AN APOS EXPLORATION OF CONCEPTUAL UNDERSTANDING OF THE CHAIN RULE IN CALCULUS BY FIRST YEAR ENGINEERING STUDENTS

By

Zingiswa Mybert Monica Jojo

(201506651)

A full dissertation for the Degree of

DOCTOR OF EDUCATION

(Mathematics Education)

In the school of Science, Mathematics and Technology Education

(SSMTE)

(Edgewood Campus)

Faculty of Education

University of Kwa-Zulu Natal

Supervisors: 1. Dr Deonarian Brijlall

2. Dr Aneshkumar Maharaj

2011

DEDICATION

This work is dedicated to

Banzo, Popo, Tazz, Fefe and Grinkqi.

My late father Mr Wiseman Thanduxolo Nodali.

Our lives are not determined by what happens to us,

But how we react to what happens to us, not by what life brings us, but by the attitude we bring to life.

A positive attitude causes a chain reaction of positive thoughts, events and outcomes. The chain rule is not used to differentiate that given attitude. It is a spark that creates extraordinary results.

Matsiritso Mazingi Mazipha

ACKNOWLEDGEMENTS

I wish to put on record my profound appreciation and sincere thanks to all those who helped and advised me in persuing this work. The diversity of experience and academic support of the 2009 Cohort group broadened my perspective on research and challenged many of my long held ideas.

I am grateful to Dr D. Brijlall and Dr A. Maharaj for their untiring inspiration, facilitation, mentoring and guidance throughout my research. I thank them for their dedication, encouragement and commitment to supervising me.

I would also like to put on record my appreciation to the 2010 first semester civil engineering students at Mangosuthu University of Technology for their willingness to allow their work to form part of this study.

I would also like to thank the University of KwaZulu Natal (Doctoral grant), Eskom, and Mangosuthu University of Technology for their financial support during my research.

To Dr Connie Israel for editing my thesis.

Finally, my sincere gratitude is extended to my children, Onesimo, Sinesipho, Siphesihle, Gcina and Lutho, for always being there for me, and bearing with my busy schedule during the course of this study.

DECLARATION

I, Zingiswa Mybert Monica Jojo declares that:

1. This research report in this thesis, except where otherwise indicated, is my original work.
2. This thesis has not been submitted for any degree or examination at any other university.
3. This thesis does not contain other person's data, pictures, graphs or other information, unless acknowledged as being sourced from other persons.
4. This thesis does not contain other person's writing, unless specifically acknowledged as being sourced from other researchers. When other written sources have been quoted, then: (i) their writing has been placed inside quotation marks, and referenced where their exact words have been used, (ii) their words have been re-written but the general information attributed to them has been referenced.
5. I have indicated in detail which part of the publication was actually written by myself alone and have fully referenced such publication where I have reproduced a publication of which I am the author, co-author or editor.
6. This thesis does not contain text, graphics or tables copied and pasted from the Internet, unless specifically acknowledged, and the source being detailed in the thesis and in the Bibliography sections.

Signed……………………………

As the candidate's Supervisor I agree to the submission of this thesis.

Signed………………………….

FIGURES USED IN THE STUDY

TABLES USED IN THE STUDY

EXTRACTS IN THIS STUDY

LIST OF APPENDICES

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

CHAPTER 2

LITERATURE REVIEW

CHAPTER 3

THEORETICAL FRAMEWORK

CHAPTER 4

RESEARCH METHODOLOGY AND PROCEDURE

CHAPTER 5

THE VALIDITY OF THE QUESTIONNAIRE

CHAPTER 6

VIDEO RECORDINGS

CHAPTER 7

INTERVIEWS AND WRITTEN QUESTIONNAIRES

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

ABSTRACT

The main issue in this study is how students conceptualise mathematical learning in the context of calculus with specific reference to the chain rule. The study focuses on how students use the chain rule in finding derivatives of composite functions (including trigonometric ones). The study was based on the APOS (Action-Process-Objects-Schema) approach in exploring conceptual understanding displayed by first year University of Technology students in learning the chain rule in calculus.

The study consisted of two phases, both using a qualitative approach. Phase 1 was the pilot study which involved collection of data via questionnaires which were administered to 23 previous semester students of known ability, willing to participate in the study. The questionnaire was then administered to 30 volunteering first year students in Phase 2. A structured way to describe an individual student's understanding of the chain rule was developed and applied to analyzing the evolution of that understanding for each of the 30 first year students. Various methods of data collection were used namely: (1) classroom observations, (2) open-ended questionnaire, (3) semi-structured and unstructured interviews, (4) video-recordings, and (5) written class work, tests and exercises.

The research done indicates that it is essential for instructional design to accommodate multiple ways of function representation to enable students to make connections and have a deeper understanding of the concept of the chain rule. Learning activities should include tasks that demand all three techniques, *Straight* form technique, *Link* form technique and *Leibniz* form technique, to cater for the variation in learner preferences. It is believed that the APOS paradigm using selected activities brought the students to the point of being better able to understand the chain rule and informed the teaching strategies for this concept.

In this way, it is believed that this conceptualization will enable the formulation of schema of the chain rule which can be applied to a wider range of contexts in calculus. There is a need to establish a conceptual basis that allows construction of a schema of the chain rule. The understanding of the concept with skills can then be augmented by instructional design based on the modified genetic decomposition. This will then subject students to a better understanding of the chain rule and hence more of calculus and its applications.

CHAPTER 1

INTRODUCTION

1.1 Overview

In this chapter, the researcher sets out to present the research process as it unfolded. This chapter provides an overview of the study. The background and purpose of this research project is detailed first. The motivation for doing this research, the nature of mathematics with respect to this study and the subject didactics of mathematics are discussed. The research questions and key terms in the study are introduced. This is followed by summaries of successive chapters. Later in this chapter, the significance of the results of this study in the current era in South Africa is indicated.

1.2 Background

In my experience and discussions held with other lecturers, despite being one of the most basic tools for a mathematician, the chain rule is also one of the most complicated, calculus tools. Calculus is one of the topics introduced to matric learners at high school, yet a large number of them receive inadequate mathematics education and join the university mostly under-prepared for the study of differential calculus. Furthermore the chain rule is not part of the South African school syllabus. In my experience many first year university students have difficulty in understanding the chain rule in differentiation. This phenomenon was also observed by Orton (1983) who indicated that students: (1) had problems in the understanding of the meaning of the derivative when it appeared as a fraction or the sum of two parts and application of the chain rule for differentiation, and (2) had little intuitive understanding of derivative as well as fundamental misconceptions about the derivative. Also in my experience some teachers at high school are less comfortable with calculus and its applications.

The engineering course done by the participants of this study does not require Mathematics as a specialization course at the University of Technology. Mathematics, as offered by the Mathematics department, is used only as a tool for servicing the engineering course. The mathematics courses offered are a semester course at levels 1, 2, and 3. The semester course outline for mathematics level 1 requires that differentiation and integration be taught within a period of 4 weeks. This relatively little time spent on teaching mathematics contributes to weak preparation and lack of appreciation and understanding of integration and differentiation in calculus. In an attempt to equip engineering students with mathematical skills, that are useful in their career, calculus is one of the many topics that are taught to them.

1.3 Purpose of the study

The main issue in this study is how students conceptualise mathematical learning in the context of calculus with specific reference to the chain rule. The study focuses on how students use the chain rule in finding derivatives of composite functions (including trigonometric ones). The study was based on the APOS (Action-Process-Objects-Schema) approach in exploring conceptual understanding displayed by first year University of Technology students in learning the chain rule in calculus. Dubinsky & McDonald (2001) suggested that APOS theory as a tool can be used objectively to explain students' difficulties with a broad range of mathematical concepts and recommended ways in which students can learn these concepts. They further argued that this theory can point us towards pedagogical strategies that lead to marked improvement in (1) student learning of complex or abstract mathematical concepts, and (2) students' use of these concepts to prove theorems, provide examples, and solve problems.

The aim of this study was *to find out whether students can construct an underlying structure of the chain rule in dealing with composition or decomposition of functions*.

This focus was accomplished by:

(1) A discussion of the types of structures constructed by students when learning the chain rule with the view to clarifying their understanding: (i) of the composition of function and (ii) of the derivative.

(2) Finding out how the lack or availability of these structures hamper or assist students' understanding of the chain rule. This was done to check whether they had a coherent understanding of composition of single-valued functions and the derivative.

(3) Determining the students' actual engagement with tasks and how these tasks link with the expected outcomes highlighted in the initial genetic decomposition.

(4) Informing possible modifications to the initial genetic decomposition.

1.4 Motivation

This study was motivated by: (1) the researcher's personal experience and interest, (2) the understanding a mathematical concept, (3) the learners' difficulties in understanding the chain rule in calculus, and (4) the learners' preference of procedural methods rather than conceptual understanding in calculus. The focus on (2), (3) and (4) is covered in Chapter 2. Below (1) is explained in detail.

1.4.1 Researcher's personal experience and interest

The researcher has been a mathematics teacher of grades 10-12 learners, from 1986 to 2006. In 2007, she had the privilege of teaching mathematics to first year engineering students at a University of Technology. She also taught engineering mathematics to Extended Curriculum Program (ECP) and Pre-tech students registered for the bridging course at the university.

The first year engineering class was composed of a mixture of students from the Pre-tech programme, (students who did a bridging course to upgrade their matric symbols for a period of six months at the university), Extended Curriculum Programme (ECP), (students who took a foundation programme for a year at the university), and students with excellent matric symbols who did not receive any prior instruction at the university. The total number of students in the researcher's class was 197 all registered for civil engineering.

The researcher noticed that in her class, whilst teaching, most students took down notes and paid little or no attention to the lesson. This made the researcher recall what Felder (1996) asserted. Students learn in many ways, either, by seeing and hearing, reflecting and acting, reasoning logically and intuitively, or memorizing and visualizing. He further suggested that how much a student learns in class is governed partly by the compatibility of his characteristic approach to learning and the instructor's characteristic approach to teaching. Some instructors lecture and others demonstrate some focus on rules and others on examples, some emphasize memory and others understanding. However, research (Claxton & Murrell, 1987) indicates that the more a person understands his or her particular strengths and weaknesses in various learning contexts, the better he or she can take appropriate action to optimize learning. What processes were used by students to build their knowledge and application of the chain rule in differential and integral calculus was of interest to this study.

The chain rule is the underlying concept in many applications of calculus: implicit differentiation, solving related rate of change problems, applying it in the fundamental theorem of calculus and solving differential equations. Research (Hassani, 1998) into the nature of students' understanding of the concepts underlying the calculus showed significant gaps between their conceptual understanding of the major ideas of calculus and their ability to perform procedures based on these ideas. The chain rule states that if $g(x)$ is a function differentiable at c and f is a function differentiable at $g(c)$, then, the composite function fog given by $(fog)(x) = f(g(x))$ is differentiable at c and that $(fog)^{'}(c) = f^{'}(g(c)).g^{'}(c)$. Cottrill (1999) asserted that: (1) conventional wisdom holds that students' conception of the chain rule (as with other rules) is that of symbol manipulation, (2) the conception of the chain rule appeared to be a straight-forward manipulation of symbols which could easily be applied in problem situations and (3) concluded that an application based on symbol manipulation carries a heavy requirement for the function to be given by an expression, fostering students' tendencies toward instrumental understanding, where they are unable to apply the chain rule. Hassani (1998) examined students' understandings on graphical, numerical (tabular), and algebraic/symbolic presentations of composition of functions and the chain rule. His study revealed that first-year undergraduate calculus students have a very meager understanding of the concept of composition of functions and their ability to explain or apply the chain rule is significantly related to their algebraic manipulative skills and their general knowledge of function concepts and function composition.

The researcher was motivated by the poor performance of first year engineering students in differentiation involving the use of the chain rule. This then culminated in an interest for the researcher to find out how students construct mathematical knowledge when learning the concept of chain rule in calculus. Conceptual knowledge as defined by Hapasaalo (2004) denotes knowledge of and a skillful "drive" along particular networks, the elements of which can be concepts, rules (for example algorithms and procedures), and even problems given in various representation forms.

The researcher's observations on students' performance in calculus using the chain rule had always revealed a difficulty in understanding and applying the concept as compared to other sections (exponential and logarithmic functions, trigonometry and complex numbers) in first year engineering mathematics. This led to an interest of how learners conceptualized the chain rule and how they could learn this concept effectively. This study therefore aims at assisting the students to understand and apply the chain rule and to inform the researcher's teaching for her future role in the classroom and the effective learning of students.

1.4.2 Learners' difficulties in understanding the chain rule in calculus

The chain rule is one of the hardest ideas conveyed to students in calculus (Gordon, 2005). It is difficult to motivate, as most students do not really understand its source. It is difficult to express in symbols even after it is developed, and it is awkward to put into words, so that many students cannot remember it, and so cannot apply it correctly.

Swanson (2006) asserted that the complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum. Despite the importance of the chain rule in calculus and its difficulty for students, the chain rule has been scarcely studied in mathematics educational research (Clark et al, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). These student difficulties include the inability to apply the chain rule to functions and also with composing and decomposing functions (Clark et al, 1997; Cottrill, 1999; Hassani, 1998).

Previous research emphasized the importance of function composition in the understanding of the chain rule, but the ability to say more than that has been elusive (Clark et al, 1997; Cottrill, 1999; Hassani, 1998). This study was therefore designed to focus on how students understand the function composition as seen through the chain rule problems using functions that are familiar, somewhat familiar and unfamiliar to them.

The derivative is an inherently difficult concept for many students (Uygur & Ozdas, 2005). This becomes clear especially when the function considered is a composite function, students' difficulties increase and get worse (Tall 1993). One of the problems is with the use of the Leibniz notation, $\frac{dy}{dx}$, whether it is a fraction or a single indivisible symbol. It causes serious conceptual problems, but this notation is indispensable in calculus. According to Tall (1993), the difficulty with the notion of the chain rule is the dilemma of whether du can be cancelled in the equation: $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$. Cottrill (1999) also studied the correlation between a student's understanding of the composition functions and understanding the chain rule. In his study, there was a small amount of evidence supporting the hypothesis which states that the understanding of composition of functions is the key to understanding the chain rule. Cottrill indicated a new study was needed to address this hypothesis.

It is evident from the above discussion that, many well-known functions have simple expressions for their derivatives while composite functions require the use of the chain rule for differentiation. Functions having fairly complicated expressions have explicit formulae for derivatives. It was the development of formulas and rules such as the chain rule enabling mathematicians to calculate the derivative that motivated the use of the name calculus for this mathematical discipline. These illustrations of learners' difficulty in using the chain rule probed my research in this direction as outcomes from this study will inform better instructional techniques. Also, it is hoped to address the hypothesis of Cotrill noted above.

1.4.3 Understanding a mathematical concept

This section presents some perceptions of understanding from a variety of theoretical perspectives. It ends by giving the operational definition of 'understanding' for this study. Duffin and Simpson (2000) have identified and named three components of understanding as (1) the building, (2) the having, and (3) the enacting. They defined 'building understanding' as the formation of connections between internal mental structures. 'Having understanding' is said to be the state of these connections at any particular time and 'enacting understanding' as the use of the connections available at a particular moment to solve a problem or construct a response to a question. Thus this is the type of understanding that may be visible from students' work when responding to mathematical tasks. Duffin and Simpson also talked about the breadth and depth of understanding. They described the breadth of understanding to be determined by the number of different possible starting points that the learner may have in solving a problem. The depth may be evidenced by the way the learners could unpack each stage of their solution in more detail by referring to more concepts.

In relating these theories of Duffin and Simpson (2000) to mathematical content an example that follows may be considered. Imagine a situation where one is given a function say, $f(x)=(x^3+5x)^2$ to differentiate. A learner may identify this function as being represented structurally as $(x^3+5x)(x^3+5x)$, and may expand the expression before differentiating. This learner might then use the product rule to differentiate the resulting expression. Another learner who recognizes the function in the form of $f(g(x))=(x^3+5x)^2$ where $f(x)=x^2$ and $g(x)=x^3+5x$, he/she may the use the chain rule to differentiate $f(x)$. A learner who sees this function to be represented structurally in one form only lacks breadth of understanding. Such a learner may even deny that the given equivalent forms of the function to be differentiated represent the same function. The depth of understanding in this case could be determined by the learner’s ability to state at each stage what is happening in mathematical terms. For example, a learner could indicate the stages at which the power rule, the product rule or the chain rule have been applied, that is, alongside the work shown in performing the mathematical task. This demonstrates a deeper understanding of solving the task than in a case where the structures will be manipulated by applying a rule with no explanations at all. Reasons for applying or doing certain procedures could also be given. A learner who instrumentally carries out manipulations is likely to be unaware of the mistakes he or she has committed.

On the contrary the understanding of a mathematical concept is explained in this study with the help and adoption of APOS. APOS ascertains that to understand a mathematical concept, one must begin with manipulating previously constructed mental or physical objects in the learner’s mind to form actions; actions would then be interiorised to form processes which are then encapsulated to form objects. These objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in schemas.

This philosophy of understanding initiated in this study led to understanding of the concept of the chain rule which was explored through the development of schema relevant to it. These led to the design of relevant activities. Students were provided with activities in class that were designed to induce them to make the suitable mental constructions suggested in the initial genetic decomposition. The tasks used in this study helped students gain experience in constructing actions corresponding to the chain rule.

This experience was built upon in subsequent activities where students were asked to reconstruct familiar actions as general processes. Lastly, the students were then provided with complicated activities where they needed to organize a variety of previously constructed objects, like functions and derivatives of composition of functions, into a schema that could be applied to chain rule problem situations. More specifically the researcher examined students' attempts to answer the tasks given in class, their tests, exercises with regard to their understanding of functions, composition of functions and the chain rule.

Although this was in the form of pencil and paper work, of importance to the researcher was the procedure used to answer the questions and not whether answers were correct or wrong. In addition to the tasks, interviews were conducted so as to substantiate the level of student understanding of the concept. These interviews were more valuable than the written assessment instruments because one student could have displayed correct written work while the transcript revealed little understanding and *vice versa.* A full range of understanding was obtained by selecting students who gave correct, partially correct and incorrect responses on the written work. Some students showed evidence of constructing very little mental connections while others had constructed bits and pieces and others seemed to have made all the constructions proposed in the genetic decomposition. The latter showed evidence of understanding the concept. They showed evidence of possessing a schema for the chain rule. This interrogation of what understanding is will imply how students learn the chain rule and hence inform my teaching of the chain rule.

1.4.4 The students' preference of procedural methods rather than conceptual understanding in calculus.

Tall (1997) refers to students developing coping strategies, like computational and manipulative skills when they are faced with conceptual difficulties. Students at the University of Technology spent little time in studying mathematics since mathematics was not required as a specialization. Mathematics 1 for engineering students is a semester course. Also large classes contributed to the weak preparation and disinterest in the subject. Students then resorted to methods and techniques that would just help them to pass mathematics. These included manipulative approaches and drill which helped the students to pass the examinations without engaging in problems that involved insight and understanding. This concurs with Smith & Moore (1991) who assert that much of what students have actually leant precisely, is a set of coping skills for getting past the next assignment, the next quiz and the next examination. They therefore have no real advantage of understanding mathematics. Naidoo (2007) recommended that a need for alternate methods of instruction to enhance teaching and understanding of calculus was essential.

Amongst other things that were identified by Tall (1992) as some difficulties students encountered were (a) preference for procedural methods rather than conceptual understanding and (b) restricted mental images of functions. Perhaps this was because mathematics lessons at the University of Technology focussed on standard methods, rules and procedures. Students' understanding is the key factor to how understanding of a concept unfolds itself. When students see a concept for the first time, they are limited to an action conception of that concept. For example, first year students may understand differentiation as an action on polynomials following certain rules applied in a particular sequence. As the student interacts more with differentiation, he or she would then understand differentiation as a more general process that is not limited to a set of rules applied to individual functions. These tasks designed in this study were designed to allow for both acquisitions of conclusions about both procedural and conceptual understanding.

1.5 What is learning in mathematics?

In this study an assumption about the nature of mathematical knowledge and how it is acquired grounds the theory (APOS) implemented. Individuals learnt mathematical concepts indirectly. They applied mental structures to make sense of each concept. Dubinsky (2010) claimed that appropriate mental structures for a given mathematical concept led to automatic, easy learning of the said concepts, while in their absence learning was almost impossible. Pedagogy should then aim at helping students build relevant mental constructions. The APOS theory presents the afore-mentioned constructions as actions, processes, objects and schemas.

A description of actions, processes, objects and schemas and their relationships that might be involved in a construction of a mathematical concept is called the *genetic decomposition* of the particular concept. In the case of the chain rule for example, this could begin with students understanding the composition of two or more functions. These would then be transformed to one composite function. They would also have to understand the derivative concept of the composite functions. The two processes would then be coordinated to obtain this derivative which would then be encapsulated to using the chain rule for differentiation.

1.6 Subject Didactics of Mathematics

Ernest (1991) asserts that Mathematics Education understood in its simplest and most concrete sense concerns the activity or practice of teaching mathematics. He further asserts that learning is inseparable from teaching. This process involves the exercise of the mind and intellect in thought, enquiry, and reasoning. Similarly, the *interpretive research paradigm* seeks to explore real human and social situations and uncover the meanings, understandings and interpretations of the actors involved. It was therefore evident that in exploring how students conceptualized the understanding of the chain rule, APOS could be used objectively to (1) explain students' difficulties with the chain rule and (2) suggest ways that students can learn the chain rule. More specifically APOS could lead us towards pedagogical strategies that in turn lead to marked improvement in (1) student learning of the chain rule and (2) students' use of this concept to solve problems in calculus.

The principles of effective mathematics teaching drawn from educational theories of Piaget illustrated that learning required interaction to develop: (1) a deep conceptual understanding, (2) positive relationships and (3) a classroom community. This social interaction leads to gradual, incremental changes in thought and behaviour of learners and through which interaction with other learners, allows them to examine, clarify and change their conceptual understanding. This study sought to explore how actions, processes and objects of the chain rule schema could be coordinated as mental structures to enhance the learning of the concept and access it in situations where it needs to be applied.

1.7 Research questions and focus of enquiry

This study used the APOS (Action-Process-Objects-Schema) approach in exploring conceptual understanding displayed by first year University of Technology students in learning the chain rule in calculus. The research question addressed by this study is:

How do students construct various structures to recognize and apply the chain rule to functions in the context of calculus?

This is with the view of clarifying:

1. *students' understanding of the function concept*
2. *students understanding of function composition*
3. *students' understanding of the derivative*
4. *students' difficulties in using the chain rule*
5. *students' schema alignment with the genetic decomposition of the chain rule*
6. *the triad stage of schema development in which students are operating with respect to the chain rule, and*
7. *students' identification of the reverse application of the chain rule in the substitution technique for integration.*

1.8 Terminology and concepts

The terms and concepts used in this study are outlined here. Further on in the study, these terms are discussed in detail.

1.8.1 Nature of Mathematics

Mathematics in this study refers to the language of engineering essential to understand how engineering mathematics works in order to master the complex relationships present in modern engineering systems and products. It is a human activity that deals with patterns, problem solving, logical thinking, in an attempt to understand the world and make use of that understanding.

1.8.2 Calculus

Calculus refers to the study in mathematics of the behaviour of function, for example, limits, rate of change, the functions' maxima and minima composition. It involves operations of differential and integral calculus.

1.8.2 Chain rule

The chain rule is the underlying concept in many applications of calculus: implicit differentiation, solving related rate problems, and solving differential equations. The rule states that, if $g(x)$ is a function differentiable at c and f is a function differentiable at $g(c)$, then, the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is differentiable at c and that $(f \circ g)^{'}(c) = f^{'}(g(c)).g^{'}(c)$.

1.8.4 APOS

APOS refers to the main mental mechanisms interiorisation and encapsulation, for building mental structures of actions, processes, objects and schema (Dubinsky, 2010; Weller et al., 2003). More concise definitions of APOS in provided in Chapter 3.

1.8.5 Genetic decomposition

Genetic decomposition is a set of mental constructions that a student might make to learn the concept and accessing it when needed. These are a result of a set of instructions which are designed to help students make the mental constructions and relate them to the mathematical concept desired.

