ON STEPHANI UNIVERSES

by

Selvandren Moopanar

Submitted in fulfilment of the
requirements for the degree of
Master of Science,
in the
Department of Mathematics and Applied Mathematics,
University of Natal

Durban 1992

Abstract

In this dissertation we study conformal symmetries in the Stephani universe which is a generalisation of the Robertson–Walker models. The kinematics and dynamics of the Stephani universe are discussed. The conformal Killing vector equation for the Stephani metric is integrated to obtain the general solution subject to integrability conditions that restrict the metric functions. Explicit forms are obtained for the conformal Killing vector as well as the conformal factor. There are three categories of solution. The solution may be categorized in terms of the metric functions k and R. As the case $\dot{k}R - k\dot{R} \neq 0$ is the most complicated, we provide all the details of the integration procedure. We write the solution in compact vector notation. As the case $k = 0$ is simple, we only state the solution without any details. In this case we exhibit a conformal Killing vector normal to hypersurfaces $t =$ constant which is an analogue of a vector in the $k = 0$ Robertson–Walker spacetimes. The above two cases contain the conformal Killing vectors of Robertson–Walker spacetimes. For the last case $\dot{k}R - k\dot{R} = 0$, $k \neq 0$ we provide an outline of the integration process. This case gives conformal Killing vectors which do not reduce to those of Robertson–Walker spacetimes. A number of the calculations performed in finding the solution of the conformal Killing vector equation are extremely difficult to analyse by hand. We therefore utilise the symbolic manipulation capabilities of Mathematica (Ver 2.0) (Wolfram 1991) to assist with calculations.

To my family

for their tremendous support and encouragement

Preface

The study described in this dissertation was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period January 1992 to December 1992. It was completed under the excellent supervision of Dr. S. D. Maharaj and Prof. P. G. L. Leach.

This study represents original work by the author except where use of the work of others has been duly acknowledged in the text and it has not been submitted in any form to another University.

Acknowledgements

The author would like to sincerely thank the following for their courteous cooperation in contributing to the success of this dissertation:

- Dr. S. D. Maharaj and Prof. P. G. L. Leach for their expert assistance spontaneously provided while planning and writing this dissertation.
- All authors of books and articles, whether mentioned by name or not, whose ideas this dissertation contains.
- The financial support of the University of Natal through a Graduate Assistantship, the Foundation for Research and Development and the German Academic Exchange Service (DAAD).
- The valuable computer assistance provided by Keshlan Govinder.
- Siva Naidoo, Darren Lortan and Pradhavin Moopanar for their careful proof-reading of this dissertation.
- The administrative assistance rendered by both Mrs. Joyce Ross, Secretary of the Department of Mathematics and Applied Mathematics, and Mrs. Barbara Becker, Faculty Officer, Faculty of Science.

Contents

1 Introduction

We take spacetime to be a 4–dimensional pseudo–Riemannian manifold endowed with a symmetric, non–degenerate metric tensor field. The spacetime geometry resembles the geometry of special relativity in the neighbourhood of a point in the manifold. The gravitational field is described by the metric tensor field. For many cosmological and astrophysical applications it is important to solve the Einstein field equations explicitly to obtain the gravitational field. The Einstein field equations are a non–linear coupled system of partial differential equations. The matter content is given by the symmetric energy–momentum tensor. The spacetime geometry is described by the curvature tensor which is used to construct the Einstein tensor. The Einstein field equations relate the matter content (energy–momentum tensor) to the spacetime geometry (Einstein tensor). Einstein obtained the field equations using a physical argument; however, the field equations may be mathematically rigorously generated utilising a variational argument. The field equations satisfy conservation laws called the Bianchi identities. In the quasi–static limit we regain the Newtonian potential for weak gravitational fields.

Exact solutions to the Einstein field equations are important as they allow for a discussion of the physical properties of specific models. They also throw light on more general qualitative features of the gravitational field. Exact solutions also have extensive applicability in cosmology and astrophysics. A comprehensive list of exact solutions to the Einstein field equations is given by Kramer *et al* (1980). Solutions to the field equations may be generated using different procedures. For example we could define the energy–momentum tensor by specifying an equation of state and solve the field equations for the metric tensor. Another possibility is to suppose that the gravitational field possesses a symmetry, e.g. a Killing vector. This assumption leads to a simplification of the field equations. Most solutions known are spacetimes of high symmetry such as the Robertson–Walker spacetimes. With less symmetry the field equations are more difficult to integrate. Thus the form of the energy–momentum tensor is specified or a restriction is imposed on the spacetime geometry in an attempt to find the solution.

A recent approach used to find exact solutions is to impose a conformal symmetry requirement on the manifold. This leads to a deeper understanding of the spacetime geometry and has the added advantage of simplifying the field equations and possibly, their solutions. A number of exact solutions have been found in various models with the assumption that the spacetime is invariant under a conformal Killing vector. In particular conformal Killing vectors have been studied in perfect fluids and anisotropic fluids by Herrera and Ponce de Leon (1985a,b,c), Herrera *et al* (1984), Maartens and Maharaj (1990), Maartens *et al* (1986), Mason and Maartens (1987), Mason and Tsamparlis (1985) and Saridakis and Tsamparlis (1991). Dyer *et al* (1987) and Maharaj *et al* (1991) have analysed spherically symmetric cosmological models with a conformal symmetry. The physical properties of the solutions to the field equations with a conformal symmetry have been extensively investigated by a number of authors. In particular the inheritance properties of kinematical and dynamical quantities in spacetimes with an inheriting conformal Killing vector, fluid flow lines are mapped conformally into fluid flow lines, have been analysed. Results in specific spacetimes have been obtained by Coley (1991) and Coley and Tupper (1989, 1990a,b,c,d). Furthermore conformal Killing vectors may be important in relativistic kinetic theory, e.g. for a distribution function for massless particles in equilibrium the inverse temperature function is a conformal Killing vector (Israel 1972). This demonstrates that it is important to find explicitly conformal Killing vectors in general relativity. The G_{15} Lie algebra of conformal Killing vectors in Minkowski spacetime is listed by Choquet-Bruhat *et al* (1977). Maartens and Maharaj (1986) found the G_{15} Lie algebra of conformal Killing vectors in the Robertson–Walker spacetimes. The conformal Killing vector equation was integrated by Maartens and Maharaj (1991) to obtain the conformal geometry of plane fronted gravitational waves with parallel rays, the *pp*-wave spacetimes. Moodley (1991) obtained solutions to the conformal Killing vector equation in certain locally rotationally symmetric spacetimes.

In this dissertation we investigate the conformally flat solution of the Einstein field equations with a perfect fluid source. This spacetime is called the Stephani universe and is a generalisation of the standard cosmological models, the Robertson–Walker models (Robertson 1929, Walker 1935). The Stephani universe differs from the Robertson–Walker models in that

(i) it is generally not homogeneous or isotropic,

(ii) the curvature index is an arbitrary function of the time coordinate and is therefore not fixed.

However the Stephani universe does reduce to Robertson–Walker under the appropriate conditions. The Robertson–Walker spacetimes have six Killing vectors; in contrast the Stephani universes do not possess Killing symmetries in general. We will solve the conformal Killing vector equation for the Stephani metric thus obtaining the conformal Killing vector and the conformal factor in general. Parts of the calculation involved in finding this solution are complicated and difficult to perform by hand. We therefore utilise the symbolic manipulation capabilities of Mathematica (Ver 2.0) (Wolfram 1991) to help with calculations. We also verify our solutions obtained using this package. It is interesting to observe that although the Stephani universe may not admit Killing vector symmetries, it does contain conformal Killing symmetries.

In chapter 2 we briefly consider those concepts in differential geometry of general relativity necessary for this dissertation. We begin by introducing manifolds. In general relativity spacetime is taken to be a 4–dimensional pseudo–Riemannian manifold with a symmetric tensor field. Vector fields and tensor fields obeying the tensor transformation law are defined on the manifold. In particular we consider the indefinite symmetric metric tensor field of rank two which describes the gravitational field. We define the covariant derivative by introducing the additional structure of a connection on the manifold. The curvature tensor, the Ricci tensor, Ricci scalar, Einstein tensor and energy–momentum tensor are defined. We are then in a position to motivate the Einstein field equations. The Lie derivative is a geometrical object which is naturally defined on the manifold. The Lie bracket, Lie algebras and Lie groups are briefly discussed. We define the conformal Killing vector and list the special cases of Killing, homothetic, special and nonspecial conformal Killing vectors.

In chapter 3 we consider the spacetime geometry of the Stephani universes in detail. We briefly discuss some of the kinematical and dynamical features of the Stephani metric and its relationship to the Robertson–Walker models. The Einstein field equations are derived in full as the details of the spacetime curvature are not well known. We calculate the nonzero connection coefficients, the components of the Ricci tensor, the Ricci scalar and the components of the Einstein tensor. The

Einstein field equations reduce to a system of two differential equations. We note that the field equations obtained agree with the equations given in the literature. The conditions under which the Stephani metric reduces to Robertson–Walker are also discussed. We observe that the field equations may also be derived via the Gauss–Codazzi equations (Stephani 1967) or by utilising the notion of infinitesimal null isotropy (Koch–Sen 1985)

In chapter 4 we solve the conformal Killing vector equation for the Stephani metric in the case $\dot{k}R - k\dot{R} \neq 0$ where k and R are metric functions. This is the most difficult case considered and we provide all the details of the integration process. We obtain the spacelike components and the timelike component of the conformal Killing vector. The conformal factor is also found. The solution obtained is subject to integrability conditions. The integrability conditions relate the functions of integration to the metric functions. Due to the complexity of parts of the calculation we utilise the symbolic manipulation capabilities of Mathematica (Ver 2.0) to assist with the calculations. Mathematica is also used to check that the conformal Killing vector equations are in fact satisfied by our solution.

In chapter 5 we provide a complete analysis of conformal symmetries in the Stephani universe. We collect the results obtained in chapter 4 for the case $\dot{k}R - k\dot{R} \neq 0$ and express the results in the more compact vector notation. The two cases omitted in chapter 4 are also considered. We solve the conformal Killing vector equation for the special case $k = 0$. As the integration process is trivial, we do not provide any details. We also integrate the conformal Killing vector equations in the remaining case $\dot{k}R - k\dot{R} = 0$, $k \neq 0$. An outline of the integration procedure is sketched. We again use Mathematica to verify that the solution obtained satisfies the conformal Killing vector equations. Wherever possible we have attempted to regain the conformal Killing vectors of the Robertson–Walker spacetimes.

In the conclusion we summarise the results obtained in this dissertation. Some areas for future investigation are pointed out. We note that the results obtained in this dissertation are original. We have not found any published work in the literature on the solution of the conformal Killing vector equation in the Stephani universes.

2 Tensors, Field Equations and Lie Algebras

2.1 Introduction

In this chapter we briefly introduce some fundamental concepts of differential geometry in general relativity relevant to this dissertation. For a more thorough treatment see Bishop and Goldberg (1968), Choquet–Bruhat *et al* (1977), Hawking and Ellis (1973) and Misner *et al* (1973). We begin by introducing the concept of an n–dimensional differentiable manifold. In general relativity we take spacetime to be a 4–dimensional pseudo–Riemannian manifold. The differentiability of the manifold allows us to define vector fields and tensor fields on the manifold which obey the general invariant tensor transformation law. The indefinite symmetric metric tensor field of rank two describes the gravitational field. The metric tensor field of general relativity reduces to the Lorentzian metric of special relativity in the neighbourhood of a point in the manifold. The above material is covered in §2.2. In §2.3 we introduce the additional structure of a connection on the manifold which enables us to define the covariant derivative. We define the concept of parallel transport and introduce curvature on the manifold by defining the Riemann tensor. In addition we define the Ricci tensor, the Ricci scalar and the Einstein tensor. The matter content of the universe is described by the general energy–momentum tensor. We relate the curvature of Riemannian space to the distribution of matter via the Einstein field equations. In §2.4 we consider the Lie derivative and its properties. The Lie bracket is defined and a Lie algebra is introduced. We briefly discuss the relationship between a Lie group and a Lie algebra. We define the conformal Killing vector and list the special cases of Killing, homothetic, special and nonspecial conformal Killing vectors. In later chapters we seek the conformal Killing vectors in Stephani universes.

2.2 Manifolds and Tensors

The concept of a manifold is central to the study of differential geometry and for the formulation of the theory of general relativity. The surface of a sphere is a classic example of a manifold. In fact any m–dimensional hypersurface in an n–dimensional Euclidean space ($m \leq n$) is a manifold. A more abstract example is the set of all rigid rotations of Cartesian coordinates in 3–dimensional Euclidean space. In general we may consider a manifold to be any set that can be continuously parametrised. The dimension of a manifold is given by the number of independent parameters and these parameters are the coordinates of the manifold. An n–dimensional differentiable manifold M has the property that it is mapped locally into $\Re^n$ so that locally its features are similar to $\Re^n$. It is important to note that although the local structure of a manifold is similar to $\Re^n$, the global topology of M may be very different from that of $\Re^n$. For a detailed treatment of manifolds and physical applications the reader is referred to Bishop and Goldberg (1968), Burke (1985), Choquet–Bruhat *et al* (1977), Dubrovin *et al* (1985), Misner *et al* (1973) and Straumann(1984).

Let M be any set and U an open subset of M. We define a bijective function $\psi : U \longrightarrow \Re^n$ that maps points $p \in U$ to $\Re^n$. The pair (U, ψ) is called a chart. The coordinate functions of the chart are the real–valued functions on U given by the mapping ψ, called the coordinate map. The purpose of ψ is to attach coordinates to the point $p \in U$. The set U is called the coordinate neighbourhood and a point $p \in U$ has coordinates $(x^0, x^1, x^2, \ldots, x^{n-1})$. Consider the collection of charts $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$, where I is an index set. This collection forms an atlas if the following properties are satisfied:

(i) The set of coordinate neighbourhoods $\{U_\alpha\}$ covers M. Thus it is possible to write

$$M = \bigcup_{\alpha \in I} U_\alpha$$

(ii) For each α we have a bijective map $\psi_\alpha : U_\alpha \longrightarrow \Re^n$. Each function ψ_α takes the open set U_α into a region of $\Re^n$ for the same n.

(iii) If $U_\alpha \cap U_\beta \neq \emptyset$, for some α and β, then the composite maps

$$\psi_\alpha \circ \psi_\beta^{-1} : \Re^n \longrightarrow \Re^n$$

$$\psi_\beta \circ \psi_\alpha^{-1} : \Re^n \longrightarrow \Re^n$$

are differentiable functions from $\Re^n$ to $\Re^n$. The inverse maps ψ_α^{-1} and ψ_β^{-1} are defined as ψ_α and ψ_β are injective. Therefore in the overlap of the two coordinate neighbourhoods U_α and U_β the coordinates in one neighbourhood are continuously differentiable functions of the coordinates in the other neighbourhood.

(iv) The collection $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$ is maximal. Thus any other chart will be contained in this collection.

For a well–defined atlas the properties (i)–(iv) listed above must be satisfied. We define an n–dimensional differentiable manifold to be the set M together with the atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in I}$.

The differentiability of a manifold allows us to introduce continuous coordinate systems, at least locally. This also permits the definition of curves, vector fields and tensor fields. We define a regularly parametrised smooth curve on the manifold as the map $x^a : \Re \longrightarrow M$. In relativity we consider only 4–dimensional manifolds where points in the manifold are labelled by the real coordinates

$$(x^0, x^1, x^2, x^3) = (ct, x^1, x^2, x^3)$$

where we set the speed of light $c = 1$. Consider two charts $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\beta, \psi_\beta)\}$ where $U_\alpha \cap U_\beta \neq \emptyset$. The coordinates in U_α and U_β, (x^0, x^1, x^2, x^3) and $(x^{0'}, x^{1'}, x^{2'}, x^{3'})$, respectively, are generated by the charts $\{(U_\alpha, \psi_\alpha)\}$ and $\{(U_\beta, \psi_\beta)\}$. We consider only orientable manifolds so that in the overlapping region $U_\alpha \cap U_\beta \neq \emptyset$ the Jacobians are positive:

$$\left|\frac{\partial x^a}{\partial x^{b'}}\right| > 0 \qquad \text{and} \qquad \left|\frac{\partial x^{a'}}{\partial x^b}\right| > 0$$

In an orientable manifold all overlapping domains of the various coordinate neighbourhoods in the atlas admit invertible coordinate transformations such that the Jacobians are positive.

Let $T_p(M)$ denote the space of all tangent vectors at the point p on a curve in the manifold M. The set of vectors $T_p(M)$ is a vector space called the tangent vector space to M at p. We construct the dual vector space $T_p^*(M)$ by defining the real–valued linear functional $T_p^*(M) : T_p(M) \longrightarrow \Re$ at p. From the space $T_p(M)$ and its dual $T_p^*(M)$ we form the Cartesian product

$$\Pi_r^s = \underbrace{T^* \times T^* \times \ldots \times T^*}_{r \text{ times}} \times \underbrace{T \times T \times \ldots \times T}_{s \text{ times}}$$

at $p \in M$. A tensor $\mathbf{T}$ of type (r, s) at p is a multilinear functional which maps elements of Π_r^s into $\Re$. The vector space of all such tensors, denoted by T_s^r, is formed by the tensor product

$$T^r_s \equiv \underbrace{T \otimes T \otimes \ldots \otimes T}_{r \text{ times}} \otimes \underbrace{T^* \otimes T^* \otimes \ldots \otimes T^*}_{s \text{ times}}$$

at $p \in M$. For further details see Hawking and Ellis (1973), Misner *et al* (1973) and Stephani (1990). By assigning a tensor of type (r,s) to each point along a curve in the manifold we generate a type (r,s) tensor field. If $\{\mathbf{e}_a\}$ and $\{\mathbf{e}^a\}$ are dual bases of $T_p(M)$ and $T^*_p(M)$, respectively, then a tensor field $\mathbf{T} \in T^r_s$ can be expressed as:

$$\mathbf{T} = T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}\, \mathbf{e}_{a_1} \otimes \mathbf{e}_{a_2} \otimes \ldots \otimes \mathbf{e}_{a_r} \otimes \mathbf{e}^{b_1} \otimes \mathbf{e}^{b_2} \otimes \ldots \otimes \mathbf{e}^{b_s}$$

where $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ are the components of $\mathbf{T}$. The components $T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s}$ of the tensor field $\mathbf{T}$ with respect to new dual bases $\{\mathbf{e}_{a'}\}$ and $\{\mathbf{e}^{a'}\}$ are related to the components $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ by the following transformation law:

$$T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} = X^{a'_1}_{c_1} X^{a'_2}_{c_2} \ldots X^{a'_r}_{c_r} X^{d_1}_{b'_1} X^{d_2}_{b'_2} \ldots X^{d_s}_{b'_s}\; T^{c_1 c_2 \ldots c_r}{}_{d_1 d_2 \ldots d_s} \tag{2.1}$$

where $X^a_{b'}$ represents the Jacobian matrix $\partial x^a / \partial x^{b'}$.

In order to consider metrical properties on the manifold we define a symmetric, nonsingular, covariant tensor field $\mathbf{g}$ of rank two on M called the metric tensor field. Thus we have that $\mathbf{g} \in T^* \otimes T^*$ and $\mathbf{g} = g_{ab}\mathbf{e}^a \otimes \mathbf{e}^b$ where $\mathbf{g}$ is the bilinear functional

$$\mathbf{g} : (\mathbf{e}_a, \mathbf{e}_b) \longrightarrow \Re$$

relative to the basis $\{\mathbf{e}_a\}$. The metric tensor field $\mathbf{g}$ endows M with the inner product

$$\begin{aligned} \langle \mathbf{X}, \mathbf{Y} \rangle &= \mathbf{g}(\mathbf{X}, \mathbf{Y}) \\ &= g_{ab} X^a Y^b \end{aligned}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are vector fields. The manifold M, in which the indefinite metric tensor $\mathbf{g}$ is defined, is called a pseudo–Riemannian manifold. The invariant quantity

$$s = \int_{t_1}^{t_2} |g_{ab}\dot{x}^a\dot{x}^b|^{\frac{1}{2}} dt$$

defines the length along a curve on M between t_1 and t_2 which represent the values of the parameter t at the endpoints of the curve. This definition is independent of the coordinates used and does not depend on the way the curve is parametrised. The infinitesimal distance between neighbouring points with coordinates x^a and $x^a + dx^a$ is defined by the invariant relativistic quantity

$$ds^2 = g_{ab}dx^a dx^b \tag{2.2}$$

called the line element or Riemannian fundamental form. Many treatments in general relativity start by defining (2.2) in terms of the metric tensor field $\mathbf{g}$ which satisfies the tensor transformation law (2.1). The spacetime of special relativity has the property that at any point in the manifold we can introduce a global coordinate system such that g_{ab} takes the Lorentzian form

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Such coordinate systems are called inertial or Cartesian coordinate systems. This coordinate system is global in special relativity because we can find a coordinate neighbourhood that covers the whole of spacetime. In general relativity we can introduce coordinates where the metric tensor always assumes the Lorentzian form at a point. For local regions in the neighbourhood of a point in the spacetime of general relativity we can introduce coordinate systems so that it is like that of special relativity. In the spacetime of general relativity there exists only local inertial or local Cartesian coordinate systems where g_{ab} approximates η_{ab}.

