DEGREE THEORY IN NONLINEAR FUNCTIONAL ANALYSIS

by

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FOR SURESH

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ABSTRACT

The objective of this dissertation is to expand on the proofs and concepts of Degree Theory, dealt with in chapters 1 and 2 of Deimling [28], to make it more readable and accessible to anyone who is interested in the field.

Chapter 1 is an introduction and contains the basic requirements for the subsequent chapters.

The remaining chapters aim at defining a $\mathbb{Z}$–valued map D (the degree) on the set

$$\mathscr{M} = \{(F, \Omega, y) \;/\; \Omega \subseteq X \text{ open}, F : \bar{\Omega} \longrightarrow X, y \notin F(\partial\Omega)\}$$

(each time, the elements of $\mathscr{M}$ satisfying extra conditions)

that satisfies :

(D1) $D(I, \Omega, y) = 1$ if $y \in \Omega$.

(D2) $D(F, \Omega, y) = D(F, \Omega_1, y) + D(F, \Omega_2, y)$ if Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin F(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$.

(D3) $D(I - H(t, .), \Omega, y(t))$ is independent of t if $H : J \times \bar{\Omega} \longrightarrow X$ and $y : J \longrightarrow X$.

An important property that follows from these three properties is

(D4) $F^{-1}(y) \neq \emptyset$ if $D(F, \Omega, y) \neq 0$.

This property ensures that equations of the form $Fx = y$ have solutions if $D(F, \Omega, y) \neq 0$.

Another property that features in these chapters is the Borsuk property which gives us conditions under which the degree is odd and hence nonzero.

TABLE OF NOTATIONS

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{C}$	set of complex numbers
$\mathbf{K}$	either $\mathbb{R}$ or $\mathbb{C}$
nls.	normed linear space
t.v.s.	topological vector space
MNC	measure of noncompactness
α	Kuratowski–MNC
β	Hausdorff–MNC
γ	either α or β
$\mathscr{B}$	collection of all bounded sets of a space
$\bar{A}$	closure of A
A^0	interior of A
∂A	boundary of A
$\mathscr{K}(\Omega, Y)$	class of compact maps
$\mathscr{F}(\Omega, Y)$	class of finite dimensional maps
conv A	convex hull of A
$\lvert . \rvert$	norm

$\rho(x, A) = \inf \{ \lvert x - a \rvert \ / \ a \in A \}$

$\lvert f \rvert_0 = \sup \{ \lvert f(x) \rvert \ / \ x \in \Omega \}$ if f is a map on Ω

$J_f(x) = \det f'(x)$

[ ]	numbers in square brackets refer to bibliography
♠	end of a theorem, lemma, corollary, remark or example

CHAPTER 1

1.1 INTRODUCTION

Nonlinear functional analysis developed mainly because many of the problems in nature are represented by nonlinear models. In practise, one would like to know whether a nonlinear equation of the form $Fx = y$ has a solution. If a solution does exist, then one would want to know if it is unique, and have some way of locating the solution. A field in nonlinear functional analysis which addresses the question of the existence of solutions to such equations is Degree Theory.

To motivate the definition and properties of the degree that uniquely define it, we consider first, the concept of the winding number of plane curves, which indicates how many times a closed curve winds around a fixed point not on the curve.

Let $\Gamma \subseteq \mathbb{C}$ be a continuously piecewise differentiable closed curve with $a \in \mathbb{C} \setminus \Gamma$. If $z(t)$, $t \in [0, 1]$ is a representation of Γ, (since Γ is closed, $z(0) = z(1)$), then $\omega(\Gamma, a) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z-a}$ is an integer and is called the winding number (index) of the point a with respect to the curve Γ. (see Alfors [25]) . It is possible to define $\omega(\Gamma, a)$ for any continuous closed curve Γ that does not pass through a (need not be piecewise differentiable) . We divide Γ into subarcs $\Gamma_1, \ldots, \Gamma_n$, each contained in a ball that does not contain a . Let σ_k be the directed line segment from the initial point to the terminal point of Γ_k and set $\sigma = \sigma_1 + \sigma_2 + \ldots + \sigma_n$. Then σ is piecewise differentiable and $\omega(\sigma, a)$ is defined. We define $\omega(\Gamma, a)$ by $\omega(\sigma, a)$. It can be shown that this definition is independent of the subdivision. More precisely, if $z_1(t)$ and $z_2(t)$ are continuous piecewise differentiable representations of Γ_1 and Γ_2 respectively, such that

$\max\{|z_j(t) - z(t)| \;/\; t \in [0, 1]\} < \min\{|a - z(t)| \;/\; t \in [0, 1]\}$ for $j = 1, 2$, then $\omega(\Gamma_1, a) = \omega(\Gamma_2, a)$. Thus we have defined

$$\omega : \{(\Gamma, a) \;/\; \Gamma \text{ is closed continuous, } a \in \mathbb{C} \setminus \Gamma\} \longrightarrow \mathbb{Z}$$

and this satisfies the following properties (which are not hard to see) :

(a) ω is constant on some neighbourhood of (Γ, a).

(b) $\omega(\Gamma, .)$ is constant on every connected component of $\mathbb{C} \setminus \Gamma$; in particular, it is equal to zero on the unbounded component.

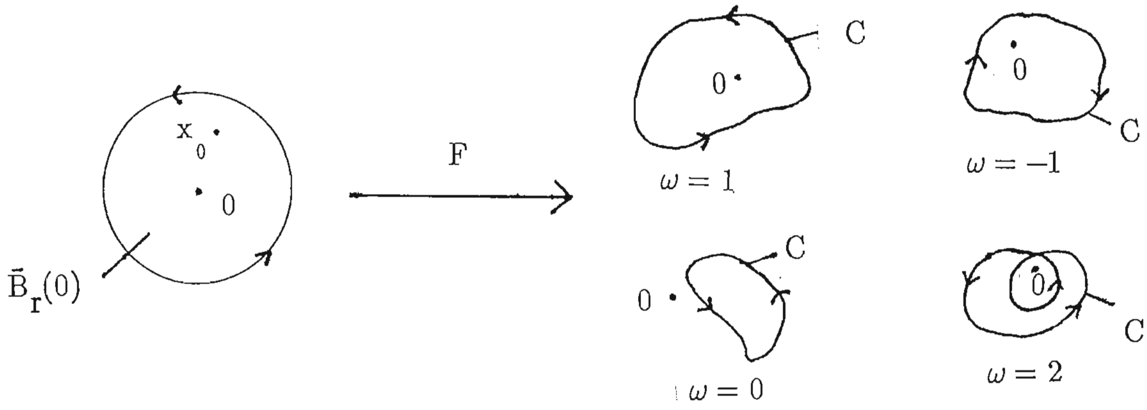
(c) If Γ_0 can be continuously deformed to Γ_1 without passing through a, then $\omega(\Gamma_0, a) = \omega(\Gamma_1, a)$. More precisely, let $z_0(t)$ and $z_1(t)$ be representations for Γ_0 and Γ_1 respectively, such that there exists a continuous $h : [0, 1] \times [0, 1] \longrightarrow \mathbb{C} \setminus \{a\}$ satisfying $h(0, t) = z_0(t)$ and $h(1, t) = z_1(t)$ in $[0, 1]$ and $h(s, 0) = h(s, 1)$ for all $s \in [0, 1]$; then $\omega(\Gamma_s, a)$ is constant for all $s \in [0, 1]$ where Γ_s is the closed curve represented by $h(s, .)$.

(d) If $-\Gamma$ denotes the curve Γ with its orientation reversed, then $\omega(-\Gamma, a) = -\omega(\Gamma, a)$.

Property (c) is most important since it allows us to calculate the winding number of a complicated curve by finding the winding number of a possibly simpler curve.

To get a more geometric feel for this, consider the following : –

Let $\bar{B}_r(0)$ be the closed ball of radius $r > 0$ centred at the origin in $\mathbb{R}$ and consider a continuous $F : \bar{B}_r(0) \longrightarrow \mathbb{R}$. As x travels once around the boundary of the ball, in a positive direction, the image points Fx travel along an oriented curve C. We assume that $0 \notin C$. Let ω_+ and ω_- denote the number of windings about the origin in a positive and negative direction, respectively, and define $\omega = \omega_+ - \omega_-$.



$\omega =$ winding number of F

It is intuitively clear that this definition leads to the following important results.

(i) If $\omega \neq 0$, then there exists $x_0 \in \bar{B}_r(0)$ such that $Fx_0 = 0$. (Kronecker's existence principle)

(ii) If F is changed continuously in such a way that none of the corresponding curves C pass through the origin, then ω remains unchanged. (Homotopy invariance)

The degree is defined so that it satisfies these nice properties.

There has been much development in Degree Theorey since the work of Brouwer in his paper published in 1912 [37]. Much effort has been made to establish the properties of the degree using analytic methods instead of algebraic topological methods. In 1934, Leray and Schauder [36] extended the degree for finite dimensional operators (of Brouwer) to infinite dimensional operators (compact perturbations of the identity). A lot of work has been done by Nussbaum and Schöneberg in extending the degree to other kinds of operators.

1.2 PRELIMINARIES

In the sequel $\mathbf{K}$ denotes either $\mathbb{C}$ or $\mathbb{R}$.

1.2.1 Definition

Let X be a linear space over K. A *norm* on X is a function $|.| : X \longrightarrow \mathbb{R}$ such that for x, y $\in$ X and k $\in$ **K**,

(i) $|x| \geq 0$ and $|x| = 0$ iff $x = 0$

(ii) $|x + y| \leq |x| + |y|$

(iii) $|k\,x| = |k|\;|x|$.

A *normed linear space* (nls.) X is a linear space X together with a norm $|.|$ on it. A *Banach* space is a nls. in which every Cauchy sequence is convergent.

In the sequel X will denote a Banach space unless otherwise stated.

If X is a nls., $x_0 \in X$ and $r > 0$, then $B_r(x_0) = \{x \in X \,/\, |x - x_0| < r\}$ is the ball of centre x_0 with radius r.

If $\|.\|$ is another norm on the nls. X, then it is useful to note that the two norms $|.|$ and $\|.\|$ are equivalent if they generate the same topology, i.e. if every $|.|$–ball contains a $\|.\|$–ball, and every $\|.\|$–ball contains a $|.|$–ball.

An equivalent condition is : there exist $\alpha, \beta > 0$ such that $\alpha|x| \leq \|x\| \leq \beta|x|$ for all $x \in X$.

If $F : X \longrightarrow Y$ is a map between two nlss. X and Y, then we write Fx instead of F(x) and we speak of the *operator* F.

Every **K**–valued operator will be called a *functional.*

The set of all bounded linear operators from a nls X to a nls Y will be denoted by BL(X, Y).

BL(X, Y) is a Banach space iff Y is a Banach space.

BL(X, X) will simply be denoted by BL(X) and BL(X, **K**) denoted by X^*, the Banach space of all continuous linear functionals $x^* : X \longrightarrow \mathbf{K}$.

The simplest element of BL(X) is I, the identity on X, i.e. $Ix = x$ for all $x \in X$.

If $\Omega \subseteq X$, then $\bar{\Omega}$ and $\partial\Omega$ will denote the closure and boundary of Ω, respectively.

If A, B $\subseteq$ X, then $A \backslash B = \{x \in A \ / \ x \notin B\}$.

We let $\mathbb{R}^n = \{ x = (x_1, \ldots, x_n) \ / \ x_i \in \mathbb{R} \text{ for } i = 1, 2, \ldots, n \}$ with $|x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$.

The identity of $\mathbb{R}^n$ will be denoted by id, i.e. $id(x) = x$ for all $x \in \mathbb{R}^n$.

Linear maps in $\mathbb{R}^n$ will be identified with their matrices $A = (a_{ij})$.

If $\delta_{ij} = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases}$ (L.Kronecker's symbol), then $id = (\delta_{ij})$.

$C_X(B)$ will denote the collection of all continuous functions from B to X and we write C(B) if B $\subseteq$ X.

We will let J denote the interval [0, 1] in $\mathbb{R}$.

1.2.2 Definition

D $\subseteq$ X is said to be *convex* if $\lambda x + (1-\lambda)y \in D$ for all $x, y \in D$ and all $\lambda \in [0, 1]$.

The *convex hull* of A $\subseteq$ X is the intersection of all convex sets that contain A, and is denoted by conv A.

It is easy to verify that

$$\text{conv } A = \{ \sum_{i=1}^{n} \lambda_i x^i \ / \ x^i \in A, \lambda_i \in [0, 1] \text{ and } \sum_{i=1}^{n} \lambda_i = 1, n \in \mathbb{N} \}.$$

1.2.3 Definitions

Let X be a topological space.

(i) A subset M of X is said to be *compact* if every open covering of M can be reduced to a finite open covering of M, i.e. if $M \subseteq \bigcup_{\lambda \in \Lambda} A_\lambda$, $A_\lambda \subseteq X$

is open, then there exist $\lambda_1, \ldots, \lambda_n$, say, such that $M \subseteq \bigcup_{i=1}^{n} A_{\lambda_i}$.

(ii) A set M is *separable* if it contains a countably dense set. (recall: $M_1 \subseteq M_2$ is dense in M_2 if $\bar{M}_1 = M_2$)

(iii) A subset M of X is *relatively compact* if $\bar{M}$ is compact. A subset M of X is precompact (totally bounded) if to every $\epsilon > 0$, there exist finitely many balls $B_\epsilon(x_i) \subseteq X$, $i = 1, \ldots, n$, such that $M \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$.

The following equivalent conditions for compactness are often useful and convenient, and the proofs may be found in any text on general topology, Willard [30] for example.

1.2.4. Theorem

Let X be a topological space and $M \subseteq X$. Then the following are equivalent :–

(i) M is compact.

(ii) $\bigcap_{\lambda \in \Lambda} A_\lambda \neq \emptyset$ whenever $(A_\lambda)_{\lambda \in \Lambda}$ is a family of closed subsets of M such that the intersection of any finite subfamily is nonempty (the finite intersection property).

(iii) Every net in M has a convergent subnet with limit in M.

The next two results relates the concepts defined above. Again, no proofs are included.

1.2.5 Theorem

Every compact set is separable.

1.2.6 Theorem

In a complete space, relative compactness is equivalent to precompactness.

We state the following useful theorem.

1.2.7 Theorem

Let $(X, |.|)$ be a nls. with $\dim X = \infty$. Then there exists a sequence $(x_n) \subseteq \partial B_1(0)$ such that $|x_m - x_n| \geq 1$ for $n \neq m$.

1.2.8 Theorem

The closed unit ball in a nls. X is compact iff $\dim X < \infty$.

From the previous results we obtain the following :–

1.2.9 Theorem

If $(X, |.|)$ is a nls., then $\partial B_1(0)$ is compact iff $\dim X < \infty$.

Since we have so many sets that are not relatively compact in infinite dimensional spaces, we introduce the concept of a *measure of noncompactness.*

Let $\mathscr{B}$ denote the collection of all bounded subsets of X. (Recall : B is a bounded subset of X if B is contained in some ball in X).

If $B \in \mathscr{B}$ is not relatively compact (precompact), then there exists an $\epsilon > 0$ such that B cannot be covered by finitely many ϵ–balls.

1.2.10 Definition

If B is a bounded subset of a nls. X, then $\operatorname{diam} B = \sup \{|x - y| \ / \ x, y \in B\}$ is called the *diameter* of B.

It seems natural to introduce the following definition which is due to Kuratowski.

1.2.11 Definition

Let X be a Banach space and $\mathscr{B}$ its bounded sets. Then

$\alpha : \mathscr{B} \longrightarrow \mathbb{R}^+$ defined by

$$\alpha(B) = \inf \{ d > 0 \; / \; B \subseteq \bigcup_{i=1}^{n} B_i, \; n \in \mathbb{N}, \; \text{diam } B_i \leq d \}$$

is called the *Kuratowski–measure of noncompactness* (α–MNC) and

$\beta : \mathscr{B} \longrightarrow \mathbb{R}^+$ defined by

$$\beta(B) = \inf \{ r > 0 \; / \; B \subseteq \bigcup_{i=1}^{n} B_r(x_i), \; n \in \mathbb{N} \}$$

is called the *Hausdorff (ball)–measure of noncompactness.*

We can regard $\alpha(B)$ and $\beta(B)$ as the extents to which B is not compact.

Sadovskii [9] also introduced a measure of noncompactness, but his was more general. It seems that Sadovskii was not aware of the work of Kuratowskii and Darbo (who proved some of the properties of the α–MNC).

Although the above definitions, which were introduced in 1930, seem quite natural, they were only taken up, 37 years later, in 1967.

Darbo has shown that if we work in a Banach space, we obtain the following useful results.

1.2.12 Theorem

Let X be a Banach space, $\mathscr{B}$ its bounded sets and $\gamma : \mathscr{B} \longrightarrow \mathbb{R}^+$ be either α or β. Then

(a) $\gamma(B) = 0$ iff $\bar{B}$ is compact for all $B \in \mathscr{B}$.

(b) γ is a seminorm, i.e. $\gamma(\lambda B) = |\lambda| \; \gamma(B)$ and $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$.

(c) $B_1 \subseteq B$ implies $\gamma(B_1) \leq \gamma(B_2)$ and $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$.

(d) $\gamma(\text{conv } B) = \gamma(B)$.

(e) $\gamma(\bar{B}) = \gamma(B)$.

Proof:

(a) "$\Longrightarrow$" : Suppose $\gamma(B)=0$. Take any $\epsilon > 0$. Then by definition, $B \subseteq \bigcup_{i=1}^{m} M_i$ where $\text{diam } M_i \leq \epsilon$ if $\gamma = \alpha$ and $M_i = B_\epsilon(x_i)$ if $\gamma = \beta$. If M_i are not ϵ–balls, then for each i, choose $x_i \in M_i$. Then we have $M_i \subseteq B_\epsilon(x_i)$. Thus B is precompact and relatively compact. So $\bar{B}$ is compact.

"$\Longleftarrow$" : Suppose $\bar{B}$ is compact. Then B is relatively compact and hence precompact. Let $\epsilon > 0$. Then B admits a finite cover by ϵ–balls. (These have radius ϵ and diameter 2ϵ.) Thus $\alpha(B) \leq 2\epsilon$ and $\beta(B) \leq \epsilon$. Since ϵ was arbitrary, we must have $\gamma(B) = 0$.

(b) Let $d > 0$ and let $B \subseteq \bigcup_{i=1}^{m} M_i \leq d$ with $\text{diam } M_i \leq d$ if $\gamma = \alpha$ and $M_i = B_\epsilon(x_i)$ if $\gamma = \beta$. Then $\lambda B \subseteq \bigcup_{i=1}^{m} \lambda M_i$ with $\text{diam } \lambda M_i \leq |\lambda| d$ if $\gamma = \alpha$ and $\lambda M_i = B_{|\lambda|d}(x_i)$ if $\gamma = \beta$. Hence $\gamma(\lambda B) \leq |\lambda| \gamma(B)$.

Now let $d > 0$ and let $\lambda B \subseteq \bigcup_{i=1}^{m} M_i$ with $\text{diam } M_i \leq d$ if $\gamma = \alpha$ and $M_i = B_d(x_i)$ if $\gamma = \beta$. Then $B \subseteq \bigcup_{i=1}^{m} \frac{1}{\lambda} M_i$ if $\lambda \neq 0$ and $\text{diam } \frac{1}{\lambda} M_i \leq \frac{1}{|\lambda|} d$ if $\gamma = \alpha$ or $\frac{1}{\lambda} M_i = B_{\frac{1}{\lambda} d}(x_i)$. Thus $|\lambda| \gamma(B) \leq \gamma(\lambda B)$ for $\lambda \neq 0$, and this is trivial for $\lambda = 0$. So we have $\gamma(\lambda B) = |\lambda| (B)$.

Now let $d_1, d_2 > 0$ and let $B_1 \subseteq \bigcup_{i=1}^{m} M_i$ and $B_2 \subseteq \bigcup_{j=1}^{n} N_j$ with $\text{diam } M_i \leq d_1$ and $\text{diam } N_j \leq d_2$ if $\gamma = \alpha$ or $M_i = B_{d_1}(x_i)$ and $N_j = B_{d_2}(y_j)$ if $\gamma = \beta$. Then $B_1 + B_2 \subseteq \bigcup_{i,j} (M_i + N_j)$ with $\text{diam } (M_i + N_j) \leq d_1 + d_2$ if $\gamma = \beta$ or $M_i + N_j \subseteq B_{d_1 + d_2}(x_i + y_j)$ if $\gamma = \beta$. Then $\gamma(B_1 + B_2) \leq \gamma(B_1) + \gamma(B_2)$. Hence γ is a seminorm.

(c) Let $d > 0$ and let $B_1 \subseteq \bigcup_{i=1}^{m} M_i$ with $\text{diam } M_i \leq d$ if $\gamma = \alpha$ or $M_i = B_d(x_i)$ if $\gamma = \beta$. Then $B_1 \subseteq B_2 \subseteq \bigcup_{i=1}^{m} M_i$. Thus by definition, $\gamma(B_1) \leq \gamma(B_2)$.

Now assume, without loss of generality, that

$\max\{\gamma(B_1), \gamma(B_2)\} = \gamma(B_2)$. Since $B_2 \subseteq B_1 \cup B_2$, we have

$\gamma(B_2) \le \gamma(B_1 \cup B_2)$. Let $d > 0$ with $B_2 \subseteq \bigcup_{i=1}^{m} M_i$ with diam $M_i \le d$ if $\gamma = \alpha$ or $M_i = B_d(x_i)$ if $\gamma = \beta$. Since $\gamma(B_1) \le \gamma(B_2)$, we can find N_j such that $B_1 \subseteq \bigcup_{j=1}^{n} N_j$ with diam $N_j \le d$ if $\gamma = \alpha$ or $N_j = B_d(x_j)$ if $\gamma = \beta$. So $B_1 \cup B_2 \subseteq (\bigcup_{j=1}^{n} N_j) \cup (\bigcup_{i=1}^{m} M_i)$ and hence by definition, $\gamma(B_1 \cup B_2) \le \gamma(B_2)$. Thus $\gamma(B_1 \cup B_2) = \max\{\gamma(B_1), \gamma(B_2)\}$.

(d) Since $B \subseteq \operatorname{conv} B$, we have by (c) that $\gamma(B) \le \gamma(\operatorname{conv} B)$.

Now let $d > 0$ with $B \subseteq \bigcup_{i=1}^{m} M_i$ with diam $M_i \le d$ if $\gamma = \alpha$ or $M_i = B_d(x_i)$ if $\gamma = \beta$. Since diam (conv M_i) $\le d$ and $B_d(x_i)$ is convex, we may assume that the M_i are convex. Now

$$\begin{aligned}
\operatorname{conv} B &\subseteq \operatorname{conv}\,[M_1 \cup \operatorname{conv}(\bigcup_{i=2}^{m} M)] \\
&\subseteq \operatorname{conv}\,[M_1 \cup \operatorname{conv}\,[M_2 \cup \operatorname{conv}(\bigcup_{i=3}^{m} M_i)]] \\
&\subseteq \ldots
\end{aligned}$$

So if we can show that $\gamma(\operatorname{conv}(C_1 \cup C_2)) \le \max\{\gamma(C_1), \gamma(C_2)\}$ for convex C_1 and C_2, then we would have

$$\gamma(\operatorname{conv} B) \le \max\{\gamma(M_1), \ldots, \gamma(M_m)\} \le d$$

and so $\gamma(\operatorname{conv} B) \le \gamma(B)$.

We would first like to show that

$$\operatorname{conv}(C_1 \cup C_2) \subseteq \bigcup_{0 \le \lambda \le 1} [\lambda C_1 + (1-\lambda) C_2] = S.$$

Now $C_1 \cup C_2 \subseteq S$, so we just need to verify that S is convex.

Let $x, y \in S$ and $\mu \in [0, 1]$. Then $x = \lambda c_1 + (1-\lambda) c_2$ and $y = \lambda' c' + (1-\lambda') c'$ for some $\lambda, \lambda' \in [0, 1]$ and $c_1, c'_1 \in C_1$, $c_2, c'_2 \in C_2$. We must show that $\mu x + (1-\mu) y \in S$.

$$\mu x + (1-\mu) y = \mu\lambda c_1 + \mu(1-\lambda)c_2 + (1-\mu)\lambda' c'_1 + (1-\mu)(1-\lambda')c'_2$$

$$= [\mu\lambda c_1 + (1-\mu)\lambda' c_1'] + [\mu(1-\lambda)c_2 + (1-\mu)(1-\lambda')c_2'].$$

Since λ, λ', $\mu \in [0, 1]$, $\mu\lambda + (1-\mu)\lambda' \in [0, 1]$.

If $0 < \mu\lambda + (1-\mu)\lambda' < 1$, then $0 < \mu(1-\lambda) + (1-\mu)(1-\lambda') < 1$ and so

$$\begin{aligned}
\mu x+(1-\mu)y &= (\mu\lambda+(1-\mu)\lambda')\left[\frac{\mu\lambda}{\mu\lambda+(1-\mu)\lambda'}c_1+\frac{(1-\mu)\lambda'}{\mu\lambda+(1-\mu)\lambda'}c_1'\right] \\
&\quad + (\mu(1-\lambda)+(1-\mu)(1-\lambda'))\left[\frac{\mu(1-\lambda)}{\mu(1-\lambda)+(1-\mu)(1-\lambda')}c_2\right. \\
&\quad \left. + \frac{(1-\mu)(1-\lambda')}{\mu(1-\lambda)+(1-\mu)(1-\lambda')}c_2'\right] \\
&\in (\mu\lambda+(1-\mu)\lambda')C_1+(\mu(1-\lambda)+(1-\mu)(1-\lambda'))C_2 \\
&\subseteq S
\end{aligned}$$

If $\mu\lambda + (1-\mu)\lambda' = 0$, then $\mu\lambda = 0 = (1-\mu)\lambda'$ and so $\mu x + (1-\mu)y = \mu c_2 + (1-\mu)c_2' \in C_2 \subseteq S$, and if $\mu\lambda + (1-\mu)\lambda' = 1$, then $\mu(1-\lambda) + (1-\mu)(1-\lambda') = 0$ and so $\mu(1-\lambda) = 0 = (1-\mu)(1-\lambda')$.

Thus $\mu x + (1-\mu)y = \mu c_1 + (1-\mu)c_1' \in C_1 \subseteq S$. Hence S is a convex set containing $C_1 \cup C_2$ and so $\operatorname{conv}(C_1 \cup C_2) \subseteq S = \bigcup_{0\le\lambda\le 1} [\lambda C_1 + (1-\lambda)C_2]$.

Since $C_1 - C_2$ is bounded, there exists $r > 0$ such that $|x| < r$ for all $x \in C_1 - C_2$.

Given $\epsilon > 0$, we can find $\lambda_1, \ldots, \lambda_p \in [0, 1]$ such that

$[0, 1] \subseteq \bigcup_{i=1}^{p} (\lambda_i - \frac{\epsilon}{r}, \lambda_i - \frac{\epsilon}{r})$ since $[0, 1]$ is compact. Now let $x \in \operatorname{conv}(C_1 \cup C_2)$. Then $x = \lambda c_1 + (1-\lambda)c_2$ for some $\lambda \in [0, 1]$, $c_1 \in C_1$, $c_2 \in C_2$. Since $\lambda \in [0, 1]$, we can find i such that $|\lambda - \lambda_i| < \frac{\epsilon}{r}$. So $x = \lambda_i c_1 + (1-\lambda_i)c_2 + [(\lambda-\lambda_i)c_1 - (\lambda-\lambda_i)c_2]$ and

$$|(\lambda - \lambda_i)c_1 - (\lambda-\lambda_i)c_2| = |\lambda-\lambda_i|\,|c_1 - c_2| < \frac{\epsilon}{r} r = \epsilon.$$

Thus $x \in \lambda_i C_1 + (1-\lambda_i)C_2 + \bar{B}_\epsilon(0)$.

So $\operatorname{conv}(C_1 \cup C_2) \subseteq \bigcup_{i=1}^{p} [\lambda_i C_1 + (1-\lambda_i)C_2 + \bar{B}_\epsilon(0)]$ and hence by (b) and (c) and the obvious statement that $\gamma(\bar{B}_\epsilon(0)) \leq 2\epsilon$, we have

$$\begin{aligned}
\gamma(\operatorname{conv}(C_1 \cup C_2)) &\leq \max_{1\leq i\leq p} \gamma(\lambda_i C_1 + (1-\lambda_i)C_2 + \bar{B}_\epsilon(0)) \\
&\leq \max_{1\leq i\leq p} [\gamma(\lambda_i C_1)+\gamma((1-\lambda_i)C_2)+\gamma(\bar{B}_\epsilon(0))] \\
&\leq \max_{1\leq i\leq p} [|\lambda_i|\gamma(C_1)+|1-\lambda_i|\gamma(C_2)+2\epsilon] \\
&\leq \max_{1\leq i\leq p} [|\lambda_i| \max\{\gamma(C_1),\gamma(C_2)\} \\
&\quad + |1-\lambda_i| \max\{\gamma(C_1),\gamma(C_2)\}+2\epsilon]. \\
&= \max\{\gamma(C_1),\gamma(C_2)\} + 2\epsilon \text{ for all } \epsilon > 0.
\end{aligned}$$

Hence $\gamma(\operatorname{conv}(C_1 \cup C_2)) \leq \max\{\gamma(C_1), \gamma(C_2)\}$ and we are done.

(e) By (c) we have $\gamma(B) \leq \gamma(\bar{B})$. If $d > 0$ with $B \subseteq \bigcup_{i=1}^{m} M_i$ with diam $M_i \leq d$ if $\gamma = \alpha$ or $M_i = B_d(x_i)$ if $\gamma = \beta$, then $\bar{B} \subseteq \bigcup_{i=1}^{m} \bar{M}_i$ and diam $\bar{M}_i$ = diam $M_i \leq d$.
So $\gamma(\bar{B}) \leq \gamma(B)$ and hence $\gamma(\bar{B}) = \gamma(B)$. ♠

Now let us compare the α–MNC and the β–MNC. Let $B \in \mathscr{B}$. If $d > 0$ with $B \subseteq \bigcup_{i=1}^{m} B_d(x_i)$, then diam $B_d(x_i) \leq 2d$ and so $\alpha(B) \leq 2\,\beta(B)$. Now let $d > 0$ with $B \subseteq \bigcup_{i=1}^{m} M_i$ such that diam $M_i \leq d$. Choose $x_i \in M_i$. Then $|x - x_i| \leq$ diam $M_i \leq d$ for all $x \in M_i$. Thus $M_i \subseteq B_d(x_i)$ for each i. So $B \subseteq \bigcup_{i=1}^{m} B_d(x_i)$ and hence $\beta(B) \leq \alpha(B)$. Thus we obtain the inequality $\beta(B) \leq \alpha(B) \leq 2\,\beta(B)$ for all $B \in \mathscr{B}$.

Strict inequalities hold in the following subsets of C(J):

$$B_1 = \{x \in C(J) \,/\, x(0) = 0,\ x(1) = 1,\ 0 \leq x(t) \leq 1 \text{ in } J\}$$
$$B_2 = \{x \in B_1 \,/\, 0 \leq x(t) \leq \tfrac{1}{2} \text{ in } [0, \tfrac{1}{2}] \text{ and } \tfrac{1}{2} \leq x(t) \leq 1 \text{ in } [\tfrac{1}{2}, 1]\}$$
$$B_3 = \{x \in B_1 \,/\, 0 \leq x(t) \leq \tfrac{2}{3} \text{ in } [0, \tfrac{1}{2}] \text{ and } \tfrac{1}{3} \leq x(t) \leq 1 \text{ in } [\tfrac{1}{2}, 1]\}.$$

We would now like to calculate the measures of the ball $B_r(x_0)$.

N.B.: If X is a finite–dimensional space, then $\bar{B}_r(x_0)$ is closed and bounded, hence compact. Thus $\gamma(B_r(x_0)) = \gamma(\bar{B}_r(x_0)) = 0$.

We will consider X to be an infinite–dimensional space. Since $B_r(x_0) = B_r(0) + x_0$, we have $\gamma(B_r(x_0)) = \gamma(B_r(0))$. Also, $B_r(0) = r\, B_1(0)$. So $\gamma(B_r(x_0)) = r\,\gamma(B_1(0)) = r\,\gamma(\bar{B}_1(0))$. Thus we need only compute $\gamma(\bar{B}_1(0))$.

Let $S = \partial B_1(0)$. Then $S \subseteq \bar{B}_1(0)$ and $\bar{B}_1(0)$ is convex. So conv $S \subseteq \bar{B}_1(0)$. For $x \in S$, $|x| = 1 = |-x|$ and so $-x \in S$. Thus $0 = \frac{1}{2}x + \frac{1}{2}(-x) \in$ conv S. Now take any $x \in \bar{B}_1(0) \setminus \{0\}$. Then $\frac{x}{|x|} \in S$. So $x = |x|(\frac{x}{|x|}) + (1 - |x|)(0) \in$ conv S. Thus we have shown that conv $S = \bar{B}_1(0)$. So $\gamma(S) = \gamma(\text{conv } S) = \gamma(\bar{B}_1(0))$.

By definition of α and β, $\alpha(S) \leq 2$ and $\beta(S) \leq 1$. Suppose $\alpha(S) < 2$. Then $S = \bigcup_{i=1}^{m} M_i$ with closed sets M_i and diam $M_i < 2$. Let X_n be an n–dimensional subspace of X. Then $S \cap X_n = \bigcup_{i=1}^{m} (M_i \cap X_n)$ is the boundary of the unit ball in X_n. By theorem 2.13, which is proved later in chapter 2, we find that one of the sets $M_i \cap X_n$ must contain a pair of antipodal points, x and $-x$. Hence diam $M_i \geq$ diam $(M_i \cap X_n) \geq 2$, a contradiction. So $\alpha(S) = 2$ and $1 = \frac{\alpha(S)}{2} \leq \beta(S) \leq 1$, giving us $\beta(S) = 1$. Thus in an infinite–dimensional space,

$$\alpha(B_r(x_0)) = 2r \quad\text{and}\quad \beta(B_r(x_0)) = r.$$

1.2.13 Definition

Let X, Y be Banach spaces and $\Omega \subseteq X$. A subset B of $C_Y(\Omega)$ is said to be *equicontinuous* at $\xi \in \Omega$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for every $\eta \in \Omega$ with $|\xi - \eta| < \delta$, we have $\sup\{|u(\xi)-u(\eta)| \,/\, u \in B\} \leq \epsilon$. B is equicontinuous on Ω if it is equicontinuous at each $x \in \Omega$.

Let $B \subseteq C_Y(\Omega)$ be a bounded equicontinuous set and let $B(\xi) = \{u(\xi) \,/\, u \in B\}$ be the slice at $\xi \in \Omega$. We now prove the following result.

1.2.14 Theorem

Let X be a Banach space, $D \subseteq \mathbb{R}^n$ be compact and $B \subseteq C_X(D)$. Then

(a) $\alpha(B) = \sup_D \alpha(B(\xi))$ if B is bounded and equicontinuous.

(b) B is relatively compact iff B is equicontinuous and $B(\xi)$ is relatively compact for every $\xi \in D$.

Proof:

(a) Let $d > 0$ with $B \subseteq \bigcup_{i=1}^{p} M_i$ and diam $M_i \leq d$. Hence $B(\xi) \subseteq \bigcup_{i=1}^{p} M_i(\xi)$ with

$$\begin{aligned} \text{diam } M_i(\xi) &= \sup\{|u(\xi)-v(\xi)| \ / \ u, v \in M_i\} \\ &= \sup\{|(u-v)(\xi)| \ / \ u, v \in M_i\} \\ &\leq \sup\{|u-v|_0 \ / \ u, v \in M_i\} \\ &= \text{diam } M_i \\ &\leq d. \end{aligned}$$

Thus $\alpha(B(\xi)) \leq \alpha(B)$ for all $\xi \in D$ and so $\sup_D \alpha(B(\xi)) \leq \alpha(B)$.

Now to obtain the opposite inequality we let $\epsilon > 0$, $u \in B$, and $\xi \in D$. To show $u(\xi) \in \bigcup_{i=1}^{p} (B(\xi^i) + B_\epsilon(0))$. Since B is equicontinuous, there exists a $\delta > 0$ such that for every $\eta \in D$ with $|\xi - \eta| < \delta$ we have $|v(\xi) - v(\eta)| < \epsilon$ for all $v \in B$. Since D is compact, we can find $\xi^1, \ldots, \xi^p \in D$ such that $D \subseteq \bigcup_{i=1}^{p} B_\delta(\xi^i)$. Now $\xi \in D$, so there exists i such that $|\xi - \xi^i| < \delta$. Thus $|u(\xi) - u(\xi^i)| < \epsilon$. So $u(\xi) = u(\xi^i) + (u(\xi) - u(\xi^i)) \in B(\xi^i) + B_\epsilon(0)$. Thus

$B(\xi) \subseteq \bigcup_{i=1}^{p} [B(\xi^i) + B_\epsilon(0)]$ for all $\xi \in D$. Let $d > \sup_D \alpha(B(\xi))$. Then we can find $M_1, \ldots, M_m$, diam $M_j \leq d$ and $\bigcup_{i=1}^{p} B(\xi^i) \subseteq \bigcup_{j=1}^{m} M_j$. Now B is the union of the finitely many sets $\{u \in B \ / \ u(\xi^1) \in M_{j_1}, \ldots, u(\xi^p) \in M_{j_p}$, where $(j_1, j_2, \ldots, j_p)$ is a permutation of $(1, 2, \ldots, p)\}$, each of which has diameter $\leq d + 2\epsilon$. Thus $\alpha(B) \leq \sup_D \alpha(B(\xi))$ and we are done.

(b) If B is equicontinuous and $B(\xi)$ is relatively compact for every $\xi \in D$, then by (a), $\alpha(B) = \sup_D \alpha(B(\xi)) = \sup_D 0 = 0$ and so B is relatively compact. Now, suppose B is relatively compact. Then $\alpha(B)=0$. Since the map $u \longrightarrow u(\xi)$ is continuous, we must also have $\alpha(B(\xi)) = 0$. So $B(\xi)$ is relatively compact for every $\xi \in D$. Now take $\epsilon > 0$. We can find $u_1, \ldots, u_p$ in $C_X(D)$ such that $B \subseteq \bigcup_{i=1}^{p} B_\epsilon(u_i)$ and $\{u_1, \ldots, u_p\}$ is equicontinuous. Therefore, there exists $\delta > 0$ such that $|\xi - \eta| < \delta$ implies that $\sup \{|u_i(\xi) - u_i(\eta)| \ / \ i = 1, \ldots, p\} < \epsilon$. Thus for $|\xi - \eta| < \delta$ we have $\sup \{|u(\xi) - u(\eta)| \ / \ u \in B\} \leq 3\epsilon$ and so B is equicontinuous. ♠

The following is an important extension theorem and is a special case of Dugundji's extension theorem.

1.2.15 Theorem

Let X and Y be nlss., $A \subseteq X$ closed and $F : A \longrightarrow Y$ continuous. Then F has a continuous extension $\tilde{F} : X \longrightarrow Y$ such that $\tilde{F}(X) \subseteq \mathrm{conv}(F(A))$.

Proof:

The idea of the proof is simple. We first construct a locally finite covering $(U_\lambda)_{\lambda \in \Lambda}$ of $X \setminus A$, i.e. $X \setminus A = \bigcup_{\lambda \in \Lambda} U_\lambda$, U_λ is open and to every $x \in X \setminus A$ there exists a neighbourhood $V(x)$ which meets only finitely many U_λ. Then we define

$$\varphi_\lambda(x) = \begin{cases} 0 & \text{if } x \notin U_\lambda \\ \rho(x, \partial U_\lambda) & \text{if } x \in U_\lambda \end{cases} \quad \text{and} \quad \psi(x) = \frac{\varphi_\lambda(x)}{\sum\limits_{\mu\in\Lambda} \varphi_\mu(x)} .$$

Notice that since the covering is locally finite, each $x \in X\backslash A$ can only belong to finitely many U_λ and so $\sum\limits_{\mu\in\Lambda} \varphi_\mu(x)$ is a finite sum and $\sum\limits_{\mu\in\Lambda} \varphi_\mu(x) > 0$. Hence ψ_λ is continuous in $X\backslash A$. Furthermore, $0 \leq \psi_\lambda(x) \leq 1$ and $\sum\limits_{\lambda\in\Lambda} \psi_\lambda(x) = 1$.

Next we choose suitable points $a_\lambda \in A$ and we let

$$\tilde{F}x = \begin{cases} Fx & \text{if } x \in A \\ \sum \psi_\lambda(x)\, Fa_\lambda & \text{if } x \notin A \end{cases} .$$

Obviously $\tilde{F}$ is an extension of F with $\tilde{F}(X) \subseteq \text{conv}(F(A))$, $\tilde{F}$ is continuous in $X\backslash A$ and at interior points of A (if there are any), and

$\tilde{F}x - Fx_0 = \sum\limits_\lambda \psi_\lambda(x)[Fa_\lambda - Fx_0]$, hence $|\tilde{F}x - Fx_0| \leq \sum\limits_\lambda \psi_\lambda(x)\, |Fa_\lambda - Fx_0|$ for $x \notin A$ and $x_0 \in A$.

It must be shown that F is continuous on $\partial A \subseteq A$. Let $x_0 \in \partial A$. Given $\epsilon > 0$, we then find $\delta > 0$ such that $|Fz - Fx_0| < \epsilon$ in $A \cap B_\delta(x_0)$, since F is continuous.

To prove continuity of $\tilde{F}$ at x_0, we should have that $\psi_\lambda(x) \neq 0$ (i.e. $x \in U_\lambda$) with $|x - x_0|$ sufficiently small implies that a_λ must be in $B_\delta(x_0)$, since then $|\tilde{F}x - Fx_0| \leq \sum\limits_\lambda \psi_\lambda(x)\, |Fa_\lambda - Fx_0| < \sum\limits_\lambda \psi_\lambda(x)\, \epsilon = \epsilon$.

We must now find appropriate U_λ and a_λ. Let B_x be a ball with centre $x \in X\backslash A$ such that diam $B_x \leq \rho(B_x, A)$, for example, $B_x = B_r(x)$ with $r = \frac{\rho(x,\ A)}{6}$. Then $X\backslash A = \bigcup\limits_{x\in X\backslash A} B_x$. $X\backslash A$ is a metric space and hence is paracompact (see Willard [30]). Thus $X\backslash A$ admits a locally finite refinement $(U_\lambda)_{\lambda\in\Lambda}$ (i.e. a locally finite open covering such that every U_λ is contained in some B_x). Now $U_\lambda \subseteq B_z$ implies $\rho(U_\lambda, A) \geq \rho(B_z, A) > 0$ and therefore we can choose $a_\lambda \in A$ such that $\rho(a_\lambda, U_\lambda) < 2\, \rho(U_\lambda, A)$ for every $\lambda \in \Lambda$. Then

$$
\begin{aligned}
|x - x_0| < \tfrac{\delta}{4} \text{ and } \psi_\lambda(x) \neq 0 \text{ (i.e. } x \in U_\lambda \subseteq B_z \text{ for some } z \in X \setminus A) \text{ imply}\\
|x - a_\lambda| &\le \rho(a_\lambda, U_\lambda) + \operatorname{diam} U_\lambda \\
&\le 2\,\rho(a_\lambda, A) + \operatorname{diam} B_z \\
&\le 3\,|x - x_0| \\
&< 3\tfrac{\delta}{4} \\
&< \delta \qquad \text{and we are done.}
\end{aligned}
$$

♠

1.2.16 Definition

A subset D of X is said to be a *retract* of X if there exists a continuous map $R : X \longrightarrow D$ such that $Rx = x$ for all $x \in D$.

i.e. D is a retract of X if $I|_D$ has a continuous extension to X.

R is called a *retraction* of D.

If $D \subseteq X$ is closed convex, then by theorem 1.2.15, $I|_D$ has a continuous extension F to X such that $F(X) \subseteq \operatorname{conv}(I(D)) = \operatorname{conv}(D) = D$. So $F : X \longrightarrow D$ is continuous such that $Fx = x$ for all $x \in D$. Thus every closed convex subset of a nls. X is a retract of X.

Differentiability

To differentiate a nonlinear operator, we have to use local approximations to the operator by linear operators. More about this can be found in Kantorovich [33].

We say that $\omega(h) = o(h)$ as $h \longrightarrow 0$ if $\frac{|\omega(h)|}{|h|} \longrightarrow 0$ as $|h| \longrightarrow 0$.

1.2.17 Definition

Let X and Y be Banach spaces over **K**, $\Omega \subseteq X$ be open and $F : \Omega \longrightarrow Y$.

F is said to be *Fréchet–differentiable* at $x_0 \in \Omega$ if there exists an $F'(x_0) \in BL(X, Y)$ such that $F(x_0 + h) = F(x_0) + F'(x_0)\,h + \omega(x_0, h)$ and $\omega(x_0, h) = o(|h|)$ as $h \longrightarrow 0$.

F is said to be the Frechet –(strong–) derivative of F at x_0.

F is said to be *Gateaux–differentiable* at $x_0 \in \Omega$ if there exists $F'(x_0) \in BL(X, Y)$ such that $\lim\limits_{t \to 0} \dfrac{F(x_0+th)-F(x_0)}{t} = F'(x_0)\,h$ for all $h \in X$.

$F'(x_0)$ is often called the Gâteaux– (weak–) derivative of F at x_0.

In the special case of functionals $\varphi : \Omega \subseteq X \longrightarrow \mathbf{K}$, we say that φ is Gâteaux–differentiable at $x_0 \in \Omega$ if there exists $\varphi'(x_0) \in X^*$ such that $\lim\limits_{t \to 0} \dfrac{\varphi(x_0+th)-\varphi(x_0)}{t} = \varphi'(x_0)\,h$ for all $h \in X$, and $\varphi'(x_0)$ is called the gradient of φ at x_0, denoted by grad $\varphi(x_0)$.

We now give some properties of the derivative, the proofs of which can be found in Kantorovich [33].

(1) If the operator F is Fréchet–differentiable at x, then it is continuous at x .

(2) Let $F = \alpha_1 F_1 + \alpha_2 F_2$. If $F_1'(x_0)$ and $F_2'(x_0)$ exist, then so does $F'(x_0)$ and we have $F'(x_0) = \alpha_1 F_1'(x_0) + \alpha_2 F_2'(x_0)$.

(3) If $F \in BL(X, Y)$, then F is Fréchet–differentiable at every point $x_0 \in X$ and $F'(x_0) = F$.

(4) Let X, Y, Z be Banach spaces with $\Omega_1 \subseteq X$ and $\Omega_2 \subseteq Y$ open. If $F : \Omega_1 \longrightarrow \Omega_2$ has a Gâteaux–derivative at $x_0 \in \Omega_1$ and $F_2 : \Omega_2 \longrightarrow Z$ is Fréchet–differentiable at $y_0 = F_1(x_0)$, $F = F_2 F_1$ has a Gâteaux derivative at x_0 and $F'(x_0) = F_2'(F_1(x_0))\, F'(x_0) = F_2'(y_0)\, F_1'(x_0)$.

Let $F : X \longrightarrow Y$ have a derivative P' in Ω. Then P' can be regarded as a mapping of the set Ω into the space BL(X, Y). Thus it is reasonable to speak of the derivative of this operator, if it exists. Then $P''(x_0) \in BL(X, BL(X, Y))$. We identify the space BL(X, BL(X, Y)) with the space $BL(X^2, Y)$, the set of all bilinear operators

$A : X \times X \longrightarrow Y$, i.e. operators A such that $A(x, .)$ and $A(., x)$ are linear for all $x \in X$ and $|A| = \sup \{|A(x, \bar{x})| \ / \ |x| \leq 1, |\bar{x}| \leq 1\} < \infty$.

CHAPTER 2

DEGREE IN FINITE DIMENSIONAL SPACES

Before we define the degree we give some definitions, notations and important results.

Let $\Omega \subseteq \mathbb{R}^n$ be open.

$C^k(\Omega)$ will denote the set of all $f : \Omega \longrightarrow \mathbb{R}^n$ which are k–times continuously differentiable in Ω, while $\bar{C}^k(\Omega) = C^k(\Omega) \cap C(\bar{\Omega})$ and $\bar{C}^{\infty}(\Omega) = \bigcap_{k \geq 1} \bar{C}^k(\Omega)$. If $f'(x_0)$ exists, then $J_f(x_0) = \det f'(x_0)$ is called the *Jacobian* of f at x_0 and x_0 is called a *critical point* of f if $J_f(x_0) = 0$. These points will play an important role later, and so we introduce $S_f(\Omega) = \{x \in \Omega \;/\; J_f(x) = 0\}$ and we write S_f whenever Ω is clear from the context.

A point $y \in \mathbb{R}^n$ will be called a *regular value* of f if $f^{-1}(y) \cap S_f(\Omega) = \emptyset$, and a *singular value* otherwise.

The following theorem is absolutely vital since it allows us to approximate continuous maps by differentiable maps. It is a special case of theorem 3.5 and so we do not prove it.

2.1 Theorem

Let $A \subseteq \mathbb{R}^n$ be compact, $f \in C(A)$ and $\epsilon > 0$. Then there exists a function $g \in C^{\infty}(\mathbb{R}^n)$ such that $|f(x) - g(x)| \leq \epsilon$ on A.

The next result, which is a special case of Sard's lemma, tells us that the regular values of a differentiable function form a dense subset of $\mathbb{R}^n$. The proof can be found in Schwartz [31].

2.2 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f \in C^1(\Omega)$. Then $\mu_n(f(S_f)) = 0$, where μ_n denotes the n–dimensional Lebesgue measure.

2.3 Theorem (Inverse function theorem)

Let Ω be open, $f \in C^1(\Omega)$ and $J_f(x_0) \neq 0$ for some $x_0 \in \Omega$. Then there exists a neighbourhood U of x_0 such that $f|_U$ is a homeomorphism onto a neighbourhood of $f(x_0)$.

The proof of this is standard, via Banach's fixed point theorem. We will use this result to show that if Ω is open and bounded and y is a regular value of f, then $f^{-1}(y)$ is finite. By theorem 2.3, for each $x_0 \in f^{-1}(y)$, there exists a neighbourhood $U(x_0)$ of x_0 such that $f^{-1}(y) \cap U(x_0) = \{x_0\}$. Consequently $f^{-1}(y)$ must be finite. Otherwise, there would be an accumulation point $x_0 \in \bar{\Omega}$ of solutions by the compactness of $\bar{\Omega}$. Thus we have a contradiction to x_0 being an isolated solution. So we must have $f^{-1}(y)$ to be finite.

The construction of a unique degree in finite dimensions can be found in Heinz [8], Nagumo [15] and Deimling [28].
We state this formally in the following theorem.

2.4 Theorem

Let $\mathcal{M} = \{(f, \Omega, y) \ / \ \Omega \subseteq \mathbb{R}^n$ open bounded, $f \in C(\bar{\Omega})$ and $y \in \mathbb{R}^n \setminus f(\partial\Omega)\}$.

(a) Then there is a unique function $d : \mathcal{M} \longrightarrow \mathbb{Z}$ satisfying the following properties:–

(d1) $d(id, \Omega, y) = 1$ if $y \in \Omega$.

(d2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ if Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \in \mathbb{R}^n \setminus f(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$.

(d3) $d(h(t, .), \Omega, y(t))$ is independent of t if $h : J \times \bar{\Omega} \longrightarrow \mathbb{R}^n$, $y : J \longrightarrow \mathbb{R}^n$ are continuous such that $y(t) \notin h(t, \partial\Omega)$ on J.

(b) If $(f, \Omega, y) \in \mathscr{M}$ with $f \in \bar{C}^1(\Omega)$ and y is a regular value of f, then we define

$$d(f, \Omega, y) = \sum_{x \in f^{-1}(x)} \operatorname{sgn} J_f(x) \text{ and we agree that } \sum_{\emptyset} = 0.$$

(c) If $(f, \Omega, y) \in \mathscr{M}$ with $f \in \bar{C}^2(\Omega)$, then we define $d(f, \Omega, y) = d(f, \Omega, y')$ where y' is any regular value of f such that $|y - y'| < \rho(y, f(\partial\Omega))$, and $d(f, \Omega, y')$ is given by (b).

(d) If $(f, \Omega, y) \in \mathscr{M}$, then we define $d(f, \Omega, y) = d(g, \Omega, y)$ where $g \in \bar{C}^2(\Omega)$ is a map such that $|g - f|_0 < \rho(y, f(\partial\Omega))$ and $d(g, \Omega, y)$ is given by (c).

This degree is often called the Brouwer degree. (f, Ω, y) will be called an admissible triplet for the Brouwer degree if $(f, \Omega, y) \in \mathscr{M}$.

Of course, the usefulness of a degree theory stems from the properties it satisfies. Apart from the three properties (d1)–(d3) that uniquely define the degree, we also have some simple consequences which we call (d4)–(d7). We write these properties down formally in the following theorem.

2.5 Theorem

Let $\mathscr{M} = \{(f, \Omega, y) \,/\, \Omega \subseteq \mathbb{R}^n$ open bounded, $f \in C(\bar{\Omega})$ and $y \in \mathbb{R}^n \setminus f(\partial\Omega)\}$ and $d : \mathscr{M} \longrightarrow \mathbb{Z}$ the Brouwer degree defined in Theorem 2.4. Then d has the following properties:–

(d1) $d(id, \Omega, y) = 1$ if $y \in \Omega$.

(d2) $d(f, \Omega, y) = d(f, \Omega_1, y) + d(f, \Omega_2, y)$ whenever Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \notin f(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$.

(d3) $d(h(t, .), \Omega, y(t))$ is independent of t whenever $h : J \times \bar{\Omega} \longrightarrow \mathbb{R}^n$ and $y : J \longrightarrow \mathbb{R}^n$ are continuous and $y(t) \notin h(t, \partial\Omega)$ for every $t \in J$.

(d4) $d(f, \Omega, y) \neq 0$ implies that $f^{-1}(y) \neq \emptyset$.

(d5) $d(., \Omega, y)$ and $d(f, \Omega, .)$ are constant on $\{g \in C(\bar{\Omega}) \,/\, |g-f|_0 < r\}$ and $B_r(y) \subseteq \mathbb{R}^n$, respectively, where $r = \rho(y, f(\partial\Omega))$. Moreover, $d(f, \Omega, .)$ is constant on every connected component of $\mathbb{R}^n \setminus f(\partial\Omega)$.

(d6) $d(g, \Omega, y) = d(f, \Omega, y)$ whenever $g|_{\partial\Omega} = f|_{\partial\Omega}$.

(d7) $d(f, \Omega, y) = d(f, \Omega_1, y)$ for every open subset Ω_1 of Ω such that $y \notin f(\bar{\Omega} \setminus \Omega_1)$.

No proofs are included here, but they are along the lines of those given in chapter 3 for the Leray–Schauder degree.

Sometimes we would like to solve equations of the type $f(x) = x$. Such points are called fixed points of the map f. The next theorem is Brouwer's fixed point theorem. It can be proved using other techniques, but we will use degree theory to prove it.

D^0 denotes the interior of the set D.

2.6 Theorem (Brouwer's fixed point theorem)

Let $D \subseteq \mathbb{R}^n$ be a nonempty compact convex set and $f : D \longrightarrow D$ continuous. Then f has a fixed point. The same is true if D is only homeomorphic to a compact convex set.

Proof:

First suppose $D = \bar{B}_r(0)$. We may assume that $f(x) \neq x$ on $\partial\Omega$, else we are done. Let $h(t, x) = x - t\, f(x)$. Then $h : J \times D \longrightarrow \mathbb{R}^n$ is continuous. For any

$(t, x) \in [0, 1) \times \partial D$ we have

$|h(t, x)| = |x - t f(x)| \geq |x| - t |f(x)| \geq (1 - t) r > 0.$ Also $f(x) \neq x$ on ∂D and so $|h(1, x)| > 0$ on ∂D. Thus $0 \notin h(t, \partial D)$ for all $t \in J$. So by (d3), $d(id - f, D^0, 0) = d(id, B_r(0), 0) = 1$ by (d1). By (d4), since $d(id - f, D^0, 0) \neq 0$, we can find $x \in B_r(0)$ such that $x - f(x) = 0$.

Next we consider D to be a general compact convex set. By Theorem 1.2.15 we have a continuous extension $\tilde{f} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $\tilde{f}(\mathbb{R}^n) \subseteq \text{conv } f(D) \subseteq D$. Since D is compact, it is also bounded, and so we can find $r > 0$ such that $D \subseteq B_r(0)$. So $\tilde{f}|_{\bar{B}_r(0)} : \bar{B}_r(0) \longrightarrow \bar{B}_r(0)$. By the first step, we can find $x \in \bar{B}_r(0)$ such that $\tilde{f}(x) = x$. But $\tilde{f}(x) \in D$. So $x \in D$. Hence $f(x) = \tilde{f}(x) = x$.

Lastly, let $h : D_0 \longrightarrow D$ be a homeomorphism with D_0 compact convex. Then $h^{-1}fh : D_0 \longrightarrow D_0$ is continuous. By the second step, we can find $x \in D_0$ such that $h^{-1}fh\,(x) = x$. Thus $f(h(x)) = h(x) \in D$ and so f has a fixed point. ♠

The following examples illustrate the above theorem.

2.7 Example

Let $A = (a_{ij})$ be an $n \times n$ – matrix such that $a_{ij} \geq 0$ for all i, j. Then there exist $\lambda \geq 0$ and $x \neq 0$ such that $x_i \geq 0$ for all i and $Ax = \lambda x$. (In other words, A has a nonzero eigenvector corresponding to a nonnegative eigenvalue).

To prove this, let $D = \{x \in \mathbb{R} \,/\, x_i \geq 0$ for all i and $\sum_{i=1}^{n} x_i = 1\}$. If $Ax = 0$ for some $x \in D$, then we are done with $\lambda = 0$. If $Ax \neq 0$ for all $x \in D$, then for $x \in D$

$\sum_{i=1}^{n} (Ax)_i \geq \alpha$ for some $\alpha > 0$. Thus $f : x \longmapsto {}^{Ax}/\sum_{i=1}^{n} (Ax)_i$ is continuous on D. If $x \in D$, then $x_i \geq 0$ for all i and $a_{ij} \geq 0$ for all i, j. So $(Ax)_i \geq 0$ for all i. Also $\sum_{j=1}^{n} \left[\dfrac{Ax}{\sum_{i=1}^{n} (Ax)_i}\right]_j = 1$. Thus ${}^{Ax}/\sum_{i=1}^{n} (Ax)_i \in D$ if $x \in D$. So $f(D) \subseteq D$.