1.9 Overview of this study

In finding a suitable approach to the theses, the following structure has been adopted and used. The thesis comprises eight chapters, the bibliography and appendices. The chapters are as follows:

Chapter One introduces the background and purpose of this study. In addition the motivation for doing this research, the nature of mathematics with respect to this study and the subject didactics in mathematics is discussed. The research questions and key terms in the study are introduced. The conclusion gives the significance of the results of this study in the current era in South Africa.

Chapter Two presents the relevant literature reviewed based on the area of exploration. Functions, processes on functions and their relevance to the chain rule, misconceptions on the chain rule, classroom instructional design and the implications of literature reviewed is explored. Studies based on APOS in the South African context are summarized.

Chapter Three presents the theoretical framework for this research study. The theory that impacts on this study is discussed. More specifically the APOS theory, as an extension to reflective abstractions forms the framework for this research study. The triad mechanism that is used to explain students' understanding is presented. The relevance of APOS to the study is clearly indicated.

Chapter Four focuses on the research design, the research methodology and the procedures undertaken to conduct this study. The outline, the summary of the research design and the research instruments used are presented. The preliminary process which involved the pilot study, research paradigm and how it fits with the study, data sources, data collection and analysis methods together with the study limitations are also discussed.

Chapter Five concentrates on the validation of the research instrument used in the Phase 1 of the study. The findings on this preliminary study are outlined in detail and how this affected the initial genetic decomposition is indicated in this chapter.

Chapter Six discusses analysis of video recordings.

Chapter Seven presents an analysis based on interviews and written questionnaires where collaborative learning versus individual learning is indicated in contrast.

Chapter Eight presents the conclusions that were drawn based on the overall study. The limitations and recommendations of the study are also presented.

1.10 Synopsis

In this chapter, the researcher has discussed the rationale that prompted my interest in conducting this research. The researcher has shown what led her in choosing the APOS approach in exploring conceptual understanding displayed by first year, University of Technology students in learning the chain rule in calculus. In the next chapter the literature reviewed is discussed while the theory that informs this study is reviewed.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In this chapter the literature and some relevant studies on the learning and teaching of the chain rule are cited and discussed. At the time of the study evidence presented by an American study supported the principle that understanding of composition of functions was the key to understanding of the chain rule (Clarke et al, 1997). In the study by Cotrill et al (1999), cooperative learning was used. Also computers were used to assist students to make the mental constructions proposed by the APOS framework. The present study on the chain rule has not been conducted in South Africa before. Other recent South African studies on APOS are summarised later in the chapter.

This chapter commences with defining the perspective and context of this study. This is followed by a discussion of the function as a concept, its importance and how it informs the understanding of the chain rule. Moreover composition of functions, whose understanding has served a purpose in understanding the chain rule in differentiation, is discussed. Thereafter rules of differentiation and the complexity of the chain rule which lead to misconceptions are outlined. The chapter concludes with a summary of other recent studies on APOS, a discussion on designing classroom instruction whilst teaching and the implications of the literature reviewed for this research.

2.2 The perspective and context of the study

In this study an interactive lecture method, instead of computers was used. At the University of Technology in this study there were no mathematics computers that were used by mathematics students due to the historically disadvantaged nature of the university. Through experience as a lecturer at this university, the researcher was aware that the lecture method used in the lecture classrooms was not very interactive. Only a few students could ask questions in a lecture due to the space constraint of the classroom and the short time allocated to each topic in the curriculum. Most of the time, the talking was done by the lecturer. However, the interaction became possible when the students attended tutorials in smaller divisions of an average of 30 in a classroom in a $1\frac{1}{2}$ hour slot every Friday afternoon where exercises on the concept taught were revisited. In the studies done on APOS using computers by (Cotrill et al, 1996), it was shown that students were in a better position to make mental constructions using computers when finding graphically the limits of certain functions. The study was not sure about certain mental constructions not being constructed. It is hoped that this engagement would fill in doubtful gaps of such a study.

2.3 Functions

Functions have been discussed under the following headings:

2.3.1 The function concept

The concept of function is central to undergraduate mathematics, a foundation to modern mathematics, and essential in related areas of sciences. Formally, a function f is defined as ‘ $f = (x; y) / y$ (the dependent variable) is assigned, by prescription or rule, to each element of the domain (independent variable) one and only one element of the range' (Dreyer, 1985, p 73). Dreyer (1985) further asserts that the elements of the domain are usually referred to as 'objects' while the process which associates each object with its image is called a function. It has also been a major focus of attention for the mathematics education research community over the past decades (Evangelidou et al, (2004); Sfard, (1992); Sierpinska, (1992), Vinner & Dreyfus, (1989). Various domains regarding functions have been addressed. Elia & Spyrou, (2006) had interest on (1) the concept image of the function definition in the students' minds and (2) the students' ability to connect different representations of the function, based on students' construction of meaning and understanding of mathematical concepts. Sierpinska, (1992) was more interested in (1) exploring university students' conceptions of functions on the basis of their concept definitions and examples of the notion, (2) students' performance in recognizing functions in their different forms of representation and transfer from one representation to the other, (3) exploring the relationship between their conceptions of functions and (4) their ability to represent different representations of functions. Thus in this study (Sierpinska, 1992), students' constructions of definition and examples for the concept of function were distinguished from the transformation of representations.

Several studies show that learning of the function concept is often facilitated by the early consideration of an action and its interpretation as a process (Briedenbach et al, 1992). According to Sfard (1989), the development of abstract mathematical objects was the product of the comprehension of processes; although some researchers suggested models that were not strictly sequential (Slavit, 1997, p. 268; Artigue, 1998); the important notion of *procept* (process-concept) underlines symbols' roles (Gray &Tall, 1994). Cotrill (1996) suggested a review of the notation used to represent functions and that functions should be presented as both arrows and relations between variables that should be used to illustrate the complementary roles of functions, such as $x!$ (read as x factorial) and variable expressions, such as x_2 (implying the second entry of x). He further asserted that while many concepts were best described in functional notation, many calculations were best done in variable notation and that while the arrow diagrams provided a good way to introduce and work with the concepts early, relations between variable expressions should not be neglected because many difficult calculations are more amenable to those techniques.

Dubinsky and Harel (1992) presented an exhaustive review of research papers which focus on student conceptions of function, understanding the notion of function, algebraic and graphical functions and the use of pedagogical software in understanding the functions. They further asserted that functions can be viewed either in terms of the relationship between dependent and independent variables, using graphs to represent functions or a special kind of correspondence between two sets.

Tall (2004) distinguished different modes of mathematical thinking, through the theory of the three worlds in mathematics emphasizing (1) the construction of mental representations of concepts that emerged from several theories on concept development, such as Sfard's (1991) work on encapsulation of processes to objects and (2) Piaget's abstraction theories (Tall, 2004). He described these worlds as hierarchical since there is a development from just perceiving a concept through actions to formal comprehension of the concept. He identified the following: (1) the 'embodied world' where individuals use their physical perceptions of the real world to perform mental experiments to build mental conceptions of mathematical concepts e.g. students' explorations of intuitive perceptions of limits of functions, (2) the 'proceptual world', where individuals start with procedural actions on mental conceptions from the first world, which by using symbols become encapsulated as concepts, and (3) the 'formal world' where properties are expressed with formal definitions as axioms. Nevetheless Tall (2004) found that concept images change on account of outer and inner stimuli, such as discussions, thoughts and problem solving, and that there is no static model that can constantly describe students' concept developments of limits of functions.

2.3.2 The action and process views of functions

Dubinsky and Harel, (1992) chose to view functions as actions or process conceptions. Here actions would be referring to manipulations required to obtain the value of the function from its definition. In a study that they conducted with undergraduate students, they found that students possess the following restrictions about what a function is, (1) *manipulation restriction* occurs when one is unable to perform manipulations or you do not have a function, (2) *quantity restriction* occurs when the inputs and outputs must be numbers and (3) *the continuity restriction* that a graph representing a function must be continuous. Due to the complex nature of the concept of function, they found that the students' prior experiences with specific situations that involve functions and the level of their abstraction were essential to construct the function schema. They further added that students start with an action conception of function prior to moving to process and object.

Dubinsky (2010) asserted that an individual requires an explicit mathematical expression before she/he considers the presence of a function at an action understanding stage. The only transformations that could be performed consisted of substitution of numerical values or other expressions for variables in the expression and calculating and simplifying. Dubinsky further asserted that it is after such actions and reflection on them, that an individual may construct mental structures abstractly in the process conception of the function. This led to a situation where the individual reverses and transforms operations on functions leading to conception of their inverses and composition of two or more functions. The coherence might then lie in the understanding that to have a function, there must be a domain, a range set, and a means of transforming elements of the domain set to elements of the range set.

2.3.3 The importance of the function concept

Functions occur throughout mathematics and are used in very diverse ways, and much has been written on learning and teaching this concept. Cotrill (1996) asserted that functions were a central part of the pre-calculus and calculus curriculum and that three representations for functions were frequently presented: algebraic, graphic and numeric (tabular). While Thompson (1994) presented valid criticism of this presentation, there was also evidence that working with students to develop connections between these representations had helped students to understand the function concepts (Romberg et al, 1992).

Carlson & Jacobs (2000) asserted that functions are fundamental and foundational for understanding major concepts in advanced mathematics including calculus. It is therefore important for students to understand both the symbolic manipulations and procedural techniques of functions to comprehend a mapping of input values to a set of output values. Carlson, Jacobs, Coe, Larson, & Hsu (2002) saw a function as an entity that accepts input and produces output enabling reasoning about dynamic mathematical content and scientific contexts. Carlson et al (2002) suggest that algebraic and procedural methods of function should be connected to conceptual learning, such that students would be better equipped to apply their algebraic techniques appropriately in solving problems and tasks.

Students needed to understand functions as general processes that accept input and produce output. It was important that they attend to the changing value of output as the independent variable varied through an interval in the domain. Also understanding limits and continuity requires a student to make judgments about the value of a function on intervals on the graphs. Definitions of derivatives, integral functions, the relationships between average and instantaneous rates of change, graphical analog between secant and tangent lines and many other topics in calculus all require students to have a clear understanding of the concept of a function.

2.3.4 Functions and properties

With regard to a property-oriented approach to function concept Kieran (1990), (1) noted that (1) a function can be described with reference to its local and global properties and (2) the study of properties is fundamental in order to characterize classes of functions. A property-oriented approach dealt with learners' ability to establish connections between different function representations (Monk & Nemirowsky, 1994). Different features of visual and symbolic representations can bring learners to different possibilities of the interpretation of functions. According to Slavit (1997) learners frequently used either approach based on the consideration of a real correspondence (*action view*- repeatable physical or mental manipulation that transforms mental or physical objects to obtain other objects, *operational view*-looking at a given notion as referring to a certain process rather than to an object) or property-oriented approaches. Nonetheless it has been observed that there were no studies proving whether a property-oriented approach effectively improves the development of an object-oriented conception of function (Slavit, 1997).

Functions are used in the comparison of abstract mathematical structures, for example in calculus, $y = (\sin x)^4$, is a typical trigonometric function relevant to first year differentiation. Selden and Selden (1994) asserted that the function concept, having evolved with mathematics, now plays a central and unifying role. Thus, the importance of attaining a broad understanding of the function concept is greater than might be apparent from considering the use of functions in a standard beginning calculus course.

Brijlall and Maharaj (2009b) in their analysis of students' constructions of the concept of continuity of a single-valued function assert that on perceiving functions as mathematical entities, students could manipulate these entities, which were understood as a system of operations. They further asserted that verifying and refining the construction of the continuity concept required conceptualization of the concept of continuity as a meaningful mathematical entity. This conceptualization enabled formulation of new mathematical ideas which can be applied in a wider range of contexts. Indeed conceptualization of the function concept enables understanding of the composition of functions, which can be applied in a wider range of contexts, including the chain rule.

2.4 Students understanding of functions

Naidoo (1996) noted that (1) first year mathematics students studied by rules, (2) the students did not enjoy mathematics, (3) the students were de-motivated and (4) lecturers tended to teach mechanistically and do standard type solutions to standard type problems. Thompson (1994) said that it may be incorrect to focus on graphs, expressions or tables as representations of function, rather functions should be seen as representations of something that is representable, such as aspects of a specific situation from students' perspective, with context based functions. He further claimed that if students do not see something remain the same as they move among different representations, then they see each representation as learnt in isolation.

Akkoc & Tall (2003) contrasted the mathematical simplicity of the function concept that is appreciated by some students with the spectrum of cognitive complication that most students have in coping with function definition in its many representations. They distinguished categories of students who have a simple grasp of the core function applicable to the full range of representations to those who see only complicated details in different contexts without any grasp of the conceptual structure.

Tall (1996) has noted that (1) the function concept is the underlying concept for the whole mathematics curriculum, from primary school through to university and (2) that this might only be possible for an expert who is able to see the role of function concept throughout the whole of mathematics. However, the story is different for a student. A student needs to construct new ideas on previously constructed mathematical concepts. For instance, students first meet functions in the form of a linear assignment for example $y = 2x + 1$ where the value of y is found from x by doubling x and then adding one. By having such experiences, students develop conceptual understanding which identify functions as formulas in which values of x are entered to calculate the value of y (Akkoc & Tall, 2005).

Tall (2004), in his theory of the three worlds emphasized (1) the construction of mental representations of concepts that emerged from several theories on concept development, such as Sfard's (1991) work on encapsulation of processes to objects and (2) Piaget's abstraction theories (Tall, 2004). Uygur (2010) in his study on cognitive development of applying the chain rule through the three worlds of mathematics noted that to progress in cognitive development of the chain rule, the formulas of the chain rule in the function and the Leibniz notations should be related in the symbolic world and that this relation should be embodied. Incorporating the theory of the three worlds in mathematics, (Tall, 2004), this study designed a 'chain rule' model highlighted in Figure 2.1.

According to Dubinsky (1991), understanding a concept has to do with relations between the mental constructs together with the interconnections that an individual uses to understand a concept, and the way in which an individual uses (or fails to use) them in problem situations. Dubinsky (1991) indicated that an understanding of functions as objects and as processes is necessary for understanding function composition beyond evaluating each function at specific points with a formula. For example to understand a function such as $f(x) = \cos x$, one needs a process conception of associating a real number x with its cosine function value.

Embodied World

Experimenting with the relationships in derivatives of, for example, $f(x) = \sqrt{x}$, $g(x) = 2x^3$ and thus $h(x) = \sqrt{2x^3}$.

Proceptual World

Procedural action leading to an algorithm, for example, the chain rule.

Formal World

Proving the chain rule by first principles or deducing use of $\frac{d}{dx}$ in the Fundamental Theorem of Calculus

Figure 2.1: A ‘chain rule’ model

Calculus is the mathematical foundation for much of the science, mathematics, and engineering curriculum at a university. The students’ starting point for building differential calculus is their knowledge of algebra. Thus the algebraic objects which include various functions contained in trigonometric expressions to be differentiated in a mathematical activity are the building blocks. Now the student has to process these functions and differentiate them one by one to give the required derivative. It is therefore appropriate to suggest that this procedure could be described as actions, which according to Dubinsky (1991a), are step-by-step procedures related by routines external to the mind of the subject. It is imperative though, that procedures learnt with meaning are those linked to conceptual knowledge-concept image and definition which is everything associated in somebody's mind related to mental pictures, properties, mental representation, contexts of applications and even statements (Hiebert & Lefevre, 1986). The above implies that understanding students and how they learn is a priority and by no means an easy endeavour.

Tall et al (2000) agreed (1) to the complexity of the function concept and (2) to explore a function machine as a cognitive root for it which embodies both its process-object duality and also its multiple representations. They thought that the function machine already had iconic, visual aspects, embodying both an object-like status and also the process aspect from input to output. The usual representations of function (table, graph, formula, procedure, verbal formulation, etc) could also be seen as ways of representing or calculating the input-output relationship. In developing their theory, they noted that the function concept itself is rarely a concept of study. Instead, the term "function" usually applies to a special kind of function, e.g. linear, quadratic, trigonometric, given by a formula and differentiable functions. They refer to such concepts as "function plus", where the "plus" refers to the relevant additional properties which significantly change the nature of a function. (For instance, a linear function only requires two distinct pairs of input-output values to determine its defining rule uniquely).

Zandieh (2000) viewed the concept of derivative in three layers; ratio, function and limit. However, she viewed the limit and function as process-object pairs. In describing the process –object framework, she indicated that the underlying structure of any representation of the derivative concept could be seen as a function whose value at any point x is the limit of the ratio of differences $f^{'}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

These three layers can be viewed as dynamic processes and as static objects. Also, each of these layers can be represented graphically, verbally physically and symbolically. When a student lacks a structural conception of one of the layers the pseudo-structural term is used to describe an object with no internal structure.

Likwambe & Christiansen, (2008) extended Zandieh's framework which only dealt with perceived connections between the various components of the concept image of the derivative existing for an individual. Likwambe & Christiansen, (2008) included the exposure of strong concept images that exist in connections across layers and representations of the derivative concept. They therefore expanded Zandieh (2000) to include instrumental understanding where they argue that some learners master rules or procedures without any insights or reasons that make the rule work. The instrumental understanding serves to explain the ability to use a rule. Thus Likwambe & Christiansen, (2008) used Zandieh's framework as a starting point and added three aspects namely (i) instrumental understanding, (ii) reflection of connections among representations and (iii) a non- layer added to reflect situations where the responses they got could not be classified in any of Zandieh's three layers.

2.4 Composition of functions

The chain rule is used to find the derivatives of composite functions. Kaplan (1984) referred to it as a function of functions. A composite function is a function that is composed of two or more functions. For the two functions f and g, the composite function or the composition of f and g, is defined by $(f \circ g)(x) = f(g(x))$.

The function $g(x)$ is substituted for x into the function $f(x)$. For example, the function $h(x) = (3x-9)^4$ could be considered as a composition of the functions, $f(x) = x^4$ and $g(x) = 3x - 9$. However, it could also be written as a composition

$f(x)=(3x)^4$ and $g(x)=x-3$. Often, a function can be written as a composition of several, different combinations of functions. One must be careful to consider the domain of the respective functions.

The chain rule allows us to find the derivative of composite functions. The chain rule states that if f and g are differentiable functions and $F(x)=f(g(x))$, then F is differentiable and the derivative of F is given by $F^{'}(x)=f'(g(x))g'(x)=(fog)'(x)$ In Leibniz notation, if $y=f(u)$, $u=g(x)$ and y and u are differentiable functions, then $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$. Kaplan (1984) chose to call this rule, the composition rule since the function to be differentiated is a composition of other functions. The same applies when a function is a product we use the product rule to get its derivative. The first year syllabus deals with a combination of a maximum of five functions that can be used in the composition. We can have more than one composition in a problem. The students should now be able to decompose the given function into its elementary pieces one step at a time. Kaplan then proposed the following table of derivatives with all possible compositions of functions. All of the formulas in the table were derived from the general chain rule with *f(x)* as one of the main functions, $x^n; e^x; \ln x; \sin x; \cos x$ and an arbitrary function *g(x)*.

Power rule	$\frac{d}{dx}[g^n(x)]=n.g^{n-1}(x).\frac{d}{dx}[g(x)]$
Exponential rule	$\frac{d}{dx}[e^{g(x)}]=e^{g(x)}.\frac{d}{dx}[g(x)]$
Natural Logarithm rule	$\frac{d}{dx}[\ln g(x)]=\frac{1}{g(x)}.\frac{d}{dx}[g(x)]$
Sine rule	$\frac{d}{dx}[\sin g(x)]=\cos g(x).\frac{d}{dx}[g(x)]$
Cosine rule	$\frac{d}{dx}[\cos g(x)]=-\sin g(x).\frac{d}{dx}[g(x)]$

Table 2.1: Rules for differentiation of functions

The chain rule is of important use to other areas of calculus. These include: (1) Finding the marginal Physical Productivity Function of the workers in Business economics ($\frac{dP}{dx}$) for $P = 10(3x+2)^3 - 10$, (2) Revenue changing when given a revenue function like $R(x) = 25(x+2)^2 + 20x - 5$, (3) Higher order differentiation used to calculate demand, cost and profit in business and (4) Calculations of rates on physical body relationships including body weight and surface area, cell growth, blood flow and other physical quantities. It is important for the students at this stage to know which formula to use and how to use it without computing the derivatives of the component functions. They must be able to identify whether a constant times a function, sum of functions, product, quotient, composition or piecewise functions are given in the problem. The implication here is that they should be well versed with function algebra.

2.6 Misconceptions about the chain rule

Oehrtman (2002) said that learning new ideas does not necessarily obliterate old ones, so students may often retain early misconceptions alongside more acceptable, subsequently developed interpretations. This relates to Piaget's concept of accommodation and assimilation. *Accommodation* is Piaget's term for modifying existing concepts or adoption of new ways of thinking in order to encompass new information, (Mwamwenda, 1989). Assimilation is defined as the process of incorporating new information to fit existing categories or ways of thinking. Thus, when students present incorrect conceptualization of the chain rule for example, they do not necessarily lack the correct one. Rather the issue is often the selection of which idea (or combination of ideas) to retrieve. The chain rule is one of the hardest ideas to convey to students in calculus (Gordon, 2005). It is difficult to motivate, such that most students do not really understand where it comes from. It is difficult to express in symbols even after it is developed, and it is awkward to put into words, so that many students cannot remember it, and so cannot apply it correctly.

We recall that Swanson (2006) asserted that the complexity of the chain rule deserves exploration because students struggle to understand it and because of its importance in the calculus curriculum as indicated previously on page six. Despite the importance of the chain rule in differential calculus and its difficulty for students, the chain rule has been scarcely studied in mathematics educational research (Clark et al, 1997; Gordon, 2005; Uygur & Ozdas, 2007; Webster, 1978). Students' difficulties included the inability to apply the chain rule to functions and also with composing and decomposing functions (Clark et al, 1997; Cottrill, 1999, Hassani, 1998). It cannot be disputed that even at the University of Technology students experience most problems in differential calculus.

Previous research emphasized the importance of function composition in the understanding of the chain rule, but the ability to say more than that has been elusive (Clark et al, 1997; Cottrill, 1999; Hassani, 1998). Hassani, (1998) in his research into the nature of students' understanding of the concepts underlying calculus has shown significant gaps between their conceptual understanding of the major ideas of calculus and their ability to perform procedures based on these ideas. The results showed that first-year undergraduate calculus students' ability to explain or apply the chain rule is significantly related to their algebraic manipulative skills and their general knowledge of function concepts and function composition.

This exploratory study was therefore designed to focus on how students understand function composition as seen through the chain rule problems using functions that are familiar, somewhat familiar and unfamiliar to them. Examples of familiar functions include polynomials and trigonometric functions. Exponential and logarithmic functions are somewhat familiar to students because they have experienced their use outside the calculus paradigm and they have had no direct experience yet in calculus. Unfamiliar ones are the compositional functions, for example, $y = \ln\left(\tan(x^2)\right)$.

The inherent difficulty of the derivative concept experienced by many students (Uygur & Ozdas, 2005) and the students' difficulties that increase and get worse (Tall 1993) becomes clear, especially when the function is a composite function. Students experience problems in with the use of the Leibniz notation, $\frac{dy}{dx}$, whether it is a fraction or a single indivisible symbol. Although this notation is indispensable in the calculus, it remains as a serious misconception. Tall further associates this difficulty regarding use of the chain rule with the dilemma of whether the du can be cancelled in the equation: $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$. A small amount of evidence supporting the hypothesis which states that, the understanding of composition of functions is the key to understanding of the chain rule emerged in the study by Cottrill (1999). His study investigated the correlation between a student's understanding of the composition of functions and understanding the chain rule.

2.7 Designing classroom instruction

Burke, Erickson, Lott & Obert (2001), assert that there is growing research support for designing classroom instruction that focuses on developing deep knowledge about mathematics procedures. When instruction is focused only on skillful execution, students develop automated procedural knowledge that is not strongly connected to any conceptual knowledge network (Star, 2000). This instruction resulted in procedures not executed "intelligently" and systematically. Understanding could be achieved, however, if students were given opportunities to develop a framework for understanding appropriate relationships, extended and applied what they knew, reflected on their experiences, and made mathematical knowledge their own (Carpenter & Lehrer, 1999). They further asserted that (1) when mathematical knowledge is understood, that knowledge is more easily remembered and more readily applied in a variety of situations (Hiebert & Carpenter, 1992; Kieran, 1992), (2) when a unit of knowledge is part of a well-connected network of mathematical understandings, parts of the network can facilitate recall (and even recreation) of other parts and (3) when knowledge is understood it becomes easier to incorporate new knowledge into existing networks, so that current understanding facilitates future learning (Hiebert & Carpenter, 1992). It is therefore important to develop teaching methods that help students develop mathematical understanding.

