= F_2 r\left[\frac{3}{2}\left(2\left(1-\frac{2r}{R_s}\right)+1\right)\ln\left|\frac{1+\left(1-\frac{2r}{R_s}\right)}{1-\left(1-\frac{2r}{R_s}\right)}\right|\right. \\
&\qquad\left. -\frac{6\left(1-\frac{2r}{R_s}\right)^2-9\left(1-\frac{2r}{R_s}\right)+2}{\left(\left(1-\frac{2r}{R_s}\right)-1\right)^2}\right] \\
&= F_2 r\left[\frac{3}{2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 - \frac{R_s^2}{4r^2}\left(6-\frac{24r}{R_s}+\frac{24r^2}{R_s^2}-9+\frac{18r}{R_s}+2\right)\right] \\
&= F_2 r\left[\frac{3}{2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 - 6 + \frac{3R_s}{2r}+\frac{R_s^2}{4r^2}\right] \\
&= F_2\frac{R_s^2}{4r}\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right]
\end{aligned} \tag{4.13}$$

and taking $\lim_{r\to\infty}$

$$
\begin{aligned}
\lim_{r\to\infty} f^D_{2,0}(r) &= \lim_{r\to\infty} F_2 \frac{R_s^2}{4r}\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \\
&= \lim_{r\to\infty} F_2 \frac{R_s^2}{4r}\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\left(-\frac{R_s}{r}-\frac{R_s^2}{2r^2}-\frac{R_s^3}{3r^3}-h.o.\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. +1+6\frac{r}{R_s}-24\frac{r^2}{R_s^2}\right] \\
&= \lim_{r\to\infty}\frac{F_2}{4}\left[\left(18r-24\frac{r^2}{R_s}\right)\left(-\frac{R_s}{r}-\frac{R_s^2}{2r^2}-\frac{R_s^3}{3r^3}-h.o.\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. +\frac{R_s^2}{r}+6R_s-24r\right] \\
&= \lim_{r\to\infty}\frac{F_2}{4}\left[-18\left(R_s+\frac{R_s^2}{2r}+\frac{R_s^3}{3r^2}+h.o.\right)\right. \\
&\qquad \left. +24\left(r+\frac{R_s}{2}+\frac{R_s^2}{3r}+\frac{R_s^3}{4r^2}+h.o.\right)+\frac{R_s^2}{r}+6R_s-24r\right] \\
&= \frac{F_2}{4}\left[-18R_s+12R_s+6R_s\right] \\
&= 0,
\end{aligned}
\tag{4.14}
$$

we see that the solution does indeed vanish at infinity. We now need to solve for F_2 using the boundary conditions at the surface of the neutron star. For the inner boundary condition, we know that inside a perfectly conducting star the Lorentz force is zero and hence

$$
\vec{E} + r\Omega\sin\theta\hat{\phi}\times\vec{B} = 0 \tag{4.15}
$$

where Ω is the rotation rate of the neutron star, as mentioned in chapter (2). Using equations (2.12), (2.19) and (2.35a) we find the electric field measured by a FIDO to be

$$
\begin{aligned}
\vec{D} &= \frac{\varepsilon}{\alpha}(\vec{E} - c\vec{\beta}\times\vec{B}) \\
&= \frac{\varepsilon}{\alpha}(-r\Omega\sin\theta\hat{\phi}\times\vec{B} + r(\Omega-\tilde{\omega})\sin\theta\hat{\phi}\times\vec{B}) \\
&= -\varepsilon_0\frac{\tilde{\omega}}{\alpha}r\sin\theta\hat{\phi}\times\vec{B} = \varepsilon_0 c\frac{\tilde{\omega}}{\alpha}\frac{\vec{\beta}}{\omega}\times\vec{B}
\end{aligned}
\tag{4.16}
$$

where we have used the relative rotation $\tilde{\omega}$, defined in chapter (2). We have to satisfy the jump conditions at the surface of the star, meaning that the magnetic field component normal to the surface ($B^{\hat{r}}$) and the electric field components which lies in the plane of the surface($D^{\hat{\theta}}$ and $D^{\hat{\phi}}$) have to be continuous across the stellar surface. The $B^{\hat{r}}$ component is continuous as was confirmed in chapter (3). From equation (4.16) we find the $D^{\hat{\phi}}$ component to be zero both inside and outside the star, making it continuous. Hence, we only need to satisfy the continuity of the $D^{\hat{\theta}}$ component. Using equation (4.16) we have the condition that the inner $D^{\hat{\theta}}$ must satisfy

$$D^{\hat{\theta}} = -\varepsilon_0 \frac{\tilde{\omega}}{\alpha} r \sin\theta B^{\hat{r}}. \tag{4.17}$$

Equation (4.17) has to be compared to the outer $D^{\hat{\theta}}$ component of equation (4.2a) (see equation (4.7)) which is

$$D^{\hat{\theta}} = \frac{\alpha}{r}\sqrt{\frac{15}{8\pi}}\frac{\partial}{\partial r}(r f^D_{2,0}(r)) \cos\theta \sin\theta. \tag{4.18}$$

We use equation (3.35a) for $B^{\hat{r}}$ and let equation (4.17) equal to equation (4.18), setting the radius to $r = R$ (at the stellar surface) in order to determine our constant, F_2 in equation (4.13). First we determine $\frac{\partial}{\partial r}(r f^D_{2,0}(r))$ from equation (4.13) as

$$\begin{aligned}
\left.\frac{\partial}{\partial r}(r f^D_{2,0}(r))\right|_{r=R} &= F_2 \frac{R_s^2}{4}\left[\left(36\frac{R}{R_s^2} - 72\frac{R^2}{R_s^3}\right)\ln\alpha_R^2 + 18\frac{1}{\alpha_R^2 R_s} - 24\frac{R}{\alpha_R^2 R_s^2}\right. \\
&\qquad\qquad \left. + 6\frac{1}{R_s} - 48\frac{R}{R_s^2}\right] \\
\Rightarrow &= 36F_2\frac{R}{4}\left[\left(1 - 2\frac{R}{R_s}\right)\ln\alpha_R^2 + \frac{R_s}{2\alpha_R^2 R} - \frac{2}{3\alpha_R^2} + \frac{R_s}{6R} - \frac{4}{3}\right] \\
\Rightarrow &= 36F_2\frac{R}{4}\left[\left(1 - 2\frac{R}{R_s}\right)\ln\alpha_R^2 + \frac{-4R^2 + 4RR_s - R_s^2}{6\alpha_R^2 R^2} - \frac{4}{3}\right] \\
\Rightarrow &= 36F_2\frac{R}{4}\left[\left(1 - 2\frac{R}{R_s}\right)\ln\alpha_R^2 - \frac{4R^2(1 - R_s/R)}{6\alpha_R^2 R^2} - \frac{4}{3} - \frac{R_s^2}{6\alpha_R^2 R^2}\right] \\
\Rightarrow &= 36F_2\frac{R}{4}\left[\left(1 - 2\frac{R}{R_s}\right)\ln\alpha_R^2 - 2 - \frac{R_s^2}{6\alpha_R^2 R^2}\right]
\end{aligned} \tag{4.19}$$

where

$$\alpha_R = \sqrt{1 - \frac{R_s}{R}}. \tag{4.20}$$

Now equating equations (4.17) and (4.18), substituting in for $B^{\hat{r}}(R)$ and $\frac{\partial}{\partial r}(rf^{D}_{2,0}(r))\big|_{r=R}$, we have

$$\begin{aligned}
&-\varepsilon_0\frac{\tilde{\omega}_R}{\alpha_R}R\sin\theta\left[-6\frac{\mu_0\mu}{4\pi R_s^3}\left[\ln\alpha_R^2+\frac{R_s}{R}+\frac{R_s^2}{2R^2}\right]\cos\theta\right]\\
&\qquad=\frac{\alpha_R}{R}\sqrt{\frac{15}{8\pi}}36F_2\frac{R}{4}\left[\left(1-2\frac{R}{R_s}\right)\ln\alpha_R^2-2-\frac{R_s^2}{6\alpha_R^2R^2}\right]\cos\theta\sin\theta\\
\Rightarrow\quad&\varepsilon_0\frac{\tilde{\omega}_R}{\alpha_R}\frac{\mu_0\mu}{4\pi}\frac{R}{R_s^3}\left[\ln\alpha_R^2+\frac{R_s}{R}+\frac{R_s^2}{2R^2}\right]\\
&\qquad=\sqrt{\frac{15}{8\pi}}F_2\frac{3\alpha_R}{2}\left[\left(1-2\frac{R}{R_s}\right)\ln\alpha_R^2-2-\frac{R_s^2}{6\alpha_R^2R^2}\right]
\end{aligned}\tag{4.21}$$

where

$$\tilde{\omega}_R=\Omega-\omega_R \tag{4.22a}$$

$$\omega_R=\frac{aR_sc}{R^3} \tag{4.22b}$$

are the values taken at the stellar surface R. To make things simpler, let us introduce two constants

$$\mathcal{Z}_1=\ln\alpha_R^2+\frac{R_s}{R}+\frac{R_s^2}{2R^2} \tag{4.23a}$$

$$\mathcal{Z}_2=\left[\left(1-2\frac{R}{R_s}\right)\ln\alpha_R^2-2-\frac{R_s^2}{6R^2\alpha_R^2}\right]^{-1}. \tag{4.23b}$$

Now solving for F_2 we have

$$F_2=\frac{\varepsilon_0\mu_0\mu}{\pi}\frac{1}{9}\sqrt{\frac{6\pi}{5}}\frac{R}{R_s^3}\frac{\tilde{\omega}_R}{\alpha_R^2}\mathcal{Z}_1\mathcal{Z}_2 \tag{4.24}$$

and substituting back into equations (4.13) and (4.26)

$$\begin{aligned} f^D_{2,0}(r) =& \frac{\varepsilon_0\mu_0\mu}{4\pi r}\frac{1}{9}\sqrt{\frac{6\pi}{5}}\frac{R}{R_s}\frac{\tilde{\omega}_R}{\alpha_R^2}\mathcal{Z}_1\mathcal{Z}_2\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2\right. \\ &\left. +1+6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \end{aligned} \tag{4.25}$$

and

$$\frac{\partial}{\partial r}(rf^D_{2,0}(r)) = \frac{\varepsilon_0\mu_0\mu r}{\pi}\sqrt{\frac{6\pi}{5}}\frac{R}{R_s^3}\frac{\tilde{\omega}_R}{\alpha_R^2}\mathcal{Z}_1\mathcal{Z}_2\left[\left(1-2\frac{r}{R_s}\right)\ln\alpha^2-2-\frac{R_s^2}{6\alpha^2r^2}\right]. \tag{4.26}$$

Then from equation (4.7) we find the electric field to be

$$\begin{aligned} D^{\hat{r}} &= -\frac{\varepsilon_0\mu_0\mu}{4\pi r^2}\frac{1}{6}\frac{R}{R_s}\frac{\tilde{\omega}_R}{\alpha_R^2}\mathcal{Z}_1\mathcal{Z}_2\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+1+6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \\ &\qquad\times(3\cos^2\theta-1) \\ &= -\frac{\varepsilon_0\mu_0\mu}{4\pi}\frac{R}{R_s^3}\frac{\tilde{\omega}_R}{\alpha_R^2}\mathcal{Z}_1\mathcal{Z}_2\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right](3\cos^2\theta-1) \end{aligned} \tag{4.27a}$$

$$D^{\hat{\theta}} = 6\frac{\varepsilon_0\mu_0\mu}{4\pi}\frac{R}{R_s^3}\frac{\tilde{\omega}_R}{\alpha_R^2}\alpha\mathcal{Z}_1\mathcal{Z}_2\left[\left(1-2\frac{r}{R_s}\right)\ln\alpha^2-2-\frac{R_s^2}{6\alpha^2r^2}\right]\cos\theta\sin\theta \tag{4.27b}$$

$$D^{\hat{\phi}} = 0 \tag{4.27c}$$

and the magnetic field remains the same as in the static dipole case.