D is convex and easily a closed bounded subset of $\mathbb{R}^n$, hence it is compact. By Brouwer's fixed point theorem, we can find $x_0 \in D$ such that $f(x_0) = x_0$. Thus $Ax_0 = (\sum_{i=1}^{n} (Ax)_i) x_0$ and $\lambda = \sum_{i=1}^{n} (Ax)_i > 0$. ♠

2.8 Example

It is impossible to retract the closed unit ball continuously onto its boundary such that the boundary remains pointwise fixed, i.e. there is no continuous map $f : \bar{B}_1(0) \longrightarrow \partial B_1(0)$ such that $f(x) = x$ for all $x \in \partial B_1(0)$. Suppose we can find a map f satisfying these properties. Then by Brouwer's fixed point theorem, $g = -f$ has a fixed point $x_0 \in \bar{B}_1(0)$. Thus $x_0 \in \partial B_1(0)$ and we have the ridiculous situation $x_0 = f(x_0) = -x_0$. ♠

We have been using the homotopy invariance up to now, i.e. if f and g are homotopic maps, then their degrees are the same. It is also useful to use the fact that if two maps have different degrees, then they cannot be homotopic. We use this in proving the following theorem (the Hedgehog theorem).

2.9 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open bounded with $0 \in \Omega$ and let $f : \partial\Omega \longrightarrow \mathbb{R}^n \setminus \{0\}$ be continuous. Suppose also that the dimension n is odd. Then there exist $x \in \partial\Omega$ and $\lambda \neq 0$ such that $f(x) = \lambda x$.

Proof:

We may assume, without loss of generality, that $f \in C(\bar{\Omega})$, by Theorem 1.2.15. By definition we have

$$\begin{aligned} d(-id, \Omega, 0) &= \operatorname{sgn} \det\, (-id)'(0) \\ &= \operatorname{sgn} \det\, (-id) \\ &= \operatorname{sgn}\, (-1)^n \\ &= -1 \quad \text{since n is odd.} \end{aligned}$$

If $d(f, \Omega, 0) \neq -1$, then f and $-id$ cannot be homotopic and so $0 \in h(J \times \partial\Omega)$ where $h(t, x) = (1 - t)\, f(x) - t\, x$. Thus there exists $(t_0, x_0) \in J \times \partial\Omega$ such that $0 = h(t_0, x_0)$. If $t_0 = 1$, then $-x_0 = 0$ and if $t_0 = 0$, we have $f(x_0) = 0$. So $t_0 \in (0, 1)$. Thus $f(x_0) = t_0(1 - t_0)^{-1}x_0$. If $d(f, \Omega, 0) = -1$, then f and id cannot be homotopic. So again, by the same argument as above, $h(t, x) = (1 - t)\, f(x) + t\, x$ must have a zero $(t_0, x_0) \in (0, 1) \times \partial\Omega$. And so,
$f(x_0) = -t_0(1 - t_0)^{-1}x_0$ as required. ♠

Since the dimension n is odd, the theorem does not apply to $\mathbb{C}^n$. A simple counterexample is the following rotation by $\frac{\pi}{2}$ of the unit circle in $\mathbb{C}$:

$$f(x_1, x_2) = (-x_1, x_2).$$

If $\Omega = B_1(0)$, then the theorem tells us that there is at least one normal such that f changes at most its orientation. In other words, there is no continuous $f : S \to \mathbb{R}$ where $S = \partial B_1(0)$ such that $f(x) \neq 0$ and $(f(x), x) = 0$ on S. In particular, if $n = 3$, this means that a 'hedgehog cannot be combed without leaving tufts or whorls'.

Whenever we want to show that $f(x) = y$ has a solution using degree theory, we have to verify that $d(f, \Omega, y) \neq 0$. Borsuk's Theorem is important in this respect.

2.10 Theorem (Borsuk's Theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded and symmetric with respect to $0 \in \Omega$. Let $f \in C(\bar{\Omega})$ be odd and $0 \notin f(\partial\Omega)$. Then $d(f, \Omega, 0)$ is odd.

Proof:

Step 1:

Here we show that we may assume that $f \in \bar{C}^1(\Omega)$ and $J_f(0) \neq 0$. Choose $g_1 \in \bar{C}^1(\Omega)$ such that $|f - g_1|_0 < \frac{1}{2}\rho(0, f(\partial\Omega))$. Let $g_2(x) = \frac{1}{2}(g_1(x) - g_1(-x))$ and choose $\delta < \frac{1}{2M}\rho(0, f(\partial\Omega))$ where M is a bound for Ω and δ is not an eigenvalue of $g_2'(0)$. Then $\tilde{f} = g_2 - \delta\, \mathrm{id}$ is in $\bar{C}^1(\Omega)$, odd and

$J_{\tilde{f}}(0) = \det \tilde{f}'(0) = \det [g_2'(0) - \delta I_n] \neq 0$. Also

$$
\begin{aligned}
|f - \tilde{f}|_0 &= |f - (g_2 - \delta\, \mathrm{id})|_0 \\
&= \sup_{\bar{\Omega}} |f(x) - \tfrac{1}{2}(g_1(x) - g_1(-x)) + \delta x| \\
&\leq \tfrac{1}{2}\sup_{\bar{\Omega}} |f(x) - g_1(x)| + \tfrac{1}{2}\sup_{\bar{\Omega}} |f(-x) - g_1(-x)| + \delta \sup_{\bar{\Omega}} |x| \\
&\leq |f - g_1|_0 + \delta M \\
&< \tfrac{1}{2}\rho(0, f(\delta\Omega)) + \tfrac{1}{2}\rho(0, f(\partial\Omega)) \\
&= \rho(0, f(\partial\Omega)).
\end{aligned}
$$

Thus by (d5), $d(f, \Omega, 0) = d(\tilde{f}, \Omega, 0)$ with $\tilde{f} \in \bar{C}^1(\Omega)$ and $J_{\tilde{f}}(0) \neq 0$.

Step 2:

Now let $f \in \bar{C}^1(\Omega)$ and $J_f(0) \neq 0$. Suppose we can find an odd $g \in \bar{C}^1(\Omega)$, $|f - g|_0 < \rho(0, f(\partial\Omega))$ such that $0 \notin g(S_g)$. Then we will have by (d5) and by definition, $d(f, \Omega, 0) = d(g, \Omega, 0) = \operatorname{sgn} J_g(0) + \sum_{0 \neq x \in g^{-1}(0)} \operatorname{sgn} J_g(x)$

Now $g(x) = 0 \Leftrightarrow g(-x) = 0$ since g is odd. So $x \in g^{-1}(0) \Leftrightarrow -x \in g^{-1}(0)$. We also have,

$$
\begin{aligned}
& g(-x + h) - g(-x) - g'(x)\, h \\
= \quad & -g(x - h) + g(x) + g'(x)(-h) \\
= \quad & -[g(x - h)\, g(x) - g'(x)(-h) = o(|h|).
\end{aligned}
$$

Thus $g'(x) = g'(-x)$. And so $J_g(-x) = \det g'(-x) = \det g'(x) = J_g(x)$. Thus $\sum_{0 \neq x \in g^{-1}(0)} \operatorname{sgn} J_g(x)$ is even. Now if $\operatorname{sgn} J_g(0) = 0$, then $J_g(0) = 0$ and so $0 \in g(S_g)$, a contradiction. Thus $\operatorname{sgn} J_g(0) \neq 0$ and hence $\operatorname{sgn} J_g(0) \in \{1, -1\}$. So $\operatorname{sgn} J_g(0) + \sum_{0 \neq x \in g^{-1}(0)} \operatorname{sgn} J_g(x)$ is odd. Thus $d(f, \Omega, 0)$ is odd.

Step 3:

We need to find an odd $g \in \bar{C}^1(\Omega)$, such that $|f - g|_0 < \rho(0, f(\partial\Omega))$ and $0 \notin g(S_g)$. Such a map g will be defined by induction. Define

$$\Omega_k = \{x \in \Omega \ / \ x_i \neq 0 \ \text{ for some } i \leq k\}$$

and choose an odd $\varphi \in C^1(\mathbb{R})$ such that $\varphi'(0) = 0$ and $\varphi(t) = 0$ iff $t = 0$. (For example $\varphi(t) = t^3$). Let $P_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $P_k(x) = x_k$ for $x \in \mathbb{R}^n$. Clearly P_k is linear. Define $\varphi_k = \varphi P_k$. Then $\varphi_k'(x) = \varphi'(P_k(x))\, P_k'(x) = \varphi'(P_k(x))\, P_k'$. Define $\bar{f}(x) = \dfrac{f(x)}{\varphi_1(x)}$ on $\Omega_1 = \{x \in \Omega \ / \ x_1 \neq 0\}$. By theorem 2.2, we choose $y^1 \notin \bar{f}(S_{\bar{f}}(\Omega_1))$ with $|y^1| < \dfrac{\delta}{M\, n}$ where $M = \sup_{[-a,\, a]} |\varphi|$, $\Omega \subseteq [-a, a]^n$.

Define $g_1(x) = f(x) - \varphi_1(x)\, y^1$ for $x \in \bar{\Omega}$. Note $g_1'(0) = f'(0)$ since $\varphi_1{}'(0) = 0$. If $x \in \Omega_1$ with $g_1(x) = 0$, then $f(x) = \varphi_1(x)\, y^1$ and $\bar{f}(x) = y^1$. Ω_1 is open and $f = \varphi_1 \bar{f}$ on Ω_1. Thus $f'(x) = \varphi_1'(x)\, \bar{f}(x) + \varphi_1(x)\, \bar{f}'(x)$ for all x in Ω_1. Therefore

$g_1'(x) = \varphi_1'(x)\, \bar{f}(x) + \varphi_1(x)\, \bar{f}'(x) - \varphi_1'(x)\, y^1 = \varphi_1(x)\, \bar{f}'(x)$ and so $\det g_1'(x) = [\varphi_1(x)]^n \det \bar{f}'(x)$. Now we have $\bar{f}(x) = y^1$ and $y^1 \notin \bar{f}(S_{\bar{f}}(\Omega_1))$. So $x \notin S_{\bar{f}}(\Omega_1)$. This means that $\det \bar{f}'(x) = J_{\bar{f}}(x) \neq 0$. Thus $J_{g_1}(x) = \det g_1'(x) \neq 0$ since $\varphi_1(x) \neq 0$ on Ω_1. Therefore 0 is a regular value of $g_1|_{\Omega_1}$. Also for all $x \in \bar{\Omega}$ $|f(x) - g_1(x)| = |\varphi_1(x)|\ |y^1| \leq M \dfrac{\delta}{M\, n} = \dfrac{\delta}{n}$. So $|f - g_1|_0 \leq \dfrac{\delta}{n}$.

Now suppose that for some $k < n$, we have an odd $g_k \in \bar{C}^1(\Omega)$ such that

$0 \notin g_k(S_{g_k}(\Omega_k))$, $|f - g_k|_0 < \frac{k\ \delta}{n}$ and $f'(0) = g_k'(0)$.

Define $\Omega'_{k+1} = \{x \in \Omega \,/\, x_{k+1} \neq 0\}$. Then $-\Omega'_{k+1} = \Omega'_{k+1}$ and $\Omega'_{k+1} \subseteq \Omega_{k+1}$. Also $\varphi_{k+1} \neq 0$ on Ω'_{k+1}. So we can define $\bar{g}_k(x) = \frac{1}{\varphi_{k+1}(x)} g_k(x)$ on Ω'_{k+1}. Find $y^{k+1} \in \Omega'_{k+1}$ such that $y^{k+1} \notin \bar{g}_k(S_{\bar{g}_k}(\Omega'_{k+1}))$ and $|y^{k+1}| < \frac{\delta}{M\,n}$.

$g_{k+1}(x) = g_k(x) - \varphi_{k+1}(x)\, y^{k+1}$ is odd. Now with the roles of Ω_1, f, $\bar{f}$, y^1, g_1 played respectively by Ω'_{k+1}, g_k, $\bar{g}_k$, y^{k+1}, g_{k+1}, we can prove that 0 is a regular value of $g_{k+1}|_{\Omega'_{k+1}}$.

Proof: Let $x \in \Omega'_{k+1}$ and $g_{k+1}(x) = 0$. We want to show that $J_{g_{k+1}}(x) \neq 0$.

Now $\varphi_{k+1}(x)\, y^{k+1} = g_k(x) = \varphi_{k+1}(x)\, \bar{g}_k(x)$ and $\varphi_{k+1}(x) \neq 0$ (since $x \in \Omega'_{k+1}$). Thus $\bar{g}_k(x) = y^{k+1}$. Since $y^{k+1} \notin \bar{g}_k(S_{\bar{g}_k}(\Omega'_{k+1}))$ we have $J_{\bar{g}_k}(x) \neq 0$. Now $g_k = \varphi_{k+1}\bar{g}_k$ near x, so

$g_k'(x) = \varphi'_{k+1}(x)\, \bar{g}_k(x) + \varphi_{k+1}(x)\, \bar{g}_k'(x) = \varphi'_{k+1}(x)\, y^{k+1} + \varphi_{k+1}(x)\, \bar{g}_k'(x)$. Therefore

$g'_{k+1}(x) = g_k'(x) - \varphi'_{k+1}(x)\, y^{k+1} = \varphi_{k+1}(x)\, \bar{g}_k'(x) = \varphi(x_{k+1})\, \bar{g}_k'(x)$. Therefore

$J_{g_{k+1}}(x) = [\varphi(x_{k+1})]^n J_{\bar{g}_k}(x) \neq 0$ (since $\varphi(x_{k+1}) \neq 0$ and $J_{\bar{g}_k}(x) \neq 0$). Now suppose that $x \in \Omega_{k+1} \backslash \Omega'_{k+1}$ and $g_{k+1}(x) = 0$. Then $x \in \Omega_k$ with $x_{k+1} = 0$, implying that $\varphi'_{k+1}(x) = \varphi'(0) = 0$. Therefore

$g'_{k+1}(x) = g_k'(x)$ and hence $J_{g_{k+1}}(x) = J_{g_k}(x)$. Also $\varphi_{k+1}(x) = \varphi(0) = 0$ and $g_k(x) = g_{k+1}(x) + \varphi_{k+1}(x)\, y^{k+1} = 0$. Since $0 \notin g_k(S_{g_k}(\Omega_k))$ (by the induction assumption), we must have $x \notin S_{g_k}(\Omega_k)$. So $J_{g_k}(x) \neq 0$ and hence $J_{g_{k+1}}(x) \neq 0$.

Thus we have proved that if $x \in \Omega_{k+1}$ and $g_{k+1}(x) = 0$, then $J_{g_{k+1}}(x) \neq 0$. So

$0 \notin g_{k+1}(S_{g_{k+1}}(\Omega_{k+1}))$. Also $g'_{k+1}(0) = g'_k(0) - \varphi'_{k+1}(0)\, y^{k+1} = g'_k(0) = f'(0)$, and

$|g_k - g_{k+1}|_0 = |\varphi_{k+1}\, y^{k+1}|_0 \leq M\, |y^{k+1}| < M \frac{\delta}{M\, n} = \frac{\delta}{n}$. Therefore

$|f - g_{k+1}|_0 \leq |f - g_k|_0 + |g_k - g_{k+1}|_0 < \frac{k\ \delta}{n} + \frac{\delta}{n} = (k+1)\frac{\delta}{n}$.

By induction, we deduce the existence of an odd $g = g_n \in \bar{C}^1(\Omega)$ such that $|f - g|_0 < \frac{n\ \delta}{n} = \delta$ and $0 \notin g(S_g(\Omega_n))$ ($\Omega_n = \Omega\backslash\{0\}$) and $g'(0) = f'(0)$. Therefore $J_g(0) = J_f(0) \neq 0$ which implies that $0 \notin S_g$, and so $0 \notin g(S_g)$. ♠

The following is a generalisation of Borsuk's theorem and is a consequence of Borsuk's theorem and the homotopy invariance.

2.11 Corollary

Let $\Omega \subseteq \mathbb{R}^n$ be open bounded and symmetric with respect to $0 \in \Omega$. Let $f \in C(\bar{\Omega})$ be such that $0 \notin f(\partial\Omega)$ and $f(-x) \neq \lambda\, f(x)$ on $\partial\Omega$ for all $\lambda \geq 1$. Then $d(f, \Omega, 0)$ is odd.

Proof:

Let $h(t, x) = (1 - t)\, f(x) + t\, g(x)$ where $g(x) = f(x) - f(-x)$. Suppose that there exists $(t_0, x_0) \in J \times \partial\Omega$ such that $f(x_0) = t_0 f(-x_0)$.

$t_0 = 0$ implies that $0 \in f(\partial\Omega)$.

$t_0 \neq 0$ implies that $f(-x_0) = \frac{1}{t_0} f(x_0)$ and $\frac{1}{t_0} \geq 1$, contrary to the hypothesis.

Thus $0 \notin h(J \times \partial\Omega)$ and so by (d3), $d(f, \Omega, 0) = d(g, \Omega, 0)$ and this is odd by Borsuk's theorem. ♠

We now give some applications of Borsuk's theorem. The first result is known as the Borsuk–Ulam theorem.

2.12 Corollary

Let $\Omega \subseteq \mathbb{R}^n$ be open bounded and symmetric with respect to $0 \in \Omega$. Let $f : \partial\Omega \longrightarrow \mathbb{R}^m$ be continuous with $m < n$. Then $f(x) = f(-x)$ for some $x \in \partial\Omega$.

Proof:

Suppose $g(x) = f(x) - f(-x) \neq 0$ on $\partial\Omega$ and let g be any continuous extension to $\bar{\Omega}$ of the boundary values, by theorem 1.2.15. By (d5), $d(g, \Omega, y) = d(g, \Omega, 0)$ for all $y \in B_r(0)$ where $r = \rho(g(\partial\Omega), 0)$. [**N.B.**: $B_r(0)$ is in $\mathbb{R}^n$]. By corollary 2.11, $d(g, \Omega, 0)$ is odd. Thus $d(g, \Omega, y) \neq 0$ for all $y \in B_r(0)$. And so by (d4), $y \in g(\bar{\Omega})$ for all $y \in B_r(0)$. Thus $B_r(0) \subseteq g(\bar{\Omega}) \subseteq \mathbb{R}^m$. So we arrive at the ridiculous situation where the $\mathbb{R}^n$–ball is contained in $\mathbb{R}^m$. Thus

$f(x) = f(-x)$ for some $x \in \partial\Omega$. ♠

This result has applications in meteorology. Here $n = 3$, and $\Omega \subseteq \mathbb{R}^n$ is the earth, and $\partial\Omega$ the surface of the earth. Let $f : \partial\Omega \longrightarrow \mathbb{R}^2$ be such that $f(x)$ is the weather at x (i.e. temperature and pressure, and $m = 2$). Then we can conclude, from the above result, that we can find two opposite points on the earth's surface having the same weather. The next result tells us something about the coverings of the boundary $\partial\Omega$ and it is sometimes referred to as the Lusternik–Schnirelman–Borsuk theorem. It will be required in our work later on.

2.13 Theorem

Let $\Omega \subseteq \mathbb{R}^n$ be open bounded and symmetric with respect to $0 \in \Omega$ and let $\{A_1, \ldots, A_p\}$ be coverings of $\partial\Omega$ by closed sets $A_i \subseteq \partial\Omega$ such that $A_i \cap (-A_i) = \emptyset$ for $i = 1, 2, \ldots, p$. Then $p \geq n + 1$.

Proof:

Suppose that $p \leq n$. Let

$$f_i(x) = \begin{cases} 1 & \text{on } A_i \\ -1 & \text{on } -A_i \end{cases} \qquad \text{for } i = 1, \ldots, p-1,$$

and

$$f_i(x) = 1 \quad \text{on } \bar{\Omega} \quad \text{for } i = p, \ldots, n.$$

For $i = 1, 2, \ldots, p-1$, extend f_i continuously to $\bar{\Omega}$ by theorem 1.2.15. We will show that f satisfies $f(-x) \neq \lambda f(x)$ on $\partial\Omega$ for every $\lambda \geq 0$.

[N.B.: $f(x) = (f_1(x), \ldots, f_n(x))$] Then by corollary 2.11, we would have $d(f, \Omega, 0) \neq 0$ since $0 \notin f(\partial\Omega)$. This would mean that we can find $x \in \Omega$ such that $f(x) = 0$, a contradiction to $f_n(x) = 1$.

Now, $x \in A_p$ implies that $-x \notin A_p$. Thus $-x \in A_i$ for some $i \leq p-1$, i.e. $x \in -A_i$. Thus $\partial\Omega \subseteq \bigcup_{i=1}^{p-1} [A_i \cup (-A_i)]$. Let $x \in \partial\Omega$. Then $x \in A_i$ implies $f_i(x) = 1$ and $f_i(-x) = -1$, and $x \in -A_j$ implies $f_j(x) = -1$ and $f_j(-x) = 1$. Thus $f(x)$ and $f(-x)$ do not point in the same direction in both cases. So $f(-x) \neq \lambda f(x)$ on $\partial\Omega$ for all $\lambda \geq 0$.

Thus, we must have $p \geq n+1$. ♠

This theorem tells us that we need at least $n + 1$ closed subsets A_i containing no antipodal points, if we want to cover $\partial B_r(0) \subseteq \mathbb{R}^n$ by such sets. Finally we apply Borsuk's theorem, to the problem of finding sufficient conditions for a continuous function to be open. This result is known as the *Domain–Invariance theorem* for maps which are locally one–to–one, i.e. to every x in the domain of f, there exists a neighbourhood $U(x)$ of x such that $f|_{U(x)}$ is one–to–one.

2.14 Theorem (Domain invariance theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be open and $f : \Omega \longrightarrow \mathbb{R}^n$ continuous and locally one–to–one. Then f is an open map.

Proof:

It is sufficient to show that for $x_0 \in \Omega$, there exists a ball $B_r(x_0)$ such that $f(B_r(x_0))$ contains a ball with centre $f(x_0)$.

Step 1 :

We will first assume that $x_0 = 0$ and $f(0) = 0$. Choose $r > 0$ such that $f|_{\bar{B}_r(0)}$ is one–to–one and consider $h(t, x) = f(\frac{1}{1+t}x) - f(-\frac{t}{1+t}x)$ for $(t, x) \in J \times \bar{B}_r(0)$. h is easily a continuous function of (t, x) with $h(0, x) = f(x)$ and $h(1, x) = f(\frac{1}{2}x) - f(-\frac{1}{2}x)$. So $h(0, .) = f$ and $h(1, .)$ is an odd function. We need to verify that $0 \notin h(J \times \partial B_r(0))$. Suppose $0 = h(t, x)$ for some $(t, x) \in J \times \partial B_r(0)$. Then $f(\frac{1}{1+t}x) = f(-\frac{t}{1+t}x)$. Since $\frac{1}{1+t}x$ and $-\frac{t}{1+t}x$ are both in $\bar{B}_r(0)$ and $f|_{\bar{B}_r(0)}$ is one–to one, we must have $\frac{1}{1+t}x = -\frac{t}{1+t}x$. Thus $x = 0$, a contradiction. So $0 \notin h(J \times \partial B_r(0))$ and by (d3) we obtain $d(h(0, .), B_r(0), 0) = d(h(1, .), B_r(0), 0)$, i.e. $d(f, B_r(0), 0) = d(h(1, .), B_r(0), 0)$. Since $h(1, .)$ is odd, we can apply Borsuk's theorem to get $d(h(1, .), B_r(0), 0) \neq 0$. If $s = \rho(f(\partial B_r(0)), 0)$, then for all $y \in B_s(0)$, we have $d(f, B_r(0), y) = d(f, B_r(0), 0)$, by (d5). So $d(f, B_r(0), y) \neq 0$ for all $y \in B_s(0)$. (d4) yields $y \in f(B_r(0))$ for all $y \in B_s(0)$. So $B_s(0) \subseteq f(B_r(0))$ as required.

Step 2 :

We will now show why we may take $x_0 = 0$ and $f(0) = 0$. Let $\tilde{\Omega} = \Omega - x_0$ and $\tilde{f}(x) = f(x + x_0) - f(x_0)$ for $x \in \tilde{\Omega}$. Then $0 \in \tilde{\Omega}$ and $\tilde{f}(0) = 0$. Also, $\tilde{\Omega}$ is open and $\tilde{f} : \tilde{\Omega} \longrightarrow \mathbb{R}^n$ is continuous and locally one–to–one. So by step 1, there exist $r > 0$ and $s > 0$ such that $B_s(0) \subseteq \tilde{f}(B_r(0))$. So $B_s(0) \subseteq f(B_r(0) + x_0) - f(x_0)$ and hence we have

$$B_s(f(x_0)) = B_s(0) + f(x_0) \subseteq f(B_r(0) + x_0) = f(B_r(x_0)). \qquad \spadesuit$$

The above theorem can be used to prove surjectivity results for continuous maps $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Suppose f is locally one–to–one and $|f(x)| \longrightarrow \infty$ as $|x| \longrightarrow \infty$. By

theorem 2.14, f is an open map and so $f(\mathbb{R}^n)$ is open. We will show that $f(\mathbb{R}^n)$ is closed. Let (x_n) be a sequence in $\mathbb{R}^n$ such that $f(x_n) \longrightarrow y$. Since $|f(x)| \longrightarrow \infty$ as $|x| \longrightarrow \infty$ we must have (x_n) to be bounded. Thus $\overline{\{x_n \,/\, n \in \mathbb{N}\}}$ is closed bounded and hence compact. So (x_n) has a convergent subsequence. Without loss of generality, we may assume that $x_n \longrightarrow x_0$. Thus
$f(x_n) \longrightarrow f(x_0)$ and so $y = f(x_0)$. Thus $f(\mathbb{R}^n)$ is an open and closed subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is connected, $\mathbb{R}^n$ and $\emptyset$ are its only open and closed subsets, and so $f(\mathbb{R}^n) = \mathbb{R}^n$.

We shall now state a theorem, due to Leray, on the degree of the composition of two continuous maps. We prove the product formula in infinite dimensional spaces and so we do not include the proof here. Before we state it, we need some preliminaries.
If $\Omega \subseteq \mathbb{R}^n$ is open bounded, $f : \bar{\Omega} \longrightarrow \mathbb{R}^n$ is continuous, then by (d5), $d(f, \Omega, y)$ is the same integer for every y in a connected component K of $\mathbb{R}^n \setminus f(\partial\Omega)$. We will denote this integer by $d(f, \Omega, K)$. Since $f(\partial\Omega)$ is compact we have one unbounded component K_∞ if $n > 1$ and two unbounded components if $n = 1$, and in this case K_∞ will denote the union of these two. K_∞ will not play a role later, since it contains points $y \notin f(\bar{\Omega})$ and so $d(f, \Omega, K_\infty) = 0$. We write gf to mean $gf(x) = g(f(x))$.

2.15 Theorem (Product formula)

Let $\Omega \subseteq \mathbb{R}^n$ be open bounded, $f \in C(\bar{\Omega})$, $g \in C(\mathbb{R}^n)$ and K_i the bounded connected components of $\mathbb{R}^n \setminus f(\partial\Omega)$. Suppose that $y \notin (gf)(\partial\Omega)$. Then
$d(gf, \Omega, y) = \sum_i d(f, \Omega, K_i)\, d(g, K_i, y)$ where only finitely many terms are different from zero.

Leray has shown that the product formula for the degree can be generalised to infinite dimensional spaces and it yields short and elegant proofs of some fundamental propositions of topology, for example the Jordan's–separation theorem. We can extend Jordan's curve theorem to $\mathbb{R}^n$ as follows.

2.16 Theorem

Let $\Omega_1 \subseteq \mathbb{R}^n$ and $\Omega_2 \subseteq \mathbb{R}^n$ be compact sets which are homeomorphic to each other. Then $\mathbb{R}^n \setminus \Omega_1$ and $\mathbb{R}^n \setminus \Omega_2$ have the same number of connected components.

Proof:

Let $h : \Omega_1 \longrightarrow \Omega_2$ be a homeomorphism onto Ω_2; $\tilde{h}$ a continuous extension of h to $\mathbb{R}^n$; $\tilde{h}^{-1}$ a continuous extension of h^{-1} to $\mathbb{R}^n$; K_j the bounded conponents of $\mathbb{R}^n \setminus \Omega_1$ and L_i the bounded components of $\mathbb{R}^n \setminus \Omega_2$. Since $\partial K_j \cap K_i = \emptyset$ for all i, we must have $\partial K_j \subseteq \Omega_1$. Similarly $\partial L_i \subseteq \Omega_2$.

Fix j and let G_q denote the components of $\mathbb{R}^n \setminus h(\partial K_j)$. Since $\bigcup_i L_i = \mathbb{R}^n \setminus \Omega_2 \subseteq \mathbb{R}^n \setminus h(\partial K_j) = \bigcup_q G_q$, we see that to every i there exists a q such that $L_i \subseteq G_q$ (components are maximal connected sets). In particular $L_\infty \subseteq K_\infty$.

Let $x \in \partial K_j$. Then since $\partial K_j \subseteq \Omega_1$, we have $\tilde{h}^{-1}\tilde{h}(x) = h^{-1}h(x) = id(x)$ since $h(x) \in \Omega_2$. So $\tilde{h}^{-1}\tilde{h}|_{\partial K_j} = id|_{\partial K_j}$ and so by (d6) $d(id, K_j, y) = d(\tilde{h}^{-1}\tilde{h}, K_j, y)$.

Consider any $y \in K_j$. Then $d(\tilde{h}^{-1}\tilde{h}, K_j, y) = 1$. By the product formula (2.12), $1 = d(\tilde{h}^{-1}\tilde{h}, K_j, y) = \sum_q d(\tilde{h}, K_j, G_q)\, d(\tilde{h}^{-1}, G_q, y)$. If $N_q = \{i \,/\, L_i \subseteq G_q\}$, then by

(d2), $d(\tilde{h}^{-1}, G_q, y) = \sum_{i \in N_q} d(\tilde{h}^{-1}, L_i, y)$ and $d(\tilde{h}, K_j, G_q) = d(\tilde{h}, K_j, L_i)$ for every

$i \in N_q$. Thus

$$1 = \sum_q \sum_{i \in N_q} d(\tilde{h}, K_j, L_i)\, d(\tilde{h}^{-1}, L_i, y)$$

$$= \sum_i d(\tilde{h}, K_j, L_i)\, d(\tilde{h}^{-1}, L_i, K_j) \tag{1}$$

since $y \in K_j \subseteq \mathbb{R}^n \setminus h^{-1}(\Omega_2) \subseteq \mathbb{R}^n \setminus h^{-1}(\partial L_i)$.

We may repeat the same argument with fixed L_i instead of K_j, to obtain

$$\begin{aligned} 1 &= \sum_j d(\tilde{h}^{-1}, L_i, K_j)\, d(\tilde{h}, K_j, L_i) \\ &= \sum_j d(\tilde{h}, K_j, L_i)\, d(\tilde{h}^{-1}, L_i, K_j). \end{aligned} \tag{2}$$

If there are only m components L_i, then (1) and summation over i in (2) yields

$$\begin{aligned} m &= \sum_{i=1}^{m} 1 = \sum_j \sum_{i=1}^{m} d(\tilde{h}, K_j, L_i)\, d(\tilde{h}^{-1}, L_i, K_j) \\ &= \sum_j 1 . \end{aligned}$$

Therefore we must also have m components K_j, and conversely. Thus $\mathbb{R}^n \setminus \Omega_1$ and $\mathbb{R}^n \setminus \Omega_2$ either have the same finite number of components or they both have countably many. ♠

We conclude this chapter with some extensions to earlier results and some final remarks.

Degree on unbounded sets

Up to this point we assumed that the open sets $\Omega \subseteq \mathbb{R}^n$, used in the degree , were also bounded, so as to ensure that $f^{-1}(y)$ was compact. Now suppose $\Omega \subseteq \mathbb{R}^n$ is open (not necessarily bounded), $f : \bar{\Omega} \longrightarrow \mathbb{R}^n$ continuous and $y \in \mathbb{R}^n \setminus f(\partial\Omega)$. Also assume that $\sup_{\bar{\Omega}} |x - f(x)| < \infty$. Let $x \in f^{-1}(y)$ and let $\sup_{\bar{\Omega}} |x - f(x)| = M$. Then $f(x) = y$ and so $|x| \le |x - f(x)| + |f(x)| \le M + |y|$. Thus $f^{-1}(y)$ is a closed bounded set and hence is compact. Let Ω_0 be any open bounded set such that $f^{-1}(y) \subseteq \Omega_0$. Thus $d(f, \Omega \cap \Omega_0, y)$ is defined, where d represents the Brouwer degree. Now let Ω_1 be another open bounded set such that $f^{-1}(y) \subseteq \Omega_1$. We need to show that $d(f, \Omega \cap \Omega_0, y) = d(f, \Omega \cap \Omega_1, y)$. Now $\Omega_0 \cap \Omega_1$ is an open bounded set such that $f^{-1}(y) \subseteq \Omega_0 \cap \Omega_1$. Thus

$y \notin f(\overline{\Omega \cap \Omega_i} \setminus \Omega \cap (\Omega_0 \cap \Omega_1))$ for $i = 0, 1$. So by (d7), we have

$d(f, \Omega \cap \Omega_i, y) = d(f, \Omega \cap (\Omega_0 \cap \Omega_1), y)$ for $i = 0, 1$. Thus

$d(f, \Omega \cap \Omega_0, y) = d(f, \Omega \cap \Omega_1, y)$. This enables us to make the following definition.

2.17 Definition

For $\Omega \subseteq \mathbb{R}^n$ open, let $\tilde{C}(\bar{\Omega})$ be the collection of all $f \in C(\bar{\Omega})$ satisfying $\sup_{\Omega} |x - f(x)| < \infty$. Let $\tilde{\mathscr{M}} = \{(f, \Omega, y) \,/\, \Omega \subseteq \mathbb{R}^n \text{ open}, f \in \tilde{C}(\bar{\Omega}), y \notin f(\partial\Omega)\}$. Then we define $\tilde{d} : \tilde{\mathscr{M}} \longrightarrow \mathbb{Z}$ by $\tilde{d}(f, \Omega, y) = d(f, \Omega \cap \Omega_0, y)$ where Ω_0 is any open bounded set containing $f^{-1}(y)$ and d is the Brouwer degree.

If $\Omega \subseteq \mathbb{R}$ is open and bounded, then it is easy to see that we obtain the Brouwer degree. We will now show that we obtain ($\tilde{d}$1)–($\tilde{d}$3).

($\tilde{d}$1) *$\tilde{d}(id, \Omega, y) = 1$ if $y \in \Omega$:*

Let Ω_0 be an open bounded set containing $id^{-1}(y) = \{y\}$. Then $y \in \Omega \cap \Omega_0$ and so by (d1), $\tilde{d}(id, \Omega, y) = d(id, \Omega \cap \Omega_0, y) = 1$.

($\tilde{d}$2) *Let Ω_1 and Ω_2 be disjoint open subsets of Ω such that $y \notin f(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. Then $\tilde{d}(f,\Omega, y) = \tilde{d}(f, \Omega_1, y) + \tilde{d}(f, \Omega_2, y)$:*

Let Ω_0 be any open bounded set such that $f^{-1}(y) \subseteq \Omega_0$. By definition,

$\tilde{d}(f, \Omega, y) = d(f, \Omega \cap \Omega_0, y) = d(f, \Omega_1 \cap \Omega_0, y) + d(f, \Omega_2 \cap \Omega_0, y)$ by (d2) since $\Omega_1 \cap \Omega_0$ and $\Omega_2 \cap \Omega_0$ are disjoint open subsets of $\Omega \cap \Omega_0$ and

$\overline{\Omega \cap \Omega_0} \setminus (\Omega_1 \cap \Omega_0) \cup (\Omega_2 \cap \Omega_0) \subseteq \bar{\Omega} \setminus \Omega_1 \cup \Omega_2$.

So $y \notin f(\overline{\Omega \cap \Omega_0} \setminus (\Omega_1 \cap \Omega_0) \cup (\Omega_2 \cap \Omega_0))$. Again by definition

$d(f, \Omega_i \cap \Omega_0, y) = \tilde{d}(f, \Omega_i, y)$, for $i = 1, 2$, and we are done.

($\tilde{d}3$) *Let* $h : J \times \Omega \longrightarrow \mathbb{R}$ *and* $y : J \longrightarrow \mathbb{R}$ *be continuous,* $\sup \{|x - h(t, x)| \ / \ (t, x) \in J \times \bar{\Omega}\} < \infty$ *and* $y(t) \notin h(t, \partial\Omega)$ *on* J. *Then* $\tilde{d}\,(h(t, .), \Omega, y(t))$ *is constant on* J:

Let $M = \sup \{|x - h(t, x)| \ / \ (t, x) \in J \times \bar{\Omega}\}$ and $M' = \max \{|y(t)| \ / \ t \in J\}$. If $x \in \bigcup_J (h(t, .))^{-1}(y(t)) = A$, then $h(t, x) = y(t)$ for some $t \in J$. Then $|x| \leq |x - h(t, x)| + |h(t, x)| \leq M + M'$, and so A is a bounded set. Let Ω_0 be any open bounded set containing A. Then $(h(t, .))^{-1}(y(t)) \subseteq A \subseteq \Omega_0$ for all $t \in J$. Thus by definition $\tilde{d}(h(t, .), \Omega, y(t)) = d(h(t, .), \Omega \cap \Omega_0, y(t))$ and this is independent of t by (d3).

Thus $\tilde{d}$ satisfies ($\tilde{d}1$)–($\tilde{d}3$). We will denote this by d and will also call it the Brouwer degree.

Degree in Finite Dimensional Topological Vector Spaces

Up to this point we used the standard basis $\{e^1, e^2, \ldots, e^n\}$ in $\mathbb{R}^n$ to define the degree [N.B. : $J_f(x) = \det f'(x)$ is dependent on the basis]. Let $\{\tilde{e}^1, \tilde{e}^2, \ldots, \tilde{e}^n\}$ be another basis for $\mathbb{R}^n$. Then there exists a matrix A, $\det A \neq 0$ such that $\tilde{x} = Ax$, $\tilde{\Omega} = A\Omega$, $g(\tilde{x}) = AfA^{-1}(\tilde{x})$, $\tilde{x} \in \tilde{\Omega}$. We want to show that $d(f, \Omega, y) = d(g, \tilde{\Omega}, Ay)$.

First suppose $f \in C^1(\bar{\Omega})$ and y is a regular value of f. Then for $\tilde{x} = Ax$,

$$\begin{aligned}
J_g(\tilde{x}) &= \det g'(\tilde{x}) \\
&= \det (AfA^{-1})'(\tilde{x}) \\
&= \det (Af'(A^{-1}\tilde{x})A^{-1}) \\
&= \det A \det f'(A^{-1}\tilde{x}) \det A^{-1} \\
&= \det f'(x) \\
&= J_f(x).
\end{aligned}$$

We need to check that Ay is a regular value of g. Let $g(\tilde{x}) = Ay$, and $\tilde{x} = Ax$. Then $AfA^{-1}(Ax) = Ay$ and so $f(x) = y$. Since y is a regular value of f, $J_f(x) \neq 0$ and so $J_g(\tilde{x}) \neq 0$, proving that Ay is a regular value of g.

Let $\tilde{x} = Ax$. Then

$$\tilde{x} \in g^{-1}(Ay) = (AfA^{-1})^{-1}(Ay) = (Af^{-1}A^{-1})(Ay) = Af^{-1}(y)$$

$$\Leftrightarrow \quad Ax \in Af^{-1}(y)$$

$$\Leftrightarrow \quad x \in f^{-1}(y)$$

$$\Leftrightarrow \quad A^{-1}\tilde{x} \in f^{-1}(y).$$

Thus,
$$\begin{aligned} d(f, \Omega, y) &= \sum_{x \in f^{-1}(y)} \operatorname{sgn} J_f(x) \\ &= \sum_{\tilde{x} \in g^{-1}(Ay)} \operatorname{sgn} J_g(\tilde{x}) \\ &= d(g, \tilde{\Omega}, Ay). \end{aligned}$$

Now take $f \in C(\bar{\Omega})$, $y \in \mathbb{R}^n \setminus f(\partial\Omega)$. Choose $f_1 \in \bar{C}^2(\bar{\Omega})$ such that $|f - f_1|_0 < \rho(y, f(\partial\Omega))$. Then

$$\begin{aligned} |g - Af_1A^{-1}|_0 &= |AfA^{-1} - Af_1A^{-1}|_0 \\ &= \sup\{|AfA^{-1}(\tilde{x}) - Af_1A^{-1}(\tilde{x})| \ / \ \tilde{x} \in \bar{\tilde{\Omega}}\} \\ &= \det A \sup\{|fA^{-1}(\tilde{x}) - f_1A^{-1}(\tilde{x})| \ / \ \tilde{x} = Ax, x \in \bar{\Omega}\} \\ &= \det A \sup\{|f(x) - f_1(x)| / x \in \bar{\Omega}\} \\ &= \det A \ |f - f_1|_0 \\ &< \det A \ \rho(y, f(\partial\Omega)), \end{aligned}$$

and

$$\begin{aligned} \rho(y, f(\partial\Omega)) &= \inf\{|y - f(x)| \ / \ x \in \partial\Omega\} \\ &= \inf\{|A^{-1}(Ay - Af(x))| \ / \ x = A^{-1}\tilde{x}, \tilde{x} \in \partial\tilde{\Omega}\} \\ &= \det A^{-1} \inf\{|Ay - AfA^{-1}(\tilde{x})| \ / \ \tilde{x} \in \partial\tilde{\Omega}\} \end{aligned}$$

$$= \quad \det A^{-1}\rho(Ay, g(\partial\tilde{\Omega})).$$

So $|g - Af_1A^{-1}|_0 \quad < \det A \ \det A^{-1}\rho(Ay, g(\partial\tilde{\Omega}))$

$$= \rho(Ay, g(\partial\tilde{\Omega})).$$

Thus $d(f, \Omega, y) = d(f_1, \Omega, y)$ and $d(g, \tilde{\Omega}, Ay) = d(Af_1A^{-1}, \tilde{\Omega}, Ay)$.

We can now reduce y to a regular value of f_1, and by what was done earlier, Ay can be reduced to a regular value of Af_1A^{-1}. Thus $d(f, \Omega, y) = d(g, \tilde{\Omega}, Ay)$ where $g = AfA^{-1}$, $\tilde{\Omega} = A\Omega$, $\tilde{x} = Ax$, $\det A \neq 0$. [**N.B.**: $\tilde{\Omega}$ is the representation of Ω, where Ω is given by the standard basis, using the new basis. Also, Ω need not be bounded.] Thus we have shown that the degree, defined on $\mathbb{R}^n$, is independent of our choice of basis for $\mathbb{R}^n$.

Our degree, defined up to this point, is only defined on $\mathbb{R}^n$. We would like to define a degree on X, where X is an n–dimensional Hausdorff real topological vector space. (i.e.: a real vector space where addition and scalar multiplication are continuous.)

Now X is homeomorphic to $\mathbb{R}^n$ and may be regarded as normed.

In fact if $\{x^1, ..., x^n\}$ is a basis for X, $h : X \longrightarrow \mathbb{R}^n$ defined by $h(\sum_{i=1}^{n}\alpha_i(x)x^i) = \sum_{i=1}^{n}\alpha_i(x)e^i$ is a homeomorphism (see Schaefer [29]) and $|h(x)|$ may be taken as $|x|$.

Let $\Omega \subseteq X$ be open, $F : \bar{\Omega} \longrightarrow X$ continuous, $(id - F)(\bar{\Omega})$ relatively compact and $y \in X \setminus F(\partial\Omega)$. We want to define a degree for the triplet (F, Ω, y). Let $f = hFh^{-1}$. We want to show that $(f, h(\Omega), h(y))$ is an admissible Brouwer triplet:

(i) Since h is a homeomorphism, $h(\Omega) \subseteq \mathbb{R}^n$ is open.

(ii) h a homeomorphism implies $h(\partial\Omega) = \partial(h(\Omega))$.

(iii) $f(\partial(h(\Omega))) = (hFh^{-1})(h(\partial\Omega)) = hF(\partial\Omega)$.

So $y \notin F(\partial\Omega) \quad \Leftrightarrow \quad h(y) \notin hF(\partial\Omega)$. Thus $h(y) \notin f(\partial(h(\Omega)))$.

(iv) f is continuous.

(v) $(id - f)(h(\bar{\Omega}))$ is relatively compact, hence bounded.

Hence $(f, h(\Omega), h(y))$ is a Brouwer triplet.

If $\{\tilde{x}^1, \ldots, \tilde{x}^n\}$ is another basis for X, then we obtain a corresponding homeomorphism $\tilde{h}$. There exists a matrix A, det A $\neq 0$ such that $\tilde{h} = Ah$. Then

$\tilde{h}(\Omega) = A(h(\Omega))$; $f = \tilde{h}F\tilde{h}^{-1} = AhF(Ah)^{-1} = Af_1A^{-1}$.

Since the degree in $\mathbb{R}^n$ is independent of the choice of basis,

$d(f, h(\Omega), h(y)) = d(f, \tilde{h}(\Omega), \tilde{h}(y))$. Thus the degree defined by

$d'(F, \Omega, y) = d(hFh^{-1}, h(\Omega), h(y))$ is well–defined.

As before, we can show that d' satisfies (d′1)–(d′3). To show that the degree is unique, we define $d_0(f, \Omega, y) = d'(h^{-1}fh, h^{-1}(\Omega), h^{-1}(y))$ for (f, Ω, y) a Brouwer triplet. Easily d_0 satisfies ($d_0$1)–($d_0$3) and so it must be the Brouwer degree (the Brouwer degree is unique, satisfying (d1)–(d3)). So if (F, Ω, y) is the triplet we are considering, then $(hFh^{-1}, h(\Omega), h(y))$ is a Brouwer triplet and so

$d(hFh^{-1}, h(\Omega), h(y)) = d_0(hFh^{-1}, h(\Omega), h(y)) = d'(F, \Omega, y)$.

Formally we have the following definition.

2.18 Definition

Let X be a real n–dimensional Hausdorff topological vector space and

$\mathcal{M} = \{(F, \Omega, y) \,/\, \Omega \subseteq X$ open, $F : \bar{\Omega} \longrightarrow X$ continuous, $\overline{F(\bar{\Omega})}$ compact and $y \in X \setminus F(\partial\Omega)\}$. Then we define $d(F, \Omega, y) = d_B(hFh^{-1}, h(\Omega), h(y))$, where $h : X \longrightarrow \mathbb{R}^n$ is the linear homeomorphism defined by $h(x^i) = e^i$, with $\{x^1, \ldots, x^n\}$ a basis for X and $\{e^1, \ldots, e^n\}$ the standard basis of $\mathbb{R}^n$ and d_B is the Brouwer degree.

We denote this degree by d and again call it the Brouwer degree.

A Relation Between the Degrees for Spaces of Different Dimension

Suppose $\Omega \subset \mathbb{R}^n$ is open bounded, $f : \Omega \longrightarrow \mathbb{R}^m$ with $m < n$ is continuous and

$y \in \mathbb{R}^m \setminus g(\partial\Omega)$ where $g = id - f$. Then $g(x) = y$ with $x \in \Omega$ implies $x = y + f(x) \in \mathbb{R}^m$. So all solutions of $g(x) = y$ are already in $\Omega \cap \mathbb{R}^m$. Thus we may expect $d(g, \Omega, y)$ to be computed by $d(g|_{\overline{\Omega \cap \mathbb{R}^m}}, \Omega \cap \mathbb{R}^m, y)$. We prove this in the following theorem.

2.19 Theorem

Let X_n be a real Hausdorff topological vector space with $\dim X_n = n$, X_m a subspace with $\dim X_m = m < n$, $\Omega \subseteq X_n$ open bounded, $f : \bar{\Omega} \longrightarrow X_m$ continuous, $g(\bar{\Omega})$ relatively compact and $y \in X_m \setminus g(\partial\Omega)$ where $g = id - f$. Then

$d(g, \Omega, y) = d(g|_{\overline{\Omega \cap X_m}}, \Omega \cap X_m, y)$.

Proof:

By definition 2.18, assume that $X_n = \mathbb{R}^n$ and

$X_m = \mathbb{R}^m = \{x \in \mathbb{R}^n \,/\, x_{m+1} = \dots = x_n = 0\}$. Since the reduction to the regular case presents no difficulty, we may assume that $f \in \bar{C}^1(\Omega)$ and $y \notin g(S_g)$. We need to verify that $y \notin g_m(S_{g_m})$ $(g_m = g|_{\bar{\Omega}_m}$, where $\Omega_m = \Omega \cap X_m)$.

Let $y = g_m(x) = g(x)$. Then $J_g(x) = \det g'(x) = \det \left[\begin{array}{c:c} I_m - \partial_j f_i(x) & -\partial_j f_i(x) \\ \hdashline (0) & I_{n-m} \end{array}\right]$

and evaluating by the last n–m rows, we obtain $J_g(x) = J_{g_m}(x)$. But $J_g(x) \neq 0$. So $J_{g_m}(x) \neq 0$ and hence $y \notin g_m(S_{g_m})$. By definition,

$d(g, \Omega, y) = \sum_{x \in g^{-1}(y)} \operatorname{sgn} J_g(x)$ and

$d(g_m, \Omega_m, y) = \sum_{x \in g_m^{-1}(y)} \operatorname{sgn} J_{g_m}(x)$.

Also $x \in g^{-1}(y) \Leftrightarrow x = y + f(x) \in \mathbb{R}^m \Leftrightarrow x \in g_m^{-1}(y)$.

So $d(g, \Omega, y) = \sum_{x \in g^{-1}(y)} \operatorname{sgn} J_g(x) = \sum_{x \in g_m^{-1}(y)} \operatorname{sgn} J_{g_m}(x) = d(g_m, \Omega_m, y)$

as required. ♠

CHAPTER 3

COMPACT MAPS

In this chapter we consider an extension of the Brouwer degree to compact perturbations of the identity.

Preliminaries

3.1 Definitions

Let X and Y be Banach spaces, $\Omega \subseteq X$ and $F : \Omega \longrightarrow Y$.

(a) F is said to be *compact* if it is continuous and $F(\Omega)$ is relatively compact, i.e. $\overline{F(\Omega)}$ is compact. We will let $\mathscr{K}(\Omega, Y)$ denote the class of all compact maps and write $\mathscr{K}(\Omega)$ instead of $\mathscr{K}(\Omega, X)$.

(b) F is said to be *completely continuous* if it is continuous and maps bounded subsets of Ω into relatively compact subsets of Y.

(c) F is said to be *finite dimensional* if $F(\Omega)$ is contained in a finite dimensional subspace of Y.

The class of all finite dimensional, compact maps will be denoted by $\mathscr{F}(\Omega, Y)$ and again we write $\mathscr{F}(\Omega)$ instead of $\mathscr{F}(\Omega, X)$.

In the linear case, a map that takes bounded sets into relatively compact sets is automatically continuous and a finite dimensional map is automatically compact.
But, consider the following example.

3.2 Example

Let $\dim X = \infty$. By theorem 1.2.7, there exists a sequence $(x_n) \subseteq \partial B_1(0)$ such that $|x_n - x_m| \geq 1$ for $n \neq m$. Let

$$\varphi(x) = \begin{cases} k(1-2|x-x_k|) & \text{if } x \in \bar{B}_{1/2}(x_k) \\ 0 & \text{otherwise} \end{cases}.$$

The functional φ is continuous and unbounded since $\varphi(x_k) = k$ for each $k \in \mathbb{N}$. If $Fx = \varphi(x)\, x_1$, then F is continuous and finite dimensional. Now $(x_k) \subseteq \bar{B}_2(0)$ and $Fx_k = kx_1$ for each $k \in \mathbb{N}$. Hence $F(\bar{B}_2(0))$ is unbounded and thus not relatively compact. So $F : \bar{B}_2(0) \longrightarrow X$ is continuous and finite dimensional, but not compact. ♠

3.3 Definition

Let $\Omega \subseteq X$ be closed and bounded. Then $F : \Omega \longrightarrow Y$ is said to be *proper* if $F^{-1}(K)$ is compact in X whenever K is compact in Y.

3.4 Theorem

Let $\Omega \subseteq X$ be closed, bounded and $F : \Omega \longrightarrow Y$ continuous and proper. Then F is also closed.

Proof:

Let A be closed in Ω. To show F(A) closed, we let (x_n) be a sequence in A such that $Fx_n \longrightarrow y$ and we show that $y \in F(A)$. Using the third equivalent property for compactness, we see that $\{Fx_n \,/\, n \in \mathbb{N}\} \cup \{y\}$ is compact. Since F is proper, $F^{-1}(\{F(x_n) \,/\, n \in \mathbb{N}\} \cup \{y\})$ is also compact and (x_n) is contained in it. Thus (x_n) has a convergent subsequence, say $x_{n_k} \longrightarrow x_0$. But A is closed, so $x_0 \in$ A and F continuous gives $Fx_{n_k} \longrightarrow Fx_0$. But $Fx_{n_k} \longrightarrow y$. Thus $y = Fx_0 \in F(A)$, proving that $F(A)$ is closed. ♠

The next result is very useful since it approximates compact maps by finite dimensional maps in some sense. It is absolutely essential in order to define a degree for compact perturbations of the identity.

3.5 Theorem

Let X and Y be Banach spaces and $B \subseteq X$ be closed bounded. Then

(a) $\mathscr{F}(B, Y)$ is dense in $\mathscr{K}(B, Y)$, i.e. for $F \in \mathscr{K}(B, Y)$ and $\epsilon > 0$, there exists $F_\epsilon \in \mathscr{F}(B, Y)$ such that $\sup_B |Fx - F_\epsilon x| < \epsilon$.

(b) If $F \in \mathscr{K}(B)$, then I–F is proper.

Proof:

(a) Let $F \in \mathscr{K}(B, Y)$ and $\epsilon > 0$. Since $\overline{F(B)}$ is compact, there exists $y_1, \ldots, y_p \in Y$ such that $\overline{F(B)} \subseteq \bigcup_{i=1}^{p} B_\epsilon(y_i)$. Define

$$\varphi_i(y) = \max\{0, \epsilon - |y - y_i|\}$$

and

$$\psi_i(y) = \frac{\varphi_i(y)}{\sum_{j=1}^{p} \varphi_j(y)}.$$

Now φ_i is continuous. For $y \in \overline{F(B)}$, we must have $y \in B_\epsilon(y_i)$ for some i, and hence $\varphi_i(y) > 0$ and $\sum_{j=1}^{p} \varphi_j(y) > 0$. Thus ψ_i is also continuous.

Define $F_\epsilon x = \sum_{i=1}^{p} \psi_i(Fx)\, y_i$, for $x \in B$. Then F_ϵ is continuous and finite dimensional.

$$\begin{aligned}\gamma(F_\epsilon B) &= \gamma(\sum_{i=1}^{p} \psi_i(FB)\, y_i) \\ &\leq \sum_{i=1}^{p} \gamma(\psi_i(FB)\, y_i).\end{aligned}$$

Now $\sum_{i=1}^{p} \psi_i(y) = 1$, and $\psi_i(y) \geq 0$ for all $y \in \overline{F(B)}$.

So $\psi_i(\overline{F(B)}) \subseteq [0, 1]$ and $[0, 1]$ is compact. Therefore $\gamma(\psi_i(\overline{F(B)})) = 0$. But

$\gamma(\psi_i(\overline{F(B)})\, y_i) = \gamma(\psi_i(\overline{F(B)})\, |y_i| = 0$.

Hence $\gamma(F_\epsilon B) = 0$. So $F_\epsilon \in \mathscr{F}(B, Y)$.

Take $x \in B$. Then

$$\begin{aligned}|F_\epsilon x - Fx| &= |\sum_{i=1}^{p} \psi_i(Fx)\, y_i - \sum_{i=1}^{p} \psi_i(Fx)\, Fx\,| \\ &\leq \sum_{i=1}^{p} \psi_i(Fx)\, |\, y_i - Fx\,|.\end{aligned}$$

If $\psi_i(Fx) > 0$, then $\varphi_i(Fx) > 0$. So $|Fx - y_i| < \epsilon$.

Thus $|F_\epsilon x - Fx| \leq \sum_{i=1}^{p} \psi_i(Fx)\, \epsilon = \epsilon$ and $\sup_{x \in B} |F_\epsilon x - Fx| \leq \epsilon$.

(b) Let $F \in \mathscr{K}(B)$ and $K \subseteq X$ compact. Must show that $A = (I-F)^{-1}(K)$ is compact. Since $I-F$ is continuous and K closed, A must also be closed.

Now $K = (I-F)(A)$. So $A \subseteq F(A) + K$ and

$\gamma(A) \leq \gamma(F(A)) + \gamma(K) = \gamma(F(A)) \leq \gamma(F(B)) = 0$. So $\gamma(A) = 0$ proving that A is relatively compact. Since A is closed, A is compact and hence $I-F$ is proper. ♠

3.6 Theorem

Let X, Y be Banach spaces, $\Omega \subseteq X$ open, $F \in \mathscr{K}(\Omega, Y)$ and F differentiable at x_0. Then $F'(x_0)$ is completely continuous.

Proof:

Let $B \subseteq \Omega$ be bounded. Must show that $F'(x_0)(B)$ is relatively compact. B bounded means that there exists $M \in \mathbb{R}$ such that $|x| \leq M$ for all $x \in B$. Suppose we have already shown that $F'(x_0)(B_1(0))$ is relatively compact.

Now $x \in B$ implies that $|\frac{1}{M}x| < 1$, so $\frac{1}{M}x \in B_1(0)$. Therefore $B \subseteq M\, B_1(0)$. So

$$\begin{aligned}
\gamma((F'(x_0))(B)) &\leq \gamma((F'(x_0))(M\, B_1(0))) \\
&= \gamma(M\, F'(x_0)(B_1(0))) \quad \text{(since } F'(x_0) \text{ is linear)} \\
&= M\, \gamma(F'(x_0)(B_1(0))) \\
&= 0.
\end{aligned}$$

Thus $F'(x_0)(B)$ is relatively compact.