When students deal with functions, they need to understand relationships and mathematical connections between them as composite functions, and this knowledge can be easily incorporated into future learning of the chain rule. Kendal (2001) used Computer Algebra Systems software to (1) identify teachers' privileged characteristics and (2) facilitated analysis of learning in relation to the teaching of derivatives. He found that a conceptual teaching method and student-centered style supported the development of the understanding of the concept of derivatives. Felder (1996) argued that it is good for educators to be aware of the importance of learning styles to encourage a more flexible and empowering approach to diverse learning contexts. Studies by Claxton & Murrell (1987) suggested, that (1) helping the students to expand their repertoire of learning ways, assists students to take more responsibility for their own learning and (2) they become actively involved in the learning process and hopefully thereby build a positive attitude towards lifelong learning.

Vygotsky (1986) noted that the possibilities of genuine education depend both on the knowledge and experience already existing within the student (level of development) as well as on the student's potential to learn. Engelbregcht, Harding & Potgieter (2010) interpreted this as approaching mathematics from a conceptual system rather than as a collection of discrete procedures. Students used conceptual understanding of mathematics when they identified and applied principles, knew and applied facts and definitions and compared and contrasted related concepts.

Clark, et al (1997) who studied students' understanding of the chain rule and its applications concluded that the difficulties with the chain rule for a large number of students could be attributed to student difficulties in dealing with the composition and decomposition of functions. This hypothesis was confirmed by Cotrill (1999) in his study of correlation between a student's understanding of composition of functions and understanding the chain rule; that understanding of the composition of functions is the key to understanding the chain rule. Both studies conducted in an American context and using computer programming to help students to make relevant mental constructs indicated a need for further research. This study is further research on how students understand the concept of the chain rule.

The chain rule is included in several studies in mathematics education literature. Some of them are about teaching of the chain rule (Lutzer, 2003; Mathews, 1989; Thoo 1995; Uygur & Ozdas, 2007) while others are on understanding the rule. Uygur (2010) who studied the cognitive development of applying the chain rule through the three worlds of mathematics suggested that the instructional way of presenting the chain rule changed focus to encourage students to obtain the chain rule with some life-related problem situations. In contrast, verifying the chain rule by using either or both graphing software or graphics calculator and an algebraic approach was considered for developing teaching and learning strategies of the chain rule in the mathematics teaching program of South Australia (SACE Board of South Australia, 2009). Uygur (2010) further noted that as much as there was an absence of studies on structural development of the chain rule, there was also a need for a study on students' applying the chain rule to second order derivatives and to two-variable composite functions. It was noted also, that the prerequisite knowledge of composite function is another significant notion for applying the chain rule by raising awareness of the relation among various cases. Uygur inferred that variable notion is another significant prerequisite knowledge in the embodied world of the cognitive development of the chain rule. Novotna and Hoch (2008) had indicated the importance of structural knowledge in applying the chain rule in the cognitive development of mathematical concepts. Students' application of the chain rule was analyzed within Tall's (2007) framework containing three levels of understanding which considered symbolic development. Their study addressed the structural development of the chain rule. On the contrary this study focused on the discussion of the types of structures constructed by students when learning the chain rule with the view to clarifying their understanding: (i) of the composition of function and (ii) of the derivative.

Anderson et al (2001) suggested a refined form of Bloom's taxonomy where he described some cognitive processes that are likely to occur in a mathematics classroom. These processes were categorized by Anderson et al (2001) as: (1) Remember, (2) understand, (3) Apply, (4) Analyze, (5) Evaluate and (6) Create. He further outlines the category of remember as having two sub-categories, namely, recognising and recalling. The category of understand is about processes of constructing meaning from instructional messages, including verbal, written and graphical representations. There are seven subcategories attached to understand, namely, interpreting, exemplifying, classifying, summarizing, inferring comparing and explaining. The category of apply concerns the processes of carrying out or using a procedure in a given situation with subcategories, executing and implementing. The category of analyze deals with the processes of breaking down material into its constituent parts and determining how the parts relate to each other and to an overall structure and purpose. It has three subcategories, namely, differentiating, organising and attributing. The category of evaluate concerns the processes of making judgements based on criteria and standards, where checking and critiquing are involved. Lastly the category creates deals with putting elements together to form a coherent or functional whole; re-organising elements into a new pattern or structure with subcategories, generate, planning and producing. Remembering encompasses retention of knowledge whilst the other categories mentioned are related to furthering and transferring knowledge.

Maharaj (2010) asserted that the main use of an APOS analysis is to point to possible pedagogic strategies. This is true even though explanations given by APOS are limited to descriptions of the thinking which an individual might be capable of making. This does not explain what is happening in an individual's mind at a particular instance. There is also no guarantee that an individual possessing a certain mental structure will access it and apply it when necessary in a given situation. Classroom instruction should therefore be designed so as to foster the students' development of mental structures called for by the APOS analysis of the chain rule suggested in the chapter three. The pedagogical approach based on APOS theory and the hypothesis on learning and teaching, is a repeated cycle consisting of three components: (1) Activities, (2) Classroom discussions and (3) Exercises. The activities, discussions and exercises in this study followed after a formal lecture instruction given in the three sequential lessons.

2.8 Learning mathematics

Various frameworks on how students learn mathematics have been suggested. Conceptual and procedural learning of mathematics have been addressed by for example, Piaget cited in (Baker and Czarnocha, 2002). Engelbrecht, Harding and Potgieter, (2010) argued that in the learning process, the process of understanding is an ongoing process that converges to full understanding but does not reach the limit of full understanding. He further asserts that the dynamic process of understanding new mathematics takes place in layers in which with every layer the student understands deeper. He claims that the students have got to expose themselves repeatedly to gain deeper understanding of a particular concept. He sees mathematics learning as consisting of two processes namely (1) first time exposure and (2) consolidation process. He believes that doing more problems of a certain type brings repeated exposure and deeper understanding. This study defines learning in mathematics according to APOS proposed by Dubinsky (1991). These repeated exposures would be necessary for students operating in the action stage regarding understanding the chain rule concept.

2.9 Recent South African studies using APOS

To come up with the theoretical framework, I was motivated by literature that used APOS in studies in South Africa. Below I discuss some of those studies.

Brijlall and Maharaj (2009a) used the APOS theory in a study where they investigated fourth-year undergraduate teacher trainee students' understanding of the two fundamental concepts monotonicity and boundedness of infinite real sequences. This was research done at the Edgewood Campus of the University of KwaZulu Natal in South Africa. They designed worksheets based on examples and non-examples approach to foster collaborative learning. They (Brijlall and Maharaj) were interested to find out how the implementation of a structured worksheet design using APOS theory, to promote collaborative learning, influence the construction of concepts in real analysis.

An examples and non-examples approach discussed by Cangelosi (1996) was used in the structured design. They focused on (1) sorting, (2) reflecting and explaining, (3) generalizing, (4) verifying and refining, and (5) extension of generalization. Their method of data collection involved a few stages: (1) design of worksheets, (2) facilitation of group-work, (3) capturing of written responses and (4) interviews. Worksheets were designed to allow students to talk about their thoughts and support each other in constructing new mathematical knowledge. The worksheets were designed in accordance with ideas postulated for a guided problem solving linear model suggested by the work of Cangelosi (1996). The model had three levels captured in Figure 2.2. (1) inductive reasoning, (conceptual level), (2) inductive and deductive reasoning (simple knowledge and knowledge of a process level) and (3) deductive reasoning (application level).

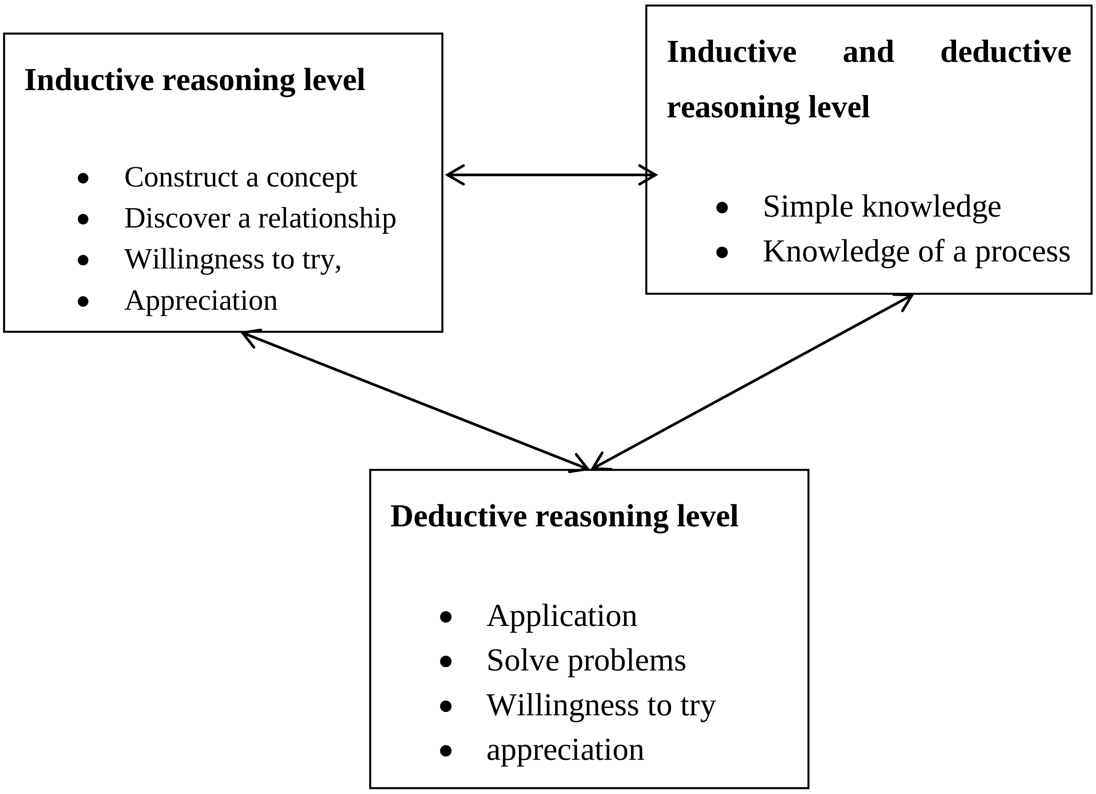


Figure 2.2: Guided problem-solving linear model

In conclusion, Brijlall & Maharaj (2009b) found that: (1) the structured worksheets encouraged group work and fostered an environment conducive to reflective abstraction, (2) the students demonstrated the ability to apply symbols, language, and mental images to construct internal processes as a way of making sense of the concepts of monotonocity and boundedness of sequences, (3) the students could apply actions on objects (sequences) which were interiorized into a system of operations, and (4) the conceptualization of the concept of boundedness of sequences and monotonocity enabled the formulation of new schema which could be applied in various contexts.

Engelbrecht, Harding & Potgieter (2010) investigated students' performance and confidence in the conceptual and procedural skills of first year calculus students. They asserted that conceptual understanding consisted of those relationships constructed internally and connected to already existing ideas. In their discussion of comparisons of the relationship between procedural and conceptual thinking, they claim that students' conceptual knowledge will necessarily increase procedural proficiency. Reference is made in this study to Andersons (1995)'s model of learning. Anderson asserted that, (1) learning begins with actions existing on conceptual knowledge and (2) the student begins to internalize the procedures involved, leaving aside the conceptual knowledge from which the procedures arose. Byrnes and Wasik (1991) cited in this study noted that for Piaget, after the student has gained proficiency in procedural knowledge, a process of reflection begins resulting in acquisition of new conceptual knowledge. Baker and Czarnocha (2002) argued that in the Piaget model procedural efficiency is a requirement for meta-cognition and conceptual thought. This was contrary to what culminated out in their study. They found that conceptual thought is independent of an individual's ability to apply his or her procedural knowledge, supporting Vygotsky (1986)'s view that the development can proceed through reflection upon existing conceptual knowledge independently of the reflection due to repeated actions.

Engelbrecht et al (2010) noted that Dubinsky recognized the distinction of conceptual knowledge to procedural proficiency and introduced the idea of processes being encapsulated as objects. They describe the features of APOS model as representing an increasing level of learning. They define a schema as when objects and processes from more than one area can be combined in more than one way. Their study indicated that students performed better in conceptual problems than procedural problems. The general opinion that 'doing' was easier than 'thinking' was disputed. They emphasized the thought that the teaching approach must be directed to cultivating conceptual thinking.

Brijlall and Maharajh (2010) explored APOS to develop insight into pre-service mathematics students' mathematical reasoning about aspects of the derivative concept. The study tried to address the question: how is the concept of continuity understood by pre-service mathematics students or does the continuity concept becomes for each of them a complete mathematical object? Ideas of student centeredness, collaborative learning and self discovery were used to strengthen the qualitative nature of the study. APOS theory guided the study in finding whether pre-service mathematics students use different interpretations of the concept of continuity correctly and effectively in their reasoning. APOS was used specifically to establish whether these students were able to construct a coherent view of continuity or if they presented different separate interpretations as pieces of knowledge.

APOS theory offered direction and became the basis for generalization (Brijlall & Maharajh, 2010) and provided the researchers with an opportunity to develop the genetic decomposition of the continuity concept. Pedagogy based on collaborative instructional design worksheets helped the students to make mental constructs. They further assert that the important aspect in preparing student teachers for mathematics education is to ensure that the students have the necessary content knowledge of the derivative concept. They also found that the instructional treatment and methods used in their study had a positive influence on the student teachers' derivative schema development. This echoed what Maharajh, Brijlall & Govender (2008, p104) noted that' the explicit pedagogic strategy ensured that most student teachers realized that to make sense of the visual tasks presented on the design worksheets, they needed to 'dissect' the tasks and bring the previously leant ideas in small 'packets' to find a solution. Results in their study indicated that the participants had good potential for learning the concept of continuity when they are allowed to collaborate in small groups using their tools to think with. Brijlall and Maharajh (2010) claim that the structured worksheets encouraged group work which fostered an environment conducive to reflective abstractions.

Bowie (2000) analyzed students' errors through the lens of existing theories in order to begin to build a picture of the processes students use to build knowledge. Her study explored the errors that students made in the special mathematics course at the University of Cape Town in 1996. She also claimed a duality of mathematical concepts as both 'processes' and 'objects' was central to the work on advanced mathematical thinking over the past two decades (Dubinsky (1991a; Harel & Kaput, 1991; Sfard, 1987; Dubinsky & Harel, 1992; Tall, 1991). She agreed that the various proponents of these theories differ in their details of approach, but claims that their theoretical perspective is similar.

She then offered an outline of Sfard's reification theory and Dubinsky's APOS theory. These are discussed in detail in the following chapter under construction in reflective abstraction. 'The learning theories of Sfard and Dubinsky highlight the importance of both an operational and structural understanding of mathematical concepts, arguing that encapsulation of processes into objects is an important step in being able to build further mathematical knowledge.' She thus argued that the students' errors in algebra demonstrate a pseudo-structural approach to algebra. The modes of construction employed by the students resonated with the five kinds of construction which explain how new objects, processes and schemas can be built from existing ones, as outlined by Dubinsky (1991a).

Bowie, (2000) define pseudo-objects as the algebraic 'objects' that students employ as their building blocks that seem totally arbitrary. She then associated these with *actions*, which in the sense that Dubinsky uses them are step-by-step procedures with the steps only related by routines and not by any relationships that exist in the mind of the subject. She then claims that an action that works in a number of cases is rehearsed into a rule and many correct rules emerge from the process. A rule is applied as a mechanistic manipulation of pseudo-objects according to Bowie (2000), and differs from a process, in that the latter is given meaning through a familiarity with the objects on which it is performed. She also found that concatenation, reversal and overgeneralization were three strategies that students used to reduce the number of rules required. Concatenation represents the grouping of concepts on the basis of surface level cues, (p10). In conclusion, she suggested that (1) learning material that fosters the development of an object conception of a mathematical concept must be continually developed, and (2) a learning environment that encourages students to persevere with the intellectual effort that is required must be provided.

APOS theory used in a study by Maharaj (2010) culminated in a proposed genetic decomposition of students' understanding of the limit concept in calculus. His study focused on: (1) how the teaching of the limit of functions should be approached, and (2) the insight that an APOS analysis of students' understanding of the limit of a function would reveal. In addition to the assumption made by Asiala (2004) cited and quoted in Jojo et al (2011, p338), Maharaj (2010) noted the hypothesis on learning, that an individual does not learn mathematical concepts directly. This is drawn from Piaget (1964) who asserted that the individual applies mental structures to make sense of a concept. Maharaj claims that learning is facilitated if the individual possesses mental structures appropriate for a given mathematical concept, and that if the appropriate mental constructs are absent, learning the concept becomes almost impossible.

Maharaj (2010) detected the following genetic decomposition relevant to both the limit of a function concept and types of limit problems encountered by participants in his study. At an action level a student confronted with the limit of a function, $\lim_{x\to a} f(x),$ can do little than substitute values of *x* close to *a*, for the variable in the expression *f(x)* and manipulate it, and may or may not see the pattern emerging. A process understanding of the function $\lim_{x\to a} f(x),$; emerges as a student constructs a mental process for values of *x* close to *a* and thinks in terms of inputs, possibly unspecified and transformations of those inputs to produce outputs. At the object level, the student sees the string as a totality and can perform mental or written actions on one-sided limits of given functions, the process understanding is encapsulated and converted to an object, $\lim_{x\to a} f(x),$ which may or may not exist. At the schema level, the actions, processes, and objects are organized and linked into a coherent framework. The framework includes possible techniques for evaluating $\lim_{x\to a} f(x),$ where *a*, could be ∞ or $-\infty$.

Data were analyzed based on APOS using the above genetic decomposition. The findings in the study included the following: (1) the limit of a function concept is one that students found difficult to understand suggesting why many students could not produce appropriate mental constructions suitable for the process, object or schema levels, and (2) the teaching design needs to be given more time. The teaching process according to Maharaj (2010) should focus on: (1) verbal and graphical approaches to finding limits of functions, (including split-functions in symbolic form) (2) the unpacking of structures given in symbolic form, and (3) modeling possible schema.

All these studies indicated that more work needed to be done in exploring APOS as a theory relevant to analyzing mental constructions that could be made by students in learning various concepts in calculus. Also pedagogic methods are emphasized in each study as the main focus in prompting the relevant mental constructs. Many other studies done exploring APOS theory in the understanding of different concepts have been read. No study focused on analysis of the understanding of mental constructions made when learning the chain rule in calculus. This study seeks to address that gap.

2.10 Implications of the literature review for this research

The concept of function is important to mathematics and not well understood by our students. These studies and those about which they report were searched for evidence of explorations of composition of functions and for discussions of the chain rule. While many of the studies of students' understanding of function relate to this study, little was found (Cotrill 1996; Clark et al, 1997 and Horvath, 2007); regarding students' understanding of composition of functions explicitly, and no studies were found on their understanding of the chain rule in the Southern African countries.

It is evident from the above discussion that, many well- known functions have simple expressions for their derivatives while composite functions require the use of the chain rule for differentiation. Functions having fairly complicated expressions have explicit formulas for derivatives. It was the development of formulas and rules such as the chain rule enabling mathematicians to calculate derivative that motivated the use of the name calculus for this mathematical discipline.

2.11 Conclusion

In this chapter the researcher provided an overview of literature on functions, their properties, understanding of functions as concepts and the importance of instructional design in general. Following this discussion, it became clear that for current understanding to facilitate future learning, new knowledge should be incorporated into existing related networks. Key aspects relating to composition of functions have been explored. The chapter concluded by reflecting on the implication of the literature reviewed for this study. Chapter three focuses on the theoretical framework informing this study.

CHAPTER 3

THEORETICAL FRAMEWORK

3.1 Introduction

In this chapter the framework used in this study that was developed through attempts to understand the ideas of Piaget concerning reflective abstraction, is discussed. In the previous chapter, literature informing this research was introduced and discussed. This chapter begins with an initial theoretical analysis focussing on: (a) what it means to understand a concept, (b) how understanding of a concept can be constructed by a learner, (c) a description of schema, and (d) the statement of the initial genetic decomposition.

A description of the specific framework for research and its components, according to which this study has been conducted is outlined together with previous research conducted using the APOS theory. Next, the researcher discusses the concept of reflective abstraction as proposed by Dubinsky (1991a) as a powerful tool in the study of advanced mathematical thinking. A description of the heart of the theoretical framework used in this study, the APOS theory, is presented together with an outline of previous research conducted using the APOS theory. Finally, because the APOS theory and Piaget's triad mechanism provided the theoretical framework for the presentation and analysis of data in chapters five, six and seven respectively, a detailed account of how APOS theory has emerged in this study is outlined.

3.2 Mathematical knowledge and its construction

Piaget's work concentrated on the development of mathematics knowledge with children in early ages, and rarely going beyond adolescence. Also, pedagogical strategies are almost absent from the totality of Piaget's work (Asiala et al, 2004). He suggested though that the same approach can be extended to more advanced topics in higher mathematics and beyond. Dubinsky (1991) suggested that usually it becomes necessary that the genetic decomposition in the original theoretical analysis is revised as a result of empirical data. He therefore believes that the incorporation of the triad concept of Piaget and Garcia (1989) led to better understanding of the construction of schema. The understanding of schemas as described in reflective abstractions was not adequate to provide a satisfactory explanation of the collected data on the genetic decomposition of the chain rule. This is revealed by (Clark et al, 1997), who report on students' understanding of the chain rule; Cottrill, (1999) on the chain rule and its relation to composition of functions and Baker, et al (submitted), on the relations between the graph of a function and properties of its first and second derivatives. The elaboration of a deeper understanding of schemas and better explanations of the data was explained by the introduction of the triad.

At the University of Technology, students do first year mathematics over a period of six months. This comes after spending a year or a semester being prepared in the foundation course. This allows more time for developing concepts and the prerequisite knowledge required for first year mathematics. This study also considers the fact that students are not taught the concepts of limits and continuity in full, which to me forms the basis of calculus. Also emphasis is placed on the use of standard formulae for derivatives without deriving them due to time constraints, and thus the notion of limits recede into the background. The researcher felt that the emphasis on exploring the use of chain rule with trigonometric functions and verbal representations of calculus concepts can be fostered through reflective abstractions following Dubinsky's (1991b) model of conceptual understanding. He believes that the concept of reflective abstraction that was introduced by Piaget (Berth & Piaget, 1966) can be a powerful tool to describe the development of the study of advanced mathematical thinking and that it could be used to analyze any mathematical knowledge applicable to higher education. This study has therefore adopted the APOS approach (Dubinky, 1991a), based on intuitive appeal as there has been little empirical research done documenting its impact on students' conceptual understanding of the chain rule in the African continent. Also, APOS has been used in research focusing on understanding of various mathematical concepts, (Pascual, 2004; Sfard, 1991; Tall, 1994; Dubinsky, 1991a; De Vries, 2001; Gray & Tall, 2002; Clark et al, 1997).

The argument is that this theoretical perspective has been very useful in attempting to understand students' learning of a broad range of topics in calculus, abstract algebra, statistics, discrete mathematics, and other areas of undergraduate mathematics (Dubinsky et al, 1991). Tall, (1999) agreed that APOS could be used to explain topics in calculus but would fall short in describing how students constructed concepts in Geometry. He argued that the sequence is not actions-process-objects-schema, but rather that geometry starts as an object based and in such a case there are processes involved (e.g. drawing, measuring, construction), and that those processes focus on gaining knowledge about the objects. The researcher is of the opinion that this view has to be explored in empirical research before coming to such conclusions. She further agrees though that the growth of knowledge in algebra and as such calculus begins with empirical abstraction and hence follows an APOS sequence. Tall (1999) also asserted that APOS theory had already shown its strength in designing undergraduate mathematical curricula but questions its universal applicability in the formal construction of knowledge from definitions to deductions in advanced mathematical thinking. He agreed though that APOS formed the basis of the only curriculum project that had a coherent cognitive perspective in the calculus reform and offers a major contribution to the analysis of various topics in advanced mathematics. The researcher used APOS to describe the observations of student behaviors when learning chain rule and to develop instruction to help students make the relevant mental constructions in developing the chain rule schema. She believes that it can provide a theoretical basis that supports and contributes to the understanding of how the students think and can suggest explanations of the difficulty experienced by students with mathematical concepts, including the chain rule.