4.2 With frame-dragging ($\vec{\beta} \neq 0$)

Let us now introduce the frame-dragging effects. As previously mentioned, these effects come in with the cross product of $\vec{\beta}$ in the constitutive relations, where previously we had set $\vec{\beta} = 0$. We shall now go on to include these effects so that $\vec{\beta} \neq 0$. First we will rewrite the vector fields expanding them in a power series for when the vector fields are of a higher order. It is possible to do this because Maxwell's equations and the constitutive relations are linear. The vector fields are thus expanded as follows

$$\vec{V} = \sum_{k\geq 0} \epsilon^k \vec{V}_k = \vec{V}_0 + \sum_{k\geq 1} \epsilon^k \vec{V}_k \tag{4.28}$$

where $\vec{V}$ is any vector field, $\vec{V}_0$ is the static vector field and ϵ is a small dimensionless parameter which is related to the spin by $\epsilon = O(\Omega)$. The electric field only occurs in the rotating case and hence is at least of first order in Ω. Thus we have $\vec{D}_0 = \vec{E}_0 = 0$. The shift vector is of first order only (see equation (2.12)) and can be written as $\vec{\beta} = \epsilon\vec{\beta}_1$. We now find the k^{th} order of the auxiliary electric field from the constitutive relation equation (2.35a) for $k \geq 1$ as

$$\varepsilon_0 \vec{E}_k = \alpha \vec{D}_k + \varepsilon_0 c \vec{\beta}_1 \times \vec{B}_{k-1}. \tag{4.29}$$

Proof:

$$\begin{aligned}
&\varepsilon_0 \vec{E} = \alpha \vec{D} + \varepsilon_0 c \vec{\beta} \times \vec{B} \\
\Rightarrow &\varepsilon_0 \left(\vec{E}_0 + \sum_{k\geq 1} \epsilon^k \vec{E}_k \right) = \alpha \left(\vec{D}_0 + \sum_{k\geq 1} \epsilon^k \vec{D}_k \right) + \varepsilon_0 c \left(\epsilon \vec{\beta}_1 \right) \times \left(\vec{B}_0 + \sum_{k\geq 1} \epsilon^k \vec{B}_k \right) \\
\Rightarrow &\varepsilon_0 \left(\epsilon \vec{E}_1 + \epsilon^2 \vec{E}_2 + \epsilon^3 \vec{E}_3 + h.o. \right) = \alpha \left(\epsilon \vec{D}_1 + \epsilon^2 \vec{D}_2 + \epsilon^3 \vec{D}_3 + h.o. \right) \\
&\qquad\qquad + \varepsilon_0 c \left(\epsilon \vec{\beta}_1 \right) \times \left(\vec{B}_0 + \epsilon \vec{B}_1 + \epsilon^2 \vec{B}_2 + h.o. \right) \\
\Rightarrow &\epsilon(\varepsilon_0 \vec{E}_1) + \epsilon^2(\varepsilon_0 \vec{E}_2) + \epsilon^3(\varepsilon_0 \vec{E}_3) + h.o. = \epsilon(\alpha \vec{D}_1) + \epsilon^2(\alpha \vec{D}_2) + \epsilon^3(\alpha \vec{D}_3) + h.o. \\
&\qquad\qquad + \epsilon(\varepsilon_0 c \vec{\beta}_1 \times \vec{B}_0) + \epsilon^2(\varepsilon_0 c \vec{\beta}_1 \times \vec{B}_1) + \epsilon^3(\varepsilon_0 c \vec{\beta}_1 \times \vec{B}_3) + h.o. \\
\Rightarrow &\varepsilon_0 \vec{E}_k = \alpha \vec{D}_k + \varepsilon_0 c \vec{\beta}_1 \times \vec{B}_{k-1}.
\end{aligned} \tag{4.30}$$

The k^{th} order of the auxiliary magnetic field for $k \geq 1$ is found to be

$$\mu_0 \vec{H}_k = \alpha \vec{B}_k - \frac{\vec{\beta}_1 \times \vec{D}_{k-1}}{\varepsilon_0 c}. \tag{4.31}$$

Proof:

$$
\begin{aligned}
&\mu_0\vec{H} = \alpha\vec{B} - \frac{\vec{\beta}\times\vec{D}}{\varepsilon_0 c} \\
\Rightarrow &\mu_0\left(\vec{H}_0 + \sum_{k\geq 1}\epsilon^k\vec{H}_k\right) = \alpha\left(\vec{B}_0 + \sum_{k\geq 1}\epsilon^k\vec{B}_k\right) - \frac{\epsilon\vec{\beta}_1\times\left(\vec{D}_0 + \sum_{k\geq 1}\epsilon^k\vec{D}_k\right)}{\varepsilon_0 c} \\
\Rightarrow &\mu_0\left(\vec{H}_0 + \epsilon\vec{H}_1 + \epsilon^2\vec{H}_2 + h.o.\right) = \alpha\left(\vec{B}_0 + \epsilon\vec{B}_1 + \epsilon^2\vec{B}_2 + h.o.\right) \\
&\qquad\qquad - \frac{\epsilon\vec{\beta}_1\times\left(\epsilon\vec{D}_1 + \epsilon^2\vec{D}_2 + \epsilon^3\vec{D}_3 + h.o.\right)}{\varepsilon_0 c} \qquad (4.32) \\
\Rightarrow &\mu_0\vec{H}_0 + \epsilon(\mu_0\vec{H}_1) + \epsilon^2(\mu_0\vec{H}_2) + h.o. = \alpha\vec{B}_0 + \epsilon(\alpha\vec{B}_1) + \epsilon^2(\alpha\vec{B}_2) + h.o. \\
&\qquad\qquad - \epsilon^2\frac{\vec{\beta}_1\times\vec{D}_1}{\varepsilon_0 c} - \epsilon^3\frac{\vec{\beta}_1\times\vec{D}_2}{\varepsilon_0 c} - h.o. \\
\Rightarrow &\mu_0\vec{H}_k = \alpha\vec{B}_k - \frac{\vec{\beta}_1\times\vec{D}_{k-1}}{\varepsilon_0 c}.
\end{aligned}
$$

From Maxwell's equations in a vacuum, and the fact that we have used a $3+1$ splitting and are only looking at one particular time slice hence $\partial/\partial t = 0$, we have the conditions

$$\vec{\nabla}\cdot\vec{B}_k = \vec{\nabla}\cdot\vec{D}_k = 0 \qquad (4.33a)$$

$$\vec{\nabla}\times\vec{H}_k = \vec{\nabla}\times\vec{E}_k = 0 \qquad (4.33b)$$

Then using equations (4.33a) and (4.33b) and equations (4.29) and (4.31) we find

$$\vec{\nabla}\times(\alpha\vec{D}_k) = -\varepsilon_0 c\vec{\nabla}\times(\vec{\beta}_1\times\vec{B}_{k-1}) \qquad (4.34a)$$

$$\vec{\nabla}\times(\alpha\vec{B}_k) = \frac{1}{\varepsilon_0 c}\vec{\nabla}\times(\vec{\beta}_1\times\vec{D}_{k-1}) \qquad (4.34b)$$

which is a hierarchical set of partial differential equations for fields $\vec{B}_k$ and $\vec{D}_k$ for $k \geq 1$. For $k = 0$, we have the static dipole case ($\vec{B}_0$). We find from equation (4.34b) that $\vec{B}_1 = 0$ since $\vec{D}_0 = 0$ and the first perturbation in the magnetic field only occurs in the second order, $\vec{B}_2$. We now wish to find the first perturbation in the electric field which is of first order, $\vec{D}_1$, corresponding to an electric quadrupole of $(l, m) = (2, 0)$ such that

$$\vec{D}_1 = \mathrm{Re}\left[\vec{\nabla} \times (f^D_{2,0}\vec{\Phi}_{2,0})\right]. \tag{4.35}$$

We now insert equation (4.35) into equation (4.34a) with $k = 1$ and inserting equation (3.4) for $\vec{B}_0$ we get

$$\vec{\nabla} \times \left(\alpha\vec{\nabla} \times \left(f^D_{2,0}(r)\vec{\Phi}_{2,0}\right)\right) = -\varepsilon_0 c\vec{\nabla} \times \left(\vec{\beta} \times \vec{\nabla} \times \left(f^{B(\mathrm{dip})}_{1,0}(r)\vec{\Phi}_{1,0}\right)\right) \tag{4.36}$$

Simplifying the LHS using equation (2.75) and equation (4.6) for $\vec{\Phi}_{2,0}$ we find

$$\begin{aligned}
&\vec{\nabla} \times \left(\alpha\vec{\nabla} \times \left(f^D_{2,0}(r)\vec{\Phi}_{2,0}\right)\right) \\
&\qquad = -\alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}\left(rf^D_{2,0}(r)\right)\right) - \frac{6}{r^2}f^D_{2,0}(r)\right]\vec{\Phi}_{2,0} \\
&\qquad = \sqrt{\frac{15}{8\pi}}\frac{\alpha}{r}\left[\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}\left(rf^D_{2,0}(r)\right)\right) - \frac{6}{r}f^D_{2,0}(r)\right]\cos\theta\sin\theta\hat{\phi}.
\end{aligned} \tag{4.37}$$

To simplify the RHS we use equation (2.79), and again using equation (4.6) for $\vec{\Phi}_{2,0}$ we find

$$-\varepsilon_0 c\vec{\nabla} \times \left(\vec{\beta} \times \vec{\nabla} \times \left(f^{B(\mathrm{dip})}_{1,0}(r)\vec{\Phi}_{1,0}\right)\right) = 6\sqrt{\frac{3}{8\pi}}\frac{\varepsilon_0\alpha\omega}{r}f^{B(\mathrm{dip})}_{1,0}(r)\cos\theta\sin\theta\hat{\phi}. \tag{4.38}$$