Now to show that $F'(x_0)(B_1(0))$ is relatively compact. Since F is differentiable at x_0,

$F(x_0+h) = Fx_0 + F'(x_0)\, h + \omega(x_0;\, h)$ where $\dfrac{|\omega(x_0;\, h)|}{|h|} \longrightarrow 0$ as $|h| \longrightarrow 0$.

Given $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |h| < \delta$ implies that $\dfrac{|\omega(x_0;\, h)|}{|h|} < \epsilon$

and so $|\omega(x_0;\, h)| < \epsilon\, |h| < \epsilon\, \delta$. Now $F'(x_0)\, h = F(x_0 + h) - Fx_0 - \omega(x_0;\, h)$.

For $|h| < \delta$, $|-\omega(x_0;\, h)| = |\omega(x_0;\, h)| < \epsilon\, \delta$. Therefore

$-\omega(x_0;\, h) \in B_{\epsilon\delta}(0) = \delta\, B_\epsilon(0)$. Now $F'(x_0)(B_\delta(0)) \subseteq F(x_0 + B_\delta(0)) - Fx_0 + \delta\, B_\epsilon(0)$,

and so $F'(x_0)(\delta\, B_1(0)) \subseteq F(B_\delta(x_0)) - Fx_0 + \delta\, B_\epsilon(0)$. Since $F'(x_0)$ is linear,

$\delta\, F'(x_0)(B_1(0)) \subseteq F(B_\delta(x_0)) - Fx_0 + \delta\, B_\epsilon(0)$ and so

$\delta\, \gamma(F'(x_0)(B_1(0))) \leq \gamma(F(B_\delta(x_0))) + \gamma(\{-Fx_0\}) + \delta\, \gamma(B_\epsilon(0))$. Hence

$\gamma(F'(x_0)(B_1(0))) \leq 2\epsilon$ for all $\epsilon > 0$. Thus $\gamma(F'(x_0)(B_1(0))) = 0$ and so

$F'(x_0)(B_1(0))$ is relatively compact. ♠

The following theorem is an easy consequence of theorem 1.2.15 , but we state it nevertheless.

3.7 Theorem

Let X, Y be Banach spaces, $A \subseteq X$ closed bounded and $F \in \mathscr{K}(A, Y)$. Then F has an extension $\tilde{F} \in \mathscr{K}(X, Y)$ and $\tilde{F}(X) \subseteq \operatorname{conv}(F(A))$.

Proof:

By theorem 1.2 15 , there exists a continuous extension $\tilde{F}$ with $\tilde{F}(X) \subseteq \text{conv}\,(F(A))$.

Then $\alpha(\tilde{F}(X)) \leq \alpha(\text{conv}\,(F(A))) = \alpha(F(A)) = 0$. ♠

The Degree

We are now ready to define the Leray–Schauder degree, a $\mathbb{Z}$–valued function D defined on the triplets $(I - F, \Omega, y)$ where $\Omega \subseteq X$ is open bounded, $F : \bar{\Omega} \longrightarrow X$ is compact and $y \in X \setminus (I - F)(\partial\Omega)$, and satisfying the following conditions :–

(D1) $D(I, \Omega, y) = 1$ if $y \in \Omega$.

(D2) If Ω_1, Ω_2 are disjoint open subsets of Ω such that $y \in X \setminus (I - F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$, then $D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$.

(D3) If $H : J \times \Omega \longrightarrow X$ and $y : J \longrightarrow X$ are continuous, H compact and $y(t) \in X \setminus (I - H(t, .))(\partial\Omega)$, then $D(I - H(t, .), \Omega, y(t))$ is independent of t.

The above triplets will be referred to as admissible LS–triplets.

We follow the following steps.

Step 1 : We show that if any $\mathbb{Z}$–valued function D defined on the collection of admissible triplets satisfies (D1) – (D3), it is unique.

Step 2 : We define a function D and show that it satisfies (D1)–(D3).

Uniqueness :

Let $\Omega \subseteq X$ be open bounded, $F \in \mathscr{K}(\bar{\Omega})$, and $y \in X \setminus (I - F)(\partial\Omega)$.

By theorem 3.5(b), $I - F$ is proper and it is also continuous, and so by theorem 3.4 it must be closed. Therefore $(I - F)(\partial\Omega)$ is closed with $y \notin (I - F)(\partial\Omega)$. Hence $\rho = \rho(y, (I - F)(\partial\Omega)) > 0$. By theorem 3.5(a), there exists $F_1 \in \mathscr{F}(\bar{\Omega})$ such that

$|F - F_1|_0 < \rho$, i.e. $\sup_{x \in \Omega} |Fx - F_1 x| < \rho$. Define $H : J \times \bar{\Omega} \longrightarrow X$ by

$H(t, x) = t\, F_1(x) + (1-t)\, Fx = Fx + t\, (F_1 x - Fx)$ for $(t, x) \in J \times \bar{\Omega}$.

F_1 and F continuous implies that H is continuous.

For each $(t, x) \in J \times \bar{\Omega}$, $H(t, x) \in \text{conv}\,(F_1(\bar{\Omega}) \cup F(\bar{\Omega}))$ and so

$H(J \times \bar{\Omega}) \subseteq \text{conv}\,(F_1(\bar{\Omega}) \cup F(\bar{\Omega}))$. Hence

$$\begin{aligned}
\gamma(H(J \times \bar{\Omega})) &\leq \gamma(\text{conv}\,(F_1(\bar{\Omega}) \cup F(\bar{\Omega}))) \\
&= \gamma(F_1(\bar{\Omega}) \cup F(\bar{\Omega})) \\
&= \max \{ \gamma(F_1(\bar{\Omega})), \gamma(F(\bar{\Omega})) \} \\
&= 0,
\end{aligned}$$

since F_1 and F are both compact.

Therefore H is compact.

Suppose $y \in (I - H(t, .))(\partial\Omega)$ for some $t \in J$. Then $y = (I - H(t, .))(x)$ for some $x \in \partial\Omega$. So $y = x - H(t, x) = x - Fx - t\, (F_1 x - Fx)$. Thus

$y - (I - F)x = -t\, (F_1 x - Fx)$, and so

$|y - (I - F)x| = |t\, (F_1 x - Fx)| \leq |F_1 x - Fx| < \rho$.

But $|y - (I - F)x| \geq \rho(y, (I - F)(\partial\Omega)) = \rho$, a contradiction.

Hence $y \notin (I - H(t, .))(\partial\Omega)$ on J.

The hypotheses of (D3) are thus satisfied, proving that

$$D(I - F, \Omega, y) = D(I - F_1, \Omega, y)\ . \tag{1}$$

Since F_1 is finite dimensional, we can find a finite dimensional subspace X_1 of X that contains $F_1(\bar{\Omega})$ and y (for example : $X_1 = \text{span}\,(F_1(\bar{\Omega}) \cup \{y\})$).

Suppose $\Omega_1 = \Omega \cap X_1 \neq \emptyset$. By theorem 1.2.15 , we can find a continuous extension of $F_1|_{\bar{\Omega} \cap X_1}$ to X_1, say $\tilde{F}_1 : X_1 \longrightarrow X_1$. By theorem 3.7 , $\tilde{F}_1$ is also compact. Since X_1 is closed in X, there exists a continuous projection P_1 from X onto X_1. Then $X = X_1 \oplus X_2$, where $X_2 = P_2(X)$, $P_2 = I - P_1$, and X_2 is closed since P_2 is continuous.

So the map $H : J \times \bar{\Omega} \longrightarrow X_1$ defined by $H(t, x) = t\, F_1(x) + (1 - t)\, \tilde{F}_1 P_1 x$, for

$(t, x) \in J \times \bar{\Omega}$, is compact. We must now show that $y(t) \equiv y \notin (I - H(t, .))(\partial\Omega)$ for $t \in J$. Let $y = (I - H(t, .))(x)$ for $x \in \bar{\Omega}$ and $t \in J$. Then $x = y + H(t, x) \in X_1$. So $P_1 x = x$ and $\tilde{F}_1 P_1 x = \tilde{F}_1 x = F_1 x$. So
$y = x - t\, F_1 x - (1 - t)\, \tilde{F}_1 P_1 x = x - t\, F_1 x - (1 - t)\, F_1 x = x - F_1 x = (I - F_1)x$. Since $y \notin (I - F_1)(\partial\Omega)$, we must have $x \notin \partial\Omega$. Therefore $y \notin (I - H(t, .))(\partial\Omega)$ for $t \in J$. Thus by (D3),

$$D(I - F_1, \Omega, y) = D(I - \tilde{F}_1 P_1, \Omega, y). \tag{2}$$

Now consider $\Omega' = \Omega_1 + B_1^2(0)$, where $B_1^2(0)$ is the unit ball of X_2. Then $\Omega_1 \subseteq \Omega'$, and $\Omega_1 \subseteq \Omega$. So $\Omega_1 \subseteq \Omega \cap \Omega'$. If $x \in \bar{\Omega}$ with $(I - \tilde{F}_1 P_1)x = y$, then $x = y + \tilde{F}_1 P_1 x \in X_1$. So $x \in \bar{\Omega} \cap X_1$ and $P_1 x = x$. Hence $y = (I - \tilde{F}_1 P_1)x = (I - \tilde{F}_1)x = (I - F_1)x$. Thus $x \in \Omega$ and hence $x \in \Omega_1 \subseteq \Omega \cap \Omega'$, proving that $y \notin (I - \tilde{F}_1 P_1)(\bar{\Omega} \setminus \Omega \cap \Omega')$ and since $\Omega \cap \Omega'$ is an open subset of Ω, we have by (D2),

$$D(I - \tilde{F}_1 P_1, \Omega, y) = D(I - \tilde{F}_1 P_1, \Omega \cap \Omega', y). \tag{3}$$

We now show that $y \notin (I - \tilde{F}_1 P_1)(\bar{\Omega}' \setminus \Omega \cap \Omega')$. Suppose $y = (I - \tilde{F}_1 P_1)(x)$ for $x \in \bar{\Omega}'$. Since $x \in \bar{\Omega}'$, there exists a sequence (y_n) in Ω' such that $y_n \longrightarrow x$. Let $y_n = x_n + b_n$ where $x_n \in \Omega_1$ and $b_n \in B_1^2(0)$, for all n. Now $P_1 y_n \longrightarrow P_1 x$ since P_1 is continuous. Thus $x_n \longrightarrow P_1 x$. Also $P_2 y_n \longrightarrow P_2 x$, and so $b_n \longrightarrow P_2 x$. $x_n \in \Omega_1$ implies that $P_1 x \in \bar{\Omega}_1$ and $b_n \in B_1^2(0)$ implies that $P_2 x \in \bar{B}_1^2(0)$. So $x_n + b_n \longrightarrow P_1 x + P_2 x = x$. Thus $x \in \bar{\Omega}_1 + \bar{B}_1^2(0) \subseteq \bar{\Omega} \cap X_1 + \bar{B}_1^2(0)$. Now $P_1 x \in \bar{\Omega} \cap X_1$. Thus $\tilde{F}_1 P_1 x = F_1 P_1 x$. So $y = (I - F_1 P_1)x$, and $x = y + F_1 P_1 x \in X_1$. Thus $x \in \bar{\Omega}_1 \subseteq \bar{\Omega} \cap X_1$. But if $x \in \partial\Omega$, then $y \in (I - \tilde{F}_1 P_1)(\partial\Omega)$. So $x \in \Omega \cap X_1 = \Omega_1 \subseteq \Omega \cap \Omega'$. Thus we have shown that $x \in \bar{\Omega}'$ with $y = (I - \tilde{F}_1 P_1)(x)$ implies that $x \in \Omega \cap \Omega'$. So $y \notin (I - \tilde{F}_1 P_1)(\bar{\Omega}' \setminus \Omega \cap \Omega')$.
By (D2),

$$D(I - \tilde{F}_1 P_1, \Omega', y) = D(I - \tilde{F}_1 P_1, \Omega \cap \Omega', y). \tag{4}$$

(3) and (4) give

$$D(I - \tilde{F}_1 P_1, \Omega, y) = D(I - \tilde{F}_1 P_1, \Omega', y). \tag{5}$$

(1), (2) and (5) give us

$$D(I - F, \Omega, y) = D(I - \tilde{F}_1 P_1, \Omega', y). \tag{6}$$

Now let $x \in \bar{\Omega}'$. Then $x \in \bar{\Omega}_1 + \bar{B}_1^2(0)$. So $P_1 x \in \bar{\Omega}_1 \subseteq \bar{\Omega} \cap X_1$ and hence $\tilde{F}_1 P_1 x = F_1 P_1 x$. Thus $(I - \tilde{F}_1 P_1)x = (I - F_1 P_1)x$ for $x \in \bar{\Omega}'$, giving us $(I - \tilde{F}_1 P_1)|_{\bar{\Omega}'} = (I - F_1 P_1)|_{\bar{\Omega}'}$. So

$$D(I - \tilde{F}_1 P_1, \Omega', y) = D(I - F_1 P_1, \Omega', y). \tag{7}$$

(6) and (7) give

$$D(I - F, \Omega, y) = D(I - F_1 P_1, \Omega', y). \tag{8}$$

Let $\Omega_1 \subseteq X_1$ be open bounded, $f : \bar{\Omega}_1 \longrightarrow X_1$ be continuous and $y \in X_1 \setminus f(\partial\Omega_1)$ (i.e. (f, Ω_1, y) is a Brouwer triplet). Now $P_1(\Omega_1 + B_1^2(0)) \subseteq \Omega_1$ and if $x \in \Omega_1$, then $x \in \Omega_1 + B_1^2(0)$ and $P_1 x = x$. So $x \in P_1(\Omega_1 + B^2_1(0))$, and hence $P_1(\Omega_1 + B_1^2(0)) = \Omega_1$. Thus $P_1(\Omega') = \Omega_1$ where $\Omega' = \Omega_1 + B_1^2(0)$. Also $P_1(\bar{\Omega}') \subseteq \bar{\Omega}_1$ and if $x \in \bar{\Omega}_1$, then there exists a sequence (x_n) in Ω_1 such that $x_n \longrightarrow x$. Now $x \in X_1$, so $P_1 x = x$. Also $x_n \in \Omega_1 \subseteq \Omega'$, so $x \in \bar{\Omega}'$. Hence $x = P_1 x \in P_1(\bar{\Omega}')$, proving that $\bar{\Omega}_1 \subseteq P_1(\bar{\Omega}')$. Thus we have $P_1(\bar{\Omega}') = \bar{\Omega}_1$.

Now $P_1|_{\bar{\Omega}'} : \bar{\Omega}' \longrightarrow \bar{\Omega}_1$ and $(I - f) : \bar{\Omega}_1 \longrightarrow X_1$. So $(I - f)P_1|_{\bar{\Omega}'} : \bar{\Omega}' \longrightarrow X_1$

In order for $(I - (I - f)P_1, \Omega', y)$ to be an admissible LS–triplet, the following conditions must be satisfied :

(a) Ω' is open bounded in X.

(b) $(I - f)P_1|_{\bar{\Omega}'}$ is compact.

(c) $y \notin (I - (I - f)P_1)(\partial\Omega')$.

We now show that these conditions are satisfied.

(a) Ω_1 and $B_1^2(0)$ are open and bounded in X_1 and X_2 respectively, so $\Omega' = \Omega_1 + B_1^2(0)$ is open and bounded in X.

(b) $((I - f)P_1)(\bar{\Omega}') = (I - f)(\bar{\Omega}_1) \subseteq \bar{\Omega}_1 - f(\bar{\Omega}_1)$. Now $\bar{\Omega}_1 \subseteq X_1$ is compact since it is a closed bounded subset of a finite dimensional space. So $f(\bar{\Omega}_1)$ must also be compact in X_1 (and hence both are compact in X).

$$\gamma(\, ((I - f)P_1)(\bar{\Omega}')\,) \le \gamma(\, \bar{\Omega}_1 - f(\bar{\Omega}_1)\,) \le \gamma(\, \bar{\Omega}_1) + \gamma(\, f(\bar{\Omega}_1)\,) = 0.$$

So $(I-f)P_1|_{\bar{\Omega}'}$ is compact.

(c) Suppose $y = (I-(I-f)P_1)x$ for $x \in \bar{\Omega}' = \bar{\Omega}_1 + \bar{B}^2_1(0)$. Then $x = y + (I-f)P_1 x \in X_1$. So $x \in \bar{\Omega}_1$ and $P_1 x = x$. Thus $y = (I-(I-f)P_1)x = (I-(I-f))x = fx$. Since $y \notin f(\partial\Omega_1)$, we must have $x \in \Omega_1$, and so $x \in \Omega_1 + B^2_1(0) = \Omega'$. Thus $y \notin (I-(I-f)P_1)(\partial\Omega')$.

Hence $(I-(I-f)P_1, \Omega', y)$ is an admissible LS–triplet.

We will now show that d_0, defined by $d_0(f, \Omega_1, y) = D(I-(I-f)P_1, \Omega_1 + B^2_1(0), y)$ satisfies $(d_0 1)$–$(d_0 3)$, where (f, Ω_1, y) is a Brouwer triplet. If it does, then it must be the Brouwer degree, since the Brouwer degree is unique.

$(d_0 1)$ For $y \in \Omega_1$,

$d_0(id, \Omega_1, y) = D(I-(I-id)P_1, \Omega_1 + B^2_1(0), y) = D(I, \Omega_1 + B^2_1(0), y) = 1$ by (D1).

$(d_0 2)$ Let Ω^1 and Ω^2 be disjoint open subsets of $\Omega_1 \subseteq X_1$ such that $y \notin f(\bar{\Omega}_1 \setminus \Omega^1 \cup \Omega^2)$. Then $d_0(f, \Omega_1, y) = D(I-(I-f)P_1, \Omega_1 + B^2_1(0), y)$. Now Ω^1 and Ω^2 disjoint open in X_1, imply that $\Omega^1 + B^2_1(0)$ and $\Omega^2 + B^2_1(0)$ are disjoint open in X.

Consider $y = (I-(I-f)P_1)x$ where $x \in \bar{\Omega}' = \bar{\Omega}_1 + \bar{B}^2_1(0)$. Then $x = y + (I-f)P_1 \in X_1$. So we get $P_1 x = x$ and $x \in \bar{\Omega}_1$. Thus $y = fx$. Since $y \notin f(\bar{\Omega}_1 \setminus \Omega^1 \cup \Omega^2)$, we must have

$x \in \Omega^1 \cup \Omega^2 \subseteq (\Omega^1 \cup \Omega^2) + B^2_1(0) = (\Omega^1 + B^2_1(0)) \cup (\Omega^2 + B^2_1(0))$. Therefore

$y \notin (I-(I-f)P_1)(\overline{\Omega_1 + B^2_1(0)} \setminus (\Omega^1 + B^2_1(0)) \cup (\Omega^2 + B^2_1(0)))$.

Thus by (D2),

$D(I-(I-f)P_1, \Omega_1 + B^2_1(0), y)$

$= D(I-(I-f)P_1, \Omega^1 + B^2_1(0), y) + D(I-(I-f)P_1, \Omega^2 + B^2_1(0), y)$

$= d_0(f, \Omega^1, y) + d_0(f, \Omega^2, y)$.

Therefore $d_0(f, \Omega, y) = d_0(f, \Omega^1, y) + d_0(f, \Omega^2, y)$

$(d_0 3)$ Let $\Omega_1 \subseteq X_1$ be open bounded, $h : J \times \bar{\Omega}_1 \to X_1$ and $y : J \to X_1$ be continuous and $y(t) \in X_1 \setminus h(t, \partial\Omega_1)$ for $t \in J$. Then

$d_0(h(t, .), \Omega, y(t)) = D(I - (I - h(t, .))P_1, \Omega_1 + B_1^2(0), y(t))$.

Define $H : J \times \overline{\Omega_1 + B_1^2(0)} \to X_1$ by

$H(t, x) = (I - h(t, .))P_1 x = P_1 x - h(t, P_1 x)$.

H is easily continuous.

$$\begin{aligned} H(J \times \overline{\Omega_1 + B_1^2(0)}) &\subseteq P_1(\overline{\Omega_1 + B_1^2(0)}) - h(J \times P_1(\overline{\Omega_1 + B_1^2(0)})) \\ &= \bar{\Omega}_1 - h(J \times \bar{\Omega}_1). \end{aligned}$$

Now $\bar{\Omega}_1$ is closed and bounded in X_1 and hence is compact. Therefore $h(J \times \bar{\Omega}_1)$ is also compact in X_1. Thus both are compact in X.

Therefore $\gamma(H(J \times \overline{\Omega_1 + B_1^2(0)})) \leq \gamma(\bar{\Omega}_1) + \gamma(h(J \times \bar{\Omega}_1)) = 0$, proving that H is compact.

Now let $y(t) = (I - H(t, .))x$ for $(t, x) \in J \times \overline{\Omega_1 + B_1^2(0)} = J \times (\bar{\Omega}_1 + \bar{B}_1^2(0))$.

Then $x = y(t) + H(t, x) \in X_1$. So $P_1 x = x$ and $x \in \bar{\Omega}_1$.

Thus $y(t) = x - H(t, x) = x - [x - h(t, x)] = h(t, x)$.

But $y(t) \notin h(t, \partial\Omega_1)$. Therefore $x \in \Omega_1 \subseteq \Omega_1 + B_1^2(0)$, and so

$y(t) \notin (I - H(t, .))(\partial(\Omega_1 + B_1^2(0)))$. Hence by (D3),

$D(I - H(t, .), \Omega_1 + B_1^2(0), y(t))$ is independent of t. Therefore

$d_0(h(t, .), \Omega_1, y(t))$ is independent of t.

Since d_0, defined on the Brouwer triplets, satisfies $(d_0 1)$–$(d_0 3)$, and since the Brouwer degree is unique, we must have d_0 to be the Brouwer degree. Therefore $d_0 = d$. Thus

$$\begin{aligned} &D(I - F_1 P_1, \Omega', y) \\ &= d_0((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y). \end{aligned} \tag{9}$$

(8) and (9) give us

$$D(I-F, \Omega, y) = d((I-F_1)|_{\bar{\Omega}_1}, \Omega_1, y).$$

Thus if a degree on our admissible LS–triplets exists , then it must be unique.

We now show the existence of a degree on admissible LS–triplets.

For $\Omega \subseteq X$ open bounded, $F \in \mathscr{K}(\bar{\Omega})$ and $y \in X \setminus (I-F)(\partial\Omega)$, define

$D(I-F, \Omega, y) = d((I-F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$ where $F_1 \in \mathscr{F}(\bar{\Omega})$, $F_1 : \bar{\Omega} \longrightarrow X_1$, $\dim X_1 < \infty$,

$y \in X_1$, $\Omega_1 = \Omega \cap X_1$ and $|F - F_1|_0 < \rho(y, (I-F)(\partial\Omega))$.

We must first show that this definition is independent of the choice of F_1 and X_1, and then show that D satisfies (D1)–(D3).

well–defined :

Suppose F_2, X_2 satisfy all the conditions that F_1, X_1 do. Let $X_0 = \mathrm{span}(X_1 \cup X_2)$. Since $\dim X_1 < \infty$ and $\dim X_2 < \infty$, we must also have $\dim X_0 < \infty$. Also let $\Omega_0 = \Omega \cap X_0$. Since Ω is open bounded in X, Ω_0 must be open bounded in X_0. Also $F_i|_{\bar{\Omega}_0} : \bar{\Omega}_0 \longrightarrow X_i$ is continuous for i = 1, 2. Let $h : J \times \bar{\Omega}_0 \longrightarrow X_0$ be defined by $h(t, x) = t\,F_1x + (1-t)\,F_2x$ for $(t, x) \in J \times \bar{\Omega}_0$.

Then h is continuous.

[**N.B.**: $\Omega_0 \subseteq \Omega$ implies that $\partial\Omega_0 \subseteq \partial\Omega$: Let $x \in \partial\Omega_0 \subseteq \bar{\Omega}_0 \subseteq \bar{\Omega}$ and suppose $x \notin \partial\Omega$. Then $x \in \Omega$. Also, $\bar{\Omega}_0 \subseteq \bar{\Omega} \cap X_0$. Therefore $x \in X_0$ and hence $x \in \Omega \cap X_0 = \Omega_0$, a contradiction. Thus we must have $x \in \partial\Omega$ and so $\partial\Omega_0 \subseteq \partial\Omega$.]

If $t \in J$ and $x \in \partial\Omega_0$, then

$$\begin{aligned} |y - h(t, x)| &= |y - (I-F)x - (Fx - h(t, x))| \\ &\geq |y - (I-F)x| - |Fx - h(t, x)| \end{aligned}$$

Now

$$|Fx - h(t, x)| = |Fx - t\,F_1x - (1-t)\,F_2x\,|$$

$$= |t\,(F - F_1)x + (1-t)(F - F_2)x|$$
$$\leq t\,|(F - F_1)x| + (1-t)\,|(F - F_2)x|$$
$$< t\,\rho + (1-t)\,\rho$$
$$= \rho.$$

Also, $x \in \partial\Omega_0$ implies $x \in \partial\Omega$, and so

$|y - (I - F)x| > \rho(y, (I - F)(\partial\Omega)) = \rho.$

Thus $|y - h(t, x)| > \rho - \rho = 0$ for all $t \in J$ and $x \in \partial\Omega_0$. Therefore $y \notin h(t, \partial\Omega_0)$ for all $t \in J$. By (d3),

$$d((I - F_1)|_{\bar{\Omega}_0}, \Omega_0, y) = d((I - F_2)|_{\bar{\Omega}_0}, \Omega_0, y). \tag{10}$$

Therefore, $y \in X_i \setminus (I - F_i)(\partial\Omega_0)$ for $i = 1,2$.

By theorem 2.19,

$$\begin{aligned} d((I - F_i)|_{\bar{\Omega}_0}, \Omega_0, y) &= d((I - F_i)|_{\overline{\Omega_0 \cap X_i}}, \Omega_0 \cap X_i, y) \\ &= d((I - F_i)|_{\overline{\Omega \cap X_i}}, \Omega \cap X_i, y) \\ &= d((I - F_i)|_{\bar{\Omega}_i}, \Omega_i, y) \quad \text{for } i = 1, 2. \end{aligned} \tag{11}$$

By (10) and (11),

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d((I - F_2)|_{\bar{\Omega}_2}, \Omega_2, y).$$

Hence our definition is independent of the choice of F_1 and X_1.

We must now show that D satisfies (D1)–(D3).

(D1):

Let $\Omega \subsetneq X$ be open bounded, and $y \in \Omega$. Then let $X_1 = \text{span}\,\{y\}$, $\Omega_1 = \Omega \cap X_1$. Then $y \in X_1$ and hence

$$\begin{aligned} D(I, \Omega, y) &= d((I - 0)|_{\bar{\Omega}_1}, \Omega_1, y) \\ &= d(I, \Omega_1, y) \\ &= 1 \qquad \text{by (d1).} \end{aligned}$$

(D2):

Let Ω^1 and Ω^2 be disjoint open subsets of Ω with $y \notin (I-F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2)$. Since $F \in \mathscr{K}(\bar{\Omega})$, $I - F$ is proper and continuous, hence it must be closed. Therefore $(I-F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2)$ is closed and $\rho_1 = \rho(y, (I-F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2)) > 0$.

Choose $F_1 \in \mathscr{F}(\bar{\Omega})$ such that $\sup\{|F_1x - Fx| \;/\; x \in \bar{\Omega}\} < \rho_1$. Then choose, as we may, X_1 a subspace of X with $\dim X_1 < \infty$, $F_1(\bar{\Omega}) \subseteq X_1$ and $y \in X_1$. Let $\Omega_1 = \Omega \cap X_1$. But $\rho_1 = \rho(y, (I-F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2)) < \rho(y, (I-F)(\partial\Omega)) = \rho$. Therefore $\sup\{|F_1x - Fx| \;/\; x \in \bar{\Omega}\} < \rho$, and by definition,

$$D(I-F, \Omega, y) = d((I-F_1)|_{\bar{\Omega}_1}, \Omega_1, y). \tag{12}$$

Now $\Omega^1 \cap X_1$ and $\Omega^2 \cap X_1$ are disjoint open subsets of Ω_1. We need to show that
$y \notin (I-F_1)|_{\bar{\Omega}_1}(\bar{\Omega}_1 \setminus (\Omega^1 \cap X_1) \cup (\Omega^2 \cap X_1)) = (I-F_1)|_{\bar{\Omega}_1}(\bar{\Omega}_1 \setminus (\Omega^1 \cup \Omega^2) \cap X_1)$.

Suppose $y = (I-F_1)x$ for $x \in \bar{\Omega}_1 \setminus (\Omega^1 \cup \Omega^2) \cap X_1$. Then $x = y + F_1x \in X_1$. This must mean that $x \notin \Omega^1 \cup \Omega^2$. Therefore $x \in \bar{\Omega} \setminus \Omega^1 \cup \Omega^2$, and

$$\begin{aligned}
|F_1x - Fx| &= |(I-F)x - (I-F_1)x| \\
&= |(I-F)x - y| \\
&\geq \rho(y, (I-F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2)) \\
&= \rho_1,
\end{aligned}$$

contrary to the way F_1 was chosen .

Therefore $y \notin (I-F_1)(\bar{\Omega}_1 \setminus (\Omega^1 \cup \Omega^2) \cap X_1)$ and by (d2) we obtain

$$\begin{aligned}
&d((I-F_1)|_{\bar{\Omega}_1}, \Omega_1, y) \\
&= d((I-F_1)|_{\bar{\Omega}_1}, \Omega^1 \cap X_1, y) + d((I-F_1)|_{\bar{\Omega}_1}, \Omega^2 \cap X_1, y).
\end{aligned} \tag{13}$$

We now need to check that $\sup\{|F_1x - Fx| \;/\; x \in \bar{\Omega}^i\} < \rho(y, (I-F_1)(\partial\Omega^i))$ for $i = 1, 2$. We claim that for $i \neq j$, $\partial\Omega^i \cap \Omega^j = \emptyset$. For if $x \in \partial\Omega^i \cap \Omega^j$, then since Ω^j is open , there exists an open neighbourhood U of x contained in Ω^j. But $x \in \partial\Omega^i$, therefore every neighbourhood of x meets Ω^i as well as its boundary, a contradiction to $\Omega^i \cap \Omega^j = \emptyset$. Thus $\partial\Omega^i \cap \Omega^j = \emptyset$. Therefore $\partial\Omega^i \subseteq \bar{\Omega} \setminus \Omega^1 \cup \Omega^2$. So

$$\sup\{|F_1x - Fx| \;/\; x \in \bar{\Omega}^i\} \le \sup\{|F_1x - Fx| \;/\; x \in \bar{\Omega}\}$$
$$< \rho(y, (I - F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2))$$
$$< \rho(y, (I - F)(\partial\Omega^i)) \quad \text{for } i = 1, 2.$$

Again by definition we have for $i = 1, 2$

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega^i \cap X_1, y) = D(I - F, \Omega^i, y). \tag{14}$$

(12), (13) and (14) give us

$$D(I - F, \Omega, y) = D(I - F, \Omega^1, y) + D(I - F, \Omega^2, y).$$

Before we prove (D3), we need the following lemma.

3.8 Lemma

Let X be a real Banach space, $\Omega \subseteq X$ open and bounded, $F \in \mathscr{K}(\bar{\Omega})$ and $y \notin (I - F)(\partial\Omega)$. Then $D(I - F, \Omega, y) = D(I - F - y, \Omega, 0)$.

Proof :

By theorem 3.5, there exists $F_1 \in \mathscr{F}(\bar{\Omega})$ such that $\sup\{|Fx - F_1x| \;/\; x \in \bar{\Omega}\} < \rho$, where $\rho = \rho(y, (I - F)(\partial\Omega))$. Let X_0 be a finite–dimensional subspace of X such that $y \in X_0$, $F_1(\bar{\Omega}) \subseteq X_0$ and $\Omega_0 = \Omega \cap X_0$. Then by definition,

$$D(I - F, \Omega, y) = d((id - F_1)|_{\bar{\Omega}_0}, \Omega_0, y). \tag{15}$$

Define $h : J \times \bar{\Omega}_0 \longrightarrow X_0$ by $h(t, x) = (I - F_1)|_{\bar{\Omega}_0}(x) - t\,y$ for $(t, x) \in J \times \bar{\Omega}_0$, and $y : J \longrightarrow X_0$ by $y(t) = (1 - t)\,y$ for $t \in J$. Then h and y are continuous. Since $\partial\Omega_0 \subseteq \partial\Omega$, for $(t, x) \in J \times \partial\Omega_0$, we have

$$\begin{aligned}
|y(t) - h(t, x)| &= |(1 - t)\,y - (I - F_1)x + t\,y| \\
&= |y - (I - F_1)x| \\
&\ge |y - (I - F)x| - |(I - F)x - (I - F_1)x| \\
&= |y - (I - F)x| - |Fx - F_1x|
\end{aligned}$$

$$\geq \rho(y, (I - F)(\partial\Omega_0)) - |Fx - F_1x|$$
$$\geq \rho(y, (I - F)(\partial\Omega)) - \sup\{|Fx - F_1x| \;/\; x \in \bar{\Omega}_0\}$$
$$\geq \rho - \sup\{|Fx - F_1x| \;/\; x \in \bar{\Omega}\}$$
$$> \rho - \rho$$
$$= 0.$$

Therefore $y(t) \notin h(t, \partial\Omega_0)$ for $t \in J$, and by (d3), $d(h(t, .), \Omega_0, y(t))$ is independent of t.

So we have $d(h(0, .), \Omega_0, y(0)) = d(h(1, .), \Omega_0, y(1))$, which is the same as

$$d((I - F_1)|_{\bar{\Omega}_0}, \Omega_0, y) = d((I - F_1)|_{\bar{\Omega}_0} - y, \Omega_0, 0). \tag{16}$$

$$\begin{aligned}\rho(0, (I - F - y)(\partial\Omega)) &= \inf\{|0 - (I - F - y)\,x| \;/\; x \in \partial\Omega\} \\ &= \inf\{|(I - F - y)(x)| \;/\; x \in \partial\Omega\} \\ &= \inf\{|(I - F)\,x - y| \;/\; x \in \partial\Omega\} \\ &= \rho(y, (I - F)(\partial\Omega)) \\ &= \rho.\end{aligned}$$

So for $x \in \bar{\Omega}$,

$|(F - y)\,x - (F_1 - y)\,x| = |Fx - F_1x| \leq \sup\{|Fx - F_1x| \;/\; x \in \bar{\Omega}\} < \rho.$

Again by definition,

$$D(I - F - y, \Omega, 0) = d((I - F_1 - y)|_{\bar{\Omega}_0}, \Omega_0, 0). \tag{17}$$

(15), (16) and (17) imply that

$$D(I - F, \Omega, y) = D(I - F - y, \Omega, 0).$$ ♠

We are now able to prove (D3).

(D3) :

Let $H : J \times \bar{\Omega} \longrightarrow X$ be compact, $y : J \longrightarrow X$ continuous such that

$y(t) \notin (I - H(t, .))(\partial\Omega)$ for all $t \in J$. We have already shown that

$$D(I - H(t, .), \Omega, y(t)) = D(I - H(t, .) - y(t), \Omega, 0).$$

Let $H_0(t, x) = H(t, x) + y(t)$. Then $H_0 : J \times \bar{\Omega} \longrightarrow X$ is compact, $0 \notin (I - H_0(t, .))(\partial\Omega)$ and $D(I - H(t, .), \Omega, y(t)) = D(I - H_0(t, .), \Omega, 0)$ by lemma 3.8.

$J \times \bar{\Omega}$ is closed bounded. Let $\delta = \inf_{t \in J} \rho((I - H_0(t, .))(\partial\Omega), 0)$. We will now prove that $\delta > 0$. If $\delta = 0$, then there is a sequence (t_n, x_n) in $J \times \partial\Omega$ such that $|x_n - H_0(t_n, x_n) - 0| \longrightarrow 0$, i.e. $x_n - H_0(t_n, x_n) \longrightarrow 0$. Since J is compact, by taking a suitable subsequence we may suppose that $t_n \longrightarrow t_0$ for some $t_0 \in J$. Similarly, since $H_0(J \times \partial\Omega) \subseteq H_0(J \times \bar{\Omega})$ is relatively compact, we may also suppose that $H_0(t_n, x_n) \longrightarrow x_0$ for some $x_0 \in X$. Therefore $x_n \longrightarrow x_0$. Since $\partial\Omega$ is closed, $x_0 \in \partial\Omega \subseteq \bar{\Omega}$. Hence by continuity of H_0, $x_0 = H_0(t_0, x_0) = H(t_0, x_0) + y(t_0)$. So $y(t_0) = (I - H(t_0, .))x_0 \in (I - H(t_0, .))(\partial\Omega)$, contrary to hypothesis. Thus $\delta > 0$.

By theorem 3.5, there exists $F \in \mathscr{F}(J \times \bar{\Omega}, X)$ such that

$$\sup_{J \times \bar{\Omega}} |F(t, x) - H_0(t, x)| < \inf_{t \in J} \rho((I - H_0(t, .))(\partial\Omega), 0).$$

So for each t,

$$\begin{aligned} \sup_{x \in J} |F(t, x) - H_0(t, x)| &\leq \sup_{J \times \bar{\Omega}} |F(s, x) - H_0(s, x)| \\ &\leq \inf_{s \in J} \rho((I - H_0(s, .))(\partial\Omega), 0) \\ &\leq \inf_{s \in J} |(I - H_0(s, .))x|. \end{aligned}$$

Therefore by definition,

$$D(I - H_0(t, .), \Omega, 0) = d((I - F(t, .))|_{\Omega_0}, \Omega_0, 0)$$

where X_0 is a subspace of X, $\dim X_0 < \infty$, $0 \in X_0$, $F(J \times \bar{\Omega}) \subseteq X_0$ and $\Omega_0 = \Omega \cap X_0$. Then $d((I - F(t, .))|_{\Omega_0}, \Omega_0, 0)$ is independent of t by (d3), proving (D3).

We have thus proved the following result.

3.9 Theorem

Let X be a real Banach space and

$\mathscr{M} = \{ (I - F, \Omega, y) \ / \ \Omega \subseteq X$ open bounded, $F \in \mathscr{K}(\bar{\Omega})$ and $y \in X \setminus (I - F)(\partial\Omega) \}$.

Then there exists a unique function $D : \mathscr{M} \longrightarrow \mathbb{Z}$ (the *Leray–Schauder degree*) satisfying (D1)–(D3). This function is defined by

$D(I - F, \Omega, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$ where $F_1 \in \mathscr{F}(\bar{\Omega})$ such that $\sup_{\bar{\Omega}} |F_1 x - Fx| < \rho(y, (I - F)(\partial\Omega))$, X_1 is a subspace of X such that $F_1(\bar{\Omega}) \subseteq X_1$, $y \in X_1$, $\dim X_1 < \infty$, $\Omega_1 = \Omega \cap X_1$ and d is the *Brouwer degree* of X_1 (defined in chapter 2).

We now obtain the following extension of (D2).

3.10 Lemma

Let $\Omega \subseteq X$ be open bounded, $F : \bar{\Omega} \longrightarrow X$ compact, $y \in X \setminus (I - F)(\partial\Omega)$.

Let $\{\Omega_k \ / \ k = 1, 2, ...\}$ be an infinite disjoint sequence of open subsets of Ω such that $y \notin (I - F)(\Omega \setminus \bigcup_{i=1}^{\infty} \Omega_i)$. Then for each k, $D(I - F, \Omega_k, y)$ is defined, only finitely many of them are non–zero, and

$$D(I - F, \Omega, y) = \sum_{k=1}^{\infty} D(I - F, \Omega_k, y).$$

Proof :

Let $x \in \partial\Omega_k$. Since Ω_k is open, we have $x \notin \Omega_k$. If $x \in \partial\Omega_i$ for some $i \neq k$, then Ω_i must meet Ω_k (since $x \in \partial\Omega_k$, every neighbourhood of x must meet Ω_k), a contradiction. So $\partial\Omega_k \cap (\bigcup_{i \neq k} \Omega_i) = \emptyset$. Also $\partial\Omega_k \cap \Omega_k = \emptyset$. Hence

$\partial\Omega_k \cap (\bigcup_{i=1}^{\infty} \Omega_i) = \emptyset$ and so $\partial\Omega_k \subseteq \bar{\Omega} \setminus \bigcup_{i=1}^{\infty} \Omega_i$. Since $y \notin (I - F)(\bar{\Omega} \setminus \bigcup_{i=1}^{\infty} \Omega_i)$, we must have $y \notin (I - F)(\partial\Omega_k)$ and so $D(I - F, \Omega_k, y)$ is defined for each k.

Let $M = (I - F)^{-1}(y)$. By theorem 3.5, $I - F$ is proper and hence $(I - F)^{-1}(y)$

is compact. So M is closed.

Now if $x \in M$, then $x \in \bar{\Omega}$ and $(I - F)\,x = y$. Since $y \notin (I - F)(\bar{\Omega} \setminus \bigcup_{i=1}^{\infty} \Omega_i)$, we must have $x \in \bigcup_{i=1}^{\infty} \Omega_i$. Thus $M \subseteq \bigcup_{i=1}^{\infty} \Omega_i$. Since M is compact, we can find a finite subset N of $\mathbb{N}$ such that $M \subseteq \bigcup_{i \in N} \Omega_i$. Since the Ω_i are disjoint,

$M \cap (\bigcup_{i \in \mathbb{N} \setminus N} \Omega_i) = \emptyset$

Therefore $(I - F)^{-1}(y) \cap \Omega_i = \emptyset$ for all $i \in \mathbb{N} \setminus N$. So $D(I - F, \Omega_i, y) = 0$ for all $i \in \mathbb{N} \setminus N$. Now $(I - F)^{-1}(y) \subseteq \bigcup_{i \in N} \Omega_i$ and so $y \notin (I - F)(\bar{\Omega} \setminus \bigcup_{i \in N} \Omega_i)$. Since N is finite, (D2) yields $D(I - F, \Omega, y) = \sum_{i \in N} D(I - F, \Omega_i, y)$ and for $i \notin N$, $D(I - F, \Omega_i, y) = 0$. Thus

$$D(I - F, \Omega, y) = \sum_{i=1}^{\infty} D(I - F, \Omega_i, y).$$

♠

Now we obtain more properties of the Leray–Schauder degree whose analogues for the Brouwer degree follow by similar proofs and were stated in theorem 2.5 without proof.

3.11 Theorem

The Leray–Schauder degree satisfies the following properties in addition to (D1)–(D3).

(D4) $D(I - F, \Omega, y) \neq 0$ implies $(I - F)^{-1}(y) \neq \emptyset$.

(D5) $D(I - G, \Omega, y) = D(I - F, \Omega, y)$ for $G \in \mathscr{K}(\bar{\Omega}) \cap B_\rho(F)$ and

$D(I - F, \Omega, y) = D(I - F, \Omega, y_1)$ for $y_1 \in B_\rho(y)$,

where $\rho = \rho(y, (I - F)(\partial\Omega)) > 0$.

Also $D(I - F, \Omega, .)$ is constant on every connected component of $X \setminus (I - F)(\partial\Omega)$.

(D6) $D(I - G, \Omega, y) = D(I - F, \Omega, y)$ if $G|_{\partial\Omega} = F|_{\partial\Omega}$, $G \in \mathscr{K}(\bar{\Omega})$.

(D7) $D(I - F, \Omega, y) = D(I - F, \Omega_1, y)$ for any open set Ω_1 of Ω such that

$y \in X \setminus (I - F)(\bar{\Omega} \setminus \Omega_1)$.

Proof :

(D4) $y \in X \setminus (I - F)(\partial\Omega) = X \setminus (I - F)(\bar{\Omega} \setminus \Omega \cup \emptyset)$. Hence by (D2), $D(I - F, \Omega, y) = D(I - F, \Omega, y) + D(I - F, \emptyset, y)$. Thus $D(I - F, \emptyset, y) = 0$. If $(I - F)^{-1}(y) = \emptyset$, then $y \in X \setminus (I - F)(\bar{\Omega} \setminus \emptyset \cup \emptyset)$ and again by (D2),

$D(I - F, \Omega, y) = D(I - F, \emptyset, y) + D(I - F, \emptyset, y) = 0 + 0 = 0.$

Thus $D(I - F, \Omega, y) \neq 0$ implies that $(I - F)^{-1}(y) \neq \emptyset$.

(D7) If $\Omega_1 \subseteq \Omega$ is open such that $y \in X \setminus (I - F)(\bar{\Omega} \setminus \Omega_1)$, then $y \in X \setminus (I - F)(\bar{\Omega} \setminus \Omega_1 \cup \emptyset)$. So

$D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \emptyset, y) = D(I - F, \Omega_1, y)$.

(D6) Let $H(t, x) = t\,Fx + (1 - t)\,Gx$. Then

$$\begin{aligned}
\gamma(H(J \times B)) &\leq \gamma(\mathrm{conv}\,(FB \cup GB)) \\
&= \gamma(FB \cup GB) \\
&= \max\,\{\gamma(FB), \gamma(GB)\} \\
&= 0.
\end{aligned}$$

Therefore $H \in \mathscr{K}(J \times \bar{\Omega}, X)$. If $y \in (I - H(t, .))(\partial\Omega)$, then there exists $x \in \partial\Omega$ such that

$$\begin{aligned}
y &= (I - H(t, .))\,x \\
&= x - t\,Fx - (1 - t)\,Gx \\
&= x - t\,Fx - (1 - t)\,Fx \\
&= (I - F)x \qquad \text{since } F|_{\partial\Omega} = G|_{\partial\Omega}.
\end{aligned}$$

Thus $y \notin (I - H(t, .))(\partial\Omega)$. Hence by (D3),

$$D(I - F, \Omega, y) = D(I - G, \Omega, y).$$

(D5) Let $G \in \mathscr{K}(\bar{\Omega}) \cap B_\rho(F)$, and $H(t, x) = (1 - t)\,Fx + t\,Gx$ with $(t, x) \in J \times \bar{\Omega}$. Easily, $H \in \mathscr{K}(J \times \bar{\Omega}, X)$. Suppose $y \in (I - H(t, .))(\partial\Omega)$ for some $t \in J$. Then for some $x \in \partial\Omega$

$y = (I - H(t, .))x = x - (1 - t)\, Fx - t\, Gx = x - Fx + t\,(Fx - Gx)$.

So $|Fx - Gx| \geq |t(Fx - Gx)| = |y - (I - F)x| \geq \rho(y, (I - F)(\partial\Omega)) = \rho$.

Therefore $|F - G|_0 \geq \rho$, a contradiction. Hence $y \notin (I - H(t, .))(\partial\Omega)$ and so by (D3),

$$D(I - F, \Omega, y) = D(I - G, \Omega, y).$$

Now let $y_1 \in B_\rho(y)$, and $H(t, x) = Fx$ where $(t, x) \in J \times \bar{\Omega}$ and $y(t) = (1 - t)\, y + t\, y_1$.

Now $H \in \mathscr{K}(J \times \bar{\Omega}, X)$.

Suppose $y(t) \in (I - H(t, .))(\partial\Omega)$. Then $y(t) = (I - H(t, .))x$ for some $x \in \partial\Omega$. This implies that $(1 - t)\, y + t\, y_1 = (I - F)x$ which means that $t\,(y_1 - y) = (I - F)x - y$. Therefore

$|y_1 - y| \geq |t\,(y_1 - y)| = |(I - F)x - y| \geq \rho(y, (I - F)(\partial\Omega)) = \rho$,

a contradiction. Hence $y(t) \notin (I - H(t, .))(\partial\Omega)$ and by (D3)

$$D(I - F, \Omega, y) = D(I - F, \Omega, y_1).$$

Now we show that $D(I - F, \Omega, .)$ is constant on every connected component C of $X \setminus (I - F)(\partial\Omega)$. Since $X \setminus (I - F)(\partial\Omega)$ is open, C is open and nonempty. Let $y \in C$. By what has just been proved, $D(I - F, \Omega, .)$ is constant on some ball neighbourhood in C of y. Thus regarded as a mapping from C to $\mathbb{R}$, $D(I - F, \Omega, .)|_C$ is continuous at each $y \in C$. Therefore it is continuous. But a continuous image of a connected set is connected. Thus $D(I - F, \Omega, C)$, as a set, is a nonempty connected subset of $\mathbb{R}$. But it is a subset of $\mathbb{Z}$, and the only nonempty connected subsets of $\mathbb{Z}$ are the one point sets. Thus $D(I - F, \Omega, C)$ is a one point set, and so $D(I - F, \Omega, .)$ is constant on C. ♠

Below, we have an extension to Borsuk's theorem. The usefulness of this theorem lies in fact that it gives conditions under which the degree is odd and hence nonzero.

3.12 Theorem

Let $\Omega \subseteq X$ be open bounded and symmetric with respect to $0 \in \Omega$, $F \in \mathscr{K}(\bar{\Omega})$, $G = I - F$, $0 \notin G(\partial\Omega)$, $G(-x) \neq \lambda\, Gx$ on $\partial\Omega$ for all $\lambda \geq 1$. Then $D(I-F, \Omega, 0)$ is odd. In particular, this is true if $F|_{\partial\Omega}$ is odd.

Proof :

Let $H(t, x) = \frac{1}{1+t} Fx - \frac{t}{1+t} F(-x)$ for $(t, x) \in J \times \bar{\Omega}$. If $B \subseteq \bar{\Omega}$, then

$$\begin{aligned}
\gamma(H(J \times B)) &\leq \gamma(\mathrm{conv}\,(FB \cup (-F(-B)))) \\
&= \gamma(FB \cup (-F(-B))) \\
&= \max\{\gamma(FB), \gamma(-F(-B))\} \\
&\leq \gamma(F(\bar{\Omega})) \\
&= 0.
\end{aligned}$$

Hence $H \in \mathscr{K}(J \times \bar{\Omega}, X)$.

Suppose $0 = (I - H(t, .))x$ with $(t, x) \in J \times \partial\Omega$. Then

$x = \frac{1}{1+t} Fx - \frac{t}{1+t} F(-x)$ and so $\frac{1}{1+t}(I-F)x = \frac{t}{1+t}(I-F)(-x)$. Therefore $(I-F)x = t\,(I-F)(-x)$.

If $t = 0$, then $G(x) = 0$ for $x \in \partial\Omega$, and if $t \neq 0$, then $\frac{1}{t} Gx = G(-x)$ with $x \in \partial\Omega$ and $\frac{1}{t} \geq 1$, contradicting the hypotheses.

Thus $0 \notin (I - H(t, .))(\partial\Omega)$, and by (D3),

$$D(I-F, \Omega, 0) = D(I-F_0, \Omega, 0), \tag{18}$$

where $F_0 x = \frac{1}{2}(Fx - F(-x))$ is odd.

Choose $F_1 \in \mathscr{F}(\bar{\Omega})$ such that $\sup_{\Omega} |F_1 x - F_0 x| < \rho(0, (I-F_0)(\partial\Omega))$ and let $F_2 x = \frac{1}{2}(F_1 x - F_1(-x))$. Then $F_2 \in \mathscr{F}(\bar{\Omega})$ is odd and for $x \in \bar{\Omega}$,

$$\begin{aligned}
|F_2 x - F_0 x| &= \left|\tfrac{1}{2} F_1 x - \tfrac{1}{2} F_1(-x) - F_0 x\right| \\
&= \left|\tfrac{1}{2}(F_1 x - F_0 x) - \tfrac{1}{2}(F_1 - F_0)(-x)\right| \\
&\leq \tfrac{1}{2}|F_1 x - F_0 x| + \tfrac{1}{2}|F_1(-x) - F_0(-x)| \\
&\leq \tfrac{1}{2}\sup_{\Omega}|F_1 x - F_0 x| + \tfrac{1}{2}\sup_{\Omega}|F_1(-x) - F_0(-x)|
\end{aligned}$$

$$= \sup_{\bar{\Omega}} |F_1 x - F_0 x|$$
$$< \rho(0, (I - F_0)(\partial\Omega)).$$

Thus by definition,

$$D(I - F_0, \Omega, 0) = d((I - F_2)|_{\bar{\Omega}_2}, \Omega_2, 0), \tag{19}$$

where X_2 is a subspace of X such that $\dim X_2 < \infty$, $F_2(\bar{\Omega}) \subseteq X_2$ and $\Omega_2 = \Omega \cap X_2$.

Since $0 \in \Omega$ and $0 \in X_2$ we must have $0 \in \Omega_2$. Ω_2 is also open, bounded and symmetric with respect to $0 \in \Omega_2$. By Borsuk's theorem (2.10), $d((I - F)|_{\bar{\Omega}_2}, \Omega_2, 0)$ is odd. Thus by (18) and (19),

$$D(I - F, \Omega, 0) \text{ is odd.}$$

If $F|_{\partial\Omega}$ is odd, then

$(I - F)(-x) = -x - F(-x) = -x + Fx = -(I - F)(x) \neq \lambda(I - F)x$ for $\lambda \geq 1$ and for all $x \in \partial\Omega$.

Hence $D(I - F, \Omega, 0)$ is odd by above. ♠

3.13 Theorem

Let $\Omega \subseteq X$ be open, $F : \Omega \longrightarrow X$ completely continuous and $I - F$ locally one–to–one. Then $I - F$ is open.

Proof :

It is sufficient to show that to $x_0 \in \Omega$, there exists a ball $B_r(x_0)$ such that $(I - F)(B_r(x_0))$ contains a ball with centre $(I - F)(x_0)$.

We will first consider the case $x_0 = 0$ and $F(0) = 0$. Since $I - F$ is locally one–to–one, we can choose $r > 0$ such that $(I - F)|_{\bar{B}_r(0)}$ is one–to–one.

Define $H(t, x) = F(\frac{1}{1+t} x) - F(-\frac{t}{1+t} x)$ for $(t, x) \in J \times \bar{B}_r(0)$.

Let $B \subseteq \bar{B}_r(0)$. Then $H(J \times B) \subseteq F(\bar{B}_r(0)) - F(\bar{B}_r(0))$. So

$\gamma(H(J \times B)) \leq \gamma(F(\bar{B}_r(0))) + \gamma(F(\bar{B}_r(0))) = 0$ since F is completely continuous.

Thus $H \in \mathscr{K}(J \times \bar{B}_r(0), X)$.

Now suppose $0 \in (I - H(t, .))(\bar{B}_r(0))$ for some $t \in J$. Then $0 = (I - H(t, .))x$ for some $x \in \bar{B}_r(0)$. So $x = F(\frac{1}{1+t}x) - F(-\frac{t}{1+t}x)$. Therefore $(I - F)(\frac{1}{1+t}x) = (I - F)(-\frac{t}{1+t}x)$. Since $x \in \bar{B}_r(0)$, we must have $\frac{1}{1+t}x \in \bar{B}_r(0)$ and $-\frac{t}{1+t}x \in \bar{B}_r(0)$. Also $(I - F)|_{\bar{B}_r(0)}$ is one–to–one. So $\frac{1}{1+t}x = -\frac{t}{1+t}x$ giving us $x = 0$. Thus $0 \notin (I - H(t, .))(\partial B_r(0))$ for all $t \in J$, and we can apply (D3) to give us

$$D(I - H(0, .), B_r(0), 0) = D(I - H(1, .), B_r(0), 0).$$

But $H(0, x) = Fx$ and $H(1, x) = F(\frac{1}{2}x) - F(-\frac{1}{2}x)$. So

$$D(I - F, B_r(0), 0) = D(I - H(1, .), B_r(0), 0).$$

By theorem 3.12, this is odd, and hence nonzero.

If $\rho = \rho(0, (I - F)(\partial B_r(0))$, then by (D5) we have for all $y \in B_\rho(0)$, $D(I - F, B_r(0), y) = D(I - F, B_r(0), 0) \neq 0$. So by (D4), $y \in (I - F)(B_r(0))$ for all $y \in B_\rho(0)$. Hence we have $B_\rho(0) \subseteq (I - F)(B_r(0))$ as required.

Now take $x_0 \in \Omega$. Passing to $\Omega - x_0$ and $\tilde{F}x = F(x + x_0) - Fx_0$ for $x \in \Omega - x_0$, we obtain, by the first part, $r > 0$ and $\rho > 0$ such that $B_\rho(0) \subseteq (I - \tilde{F})(B_r(0))$. Let $x \in B_\rho((I - F)(x_0))$. Then $|x - (I - F)x_0| < \rho$. So $x - (I - F)x_0 \in (I - \tilde{F})(B_r(0))$. Thus we can find $y \in B_r(0)$ such that $x - (I - F)(x_0) = (I - \tilde{F})(y) = y - F(y + x_0) + Fx_0$. So $x = x_0 + y - F(y + x_0) = (I - F)(y + x_0)$ and $y + x_0 \in B_r(x_0)$. Hence $x \in (I - F)(B_r(x_0))$ and so we have $B_\rho((I - F)x_0) \subseteq (I - F)(B_r(x_0))$ and we are done. ♠

The above theorem can be used to prove surjectivity results. Now we show that we also obtain a product formula for the degree.

3.14 Theorem (Product Formula)

Let $\Omega \subseteq X$ be open bounded, $F_0 \in \mathscr{K}(\bar{\Omega})$, $F = I - F_0$, $G_0 : X \longrightarrow X$ completely continuous, $G = I - G_0$, $y \notin GF(\partial\Omega)$, and $(K_\lambda)_{\lambda\in\Lambda}$ the connected components of $X \setminus F(\partial\Omega)$. Then

$$D(GF, \Omega, y) = \sum_{\lambda\in\Lambda} D(F, \Omega, K_\lambda)\, D(G, K_\lambda, y)$$

where only finitely many terms are nonzero and $D(F, \Omega, K_\lambda) = D(F, \Omega, z)$ for any $z \in K_\lambda$.

Proof:

We first verify that (GF, Ω, y) is an LS–triplet, i.e. $I - GF \in \mathscr{K}(\bar{\Omega})$. Now $I - GF$ means $(I - GF)|_{\bar{\Omega}}$.