3.3 Research Framework

This study was conducted according to a specific framework for research and curriculum development in advanced mathematics education, which guided the systematic enquiry of how students acquire mathematical knowledge and what instructional interventions contribute to student learning. The framework consists of three components: theoretical analysis, instructional treatment, and observations and assessment of student learning as proposed by Asiala et al (2004) and illustrated in Figure 3.1.

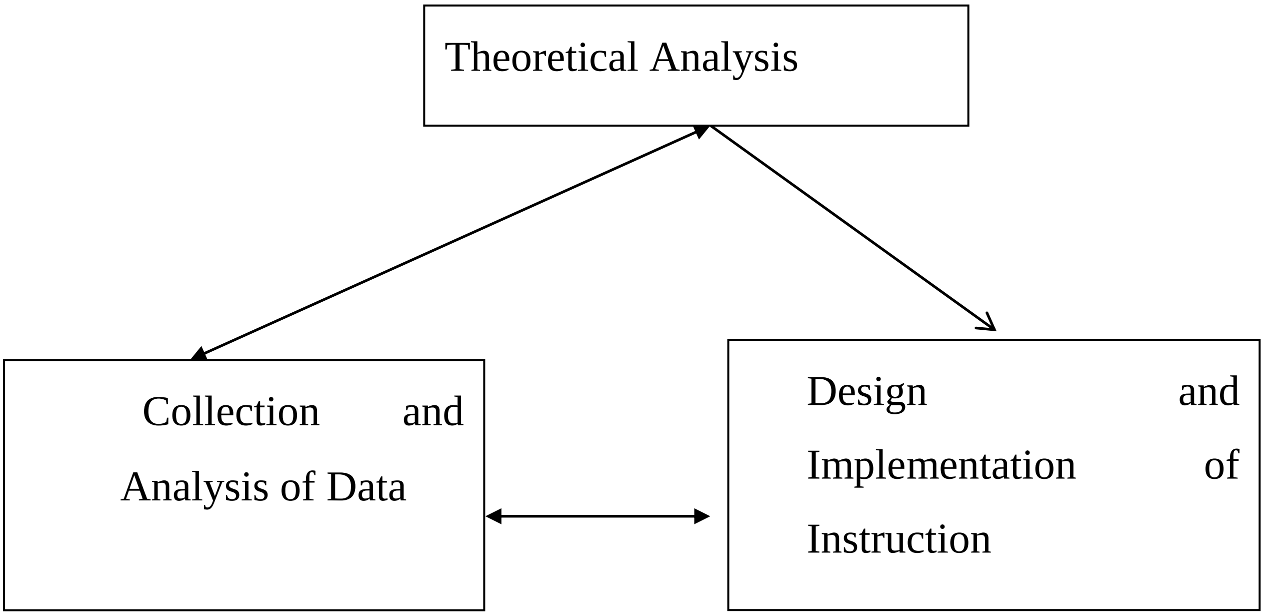


Figure 3.1: Theoretical Framework for Research

3.3.1 Theoretical analysis

The first step in this approach was to conduct an initial theoretical analysis of the chain rule concept relative to specific mental constructions that a learner made in order to develop his understanding of the chain rule. The result of this analysis was a model of change called genetic decomposition of the concept in this case, the chain rule, (Clark et al, 1997). The genetic decomposition defined what it means to understand the chain rule concept and how that understanding can be constructed by the student. This led to a design of instructional treatment that focused directly on trying to get students to make constructions in the initial genetic decomposition of the chain rule. The last step engaged implementation of instruction leading to collection of data, which was analysed in the context of the theoretical perspective. More complete description of the research framework used in this study can be found in Asiala, Brown, De Vries, Dubinsky, Mathews, and Thomas (2004). Hence we define the following APOS concepts for clear understanding of the genetic decomposition of the chain rule.

***Actions**- This refers to a repeatable physical or mental manipulation that transforms objects by reacting to external cues that give precise details on what steps to take. For example, for most students who interpret the idea of a function as contained in the 'formula' Dubinsky (1991a) for computing values, is restricted to the action concept of function.

***Process**- When the action is repeated and the student reflects upon it, an action that takes place entirely in the mind is internal, and may be interiorised as the process. With the process conception of a function, an individual can link two or more processes to construct a composition, or reverse the process to obtain inverse functions. It is expected that students attempting the chain rule at least work at this level.

***Objects**- This process is perceived as an entity upon which actions and processes can be made. The student here can reflect on operations applied to a particular process, becomes aware of the process as a totality, realizes those transformations and is able to construct such transformations. We say that the process has been encapsulated to an object.

*Schema- This is a collection of cognitive objects, their connections and internal processes for manipulating these processes. Schemas themselves can be treated as objects and included in the organization of higher order schemas. Asiala et al, (2004) asserts that an individual's schema is the totality of knowledge which for him is connected consciously or unconsciously to a particular mathematical topic, for example an individual may have a function schema, derivative schema, or a chain rule schema.

3.3.1.1 Understanding a mathematical concept

Hiebert & Carpenter (1992) asserted that learning mathematics with understanding involves making connections among ideas, and that those connections are considered to facilitate the transfer of prior-knowledge to novel situations. With regard to the psychological approach to learning, the constructivist idea is that understanding is a continuing activity of individuals organizing their own knowledge structures, a dynamic process rather than an acquisition of categories of knowing (Confrey, 1994; Gagnon & Collay, 2001; Pirie & Kieren, 1994). According to Bransford, Brown & Cocking, (2000), a mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. They further asserted that the degree of understanding is determined by the number and the strength of the connections made with previously acquired mathematics. Thus a mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. It is therefore assumed that well-connected and conceptually grounded ideas enable their holders to both remember them and see them as part of a larger whole within which each part shares reciprocal relationships with other parts, (Pirie & Kieren, 1994).

Wiggins, (1993) defined understanding as something different that emerges when we are required to reflect upon achievement, in verifying or criticizing, re-thinking and re-learning what we know. In this process we question the assumptions upon which prior learning was based. The new definitions of understanding a mathematical concept involve knowing facts and concepts and how they connect and this should be related to knowing how and when to use skills and strategies. Wiggins & McTighe (1998) further identified several inter-related aspects of understanding including (1) explanation, (2) interpretation, (3) contextual applications, (4) perspective, (5) empathy and (6) self-knowledge. Not all of these apply to each learning situation and they are not hierarchical or mutually exclusive. Students who can explain their ideas justify understanding by making connections and inferences. Those who apply knowledge demonstrate their ability to use what they have learnt in complex situations. Lastly, those who show self-knowledge recognize the limits of their understanding.

Conceptual knowledge as defined by Hapasaalo (2004) denotes knowledge of and a skillful "drive" along particular networks, the elements of which can be concepts, rules (algorithms, procedures, etc.), and even problems (a solved problem may introduce a new concept or rule) given in various representation forms. Dubinsky (1991) asserted that understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are interiorised to form processes which are then encapsulated to form objects. Those objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in schemas.

Understanding of the concept of the chain rule was explored in relation to the development of schema relevant to it. Students were provided with activities in class that were designed to induce them to make the suitable mental constructions as suggested by the initial genetic decomposition. The tasks used in this study helped students gain experience in constructing actions corresponding to the chain rule.

This experience was built upon in subsequent activities where students were asked to reconstruct familiar actions as general processes. Lastly the students were provided with complicated activities where they needed to organize a variety of previously constructed objects, like functions and derivatives of composition of functions, into a schema applied to chain rule problem situations. More specifically the researcher examined students' attempts to answer the tasks given in class, their tests, and exercises with regard to their understanding of functions, composition of functions and the chain rule.

Although this was in the form of pencil and paper work, of importance to the researcher was the procedure used to answer the questions and not whether answers were correct or not. In addition to the tasks, interviews were conducted to substantiate the level of student understanding of the concept. These interviews were more valuable than the written assessment instruments because one student could have displayed correct written work while the transcript revealed little understanding and *vice versa*. A full range of understanding was obtained by selecting students who gave correct, partially correct and incorrect responses on the written work. Some students showed evidence of constructing very little while others constructed bits and pieces and yet others seemed to have made all the constructions proposed in the genetic decomposition. The latter showed evidence of understanding the concept and of possessing a schema for the chain rule.

3.3.1.2 Schema

A schema is a more or less logical connected collection of objects and processes (Dubinsky, 1991). Each individual may have a number of different schemas in order to deal with, understand, organize or make sense of a perceived problem situation using her knowledge of an individual concept in mathematics. A schema for a certain mathematical concept is an individual's collection of actions, processes, objects and other schemas which are linked by some general principles to form a framework in the individual's mind that may be brought to bear upon a problem situation involving that concept (Dubinsky & Mc Donald, 2000). Asiala et al, (2004) asserted that an individual's schema is the totality of knowledge which for her is connected consciously or unconsciously to a particular mathematical topic, for example an individual may have a function schema, derivative schema, chain rule schema. Those schemas were interrelated in a large complex organization within the individual's mind.

They further asserted that an individual's schema for a concept includes his/her version of the concept described by the genetic decomposition, as well as other concepts that are perceived to be linked to the concept in the concept of problem situation. A genetic decomposition representing one reasonable way that students might use to construct a particular concept is formed by isolating small portions of the complex structure and giving explicit descriptions of possible relations between schemas.

The existence of a schema is inseparable from its continuous construction and reconstruction. A calculus student may have interiorized the action of taking the derivative of a function and may be able to do this with a vast number of examples using various techniques that are often taught and occasionally learnt in calculus courses. If the process is interiorized from relevant actions, the student might be able to reverse it to solve problems in which a function is given and it is desired to find a function whose derivative is the original function, (integration). A process of understanding a certain mathematical concept using APOS can therefore be attributed to a successful students' construction of schemas for that concept. The essential tool to induce these constructions was chosen instructions and activities given in class that were relevant to the chain rule.

3.3.1.3 The development of the chain rule schema

A structured set of mental constructs which might describe how the concept can develop in the mind of an individual is called the **genetic decomposition** of that particular concept. The mental constructions that the learner might make include actions, processes, objects and schemas. The genetic decomposition of the concept of the chain rule given here guided my teaching instruction in class and the construction of the interview tasks. The chain rule schema develops through the levels of the Triad: *Intra-*, *Inter-*, and *Trans-*.

3.3.1.3.1 The Intra- stage

In the first stage, the Intra- stage, the student had a collection of rules for finding derivatives of functions in various situations, but had no recognition of the relationships between them. This collection might include some special cases of the chain rule, and perhaps even the general formula which was perceived as a separate rule rather than a generalization of the others. Students who were in the Intra- stage of chain rule schema development were those who saw the various rules for differentiation as not related. They were able to solve some of the problems by simply applying rules which had been memorized and in some cases not remembered correctly. Those were students who were skilled at algebraic manipulations, easily able to assimilate rules and procedures in a cognitive structure that consisted of a list of unconnected actions, processes and objects to produce correct answers.

3.3.1.3.2 The Inter-stage

The Inter-stage was characterized by the student's ability to begin to (mentally) collect all different cases and recognized that those were related. At that stage the collection of elements in the chain rule schema was being formed, and the collection is called a pre-schema. Students in the Inter- stage showed evidence of having collected some or all the differentiation and integration rules in a group and perhaps provided the general statement of the chain rule without yet constructing the underlying structure of the relationships, Jojo et al.. (2010). A student could tackle

$\frac{dy}{dx}$ of $y = (\cos ec^3 x + e^{\tan x})^2$, by applying the power rule, not sure that she was using the chain rule. This student during interviews and with further questioning explained the connection between his/her general statement of the chain rule and its applicability.

3.3.1.3.3 The Trans- stage

At the Trans- stage a student had constructed the underlying structure of the chain rule. He/she linked the composition and decomposition of functions to differentiation, and recognized various forms of the chain rule as linked in the sense that they followed from the same general rule through function composition. It was only at this stage of development that the underlying structure of the chain rule schema was constructed through reflection on relationships between various objects from previous stages. At Trans- level the elements in the schema must go beyond being described essentially by a list, to being described by a single rule (Clark et al., 1997). A student who displayed coherence of understanding of a collection of derivative rules and understanding of composition of functions as a schema had moved to the Trans- stage of development. He/she was able to reflect on the explicit structure of the chain rule and he/she was capable of operating on the mental constructions which made up his/her collection. Without stating the chain rule this student was able to use it proficiently. Jojo et al (2010) assert that this student was at this stage able to link function composition and decomposition to differentiation and integration and was able to link the two. Based on the above, the researcher arrived at the following genetic decomposition

For a student to have his or her function schema

(i) He/she had developed a process or object conception of a function and

(ii) Has developed a process or object conception of a composition of functions.

For a derivative schema,

(iii) He/she had developed a process conception of differentiation

(iv) The student then uses the previously constructed schemas of functions, composition of functions and derivative to define the chain rule. In this process the student recognized a given function as the composition of two functions, took their derivatives separately and the multiplied them.

(v) The student recognized and applied the chain rule to specific situations. The initial genetic decomposition is modeled in Figures 3.2 and 3.3 in the following section.

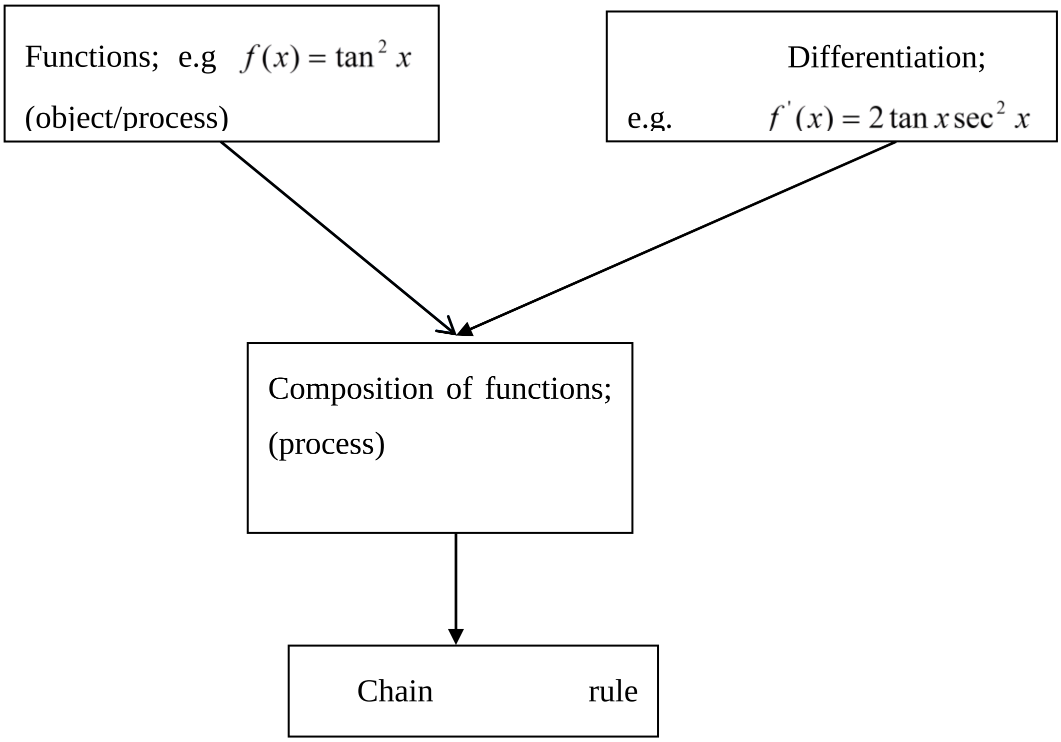


Figure 3.2: Initial genetic decomposition of the chain rule

3.3.2 Design and implementation of instruction

Once a week, the class met in groups of 30 for tutorial lessons and on the other days in a regular classroom as a cluster of 197 for lessons. Homework was given at the end of each lesson and was completed during class supervision. Understanding of the concept of the chain rule was explored through the development of schema relevant to it. Students were provided with activities in class that were designed to induce them to make the suitable mental constructions as suggested by the initial genetic decomposition. The tasks used in this study helped students to gain experience in constructing actions corresponding to the chain rule.

Subsequent activities were provided to help students build experiences of reconstructing familiar actions as general processes. Those activities were followed by higher order activities where the students needed to organize a variety of previously constructed objects, like functions and derivatives of composition of functions, into a schema that could be applied to problem situations involving the chain rule. More specifically their understanding of functions, composition of functions and the chain rule were evident from the students' attempts to answer the tasks given in class, their tests, and exercises.

It was not important whether the answers given in the form of pencil and paper work were correct or wrong, of importance to the researcher was the procedure used to answer the questions. In addition to the tasks, to substantiate the level of student understanding of the concept, interviews were conducted. Those interviews were more valuable than the written assessment instruments because one student could have displayed correct written work while the transcript might reveal little understanding and vice versa. Students who gave correct, partially correct and incorrect responses on the written work were selected to cover a full range of understanding. Some students showed evidence of possessing a schema for the chain rule. They had made all the constructions proposed in the genetic decomposition, while others, showed evidence of constructing very little and others had constructed bits and pieces. The first group showed evidence of understanding the concept.

3.3.3 Collection and analysis of data

The pedagogical strategy used in this study was embodied in the ACE (Activities, Class discussions and Exercises) teaching cycle.

3.3.3.1 Activities

Activities designed to help students make the mental constructions in the proposed genetic decomposition were given to students in the form of a tutorial test. Students worked collaboratively and some individually. Through these activities the students gained experience with mathematical issues related to understanding the chain rule.

3.3.3.2 Classroom Discussions

Class discussions came after the students had received three consecutive lessons on the chain rule and its applications. Those lessons and instruction were video-taped and later video clips of discussions related to mental constructions of the chain rule and the thought process were recorded. They form part of Appendix B. Here classroom tasks were given and discussed with the students. A questionnaire consisting of 12 items was then presented to a sample of 30 students who volunteered to participate in the study. Some students of the 30 volunteers were then interviewed and encouraged to describe his/her solution displayed on some of the tasks. The interviews were audio-taped and later transcribed and all written work was collected. These interview transcriptions are analyzed in detail in Chapter 7. The transcripts and written work were later coded and searched for issues related to understanding of the chain rule in calculus.

3.3.3.3 Exercises

Exercises were assigned to the students in the form of homework done outside of classroom supervision at the end of each of the consecutive lessons taught on chain rule definition and use in calculus. The purpose of the exercises was for students to reinforce the ideas and concepts constructed in class and to use and apply the chain rule they had learnt. Exercises in the learning of the chain rule were also given in the form of written questions and answers in the form of tests, the specially designed questionnaire and in-depth interviews of students chosen according to answers given in the pen and paper exercises.

3.4 The structure of the genetic decomposition of the chain rule

Based on the theoretical analysis in the previous sections, a structure of the genetic decomposition of the chain rule is presented in this section. Differentiation also depended on the kinds of functions we had. This is a description often displayed as a diagram, of the actions, processes, objects and schemas and their relationships, that might be involved in the formation of mental constructs. Analysis of results in this study incorporate APOS extended with the use of the Piaget's Triad mechanism to explain constructs which cannot be explained as actions, processes, objects and or schema. Figure 3.3 illustrates this extension.

3.5 Reflective abstraction in advanced mathematical thinking

Reflective abstraction is a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development. He made two important observations: first, reflective abstraction has no absolute beginning, but is present at the earliest ages in the coordination of the sensori-motor structures (Beth & Piaget, 1966) and second, that it continues on up through higher mathematics to the extent that the entire history of the development of mathematics from antiquity to the present day may be considered as an example of the process of reflective abstraction.

Piaget distinguished three major kinds of abstraction which are not independent of each other. He talks of *empirical knowledge* which derives knowledge from properties of objects, (Beth & Piaget, 1966). According to Piaget, (Piaget & Garcia, 1983), this kind of abstraction leads to *extraction of common properties of objects* and *extensional generalizations* that is, the passage from specific to general. *Pseudo-empirical* is intermediate between empirical and reflective abstraction and it spells out properties that the actions of the subject have introduced into objects, (Piaget, 1985). Finally, *reflective abstraction* is drawn from what Piaget (1980) called general co-ordinations of actions by the subject internally. He asserts that this kind of abstraction leads to a very different sort of generalization which is constructive and results in new syntheses in the midst of which particular laws acquire new meaning (Piaget & Garcia,1983).

There were many examples of instances of reflective abstractions, but we can site that when a student was given a problem like, Differentiate: $y = \left(\cos ec^3 x + e^{\tan x}\right)^2$, he performed several individual actions in his mind to identify the different functions involved. He would then interiorise and coordinate the actions to form a total ordering of where to start differentiating. Those actions would form new actions, and ultimately new objects (which may no longer be physical but rather mathematical such as a new function derived from the original one). Piaget (1972) further asserted that it is reflective abstractions in its most advanced form that leads to the kind of mathematical thinking by which form or process is separated from the content and that processes themselves are converted, in the mind of the mathematician, to objects of content, (Piaget, 1972). Empirical abstraction therefore deals with action as opposed to objects and it differs from pseudo-empirical abstraction in that it is concerned, not so much with the actions themselves, but with interrelationships among actions, which Piaget (1975) called 'general co-ordinations.'

According to Piaget, the first part of reflective abstractions consists of drawing properties from mental or physical actions at a particular level of thought (Beth & Piaget, 1966). He says that this involves consciousness of the actions, and can include the act of separating a form from its content. Whatever is abstracted is projected onto a higher plane of thought where other actions are present as well as more powerful modes of thought.

3.5.1 Construction in reflective abstraction

Piaget's notion of reflective abstraction is central to APOS which is Dubinsky's theory which would be important for advanced mathematical thinking (Dubinsky 1991b). Reflective abstraction has two components: (a) a projection of existing knowledge onto a higher plane of thought and (b) the reorganization of existing knowledge structures (Dubinsky, 1991a). Reflective abstraction is therefore a process of construction and Dubinsky outlines five kinds of construction in reflective abstraction:

1. ***Interiorisation***: Here actions and objects are interiorised into a system of operations. This is parallel to Sfard's (1991) theory of reification, who, is of the idea that different mathematical notions can be conceived, either, structurally, as objects and procedurally as processes. She describes the route from processes to objects as involving three stages: interiorisation, condensation and reification. For Sfard (1991), during the phase of interiorisation a student becomes familiar with a process and can carry it out through mental representations.