Now equating the LHS to RHS and substituting in for $f^{B(\mathrm{dip})}_{1,0}(r)$ from equation (3.33) we obtain

$$\begin{aligned}
&\sqrt{\frac{15}{8\pi}}\frac{\alpha}{r}\left[\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}\left(rf^D_{2,0}(r)\right)\right) - \frac{6}{r}f^D_{2,0}(r)\right]\cos\theta\sin\theta\hat{\phi} \\
&\qquad = 6\sqrt{\frac{3}{8\pi}}\frac{\varepsilon_0\alpha\omega}{r}\frac{\mu_0\mu}{4\pi}\sqrt{\frac{8\pi}{3}}\frac{3r}{R_s^3}\left[\ln\alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]\cos\theta\sin\theta\hat{\phi} \\
\Rightarrow\quad &\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}\left(rf^D_{2,0}(r)\right)\right) - \frac{6}{r}f^D_{2,0}(r) \\
&\qquad = 12\sqrt{\frac{6\pi}{5}}\frac{\varepsilon_0\mu_0\mu}{4\pi}\frac{ac}{R_s^2r^2}\left[\ln\alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]
\end{aligned} \tag{4.39}$$

where we have substituted in equation (2.13) for ω. Hence we now have a second-order inhomogeneous linear ordinary differential equation

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^D_{2,0}(r))) - \frac{6}{r} f^D_{2,0}(r) = 12\sqrt{\frac{6\pi}{5}} \frac{\varepsilon_0 \mu_0 \mu}{4\pi} \frac{ac}{R_s^2 r^2} \left[\ln \alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2} \right] \tag{4.40}$$

which will have a new solution for $f^D_{2,0}(r)$. Now in order to solve equation (4.40) we first solve for the homogeneous solution obtained previously in equation (4.13) which we shall now refer to as $f^{D(h)}_{2,0}(r)$ as follows

$$f^{D(h)}_{2,0}(r) = F_2' \frac{R_s^2}{4r} \left[6\frac{r^2}{R_s^2} \left(3 - 4\frac{r}{R_s} \right) \ln \alpha^2 + 1 + 6\frac{r}{R_s} \left(1 - 4\frac{r}{R_s} \right) \right]. \tag{4.41}$$

Here we have relabelled the constant F_2' as it will have a new value in the final solution of equation (4.40). We then solve for the particular solution of the inhomogeneous equation (4.40) vanishing at infinity. We use the method of undetermined coefficients to solve for the particular solution, which is essentially a "guess and check" method. We want the LHS of equation (4.40) to look like the RHS after it has been differentiated and summed up. We take our educated guess to be

$$f^{D(p)}_{2,0} = \mathcal{P}\frac{1}{r} \left[\ln \alpha^2 + \frac{R_s}{r} \right] \tag{4.42}$$

where $\mathcal{P}$ is our undetermined coefficient. This guess makes sense, as we can see on the LHS of equation (4.40), the $\left[(6/r) f^D_{2,0}(r)\right]$ would provide us with the $1/r^2$ out in front, and the derivative of $\left[r f^D_{2,0}(r)\right]$ would provide us with the extra term needed in the square brackets on the RHS. First finding the derivative of $\left[r f^{D(p)}_{2,0}(r)\right]$ we have

$$\begin{aligned} \frac{\partial}{\partial r}\left(r\mathcal{P}\frac{1}{r} \left[\ln \alpha^2 + \frac{R_s}{r} \right] \right) &= \mathcal{P} \left[\frac{R_s}{\alpha^2 r^2} - \frac{R_s}{r^2} \right] \\ &= \mathcal{P} \frac{R_s^2}{\alpha^2 r^3} \end{aligned} \tag{4.43}$$

Now substituting into the LHS of equation (4.40) we find

$$\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}\left(rf^D_{2,0}(r)\right)\right)-\frac{6}{r}f^D_{2,0}(r)=-3\mathcal{P}\frac{R_s}{r^4}-6\mathcal{P}\frac{1}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}\right]$$
$$=-6\mathcal{P}\frac{1}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}+\frac{R_s^2}{2r^2}\right] \tag{4.44}$$

which has the same form as the RHS of equation (4.40). Now comparing the RHS of equation (4.44) with the RHS of equation (4.40) we find $\mathcal{P}$ as:

$$-6\mathcal{P}\frac{1}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}+\frac{R_s^2}{2r^2}\right]=12\sqrt{\frac{6\pi}{5}}\frac{\varepsilon_0\mu_0\mu}{4\pi}\frac{ac}{R_s^2r^2}\left[\ln\alpha^2+\frac{R_s}{r}+\frac{R_s^2}{2r^2}\right]$$
$$\Rightarrow\quad \mathcal{P}=-2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{R_s^2}. \tag{4.45}$$

Hence, substituting equation (4.45) back into equation (4.42), we find the peculiar solution to be

$$f^{D(p)}_{2,0}=-2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{R_s^2r}\left[\ln\alpha^2+\frac{R_s}{r}\right]. \tag{4.46}$$

Now, the total solution to equation (4.40) is the sum of the homogeneous solution equation (4.41) and the particular solution equation (4.46) such that

$$f^D_{2,0}(r)=f^{D(h)}_{2,0}(r)+f^{D(p)}_{2,0}(r). \tag{4.47}$$

As before, in the case excluding frame-dragging, we need to satisfy the boundary conditions in order to solve for the constant F_2'. For equation (4.18), we need to know $\frac{\partial}{\partial r}\left(rf^D_{2,0}(r)\right)\big|_{r=R}$. We have $\frac{\partial}{\partial r}\left(rf^{D(h)}_{2,0}(r)\right)\Big|_{r=R}$ from equation (4.26). For $\frac{\partial}{\partial r}\left(rf^{D(p)}_{2,0}(r)\right)\Big|_{r=R}$ we have

$$\frac{\partial}{\partial r}\left(rf^{D(p)}_{2,0}(r)\right)\Big|_{r=R}=-2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{\alpha_R^2R^3} \tag{4.48}$$

hence

$$
\begin{aligned}
\frac{\partial}{\partial r}\left(r f^{D}_{2,0}(r)\right)\bigg|_{r=R} &= 36F'_2\frac{R}{4}\left[\left(1-2\frac{R}{R_s}\right)\ln\alpha_R^2 - 2 - \frac{R_s^2}{6\alpha_R^2R^2}\right] - 2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{\alpha_R^2R^3} \\
&= 36F'_2\frac{R}{4}\mathcal{Z}_2^{-1} - 2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{\alpha_R^2R^3}
\end{aligned} \tag{4.49}
$$

Now equating equations (4.17) and (4.18) we have

$$
\begin{aligned}
& 6\frac{\tilde{\omega}_R}{\alpha_R}\frac{\varepsilon_0\mu_0\mu}{4\pi}\frac{R}{R_s^3}\mathcal{Z}_1\cos\theta\sin\theta = \sqrt{\frac{15}{8\pi}}\frac{\alpha_R}{R}\left[36F'_2\frac{R}{4}\mathcal{Z}_2^{-1} - 2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{\alpha_R^2R^3}\right]\cos\theta\sin\theta \\
\Rightarrow\quad & 36F'_2\frac{R}{4}\mathcal{Z}_2^{-1} = 6\sqrt{\frac{8\pi}{15}}\frac{\tilde{\omega}_R}{\alpha_R^2}\frac{\varepsilon_0\mu_0\mu}{4\pi}\frac{R^2}{R_s^3}\mathcal{Z}_1 + 2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{\alpha_R^2R^3} \\
\Rightarrow\quad & 36F'_2\frac{R}{4}\mathcal{Z}_2^{-1} = \frac{4}{\alpha_R^2R_s^4}\sqrt{\frac{6\pi}{5}}\frac{\varepsilon_0\mu_0\mu}{4\pi}\left[R_sR^2\tilde{\omega}_R\mathcal{Z}_1 + \frac{acR_s^4}{2r^3}\right] \\
\Rightarrow\quad & F'_2 = \frac{4\mathcal{Z}_2}{9\alpha_R^2R_s^4}\sqrt{\frac{6\pi}{5}}\frac{\varepsilon_0\mu_0\mu}{4\pi}\left[R_sR\tilde{\omega}_R\mathcal{Z}_1 + \frac{\omega_RR_s^3}{2R}\right].
\end{aligned} \tag{4.50}
$$

Putting it all together we obtain the electric quadrupole function solution as

$$
\begin{aligned}
f_{2,0}^{D(\mathrm{quad})}(r) = F_2' \frac{R_s^2}{4r} & \left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \\
& \qquad - 2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{R_s^2 r}\left[\ln\alpha^2 + \frac{R_s}{r}\right] \\
= \frac{\mathcal{Z}_2}{9\alpha_R^2 R_s^2}\sqrt{\frac{6\pi}{5}}\frac{\varepsilon_0\mu_0\mu}{4\pi r} & \left[R_s R\tilde{\omega}_R\mathcal{Z}_1 + \frac{\omega_R R_s^3}{2R}\right] \\
& \times\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \\
& \qquad - 2\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\frac{ac}{R_s^2 r}\left[\ln\alpha^2 + \frac{R_s}{r}\right] \\
= \frac{\varepsilon_0\mu_0\mu}{4\pi r}\sqrt{\frac{6\pi}{5}} & \left\{\frac{\mathcal{Z}_2}{9\alpha_R^2 R_s^2}\left[R_s R\tilde{\omega}_R\mathcal{Z}_1 + \frac{\omega_R R_s^3}{2R}\right]\right. \\
& \times\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \\
& \qquad \left. - 2\frac{ac}{R_s^2}\left[\ln\alpha^2 + \frac{R_s}{r}\right]\right\} \\
= \frac{\varepsilon_0\mu_0\mu}{4\pi r}\sqrt{\frac{6\pi}{5}} & \left\{\frac{\mathcal{Z}_2}{18\alpha_R^2}\left[\frac{\omega_R R_s}{R} + 2\mathcal{Z}_1\frac{\tilde{\omega}_R R}{R_s}\right]\right. \\
& \times\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] \\
& \qquad \left. - 2\frac{\omega r^3}{R_s^3}\left[\ln\alpha^2 + \frac{R_s}{r}\right]\right\}
\end{aligned}
\tag{4.51}
$$