2.3 Covariant Derivatives and Field equations

In order to define covariant derivatives we need to impose the additional structure of a connection on the manifold M. A connection ∇ at a point $p \in M$ assigns a differential operator $\nabla_{\mathbf{X}}$ to each vector field $\mathbf{X}$ at p. The operator $\nabla_{\mathbf{X}}$ maps a vector field $\mathbf{Y}$ to a vector field $\nabla_{\mathbf{X}}\mathbf{Y}$ and satisfies the following properties:

(i) $\nabla_{\mathbf{X}}\mathbf{Y}$ is a tensor in the argument $\mathbf{X}$. For functions f, g and vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$ we have

$$\nabla_{(f\mathbf{X}+g\mathbf{Y})}\mathbf{Z} = f\nabla_{\mathbf{X}}\mathbf{Z} + g\nabla_{\mathbf{Y}}\mathbf{Z}$$

Therefore the derivative $\nabla_{\mathbf{X}}$ at p depends only on the direction of $\mathbf{X}$ at p.

(ii) For $\alpha, \beta \in \Re$ and vector fields $\mathbf{X}, \mathbf{Y}, \mathbf{Z}$:

$$\nabla_{\mathbf{X}}(\alpha\mathbf{Y} + \beta\mathbf{Z}) = \alpha\nabla_{\mathbf{X}}\mathbf{Y} + \beta\nabla_{\mathbf{X}}\mathbf{Z}$$

(iii) For a continuous function f and vector field $\mathbf{Y}$:

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = \mathbf{X}(f)\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y}$$

Then $\nabla_{\mathbf{X}}\mathbf{Y}$ is the covariant derivative (with respect to ∇) of $\mathbf{Y}$ in the direction $\mathbf{X}$ at p. Since $\nabla_{\mathbf{X}}\mathbf{Y}$ is a tensor in $\mathbf{X}$ we can define $\nabla\mathbf{Y}$, the covariant derivative of $\mathbf{Y}$, as that tensor field of type (1,1) which when contracted with $\mathbf{X}$ produces the vector $\nabla_{\mathbf{X}}\mathbf{Y}$. A connection ∇ on M is a rule which assigns a connection ∇ to each point such that, if $\mathbf{Y}$ is a vector field on M, then $\nabla\mathbf{Y}$ is a tensor field on M.

Since $\nabla\mathbf{Y}$ is a (1,1) tensor field, given a basis $\{\mathbf{e}_a\}$ and its dual $\{\mathbf{e}^a\}$ in a coordinate neighbourhood in M, the covariant derivative of $\mathbf{Y}$ can be written as

$$\nabla\mathbf{Y} = Y^a{}_{;b}\mathbf{e}_a \otimes \mathbf{e}^b$$

where $Y^a{}_{;b}$ are the components of $\nabla \mathbf{Y}$. These components are defined by

$$Y^a{}_{;b} \;=\; Y^a{}_{,b} + \Gamma^a{}_{bc} Y^c$$

where the $\Gamma^a{}_{bc}$ are the connection coefficients. The connection coefficients, $\Gamma^a{}_{bc}$, are defined as

$$\Gamma^a{}_{bc} \;=\; \langle \mathbf{e}^a, \nabla_{\mathbf{e}_b} \mathbf{e}_c \rangle$$

relative to the bases $\{\mathbf{e}_a\}$ and $\{\mathbf{e}^a\}$. We consider only symmetric connection coefficients. The transformation properties of the functions $\Gamma^a{}_{bc}$ are determined by the connection properties (i)–(iii). If the bases are coordinate bases defined by the coordinates $\{x^a\}$ and $\{x^{a'}\}$, then the transformation law is given by

$$\Gamma^{a'}{}_{b'c'} \;=\; X^{a'}_{d} X^{e}_{b'} X^{f}_{c'} \Gamma^d{}_{ef} + X^{a'}_{d} X^{d}_{c'b'}$$

Thus the connection coefficients are not tensorial.

The definition of a covariant derivative can be extended to arbitrary tensor fields subject to the following conditions:

(i) If $\mathbf{T}$ is a tensor field of type (r, s), then $\nabla \mathbf{T}$ is a tensor field of type $(r, s+1)$,

(ii) ∇ is linear and commutes with the operation of contraction,

(iii) The Leibniz product rule holds and

(iv) $\nabla f = df$ for continuous functions f.

If $\mathbf{T}$ is a type (r, s) tensor field, then the components of the covariant derivative $\nabla \mathbf{T}$ are given by

$$\begin{aligned} T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s;c} \;=\; & T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s,c} + T^{d a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} \Gamma^{a_1}{}_{cd} + \cdots + T^{a_1 a_2 \ldots d}{}_{b_1 b_2 \ldots b_s} \Gamma^{a_r}{}_{cd} \\ & - T^{a_1 a_2 \ldots a_r}{}_{d b_2 \ldots b_s} \Gamma^d{}_{c b_1} - \cdots - T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots d} \Gamma^d{}_{c b_s} \end{aligned}$$

The statement that given a metric tensor field $\mathbf{g}$ there exists a unique symmetric connection ∇ such that $\nabla\mathbf{g} = 0$ is the fundamental theorem of Riemannian geometry. The tensor field $\mathbf{T}$ is parallel transported along the integral curves of a vector field $\mathbf{X}$ if

$$\nabla_{\mathbf{X}}\mathbf{T} = 0$$

Thus the metric tensor $\mathbf{g}$ is parallel transported along all smooth curves. The above unique connection ∇ is called the metric connection and we can show that the connection coefficients are expressed in terms of the metric tensor $\mathbf{g}$ and its derivatives as follows

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d}) \tag{2.3}$$

Parallel transport is path dependent and provides a measure of the curvature of the manifold. This path dependence corresponds to the fact that the covariant derivatives do not generally commute. The Riemann, or curvature, tensor $\mathbf{R}$ gives a measure of this noncommutation via the Ricci identity

$$X_{b;cd} - X_{b;dc} = R^a{}_{bcd} X_a$$

for vector fields $\mathbf{X}$. The components $R^a{}_{bcd}$ of the Riemann tensor $\mathbf{R}$ are expressed in terms of the connection coefficients by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^e{}_{bd}\Gamma^a{}_{ec} - \Gamma^e{}_{bc}\Gamma^a{}_{ed} \tag{2.4}$$

It can be established that the components of the curvature tensor satisfy the following symmetry properties (Wald 1984):

$$R_{abcd} = R_{cdab}$$

$$R_{abcd} = -R_{abdc}$$

$$R_{abcd} = -R_{bacd}$$

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0$$

$$R^a{}_{bcd;e} + R^a{}_{bec;d} + R^a{}_{bde;c} = 0 \qquad (2.5)$$

The last property (2.5) is called the Bianchi identity. Contraction of the Riemann tensor (2.4) results in a tensor field of type (0,2) called the Ricci tensor with components

$$R_{ab} = R^c{}_{acb} \qquad (2.6)$$

The property $R_{abcd} = R_{cdab}$ implies that the Ricci tensor is symmetric. By contracting the Ricci tensor we obtain the Ricci, or curvature, scalar

$$R = R^a{}_a \qquad (2.7)$$

The symmetric Einstein tensor **G** is defined in terms of the Ricci tensor (2.6) and the Ricci scalar (2.7) by

$$G_{ab} = R_{ab} - \tfrac{1}{2}Rg_{ab} \qquad (2.8)$$

On setting $c = a$ in the Bianchi identity (2.5) and utilising equations (2.6)–(2.8) we obtain the result

$$G^{ab}{}_{;b} = 0 \qquad (2.9)$$

which proves that the Einstein tensor (2.8) has zero divergence.

In general relativity the matter distribution is described by the symmetric energy–momentum tensor

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} + q_a u_b + q_b u_a + \pi_{ab} \qquad (2.10)$$

where μ is the proper density, p is the isotropic pressure, q_a is the heat flow and π_{ab} is the anisotropic stress tensor relative to the fluid 4–velocity $\mathbf{u}$ where $< \mathbf{u}, \mathbf{u} >= -1$. For a perfect fluid (2.10) becomes

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab} \tag{2.11}$$

due to the absence of heat conduction terms and stress terms corresponding to viscosity. The spacetime curvature of Riemannian space is related to the distribution of matter by the Einstein field equations. These field equations relate the energy–momentum tensor (2.10) to the Einstein tensor (2.8):

$$\begin{aligned} G_{ab} &= R_{ab} - \tfrac{1}{2} R g_{ab} \\ &= \kappa T_{ab} \end{aligned} \tag{2.12}$$

where $\kappa = 8\pi G$ is the coupling constant. Here G is the conventional gravitational constant. The field equations (2.12) form a set of ten coupled nonlinear partial differential equations. Since the covariant divergence of the Einstein tensor $\mathbf{G}$ vanishes by (2.9), the ten field equations are not all independent. As a consequence the field equations provide only six independent differential equations for the gravitational field $\mathbf{g}$. From (2.9) and the Einstein field equations (2.12) we obtain

$$T^{ab}{}_{;b} = 0$$

which is the conservation law for energy–momentum.

2.4 Lie Derivatives, Lie Algebras and Conformal Motions

The Lie derivative plays an important role in describing symmetries of physical fields in general and the gravitational field in particular. The Lie derivative provides a coordinate independent description of a symmetry property in the manifold M and is defined naturally by the manifold structure. Therefore, unlike the covariant derivative, the Lie derivative is an operation defined on a differentiable manifold without imposing additional structure on the manifold. Let $\mathbf{X}$ be a vector field in M such that $\mathbf{X}$ operates on differentiable scalar fields f producing scalar fields $\mathbf{X}f$. The Lie derivative with respect to $\mathbf{X}$ is an extension of this operation to the action of an operator $\mathcal{L}_{\mathbf{X}}$ on all differentiable tensor fields. The operator $\mathcal{L}_{\mathbf{X}}$ satisfies the following properties:

(i) $\mathcal{L}_{\mathbf{X}}$ preserves tensor type. Therefore, if $\mathbf{T}$ is a tensor field of type (r, s) then $\mathcal{L}_{\mathbf{X}}\mathbf{T}$ is also a tensor field of type (r, s),

(ii) $\mathcal{L}_{\mathbf{X}}$ maps tensors linearly and commutes with the operation of contraction,

(iii) For arbitrary tensors $\mathbf{S}$ and $\mathbf{T}$

$$\mathcal{L}_{\mathbf{X}}(\mathbf{S} \otimes \mathbf{T}) = (\mathcal{L}_{\mathbf{X}}\mathbf{S}) \otimes \mathbf{T} + \mathbf{S} \otimes (\mathcal{L}_{\mathbf{X}}\mathbf{T})$$

which is the Leibniz product rule and

(iv) $\mathcal{L}_{\mathbf{X}}f = \mathbf{X}f$ for functions f.

The components of the Lie derivative of $\mathbf{T}$ with respect to $\mathbf{X}$ are given by

$$\begin{aligned}
\mathcal{L}_{\mathbf{X}}T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s} &= T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots b_s,c}X^c - T^{ca_2\ldots a_r}{}_{b_1b_2\ldots b_s}X^{a_1}{}_{,c} - \cdots \\
&\quad - T^{a_1a_2\ldots c}{}_{b_1b_2\ldots b_s}X^{a_r}{}_{,c} + T^{a_1a_2\ldots a_r}{}_{cb_2\ldots b_s}X^c{}_{,b_1} + \cdots \\
&\quad + T^{a_1a_2\ldots a_r}{}_{b_1b_2\ldots c}X^c{}_{,b_s}
\end{aligned}$$

for a type (r, s) tensor field $\mathbf{T}$.

The Lie bracket or commutator of two vector fields $\mathbf{X}$ and $\mathbf{Y}$ is defined by

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX} \tag{2.13}$$

As $[\mathbf{X}, \mathbf{Y}]$ inherits the linearity properties of $\mathbf{X}$ and $\mathbf{Y}$ the Lie bracket $[\mathbf{X}, \mathbf{Y}] \in T(M)$. A finite dimensional vector space on which the bracket operation (2.13) has been defined is called a Lie algebra. The Lie bracket operation (2.13) is skew–symmetric and bilinear. It is not associative and satisfies the Jacobi identity

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0 \tag{2.14}$$

for arbitrary vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$. The Lie derivative and the Lie bracket are related by the identities

$$\begin{aligned} \mathcal{L}_{\mathbf{X}}\mathbf{Y} &= [\mathbf{X}, \mathbf{Y}] \\ \mathcal{L}_{[\mathbf{X}, \mathbf{Y}]} &= [\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] \end{aligned}$$

for vector fields $\mathbf{X}$ and $\mathbf{Y}$.

Every Lie algebra defines a unique, simply connected Lie group. An r–dimensional Lie group G_r is a group which is also a smooth r–dimensional differentiable manifold whose structure is such that the group composition $G_r \times G_r \longrightarrow G_r$ and the group inverse $G_r \longrightarrow G_r$ are smooth maps. We do not pursue the subject of Lie groups further as it is not relevant to this dissertation. For more information on Lie groups the reader is referred to Choquet–Bruhat *et al* (1977), Dubrovin *et al* (1984), Sattinger and Weaver (1986) and Schutz (1980). Kramer *et al* (1980) comprehensively discuss the relevance of Lie groups and Lie algebras to various classes of solutions to the Einstein field equations.

Manifolds, including the pseudo–Riemannian manifold of general relativity, may admit continuous groups G_r of infinitesimal transformations. A conformal motion of a manifold M preserves the metric up to a factor. In this dissertation we are concerned with the conformal motion of the Stephani universes. A conformal

Killing vector $\mathbf{X}$ is defined by

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\phi g_{ab} \tag{2.15}$$

where $\phi = \phi(x^a)$ is the conformal factor and $\mathbf{g}$ is the metric tensor field. If $\mathbf{g}$ is specified, then we solve (2.15) to obtain the conformal Killing vector $\mathbf{X}$ (see chapter 4). There are four cases associated with the equation (2.15):

(i) $\phi = 0$: $\mathbf{X}$ is a Killing vector,

(ii)$\phi_{,a} = 0 \neq \phi$: $\mathbf{X}$ is a homothetic Killing vector,

(iii) $\phi_{;ab} = 0 \neq \phi_{,a}$: $\mathbf{X}$ is a special conformal Killing vector and

(iv) $\phi_{;ab} \neq 0$: $\mathbf{X}$ is a nonspecial conformal Killing vector.

Killing vectors generate constants or first integrals of the motion along geodesics. The Killing vectors span a group of isometries which may be used to characterise systematically and invariantly solutions of the Einstein field equations. A homothetic Killing vector scales all distances by the same constant factor and preserves the null geodesic affine parameters. Homothetic Killing vectors lead to self–similar spacetimes. Conformal Killing vectors generate constants of the motion along null geodesics for massless particles. Suppose that G_r is a group of conformal motions with generators $\{\mathbf{X}_I\} = \{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_r\}$. The elements of the basis $\{\mathbf{X}_I\}$ are related by

$$[\mathbf{X}_I, \mathbf{X}_J] = C^K{}_{IJ} \mathbf{X}_K \tag{2.16}$$

where the $C^K{}_{IJ}$ are the structure constants of the group and satisfy

$$C^K{}_{IJ} = -C^K{}_{JI}$$

The Lie identity

$$C^K{}_{LM} C^M{}_{IJ} + C^K{}_{IM} C^M{}_{JL} + C^K{}_{JM} C^M{}_{LI} = 0$$

is obtained by substituting the relation (2.16) into the Jacobi identity (2.14). The dimension r of the maximal group of conformal motions G_r is given as

$$r = \tfrac{1}{2}(n+1)(n+2)$$

by Choquet–Bruhat *et al* (1977) for an n–dimensional manifold. In the spacetime manifold of classical general relativity we have a maximal G_{15} of conformal motions since $n = 4$.

3 The Stephani Universe

3.1 Introduction

The Robertson–Walker cosmological models satisfy the cosmological principle, i.e. they are homogeneous and isotropic. The Stephani universe is a generalisation of the Robertson–Walker models. In contrast to the Robertson–Walker model the Stephani universe is generally not homogeneous or isotropic, but does reduce to the Robertson–Walker model in the appropriate limit. The Stephani universe is the conformally flat solution of the Einstein field equations with a perfect fluid source. As in the Robertson–Walker models the hypersurfaces orthogonal to matter world lines have constant curvature. However, since the curvature index in the Stephani model is an arbitrary function of time, its sign can change from one hypersurface to another. Therefore the Stephani model may appear to have a positive spatial curvature at one moment and a negative spatial curvature at another. In contrast the spatial curvature of the Robertson–Walker model remains fixed to describe always a flat, closed or open spacetime. In §3.2 we introduce the Stephani line element and briefly discuss some of its features. We derive the field equations in full as the details of the spacetime curvature are not well known. We list the non–zero connection coefficients, the components of the Ricci tensor, the Ricci scalar and the Einstein tensor components. In §3.3 we present the field equations for the Stephani universe for an energy–momentum tensor describing a perfect fluid. We also give the conditions for the Stephani line element to reduce to Robertson–Walker form. Finally we observe that the field equations may be derived with the help of the Gauss–Codazzi equations (Stephani 1967) or may also be obtained by utilising the notion of infinitesimal null isotropy (Koch–Sen 1985).

3.2 Spacetime Geometry

The Robertson–Walker models are the simplest models of the universe obtained from the cosmological principle. In coordinates $(x^a) = (t, x, y, z)$ the Robertson–Walker line element is given by

$$ds^2 = -dt^2 + \frac{R^2(t)}{\left[1 + \frac{1}{4}k\left(x^2 + y^2 + z^2\right)\right]^2}(dx^2 + dy^2 + dz^2)$$

where $k = 0, 1, -1$ and $R(t)$ is the scale factor. The Robertson–Walker models are the standard cosmological models describing a homogeneous and isotropic universe. The hypersurfaces orthogonal to the matter world lines have constant curvature and the curvature index k is fixed.

A generalisation of the Robertson–Walker model is given by the Stephani universe. As for the Robertson–Walker spacetimes the hypersurfaces orthogonal to the matter world lines have constant curvature, but the difference in the Stephani universe is that the curvature index k is an arbitrary function of the time coordinate. As a consequence the index k can change its sign from one hypersurface to another. Thus the spacetime may have curvature index $k > 0$ at one instance and curvature index $k < 0$ at another time. In coordinates $(x^a) = (t, x, y, z)$ the Stephani line element has the form

$$ds^2 = -D^2 dt^2 + \frac{R^2(t)}{V^2}(dx^2 + dy^2 + dz^2) \tag{3.1}$$

where

$$\begin{aligned}
V &= 1 + \frac{1}{4}k(t)\left\{[x - x_0(t)]^2 + [y - y_0(t)]^2 + [z - z_0(t)]^2\right\} \\
D &= F(t)\left(\frac{V_t}{V} - \frac{\dot{R}}{R}\right) = F\frac{R}{V}\left(\frac{V}{R}\right)_t \\
k &= \left[C^2(t) - \frac{1}{F^2(t)}\right] R^2(t)
\end{aligned}$$

with subscripts t, x, y, z denoting partial differentiation and dots representing differentiation with respect to time. The functions C, F, R, x_0, y_0, z_0 are arbitrary functions of time. The Stephani metric (3.1) is conformally flat. This is analogous to the Robertson–Walker spacetimes as the Weyl tensor also vanishes there. Therefore the vacuum solution, obtained by taking $C = 0$ as may be verified by analysing the field equations given later in this chapter, is Riemann flat. The matter source in this solution is a perfect fluid having zero shear and rotation. Since the fluid 4–velocity is given by $u^a = D^{-1}\delta^a_0$, the 4–acceleration vector has the form

$$\dot{u}^a = \begin{cases} 0 & \text{for } a = 0 \\ \left(\dfrac{V^2}{DR^2}\right) D_a & \text{for } a = 1, 2, 3 \end{cases}$$

The expansion of the Stephani universe is given by

$$\theta = \frac{3}{F}$$

The above expressions characterise the kinematics of the Stephani solution. We investigate the dynamics later in this chapter. The Stephani universe was originally obtained by Stephani (1967) by studying solutions of the Einstein field equations that can be embedded in a flat 5–dimensional space. It is interesting to observe that the Stephani universe was also obtained by Krasiński (1981) in his search for intrinsically spherically symmetric solutions, i.e. solutions which are composed of spherically symmetric subspaces. The physical properties of the Stephani universe have been investigated by Krasiński (1983). In particular he investigated the geometry of the Stephani solution in the vicinity of a spatial hypersurface in which k changes from positive to negative. The general Stephani solution shares several qualitative features with the de Sitter solution (a special Stephani solution in which the metric is de Sitter) in which an example of foliation is used where the curvature k changes in moving from one hypersurface to another (Krasiński 1983). It should be pointed out that the Stephani universe is not inconsistent with the standard observational tests and is therefore a viable cosmological model.