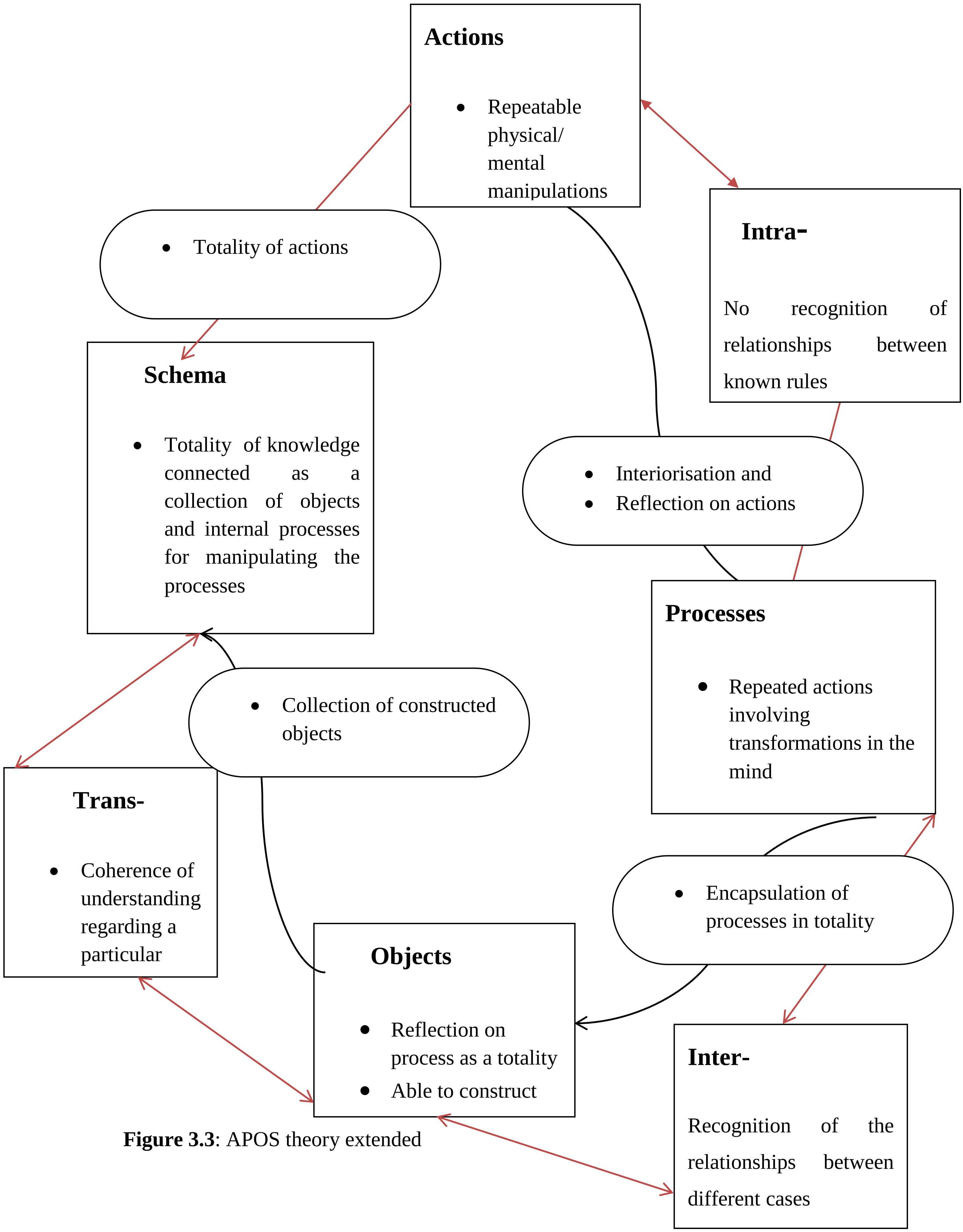


Figure 3.3: APOS theory extended

2. ***Co-ordination***: Two or more processes are co-ordinated in order to form a new process, e.g. the chain rule for differentiation requires the co-ordination of composition of functions with derivatives (Bowie, 2003). Sfard (1992) refers to this as condensation where there is a gradual quantitative change in which a sequence of mathematical operations is dealt with in terms of 'input and output without necessarily considering its component steps'.

3. ***Encapsulation***: This is where the construction of mathematical understanding extends from one level to the other, where new forms of the process are built drawing from the previous ones to form an object. This echoes Sfard's notion of reification which involves the student's 'mind's eyes ability to envisage the results of the processes as permanent entities in their own right'. (Sfard & Linchevski, 1994). This shift occurs when the student is able to detach the understanding from the processes that produced it and see it as an object. (Dubinky et al, 1989) proposed that composition of functions is a binary operation that acts on two functions, considered as objects to form a third one. They then assert that the student has to unpack these functions, reflect on corresponding processes, and interiorise them. Then the two processes can be encapsulated into an object which is the function that results from the composition. Dubinsky rates this process as the most complicated one than simple substitution and agrees that students have a problem with the chain rule for differentiation where it is necessary to co-ordinate the view of composition of the function with the notion of derivative.

4. ***Generalisation*:** An existing schema is applied to a wide range of contexts. This would happen for example when the student is able to see that after finding the derivatives of the various functions in a composition, they now have to be multiplied to put the chain rule into application.

5. ***Reversal*:** An interiorised process can be thought of in reverse. A new process can be constructed by means of reversing the existing one. For example, $\int (3x-1)^2\,dx$ where the reversal of the chain rule would be used.

3.6 The APOS framework

The theory on reflective abstraction and the triad suggested by Piaget and Garcia (1983) are important for higher mathematics as they were useful to explain children's logical thinking. In extension of this theory, Dubinsky et al (1991) isolated some essential features of reflective abstractions reorganized and reconstructed them and formed a coherent theory of mathematical knowledge and its construction, APOS. This study adopted the APOS approach (Dubinsky 1991a), based on its intuitive appeal as there has been little empirical research done documenting the use of it on students' conception of various mathematical concepts. This approach, through which this study was conducted, began with a statement of an overall perspective of what it means to learn and know something in mathematics as prescribed by Asiala et al (2004): 'An individual's mathematical knowledge is his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing and reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.' They further believe that understanding a mathematical concept begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorised to form processes which are then encapsulated to form objects. They say that these objects could be de-encapsulated back to the processes from which they are formed, which would be finally organized in schemas. The formation of these objects in understanding the chain rule is explained in Figure 3.4.

The overview of the following model is that one begins with two functions F and G and transforms them into a single function, FoG. The transformation begins by de-encapsulation of F and G back to the process F(x) and G(x) from which it came. The two processes are then coordinated to obtain the process *x* on F(G(*x*)), which is then encapsulated to the object FoG.

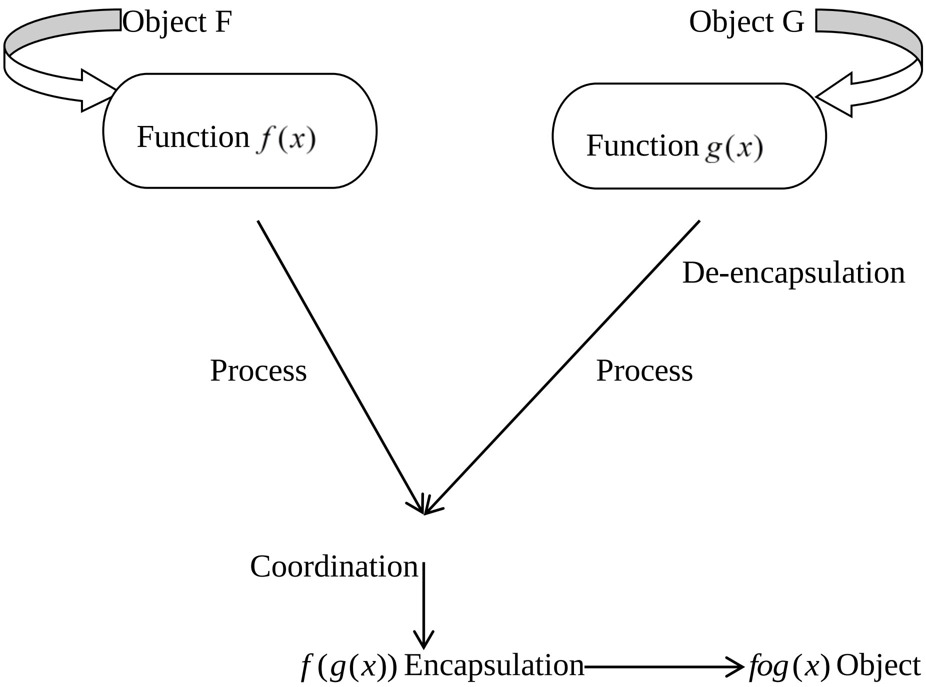


Figure 3.4: Illustration of the composition of functions

For the chain rule the above diagram is taken further to a process of finding the derivative of F and G separately and then multiplying the result represented as $f'(g(x)).g'(x)$.

Figure 3.5 illustrates that to find the derivative of a composite function, $f \circ g(x)$ we find the derivative of*f*, and the derivative of *g*, and then we multiply them together. The tricky part is in identifying the function $g(x)$ as a single entity in $f(x)$. The *x* in $f(x)$ is not *x* but another function in *x*. Thus the derivative $g'(x)$ has to evaluated first and then multiplied by the derivative, $f'(g(x))$ to get $f'(g(x)).g'(x)$. To avoid the confusion of the two functions in *x*, one may represent the function *g(x)* as *u*. This will then enable us to find separately the derivatives $g'(x)$ and $f'(u)$. Result would then be found when we multiply $g'(x) \times f'(u)$.

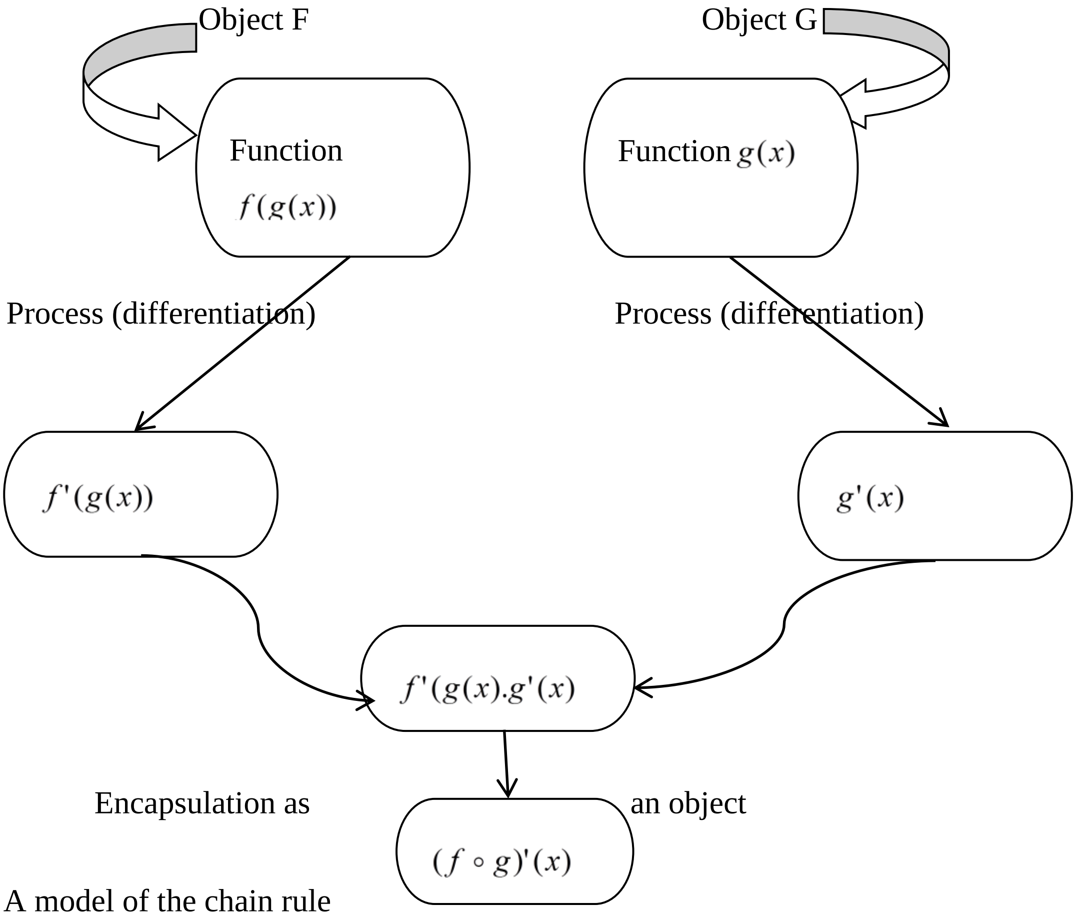


Figure 3.5: A model of the chain rule

This would safe guard against two errors which the students usually make, namely: (1) Finding $f'(x) \times g'(x)$, for example the derivative, $\cos(3x^2 + 4x - 5) \neq -\sin x.(6x+4)$ but is $-\sin(3x^2 + 4x - 5).(6x+4)$. (2) Finding $f'(g'(x))$ where one derivative is plugged onto the other one. For example the derivative of $\tan(3x^4) \neq \sec^2 12x^3$ but is $\sec^2(3x^4).12x^3$.

The students interiorise their actions by discussing their actions with others collaboratively. They must also write verbal descriptions of their actions using their own words. Mental constructions that a learner makes include actions, processes, objects and schema of the chain rule. The following figure illustrates a model developed to explain within the APOS context.

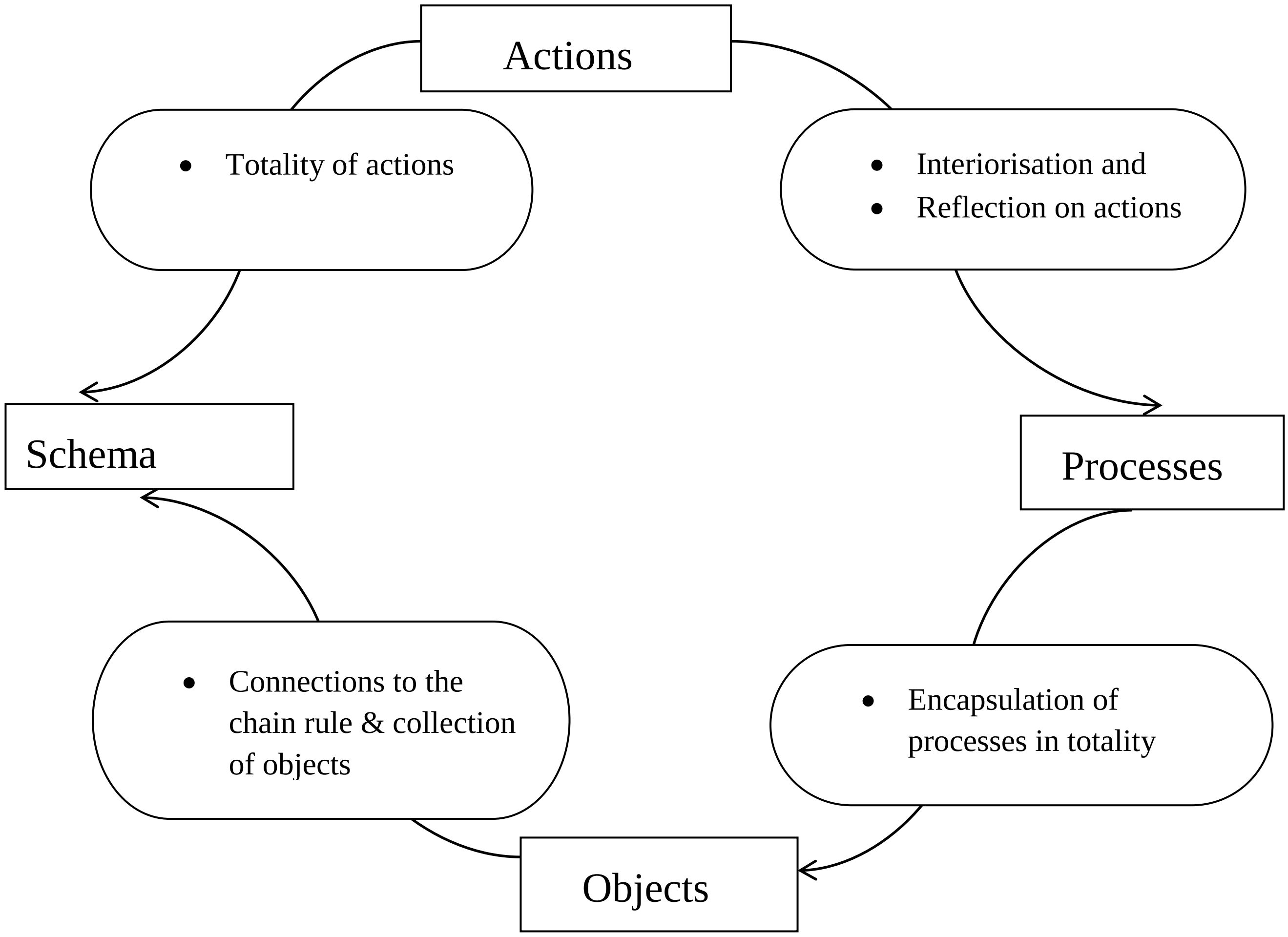


Figure 3.6: A model for the chain rule in the context of APOS

3.7 Piaget's Triad mechanism

The Triad mechanism consists of three stages, referred to as Intra, Inter, and Trans in the development of connections an individual can make between particular constructs within the schema, as well as the coherence of these connections, (Dubinsky, 1991). These stages have been discussed in detail (pp 42-43) as part of the proposed genetic decomposition of the chain rule.

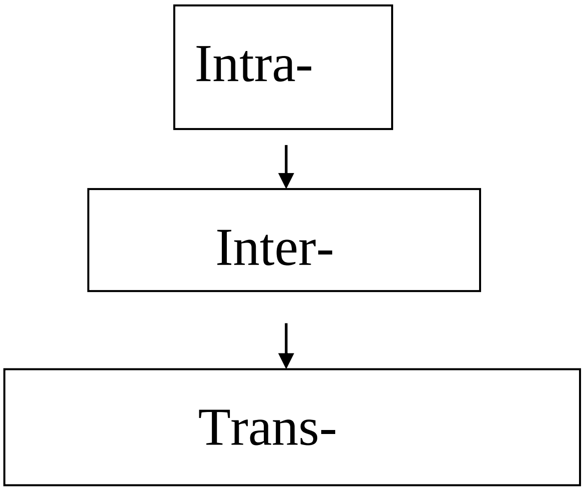


Figure 3.7: The stages of Piaget's Triad mechanism

The Intra stage focuses on 'a single object', followed by inter which is 'study of transformations between objects' and Trans- noted as 'schema development connecting actions, processes and objects'. Tall (1997) argues though that the confinement by Dubinsky to the order of 'action, process and objects, was mean, in the sense that it avoids other widely used terms such as, 'Inter-' including the study of the properties of objects, or 'Intra-' being concerned with the relationship between them. He argues that there is no transformation that takes place when a child compares the sizes of objects. Zingiswa, et al, (2005) assert that the idea of transformation is a tool the teacher can use, a tool that does not dehumanise learning because it corresponds to one aspect of the voluntary activities of the students' mind. Dubinsky (1991) believes that an individual at the Trans- stage for the function concept could construct various systems of transformations such as rings of functions, together with the operations included in such mathematical structures.

3.8 Conclusion

Based on the initial genetic decomposition of how the chain rule concept may be learned, an interpretation of the mental constructions using APOS were made in this chapter. A student was said to understand the chain rule once his/her collection of derivative rules and understanding of function composition was capable of operating on mental constructions acquired, and was able to reflect on the explicit structure of the chain rule which these constructions were implicitly containing. The existence of the chain rule schema is inseparable from its construction and reconstruction. This schema is a collection of actions, processes and objects conceptions and other previously constructed schema which were coordinated and synthesized to form mathematical structures that were applied when chain rule is needed.

The students' mathematical knowledge of the chain rule depends on relations between mental constructs together with the interconnections that the student uses to understand the concept or rule, and the way in which he/she uses or fails to use that concept in problem situations. In this statement it is acknowledged that what a person knows and is capable of doing is not always available to him at a given moment and in a given situation. It may happen that a student misses a question in a test or examination not because he doesn't know it but perhaps because he could not access it at that moment. With a question or a little assistance the answer may be accessed and used correctly by the student. Also the student would respond to perceived problem situations by applying and using the knowledge of chain rule acquired.

The APOS theory and Piaget's Triad mechanism which provided the theoretical framework for the presentation and analysis of data in the following three chapters has been discussed in full in this chapter. The following chapter gives an outline of the methodology used in this study, the participants and limitations of the study.

CHAPTER 4

RESEARCH METHODOLOGY AND PROCEDURE

4.1 Introduction

Chapter 3 provided an overview of APOS theory as the theoretical framework used in this study. This chapter resumes with the reintroduction of the critical research questions and presents the subjects and instrument used in this study. Further on, methodological framework used, and research paradigm within which the study was located is described. How the paradigm fits with this study, data sources and data collection processes and analysis in Phase 2 are presented. In addition the chapter outlines the limitations governing the study and then concludes with a brief summary of the methodology used.

4.2 The critical research questions

The study proposed the APOS (Action-Process-Objects-Schema) approach in exploring the conceptual understanding displayed by first year the University of Technology students in learning the chain rule in calculus. The research question addressed by this study was:

How do students construct various structures to recognize and apply the chain rule in the context of calculus?

This was with the view of clarifying:

8. *the students' understanding of the function concept,*
9. *the students understanding of function composition,*
10. *their understanding of the derivative,*
11. *the students' difficulties in using the chain rule,*

12. *students' schema alignment with a genetic decomposition of the chain rule,*

13. *the triad stage of schema development in which students are operating with respect to the chain rule, and*

14. *the students' identification of the reverse application of the chain rule in the substitution technique for integration.*

4.3 Subjects

The subjects for this study were first year engineering students (197) at the University of Technology who had been taught more than half of calculus concepts. Sections done included limits of functions, the rate of change of a function, finding the derivatives of polynomials and algebraic together with trigonometric functions and also use of product together with quotient rules in calculus. Some of these students had been through a foundation course including introduction to calculus for a period of six months at the university (Pre-tech students), Extended Curriculum Programme (ECP), (students who have undergone a foundation programme for a year at the university), while others had good matric symbols and were registered for first year without going through any foundation course. All students in class, (197) had given written consent and allowed the researcher to conduct the study. This was done because video-taping of the three sequential lessons took place in class. A small sample of 30 volunteering students were selected, because (1) interpretive case studies depend on descriptive foundation, and (2) this type of research takes a lot of time. All of the students were registered for the engineering course and mathematics was not one of their majors. They registered for mathematics as an aid to assist them in understanding their other engineering courses.

4.4 Instruments for collection of data

4.4.1 Phase 1

The study was composed of two Phases. Phase 1 was comprised of the pilot study. The subjects in this phase were second year students who volunteered to participate in the study. Those students had already written and passed an examination in first year calculus. A questionnaire was administered to 23 volunteered subjects. The pilot study was conducted for purposes of validating the questionnaire for purposes of the main study. Some of those participants were selected and then interviewed to explain their responses to the questionnaires. The participants interviewed were chosen based on their written responses on the questionnaire. The results of the pilot study are included in this study in Appendix A and discussed in full in Chapter 5.

4.4.2 Phase 2

There were no revisions done on the questionnaire used in the Pilot study. The questionnaire was then administered to 30 first year students covering categories including definition of functions, composition and decomposition of functions, derivatives and chain rule embedded in the structure of the integrand. The other instruments included activities, class work exercises and tutorial tests. After each written exercise, a sample of six participants were selected, interviewed and asked to explain some of their responses to selected questions.

The questions in the questionnaire tested the understanding of: (1) definition of function using domain and range, (2) definition of function using graphical methods, (3) composition of functions, (4) decomposition of composed functions, (5) the derivative, and (6) the applications of chain rule.

The students volunteered their time up to $1\frac{1}{2}$ hours to complete the questionnaire. A student-numeric code was used to identify all of the written work collected as data on each questionnaire. This code was also used on all audio–tape transcriptions made during interviews on selected students. This code is used in this report with confidentiality not compromised.

4.4.3 Tools used for data collection

Data was collected using a questionnaire comprised of 12 questions on functions, their composition, derivative and the structure of the integrand. Interviews of certain subjects based on their responses to the questionnaire provided the main source of understanding as the participants justified their responses. Some questions used in the interview included: (1) There you are, just working it out. Were you perhaps using any rule in your differentiation? (2) Can you state the chain rule? (3) What do you mean by right through? (4) I want you to be explicit in explaining your understanding of the chain rule. What does it say? Can you write it down maybe? (5) If for example I give you, this function, $y = \tan^2(3x + e^{\sqrt{x^2+1}}$ to differentiate, what would be the derivative of y?

4.5 Qualitative research

By its nature, qualitative research methodology allows one to use different research strategies to collect data. It also allows for the voice of the participants to be heard. Romberg (1992) asserted that when no numbers are used in categorising, organizing and interpreting the relevant information that have been gathered, then the method is said to be qualitative. Qualities of this type of research were outlined by Fouche' & Delport (2002, p.79) as follows (1) Qualitative research elicits participants accounts of meaning, experience or perceptions about a concept, (2) It produces descriptive data, (3) Qualitative approaches allow for more diversity in responses as well as the capacity to adapt to new developments or issues, (4) In qualitative methods, forms of the data collected can include interviews and group discussions, observation and reflection field notes, various texts, pictures, and other materials.