We now separate the frame-dragging effect ω of the electric quadrupole function from the pure rotation Ω as follows

$$
\begin{aligned}
f_{2,0}^{D(\mathrm{quad})}(r) &= \frac{\varepsilon_0\mu_0\mu}{4\pi r}\sqrt{\frac{6\pi}{5}}\left\{\frac{\mathcal{Z}_2}{18\alpha_R^2}\left[\frac{\omega_R R_s}{R} + 2\mathcal{Z}_1\frac{\tilde{\omega}_R R}{R_s}\right]\right. \\
& \left.\times\left[6\frac{r^2}{R_s^2}\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + 1 + 6\frac{r}{R_s}\left(1-4\frac{r}{R_s}\right)\right] - 2\frac{\omega r^3}{R_s^3}\left[\ln\alpha^2 + \frac{R_s}{r}\right]\right\} \\
&= \frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\left\{\frac{\mathcal{Z}_2 r}{3\alpha_R^2 R_s^2}\left[\frac{\omega_R R_s}{R} + 2\mathcal{Z}_1\frac{R}{R_s}(\Omega-\omega_R)\right]\right. \\
& \left.\times\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2 + \frac{R_s^2}{6r^2} + \frac{R_s}{r} - 4\right] - 2\frac{\omega r^2}{R_s^3}\left[\ln\alpha^2 + \frac{R_s}{r}\right]\right\} \\
&= \frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\left\{\left[\frac{\mathcal{Z}_2\omega_R r}{3\alpha_R^2 R_s R} - \frac{2\mathcal{Z}_1\mathcal{Z}_2\omega_R R r}{3\alpha_R^2 R_s^3} + \frac{2\mathcal{Z}_1\mathcal{Z}_2\Omega R r}{3\alpha_R^2 R_s^3}\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right]-2\frac{\omega r^2}{R_s^3}\left[\ln\alpha^2+\frac{R_s}{r}\right]\Bigg\} \\
=\;&\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\Bigg\{\frac{2\mathcal{Z}_1\mathcal{Z}_2\Omega Rr}{3\alpha_R^2R_s^3}\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right] \\
&-\frac{2\omega r^4}{R_s^5}\left[\left[\frac{\mathcal{Z}_1\mathcal{Z}_2R_s^2R}{3\alpha_R^2r^3}\frac{\omega_R}{\omega}-\frac{\mathcal{Z}_2R_s^4}{6\alpha_R^2Rr^3}\frac{\omega_R}{\omega}\right]\right. \\
&\left.\times\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right]+\frac{R_s^2}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}\right]\right]\Bigg\} \\
=\;&\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\Bigg\{\frac{2\mathcal{Z}_1\mathcal{Z}_2\Omega Rr}{3\alpha_R^2R_s^3}\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right] \\
&-\frac{2\omega r^4}{R_s^5}\left[\left[\frac{\mathcal{Z}_1\mathcal{Z}_2R_s^2}{3\alpha_R^2R^2}-\frac{\mathcal{Z}_2R_s^4}{6\alpha_R^2R^4}\right]\right. \\
&\left.\times\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right]+\frac{R_s^2}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}\right]\right]\Bigg\} \\
=\;&\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\Bigg\{\frac{2\mathcal{Z}_1\mathcal{Z}_2\Omega Rr}{3\alpha_R^2R_s^3}\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right] \\
&-\frac{2\omega r^4}{R_s^5}\left[\frac{\mathcal{Z}_2R_s^2}{3\alpha_R^2R^2}\left[\mathcal{Z}_1-\frac{R_s^2}{2R^2}\right]\right. \\
&\left.\times\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right]+\frac{R_s^2}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}\right]\right]\Bigg\} \\
=\;&\frac{\varepsilon_0\mu_0\mu}{4\pi}\sqrt{\frac{6\pi}{5}}\Bigg\{\frac{2\mathcal{Z}_1\mathcal{Z}_2\Omega Rr}{3\alpha_R^2R_s^3}\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right] \\
&-\frac{2\omega r^4}{R_s^5}\left[\frac{R_s^2}{r^2}\left[\ln\alpha^2+\frac{R_s}{r}\right]+\frac{\mathcal{Z}_2R_s^2}{3\alpha_R^2R^2}\left[\ln\alpha_R^2+\frac{R_s}{R}\right]\right. \\
&\left.\times\left[\left(3-4\frac{r}{R_s}\right)\ln\alpha^2+\frac{R_s^2}{6r^2}+\frac{R_s}{r}-4\right]\right]\Bigg\}. \qquad (4.52)
\end{aligned}
$$

Now to obtain the electric field components, we again use equation (4.7) and find that

$$D^{\hat{r}}=-\frac{1}{r}\sqrt{\frac{15}{8\pi}}f_{2,0}^{D(\mathrm{quad})}(3\cos^2\theta-1) \qquad (4.53a)$$

$$=-\frac{1}{r}\sqrt{\frac{15}{2\pi}}f_{2,0}^{D(\mathrm{quad})}P_2(\cos\theta) \qquad (4.53b)$$

$$D^{\hat{\theta}}=\frac{\alpha}{r}\sqrt{\frac{15}{8\pi}}\frac{\partial}{\partial r}(rf_{2,0}^{D(\mathrm{quad})})\cos\theta\sin\theta \qquad (4.53c)$$

$$D^{\hat{\phi}}=0 \qquad (4.53d)$$

where we have used the Legendre polynomial $P_2(x) = (3x^2 - 1)/2$ for the $D^{\hat{r}}$ component. The radial derivative is given by

$$\begin{aligned}
&\frac{\partial}{\partial r}(r f_{2,0}^{D(\text{quad})}) \\
&\quad = \sqrt{\frac{8\pi}{15}} \frac{\varepsilon_0 \mu_0 \mu r}{4\pi} \left\{ \frac{6\mathcal{Z}_1 \mathcal{Z}_2 \Omega R}{\alpha_R^2 R_s^3} \left[\left(1 - 2\frac{r}{R_s}\right) \ln \alpha^2 - 2 - \frac{R_s^2}{6\alpha^2 r^2} \right] - \frac{\omega r^3}{R_s^5} \times \right. \\
&\quad \left. \left[\frac{6\mathcal{Z}_2 R_s^2}{\alpha_R^2 R^2} \left(\ln \alpha_R^2 + \frac{R_s}{R} \right) \left[\left(1 - 2\frac{r}{R_s}\right) \ln \alpha^2 - 2 - \frac{R_s^2}{6\alpha^2 r^2} \right] \frac{3R_s^4}{\alpha^2 r^4} \right] \right\}.
\end{aligned} \tag{4.54}$$

We now have solutions for the magnetic and electric fields in a curved space up to the first order of expansion in the spin parameter. The order of expansion shall be explained in the next chapter where we shall derive a general formalism for the fields relating their expansion coefficients.

Chapter 5

General formalism of an aligned dipole with rotation

In this chapter we will derive a general relation between the expansion coefficients of the magnetic and electric fields, $\vec{B}_k$ and $\vec{D}_k$, for any order of expansion k. Neither the magnetic field nor the electric field contain any toroidal components in our problem as the fields are axisymmetric, hence any coefficient with $m > 0$ is zero. We, therefore, expand the fields as

$$\vec{D}_k = \sum_{l\geq 1} \vec{\nabla} \times (f_{l,0}^{D(k)} \vec{\Phi}_{l,0}) \tag{5.1a}$$

$$\vec{B}_k = \sum_{l\geq 1} \vec{\nabla} \times (f_{l,0}^{B(k)} \vec{\Phi}_{l,0}). \tag{5.1b}$$

The expansion in $f_{l,0}^{D(k)}$ is in turn an expansion in the spin parameter and is thus related to the frame-dragging effect. Substituting equations (5.1a) and (5.1b) into equations (4.34a) and (4.34b) for $(l,k) \geq 1$ we find

$$\begin{aligned} &\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f_{l,0}^{D(k)})) - \frac{l(l+1)}{r} f_{l,0}^{D(k)} \\ &\qquad = 3\varepsilon_0 \omega \left[l\sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}} f_{l-1,0}^{B(k-1)} - (l+1)\sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}} f_{l+1,0}^{B(k-1)} \right] \end{aligned} \tag{5.2a}$$

$$\begin{aligned}
&\frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(rf_{l,0}^{B(k)})) - \frac{l(l+1)}{r}f_{l,0}^{B(k)} \\
&\qquad = -3\mu_0\omega\left[l\sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}}f_{l-1,0}^{D(k-1)} - (l+1)\sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}}f_{l+1,0}^{D(k-1)}\right]
\end{aligned} \tag{5.2b}$$

We obtain equation (5.2a) using equations (2.75) and (2.79), as follows

$$\begin{aligned}
&\vec{\nabla}\times(\alpha\sum_{l\geq 1}\vec{\nabla}\times(f_{l,0}^{D(k)}\vec{\Phi}_{l,0})) = -\varepsilon_0 c\vec{\nabla}\times(\vec{\beta}_1\times\sum_{l\geq 1}\vec{\nabla}\times(f_{l,0}^{B(k-1)}\vec{\Phi}_{l,0}) \\
&\Rightarrow -\alpha\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}(rf_{l,0}^{D(k)})\right) - \frac{l(l+1)}{r^2}f_{l,0}^{D(k)}\right]\vec{\Phi}_{l,0} \\
&\qquad = -3\varepsilon_0 c\alpha\frac{\omega}{cr}f_{l,0}^{B(k-1)}\left[\sqrt{l(l+2)}J_{l+1,0}\vec{\Phi}_{l+1,0} - \sqrt{(l+1)(l-1)}J_{l,0}\vec{\Phi}_{l-1,0}\right] \\
&\Rightarrow \left[\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}(rf_{l,0}^{D(k)})\right) - \frac{l(l+1)}{r}f_{l,0}^{D(k)}\right]\vec{\Phi}_{l,0} \\
&\qquad = 3\varepsilon_0\omega\left[(l+1)\sqrt{\frac{l(l+2)}{(2l+1)(2l+3)}}\vec{\Phi}_{l+1,0}f_{l,0}^{B(k-1)} - l\sqrt{\frac{(l+1)(l-1)}{(2l+1)(2l-1)}}\vec{\Phi}_{l-1,0}f_{l,0}^{B(k-1)}\right] \\
&\Rightarrow \left[\frac{\partial}{\partial r}\left(\alpha^2\frac{\partial}{\partial r}(rf_{l,0}^{D(k)})\right) - \frac{l(l+1)}{r}f_{l,0}^{D(k)}\right]\vec{\Phi}_{l,0} \\
&\qquad = 3\varepsilon_0\omega\left[l\sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}}\vec{\Phi}_{l,0}f_{l-1,0}^{B(k-1)} - (l+1)\sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}}\vec{\Phi}_{l,0}f_{l+1,0}^{B(k-1)}\right] \\
&\Rightarrow \frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(rf_{l,0}^{D(k)})) - \frac{l(l+1)}{r}f_{l,0}^{D(k)} \\
&\qquad = 3\varepsilon_0\omega\left[l\sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}}f_{l-1,0}^{B(k-1)} - (l+1)\sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}}f_{l+1,0}^{B(k-1)}\right].
\end{aligned} \tag{5.3}$$