For the Stephani line element (3.1) the nonvanishing connection coefficients (2.3) are given by

$$\Gamma^0{}_{00} = \frac{D_t}{D} \qquad \Gamma^0{}_{01} = \frac{D_x}{D}$$

$$\Gamma^0{}_{02} = \frac{D_y}{D} \qquad \Gamma^0{}_{03} = \frac{D_z}{D}$$

$$\Gamma^0{}_{11} = \left(\frac{1}{D^2}\right)\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t \qquad \Gamma^0{}_{22} = \left(\frac{1}{D^2}\right)\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t$$

$$\Gamma^0{}_{33} = \left(\frac{1}{D^2}\right)\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t \qquad \Gamma^1{}_{00} = \left(\frac{V^2}{R^2}\right)DD_x$$

$$\Gamma^1{}_{01} = \left(\frac{V}{R}\right)\left(\frac{R}{V}\right)_t \qquad \Gamma^1{}_{11} = -\left(\frac{V_x}{V}\right)$$

$$\Gamma^1{}_{12} = -\left(\frac{V_y}{V}\right) \qquad \Gamma^1{}_{13} = -\left(\frac{V_z}{V}\right)$$

$$\Gamma^1{}_{22} = \frac{V_x}{V} \qquad \Gamma^1{}_{33} = \frac{V_x}{V}$$

$$\Gamma^2{}_{00} = \left(\frac{V^2}{R^2}\right) DD_y \qquad \Gamma^2{}_{02} = \left(\frac{V}{R}\right)\left(\frac{R}{V}\right)_t$$

$$\Gamma^2{}_{11} = \frac{V_y}{V} \qquad \Gamma^2{}_{12} = -\left(\frac{V_x}{V}\right)$$

$$\Gamma^2{}_{22} = -\left(\frac{V_y}{V}\right) \qquad \Gamma^2{}_{23} = -\left(\frac{V_z}{V}\right)$$

$$\Gamma^2{}_{33} = \frac{V_y}{V} \qquad \Gamma^3{}_{00} = \left(\frac{V^2}{R^2}\right) DD_z$$

$$\Gamma^3{}_{03} = \left(\frac{V}{R}\right)\left(\frac{R}{V}\right)_t \qquad \Gamma^3{}_{11} = \frac{V_z}{V}$$

$$\Gamma^3{}_{13} = -\left(\frac{V_x}{V}\right) \qquad \Gamma^3{}_{22} = \frac{V_z}{V}$$

$$\Gamma^3{}_{23} = -\left(\frac{V_y}{V}\right) \qquad \Gamma^3{}_{33} = -\left(\frac{V_z}{V}\right)$$

With the above connection coefficients we determine that the components of the

Ricci tensor (2.6) for the metric (3.1) are given by

$$R_{00} = D\left(\frac{V^2}{R^2}\right)(D_{xx}+D_{yy}+D_{zz}) - D\left(\frac{V}{R^2}\right)(D_xV_x+D_yV_y+D_zV_z)$$

$$+3\left(\frac{1}{D}\right)\left(\frac{V}{R}\right)\left[D_t\left(\frac{R}{V}\right)_t - D\left(\frac{R}{V}\right)_{tt}\right] \tag{3.2}$$

$$R_{11} = -\frac{D_{xx}}{D} + 2\left(\frac{V_x}{V}\right)^2 - \frac{D_x}{D}\frac{V_x}{V} + \frac{D_y}{D}\frac{V_y}{V} + \frac{D_z}{D}\frac{V_z}{V}$$

$$-\left(\frac{1}{D^3}\right)\left[D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - 2D\left(\frac{R}{V}\right)_t^2 - D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$-\left(\frac{V}{R}\right)\left[2\left(\frac{R}{V}\right)_{xx} + \left(\frac{R}{V}\right)_{yy} + \left(\frac{R}{V}\right)_{zz}\right] \tag{3.3}$$

$$R_{22} = -\frac{D_{yy}}{D} + 2\left(\frac{V_y}{V}\right)^2 + \frac{D_x}{D}\frac{V_x}{V} - \frac{D_y}{D}\frac{V_y}{V} + \frac{D_z}{D}\frac{V_z}{V}$$

$$-\left(\frac{1}{D^3}\right)\left[D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - 2D\left(\frac{R}{V}\right)_t^2 - D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$-\left(\frac{V}{R}\right)\left[\left(\frac{R}{V}\right)_{xx} + 2\left(\frac{R}{V}\right)_{yy} + \left(\frac{R}{V}\right)_{zz}\right] \tag{3.4}$$

$$R_{33} = -\frac{D_{zz}}{D} + 2\left(\frac{V_z}{V}\right)^2 + \frac{D_x}{D}\frac{V_x}{V} + \frac{D_y}{D}\frac{V_y}{V} - \frac{D_z}{D}\frac{V_z}{V}$$

$$-\left(\frac{1}{D^3}\right)\left[D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - 2D\left(\frac{R}{V}\right)_t^2 - D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$-\left(\frac{V}{R}\right)\left[\left(\frac{R}{V}\right)_{xx} + \left(\frac{R}{V}\right)_{yy} + 2\left(\frac{R}{V}\right)_{zz}\right] \tag{3.5}$$

$$R_{ab} = 0, \quad a \neq b$$

Substituting the components (3.2)–(3.5) of the Ricci tensor (2.6) into (2.7) we obtain the Ricci scalar

$$R = -2\left(\frac{V^2}{R^2}\right)\left(\frac{D_{xx}}{D} + \frac{D_{yy}}{D} + \frac{D_{zz}}{D}\right) + 2\left(\frac{V^2}{R^2}\right)\left(\frac{D_x}{D}\frac{V_x}{V} + \frac{D_y}{D}\frac{V_y}{V} + \frac{D_z}{D}\frac{V_z}{V}\right)$$

$$-6\left(\frac{1}{D^3}\right)\left(\frac{V^2}{R^2}\right)\left[D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - D\left(\frac{R}{V}\right)_t^2 - D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$-4\left(\frac{V^3}{R^3}\right)\left[\left(\frac{R}{V}\right)_{xx} + \left(\frac{R}{V}\right)_{yy} + \left(\frac{R}{V}\right)_{zz}\right]$$

$$+2\left(\frac{V^2}{R^2}\right)\left[\left(\frac{V_x}{V}\right)^2 + \left(\frac{V_y}{V}\right)^2 + \left(\frac{V_z}{V}\right)^2\right] \tag{3.6}$$

The Ricci tensor components (3.2)–(3.5) and the Ricci scalar (3.6) generate the components of the Einstein tensor (2.8) for the line element (3.1):

$$G_{00} = 3\left(\frac{V^2}{R^2}\right)\left(\frac{R}{V}\right)_t^2 - 2D^2\left(\frac{V^3}{R^3}\right)\left[\left(\frac{R}{V}\right)_{xx} + \left(\frac{R}{V}\right)_{yy} + \left(\frac{R}{V}\right)_{zz}\right]$$

$$+ D^2\left(\frac{V^2}{R^2}\right)\left[\left(\frac{V_x}{V}\right)^2 + \left(\frac{V_y}{V}\right)^2 + \left(\frac{V_z}{V}\right)^2\right] \tag{3.7}$$

$$G_{11} = \frac{D_{yy}}{D} + \frac{D_{zz}}{D} - 2\frac{D_x}{D}\frac{V_x}{V} + \left(\frac{V_x}{V}\right)^2 - \left(\frac{V_y}{V}\right)^2 - \left(\frac{V_z}{V}\right)^2$$

$$+\left(\frac{1}{D^3}\right)\left[2D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - D\left(\frac{R}{V}\right)_t^2 - 2D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$+\left(\frac{V}{R}\right)\left[\left(\frac{R}{V}\right)_{yy} + \left(\frac{R}{V}\right)_{zz}\right] \tag{3.8}$$

$$G_{22} = \frac{D_{xx}}{D} + \frac{D_{zz}}{D} - 2\frac{D_y}{D}\frac{V_y}{V} - \left(\frac{V_x}{V}\right)^2 + \left(\frac{V_y}{V}\right)^2 - \left(\frac{V_z}{V}\right)^2$$

$$+\left(\frac{1}{D^3}\right)\left[2D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - D\left(\frac{R}{V}\right)_t^2 - 2D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$+\left(\frac{V}{R}\right)\left[\left(\frac{R}{V}\right)_{xx} + \left(\frac{R}{V}\right)_{zz}\right] \tag{3.9}$$

$$G_{33} = \frac{D_{xx}}{D} + \frac{D_{yy}}{D} - 2\frac{D_z}{D}\frac{V_z}{V} - \left(\frac{V_x}{V}\right)^2 - \left(\frac{V_y}{V}\right)^2 + \left(\frac{V_z}{V}\right)^2$$

$$+\left(\frac{1}{D^3}\right)\left[2D_t\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_t - D\left(\frac{R}{V}\right)_t^2 - 2D\left(\frac{R}{V}\right)\left(\frac{R}{V}\right)_{tt}\right]$$

$$+\left(\frac{V}{R}\right)\left[\left(\frac{R}{V}\right)_{xx} + \left(\frac{R}{V}\right)_{yy}\right] \tag{3.10}$$

$$G_{ab} = 0, \quad a \neq b$$

Thus far we have been treating D and V as arbitrary functions of t, x, y, and z. On utilising the explicit forms of D and V for the Stephani metric (3.1) we find that

(3.7)–(3.10) reduce to the system

$$G_{00} = 3C^2F^2(t)\left(\frac{V_t}{V} - \frac{\dot{R}}{R}\right)^2$$

$$= 3C^2D^2 \quad (3.11)$$

$$G_{11} = -3C^2\left(\frac{R^2}{V^2}\right) + 2C\dot{C}\frac{(V/R)}{(V/R)_t}\left(\frac{R^2}{V^2}\right) \quad (3.12)$$

$$G_{22} = G_{11} \quad (3.13)$$

$$G_{33} = G_{11} \quad (3.14)$$

$$G_{ab} = 0, \quad a \neq b$$

Thus as for the Robertson–Walker spacetime only two of the components of the Einstein tensor are independent. We have explicitly calculated the above quantities as they do not appear explicitly in the literature and would be helpful to others studying the Stephani spacetimes.

3.3 Field Equations

We formulate the field equations for the case of a perfect fluid energy–momentum tensor (2.11) with a comoving fluid 4–velocity $\mathbf{u}$ given by $u^a = D^{-1}\delta^a_0$. The energy–momentum tensor (2.11) becomes

$$T_{ab} = D^2(\mu + p)\,\delta^0_a\,\delta^0_b + pg_{ab} \quad (3.15)$$

With the help of the Einstein tensor components (3.11)–(3.14) and the energy–momentum tensor (3.15) we find that the Einstein field equations (2.12) can be written as the system

$$\kappa\mu = 3C^2(t) \tag{3.16}$$

$$\begin{aligned}\kappa p &= -3C^2(t) + 2C\dot{C}\frac{(V/R)}{(V/R)_t} \\ &= -\kappa\mu + 2C\dot{C}\frac{(V/R)}{(V/R)_t}\end{aligned} \tag{3.17}$$

We note that our field equations (3.16)–(3.17) agree with the equations given by Kramer *et al* (1980), Krasiński (1983) and Stephani (1967). This provides a check on our calculations in §3.2. The field equations (3.16)–(3.17) constitute a system of two equations in the variables μ, p, C, F, R, x_0, y_0, z_0. Therefore the Stephani universe allows for more general behaviour than the simpler Robertson–Walker model. As was pointed out in §3.2 the vacuum solution of the Stephani universe corresponds to $C = 0$ from (3.16) and (3.17). The physical properties of the de Sitter solution as a special case of the Stephani field equations were analysed by Krasiński (1983). The Stephani universe also arises in the more general class of perfect fluid spacetimes analysed by Krasiński (1989).

The Stephani line element reduces to the Robertson–Walker models if and only if one of the following three conditions applies:

(i) k, x_0, y_0 and z_0 are constants,

(ii) $\dot{u}^a = 0$,

(iii) $p = f(\mu)$, where the function f is independent of the spacelike coordinates x, y, z.

In general the Stephani solution has no symmetry. If x_0, y_0 and z_0 are constants the solution becomes symmetric about the line $x = x_0$, $y = y_0$, $z = z_0$ but still lacks the homogeneity of the Robertson–Walker models. Thus we also require k to be constant in condition (i) to obtain the Robertson–Walker spacetimes. Condition (ii) implies that the matter flow lines in the Stephani model are nongeodesic and only become geodesics when the Stephani line element reduces to the Robertson–Walker model. The last condition (iii) means that the equation of state in the Stephani model is position dependent. If the equation of state is of the barotropic form $p = f(\mu)$, then the Stephani model reduces to Robertson–Walker.

The Stephani universe was obtained by Stephani (1967) by seeking solutions to the Einstein field equations that are embedded in a flat 5–dimensional space. The field equations (3.16) and (3.17) can then be derived via the Gauss–Codazzi equations

$$R_{abcd} = e\,(\Omega_{ac}\Omega_{bd} - \Omega_{ad}\Omega_{bc}) \qquad \text{(Gauss)} \tag{3.18}$$

$$\Omega_{ab;c} = \Omega_{ac;b} \qquad \text{(Codazzi)}$$

where $e = \pm 1$ and Ω_{ab} are symmetric tensors defined on the 4–dimensional manifold M. These tensors are generalisations to higher dimensions of the tensor of the second fundamental form (Hawking and Ellis 1973, Kramer *et al* 1980). The Gauss equations (3.18) and the field equations (2.12) yield

$$\kappa T_{ab} = e\,(\Omega_{ab}\Omega^c_{\ c} - \Omega_{ac}\Omega^c_{\ b}) \tag{3.19}$$

For the Stephani metric $e = +1$ and the symmetric tensors Ω_{ab} assume the form

$$\Omega_{ab} = \dot{C}\frac{(V/R)}{(V/R)_t}u_a u_b + C(t) g_{ab} \tag{3.20}$$

Substituting (3.20) in (3.19) we obtain the field equations (3.16) and (3.17). The Stephani universe may also be found by utilising the approach of solving the Einstein field equations by introducing the notion of infinitesimal null isotropy. The concept of infinitesimal null isotropy in terms of null sectional curvature was defined by Karcher (1982). Koch–Sen (1985) showed that null isotropic spacetimes are conformally flat perfect fluid solutions to the Einstein field equations and consequently are just the Stephani universes. Recently Jiang (1992) generalised the Stephani universe to include nonvanishing heat flow q^a in the energy–momentum tensor (2.10), with vanishing anisotropic stress π_{ab}, using a tetrad formalism.

4 Conformal Killing Vector Equation

4.1 Introduction

In this chapter we solve the conformal Killing vector equation for the Stephani metric (3.1). There are three cases that arise in the integration process; we consider only the most general case $\dot{k}R - k\dot{R} \neq 0$ in this chapter. We obtain the conformal Killing vector $\mathbf{X}$ and the conformal factor ϕ subject to integrability conditions that place restrictions on the solution. As the integration process is lengthy and complicated we provide only the details of the calculation in this chapter. We analyse the solution obtained further in chapter 5. In §4.2 we write the conformal Killing vector equation as a coupled system of ten first order partial differential equations. We briefly review other solutions of relevance with a conformal Killing vector in the literature. In §4.3 we partially integrate the conformal Killing vector equations to obtain the spacelike components of the conformal Killing vector. We have obtained expressions for these components in which the dependence on the spatial coordinates x, y and z is explicitly known. The timelike component of the conformal Killing vector is found in §4.4 by utilising the method of characteristics. In the process of finding the timelike component six integrability conditions have to be satisfied. In §4.5 we find the conformal factor thereby generating another five integrability conditions. It is important to note that our solutions are applicable only if the condition $\dot{k}R - k\dot{R} \neq 0$

holds. We consider the remaining special cases in chapter 5. Parts of the calculation are complex and extremely difficult to analyse by hand. We therefore use the symbolic manipulation capabilities of Mathematica (Ver 2.0) (Wolfram 1991) to assist with calculations.

4.2 Conformal Equation

In this section we list the ten independent partial differential equations that arise from the conformal Killing vector equation (2.15). As the integration process is lengthy the details of the solution are given in the subsequent sections of this chapter. For the metric (3.1) the conformal equation (2.15) reduces to the following system of ten equations:

$$\frac{D_t}{D}X^0 + \frac{D_x}{D}X^1 + \frac{D_y}{D}X^2 + \frac{D_z}{D}X^3 + X^0_t = \phi \tag{4.1}$$

$$\frac{R^2}{V^2}X^1_t - D^2X^0_x = 0 \tag{4.2}$$

$$\frac{R^2}{V^2}X^2_t - D^2X^0_y = 0 \tag{4.3}$$

$$\frac{R^2}{V^2}X^3_t - D^2X^0_z = 0 \tag{4.4}$$

$$\left(\frac{\dot{R}}{R} - \frac{V_t}{V}\right)X^0 - \frac{V_x}{V}X^1 - \frac{V_y}{V}X^2 - \frac{V_z}{V}X^3 + X^1_x = \phi \tag{4.5}$$

$$X^2_x + X^1_y = 0 \tag{4.6}$$

$$X^3_x + X^1_z = 0 \tag{4.7}$$

$$\left(\frac{\dot{R}}{R} - \frac{V_t}{V}\right) X^0 - \frac{V_x}{V} X^1 - \frac{V_y}{V} X^2 - \frac{V_z}{V} X^3 + X^2_y = \phi \tag{4.8}$$

$$X^3_y + X^2_z = 0 \tag{4.9}$$

$$\left(\frac{\dot{R}}{R} - \frac{V_t}{V}\right) X^0 - \frac{V_x}{V} X^1 - \frac{V_y}{V} X^2 - \frac{V_z}{V} X^3 + X^3_z = \phi \tag{4.10}$$

The system (4.1)–(4.10) is a coupled system of ten first order partial differential equations. We need to integrate this system to obtain the conformal Killing vector $\mathbf{X} = (X^0, X^1, X^2, X^3)$ and the conformal factor ϕ in terms of the arbitrary functions k, F, R, x_0, y_0 and z_0. The solution found is subject to integrability conditions that contain the metric functions k, F, R, x_0, y_0, z_0 and their derivatives.

The general solution of the conformal Killing vector equation has been found in some other spacetimes. The G_{15} Lie algebra of conformal Killing vectors in Minkowski spacetime is given by Choquet–Bruhat *et al* (1977). Maartens and Maharaj (1986) found the G_{15} Lie algebra of conformal Killing vectors in the Robertson–Walker spacetimes for all three cases of the spatial geometry $k = 0, 1, -1$. The conformal Killing vector equation (2.15) was integrated by Maartens and Maharaj (1991) to obtain the conformal geometry of the *pp*–wave spacetimes. Moodley (1991) found solutions to the conformal Killing vector equation in certain locally rotationally symmetric spacetimes. We expect the conformal geometry of the Stephani spacetimes to be more complicated than the previous cases because it lacks symmetry. Many authors impose the restriction of a conformal symmetry to simplify the nonlinear Einstein field equations in attempts to find exact solutions. For an application of conformal symmetry to spherically symmetric gravitational fields the reader is referred to Dyer *et al* (1987), Herrera and Ponce de Leon (1985a,b,c), Maartens and Maharaj (1990) and Maharaj *et al* (1991).