This study is qualitative in nature. There is a wide range of approaches to qualitative research used at present. Asiala et al, (2004) suggested that in considering the variety of approaches being used, two aspects have to be addressed: (1) the theoretical perspective taken by researchers using that approach, and (2) the actual methods by which data is collected and analysed. Woods (2006) in contrast outlines four features of a qualitative research as: (1) focussing on natural settings, (2) interest in meanings, perspectives and understandings, (3) process and inductive analysis, and (4) grounded theory. In this study the researcher describes a qualitative methodological framework that has been used with the concern for theoretical and empirical aspects together with applicability to the real classroom situation in the form of instructional treatments.

4.5.1 The theoretical perspective

The perspective taken in this study seeks to describe a set of specific mental constructions that a student might make in order to develop his/her understanding of the chain rule concept. The result of this analysis is known as a **genetic decomposition** of the concept. This analysis was influenced by the researcher's own understanding of the chain rule and the previous experience in learning and teaching it.

The theoretical perspective used in this study of what it means to learn and to know something in Mathematics is based on the following assumption outlined in Asiala, et al (2004, p 7).

''An individual's mathematical knowledge is his/ her tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing and reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations.''

A full discussion on this statement can be found in Asiala et al (2004). The main issue in this study was how students learn the chain rule concept and access it when needed. Byers (1980, pp 5-6) believed that it is impossible to understand a piece of mathematics in the absence of pre-requisite knowledge. He further asserted that the understanding of mathematics deepens with the acquisition of new mathematical knowledge and that understanding involves availability for ready retrieval. Thus to understand the chain rule goes beyond the ability to use it in calculations, but to be able to remember or reconstruct its knowledge to form a collection of processes and objects that can be organized in a structured manner to form a schema of the chain rule.

4.5.2 The actual methods used for data collection and analysis

The study consisted of two phases, both using a qualitative approach. Phase 1 was the pilot study which involved collection of data via questionnaires which were administered to 23 previous semester students of known ability, willing to participate in the study. These were students who had already written an examination on first year calculus and passed it. This was done to check for errors, validity and reliability in the instrument, (Jojo et al, (2011)). Some of the students were interviewed for clarity on their written responses. The results of the pilot study are outlined in the next chapter.

The questionnaire was then administered to 30 volunteering first year students in Phase 2. A structured way to describe an individual student's understanding of the chain rule was developed and applied to analyzing the evolution of that understanding for each of the 30 first year students. Various methods of data collection were used namely: (1) classroom observations, (2) open-ended questionnaire, (3) semi-structured and unstructured interviews, (4) video-recordings, and (5) written class work, tests and exercises.

The study followed an APOS approach where the participants were observed in action. This approach allowed the researcher to select the examples that illustrate the points that he or she wished to make (Cohen & Manion, 2000). Those researchers asserted that the analysis of the individual student answers consists of a construction of taxonomies, resulting from the various observation sessions of each student’s work.

Since the main aim of this study was to analyze students' mathematical thinking in the context of the chain rule, an interpretive paradigm was used.

4.5.3 The interpretive paradigm

Schultz (1962) suggested that interpretivists share the following beliefs about the nature of knowing and reality:

(1) Relativist ontology - assumes that reality as we know it is constructed inter-subjectively through the meanings and understandings developed socially and experientially.

(2) Transactional or subjectivist epistemology - assumes that we cannot separate ourselves from what we know. The investigator and the object of investigation are linked such that who we are and how we understand the world is a central part of how we understand ourselves, others and the world.

(3) Findings emerge through dialogue in which conflicting interpretations are negotiated among members of a community.

(4) Fostering a dialogue between researchers and respondents is critical. It is through this dialectical process that a more informed and sophisticated understanding of the social world can be created.

Angen (2000), claimed that the interpretivists assume that the researcher's values are inherent in all phases of the interview and the truth is negotiated right through the interview process. He further outlined the following as characteristics of the interpretive paradigm: (1) interpretive approaches rely heavily on naturalistic methods (interviewing, observations and analysis of existing texts), (2) these methods ensure an adequate dialogue between the researchers and those with whom they interact in order to collaboratively construct a meaningful reality, (3) generally, meanings are emergent from the research process, and (4) qualitative methods are used.

Cohen et al (2000) asserted that an interpretive paradigm model works explicitly from within the human perspective. As noted earlier, the nature of research questions in this study indicated an interpretive action research design with the unit of analysis being the students work. According to Cohen et al (2000), the interpretive enquiry interprets and discovers the perspectives of the participants in the study and the answers to the enquiry are practically dependent on the context. More specifically the researcher examined students' attempts to answer the tasks given in class, their tests and exercises with regard to their understanding of functions, composition of functions and the chain rule.

4.5.4 How the paradigm fits with my study

Interpretive approaches rely heavily on interviewing, observations and analysis of existing texts (Cohen, et al 2000). They also asserted that these approaches ensure an adequate dialogue between the researcher and those with whom he/she interacts to construct a meaningful reality and derive meanings from the research process. It was of importance to this study to analyze students' mathematical thinking in the context of the chain rule with the aim of explaining students' misconceptions and difficulties identified. It was also of significance to note whether students understand and could correctly interpret the structural representations of embedded functions. This concerns differentiation of composition of different functions like exponential and trigonometric functions in calculus. It was of importance here to see how students used the chain rule when constructing their knowledge of derivatives.

4.6 Methodological framework

The instructional approach to fostering conceptual thinking in mathematics was seen by Dubinsky (1991) as having four steps, namely: (1) observation of students in the process of learning a particular topic to see their developing conceptual structures or images, (2) analyze the data, using these observations along with APOS theory to develop the genetic decomposition of the concept under study, (3) design of instruction in relation to the proposed genetic decomposition, and (4) develop activities suitable to induce students to make the specific reflective abstractions relevant to the concept. This approach was used in this study. Three sequential lessons on learning the chain rule by first year engineering students were presented and video-taped to help students develop an understanding of the concept of the chain rule. The instruction designed for the lessons followed aspects repeated by the proposed genetic decomposition. Activities suitable for students to make abstractions regarding the chain rule were then selected and given to students to do during three consecutive tutorial lessons which took place on Friday afternoons. This was then followed by two written test exercises on differentiation using the chain rule and integral calculus.

4.7 Data sources

Data for this analysis included results from the Pilot study. It also included analysis of students' performances in the tasks given regarding their understanding of the function, derivative, composition of functions and the chain rule. Interviews were conducted to analyze students' responses on written tasks and class discussions. After the analysis of the tasks in Phase1, there was a video recording of three sequential lessons on the chain rule. This time students' engagement with the lesson was of importance. Audio-recording was employed while interacting with the students, and when guiding individual students. Students worked individually and collaboratively. There were also classroom observations where the researcher observed how students worked through the tasks in providing answers to class works, tests and exercises on the chain rule concepts.

4.8 Data collection procedures

Data was collected using a multi-method approach. A Pilot study was used to validate the questionnaire to be used in the study. Reports on it are found in the next chapter. Three sequential lessons on derivatives, the chain rule and its applications were then video-taped. Also, class observations, questionnaires, interviews audio-taped, tests and exercises were other sources of data. In the next subsections the researcher discusses each data collection tool that was used in the study.

4.8.1 Classroom observations

Observing is the process of studying classroom activities to determine teaching strategies and student responsiveness. It can be used to gain insight into planning, organization, methods of presentation, behaviour management techniques and individual student differences (Olsen, 2008). Observation in this study was used as a tool to obtain information that could later be analyzed to gain a better understanding of instructional procedures and classroom interactions. Such understanding helps to inform and refine the teaching skills.

The researcher through observations noted methods to motivate students and kept them focussed on instructional classroom activities. Observations in this study were structured such that the researcher and student movement was documented using a seating chart and arrows to show movement throughout the lesson. Notes were made on the chart to record conversations and activities at various locations in the classroom. Also, a question and answer record sheet was used to record all the questions the researcher asked and the corresponding student responses. Data from this observation gave insight into the type of questioning used to elicit higher level thought from the student. Observations are theory driven. They allow the researcher to see substance in the data (Olsen, 2008).

4.8.2 Interviews

Qualitative interviews are defined by Sewell (2002, p 27) as ‘‘an attempt to understand the world from the participants’ point of view, to unfold the meaning of the people’s experiences and to uncover their world prior to scientific explanations’’. In this study, selected participants were interviewed for clarity and explanations on their written responses. Those were responses in the form of answers given to classroom activities, written exercises, tutorials and tests. Interviews engaged in meta-analysis and extracted more data. The extent of abstraction, critical thinking, insightfulness conceptualization and imagination levels were measured through interviews. The interviews in this study were used as a research tool and open-ended questions were prepared in advance and used to extract information from the participants.

Kvale (1996) agrees that in qualitative research interviews help in understanding something from the subjects' point of view and to uncover the meaning of their experiences. Frey (1994) distinguishes various forms of interviewing as (1) individual or group face-to face verbal exchange, (2) mailed or self-administered questionnaires, and (3) telephone surveys. Lankshear and Knobel (2004) assert that structured interviews aim to maximise comparisons across responses to questions of the interview, characterised by a pre-set list of questions asked in a fixed order with no deviations from the list regardless of the response to the question asked. Unstructured interviews aim to solicit as much information as possible without confining the respondent to particular themes or topics. Semi-structured interviews included a list of prepared questions that were used as guide only and follows up on relevant comments made by the interviewee. The duration of an interview could be five minutes or hours or span over days depending on the purpose of the interview.

Phase 2 used in-depth, task based interviews with six of the students, for clarity on some of their written responses. Two fold types of interviews were conducted: (1) to get feedback on how students perceive chain rule, and (2) to fulfil a verification purpose where the individual student's written response was clarified. This was done in an attempt to describe how these students constructed the concept of the chain rule. Participants were chosen based on their scores on the instrument. The interviews followed a guide designed to elicit the students' understanding of the chain rule based on the tasks given in the questionnaire.

In this study the interviews were conducted at the university over a number of days depending on the availability of the participants. Those interviews were semi-structured and the questions were mostly open-ended. Open ended questions allowed the participants to express freely their way of thinking when they wrote the answers and allowed them to change or add some information, they deemed fit.

Class work exercises given for homework included the following problems. Some of the students' responses to the given exercises are included in Appendix C. These exercises included:

1. Determine the derivatives of: (i) $y = \tan(2x+1)$

(ii) $f(x) = \sin^2 2x$

(iii) $y = -4\sin 2x - 4x\cos 2x$

(iv) $f(x) = \ln e^{\tan x}$

2. Determine $\int e^{\cos x} \sin(2x)dx$

This was done to minimise the mismatches that existed between the learning styles of most students in the class and the teaching style of the lecturer. Successful achievement of the latter could increase the students' comfort level and willingness to learn and understand the chain rule when these are later assessed.

4.9 Challenges of doing classroom based research.

These are discussed under the following headings.

4.9.1 Validity and reliability

Validity determines whether a research instrument investigates what is intended to be investigated, while reliability refers to how consistent the results are. The issues of validity concerns in my research study span over the two broad areas: Triangulation and Participation.

4.9.2 Triangulation

Triangulation is a process of verification that increases the validity by incorporating three different viewpoints and methods (Rubin & Rubin 1995). It is often used to indicate that more than two methods are used in a study with a view to double (or triple) checking results. Several researchers have given different definitions of triangulation. Cohen and Manion (2000, p37) define triangulation as an "attempt to map out, or explain more fully, the richness and complexity of human behavior by studying it from more than one standpoint." Altrichter et al. (2008, p46) contend that triangulation "gives a more detailed and balanced picture of the situation." According to O'Donoghue and Punch (2003, 54), triangulation is a "method of cross-checking data from multiple sources to search for regularities in the research data."

Denzin (1978) distinguished four types of triangulation: (1) *data triangulation* which involves time, space and persons, (2) *investigator triangulation* which involves multiple researchers in an investigation, (3) *theory triangulation* which involves using more than one theoretical scheme in the interpretation of the phenomenon and (4) *methodological triangulation* which involves more than one method to gather data, such as interviews, observations, questionnaires and documents.

This study triangulated data over time, space (classroom and interviews), different learners interviewed and different research methods used. Data were collected over a period of time. Three sequential lessons covering the teaching of derivatives were video-taped. Questionnaires were used to collect data. It was not important whether the answers were correct or wrong, but it was essential to get the thinking behind the displayed answers. Selected students were also interviewed based on their scores on the written instrument. This was done to validate the answers given by the students in the written instrument.

4.9.3 Participation

The subjects for this study were 197 first year engineering students at a University of Technology registered from different backgrounds as outlined in Chapter One. A small sample of 30 volunteering students were selected as participants in this study because (1) interpretive action research depends on descriptive foundation, and (2) this type of research takes a lot of time. The students participated on their own free will in the study to ensure that the data collected had validity. Those who did not want to be part of the sample were not forced to be included.

4.10 Quality standards and procedures

A number of factors and procedures to ensure that the data accurately reflect students' thinking were considered. Data was collected throughout an academic year in two semesters of the first year course. The informed consent form was read and explained to all subjects (197) in this study. The volunteers were expected to complete the questionnaire in a one and a half hour duration slot. Students were assured that all their responses would be treated with strict confidentiality and that their performance in the study was in no way to be used to determining their expertise in the course. The same applied to the classroom exercises and tutorial tests given as they were not used for their regression or progression in the course but only served as data for the study.

4.11 Trustworthiness and Ethical issues

The criteria for judging the overall trustworthiness of a qualitative study were selected. The table below adapted from Krefting (1991, p 217) gives a summary of the strategies and criteria used in this study to establish trustworthiness.

Ethical clearance was granted for my study. The ethical clearance approval number is HSS/0862/09D. Thereafter written permission for conducting this study in the institution was granted by the research directorate of the institution. This was granted after a summary of the proposal was presented to the institution's research committee.

Participants in this study were notified that their participation in the study was completely voluntary and they could withdraw at any stage, if they wished to. All the participants in the study were promised confidentiality and anonymity. The nature, process and purpose of the study were outlined to all the subjects. Fictitious names were used to protect the identity of the participants. Students were also invited to ask questions to seek clarity on any issue or any uncertainty they were experiencing during the course of the study. A letter of consent which is included in Appendix A2 was presented and read to all subjects. The letter requested permission from the all students to participate in the study and to be interviewed if for clarity of the written responses where necessary. This was ensured since the sequential video-tapes were done in class in the presence of volunteers and non-volunteers.

Delport (2002) asserts that qualitative data methods include questionnaires, checklists, indexes and scales. He further notes that certain principles and procedures have to be followed in the construction of the various methods. This would afford these measuring instruments reliability and validity. These principles and procedures provide strategies for improving the trustworthiness of the inquiry.

Strategy	Criteria	Application
Credibility	Prolonged and varied field experience Triangulation Interview Technique	Participants' written responses from the questionnaire, written class activities Digital voice recordings of interviews
Transferability	Dense description	Extraction of participant's written responses to questions used in unstructured interviews Verbatim quotes from interviews
Dependability	Dependability audit Triangulation	Interview transcripts Participants' responses to questionnaire
Confirmability	Confirmability audit Triangulation	Checking of transcripts Checking of participants' responses Verbal questions to be checked

Table 4.1 Strategies for establishment of trustworthiness in the study

4.12 Limitations of the study

Since the subjects were volunteers, there was no control over their range of abilities. Also since the study followed the interpretive paradigm, the samples were small. Only a set of students' understanding was described and not the whole population of first year engineering students. Issues regarding students' characteristics, like health and their industrial specialization, were not considered by this study.

4.13 Conclusion

In this chapter, methodological issues pertaining to this study were considered. The critical research questions and research instruments were shown to be in line with the dictates of some experts in the field of education research. The qualitative paradigm was discussed and shown to co-inside with the theoretical framework adopted. The data capture methods are aligned to a qualitative approach. Certain mechanisms are identified and applied to assure reliability and validity compliance. In the following chapter, the researcher discusses the preliminary results of Phase 1 conducted to validate the research instrument.

CHAPTER 5

THE VALIDITY OF THE QUESTIONNAIRE

5.1 Introduction

In the previous chapter, the research design, methodology and procedures used in this study were discussed. A detailed description of each tool of inquiry and data sources was provided. In addition the data analysis process was revealed. In this chapter, discussions and results on the pilot study conducted are presented.

Qualitative methods were employed and data collected via a questionnaire administered to a group of calculus students' (n = 23). The questionnaire was designed to give an insight into their knowledge of and skill with functions, composition of functions, differentiation and the chain rule. The data were analyzed to investigate their performance on the composition of functions items as related to that of the chain rule. Follow-up interviews based on some questionnaire responses were conducted with five subjects. The second part of the pilot study involved interviews with some participants who answered the questionnaire. This was done to describe how those students constructed the concept of the chain rule. The six participants in the interviews were chosen based on their scores in Part 1, covering a range of chain rule scores and a range of overall scores. The interview followed a structure designed to elicit the student's understanding of the rule based on tasks from the previous instrument.

5.2 Analysis and discussion of items from questionnaire

A questionnaire consisting of twelve items was administered to the 23 students. The items addressed the following skills in the given sequence: (1) Items 1 and 2 focused on whether a given graph represented a function or not, (2) Items 3 and 4 focused on the understanding of composition of functions, (3) Items 5.1 to 5.6 dealt with students’ applications of rules for derivatives, including the chain rule, and (4) Items 6.1and 6.2 focused on integration where the chain rule is embedded in the structure of the integrand. The 12 items were coded (scored) using a 5 point rubric based on the following guidelines adapted and modified from Carlson (1998).

Score	**Description of mental action**	**Behaviors**
5	Made all the mental constructions proposed in the genetic decomposition regarding the concept tested	A complete response to all aspects of the item and indicating complete mathematical understanding of the concept
4	Made most of the bits and pieces of mental constructions of the concept	A partially complete response with minor computational errors, demonstrating understanding of the main idea of the problem
3	Displaying few mental constructions of the concept, with some explanations	Not totally complete in response to all aspects of the item and incomplete reasoning.
2	Displaying few mental constructions with no explanations	No reasoning to justify written response
1	Showed no mental constructions of concept at all	No written response

Table 5.1: Scoring codes used

These guidelines were used to construct specific rubrics for each item. The analysis of the results was based on two considerations: (1) the initial genetic decomposition of the concept of the chain rule was used to guide the researcher's teaching instruction in class and guided the construction of the interview tasks used, and (2) Piaget's Triad mechanism which consists of three stages. These are referred to as Intra-; Inter- and Trans- which display the development of connections an individual can make between particular constructs within the schema, as well as the coherence of these constructions, Dubinsky (1991). The Intra stage focuses on 'a single object' followed by Inter which is 'study of transformations between objects' and Trans noted as 'schema development connecting actions, processes and objects', Dubinsky (1991).

Learners' written work served as a critical source of validation for the questionnaire. By analysing what the learners wrote, the researcher gained an understanding of how students negotiated the mathematics embedded in the context. Figure 5.1 gives the summary of the scores gained by the participants in each category based on the above description.

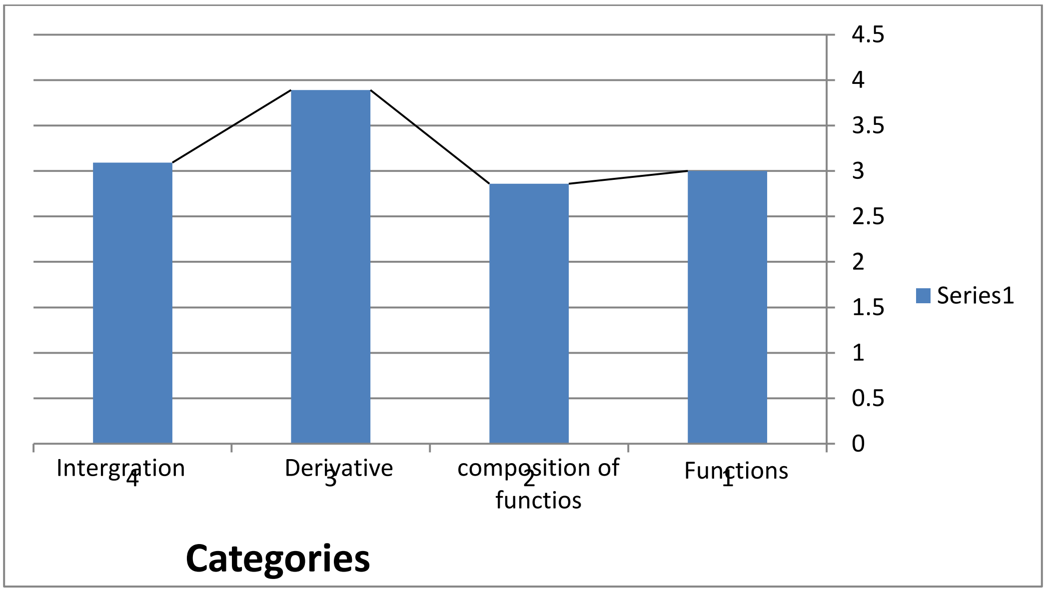


Figure 5.1: Bar graph displaying mean scores for each category

From the above graph the scores revealed a lower mean score on answers displayed for composition of functions and a higher mean score for the derivative category. This indicates that most students presented correct answers for the latter category even though they did not clearly understand the composition and decomposition of functions.

5.3 Category A: Functions

This category had two items focusing on whether a given graph indicated a function or not. The scoring codes indicated in Table 1 were adapted for this category, as indicated in Table 5.2.

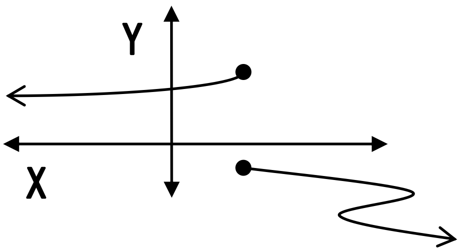
Score	1	2	3	4	5
Indicator	Yes, with incorrect or no response	Yes, but shows some understanding of function concept	No, with incorrect explanation	No, with use of vertical line test, without elaboration	No, with a correct explanation

Table 5.2: Category A: Attainment of mean scores

Scores were allocated as follows: 5 for “No” with a correct explanation; 4 for “No” and use of the vertical line test (without elaboration); 3 for “No” and incorrect explanations; 2 for “Yes” but response shows knowledge of function concept and 1 for “Yes” and any other incorrect or no explanation.

5.3.1 Item 1

Is a student correct to identify the following as a function?



Explain.

Figure 5.2: Graph for item 1

The item was designed to test students' understanding of the concept of functions represented graphically. It was of interest to know if they would mention the vertical test line with elaboration, continuity of the functions or show misunderstanding of the closed circles in their explanations.

No, the Student is not correct, If you draw a verti
line (vertical line test for a function, which states th
a vertical line drawn must cut the graph at one
and only one point, and that graph is then said
to be function) touches the graph more than once
since the graph are continous. A vertical line touches
Both the dots that signifies that the graph is contin
thus the graph is said to be an a function not

Extract 5.1: Written response of S 17

The results showed that 13 students out of 23 displayed a clear understanding of graphical representation of a function. They gave a complete response to all aspects of the item indicating a good mathematical understanding of functions represented graphically. For example the response of S17, who was given a score of 3, is shown in Figure 5.1. He had one reason though for the graph to be classified as a non-function. Two students also mentioned that the presence of the zigzag part in the graph as a non- function. The revised questionnaire would therefore still have closed dots in the graph, and a zigzag on the right lower side. We could claim that according to APOS, the students showed a development of process or object conception of a function concept. According to the Triad, these students were operating in the Trans- stage since they were able to explain the features of a function both graphically and by use of ordered pairs. These were not separate entities for them. Here the action had been interiorised into a function object.

In an interview with S4, she indicated that if a vertical test line was done, it would touch the graph more than once on the zigzag part of the graph. Questioned further, she said:

Researcher: Would it make any difference if one of the dots was opened?

S4: *To me no difference, I don't really understand what the dots mean*.

In the main study questionnaire it was decided that Item 1 would not be revised and a diagram with the closed dots will be presented with the zigzag part of the graph. This would ensure understanding of the vertical line test and the understanding of continuity of the function.

5.3.2 Item 2

A given correspondence associates 3 with each positive number, -3 with each negative number and 1 with 0. A student has marked the afore-mentioned relationship as a function. Is that correct, support your answer?

This item focused on the students' understanding of the concept of a function represented as a set of ordered pairs.