Similarly, we obtain equation (5.2b) as

$$
\begin{aligned}
&\vec{\nabla} \times (\alpha \sum_{l\geq 1} \vec{\nabla} \times (f_{l,0}^{B(k)} \vec{\Phi}_{l,0}) = \frac{1}{\varepsilon_0 c} \vec{\nabla} \times (\vec{\beta}_1 \times \sum_{l\geq 1} \vec{\nabla} \times (f_{l,0}^{D(k-1)} \vec{\Phi}_{l,0})) \\
&\Rightarrow - \alpha \left[\frac{1}{r} \frac{\partial}{\partial r} \left(\alpha^2 \frac{\partial}{\partial r} (r f_{l,0}^{B(k)}) \right) - \frac{l(l+1)}{r^2} f_{l,0}^{B(k)} \right] \vec{\Phi}_{l,0} \\
&\quad = 3\alpha \frac{\omega}{\varepsilon_0 c^2 r} f_{l,0}^{D(k-1)} \left[\sqrt{l(l+2)} J_{l+1,0} \vec{\Phi}_{l+1,0} - \sqrt{(l+1)(l-1)} J_{l,0} \vec{\Phi}_{l-1,0} \right] \\
&\Rightarrow \left[\frac{\partial}{\partial r} \left(\alpha^2 \frac{\partial}{\partial r} (r f_{l,0}^{B(k)}) \right) - \frac{l(l+1)}{r} f_{l,0}^{B(k)} \right] \vec{\Phi}_{l,0} \\
&\quad = -3\mu_0 \omega \left[(l+1) \sqrt{\frac{l(l+2)}{(2l+1)(2l+3)}} \vec{\Phi}_{l+1,0} f_{l,0}^{D(k-1)} - l \sqrt{\frac{(l+1)(l-1)}{(2l+1)(2l-1)}} \vec{\Phi}_{l-1,0} f_{l,0}^{D(k-1)} \right] \\
&\Rightarrow \left[\frac{\partial}{\partial r} \left(\alpha^2 \frac{\partial}{\partial r} (r f_{l,0}^{B(k)}) \right) - \frac{l(l+1)}{r} f_{l,0}^{B(k)} \right] \vec{\Phi}_{l,0} \\
&\quad = -3\mu_0 \omega \left[l \sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}} \vec{\Phi}_{l,0} f_{l-1,0}^{D(k-1)} - (l+1) \sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}} \vec{\Phi}_{l,0} f_{l+1,0}^{D(k-1)} \right] \\
&\Rightarrow \frac{\partial}{\partial r} (\alpha^2 \frac{\partial}{\partial r} (r f_{l,0}^{B(k)})) - \frac{l(l+1)}{r} f_{l,0}^{B(k)} \\
&\quad = -3\mu_0 \omega \left[l \sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}} f_{l-1,0}^{D(k-1)} - (l+1) \sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}} f_{l+1,0}^{D(k-1)} \right].
\end{aligned}
\tag{5.4}
$$

For $(l, m) = (0, 0)$ we take the expansion coefficients to be zero, ie $f_{0,0}^{D(k)} = f_{0,0}^{B(k)} = 0$, as, in this case, we have no electric or magnetic field. It is clear that the coefficients $f_{l,0}^{D(k)}$ and $f_{l,0}^{B(k)}$ are related by the coefficients one order below themselves, ie $f_{l,0}^{D(k)}$ and $f_{l,0}^{B(k)}$ are related to $f_{l,0}^{B(k-1)}$ and $f_{l,0}^{D(k-1)}$ respectively. It is important to note, from this, that $f_{l,0}^{D(k)}$ and $f_{l,0}^{B(k)}$ are uncoupled in nature. We are thus able to solve for any order expansion coefficient provided we solve for the coefficients preceding the one we want. The approximate solutions to third order are shown below as examples of the recurrence nature of the solutions. These solutions are useful if ever higher degrees of accuracy are needed, as will be seen.

As before, the electric field only occurs in the first order as it requires the rotation of the star, hence for the zeroth order we have $f_{l,0}^{D(0)} = 0$. For the zeroth order of the magnetic field we have $f_{1,0}^{B(0)} = f_{1,0}^{B(\mathrm{dip})}$ from equation (3.33). For $l \geq 2$ the magnetic field coefficients in the zeroth order are zero, ie $f_{l\geq 2,0}^{B(0)} = 0$. We do not have any corrections in the first order to the magnetic field. This is because $f_{l,0}^{D(0)} = 0$ implying that the

RHS of equation (5.2b) is zero for $f_{l,0}^{B(1)}$. We then have a linear homogeneous second-order partial differential equation with the boundary conditions vanishing at infinity, hence the solution also vanishes meaning $f_{l,0}^{B(1)} = 0$. We then have our first correction from the coefficient $f_{l,0}^{D(1)}$ which is solved using equation (5.2a). For this case, the only inhomogeneous equation occurs for $l = 2$, since for $f_{l,0}^{B(0)}$ we can only have $l = 1$, and we obtain equation (4.40) in the following form

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f_{2,0}^{D(1)})) - \frac{6}{r} f_{2,0}^{D(1)} = \frac{6}{\sqrt{5}} \varepsilon_0 \omega f_{1,0}^{B(0)} \tag{5.5}$$

with solution equation (4.52). We next evaluate $f_{l,0}^{B(2)}$ using equation (5.2b) where the RHS can only use the coefficient $f_{2,0}^{D(1)}$ and the boundary conditions again vanish at infinity. We obtain two inhomogeneous equations, one for $l = 1$ and one for $l = 3$,

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f_{1,0}^{B(2)})) - \frac{2}{r} f_{1,0}^{B(2)} = \frac{6}{\sqrt{5}} \frac{\omega}{\varepsilon_0 c^2} f_{2,0}^{D(1)} \tag{5.6a}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f_{3,0}^{B(2)})) - \frac{12}{r} f_{3,0}^{B(2)} = -18\sqrt{\frac{2}{35}} \frac{\omega}{\varepsilon_0 c^2} f_{2,0}^{D(1)}. \tag{5.6b}$$

Lastly, the second order magnetic field coefficients are used to solve for the third order electric field coefficients where we again obtain two inhomogeneous equations, now for $l = 2$ and $l = 4$, with the boundary conditions at the stellar surface as prescribed by equation (4.16). We obtain the equations

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f_{2,0}^{D(3)})) - \frac{6}{r} f_{2,0}^{D(3)} = \frac{6}{\sqrt{5}} \varepsilon_0 \omega \left[f_{1,0}^{B(2)} - 3\sqrt{\frac{2}{7}} f_{3,0}^{B(2)} \right] \tag{5.7a}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f_{4,0}^{D(3)})) - \frac{20}{r} f_{4,0}^{D(3)} = 4\sqrt{\frac{15}{7}} \varepsilon_0 \omega f_{3,0}^{B(2)}. \tag{5.7b}$$

We can now see, from equations (5.5) to (5.7b), the hierarchical nature used to increase the accuracy of the solutions of the fields step by step by increasing the number of multipoles of order l corresponding to the required degree of approximation in the spin parameter.

Solving for these equations is beyond the scope of this work. Only the first few coefficients can be solved analytically, thereafter, the equations become too complicated. It is possible, however, to solve the equations numerically using spectral methods by expanding the solutions into rational Chebyshev functions[32, 46]. The initial outline on how to do this will be given below.

To begin with, the fields of $\vec{D}$ and $\vec{B}$ are expanded using a finite number of multipolar coefficients, N_D and N_B respectively, and we drop the order k, such that

$$\vec{D} = \sum_{l=1}^{N_D} \vec{\nabla} \times (f^D_{l,0}\vec{\Phi}_{l,0}) \tag{5.8a}$$

$$\vec{B} = \sum_{l=1}^{N_B} \vec{\nabla} \times (f^B_{l,0}\vec{\Phi}_{l,0}). \tag{5.8b}$$

The coefficients $f^D_{l,0}$ and $f^B_{l,0}$ now have to satisfy

$$\begin{aligned}\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^D_{l,0})) - \frac{l(l+1)}{r} f^D_{l,0} \\ = 3\varepsilon_0 \omega \left[l\sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}} f^B_{l-1,0} - (l+1)\sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}} f^B_{l+1,0} \right]\end{aligned} \tag{5.9a}$$

$$\begin{aligned}\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^B_{l,0})) - \frac{l(l+1)}{r} f^B_{l,0} \\ = -3\frac{\omega}{\varepsilon_0 c^2} \left[l\sqrt{\frac{(l-1)(l+1)}{(2l-1)(2l+1)}} f^D_{l-1,0} - (l+1)\sqrt{\frac{l(l+2)}{(2l+3)(2l+1)}} f^D_{l+1,0} \right]\end{aligned} \tag{5.9b}$$

Once again, we have the boundary conditions at infinity enforcing vanishing coefficients and we now need to enforce the boundary conditions at the neutron star surface as we did in chapter (4). We insert equations (5.8a) and (5.8b) into equation (4.16), make use of equations (2.74c) and (2.78) and equate the $\hat{\theta}$ components to obtain the relationship between the $\vec{D}$ and $\vec{B}$ coefficients as follows:

Substituting in equations (5.8a) and (5.8b) into equation (4.16):

$$\begin{aligned} \vec{D} &= \varepsilon_0 c \frac{\tilde{\omega}}{\alpha\omega} \vec{\beta} \times \vec{B} \\ \Rightarrow \sum_{l=1}^{N_D} \vec{\nabla} \times (f_{l,0}^D \vec{\Phi}_{l,0}) &= \varepsilon_0 c \frac{\tilde{\omega}}{\alpha\omega} \vec{\beta} \times \sum_{l=1}^{N_B} \vec{\nabla} \times (f_{l,0}^B \vec{\Phi}_{l,0}). \end{aligned} \tag{5.10}$$

Making use of equations (2.74c), (2.66), (2.69) and (2.78):

$$\begin{aligned} &\sum_{l=1}^{N_D} \left[-\frac{\sqrt{l(l+1)}}{r} f_{l,0}^D Y_{l,0} \hat{r} - \frac{\alpha}{r} \frac{\partial}{\partial r}(r f_{l,0}^D) \frac{r}{\sqrt{l(l+1)}} \vec{\nabla} Y_{l,0} \right] \\ &\quad = \varepsilon_0 c \frac{\tilde{\omega}}{\alpha\omega} \frac{\omega}{c} \sin\theta \sum_{l=1}^{N_B} \left[\sqrt{l(l+1)} f_{l,0}^B Y_{l,0} \hat{\theta} - \frac{\alpha}{\sqrt{l(l+1)}} \frac{\partial}{\partial r}(r f_{l,0}^B) \frac{\partial}{\partial \theta} Y_{l,0} \hat{r} \right] \\ \Rightarrow &\sum_{l=1}^{N_D} \left[-\frac{\sqrt{l(l+1)}}{r} f_{l,0}^D Y_{l,0} \hat{r} - \frac{\alpha}{r\sqrt{l(l+1)}} \frac{\partial}{\partial r}(r f_{l,0}^D) \frac{\partial}{\partial \theta} Y_{l,0} \hat{\theta} \right] \\ &\quad = \frac{\varepsilon_0 \tilde{\omega}}{\alpha} \sin\theta \sum_{l=1}^{N_B} \left[\sqrt{l(l+1)} f_{l,0}^B Y_{l,0} \hat{\theta} - \frac{\alpha}{\sqrt{l(l+1)}} \frac{\partial}{\partial r}(r f_{l,0}^B) \frac{\partial}{\partial \theta} Y_{l,0} \hat{r} \right]. \end{aligned} \tag{5.11}$$

Equating the $\hat{\theta}$ components and multiplying both sides by $\sin\theta$:

$$-\alpha^2 \frac{1}{\sqrt{l(l+1)}} \frac{\partial}{\partial r}(r f_{l,0}^D) \sin\theta \frac{\partial}{\partial \theta} Y_{l,0} = \varepsilon_0 r \tilde{\omega} \sqrt{l(l+1)} f_{l,0}^B \sin^2\theta Y_{l,0}. \tag{5.12}$$