4.3 The Spacelike Components

In this section we will partially integrate the system (4.1)–(4.10) to obtain the spacelike components X^1, X^2 and X^3 of the conformal Killing vector $\mathbf{X}$. The remaining timelike component X^0 will be found in the next section. On taking differences of (4.5), (4.8) and (4.10) we obtain

$$X^1_x = X^2_y = X^3_z \tag{4.11}$$

The sum of the derivative of (4.6) with respect to z, the derivative of (4.7) with respect to y and the derivative of (4.9) with respect to x implies the following identities

$$X^1_{yz} = 0 \tag{4.12}$$

$$X^2_{xz} = 0 \tag{4.13}$$

$$X^3_{xy} = 0 \tag{4.14}$$

On differentiating (4.6) with respect to y and (4.7) with respect to z and using (4.11) we obtain for the spacelike component X^1:

$$X^1_{xx} + X^1_{yy} = 0 \tag{4.15}$$

$$X^1_{xx} + X^1_{zz} = 0 \tag{4.16}$$

Similarly (4.6), (4.9) and (4.11) give for the spacelike component X^2:

$$X^2_{yy} + X^2_{xx} = 0 \tag{4.17}$$

$$X^2_{yy} + X^2_{zz} = 0 \tag{4.18}$$

Also from (4.7), (4.9) and (4.11) we obtain for the spacelike component X^3:

$$X^3_{zz} + X^3_{xx} = 0 \tag{4.19}$$

$$X^3_{zz} + X^3_{yy} = 0 \tag{4.20}$$

These identities help in the integration process.

Then differentiation of (4.11) with respect to x and y, x and z, y and z, in turn and using (4.12), (4.13) and (4.14) implies the following identities:

$$X^1_{xxy} = 0 \tag{4.21}$$

$$X^1_{xxz} = 0 \tag{4.22}$$

$$X^2_{yyx} = 0 \tag{4.23}$$

$$X^2_{yyz} = 0 \tag{4.24}$$

$$X^3_{zzx} = 0 \tag{4.25}$$

$$X^3_{zzy} = 0 \tag{4.26}$$

On differentiating (4.11) twice with respect to x and utilising (4.17) and (4.19) we obtain the following relationships:

$$X^1_{xxx} = -X^2_{yyy} = -X^3_{zzz} \tag{4.27}$$

Then differentiation of (4.18) with respect to y and applying (4.11) gives the result $-X^2_{yyy} = X^1_{zzx}$. Substitution of this result in (4.27) gives

$$X^1_{xxx} = 0 \tag{4.28}$$

where we have utilised the derivative of (4.16) with respect to x. Thus by equation (4.27) we also have

$$X^2_{yyy} = 0 \tag{4.29}$$

$$X^3_{zzz} = 0 \tag{4.30}$$

Equations (4.21), (4.22) and (4.28) imply that X^1_{xx} is a function only of time. Similarly X^2_{yy} is a function only of time by equations (4.23), (4.24) and (4.29). Also X^3_{zz} is a function only of time by (4.25), (4.26) and (4.30).

Since X^1_{xx} is a function of time only we may write

$$X^1_{xx} = -\mathcal{A}(t) \tag{4.31}$$

where $\mathcal{A}(t)$ is an arbitrary function. Then from equations (4.15) and (4.16) we obtain

$$X^1_{yy} = \mathcal{A}(t) \tag{4.32}$$

$$X^1_{zz} = \mathcal{A}(t) \tag{4.33}$$

Integrating equation (4.32) and using the result (4.12) we obtain :

$$X^1 = \tfrac{1}{2}\mathcal{A}(t)y^2 + \mathcal{D}(t,x)y + \tilde{\mathcal{D}}(t,x,z) \tag{4.34}$$

where $\mathcal{D}(t,x)$ and $\tilde{\mathcal{D}}(t,x,z)$ are functions of integration. Substitution of (4.34) in (4.33) yields

$$\tilde{\mathcal{D}}_{zz} = \mathcal{A}(t)$$

which is integrated to give

$$\tilde{\mathcal{D}}(t,x,z) = \tfrac{1}{2}\mathcal{A}(t)z^2 + \mathcal{E}(t,x)z + \mathcal{G}(t,x)$$

where $\mathcal{E}(t,x)$ and $\mathcal{G}(t,x)$ are functions resulting from integration. Then the spacelike component X^1 becomes

$$X^1 = \tfrac{1}{2}\mathcal{A}(t)(y^2+z^2) + \mathcal{D}(t,x)y + \mathcal{E}(t,x)z + \mathcal{G}(t,x) \tag{4.35}$$

We substitute the form of X^1 given by (4.35) in (4.31) to obtain the differential equation

$$\mathcal{D}_{xx}y + \mathcal{E}_{xx}z + \mathcal{G}_{xx} = -\mathcal{A}(t)$$

For consistency we must have

$$\mathcal{D}_{xx} = 0$$

$$\mathcal{E}_{xx} = 0$$

$$\mathcal{G}_{xx} = -\mathcal{A}(t)$$

which gives upon integration

$$\mathcal{D}(t,x) = \mathcal{D}_1(t)x + \mathcal{D}_2(t)$$

$$\mathcal{E}(t,x) = \mathcal{E}_1(t)x + \mathcal{E}_2(t)$$

$$\mathcal{G}(t,x) = -\tfrac{1}{2}\mathcal{A}(t)x^2 + \mathcal{G}_1(t)x + \mathcal{G}_2(t)$$

where $\mathcal{D}_1(t)$, $\mathcal{D}_2(t)$, $\mathcal{E}_1(t)$, $\mathcal{E}_2(t)$, $\mathcal{G}_1(t)$ and $\mathcal{G}_2(t)$ are arbitrary functions. Substituting the above results in (4.35) we obtain

$$X^1 = \tfrac{1}{2}\mathcal{A}(t)(y^2 + z^2 - x^2) + \mathcal{D}_1(t)xy + \mathcal{D}_2(t)y + \mathcal{E}_1(t)xz + \mathcal{E}_2(t)z + \mathcal{G}_1(t)x + \mathcal{G}_2(t) \tag{4.36}$$

for the spacelike component X^1.

Similarly X^2_{yy} is defined as

$$X^2_{yy} = -\mathcal{B}(t) \tag{4.37}$$

where $\mathcal{B}(t)$ is an arbitrary function. Then equations (4.17) and (4.18) imply

$$X^2_{xx} = \mathcal{B}(t) \tag{4.38}$$

$$X^2_{zz} = \mathcal{B}(t) \tag{4.39}$$

Integration of (4.38) and applying (4.13) yields

$$X^2 = \tfrac{1}{2}\mathcal{B}(t)x^2 + \mathcal{J}(t,y)x + \tilde{\mathcal{J}}(t,y,z) \tag{4.40}$$

where $\mathcal{J}(t,y)$ and $\tilde{\mathcal{J}}(t,y,z)$ are arbitrary functions obtained via the integration process. On substituting (4.40) in (4.39) we obtain

$$\tilde{\mathcal{J}}_{zz} = \mathcal{B}(t)$$

which gives the result

$$\tilde{\mathcal{J}}(t,y,z) = \tfrac{1}{2}\mathcal{B}(t)z^2 + \mathcal{F}(t,y)z + \mathcal{H}(t,y)$$

where $\mathcal{F}(t,y)$ and $\mathcal{H}(t,y)$ are functions of integration. Thus the spacelike component X^2 becomes

$$X^2 = \tfrac{1}{2}\mathcal{B}(t)(x^2+z^2) + \mathcal{J}(t,y)x + \mathcal{F}(t,y)z + \mathcal{H}(t,y) \qquad (4.41)$$

By substituting (4.41) in (4.37) we obtain

$$\mathcal{J}_{yy}x + \mathcal{F}_{yy}z + \mathcal{H}_{yy} = -\mathcal{B}(t)$$

Thus we must have

$$\mathcal{J}_{yy} = 0$$

$$\mathcal{F}_{yy} = 0$$

$$\mathcal{H}_{yy} = -\mathcal{B}(t)$$

which upon integration yields

$$\mathcal{J}(t,y) = \mathcal{J}_1(t)y + \mathcal{J}_2(t)$$

$$\mathcal{F}(t,y) = \mathcal{F}_1(t)y + \mathcal{F}_2(t)$$

$$\mathcal{H}(t,y) = -\tfrac{1}{2}\mathcal{B}(t)y^2 + \mathcal{H}_1(t)y + \mathcal{H}_2(t)$$

where $\mathcal{J}_1(t)$, $\mathcal{J}_2(t)$, $\mathcal{F}_1(t)$, $\mathcal{F}_2(t)$, $\mathcal{H}_1(t)$ and $\mathcal{H}_2(t)$ are arbitrary functions of time. Substituting these equations in (4.41) gives

$$X^2 = \tfrac{1}{2}\mathcal{B}(t)(x^2+z^2-y^2) + \mathcal{J}_1(t)xy + \mathcal{J}_2(t)x + \mathcal{F}_1(t)yz + \mathcal{F}_2(t)z + \mathcal{H}_1(t)y$$

$$+\mathcal{H}_2(t) \tag{4.42}$$

for the spacelike component X^2.

Again as X^3_{zz} is a function of time we may write

$$X^3_{zz} = -\mathcal{C}(t) \tag{4.43}$$

where $\mathcal{C}(t)$ is an arbitrary function. We obtain from (4.19) and (4.20)

$$X^3_{xx} = \mathcal{C}(t) \tag{4.44}$$

$$X^3_{yy} = \mathcal{C}(t) \tag{4.45}$$

Utilisation of (4.14) when integrating (4.44) gives

$$X^3 = \tfrac{1}{2}\mathcal{C}(t)x^2 + \mathcal{K}(t,z)x + \tilde{\mathcal{K}}(t,y,z) \tag{4.46}$$

where $\mathcal{K}(t,z)$ and $\tilde{\mathcal{K}}(t,y,z)$ are arbitrary functions resulting from the integration process. Substitution of (4.46) in (4.45) yields

$$\tilde{\mathcal{K}}_{yy} = \mathcal{C}(t)$$

which upon integrating gives

$$\tilde{\mathcal{K}}(t,y,z) = \tfrac{1}{2}\mathcal{C}(t)y^2 + \mathcal{L}(t,z)y + \mathcal{I}(t,z)$$

where $\mathcal{L}(t,z)$ and $\mathcal{I}(t,z)$ are functions of integration. Thus the spacelike component X^3 becomes

$$X^3 = \tfrac{1}{2}\mathcal{C}(t)(x^2+y^2) + \mathcal{K}(t,z)x + \mathcal{L}(t,z)y + \mathcal{I}(t,z) \tag{4.47}$$

Substituting (4.47) in (4.43) gives the second order differential equation

$$\mathcal{K}_{zz}x + \mathcal{L}_{zz}y + \mathcal{I}_{zz} = -\mathcal{C}(t)$$

This gives the consistency conditions

$$\mathcal{K}_{zz} = 0$$

$$\mathcal{L}_{zz} = 0$$

$$\mathcal{I}_{zz} = -\mathcal{C}(t)$$

On integrating the above set of equations we obtain

$$\mathcal{K}(t,z) = \mathcal{K}_1(t)z + \mathcal{K}_2(t)$$

$$\mathcal{L}(t,z) = \mathcal{L}_1(t)z + \mathcal{L}_2(t)$$

$$\mathcal{I}(t,z) = -\tfrac{1}{2}\mathcal{C}(t)z^2 + \mathcal{I}_1(t)z + \mathcal{I}_2(t)$$

where $\mathcal{K}_1(t)$, $\mathcal{K}_2(t)$, $\mathcal{L}_1(t)$, $\mathcal{L}_2(t)$, $\mathcal{I}_1(t)$ and $\mathcal{I}_2(t)$ are arbitrary functions. Thus (4.47) becomes

$$\begin{aligned} X^3 &= \tfrac{1}{2}\mathcal{C}(t)(x^2+y^2-z^2) + \mathcal{K}_1(t)xz + \mathcal{K}_2(t)x + \mathcal{L}_1(t)yz + \mathcal{L}_2(t)y + \mathcal{I}_1(t)z \\ &\quad + \mathcal{I}_2(t) \end{aligned} \tag{4.48}$$

for the spacelike component X^3.

Substitution of (4.36), (4.42) and (4.48) in equation (4.11) gives the following two independent equations:

$$-\mathcal{A}x + \mathcal{D}_1 y + \mathcal{E}_1 z + \mathcal{G}_1$$

$$= \; -\mathcal{B}y + \mathcal{J}_1 x + \mathcal{F}_1 z + \mathcal{H}_1$$

$$. \quad = \; -\mathcal{C}z + \mathcal{K}_1 x + \mathcal{L}_1 y + \mathcal{I}_1$$

This implies that

$$\mathcal{J}_1(t) \; = \; \mathcal{K}_1(t) \; = \; -\mathcal{A}(t)$$

$$\mathcal{D}_1(t) \; = \; \mathcal{L}_1(t) \; = \; -\mathcal{B}(t)$$

$$\mathcal{E}_1(t) \; = \; \mathcal{F}_1(t) \; = \; -\mathcal{C}(t)$$

$$\mathcal{I}_1(t) \; = \; \mathcal{H}_1(t) \; = \; \mathcal{G}_1(t)$$

Note that the above results and equations (4.6), (4.7) and (4.9) further reduce the number of arbitrary functions :

$$\mathcal{J}_2 = -\mathcal{D}_2\,, \quad \mathcal{K}_2 = -\mathcal{E}_2\,, \quad \mathcal{L}_2 = -\mathcal{F}_2$$

Then the spacelike components (4.36), (4.42) and (4.48) become

$$\begin{aligned} X^1 \; = \; & \tfrac{1}{2}\mathcal{A}(t)(y^2 + z^2 - x^2) - \mathcal{B}(t)xy + \mathcal{D}_2(t)y - \mathcal{C}(t)xz + \mathcal{E}_2(t)z + \mathcal{G}_1(t)x \\ & + \mathcal{G}_2(t) \end{aligned} \tag{4.49}$$

$$\begin{aligned} X^2 \; = \; & \tfrac{1}{2}\mathcal{B}(t)(x^2 + z^2 - y^2) - \mathcal{A}(t)xy - \mathcal{D}_2(t)x - \mathcal{C}(t)yz + \mathcal{F}_2(t)z + \mathcal{G}_1(t)y \\ & + \mathcal{H}_2(t) \end{aligned} \tag{4.50}$$

$$\begin{aligned} X^3 \; = \; & \tfrac{1}{2}\mathcal{C}(t)(x^2 + y^2 - z^2) - \mathcal{A}(t)xz - \mathcal{E}_2(t)x - \mathcal{B}(t)yz - \mathcal{F}_2(t)y + \mathcal{G}_1(t)z \\ & + \mathcal{I}_2(t) \end{aligned} \tag{4.51}$$

We have therefore found expressions for the spacelike components X^1, X^2 and X^3 in which the x, y and z dependence is explicitly known. These expressions will be of assistance in later calculations. Note that in the process of obtaining the spacelike components (4.49)–(4.51) we have completely solved the conformal Killing vector equations (4.6)–(4.7) and (4.9). It remains to obtain an expression for X^0 and ϕ and to solve the remaining equations of the system (4.1)–(4.10).

4.4 The Timelike Component

In order to complete the solution of the conformal Killing vector **X** we need to obtain the timelike component X^0. It is convenient to introduce the quantity

$$\alpha = \frac{D}{R/V} = F\left(\frac{V}{R}\right)_t$$

which on substituting the explicit value of V becomes:

$$\begin{aligned}\alpha &= \frac{F}{R^2}\Big\{\tfrac{1}{4}(\dot{k}R - k\dot{R})\left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right] \\ &\quad - \tfrac{1}{2}kR\left[(x-x_0)\dot{x}_0+(y-y_0)\dot{y}_0+(z-z_0)\dot{z}_0\right] - \dot{R}\Big\}\end{aligned} \tag{4.52}$$

With the above definition for α, equations (4.2)–(4.4) can be written in the following compact form :

$$X^1_t = \alpha^2 X^0_x \tag{4.53}$$

$$X^2_t = \alpha^2 X^0_y \tag{4.54}$$

$$X^3_t = \alpha^2 X^0_z \tag{4.55}$$

Differentiation of (4.53) with respect to y and (4.54) with respect to x and taking the difference gives

$$\alpha\alpha_y X^0_x - \alpha\alpha_x X^0_y = \dot{\mathcal{A}}y - \dot{\mathcal{B}}x + \dot{\mathcal{D}}_2 \tag{4.56}$$

where we have used (4.49) and (4.50). Similarly we obtain

$$\alpha\alpha_z X^0_x - \alpha\alpha_x X^0_z = \dot{\mathcal{A}}z - \dot{\mathcal{C}}x + \dot{\mathcal{E}}_2 \tag{4.57}$$

by differentiating (4.53) and (4.55) and substituting (4.49) and (4.51) in the difference. In addition subtraction of the appropriate derivatives of (4.54) and (4.55) and using the components (4.50) and (4.51) yields

$$\alpha\alpha_z X^0_y - \alpha\alpha_y X^0_z = \dot{\mathcal{B}}z - \dot{\mathcal{C}}y + \dot{\mathcal{F}}_2 \tag{4.58}$$

for the timelike component X^0.

Equations (4.56)–(4.58) are a system of first order partial differential equations which can be solved for X^0 by utilising the method of characteristics in which a partial differential equation is converted to an equivalent system of ordinary differential equations. For the partial differential equation (4.56) we obtain the four characteristic equations

$$\frac{dx}{\alpha\alpha_y} = \frac{dy}{-\alpha\alpha_x} = \frac{dz}{0} = \frac{dt}{0} = \frac{dX^0}{(\dot{\mathcal{A}}y - \dot{\mathcal{B}}x + \dot{\mathcal{D}}_2)} \tag{4.59}$$

We need to obtain four characteristics for this system of differential equations to generate the general solution. It can be easily established that

$$t = \mathrm{C}_1$$

$$z = \mathrm{C}_2$$

$$\alpha = \mathrm{C}_3$$

where C_1, C_2 and C_3 are three characteristics. To obtain the fourth characteristic C_4 we use the fact that t, z and α are characteristics for the partial differential equation (4.56). We note that α given by (4.52) is a quadratic in $y - y_0(t)$ and has the real root

$$y - y_0(t) = \frac{-\tilde{Y} + \sqrt{\tilde{Y}^2 - 4\tilde{X}\tilde{Z}}}{2\tilde{X}} \tag{4.60}$$

where we have set

$$\begin{aligned}
\tilde{X} &= \frac{F}{4R^2}(\dot{k}R - k\dot{R}) \\
\tilde{Y} &= -\frac{F}{2R}k\dot{y}_0 \\
\tilde{Z} &= \frac{F}{4R^2}(\dot{k}R - k\dot{R})\left[(x - x_0)^2 + (z - z_0)^2\right] - \frac{F}{2R}k\left[(x - x_0)\dot{x}_0 + (z - z_0)\dot{z}_0\right] \\
&\quad - \frac{F\dot{R}}{R^2} - \alpha
\end{aligned}$$

[**Note**: We take only the positive square root in (4.60) without any loss in generality. For the negative root we find that when X^0, given by (4.62), is substituted into (4.56) restrictions are placed on the functions of integration unlike in the case for the positive root. Thus the solution for the negative root is contained in that of the positive root.] As $\tilde{X} \neq 0$ we must have

$$\dot{k}R - k\dot{R} \neq 0$$

for a consistent solution.