It was important to know whether students would associate elements in the domain with those in the range using mapping and interpreting. This would indicate whether they are the process, object, or schema stage of APOS. The responses to Item 2 indicated that most students had forgotten about the mapping concept of a function. Even those students, who knew that the relationship was not a function, could not substantiate their decisions with explanations reflecting knowledge of functions using mapping. Only 5 out of 23 students displayed understanding of the function concept for this item. Interestingly all 5 students gave full explanations for Item1. This suggests that they had adequately connected schema which incorporated vertical line test, continuity and mappings. Item 2 is a very suitable item and it was decided that it would be included without changes in the main instrument.

5.4 Category B: Composition of functions

This category consisted of two items focusing on composition of functions. This category is included to establish whether the students had a process or object understanding with regards to composition of functions. Item 3 dealt with composition and of functions using the '∘' notation.

5.4 1 Item 3

Given two functions, $f(x)$, and $g(x)$ such that $f(x)=4e^{x}$ and $g(x)=3\sin x$. Find $(f \circ g)\pi$.

The scores were allocated as indicated in Table 5.3.

Score	1	2	3	4	5
Indicator	for evaluating $f(\pi)$ and $g(\pi)$, and stopped without further computation	for subtraction of functions and then evaluating at π	for finding $(g \circ f)(\pi)$	for arriving at $f(g(\pi))=4e^{3\sin\pi}$	a completely correct response

Table 5.3: Item 3 attainment of mean scores

The results showed that most students experienced difficulty in dealing with the composition of functions (only 8 out of 23 displayed complete understanding of composition of functions). Even more students experienced problems with their decompositions. Most of them indicated that they did not understand the ‘ ’ in between the functions and hence could not come up with the correct computations. It was noticed that the few students who showed evidence of working with composite functions at the object stage of APOS were operating in the action stage when they had to decompose the functions. They lacked the ability to reverse their thought processes of previously interiorised actions. Those students were restricted to the action stage of the concept of a function concept. Their idea of a function was contained in a ‘formula’ (Dubinsky, 1991a). The process of co-ordination of composition of functions with derivatives was incomplete and thus the new process of decomposition could not be formed. Sfard (1992) refers to this as condensation where there is a gradual quantitative change in which a sequence of mathematical operations component steps’ (p 28).

5.4.2 Item 4

Given that $(f \circ g)(x) = -10 \sin 5x \cos 5x$

4.1 Find functions f and g that satisfy this condition.

4.2 Is there more than one answer to part (a)? Explain.

This item required decomposition of a composite function and was scored Table 5.4.

Score	1	2	3	4	5
Indicator	left blank	for correct expressions unlabeled and 1 for 4	for “No” for part 4.2; reversed labels (process); or incorrect answer for 4.1 and “Yes” for 4.2	misunderstanding the ‘$\circ$’notation but indicated understanding of composition	for two correctly labeled pairs for 4.1 and 4.2

Table 5.4: Item 4 attainment of mean scores

Only 3 out of 23 students found correct functions for 4.1 and displayed complete understanding of decomposition of the given composite function. With the process conception of a function, an individual can link two or more processes to construct a composition, or reverse the process to obtain the original functions. It was expected that students attempting the chain rule at least work at this level. It was found that 15 students operated in the action stage of APOS regarding decomposing the given function. They did not know which steps to take because they were restricted to the formula interpretation of the composed function. Those students were unable to come up with two or more functions to reverse the given composed function. It was decided that this item would be included in the main instrument for the study, without changes.

5.5 Category C: Derivative

This category consisted of six items dealing with differentiation of functions.

5.5.1 Item 5.1 to 5.6

Differentiate the following, with respect to x:

5.1 $y = -3\sin x + 2e^{\cos x} - 5e^{x}$	5.4 $y = \sin^2(4x^2 + e^{\sqrt{2x - \cos e}})$
5.2 $y = \cos(2x-5)^3$	5.5 $f(x) = \ln[\cos(7x)]$
5.3 $f(x) = \sin^3(4x)$	5.6 $y = (\cos ec^3 x + e^{\tan x})^2$

For item 5.1 score allocation followed the guide in Table 5.5.

Score	1	2	3	4	5
Indicator	a guessed answer	not using appropriate rule; or compound errors with rules	involving elementary differentiation rules	error with derivative of trigonometric functions	correct answer

Table 5.5: Item 5.1 attainment of mean scores

Most students differentiated correctly in this item, (16 out of 23). They mostly operated in the object stage of APOS regarding this item with minor errors of signs occasionally. One would say that the action of differentiating was applied to the process and the process of differentiation was correctly encapsulated to form an object for each student.

Items 5.2 to 5.6 dealt with evaluation of the derivative using the chain rule. Scores were allocated as indicated in Table 5.6.

Score	1	2	3	4	5
Indicator	no evidence of considering chain rule.	applied chain rule indiscriminately; or attempted to avoid chain rule by expanding/rearranging terms (Item 5.2)	If computed mixing composition	a minor error such as dropping (−) sign or arithmetic errors; or applied chain rule but error with derivative rule	a correct answer

Table 5.6: Attainment of mean scores for items 5.2 to 5.6

S16 (Mzi') wrote an explanation of how he got to his answers.

5.2 $y = \cos(2x-5)^3$

$$\frac{dy}{dx} = 3[\cos(2x-5)]^2 . -\sin(2x-5) . 2$$

CHAIN RULE

DIFFERENTIATE EVERY THING "PEEL THE SUM LIKE AN ONION." STARTING WITH THE POWER.

5.3 $f(x) = \sin^3(4x)$

$$f(x) = \sin^3(4x)$$

$$f(x) = 3[\sin(4x)]^2 . 4$$

I took THE EXPONENT AND move it to the coefficient.

I do not QUIET REMEMBER

Extract 5.2: Mzi's response to Items 5.2 and 5.3

When Mzi was asked what he meant in question 5.2 when he wrote differentiation has to be like peeling an onion, he said:

Mzi: *To me differentiating such a function that is so loaded with many other functions is like peeling an onion, taking it layer by layer until you get to the inner one.*

Researcher: What do you mean?

Mzi: *In 5.6 for example, you can start off with the power outside, differentiate with respect to it. Jaa…., You have to imagine everything inside the bracket as one function, effect the power first and then come inside the bracket.*

Researcher: Then what?

Mzi: *Oh Ma….am, you see now, all you do is to attend to each function, find its derivative and keep on multiplying with every result until you finish.*

Researcher: Is there some rule that guided you in your differentiation?

Mzi: *Ja, the chain rule.*

Researcher: What does it say?

Mzi: *Do you want me to put it in symbols*?

Researcher: Yes if you can.

Mzi: *I cannot be able to put it in symbols. But I can show you another way of doing the same problem.*

Researcher: How?

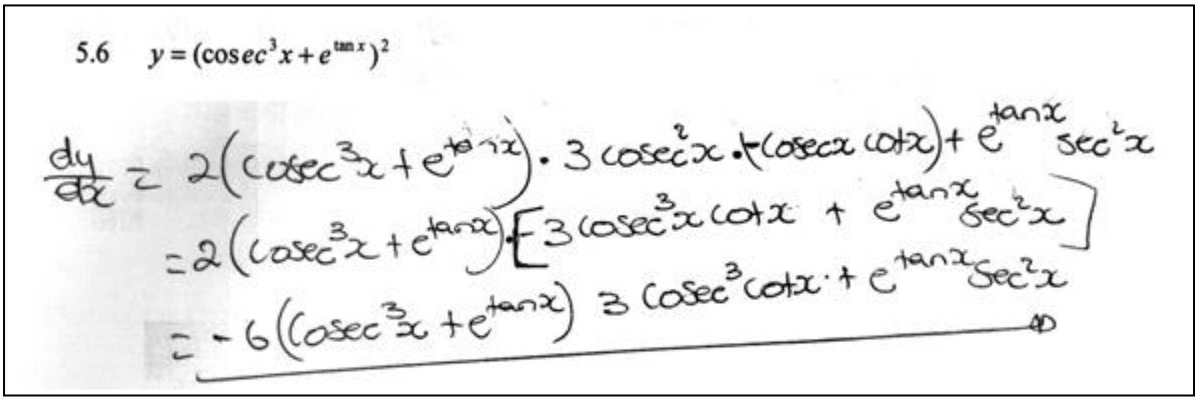
Mzi: *Where you substitute all the functions inside with symbols u, v. w. etc and then find the different derivatives, after which you multiply each result you get, it works just like the chain rule, but it's too long, I don't like it.*

Mzi was scored 2 for 5.2 and 3 for 5.3. Figure 5.2 shows some of Mzi's written responses. He knew the chain rule but did not apply it correctly in 5.2. Indeed in his explanation, one could say he displayed an object stage understanding of the chain rule. He seemed to have learnt and knew the chain rule and was applying it with understanding to different problems except 5.2, as he explained. He was (1) able to access the chain rule as per need, (2) able to reflect on it by paying conscious attention to techniques and algorithms used in dealing with chain rule, and (3) able to understand all the procedures involved in performing calculations involving the chain rule when interviewed.

Three of the students interviewed acknowledged learning the chain rule, and could not express it though they were able to apply it correctly. Also during interviews it was clear that, about 60% of the students except Mzi, interchanged function composition with function multiplication.

Claire (S20) only thought about revisiting her answer in function decomposition and wrote it correctly. She said it then came to her that the previous question dealt with function compositions. Even though she had a good idea of how the chain rule worked, she regarded the structure of the Items 5.2 and 5.3 as the same. She was therefore operating in the action stage of APOS where she had a problem with how the different composition of functions was displayed. Thus although she seemed okay in explaining how differentiation could be done, she could not do it because she did not understand the composition of functions. This supports the finding of Frid (2004), who found that some subjects committed errors in applying the differentiation rules. He further found that in differentiating a composite function, the subjects did not apply the chain rule appropriately as they could not recognize a composite function, and the subjects did not differentiate an inner function after the outer function.

S23, S8 and S13 displayed a schema understanding of differentiation. S8 chose to use natural logarithms after which she applied the chain rule correctly. When she was asked why she used this method, she explained that she had problems dealing with the cosine of a function cubed, and wanted to be sure that the cosine itself was not cubed. She also claimed that Items 5.3, 5.4 and 5.5 were simple for her, as she just had to use the chain rule. She had developed a process conception of differentiation. She then used the previously constructed schemas of functions, composition of functions and derivative to apply the chain rule. This student also recognized a given function as the composition of two or more functions, took their derivatives separately and then multiplied them. Her work for Item 5.6 is displayed in the following extract.



5.6 $y=(\text{cosec}^3 x+e^{\tan x})^2$

$$\frac{dy}{dx}=2(\text{cosec}^3x+e^{\tan x})\cdot 3\,\text{cosec}^2x\cdot(-\text{cosec}\,x\cot x)+e^{\tan x}\sec^2x$$

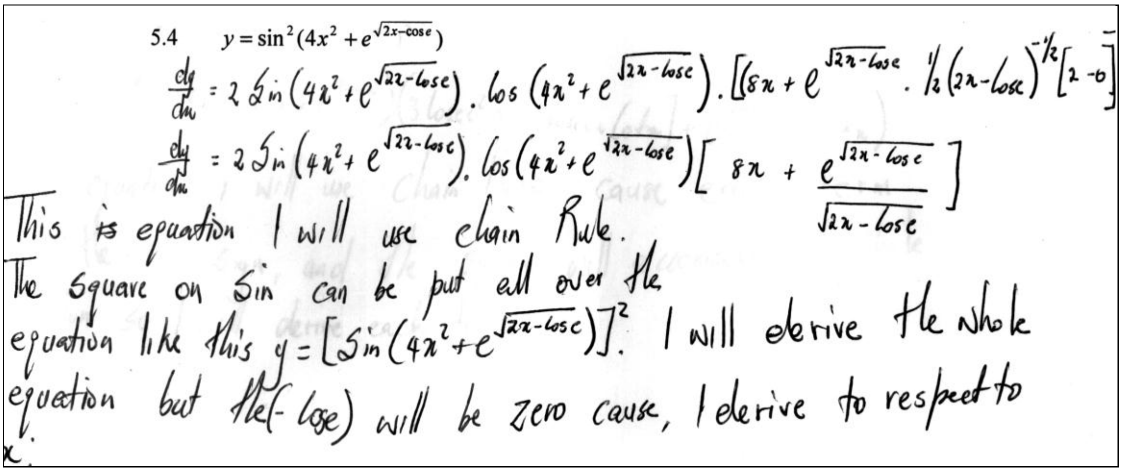
$$=2(\text{cosec}^3x+e^{\tan x})[3\,\text{cosec}^3x\cot x+e^{\tan x}\sec^2x]$$

$$=2(\text{cosec}^3x+e^{\tan x})\,3\,\text{cosec}^3\cot x\cdot+e^{\tan x}\sec^2x$$

$$=-6((\text{cosec}^3x+e^{\tan x})$$

Extract 5.3: S 8's response to item 5.6

S13 presented the use of the chain rule as in Figure 13



5.4 $y=\sin^2(4x^2+e^{\sqrt{2x-\cos e}})$

$$\frac{dy}{dx}=2\sin(4x^2+e^{\sqrt{2x-\cos e}})\,.\,\cos(4x^2+e^{\sqrt{2x-\cos e}})\,.\,[(8x+e^{\sqrt{2x-\cos e}}\,.\,\tfrac{1}{2}(2x-\cos e)^{-1/2}[2-0]]$$

$$\frac{dy}{dx}=2\sin(4x^2+e^{\sqrt{2x-\cos e}})\cos(4x^2+e^{\sqrt{2x-\cos e}})\left[8x+\frac{e^{\sqrt{2x-\cos e}}}{\sqrt{2x-\cos e}}\right]$$

This ~~is~~ equation I will use chain Rule.
The Square on Sin can be put all over the equation like this $y=[\sin(4x^2+e^{\sqrt{2x-\cos c}})]^2$. I will ederive the whole equation but the(- cose) will be zero cause, I ederive to respect to x.

Extract 5.4: S13's response to item 5.4

When interviewed student S13 had difficulty in stating the chain rule in symbols, but was able to explain how he applied it. When asked why he could not state the rule, he responded:

S13: *Why should I learn it? It's never asked for in any examination or test. As long as I know how it works and can get my sums correctly, why bother?*

Thus, in terms of understanding the chain rule and relating it to function-composition, S13 was operating in the process stage of APOS. His was able to interpret functions as objects to develop processes corresponding to differentiation and to put it all together and apply the chain rule.

Loyiso's (S23) responses on this category were captured as follows:

Researcher: Tell me, or explain how did you come up with your answer in question 5.2?

Loyiso: *You see, eh, in* $y = \cos(2x-5)^3$*, I simply expanded the angle, and multiplied out.*

Researcher: Why would you do that?

Loyiso: *Because it was easier for me that way.*

Loyiso was scored 5 in this question as he was able to differentiate the resulting function after expansion correctly. He avoided using the chain rule in finding the desired derivative and there was no evidence of interiorisation of actions in that regard. Loyiso displayed encapsulation of the processes of differentiation in totality besides using the chain rule. This is a feature that was identified by Bransford et al (2000), that understanding a mathematical concept involves a connection to mental representations which are part of an internal network. These networks of previously acquired mathematical knowledge were used by Loyiso in responding to Item 5.2. The researcher's continued interview with Loyiso in the next tasks revealed more:

Researcher: And in 5.3?

Loyiso: *I did the same thing, I tried to expand the function, so I split it up as,* $f(x) = \sin^2(4x)\sin(4x)$.

Researcher: And then?

Loyiso: *Ngabon' ukuthi ngisebenzise ama identities, lapho u* $\sin^2 4x$ *ngimenze u* $1-2\cos(8x)$*, ngabe sengiya multiplaya ngo* $\sin 4x$*, maqede ke ekugcineni ngathola iderivative. (I thought that I must use the identities where I changed* $\sin^2 4x$ *to be* $1-2\cos(8x)$*, I then multiplied with* $\sin^2 4x$ *, worked out until I got the derivative.*

Researcher: So did you do the expansion in order to avoid the chain rule? Do you know the chain rule?

Loyiso*: Chain rule, I don't know that, I remember it vividly, I don't know it.*

Researcher: Have you ever heard about it?

Loyiso: *Oh mehm, that was last year, I have forgotten about it and moreover ngangingayizwanga kahle ukuthi ithini noma isebenza kanjani. (I never really understood what it said or how it works).*

Unfortunately, it became explicit as Loyiso was interviewed further that the chain rule was not a popular concept for him. He deliberately avoided it in his calculations. The following functions in Items, 5.4, 5.5 and 5.6 were all composite functions where no substitutions could be made other than using the chain rule. He was scored 2 in both Items 5.2 and 5.3.

Researcher: And in 5.4, what was your approach?

Loyiso: *That was too difficult for me, I just differentiated* $4x^2$ *to get 8x. I then moved to the other function and differentiated it bit by bit.*

Researcher: What do you mean bit by bit?

Loyiso: *The other functions had to be differentiated too one by one.*

Researcher: Are you sure you did not use the chain rule?

Loyiso: *I dont know, I just differentiated*.

He thus differentiated each function and multiplied the result. His work in Item 5.4 displayed a correct complete response. In other words he applied the chain rule without knowing it. This was not evident from the written response and he was scored 5 for Item 5.4. It was therefore interesting to find out why he was not using the chain rule in the other tasks. Loyiso' concept of differentiation using all other rules except the chain rule was complete. His actions of differentiation were not transformed in his mind such that the processes of differentiation could not be encapsulated in totality. He had no process or object conception of a composition of functions. This became most evident as he was interviewed further.

Researcher: Alright then, tell me how you approached 5.5

Loyiso: *Well in 5.5, there were three functions, ln, cos7x and 7x itself. You just have to differentiate each one of them and multiply.*

Researcher: If that is the case, why did you not differentiate *7x?*

Loyiso: '*Eish……… I simply forgot, but there are three functions there and eh…, aah, there should have been a seven multiplying, I overlooked that.*

Researcher: Tell me what you did in question 5.6

Loyiso: *In 5.6, I thought I must square the function first, expand it. But now when I got* $cosec^6x$*, I thought there must be a problem and stopped.*

Researcher: Why did you not differentiate further?

Loyiso: *I wasn't sure what to do, so I left it.*

He could not differentiate $y = (\cos ec^3 x + e^{\tan x})^2$ further, after expanding it to be $y = \cos ec^6 x + 2\cos ec^3 x.e^{\tan x} + (e^{\tan x})^2$. One would say that Loyiso has interiorized his actions and objects into a system of operations. He is familiar with a process of differentiation and can carry it out through mental representations. His co-ordination is failing though because he is unable to co-ordinate the composition of functions with derivatives and this is required for the chain rule. Hence he could not use it and tried to avoid it each time. With him the object stage of the chain rule could not be formed because the construction of the chain rule concept understanding was not complete. Although he has a function schema and recognizes the composition of functions, he does not have the chain rule schema. According to the Triad mechanism of Piaget, he is operating in the Inter stage.

Over 60% of students scored 4 and above in this category. Only 42% of those students used the chain rule in their responses though. Quite a number of students tried to avoid use of the chain rule in their responses. These items will be included without change in the main instrument.

5.6 Category D: The structure of the integrand

This category focused on integration where the chain rule is embedded in the structure of the integrand. It consisted of two items.

5.6.1 Items 6.1 and 6.2

Evaluate:

6.1 $\int 2x\sqrt{1+x^2}\,dx$

6.2 $\int (3x+2)^6\,dx$

It was important to find out how students would use the chain rule, to find the answer to the given problems. The table of standard integrals was therefore not given to them. Scores were allocated as indicated in Table 5.7.

Score	1	2	3	4	5
Indicator	incorrect answer	not interpreting the root sign correctly	minor errors of putting incorrect signs	identifying the function, its derivative and then applying standard integrals	for correct answer using chain rule

Table 5.7: Allocation of scores for category D

For Items 6.1 and 6.2, S13 displayed what Piaget refers to as applying an existing schema to a wide range of contexts. He could deal with reversal of the chain rule where the new process response required by questions 6.1 and 6.2 was constructed.

But for Loyiso, it was a different story. In a continuation of the interview the researcher had with Loyiso, he said:

Researcher: Would you explain to me how you did 6.1?

Loyiso: *I just looked at what is inside the bracket*

Researcher: Which bracket?

Loyiso: *This root sign (pointing at the question). I think of 2x, the derivative of* x^2.

Researcher: What about it?

Loyiso: *If it was* $\int 3x\sqrt{1+x^2}\,dx$ *(he writes down roughly), I would write down* $\frac{1}{3}\int 3x\sqrt{1+x^2}\,dx$ *to make up for the derivative, uuhm… no, I will put* $\frac{2}{3}\int 2x\sqrt{1+x^2}\,dx$, *now it would be right because it will be like that one in the table.*

Researcher: Which table are you talking about? Also, aren't you referring to 6.2?

Loyiso: *Yes, I am now confused. The table is the one that is always given when we write, with standard integrals.*

Researcher: Would you by any chance use the chain rule?

Loyiso: *No, I told you I don't remember anything about the chain rule. Why should I use it anyway, because I think, ekufanele ngikwenze wuku- organiz(a) (what I should do is to organize) what has been given to me, suit one of the integrals given in the list (standard integrals) and when I find it, just write down the answer.*

Loyiso seems to be relying mostly on the table of standard integrals and thus it is not easy to assert that he used the chain rule in his calculations even though he got correct answers to Items, 6.1 and 6.2.

Students in the Inter- stage will show evidence of having collected some or all the differentiation and integration rules in a group and perhaps provide the general statement of the chain rule without yet constructing the underlying structure of the relationships. That student would tackle $\frac{dy}{dx}$ of $y=(\cos ec^3 x + e^{\tan x})^2$, by applying the power rule, not sure that he/ she is using the chain rule. This student during interviews, and further questioning would explain the connection between his general statement of the chain rule and its applicability. Lastly a student who displays coherence of understanding of a collection of derivative rules and of composition of functions as a schema will have moved to the Trans- stage of development. He will be able to reflect on the explicit structure of the chain rule and he/she will be capable of operating on the mental constructions which makes up his collection. Without stating the chain rule, this student will be able to use it proficiently. The student should at this stage be able to link function composition and decomposition to differentiation and integration and to link the two.

5.7 Conclusions and Implications

This chapter provided the preliminary results on the pilot study conducted to validate the research instrument. It was evident from the results displayed and interviews that followed that even though students wrote down the correct answer, they were not always thinking of the correct answer. There was no need for major revisions to the questionnaire. The data obtained from the written responses to the items provided sufficient information to understand the mental constructions students made in conceptualizing the chain rule. The responses also provided evidence that coincided with the proposed genetic decomposition of the chain rule.

In the next chapter, the report on three sequential video-taped lessons presented by the researcher on the chain rule and its applications is discussed. The analysis of data is discussed in detail in the chapters that follow.

CHAPTER 6

ANALYSIS OF VIDEO RECORDINGS

6.1 Introduction

This chapter reports on instructional design video-recorded in three sequential lessons based on the introduction and use of the chain rule to first year university engineering students in the first semester of 2011. The lessons were video recorded during class time in slots of 1 hour duration. Those lessons followed immediately after the students had done an introduction to calculus, differentiation using the product and quotient rules. Here the researcher discusses the presentation of those lessons and the interactions between the researcher and the students, in the classrooms. Confidentiality and anonymity has been ascertained by changing the names of the subjects and using fictitious names for all participants in the study.

6.2 The structure of the lessons

The purpose of the first lesson was to introduce the learners to the concept of the chain rule and how it could be used to differentiate composite trigonometric functions. This was done using instructional design as part of the research framework, proposed in chapter three. It was important to know how an APOS analysis could help the researcher to understand the learning process by providing explanations on phenomena observed when students tried to construct understanding of the chain rule. Weyer (2010) asserts that the first circumstance to happen should be something that happens as a result of instructional practice. Also constructivists, Piaget (1966) and Vygotsky (1986) have insisted that the mental constructions that a person uses to understand a mathematical concept are made in a social context and with considerable intervention from teachers and fellow students. Thus mathematics pedagogy based on the said theories approaches mathematics as a conceptual system rather than discrete procedures. Genuine education depends both on the knowledge and experience already existing within a student's level of development as well as on the student's potential to learn (Vygotsky, 1986). Beth and Piaget, (1966) argue that if we observe a person moving from 'not understanding a concept' to 'understanding it', we cannot be able to see what mental mechanisms are used, also we cannot tell if these mechanisms are used to construct the concept or to gain access to it. Thus we felt it necessary to video-record the students in action so that we could go beyond observation and consider their conversations, written texts and their interactions with peers and instructor.