We now make use of the following recurrence relations

$$\sin\theta \frac{\partial}{\partial \theta} Y_{l,m} = l J_{l+1,m} Y_{l+1,m} - (l+1) J_{l,m} Y_{l-1,m} \tag{5.13a}$$

$$\cos\theta Y_{l,m} = J_{l+1,m} Y_{l+1,m} + J_{l,m} Y_{l-1,m} \tag{5.13b}$$

$$\cos^2\theta Y_{l,m} = J_{l+1,m} J_{l+2,m} Y_{l+2,m} + (J_{l+1,m}^2 + J_{l,m}^2) Y_{l,m} + J_{l,m} J_{l-1,m} Y_{l-2,m}. \tag{5.13c}$$

Note that $\sin^2\theta Y_{l,0}$ can be rewritten as $Y_{l,0} - \cos^2\theta Y_{l,0}$. Thus we have

$$\begin{aligned}
&- \alpha^2 \frac{1}{\sqrt{l(l+1)}} \frac{\partial}{\partial r}(r f^D_{l,0}) \left[l J_{l+1,0} Y_{l+1,0} - (l+1) J_{l,0} Y_{l-1,0} \right] \\
&\qquad = \varepsilon_0 r \tilde{\omega} \sqrt{l(l+1)} f^B_{l,0} \left[Y_{l,0} - J_{l+1,0} J_{l+2,0} Y_{l+2,0} - (J^2_{l+1,0} + J^2_{l,0}) Y_{l,0} \right. \\
&\qquad\qquad \left. - J_{l,0} J_{l-1,0} Y_{l-2,0} \right].
\end{aligned} \tag{5.14}$$

Multiplying out all variables containing l and reducing or increasing the value of l for each component to obtain only $Y_{l,0}$ which will be subsequently cancelled off, we finally have the relationship between the $\vec{D}$ and $\vec{B}$ coefficients as

$$\begin{aligned}
&- \alpha^2 \left[\sqrt{\frac{l}{l+1}} J_{l+1,0} \frac{\partial}{\partial r}(r f^D_{l,0}) Y_{l+1,0} - \sqrt{\frac{l+1}{l}} J_{l,0} \frac{\partial}{\partial r}(r f^D_{l,0}) Y_{l-1,0} \right] \\
&= \varepsilon_0 r \tilde{\omega} \left[\sqrt{l(l+1)} f^B_{l,0} Y_{l,0} - \sqrt{l(l+1)} J_{l+1,0} J_{l+2,0} f^B_{l,0} Y_{l+2,0} \right. \\
&\qquad \left. - \sqrt{l(l+1)} (J^2_{l+1,0} + J^2_{l,0}) f^B_{l,0} Y_{l,0} - \sqrt{l(l+1)} J_{l,0} J_{l-1,0} f^B_{l,0} Y_{l-2,0} \right] \\
\Rightarrow \quad & \alpha^2 \left[\sqrt{\frac{l+2}{l+1}} J_{l+1,0} \frac{\partial}{\partial r}(r f^D_{l+1,0}) - \sqrt{\frac{l-1}{l}} J_{l,0} \frac{\partial}{\partial r}(r f^D_{l-1,0}) \right] \\
&= \varepsilon_0 r \tilde{\omega} \left[\sqrt{l(l+1)} (1 - J^2_{l,0} - J^2_{l+1,0}) f^B_{l,0} - \sqrt{(l-2)(l-1)} J_{l,0} J_{l-1,0} f^B_{l-2,0} \right. \\
&\qquad \left. - \sqrt{(l+2)(l+3)} J_{l+1,0} J_{l+2,0} f^B_{l+2,0} \right]
\end{aligned} \tag{5.15}$$

which has to be evaluated at the star's surface $r = R$. Below we shall show the system of partial differential equations of the first three coefficients for each the magnetic and electric field obtained from equations (5.9a) and (5.9b) and their associated boundary conditions which are obtained from equation (5.15). The partial differential equations for the field coefficients read as follows

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^B_{1,0})) - \frac{2}{r} f^B_{1,0} = \frac{6}{\sqrt{5}} \frac{\omega}{\varepsilon_0 c^2} f^D_{2,0} \tag{5.16a}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^B_{3,0})) - \frac{12}{r} f^B_{3,0} = \frac{1}{\sqrt{7}} \frac{\omega}{\varepsilon_0 c^2} \left[-18\sqrt{\frac{2}{5}} f^D_{2,0} + 4\sqrt{15} f^D_{4,0} \right] \tag{5.16b}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^B_{5,0})) - \frac{30}{r} f^B_{5,0} = \frac{2}{\sqrt{11}} \frac{\omega}{\varepsilon_0 c^2} \left[-5\sqrt{6} f^D_{4,0} + 9\sqrt{\frac{35}{13}} f^D_{6,0} \right] \tag{5.16c}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^D_{2,0})) - \frac{6}{r} f^D_{2,0} = \frac{6}{\sqrt{5}} \varepsilon_0 \omega \left[f^B_{1,0} - 3\sqrt{\frac{2}{7}} f^B_{3,0} \right] \tag{5.16d}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^D_{4,0})) - \frac{20}{r} f^D_{4,0} = 2\sqrt{3} \varepsilon_0 \omega \left[2\sqrt{\frac{5}{7}} f^B_{3,0} - 5\sqrt{\frac{2}{11}} f^B_{5,0} \right] \tag{5.16e}$$

$$\frac{\partial}{\partial r}(\alpha^2 \frac{\partial}{\partial r}(r f^D_{6,0})) - \frac{42}{r} f^D_{6,0} = 18\sqrt{\frac{35}{143}} \varepsilon_0 \omega f^B_{5,0} \tag{5.16f}$$

The associated boundary conditions evaluated at $r = R$ read as

$$\sqrt{\frac{2}{5}} \alpha^2 \frac{\partial}{\partial r}(r f^D_{2,0}) = \varepsilon_0 r \tilde{\omega} \left[\frac{2\sqrt{2}}{5} f^B_{1,0} - \frac{12}{5\sqrt{7}} f^B_{3,0} \right] \tag{5.17a}$$

$$\alpha^2 \left[\frac{2}{3}\sqrt{\frac{5}{7}} \frac{\partial}{\partial r}(r f^D_{4,0}) - \sqrt{\frac{6}{35}} \frac{\partial}{\partial r}(r f^D_{2,0}) \right] = \varepsilon_0 r \tilde{\omega} \left[\frac{44}{15\sqrt{3}} f^B_{3,0} - \frac{2}{5}\sqrt{\frac{6}{7}} f^B_{1,0} - \frac{20}{3}\sqrt{\frac{10}{231}} f^B_{5,0} \right] \tag{5.17b}$$

$$\alpha^2 \left[\sqrt{\frac{42}{143}} \frac{\partial}{\partial r}(r f^D_{6,0}) - \frac{2}{3}\sqrt{\frac{5}{11}} \frac{\partial}{\partial r}(r f^D_{4,0}) \right] = \varepsilon_0 r \tilde{\omega} \left[\frac{58}{39}\sqrt{\frac{10}{3}} f^B_{5,0} - \frac{40}{3\sqrt{231}} f^B_{3,0} \right] \tag{5.17c}$$

Note that for $l = 1$, the magnetic field at the surface of the star is equal to the general-relativistic static dipole equation (3.33) and for all other multipoles, $f^B_{l\neq 1,0}$ vanishes at $r = R$.

From here, one is able to use spectral methods, preferably choosing to use rational Chebyshev functions, to solve numerically for the fields. We have thus found a general relation for the aligned fields which can be solved numerically to the desired degree of accuracy.

Chapter 6

Magnetic field of an orthogonal dipole without rotation

After studying the aligned dipole magnetic field, we shall now look at the extreme case of an orthogonal dipole magnetic field in a vacuum without rotation and confirm that we obtain the orthogonal dipole magnetic field in a flat spacetime. For the orthogonal dipole without rotation, we use the same methods as before with the aligned dipole magnetic field. We again expect the magnetic field far from the neutron star to be the same as the flat spacetime magnetic field, which, for the orthogonal case, is

$$\vec{B} = \frac{\mu_0 \mu}{4\pi r^3}\left(2\sin\theta\cos\phi\hat{r} - \cos\theta\cos\phi\hat{\theta} + \sin\phi\hat{\phi}\right). \tag{6.1}$$

The full derivation of equation (6.1) can be found in appendix A. We take the definition of the orthogonal dipole magnetic field in a curved spacetime as being expressed with only the spherical harmonic $\vec{\Phi}_{1,1}$ corresponding to $(l, m) = (1, 1)$ since, again, $l = 1$ corresponds to the dipole field and now $m = 1$ corresponds to the orthogonal alignment. We thus expand the magnetic field as

$$\vec{B} = \mathrm{Re}\left[\vec{\nabla} \times (f^B_{1,1}(r)\vec{\Phi}_{1,1})\right]. \tag{6.2}$$

In order to find the flat spacetime magnetic field of equation (6.2) and satisfy equation (6.1), we take the limit of equation (6.2) as $r \to \infty$ and equate it to equation (6.1) to solve for $\lim_{r\to\infty} f^B_{1,1}(r)$. First we calculate the value of $\vec{\Phi}_{1,1}$. From equation (2.65) we have

$$\vec{Y}_{1,1} = -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}\hat{r}. \tag{6.3}$$

Taking the gradient of $\vec{Y}_{1,1}$ we find

$$\vec{\nabla}\vec{Y}_{1,1} = -\frac{1}{r}\sqrt{\frac{3}{8\pi}}\left[\cos\theta e^{i\phi}\hat{\theta} + ie^{i\phi}\hat{\phi}\right]. \tag{6.4}$$

Hence, from equation (2.67), we obtain

$$\begin{aligned}\vec{\Phi}_{1,1} &= \frac{r\hat{r}}{\sqrt{2}} \times -\frac{1}{r}\sqrt{\frac{3}{8\pi}}\left[\cos\theta e^{i\phi}\hat{\theta} + ie^{i\phi}\hat{\phi}\right] \\ &= \sqrt{\frac{3}{16\pi}}e^{i\phi}\left[i\hat{\theta} - \cos\theta\hat{\phi}\right].\end{aligned} \tag{6.5}$$