$I - GF = (I - (I - G_0)(I - F_0))|_{\bar{\Omega}} = F_0 + G_0|_{\bar{\Omega}} - G_0F_0$. Thus

$(I - GF)(\bar{\Omega}) \subseteq F_0(\bar{\Omega}) + G_0(\bar{\Omega}) - G_0F_0(\bar{\Omega})$ and

$\gamma((I - GF)(\bar{\Omega})) \leq \gamma(F_0(\bar{\Omega})) + \gamma(G_0(\bar{\Omega})) + \gamma(-G_0F_0(\bar{\Omega}))$. $F_0 \in \mathscr{K}(\bar{\Omega})$, so $\gamma(F_0(\bar{\Omega})) = 0$. G_0 is completely continuous, so since $\bar{\Omega}$ is bounded, $\gamma(G_0(\bar{\Omega})) = 0$. $F_0(\bar{\Omega})$ is relatively compact, hence bounded, and so $\gamma(-G_0F_0(\bar{\Omega})) = \gamma(G_0(F_0(\bar{\Omega}))) = 0$. Thus $\gamma((I - GF)(\bar{\Omega})) = 0$. Since $I - GF$ is continuous, we get $I - GF \in \mathscr{K}(\bar{\Omega})$.

Step 1:

$F(\bar{\Omega})$ is bounded, so there exists $r > 0$ such that $F(\bar{\Omega}) \subseteq B_r(0)$. Let $x \in G^{-1}(y) \cap \bar{B}_r(0)$. Then $x \in \bar{B}_r(0)$ and $x = G_0x + y$. Thus $G^{-1}(y) \cap \bar{B}_r(0) \subseteq G_0(\bar{B}_r(0)) + y$. G_0 is completely continuous, so $G_0(\bar{B}_r(0))$ is relatively compact, and hence $G_0(\bar{B}_r(0)) + y$ is relatively compact. So $G^{-1}(y) \cap \bar{B}_r(0)$ must be relatively compact. But $G^{-1}(y) \cap \bar{B}_r(0)$ is closed, hence it must be compact. Let $M = G^{-1}(y) \cap \bar{B}_r(0)$. If $x \in M$, then $Gx = y$ and so $x \notin F(\partial\Omega)$ since $y \notin GF(\partial\Omega)$. Thus $x \in X \setminus F(\partial\Omega)$. So $M \subseteq X \setminus F(\partial\Omega) = \bigcup_{\lambda\in\Lambda} K_\lambda$. Since M is compact, we can find finitely many i,

$i = 1, 2, ..., p$ such that $\bigcup_{i=1}^{p} K_i$ together with $K_{p+1} = K_\infty \cap B_{r+1}(0)$ cover M. Since K_∞ is the unbounded component of $X \setminus F(\partial\Omega)$, it contains points $y \notin F(\bar{\Omega})$ and so $D(F, \Omega, K_\infty) = 0$. Hence $D(F, \Omega, K_{p+1}) = 0$.

Now suppose $\lambda \notin \{1, 2, ..., p\}$ and $\lambda \neq \infty$. $X \setminus B_r(0) \subseteq X \setminus F(\bar{\Omega}) \subseteq X \setminus F(\partial\Omega)$ and $X \setminus B_r(0)$ is unbounded and connected. Hence $X \setminus B_r(0) \subseteq K_\infty$. Since $K_\lambda \cap K_\infty = \emptyset$ we must have $K_\lambda \subseteq B_r(0)$. Since the connected components are disjoint and since $M \subseteq \bigcup_{i=1}^{p+1} K_i$ we must have $K_\lambda \cap M = \emptyset$ and so $K_\lambda \cap G^{-1}(y) = \emptyset$. Hence $D(G, K_\lambda, y) = 0$ for $\lambda \notin \{1, 2, ..., p\}$, proving that the sum is finite.

Step 2 :

Let $S_m = \{z \in B_{r+1}(0)\setminus F(\partial\Omega) \mid D(F, \Omega, z) = m\}$ and

$N_m = \{\lambda \in \Lambda \mid D(F, \Omega, K_\lambda) = m\}$ for $m \in \mathbb{Z} \setminus \{0\}$. Now

$S_m \subseteq B_{r+1}(0) \setminus F(\partial\Omega) \subseteq X \setminus F(\partial\Omega) = \bigcup_{\lambda\in\Lambda} K_\lambda$.

If $x \in S_m$, then $x \in K_\lambda$ for some $\lambda \in \Lambda$ and $D(F, \Omega, x) = m$. Hence $D(F, \Omega, K_\lambda) = m$ and so $\lambda \in N_m$. Thus $S_m \subseteq \bigcup_{\lambda\in N_m} K_\lambda$. Now if $x \in \bigcup_{\lambda\in N_m} K_\lambda$, then $x \in K_\lambda$ for some $\lambda \in N_m$. So

$D(F, \Omega, x) = D(F, \Omega, K_\lambda) = m$. We must still show that $x \in B_{r+1}(0) \setminus F(\partial\Omega)$. Now $\bigcup_{\lambda\in N_m} K_\lambda \subseteq X \setminus F(\partial\Omega)$ and so $x \notin F(\partial\Omega)$. Since $m \neq 0$, K_λ is not the unbounded component K_∞ and so x must be an element of $B_{r+1}(0)$ and hence $x \in S_m$. Thus $S_m = \bigcup_{\lambda\in N_m} K_\lambda$. Hence S_m is open. So by lemma 3.10,

$$\begin{aligned}\sum_{\lambda\in\Lambda} D(F, \Omega, K_\lambda)\, D(G, K_\lambda, y) &= \sum_m m \Big(\sum_{\lambda\in N_m} D(G, K_\lambda, y) \Big) \\ &= \sum_m m\, D(G, S_m, y). \qquad (20)\end{aligned}$$

Thus we have to show that

$$D(GF, \Omega, y) = \sum_m m\, D(G, S_m, y). \tag{20'}$$

Now $\partial S_m = \overline{\bigcup_{\lambda \in N_m} K_\lambda} \setminus \bigcup_{\lambda \in N_m} K_\lambda \subseteq \bigcup_{\lambda \in N_m} \bar{K}_\lambda \setminus \bigcup_{\lambda \in N_m} K_\lambda \subseteq \bigcup_{\lambda \in N_m} \partial K_\lambda$ and

$\partial K_\lambda \subseteq F(\partial\Omega)$ for all $\lambda \in N_m$.

Thus $\partial S_m \subseteq F(\partial\Omega)$. By theorem 3.5, we can find $G_1 \in \mathscr{F}(\bar{B}_{r+1}(0))$ such that

$\sup_{x \in \bar{B}_{r+1}(0)} |G_1 x - G_0 x| < \rho(y, GF(\partial\Omega))$. Let $\tilde{G} = I - G_1$. Then

$|\tilde{G}F - GF|_0 = |\, (\tilde{G} - G)|_{F(\bar{\Omega})}\, |_0 \leq |\tilde{G} - G|_0 = |G_1 - G_0|_0 < \rho(y, GF(\partial\Omega))$

and

$|\tilde{G} - G|_0 < \rho(y, GF(\partial\Omega)) \leq \rho(y, G(\partial S_m))$. Thus by (D5),

$$D(\tilde{G}F, \Omega, y) = D(GF, \Omega, y) \tag{21}$$

and

$$D(\tilde{G}, S_m, y) = D(G, S_m, y) \tag{22}$$

for all m.

If $M_0 = \bar{B}_{r+1}(0) \cap \tilde{G}^{-1}(y) = \emptyset$, then both sides of (20′) are zero, so we may assume that $M_0 \neq \emptyset$.

Since M_0 is compact and $y \notin \tilde{G}F(\partial\Omega)$, we have

$\rho(M_0, F(\partial\Omega)) = \inf\{|x - z| \;/\; x \in M_0, z \in F(\partial\Omega)\} > 0$ because

$F(\partial\Omega) = (I - F_0)(\partial\Omega)$ is closed.

Again by theorem 3.5, we can find $F_1 \in \mathscr{F}(\bar{\Omega})$ such that

$|F_1 - F_0|_0 < \min\{1, \rho(M_0, F(\partial\Omega))\}$.

Let $\tilde{F} = I - F_1$. For $x \in \bar{\Omega}$,

$|\tilde{F}x| \leq |\tilde{F}x - Fx| + |Fx| \leq |\tilde{F} - F|_0 + |Fx| < 1 + r$,

so $\tilde{F}(\bar{\Omega}) \subseteq B_{r+1}(0)$.

Define $\tilde{S}_m = \{z \in B_{r+1}(0) \setminus \tilde{F}(\partial\Omega) \;/\; D(\tilde{F}, \Omega, z) = m\}$. Since

$|F_1 - F_0|_0 < \rho(M_0, F(\partial\Omega)) \leq \rho(z, F(\partial\Omega))$ for all $z \in M_0$, we have

$D(F_1, \Omega, z) = D(F, \Omega, z)$ for all $z \in M_0$. Thus $S_m \cap M_0 = \tilde{S}_m \cap M_0$. Now $y \notin \tilde{G}(\partial S_m)$. We want to show that $y \notin \tilde{G}(S_m \setminus S_m \cap \tilde{S}_m)$. If $y = \tilde{G}x$ with $x \in S_m$ then $x \in \tilde{G}^{-1}(y) \cap \bar{B}_{r+1}(0) = M_0$ and so $x \in M_0 \cap S_m = M_0 \cap \tilde{S}_m$. Hence $x \in S_m \cap \tilde{S}_m$. Thus $y \notin \tilde{G}(S_m \setminus S_m \cap \tilde{S}_m)$, and so by (D7)

$$D(\tilde{G}, S_m, y) = D(\tilde{G}, S_m \cap \tilde{S}_m, y). \tag{23}$$

Similarly,

$$D(\tilde{G}, \tilde{S}_m, y) = D(\tilde{G}, S_m \cap \tilde{S}_m, y). \tag{24}$$

From (20), (22), (23) and (24) we obtain

$$\sum_\lambda D(F, \Omega, K_\lambda)\, D(G, K_\lambda, y) = \sum_m m\, D(\tilde{G}, \tilde{S}_m, y)$$

and from (21) we obtain

$$D(GF, \Omega, y) = D(\tilde{G}F, \Omega, y).$$

Now choose a subspace X_1 of X such that $\dim X_1 < \infty$, $y \in X_1$, $F_1(\bar{\Omega}) \subseteq X_1$ and $G_1(\bar{B}_{r+1}(0)) \subseteq X_1$. By the product formula in finite dimensions and the definition of the Leray–Schauder degree, we have

$$\begin{aligned} \sum_m m\, D(\tilde{G}, \tilde{S}_m, y) &= \sum_m m\, d(\tilde{G}|_{X_1}, \tilde{S}_m \cap X_1, y) \\ &= d(\tilde{G}\,\tilde{F}|_{\overline{\Omega \cap X_1}}, \Omega \cap X_1, y) \\ &= D(\tilde{G}\,\tilde{F}, \Omega, y). \end{aligned}$$

So we have $\sum_\lambda D(F, \Omega, K_\lambda)\, D(G, K_\lambda, y) = D(\tilde{G}\,\tilde{F}, \Omega, y)$ and $D(GF, \Omega, y) = D(\tilde{G}F, \Omega, y)$. Thus, we just need to show that

$$D(\tilde{G}\,\tilde{F}, \Omega, y) = D(\tilde{G}F, \Omega, y).$$

Now $\tilde{G}\,\tilde{F} = I - (G_1\tilde{F} + F_1)$ $\quad$ (25)

and $\tilde{G}F = I - (F_0 + G_1F)$. $\quad$ (26)

Consider $H(t, x) = F_0x + t\,(F_1x - F_0x) + G_1(Fx + t(\tilde{F}x - Fx))$ for $(t, x) \in J \times \bar{\Omega}$. Then $H \in \mathscr{K}(J \times \bar{\Omega}, X)$ and $x - H(t, x) = \tilde{G}(Fx + t\,(\tilde{F}x - Fx))$.

If $y = (I - H(t, .))x$ with $x \in \partial\Omega$ and $t \in J$, then $\tilde{G}(Fx + t\,(\tilde{F}x - Fx)) = y$.

Also $Fx + t(\tilde{F}x - Fx) = (1-t)Fx + t\tilde{F}x \in \bar{B}_{r+1}(0)$. Thus

$z = Fx + t(\tilde{F}x - Fx) \in M_0$. Since $x \in \partial\Omega$,

$|z - Fx| \geq \rho(M_0, F(\partial\Omega)) > |F_1 - F_0|_0$. But

$|z - Fx| = |Fx + t(\tilde{F}x - Fx) - Fx| = t\,|F_0 x - F_1 x|$.

Thus $|F_0 x - F_1 x| \geq t\,|F_0 x - F_1 x| > |F_1 - F_0|_0$, a contradiction.

Hence $y \notin (I - H(t, .))(\partial\Omega)$. Applying (D3), we get

$$D(I - H(1, .), \Omega, y) = D(I - H(0, .), \Omega, y)$$

which is the same as

$$D(I - (F_1 + G_1\tilde{F}), \Omega, y) = D(I - (F_0 + G_1 F), \Omega, y).$$

By (25) and (26) we obtain

$$D(\tilde{G}\,\tilde{F}, \Omega, y) = D(\tilde{G}F, \Omega, y)$$

which was what we were required to show. ♠

We obtain the following version of Jordan's separation theorem.

3.15 Theorem

Let A and B be closed bounded subsets of the real Banach space X such that there exists a homomorphism $G = I - F$ from A onto B, with $F \in \mathscr{F}(A)$. Then $X \backslash A$ and $X \backslash B$ have the same number of components.

We do not include a proof here because it is along the lines of theorem 2.16.

Now we prove a result that reduces a degree on some space to a degree on a subspace.

3.16 Theorem

Let X_0 be a closed subspace of X, $\Omega \subseteq X$ open bounded, $F : \bar{\Omega} \longrightarrow X_0$ compact, $G = I - F$, $y \in X_0 \setminus G(\partial\Omega)$. Then

$$D(G, \Omega, y) = D(G|_{\overline{\Omega \cap X_0}}, \Omega \cap X_0, y).$$

Proof :

Since $G(\partial\Omega)$ is closed $\rho = \rho(y, G(\partial\Omega)) > 0$. By theorem 3.5, we can find $F_1 \in \mathscr{F}(\bar{\Omega}, X_0)$ such that $\sup_{\bar{\Omega}} |F_1 x - Fx| < \rho$.

Let X_1 be a subspace of X such that $\dim X_1 < \infty$, $F_1(\bar{\Omega}) \subseteq X_1$, $y \in X_1$, $\Omega_0 = \Omega \cap X_0$ and $\Omega_1 = \Omega \cap X_1$.

Now $X_0 \cap X_1$ is a subspace of X, $\dim (X_0 \cap X_1) < \infty$, $y \in X_0 \cap X_1$, $F_1(\bar{\Omega}) \subseteq X_0 \cap X_1$.

So by definition

$$\begin{aligned} D(G, \Omega, y) &= d((I - F_1)|_{\overline{\Omega \cap (X_0 \cap X_1)}}, \Omega \cap X_0 \cap X_1, y) \\ &= d((I - F_1)|_{\overline{\Omega_0 \cap X_1}}, \Omega_0 \cap X_1, y). \end{aligned} \tag{27}$$

Also $\partial\Omega_0 = \overline{\Omega \cap X_0} \setminus \Omega \cap X_0 \subseteq \bar{\Omega} \cap X_0 \setminus \Omega \cap X_0 = (\partial\Omega) \cap X_0 \subseteq \partial\Omega$ and so $\sup_{\bar{\Omega}_0} |F_1 x - Fx| \le \sup_{\bar{\Omega}} |F_1 x - Fx| < \rho(y, G(\partial\Omega)) \le \rho(y, G(\partial\Omega_0))$.

Hence by definition

$$D(G|_{\bar{\Omega}_0}, \Omega_0, y) = d((I - F_1)|_{\overline{\Omega_0 \cap X_1}}, \Omega_0 \cap X_1, y). \tag{28}$$

From (27) and (28), we get

$$D(G, \Omega, y) = D(G|_{\bar{\Omega}_0}, \Omega_0, y).$$ ♠

The fixed point theorem corresponding to Brouwer's fixed point theorem is Schauder's fixed point theorem, which follows. It was extended by Schauder in 1930.

3.17 Theorem

Let X be a real Banach space, $C \subseteq X$ nonempty closed bounded and convex, $F : C \longrightarrow C$ compact. Then F has a fixed point.

Proof :

By the remarks after definition 1.2.16, C is a retract of X. So there exists a retraction $R : X \longrightarrow C$. Since C is bounded, there exists $r > 0$ such that $C \subseteq B_r(0)$. Now $FR : X \longrightarrow C$ is continuous. Let $H(t, x) = t\, FRx$ for $(t, x) \in J \times \bar{B}_r(0)$. If $0 = (I - H(t, .))x$ for some $(t, x) \in J \times \partial B_r(0)$, then $x = t\, FRx$.

$t = 0$ implies that $x = 0$, a contradiction.

$t \neq 0$ implies $\frac{1}{t} x = FRx \in C$ and so $| \frac{1}{t} x| < r$. But $| \frac{1}{t} x | = \frac{1}{t} |x| \geq r$, a contradiction.

So $0 \notin (I - H(t, .))(\partial B_r(0))$ on J.

Thus by (D3),

$D(I - FR, B_r(0), 0) = D(I, B_r(0), 0) = 1$ by (D1).

By (D4), there exists $x \in B_r(0)$ such that $(I - FR)x = 0$. So $x = FRx \in C$ and hence $FRx = Fx$. Thus

$$x = Fx.$$

♠

Given a problem where we want to use Schauder's fixed point theorem or a degree argument, we first look for a suitable Banach space X. Then we formulate the problem as $x - Fx = 0$ such that F is completely continuous, if we can. Thereafter we apply the homotopy $H(t, x)$ to reduce $I - F$ to a simpler map $I - F_0$. In most examples, the most difficult part is finding a suitable open bounded $\Omega \subseteq X$ such that $H(t, x) \neq x$ on $\partial\Omega$, or finding a closed bounded convex set C such that C is invariant under F.

This is the question of finding *a priori bounds* for the possible solutions, i.e. in the simplest case, find $r > 0$ such that

$$\{x \,/\, x - \lambda\, Fx = 0 \text{ for some } \lambda \in [0, 1]\} \subseteq \bar{B}_r(0).$$

This can be illustrated by the following example.

3.18 Example

Let X be a real Banach space, $J = [0, a] \subseteq \mathbb{R}$, $f : J \times X \longrightarrow X$ completely continuous and $|f(t, x)| \leq c\,(1 + |x|)$ on $J \times X$, for some $c \geq 0$. Then the initial value problem,

$$x' = f(t, x),\ x(0) = x_0, \tag{29}$$

has at least one solution on J.

It is useful to note that (29) is equivalent to the existence of a continuous function $x : J \longrightarrow X$ such that

$$x(t) = x_0 + \int_0^t f(s, x(s))\, ds. \tag{30}$$

The natural space for (30) is $Y = C_X(J)$.

Define $F : Y \longrightarrow Y$ by $(Fx)(t) = x_0 + \int_0^t f(s, x(s))\, ds$ for $x \in Y$ and $t \in J$.

To show that F is completely continuous, we must show that for every bounded $B \subseteq Y$, FB is relatively compact. Now

$F(B)(t) \subseteq \{x_0\} + \{ \int_0^t f(s, x(s))\, ds \;/\; x \in B \}$ for $t \in J$.

Since $\int_0^t g(s)\, ds$ is the limit of Riemann sums $t \sum_i g(s_i)\,(s_i - s_{i-1})/t$, we have

$\{ \int_0^t f(s, x(s))\, ds \;/\; x \in B \} \subseteq t\, \overline{\text{conv}}\, \{ f(s, x(s)) \;/\; s \in J, x \in B \}$. So

$\alpha(FB(t)) \leq \alpha(\{ \int_0^t f(s, x(s)\, ds \;/\; x \in B \}) \leq t\, \alpha(\{f(s, x(s)) \;/\; s \in J, x \in B \})$.

Since $J \times B$ is bounded, and f is completely continuous, we must have

$\alpha(\{f(s, x(s)) \;/\; s \in J, x \in B\}) = 0$, and so $\alpha(FB(t)) = 0$ for all $t \in J$. Thus

$$\sup_{t \in J} \alpha(FB(t)) = 0. \tag{31}$$

Now FB is bounded and for $x \in B$, $t_1, t_2 \in J$,

$$\begin{aligned} |Fx(t_1) - Fx(t_2)| &= \left| \int_0^{t_1} f(s, x(s))\, ds - \int_0^{t_2} f(s, x(s))\, ds \right| \\ &= \left| \int_{t_2}^{t_1} f(s, x(s))\, ds \right| \quad (\text{assume } t_1 \geq t_2) \end{aligned}$$

$$\leq \int_{t_2}^{t_1} |f(s, x(s))| \, ds$$
$$\leq |t_1 - t_2| \, c \, (1 + |x|)$$
$$\leq c \, |t_1 - t_2| \, (1 + M) \quad \text{if M is a bound of B,}$$

and so FB is easily equicontinuous.

Therefore by theorem ,

$$\alpha(FB) = \sup_J \alpha(FB(t)). \tag{32}$$

So by (31) and (32), $\alpha(FB) = 0$ and hence F is completely continuous.

Now suppose x is a solution of $(I - \lambda F)x = 0$ for some $\lambda \in [0, 1]$. Then

$$|x(t)| \leq |x_0| + c \int_0^t (1 + |x(s)|) \, ds \leq c_1 + c \int_0^t |x(s)| \, ds = \varphi(t)$$

with $c_1 = |x_0| + c\,a$.

Now $\varphi'(t) = c \, |x(t)| \leq c \, \varphi(t)$.

$$\begin{aligned} \text{So } (\varphi(t) \, e^{-ct})' &= \varphi'(t) \, e^{-ct} - c \, \varphi(t) \, e^{-ct} \\ &= [\varphi'(t) - c \, \varphi(t)] \, e^{-ct} \\ &\leq 0. \end{aligned}$$

Therefore $\int_0^t (\varphi(s) \, e^{-cs})' ds \leq 0$, which is the same as $\varphi(t) \, e^{-ct} \leq \varphi(0) = c_1$ for all $t \in J$. Hence we have the *a priori* estimate,

$$|x|_0 \leq \sup_{t \in J} \varphi(t) \leq \sup_{t \in J} c_1 \, e^{ct} = c_1 \, e^{ca} = c_2.$$

Choose $r > c_2$. If $H(t, x) = t\,Fx$, $(t, x) \in [0, 1] \times \bar{B}_r(0)$, then H is compact (since $H([0, 1] \times \bar{B}_r(0) \subseteq \text{conv}\,(F\,\bar{B}_r(0) \cup \{0\}\,)\,)$ and $(I - H(t, .))x = 0$ implies that $(I - t\,F)x = 0$ and so $(I - t\,F)x = 0$ with $t \in [0, 1]$. Therefore $|x|_0 \leq c_2 < r$. Thus $x \in B_r(0)$ which means that $x \notin \partial B_r(0)$, and hence by (D3),

$$D(I - F, B_r(0), 0) = D(I, B_r(0), 0), \tag{33}$$

and by (D1),

$$D(I, B_r(0), 0) = 1. \tag{34}$$

(33) and (34) give us $D(I - F, B_r(0), 0) = 1$. By (D4), there exists $x \in B_r(0)$ such

that $(I-F)x = 0$. Thus, (30) has a continuous solution. ♠

The following is a result of Schäfer concerning the homotopy $H(t, x) = t\,Fx$.

3.19 Corollary

Let $F : X \longrightarrow X$ be completely continuous. Then the following alternative holds: Either $x - t\,Fx = 0$ has a solution for every $t \in [0, 1]$, or $S = \{x \,/\, x = t\,Fx \text{ for some } t \in (0, 1)\}$ is unbounded.

Proof :

Suppose $x - t_0 Fx = 0$ has no solution for some $t_0 \in (0, 1]$ and let $F_0 = t_0 F$. Now take any $r > 0$ and consider the radial retraction $R : X \longrightarrow \bar{B}_r(0)$ defined by

$$Rx = \begin{cases} x & \text{if } |x| \leq r \\ \frac{rx}{|x|} & \text{if } |x| > r \end{cases}$$

Then $RF_0|_{\bar{B}_r(0)} : \bar{B}_r(0) \longrightarrow \bar{B}_r(0)$ is continuous.

Let $(x_n) \subseteq \bar{B}_r(0)$. To show that $RF_0(x_n)$ has a convergent subsequence. Since F is completely continuous, so is F_0. Thus $(F_0 x_n)$ has a convergent subsequence, say $F_0 x_{n_k} \longrightarrow y$ and continuity of R gives us $RF_0 x_{n_k} \longrightarrow Ry$. Thus RF_0 is a compact operator. Since $\bar{B}_r(0)$ is closed, bounded and convex, we can apply Schauder's fixed point theorem (theorem 3.17), to obtain a point $x \in \bar{B}_r(0)$ such that $RF_0 x = x$. If $F_0 x \in \bar{B}_r(0)$, then $RF_0 x = F_0 x$ and then we get $t_0 Fx = x$, a contradiction to this equation having no solution. Hence $|F_0 x| > r$ and so $x = RF_0 x = \frac{rF_0 x}{|F_0 x|}$. So $x = \mu\,Fx$ with $\mu = \frac{rt_0}{|F_0 x|}$ and $0 < \mu < 1$, i.e. $x \in S$. We also obtain $|x| = \left| \frac{rF_0 x}{|F_0 x|} \right| = r$. Thus S is unbounded. ♠

Compact Linear Operators

Up to this point we have considered arbitrary nonlinear operators. In applications, we sometimes encounter nonlinearities of the form $F = L + R$ where L is linear and R is nonlinear, but small in some sense. Then we would like to know whether the nice properties of L carry over to F.

Among the linear operators of a Banach space into itself, the compact linear operators, are quite simple, since the results from linear algebra can be extended to this class. We denote this class by $CL(X)$ and if $L \in CL(X)$, then L is a completely continuous operator, but we will call it compact. The aim in this section is to obtain a formula similar to $d(A, \Omega, 0) = \operatorname{sgn} \det A$ from chapter 2.

3.20 Theorem

Let X be a real Banach space, $L_0 \in CL(X)$ and $L = I - L_0$. Then we have

(a) Let $M = I - M_0$ with $M_0 \in CL(X)$. Suppose also that L and M are one–to–one. Then $D(LM, \Omega, 0) = D(L, \Omega, 0)\, D(M, \Omega, 0)$ for every bounded $\Omega \subseteq X$ such that $0 \in \Omega$.

(b) Let $X = \bigoplus_{i=1}^{m} X_i$ be the topological direct sum of closed subspaces $X_1, \ldots, X_m$ such that $L_0(X_i) \subseteq X_i$. Let L be one–to–one. Then
$D(L, B_1(0), 0) = \prod_{i=1}^{m} D(L|_{X_i}, B_1(0) \cap X_i, 0)$.

Proof :

(a) Let K_λ be the connected components of $X \backslash M(\partial\Omega)$. Now $0 \in \Omega$, so $0 \notin \partial\Omega$. Now M is one–to–one and $M0 = 0$. So $0 \notin M(\partial\Omega)$. Thus $0 \in K_\alpha$ for some α and $0 \notin K_\lambda$ for all $\lambda \neq \alpha$.

Also, since L is linear, we must have $L0 = 0$. Thus $L^{-1}0 = \{0\}$ (since L is one–to–one). So $L^{-1}0 \cap K_\lambda = \emptyset$ for all $\lambda \neq \alpha$ and hence

$$D(L, K_\lambda, 0) = 0. \tag{35}$$

Now by the product formula, we get

$$D(LM, \Omega, 0) = \sum_{\lambda} D(M, \Omega, K_\lambda)\, D(L, K_\lambda, 0)$$

and by (35) we get

$$D(LM, \Omega, 0) = D(M, \Omega, K_\alpha)\, D(L, K_\alpha, 0). \tag{36}$$

Since $0 \in K_\alpha$, we have by definition

$$D(M, \Omega, K_\alpha) = D(M, \Omega, 0). \tag{37}$$

Also, since $0 = L0 \in L(K_\alpha)$, we have $0 \notin L(\bar{\Omega} \setminus K_\alpha)$, and hence by (D7)

$$D(L, \Omega, 0) = D(L, K_\alpha, 0). \tag{38}$$

By (36), (37) and (38) we have

$$D(LM, \Omega, 0) = D(M, \Omega, 0)\, D(L, \Omega, 0),$$

as required.

(b) It is sufficient to prove the case $m = 2$, for the result, will follow by induction.

Consider the projections $P_i : X \to X_i$ and $M_1 = LP_1 + P_2$ and $M_2 = P_1 + LP_2$.

$M_1M_2 = (LP_1 + P_2)(P_1 + LP_2) = LP_1^2 + P_2P_1 + LP_1LP_2 + P_2LP_2$.

Now $LP_1LP_2(X) = LP_1L(X_2) \subseteq LP_1X_2 = 0$, $P_1^2 = P_1$, $P_2P_1 = 0$

and $LP_2(X) \subseteq X_2$ and so $P_2LP_2 = LP_2$.

Thus $M_1M_2 = LP_1 + LP_2 = L(P_1 + P_2) = L$.

$$\begin{aligned}
(I - M_i)(X_i) &= [I - (P_j + LP_i)](X_i) \quad i \neq j \\
&= (I - LP_i)(X_i) \\
&= (I - L)(X_i) \\
&= L_0(X_i) \\
&\subseteq X_i \text{ for } i = 1, 2,
\end{aligned}$$

and so

$I - M_1 = L_0P_1 \in CL(X)$ and $I - M_2 = L_0P_2 \in CL(X)$.

Suppose $M_2x = 0$ for some $x \in X$. Let $x_i = P_ix$, $i = 1, 2$. So $x = x_1 + x_2$

and $x_i \in X_i$.

$M_i(x_1 + x_2) = 0$ implies that $(P_j + LP_i)(x_1 + x_2) = 0$

and so $x_j + Lx_i = 0$.

But $Lx_i \in X_i$ and $x_j \in X_j$ with $i \neq j$.

By uniqueness of the representation $x_j = 0$ and $Lx_i = 0$. But L is one–to–one and so $x_1 = 0 = x_2$. Thus $x = 0$.

Thus M_i is one–to–one for i = 1, 2. With $\Omega = B_1(0)$,

$$\begin{aligned} D(L, \Omega, 0) &= D(M_1M_2, \Omega, 0) \\ &= D(M_1, \Omega, 0)\, D(M_2, \Omega, 0) \quad \text{(by part (a))} \\ &= D(M_1|_{\overline{\Omega\cap X_1}}, \Omega \cap X_1, 0)\, D(M_2|_{\overline{\Omega\cap X_2}}, \Omega \cap X_2, 0) \\ &= \prod_{i=1}^{2} D(M_i|_{\overline{\Omega\cap X_i}}, \Omega \cap X_i, 0) \\ &= \prod_{i=1}^{2} D(L|_{X_i}, \Omega \cap X_i, 0). \end{aligned}$$

♠

The next two results can be found in basic texts in functional analysis and no proofs will be given.

3.21 Theorem

Let X be a Banach space, $L_0 \in CL(X)$ and $L = I - L_0$. Then

(a) $N(L) = \{x \in X \,/\, Lx = 0\}$ is finite dimensional and

$R(L) = \{(x \,/\, x \in X\}$ is closed.

(b) Suppose that V and W are closed subspaces of X such that

$V \subset W$, $V \neq W$ and $L(W) \subseteq V$. Then there exists $w \in W \setminus V$ such that $|w| = 1$ and $\rho(L_0w, L_0(V)) \geq \frac{1}{2}$.

The next result is a spectral theorem.

3.22 Theorem

Let X be a Banach space over $\mathbf{K} = \mathbb{R}$ or $\mathbf{K} = \mathbb{C}$, $L_0 \in CL(X)$, $L_\lambda = L_0 - \lambda I$ for $\lambda \in K$, and let Λ be the set of all eigenvalues of L_0. Then

(S1) $\Lambda \subseteq \{\mu \in \mathbf{K} \,/\, |\mu| \leq |L_0|\}$, Λ is at most countable and only $\mu = 0$ may be a cluster point of Λ.

(S2) L_λ is a homeomorphism onto X for every $\lambda \notin \Lambda \cup \{0\}$.

(S3) To every $\lambda \in \Lambda \setminus \{0\}$ there exists a smallest natural number $k = k(\lambda)$ such that we have, with $R(\lambda) = R(L_\lambda^k)$ and $N(\lambda) = N(L_\lambda^k)$

(a) $X = R(\lambda) \oplus N(\lambda)$, $\dim N(\lambda) < \infty$ and $R(\lambda)$ is closed.

(b) $R(\lambda)$ and $N(\lambda)$ are invariant under L_0 and $L_\lambda|_{R(\lambda)}$ is a homeomorphism onto $R(\lambda)$.

(c) $N(\mu) \subseteq R(\lambda)$ whenever $\lambda, \mu \in \Lambda \setminus \{0\}$ and $\lambda \neq \mu$.

As in linear algebra, $\dim N(\lambda)$ is called the *algebraic multiplicity* of the eigenvalue λ while $\dim N(L_\lambda) \leq \dim N(\lambda)$ is called the *geometric multiplicity* of λ. We now prove the analogue of

$$d(A, \Omega, 0) = \operatorname{sgn} \det A = (-1)^{\dim N}$$

from chapter 2.

3.23 Theorem

Let X be a real Banach space, $L \in CL(X)$, $\lambda \neq 0$ and λ^{-1} not an eigenvalue of L. Let $\Omega \subseteq X$ be open bounded and $0 \in \Omega$. Then $D(I - \lambda L, \Omega, 0) = (-1)^{m(\lambda)}$, where $m(\lambda)$ is the sum of the algebraic multiplicities of the eigenvalues μ of L satisfying $\mu\,\lambda > 1$, and $m(\lambda) = 0$ if L has no eigenvalues of this kind.

Proof :

Let $M = I - \lambda L = -\lambda(L - \lambda^{-1}I)$. By (S2), $L - \lambda^{-1}I$ is a homeomorphism onto X since λ^{-1} is not an eigenvalue of L. Hence M is a homeomorphism onto X. Thus $D(I - \lambda L, \Omega, 0) = D(I - \lambda L, B_1(0), 0)$ by (D7) and so it is sufficient to consider $\Omega = B_1(0)$. By (S1), there are at most finitely many $\mu \in \Lambda$ such that $\mu\lambda > 1$, i.e. $\operatorname{sgn}\mu = \operatorname{sgn}\lambda$ and $|\mu| > |\lambda|^{-1}$, say $\mu_1, \ldots, \mu_p$.

Let $V = \bigoplus_{i=1}^{p} N(\mu_i)$ and $W = \bigcap_{j=1}^{p} R(\mu_j)$. We will now show that $X = V \oplus W$.

If $x \in V \cap W$, then $x = \sum_{j=1}^{p} x_j$, $x_j \in N(\mu_j)$ and $x \in R(\mu_j)$ for $j = 1, 2, \ldots, p$.

By (S3)(c), $N(\mu_j) \subseteq R(\mu_1)$ for $j = 2, \ldots, p$ and we have $\sum_{j=2}^{p} x_j \in R(\mu_1)$. Hence we have $x_1 = x - \sum_{j=2}^{p} x_j \in R(\mu_1) \cap N(\mu_1) = \{0\}$, and similarly we may obtain $x_2 = \ldots = x_p = 0$. Thus $V \cap W = \{0\}$.

Now take any $x \in X$. Then $x = x_j + y_j$ for $x_j \in N(\mu_j)$ and $y_j \in R(\mu_j)$ by (a) of (S3).

$$\begin{aligned} \text{So } x - \sum_{j=1}^{p} x_j &= x - x_k - \sum_{j \neq k} x_j \\ &= y_k - \sum_{j \neq k} x_j \in R(\mu_k) \text{ by (c) of (S3) for } k = 1, 2, \ldots, p. \end{aligned}$$

Thus $x - \sum_{j=1}^{p} x_j \in W$ and so $x = \sum_{j=1}^{p} x_j + w$ for some $w \in W$. Hence we get $X = V \oplus W$.

$L(V) \subseteq V$ and $L(W) \subseteq W$ since V and W are invariant under L by (b) of (S3). Thus $\lambda L(V) \subseteq V$ and $\lambda L(W) \subseteq W$. Also, M is a homeomorphism, hence one–to–one. So we may apply theorem 3.20 to get

$$D(M, \Omega, 0) = D(M|_V, \Omega \cap V, 0).D(M|_W, \Omega \cap W, 0) \tag{39}$$

with $\Omega = B_1(0)$.

Consider $H(t, x) = t\lambda Lx$ for $(t, x) \in J \times \overline{\Omega \cap W}$.

and suppose $0 = (I - H(t, .))(x)$ for $(t, x) \in J \times \partial(\Omega \cap W)$.

So $0 = (I - t\lambda L)x$.

If $t = 0$, then $x = 0$.

If $t = 1$, then $(I - \lambda L)x = 0$ and so $(L - \lambda^{-1}I)x = 0$. Since λ^{-1} is not an eigenvalue of L, we must have $x = 0$. Suppose $0 < t < 1$. Then $(L - (t\lambda)^{-1}I)x = 0$ with $(t\lambda)^{-1}\lambda = t^{-1} > 1$. Hence if $x \neq 0$ $(t\lambda)^{-1}$ is one of $\mu_1, \ldots, \mu_p$, say μ_j and $x \in N(\mu_j)$. But $x \in W$. So we have $x \in V \cap W = \{0\}$. Thus $x = 0$.

So for all $t \in J$, $0 \notin (I - H(t, .))(\partial(\Omega \cap W))$. Since H is compact, we have by (D3), $D(M|_W, \Omega \cap W, 0) = D(I, \Omega \cap W, 0)$ and $D(I, \Omega \cap W, 0) = 1$ by (D1). So

$$D(M|_W, \Omega \cap W, 0) = 1. \tag{40}$$

Since $N(\mu_i)$ is finite dimensional for each i and by theorem 3.20, again, we have

$$D(M|_V, \Omega \cap V, 0) = \prod_{i=1}^{p} d(M|_{N(\mu_i)}, \Omega \cap N(\mu_i), 0). \tag{41}$$

Now define $h(t, x) = (2t - 1)x - t\lambda Lx$ for $(t, x) \in J \times \overline{\Omega \cap N(\mu_i)}$ and let $0 = h(t, x)$. Then $0 = (2t - 1)x - t\lambda Lx$.

If $t = 0$ then $x = 0$. If $t = 1$ then $(I - \lambda L)x = 0$ and hence $x = 0$, since λ^{-1} is not an eigenvalue of L. Now suppose $0 < t < 1$. Then if $x \neq 0$, $(L - \frac{2t - 1}{t\lambda} I)x = 0$ and since μ_j is the only eigenvalue of $L|_{N(\mu_j)}$ we must have $\frac{2t - 1}{t\lambda} = \mu_j$ and $\frac{2t - 1}{t} = \lambda\mu_j > 1$ and so $t > 1$, a contradiction.

Thus $x = 0$. So $0 \notin h(t, \partial(\Omega \cap N(\mu_i)))$ and hence by (d3)

$$\begin{aligned} d(M|_{N(\mu_j)}, \Omega \cap N(\mu_i), 0) &= d(-id|_{N(\mu_j)}, \Omega \cap N(\mu_j), 0) \\ &= (-1)^{\dim N(\mu_j)}. \end{aligned} \tag{42}$$

Thus (39), (40), (41) and (42) give us

$$D(M, \Omega, 0) = \prod_{i=1}^{p} (-1)^{\dim N(\mu_i)} = (-1)^{m(\lambda)}$$

where $m(\lambda) = \sum_{i=1}^{p} \dim N(\mu_i)$. If there are no such μ, then $X = W$ and

$$D(M, \Omega, 0) = 1 = (-1)^0.$$

♠

We illustrate this theorem with the next example.

3.24 Example

Consider the boundary value problem

$$x'' + \mu x = 0 \quad \text{in } J \tag{43}$$

$$x(0) = x(1) = 0. \tag{44}$$

By standard results on boundary value problems, (43) and (44) are equivalent to

$$\left.\begin{aligned} & x(t) - \mu \int_0^1 k(s,\ t)\ x(s)\ ds \ = 0 \ \text{ in } J \\ \text{where} \qquad & k(s,\ t) = \begin{cases} s(1-t) & 0 \le s \le t \le 1 \\ t(1-s) & 0 \le t \le s \le 1 \end{cases} \end{aligned}\right\}. \tag{45}$$

Let $X = C(J)$ and $(Lx)(t) = \int_0^1 k(t,\ s)\ x(s)\ ds$. Then $L \in CL(X)$.

Thus (45) becomes

$$x - \mu\, Lx = 0. \tag{46}$$

Now (46) has nontrivial solutions <=> μ^{-1} is an eigenvalue of L.

If $\mu \le 0$, then the general solution to (43) is

$$x(t) = \begin{cases} c\ e^{\sqrt{-\mu}\ t} + d\ e^{-\sqrt{-\mu}\ t} & , \mu < 0 \\ c + \ d\ t & , \mu = 0. \end{cases}$$

The boundary conditions in (44) give us $c = d = 0$. Thus $x(t) = 0$ for $\mu \le 0$.

If $\mu > 0$ then the general solution to (43) is $x(t) = c \sin(\sqrt{\mu}\ t) + d \cos(\sqrt{\mu}\, t)$.

Again the boundary conditions give us $d = 0$ and $c \sin \sqrt{\mu} = 0$.

For $c \neq 0$,

$$\begin{aligned} & c \sin(\sqrt{\mu}) = 0 \\ <=> \ & \sin(\sqrt{\mu}) = 0 \\ <=> \ & \sqrt{\mu} = n\pi \text{ for some } n \in \mathbb{N} \\ <=> \ & \mu = n^2\pi^2 \text{ for some } n \in \mathbb{N}. \end{aligned}$$

$\lambda \neq 0$ is an eigenvalue of L <=> $x - \lambda^{-1}Lx = 0$ has nontrivial solutions

<=> $x(t) = c \sin(\sqrt{\lambda^{-1}}\, t)$ $c \neq 0$ and $\lambda^{-1} = n^2\pi^2$ for some $n \in \mathbb{N}$.

Thus $\lambda_n = (n^2\pi^2)^{-1}$ for $n \in \mathbb{N}$ are the eigenvalues of L.

$$
\begin{aligned}
N(L-\lambda_n I) &= \{x \in X \,/\, (L-\lambda_n I)\,x = 0\} \\
&= \{x \in X \,/\, x(t) = c \sin(\sqrt{\lambda_n^{-1}}\, t),\ c \in \mathbb{R}\} \\
&= \operatorname{span}\{\, x_n(t) = \sin(\sqrt{\lambda_n^{-1}}\, t)\,\} \\
&= \operatorname{span}\{\, x_n(t) = \sin(n\pi t)\,\}.
\end{aligned}
$$

Thus $\dim N(L-\lambda_n I) = 1$.

We now want to show that $k(\lambda_n) = 1$ (i.e. the algebraic multiplicity of λ_n is 1).

Let $x \in N((L-\lambda_n I)^2)$. Then $(L-\lambda_n I)^2 x = 0$ i.e. $(L-\lambda_n I)((L-\lambda_n I)x) = 0$.

So $(L-\lambda_n I)\,x \in N(L-\lambda_n I) = \operatorname{span}\{\sin(n\pi t)\}$. Thus $(L-\lambda_n I)x = c\sin(n\pi t)$ for some $c \in R$ and so

$$x(t) = \lambda_n^{-1}\,[Lx(t) - c\sin(n\pi t)] \tag{47}$$

$$
\begin{aligned}
\text{Now } (Lx)(t) &= \int_0^1 k(t,s)\,x(s)\,ds \\
&= \int_0^t k(t,s)\,x(s)\,ds + \int_t^1 k(t,s)\,x(s)\,ds \\
&= \int_0^t s(1-t)\,x(s)\,ds + \int_t^1 t(1-s)\,x(s)\,ds \\
&= (1-t)\int_0^t s\,x(s)\,ds + t\int_t^1 (1-s)\,x(s)\,ds
\end{aligned}
\tag{48}
$$

And so

$$
\begin{aligned}
(Lx)'(t) &= -\int_0^t s\,x(s)\,ds + (1-t)\,t\,x(t) + \int_t^1 (1-s)\,x(s)\,ds - t\,(1-t)\,x(t) \\
&= -\int_0^1 s\,x(s)\,ds + \int_t^1 x(s)\,ds
\end{aligned}
$$

and $(Lx)''(t) = -x(t)$.

By (47), $x'(t) = \lambda_n^{-1}[(Lx)'(t) - cn\pi\cos(n\pi t)]$

and $x''(t) = \lambda_n^{-1}[(Lx)''(t) + cn^2\pi^2\sin(n\pi t)]$.

So $x''(t) + \lambda_n^{-1}\,x(t) = c\,n^2\pi^2\sin(n\pi t)$. (49)

By (47) and (48) we get $x(0) = 0 = x(1)$.

$$
\begin{aligned}
\text{Now } \int_0^1 \sin^2(n\pi t)\,dt &= \int_0^1 \frac{1-\cos 2n\pi t}{2}\,dt \\
&= \frac{1}{2}\left[t - \frac{\sin 2n\pi t}{2n\pi t}\right]_0^1 \\
&= \frac{1}{2}.
\end{aligned}
$$

$$\text{Thus } \tfrac{c}{2} n^2\pi^2 = cn^2\pi^2 \int_0^1 \sin^2(n\pi t)\, dt$$
$$= \int_0^1 (x''(t) + \lambda_n^{-1} x(t)) \sin(n\pi t)\, dt \quad \text{by (49).}$$

But $\int_0^1 x''(t) \sin(n\pi t)\, dt$

$$= \sin(n\pi t)\, x'(t)\Big|_0^1 - \int_0^1 x'(t)\, n\pi \cos(n\pi t)\, dt$$
$$= -n\pi[\cos(n\pi t)\, x(t)\Big|_0^1 + \int_0^1 x(t)\, n\pi \sin(n\pi t)\, dt\,]$$
$$= -n^2\pi^2 \int_0^1 x(t) \sin(n\pi t)\, dt$$
$$= \int_0^1 (-\lambda_n^{-1} x(t) \sin(n\pi t))\, dt.$$

Substituting in (49) we get $\frac{c}{2} n^2\pi^2 = 0$ and hence $c = 0$.

Thus $(L - \lambda_n I)x = 0$ and so $x \in N(L - \lambda_n I)$.

So $N(L - \lambda_n I) = N((L - \lambda_n I)^2)$ proving that $k(\lambda_n) = 1$ for all $n \in \mathbb{N}$.

If $\lambda < 0$ then $\lambda \lambda_n < 0$ for all $n \in \mathbb{N}$ and λ^{-1} is not an eigenvalue of L and so $m(\lambda) = 0$. Thus $D(I - \lambda L, B_1(0), 0) = 1$ for $\lambda < 0$ and

$D(I - \lambda L, B_1(0), 0) = 1$ for $\lambda = 0$.

If $0 < \lambda < \pi^2$, $\lambda^{-1} > (\pi^2)^{-1} \geq (n^2\pi^2)^{-1} = \lambda_n^{-2}$ for all $n \geq 1$.

So λ^{-1} is not an eigenvalue of L. Also $\lambda\lambda_n < \pi^2 (n^2\pi^2)^{-1} = \dfrac{1}{n^2} \leq 1$ and so

$m(\lambda) = 0$. Therefore

$$D(I - \lambda L, B_1(0), 0) = 1 \quad \text{for } -\infty < \lambda < \pi^2. \tag{50}$$

If $n^2\pi^2 < \lambda < (n+1)^2 \pi^2$, then $\lambda_{n+1} < \lambda^{-1} < \lambda_n$. So λ^{-1} is not an eigenvalue of L.

Now $\lambda \lambda_{n+1} < 1$. So $\lambda \lambda_m \leq \lambda \lambda_{n+1} < 1$ for all $m \geq n + 1$. Also $\lambda \lambda_n > 1$.

So $\lambda \lambda_i \geq \lambda \lambda_n > 1$ for $i = 1, 2, \ldots, n$. Therefore $\lambda_1, \ldots, \lambda_n$ are the only eigenvalues of L satisfying $\lambda \lambda_i > 1$. Also, the algebraic multiplicity of each λ_i is 1 since $\dim N((L - \lambda_i I)^{k(\lambda_i)}) = \dim N(L - \lambda_i I) = 1$.

So $m(\lambda) = n$. By Theorem 3.2.21 we have

$$D(L - \lambda I, B_1(0), 0) = (-1)^{m(\lambda)} \quad \text{for } (n\pi)^2 < \lambda < ((n+1)\lambda)^2$$

$$= (-1)^n \tag{51}$$

(50) and (51) give us

$$D(L - \lambda I, B_1(0), 0) = \begin{cases} 1 & \text{if } -\infty < \lambda < \pi^2 \\ (-1)^n & \text{if } n^2\pi^2 < \lambda < (n+1)^2\pi^2 \end{cases} .$$

♠

CHAPTER 4

4.1 SET CONTRACTIONS

We saw that we could extend the degree theory for finite dimensional maps to a degree for compact perturbations of the identity. Now we extend further to another type of perturbation of the identity. Before we discuss the degree, we will give some definitions.

4.1.1 Definition

In the sequel, X will denote a Banach space and $\gamma : \mathscr{B} \longrightarrow \mathbb{R}$ will be either α or β, the Kuratowski or Hausdorff measures of noncompactness, respectively.

Let $\Omega \subseteq X$ and $F : \Omega \longrightarrow X$ be continuous.

F is *Lipschitz* if $|Fx - Fy| \leq k\ |x - y|$ for some $k > 0$ and all $x, y \in \Omega$ and a *strict contraction* if $k < 1$. If $k = 1$ is the smallest Lipschitz constant, then F is called *nonexpansive.*

F is said to be *γ–Lipschitz* if $\gamma(FB) \leq k\ \gamma(B)$ for some $k \geq 0$ and all bounded $B \subseteq \Omega$.

If $k < 1$, we call F a *strict γ–contraction.*

F is *γ–condensing* if $\gamma(FB) < \gamma(B)$ whenever $B \subseteq \Omega$ is bounded and $\gamma(B) > 0$. (In other words, $\gamma(FB) \geq \gamma(B)$ implies that $\gamma(B) = 0$.)

$SC_\gamma(\Omega)$ will consist of all strict γ–contractions $F : \Omega \longrightarrow X$ and $C_\gamma(\Omega)$ all γ–condensing maps.

F is called a *k–set contraction* if it is a strict γ–contraction with constant k.

These definitions contain the condition that F is bounded; i.e F takes bounded sets into bounded sets. It is easy to see that $SC_\gamma(\Omega) \subseteq C_\gamma(\Omega)$.

Also, if Ω is closed and $F \in C_\gamma(\Omega)$, then F is γ–Lipschitz with $k = 1$. To see this, let $B \subseteq \Omega$ be bounded. If $\gamma(B) > 0$, then $\gamma(FB) < \gamma(B)$. If $\gamma(B) = 0$, then B is relatively compact. Let (y_n) be any sequence in F(B). Then $y_n = Fx_n$ for some $x_n \in B$, $n \in \mathbb{N}$. Since B is relatively compact, some subsequence (x_{k_n}) of (x_n) converges in X to say x. Since Ω is closed, we must have $x \in \Omega$ and $(y_{k_n}) = (Fx_{k_n})$ converges to Fx by the continuity of F. Thus FB is relatively compact and hence $\gamma(FB) = 0 = \gamma(B)$. We have thus shown that F is γ – Lipschitz with constant $k = 1$.

4.1.2 Example

If $F : \Omega \longrightarrow X$ is Lipschitz with constant k, then F is α–Lipschitz with constant k. To see this, we use the definition of α. Let $B \subseteq \Omega$ be bounded and suppose B admits a finite cover by sets $U_1, U_2, \ldots, U_n$, such that $\operatorname{diam} U_i \leq d$, $i = 1, \ldots, n$, $d > 0$. Then FB is covered by sets $FU_1, FU_2, \ldots, FU_n$, with

$$\begin{aligned}
\operatorname{diam} FU_i &= \sup \{|Fx - Fy| \;/\; x, y \in U_i\} \\
&\leq \sup \{k \mid x - y| \;/\; x, y \in U_i\} \\
&= k \sup \{|x - y| \;/\; x, y \in U_i\} \\
&= k \operatorname{diam} U_i \\
&\leq k\, d.
\end{aligned}$$

Thus $\alpha(FB) \leq k\, \alpha(B)$ and hence F is α–Lipschitz with constant k.

Now, if we have $G : \Omega \longrightarrow X$ to be α–Lipschitz with constant $\tilde{k}$, then F + G is α–Lipschitz with constant $k + \tilde{k}$. Indeed, if $B \subseteq \Omega$ is bounded, then

$$\begin{aligned}
\alpha((F + G)(B)) &\leq \alpha(FB + GB) \\
&\leq \alpha(FB) + \alpha(GB) \\
&\leq k\, \alpha(B) + \tilde{k}\, \alpha(B) \\
&= (k + \tilde{k})\, \alpha(B).
\end{aligned}$$

Therefore F + G is α–Lipschitz with constant $k + \tilde{k}$. ♠

4.1.3 **Example**

We know that $SC_\gamma(\Omega) \subseteq C_\gamma(\Omega)$. Nussbaum [2] gave the following example of a map that is α–condensing but not a strict α–contraction.

Let $\phi : [0, 1] \longrightarrow \mathbb{R}$ be a continuous strictly decreasing nonnegative function such that $\phi(0) = 1$ and consider the map $F : \bar{B}_1(0) \longrightarrow \bar{B}_1(0)$ defined by $Fx = \phi(|x|)\, x$, where $B_1(0)$ is the closed unit ball about 0 in an infinite dimensional space X. Let $r \in (0, 1)$. If $x \in \partial(B_{r\phi(r)}(0))$, then $|x| = r\,\phi(r)$ and $x = \frac{x}{|x|}|x| = \frac{x}{|x|} r\,\phi(r) = \frac{r\,x}{|x|}\phi(r)$. Let $y = \frac{r\,x}{|x|}$, then $|y| = \left|\frac{r\,x}{|x|}\right| = r \leq 1$. Therefore

$x = y\,\phi(r) = y\,\phi(|y|) = Fy \in F\,\bar{B}_r(0)$. Thus $\partial(B_{r\phi(r)}(0)) \subseteq F\,\bar{B}_r(0)$. So we obtain

$\alpha(F\,\bar{B}_r(0)) \geq \alpha(\partial(B_{r\phi(r)}(0))) = 2\,r\,\phi(r) = \alpha(\bar{B}_r(0))\,\phi(r)$.

If for some $k < 1$, F is a k–set–contraction, then $\alpha(F\,\bar{B}_r(0)) \leq k\,\alpha(\bar{B}_r(0))$. Since ϕ is strictly decreasing and continuous, we see that $\phi(r) \longrightarrow 1$ as $r \longrightarrow 0$, and we can therefore find $r > 0$ such that $\phi(r) > k$, giving us

$\alpha(F\,\bar{B}_r(0)) \geq \alpha(\bar{B}_r(0))\,\phi(r) > k\,\alpha(\bar{B}_r(0))$, a contradiction.

Thus F cannot be a k–set–contraction for any $k < 1$, i.e. F cannot be a strict α–contraction.

Now let $B \subseteq \bar{B}_1(0)$. Then $FB \subseteq \text{conv}\,(B \cup \{0\})$ and hence

$$\alpha(FB) \leq \alpha(\text{conv}\,(B \cup \{0\})) = \alpha(B \cup \{0\}) = \alpha(B). \tag{1}$$

However, we can say more than this. Let $B \subseteq \bar{B}_1(0)$ with $\alpha(B) = d > 0$. Select $0 < r < \frac{d}{2}$ and let $B_1 = B \cap \bar{B}_r(0)$ and $B_2 = B \setminus \bar{B}_r(0)$. Now $d = \alpha(B) \leq \alpha(\bar{B}_1(0)) = 2$. Therefore $r < \frac{d}{2} \leq 1$ and so $\bar{B}_r(0) \subseteq \bar{B}_1(0)$. By (1) we get $\alpha(FB_1) \leq \alpha(F\bar{B}_r(0)) \leq \alpha(\bar{B}_r(0)) = 2r < d = \alpha(B)$. Therefore

$$\alpha(FB_1) < \alpha(B). \tag{2}$$

If $b \in B_2 = B \setminus \bar{B}_r(0)$, then $|b| > r$ and since ϕ is strictly decreasing, $\phi(|b|) < \phi(r)$. Therefore

$$\begin{aligned} FB_2 &= \{ Fb \,/\, b \in B_2 \} \\ &= \{ \phi(|b|)\, b \,/\, b \in B_2 \} \\ &\subseteq \{ \lambda b \,/\, b \in B_2, 0 \leq \lambda < \phi(r) \} \\ &\subseteq \operatorname{conv} (\phi(r)B \cup \{0\}), \end{aligned}$$

since we have $\phi(r) > \phi(1) \geq 0$ and $\lambda b = \frac{\lambda}{\phi(r)} \phi(r)b + (1 - \frac{\lambda}{\phi(r)})\, 0$.

$$\begin{aligned} \text{Thus } \alpha(FB_2) &\leq \alpha(\operatorname{conv} (\phi(r)B \cup \{0\})) \\ &= \alpha(\phi(r)B \cup \{0\}) \\ &= \phi(r)\, \alpha(B) \\ &< \alpha(B). \end{aligned} \tag{3}$$

Now $B = B_1 \cup B_2$ and $FB \subseteq FB_1 \cup FB_2$. Therefore, by (2) and (3), we have

$$\begin{aligned} \alpha(FB) &\leq \alpha(FB_1 \cup FB_2) \\ &= \max \{\alpha(FB_1), \alpha(FB_2)\} \\ &< \alpha(B), \end{aligned}$$

showing that F is α–condensing.

Thus F is a map that is α–condensing but not a strict α–contraction. ♠

We know that every Lipschtz map with constant k, is also γ–Lipschitz with the same constant k. In the following example our map is Lipschitz with constant k, but γ–Lipschitz with a smaller constant k.