The right hand side of (4.60) depends on the characteristics t, z, α and the variable x. Thus we can treat $y - y_0(t)$ as a function only of x in the integration process. Then from (4.59) and (4.60) we have that

$$\frac{dx}{\alpha\alpha_y} = \frac{dX^0}{(\dot{\mathcal{A}}y - \dot{\mathcal{B}}x + \dot{\mathcal{D}}_2)}$$

$$\mathrm{C}_4 = X^0 - \frac{1}{\alpha}\int \frac{(\dot{\mathcal{A}}y - \dot{\mathcal{B}}x + \dot{\mathcal{D}}_2)}{(F/2R^2)[(\dot{k}R - k\dot{R})(y - y_0) - kR\dot{y}_0]}dx$$

$$\mathrm{C}_4 = X^0 - \frac{B}{\alpha}\int dx + \frac{K}{\alpha}\int \frac{(x - x_0)d(x - x_0)}{\sqrt{E + H(x - x_0) + G(x - x_0)^2}}$$
$$- \frac{A + L}{\alpha}\int \frac{d(x - x_0)}{\sqrt{E + H(x - x_0) + G(x - x_0)^2}}$$

where we have set for convenience

$$A = \frac{\dot{\mathcal{A}}kR\dot{y}_0}{(\dot{k}R - k\dot{R})}$$

$$B = \frac{\dot{\mathcal{A}}}{(F/2R^2)(\dot{k}R - k\dot{R})}$$

$$E = -\frac{F}{R^2}(\dot{k}R - k\dot{R})\left[\frac{F}{4R^2}(\dot{k}R - k\dot{R})(z - z_0)^2 - \frac{F}{2R}k(z - z_0)\dot{z}_0 - \frac{F\dot{R}}{R^2} - \alpha\right]$$
$$+ \frac{F^2}{4R^2}k^2\dot{y}_0^2$$

$$G = -\frac{F^2}{4R^4}(\dot{k}R - k\dot{R})^2$$

$$H = \frac{F^2}{2R^3}k(\dot{k}R - k\dot{R})\dot{x}_0$$

$$K = \dot{\mathcal{B}}$$

$$L = \dot{\mathcal{A}} y_0 - \dot{\mathcal{B}} x_0 + \dot{\mathcal{D}}_2$$

Note that in the above $G < 0$ and $(4EG - H^2) < 0$. Hence using Gradshteyn and Ryzhik (1980) we can express the integrals in the characteristic C_4 in terms of elementary functions:

$$\begin{aligned} C_4 &= X^0 - \frac{B}{\alpha} x + \frac{K}{\alpha G} \left[\sqrt{E + H(x - x_0) + G(x - x_0)^2} \right] \\ &+ \left(\frac{KH}{2\alpha G} + \frac{A + L}{\alpha} \right) \left[\frac{1}{\sqrt{-G}} \arcsin \frac{2G(x - x_0) + H}{\sqrt{-(4EG - H^2)}} \right] \end{aligned} \quad (4.61)$$

Thus we have found the general solution of (4.56) as the fourth characteristic (4.61) is known.

We may express the general solution of the partial differential equation (4.56) as

$$C_4 = f(C_1, C_2, C_3)$$

where f is an arbitrary function. We may write the general solution in the equivalent form:

$$\begin{aligned} X^0 &= \frac{B}{\alpha} x - \frac{K\alpha_y}{\alpha G} - \left(\frac{KH}{2\alpha G} + \frac{A + L}{\alpha} \right) \left[\frac{1}{\sqrt{-G}} \arcsin \frac{-\alpha_x}{\sqrt{\alpha_x{}^2 + \alpha_y{}^2}} \right] \\ &+ f(t, z, \alpha) \end{aligned} \quad (4.62)$$

On substituting (4.62) first in (4.57) and then in (4.58) we obtain the following conditions on the function f :

$$\alpha\alpha_x \frac{\partial f}{\partial z} = \dot{\mathcal{C}} x - \dot{\mathcal{E}}_2 - \dot{\mathcal{A}} z_0 - \frac{\dot{\mathcal{A}} k R \dot{z}_0}{(\dot{k} R - k \dot{R})} + \left(\frac{KH}{2G} + A + L \right) \frac{\alpha_z \alpha_y}{(\alpha_x{}^2 + \alpha_y{}^2)} \quad (4.63)$$

$$\alpha\alpha_y \frac{\partial f}{\partial z} = \dot{\mathcal{C}} y - \dot{\mathcal{F}}_2 - \dot{\mathcal{B}} z_0 - \frac{\dot{\mathcal{B}} k R \dot{z}_0}{(\dot{k} R - k \dot{R})} - \left(\frac{KH}{2G} + A + L \right) \frac{\alpha_z \alpha_x}{(\alpha_x{}^2 + \alpha_y{}^2)} \quad (4.64)$$

Elimination of $\partial f/\partial z$ from (4.63) and (4.64) gives the equation

$$\alpha_y\left[-\dot{\mathcal{A}}z_0 - \frac{\dot{\mathcal{A}}kR\dot{z}_0}{(\dot{k}R - k\dot{R})} + \dot{\mathcal{C}}x - \dot{\mathcal{E}}_2\right] + \alpha_x\left[\dot{\mathcal{B}}z_0 + \frac{\dot{\mathcal{B}}kR\dot{z}_0}{(\dot{k}R - k\dot{R})} - \dot{\mathcal{C}}y + \dot{\mathcal{F}}_2\right]$$
$$+\alpha_z\left(\frac{KH}{2G} + A + L\right) = 0$$

On substituting the explicit form of α from (4.52) in the above equation we obtain a polynomial equation in x, y, z. On setting the coefficients to zero in the polynomial equation we obtain the following consistency conditions:

$$\dot{\mathcal{E}}_2 + \dot{\mathcal{A}}z_0 + \frac{\dot{\mathcal{A}}kR\dot{z}_0}{(\dot{k}R - k\dot{R})} = \dot{\mathcal{C}}x_0 + \frac{\dot{\mathcal{C}}\dot{x}_0kR}{(\dot{k}R - k\dot{R})} \tag{4.65}$$

$$\dot{\mathcal{F}}_2 + \dot{\mathcal{B}}z_0 + \frac{\dot{\mathcal{B}}kR\dot{z}_0}{(\dot{k}R - k\dot{R})} = \dot{\mathcal{C}}y_0 + \frac{\dot{\mathcal{C}}\dot{y}_0kR}{(\dot{k}R - k\dot{R})} \tag{4.66}$$

$$\dot{\mathcal{D}}_2 + \dot{\mathcal{A}}y_0 + \frac{\dot{\mathcal{A}}kR\dot{y}_0}{(\dot{k}R - k\dot{R})} = \dot{\mathcal{B}}x_0 + \frac{\dot{\mathcal{B}}\dot{x}_0kR}{(\dot{k}R - k\dot{R})} \tag{4.67}$$

where the last equation is equivalent to $[(KH/2G) + A + L] = 0$. Substitution of the conditions (4.65)–(4.67) in (4.63) gives the following restriction:

$$\frac{\partial f(t,z,\alpha)}{\partial z} = \frac{\dot{\mathcal{C}}}{\alpha(F/2R^2)(\dot{k}R - k\dot{R})}$$

(Note that (4.64) gives the same restriction on $f(t,z,\alpha)$). We integrate this equation to obtain

$$f(t,z,\alpha) = \frac{\dot{\mathcal{C}}z}{\alpha(F/2R^2)(\dot{k}R - k\dot{R})} + F_1(t,\alpha)$$

where $F_1(t,\alpha)$ is a function of integration. Substitution of $f(t,z,\alpha)$ and the condition $[(KH/2G) + A + L] = 0$ in equation (4.62) yields the result

$$X^0 = \frac{\dot{\mathcal{A}}x + \dot{\mathcal{B}}(y - y_0) + \dot{\mathcal{C}}z - (\dot{\mathcal{B}}\dot{y}_0kR)/(\dot{k}R - k\dot{R})}{\alpha(F/2R^2)(\dot{k}R - k\dot{R})} + F_1(t,\alpha) \tag{4.68}$$

where we have utilised the explicit values for B, K and G.

We now solve for the function $F_1(t,\alpha)$. On substituting (4.68) and (4.49) in (4.53) we obtain

$$\begin{aligned}\alpha^2\frac{\partial F_1}{\partial\alpha}\alpha_x &= \left[\dot{\mathcal{G}}_1 - \dot{\mathcal{B}}y_0 - \frac{\dot{\mathcal{B}}kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right]x + \dot{\mathcal{G}}_2 - \tfrac{1}{2}\dot{\mathcal{A}}(x_0^2 + y_0^2 + z_0^2)\\ &\quad - \frac{\dot{\mathcal{A}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) + \frac{2\dot{\mathcal{A}}\dot{R}}{(\dot{k}R - k\dot{R})}\\ &\quad + \left[\dot{\mathcal{B}}y_0 + \frac{\dot{\mathcal{B}}kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right]\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right]\end{aligned}$$

where we have used conditions (4.65) and (4.67). Now F_1 is a function of t and α, and α_x is a function only of t and x. We observe also that the right hand side of the equation is a function of t and x. This implies that $\alpha^2\partial F_1/\partial\alpha$ is necessarily a function only of t. This condition gives after some calculation

$$\alpha^2\frac{\partial F_1}{\partial\alpha} = \frac{\dot{\mathcal{G}}_1 - \dot{\mathcal{B}}y_0 - (\dot{\mathcal{B}}kR\dot{y}_0)/(\dot{k}R - k\dot{R})}{(F/2R^2)(\dot{k}R - k\dot{R})} \tag{4.69}$$

subject to the consistency condition

$$\begin{aligned}&-\dot{\mathcal{G}}_2 + \tfrac{1}{2}\dot{\mathcal{A}}(x_0^2 + y_0^2 + z_0^2) + \frac{\dot{\mathcal{A}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{\mathcal{A}}\dot{R}}{(\dot{k}R - k\dot{R})}\\ &= \dot{\mathcal{G}}_1 x_0 + \frac{\dot{\mathcal{G}}_1\dot{x}_0 kR}{(\dot{k}R - k\dot{R})}\end{aligned} \tag{4.70}$$

It is now possible to find the function F_1 by integrating (4.69) :

$$F_1(t,\alpha) = \frac{-\dot{\mathcal{G}}_1 + \dot{\mathcal{B}}y_0 + (\dot{\mathcal{B}}kR\dot{y}_0)/(\dot{k}R - k\dot{R})}{\alpha(F/2R^2)(\dot{k}R - k\dot{R})} + \mathcal{M}(t) \tag{4.71}$$

where $\mathcal{M}(t)$ results from the integration process.

Therefore on substituting (4.71) into (4.68) we obtain the timelike component X^0:

$$X^0 = \frac{\dot{\mathcal{A}}x + \dot{\mathcal{B}}y + \dot{\mathcal{C}}z - \dot{\mathcal{G}}_1}{\alpha(F/2R^2)(\dot{k}R - k\dot{R})} + \mathcal{M}(t) \tag{4.72}$$

We substitute (4.72) into equations (4.54) and (4.55) and obtain the following two consistency conditions:

$$\begin{aligned} -\dot{\mathcal{H}}_2 + \tfrac{1}{2}\dot{\mathcal{B}}(x_0^2 + y_0^2 + z_0^2) + \frac{\dot{\mathcal{B}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{\mathcal{B}}\dot{R}}{(\dot{k}R - k\dot{R})} \\ = \dot{\mathcal{G}}_1 y_0 + \frac{\dot{\mathcal{G}}_1\dot{y}_0 kR}{(\dot{k}R - k\dot{R})} \end{aligned} \tag{4.73}$$

$$\begin{aligned} -\dot{\mathcal{I}}_2 + \tfrac{1}{2}\dot{\mathcal{C}}(x_0^2 + y_0^2 + z_0^2) + \frac{\dot{\mathcal{C}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{\mathcal{C}}\dot{R}}{(\dot{k}R - k\dot{R})} \\ = \dot{\mathcal{G}}_1 z_0 + \frac{\dot{\mathcal{G}}_1\dot{z}_0 kR}{(\dot{k}R - k\dot{R})} \end{aligned} \tag{4.74}$$

Thus we have obtained the timelike component X^0, given by equation (4.72) subject to the six integrability conditions (4.65)–(4.67), (4.70), (4.73) and (4.74). In addition we have completely solved the conformal Killing vector equations (4.2)–(4.4). Note that we have the restriction $\dot{k}R - k\dot{R} \neq 0$ in the above results. In order to complete the solution we need to obtain the conformal factor ϕ and solve the remaining four equations of the system (4.1)–(4.10).

4.5 The Conformal Factor

In this section we will completely solve the remaining conformal Killing vector equations (4.1), (4.5), (4.8) and (4.10) and thus obtain the conformal factor ϕ. In the process we obtain five more integrability conditions. As (4.11) is valid we note that (4.5), (4.8) and (4.10) are equivalent. Thus it remains to solve equations (4.1) and (4.5) and to find the conformal factor ϕ. Instead of working with (4.1) we will replace this equation by the difference of (4.1) and (4.5) so that ϕ is eliminated. On substituting the explicit value of the function D, defined in terms of the metric functions, in (4.1) and taking the difference between (4.1) and (4.5) we obtain

$$\left[\frac{\dot{F}}{F}+\frac{(V/R)_{tt}}{(V/R)_t}\right]X^0+\left[\frac{(V/R)_{tx}}{(V/R)_t}\right]X^1+\left[\frac{(V/R)_{ty}}{(V/R)_t}\right]X^2+\left[\frac{(V/R)_{tz}}{(V/R)_t}\right]X^3+X^0_t$$
$$-X^1_x = 0 \tag{4.75}$$

On substituting for X^0 from (4.72) and simplifying equation (4.75) assumes the following form :

$$\left[\left(\frac{V}{R}\right)_{tx}\right]X^1+\left[\left(\frac{V}{R}\right)_{ty}\right]X^2+\left[\left(\frac{V}{R}\right)_{tz}\right]X^3-\left(\frac{V}{R}\right)_t X^1_x$$
$$+\left[\frac{2R^2(\dot{\mathcal{A}}x+\dot{\mathcal{B}}y+\dot{\mathcal{C}}z-\dot{\mathcal{G}}_1)}{F^2(\dot{k}R-k\dot{R})}\right]_t+\frac{\dot{F}(V/R)_t\mathcal{M}}{F}+\left(\frac{V}{R}\right)_{tt}\mathcal{M}+\left(\frac{V}{R}\right)_t\dot{\mathcal{M}}=0$$
$$\tag{4.76}$$

where the spacelike components X^1, X^2 and X^3 are given by equations (4.49)–(4.51) respectively.

To analyse (4.76) we need to substitute for X^1, X^2, X^3 and obtain a polynomial equation in x, y, z. To perform this by hand is extremely complicated and we need to resort to a symbolic manipulation package. For our purposes Mathematica (Ver 2.0) (Wolfram 1991) is appropriate. On substituting (4.49)–(4.51) in (4.76) we obtain the following lengthy polynomial equation in x, y, z:

$$\Big\{\ \mathcal{G}_1(\dot{k}R-k\dot{R})\ -\ \mathcal{A}\left[x_0(\dot{k}R-k\dot{R})+kR\dot{x}_0\right]\ -\ \mathcal{B}\left[y_0(\dot{k}R-k\dot{R})+kR\dot{y}_0\right]$$

$$-\mathcal{C}\Big[z_0(\dot{k}R - k\dot{R}) + kR\dot{z}_0\Big] + \mathcal{M}\Bigg[\frac{\dot{F}}{F}(\dot{k}R - k\dot{R}) - 2\frac{\dot{R}}{R}(\dot{k}R - k\dot{R})$$

$$+ (\dot{k}R - k\dot{R})^{\cdot}\Bigg] + \dot{\mathcal{M}}(\dot{k}R - k\dot{R})\Bigg\}\,(x^2 + y^2 + z^2)$$

$$+2\Bigg\{\ \mathcal{A}\Big[\frac{1}{2}(\dot{k}R - k\dot{R})(x_0^2 + y_0^2 + z_0^2) + kR(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - 2\dot{R}\Big]$$

$$-\frac{4\dot{\mathcal{A}}R^4}{F^2(\dot{k}R - k\dot{R})^2}\Bigg[\frac{\dot{F}}{F}(\dot{k}R - k\dot{R}) - 2\frac{\dot{R}}{R}(\dot{k}R - k\dot{R}) + (\dot{k}R - k\dot{R})^{\cdot}\Bigg]$$

$$+\frac{4\ddot{\mathcal{A}}R^4}{F^2(\dot{k}R - k\dot{R})} + \mathcal{G}_2(\dot{k}R - k\dot{R}) + \mathcal{D}_2\Big[y_0(\dot{k}R - k\dot{R}) + kR\dot{y}_0\Big]$$

$$+\mathcal{E}_2\Big[z_0(\dot{k}R - k\dot{R}) + kR\dot{z}_0\Big] - \mathcal{M}\Bigg\{x_0\Bigg[\frac{\dot{F}}{F}(\dot{k}R - k\dot{R}) - 2\frac{\dot{R}}{R}(\dot{k}R - k\dot{R})$$

$$+ (\dot{k}R - k\dot{R})^{\cdot}\Bigg] + 2\dot{x}_0(\dot{k}R - k\dot{R}) + \frac{\dot{F}kR\dot{x}_0}{F} + kR\ddot{x}_0\Bigg\}$$

$$-\ \dot{\mathcal{M}}\Big[x_0(\dot{k}R - k\dot{R}) + kR\dot{x}_0\Big]\ \Bigg\}\,x$$

$$+2\Bigg\{\ \mathcal{B}\Big[\frac{1}{2}(\dot{k}R - k\dot{R})(x_0^2 + y_0^2 + z_0^2) + kR(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - 2\dot{R}\Big]$$

$$-\frac{4\dot{\mathcal{B}}R^4}{F^2(\dot{k}R - k\dot{R})^2}\Bigg[\frac{\dot{F}}{F}(\dot{k}R - k\dot{R}) - 2\frac{\dot{R}}{R}(\dot{k}R - k\dot{R}) + (\dot{k}R - k\dot{R})^{\cdot}\Bigg]$$

$$+\frac{4\ddot{\mathcal{B}}R^4}{F^2(\dot{k}R - k\dot{R})} + \mathcal{H}_2(\dot{k}R - k\dot{R}) - \mathcal{D}_2\Big[x_0(\dot{k}R - k\dot{R}) + kR\dot{x}_0\Big]$$

$$+\mathcal{F}_2\Big[z_0(\dot{k}R - k\dot{R}) + kR\dot{z}_0\Big] - \mathcal{M}\Bigg\{y_0\Bigg[\frac{\dot{F}}{F}(\dot{k}R - k\dot{R}) - 2\frac{\dot{R}}{R}(\dot{k}R - k\dot{R})$$

$$+ (\dot{k}R - k\dot{R})^{\cdot}\Bigg] + 2\dot{y}_0(\dot{k}R - k\dot{R}) + \frac{\dot{F}kR\dot{y}_0}{F} + kR\ddot{y}_0\Bigg\}$$

$$-\ \dot{\mathcal{M}}\Big[y_0(\dot{k}R - k\dot{R}) + kR\dot{y}_0\Big]\ \Bigg\}\,y$$

$$
\begin{aligned}
&+2\Bigg\{\ \mathcal{C}\left[\frac{1}{2}(\dot{k}R-k\dot{R})(x_0^2+y_0^2+z_0^2)+kR(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-2\dot{R}\right]\\
&-\frac{4\dot{\mathcal{C}}R^4}{F^2(\dot{k}R-k\dot{R})^2}\left[\frac{\dot{F}}{F}(\dot{k}R-k\dot{R})-2\frac{\dot{R}}{R}(\dot{k}R-k\dot{R})+(\dot{k}R-k\dot{R})^{\cdot}\right]\\
&+\frac{4\ddot{\mathcal{C}}R^4}{F^2(\dot{k}R-k\dot{R})}+\mathcal{I}_2(\dot{k}R-k\dot{R})-\mathcal{E}_2\left[x_0(\dot{k}R-k\dot{R})+kR\dot{x}_0\right]\\
&-\mathcal{F}_2\left[y_0(\dot{k}R-k\dot{R})+kR\dot{y}_0\right]-\mathcal{M}\Bigg\{z_0\left[\frac{\dot{F}}{F}(\dot{k}R-k\dot{R})-2\frac{\dot{R}}{R}(\dot{k}R-k\dot{R})\right.\\
&\left.+(\dot{k}R-k\dot{R})^{\cdot}\right]+2\dot{z}_0(\dot{k}R-k\dot{R})+\frac{\dot{F}kR\dot{z}_0}{F}+kR\ddot{z}_0\Bigg\}\\
&-\dot{\mathcal{M}}\left[z_0(\dot{k}R-k\dot{R})+kR\dot{z}_0\right]\ \Bigg\}\ z
\end{aligned}
$$

$$
\begin{aligned}
&+2\Bigg\{\ \mathcal{G}_1\left[\frac{1}{2}(\dot{k}R-k\dot{R})(x_0^2+y_0^2+z_0^2)+kR(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-2\dot{R}\right]\\
&-\frac{4\dot{\mathcal{G}}_1R^4}{F^2(\dot{k}R-k\dot{R})^2}\left[\frac{\dot{F}}{F}(\dot{k}R-k\dot{R})-2\frac{\dot{R}}{R}(\dot{k}R-k\dot{R})+(\dot{k}R-k\dot{R})^{\cdot}\right]\\
&+\frac{4\ddot{\mathcal{G}}_1R^4}{F^2(\dot{k}R-k\dot{R})}+\mathcal{G}_2\left[x_0(\dot{k}R-k\dot{R})+kR\dot{x}_0\right]+\mathcal{H}_2\left[y_0(\dot{k}R-k\dot{R})+kR\dot{y}_0\right]\\
&+\mathcal{I}_2\left[z_0(\dot{k}R-k\dot{R})+kR\dot{z}_0\right]-\mathcal{M}\Bigg\{\frac{1}{2}(x_0^2+y_0^2+z_0^2)\left[\frac{\dot{F}}{F}(\dot{k}R-k\dot{R})\right.\\
&\left.-2\,\frac{\dot{R}}{R}(\dot{k}R-k\dot{R})+(\dot{k}R-k\dot{R})^{\cdot}\right]+\frac{\dot{F}kR}{F}(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)\\
&+2(\dot{k}R-k\dot{R})(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)+kR(x_0\ddot{x}_0+y_0\ddot{y}_0+z_0\ddot{z}_0)\\
&+kR(\dot{x}_0^2+\dot{y}_0^2+\dot{z}_0^2)-2\left(\frac{\dot{F}\dot{R}}{F}-2\frac{\dot{R}^2}{R}+\ddot{R}\right)\Bigg\}\\
&-\dot{\mathcal{M}}\left[\frac{1}{2}(\dot{k}R-k\dot{R})(x_0^2+y_0^2+z_0^2)+kR(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-2\dot{R}\right]\ \Bigg\}\ =\ 0
\end{aligned}
$$