Now taking the limit of equation (6.2), substituting in equation (6.5) and taking the real solution we have

$$\begin{aligned}\vec{B} &= \sqrt{\frac{3}{16\pi}}\mathrm{Re}\left[\vec{\nabla}\times\left(\lim_{r\to\infty} f^B_{1,1}(r)e^{i\phi}\left[i\hat{\theta} - \cos\theta\hat{\phi}\right]\right)\right] \\ &= \sqrt{\frac{3}{16\pi}}\frac{1}{r}\mathrm{Re}\left[\lim_{r\to\infty} f^B_{1,1}(r)e^{i\phi}2\sin\theta\hat{r} + \frac{\partial}{\partial r}\left(r\lim_{r\to\infty} f^B_{1,1}(r)\right)e^{i\phi}\cos\theta\hat{\theta}\right. \\ &\qquad\left. + \frac{\partial}{\partial r}\left(r\lim_{r\to\infty} f^B_{1,1}(r)\right)ie^{i\phi}\hat{\phi}\right] \\ &= \sqrt{\frac{3}{16\pi}}\frac{1}{r}\left[\lim_{r\to\infty} f^B_{1,1}(r)2\sin\theta\cos\phi\hat{r} + \frac{\partial}{\partial r}\left(r\lim_{r\to\infty} f^B_{1,1}(r)\right)\cos\theta\cos\phi\hat{\theta}\right. \\ &\qquad\left. - \frac{\partial}{\partial r}\left(r\lim_{r\to\infty} f^B_{1,1}(r)\right)\sin\phi\hat{\phi}\right]\end{aligned} \tag{6.6}$$

Equating the $\hat{r}$ components of equations (6.1) and (6.6) we find $\lim_{r\to\infty} f^B_{1,0}(r)$ to be

$$\begin{aligned}
&\sqrt{\frac{3}{16\pi}}\frac{1}{r}\lim_{r\to\infty} f^B_{1,1}(r) 2\sin\theta\cos\phi = \frac{\mu_0\mu}{4\pi r^3} 2\sin\theta\cos\phi \\
&\Rightarrow \lim_{r\to\infty} f^B_{1,1}(r) = \frac{\mu_0\mu}{4\pi r^2}\sqrt{\frac{16\pi}{3}}
\end{aligned} \tag{6.7}$$

and hence, in the far field limit, equation (6.2) becomes

$$\vec{B} = \mathrm{Re}\left[\frac{\mu_0\mu}{4\pi}\sqrt{\frac{16\pi}{3}}\vec{\nabla}\times(\frac{\vec{\Phi}_{1,1}}{r^2})\right]. \tag{6.8}$$

As before, we wish to find a separable solution for equation (6.2) using the boundary condition in equation (6.7). Again, we wish to satisfy (3.1b) and so we insert equation (6.2) into equation (3.1b) and solve, as before, to obtain the second-order linear ordinary differential equation for the scalar $f^B_{1,1}(r)$ as

$$\frac{\partial}{\partial r}(\alpha^2\frac{\partial}{\partial r}(rf^B_{1,1})) - \frac{2}{r}f^B_{1,1} = 0. \tag{6.9}$$

Solving equation (6.9) is exactly the same as how we solved for equation (3.14). We find that

$$f^B_{1,1}(r) = F_3 r\left[\ln\alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right]. \tag{6.10}$$

Using equation (6.7) to solve for F_3 we have

$$\begin{aligned}
\lim_{r\to\infty} f^B_{1,1}(r) &= \lim_{r\to\infty} F_3 r q_1 \\
\Rightarrow \frac{\mu_0\mu}{4\pi r^2}\sqrt{\frac{16\pi}{3}} &= \lim_{r\to\infty} F_3 r\left[-\frac{R_s}{r} - \frac{R_s^2}{2r^2} - \frac{R_s^3}{3r^3} - \frac{R_s^4}{4r^4}... + \frac{R_s}{r} + \frac{R_s^2}{2r^2}\right] \\
&= \lim_{r\to\infty} -F_3 r\left[\frac{R_s^3}{3r^3} + \frac{R_s^4}{4r^4} + ...\right] \\
&= \lim_{r\to\infty} -F_3 r\frac{R_s^3}{3r^3}\left[1 + \frac{3R_s}{4r} + \frac{3R_s^2}{5r^2} + ...\right] \\
&= -F_3\frac{R_s^3}{3r^2}.
\end{aligned} \tag{6.11}$$

and hence

$$F_3 = -\frac{\mu_0 \mu}{4\pi} \sqrt{\frac{16\pi}{3}} \frac{3}{R_s^3}. \tag{6.12}$$

The solution to equation (6.9) is thus

$$f_{1,1}^{B(\mathrm{dip})} = -\frac{\mu_0 \mu}{4\pi} \sqrt{\frac{16\pi}{3}} \frac{3r}{R_s^3} \left[\ln \alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2} \right] \tag{6.13}$$

which is related to the aligned dipole magnetic field, equation (3.33), by

$$f_{1,1}^{B(\mathrm{dip})} = -\sqrt{2} f_{1,0}^{B(\mathrm{dip})}. \tag{6.14}$$

We find the magnetic field components by solving equation (6.2). First we find $\frac{\partial}{\partial r}(r f_{1,1}^{B(\mathrm{dip})})$ as

$$\frac{\partial}{\partial r}(r f_{1,1}^{B(\mathrm{dip})}(r)) = -\frac{\mu_0 \mu}{4\pi} \sqrt{\frac{16\pi}{3}} \frac{3r}{\alpha R_s^3} \left[2\alpha \ln \alpha^2 + \frac{R_s}{r} \frac{2r - R_s}{\sqrt{r(r - R_s)}} \right]. \tag{6.15}$$

The magnetic field is found to be

$$\vec{B} = \frac{1}{r} \sqrt{\frac{3}{16\pi}} \left[f_{1,1}^{B(\mathrm{dip})} 2 \sin\theta \cos\phi \hat{r} + \alpha \frac{\partial}{\partial r}(r f_{1,1}^{B(\mathrm{dip})}(r))(\cos\theta \cos\phi \hat{\theta} - \sin\phi \hat{\phi}) \right] \tag{6.16}$$

and hence the magnetic field components are

$$B^{\hat{r}} = -6 \frac{\mu_0 \mu}{4\pi R_s^3} \left[\ln \alpha^2 + \frac{R_s}{r} + \frac{R_s^2}{2r^2} \right] \sin\theta \cos\phi \tag{6.17a}$$

$$B^{\hat{\theta}} = -3 \frac{\mu_0 \mu}{4\pi R_s^3} \left[2\alpha \ln \alpha^2 + \frac{R_s}{r} \frac{2r - R_s}{\sqrt{r(r - R_s)}} \right] \cos\theta \cos\phi \tag{6.17b}$$

$$B^{\hat{\phi}} = 3 \frac{\mu_0 \mu}{4\pi R_s^3} \left[2\alpha \ln \alpha^2 + \frac{R_s}{r} \frac{2r - R_s}{\sqrt{r(r - R_s)}} \right] \sin\phi. \tag{6.17c}$$

As with equations (3.37a) and (3.37b), we take the limits on equations (6.17a) to (6.17c) as $r \to \infty$ to compare to the flat spacetime magnetic field of an orthogonal dipole in equation (6.1) and find

$$B^{\hat{r}} = \frac{\mu_0 \mu}{4\pi r^3} 2 \sin\theta \cos\phi \tag{6.18a}$$

$$B^{\hat{\theta}} = -\frac{\mu_0 \mu}{4\pi r^3} \cos\theta \cos\phi \tag{6.18b}$$

$$B^{\hat{\phi}} = \frac{\mu_0 \mu}{4\pi r^3} \sin\phi \tag{6.18c}$$

as expected.

In this chapter we have solved for the magnetic field of an orthogonal dipole without rotation in a curved space. We have taken the far field limit as $r \to \infty$, as we did for the aligned dipole, and proven that it is equal to the orthogonal flat spacetime magnetic field, as it should be. We shall omit the solutions to the orthogonal electric field for now as they are solved using spectral methods which will not be covered in this work.

Chapter 7

Conclusion

It is clear that general relativistic effects cannot be ignored when considering a neutron star's magnetosphere, particularly near the surface. We used the approach of a $3 + 1$ split of the spacetime metric in order to solve for the Maxwell's equations of an isolated neutron star in a vacuum. A concise breakdown of the equations and formalisms used was given in chapter (2). Our formalism differed slightly from most in that we used a $(+, -, -, -)$ metric signature instead of the $(-, +, +, +)$ metric signature.

We began, in chapter (3), by solving for the magnetic field of an aligned dipole without rotation. Solving for equation (3.14) proved to be a difficult task and many different methods of solving second-order linear ordinary differential equations were employed before finding the correct method, aided greatly by Rezzolla, Ahmedov and Miller (2001)[25]. The solution was obtained using Legendre's functions and the boundary condition at infinity in equation (3.9). Once the solution to equation (3.14) was obtained, the magnetic field solution equation (3.35) was found fairly smoothly with the far field limit equation (3.37) matching the standard aligned flat spacetime dipole magnetic field solution, hence confirming our solution to be mathematically sound.

Chapter (4) introduced rotation to the system in order to solve for the quadrupole electric field. We knew, from the constitutive relations, equations (2.35a) and (2.35b), that the electric and magnetic field would mix from the frame-dragging effects. We thus simplified the method of solving for the electric field and initially excluded the frame-dragging effects by setting $\beta = 0$. Once the electric field was found without any frame-dragging effects, we then went on to include the frame-dragging effects and adapted the electric field solution. The electric field without frame-dragging effects was found again using the Legendre functions, this time with the boundary condition at the surface of the neutron star, with its solution being equation (4.25) with the electric field solution equation (4.27). We then introduce the frame-dragging effects by expanding the fields in a power series and solving for the first order electric field (with the magnetic field remaining the same as our solution in chapter (3) which is of zeroth order as the first order magnetic field is zero). We obtain a second-order inhomogeneous linear ordinary differential equation (4.40) with the homogeneous solution being equation (4.41). We tried many different methods, such as Green's function, to solve for the particular solutions before settling on the method of undetermined coefficients and found the solution to be equation (4.46). The final solution for the electric quadrupole function, using the boundary condition at the surface of the neutron star, was found to be equation (4.51) and separating the frame-dragging effect ω from the pure rotation Ω we found it to be equation (4.52) with the electric field solution equation (4.53).

In chapter (5) we provided a general relationship between the fields of an aligned dipole for any order in the spin such that one can evaluate the fields to any desired degree of accuracy. In chapters (3) and (4) we solved for the zeroth order magnetic field and first order electric field respectively. With the relationship between the magnetic and electric field coefficients obtained in chapter (5), equation (5.15), one can thus determine the field solutions of higher order multipoles as can be see in equations (5.16a) to (5.16f) where an increase in l indicates an increase in the multipole order. Solving for these equations requires the use of spectral methods, such as rational Chebyshev functions, to solve numerically for the fields, which is beyond the scope of this dissertation.

Chapter (6) began looking at the case of the orthogonal dipole where we solved for the non-rotating magnetic field. The orthogonal flat spacetime dipole magnetic field had to be calculated from scratch as the standard flat spacetime dipole magnetic field solution is given for an aligned dipole. The orthogonal magnetic field was found using the same methods as for the aligned magnetic field with orthogonal magnetic dipole function being equation (6.13) and magnetic field solution equation (6.17). The far field limit of equation (6.17) matched the orthogonal flat spacetime dipole magnetic field solution, again confirming our solution to be mathematically sound.