4.1.4 Example

Consider the ball–retraction $R : X \longrightarrow \bar{B}_1(0)$ given by

$$Rx = \begin{cases} x & \text{if } |x| \leq 1 \\ \frac{x}{|x|} & \text{if } |x| \geq 1 \end{cases}.$$

Let $x, y \in X$. If $|x| \leq 1$ and $|y| \leq 1$, then $|Rx - Ry| = |x - y| \leq 2|x - y|$. If $|x| \geq 1$ and $|y| \geq 1$, then

$$
\begin{aligned}
|Rx - Ry| &= \left|\frac{x}{|x|} - \frac{y}{|y|}\right| \\
&= \left|\frac{x}{|x|} - \frac{x}{|y|} + \frac{x}{|y|} - \frac{y}{|y|}\right| \\
&= \left|\left(\frac{1}{|x|} - \frac{1}{|y|}\right)x + \frac{1}{|y|}(x - y)\right| \\
&\leq \left|\frac{1}{|x|} - \frac{1}{|y|}\right| \; |x| + \frac{1}{|y|} \; |x - y| \\
&= \left|\frac{|y| - |x|}{|x| \; |y|}\right| \; |x| + \frac{1}{|y|} \; |x - y| \\
&= \left|\frac{|x| - |y|}{|y|}\right| + \frac{1}{|y|} \; |x - y| \\
&\leq \Big||x| - |y|\Big| + |x - y| \\
&\leq |x - y| + |x - y| \\
&= 2\,|x - y|.
\end{aligned}
$$

If $|x| \geq 1$ and $|y| \leq 1$, then

$$
\begin{aligned}
|Rx - Ry| &= \left|\frac{x}{|x|} - y\right| \\
&= \left|\frac{x}{|x|} - \frac{y}{|x|} + \frac{y}{|x|} - y\right| \\
&\leq \frac{1}{|x|} \; |x - y| + \left|\frac{1}{|x|} - 1\right| \; |y| \\
&\leq |x - y| + \Big||x| - 1\Big| \; |y|
\end{aligned}
$$

$$\leq \quad |x-y| + (|x| - 1)$$

$$\leq \quad |x-y| + \Big| |x| - |y| \Big|$$

$$\leq \quad |x-y| + |x-y|$$
$$= \quad 2\,|x-y|.$$

Thus for all $x, y \in X$, $|Rx - Ry| \leq 2\,|x-y|$ and so R is Lipschitz with constant 2.

Hence R is γ–Lipschitz with constant 2. But this constant can be improved upon.

Let $B \subseteq X$ be bounded. For any $x \in B$, $x = 1\,x + 0.0 \in \text{conv}\,(B \cup \{0\})$ and $\frac{1}{|x|}\,x + (1 - \frac{1}{|x|})\,0 \in \text{conv}\,(B \cup \{0\})$. Hence $Rx \in \text{conv}\,(B \cup \{0\})$ and so $RB \subseteq \text{conv}\,(B \cup \{0\})$. Thus $\gamma(RB) \leq \gamma(\text{conv}\,(B \cup \{0\})) = \gamma(B \cup \{0\}) = \gamma(B)$.

So R is γ–Lipschitz with constant 1. ♠

Theorems 3.5(b) and 3.6 can now both be extended to γ–Lipschitz maps.

4.1.5 Theorem

(a) Let $B \subseteq X$ be closed bounded and $F \in C_\gamma(B)$. Then $I - F$ is proper and maps closed subsets of B onto closed sets.

(b) Let $\Omega \subseteq X$ be open, $F : \Omega \longrightarrow X$ be γ–Lipschitz with constant k and differentiable at x_0. Then $F'(x_0)$ is γ–Lipschitz with the same constant k.

Proof :

(a) Let K be compact and $A = (I - F)^{-1}(K)$. Then $(I - F)A = K$ and $A \subseteq K + FA$. Therefore $\gamma(A) \leq \gamma(K + FA) = \gamma(FA)$. Since $F \in C_\gamma(B)$, $\gamma(A) = 0$ and so A is relatively compact. But $I - F$ is continuous and K is compact, hence closed. Thus A is closed, hence compact. Thus $I - F$ is proper and since it is continuous, it must also be closed.

(b) Since F is differentiable at x_0, $F(x_0+h) = F(x_0) + F'(x_0)\,h + \omega(x_0, h)$ where

$\left|\frac{\omega(x_0, h)}{h}\right| \longrightarrow 0$ as $|h| \longrightarrow 0$, i.e. for $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $|\omega(x_0, h)| \le \epsilon\,|h|$ when $|h| \le \delta$. If $B \subseteq X$ is bounded, then $B \subseteq B_r(0)$ for some $r > 0$. Therefore $\lambda B \subseteq \lambda\, B_r(0) = B_\delta(0)$ where $\lambda = \frac{\delta}{r}$. Hence

$$\begin{aligned}
\gamma(F'(x_0)\,\lambda B) &\le \gamma(F(x_0 + \lambda B)) + \gamma(F(x_0)) + \gamma(\omega(x_0, \lambda B)) \\
&= \gamma(F(x_0 + \lambda B)) + \gamma(\omega(x_0, \lambda B)).
\end{aligned}$$

Now if $x \in B$, then $|\lambda x| = \lambda\,|x| \le \lambda\, r = \delta$ and so $|\omega(x_0, \lambda x)| \le \epsilon\,\delta$.

So for $x, y \in B$,

$|\omega(x_0, \lambda x) - \omega(x_0, \lambda y)| \le |\omega(x_0, \lambda x)| + |\omega(x_0, \lambda y)| \le 2\epsilon\delta.$

Therefore $\gamma(\omega(x_0, \lambda B)) \le 2\epsilon\delta$.

$$\begin{aligned}
\text{So } \lambda\ \gamma(F'(x_0)B) &= \gamma(F'(x_0)\,\lambda B) \\
&\le \gamma(F(x_0 + \lambda B)) + 2\epsilon\delta \\
&\le k\ \gamma(x_0 + \lambda B) + 2\epsilon\delta \\
&= k\ \gamma(\lambda B) + 2\epsilon\delta \\
&= \lambda k\ \gamma(B) + 2\epsilon\delta.
\end{aligned}$$

Thus $\gamma(F'(x_0)B) \le k\ \gamma(B) + \frac{2\epsilon\delta}{\lambda} = k\ \gamma(B) + 2\epsilon r \longrightarrow k\ \gamma(B)$ as $\epsilon \longrightarrow 0$.

So $\gamma(F'(x_0)B) \le k\ \gamma(B)$ showing that $F'(x_0)$ is γ–Lipschitz with constant k. ♠

Dugundji's extension theorem yields, as an easy exercise, that every compact map on a closed subset of X has a compact extension (see theorem 3.7). We cannot obtain such a result for γ–Lipschitz maps.

If $F : \bar{B}_r(0) \longrightarrow X$ is γ–Lipschitz with constant k, then there exists a γ–Lipschitz extension, with the same k, to all of X, namely $F \circ R$ where $R : X \longrightarrow \bar{B}_r(0) \subseteq X$ is defined by

$$Rx = \begin{cases} x & |x - x_0| \leq r \\ x_0 + r \dfrac{x - x_0}{|x - x_0|} & |x - x_0| > r \end{cases}.$$

In a Hilbert space, any γ–Lipschitz map defined on a closed convex set has a γ–Lipschitz extension with the same constant. This follows from the next theorem, which we state without proof.

4.1.6 Theorem

Let X be a Hilbert space and $C \subseteq X$ be closed and convex. Then the metric projection $P : X \longrightarrow C$ is nonexpansive, in particular α–Lipschitz with constant $k = 1$.

The following lemma, which will play an important role in the sequel, is due to Kuratowski (1930).

4.1.7 Lemma

Let X be a Banach space, (B_i) a decreasing sequence of nonempty closed subsets such that $\alpha(B_i) \longrightarrow 0$ as $i \longrightarrow \infty$. Then $\bigcap_i B_i$ is nonempty and compact.

Proof :

Since each B_i is closed, $\bigcap_i B_i$ is also closed. We just need to show it to be relatively compact. Suppose $\alpha(\bigcap_i B_i) > 0$ and let $\epsilon = \alpha(\bigcap_i B_i)$. Since $\alpha(B_i) \longrightarrow 0$ as $i \longrightarrow \infty$, there exists $N \in \mathbb{N}$ such that $i \geq N$ implies that $\alpha(B_i) < \epsilon$. So for $i \geq N$, we have $\alpha(B_i) < \alpha(\bigcap_i B_i) \leq \alpha(B_i)$, a contradiction. Thus $\alpha(\bigcap_i B_i) = 0$ and so it is relatively compact. Being closed, it must also be compact.

We must now show that $\bigcap_i B_i \neq \emptyset$. Since each B_i is nonempty, for each i we ca

choose an $x_i \in B_i$. Then

$$\alpha(\{x_i / i \geq 1\}) = \alpha(\{x_i / i \geq p\} \cup \{x_i / i = 1, 2, \ldots, p-1\})$$
$$= \alpha(\{x_i / i \geq p\}) \quad \text{for all p.}$$

Now $\{x_i / i \geq p\} \subseteq B_p$. Hence $\alpha(\{x_i / i \geq p\}) \leq \alpha(B_p) \longrightarrow 0$ as $p \longrightarrow$

Therefore $\alpha(\{x_i / i \geq p\}) \longrightarrow 0$ as $p \longrightarrow \infty$ and so $\alpha(\{x_i / i \geq 1\}) = 0$. Th

$\{x_i / i \geq 1\}$ is relatively compact. Therefore (x_n) has a convergent subsequen

say $x_{k_i} \longrightarrow x_0$. We claim that $x_0 \in \bigcap_i B_i$. To show this, take $n \in \mathbb{N}$. $k_i \geq$

implies that $x_{k_i} \in B_{k_i} \subseteq B_n$. So $\{x_{k_i} / k_i \geq n\} \subseteq B_n$. Now $x_{k_i} \longrightarrow x_0$ and B_n

closed. Thus $x_0 \in B_n$. But $n \in \mathbb{N}$ was arbitrary. Hence we must have $x_0 \in \bigcap_i B_i$.

Thus B is a nonempty compact set.

Now we obtain a generalisation of Schauder's fixed point theorem.

4.1.8 Theorem

Let $C \subseteq X$ be nonempty, closed, bounded and convex, and $F : C \longrightarrow$

γ–condensing. Then F has a fixed point.

Proof :

We will assume, for now, that $0 \in C$.

(1) Suppose the result is true for strict γ–contractions. Choose $k_n < 1$ su

that $k_n \longrightarrow 1$ (for example $k_n = \sum_{i=1}^{n} \frac{1}{2^i}$) and consider $k_n F$. For $x \in$

we need to have $k_n Fx \in C$. Since $0 \in C$ and $Fx \in C$,

$k_n Fx = k_n Fx + (1 - k_n) 0 \in C$ because C is convex. Therefore

$k_n F : C \longrightarrow C$. Let $B \subseteq C$. Then

$\gamma((k_n F)B) = k_n \gamma(FB) < k_n \gamma(B)$ if $\gamma(B) > 0$. If $\gamma(B) = 0$, then B

relatively compact and so is FB. Thus

$\gamma(k_n(FB)) = k_n\gamma(FB) = 0 = \gamma(B)$. Therefore
$k_nF : C \longrightarrow C$ is a strict γ-contraction. Since the result is true for these maps, k_nF has a fixed point $x_n \in C$, i.e. $k_nF(x_n) = x_n$.
So $x_n - Fx_n = k_nFx_n - Fx_n = (k_n - 1)Fx_n \longrightarrow 0$ as $n \longrightarrow \infty$. But $x_n - Fx_n = (I-F)x_n \in (I-F)(C)$ and $I-F$ is closed (by theorem 4.5), so $(I-F)(C)$ is closed. Thus $0 \in (I-F)(C)$. So there exists $x_0 \in C$ such that $0 = (I-F)x_0$, i.e. $Fx_0 = x_0$.

(2) Now suppose $F : C \longrightarrow C$ is a strict γ-contraction with constant $k < 1$. Define a sequence (C_n) by $C_0 = C$, $C_n = \overline{\text{conv}}\,(FC_{n-1})$, $n \geq 1$.
$C_1 = \overline{\text{conv}}\,(FC_0) = \overline{\text{conv}}\,(FC) \subseteq \overline{\text{conv}}\,(C) = C = C_0$. Suppose $C_k \subseteq C_{k-1}$.
Then $C_{k+1} = \overline{\text{conv}}\,(FC_k) \subseteq \overline{\text{conv}}\,(FC_{k-1}) = C_k$. Hence by induction,

$C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$.

Thus we have a decreasing sequence of closed convex sets. We also have,

$$\begin{aligned}
\gamma(C_n) &= \gamma(\overline{\text{conv}}\,(FC_{n-1})) \\
&= \gamma(FC_{n-1}) \\
&\leq k\,\gamma(C_{n-1}) \\
&\leq k^2\gamma(C_{n-2}) \\
&\leq . \\
&\quad . \\
&\quad . \\
&\leq k^n\gamma(C_0) \longrightarrow 0 \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Thus $\gamma(C_n) \longrightarrow 0$ as $n \longrightarrow \infty$. Let $\tilde{C} = \bigcap_{n=0}^{\infty} C_n$. Then $\tilde{C}$ is closed, bounded and convex. By lemma 4.17, $\tilde{C}$ is compact. For any $x \in \tilde{C}$, we have $x \in C_n$ for all n. So $Fx \in FC_{n-1} \subseteq (\overline{\text{conv}}\,FC_{n-1}) = C_n$ for all n. Thus $Fx \in \tilde{C}$ and hence $F|_{\tilde{C}} : \tilde{C} \longrightarrow \tilde{C}$. Since $\tilde{C}$ is compact, $F|_{\tilde{C}}$ is a compact map. Hence by Schauder's fixed point theorem, there exists

$x_0 \in \tilde{C}$ such that $Fx_0 = x_0$. Since $\tilde{C} \subseteq C$, $F : C \longrightarrow C$ has a fixed point.

Now suppose $0 \notin C$. Since $C \neq \emptyset$, there exists $y_0 \in C$. Now consider $C' = C - y_0$ and $F'(x) = F(x + y_0) - y_0$. Then $0 \in C'$ and $F' : C' \longrightarrow C'$ is also γ–condensing. So by part 1, F' has a fixed point, i.e. there exists $x_0 \in C'$ such that $F'x_0 = x_0$. Therefore $F(x_0 + y_0) - y_0 = x_0$, and so $F(x_0 + y_0) = x_0 + y_0$ and F has a fixed point. ♠

The previous theorem is a result of Darbo's theorem and a fixed point theorem of Krasnoselskii [35].

We are now ready to define a degree for γ–condensing maps. As in the case of the Leray–Schauder degree, we consider the triplets $(I - F, \Omega, y)$ where X is a Banach space, $\Omega \subseteq X$ open bounded, $F : \bar{\Omega} \longrightarrow X$ is γ–condensing and $y \in X \setminus (I - F)(\partial\Omega)$, and we define a unique $\mathbb{Z}$–valued map on these triplets that satisfies the properties :

(D1) $D(I, \Omega, y) = 1$ if $y \in \Omega$.

(D2) If Ω_1 and Ω_2 are disjoint open subsets of Ω such that $y \in X \setminus (I - F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$, then $D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$.

(D3) Let $H : J \times \bar{\Omega} \longrightarrow X$, $y : J \longrightarrow X$ be continuous, $\gamma(H(J \times B)) < \gamma(B)$ for $B \subseteq \bar{\Omega}$ and $\gamma(B) > 0$ (i.e. H is γ–condensing) and $y(t) \in X \setminus (I - H(t, .))(\partial\Omega)$. Then $D(I - H(t, .), \Omega, y(t))$ is independent of t.

Degree for Strict γ–Contractions

Let $\mathscr{M} = \{(I - F, \Omega, y) /\ \Omega \subseteq X$ is open bounded, $F : \bar{\Omega} \longrightarrow X$ is a srtict γ–contraction, and $y \in X \setminus (I - F)(\partial\Omega)\ \}$

We first show that if there exists a $\mathbb{Z}$–valued function on $\mathcal{M}$ satisfying (D1)–(D3), then it must be unique.

Uniqueness:

Since D satisfies (D1)–(D3), it must also satisfy (D4)–(D7). So by (D4) we have $D(I - F, \Omega, y) = 0$ if $(I - F)^{-1}(y) = \emptyset$. Therefore we assume that $(I - F)^{-1}(y) \neq \emptyset$. Let $C_0 = \overline{\operatorname{conv}}\,(F(\bar{\Omega}) + y)$ and $C_n = \overline{\operatorname{conv}}\,(F(\bar{\Omega} \cap C_{n-1}) + y)$. Now $C_0 = \overline{\operatorname{conv}}\,(F(\bar{\Omega}) + y) \supseteq \overline{\operatorname{conv}}\,(F(\bar{\Omega} \cap C_0) + y) = C_1$. Therefore $C_0 \supseteq C_1$.

Suppose $C_{k-1} \supseteq C_k$. Then

$C_{k+1} = \overline{\operatorname{conv}}\,(F(\bar{\Omega} \cap C_k) + y) \subseteq \overline{\operatorname{conv}}\,(F(\bar{\Omega} \cap C_{k-1}) + y) = C_k$.

Therefore C_n is a decreasing sequence of nonempty closed convex sets. Also

$$\begin{aligned}
\gamma(C_n) &= \gamma(\overline{\operatorname{conv}}\,(F(\bar{\Omega} \cap C_{n-1}) + y) \\
&= \gamma((F(\bar{\Omega} \cap C_{n-1}) + y) \\
&= \gamma((F(\bar{\Omega} \cap C_{n-1})) \\
&\leq \gamma(F(C_{n-1})) \\
&\leq k\,\gamma(C_{n-1}) \\
&\leq \;. \\
&\quad\;\; . \\
&\quad\;\; . \\
&\leq k^n \gamma(C\;) \longrightarrow 0 \text{ as } n \longrightarrow \infty, \text{ since } k < 1.
\end{aligned}$$

By lemma 4.7, $C_\infty = \bigcap_{n=0}^{\infty} C$ is nonempty, compact and convex.

We will now show that $(I - F)^{-1}(y) \subseteq C_\infty \cap \Omega$. Let $x \in (I - F)^{-1}(y)$. Then $(I - F)x = y$. Since $y \notin (I - F)(\partial\Omega)$, we must have $x \in \Omega$. Also $x = Fx + y \in C_0$, therefore $x = Fx + y \in F(\bar{\Omega} \cap C_0) + y \subseteq C_1$, and so $x = Fx + y \in F(\bar{\Omega} \cap C_1) + y \subseteq C_2$. If $x \in C_n$, then $x = Fx + y \in F(\bar{\Omega} \cap C_n) + y \subseteq C_{n+1}$. Therefore $x \in C_n$ for all n, so $x \in C_\infty$, and hence $x \in C_\infty \cap \Omega$.

From the definitions of the sets C_n we have that $F(\bar{\Omega} \cap C_\infty) + y \subseteq C_\infty$.

Since C_∞ is a closed convex subset of the nls. X, C_∞ is a retract (from the remark after definition 1.2.16). Let $R : X \longrightarrow C_\infty$ be a retraction.

We will show that $(I - F)^{-1}(y) \subseteq R^{-1}(\Omega) \cap \Omega$. Since $(I - F)^{-1}(y) \subseteq C_\infty \cap \Omega$, we will show that $C_\infty \cap \Omega \subseteq R^{-1}(\Omega) \cap \Omega$. Let $x \in C_\infty \cap \Omega$. Then $x \in C_\infty$ and $x \in \Omega$, which means that $Rx = x \in \Omega$. Therefore $x \in R^{-1}(x) \subseteq R^{-1}(\Omega)$, and hence $x \in R^{-1}(\Omega) \cap \Omega$.

$(I - F)^{-1}(y) \subseteq R^{-1}(\Omega) \cap \Omega$ implies that $y \notin (I - F)(\bar{\Omega} \setminus (R^{-1}(\Omega) \cap \Omega))$. Since R is continuous, $R^{-1}(\Omega) \cap \Omega$ is open and by (D7) we have

$$D(I - F, \Omega, y) = D(I - F, R^{-1}(\Omega) \cap \Omega, y). \tag{4}$$

We now show that $D(I - F, R^{-1}(\Omega) \cap \Omega, y) = D(I - FR, R^{-1}(\Omega) \cap \Omega, y)$.

Define $H : J \times \overline{R^{-1}(\Omega) \cap \Omega} \longrightarrow X$ by $H(t, x) = t\,FRx + (1 - t)Fx = Fx + t(FRx - Fx)$. Then H is continuous.

Suppose $y = (I - H(t, .))x$ for $t \in J$ and $x \in \overline{R^{-1}(\Omega) \cap \Omega} \subseteq R^{-1}(\bar{\Omega}) \cap \bar{\Omega}$. Then
$x = y + H(t, x) = y + Fx + t(FRx - Fx) = (1 - t)(Fx + y) + t(FRx + y)$.
Now $x \in R^{-1}(\bar{\Omega}) \cap \bar{\Omega}$. So $Rx \in \bar{\Omega}$ and $x \in \bar{\Omega}$. Therefore $Rx \in \bar{\Omega} \cap C_n$ for all n.
Hence $FRx + y \in F(\bar{\Omega} \cap C_n) + y$ for all n. Now $x \in \bar{\Omega}$ implies $Fx + y \in F(\bar{\Omega}) + y$.
Therefore $x = (1 - t)(Fx + y) + t(FRx + y) \in \overline{\text{conv}}\,(F(\bar{\Omega}) + y) = C_0$. So
$Fx + y \in F(\bar{\Omega} \cap C_0) + y$, and hence
$x = (1 - t)(Fx + y) + t(FRx + y) \in \overline{\text{conv}}\,(F(\bar{\Omega} \cap C_0) + y) = C_1$. Again
$Fx + y \in F(\bar{\Omega} \cap C_1) + y$, and so
$x = (1 - t)(Fx + y) + t(FRx + y) \in \overline{\text{conv}}\,(F(\bar{\Omega} \cap C_1) + y) = C_2$, etc.. Thus we have $x \in C_n$ for all n, and hence $x \in C_\infty$. Therefore $Rx = x$ and so $x \in (I - F)^{-1}(y)$. But $(I - F)^{-1}(y) \subseteq R^{-1}(\Omega) \cap \Omega$ which is open. Therefore $x \in R^{-1}(\Omega) \cap \Omega$, and so $x \notin \partial(R^{-1}(\Omega) \cap \Omega)$. Hence $y \notin (I - H(t, .))(\partial(R^{-1}(\Omega) \cap \Omega))$.

Let $B \subseteq \overline{R^{-1}(\Omega) \cap \Omega}$. We will now show that $\gamma(H(J \times B)) \leq k\,\gamma(B)$.

Now $H(t, x) = (1 - t)\,Fx + t\,FRx$. Therefore $H(J \times B) \subseteq \text{conv}\,(F(B) \cup FR(B))$. Now

$R(X) \subseteq C_\infty$ and C_∞ is compact. Hence $\gamma(R(X)) \leq \gamma(C_\infty) = 0$. Therefore $\gamma(R(X)) = 0$, and so $R(X)$ is relatively compact. Thus $R \in \mathscr{K}(X)$. So $R(B)$ is relatively compact, and therefore $FR(B)$ is relatively compact, which implies that $\gamma(FR(B)) = 0$. Therefore

$$\begin{aligned}
\gamma(H(J \times B)) \quad & \leq \gamma(\mathrm{conv}\,(F(B) \cup FR(B))) \\
& = \gamma(F(B) \cup FR(B)) \\
& = \max \{\gamma(F(B)), \gamma(FR(B))\} \\
& = \gamma(F(B)) \\
& \leq k\, \gamma(B)
\end{aligned}$$

By (D3),

$$D(I - F, R^{-1}(\Omega) \cap \Omega, y) = D(I - FR, R^{-1}(\Omega) \cap \Omega, y). \tag{5}$$

Now $FR(R^{-1}(\Omega) \cap \Omega) \subseteq FR(X)$. Now $R(X)$ relatively compact implies that $F(R(X))$ is relatively compact. So $\gamma(FR(R^{-1}(\Omega) \cap \Omega)) \leq \gamma(F(R(X))) = 0$. Therefore $FR(R^{-1}(\Omega) \cap \Omega)$ is relatively compact, implying that $FR \in \mathscr{K}(R^{-1}(\Omega) \cap \Omega)$. Thus $(I - FR, R^{-1}(\Omega) \cap \Omega, y)$ is a LS–triplet. By the same procedure used in chapter 3, using the uniqueness of the Leray–Schauder degree, we can conclude that

$$D(I - FR, R^{-1}(\Omega) \cap \Omega, y) = D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y). \tag{6}$$

Thus we have shown that

$$D(I - F, \Omega, y) = \begin{cases} D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y) & \text{if } (I - F)^{-1}(y) \neq \emptyset \\ 0 & \text{if } (I - F)^{-1}(y) = \emptyset \end{cases}.$$

We now show that this formula can be used to define the degree, i.e. we show that R can be replaced by $\tilde{R}$ in (6) (where $\tilde{R}$ is any retraction of X onto any closed subset C satisfying $C_\infty \subseteq C$, $F(\bar{\Omega} \cap C) + y \subseteq C$ and $F(\bar{\Omega} \cap C)$ is relatively compact).

Well–defined:

A set C satisfying all the above properties does exist, namely C_∞ itself. Since C is closed convex, it must be a retract (follows from the remark after definition 1.2.16). Let $\tilde{R} : X \longrightarrow C$ be a retraction, and $\Omega_1 = R^{-1}(\Omega) \cap \Omega$, $\Omega_2 = \tilde{R}^{-1}(\Omega) \cap \Omega$, and $\Omega_3 = \Omega_1 \cap \Omega_2$. We show that $(I - F\tilde{R}, \Omega_2, y)$ is a LS–triplet. Easily, we have Ω_2 open bounded and $F\tilde{R} \in \mathscr{K}(\bar{\Omega}_2)$. We need to check that $y \notin (I - F\tilde{R})(\partial\Omega_2)$. Suppose $y = (I - F\tilde{R})x$ for some $x \in \bar{\Omega}_2 \subseteq \tilde{R}^{-1}(\bar{\Omega}) \cap \bar{\Omega}$. Then $x = y + F\tilde{R}x$. Now $\tilde{R}x \in \bar{\Omega} \cap C$. So $x = y + F\tilde{R}x \in y + F(\bar{\Omega} \cap C) \subseteq C$. Therefore $x \in C$ and hence $\tilde{R}x = x$. So we have $y = (I - F)x$. Since $y \notin (I - F)(\partial\Omega)$, we must have $x \in \Omega$. But $\tilde{R}x = x$ and so $x \in \tilde{R}^{-1}x \in \tilde{R}^{-1}(\Omega)$. Therefore $x \in \tilde{R}^{-1}(\Omega) \cap \Omega = \Omega_2$. Thus $x \notin \partial\Omega_2$. So $y \notin (I - F\tilde{R})(\partial\Omega_2)$, proving that $(I - F\tilde{R}, \Omega_2, y)$ is a LS–triplet.

We now show that $D_{LS}(I - FR, \Omega_1, y) = D_{LS}(I - FR, \Omega_3, y)$.

Suppose $y = (I - FR)x$ with $x \in \bar{\Omega}_1 \setminus \Omega_3$. Then $x = FRx + y$. Now

$$\begin{aligned}
\bar{\Omega}_1 \setminus \Omega_3 &= \overline{R^{-1}(\Omega) \cap \Omega} \setminus ((R^{-1}(\Omega) \cap \Omega) \cap (\tilde{R}^{-1}(\Omega) \cap \Omega)) \\
&= \overline{R^{-1}(\Omega) \cap \Omega} \setminus (R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega) \\
&\subseteq \overline{R^{-1}(\Omega)} \cap \bar{\Omega} \setminus (R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega) \\
&\subseteq R^{-1}(\bar{\Omega}) \cap \bar{\Omega} \setminus (R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega).
\end{aligned}$$

Therefore $x \in R^{-1}(\bar{\Omega}) \cap \bar{\Omega}$, implying that $Rx \in \bar{\Omega}$ and $x \in \bar{\Omega}$. So $Rx \in \bar{\Omega} \cap C_\infty$. Since $F(\bar{\Omega} \cap C_\infty) + y \subseteq C_\infty$, we have $x = F(Rx) + y \in F(\bar{\Omega} \cap C_\infty) + y \subseteq C_\infty \subseteq C$. Hence $Rx = x$ and $\tilde{R}x = x$, which means that $x \in R^{-1}(x)$ and $x \in \tilde{R}^{-1}(x)$. Therefore $y = x - FRx = x - Fx = (I - F)x$ with $x \in \bar{\Omega}$. Since $y \notin (I - F)(\partial\Omega)$, we must have $x \in \Omega$. So $x \in R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega = \Omega_3$, a contradiction. Hence $y \notin (I - FR)(\Omega_1 \setminus \Omega_3)$. Therefore by $(D_{LS}7)$,

$$D_{LS}(I - FR, \Omega_1, y) = D_{LS}(I - FR, \Omega_3, y). \tag{7}$$

Now we show that $D_{LS}(I - F\tilde{R}, \Omega_2, y) = D_{LS}(I - F\tilde{R}, \Omega_3, y)$. We will use $(D_{LS}7)$, by verifying that $y \notin (I - F\tilde{R})(\bar{\Omega}_2 \setminus \Omega_3)$. Suppose $y = (I - F\tilde{R})x$ where $x \in \bar{\Omega}_2 \setminus \Omega_3$. Then

$x = F\tilde{R}x + y$. Now

$\bar{\Omega}_2 \setminus \Omega_3 = \overline{\tilde{R}^{-1}(\Omega) \cap \Omega} \setminus (R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega) \subseteq (\tilde{R}^{-1}(\bar{\Omega}) \cap \bar{\Omega}) \setminus (R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega)$.

Thus $x \in \tilde{R}^{-1}(\bar{\Omega}) \cap \bar{\Omega}$, and so $\tilde{R}x \in \bar{\Omega}$ and $x \in \bar{\Omega}$. Therefore

$x = F\tilde{R}x + y \in F(\bar{\Omega} \cap C) + y \subseteq C$. So $\tilde{R}x = x$ and $x \in \tilde{R}^{-1}(x)$. Therefore

$y = x - F\tilde{R}x = x - Fx = (I - F)x$. Since $y \notin (I - F)(\partial\Omega)$, we must have $x \in \Omega$. Therefore

$x \in \tilde{R}^{-1}(x) \subseteq \tilde{R}^{-1}(\Omega)$. Now $x = F\tilde{R}x + y \in F(\bar{\Omega}) + y \subseteq C_0$, which means that $\tilde{R}x = x \in C_0$.

So $x = F\tilde{R}x + y \in F(\bar{\Omega} \cap C_0) + y \subseteq C_1$, which means that $\tilde{R}x = x \in C_1$. Again

$x = F\tilde{R}x + y \in F(\bar{\Omega} \cap C_1) + y \subseteq C_2$, etc. . Thus $x \in C_n$ for all n, and hence $x \in C_\infty$.

Therefore $Rx = x$, and so $x \in R^{-1}(x) \subseteq R^{-1}(\Omega)$. This gives us $x \in \tilde{R}^{-1}(\Omega) \cap R^{-1}(\Omega) \cap \Omega = \Omega_3$,

a contradiction. Thus we must have $y \notin (I - F\tilde{R})(\bar{\Omega}_2 \setminus \Omega_3)$. By ($D_{LS}$7), we obtain

$$D_{LS}(I - F\tilde{R}, \Omega_2, y) = D_{LS}(I - F\tilde{R}, \Omega_3, y). \tag{8}$$

Now we show that $D_{LS}(I - FR, \Omega_3, y) = D_{LS}(I - F\tilde{R}, \Omega_3, y)$.

Consider $H : J \times \bar{\Omega}_3 \longrightarrow X$ defined by $H(t, x) = t\, FRx + (1 - t)\, F\tilde{R}x$.

$\bar{\Omega}_3 = \overline{R^{-1}(\Omega) \cap \tilde{R}^{-1}(\Omega) \cap \Omega} \subseteq R^{-1}(\bar{\Omega}) \cap \tilde{R}^{-1}(\bar{\Omega}) \cap \bar{\Omega}$, so if $x \in \bar{\Omega}_3$, then $Rx \in \bar{\Omega}$, $\tilde{R}x \in \bar{\Omega}$ and

$x \in \bar{\Omega}$. So $FRx \in F(\bar{\Omega} \cap C_\infty)$ and $F\tilde{R}x \in F(\bar{\Omega} \cap C)$. Therefore

$H(t, x) = t\, FRx + (1 - t)\, F\tilde{R}x \in \operatorname{conv}(\, F(\bar{\Omega} \cap C_\infty) \cup F(\bar{\Omega} \cap C)\,)$, and hence

$H(J \times \bar{\Omega}_3) \subseteq \operatorname{conv}(\, F(\bar{\Omega} \cap C_\infty) \cup F(\bar{\Omega} \cap C)\,)$. Therefore,

$$\begin{aligned}
\gamma(H(J \times \bar{\Omega}_3)) &\leq \gamma(\operatorname{conv}(\, F(\bar{\Omega} \cap C_\infty) \cup F(\bar{\Omega} \cap C)\,)) \\
&= \gamma(F(\bar{\Omega} \cap C_\infty) \cup F(\bar{\Omega} \cap C)) \\
&= \max\{\gamma(F(\bar{\Omega} \cap C_\infty)), \gamma(F(\bar{\Omega} \cap C))\} \\
&= 0\text{, since } F(\bar{\Omega} \cap C_\infty) \text{ and } F(\bar{\Omega} \cap C) \text{ are relatively compact. Therefore}
\end{aligned}$$

$H(J \times \bar{\Omega}_3)$ is relatively compact. We need to show that $y \notin (I - H(t, .))(\partial\Omega_3)$ on J.

Let $y = (I - H(t, .))x$ for $(t, x) \in J \times \bar{\Omega}_3$. Then

$x = y + H(t, x) = t(FRx + y) + (1 - t)(F\tilde{R}x + y)$. Now $x \in \bar{\Omega}_3 \subseteq \bar{\Omega}_1 \cap \bar{\Omega}_2$, so $x \in \bar{\Omega}_1$ and

$x \in \bar{\Omega}_2$. $\bar{\Omega}_1 = \overline{R^{-1}(\Omega) \cap \Omega} \subseteq R^{-1}(\bar{\Omega}) \cap \bar{\Omega}$. Therefore $Rx \in \bar{\Omega}$ and $x \in \bar{\Omega}$. So $Rx \in \bar{\Omega} \cap C_\infty$

and hence $FRx + y \in C_\infty \subseteq C$.

Similarly, since $x \in \bar{\Omega}_2$, we obtain $F\tilde{R}x + y \in C$. Therefore $x = t(FRx + y) + (1 - t)(F\tilde{R}x + y) \in \text{conv } C = C$ since C is convex. Thus $\tilde{R}x = x$. Since $x \in \bar{\Omega}$, $Fx + y \in C_0$. So $x = t(FRx + y) + (1 - t)(Fx + y) \in \text{conv } C_0 = C_0$. Therefore $x \in \bar{\Omega} \cap C_0$ and so $x = t(FRx + y) + (1 - t)(Fx + y) \in \text{conv } C_1 = C_1$, etc. . Hence we get $x \in C_\infty$, and so $Rx = x$. Thus $x = Fx + y$. As before $(I - F)^{-1}(y) \subseteq R^{-1}(\Omega) \cap \Omega$ and $(I - F)^{-1}(y) \subseteq \tilde{R}^{-1}(\Omega) \cap \Omega$, so $(I - F)^{-1}(y) \subseteq \Omega_3$. Thus $x \notin \partial\Omega_3$. Therefore $y \notin (I - H(t, .))(\partial\Omega_3)$.

By $(D_{LS}3)$,

$$D_{LS}(I - FR, \Omega_3, y) = D_{LS}(I - F\tilde{R}, \Omega_3, y). \tag{9}$$

By (7), (8) and (9) we have

$$D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y) = D_{LS}(I - F\tilde{R}, \tilde{R}^{-1}(\Omega) \cap \Omega, y).$$

Thus we have shown that on our definition, the degree is well–defined.

We now go on to show the existence of such a map.

Existence:

If $F \in SC_\gamma(\Omega)$, and $y \notin (I - F)(\partial\Omega)$, define

$$D(I - F, \Omega, y) = D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y),$$

where C is any closed convex subset of X satisfying $C_\infty \subseteq C$, $F(\bar{\Omega} \cap C) + y \subseteq C$, $F(\bar{\Omega} \cap C)$ is relatively compact and $R : X \longrightarrow C$ is any retraction.

We must now show that D satisfies (D1)–(D3).

(D1) :

Let $y \in \Omega$. If $F = 0$, then $C_\infty = \{y\}$ and $R : X \longrightarrow \{y\}$ defined by $Rx = y$ for all $x \in X$ is the retraction. So $D(I, \Omega, y) = D_{LS}(I, R^{-1}(\Omega) \cap \Omega, y)$. Since $y \in \{y\}$, we must have $Ry = y$. Thus $y \in R^{-1}(y)$. We also have $y \in \Omega$. So $y \in R^{-1}(y) \cap \Omega$. By $(D_{LS}1)$,

$$D(I, \Omega, y) = D_{LS}(I, R^{-1}(\Omega) \cap \Omega, y) = 1.$$

(D2) :

Let $F \in SC_\gamma(\bar{\Omega})$, $y \notin (I-F)(\partial\Omega)$ and Ω_1, Ω_2 disjoint open subsets of Ω such that $y \notin (I-F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. Now

$$D(I-F, \Omega, y) = D_{LS}(I-FR, R^{-1}(\Omega) \cap \Omega, y) \tag{10}$$

where $R : X \longrightarrow C_\infty$ is a retraction.

$R^{-1}(\Omega_1) \cap \Omega_1$ and $R^{-1}(\Omega_2) \cap \Omega_2$ are disjoint open subsets of $R^{-1}(\Omega) \cap \Omega$. We need to show that

$y \notin (I-FR)(\overline{R^{-1}(\Omega) \cap \Omega} \setminus ((R^{-1}(\Omega_1) \cap \Omega_1) \cup (R^{-1}(\Omega_2) \cap \Omega_2)))$. Suppose $y = (I-FR)x$ for $x \in \overline{R^{-1}(\Omega) \cap \Omega} \setminus ((R^{-1}(\Omega_1) \cap \Omega_1) \cup (R^{-1}(\Omega_2) \cap \Omega_2))$. Then $x = FRx + y$. Also $\overline{R^{-1}(\Omega) \cap \Omega} \subseteq R^{-1}(\bar{\Omega}) \cap \bar{\Omega}$. So $Rx \in \bar{\Omega}$ and $x \in \bar{\Omega}$. Therefore $x = FRx + y \in F(\bar{\Omega} \cap C_n) + y$ for all n. So $x \in C_\infty$ and hence $Rx = x$. Therefore $y = (I-F)x$. Since $y \notin (I-F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$ and $x \in \bar{\Omega}$, we must have $x \in \Omega_1 \cup \Omega_2$. Suppose (without loss of generality) that $x \in \Omega_1$. Since $Rx = x$, we have $x \in R^{-1}(\Omega_1) \cap \Omega_1$, a contradiction.

Therefore $y \notin (I-FR)(\overline{R^{-1}(\Omega) \cap \Omega} \setminus ((R^{-1}(\Omega_1) \cap \Omega_1) \cup (R^{-1}(\Omega_2) \cap \Omega_2)))$. Applying $(D_{LS}2)$, we get

$$D_{LS}(I-FR, R^{-1}(\Omega) \cap \Omega, y)$$
$$= D_{LS}(I-FR, R^{-1}(\Omega_1) \cap \Omega_1, y) + D_{LS}(I-FR, R^{-1}(\Omega_2) \cap \Omega_2, y) \tag{11}$$

Let C_∞^i be constructed just as C_∞ was, with Ω replaced by Ω_i, where $i = 1, 2$.

If $(I-F)^{-1}(y) \cap \Omega_i \neq \emptyset$, then $C_\infty^i \subseteq C_\infty$, $R : X \longrightarrow C_\infty$ and C_∞ is admissible. So

$$D(I-F, \Omega_i, y) = D_{LS}(I-FR, R^{-1}(\Omega_i) \cap \Omega_i, y).$$

Suppose $(I-F)^{-1}(y) \cap \bar{\Omega}_i = \emptyset$. Then $D(I-F, \Omega_i, y) = 0$. We must show that $D_{LS}(I-FR, R^{-1}(\Omega_i) \cap \Omega_i, y) = 0$. Suppose $(I-FR)^{-1}(y) \cap \overline{R^{-1}(\Omega_i) \cap \Omega_i} \neq \emptyset$. Then there

exists $x \in (I-FR)^{-1}(y) \cap \overline{R^{-1}(\Omega_i) \cap \Omega_i} \subseteq (I-FR)^{-1}(y) \cap R^{-1}(\bar{\Omega}_i) \cap \bar{\Omega}_i$. This implies that $y = (I-FR)x$ and $x \in R^{-1}(\bar{\Omega}_i) \cap \bar{\Omega}_i$. Therefore $x = FRx + y$ and $Rx \in \bar{\Omega} \cap C_\infty$. Thus $x \in C_\infty$ and so $Rx = x$. Therefore $y = (I-F)x$. Since $y \notin (I-F)(\partial\Omega)$, we must have $x \in \Omega$. So we obtain $x \in (I-F)^{-1}(y) \cap \bar{\Omega}_i$, a contradiction.
Therefore $(I-FR)^{-1}(y) \cap \overline{R^{-1}(\Omega_i) \cap \Omega_i} = \emptyset$, resulting in $D_{LS}(I-FR, R^{-1}(\Omega_i) \cap \Omega_i, y) = 0$. So $D(I-F, \Omega_i, y) = 0 = D_{LS}(I-FR, R^{-1}(\Omega_i) \cap \Omega_i, y)$.
So we obtain in either case,

$$D(I-F, \Omega_i, y) = D_{LS}(I-FR, R^{-1}(\Omega_i) \cap \Omega_i, y). \tag{12}$$

By (10), (11) and (12), we have

$$D(I-F, \Omega, y) = D(I-F, \Omega_1, y) + D(I-F, \Omega_2, y),$$

proving (D2).

(D3) :

Let $H : J \times \bar{\Omega} \longrightarrow X$ be a strict γ–contraction with constant $k < 1$, $y : J \longrightarrow X$ continuous and $y(t) \notin (I - H(t, .))(\partial\Omega)$ for all $t \in J$.
Let $C_0 = \overline{\text{conv}}\,(H(J \times \bar{\Omega}) + y(J))$, $C_n = \overline{\text{conv}}\,(H(J \times \bar{\Omega} \cap C_{n-1}) + y(J))$ for $n \geq 1$ and $C_\infty(H) = \bigcap_{n \geq 0} C_n$. As before $C_\infty(H)$ is compact, convex, and closed.
Suppose $(I - H(t, .))^{-1}(y(t)) \neq \emptyset$. Then there exists $x \in (I - H(t, .))^{-1}(y(t))$. Therefore $x = H(t, x) + y(t) \in C_0$. So $x = H(t, x) + y(t) \in C_1$, etc. . Therefore $x \in C_\infty(H)$. So $C_\infty(H) = \emptyset$ implies that $(I - H(t, .))^{-1}(y(t)) = \emptyset$, and so $D(I - H(t, .), \Omega, y(t)) = 0$ for all $t \in J$.
Suppose $C_\infty(H) \neq \emptyset$ and let $R : X \longrightarrow C_\infty(H)$ be a retraction. We need to check that $C_\infty(H)$ is admissible. Consider $H(t, .)$ for some $t \in J$. Then $C^t_\infty \subseteq C_\infty$, where C^t_∞ is constructed as C_∞ was, with F replaced by $H(t, .)$ and y replaced by $y(t)$.
$H(t, \bar{\Omega} \cap C_\infty(H)) + y(t) \subseteq C_\infty(H)$ by definition of the sets C_n and
$H(t, \bar{\Omega} \cap C_\infty(H)) \subseteq H(J \times \bar{\Omega} \cap C_\infty(H))$. So

$\gamma(H(t, \bar{\Omega} \cap C_\infty(H))) \leq \gamma(H(J \times \bar{\Omega} \cap C_\infty(H))) \leq k\ \gamma(\bar{\Omega} \cap C_\infty(H)) = 0$ since $C_\infty(H)$ is compact. Therefore $H(t, \bar{\Omega} \cap C_\infty(H))$ is relatively compact. Thus $C_\infty(H)$ is admissible. Hence $D(I - H(t, .), \Omega, y(t)) = D_{LS}(I - H(t, R(.)), R^{-1}(\Omega) \cap \Omega, y(t))$.
$\gamma(H(J \times R(R^{-1}(\Omega) \cap \Omega))) \leq \gamma(H(J \times (R(\Omega) \cap \Omega))) \leq k\ \gamma(R(\Omega) \cap \Omega) = 0$ since $R(\Omega) \subseteq C_\infty(H)$ which is compact. So $H \in \mathscr{K}(J \times \bar{\Omega}, X)$, and by ($D_{LS}3$) we have $D_{LS}(I - H(t, R(.)), R^{-1}(\Omega) \cap \Omega, y(t))$ is independent of t.

The following theorem ensures that the degree for set contractions is in fact an extension of the LS–degree.

4.1.9 Theorem

If $F \in \mathscr{K}(\bar{\Omega})$ and $y \in X\backslash(I - F)(\partial\Omega)$, then

$$D(I - F, \Omega, y) = D_{LS}(I - F, \Omega, y).$$

Proof :

By the same procedure as that used in getting equations (4) and (5) in the uniqueness proof, we get

$$D_{LS}(I - F, \Omega, y) = D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y)$$

where R is defined as before. Also by definition

$$D(I - F, \Omega, y) = D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y).$$

Thus

$$D(I - F, \Omega, y) = D_{LS}(I - F, \Omega, y).$$ ♠

Degree For γ–Condensing Maps

We first show that if F is γ–condensing, then k F is a strict γ–contraction for all positive $k < 1$.

Let $F : \bar{\Omega} \longrightarrow X$ be a γ–condensing map. Then $k\,F : \bar{\Omega} \longrightarrow X$. Take any bounded set $B \subseteq \bar{\Omega}$. Then $\gamma((k\,F)(B)) = \gamma(k\,F(B)) = k\,\gamma(FB)$.

If $\gamma(B) > 0$, then $k\,\gamma(FB) < k\,\gamma(B)$ since F is γ–condensing.

If $\gamma(B) = 0$, then B is relatively compact. Since k F is continuous, k F(B) is relatively compact, and so $\gamma(k\,F(B)) = 0$. Therefore $\gamma((k\,F)(B)) = 0 = k\,\gamma(B)$.

Thus in either case, $\gamma((k\,F)(B)) \leq k\,\gamma(B)$, proving that k F is a strict γ–contraction.

We now establish the uniqueness of the degree, if it exists.

Uniqueness:

Let

$\mathscr{M} = \{(I - F, \Omega, y) \;/\; \Omega \subseteq X$ open bounded, $F : \bar{\Omega} \longrightarrow X$ γ–condensing and $y \notin (I - F)(\partial\Omega)\}$ and suppose that that there exists a map $D : \mathscr{M} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3).

Let $(I - F, \Omega, y) \in \mathscr{M}$. By theorem 4.5, I − F is proper, hence $(I - F)(\partial\Omega)$ is closed. Therefore $\rho = \rho(y, (I - F)(\partial\Omega)) > 0$. Let $0 \leq r = \sup\{|Fx| \;/\; x \in \bar{\Omega}\}$. Choose $0 \leq k < 1$ such that $(1 - k)\,r < \rho$.

Define $H : J \times \bar{\Omega} \longrightarrow X$ by $H(t, x) = (1 - t)\,Fx + t\,k\,Fx = (1 - t(1 - k))\,Fx$ for $(t, x) \in J \times \bar{\Omega}$.

Then H is continuous.

Let $B \subseteq \bar{\Omega}$ with $\gamma(B) > 0$. Then $H(J \times B) \subseteq \operatorname{conv}(FB \cup k\,FB)$. Therefore

$\gamma(H(J \times B)) \leq \gamma(\operatorname{conv}(FB \cup k\,FB)) = \gamma(FB \cup k\,FB) = \max\{\gamma(FB), \gamma(k\,FB)\}$

$= \max\{\gamma(FB), k\,\gamma(FB)\} < \gamma(B)$.

Also, for $(t, x) \in J \times \partial\Omega$,

$$\begin{aligned} |y - (I - H(t, .))x| &= |y - (I - F)x + t(1 - k)\,Fx| \\ &\geq |y - (I - F)x| - t(1 - k)|Fx| \\ &\geq \rho - t(1 - k)r \\ &> \rho - \rho \end{aligned}$$

$$= \quad 0.$$

Therefore $y \notin (I - H(t, .))(\partial\Omega)$ for all $t \in J$. Thus by (D3),we have

$$D(I - F, \Omega, y) = D(I - k\, F, \Omega, y).$$

By above k F is a strict γ–contraction.

As was done in the Leray–Schauder degree, we can show that

$D(I - k\, F, \Omega, y) = D_{SC_\gamma}(I - k\, F, \Omega, y)$ where D_{SC_γ} is the degree for strict γ–contactions.

Therefore $D(I - F, \Omega, y) = D_{SC_\gamma}(I - k\, F, \Omega, y)$, showing the uniqueness of the degree.

Now to show the existence of the degree.

Existence:

For $F \in C_\gamma(\bar{\Omega})$ and $y \notin (I - F)(\partial\Omega)$ define the degree by

$$D(I - F, \Omega, y) = D_{SC_\gamma}(I - k\, F, \Omega, y),$$

where $(1 - k) \sup_{x \in \bar{\Omega}} |Fx| < \rho(y, (I - F)(\partial\Omega)) = \rho$, and $k < 1$. Let $r = \sup_{x \in \bar{\Omega}} |Fx|$.

We now want to show that for k_1 and k_2 satisfying these conditions,

$$D_{SC_\gamma}(I - k_1 F, \Omega, y) = D_{SC_\gamma}(I - k_2 F, \Omega, y).$$

Define $H : J \times \bar{\Omega} \longrightarrow X$ by $H(t, x) = (1 - t)k_1 Fx + tk_2 Fx$ for $(t, x) \in J \times \bar{\Omega}$.

H is continuous. For $B \subseteq \bar{\Omega}$,

$$\begin{aligned}
\gamma(H(J \times B)) &\leq \gamma(\text{conv}\,(k_1 FB \cup k_2 FB)) \\
&= \gamma(k_1 FB \cup k_2 FB\,) \\
&= \max\{\, \gamma(k_1 FB), \gamma(k_2 FB)\,\} \\
&= k\, \gamma(FB) \qquad \text{for } k = \max\{k_1, k_2\} \\
&\leq \gamma(B) \qquad \text{since } 0 \leq k < 1.
\end{aligned}$$

Lastly, we must show that $y \notin (I - H(t, .))(\partial\Omega)$ for $t \in J$. Let $(t, x) \in J \times \partial\Omega$, then

$$\begin{aligned}
|y - (I - H(t, .))x| &= |y - x + H(t, x)| \\
&= |y - (I - F)x - ((1 - t)(1 - k_1) + t(1 - k_2))\, Fx\,|
\end{aligned}$$

$$
\begin{aligned}
&\geq \rho - ((1-t)(1-k_1) + t(1-k_2))\, r \\
&> \rho - ((1-t)\rho + t\,\rho) \\
&= 0.
\end{aligned}
$$

Therefore $y \notin (I - H(t, .))(\partial\Omega)$ for $t \in J$. Thus $(D_{SC_\gamma} 3)$ is satisfied, proving that

$$D_{SC_\gamma}(I - k_1 F, \Omega, y) = D_{SC_\gamma}(I - k_2 F, \Omega, y).$$

We will now show that (D1)–(D3) are satisfied.

(D1) :

Let $y \in \Omega$.

$D(I, \Omega, y) = D_{SC_\gamma}(I - k\,F, \Omega, y)$ where $F \equiv 0$ and any $k \in [0, 1)$. By $(D_{SC_\gamma} 1)$, $D_{SC_\gamma}(I - k\,F, \Omega, y) = 1$, proving (D1).

(D2) :

Let $F \in C_\gamma(\bar{\Omega})$, Ω_1 and Ω_2 disjoint open subsets of Ω with $y \notin (I - F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. Let $\rho = \rho(y, (I - F)(\partial\Omega))$, $\delta = \rho(y, (I - F)(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)))$ and choose $k \in [0, 1)$ such that $(1 - k)\, r < \delta$ where $r = \sup \{|Fx| \;/\; x \in \bar{\Omega}\}$. Now $\delta \leq \rho$, so $(1 - k)\, r < \rho$. Therefore by definition, $D(I - F, \Omega, y) = D_{SC_\gamma}(I - k\,F, \Omega, y)$. We now need to show that $y \notin (I - k\,F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. Let $x \in \bar{\Omega} \setminus \Omega_1 \cup \Omega_2$. Then

$$
\begin{aligned}
|y - (I - k\,F)x| &= |y - (I - F)x - (1 - k)Fx| \\
&\geq |y - (I - F)x| - (1 - k)\,|Fx| \\
&\geq \delta - (1 - k)\, r \\
&> \delta - \delta \\
&= 0.
\end{aligned}
$$

So $y \notin (I - k\,F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. Thus

$$D_{SC_\gamma}(I - k\,F, \Omega, y) = D_{SC_\gamma}(I - k\,F, \Omega_1, y) + D_{SC_\gamma}(I - k\,F, \Omega_2, y).$$

Now to show that

$(1 - k) \sup \{|Fx| \;/\; x \in \bar{\Omega}_i\} < \rho(y, (I - F)(\partial\Omega_i))$ for $i = 1, 2$.

Now $\sup \{|Fx| \;/\; x \in \bar{\Omega}_i\} \leq \sup \{|Fx| \;/\; x \in \bar{\Omega}\} = r$ and

$\rho(y, (I - F)(\partial\Omega_i)) \geq \rho(y, (I - F)(\partial\Omega)) = \rho$ since $\partial\Omega_i \subseteq \partial\Omega$ for $i = 1, 2$. Therefore

$$\begin{aligned}(1 - k) \sup \{|Fx| \;/\; x \in \bar{\Omega}_i\} &\leq (1 - k) \sup \{|Fx| \;/\; x \in \bar{\Omega}\} \\ &= (1 - k)\, r \\ &< \rho \\ &\leq \rho(y, (I - F)(\partial\Omega_i)) \text{ for } i = 1, 2.\end{aligned}$$

Thus $D_{SC_\gamma}(I - k\,F, \Omega_i, y) = D(I - F, \Omega_i, y)$ for $i = 1, 2$.

Hence $D(I - F, \Omega, y) = D(I - F, \Omega_1, y) + D(I - F, \Omega_2, y)$, proving (D2) for γ–condensing maps.

(D3) :

Let $H \in C(J \times \bar{\Omega}, X)$, $y \in C(J)$, $y(t) \notin (I - H(t, .))(\partial\Omega)$ on J and for $B \subseteq \bar{\Omega}$ with $\gamma(B) > 0$, $\gamma(H(J \times B)) < \gamma(B)$.

Let $\rho_t = \rho(y(t), (I - H(t, .))(\partial\Omega)) > 0$ and $\rho = \inf\limits_{t \in J} \rho_t$,

$r = \sup \{|H(t, x)| \;/\; (t, x) \in J \times \bar{\Omega}\}$. Assume that $\rho > 0$, and choose $k \in [0, 1)$ such that $(1 - k)\, r < \rho$. Then

$$\begin{aligned}(1 - k) \sup \{|H(t, x)| \;/\; x \in \bar{\Omega}\} &\leq (1 - k) \sup \{|H(t, x)| \;/\; (t, x) \in J \times \bar{\Omega}\} \\ &= (1 - k)\, r \\ &< \rho \\ &= \inf_{t \in J} \rho_t \\ &\leq \rho_t \text{ for all } t \in J.\end{aligned}$$

Therefore,

$$D(I - H(t, .), \Omega, y(t)) = D_{SC_\gamma}(I - k\,H(t, .), \Omega, y(t)),$$

and this is independent of t since $\gamma(k\,H(J \times B)) = k\,\gamma(H(J \times B)) \leq k\,\gamma(B)$.

Thus (D3) holds.

We still have to show that $\rho = \inf_{t \in J} \rho_t > 0$.

Suppose $\rho = 0$. Then there exists a sequence $(t_n) \subseteq J$ such that $\rho_{t_n} \longrightarrow 0$. For each n, ther

exists $x_n \in \partial\Omega$ such that $|y(t_n) - (I - H(t_n, .))(x_n)| \leq \rho_{t_n} + \frac{1}{n}$ and $\rho_{t_n} + \frac{1}{n} \longrightarrow$

Therefore

$|y(t_n) - (I - H(t_n, .))(x_n)| \longrightarrow 0$, and hence $y(t_n) - (x_n - H(t_n, x_n)) \longrightarrow 0$. Since $(t_n) \subseteq$ with J compact, there exists a subsequence of t_n converging in J, say $t_n \longrightarrow t_0$ (without lo of generality). y continuous implies that $y(t_n) \longrightarrow y(t_0)$. Now

$\{x_n / n \in \mathbb{N}\} \subseteq \{x_n - H(t_n, x_n) / n \in \mathbb{N}\} + \{H(t_n, x_n) / n \in \mathbb{N}\}$.

$x_n - H(t_n, x_n) = y(t_n) - (y(t_n) - (x_n - H(t_n, x_n))) \longrightarrow y(t_0) - 0 = y(t_0)$.

So $\{x_n - H(t_n, x_n) / n \in \mathbb{N}\} \cup \{y(t_0)\}$ is compact. Hence

$\gamma(\{x_n - H(t_n, x_n) / n \in \mathbb{N}\} = 0$, and so

$\gamma(\{x_n / n \in \mathbb{N}\}) \leq 0 + \gamma(\{H(t_n, x_n) / n \in \mathbb{N}\})$. Now

$\gamma(\{H(t_n, x_n) / n \in \mathbb{N}\}) \leq \gamma(H(J \times \{x_n / n \in \mathbb{N}\})) < \gamma(\{x_n / n \in \mathbb{N}\})$ if

$\gamma(\{x_n / n \in \mathbb{N}\}) > 0$.