For the above equation to be valid the coefficients of the powers of x, y, z must separately vanish. Hence we obtain the following consistency conditions

$$
\begin{aligned}
&-\mathcal{G}_1 + \mathcal{A}\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{B}\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{C}\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right] \\
= \;& \mathcal{M}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \dot{\mathcal{M}}
\end{aligned} \tag{4.77}
$$

$$
\begin{aligned}
&\mathcal{G}_2 + \mathcal{D}_2\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{E}_2\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right] \\
&+\mathcal{A}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{R}}{(\dot{k}R - k\dot{R})}\right] \\
&- \frac{4\dot{\mathcal{A}}R^4}{F^2(\dot{k}R - k\dot{R})^2}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \frac{4\ddot{\mathcal{A}}R^4}{F^2(\dot{k}R - k\dot{R})^2} \\
= \;& \mathcal{M}\left\{x_0\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + 2\dot{x}_0 + \frac{\dot{F}kR\dot{x}_0}{F(\dot{k}R - k\dot{R})} + \frac{kR\ddot{x}_0}{(\dot{k}R - k\dot{R})}\right\} \\
&+\dot{\mathcal{M}}\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right]
\end{aligned} \tag{4.78}
$$

$$
\begin{aligned}
&\mathcal{H}_2 - \mathcal{D}_2\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{F}_2\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right] \\
&+\mathcal{B}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{R}}{(\dot{k}R - k\dot{R})}\right] \\
&- \frac{4\dot{\mathcal{B}}R^4}{F^2(\dot{k}R - k\dot{R})^2}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \frac{4\ddot{\mathcal{B}}R^4}{F^2(\dot{k}R - k\dot{R})^2} \\
= \;& \mathcal{M}\left\{y_0\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + 2\dot{y}_0 + \frac{\dot{F}kR\dot{y}_0}{F(\dot{k}R - k\dot{R})} + \frac{kR\ddot{y}_0}{(\dot{k}R - k\dot{R})}\right\} \\
&+\dot{\mathcal{M}}\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right]
\end{aligned} \tag{4.79}
$$

$$
\begin{aligned}
&\mathcal{I}_2 - \mathcal{E}_2\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right] - \mathcal{F}_2\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right] \\
&+\mathcal{C}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{R}}{(\dot{k}R - k\dot{R})}\right] \\
&- \frac{4\dot{\mathcal{C}}R^4}{F^2(\dot{k}R - k\dot{R})^2}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \frac{4\ddot{\mathcal{C}}R^4}{F^2(\dot{k}R - k\dot{R})^2} \\
= \;& \mathcal{M}\left\{z_0\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + 2\dot{z}_0 + \frac{\dot{F}kR\dot{z}_0}{F(\dot{k}R - k\dot{R})} + \frac{kR\ddot{z}_0}{(\dot{k}R - k\dot{R})}\right\} \\
&+\dot{\mathcal{M}}\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right] \qquad (4.80)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{G}_2\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{H}_2\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{I}_2\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right] \\
&+\mathcal{G}_1\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{R}}{(\dot{k}R - k\dot{R})}\right] \\
&- \frac{4\dot{\mathcal{G}}_1R^4}{F^2(\dot{k}R - k\dot{R})^2}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \frac{4\ddot{\mathcal{G}}_1R^4}{F^2(\dot{k}R - k\dot{R})^2} \\
= \;& \mathcal{M}\left\{\frac{1}{2}(x_0^2 + y_0^2 + z_0^2)\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right]\right. \\
&\quad + \frac{\dot{F}kR}{F(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) + 2(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) \\
&\quad + \frac{kR}{(\dot{k}R - k\dot{R})}(\dot{x}_0^2 + \dot{y}_0^2 + \dot{z}_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\ddot{x}_0 + y_0\ddot{y}_0 + z_0\ddot{z}_0) \\
&\quad \left. - \frac{2}{(\dot{k}R - k\dot{R})}\left(\frac{\dot{F}\dot{R}}{F} - 2\frac{\dot{R}^2}{R} + \ddot{R}\right)\right\} \\
&+\dot{\mathcal{M}}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{R}}{(\dot{k}R - k\dot{R})}\right] \quad (4.81)
\end{aligned}
$$

Therefore we have obtained another five integrability conditions (4.77)-(4.81) for a consistent solution.

Finally we obtain the conformal factor ϕ using equation (4.5). Substitution of (4.72) into (4.5) and simplifying yields

$$\begin{aligned} V\phi &= -V_x X^1 - V_y X^2 - V_z X^3 + V X^1_x - \frac{2R^3(\dot{\mathcal{A}}x + \dot{\mathcal{B}}y + \dot{\mathcal{C}}z - \dot{\mathcal{G}}_1)}{F^2(\dot{k}R - k\dot{R})} \\ &\quad + \frac{\mathcal{M}(V\dot{R} - \dot{V}R)}{R} \end{aligned} \tag{4.82}$$

where X^1, X^2 and X^3 are given by (4.49)-(4.51). Again to evaluate the right hand side of (4.82) is extremely complicated to be done by hand. We resort again to the computer package Mathematica (Ver 2.0) (Wolfram 1991) to evaluate the right hand side of (4.82). Substituting (4.49)-(4.51) in (4.82) and after some simplification we obtain the following

$$\begin{aligned} V\phi &= (x^2+y^2+z^2)\left[\frac{k}{4}(\mathcal{A}x_0 + \mathcal{B}y_0 + \mathcal{C}z_0 - \mathcal{G}_1) - \frac{\mathcal{M}(\dot{k}R - k\dot{R})}{4R}\right] \\ &\quad + x\left\{-\mathcal{A}\left[1 + \frac{k}{4}({x_0}^2 + {y_0}^2 + {z_0}^2)\right] - \frac{k}{2}(\mathcal{D}_2 y_0 + \mathcal{E}_2 z_0 + \mathcal{G}_2)\right. \\ &\quad \left. + \frac{\mathcal{M}\left[(\dot{k}R - k\dot{R})x_0 + kR\dot{x}_0\right]}{2R} - \frac{2\dot{\mathcal{A}}R^3}{F^2(\dot{k}R - k\dot{R})}\right\} \\ &\quad + y\left\{-\mathcal{B}\left[1 + \frac{k}{4}({x_0}^2 + {y_0}^2 + {z_0}^2)\right] - \frac{k}{2}(-\mathcal{D}_2 x_0 + \mathcal{F}_2 z_0 + \mathcal{H}_2)\right. \\ &\quad \left. + \frac{\mathcal{M}\left[(\dot{k}R - k\dot{R})y_0 + kR\dot{y}_0\right]}{2R} - \frac{2\dot{\mathcal{B}}R^3}{F^2(\dot{k}R - k\dot{R})}\right\} \\ &\quad + z\left\{-\mathcal{C}\left[1 + \frac{k}{4}({x_0}^2 + {y_0}^2 + {z_0}^2)\right] - \frac{k}{2}(-\mathcal{E}_2 x_0 - \mathcal{F}_2 y_0 + \mathcal{I}_2)\right. \\ &\quad \left. + \frac{\mathcal{M}\left[(\dot{k}R - k\dot{R})z_0 + kR\dot{z}_0\right]}{2R} - \frac{2\dot{\mathcal{C}}R^3}{F^2(\dot{k}R - k\dot{R})}\right\} \end{aligned}$$

$$+\mathcal{G}_1\left[1+\frac{k}{4}({x_0}^2+{y_0}^2+{z_0}^2)\right]+\frac{2\dot{\mathcal{G}}_1R^3}{F^2(\dot{k}R-k\dot{R})}+\frac{k}{2}(\mathcal{G}_2x_0+\mathcal{H}_2y_0+\mathcal{I}_2z_0)$$
$$-\frac{\mathcal{M}}{4R}\left[(\dot{k}R-k\dot{R})({x_0}^2+{y_0}^2+{z_0}^2)+2kR(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-4\dot{R}\right] \tag{4.83}$$

Thus we have completely solved the conformal Killing vector equations (4.1)–(4.10) subject to the restriction $\dot{k}R - k\dot{R} \neq 0$ to obtain the conformal Killing vector $\mathbf{X} = (X^0, X^1, X^2, X^3)$ and the conformal factor ϕ. We have found expressions for $\mathbf{X}$ and ϕ in which the dependence on the spacelike coordinates x, y, z is explicitly known; the resulting functions of integration involve only the variable t. The timelike component X^0 is given by (4.72) and the spacelike components X^1, X^2 and X^3 are given by (4.49)–(4.51). This solution is subject to the eleven integrability conditions (4.65)–(4.67), (4.70), (4.73)–(4.74) and (4.77)–(4.81). The integrability conditions involve only the timelike coordinate t for the eleven functions of integration $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_2$, $\mathcal{E}_2$, $\mathcal{F}_2$, $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{H}_2$, $\mathcal{I}_2$, $\mathcal{M}$. The integrability conditions relate the functions of integration to the metric functions k, F, R, x_0, y_0, z_0. We discuss this solution further in §5.2 where the equations are also put into compact vector notation. Note that the solution presented in this chapter holds only if $\dot{k}R - k\dot{R} \neq 0$. The remaining special cases with $k = 0$ and $\dot{k}R - k\dot{R} = 0$ with $k \neq 0$ are considered in §5.3 and §5.4 respectively.

5 Conformal Symmetries in Stephani Universes

5.1 Introduction

In this chapter we provide a complete analysis of conformal symmetries in the Stephani universe. The most complex case $\dot{k}R - k\dot{R} \neq 0$ was considered in detail in chapter 4. In §5.2 we collect the results obtained for the conformal Killing vector equation in chapter 4 to facilitate easy reference. Since the form of this solution is difficult to analyse we simplify the solution using vector notation. The integrability conditions may be expressed in a more fundamental form on introduction of a new time variable. Two further cases remain to be considered to generate the general solution of the conformal Killing vector equations. In §5.3 we solve the conformal Killing vector equation for the first case, $k = 0$. As the integration process is simple we do not provide details and only list the results obtained. A conformal Killing vector in the Stephani universe, which is the analogue to the conformal Killing vector normal to the hypersurface $t = \text{constant}$ in the $k = 0$ Robertson–Walker spacetimes, is explicitly obtained. The remaining case governed by $\dot{k}R - k\dot{R} = 0$, $k \neq 0$ is considered in §5.4. As the integration process is similar to chapter 4 we give only an outline of the details of the integration procedure. In the process of solving the conformal Killing vector equation we use the same symbols to denote the functions of integration for easy comparison between the three cases that arise. This is an abuse of notation but should not lead to any confusion. Note that in some parts of the calculation we utilise the symbolic manipulation capabilities of Mathematica (Ver 2.0) (Wolfram 1991). We check that the conformal Killing vector equations are in fact satisfied using this package.

5.2 The Case $\dot{k}R - k\dot{R} \neq 0$

The general solution of the conformal Killing vector equations (4.1)–(4.10) for the case $\dot{k}R - k\dot{R} \neq 0$ was found in chapter 4. The conformal Killing vector $\mathbf{X} = (X^0, X^1, X^2, X^3)$ is given by the equations (4.49)–(4.51), (4.72) and the conformal factor ϕ has the form (4.83). This solution is subject to the integrability conditions (4.65)–(4.67), (4.70), (4.73)–(4.74) and (4.77)–(4.81). We bring the various results together for easy reference. Collecting the appropriate equations from chapter 4 we have the following solution for the case $\dot{k}R - k\dot{R} \neq 0$:

$$X^0 = \frac{\dot{\mathcal{A}}x + \dot{\mathcal{B}}y + \dot{\mathcal{C}}z - \dot{\mathcal{G}}_1}{\alpha(F/2R^2)(\dot{k}R - k\dot{R})} + \mathcal{M}(t)$$

$$X^1 = \tfrac{1}{2}\mathcal{A}(t)(y^2 + z^2 - x^2) - \mathcal{B}(t)xy + \mathcal{D}_2(t)y - \mathcal{C}(t)xz + \mathcal{E}_2(t)z + \mathcal{G}_1(t)x + \mathcal{G}_2(t)$$

$$X^2 = \tfrac{1}{2}\mathcal{B}(t)(x^2 + z^2 - y^2) - \mathcal{A}(t)xy - \mathcal{D}_2(t)x - \mathcal{C}(t)yz + \mathcal{F}_2(t)z + \mathcal{G}_1(t)y + \mathcal{H}_2(t)$$

$$X^3 = \tfrac{1}{2}\mathcal{C}(t)(x^2 + y^2 - z^2) - \mathcal{A}(t)xz - \mathcal{E}_2(t)x - \mathcal{B}(t)yz - \mathcal{F}_2(t)y + \mathcal{G}_1(t)z$$

$$+ \mathcal{I}_2(t)$$

$$\begin{aligned}
V\phi &= (x^2+y^2+z^2)\left[\frac{k}{4}(\mathcal{A}x_0+\mathcal{B}y_0+\mathcal{C}z_0-\mathcal{G}_1) - \frac{\mathcal{M}(\dot{k}R-k\dot{R})}{4R}\right] \\
&+ x\left\{-\mathcal{A}\left[1+\frac{k}{4}({x_0}^2+{y_0}^2+{z_0}^2)\right] - \frac{k}{2}(\mathcal{D}_2y_0+\mathcal{E}_2z_0+\mathcal{G}_2)\right. \\
&\quad \left. + \frac{\mathcal{M}\left[(\dot{k}R-k\dot{R})x_0+kR\dot{x}_0\right]}{2R} - \frac{2\dot{\mathcal{A}}R^3}{F^2(\dot{k}R-k\dot{R})}\right\} \\
&+ y\left\{-\mathcal{B}\left[1+\frac{k}{4}({x_0}^2+{y_0}^2+{z_0}^2)\right] - \frac{k}{2}(-\mathcal{D}_2x_0+\mathcal{F}_2z_0+\mathcal{H}_2)\right. \\
&\quad \left. + \frac{\mathcal{M}\left[(\dot{k}R-k\dot{R})y_0+kR\dot{y}_0\right]}{2R} - \frac{2\dot{\mathcal{B}}R^3}{F^2(\dot{k}R-k\dot{R})}\right\} \\
&+ z\left\{-\mathcal{C}\left[1+\frac{k}{4}({x_0}^2+{y_0}^2+{z_0}^2)\right] - \frac{k}{2}(-\mathcal{E}_2x_0-\mathcal{F}_2y_0+\mathcal{I}_2)\right. \\
&\quad \left. + \frac{\mathcal{M}\left[(\dot{k}R-k\dot{R})z_0+kR\dot{z}_0\right]}{2R} - \frac{2\dot{\mathcal{C}}R^3}{F^2(\dot{k}R-k\dot{R})}\right\} \\
&+ \mathcal{G}_1\left[1+\frac{k}{4}({x_0}^2+{y_0}^2+{z_0}^2)\right] + \frac{2\dot{\mathcal{G}}_1R^3}{F^2(\dot{k}R-k\dot{R})} + \frac{k}{2}(\mathcal{G}_2x_0+\mathcal{H}_2y_0+\mathcal{I}_2z_0) \\
&- \frac{\mathcal{M}}{4R}\left[(\dot{k}R-k\dot{R})({x_0}^2+{y_0}^2+{z_0}^2)+2kR(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-4\dot{R}\right]
\end{aligned}$$

subject to the eleven integrability conditions

$$\dot{\mathcal{E}}_2 + \dot{\mathcal{A}}z_0 + \frac{\dot{\mathcal{A}}kR\dot{z}_0}{(\dot{k}R-k\dot{R})} = \dot{\mathcal{C}}x_0 + \frac{\dot{\mathcal{C}}\dot{x}_0kR}{(\dot{k}R-k\dot{R})}$$

$$\dot{\mathcal{F}}_2 + \dot{\mathcal{B}}z_0 + \frac{\dot{\mathcal{B}}kR\dot{z}_0}{(\dot{k}R-k\dot{R})} = \dot{\mathcal{C}}y_0 + \frac{\dot{\mathcal{C}}\dot{y}_0kR}{(\dot{k}R-k\dot{R})}$$

$$\dot{\mathcal{D}}_2 + \dot{\mathcal{A}}y_0 + \frac{\dot{\mathcal{A}}kR\dot{y}_0}{(\dot{k}R-k\dot{R})} = \dot{\mathcal{B}}x_0 + \frac{\dot{\mathcal{B}}\dot{x}_0kR}{(\dot{k}R-k\dot{R})}$$

$$-\dot{\mathcal{G}}_2 + \tfrac{1}{2}\dot{\mathcal{A}}(x_0^2 + y_0^2 + z_0^2) + \frac{\dot{\mathcal{A}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{\mathcal{A}}\dot{R}}{(\dot{k}R - k\dot{R})}$$

$$= \dot{\mathcal{G}}_1 x_0 + \frac{\dot{\mathcal{G}}_1\dot{x}_0 kR}{(\dot{k}R - k\dot{R})}$$

$$-\dot{\mathcal{H}}_2 + \tfrac{1}{2}\dot{\mathcal{B}}(x_0^2 + y_0^2 + z_0^2) + \frac{\dot{\mathcal{B}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{\mathcal{B}}\dot{R}}{(\dot{k}R - k\dot{R})}$$

$$= \dot{\mathcal{G}}_1 y_0 + \frac{\dot{\mathcal{G}}_1\dot{y}_0 kR}{(\dot{k}R - k\dot{R})}$$

$$-\dot{\mathcal{I}}_2 + \tfrac{1}{2}\dot{\mathcal{C}}(x_0^2 + y_0^2 + z_0^2) + \frac{\dot{\mathcal{C}}kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{\mathcal{C}}\dot{R}}{(\dot{k}R - k\dot{R})}$$

$$= \dot{\mathcal{G}}_1 z_0 + \frac{\dot{\mathcal{G}}_1\dot{z}_0 kR}{(\dot{k}R - k\dot{R})}$$

$$-\mathcal{G}_1 + \mathcal{A}\left[x_0 + \frac{kR\dot{x}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{B}\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{C}\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right]$$

$$= \mathcal{M}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \dot{\mathcal{M}}$$

$$\mathcal{G}_2 + \mathcal{D}_2\left[y_0 + \frac{kR\dot{y}_0}{(\dot{k}R - k\dot{R})}\right] + \mathcal{E}_2\left[z_0 + \frac{kR\dot{z}_0}{(\dot{k}R - k\dot{R})}\right]$$

$$+\mathcal{A}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot{k}R - k\dot{R})}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \frac{2\dot{R}}{(\dot{k}R - k\dot{R})}\right]$$

$$-\frac{4\dot{\mathcal{A}}R^4}{F^2(\dot{k}R - k\dot{R})^2}\left[\frac{\dot{F}}{F} - 2\frac{\dot{R}}{R} + \frac{(\dot{k}R - k\dot{R})^{\cdot}}{(\dot{k}R - k\dot{R})}\right] + \frac{4\ddot{\mathcal{A}}R^4}{F^2(\dot{k}R - k\dot{R})^2}$$

$$= \mathcal{M}\left\{x_0\left[\frac{\dot F}{F} - 2\frac{\dot R}{R} + \frac{(\dot kR - k\dot R)^{\cdot}}{(\dot kR - k\dot R)}\right] + 2\dot x_0 + \frac{\dot F kR\dot x_0}{F(\dot kR - k\dot R)} + \frac{kR\ddot x_0}{(\dot kR - k\dot R)}\right\}$$