This work provides us with a great foundation and understanding of the concepts involved in solving for the fields under general relativistic considerations. It can be applied to future work when solving for an oblique rotator and subsequently apply this knowledge to particle motion and generation, particularly at the polar caps and hence pulsar emission mechanisms.

Appendix A

Orthogonal Flat Spacetime Dipole Magnetic Field Derivation

The derivation of the aligned dipole magnetic field in a flat spacetime can be found in [45]. This derivation can be altered to find the orthogonal dipole magnetic field in a flat spacetime.

We align the dipole along the x-axis which consists of two charges, $+b$ and $-b$, separated by an infinitesimal distance l. This dipole produces the same field as would an infinitesimal current loop dipole would such that $bl = \mathcal{I}A = \mu$, where $\mu = \|\vec{\mu}\|$ and $\vec{\mu}$ is the magnetic dipole moment, A is the loop area and $\mathcal{I}$ is the current in the loop. We now wish to find the field of the dipole at point P which is shown in figure (A.1) and lies in the xz-plane.

From figure (A.1), we can define r, r_1 and r_2 as

$$r = \sqrt{z^2 + x^2} \tag{A.1a}$$

$$r_1 = \sqrt{z^2 + \left(x + \frac{l}{2}\right)^2} \tag{A.1b}$$

$$r_2 = \sqrt{z^2 + \left(x - \frac{l}{2}\right)^2}. \tag{A.1c}$$

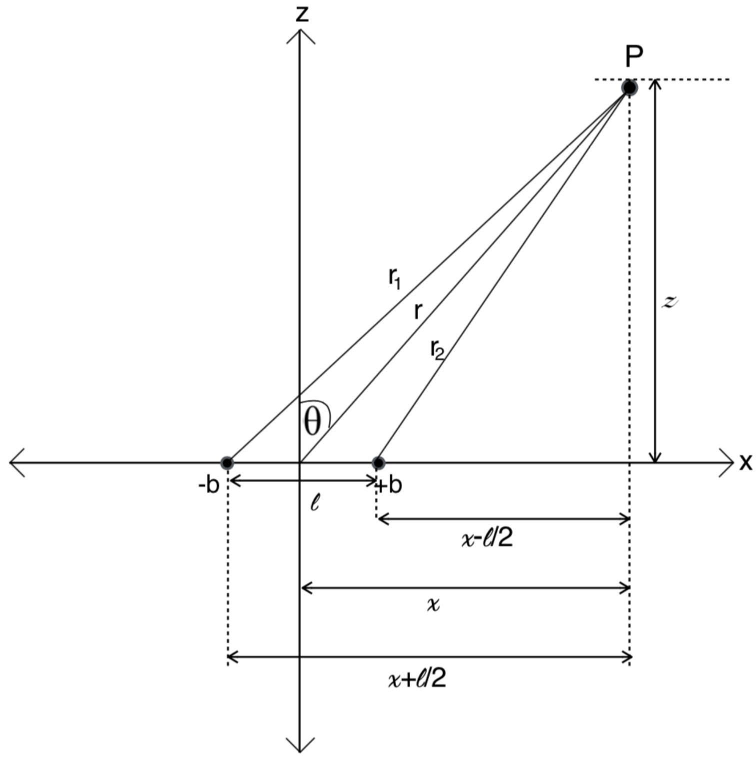

FIGURE A.1: Distance to point P from a magnetic dipole aligned along the x-axis in a flat spacetime.

We want our final answer to be in spherical coordinates, hence we note the relationship between the cartesian coordinates and the spherical coordinates as

$$x = r \sin\theta \cos\phi \tag{A.2a}$$

$$y = r \sin\theta \sin\phi \tag{A.2b}$$

$$z = r \cos\theta \tag{A.2c}$$

with the relationship between the unit vectors being

$$\hat{x} = \sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi} \tag{A.3a}$$

$$\hat{y} = \sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi} \tag{A.3b}$$

$$\hat{z} = \cos\theta\hat{r} - \sin\theta\hat{\theta}. \tag{A.3c}$$

To determine the field at point P for each field component, we sum the fields due to each of the charges $+b$ and $-b$ in the component direction. For the B_z component, we thus have

$$\begin{aligned} B_z &= B_{z(-b)} + B_{z(+b)} \\ &= \frac{-bz}{r_1^3} + \frac{bz}{r_2^3} \\ &= -bz\left[\frac{1}{r_1^3} - \frac{1}{r_2^3}\right]. \end{aligned} \tag{A.4}$$

We are interested only in the limit of small l, hence all higher orders in l vanish. For $1/r_1^3$ we have

$$\begin{aligned} \lim_{l\to 0}\frac{1}{r_1^3} &= \lim_{l\to 0}\left[z^2 + \left(x + \frac{l}{2}\right)^2\right]^{-\frac{3}{2}} \\ &= \lim_{l\to 0}\left[z^2 + x^2 + xl\right]^{-\frac{3}{2}} \\ &= \lim_{l\to 0}\frac{1}{r^3}\left[1 + \frac{xl}{r^2}\right]^{-\frac{3}{2}} \\ &= \frac{1}{r^3}\left[1 - \frac{3xl}{2r^2}\right] \end{aligned} \tag{A.5}$$

and for $1/r_2^3$ we have

$$
\begin{aligned}
\lim_{l \to 0} \frac{1}{r_2^3} &= \lim_{l \to 0} \left[z^2 + \left(x - \frac{l}{2} \right)^2 \right]^{-\frac{3}{2}} \\
&= \lim_{l \to 0} \left[z^2 + x^2 - xl \right]^{-\frac{3}{2}} \\
&= \lim_{l \to 0} \frac{1}{r^3} \left[1 - \frac{xl}{r^2} \right]^{-\frac{3}{2}} \\
&= \frac{1}{r^3} \left[1 + \frac{3xl}{2r^2} \right].
\end{aligned} \tag{A.6}
$$

Substituting equations (A.5) and (A.6) into equation (A.4) and using equation (A.2a) we find B_z in the limit of small l to be

$$
\begin{aligned}
\lim_{l \to 0} B_z &= -\frac{bz}{r^3} \left[1 - \frac{3xl}{2r^2} - 1 - \frac{3xl}{2r^2} \right] \\
&= \frac{bz}{r^3} \frac{3xl}{r^2} \\
&= 3 \frac{bl}{r^3} \frac{zx}{r^2} \\
&= 3 \frac{\mu}{r^3} \cos\theta \sin\theta \cos\phi.
\end{aligned} \tag{A.7}
$$

For B_x component, summing the fields of the two point charges in the x direction and using equations (A.5), (A.6) and (A.2a) we find B_x in the limit of small l to be

$$
\begin{aligned}
\lim_{l \to 0} B_x &= B_{x(-b)} + B_{x(+b)} \\
&= \frac{-b}{r_1^2} \frac{x + \frac{l}{2}}{r_1} + \frac{b}{r_2^2} \frac{x - \frac{l}{2}}{r_2} \\
&= -b \left[\frac{x + \frac{l}{2}}{r_1^3} - \frac{x - \frac{l}{2}}{r_2^3} \right] \\
&= -\frac{b}{r^3} \left[\left(x + \frac{l}{2} \right) \left(1 - \frac{3xl}{2r^2} \right) - \left(x - \frac{l}{2} \right) \left(1 + \frac{3xl}{2r^2} \right) \right] \\
&= -\frac{b}{r^3} \left[l - \frac{3x^2 l}{r^2} \right] \\
&= 3 \frac{bl}{r^3} \left[\frac{x^2}{r^2} - \frac{1}{3} \right] \\
&= 3 \frac{\mu}{r^3} \left(\sin^2\theta \cos^2\phi - \frac{1}{3} \right).
\end{aligned} \tag{A.8}
$$

If we now place our point P in the xy-plane such that the configuration looks exactly the same as that in figure (A.1) except now, instead of z, we have y, we can find the y component of the magnetic field which looks exactly the same as the z component. Hence for B_y in the limit of small l we find

$$\begin{aligned}\lim_{l\to 0} B_y &= 3\frac{bl}{r^3}\frac{yx}{r^2} \\ &= 3\frac{\mu}{r^3}\sin^2\theta\cos\phi\sin\phi.\end{aligned} \tag{A.9}$$

The total magnetic field is now found by summing all three magnetic field components. Hence using equations (A.7), (A.8), (A.9) and (A.3a) we find the total field as

$$\begin{aligned}\vec{B} &= B_x\hat{x} + B_y\hat{y} + B_z\hat{z} \\ &= 3\frac{\mu}{r^3}\left[\left(\sin^2\theta\cos^2\phi - \frac{1}{3}\right)\left(\sin\theta\cos\phi\hat{r} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi}\right) + \sin^2\theta\cos\phi\sin\phi\right. \\ &\qquad \left.\times\left(\sin\theta\sin\phi\hat{r} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi}\right) + \cos\theta\sin\theta\cos\phi\left(\cos\theta\hat{r} - \sin\theta\hat{\theta}\right)\right] \\ &= 3\frac{\mu}{r^3}\left[\left(\sin^3\theta\cos^3\phi - \frac{1}{3}\sin\theta\cos\phi + \sin^3\theta\cos\phi\sin^2\phi + \cos^2\theta\sin\theta\cos\phi\right)\hat{r} + \right. \\ &\quad \left(\cos\theta\sin^2\theta\cos^3\phi - \frac{1}{3}\cos\theta\cos\phi + \cos\theta\sin^2\theta\cos\phi\sin^2\phi - \cos\theta\sin^2\theta\cos\phi\right)\hat{\theta} \\ &\qquad \left. + \left(-\sin^2\theta\cos^2\phi\sin\phi + \frac{1}{3}\sin\phi + \sin^2\theta\cos^2\phi\sin\phi\right)\hat{\phi}\right] \\ &= 3\frac{\mu}{r^3}\left[\sin\theta\cos\phi\left(\sin^2\theta\cos^2\phi - \frac{1}{3} + \sin^2\theta\sin^2\phi + \cos^2\theta\right)\hat{r}\right. \\ &\qquad \left. + \cos\theta\cos\phi\left(\sin^2\theta\cos^2\phi - \frac{1}{3} + \sin^2\theta\sin^2\phi - \sin^2\theta\right)\hat{\theta} + \frac{1}{3}\sin\phi\hat{\phi}\right] \\ &= 3\frac{\mu}{r^3}\left[\frac{2}{3}\sin\theta\cos\phi\hat{r} - \frac{1}{3}\cos\theta\cos\phi\hat{\theta} + \frac{1}{3}\sin\phi\hat{\phi}\right] \\ &= \frac{\mu}{r^3}\left[2\sin\theta\cos\phi\hat{r} - \cos\theta\cos\phi\hat{\theta} + \sin\phi\hat{\phi}\right].\end{aligned} \tag{A.10}$$

For our expression of the orthogonal flat spacetime dipole magnetic field, we thus have

$$\vec{B} = \frac{\mu_0\mu}{4\pi r^3}\left(2\sin\theta\cos\phi\hat{r} - \cos\theta\cos\phi\hat{\theta} + \sin\phi\hat{\phi}\right). \tag{A.11}$$

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