Thus $\gamma(\{x_n / n \in \mathbb{N}\}) = 0$ and so there exists a subsequence of x_n that converges, say $x_n \longrightarrow x_0$ (without loss of generality). But $(x_n) \subseteq \partial\Omega$ which is closed, and so $x_0 \in \partial\Omega \subseteq \bar{\Omega}$. Therefore $H(t_n, x_n) \longrightarrow H(t_0, x_0)$. Hence $x_0 - H(t_0, x_0) = y(t_0)$, implyi that

$y(t_0) \in (I - H(t_0, .))(\partial\Omega)$, a contradiction to the hypothesis. Thus $\rho > 0$.

Thus we have proved the following theorem.

4.1.10 Theorem

Let X be a Banach space and

$\mathscr{M} = \{(I - F, \Omega, y) \mathrel{/} \Omega \subseteq X$ open bounded, $F \in C_\gamma(\bar{\Omega}),\ y \in X \setminus (I - F)(\partial\Omega)\ \}$.

(a) Then there exists a unique map $D : \mathscr{M} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3), the *degree for γ–condensing maps*.

(b) Let $F \in SC_\gamma(\bar{\Omega})$. If there exists a closed convex $C \subseteq X$ such that $C_\infty \subseteq C$, $F(\bar{\Omega} \cap C) + y \subseteq C$ and $F(\bar{\Omega} \cap C)$ is relatively compact (C_∞ is defined above), and if R is any retraction onto C, then

$$D(I - F, \Omega, y) = D_{LS}(I - FR, R^{-1}(\Omega) \cap \Omega, y)$$

where D_{LS} is the Leray–Schauder degree and $D(I - F, \Omega, y) = 0$ if no such C exists.

(c) If $F \in C_\gamma(\bar{\Omega})$, then

$$D(I - F, \Omega, y) = D_{SC_\gamma}(I - k\,F, \Omega, y)$$

where $k \in [0, 1)$ and $(1 - k) \sup \{|Fx| \mathrel{/} x \in \bar{\Omega}\ \} < \rho(y, (I - F)(\partial\Omega))$, and D_{SC_γ} is the degree defined in (b).

Again we obtain the properties (D4)–(D7) of the degree.

4.1.11 Theorem

Besides (D1)–(D3), the degree defined above has the following properties.

(D4) $D(I - F, \Omega, y) \neq 0$ implies $(I - F)^{-1}(y) \neq \emptyset$.

(D5) $D(I - G, \Omega, y) = D(I - F, \Omega, y)$ for $G \in C_\gamma(\bar{\Omega}) \cap B_\rho(F)$ and $D(I - F, \Omega, .)$ is constant on $B_\rho(y)$ where $\rho = \rho(y, (I - F)(\partial\Omega))$. More than that, we have $D(I - F, \Omega, .)$ is constant on every connected component of $X \setminus (I - F)(\partial\Omega)$.

(D6) $D(I - G, \Omega, y) = D(I - F, \Omega, y)$ whenever $G|_{\partial\Omega} = F|_{\partial\Omega}$ and

$G \in C_\gamma(\bar{\Omega})$.

(D7) $D(I - F, \Omega, y) = D(I - F, \Omega_1, y)$ for every open subset Ω_1 of Ω satisfying $y \notin (I - F)(\bar{\Omega} \setminus \Omega_1)$.

The proofs go exactly like those in theorem 3.11, since they follow from (D1)–(D3).

In (D6), for $H(t, x) = t\,Fx + (1 - t)\,Gx$ we have for $B \subseteq \bar{\Omega}$, $\gamma(B) > 0$,

$\gamma(H(J \times B)) \leq \max\{\gamma(FB), \gamma(GB)\} < \gamma(B)$.

The next theorem shows that the γ–condensing degree is in fact an extension of the degree for strict γ–contractions.

4.1.12 Theorem

If $F \in SC_\gamma(\bar{\Omega})$ with $y \in X \setminus (I - F)(\partial\Omega)$, then

$$D(I - F, \Omega, y) = D_{SC_\gamma}(I - F, \Omega, y).$$

Proof :

As in the uniqueness proof, we can show that

$$D_{SC_\gamma}(I - F, \Omega, y) = D_{SC_\gamma}(I - k\,F, \Omega, y),$$

where $(1 - k) \sup_{x \in \Omega} |Fx| < \rho(y, (I - F)(\partial\Omega))$.

By definition,

$$D(I - F, \Omega, y) = D_{SC_\gamma}(I - k\,F, \Omega, y).$$

Thus we have

$$D(I - F, \Omega, y) = D_{SC_\gamma}(I - F, \Omega, y).$$

♠

4.1.13 Theorem

Let X_0 be a closed subspace of X, $\Omega \subseteq X$ open bounded, $F : \bar{\Omega} \longrightarrow X_0$ a γ–condensing map, $y \in X_0 \setminus (I - F)(\partial\Omega)$. Then

$$D(I - F, \Omega, y) = D((I - F)|_{\overline{\Omega \cap X_0}}, \Omega \cap X_0, y).$$

Proof :

Let $r = \sup \{|Fx| \ / \ x \in \bar{\Omega}\}$, $\rho = \rho(y, (I - F)(\partial\Omega)) > 0$. Choose $k \in [0, 1)$ such that $(1 - k)\, r < \rho$. Then by definition

$$D(I - F, \Omega, y) = D(I - k F, \Omega, y).$$

Also with $\Omega_0 = \Omega \cap X_0$,

$$\begin{aligned}
(1 - k) \sup &\{|Fx| \ / \ x \in \bar{\Omega}_0\} \\
&\leq (1 - k) \sup \{|Fx| \ / \ x \in \bar{\Omega}\} \\
&< \rho \\
&= \rho(y, (I - F)(\partial\Omega)) \\
&\leq \rho(y, (I - F)(\partial\Omega_0)) \quad \text{since } \partial\Omega_0 \subseteq \partial\Omega.
\end{aligned}$$

Thus by definition again,

$$D((I - F)|_{\bar{\Omega}_0}, \bar{\Omega}_0, y) = D((I - k F)|_{\bar{\Omega}_0}, \Omega_0, y).$$

Thus we only need to show that

$$D(I - k F, \Omega, y) = D((I - k F)|_{\bar{\Omega}_0}, \Omega_0, y),$$

and hence we may assume that F is a strict γ–contraction with constant $k < 1$, and we must show that $D(I - F, \Omega, y) = D((I - F)|_{\bar{\Omega}_0}, \Omega_0, y)$.

If $(I - F)^{-1}(y) = \emptyset$, then $(I - F)^{-1}(y) \cap \bar{\Omega}_0 = \emptyset$ and so

$$D(I - F, \Omega, y) = 0 = D((I - F)|_{\bar{\Omega}_0}, \Omega_0, y).$$

Now assume that $(I - F)^{-1}(y) \neq \emptyset$. This implies that $C_\infty(F) \neq \emptyset$. Let $R : X \longrightarrow C_\infty$ be a retraction. Then

$$\begin{aligned}
D(I - F, \Omega, y) &= D((I - FR)|_{\overline{R^{-1}(\Omega) \cap \Omega}}, R^{-1}(\Omega) \cap \Omega, y) \\
&= D((I - FR)|_{\overline{R^{-1}(\Omega) \cap \Omega_0}}, R^{-1}(\Omega) \cap \Omega_0, y)
\end{aligned}$$

since this result holds for the LS–degree.

$C_\infty(F|_{\bar{\Omega}_0}) \subseteq C_\infty$. Hence C_∞ is admissible for $F|_{\bar{\Omega}_0}$. Thus

$$D((I-F)|_{\bar{\Omega}_0}, \Omega_0, y) = D((I-FR)|_{\overline{R^{-1}(\Omega_0) \cap \Omega_0}}, R^{-1}(\Omega_0) \cap \Omega_0, y).$$

So we need to show that

$$D((I-FR)|_{\overline{R^{-1}(\Omega) \cap \Omega_0}}, R^{-1}(\Omega) \cap \Omega_0, y)$$
$$= D((I-FR)|_{\overline{R^{-1}(\Omega_0) \cap \Omega_0}}, R^{-1}(\Omega_0) \cap \Omega_0, y).$$

Therefore we must show that

$$y \notin (I-FR)(\overline{R^{-1}(\Omega) \cap \Omega_0} \setminus (R^{-1}(\Omega_0) \cap \Omega_0)).$$

Suppose $y = (I-FR)x$ for

$$\begin{aligned} x \quad &\in \quad \overline{R^{-1}(\Omega) \cap \Omega_0} \setminus (R^{-1}(\Omega_0) \cap \Omega_0) \\ &\subseteq \quad R^{-1}(\bar{\Omega}) \cap \bar{\Omega}_0 \setminus (R^{-1}(\Omega_0) \cap \Omega_0) \\ &\subseteq \quad (R^{-1}(\bar{\Omega}) \setminus R^{-1}(\Omega_0) \cap \bar{\Omega}_0) \cup (R^{-1}(\bar{\Omega}) \cap \partial\Omega_0). \end{aligned}$$

Then $x \in R^{-1}(\bar{\Omega})$ and so $Rx \in \bar{\Omega} \cap C_\infty$. Therefore $x = FRx + y \in C_\infty$, and hence $Rx = x$. So $y = (I-F)x$. If $x \in \partial\Omega_0$, then $y \in (I-F)(\partial\Omega_0) \subseteq (I-F)(\partial\Omega)$, a contradiction.

If $x \notin R^{-1}(\Omega_0)$, then $x = Rx \notin \Omega_0$, a contradiction to $x = Fx + y \in X_0$. Therefore $y \notin (I-FR)(R^{-1}(\Omega) \cap \Omega_0 / R^{-1}(\Omega_0) \cap \Omega_0)$. Thus

$$D((I-FR)|_{R^{-1}(\Omega) \cap \Omega_0}, R^{-1}(\Omega) \cap \Omega_0, y)$$
$$D((I-FR)|_{R^{-1}(\Omega_0) \cap \Omega_0}, R^{-1}(\Omega_0) \cap \Omega_0, y),$$

proving the result. ♠

The following lemma can be found in Nussbaum [1].

4.1.14 Lemma

Let $H : \bar{\Omega} \to X$ be odd, continuous, Ω symmetric with respect $0 \in \Omega$, then $C_\infty(H)$ is symmetric.

Proof :

If $x \in \text{conv } H(\bar{\Omega})$, then $x = \lambda_1 H(x_1) + \dots + \lambda_k H(x_k)$ where $x_k \in \bar{\Omega}$ and $\lambda_k \in [0, 1]$ and the λ_k sum to 1.

Then $-x = \lambda_1 H(-x_1) + \dots + \lambda_k H(-x_k)$ with $-x_k \in \bar{\Omega}$ (since Ω is symmetric, $\bar{\Omega}$ is symmetric). Therefore $-x \in \text{conv } H(\bar{\Omega})$. Thus $\text{conv } H(\bar{\Omega})$ symmetric implying that $C_0 = \overline{\text{conv}}\ H(\bar{\Omega})$ is symmetric. Suppose C_{n-1} is symmetric for $n \geq 1$. Let $x \in \text{conv } H(\bar{\Omega} \cap C_{n-1})$. Then $x = \lambda_1 H(x_1) + \dots + \lambda_k H(x_k)$, where $x_k \in \bar{\Omega} \cap C_{n-1}$.

Therefore $-x \in \bar{\Omega} \cap C_{n-1}$, which implies that

$-x = \lambda_1 H(-x_1) + \dots + \lambda_k H(-x_k) \in \text{conv } H(\bar{\Omega} \cap C_{n-1})$. Therefore $\text{conv } H(\bar{\Omega} \cap C_{n-1})$ is symmetric , implying that C_n is symmetric.

Thus C_∞ is symmetric. ♠

The extension of Borsuk's theorem is simple.

4.1.15 Theorem

Let $\Omega \subseteq X$ be open bounded and symetric with respect to $0 \in \Omega$, $F \in C_\gamma(\bar{\Omega})$, $0 \notin (I - F)(\partial\Omega)$ and $(I - F)(-x) \neq \lambda\,(I - F)(x)$ on $\partial\Omega$ for all $\lambda \geq 1$. Then $D(I - F, \Omega, 0)$ is odd. In particular, this is true if $F|_{\partial\Omega}$ is odd and $x \neq Fx$ on $\partial\Omega$.

Proof :

Define $H(t, x) = \frac{1}{1+t} Fx - \frac{t}{1+t} F(-x)$ for $(t, x) \in J \times \bar{\Omega}$. Then for $B \subseteq \bar{\Omega}$ and $\gamma(B) > 0$,

$$\begin{aligned} \gamma(H(J \times B)) &\leq \gamma(\text{conv } (FB \cup (-F(-B)))) \\ &= \gamma(FB \cup (-F(-B))) \\ &= \max \{\gamma(FB), \gamma(-F(-B))\} \\ &< \gamma(B). \end{aligned}$$

Therefore $H \in C_\gamma(J \times \bar{\Omega}, X)$. If $0 \in (I - H(t, .))(\partial\Omega)$ for some $t \in J$, then $0 = (I - H(t, .))x$ for some $x \in \partial\Omega$. Therefore

$0 = \frac{1}{1+t}(I - F)x - \frac{t}{1+t}(I - F)(-x)$. If $t \neq 0$, then $(I - F)(-x) = \frac{1}{t}(I - F)x$ with $\frac{1}{t} \geq 1$, and if $t = 0$, then $(I - F)x = 0$, a contradiction.

Therefore $0 \notin (I - H(t, .))(\partial\Omega)$. Thus we may apply (D3), to obtain

$$D(I - F, \Omega, 0) = D(I - G, \Omega, 0), \tag{13}$$

where $G(x) = \frac{1}{2}(Fx - F(-x))$ is odd. Choose $k \in [0, 1]$ such that $(1 - k) \sup \{|Gx| \;/\; x \in \bar{\Omega}\} < \rho(0, (I - G)(\partial\Omega))$. Then

$$D(I - G, \Omega, 0) = D(I - k\,G, \Omega, 0). \tag{14}$$

Let $H = k\,G$. Then H is also odd and $H \in SC_\gamma(\bar{\Omega})$. Let $C_0 = \overline{\text{conv}}\,(H(\bar{\Omega}))$, $C_n = \overline{\text{conv}}\,(H(\bar{\Omega} \cap C_{n-1}))$ for $n \geq 1$. By lemma 4.1.12, each C_n is symmetric and so $C_\infty(H) = \bigcap_{n \geq 0} C_n$ is also symmetric. Let $R_0 : X \longrightarrow C_\infty(H)$ be a retraction. Then $Rx = \frac{1}{2}(R_0 x - R_0(-x))$ is odd and is also a retraction onto $C_\infty(H)$, since for $x \in C_\infty(H)$ we have $-x \in C_\infty(H)$ and
$Rx = \frac{1}{2}(R_0(x) - R_0(-x)) = \frac{1}{2}(x - (-x)) = x$.

If $x \in R^{-1}(\Omega) \cap \Omega$, then $-x \in \Omega$ and $Rx \in \Omega$ implies that $-Rx \in \Omega$, and this means that $R(-x) \in \Omega$ since R is odd. So $-x \in R^{-1}(\Omega) \cap \Omega$. Thus $R^{-1}(\Omega) \cap \Omega$ is symmetric.

Also $G(0) = 0$. So $H(0) = 0$, and hence $0 \in C_\infty(H)$. Therefore $R0 = 0$, and so $0 \in R^{-1}(\Omega)$ and $0 \in \Omega$. Thus $0 \in R^{-1}(\Omega) \cap \Omega$. By definition,

$$D(I - H, \Omega, 0) = D(I - HR, R^{-1}(\Omega) \cap \Omega, 0). \tag{15}$$

Since HR is odd, we can apply theorem 3.12 to obtain
$D(I - HR, R^{-1}(\Omega) \cap \Omega, 0)$ is odd. (13), (14) and (15) yield
$D(I - F, \Omega, 0)$ is odd. ♠

Before we prove the domain invariance theorem, we require the following lemma which is found in Nussbaum [1].

4.1.16 **Lemma**

Let V be a closed bounded set in a Banach space X. For any subset $A \subseteq V$ an any real $\epsilon > 0$, let $A_\epsilon = \{x \in V \,/\, \rho(x, A) < \epsilon\}$. Let $f : V \longrightarrow X$ be a continuou map such that for any $A \subseteq V$ with $\gamma(A) > 0$, $\overline{\lim_{\epsilon \to 0}} \gamma(f(A_\epsilon)) < \gamma(A)$. Let J = [0, and assume that we are given two homotopies, $G : J \times V \longrightarrow V$ and $H : J \times V \longrightarrow V$, such that G(t, x) and H(t, x) are uniformly continuous in t, $G_t \equiv G(t, .)$ is a k_t–set contraction and $H_t \equiv H(t, .)$ is a h_t–set contraction, and $k_t + h_t \leq 1$ for $t \in J$. Consider the homotopy $F(t, x) = f(H(t, x)) - f(G(t, x)$. Then if A is any subset of V with $\gamma(A) > 0$,

$$\gamma(F(J \times A)) < \gamma(A).$$

Proof :

Suppose $A \subseteq V$ and $\gamma(A) = d > 0$ and suppose $s \in J$. We want to find an ope interval J_s about s in J such that $\gamma(F(J_s \times A)) < \gamma(A)$. To do this consid $H_s(A)$ and $G_s(A)$.

If $H_s(A)$ and $G_s(A)$ are relatively compact, then $f(H_s(A))$ and $f(G_s(A))$ a relatively compact. By the uniform continuity of H(t, x) and G(t, x) in t, the exists $\delta > 0$ such that for $t \in J$ and $|t - s| \leq \delta$, $G_t(A) \subseteq N_{d/8}(G_s(A))$ and $H_t(A) \subseteq N_{d/8}(H_s(A))$. If we set $J_s = J \cap (s - \delta, s + \delta)$, it follows that $F(J_s \times A) \subseteq \{y - z \,/\, y \in f(N_{d/8}(H_s(A))), z \in f(N_{d/8}(G_s(A)))\}$, so that $\gamma(F(J_s \times A)) \leq \frac{d}{4} + \frac{d}{4} < d = \gamma(A)$.

If $\gamma(H_s(A)) > 0$ or $\gamma(G_s(A)) > 0$, we may assume for definiteness that $\gamma(H_s(A)) > 0$. By assumption on f, there exists $\epsilon > 0$ such that if we set $C_{s,\epsilon} = N_\epsilon(H_s(A)) \cap V$, then $\gamma(f(C_{s,\epsilon})) < \gamma(H_s(A))$. By the uniform continuity H in t, there exists $\delta_1 > 0$ such that if $t \in J$ and $|t - s| \leq \delta_1$, then $H_t(A) \subseteq C_{s,}$ If we write $4a = \gamma(H_s(A)) - \gamma(f(C_{s,\epsilon}))$, by the uniform continuity of G in there exists $\delta_2 > 0$ such that for $t \in J$ and $|t - s| \leq \delta_2$, $G_t(A) \subseteq N_a(G_s(A))$.

follows that if we take $\delta = \min\{\delta_1, \delta_2\}$ and $J_s = J \cap (s - \delta, s + \delta)$, then $\gamma(f(H(J_s \times A))) \leq \gamma(H_s(A)) - 4a$ and $\gamma(f(G(J_s \times A))) \leq \gamma(G_s(A)) + 2a$. This in turn implies that

$$\begin{aligned}
\gamma(F(J_s \times A)) &\leq \gamma(\{y - z \,/\, y \in f(H(J_s \times A)), z \in f(G(J_s \times A))\} \\
&\leq (\gamma(H_s(A)) - 4a) + (\gamma(G_s(A)) + 2a) \\
&< \gamma(H_s(A)) + \gamma(G_s(A)) \\
&< h_s\, \gamma(A) + k_s\, \gamma(A) \\
&= (h_s + k_s)\, \gamma(A) \\
&\leq \gamma(A).
\end{aligned}$$

The remainder of the proof is a simple compactness argument. As we have shown, for each $s \in J$, there is an open interval J_s about s in J such that $\gamma(F(J_s \times A)) < \gamma(A)$. By the compactness of J, J can be covered by a finite number of these subintervals, say $J_{s_1}, \ldots, J_{s_n}$. Then

$$\begin{aligned}
\gamma(F(J \times A)) &= \gamma\Big(\bigcup_{i=1}^{n} F(J_{s_i} \times A)\Big) \\
&= \max\{\, \gamma(F(J_{s_i} \times A)) \,/\, i = 1, \ldots, n \,\} \\
&< \gamma(A).
\end{aligned}$$

♠

Remarks : If f is a k–set contraction, $k < 1$, then for any $A \subseteq V$ with $\gamma(A) > 0$, $\gamma(f(A_\epsilon)) \leq k\, \gamma(A_\epsilon) \leq k\,(\gamma(A) + 2\epsilon)$, and $k\,\gamma(A) + 2\epsilon k < \gamma(A)$ for $\epsilon < [(1 - k)\,\gamma(A)]/2$. Thus the condition of Lemma 4.1.16 holds if f is a k–set contraction, $k < 1$. The hypothesis also holds if f is a condensing map. In this case, take $\delta = \gamma(A) - \gamma(f(A)) > 0$, and, by uniform continuity, select $\epsilon > 0$ so that $f(A_\epsilon) \subseteq N_{\delta/3}(f(A))$. Then we have $\gamma(f(A_{\epsilon'})) \leq \gamma(f(A)) + 2\delta/3 < \gamma(A)$ for $0 \leq \epsilon' < \epsilon$.

4.1.17 Theorem (Invariance of Domain)

Let Ω be an open subset of a Banach space X and let $f : \Omega \longrightarrow X$ be a continuous map such that $I-f$ is one–to–one. Assume that for each $x_0 \in \Omega$, there is a closed ball V about x_0, $V \subseteq \Omega$, such that for any $A \subseteq V$ with $\gamma(A) > 0$, if we set

$A_\epsilon = \{x \in V \,/\, d(x, A) < \epsilon\}$, then $\overline{\lim\limits_{\epsilon \downarrow 0}}\, \gamma(f(A_\epsilon)) < \gamma(A)$.

Then $(I-F)(\Omega)$ is open.

Proof :

Suppose $(I-f)x_0 = z_0$. Select a closed ball V about x_0 as in the statement of the theorem. We want to show that $(I-f)(V)$ contains an open neighbourhood of z_0, and since $x_0 \in \Omega$ is arbitrary, this will show that $(I-f)(\Omega)$ is open. Clearly, we can assume $x_0 = z_0 = 0$. Suppose we can show that $D(I-f, V^0, 0) \neq 0$. Since $I-f$ is one–to–one, $x - fx \neq 0$ for $x \in \partial V$; and since $I-f$ is a closed map (because $f|_V$ is γ–condensing), $|x - fx| \geq \epsilon > 0$ for $x \in \partial V$. For $|z| < \epsilon$, $I - f_z$ is homotopic to $I-f$ ($f_z x = fx + z$) by the homotopy $I - t f_z - (1-t) f$, $0 \leq t \leq 1$; and this homotopy is uniformly continuous in t and has no zeros on ∂V. Thus we see that

$$\begin{aligned} D(I-f, V^0, 0) &= D(I-f_z, V^0, 0) \\ &= D(I-f-z, V^0, 0) \\ &= D(I-f, V^0, z) \\ &= D(I-f, V^0, 0) \\ &\neq 0, \end{aligned}$$

so there exists $u \in V$ with $(I-f)u = z$. This shows that $(I-f)(V) \supseteq B_\epsilon(0)$, the open ball about 0.

To complete the proof, it thus suffices to prove that $D(I-f, V^0, 0) \neq 0$. Consider the homotopy $F(t, x) = f\left(\frac{x}{1+t}\right) - f\left(\frac{-t\,x}{1+t}\right)$, $0 \leq t \leq 1$.

If we set $H(t, x) = \frac{x}{1+t}$ and $G(t, x) = \frac{-t\,x}{1+t}$, we see that $H : J \times V \longrightarrow V$, $G : J \times V \longrightarrow V$, H_t is a $\frac{1}{1+t}$ – set contraction, G_t is a $\frac{1}{1+t}$ – set contraction, $\frac{1}{1+t} + \frac{t}{1+t} = 1$, and G and H are uniformly continuous in t. By lemma 4.1.16, if

$A \subseteq V$ and $\gamma(A) > 0$, $\gamma(F(J \times A)) < \gamma(A)$. Also, $F(t, x) \neq x$ for $(t, x) \in J \times \partial V$, for if $F(t, x) = x$, we obtain $\frac{x}{1+t} - f(\frac{x}{1+t}) = \frac{-tx}{1+t} - f(\frac{-tx}{1+t})$, which contradicts the fact that $I - f$ is one–to–one. It follows by (D3) that

$D(I - f, V^0, 0) = D(I - F_0, V^0, 0) = D(I - F_1, V^0, 0)$. However, $F_1(x) = f(\frac{x}{2}) - f(-\frac{x}{2})$, so $F_1(-x) = -F_1(x)$, and by theorem 4.1.15 we find $D(I - F_1, V^0, 0)$ is odd and hence nonzero. ♠

The last two results, with the proofs, are taken from Nussbaum.

4.2 THE NUSSBAUM DEGREE

We would like to define a degree for the triplet $(I - F, \Omega, 0)$, where X is a Banach space, $\Omega \subseteq X$ open bounded, $F : \Omega \longrightarrow X$ γ–condensing and $S = \{x \in \Omega \;/\; Fx = x\}$ compact. (The empty set is regarded as a compact set.).

Now $S \subseteq \bigcup_{x \in S} B_{r_x}(x)$ where $\bar{B}_{r_x}(x) \subseteq \Omega$. Since S is compact, we have $S \subseteq \bigcup_{i=1}^{n} B_{r_i}(x_i)$ with $\bar{B}_{r_i}(x_i) \subseteq \Omega$. If $V = \bigcup_{i=1}^{n} B_{r_i}(x_i)$, then V is an open neighbourhood of S and $\bar{V} \subseteq \bigcup_{i=1}^{n} \bar{B}_{r_i}(x_i) \subseteq \Omega$. If $0 = (I - F)x$ with $x \in \Omega$, then $Fx = x$ and so $x \in S \subseteq V$. Thus $x \in V$ and hence $x \notin \partial V$. So

$$0 \notin (I - F)(\partial V). \tag{1}$$

Since F is γ–condensing, so is $F|_{\bar{V}}$. Therefore $(I - F, V, 0)$ is an admissible triplet for the γ–condensing degree, D_{C_γ}. Thus we define

$$D(I - F, \Omega, 0) = D_{C_\gamma}(I - F, V, 0).$$

We must show that this definition is independent of V.

Let V_i be an open neighbourhood of S with $\bar{V}_i \subseteq \Omega$ for $i = 1, 2$. Then $V_1 \cap V_2$ is an open neighbourhood of S and $\overline{V_1 \cap V_2} \subseteq \bar{V}_1 \cap \bar{V}_2 \subseteq \Omega$.

Now $\Omega_1 = V_i \setminus \overline{V_1 \cap V_2}$ and $\Omega_2 = V_1 \cap V_2$ are disjoint open subsets of V_i. Let $0 = (I - F)x$ for $x \in \bar{V}_i \setminus \Omega_1 \cup \Omega_2$. Then $x \in S \subseteq V_1 \cap V_2$. But $x \notin \Omega_1 \cup \Omega_2$. So $x \notin V_i \setminus \overline{V_1 \cap V_2}$ and $x \notin V_1 \cap V_2$. Therefore $x \in \overline{V_1 \cap V_2}$ and $x \notin V_1 \cap V_2$. So $x \in \partial(V_1 \cap V_2)$, a contradiction.

Thus $0 \notin (I - F)(\bar{V}_i \setminus (\Omega_1 \cup \Omega_2))$.

By $(D_{C_\gamma} 2)$,

$$D_{C_\gamma}(I - F, V_i, 0) = D_{C_\gamma}(I - F, \Omega_1, 0) + D_{C_\gamma}(I - F, \Omega_2, 0).$$

Now $(I - F)^{-1}(0) = S \subseteq V_i \cup (V_1 \cap V_2)$. Therefore $(I - F)^{-1}(0) \cap (V_i \setminus \overline{V_1 \cap V_2}) = \emptyset$, by $(D_{C_\gamma} 4)$, $D_{C_\gamma}(I - F, \Omega_1, 0) = 0$. So

$$D_{C_\gamma}(I - F, V_i, 0) = D_{C_\gamma}(I - F, V_1 \cap V_2, 0),$$

and hence

$$D_{C_\gamma}(I - F, V_1, 0) = D_{C_\gamma}(I - F, V_2, 0).$$

So the the degree is well–defined.

Now we show that D satisfies $(D1) - (D3)$.

(D1) :

Let $F \equiv 0$ and $0 \in \Omega$. Then $S = \{x \in \Omega \ / \ Fx = x\} = \{0\}$. Let V be any
neighbourhood of S such that $\bar{V} \subseteq \Omega$. Then $0 \in V$ and so $D(I, \Omega, 0) = D_{C_\gamma}(I, V, 0) = 1$
by $(D_{C_\gamma} 1)$.

(D2) :

Let Ω_1, Ω_2 be disjoint open subsets of Ω such that $0 \notin (I - F)(\Omega \setminus \Omega_1 \cup \Omega_2)$.

$S = \{x \in \Omega \,/\, Fx = x\}$ is compact. Let $S_i = \{x \in \Omega_i \,/\, Fx = x\}$ for $i = 1, 2$. Then $S_i = \Omega_i \cap S$. Let $x \in S \cap (X \setminus \Omega_2)$. Then $x = Fx$, $x \in \Omega$ and $x \in X \setminus \Omega_2$. If $x \notin \Omega_1$, then $x \in \Omega \setminus \Omega_1 \cup \Omega_2$, a contradiction. So $x \in \Omega_1$ and hence $x \in S_1$. Therefore $S \cap (X \setminus \Omega_2) \subseteq S_1$.

Now let $x \in S_1$. Then $x \in S$ and $x \in \Omega_1$. So $Fx = x$ and $x \in \Omega_1$, and hence $x \notin \Omega_2$ since Ω_1 and Ω_2 are disjoint. Therefore $x \in S \cap (X \setminus \Omega_2)$, and hence $S_1 \subseteq S \cap (X \setminus \Omega_2)$.

Thus $S_1 = S \cap (X \setminus \Omega_2)$.

$X \setminus \Omega_2$ is closed and S is closed, so S_1 is closed, and is a subset of a compact set S, hence S_1 must be compact.

Similarly S_2 is compact.

Therefore $(I - F, \Omega_1, 0)$ and $(I - F, \Omega_2, 0)$ are admissible triplets.

Let V_i be an open neighbourhood of S_i such that $\bar{V}_i \subseteq \Omega_i$ for $i = 1, 2$, and let $V = V_1 \cup V_2$. If $x \in S$, then $Fx = x$ and $x \in \Omega$. Therefore $0 = (I - F)x$ for $x \in \Omega$. But $0 \notin (I - F)(\Omega \setminus \Omega_1 \cup \Omega_2)$. So $x \in \Omega_1 \cup \Omega_2$. Therefore $x \in \Omega_1$ or $x \in \Omega_2$, and so

$S \subseteq S_1 \cup S_2 \subseteq V_1 \cup V_2 = V$, and $\bar{V} = \overline{V_1 \cup V_2} \subseteq \bar{V}_1 \cup \bar{V}_2 \subseteq \Omega_1 \cup \Omega_2 \subseteq \Omega$. So V is an open neighbourhood of S such that $\bar{V} \subseteq \Omega$.

By definition,

$$D(I - F, \Omega, 0) = D_{C_\gamma}(I - F, V, 0), \tag{2}$$

and

$$D(I - F, \Omega_i, 0) = D_{C_\gamma}(I - F, V_i, 0), \tag{3}$$

for $i = 1, 2$.

We now need to show that $0 \notin (I - F)(\bar{V} \setminus V_1 \cup V_2) = (I - F)(\partial V)$. By (1), this is true, and hence by $(D_{C_\gamma} 2)$,

$$D_{C_\gamma}(I - F, V, 0) = D_{C_\gamma}(I - F, V_1, 0) + D_{C_\gamma}(I - F, V_2, 0). \tag{4}$$

By (2), (3), and (4), we have

$$D(I - F, \Omega, 0) = D(I - F, \Omega_1, 0) + D(I - F, \Omega_2, 0).$$

The following theorem is an extension of the ordinary (D3), found in Nussbaum [2].

4.2.1 Theorem

Let $\Omega \subseteq J \times X$ be open bounded and $H : \Omega \longrightarrow X$ be continuous such that $S = \{(t, x) \in \Omega \ / \ H(t, x) = x\}$ is compact and $\gamma(H(\Omega \cap (J \times B))) < \gamma(B)$ for all bounded $B \subseteq X$ with $\gamma(B) > 0$. Set $\Omega_t = \{x \in X \ / \ (t, x) \in \Omega\}$. Then $D(I - H(t, .), \Omega_t, 0)$ is independent of t.

Proof :

Step 1 :

Suppose we have shown that every $t \in J$ has a neighbourhood O_t such that $D(I - H(s, .), \Omega_s, 0)$ is constant for all $s \in O_t$.

Let $U = \{t \in J \ / \ D(I - H(t, .), \Omega_t, 0) = D(I - H(0, .), \Omega_0, 0)\}$ and let $t \in U$. Then $D(I - H(t, .), \Omega_t, 0) = D(I - H(0, .), \Omega_0, 0)$. But for all $s \in O_t$, $D(I - H(s, .), \Omega_s, 0)$ is constant and hence must be $D(I - H(t, .), \Omega_t, 0)$. So $D(I - H(s, .), \Omega_s, 0) = D(I - H(0, .), \Omega_0, 0)$ for all $s \in O_t$. Therefore $O_t \subseteq U$ and hence U is open in J.

Let $t \notin U$. Then $D(I - H(t, .), \Omega_t, 0) \neq D(I - H(0, .), \Omega_0, 0) = d_0$. So for all $s \in O_t$, $D(I - H(s, .), \Omega_s, 0) = D(I - H(t, .), \Omega_t, 0) \neq d_0$. Hence $s \notin U$.

So $O_t \subseteq J \setminus U$. Therefore $J \setminus U$ is open and hence U is closed in J.

But U is an open and closed set in a connected set J. Hence it is either empty or J. Since $0 \in U$, we must have $U = J$. Therefore

$$D(I - H(t, .), \Omega_t, 0) = D(I - H(0, .), \Omega_0, 0)$$

for all $t \in J$, and is thus constant on J.

Step 2 :

We must now show that for $t_0 \in J$, we can find an open neighbourhood O_{t_0} of t_0 in J such that $D(I - H(t, .), \Omega_t, 0)$ is constant for all $t \in O_{t_0}$.

Let $S_t = \{t\} \times \{x \,/\, (t, x) \in S\}$. Given $(t_0, x) \in S_{t_0}$, we can find an open neighbourhood N_x of x (in X) and $\epsilon_x > 0$ such that $\overline{J_x \cap J} \times \bar{N}_x \subseteq \Omega$ $(J_x = (t_0 - \epsilon_x, t_0 + \epsilon_x))$. S_{t_0} is easily shown to be closed and since it is a subset of the compact set S, S_{t_0} is also compact. So there exist finitely many that cover it, say $(J_{x_i} \cap J) \times N_{x_i}$, $i = 1, 2, ..., n$.

Let $\epsilon = \min\{\epsilon_{x_i} \,/\, i = 1, 2, ..., n\}$ with $I = (t_0 - \epsilon, t_0 + \epsilon)$ and $V = \bigcup_{i=1}^{n} N_{x_i}$.

So for $(t_0, x) \in S_{t_0}$, $(t_0, x) \in (J_{x_i} \cap J) \times N_{x_i}$ for some i. Therefore $x \in N_{x_i} \subseteq V$ and $t_0 \in (I \cap J)$. So $(t_0, x) \in (I \cap J) \times V$, and hence

$$S_{t_0} \subseteq (I \cap J) \times V. \tag{5}$$

$$\begin{aligned}\overline{(I \cap J)} \times \bar{V} &= \overline{(I \cap J)} \times \overline{\left(\bigcup_{i=1}^{n} N_{x_i}\right)} \\ &\subseteq \overline{(I \cap J)} \times \bigcup_{i=1}^{n} \bar{N}_{x_i} \\ &= \bigcup_{i=1}^{n} \left[\overline{(I \cap J)} \times \bar{N}_{x_i}\right] \\ &\subseteq \Omega.\end{aligned}$$

Let $J_\eta = (t_0 - \eta, t_0 + \eta)$ where $\eta > 0$.

We claim that for η small enough, $S_t \subseteq (J_\eta \cap J) \times V$ for $t \in \overline{J_\eta \cap J}$.

Suppose not. Then we can find a sequence (t_n, x_n) in S such that $t_n \longrightarrow t_0$ and $x_n \notin V$. Since S is compact, we can find a convergent subsequence, say $(t_{n_i}, x_{n_i}) \longrightarrow (t_0, x) \in S$. H continuous implies that $H(t_{n_i}, x_{n_i}) \longrightarrow H(t_0, x)$. So $x_{n_i} \longrightarrow H(t_0, x)$. But $x_{n_i} \longrightarrow x$, so $H(t_0, x) = x$. Therefore

$(t_0, x) \in S_{t_0} \subseteq (I \cap J) \times V$ by (5). But $x_{n_i} \notin V$ and V open implies that $x \notin V$, a contradiction.

Thus for η small enough, $S_t \subseteq (\overline{J_\eta \cap J}) \times V$ for $t \in \overline{J_\eta \cap J}$.

Choose η and V as above. Then $H : (\overline{J_\eta \cap J}) \times \bar{V} \longrightarrow X$ is continuous and for $t \in \overline{J_\eta \cap J}$, $S_t \subseteq \overline{J_\eta \cap J} \times V$. $x \in \partial V$ implies that $(t, x) \notin S_t$, and so $H(t, x) \neq x$. Therefore $0 \notin (I - H(t, .))(\partial V)$.

Let $S'_t = \{ x \in \Omega_t \, / \, H(t, x) = x \}$. If $x \in S'_t$, then $x \in \Omega_t$ and $H(t, x) = x$. So $(t, x) \in \Omega$ and $H(t, x) = x$.Therefore $(t, x) \in S_t \subseteq \overline{J_\eta \cap J} \times V$. So $x \in V$, and $S'_t \subseteq V$ for all $t \in \overline{J_\eta \cap J}$.

Let $t \in \overline{J_\eta \cap J}$ and $x \in \bar{V}$. Then

$(t, x) \in (\overline{J_\eta \cap J}) \times \bar{V} \subseteq (\overline{I \cap J}) \times \bar{V} \subseteq \Omega$. Therefore $x \in \Omega_t$ (η can be chosen so that $J_\eta \subseteq I$). So $\bar{V} \subseteq \Omega_t$, and hence for all

$t \in \overline{J_\eta \cap J}$,

$$D(I - H(t, .), \Omega_t, 0) = D_{C_\gamma}(I - H(t, .), V, 0)$$

and $(I - H(t, .), V, 0)$ is admissible (for D_{C_γ}). Thus

$D_{C_\gamma}(I - H(t, .), V, 0)$ is constant on $\overline{J_\eta \cap J}$ and thus

$D(I - H(t, .), \Omega_t, 0)$ is independent of t on $\overline{J_\eta \cap J}$. ♠

4.2.2 **Therorem**

Let $F : \bar{\Omega} \longrightarrow X$ be γ–condensing and $0 \notin (I - F)(\partial\Omega)$. Then

$$D_N(I - F, \Omega, 0) = D_{C_\gamma}(I - F, \Omega, 0),$$

where D_N is the Nussbaum degree.

Proof :

Let $S = \{ x \in \Omega \;/\; Fx = x \}$ and let V be an open neighbourhood of S such that $\bar{V} \subseteq \Omega$. Now S is compact since $I - F$ is proper, so $D_N(I - F, \Omega, 0)$ is defined. Then

$$D_N(I - F, \Omega, 0) = D_{C_\gamma}(I - F, V, 0).$$

Since $0 \notin (I - F)(\bar{\Omega} \setminus V)$, we have

$$D_{C_\gamma}(I - F, \Omega, 0) = D_{C_\gamma}(I - F, V, 0).$$

Thus we have

$$D_N(I - F, \Omega, 0) = D_{C_\gamma}(I - F, \Omega, 0).$$ ♠

4.2.3 **Theorem**

Let $\Omega \subseteq X$ be open bounded and symmetric with respect to $0 \in \Omega$, $F : \Omega \longrightarrow X$ be γ–condensing, $S = \{ x \in \Omega \;/\; Fx = x \}$ compact and $F(x) = -F(-x)$ for all $x \in \Omega$. Then $D(I - F, \Omega, 0)$ is odd.

Proof :

Let V be an open neighbourhood of S such that $\bar{V} \subseteq \Omega$. Then

$$D(I - F, \Omega, 0) = D_{C_\gamma}(I - F, V, 0).$$

$F|_{\partial V}$ is odd since $\partial V \subseteq \Omega$. Therefore by theorem 4.1.13, we have $D_{C_\gamma}(I - F, V, 0)$ is odd, and so $D(I - F, \Omega, 0)$ is odd. ♠

CHAPTER 5

DEGREE OF MAPS ON UNBOUNDED SETS

Up to this point, $\Omega \subseteq X$ was open and bounded. We will now consider $\Omega \subseteq X$ to be just open. Of course, we will require extra conditions on our function F . First we consider locally compact operators and then locally γ–condensing operators.

5.1 LOCALLY COMPACT OPERATORS

We will consider the triplet $(I - F, \Omega, y)$ where X is a Banach space, $\Omega \subseteq X$ is open, $F : \bar{\Omega} \longrightarrow X$ is locally compact, $y \notin (I - F)(\partial\Omega)$ and $(I - F)^{-1}(y)$ is compact. We show that there is a unique $\mathbb{Z}$–valued map defined on these triplets, the degree.

N.B.: F is locally compact if for each $x \in \Omega$, there exists a neighbourhood $U(x)$ of x such that $F|_{U(x)}$ is compact.

Firstly we will show that there exists a bounded neighbourhood $V \subseteq \Omega$ of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is compact. F is locally compact, so for each $x \in (I - F)^{-1}(y)$, there is a neighbourhood $U(x)$ of x such that $F|_{U(x)}$ is compact. Choose $r_x > 0$ small enough so that $\bar{B}_{r_x}(x) \subseteq U(x)$. Thus $F|_{\bar{B}_{r_x}}(x)$ is compact. Since $(I - F)^{-1}(y)$ is compact and

$(I - F)^{-1}(y) \subseteq \bigcup_{x \in (I-F)^{-1}(y)} B_{r_x}(x)$ we must have $(I - F)^{-1}(y) \subseteq \bigcup_{i=1}^{n} B_{r_i}(x_i) = V$ where $r_i = r_{x_i}$. Let $r = \max\{r_1, \ldots, r_n\}$. Then V is a bounded set with bound $r + \max\{|x_i| \,/\, i = 1, \ldots, n\}$.

$$\text{Now } F(\bar{V}) \quad = \quad F(\overline{\bigcup_{i=1}^{n} B_{r_i}(x_i)})$$

$$\subseteq \quad F(\bar{B}_{r_1}(x_1) \cup \dots \cup \bar{B}_{r_n}(x_n))$$
$$= \quad F(\bar{B}_{r_1}(x_1)) \cup \dots \cup F(\bar{B}_{r_n}(x_n))$$
$$\subseteq \quad \overline{F(\bar{B}_{r_1}(x_1)) \cup \dots \cup F(\bar{B}_{r_n}(x_n))}$$

But $\overline{F(\bar{B}_{r_i}(x_i))}$ is compact for each i, so $\bigcup_{i=1}^{n} \overline{F(\bar{B}_{r_i}(x_i))}$ is also compact. Thus $F(\bar{V})$ is relatively compact and so $F|_{\bar{V}}$ is compact. So we define, for our triplet $(I - F, \Omega, y)$,

$$D(I - F, \Omega, y) = D_{LS}(I - F, V, y), \tag{1}$$

where V is any bounded neighbourhood of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is compact.

Since $(I - F)^{-1}(y) \subseteq V$, we have $y \notin (I - F)(\partial V)$.

We must show that this well–defines D. Suppose we have

$V_1, V_2 \subseteq \Omega$ are bounded neighbourhoods of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}_i}$ is compact for $i = 1, 2$. Let $V = V_1 \cap V_2$ and suppose that $y \in (I - F)(\bar{V}_i \setminus V)$.

Then $y = (I - F)(x)$ for some $x \in \bar{V}_i \setminus V$. Therefore $x \in (I - F)^{-1}(y) \subseteq V_j$, $j = 1, 2$. So $x \in V_1 \cap V_2 = V$, a contradiction. Thus $y \notin (I - F)(\bar{V}_i \setminus V)$ for $i = 1, 2$. So by $(D_{LS}7)$,

$$D_{LS}(I - F, V_1, y) = D_{LS}(I - F, V, y) = D_{LS}(I - F, V_2, y),$$

proving that the degree is well–defined.

With D defined in (1), we will show that it satisfies (D1)–(D3).

(D1) :

Let $y \in \Omega$. Now $F \equiv 0 : \bar{\Omega} \longrightarrow X$ is locally compact (in fact, it is compact), $(I - F)^{-1}(y) = \{y\}$ is compact and $y \notin \partial\Omega = (I - F)(\partial\Omega)$. Let $V = B_1(y) \cap \Omega$. Then V is a bounded neighbourhood of $(I - F)^{-1}(y)$ and $F|_{\bar{V}}$ is compact. Thus

$$D(I, \Omega, y) = D_{LS}(I, V, y) = 1$$

by $(D_{LS}1)$.

(D2) :

Let Ω_1 and Ω_2 be disjoint subsets of Ω with $y \in X \setminus (I-F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. To show that $D(I-F, \Omega, y) = D(I-F, \Omega_1, y) + D(I-F, \Omega_2, y)$. Let $V \subseteq \Omega$ be a bounded neighbourhood of $(I-F)^{-1}(y)$ such that $F|_{\bar{V}}$ is compact. Then

$$D(I-F, \Omega, y) = D_{LS}(I-F, V, y). \tag{2}$$

Now let $V_i = V \cap \Omega_i$ for $i = 1, 2$. Then $V_i \subseteq \Omega_i$ is a bounded neighbourhood of $(I-F)^{-1}(y) \cap \bar{\Omega}_i$ such that $F|_{\bar{V}_i}$ is compact. Then

$$D(I-F, \Omega_i, y) = D_{LS}(I-F, V_i, y), \tag{3}$$

for $i = 1, 2$. We will now show that $y \notin (I-F)(\bar{V} \setminus V_1 \cup V_2)$. Suppose it does. Then $y = (I-F)(x)$ for some $x \in \bar{V} \setminus V_1 \cup V_2$. Now

$$V_1 \cup V_2 = (V \cap \Omega_1) \cup (V \cap \Omega_2) = V \cap (\Omega_1 \cup \Omega_2). \tag{4}$$

But $x \in \bar{V} \subseteq \bar{\Omega}$ and $y \notin (I-F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$. Thus we must have $x \in \Omega_1 \cup \Omega_2$. Therefore by (4), $x \notin V$. But $x \in (I-F)^{-1}(y) \subseteq V$, a contradiction. Thus $y \notin (I-F)(\bar{V} \setminus V_1 \cup V_2)$. So by $(D_{LS}2)$,

$$D_{LS}(I-F, V, y) = D_{LS}(I-F, V_1, y) + D_{LS}(I-F, V_2, y). \tag{5}$$

(2), (3) and (5) give us

$$D(I-F, \Omega, y) = D(I-F, \Omega_1, y) + D(I-F, \Omega_2, y).$$

(D3) :

Let $\Omega \subseteq X$ be open, $y : J \longrightarrow X$ and $H : J \times \bar{\Omega} \longrightarrow X$ be continuous. Suppose that for each $x \in \Omega$, there exists a neighbourhood $U(x)$ of x such that $H|_{J \times U(x)}$ is compact (i.e. H is locally compact) and further suppose that for each $t \in J$, $y(t) \notin X \setminus (I - H(t, .))(\partial\Omega)$ and $\bigcup_{t \in J} (I - H(t, .))^{-1}(y(t))$ is compact.

We will show that $D(I - H(t, .), \Omega, y(t))$ is independent of t.

For each $x \in \Omega$, there exists a neighbourhood $U(x)$ of x such that $H|_{J \times U(x)}$ is compact. Choose $r_x > 0$ small enough so that $\bar{B}_{r_x}(x) \subseteq U(x)$. Thus

$H|_{J\times\bar{B}_{r_x}(x)}$ is compact.

Now $A = \bigcup_{t\in J} (I - H(t, .)^{-1}(y(t))$ is compact and contained in Ω. (Since $y(t) \notin (I - H(t, .))(\partial\Omega))$. Thus we must have $x_1, ..., x_n \in A$, $r_i = r_{x_i}$ such that $A \subseteq \bigcup_i B_{r_i}(x_i) = V$. Let $r = \max\{r_1, ..., r_n\}$ and $s = \max\{|x_1|, ..., |x_n|\}$. Then for $x \in V$, we have $x \in B_{r_i}(x_i)$ for some i, and so

$|x| \leq |x - x_i| + |x_i| < r_i + |x_i| \leq r + s.$

Thus V is a bounded neighbourhood of A. Then

$$D(I - H(t, .), \Omega, y(t)) = D_{LS}(I - H(t, .), V, y(t)),$$

if the following conditions (which are proved as well) hold.

(a) *For each $t \in J$, $H(t, .)$ is locally compact* : For each $x \in \Omega$, there exists a neighbourhood $U(x)$ of x such that $H|_{J\times U(x)}$ is compact. So $H(J \times U(x))$ is relatively compact. Since $H(t, U(x)) \subseteq H(J \times U(x))$, $H(t, U(x))$ is also relatively compact. Hence $H(t, .)$ is locally compact.

(b) *$y(t) \notin (I - H(t, .))(\partial\Omega)$* : This is given.

(c) *$A_t = (I - H(t, .))^{-1}(y(t))$ is compact* : $A_t \subseteq A$, A is compact and A_t is closed. Hence A_t is compact.

(d) *V is a bounded neighbourhood of A_t for each t* : V is a bounded neighbourhood of A, hence of each A_t.

(e) *$H(t, .)|_{\bar{V}}$ is compact for each $t \in J$* :

$$H(J \times \bar{V}) = H(J \times \overline{\bigcup_{i=1}^{n} B_{r_i}(x_i)})$$
$$\subseteq H(J \times \bigcup_{i=1}^{n} \bar{B}_{r_i}(x_i))$$
$$= \bigcup_{i=1}^{n} H(J \times \bar{B}_{r_i}(x_i))$$

$H(J \times \bar{B}_{r_i}(x_i))$ is relatively compact and a finite union of relatively compact sets is again relatively compact. Hence $H(J \times \bar{V})$ is relatively compact. For each $t \in J$, $H(t, \bar{V}) \subseteq H(J \times \bar{V})$. Hence $H(t, \bar{V})$ is relatively compact. Thus $H(t, .)\,|_{\bar{V}}$ is compact.

Now $D_{LS}(I - H(t, .), V, y(t))$ is independent of t by (D_{LS}3) and by (6), $D(I - H(t, .), \Omega, y(t))$ is independent of t.

Thus D defined in (1) satisfies (D1)–(D3).

Now, we show that there is only one $\mathbb{Z}$–valued map, defined on the given triplets, satisfying (D1)–(D3).

Let $\mathscr{M} = \{I - F, \Omega, y) \,/\, \Omega \subseteq X$ open, bounded, $F : \bar{\Omega} \longrightarrow X$ compact, and $y \notin (I - F)(\partial\Omega)\}$ and $\tilde{\mathscr{M}} = \{(I - F, \Omega, y) \,/\, \Omega \subseteq X$ open, $F : \bar{\Omega} \longrightarrow X$ locally compact, $y \notin (I - F)(\partial\Omega)$ and $(I - F)^{-1}(y)$ is compact$\}$.

Let $D : \tilde{\mathscr{M}} \longrightarrow \mathbb{Z}$ satisfy (D1)–(D3) (then D also satisfies (D4)–(D7)).

Now a compact operator is locally compact and if $F : \bar{\Omega} \longrightarrow X$ is compact, then $(I - F)^{-1}(y)$ is also compact. So $\mathscr{M} \subseteq \tilde{\mathscr{M}}$. Let $D' = D|_{\mathscr{M}}$. We will show that D' satisfies (D′1)–(D′3).

(D′1) and (D′2) are trivial since (D1) and (D2) hold.

(D′3) :

Let $H : J \times \bar{\Omega} \longrightarrow X$ and $y : J \longrightarrow X$ be continuous, H compact and $y(t) \notin (I - H(t, .)(\partial\Omega)$ for each $t \in J$. In order to use (D3), we must have $A = \bigcup_{t \in J} (I - H(t, .)^{-1}(y(t))$ to be compact.

Let (x_n) be a sequence in A. Then $x_n \in (I - H(t_n, .))^{-1}(y(t_n))$ for some $t_n \in J$. Therefore $x_n = y(t_n) + H(t_n, x_n)$. (t_n) is a sequence in the compact set J, hence there exists some subsequence that converges, say $t_{k_n} \longrightarrow t_0 \in J$.

$H(J \times \bar{\Omega})$ is relatively compact. Therefore some subsequence of $H(t_{k_n}, x_{k_n})$ converges. Without loss of generality, we may assume that $H(t_{k_n}, x_{k_n}) \longrightarrow y_0$.

So $x_{k_n} = y(t_{k_n}) + H(t_{k_n}, x_{k_n}) \longrightarrow y(t_0) + y_0 = x_0$. Since $x_n \in \Omega$, we have $x_0 \in \bar{\Omega}$.

So $H(t_{k_n}, x_{k_n}) \longrightarrow H(t_0, x_0)$. Therefore $y_0 = H(t_0, x_0)$ and so $x_0 = y(t_0) + H(t_0, x_0)$, which implies that $y(t_0) = (I - H(t_0, .))(x_0)$. Therefore $x_0 \in (I - H(t_0, .))^{-1}(y(t_0)) \subseteq A$. Therefore A must be compact and so $D'(I - H(t, .), \Omega, y(t)) = D(I - H(t, .), \Omega, y(t))$ and is independent of t by (D3).

Therefore $D' : \mathscr{M} \longrightarrow \mathbb{Z}$ satisfies (D′1)–(D′3).

By uniqueness of the Leray–Schauder degree,

$$D' = D_{LS} \tag{7}$$

Now $D : \tilde{\mathscr{M}} \longrightarrow \mathbb{Z}$ satisfies (D1)–(D3) (hence it satisfies (D4)–(D7)). Let $(I - F, \Omega, y) \in \tilde{\mathscr{M}}$ and let V be any bounded neighbourhood of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is compact. Suppose

$y \in (I - F)(\bar{\Omega} \setminus V)$. Then for some $x \in \bar{\Omega} \setminus V$, $y = (I - F)(x)$ which implies that $x \in (I - F)^{-1}(y) \subseteq V$, a contradiction. So $y \notin (I - F)(\bar{\Omega} \setminus V)$. Therefore by (D7),

$$D(I - F, \Omega, y) = D(I - F, V, y). \tag{8}$$

But $(I - F, V, y) \in \mathscr{M}$, so

$$D(I - F, V, y) = D'(I - F, V, y) = D_{LS}(I - F, V, y)$$

by (7). So there is a unique $\mathbb{Z}$–valued map $D : \tilde{\mathcal{M}} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3). Thus we have proved the following theorem :

5.1.1 Theorem

Let X be a Banach space and

$\mathcal{M} = \{I - F, \Omega, y) \,/\, \Omega \subseteq X$ open, $F : \bar{\Omega} \longrightarrow X$ locally compact, $y \notin (I - F)(\partial\Omega)$ and $(I - F)^{-1}(y)$ is compact$\}$.

(a) Then there exists a unique map $D : \mathcal{M} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3) the *degree for locally compact operators.*

(b) Let $(I - F, \Omega, y) \in \mathcal{M}$. Then $D(I - F, \Omega, y) = D_{LS}(I - F, V, y)$ where V is any bounded neighbourhood of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is compact and D_{LS} is the Leray–Schauder degree.

It is easy to see that this degree is really an extension of the LS–degree. We also have the Borsuk property and the properties (D4)–(D7) holding.

5.2 LOCALLY γ–CONDENSING OPERATORS

We want to define a degree for the triplet $(I - F, \Omega, y)$ where $\Omega \subseteq X$ is open, $F : \bar{\Omega} \longrightarrow X$ is locally γ–condensing (i.e. for each $x \in \Omega$, there exists a neighbourhood U(x) of x such that $F|_{U(x)}$ is γ–condensing), $y \in X \setminus (I - F)(\partial\Omega)$ and $(I - F)^{-1}(y)$ is compact.

First we show the existence of $V \subseteq \Omega$, a bounded neighbourhood of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is γ–condensing. The procedure used to obtain a V is exactly like that used for locally compact maps.

Thus we obtain $(I - F)^{-1}(y) \subseteq \bigcup_{i=1}^{n} B_{r_i}(x_i) = V$ where $F|_{\bar{B}_{r_i}(x_i)}$ is γ–condensing for $i = 1, 2, \ldots, n$. Then V is a bounded neighbourhood of $(I - F)^{-1}(y)$. Let $B \subseteq \bar{V}$ such that

$\gamma(B) > 0$. Then $B = \bigcup_{i=1}^{n} (B \cap B_{r_i}(x_i))$ and so $FB = \bigcup_{i=1}^{n} F(B \cap B_{r_i}(x_i))$.

$$\begin{aligned}\text{So } \gamma(FB) &= \gamma\left(\bigcup_{i=1}^{n} F(B \cap B_{r_i}(x_i)\right) \\ &= \max\{\gamma(F(B \cap B_{r_i}(x_i))) \,/\, i = 1, 2, \ldots, n\} \\ &= \gamma(F(B \cap B_{r_k}(x_k))), \text{ say}\end{aligned}$$

If $\gamma(B \cap B_{r_k}(x_k)) = 0$, then $B \cap B_{r_k}(x_k)$ is relatively compact and since F is continuous with closed domain $\bar{\Omega}$, $F(B \cap B_{r_k}(x_k))$ is also relatively compact. Therefore $\gamma(F(B \cap B_{r_k}(x_k))) = 0$ and so $\gamma(FB) = 0 < \gamma(B)$.