$$+\dot{\mathcal{M}}\left[x_0 + \frac{kR\dot x_0}{(\dot kR - k\dot R)}\right]$$

$$\mathcal{H}_2 - \mathcal{D}_2\left[x_0 + \frac{kR\dot x_0}{(\dot kR - k\dot R)}\right] + \mathcal{F}_2\left[z_0 + \frac{kR\dot z_0}{(\dot kR - k\dot R)}\right]$$

$$+\mathcal{B}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot kR - k\dot R)}(x_0\dot x_0 + y_0\dot y_0 + z_0\dot z_0) - \frac{2\dot R}{(\dot kR - k\dot R)}\right]$$

$$-\frac{4\dot{\mathcal{B}}R^4}{F^2(\dot kR - k\dot R)^2}\left[\frac{\dot F}{F} - 2\frac{\dot R}{R} + \frac{(\dot kR - k\dot R)^{\cdot}}{(\dot kR - k\dot R)}\right] + \frac{4\ddot{\mathcal{B}}R^4}{F^2(\dot kR - k\dot R)^2}$$

$$= \mathcal{M}\left\{y_0\left[\frac{\dot F}{F} - 2\frac{\dot R}{R} + \frac{(\dot kR - k\dot R)^{\cdot}}{(\dot kR - k\dot R)}\right] + 2\dot y_0 + \frac{\dot F kR\dot y_0}{F(\dot kR - k\dot R)} + \frac{kR\ddot y_0}{(\dot kR - k\dot R)}\right\}$$

$$+\dot{\mathcal{M}}\left[y_0 + \frac{kR\dot y_0}{(\dot kR - k\dot R)}\right]$$

$$\mathcal{I}_2 - \mathcal{E}_2\left[x_0 + \frac{kR\dot x_0}{(\dot kR - k\dot R)}\right] - \mathcal{F}_2\left[y_0 + \frac{kR\dot y_0}{(\dot kR - k\dot R)}\right]$$

$$+\mathcal{C}\left[\frac{1}{2}(x_0^2 + y_0^2 + z_0^2) + \frac{kR}{(\dot kR - k\dot R)}(x_0\dot x_0 + y_0\dot y_0 + z_0\dot z_0) - \frac{2\dot R}{(\dot kR - k\dot R)}\right]$$

$$-\frac{4\dot{\mathcal{C}}R^4}{F^2(\dot kR - k\dot R)^2}\left[\frac{\dot F}{F} - 2\frac{\dot R}{R} + \frac{(\dot kR - k\dot R)^{\cdot}}{(\dot kR - k\dot R)}\right] + \frac{4\ddot{\mathcal{C}}R^4}{F^2(\dot kR - k\dot R)^2}$$

$$= \mathcal{M}\left\{z_0\left[\frac{\dot F}{F} - 2\frac{\dot R}{R} + \frac{(\dot kR - k\dot R)^{\cdot}}{(\dot kR - k\dot R)}\right] + 2\dot z_0 + \frac{\dot F kR\dot z_0}{F(\dot kR - k\dot R)} + \frac{kR\ddot z_0}{(\dot kR - k\dot R)}\right\}$$

$$+\dot{\mathcal{M}}\left[z_0 + \frac{kR\dot z_0}{(\dot kR - k\dot R)}\right]$$

$$\mathcal{G}_2\left[x_0+\frac{kR\dot{x}_0}{(\dot{k}R-k\dot{R})}\right]+\mathcal{H}_2\left[y_0+\frac{kR\dot{y}_0}{(\dot{k}R-k\dot{R})}\right]+\mathcal{I}_2\left[z_0+\frac{kR\dot{z}_0}{(\dot{k}R-k\dot{R})}\right]$$

$$+\mathcal{G}_1\left[\frac{1}{2}(x_0^2+y_0^2+z_0^2)+\frac{kR}{(\dot{k}R-k\dot{R})}(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-\frac{2\dot{R}}{(\dot{k}R-k\dot{R})}\right]$$

$$-\frac{4\dot{\mathcal{G}}_1R^4}{F^2(\dot{k}R-k\dot{R})^2}\left[\frac{\dot{F}}{F}-2\frac{\dot{R}}{R}+\frac{(\dot{k}R-k\dot{R})^{\cdot}}{(\dot{k}R-k\dot{R})}\right]+\frac{4\ddot{\mathcal{G}}_1R^4}{F^2(\dot{k}R-k\dot{R})^2}$$

$$=\mathcal{M}\left\{\frac{1}{2}(x_0^2+y_0^2+z_0^2)\left[\frac{\dot{F}}{F}-2\frac{\dot{R}}{R}+\frac{(\dot{k}R-k\dot{R})^{\cdot}}{(\dot{k}R-k\dot{R})}\right]\right.$$

$$+\frac{\dot{F}kR}{F(\dot{k}R-k\dot{R})}(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)+2(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)$$

$$+\frac{kR}{(\dot{k}R-k\dot{R})}(\dot{x}_0^2+\dot{y}_0^2+\dot{z}_0^2)+\frac{kR}{(\dot{k}R-k\dot{R})}(x_0\ddot{x}_0+y_0\ddot{y}_0+z_0\ddot{z}_0)$$

$$\left.-\frac{2}{(\dot{k}R-k\dot{R})}\left(\frac{\dot{F}\dot{R}}{F}-2\frac{\dot{R}^2}{R}+\ddot{R}\right)\right\}$$

$$+\dot{\mathcal{M}}\left[\frac{1}{2}(x_0^2+y_0^2+z_0^2)+\frac{kR}{(\dot{k}R-k\dot{R})}(x_0\dot{x}_0+y_0\dot{y}_0+z_0\dot{z}_0)-\frac{2\dot{R}}{(\dot{k}R-k\dot{R})}\right]$$

The integrability conditions place restrictions on the metric functions k, F, R, x_0, y_0, z_0 for the existence of a conformal symmetry $\mathbf{X}$.

We can regain the conformal Killing vectors of Robertson–Walker spacetime from the above equations. However, as $\dot{k}R-k\dot{R}\neq 0$, the conformal Killing vectors of the $k=1$ and $k=-1$ spacetimes are only possible as $k\neq 0$ in this case. The above form of the solution is not easy to work with. It is possible to express this solution in a more compact form using vector notation. We introduce the vectors

$$\mathbf{A}=\begin{pmatrix}\mathcal{A}(t)\\ \mathcal{B}(t)\\ \mathcal{C}(t)\end{pmatrix}\qquad\qquad \mathbf{r}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$

$$\mathbf{B} = T\mathbf{r}_0 \qquad \mathbf{r}_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\mathbf{U} = \begin{pmatrix} \mathcal{F}_2(t) \\ -\mathcal{E}_2(t) \\ \mathcal{D}_2(t) \end{pmatrix} \qquad \mathbf{V} = \begin{pmatrix} \mathcal{G}_2(t) \\ \mathcal{H}_2(t) \\ \mathcal{I}_2(t) \end{pmatrix}$$

where we have defined the quantity

$$T = \frac{k}{R}$$

Then the conformal Killing vector $\mathbf{X}$ is given by

$$X^0 = \frac{\dot{\mathbf{A}} \cdot \mathbf{r} - \dot{\mathcal{G}}_1}{\alpha(F/2)\dot{T}} + \mathcal{M} \tag{5.1}$$

$$\begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix} = \tfrac{1}{2}(\mathbf{r}\cdot\mathbf{r})\mathbf{A} - (\mathbf{A}\cdot\mathbf{r})\mathbf{r} + \mathbf{r}\times\mathbf{U} + \mathcal{G}_1\mathbf{r} + \mathbf{V} \tag{5.2}$$

Similarly the conformal factor ϕ is given in the equation

$$\begin{aligned} V\phi = {} & \frac{k}{4}[\mathbf{r}\cdot\mathbf{r}(\mathbf{A}\cdot\mathbf{r}_0 - \mathcal{G}_1) - \mathbf{r}_0\cdot\mathbf{r}_0(\mathbf{A}\cdot\mathbf{r} - \mathcal{G}_1)] - (\mathbf{A}\cdot\mathbf{r} - \mathcal{G}_1) \\ & - \frac{2R}{F^2\dot{T}}(\dot{\mathbf{A}}\cdot\mathbf{r} - \dot{\mathcal{G}}_1) - \frac{k}{2}(\mathbf{r} - \mathbf{r}_0)\cdot\mathbf{V} - \frac{k}{2}\mathbf{r}\cdot(\mathbf{r}_0\times\mathbf{U}) \\ & - \frac{\mathcal{M}R}{2}\left[\tfrac{1}{2}\dot{T}\mathbf{r}\cdot\mathbf{r} - \dot{\mathbf{B}}\cdot\mathbf{r} + \left(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r}_0 + \frac{2}{R}\right)^{\cdot}\right] \end{aligned} \tag{5.3}$$

The eleven integrability conditions reduce to the following vector equations :

$$\dot{\mathbf{U}} = -\frac{1}{\dot{T}}\dot{\mathbf{A}}\times\dot{\mathbf{B}} \tag{5.4}$$

$$\dot{\mathbf{V}} = \frac{1}{\dot{T}}\left[\left(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r}_0 + \frac{2}{R}\right)^{\cdot}\dot{\mathbf{A}} - \dot{\mathcal{G}}_1\dot{\mathbf{B}}\right] \tag{5.5}$$

$$\mathcal{G}_1 = \frac{1}{F\dot{T}}\left[F\mathbf{A}\cdot\dot{\mathbf{B}} - (F\mathcal{M}\dot{T})^{\cdot}\right] \tag{5.6}$$

$$\mathbf{V} = \frac{1}{F\dot{T}}\left[F\mathbf{U}\times\dot{\mathbf{B}} - 4\left(\frac{\dot{\mathbf{A}}}{F\dot{T}}\right)^{\cdot} - F\left(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r}_0 + \frac{2}{R}\right)^{\cdot}\mathbf{A} + (F\mathcal{M}\dot{\mathbf{B}})^{\cdot}\right] \tag{5.7}$$

$$\mathbf{V}\cdot\dot{\mathbf{B}} = \frac{1}{F}\left[F\mathcal{M}\left(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r}_0 + \frac{2}{R}\right)^{\cdot}\right]^{\cdot} - \mathcal{G}_1\left(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r}_0 + \frac{2}{R}\right)^{\cdot} - \frac{4}{F}\left(\frac{\dot{\mathcal{G}}_1}{F\dot{T}}\right)^{\cdot} \tag{5.8}$$

The equations (5.1)–(5.3) are an equivalent representation of the solution for the conformal Killing vector equations (4.1)–(4.10) subject to the conditions (5.4)–(5.8) in the case $\dot{k}R - k\dot{R} \neq 0$. The vectors **A**, **U**, **V** involving functions resulting from the integration process are related to the vectors **r**, $\mathbf{r}_0$, **B** which are defined in terms of the metric functions. The integrability conditions (5.4), (5.5), (5.7) are vector equations. The conditions (5.6) and (5.8) are scalar equations. Having found the above vector form of the solution we are in a position to investigate the physical properties of the Stephani universe with a conformal symmetry. It would also be interesting to consider the effect that this conformal solution would have on the general Einstein field equations (3.16)–(3.17). In particular it would be interesting to consider the foliation of the de Sitter spacetime as a special case of the Stephani solution. With the assumptions

$$x_0 = y_0 = z_0 = 0$$

$$R = \text{constant}$$

$$k = -t$$

Krasiński (1983) has attempted to describe the qualitative features of the Stephani universe. The conformal symmetry would influence the physical properties of the model. This is an area for future research.

Note that the integrability conditions (5.4)–(5.8) may be further simplified if we introduce a new time variable τ where

$$\frac{d}{d\tau} = \frac{1}{F\dot{T}}\frac{d}{dt}$$

It is also convenient to introduce the quantities

$$\mathbf{C} = F\mathbf{B}'$$

$$f = F\left(\tfrac{1}{2}\mathbf{B}\cdot\mathbf{r}_0 + \frac{2}{R}\right)'$$

$$m = F\mathcal{M}\dot{T}$$

where we use the notation that primes denote differentiation with respect to τ whereas dots denote differentiation with respect to t. Then the integrability conditions (5.4)–(5.8) may be put into the form

$$\mathbf{U}' = -\mathbf{A}' \times \mathbf{C}$$

$$\mathbf{V}' = f\mathbf{A}' - \mathcal{G}_1'\mathbf{C}$$

$$\mathcal{G}_1 = \mathbf{A}\cdot\mathbf{C} - m'$$

$$\mathbf{V} = \mathbf{U} \times \mathbf{C} - 4\mathbf{A}'' - f\mathbf{A} + (m\mathbf{C})'$$

$$\mathbf{V} \cdot \mathbf{C} = (mf)' - \mathcal{G}_1 f - 4\mathcal{G}_1''$$

which is of a more fundamental form than the equations given previously. If the metric functions f, m, $\mathbf{C}$ are specified then we can find the functions of integration $\mathbf{A}$, $\mathbf{U}$ and $\mathbf{V}$ for the conformal Killing vector $\mathbf{X}$ from the above for the given cosmological model.

5.3 The Case $k = 0$

The special case $k = 0$ has been excluded in the solution given in §5.2. The conformal Killing vector equations (4.1)–(4.10) may also be integrated in this elementary case to yield the conformal Killing vector $\mathbf{X}$ and the conformal factor ϕ. We do not provide details of the integration process in this case as the calculations are very simple. With the restriction $k = 0$ the general solution to the conformal Killing vector equations (4.1)–(4.10) is given by

$$X^0 = \frac{R^4}{F^2 \dot{R}^2}\left[\tfrac{1}{2}\dot{\mathcal{G}}_1(x^2+y^2+z^2) + \dot{\mathcal{G}}_2 x + \dot{\mathcal{H}}_2 y + \dot{\mathcal{I}}_2 z + \mathcal{M}(t)\right] \tag{5.9}$$

$$\begin{aligned} X^1 &= \tfrac{1}{2}\mathcal{A}(y^2+z^2-x^2) - \mathcal{B}xy + \mathcal{D}_2 y - \mathcal{C}xz + \mathcal{E}_2 z + \mathcal{G}_1(t)x \\ &\quad + \mathcal{G}_2(t) \end{aligned} \tag{5.10}$$

$$X^2 = \tfrac{1}{2}\mathcal{B}(x^2+z^2-y^2) - \mathcal{A}xy - \mathcal{D}_2 x - \mathcal{C}yz + \mathcal{F}_2 z + \mathcal{G}_1(t)y$$

$$+\mathcal{H}_2(t) \tag{5.11}$$

$$\begin{aligned} X^3 &= \tfrac{1}{2}\mathcal{C}(x^2+y^2-z^2) - \mathcal{A}xz - \mathcal{E}_2 x - \mathcal{B}yz - \mathcal{F}_2 y + \mathcal{G}_1(t)z \\ &\quad + \mathcal{I}_2(t) \end{aligned} \tag{5.12}$$

$$\begin{aligned} \phi &= \frac{R^3}{F^2\dot{R}}\left[\tfrac{1}{2}\dot{\mathcal{G}}_1(x^2+y^2+z^2) + \dot{\mathcal{G}}_2 x + \dot{\mathcal{H}}_2 y + \dot{\mathcal{I}}_2 z + \mathcal{M}(t)\right] - \mathcal{A}x - \mathcal{B}y \\ &\quad - \mathcal{C}z + \mathcal{G}_1(t) \end{aligned} \tag{5.13}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}_2$, $\mathcal{E}_2$, $\mathcal{F}_2$ are constants. The solution (5.9)–(5.13) is subject to the integrability conditions

$$\left(\frac{R^2}{F\dot{R}}\dot{\mathcal{G}}_1\right)^{\cdot} = 0 \tag{5.14}$$

$$\left(\frac{R^2}{F\dot{R}}\dot{\mathcal{G}}_2\right)^{\cdot} = -\frac{F\dot{R}}{R^2}\mathcal{A} \tag{5.15}$$

$$\left(\frac{R^2}{F\dot{R}}\dot{\mathcal{H}}_2\right)^{\cdot} = -\frac{F\dot{R}}{R^2}\mathcal{B} \tag{5.16}$$

$$\left(\frac{R^2}{F\dot{R}}\dot{\mathcal{I}}_2\right)^{\cdot} = -\frac{F\dot{R}}{R^2}\mathcal{C} \tag{5.17}$$

$$\left(\frac{R^2}{F\dot{R}}\mathcal{M}(t)\right)^{\cdot} = \frac{F\dot{R}}{R^2}\mathcal{G}_1(t) \tag{5.18}$$

The system (5.9)–(5.13) represents the solution to the conformal Killing vector equation, subject to the integrability conditions (5.14)–(5.18), for the case $k = 0$.

For appropriate choices of the functions of integration in the above solution we can generate the Killing vectors of the $k = 0$ Robertson–Walker spacetimes.

Similarly we can obtain the conformal Killing vectors of the Robertson–Walker spacetime (Maartens and Maharaj 1986) for the $k = 0$ case. Thus the equations of this section together with the results of §5.2 reproduce the conformal Killing vectors for all the Robertson–Walker models for the values $k = 0, 1, -1$. To illustrate the procedure we will generate, as an example, the analogue of a conformal Killing vector in Robertson–Walker spacetime which is normal to the homogeneous hypersurfaces $t =$ constant. This timelike conformal Killing vector given by

$$\mathbf{X} = (R(t), 0, 0, 0)$$

with conformal factor

$$\phi = \dot{R}$$

is listed by Maartens and Maharaj (1986) for the $k = 0$ Robertson–Walker models.

With the conditions $X^1 = X^2 = X^3 = 0$, equations (5.9)-(5.13) imply

$$\begin{aligned} \mathcal{A} &= 0 & \mathcal{B} &= 0 \\ \mathcal{C} &= 0 & \mathcal{D}_2 &= 0 \\ \mathcal{E}_2 &= 0 & \mathcal{F}_2 &= 0 \\ \mathcal{G}_1 &= 0 & \mathcal{G}_2 &= 0 \\ \mathcal{I}_2 &= 0 \end{aligned}$$

Then the conformal Killing vector is given by

$$\mathbf{X} = \left(\frac{R^4}{F^2\dot{R}^2}\mathcal{M}(t), 0, 0, 0\right)$$

with the conformal factor

$$\phi = \frac{R^3}{F^2\dot{R}}\mathcal{M}(t)$$

The integrability conditions (5.14)–(5.17) are identically satisfied. The last condition (5.18) gives the following form for the function $\mathcal{M}(t)$:

$$\left(\frac{R^2}{F\dot{R}}\mathcal{M}(t)\right)^{\cdot} = 0$$

We regain the $k = 0$ Robertson–Walker conformal Killing vector given above by an appropriate choice of the function F. Hence we have established that the Stephani conformal Killing vector is a generalisation of the normal conformal Killing vector $\mathbf{X} = (R(t), 0, 0, 0)$ in the Robertson–Walker spacetimes.

5.4 The Case $\dot{k}R - k\dot{R} = 0, \ k \neq 0$

The remaining special case for the general solution of (4.1)–(4.10) is governed by the condition $\dot{k}R - k\dot{R} = 0$, $k \neq 0$. It is also possible to integrate the conformal Killing vector equations in this last case and generate the general solution. The integration is more complicated than the case $k = 0$ of §5.3. The method of solution is similar to that of chapter 4; consequently we provide only the main points of the integration process in obtaining the solution. The spacelike components are the same as in §4.3. Therefore the conformal Killing vector equations (4.6), (4.7) and (4.9) give

$$\begin{aligned} X^1 &= \tfrac{1}{2}\mathcal{A}(t)(y^2 + z^2 - x^2) - \mathcal{B}(t)xy + \mathcal{D}_2(t)y - \mathcal{C}(t)xz + \mathcal{E}_2(t)z + \mathcal{G}_1(t)x \\ &\quad + \mathcal{G}_2(t) \end{aligned} \tag{5.19}$$

$$\begin{aligned} X^2 &= \tfrac{1}{2}\mathcal{B}(t)(x^2 + z^2 - y^2) - \mathcal{A}(t)xy - \mathcal{D}_2(t)x - \mathcal{C}(t)yz + \mathcal{F}_2(t)z + \mathcal{G}_1(t)y \\ &\quad + \mathcal{H}_2(t) \end{aligned} \tag{5.20}$$

$$X^3 = \tfrac{1}{2}\mathcal{C}(t)(x^2+y^2-z^2) - \mathcal{A}(t)xz - \mathcal{E}_2(t)x - \mathcal{B}(t)yz - \mathcal{F}_2(t)y + \mathcal{G}_1(t)z + \mathcal{I}_2(t) \qquad (5.21)$$

for the spacelike components.