If $\gamma(B \cap B_{r_k}(x_k)) > 0$, then $\gamma(F(B \cap B_{r_k}(x_k))) < \gamma(B \cap B_{r_k}(x_k)) \leq \gamma(B)$, since $F|_{\bar{B}_{r_k}(x_k)}$ is γ–condensing. Hence $\gamma(B) > 0$ implies $\gamma(FB) < \gamma(B)$ and so $F|_{\bar{V}}$ is γ–condensing.

We would like to define

$$D(I - F, \Omega, y) = D_{C_\gamma}(I - F, V, y), \tag{1}$$

where D_{C_γ} is the degree for γ–condensing maps and V is any bounded neighbourhood of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is γ– condensing.

As in the case of locally compact maps, $y \notin (I - F)(\delta\Omega)$ and $D_{C_\gamma}(I - F, V_1, y) = D_{C_\gamma}(I - F, V_2, y)$ for $V_1, V_2 \subseteq \Omega$ any bounded neighbourhoods of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}_i}$ is γ–condensing for $i = 1, 2$. Thus the degree defined above is well–defined.

Now to show that (D1)–(D3) hold. The proof of (D1) and (D2) is exactly the same as that for the locally compact operators, with *compact* replaced by *γ–condensing.* We will now prove (D3).

(D3) :

We have the following hypotheses for (D3):

Let $H : J \times \bar{\Omega} \longrightarrow X$ and $y : J \longrightarrow X$ be continuous. Suppose for each $x \in \Omega$ there exists a neighbourhood $U(x)$ of x such that $\gamma(H(J \times B)) < \gamma(B)$ for $B \subseteq U(x)$ with $\gamma(B) > 0$. Further, suppose that $y(t) \in X \setminus (I - H(t, .))(\partial\Omega)$ and $A = \bigcup_{t\in J} (I - H(t, .))^{-1}(y(t))$ is compact.

We must show that $D(I - H(t, .), \Omega, y(t))$ is independent of t. As in the proof for locally compact maps, we obtain $A \subseteq \bigcup_{i=1}^{n} B_{r_i}(x_i) = V$ where $H|_{J\times\bar{B}_{r_i}(x_i)}$ is γ–condensing.

(a) *For each $t \in J$, $H(t, .)$ is locally γ–condensing :*

For each $x \in \Omega$, there exists a neighbourhood $U(x)$ of x such that $\bar{H}|_{J\times\bar{U}(x)}$ is γ–condensing. Let $B \subseteq U(x)$ with $\gamma(B) > 0$. Then $\gamma(H(t, B) \leq \gamma(H(J \times B)) < \gamma(B)$.

So $H(t, .)$ is locally γ–condensing.

(b) $y(t) \notin (I - H(t, .))^{-1}(\partial\Omega)$:

This is part of the hypothesis.

(c) $A_t = (I - H(t, .))^{-1}(y(t))$ *is compact :*

$A_t \subseteq A$ with A_t closed and A compact. Thus A_t is compact.

(d) *V is a bounded neighbourhood of A_t for all $t \in J$:*

V is a bounded neighbourhood of A, hence of A_t.

(e) $H(t, .)|_{\bar{V}}$ *is γ–condensing :*

Let $B \subseteq \bar{V}$ with $\gamma(B) > 0$. Now $B = \bigcup_{i=1}^{n} (B \cap \bar{B}_{r_i}(x_i))$. So

$$\begin{aligned}
\gamma(H(t, B)) &\leq \gamma(H(J \times B)) \\
&= \gamma(\bigcup_{i=1}^{n} H(J \times (B \cap \bar{B}_{r_i}(x_i)))) \\
&= \max \{\gamma(H(J \times (B \cap \bar{B}_{r_i}(x_i)))) \ / \ i = 1, 2, \ldots, n\} \\
&= \gamma(H(J \times (B \cap \bar{B}_{r_k}(x_k)))), \text{ say.} \qquad (2)
\end{aligned}$$

If $\gamma(B \cap \bar{B}_{r_k}(x_k)) = 0$, then $B \cap \bar{B}_{r_k}(x_k)$ is relatively compact. Since H is continuous with closed domain, $H(J \times (B \cap \bar{B}_{r_k}(x_k)))$ is also relatively compact.

Therefore $\gamma(H(J \times (B \cap \bar{B}_{r_k}(x_k)))) = 0 < \gamma(B)$. (3)

So by (2) and (3), $\gamma(H(t, B)) < \gamma(B)$.

If $\gamma(B \cap \bar{B}_{r_k}(x_k)) > 0$, then

$$\gamma(H(J \times (B \cap \bar{B}_{r_k}(x_k)))) < \gamma(B \cap \bar{B}_{r_k}(x_k)) \leq \gamma(B). \tag{4}$$

So by (2) and (4), $\gamma(H(t, B)) < \gamma(B)$.

Thus $H(t, .)|_{\bar{V}}$ is γ–condensing.

Thus we have V to be admissible for each t, and so

$$D(I - H(t, .), \Omega, y(t)) = D_{C_\gamma}(I - H(t, .), V, y(t))$$

and this is independent of t by $(D_{C_\gamma}3)$.

Let

$\mathscr{M} = \{I - F, \Omega, y) \,/\, \Omega \subseteq X$ open bounded $F : \bar{\Omega} \longrightarrow X$ γ–condensing, $y \in X \setminus (I - F) \,/\, \partial\Omega)\}$

and

$\tilde{\mathscr{M}} = \{(I - F, \Omega, y) \,/\, \Omega \subseteq X$ open, $F : \bar{\Omega} \longrightarrow X$ locally γ–condensing, $y \in X \setminus (I - F)(\partial\Omega)$ and $(I - F)^{-1}(y)$ is compact$\}$.

We need to show that there is a unique map $D : \tilde{\mathscr{M}} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3).

Let $D : \tilde{\mathscr{M}} \longrightarrow \mathbb{Z}$ satisfy (D1)–(D3). Then it also satisfies (D4)–(D7).

Any γ–condensing map is locally γ–condensing and if $F \in C_\gamma(\bar{\Omega})$, then $(I - F)^{-1}(y)$ is compact (since $I - F$ is proper).

So $\mathscr{M} \subseteq \tilde{\mathscr{M}}$.

Let $(I - F, \Omega, y) \in \mathscr{M}$. As before, there exists an open bounded neighbourhood V in Ω of $(I - F)^{-1}(y)$ such that $F|_{\bar{V}}$ is γ–condensing. Then $y \notin (I - F)(\bar{\Omega} \setminus V)$ and hence by (D7),

$$D(I - F, \Omega, y) = D(I - F, V, y). \tag{5}$$

Now $(I - F, V, y) \in \mathscr{M}$. We will show that $D' = D|_{\mathscr{M}}$ satisfies (D'1)–(D'3). Since (D1) and (D2) hold, we also have (D'1) and (D'2) holding.

(D'3) :

Let $H : J \times \bar{\Omega} \longrightarrow X$ and $y : J \longrightarrow X$ be continuous, H γ–condensing and $y(t) \notin (I - H(t, .))^{-1}(\partial\Omega)$ for each $t \in J$. In order to use (D3), $A = \bigcup_{t \in J} (I - H(t, .))^{-1}(y(t))$ must be compact. Let (x_n) be a sequence in A. Then $x_n \in (I - H(t_n, .))^{-1}(y(t_n))$ for some $t_n \in J$. Therefore $x_n = y(t_n) + H(t_n, x_n)$. J is compact, so some subsequence of (t_n) converges, say $t_{k_n} \longrightarrow t_0 \in J$ and by continuity of y, $y(t_{k_n}) \longrightarrow y(t_0)$. So $\{y(t_{k_n}) / n \in \mathbb{N}\}$ is relatively compact (since every sequence in it is convergent). Now

$\{x_{k_n} / n \in \mathbb{N}\} \subseteq \{y(t_{k_n}) / n \in \mathbb{N}\} + \{H(t_{k_n}, x_{k_n}) / n \in \mathbb{N}\}$.

$$
\begin{aligned}
\text{So} \quad \gamma(\{x_{k_n} / n \in \mathbb{N}\}) &\le \gamma(\{y(t_{k_n}) / n \in \mathbb{N}\}) + \gamma(\{H(t_{k_n}, x_{k_n}) / n \in \mathbb{N}\}) \\
&= 0 + \gamma(\{H(t_{k_n}, x_{k_n}) / n \in \mathbb{N}\}) \\
&\le \gamma(H(J \times \{x_{k_n} / n \in \mathbb{N}\})).
\end{aligned}
$$

If $\gamma(\{x_{k_n} / n \in \mathbb{N}\}) > 0$ then

$\gamma(\{x_{k_n} / n \in \mathbb{N}\}) \le \gamma(H(J \times \{x_{k_n} / n \in \mathbb{N}\})) < \gamma(\{x_{k_n} / n \in \mathbb{N}\})$, a contradiction.

Thus $\gamma(\{x_{k_n} / n \in \mathbb{N}\}) = 0$. Therefore $\{x_{k_n} / n \in \mathbb{N}\}$ is relatively compact. So some subsequence of it converges. Without loss of generality, assume $x_{k_n} \longrightarrow x_0$.

Since $x_{k_n} \in \Omega$, we must have $x_0 \in \bar{\Omega}$, and by continuity of H, $H(t_{k_n}, x_{k_n}) \longrightarrow H(t_0, x_0)$

But $H(t_{k_n}, x_{k_n}) = x_{k_n} - y(t_{k_n}) \longrightarrow x_0 - y(t_0) = y_0$. Therefore $H(t_0, x_0) = y_0$, and so $x_0 - y(t_0) = H(t_0, x_0)$. Thus $y(t_0) = (I - H(t_0, .))(x_0)$ and hence $x_0 \in (I - H(t_0, .))^{-1}(y(t_0)) \subseteq A$.

Hence A is compact.

Thus $D'(I - H(t, .), \Omega, y(t)) = D(I - H(t, .), \Omega, y(t))$ and this is independent of t by (D3).

So $D' : M \longrightarrow \mathbb{Z}$ and it satisfies (D′1)–(D′3). By uniqueness of the degree for γ–condensing maps,

$$D' = D_{C_\gamma}. \tag{6}$$

Now $(I - F, V, y) \in \mathscr{M}$, so

$$D(I - F, V, y) = D'(I - F, V, y) = D_{C_\gamma}(I - F, V, y) \tag{7}$$

by (6).

Thus (1) and (7) give

$$D(I - F, \Omega, y) = D_{C_\gamma}(I - F, V, y),$$

and so there is a unique map, $D : \tilde{\mathscr{M}} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3).

It is again an easy exercise to check that this degree is an extension of the γ–condensing degree.

As we had in the previous chapters this unique map will satisfy (D4)– (D7) and Borsuk's property.

CHAPTER 6

DEGREE IN LOCALLY CONVEX SPACES

Before we define a degree on such spaces we give some definitions and facts about them. Proofs of the results can be found in Schäfer [29].

6.1 Definition :

(X, τ) is a *topological vector space* (t.v.s.) if X is a vector space over the field **K** with topology τ such that addition $A : (x, y) \longrightarrow x + y$ and scalar multiplication $S : (\lambda, x) \longrightarrow \lambda x$ are continuous.

The field **K** is either $\mathbb{R}$ or $\mathbb{C}$.

τ is *separated* if different points have disjoint neighbourhoods.

The following theorem gives conditions that a t.v.s. satisfies.

6.2 Theorem

Let (X, τ) be a t.v.s. with τ separated. Then there is a basic system $\mathcal{U}(0)$ of neighbourhoods of 0, with the following properties.

(a) $U \in \mathcal{U}(0)$ and $\lambda \neq 0$ imply that $\lambda U \in \mathcal{U}(0)$.

(b) For $U \in \mathcal{U}(0)$, there exists $V \in \mathcal{U}(0)$ such that $V + V \subseteq U$.

(c) $\bigcap_{U \in \mathcal{U}(0)} U = \{0\}$.

(d) Every $U \in \mathcal{U}(0)$ is open, absorbant and balanced, where U is called *absorbant* if to each $x \in X$, there exists $\lambda > 0$ such that $x \in \lambda U$, and *balanced* if $\lambda U \subseteq U$ for all λ with $|\lambda| \leq 1$.

A basic neighbourhood system of any $x_0 \in X$ is given by $\mathscr{U}(x_0) = x_0 + \mathscr{U}(0)$.

A t.v.s. (X, τ) is said to be locally convex if there exists a neighbourhood system $\mathscr{U}($

satisfying in addition

(e) Every $U \in \mathscr{U}(0)$ is convex.

An $\Omega \subseteq X$ is said to be *bounded* if to every $U \in \mathscr{U}(0)$, there exists $\lambda_U > 0$ such that $\Omega \subseteq \lambda_U U$.

6.3 Theorem

Let X be a locally convex t.v.s. and $\mathscr{U}(0)$ a basic system of neighbourhoods of with the properties (a) – (e). Let $p_U : X \longrightarrow \mathbb{R}$ be defined by $p_U(x) = \inf \{\lambda > 0 \ / \ x \in \lambda U\}$. Then p_U is a continuous seminorm, $U = \{x \in X \ / \ p_U(x) < 1\}$ and $\partial U = \{x \in X \ / \ p_U(x) = 1\}$.
(p_U is called the *Minkowski functional.*)

The above theorem is standard and so we state it without proof.

We would like to define a degree for the following triplet:
$(I - F, \Omega, y)$ where X is a locally convex t.v.s., $\Omega \subseteq X$ is open, $F : \bar{\Omega} \longrightarrow X$ is compact and $y \in X \setminus (I - F)(\partial\Omega)$.
Before we do this, we first have to give some approximation for F, the degree of which know.

6.4 Theorem

Let X be a topological space, Y a locally convex t.v.s., $\Omega \subseteq X$ and $F : \Omega \longrightarrow$ compact. Let $\mathscr{U}(0)$ be a neighbourhood base of $0 \in Y$ satisfying (a) – (e) theorem 6.2. Then we have

(a) For $U \in \mathscr{U}(0)$, there exists a finite dimensional F_U such that $F_U x - Fx \in$

on U. F_U also turns out to be compact.

(b) $I - F$ maps closed subsets of Ω onto closed sets.

Proof :

(a) Since $\overline{F(\Omega)}$ is compact, we can find $y_1, \ldots, y_m \in Y$ such that $\overline{F(\Omega)} \subseteq \bigcup_{i=1}^{m} (y_i + U)$. Let $\varphi_i(x) = \max\{0, 1 - p_U(Fx - y_i)\}$ on Ω. Then φ_i i continuous and non–negative. For $x \in \Omega$, $F(x) \in y_i + U$ for some i.
So $Fx - y_i \in U$. Thus $p_U(Fx - y_i) < 1$ and so $1 - p_U(Fx - y_i) > ($
Therefore $\sum_{i=1}^{m} \varphi_i(x) > 0$ for all $x \in \Omega$. So we may define
$\lambda_i(x) = (\sum_{j=1}^{m} \varphi_j(x))^{-1} \varphi_i(x)$ and $F_U x = \sum_{i=1}^{m} \lambda_i(x)\, y_i$ on Ω. Then F_U is continuou and finite dimensional. Easily $\sum_{i=1}^{m} \lambda_i(x) = 1$ on Ω. ($\lambda_i(x) \in [0, 1]$). So

$$
\begin{aligned}
p_U(Fx - F_U x) &= p_U(Fx - \sum_{i=1}^{m} \lambda_i(x)\, y_i) \\
&= p_U(\sum_{i=1}^{m} \lambda_i(x)\, Fx - \sum_{i=1}^{m} \lambda_i(x)\, y_i) \\
&= p_U(\sum_{i=1}^{m} \lambda_i(x)\, (Fx - y_i)\,) \\
&\leq \sum_{i=1}^{m} p_U(\lambda_i(x)(Fx - y_i)) \quad (p_U \text{ is a seminorm}) \\
&= \sum_{i=1}^{m} \lambda_i(x)\, p_U(Fx - y_i).
\end{aligned}
$$

Now $\varphi_i(x) > 0$ for some i, and so $p_U(Fx - y_i) < 1$ for some i. If $p_U(Fx - y_i) \geq 1$, then $\varphi_i(x) = 0$ and hence $\lambda_i(x) = 0$.

So $\sum_{i=1}^{m} \lambda_i(x)\, p_U(Fx - y_i) < \sum_{i=1}^{m} \lambda_i(x) = 1$. Therefore $p_U(Fx - F_U(x)) < 1$ ar this implies that $Fx - F_U x \in U$.

To show $F_U(\Omega)$ is relatively compact we just need to show that eve sequence in it has a convergent subsequence since it is contained in a fini dimensional subspace of Y.

Let $(F_U(x_n))$ be a sequence in $F_U(\Omega)$. $(\lambda_1(x_n))$ is a sequence in J, hence it h a convergent subsequence, say $\lambda_1(x_{k_n}) \longrightarrow \alpha_1$. Similarly $\lambda_2(x_{n_k})$ has convergent subsequence, etc. So we obtain a subsequence, (x_{n_k}) of (x_n) suc

that $\lambda_i(x_{n_k}) \longrightarrow \alpha_i$ for i = 1, 2, ..., m. Therefore

$F_U(x_{n_k}) = \sum_{i=1}^{m} \lambda_i(x_{n_k})\, y_i \longrightarrow \sum_{i=1}^{m} \alpha_i y_i$. So $F_U(\Omega)$ is relatively compact. Thus

$F_U : \Omega \longrightarrow Y$ is compact.

(b) Let $\Omega_0 \subseteq \Omega$ be closed. To show $(I - F)(\Omega_0)$ is closed. Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net i

Ω_0 such that $(I - F)(x_\lambda) \longrightarrow y$. Now $Fx_\lambda \in \overline{F(\Omega)}$, which is compact. Therefor

$(Fx_\lambda)_{\lambda\in\Lambda}$ has a cluster point $y_0 \in \overline{F(\Omega)}$. Thus there exists a subnet $(x_\omega)_{\omega\in}$

of $(x_\lambda)_{\lambda\in\Lambda}$ such that $Fx_\omega \longrightarrow y_0$. So $x_\omega = (x_\omega - Fx_\omega) + Fx_\omega \longrightarrow y + y$

and hence $y + y_0$ is a cluster point of $(x_\lambda)_{\lambda\in\Lambda}$. But $(x_\lambda)_{\lambda\in\Lambda} \subseteq \Omega_0$ and Ω_0 i

closed, hence $x_0 = y + y_0 \in \Omega_0$. Therefore

$(I - F)(x_0) = \lim_{\omega} (I - F)(x_\omega) = y$.

So $y = (I - F)(x_0) \subseteq (I - F)(\Omega_0)$. Thus $(I - F)(\Omega_0)$ i

closed.

The following procedure gives a way of defining the degree:

Let $(I - F, \Omega, y)$ be the triplet we are considering. By Theorem 6.4(b), I − F is close

Hence $(I - F)(\partial\Omega)$ is closed. So there exists

$$U \in \mathscr{U}(0) \text{ such that } (y + U) \cap (I - F)(\partial\Omega) = \emptyset. \quad (1$$

By Theorem 6.4 (a), there exists a finite dimensional F_1 such that $F_1x - Fx \in U$ on $\bar{\Omega}$. Le

X_1 be a subspace of X such that dim $X_1 < \infty$, $F_1(\bar{\Omega}) \subseteq X_1$, $y \in X_1$ and let $\Omega_1 = \Omega \cap X_1$. No

(a) $[I - (I - F)]\,(\bar{\Omega}_1)$ is bounded:

$[I - (I - F_1)]\,(\bar{\Omega}_1) = F_1(\bar{\Omega}_1)$.

and

(b) $y \in X_1 \setminus (I - F_1)(\partial\Omega_1)$:

Suppose $y = (I - F_1)(x)$ for $x \in \bar{\Omega}_1$. Then $F_1x = x - y$. But $F_1x - Fx \in U$. So

$x - y - Fx \in U$. Therefore $(I - F)(x) \in y + U$. Since $(I - F)(\partial\Omega) \cap (y + U) = \emptyset$

we must have $x \notin \partial\Omega$.

$$\begin{aligned} \text{But } \partial\Omega_1 &= \overline{\Omega \cap X_1} \setminus \Omega \cap X_1 \\ &\subseteq \bar{\Omega} \cap X_1 \setminus \Omega \cap X_1 \\ &= \partial\Omega \cap X_1. \end{aligned} \tag{2}$$

Thus $x \notin \partial\Omega_1$, and hence $y \notin (I - F_1)(\partial\Omega_1)$.

So (a) and (b) imply that $d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$ is defined by definition 2.17 (where d is the Brouwer degree extended to unbounded sets). Thus, it seems natural to define the degree by

$$D(I - F, \Omega, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y). \tag{3}$$

We must show that this degree is well–defined.

6.5 Theorem

Let $(I - F, \Omega, y)$ be one of the triplets we are considering. Suppose there exist finite dimensional F_i such that $F_i x - Fx \in U$ (where U is obtained by (1)) on $\bar{\Omega}$ and a subspace X_i of X such that $\dim X_i < \infty$, $F_i(\bar{\Omega}) \subseteq X_i$, $y \in X_i$ and $\Omega_i = \Omega \cap X_i$, for $i = 1, 2$. Then

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d((I - F_2)|_{\bar{\Omega}_2}, \Omega_2, y).$$

Proof :

N.B.: Ω_1 and Ω_2 come from spaces of different dimension. Hence we must use theorem 2.19.

Let $X_3 = \text{span}(X_1 \cup X_2)$ and $\Omega_3 = \Omega \cap X_3$.

Let Ω_0 be any bounded open set that contains $(I - F_i)^{-1}(y)$, $i = 1, 2$. (This can be done since $(I - F_i)^{-1}(y)$ is compact). Then by definition 2.17.

$$d((I - F_i)|_{\bar{\Omega}_i}, \Omega_i, y) = d((I - F_i)|_{\overline{\Omega_i \cap \Omega_0}}, \Omega_i \cap \Omega_0, y) \tag{4}$$

and

$$d((I-F_i)|_{\bar\Omega_3}, \Omega_3, y) = d((I-F_i)|_{\overline{\Omega_3 \cap \Omega_0}}, \Omega_3 \cap \Omega_0, y). \qquad (5$$

We will first show that

$$d((I-F_i)|_{\overline{\Omega_i \cap \Omega_0}}, \Omega_i \cap \Omega_0, y) = d((I-F_i)|_{\overline{\Omega_3 \cap \Omega_0}}, \Omega_3 \cap \Omega_0, y).$$

In order to apply Theorem 2.19, we must have $F_i|_{\overline{\Omega_3 \cap \Omega_0}} : \overline{\Omega_3 \cap \Omega_0} \longrightarrow X_i$ i

continuous, $\Omega_3 \cap \Omega_0$ open bounded and $y \in X_i \setminus ((I-F_i)|_{\overline{\Omega_3 \cap \Omega_0}})(\partial(\Omega_3 \cap \Omega_0))$

Easily, $F_i|_{\overline{\Omega_3 \cap \Omega_0}}$ is continuous.

Suppose $y = (I-F_i)(x)$ for $x \in \partial(\Omega_3 \cap \Omega_0)$.

$$\begin{aligned} \text{Now } \partial(\Omega_3 \cap \Omega_0) &= \overline{\Omega_3 \cap \Omega_0} \setminus \Omega_3 \cap \Omega_0 \\ &\subseteq \bar\Omega_3 \cap \bar\Omega_0 \setminus \Omega_3 \cap \Omega_0 \\ &= (\partial\Omega_3 \cap \bar\Omega_0) \cup (\bar\Omega_3 \cap \partial\Omega_0). \end{aligned}$$

Since $x \in (I-F_i)^{-1}(y) \subseteq \Omega_0$ we must have $x \in \partial\Omega_3 \cap \bar\Omega_0$.

$$\begin{aligned} \text{But} \quad \partial\Omega_3 &= \overline{\Omega \cap X_3} \setminus \Omega \cap X_3 \\ &\subseteq \bar\Omega \cap X_3 \setminus \Omega \cap X_3 \\ &= \partial\Omega \cap X \end{aligned}$$

So $x \in \partial\Omega$.

Now $(I-F)x = (I-F_i)x + (F_ix - Fx) \in y + U$. So $(I-F)(\partial\Omega) \cap (y+U) \neq \emptyset$,

contradiction. Thus $y \in X_i \setminus ((I-F_i)|_{\overline{\Omega_3 \cap \Omega_0}})(\partial(\Omega_3 \cap \Omega_0))$.

By Theorem 2.19, we have

$$\begin{aligned} &d((I-F_i)|_{\overline{\Omega_3 \cap \Omega_0}}, \Omega_3 \cap \Omega_0, y) \\ &\quad = d((I-F_i)|_{\overline{\Omega_3 \cap \Omega_0 \cap X_i}}, \Omega_3 \cap \Omega_0 \cap \Omega_i, y) \\ &\quad = d((I-F_i)|_{\overline{\Omega_i \cap \Omega_0}}, \Omega_i \cap \Omega_0, y). \qquad (6 \end{aligned}$$

By (4), (5) and (6),

$$d((I - F_i)|_{\bar{\Omega}_i}, \Omega_i, y) = d((I - F_i)|_{\bar{\Omega}_3}, \Omega_3, y)). \quad (7$$

We will now show that

$$d((I - F_1)|_{\bar{\Omega}_3}, \Omega_3, y) = d((I - F_2)|_{\bar{\Omega}_3}, \Omega_3, y).$$

Define $h : J \times \bar{\Omega}_3 \longrightarrow X$ by $h(t, x) = t\,(I - F_1)x + (1 - t)(I - F_2)x$ for $(t, x) \in J \times \bar{\Omega}_3$.

Then (i) h is continuous

(ii) $\sup\{|x - h(t, x)| \;/\; (t, x) \in J \times \bar{\Omega}_3\}$

$= \sup\{|x - t(I - F_1)x - (1 - t)(I - F_2)x| \;/\; (t, x) \in J \times \bar{\Omega}_3\}$

$= \sup\{|t\,F_1x + (1 - t)\,F_2x| \;/\; (t, x) \in J \times \bar{\Omega}_3\}$

$\leq \sup\{|t\,F_1x| \;/\; (t, x) \in J \times \bar{\Omega}_3\}$

$\quad + \sup\{|(1 - t)\,F_2x| \;/\; (t, x) \in J \times \bar{\Omega}_3\}$

$\leq \sup\{|F_1x| \;/\; x \in \bar{\Omega}_3\} + \sup\{|F_2x| \;/\; (t, x) \in \bar{\Omega}_3\}$

$< \infty$.

(iii) If $y = h(t, x)$ for $(t, x) \in J \times \partial\Omega_3$, then

$$\begin{aligned} y &= t\,(I - F_1)\,x + (1 - t)(I - F_2)x \\ &= x - Fx - [t\,(F_1x - Fx) + (1 - t)(F_2x - Fx)]. \end{aligned}$$

$F_1x - Fx$, $F_2x - Fx \in U$ and U is convex, so $t(F_1x - Fx) + (1 - t)(F_2x - Fx) \in U$. Therefore $(I - F)x \in y +$ with $x \in \partial\Omega_3 = \partial\Omega \cap X_3 \subseteq \partial\Omega$. A contradiction to $(I - F)(\partial\Omega) \cap (y + U) = \emptyset$. Thus $y \notin h(t, \partial\Omega_3)$, $t \in J$.

Therefore the hypotheses for (d3) in definition 2.17 are satisfied, to give us

$$d((I - F_1)|_{\bar{\Omega}_3}, \Omega_3, y) = d((I - F_2)|_{\bar{\Omega}_3}, \Omega_3, y). \quad (8$$

(7) and (8) imply that

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d((I - F_2)|_{\bar{\Omega}_2}, \Omega_2, y).$$

6.6 Theorem

Let $(I - F, \Omega, y)$ be our triplet. Suppose there exist $U_i \in \mathcal{U}(0)$ such that $(y + U_i) \cap (I - F)(\partial\Omega) = \emptyset$, $i = 1, 2$, and there exist finite dimensional F_i and subspace X_i of X such that $\dim X_i < \infty$, $F_i(\bar{\Omega}) \subseteq X_i$, $y \in X_i$, $\Omega_i = \Omega \cap X_i$, $i = 1, 2$. Then

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d((I - F_2)|_{\bar{\Omega}_2}, \Omega_2, y).$$

Proof :

Let $V \in \mathcal{U}(0)$ such that $V \subseteq U_1 \cap U_2$. By Theorem 6.4(a), there exist finite dimensional F_3 and a subspace X_3 of X such that $\dim X_3 < \infty$, $F_3(\bar{\Omega}) \subseteq X_3$, $y \in X_3$, $\Omega_3 = \Omega \cap X_3$, and $F_3x - Fx \in V$ on $\bar{\Omega}$. Hence $F_3x - Fx \in U_i$ on $\bar{\Omega}$ for $i = 1, 2$. So by Theorem 6.5,

$$d((I - F_i)|_{\bar{\Omega}_i}, \Omega_i, y) = d((I - F_3)|_{\bar{\Omega}_3}, \Omega_3, y)$$

for $i = 1, 2$. Thus

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d((I - F_2)|_{\bar{\Omega}_2}, \Omega_2, y).$$

Thus, the degree defined by

$$D(I - F, \Omega, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$$

in (3) is well–defined.

The following lemma can be found in Nagumo[16].

6.7 Lemma

Let K_i $(i = 1, 2)$ be compact sets in X. Then $K_1 + K_2$ is compact in X.

Proof :

$K_1 \times K_2$ is a compact set in the product space $X \times X$. The map $\phi : X \times X \longrightarrow X$ defined by $\phi(x, y) = x + y$, $(x,y) \in X \times X$ is continuous and $\phi(K_1 \times K_2) = K_1 + K_2$. Thus $K_1 + K_2$ is also compact.

6.8 Lemma

For our admissible triplet $(I - F, \Omega, y)$,

$$D(I - F, \Omega, y) = D(I - (F + y), \Omega, 0).$$

Proof :

$(F + y)(\bar{\Omega}) = F(\bar{\Omega}) + y$. Since F is compact, by the above lemma $F + y$ is also compact. Also $0 = (I - (F + y))(x) = x - Fx - y$ implies that $y = (I - F)(x)$. Therefore $x \notin \partial\Omega$.

So $0 \notin (I - (F + y))(\partial\Omega)$. Hence $(I - (F + y), \Omega, 0)$ is an admissible triplet.

$y \notin (I - F)(\partial\Omega)$ and $(I - F)(\partial\Omega)$ is closed, so we can find $U \in \mathscr{U}(0)$ such that $(y + U) \cap (I - F)(\partial\Omega) = \emptyset$. By Theorem 6.4, there exists a finite dimensional I such that $F_1 x - Fx \in U$ on $\bar{\Omega}$. Let X_1 be a subspace of X such that dim $X_1 <$ $F_1(\bar{\Omega}) \subseteq X_1$, $y \in X_1$ and let $\Omega_1 = \Omega \cap X_1$. Then

$$\begin{aligned} D(I - F, \Omega, y) &= d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) \\ &= d((I - (F_1 + y))|_{\bar{\Omega}_1}, \Omega_1, 0). \end{aligned} \quad (9$$

Now $(F_1 + y)(\bar{\Omega}) = F_1(\bar{\Omega}) + y \subseteq X_1$, $0 \in X_1$ and $(F_1 + y)x - (F + y)x = F_1 x - Fx \in U$ on $\bar{\Omega}$. Thus by definition

$$D(I - (F + y), \Omega, 0) = d((I - (F_1 + y))|_{\bar{\Omega}_1}, \Omega_1, 0) \quad (1$$

Thus (9) and (10) give us

$$D(I - F, \Omega, y) = D(I - (F + y), \Omega, 0).$$

6.9 Theorem

For the triplet $(I - F, \Omega, y)$, the degree defined by

$$D(I - F, \Omega, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$$

(where the triplet $(I - F_1, \Omega_1, y)$ is defined as above) satisfies (D1)–(D3).

Proof :

(D1) :

To show $D(I, \Omega, y) = 1$ if $y \in \Omega$. Here $F \equiv 0$. Since $y \notin I(\partial\Omega) = \partial\Omega$, there exists $U \in \mathscr{U}(0)$ such that $(y + U) \cap \partial\Omega = \emptyset$. If $F_1 \equiv 0$, then F_1 is finite–dimensional an $F_1x - Fx = 0 \in U$ on $\bar{\Omega}$. Let X_1 be a subspace of X such that $\dim X_1 < \infty$, $y \in X$ and let $\Omega_1 = \Omega \cap X_1$. Now $F_1(\bar{\Omega}) = 0 \in X_1$. So by definition $D(I - F, \Omega, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y) = d(I|_{\bar{\Omega}_1}, \Omega_1, y) = 1$ since $y \in \Omega_1$, an by (d1).

(D2) :

Let Ω^1, Ω^2 be disjoint open subsets of Ω such that $y \in X \setminus (I - F)(\bar{\Omega} \setminus \Omega^1 \cup \Omega^2)$. W must show that $D(I - F, \Omega, y) = D(I - F, \Omega^1, y) + D(I - F, \Omega^2, y)$. Now $\bar{\Omega} \setminus \Omega^1 \cup \Omega^2$ is closed and hence $(I - F)(\Omega^1 \cup \Omega^2)$ is also closed. Therefore the exists $U \in \mathscr{U}(0)$ such that $(y + U) \cap (I - F)(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2) = \emptyset$. By theorem 6.4(there exists a finite dimensional F_1 such that $F_1x - Fx \in U$ on $\bar{\Omega}$ Let X_1 be a subspace of X such that $F_1(\bar{\Omega}) \subseteq X_1$, $y \in X_1$ and let $\Omega_1 = \Omega \cap X_1$. Then

$$D(I - F, \Omega, y) = d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y). \qquad (1$$

$F_1|_{\bar{\Omega}^i}$ is also an approximation for $F|_{\bar{\Omega}^i}$, $i = 1, 2$. If $\Omega^i_1 = \Omega^i \cap X_1$ for $i = 1,$ then

$$D(I - F, \Omega^i, y) = d((I - F_1)|_{\bar{\Omega}^i_1}, \Omega^i_1, y), \qquad (1$$

$$\text{for } i = 1, 2.$$

Ω^1_1 and Ω^2_1 are disjoint open subsets of Ω_1.

$$\begin{aligned}
\bar{\Omega}_1 \setminus \Omega^1_1 \cup \Omega^2_1 &= \overline{\Omega \cap X_1} \setminus (\Omega^1 \cap X_1) \cup (\Omega^2 \cup X_1) \\
&\subseteq \bar{\Omega} \cap X_1 \setminus (\Omega^1 \cup \Omega^2) \cap X_1 \\
&= (\bar{\Omega} \setminus \Omega^1 \cup \Omega^2) \cap X_1.
\end{aligned}$$

Suppose $y = (I - F_1)(x)$. Then $y = (I - Fx) - (F_1x - Fx)$. Now $F_1x - Fx \in U$. S $(I - F)(x) \in y + U$. Therefore $x \notin \bar{\Omega} \setminus \Omega^1 \cup \Omega^2$ and so $x \notin \bar{\Omega}_1 \setminus \Omega^1_1 \cup \Omega^2_1$. Thus $y \notin (I - F_1)(\bar{\Omega}_1 \setminus \Omega^1_1 \cup \Omega^2_1)$ and by (d2),

$$d((I - F_1)|_{\bar{\Omega}_1}, \Omega_1, y)$$
$$= d((I - F_1)|_{\bar{\Omega}^1_1}, \Omega^1_1, y) + d((I - F_1)|_{\bar{\Omega}^2_1}, \Omega^2_1, y). \quad (13$$

Thus, (11), (12), (13) imply that

$$D(I - F, \Omega, y) = D(I - F, \Omega^1, y) + D(I - F, \Omega^2, y).$$

(D3) :

Let $H : J \times \bar{\Omega} \longrightarrow X$ and $y : J \longrightarrow X$ be continuous, ($\Omega \subseteq X$ open, X is a local convex t.v.s.) H be compact and $y(t) \in X \setminus (I - H(t, .))(\partial\Omega)$ for all $t \in J$. Then w must show that $D(I - H(t, .), \Omega, y(t))$ is independent of t. First take $y(t) \equiv 0$. F $\tau \in J$. We will show that there is some interval about τ on which the degree constant. Consider $X^* = \mathbb{R} \times X$. If $\mathscr{U}$ is a system of neighbourhoods at the orig for X then a system of neighbourhoods at the origin for X^* is given by

$$\mathscr{U}^* = \{(-\delta, \delta) \times U \;/\; \delta > 0, U \in \mathscr{U}\}.$$

Let $\Omega^* = \mathbb{R} \times \Omega$ and define

$$H^*(t, x) = (0, H(\langle t\rangle, x)),$$
$$\text{for } (t, x) \in \Omega^*,$$

where
$$\langle t\rangle = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq 1 \\ 1 & \text{if } t > 1 \end{cases}.$$

Then $\bar{\Omega}^* = \mathbb{R} \times \bar{\Omega}$, $\partial\Omega^* = \bar{\Omega}^* \setminus \Omega^* = \mathbb{R} \times \bar{\Omega} \setminus \mathbb{R} \times \Omega = \mathbb{R} \times \partial\Omega$. Also for $(t, x) \in \bar{\Omega}$ we have $H^*(t, x) = (0, H(\langle t\rangle, x)) \subseteq \{0\} \times H(J \times \bar{\Omega})$, so $H^*(\bar{\Omega}^*) \subseteq \{0\} \times H(J \times \bar{\Omega})$ and this is relatively compact, and hence $H^*(\bar{\Omega}^*)$ relatively compact. Thus $H^* : \bar{\Omega}^* \longrightarrow \{0\} \times X$ is compact. Suppose $(\tau, 0) \in (I - H^*)(\partial\Omega^*)$. Then $(\tau, 0) = (I - H^*)(t, x)$ for some $(t, x) \in \mathbb{R} \times \partial$ Therefore $(\tau, 0) = (t, x) - H^*(t, x) = (t, x) - (0, H(\langle t\rangle, x))$. So $t = \tau$ and

$0 = (I - H(\tau, .))(x)$ where $x \in \partial\Omega$, a contradiction. Thus $(\tau, 0) \notin (I - H^*)(\partial\Omega^*)$. By theorem 6.4(a), $(I - H^*)(\partial\Omega^*)$ is closed. Therefore there exists $U^* \in \mathscr{U}^*$ such that $(U^* + (\tau, 0)) \cap (I - H^*)(\partial\Omega^*) = \emptyset$. Now $U^* \in \mathscr{U}^*$ implies that there exists $\delta > 0$ and $U \in \mathscr{U}$ such that $U^* = (-\delta, \delta) \times U$. So $((\tau - \delta, \tau + \delta) \times U) \cap (I - H^*)(\partial\Omega^*) = \emptyset$. Let $|t - \tau| < \delta$. Then $(t, x) \notin (I - H^*)(\partial\Omega^*)$ for all $x \in U$. Thus $(t, x) \neq (I - H^*)(t_1, x_1)$ for all $x \in U$ and for all $(t_1, x_1) \in \mathbb{R} \times \partial\Omega$. So in particular, $(t, x) \neq (I - H^*)(t, x_1)$ for all $x \in U$ and $x_1 \in \partial\Omega$, i.e. $(t, x) \neq (t, x_1) - (0, H(<t>, x_1))$ for all $x \in U$ and $x_1 \in \partial\Omega$, implying that $U \cap (I - H(<t>, .))(\partial\Omega) = \emptyset$ whenever $|t - \tau| < \delta$. By theorem 6.4(a), there exists a finite dimensional $F : J \times \bar{\Omega} \longrightarrow X$ such that $F(t, x) - H(t, x) \in U$ on $J \times \bar{\Omega}$. If X_1 is a subspace of X such that $\dim X_1 < \infty$, $F(J \times \bar{\Omega}) \subseteq X_1$ and $\Omega_1 = \Omega \cap X_1$, then

$$D(I - H(t, .), \Omega, 0) = d((I - F(t, .))|_{\bar{\Omega}_1}, \Omega_1, 0)$$

for $|t - \tau| < \delta$ on J. Therefore there exists $\delta > 0$ such that $D(I - H(t, .), \Omega, 0)$ is constant on $(\tau - \delta, \tau + \delta) \cap J$. So every $\tau \in J$ has a neighbourhood on which the degree is constant. Since J is a connected set, $D(I - H(t, .), \Omega, 0)$ is constant on J.

Now if $y(t)$ was not a constant, then

$$D(I - H(t, .), \Omega, y(t)) = D(I - (H(t, \cdot) + y(t)), \Omega, 0) \tag{14}$$

by lemma 6.8.

Now y is continuous and J is compact so $y(J)$ is compact. Therefore $\overline{H(J \times \bar{\Omega})} + y(J)$ is compact by lemma 6.7. Therefore $(I - (H(t, .) + y(t)), \Omega, 0)$ is an admissible triplet and so by above $D(I - (H(t, .) + y(t)), \Omega, 0)$ is constant on J. So (14) implies that $D(I - H(t, .), \Omega, y(t))$ is constant on J, proving (D3).

The proof of (D3) is found in Nagumo [16]. The following are some results on subspaces and projections.

6.10 Lemma

Let X be any topological space and X_0 a subspace of X. If $K \subseteq K_0$ is compact in X_0, then K is compact in X.

Proof :

Let $\{V_\alpha\}_{\alpha\in A}$ be any open cover of K in X. Then $\{V_\alpha \cap X_0\}_{\alpha\in A}$ forms an open cover of K in X_0. By compactness, this can be reduced to a finite subcover, say, $V_1 \cap X_0, \ldots, V_n \cap X_0$. So $V_1, \ldots, V_n$ is a finite subcover of K in X. Thus K is compact in X. ♠

6.11 Theorem (Tychonoff)

Let (X, τ) be a Hausdorff real topological vector space of finite dimension n. Then X admits a norm $\|.\|$ that gives the topology τ and makes $(X, \|.\|)$ isometrically isomorphic to $(\mathbb{R}^n, |.|)$ (where $|.|$ is the usual norm in $\mathbb{R}^n$). Indeed, if $h : X \longrightarrow \mathbb{R}^n$ is any algebraic isomorphism, then it is also a homeomorphism $(X, \tau) \longrightarrow (\mathbb{R}^n, |.|)$ and $\|x\| = |h(x)|$, $x \in X$ defines a norm $\|.\|$ on X with the asserted properties.

We do not include the proof for the above theorem. If it is required, it can be found in Schäfer [29].

6.12 Lemma

Let X_0 be a finite dimensional subspace of a Hausdorff t.v.s. X. Then X_0 is closed in X.

Proof :

X_0 is a subspace of X, so X_0 has the relative topology induced by the topology on X. By Tychonoff's theorem (6.11), X_0 is homeomorphic with $\mathbb{R}^n$ and has a norm $|.|$ that gives the relative topology on X_0. Now, let $(x_\lambda)_{\lambda\in\Lambda}$ be a net i X_0 such that $x_\lambda \longrightarrow x \in X$. To show $x \in X_0$. Now $B_1(0)$ in X_0 is open in X_0 ar hence by the relative topology, there exists an open set Ω in X such that $B_1(0) = \Omega \cap X_0$. Ω open in X, and $0 \in \Omega$ implies that there exists $U \in \mathcal{U}(0)$ suc that $U \subseteq \Omega$. Since X is a t.v.s., there exists $V \in \mathcal{U}(0)$ such that $V + V \subseteq U$. $x_\lambda \longrightarrow x$ implies that there exists $\lambda_V \in \Lambda$ such that ($\lambda \geq \lambda_V$ implies $x_\lambda \in x + V$ Therefore $x_{\lambda_V} - x \in V$ and so $x - x_{\lambda_V} \in (-1)\, V \subseteq V$ (since V is balanced). So f $\lambda \geq \lambda_V$ we have

$x_\lambda - x_{\lambda_V} = (x_\lambda - x) + (x - x_{\lambda_V}) \in V + V \subseteq U \subseteq \Omega$. But $x_\lambda - x_{\lambda_V} \in X_0$. So $x_\lambda - x_{\lambda_V} \in \Omega \cap X_0 = B_1(0)$ for all $\lambda \geq \lambda_V$. Therefore $|x_\lambda - x_{\lambda_V}| < 1$ and $|x_\lambda| \leq 1 + |x_{\lambda_V}| = R$ for all $\lambda \geq \lambda_V$. So $(x_\lambda)_{\lambda \geq \lambda_V} \subseteq B_R(0)$ is a subnet $(x_\lambda)_{\lambda\in\Lambda}$ where $x_\lambda \longrightarrow x$. Now $B = \mathrm{Cl}_{X_0} B_R(0)$ is compact in X_0 (it is closed ar bounded) and so, by lemma 6.10, also compact in X. Since X is Hausdorff, B closed in X. Hence $x \in B \subseteq X_0$. Therefore $x \in X_0$, proving X_0 is closed in X.

The next result can be found in standard books on linear functional analysis (for examp Limaye [27]).

6.13 Lemma

If $\{x^1, x^2, \ldots, x^n\}$ is a linearly independent set in a nls. X, then there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ in X^* such that $\alpha_j(x^i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

Let X be a finite dimensional real nls. (dim X = n) with basis $\{x^1, x^2, \ldots, x^n\}$. Then by th above lemma, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ in X^* such that $\alpha_j(x^i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

So if $x \in X$, then there exists $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $x = \lambda_1 x^1 + \ldots + \lambda_n x^n$.

$$\begin{aligned}
\alpha_j(x) &= \alpha_j(\lambda_1 x^1 + \ldots + \lambda_n x^n) \\
&= \lambda_1 \alpha_j(x^1) + \ldots + \lambda_n \alpha_j(x^n) \\
&= \lambda_j \alpha_j(x^j) \\
&= \lambda_j .
\end{aligned}$$

Then $x = \alpha_1(x)\, x^1 + \ldots + \alpha_n(x)\, x^n = \sum_{i=1}^{n} \alpha_i(x)\, x^i$.

The next results can be found in Taylor [32].

6.14 Lemma

A T_1 t.v.s. is Hausdorff.

6.15 Lemma

Let X be a locally convex topological vector space. Let M be a subspace of X a

let f be a continuous linear functional on M. Then there exists a continuous line

functional F on X which is an extension of f.

6.16 Lemma

Let X be a locally convex T_1 t.v.s. and let X_1 be a finite dimensional subspace of (dim $X_1 = n$). Then there exists a continuous projection $P_1 : X \longrightarrow X_1$ from X on X_1, and X_1 and $X_2 = (I - P_1)(X)$ are closed complementary linear subspaces of i.e. X_1 and X_2 are closed with $X = X_1 \oplus X_2$.

Proof :

By lemma 6.14, X is Hausdorff. Therefore by Tychonoff's theorem, admits a norm which gives it precisely the relative topology in X. Let $\{x^1, \ldots, x$

be a basis for X_1. Since X_1 is a nls., by remarks after lemma 6.13 there exists $\alpha_1, \ldots, \alpha_n \in X_1^*$ such that $x = \sum_{i=1}^{n} \alpha_i(x)\, x^i$ for all $x \in X_1$.

By lemma 6.15, there exist continuous linear functionals β_i on X which a extensions of α_i. Define a mapping $P_1 : X \longrightarrow X_1$ by $P_1 x = \sum_{i=1}^{n} \beta_i(x)\, x^i$, $x \in$ X

Now P_1 is linear and continuous (since each β_i is). If $x \in X_1$, then $P_1 x = \sum_{i=1}^{n} \beta_i(x)\, x^i = \sum_{i=1}^{n} \alpha_i(x)\, x^i = x$. Thus P_1 is surjective. Also, since $P_1 x \in X_1$ for all $x \in X$, $P_1(P_1 x) = P_1 x$. So $P_1^2 = P_1$. Therefore P_1 is a continuo projection of X onto X_1 ($P_2 = I - P_1$ is easily a projection of X onto a complementary subspace of X_1). It remains to be shown that X_1 and $X_2 = P_2($X are closed.

Let $x \in \bar{X}_1$. Then there exists a net $(x_\lambda)_{\lambda \in \Lambda}$ in X_1 converging to x in X_1. But $P_1 x_\lambda = x_\lambda$ and $P_1 x_\lambda \longrightarrow P_1 x$ (P_1 continuous). Hence $x_\lambda \longrightarrow P_1 x$.

By uniqueness of limits (since X is Hausdorff), $x = P_1 x \in X_1$. Therefore X_1 is close Now $P_2 = I - P_1 : X \longrightarrow X_2$ is a continuous projection of X onto X_2 and similarly X is closed.

We are now ready to show that the degree is unique.

6.17 Theorem

Let X be a locally convex t.v.s. and

$\mathscr{M} = \{(I - F, \Omega, y) \;/\; \Omega \subseteq X \text{ open}, F : \bar{\Omega} \longrightarrow X \text{ compact}, y \in X \setminus (I - F)(\partial\Omega)\}$.

Then there is a unique map $D : \mathscr{M} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3).

Proof :

By theorem 6.9, the existence of such a map is guaranteed.

Let the map $D : M \longrightarrow \mathbb{Z}$ satisfy (D1)–(D3) and let $(I - F, \Omega, y) \in \mathscr{M}$. $I - F$ is closed operator by Theorem 6.4(b). So $(I-F)(\partial\Omega)$ is closed. Hence there exists $U \in \mathcal{U}(0)$ such that $(y+U) \cap (I-F)(\partial\Omega) = \emptyset$. By theorem 6.4(a), there exists a finite dimensional F_1 such that $F_1x - Fx \in U$ on $\bar{\Omega}$. F_1 is also a compact map by theorem 6.4(a). Consider a subspace X_1 of X such that $\dim X_1 < \infty$, $F_1(\bar{\Omega}) \subseteq X_1$, $y \in X_1$ and $\Omega_1 = \Omega \cap X_1$. Define

$$H : J \times \bar{\Omega} \longrightarrow X$$

by

$$H(t, x) = t\, F_1 x + (1-t)\, Fx = Fx + t(F_1 x - Fx)$$

for $(t, x) \in J \times \bar{\Omega}$.

Let $(H(t_\lambda, x_\lambda))$ be a net in $H(J \times \bar{\Omega})$. Since J is compact, we may assume withou loss of generality that $t_\lambda \longrightarrow t_0 \in J$. $F_1(\bar{\Omega})$ is relatively compact, so $F_1(x_\lambda)$ has convergent subnet, say $F_1(x_\alpha) \longrightarrow y_1$. $F(\bar{\Omega})$ is also relatively compact, so $F(x_\alpha)$ ha a convergent subnet, say $F(x_\beta) \longrightarrow y$. So

$H(t_\beta, x_\beta) = t_\beta F_1 x_\beta + (1-t_\beta)\, Fx_\beta \longrightarrow t_0 y_1 + (1-t_0)\, y.$

Thus H is compact (since it is continuous).

If $y = (I - H(t, .))(x)$ $(t, x) \in J \times \bar{\Omega}$, then

$y = x - F(x) - t\,(F_1x - Fx) = (I-F)(x) - t\,(F_1x - Fx)$. Now $F_1x - Fx \in U$ an since U is balanced, $t\,(F_1x - Fx) \in U$. Therefore $(I-F)(x) \in y + U$ and so $x \notin \partial\Omega$. Therefore $y \notin (I - H(t, .))(\partial\Omega)$ for all $t \in J$. So by (D3),

$$D(I-F, \Omega, y) = D(I - F_1, \Omega, y). \qquad (15$$

Since X_1 is finite dimensional, there exists a continuous projection $P_1 : X \longrightarrow X$ Then $X = X_1 \oplus X_2$ where $X_2 = P_2(X)$, $P_2 = I - P_1$. (By lemma 6.16). E Tychonoff's theorem, since X_1 is finite–dimensional, it is also a nls Now $\bar{\Omega}_1$ is a closed subset of X_1, so by theorem 1.2.15, $F_1|_{\bar{\Omega} \cap X_1} : \bar{\Omega} \cap X_1 \longrightarrow X_1$ ha a continuous extension $\tilde{F}_1 : X_1 \longrightarrow X_1$ such that $\tilde{F}_1(X_1) \subseteq \mathrm{conv}\,(F_1(\bar{\Omega} \cap X_1)) \subseteq X_1$. X is a nls., hence has a measure of noncompactness defined on it.

Hence $\gamma(\tilde{F}_1(X_1)) \leq \gamma\,(\text{conv}\,(F_1(\bar{\Omega} \cap X_1))) = \gamma(F_1(\bar{\Omega} \cap X_1)) = 0$ since $F_1(\bar{\Omega} \cap X_1) \subseteq X$ is relatively compact. Therefore $\tilde{F}_1$ is compact.

Now let $H(t, x) = t\,F_1 x + (1-t)\,\tilde{F}_1 P_1 x$ for $(t, x) \in J \times \bar{\Omega}$. $\tilde{F}_1$ is compact, hence $\tilde{F}_1 P_1$ is also compact. Since F_1 is compact, H must also be compact. Suppose $y = (I - H(t, .))(x)$ for $(t, x) \in J \times \bar{\Omega}$. Then $x = y + H(t, x) \in X_1$. Therefore $x \in \bar{\Omega} \cap X_1$. Therefore $P_1 x = x$ and $\tilde{F}_1 P_1 x = \tilde{F}_1 x = F_1 x$. So $y = x - t\,F_1 x - (1-t)\,\tilde{F}_1 P_1 x = x - t\,F_1 x - (1-t)\,F_1 x = (I - F_1)(x)$. Thus we must have $x \notin \partial\Omega$ and so $y \notin (I - H(t, .))(\partial\Omega)$ for all $t \in J$. Thus by (D3) again

$$D(I - F_1, \Omega, y) = D(I - \tilde{F}_1 P_1, \Omega, y). \tag{16}$$

Now consider $\Omega' = P_1^{-1}(\Omega_1)$. Ω_1 open in X_1 and P_1 continuous give us Ω' open in X. Also $\Omega_1 \subseteq \Omega'$. So $\Omega_1 \subseteq \Omega' \cap \Omega$ with $\Omega' \cap \Omega$ open in X. If $x \in \Omega$ with $y = (I - \tilde{F}_1 P_1)(x)$, then $x = y + \tilde{F}_1 P_1 x \in X_1$. So $x \in \Omega \cap X_1 = \Omega_1$. Therefore $y \notin (I - \tilde{F}_1 P_1)(\bar{\Omega} \setminus \Omega_1) \supseteq (I - \tilde{F}_1 P_1)(\bar{\Omega} \setminus \Omega' \cap \Omega)$. So $y \notin (I - \tilde{F}_1 P_1)(\bar{\Omega} \setminus \Omega' \cap \Omega)$. Since $\Omega' \cap \Omega$ is open in X, we have by (D7),

$$D(I - \tilde{F}_1 P_1, \Omega, y) = D(I - \tilde{F}_1 P_1, \Omega' \cap \Omega, y). \tag{17}$$

If $x \in \Omega' = P_1^{-1}(\Omega_1)$ with $y = (I - \tilde{F}_1 P_1)(x)$, then $x = y + \tilde{F}_1 P_1 x \in X_1$ and so $x = P_1 x$ and $P_1 x \in \Omega_1$. So $x \in \Omega_1$. Therefore

$$y \notin (I - \tilde{F}_1 P_1)(\Omega' \setminus \Omega_1) \tag{18}$$

Now to show $y \notin (I - \tilde{F}_1 P_1)(\partial\Omega')$: Let $x \in \bar{\Omega}'$. Then there exists a net $(x_\lambda) \subseteq \Omega'$ such that $x_\lambda \longrightarrow x$. P_1 continuous implies $P_1 x_\lambda \longrightarrow P_1 x$. But $P_1 x_\lambda \in \Omega_1$. So $P_1 x \in \bar{\Omega}_1$. Therefore $x \in P_1^{-1}(\bar{\Omega}_1)$, and therefore $\bar{\Omega}' \subseteq P_1^{-1}(\bar{\Omega}_1)$.

Now let $y = (I - \tilde{F}_1 P_1)(x)$ with $x \in \bar{\Omega}'$. Then $P_1 x \in \bar{\Omega}_1$ and hence $\tilde{F}_1 P_1 x = F_1 P_1 x$. So

$$y = (I - F_1 P_1)x \tag{19}$$

and $x = y + F_1 P_1 x \in X_1$. Therefore $x = P_1 x \in \bar{\Omega}_1 \subseteq \bar{\Omega} \cap X_1$.

If $y = (I - F_1 P_1)x_0$ for $x_0 \in \bar{\Omega}$ then $x_0 = y + F_1 P_1 x_0 \in X_1$. So $P_1 x_0 = x_0$ and $y = (I - F_1)x_0$ and so $x_0 \notin \partial\Omega$. Therefore

$$y \notin (I - F_1P_1)(\partial\Omega). \tag{20}$$

(19) and (20) give us that $x \notin \partial\Omega$ and so $x \in \Omega \cap X_1 = \Omega_1 \subseteq \Omega'$. Therefore

$$y \notin (I - \tilde{F}_1P_1)(\partial\Omega'). \tag{21}$$

(18) and (21) give $y \notin (I - \tilde{F}_1P_1)(\bar{\Omega}' \setminus \Omega_1) \supseteq (I - \tilde{F}_1P_1)(\bar{\Omega}' \setminus \Omega' \cap \Omega)$. So by (D7)

$$D(I - \tilde{F}_1P_1, \Omega' \cap \Omega, y) = D(I - \tilde{F}_1P_1, \Omega', y). \tag{22}$$

Let $x \in \bar{\Omega}' \subseteq P_1^{-1}(\bar{\Omega}_1)$. Then $P_1x \in \bar{\Omega}_1$. So $\tilde{F}_1P_1x = F_1P_1x$. Therefore $(I - \tilde{F}_1P_1)(x) = (I - F_1P_1)(x)$. Hence $(I - \tilde{F}_1P_1)|_{\bar{\Omega}'} = (I - F_1P_1)|_{\bar{\Omega}'}$. Therefore

$$D(I - \tilde{F}_1P_1, \Omega', y) = D(I - F_1P_1, \Omega', y). \tag{23}$$

(15), (16), (17), (22) and (23) give us

$$D(I - F, \Omega, y) = D(I - F_1P_1, P_1^{-1}(\Omega_1), y). \tag{24}$$

Now let (f, Ω_1, y) be an extended Brouwer triplet : i.e. $\Omega_1 \subseteq X_1$ open, $f : \bar{\Omega}_1 \longrightarrow X_1$ continuous, $y \in X_1 \setminus f(\partial\Omega_1)$ and $(id - f)(\bar{\Omega}_1)$ is bounded.