The timelike component and the conformal factor are also obtained in a similar manner as in chapter 4. Since $\dot{k}R - k\dot{R} = 0$, $k \neq 0$ we have that

$$k(t) = \epsilon R(t)$$

where $\epsilon \neq 0$ is an arbitrary constant. Thus α defined by equation (4.52) becomes

$$\alpha = -\frac{1}{2}\epsilon F\left[(x-x_0)\dot{x}_0 + (y-y_0)\dot{y}_0 + (z-z_0)\dot{z}_0\right] - \frac{F\dot{R}}{R^2} \qquad (5.22)$$

when $\dot{k}R - k\dot{R} = 0$. The integration process for this case is simpler than that in chapter 4 since α is linear in x, y, z. As before we can write (4.2)–(4.4) in the form

$$X^1_t = \alpha^2 X^0_x$$

$$X^2_t = \alpha^2 X^0_y$$

$$X^3_t = \alpha^2 X^0_z$$

Substitution of (5.19)–(5.21) in the above yields

$$\alpha\alpha_y X^0_x - \alpha\alpha_x X^0_y = \dot{\mathcal{A}}y - \dot{\mathcal{B}}x + \dot{\mathcal{D}}_2$$

$$\alpha\alpha_z X^0_x - \alpha\alpha_x X^0_z = \dot{\mathcal{A}}z - \dot{\mathcal{C}}x + \dot{\mathcal{E}}_2$$

$$\alpha\alpha_z X^0_y - \alpha\alpha_y X^0_z = \dot{\mathcal{B}}z - \dot{\mathcal{C}}y + \dot{\mathcal{F}}_2$$

Noting that

$$y - y_0(t) = -\frac{1}{\dot{y}_0}[(x - x_0)\dot{x}_0] - \frac{1}{\dot{y}_0}[(z - z_0)\dot{z}_0] - \frac{2\dot{R}}{\epsilon R^2 \dot{y}_0} - \frac{2}{\epsilon F \dot{y}_0}\alpha$$

from (5.22) and applying the method of characteristics we obtain the timelike component:

$$\begin{aligned} X^0 &= \frac{1}{\alpha \alpha_y}(\dot{\mathcal{D}}_2 x - \dot{\mathcal{F}}_2 z - \dot{\mathcal{H}}_2) + \mathcal{M}(t) \\ &= \frac{\dot{\mathcal{D}}_2 x - \dot{\mathcal{F}}_2 z - \dot{\mathcal{H}}_2}{\alpha(-\epsilon F \dot{y}_0/2)} + \mathcal{M}(t) \end{aligned} \tag{5.23}$$

where $\mathcal{M}(t)$ is a function of integration.

With X^0 given by (5.23) the conformal Killing vector equations (4.2)–(4.4) are satisfied. In the process of finding the timelike component we observe that the following conditions are generated

$$\mathcal{A}(t) = \text{constant} \qquad \mathcal{B}(t) = \text{constant}$$

$$\mathcal{C}(t) = \text{constant} \qquad \mathcal{G}_1(t) = \text{constant}$$

$$\dot{\mathcal{E}}_2 \dot{y}_0 - \dot{\mathcal{D}}_2 \dot{z}_0 - \dot{\mathcal{F}}_2 \dot{x}_0 = 0 \tag{5.24}$$

$$-\frac{2\dot{\mathcal{D}}_2 \dot{R}}{\epsilon R^2} + \dot{\mathcal{D}}_2(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) = \dot{\mathcal{H}}_2 \dot{x}_0 - \dot{\mathcal{G}}_2 \dot{y}_0 \tag{5.25}$$

$$-\frac{2\dot{\mathcal{F}}_2 \dot{R}}{\epsilon R^2} + \dot{\mathcal{F}}_2(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) = \dot{\mathcal{I}}_2 \dot{y}_0 - \dot{\mathcal{H}}_2 \dot{z}_0 \tag{5.26}$$

It remains to solve the remaining conformal Killing vector equations (4.1), (4.5), (4.8) and (4.10) and to obtain the conformal factor ϕ. Further integrability conditions will be generated in this process.

Note that (4.5), (4.8) and (4.10) are equivalent as $X^1_x = X^2_y = X^3_z$. We eliminate ϕ by taking the difference of (4.1) and (4.5). On substituting for X^0, from (5.23), in the resulting equation and simplifying we obtain the restriction

$$\alpha_x X^1 + \alpha_y X^2 + \alpha_z X^3 - \alpha X^1_x - \frac{\alpha_{ty}}{{\alpha_y}^2}(\dot{\mathcal{D}}_2 x - \dot{\mathcal{F}}_2 z - \dot{\mathcal{H}}_2) + \frac{1}{\alpha_y}(\ddot{\mathcal{D}}_2 x - \ddot{\mathcal{F}}_2 z - \ddot{\mathcal{H}}_2)$$
$$+\alpha_t \mathcal{M}(t) + \alpha\dot{\mathcal{M}} = 0$$

Substitution of (5.19)–(5.21) in the above equation yields a polynomial equation in x, y, z. This implies the following consistency conditions:

$$\mathcal{A}\dot{x}_0 + \mathcal{B}\dot{y}_0 + \mathcal{C}\dot{z}_0 = 0 \tag{5.27}$$

$$-\frac{2\mathcal{B}\dot{R}}{\epsilon R^2} + \mathcal{B}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \mathcal{D}_2(t)\dot{x}_0 + \mathcal{F}_2(t)\dot{z}_0$$
$$= \frac{1}{F}(\mathcal{M}(t)F\dot{y}_0)^{\cdot} \tag{5.28}$$

$$-\frac{2\mathcal{A}\dot{R}}{\epsilon R^2} + \mathcal{A}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) + \mathcal{D}_2(t)\dot{y}_0 + \mathcal{E}_2(t)\dot{z}_0$$
$$= \frac{1}{F}(\mathcal{M}(t)F\dot{x}_0)^{\cdot} + \frac{4}{\epsilon^2 F}\left(\frac{\dot{\mathcal{D}}_2}{F\dot{y}_0}\right)^{\cdot} \tag{5.29}$$

$$-\frac{2\mathcal{C}\dot{R}}{\epsilon R^2} + \mathcal{C}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) - \mathcal{E}_2(t)\dot{x}_0 - \mathcal{F}_2(t)\dot{y}_0$$
$$= \frac{1}{F}(\mathcal{M}(t)F\dot{z}_0)^{\cdot} - \frac{4}{\epsilon^2 F}\left(\frac{\dot{\mathcal{F}}_2}{F\dot{y}_0}\right)^{\cdot} \tag{5.30}$$

$$-\frac{2\mathcal{G}_1\dot{R}}{\epsilon R^2} + \mathcal{G}_1(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) + \mathcal{G}_2(t)\dot{x}_0 + \mathcal{H}_2(t)\dot{y}_0 + \mathcal{I}_2(t)\dot{z}_0$$

$$= \frac{1}{F}[\mathcal{M}(t)F(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0)]^{\cdot} - \frac{2}{\epsilon F}\left(\frac{\mathcal{M}(t)F\dot{R}}{R^2}\right)^{\cdot} + \frac{4}{\epsilon^2 F}\left(\frac{\dot{\mathcal{H}}_2}{F\dot{y}_0}\right)^{\cdot} \quad (5.31)$$

From equation (4.5) we obtain the conformal factor

$$\begin{aligned}
V\phi \;=\; & (x^2+y^2+z^2)\frac{\epsilon R}{4}(\mathcal{A}x_0 + \mathcal{B}y_0 + \mathcal{C}z_0 - \mathcal{G}_1) \\
& + x\left[-\frac{\mathcal{A}\epsilon R}{4}({x_0}^2 + {y_0}^2 + {z_0}^2) - \mathcal{A} + \frac{2\dot{\mathcal{D}}_2 R}{\epsilon F^2 \dot{y}_0} - \frac{\epsilon R}{2}(\mathcal{D}_2(t)y_0 + \mathcal{E}_2(t)z_0\right. \\
& \left. + \mathcal{G}_2(t) - \mathcal{M}(t)\dot{x}_0)\right] \\
& + y\left[-\frac{\mathcal{B}\epsilon R}{4}({x_0}^2 + {y_0}^2 + {z_0}^2) - \mathcal{B} - \frac{\epsilon R}{2}(-\mathcal{D}_2(t)x_0 + \mathcal{F}_2(t)z_0\right. \\
& \left. + \mathcal{H}_2(t) - \mathcal{M}(t)\dot{y}_0)\right] \\
& + z\left[-\frac{\mathcal{C}\epsilon R}{4}({x_0}^2 + {y_0}^2 + {z_0}^2) - \mathcal{C} - \frac{2\dot{\mathcal{F}}_2 R}{\epsilon F^2 \dot{y}_0} - \frac{\epsilon R}{2}(-\mathcal{E}_2(t)x_0 - \mathcal{F}_2(t)y_0\right. \\
& \left. + \mathcal{I}_2(t) - \mathcal{M}(t)\dot{z}_0)\right] \\
& + \frac{\mathcal{G}_1\epsilon R}{4}({x_0}^2 + {y_0}^2 + {z_0}^2) + \mathcal{G}_1 + \frac{\epsilon R}{2}(\mathcal{G}_2(t)x_0 + \mathcal{H}_2(t)y_0 + \mathcal{I}_2(t)z_0) \\
& - \frac{2\dot{\mathcal{H}}_2 R}{\epsilon F^2 \dot{y}_0} - \frac{\mathcal{M}(t)\epsilon R}{2}(x_0\dot{x}_0 + y_0\dot{y}_0 + z_0\dot{z}_0) + \frac{\mathcal{M}(t)\dot{R}}{R} \qquad (5.32)
\end{aligned}$$

where we have used (5.19)-(5.21) and (5.23).

We have thus solved the conformal Killing vector equations (4.1)-(4.10) for the case $\dot{k}R - k\dot{R} = 0$, $k \neq 0$ and obtained the conformal Killing vector $\mathbf{X}$ and the conformal factor ϕ. The timelike component X^0 is given by (5.23) and the spacelike components X^1, X^2, X^3 are defined by (5.19)-(5.21). Equation (5.32) gives the conformal factor ϕ. The solution obtained is subject to the eight integrability conditions (5.24)-(5.26), (5.27)-(5.31). Note that in this case $k = \epsilon R(t)$ with $\epsilon \neq 0$. Thus the quantity k is strictly a function of time and cannot be a constant. Therefore the conformal Killing vectors corresponding to $\dot{k}R - k\dot{R} = 0$, $k \neq 0$ do not reduce to the conformal Killing vectors of Robertson–Walker spacetimes as a special case. Thus we have established that the Stephani universes contain conformal symmetries that do not have an analogue in Robertson–Walker spacetimes.

The results of §5.2, §5.3, §5.4 comprise the general solution of the conformal Killing vector equation in Stephani universes. By generating the general conformal Killing vector $\mathbf{X}$ in Stephani universes we have created future avenues of research. The physical properties of the Stephani models with a conformal symmetry should be investigated. The possible solutions to the Einstein field equations (3.16)–(3.17) are restricted by the conformal Killing vector $\mathbf{X}$. In particular it would be interesting to study the special cases of the scale factors of cosmological significance that arise in the Robertson–Walker spacetimes: de Sitter, Friedmann etc.

6 Conclusion

Our objective in this dissertation was to analyse conformal symmetries in Stephani universes which are generalisations of the Robertson–Walker models. We have obtained the general solution to the conformal Killing vector equation in the Stephani universe for the three cases

(i) $\dot{k}R - k\dot{R} \neq 0$,

(ii) $k = 0$ and

(iii) $\dot{k}R - k\dot{R} = 0,\ k \neq 0$

subject to integrability conditions. The integrability conditions are essentially relationships between the functions arising from the integration process and the metric functions. In the process of finding the solution we encountered some very long calculations which were readily amenable to treatment with the symbolic manipulation capabilities of Mathematica (Ver 2.0) (Wolfram 1991). We also verified that the solution obtained was correct using this package. It is significant to note that although the Stephani universe may not admit Killing vector symmetries, it does admit conformal Killing symmetries.

In chapter 2 we introduced only those aspects of differential geometry relevant to this dissertation. In particular we studied those aspects which are necessary for the development of conformal symmetries of this dissertation. We introduced the concept of a differentiable manifold on which vector and tensor fields are defined. The covariant derivative was defined and curvature was introduced on the manifold via the Riemann tensor. Then we introduced the Einstein field equations. The Lie derivative is a natural geometric structure on the manifold. Lie algebras were defined and we briefly discussed the relationship between a Lie group and a Lie algebra. We imposed the condition of a conformal symmetry on the manifold by defining the conformal Killing vector.

The Stephani universe was introduced in chapter 3. The Stephani universe is a generalisation of the Robertson–Walker models. The kinematics and dynamics of the Stephani universe were considered. The Einstein field equations were fully derived as this model is not well known. We found that the field equations obtained agreed with those given by Kramer *et al* (1980), Krasiński (1983) and Stephani (1967). The requirements for the Stephani line element to reduce to the Robertson–Walker model were also discussed in this chapter. Our equations were obtained directly from the Einstein field equations. It is also possible to obtain the field equations using an embedding procedure (Stephani 1967) or the concept of infinitesimal null isotropy (Koch–Sen 1985).

In chapter 4 we solved the conformal Killing vector equation. The solution given is subject to the restriction $\dot{k}R - k\dot{R} \neq 0$. As this is the most complicated case we provided all the details of the integration procedure. We obtained explicit forms for the timelike component and the three spacelike components of the conformal Killing vector as well as the conformal factor. The solution obtained is subject to eleven integrability conditions that relate the functions of integration to the metric functions. Since parts of the calculation were difficult to perform by hand we used Mathematica (Ver 2.0) to assist with calculations. Mathematica (Ver 2.0) was also used to verify that the solution found satisfies the conformal Killing vector equations.

In chapter 5 we gave a complete analysis of conformal symmetries in Stephani universes. The results obtained in chapter 4 for the case $\dot{k}R - k\dot{R} \neq 0$ were expressed in vector notation. This greatly simplified the form of the solution. In this chapter we also considered the two cases omitted in chapter 4. For the case $k = 0$ we merely stated the results as the integration process is very simple. As an example we generated the conformal Killing vector in the Stephani universe which is the analogue of a conformal Killing vector in the $k = 0$ Robertson–Walker spacetime normal to the homogeneous hypersurfaces $t =$ constant. The conformal Killing vector equation was also solved for the remaining case $\dot{k}R - k\dot{R} = 0$, $k \neq 0$. As the solution process was similar to that used in chapter 4 we only gave an outline of the integration procedure. The solution found in this case was subject to eight integrability conditions that restrict the metric functions. We again utilised Mathematica (Ver 2.0) to verify that the solution obtained does indeed satisfy the conformal Killing vector equations. It is significant to note that the cases $\dot{k}R - k\dot{R} \neq 0$ and $k = 0$ contain the Robertson–Walker conformal Killing vectors. However, the case $\dot{k}R - k\dot{R} = 0$, $k \neq 0$ yields vectors which cannot be reduced to the conformal Killing vectors of Robertson–Walker spacetimes.

Having obtained the conformal Killing vectors in Stephani universes the physical properties of these models should be studied further. This would involve an analysis of the Einstein field equations for the Stephani universes. Also it would be interesting to consider more general symmetries in the Stephani universe which contain conformal Killing vectors, e.g. the curvature inheritance of Duggal (1992). The existence of other symmetries would improve our understanding of the geometry of the Stephani universes.

The results obtained in this dissertation are original. Apparently this dissertation represents the first attempt to solve the conformal Killing vector equation in the Stephani universe. We hope that we have demonstrated that the study of symmetries in the Stephani universe is a fertile area of research and warrants further investigation.

7 References

1. Bishop R L and Goldberg S I 1968 *Tensor Analysis on Manifolds* (New York: McMillan)

2. Burke W L 1985 *Applied Differential Geometry* (Cambridge: Cambridge University Press)

3. Choquet-Bruhat Y, Dewitt-Morrette C and Dillard-Bleick M 1977 *Analysis, Manifolds and Physics* (Amsterdam: North-Holland)

4. Coley A A 1991 *Class. Quantum Grav.* **8** 955

5. Coley A A and Tupper B O J 1989 *J. Math. Phys.* **30** 2616

6. Coley A A and Tupper B O J 1990a *Gen. Rel. Grav.* **22** 241

7. Coley A A and Tupper B O J 1990b *Class. Quantum Grav.* **7** 1961

8. Coley A A and Tupper B O J 1990c *Class. Quantum Grav.* **7** 2195

9. Coley A A and Tupper B O J 1990d *J. Math. Phys.* **31** 1541

10. Dubrovin B A, Fomenko A T and Novikov S P 1984 *Modern Geometry–Methods and Applications* Vol I (New York: Springer Verlag)

11. Dubrovin B A, Fomenko A T and Novikov S P 1985 *Modern Geometry–Methods and Applications* Vol II (New York: Springer Verlag)

12. Duggal K L 1992 *J. Math. Phys.* **33** 2989

13. Dyer C C, McVittie G C and Oattes L M 1987 *Gen. Rel. Grav.* **19** 887

14. Gradshteyn I S and Ryzhik I M 1980 *Tables of Integrals, Series and Products* (New York: Academic Press)

15. Hawking S W and Ellis G F R 1973 *The Large Scale Structure of Spacetime* (Cambridge: Cambridge University Press)

16. Herrera L, Jimenez J, Leal L, Ponce de Leon J, Esculpi M and Galma V 1984 *J. Math. Phys.* **25** 3274

17. Herrera L and Ponce de Leon J 1985a *J. Math. Phys.* **26** 778

18. Herrera L and Ponce de Leon J 1985b *J. Math. Phys.* **26** 2018

19. Herrera L and Ponce de Leon J 1985c *J. Math. Phys.* **26** 2847

20. Israel W 1972 *Studies in Relativity* O'Raifeartaigh L ed. (Oxford: Clarendon)

21. Jiang S 1992 *J. Math. Phys.* **33** 3503

22. Karcher H 1982 *Arch. Math.* **38** 58

23. Koch-Sen L 1985 *J. Math. Phys.* **26** 407

24. Kramer D, Stephani H, MacCallum M A H and Herlt E 1980 *Exact Solutions of Einstein's Field Equations* (Cambridge: Cambridge University Press)

25. Krasiński A 1981 *Gen. Rel. Grav.* **13** 1021

26. Krasiński A 1983 *Gen. Rel. Grav.* **15** 673

27. Krasiński A 1989 *J. Math. Phys.* **30** 433

28. Maartens R and Maharaj M S 1990 *J. Math. Phys.* **31** 151

29. Maartens R and Maharaj S D 1986 *Class. Quantum Grav.* **3** 1005

30. Maartens R and Maharaj S D 1991 *Class. Quantum Grav.* **8** 503

31. Maartens R, Mason D P and Tsamparlis M 1986 *J. Math. Phys.* **27** 2987

32. Maharaj S D, Leach P G L and Maartens R 1991 *Gen. Rel. Grav.* **23** 261

33. Mason D P and Maartens R 1987 *J. Math. Phys.* **28** 2182

34. Mason D P and Tsamparlis M 1985 *J. Math. Phys.* **26** 2881

35. Misner C W, Thorne K S and Wheeler J A 1973 *Gravitation* (San Francisco: Freeman)

36. Moodley M 1991 *Conformal Symmetries: Solutions in two classes of cosmological models* (M.Sc. dissertation:University of Natal, Durban)

37. Robertson H P 1929 *Proc. Natl. Acad. Sci.* **15** 822

38. Saridakis E and Tsamparlis M 1991 *J. Math. Phys.* **32** 1541

39. Sattinger D H and Weaver O L 1986 *Lie Groups and Algebras with Applications to Physics, Geometry and Mechanics* (New York: Springer Verlag)

40. Schutz B 1980 *Geometrical Methods of Mathematical Physics* (Cambridge: Cambridge University Press)

41. Stephani H 1967 *Commun. Math. Phys.* **4** 137

42. Stephani H 1990 *General Relativity: An Introduction to the Theory of the Gravitional Field* (Cambridge: Cambridge University Press)

43. Straumann N 1984 *General Relativity and Relativistic Astrophysics* (Berlin: Springer Verlag)

44. Wald R 1984 *General Relativity* (Chicago: University of Chicago Press)

45. Walker A G 1935 *Q. J. Math. Oxford* Ser.**6** 81

46. Wolfram S 1991 *Mathematica* (Redwood City: Addison-Wesley)