Define $d_0(f, \Omega_1, y) = D(I - (I - F)P_1, P_1^{-1}(\Omega_1), y)$.

For $(I - (I - f)P_1, P_1^{-1}(\Omega_1), y)$ to be an admissible triplet for D we must have

(i) $P_1^{-1}(\Omega_1)$ open in X.

(ii) $(I - f)P_1|_{\overline{P_1^{-1}(\Omega_1)}}$ compact.

(iii) $y \in X \setminus (I - (I - f)P_1)(\partial P_1^{-1}(\Omega_1))$.

We now prove them.

(i) Ω_1 is open in X_1 and $P_1 : X \longrightarrow X_1$ is a continuous projection. So $P_1^{-1}(\Omega_1)$ open in X.

(ii) $(I - f)P_1(\overline{P_1^{-1}(\Omega_1)}) \subseteq (I - f)P_1(P_1^{-1}(\bar{\Omega}_1)) = (I - f)(\bar{\Omega}_1) \subseteq X_1$. But $(I - f)(\bar{\Omega}_1)$ is bounded. So $\overline{(I - f)P_1(\overline{P_1^{-1}(\Omega_1)})}$ is a closed bounded subset a finite dimensional space, hence is compact. So $(I - f)P_1|_{\overline{P_1^{-1}(\Omega_1)}}$ compact.

(iii) $\partial P_1^{-1}(\Omega_1) = \overline{P_1^{-1}(\Omega_1)} \setminus P_1^{-1}(\Omega_1) \subseteq P_1^{-1}(\bar{\Omega}_1) \setminus P_1^{-1}(\Omega) = P_1^{-1}(\bar{\Omega}_1 \setminus \Omega_1) = P_1^{-1}(\partial\Omega_1)$.

Let $y = (I - (I - f)P_1)x$ with $x \in \partial P_1^{-1}(\Omega_1) \subseteq P_1^{-1}(\partial\Omega_1)$. Therefore $P_1 x \in \partial\Omega_1$.

Also $x \in X_1$. So $x = P_1 x$. Therefore $x \in \partial\Omega_1$. So $y = (I - (I - f)P_1)x = f$

and $x \in \partial\Omega_1$, a contradiction. Hence $y \notin (I - (I - f)P_1)(\partial P_1^{-1}(\Omega_1))$.

Now we must show that d_0 satisfies $(d_0 1)$–$(d_0 3)$.

$(d_0 1)$:

Let $y \in \Omega_1$. Then

$$\begin{aligned} d_0(\mathrm{id}, \Omega_1 y) &= D(I - (I - \mathrm{id})P_1, P_1^{-1}(\Omega_1), y) \\ &= D(I, P_1^{-1}(\Omega_1), y) \\ &= 1 \quad \text{since } P_1 y = y \text{ and hence } y \in P_1^{-1}(y) \text{ and by (D1).} \end{aligned}$$

$(d_0 2)$:

Let Ω^1, Ω^2 be disjoint open subsets of Ω_1 with $y \in X_1 \setminus f(\bar{\Omega}_1 \setminus \Omega^1 \cup \Omega^2)$. Then

$$d_0(f, \Omega_1, y) = D(I - (I - f)P_1, P_1^{-1}(\Omega_1), y). \qquad (25$$

Ω^1, Ω^2 are disjoint open subsets of Ω_1. Hence $P_1^{-1}(\Omega^1)$, $P_1^{-1}(\Omega^2)$ are disjoint ope subsets of $P_1^{-1}(\Omega)$. Now suppose $y = (I - (I - f)P_1)x$ where

$x \in \overline{P_1^{-1}(\Omega_1)} \setminus P_1^{-1}(\Omega^1) \cup P_1^{-1}(\Omega^2) \subseteq P_1^{-1}(\bar{\Omega}_1) \setminus P_1^{-1}(\Omega^1 \cup \Omega^2) = P_1^{-1}(\bar{\Omega}_1 \setminus \Omega^1 \cup \Omega^2)$.

Therefore $P_1 x \in \bar{\Omega}_1 \setminus \Omega^1 \cup \Omega^2$. Also

$x = y + (I - f)P_1 x \in X_1$. So $P_1 x = x$. Therefore $x \in \bar{\Omega} \setminus \Omega^1 \cup \Omega^2$ and

$y = (I - (I - f)P_1)x = f(x)$, a contradiction. So

$y \notin (I - (I - f)P_1)(\overline{P_1^{-1}(\Omega_1)} \setminus P_1^{-1}(\Omega^1) \cup P_1^{-1}(\Omega^2))$. So by (D2)

$D(I - (I - f)P_1, P_1^{-1}(\Omega_1), y)$

$= D(I - (I - f)P_1, P_1^{-1}(\Omega^1), y) + D(I - (I - f)P_1, P_1^{-1}(\Omega^2), y)$

$= d_0(f, \Omega^1, y) + d_0(f, \Omega^2, y)$. $\qquad$ (26

(25) and (26) give $d_0(f, \Omega_1, y) = d_0(f, \Omega^1, y) + d_0(f, \Omega^2, y)$.

($d_0$3) :

Let $\Omega_1 \subseteq X_1$ be open, $h : J \times \bar{\Omega}_1 \longrightarrow X_1$ and $y : J \longrightarrow X_1$ be continuous,

$\sup \{|x - h(t,x)| / (t,x) \in J \times \bar{\Omega}_1\} < \infty$.

(N.B.: we can write $|.|$, since X_1 is f.d. and hence a nls) and $y(t) \notin h(t, \partial\Omega_1)$ for a

$t \in J$.

$d_0(h(t, .), \Omega_1, y(t))$

$= D(I - (I - h(t, .))P_1, P_1^{-1}(\Omega_1), y(t))$. (2

Define $H : J \times \overline{P_1^{-1}(\Omega_1)} \longrightarrow X_1$ by $H(t, x) = (I - h(t, .))P_1x$, $(t, x) \in J \times \overline{P_1^{-1}(\Omega_1)}$.

$\{(id - h(t, .))(\bar{\Omega}_1) / t \in J\}$ is a bounded subset of a finite dimensional space. Hen

is relatively compact. Thus H is compact (easily continuous). Now suppose

$y(t) = (I - H(t, .))(x)$, $(t, x) \in J \times \overline{P_1^{-1}(\Omega_1)}$. Then $x = y(t) + H(t, x) \in X_1$.

So $P_1x = x$. Also $x \in \overline{P_1^{-1}(\Omega_1)} \subseteq P_1^{-1}(\bar{\Omega}_1)$. So $P_1x \in \bar{\Omega}_1$. Therefore $x \in \bar{\Omega}_1$, and

$y(t) = x - H(t, x) = x - (I - h(t, .))P_1x = x - (I - h(t, .))x = h(t, x)$. Hence

$x \notin \partial\Omega_1$ i.e. $P_1x \notin \partial\Omega_1$. This implies that $x \notin P_1^{-1}(\partial\Omega_1)$.

But $\partial P_1^{-1}(\Omega_1) = \overline{P_1^{-1}(\Omega_1)} \setminus P_1^{-1}(\Omega_1) \subseteq P_1^{-1}(\bar{\Omega}_1) \setminus P_1^{-1}(\Omega_1) = P_1^{-1}(\partial\Omega_1)$. Thus

$x \notin \partial P_1^{-1}(\Omega_1)$. Therefore $y(t) \notin (I - H(t, .))(\partial P_1^{-1}(\Omega_1))$ for all $t \in J$. So by (D3),

$D(I - H(t, .), P_1^{-1}(\Omega_1), y(t))$ is independent of t and by (27),

$d_0(h(t, .), \Omega_1, y(t))$ is independent of t.

Thus d_0, defined on the extended Brouwer triplets, satisfies ($d_0$1)–($d_0$3). Since t

Brouwer degree is unique, $d_0 = d$.

Therefore by (24),

$$D(I - F, \Omega, y) = D(I - F_1P_1, P_1^{-1}(\Omega_1), y) = d((id - F_1)|_{\bar{\Omega}_1}, \Omega_1, y).$$

Thus there is a unique map

$D : \mathscr{M} \longrightarrow \mathbb{Z}$ satisfying (D1)–(D3). ♠

CHAPTER 7

DEGREE FOR SEMICONDENSING VECTOR FIELDS

In this chapter, we give an extension of the degree to semicondensing vector fields. Most the work is extracted from the paper by Schöneberg [20].

Let X be a real Banach space of infinite dimension.

7.1 Definition

Let $\mathcal{M} = \{c : \mathbb{R}^+ \longrightarrow \mathbb{R}^+ \ / \ c$ is a continuous strictly increasing map such that $c(0) =$ and $c(r) \longrightarrow \infty$ as $r \longrightarrow \infty\}$.

$F : \Omega \longrightarrow X$, where $\Omega \subseteq X$, is a *c–condensing map*, for $c \in \mathcal{M}$ if it is a continuo map such that $\gamma(FB) < c(\gamma(B))$ for all bounded $B \subseteq \Omega$ with $\gamma(B) > 0$.

If $c(t) \equiv t$, then a c–condensing map is simply a γ–condensing map.

7.2 Definition

The map $\mathcal{F} : X \longrightarrow 2^{X^*}$ defined by $\mathcal{F}x = \{ x^* \in X^* \ / \ x^*(x) = |x|^2 = |x^*|^2 \}$ called the *duality map* of X.

The following is an extension of the inner product in Hilbert spaces to real Banach spaces.

7.3 Definition

The *semi–inner products* $(.,.)_\pm : X \times X \longrightarrow \mathbb{R}$ are defined by

$$(x, y)_+ = |y| \lim_{t \to 0^+} t^{-1}(|y + tx| - |y|) \quad \text{and}$$

$$(x, y)_- = |y| \lim_{t \to 0^+} t^{-1}(|y| - |y - tx|).$$

Deimling [28] shows that the semi–inner products have the representations

$$(x, y)_+ = \sup \{y^*(x) \mid y^* \in \mathscr{F}y\} \quad \text{and}$$

$$(x, y)_- = \inf \{y^*(x) \mid y^* \in \mathscr{F}y\} .$$

Deimling [28] also shows that the semi–inner products satisfy the following useful properties.

7.4 Theorem

$$(x, z)_- + (y, z)_- \leq (x + y, z)_\pm \leq (x, z)_\pm + (y, z)_+ ,$$

$$|(x, y)_\pm| \leq |x| \; |y| ,$$

$$(x + \alpha y, y)_\pm = (x, y)_\pm + \alpha \, |y|^2 \text{ for all } \alpha \in \mathbb{R} \text{ and}$$

$$(\alpha x, \beta y)_\pm = \alpha\beta \, (x, y)_\pm \text{ for } \alpha\beta \geq 0 .$$

7.5 Theorem

Let $\Omega \subseteq X$ be open, $F_1, F_2 : \Omega \longrightarrow X$ be continuous, $c : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be continuous and $\epsilon \geq 0$. Suppose that for all $y_1, y_2 \in \Omega$,

$$c(|y_1 - y_2|) \, |y_1 - y_2| \leq (F_1 y_1 - F_2 y_2, y_1 - y_2)_+ + \epsilon \, |y_1 - y_2| .$$

Then for all $y_1, y_2 \in \Omega$,

$$c(|y_1 - y_2|) \, |y_1 - y_2| \leq (F_1 y_1 - F_2 y_2, y_1 - y_2)_- + \epsilon \, |y_1 - y_2| .$$

Proof:

Let $y_1, y_2 \in \Omega$. Since Ω is open, there exists $d > 0$ such that $z_j(t) = y_j - t F_j y_j \in \Omega$ whenever $0 \leq t \leq d$ and $j = 1, 2$.

$$\begin{aligned}
& c(|z_1(t) - z_2(t)|) \, |z_1(t) - z_2(t)| \\
\leq \; & (F_1 z_1(t) - F_2 z_2(t), z_1(t) - z_2(t))_+ + \epsilon \, |z_1(t) - z_2(t)| \\
= \; & (t^{-1}(z_1(t) - z_2(t)) + F_1 z_1(t) - F_2 z_2(t) - t^{-1}(z_1(t) - z_2(t)), z_1(t) - z_2(t))_+ \\
& \qquad + \epsilon \, |z_1(t) - z_2(t)| \\
= \; & (t^{-1}(z_1(t) - z_2(t)) + F_1 z_1(t) - F_2 z_2(t), z_1(t) - z_2(t))_+ - t^{-1}|z_1(t) - z_2(t)|^2 \\
& \qquad + \epsilon \, |z_1(t) - z_2(t)|
\end{aligned}$$

$$\begin{aligned}
&\le \quad t^{-1}[|z_1(t) - z_2(t) + t(F_1z_1(t) - F_2z_2(t))| - |z_1(t) - z_2(t)| \\
&\qquad\qquad + \epsilon\, t]\; |z_1(t) - z_2(t)|
\end{aligned}$$

for $0 < t \le d$.

Suppose $y_1 \ne y_2$. Since $z_1(t) - z_2(t) \longrightarrow y_1 - y_2$ as $t \longrightarrow 0^+$, for all small enough $t > 0$, we may multiply by $\dfrac{|y_1 - y_2|}{|z_1(t) - z_2(t)|}$ to get

$$\begin{aligned}
& c(|z_1(t) - z_2(t)|)\; |y_1 - y_2| \\
\le \quad & |y_1 - y_2|\; t^{-1}[|y_1 - y_2 + t(F_2y_2 - F_2z_2(t)) + t(F_1z_1(t) - F_1y_1)| \\
& \qquad - |y_1 - y_2 - t(F_1y_1 - F_2y_2)| + \epsilon\, t\,] \\
\le \quad & |y_1 - y_2|\; t^{-1}[|y_1 - y_2| + t|F_2y_2 - F_2z_2(t)| + t|F_1z_1(t) - F_1y_1| \\
& \qquad - |y_1 - y_2 - t(F_1y_1 - F_2y_2)| + \epsilon\, t\,] \\
= \quad & |y_1 - y_2|\; t^{-1}[|y_1 - y_2| - |y_1 - y_2 - t(F_1y_1 - F_2y_2)|\,] \\
& \qquad + |y_1 - y_2|\; [|F_2y_2 - F_2z_2(t)| + |F_1z_1(t) - F_1y_1| + \epsilon\,].
\end{aligned}$$

Noting that $|y| \lim_{t \to 0^+} t^{-1}[|y| - |y - tx|\,] = (x, y)_-$, $z_j(t) \longrightarrow y_j$ as $t \longrightarrow 0^+$, and the continuity of c, $|.|$, F_1 and F_2, we obtain, by taking limits as $t \longrightarrow 0^+$,

$c(|y_1 - y_2|)\; |y_1 - y_2| \le (F_1y_1 - F_2y_2,\, y_1 - y_2)_- + \epsilon\, |y_1 - y_2|$, giving us the desired result. ♠

7.6 Definition

Let $\Omega \subseteq X$ and $F : \Omega \longrightarrow X$. Then F is said to be *accretive* if $(Fx - Fy, x - y)_+ \ge 0$ for all $x, y \in \Omega$.

If $c \in \mathscr{M}$, then F is *c–accretive* if $(Fx - Fy, x - y)_+ \ge c(|x - y|)\; |x - y|$ for all $x, y \in \Omega$.

F is *strongly accretive* if F is c–accretive for some $c \in \mathscr{M}$.

N.B. If Ω is open and F is continuous, then we can replace $(.,\, .)_+$ by $(.,\, .)_-$ in the above definition, by theorem 7.5. (In theorem 7.5, c need not belong to $\mathscr{M}$).

7.7 **Theorem**

Let $\Omega \subseteq X$ be open and $F : \bar{\Omega} \longrightarrow X$ be continuous and strongly accretive. Then the following are equivalent :–

(1) F has a zero in Ω.

(2) There exists $x_0 \in \Omega$ such that $|Fx_0| \leq |Fx|$ for all $x \in \partial\Omega$.

The proof of the above theorem can be found in Kirk and Schöneberg [23] and [24].

7.8 **Definition**

Let $\Omega \subseteq X$ be open bounded. Then $F : \bar{\Omega} \longrightarrow X$ is said to be *semicondensing* if it is continuous and if there exists a bounded continuous mapping $V : \Omega \times \Omega \longrightarrow X$ and $c \in \mathcal{M}$ such that :–

(a) $Fx = V(x, x)$ for all $x \in \Omega$.

(b) $\{V(., y) \,/\, y \in \Omega\}$ is equicontinuous.

(c) For all $A \subseteq \Omega$ with $\alpha(A) > 0$, there exists $\epsilon \in [\,0, c(\alpha(A))\,)$ and a finite covering $\{A_1, ..., A_n\}$ of A such that $c(|y_1 - y_2|)\,|y_1 - y_2| \leq (\,V(x_1, y_1) - V(x_2, y_2), y_1 - y_2)_+ + \epsilon\,|y_1 - y_2|$ for all $y_1, y_2 \in \Omega$ and all $x_1, x_2 \in A$ belonging to the same A_i.

The pair (V, c), in the above definition, is called a representation for the semicondensing vector field F on Ω.

7.9 **Remark**

Theorem 7.5 with $F_1 = V(x_1, .)$ and $F_2 = V(x_2, .)$ allows us to replace $(., .)_+$ by $(., .)_-$ in condition (c).

The following example illustrates the conditions (a), (b) and (c) in definition 7.8 .

7.10 Example

Let $\Omega \subseteq X$ be open bounded, $F_1 : \bar{\Omega} \longrightarrow X$ continuous bounded and accretive, and $F_2 : \bar{\Omega} \longrightarrow X$ α–condensing. We will show that $I + F_1 - F_2$ is semicondensing. The map $V : \Omega \times \Omega \longrightarrow X$ defined by $V(x, y) = (I + F_1)\, y - F_2 x$ is bounded and continuous.

(a) $V(x, x) = (I + F_1 - F_2)x$ for all $x \in \Omega$.

(b) Let $x \in \Omega$ and $\epsilon > 0$. Since F_2 is continuous, there exists $\delta > 0$ such that $|x - x'| < \delta$ implies $|F_2 x - F_2 x'| < \epsilon$.

So if $|x - x'| < \delta$ then for $y \in \Omega$

$$\begin{aligned} & |V(x, y) - V(x', y)| \\ = \;& |(I + F_1)y - F_2 x - (I + F_1)y + F_2 x'| \\ = \;& |F_2 x - F_2 x'| \\ < \;& \epsilon. \end{aligned}$$

Hence $\{V(., y) \,/\, y \in \Omega\}$ is equicontinuous.

(c) Let $c(t) \equiv t$ and let $x_1, x_2, y_1, y_2, \in \Omega$. Then

$$\begin{aligned} & c(|y_1 - y_2|)\, |y_1 - y_2| \\ = \;& |y_1 - y_2|^2 \\ \leq \;& |y_1 - y_2|^2 + (F_1 y_1 - F_1 y_2, y_1 - y_2)_+ \quad \text{(since } F_1 \text{ is accretive)} \\ = \;& (F_1 y_1 - F_1 y_2 + y_1 - y_2, y_1 - y_2)_+ \quad \text{(by theorem 7.4)} \\ = \;& (V(x_1, y_1) - V(x_1, y_2), y_1 - y_2)_+ \\ \leq \;& (V(x_1, y_1) - V(x_2, y_2), y_1 - y_2)_+ + (V(x_2, y_2) - V(x_1, y_2), y_1 - y_2)_+ \\ \leq \;& (V(x_1, y_1) - V(x_2, y_2), y_1 - y_2)_+ \\ & \qquad + |F_2 x_1 - F_2 x_2|\, |y_1 - y_2|. \qquad (1) \end{aligned}$$

Let $A \subseteq \Omega$ with $\alpha(A) > 0$. Since F_2 is α–condensing, $\alpha(F_2(A)) < \alpha(A)$. By definition of α, we can find $\epsilon > 0$ such that $\alpha(F_2(A)) < \epsilon\, \alpha(A)$, and a finite covering $\{A_1, \ldots, A_n\}$ of A such that $|F_2 x_1 - F_2 x_2| \leq \epsilon$ whenever x_1, x_2 belong to the same A_i. So by (1), for all $y_1, y_2 \in \Omega$ and all x_1, x_2 in the same

A_i, we have

$|y_1 - y_2|^2 \le (V(x_1, y_1) - V(x_2, y_2), y_1 - y_2)_+ + \epsilon\, |y_1 - y_2|$.

Thus (V, c) is a representation for $I + F_1 - F_2$ and so $I + F_1 - F_2$ is semicondensing. ♠

We will now give some properties of semicondensing vector fields.

7.11 Theorem

Let $\Omega \subseteq X$ be open bounded and let $F : \bar{\Omega} \longrightarrow X$ be semicondensing. Then

(1) F is bounded.

(2) F is proper.

(3) F(A) is closed whenever $A \subseteq \bar{\Omega}$ is closed.

(4) If $\Omega_1 \subseteq \Omega$ is open, then $F|_{\bar{\Omega}_1}$ is semicondensing.

(5) If $\bar{F} : \bar{\Omega} \longrightarrow X$ is semicondensing and $t, \bar{t} \ge 0$ such that $t + \bar{t} > 0$, then $tF + \bar{t}\bar{F}$ is semicondensing.

Proof:

(1) and (4) are obvious. (3) follows from (2) since F is continuous.

To prove (5), if (V, c) and $(\bar{V}, \bar{c})$ are representations for F and $\bar{F}$ respectively, then $(tV + \bar{t}\bar{V},\ tc + \bar{t}\bar{c})$ is a representation for $tF + \bar{t}\bar{F}$ (noting that in the definition of semicondensing vector fields, $(.,.)_+$ can be replaced by $(.,.)_-$).

Now to prove (2). Let (V, c) be a representation for F. Let $K \subseteq X$ be compact. To show that $F^{-1}(K)$ is compact. Let (x_n) be a sequence in $F^{-1}(K)$. Then $(Fx_n) \subseteq K$ and so we may assume that $Fx_n \longrightarrow z \in K$. By the continuity of F we may select a sequence (z_n) in Ω such that $|x_n - z_n| \le \frac{1}{n}$ and $|Fx_n - Fz_n| \le \frac{1}{n}$ for each n. Then $Fz_n = Fx_n + (Fz_n - Fx_n) \longrightarrow z + 0 = z$. Let $A = \{z_n \ / \ n \in \mathbb{N}\}$. If we can show that $\alpha(A) = 0$ then some subsequence of (z_n) will be convergent, say

$z_{n_i} \longrightarrow y$. Then we will have $x_{n_i} = z_{n_i} + (x_{n_i} - z_{n_i}) \longrightarrow y + 0 = y$ and thus $F^{-1}(K)$ will be compact

So suppose $\alpha(A) > 0$. Since F is semicondensing, there exists $\epsilon \in [\,0, c(\alpha(A))\,)$ and a finite covering $\{A_1, ..., A_m\}$ of A such that

$c(|y - \bar{y}|)\ |y - \bar{y}| \leq (V(x, y) - V(\bar{x}, \bar{y}), y - \bar{y})_+ + \epsilon\ |y - \bar{y}|$ for all $y, \bar{y} \in \Omega$ and all $x, \bar{x} \in A$ belonging to the same A_i. Choose $\delta > 0$ such that $\epsilon + 2\delta < c(\alpha(A))$ and then choose $n_0 \in \mathbb{N}$ such that $|Fz_n - z| \leq \delta$ for all $n \geq n_0$. Let $\Gamma_i \subseteq A$ be defined by $\Gamma_i = \{y \in A_i \,/\, y = z_n$ for some $n \geq n_0\}$. By definition of α, we can find some $j \in \{1, ..., m\}$ such that diam $\Gamma_j \geq \alpha(A)$. (N.B.: $\alpha(A) = \alpha(\{z_n \,/\, n \geq n_0\})$).

So for $y, \bar{y} \in \Gamma_j$, we have,

$$
\begin{aligned}
c(|y - \bar{y}|) &\leq |V(y, y) - V(\bar{y}, \bar{y})| + \epsilon \\
&= |Fy - F\bar{y}| + \epsilon \\
&\leq |Fy - z| + |z - F\bar{y}| + \epsilon \\
&\leq \delta + \delta + \epsilon \\
&= 2\delta + \epsilon
\end{aligned}
$$

So $c(\text{diam } \Gamma_j) \leq 2\delta + \epsilon < c(\alpha(A))$ and since c is strictly increasing, diam $\Gamma_j < \alpha(A) \leq$ diam Γ_j, a contradiction. Hence $\alpha(A) = 0$. ♠

7.12 Definition

Let $\Omega \subseteq X$ be open bounded and $F : \bar{\Omega} \longrightarrow X$. F is said to be *semiaccretive* if F is continuous and there exists a bounded continuous map $W : \Omega \times \Omega \longrightarrow X$ such that $Fx = W(x, x)$ for $x \in \Omega$, $W(x, .)$ is accretive for all $x \in \Omega$, and the map $x \longmapsto W(x, .)$ is a compact map of Ω into the space of bounded, continuous and accretive mappings of Ω into X, where the latter space is taken with the topology of uniform convergence.

7.13 Theorem

Let $\Omega \subseteq X$ be open bounded, $F : \bar{\Omega} \longrightarrow X$ semicondensing and $G : \bar{\Omega} \longrightarrow X$ semiaccretive. Then $F + G$ is semicondensing.

The proof of the previous theorem can be found in Schöneberg [20]. In defining the degree, the following theorem proves very useful.

7.14 Theorem

Let $\Omega \subseteq X$ be open bounded and let $F_1 : \bar{\Omega} \longrightarrow X$ and $F_2 : \bar{\Omega} \longrightarrow X$ be semicondensing with representations (V_1, c_1) and (V_2, c_2) respectively. Define

$W : J \times \Omega \times \Omega \longrightarrow X$ and $d : J \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ by

$W(t, x, y) = t\, V_1(x, y) + (1-t)\, V_2(x, y)$ and $d(t, r) = t\, c_1(r) + (1-t)\, c_2(r)$.

Then the set $G = \{ (t, x) \in J \times \Omega \;/\; W(t, x, y) = 0$ for some $y \in \Omega \}$ is open in $J \times X$ and there is a unique map $H : G \longrightarrow \Omega$ satisfying $W(t, x, H(t, x)) = 0$ for all $(t, x) \in G$.

Furthermore, H is continuous and for all bounded $A \subseteq X$ with $\alpha(A) > 0$ we have $\alpha(H(G \cap (J \times A))) < \alpha(A)$.

Proof:

We break the proof up into four parts.

(a) *For all $(t, x) \in J \times \Omega$, the map $W(t, x, .)$ is continuous and $d(t, .)$–accretive:*

$t\, F_1 + (1-t)\, F_2$ is semicondensing by theorem 7.11 and $(W(t, ., .), d(t, .))$ is easily a representation for it. Let $(t, x) \in J \times \Omega$. Since Ω is open, there exists $r > 0$ such that $B_r(x) \subseteq \Omega$. Then $\alpha(B_r(x)) = 2r > 0$. Thus there exists $\epsilon(t) \in [\, 0, d(t, \alpha(B_r(x)))\,) = [\, 0, d(t, 2r)\,)$ and a finite covering $\{A_1, \ldots, A_n\}$ of $B_r(x)$ such that

$d(t, |y_1 - y_2|)\, |y_1 - y_2|$

$\leq (\, W(t, x_1, y_1) - W(t, x_2, y_2), y_1 - y_2\,)_+ + \epsilon(t)\, |y_1 - y_2|$

for all $y_1, y_2 \in \Omega$ and all $x_1, x_2 \in B_r(x)$ belonging to the same A_i.

Take $x_1 = x_2 = x$ and let $r \longrightarrow 0$. Then we have

$$
\begin{aligned}
& d(t, |y_1 - y_2|)\, |y_1 - y_2| \\
\leq \quad & (\, W(t, x, y_1) - W(t, x, y_2), y_1 - y_2)_+ \qquad (2)
\end{aligned}
$$

for all $y_1, y_2 \in \Omega$. Thus, by definition, $W(t, x, .)$ is $d(t, .)$–accretive.

Now let $y_0 \in \Omega$ and $\epsilon > 0$. Since W is continuous, there exist neighbourhoods N_t, N_x and N_{y_0} of t, x, and y_0 respectively, such that if

$(\bar{t}, \bar{x}, \bar{y}) \in N_t \times N_x \times N_{y_0}$, then

$|W(\bar{t}, \bar{x}, \bar{y}) - W(t, x, y_0)| < \epsilon$. Thus whenever $\bar{y} \in N_{y_0}$, we have

$|W(t, x, \bar{y}) - W(t, x, y_0)| < \epsilon$, proving the continuity of $W(t, x, .)$.

(b) *G is open* :

Let $(t_0, x_0) \in G$. Then there exists $y_0 \in \Omega$ such that $W(t_0, x_0, y_0) = 0$. Since Ω is open, we can choose $R > 0$ such that $B_R(y_0) \subseteq \Omega$. From (2) we have for all $y_1, y_2 \in \Omega$,

$$
\begin{aligned}
& d(t_0, |y_1 - y_2|)\, |y_1 - y_2| \\
\leq \quad & (\, W(t_0, x_0, y_1) - W(t_0, x_0, y_2), y_1 - y_2\,)_+ \\
\leq \quad & |W(t_0, x_0, y_1) - W(t_0, x_0, y_2)|\, |y_1 - y_2|.
\end{aligned}
$$

Thus for $y_1, y_2 \in \Omega$ with $y_1 \neq y_2$,

$d(t_0, |y_1 - y_2|) \leq |W(t_0, x_0, y_1) - W(t_0, x_0, y_2)|$

If $y_1 = y_0$ and $y_2 = y \in \partial B_R(y_0)$ (and hence $y_1 \neq y_2$), then

$$
\begin{aligned}
& |W(t_0, x_0, y)| \\
& \quad = \quad |W(t_0, x_0, y) - W(t_0, x_0, y_0)| \\
& \quad \geq \quad d(t_0, |y - y_0|) \\
& \quad = \quad d(t_0, R) \\
& \quad > \quad 0. \qquad (3)
\end{aligned}
$$

By the equicontinuity of $\{V_i(\cdot, y) \ /\ y \in \Omega \}$, $i = 1, 2$, and by the boundedness of V_1 and V_2, $\{W(\cdot, \cdot, y) \,/\, y \in \Omega\}$ is equicontinuous.

Therefore there exists $\delta > 0$ such that if $\max\{\,|t - t_0|,\ |x - x_0|\,\} \le \delta$ then

$$|W(t, x, y) - W(t_0, x_0, y)| \le \tfrac{1}{2}\, d(t_0, R) \qquad (4)$$

for all $y \in \Omega$.

So if $\max\{\,|t - t_0|,\ |x - x_0|\,\} \le \delta$ and $y \in \partial B_R(y_0)$, then

$$\begin{aligned} |W(t, x, y)| &\ge |W(t_0, x_0, y)| - |W(t, x, y) - W(t_0, x_0, y)| \\ &\ge d(t_0, R) - \tfrac{1}{2}\, d(t_0, R) \quad \text{(by (3) and (4))} \\ &= \tfrac{1}{2}\, d(t_0, R). \end{aligned}$$

$$\begin{aligned} \text{Therefore } \epsilon(\delta) &= \inf\{\,|W(t, x, y)| \,/\, y \in \partial B_R(y_0),\ (t, x) \in J \times \Omega \text{ with } \\ &\qquad \max\{|t - t_0|, |x - x_0|\} \le \delta\,\} \\ &\ge \tfrac{1}{2}\, d(t_0, R) \\ &> 0. \end{aligned}$$

By continuity of W, we may assume that δ is chosen so small that $|W(t, x, y_0)| \le \epsilon(\delta)$ whenever $(t, x) \in J \times \Omega$ with $\max\{|t - t_0|, |x - x_0|\} \le \delta$. But for all such $(t, x) \in J \times \Omega$, $|W(t, x, y_0)| \le \epsilon(\delta) \le |W(t, x, y)|$ for all $y \in \partial B_R(y_0)$. Now by part (a), $W(t, x, .)$ is $d(t, .)$–accretive and hence is strongly accretive. Applying theorem 7.7, we obtain that $W(t, x, .)$ has a zero in $B_R(y_0) \subseteq \Omega$. Thus for all $(t, x) \in J \times \Omega$ with $\max\{|t - t_0|, |x - x_0|\} \le \delta$, there exists $y \in \Omega$ such that $W(t, x, y) = 0$ and hence $(t, x) \in G$. Proving that G is open.

(c) *There exists a unique map $H : G \longrightarrow \Omega$ such that $W(t, x, H(t, x)) = 0$ for $(t, x) \in G$ and this map H is continuous*:

Since for $(t, x) \in G$, there exists $y \in \Omega$ such that $W(t, x, y) = 0$, we can find a map $H : G \longrightarrow \Omega$ such that $W(t, x, H(t, x)) = 0$. Suppose H_1 was another such map. By (2) we have for $(t, x) \in G$,

$$\begin{aligned} &d(t, |H(t, x) - H_1(t, x)|)\, |H(t, x) - H_1(t, x)| \\ &\qquad \le (\, W(t, x, H(t, x)) - W(t, x, H_1(t, x)),\ H(t, x) - H_1(t, x)\,)_+ \end{aligned}$$

$= (0, H(t, x) - H_1(t, x))_+$

$= 0.$

So either $d(t, |H(t, x) - H_1(t, x)|) = 0$ or $|H(t, x) - H_1(t, x)| = 0.$

In either case $H(t, x) = H_1(t, x)$ and so $H_1 = H.$

Therefore there must be a unique map $H : G \longrightarrow \Omega$ satisfying

$W(t, x, H(t, x)) = 0$ for all $(t, x) \in G.$

We now want to show that H is continuous. Let $(t_0, x_0), (t, x) \in G$. With $y_1 = H(t, x)$ and $y_2 = H(t_0, x_0)$ we have by the result in (a),

$$\begin{aligned} d(t, |y_1 - y_2|)\, |y_1 - y_2| &\leq (W(t, x, y_1) - W(t, x, y_2), y_1 - y_2)_+ \\ &\leq |W(t, x, y_2)|\, |y_1 - y_2| \quad \text{since } W(t, x, H(t, x)) = 0. \end{aligned}$$

If $y_1 - y_2 \neq 0$, then $d(t, |y_1 - y_2|) \leq |W(t, x, y_2)|$ and if $y_1 - y_2 = 0$, then this is trivially true. Let $(t, x) \longrightarrow (t_0, x_0)$ in G. Since $W(\cdot, \cdot, H(t_0, x_0))$ is continuous, we have $W(t, x, H(t_0, x_0)) \longrightarrow W(t_0, x_0, H(t_0, x_0)) = 0.$

Since $d \in \mathscr{M}$, $|H(t, x) - H(t_0, x_0)| \longrightarrow 0$ and so $H(t, x) \longrightarrow H(t_0, x_0)$, proving that H is continuous.

(d) *For all bounded $A \subseteq X$ with $\alpha(A) > 0$, $\alpha(H(G \cap (J \times A))) < \alpha(A)$:*

Without loss of generality assume $A \subseteq \Omega$. Since $\alpha(A) > 0$, there exists a finite covering $\{A_1, \ldots, A_n\}$ of A and $\epsilon(t) \in [\, 0, d(t, \alpha(A))\,)$ such that

$$\begin{aligned} d(t, |y_1 - y_2|)\, |y_1 - y_2| \leq (W(t, x_1, y_1) - W(t, x_2, y_2), y_1 - y_2)_+ \\ + \epsilon(t)\, |y_1 - y_2| \end{aligned}$$

for all $y_1, y_2 \in \Omega$, for all $t \in J$ and for all $x_1, x_2 \in A$ belonging to the same A_i.

Since V_1, V_2 are bounded, for each $t \in J$ we can find an open neighbourhood N_t of t such that

$|W(t, x, y) - W(s, x, y)| \leq \frac{1}{3} (d(t, \alpha(A)) - \epsilon(t))$ for all $s \in N_t$ and all

$x, y \in \Omega$.

Select $t_1, \ldots, t_m \in J$ such that $\{N_{t_1}, \ldots, N_{t_m}\}$ covers J and define $\Gamma_{ij} \subseteq J \times X$ by $\Gamma_{ij} = N_{t_j} \times A_i$. Then for $(t, x), (\bar{t}, \bar{x}) \in \Gamma_{ij} \cap G$, we have

$$\begin{aligned}
& d(t_j, |H(t, x) - H(\bar{t}, \bar{x})|)\, |H(t, x) - H(\bar{t}, \bar{x})| \\
\leq\ & (W(t_j, x, H(t, x)) - W(t_j, \bar{x}, H(\bar{t}, \bar{x})), H(t, x) - H(\bar{t}, \bar{x}))_+ \\
& \quad + \epsilon(t_j)\, |H(t, x) - H(\bar{t}, \bar{x})| \\
\leq\ & |H(t, x) - H(\bar{t}, \bar{x})|\, [|W(t_j, x, H(t, x)) - W(t_j, \bar{x}, H(\bar{t}, \bar{x}))| + \epsilon(t_j)] \\
\leq\ & |H(t, x) - H(\bar{t}, \bar{x})|\, [|W(t_j, x, H(t, x)) - W(t, x, H(t, x))| \\
& \quad + |W(t_j, \bar{x}, H(\bar{t}, \bar{x})) - W(\bar{t}, \bar{x}, H(\bar{t}, \bar{x}))| + \epsilon(t_j)] \\
\leq\ & |H(t, x) - H(\bar{t}, \bar{x})|\, [\tfrac{2}{3}(d(t_j, \alpha(A)) - \epsilon(t_j)) + \epsilon(t_j)] \\
=\ & |H(t, x) - H(\bar{t}, \bar{x})|\, [\tfrac{2}{3} d(t_j, \alpha(A)) + \tfrac{1}{3} \epsilon(t_j)].
\end{aligned}$$

Therefore

$$\begin{aligned}
& d(t_j, \operatorname{diam} H(\Gamma_{ij} \cap G)) \\
& \quad \leq \tfrac{2}{3} d(t_j, \alpha(A)) + \tfrac{1}{3} \epsilon(t_j) \\
& \quad < \tfrac{2}{3} d(t_j, \alpha(A)) + \tfrac{1}{3} d(t_j, \alpha(A)) \\
& \quad = d(t_j, \alpha(A)).
\end{aligned}$$

Hence $\operatorname{diam} H(\Gamma_{ij} \cap \Omega) < \alpha(A)$ for all i, j.

But $G \cap (J \times A) \subseteq \bigcup_{i,j} (\Gamma_{ij} \cap G)$. Hence by definition of α,

$\alpha(H(G \cap (J \times A))) < \alpha(A)$. ♠

The following corollary follows easily from the above theorem.

7.15 Corollary

Let $\Omega \subseteq X$ be open bounded and let (V, c) be a representation for a semicondensing vector field on $\bar{\Omega}$. Then the set $G = \{ x \in \Omega \ / \ V(x, y) = 0 \text{ for some } y \in \Omega \}$ is open in X and there is a unique map $H : G \longrightarrow \Omega$ satisfying $V(x, H(x)) = 0$ for all

$x \in G$. This map is α–condensing.

We are now nearly ready to define a degree for semicondensing maps. The following lemma helps in this regard.

7.16 Lemma

Let $\Omega \subseteq X$ be open bounded, and for $i = 1, 2$, let $F_i : \bar{\Omega} \longrightarrow X$ be semicondensing with representation (V_i, c_i). Let $G_i \subseteq X$ be defined by $G_i = \{ x \in \Omega \ / \ V_i(x, y) = 0$ for some $y \in \Omega \}$ and let $H_i : G_i \longrightarrow \Omega$ be defined by $V_i(x, H(x)) = 0$ for all $x \in G_i$. Then $t F_1 x + (1-t) F_2 x \neq 0$ for all $t \in J$ and $x \in \partial\Omega$ implies that $D_N(I - H_i, G_i, 0)$, $i = 1, 2$ is defined and $D_N(I - H_1, G, 0) = D_N(I - H_2, G, 0)$, where D_N is the Nussbaum degree from chapter 4.

Proof:

Let W, d, G and H be as in theorem 7.14. Set $M = \{ (t, x) \in G \ / \ H(t, x) = x \}$. Then, easily, $M = \{ (t, x) \in J \times \Omega \ / \ W(t, x, x) = 0 \}$. If $((t_n, x_n)) \subseteq M$ such that $(t_n, x_n) \longrightarrow (t, x) \in J \times \bar{\Omega}$, then $V_i(x_n, x_n) = F_i x_n$, $i = 1, 2$. So $0 = W(t_n, x_n, x_n) = t_n F_1 x_n + (1 - t_n) F_2 x_n \longrightarrow t F_1 x + (1-t) F_2 x$ since F_1 and F_2 are contnuous on $\bar{\Omega}$. Thus $t F_1 x + (1-t) F_2 x = 0$ and by hypothesis $x \notin \partial\Omega$.
Let $A = \{ x \in \Omega \ / \ (t, x) \in M$ for some $t \in J \}$.
If $x \in A$, then $x \in \Omega$ and $(t, x) \in M$ for some $t \in J$. Thus $(t, x) \in G$ and $H(t, x) = x$. So $x = H(t, x) \in H(G \cap (J \times A))$ and hence we get $A \subseteq H(G \cap (J \times A))$. If $\alpha(A) > 0$, then $\alpha(A) \leq \alpha(H(G \cap (J \times A))) < \alpha(A)$, a contradiction. So $\alpha(A) = 0$. Since $M \subseteq J \times A$, we must have that $\alpha(M) = 0$ and since M is closed, it must be compact.
If $G^t = \{ x \in X \ / \ (t, x) \in G \}$

$= \{ x \in X \ / \ (t, x) \in J \times \Omega$ and $W(t, x, y) = 0$ for some $y \in \Omega \}$, then $G^0 = \{ x \in X \ / \ x \in \Omega \text{ and } W(0, x, y) = 0 \text{ for some } y \in \Omega \}$

$$= \{ x \in X \ / \ x \in \Omega \text{ and } V_2(x, y) = 0 \text{ for some } y \in \Omega \}$$

$$= G_2$$

and $G^1 = \{ x \in X \ / \ x \in \Omega \text{ and } W(1, x, y) = 0 \text{ for some } y \in \Omega \}$

$$= \{ x \in X \ / \ V_1(x, y) = 0 \text{ for some } y \in \Omega \}$$

$$= G_1.$$

Now by theorem 4.2.1,

$D_N(I - H(0, .), G^0, 0) = D_N(I - H(1, .), G^1, 0)$. So if $H_1 = H(1, .)$ and $H_2 = H(0, .)$ then

$$D_N(I-H_2, G_2, 0) = D_N(I - H_1, G_1, 0).$$ ♠

Now consider the triplet $(F, \Omega, 0)$ where $\Omega \subsetneq X$ is open bounded, $F : \bar{\Omega} \longrightarrow X$ semicondensing such that $0 \notin F(\partial\Omega)$.

Let (V, c) be a representation for F and set $G = \{ x \in \Omega \ / \ V(x, y) = 0 \text{ for some } y \in \Omega \}$ and define $H : G \longrightarrow \Omega$ by $V(x, Hx) = 0$ for all $x \in \Omega$. By lemma 7.16, $D_N(I - H, G, 0)$ is defined and hence we define the degree on the triplet $(F, \Omega, 0)$ by

$$D(F, \Omega, 0) = D_N(I - H, G, 0).$$

We must show that this is well–defined.

Let (V_j, c_j), $j = 1, 2$ be two representations for F. If G_j and H_j are defined as in lemma 7.16 with $F_1 = F_2 = F$, then, since $0 \notin F(\partial\Omega)$, we must have by the same lemma that

$D_N(I - H_1, G_1, 0) = D_N(I - H_2, G_2, 0)$. Hence $D(F, \Omega, 0)$ is well–defined.

7.17 Remark

If $F = I - H$ with $H : \bar{\Omega} \longrightarrow X$ α–condensing and $x - Hx \neq 0$ for all $x \in \delta\Omega$, then F is semicondensing by example 7.10. Here $F = I + 0 - H$ and $0 : \bar{\Omega} \longrightarrow X$ is accretive, continuous and bounded. Here (V, c) is a representation for F where

$V(x, y) = y - Hx$, $x, y \in \Omega$ and $c(t) = t$, $t \in J$.

Let $\quad G \quad = \{ x \in \Omega \ / \ V(x, y) = 0$ for some $y \in \Omega \}$

$= \{ x \in \Omega \ / \ Hx = y$ for some $y \in \Omega \}$

$= \{ x \in \Omega \ / \ Hx \in \Omega \}$.

Now for $x \in G$, we have $Hx \in \Omega$ and so $V(x, Hx) = Hx - Hx = 0$.

Therefore by definition,

$$D(I - H, \Omega, 0) = D_N(I - H, G, 0). \tag{5}$$

Since $H : \bar{\Omega} \longrightarrow X$ is α-condensing, $I - H$ is proper and hence $(I - H)^{-1}(0)$ is compact. Since $0 \notin (I - H)(\partial\Omega)$, we must have $(I - H)^{-1}(0) \subseteq \Omega$ and this is compact. Thus $(I - H|_{\Omega}, \Omega, 0)$ is a Nussbaum triplet. Let $0 = (I - H)(x)$ with $x \in \Omega$. Then $Hx = x$ and $x \in \Omega$. So $x \in G$. Thus $0 \notin (I - H)(\Omega \backslash G)$ and so by $(D_N 7)$,

$$D_N(I - H, G, 0) = D_N(I - H, \Omega, 0). \tag{6}$$

(5) and (6) give us

$$D(I - H, \Omega, 0) = D_N(I - H, \Omega, 0).$$

Thus the degree defined is in fact an extension of the Nussbaum degree. ♠

The following results show that our degree satisfies those properties that make degree theory useful.

7.18 Theorem

Let $\Omega \subseteq X$ be open bounded and $F : \bar{\Omega} \longrightarrow X$ be semicondensing with representation (V, c) such that $0 \notin F(\partial\Omega)$. Then the degree, defined above satisfies

(a) $\quad D(I, \Omega, 0) = 1$ if $0 \in \Omega$. (D1)

(b) $\quad$ If Ω_1 and Ω_2 are disjoint open subsets of Ω with $0 \notin F(\bar{\Omega} \setminus \Omega_1 \cup \Omega_2)$, then

$$D(F, \Omega, 0) = D(F, \Omega_1, 0) + D(F, \Omega_2, 0). \tag{D2}$$

(c) $\quad$ If $D(F, \Omega, 0) \neq 0$, then $F^{-1}(0) \neq \emptyset$. (D4)

(d) If F is strongly accretive and $F^{-1}(0) \neq \emptyset$, then $D(F, \Omega, 0) = 1$.

(e) If $F|_{\partial\Omega} = G|_{\partial\Omega}$ and $G : \bar{\Omega} \longrightarrow X$ is semicondensing, then

$$D(F, \Omega, 0) = D(G, \Omega, 0). \tag{D6}$$

(f) If Ω is symmetric with respect to $0 \in \Omega$ and $Fx = -F(-x)$ for all $x \in \partial\Omega$, then $D(F, \Omega, 0)$ is odd.

Proof:

(a) Follows from remark 7.17.

(b) If $G = \{ x \in \Omega \ / \ V(x, y) = 0$ for some $y \in \Omega \}$, then by corollary 7.15, let $H : G \longrightarrow \Omega$ be the unique map such that $V(x, Hx) = 0$ for $x \in G$. Then by definition,

$$D(F, \Omega, 0) = D_N(I - H, G, 0). \tag{7}$$

Let $G_i = \{ x \in \Omega_i \ / \ V(x, y) = 0$ for some $y \in \Omega_i \}$. Then G_i is open. Consider $H_i = H|_{G_i} : G_i \longrightarrow \Omega$. Then $H_i : G_i \longrightarrow \Omega_i$. So by definition again,

$$D(F, \Omega_i, 0) = D_N(I - H_i, G_i, 0) \ , i = 1, 2 . \tag{8}$$

Now G_i is an open subset of G for $i = 1, 2$ and G_1 and G_2 are disjoint. Suppose $0 = (I - H)x$ for $x \in G\backslash(G_1 \cup G_2)$. Then $x = Hx$. Since $x \in G$, $V(x, Hx) = 0$ and so $Fx = V(x, x) = 0$ with $x \in G \backslash(G_1 \cup G_2)$. Since $x \notin G_i$ and $V(x, Hx) = 0$ we must have $x \notin \Omega_i$. Hence $x \in \Omega\backslash(\Omega_1 \cup \Omega_2)$ with $Fx = 0$, a contradiction. Hence $0 \notin (I - H)(G\backslash(G_1 \cup G_2))$, and so by (D_N2),

$$\begin{aligned} D_N(I - H, G, 0) &= D_N(I - H, G_1, 0) + D_N(I - H, G_2, 0) \\ &= D_N(I - H_1, G_1, 0) + D_N(I - H_2, G_2, 0). \end{aligned} \tag{9}$$

(7), (8) and (9) give us

$$D(F, \Omega, 0) = D(F, \Omega_1, 0) + D(F, \Omega_2, 0).$$

(c) Let $D(F, \Omega, 0) \neq 0$. By corollary 7.15, if $G = \{ x \in \Omega \ / \ V(x, y) = 0$ for some $y \in \Omega \}$, then there exists a unique map

$H : G \longrightarrow \Omega$ such that $V(x, Hx) = 0$, $x \in G$. Then

$D(F, \Omega, 0) = D_N(I - H, G, 0)$. So $D_N(I - H, G, 0) \neq 0$ and by $(D_N 4)$,

$(I - H)^{-1}(0) \neq \emptyset$. Thus we can find $x_0 \in G$ such that $x_0 = Hx_0$. But $x_0 \in G$,

so $V(x_0, Hx_0) = 0$. Hence $Fx_0 = V(x_0, x_0) = V(x_0, Hx_0) = 0$. Therefore

$F^{-1}(0) \neq \emptyset$.

(d) Since F is strongly accretive, we can find $c \in \mathscr{M}$ such that F is c–accretive. So for all $x, y \in \bar{\Omega}$, $(Fx - Fy, x - y)_+ \geq c(|x - y|)\, |x - y|$. Suppose $F^{-1}(0) \neq \emptyset$ and let x_1, x_2 be zeros of F in Ω. Then

$0 = (0, x_1 - x_2)_+ = (Fx_1 - Fx_2, x_1 - x_2)_+ \geq c(|x_1 - x_2|)\, |x_1 - x_2|$.

So $c(|x_1 - x_2|)\, |x_1 - x_2| = 0$. Hence $|x_1 - x_2| = 0$ and so $x_1 = x_2$. Thus F has a unique zero in Ω, say $x_0 \in \Omega$. Let $W : \Omega \times \Omega \longrightarrow X$ be defined by

$W(x, y) = Fy$.

(W, c) is a representation for F :

(1) $W(x, x) = Fx$.

(2) Let $\epsilon > 0$ and $x \in \Omega$. Then for all $x_1 \in \Omega$

$$\sup \{ |W(x, y) - W(x_1, y)| \;/\; y \in \Omega \}$$
$$= \sup \{ |Fy - Fy| \;/\; y \in \Omega \}$$
$$= 0$$
$$< \epsilon.$$

Thus $\{ W(., y) \;/\; y \in \Omega \}$ is equicontinuous.

(3)
$$c(|y - \bar{y}|)\, |y - \bar{y}|$$
$$\leq (Fy - F\bar{y}, y - \bar{y})_+$$
$$= (W(x, y) - W(\bar{x}, \bar{y}), y - \bar{y})_+$$
$$\leq (W(x, y) - W(\bar{x}, \bar{y}), y - \bar{y})_+ + \epsilon\, |y - \bar{y}|$$

for all $x, \bar{x}, y, \bar{y} \in \Omega$.

Hence (W, c) is a representation for F.

Define $G = \{ x \in \Omega \;/\; W(x, y) = 0 \text{ for some } y \in \Omega \}$

$= \{ x \in \Omega \ / \ Fy = 0 \ \text{ for some } y \in \Omega \}$

$= \{ x \in \Omega \ / \ Fx_0 = 0 \}$ since x_0 is the unique zero of F

$= \Omega.$

There exists a unique map $H : \Omega \longrightarrow \Omega$ such that $W(x, Hx) = 0$ for $x \in \Omega$. But $W(x, Hx) = FHx$. So $FHx = 0$. But F has a unique zero x_0, hence $Hx = x_0$ for all $x \in \Omega$. So by definition,

$$\begin{aligned} D(F, \Omega, 0) &= D_N(I - H, \Omega, 0) \\ &= D_N(I - x_0, \Omega, 0) \\ &= D_N(I, \Omega, x\,) \\ &= 1 \quad \text{by } (D_N 1) \text{ since } x_0 \in \Omega. \end{aligned}$$

(e) Let F_i be semicondensing, $0 \notin F_i(\partial\Omega)$, with representation (V_i, c_i) , $i = 1, 2$, and such that $F_1|_{\partial\Omega} = F_2|_{\partial\Omega}$.

Let $G_i = \{ x \in \Omega \ / \ V_i(x, y) = 0 \ \text{ for some } y \in \Omega \}$ and $H_i : G_i \longrightarrow \Omega$ be defined by $V_i(x, H_i x) = 0$ for all $x \in G_i$.

Suppose $0 = t\, F_1 x + (1 - t)\, F_2 x$ for $x \in \partial\Omega$.

Then $0 = t\, F_1 x + (1 - t)\, F_1 x = F_1 x$. A contradiction. Therefore $0 \notin t\, F_1 x + (1 - t)\, F_2 x$ for all $t \in J$ and $x \in \partial\Omega$. Then by lemma 7.16, $D_N(I - H_1, G_1, 0) = D_N(I - H_2, G_2, 0)$ and so by definition,

$$D(F_1, \Omega, 0) = D(F_2, \Omega, 0).$$

(f) If we replace F by $\frac{1}{2}(Fx - F(-x))$, we will have $Fx = -F(-x)$ for all $x \in \bar{\Omega}$ and F will also be semicondensing. Let (V, c) be a representation for F. Then if we define $\bar{V} : \Omega \times \Omega \longrightarrow X$ by

$\bar{V}(x, y) = \frac{1}{2}(V(x, y) - V(-x, -y))$, then $(\bar{V}, c)$ is also a representation of F.

Let $G = \{ x \in \Omega \ / \ \bar{V}(x, y) = 0 \ \text{ for some } y \in \Omega \}$ and $H : G \longrightarrow \Omega$ be defined by $\bar{V}(x, Hx) = 0$ for $x \in G$.

$0 \in G$:

Now $0 \in \Omega$. So $F(0) = -F(-0)$. Thus $F(0) = 0$ and so $V(0, 0) = 0$, giving us $0 \in G$.

$G = -G$:

Let $x \in G$. Then $\bar{V}(x, y) = 0$ for some $y \in \Omega$. Then $\bar{V}(-x, -y) = -\bar{V}(x, y) = 0$ and since $y \in \Omega$ and Ω is symmetric with respect to 0, we must have $-y \in \Omega$. Since $-x \in \Omega$, we must have $-x \in G$, and so G is symmetric with respect to 0.

$Hx = -H(-x)$:

Let $x \in G$. Then $\bar{V}(x, Hx) = 0$.

Now $\bar{V}(x, -H(-x)) = -\bar{V}(-x, H(-x))$. Since $x \in G$ we must have $-x \in G$ and so $\bar{V}(-x, H(-x)) = 0$. Hence $\bar{V}(x, -H(-x)) = 0$.

But $H : G \longrightarrow \Omega$ was a unique map such that $\bar{V}(x, Hx) = 0$ for all $x \in G$. Hence $Hx = -H(-x)$ for all $x \in G$.

By Borsuk's theorem for the Nussbaum degree, $D_N(I - H, G, 0)$ is odd. But $D(F, \Omega, 0) = D_N(I - H, G, 0)$ by definition.

Hence $D(F, \Omega, 0)$ is odd. ♠

The last result is the (D3) property.

7.19 Theorem

Let $\Omega \subseteq X$ be open bounded and $H : J \times \bar{\Omega} \longrightarrow X$ be continuous such that $H(t, x) \neq 0$ for $(t, x) \in J \times \partial\Omega$, $H(t, .)$ is semicondensing for all $t \in J$ and $\{ H(., x) \,/\, x \in \partial\Omega \}$ is equicontinuous. Then $D(H(t, .), \Omega, 0)$ is independent of t.

Proof:

Since J is compact, it suffices to show that for each $t_0 \in J$, there exists some interval about t_0 on which $D(H(t, .), \Omega, 0)$ is independent of t.

Fix $t_0 \in J$. By theorem 7.11 (3), $H(t_0, .)(\partial\Omega)$ is closed. Thus, since $0 \notin H(t_0, .)(\partial\Omega)$, we can find $\epsilon > 0$ such that $B_\epsilon(0) \cap H(t_0, .)(\partial\Omega) = \emptyset$. Since

$\{ H(t, x) \ / \ x \in \partial\Omega \}$ is equicontinuous, there exists an interval $I \subseteq J$ about t_0, such that $|H(t, x) - H(t_0, x)| < \epsilon$ for all $t \in I$ and all $x \in \partial\Omega$.

Fix $t_1 \in I$. Then for $t \in J$ and $x \in \partial\Omega$,

$$
\begin{aligned}
&|t\, H(t_1, x) + (1-t)\, H(t_0, x)| \\
&\qquad \geq |H(t_0, x)| - t\, |H(t_0, x) - H(t_1, x)| \\
&\qquad \geq \epsilon - |H(t_0, x) - H(t_1, x)| \\
&\qquad > \epsilon - \epsilon \\
&\qquad = 0.
\end{aligned}
$$

Thus $0 \neq t\, H(t_1, x) + (1-t)\, H(t_0, x)$ for $x \in \partial\Omega$. So by lemma 7.16, and by definition, $D(H(t_1, .), \Omega, 0) = D(H(t_0, .), \Omega, 0)$.

Thus $D(H(t, .), \Omega, 0)$ is constant on I. ♠

CONCLUSION

A further extension of the degree, not covered in this dissertation, is the degree of multivalued maps. More about this can be found in Petryshyn and Fitzpatrick [7] and Ma [21].

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