The main features of lesson one and the other two lessons are discussed in detail in the following sections. Students in the three sequential lessons were instructed with the intention of increasing their understanding of the chain rule to the object or schema stages of APOS. Lesson one took place in a laboratory where thirty first year civil students assembled for their tutorial lesson. Other sub-groups of such students were having parallel sessions of tutorials in other classrooms conducted by other lecturers in the department of mathematics. All these subgroups were introduced to the chain rule and its applications by the different instructors. Lesson two mainly dealt with exposing students to differentiation using the chain rule in the context of logarithmic equations. In lesson three the researcher issued learners with tasks on worksheets where learners worked collaboratively with each other. Those tasks were given to (1) consolidate the work done in the two previous lessons, (2) allow reflection as a major source of mathematical knowledge, (3) find out the types of structures constructed by students when learning the chain rule, and (4) determining the students' actual engagement with tasks and how these tasks link with the expected outcomes highlighted in the initial genetic decomposition.

6.3 Lesson one: Presentation and discussions

The students were seated around small tables that were arranged to fit the rectangular shape of the laboratory. Tables were arranged in this classroom such that six students occupied seats around each table. This was a suitable venue for them when they attend their tutorial lessons.

The first lesson started with students listening to the introduction and definition of the chain rule by the researcher. After the discussion of four examples on the whiteboard, the students were given time in class to work on the solutions of problems based on the use of the chain rule. Volunteers for some examples were requested to write their answers on the whiteboard. This happened when the lecturer (researcher) moved from desk to desk looking at how the rest of the students presented the solutions in their workbooks.

Activities to construct the relevant mental structures for concepts necessary prior to the study of the chain rule were deemed covered. So our belief was that most students had the necessary mental constructions described in the genetic decomposition before they were introduced to the chain rule. The chain rule was then defined clearly for purposes that the students should not only know it but have to remember it, use it and apply it to various problems. At this stage students were discouraged from copying problems solved on the whiteboard as the emphasis was on understanding the chain rule. Students usually wrote notes while the lecturer explained because the whiteboard used in the classroom was small and they wanted to capture everything written on it before the information was erased to accommodate other examples. This partly contradicts Felder (1996) who suggested that the extent to which a student learns depend on the compatibility of his characteristic approach to learning and the instructor's characteristic approach to teaching. The instructor in this lesson used the demonstration method to enforce understanding of the concept of the chain rule in this class.

The only available resource we had for the lessons were either a whiteboard or chalkboard. Interventions and selected approaches were used to strengthen the learning aspirations of the students. As an experienced lecturer I had to use scaffolding to help the learners to move from one level of mathematical understanding to another. Tools and strategies which assist students to attain a higher level of understanding by encouraging creative and divergent thinking are known as scaffolds (Brush & Saye, 2001; Mccosker & Diezman, 2009). Anghileri (2006) asserts that students actively construct meaning as they engage significantly within established mathematical practices. These tools in a mathematics classroom could include diagrams, pictures, technology, mathematics formulas and hints for an effective solution process.

Peer collaboration was encouraged by allowing students to work on mathematical tasks given during the lesson. This collaboration, when combined with effective sequencing and pacing of the lesson, contributes to the teaching and learning of mathematics. Pacing and sequencing refers to the way the lecturer moves from one concept to the next within the mathematics topic to ensure maximum use of instruction time. Feedback was also used by the lecturer to create an appropriate milieu for effective teaching and learning. There was greater lecturer-student interaction promoted as students were encouraged to verbalise what they saw and thought. They were motivated to explain and justify their written answers. It was through the students' comments, questions and answers that their mental constructions regarding the chain rule were accessed. This was done in an attempt to answer the question, *'How do students construct various structures to recognize and apply the chain rule in the context of calculus? '*

The comparisons between three different techniques were made in chain rule differentiation. The first technique was the one using '*Leibniz form technique*'.

The second one was the one where we differentiate from the innermost function and move outwards. We shall henceforth refer to this method of chain rule differentiation as a '*link form technique*' of the chain rule. The third one involves straight application of the chain rule in differentiation. We shall refer to this method of differentiation as a *'straight form technique'*. In this technique students used the chain rule mechanically by finding the derivatives of all the functions starting with the function on the outside of the given problem and multiplying out. For example, consider differentiating $y = \ln \sin x^3$. We have characterized the three forms of the chain rule: (1) *Leibniz* form technique gives, we let $y = \ln u; then \frac{dy}{du} = \frac{1}{u};$ where $u = \sin v;$ and $v = x^3$ so that $\frac{dv}{dx} = 3x^2; and \frac{du}{dv} = \cos u,$ and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx} = \frac{1}{u} \times \cos u \times 3x^2$. This would give $\frac{3x^2 \cos x^3}{\sin x^3}$. (2) *Link* form technique gives, we get $3x^2 \times \cos x^3 \times \frac{1}{\sin x^3}$. (3) Using the *Straight* form technique we get, $\frac{1}{\sin x^3} \times \cos x^3 \times 3x^2$. Answers using the three techniques were simplified to see if they were the same.

The lesson was introduced by discussion of the differentiation of the problem, '$y = (x^3 - 3)^3$'. The given function was identified as a composite function and a *Leibniz form* of differentiation was used where $x^3 - 3$ was represented by *u*. Its derivative was then indicated as $\frac{du}{dx} = 3x^2$. The function, u^3, was then differentiated to get $\frac{dy}{du} = 3u^2$ and $\frac{dy}{dx} = \frac{dy}{du} . \frac{du}{dx}$ $= 3u^2 . 3x^2 = 3(x^3 - 3)^2 . 3x^2$. $= 9x^2 (x^3 - 3)^2$ The presentation on the whiteboard was captured in the video-clip of Figure 6.1.

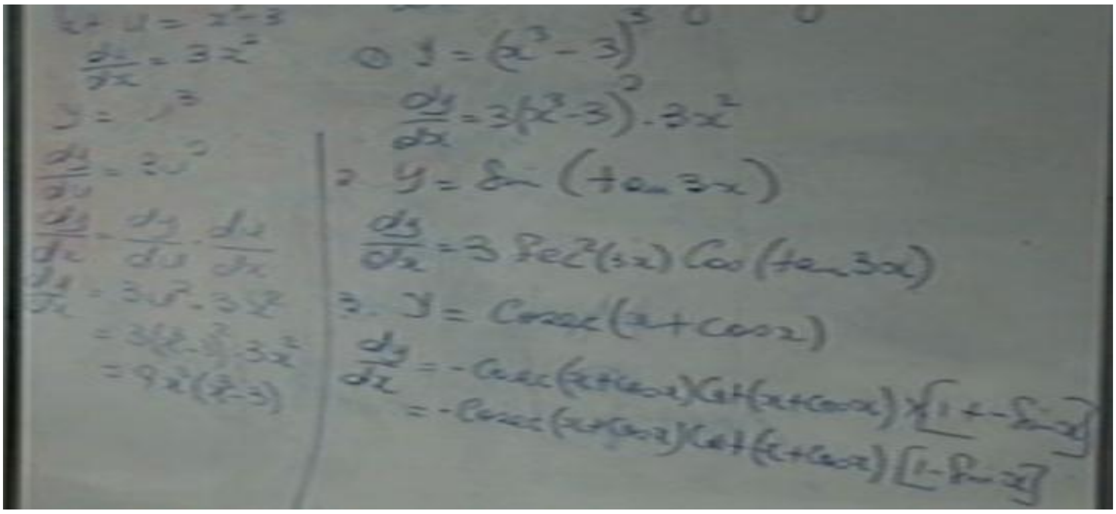


Figure 6.1: Leibniz, straight and link form techniques of the chain rule application.

Various examples on differentiation using the chain rule were discussed with the students in this class. These included differentiation of composite functions like: (1) $f(x) = \sin(\tan 3x)$; (2) $y = \cos ec(x + \cos x)$ (3) $y = \sin^3(4x)$ (4) $y = \ln(x^2.\cos x)$ (5) $f(x) = x^{2x}$ and (6) $f(x) = \ln \dfrac{x^5(1-x^2)^6}{x(\sqrt{1-x^3})}$. The other tasks that the students had to engage in appear in Appendix B3.

After the second problem, one of the students asked, '*Is it a must that all the time when differentiating we start at the end of the problem, can't we start from the beginning, will that make a difference?'* The technique used to differentiate $y = \cos ec(x + \cos x)$ was the *straight* form technique (see Figure 6.1, number 3). Differentiation using the chain rule proceeded from the outward function to the innermost function. This was then presented as a solution, $y' = -\cos ec(x + \cos x)\cot(x + \cos x).(1 + (-\sin x))$. After the researcher explained that this would come to the same answer whichever way it was done, a further question came from the same student, '*Ngamany'amazwi bowubalekel'ibracket? (In other words you did not want to use the bracket?*) To this the researcher responded: 'Not really, but I was trying to show you that you can also start differentiating the given function from the left to the right as long as you are going to differentiate each function.' This meant that differentiation using the chain rule starting from the innermost function ($x + cos\ x$) contained in the brackets and then multiplying by the derivative of the outward function *cosec* of ($x + cos\ x$) would come to the same solution.

A mental or physical transformation of mental or physical objects is considered to be an action when it is a reaction to stimuli which the subject perceives as external (Dubinsky, (1991). This student perceived and reflected on a repeated action of differentiation and wanted to establish control over it. This is why she wanted to verify the direction and steps to be followed when differentiating. The student operated in the action stage at that moment and was thinking about the problem in a step-by-step manner and was looking at one step at a time. The interiorisation of the action then started when the mental mechanism of differentiating composite functions was converted to a process that took place internally in the student's mind. Interiorisation had not been experienced by the said student at that point in time.

The students were working on their own after which they exchanged books for corrections. They then worked collaboratively as unorganized pairs as they compared their solutions. A student who had finished doing the first problem on his own would then exchange his book with another one who had also finished, for marking. The discussion on how one student, arrived at his solution, would then ensue between the two. At this stage each one of the students justified how he or she arrived at his own solution. Most of the time sounds like, '*ooohh…*' were heard as one of them realized his/her mistake. A student working individually on one of the problems given can be viewed in the video-frame of Figure 6.2.

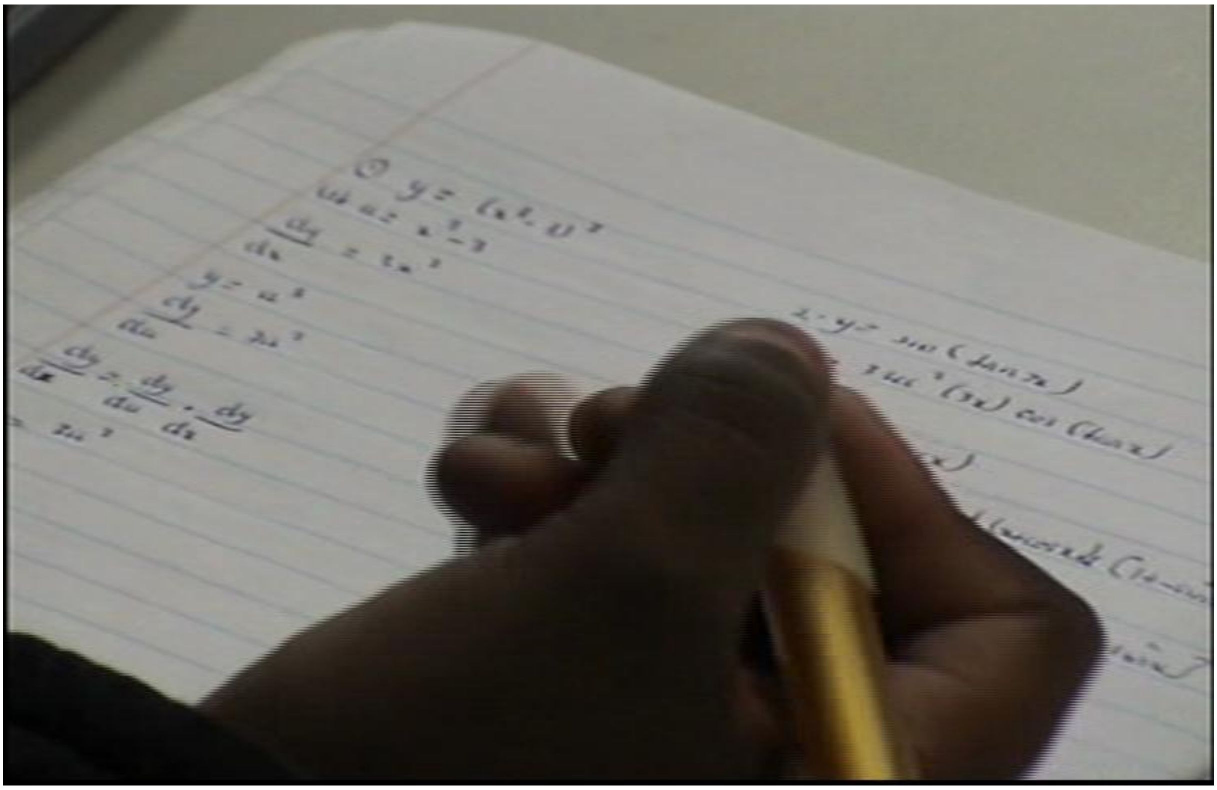


Figure 6.2: Student working individually

Of most interest was the argument that ensued as each of the students in a pair argued for the correctness of his response and the other one also claiming his presentation to be correct. They then called for the researcher to resolve their dispute. The dispute was around the different approaches that the two had followed in finding the solution to: Differentiate $y = \sin^3(4x)$. One of the students presented his solution as $4.\cos(4x).3\sin^2(4x)$ while the other one gave $3\sin^2(4x).\cos(4x).4$. They knew that the two solutions would give the same answer $12\sin^2(4x).\cos(4x)$. The main argument surrounded the direction of differentiation. The representational way of seeing the chain rule, was operational (seeing it from the inside) for one of them, while the other student saw it in a linear way, where he dealt with the differentiation one by one in a straight line. The latter student used the *straight* form technique of the chain rule while the former used the *link* form technique. Those students are said to be reacting to stimuli perceived as external in the action stage. They were treating differentiation as a formula. When they finally realised that both their solutions were correct, the action understanding was converted to process understanding where transformations were interiorised in their minds. Each one of them compared and learnt from each other the difference in approach to the same problem.

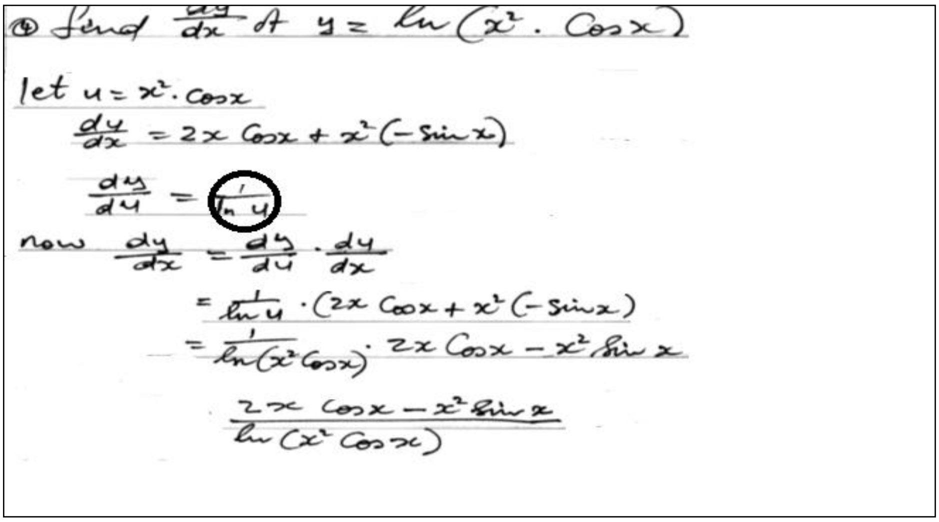
Several times the researcher stopped at the tables and assisted the students with their queries without giving them straight answers (see Figure 6.3). Students also raised their hands to get attention of the researcher to answer queries or questions.

Figure 6.3: Picture of students in the learning laboratory

In the classroom, the students worked collaboratively on mathematics tasks designed to help them apply the mental structures that they were expected to build on in understanding the chain rule. In some cases students worked individually, in pairs or as a group in trying to negotiate a group solution to the problem or sometimes compared their solutions. This is illustrated by the still photo in Figure 6.4.

Figure 6.4: Students working in pairs

After much explanation on how the chain rule works, the problem based on $y = \ln(x^2.\cos x)$ was given as an exercise to try out in class on an individual basis. As a pair of students indicated that they had finished they were instructed to exchange books and to mark each other's work. This allowed them to then work collaboratively. As the other student marked his companion's work, he would shout for the return of his book to correct his own mistake which he detected as he was marking. As the researcher went round from desk to desk checking on the answers displayed, she noticed how one student was arguing with his companion who marked him wrong for a presentation he thought was correct. The answer is displayed in Extract 6.1.



① find $\frac{dy}{dx}$ of $y = \ln(x^2 . \cos x)$

let $u = x^2.\cos x$

$$\frac{dy}{dx} = 2x \cos x + x^2(-\sin x)$$

$$\frac{dy}{du} = \frac{1}{\ln u}$$

now $$\frac{dy}{dx} = \frac{dy}{du} . \frac{dy}{dx}$$

$$= \frac{1}{\ln u} \cdot (2x \cos x + x^2(-\sin x))$$

$$= \frac{1}{\ln(x^2 \cos x)} \cdot 2x \cos x - x^2 \sin x$$

$$\frac{2x \cos x - x^2 \sin x}{\ln(x^2 \cos x)}$$

Extract 6.1: Senzo’s response

The marker, Sbu, drew a circle and question mark in the problem above arguing that Senzo (the owner of the book he marked) could not represent $\frac{dy}{dx}$ with a function in *u* and *x* at the same time. Initially he questioned even the method used by Senzo. When he was reminded of the *Leibniz* form technique used by Senzo who was comfortable with the substitution method, he (Sbu) exclaimed that ‘*It’s a long method anyway*.’ Sbu was operating in the action stage. He only knew how to perform operations on differentiation from memory or clearly given instruction. Senzo had a process conception since he displayed transformation of physical or mental objects perceived as relatively internal and totally under his control. Senzo’s error was to write the derivative of $y = \ln u$ as $\frac{dy}{du} = \frac{1}{\ln u}$ instead of $\frac{1}{u}$. He was able to convince and justify by explaining to Sbu how he found his derivative without deviating from the method used by Sbu in his own solution. It was during redoing the problem and trying to convince Sbu that he noted his mistake and rewrote the derivative as $\frac{1}{u}$. The argument was most interesting as Senzo revisited his presentation to convince Sbu that he understood the use of the *Leibniz* form technique and preferred it over the other techniques. Senzo used conceptual understanding as he identified and applied differentiation principles. The students operating in the action stage care about getting the solution correctly and cannot justify how they arrived at their answers. They think about procedure in terms of its individual steps. They also could not link and adjust their steps of operations to differentiate functions comprising of multiple compositions. Senzo had interiorized actions since he was able to use the process as an internalized procedure.

Problems involving the differentiation of a composite of a greater number of functions were then given so that those who had finished were kept occupied. Often, they would report their results to the class. Sometimes the students lifted up their hands to draw the lecturer's attention to their questions. It was interesting to see that some students pay attention to the explanations given to their companions even though they hadn't posed any questions. In this way lecturer-student interaction was enhanced. The video-frame in Figure 6.5 captures one of the interactions between the lecturer and a student during the tutorial period.

Figure 6.5: Student-lecturer interaction

Throughout the teaching process, the emphasis was on discussion, reflection, explanations by the lecturer where appropriate, success in completing the tasks and understanding the chain rule.

Some of the class work problems given are displayed in Figure 6.6.

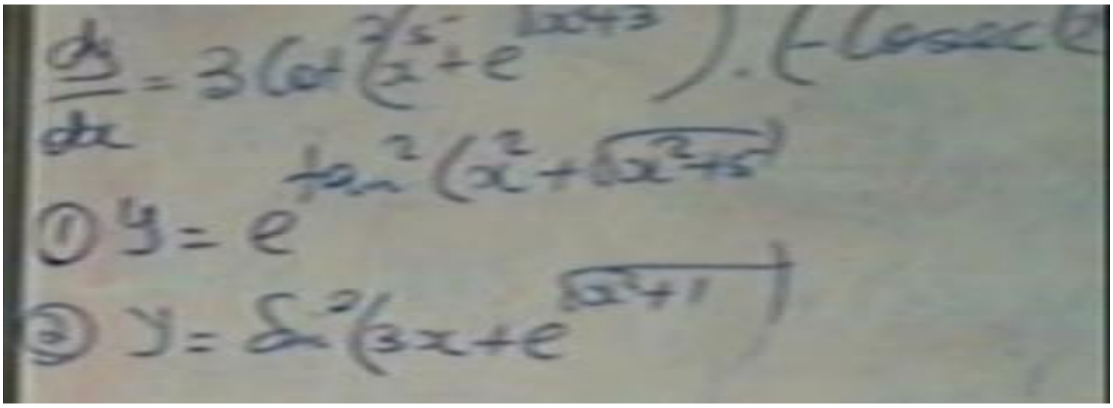


Figure 6.6: Other problems given

The greatest challenge in this classroom was the use of a small white board which was the only means to aid my teaching. After two or more problems I had to erase it time and again as the lesson developed. Buhle asked after the discussion of $f(x) = \ln \frac{x^5(1-x^2)^6}{x(\sqrt{1-x^3})}$. *'Was it going to be wrong to first differentiate with respect to ln function and then after that use the product rule for the other functions?* This would be as a result of slicing the composite function into pieces and not recognizing the *ln* function as an instructing function in the composite. According to APOS the process of differentiation had been encapsulated to an object. This student understood that the derivative with respect to *ln* function was $\frac{1}{\frac{x^5(1-x^2)^6}{x(\sqrt{1-x^3}}}$. He was prepared to then differentiate the remaining composite function using the product and quotient rules. The researcher tried to discourage students from this long chain of differentiation as she feared that they would make unnecessary mistakes, like omitting brackets where they were due. The students were mostly encouraged to first use the logarithmic laws in interpreting the function. This was because logarithmic differentiation was also tested specifically on this section. The construction of mental structures was very spontaneous at first where the student differentiated with respect to the logarithm first and seeing the $\frac{x^5(1-x^2)^6}{x(\sqrt{1-x^3})}$ as one composite function. Specific mental constructions where the encapsulation of the processes of differentiation including the product and quotient rules together with the chain rule were employed by this student successfully. This student is said to be having a schema of the chain rule. For him its' not the complexity of the problem that determines application of the chain rule but rather, actions, processes, objects and a coherence of other schema. He was able to jump back and fourth among the four stages even though they are made in a partially ordered sequence. He showed coherence relative to the concept of the chain rule in the sense that the student had devised some means, explicit or implicit of deciding to stick to the use of the chain rule to sidestep the application of logarithmic rules.

The explanation given by the researcher was that *ln* was the one that was commanding the other functions on how to behave and that the function given could not be rearranged anyhow. An example similar to the original one was given to assist in explaining the operations with logarithmic functions. Perhaps the identification of the dominant function, if there be one, would help the students to eliminate misconceptions before they differentiate.

After twenty five minutes, the problem ‘Differentiate $y = \ln\left[\dfrac{e^x.\sin x}{x\tan x}\right]$ was given as an activity to be done in class. Zonke volunteered to do its solution on the whiteboard (see Figure 6.7), but did not want to give any explanations to the class on how she did the problem. After her clear illustration to the above problem she instructed the class to inspect her solution and pose questions that she would attend to if they needed further clarity. This student is said to be thinking of the chain rule as an object because, she reflects on a need to apply the derivative as a series of manipulations. The mental constructions in the procedures involved in differentiation using the chain rule were transformed by some action into an object that could be seen as a total entity. Encapsulation of the process to object stage was complete. She was aware of the differentiation of this problem as a totality as she constructed transformations on the given problem.

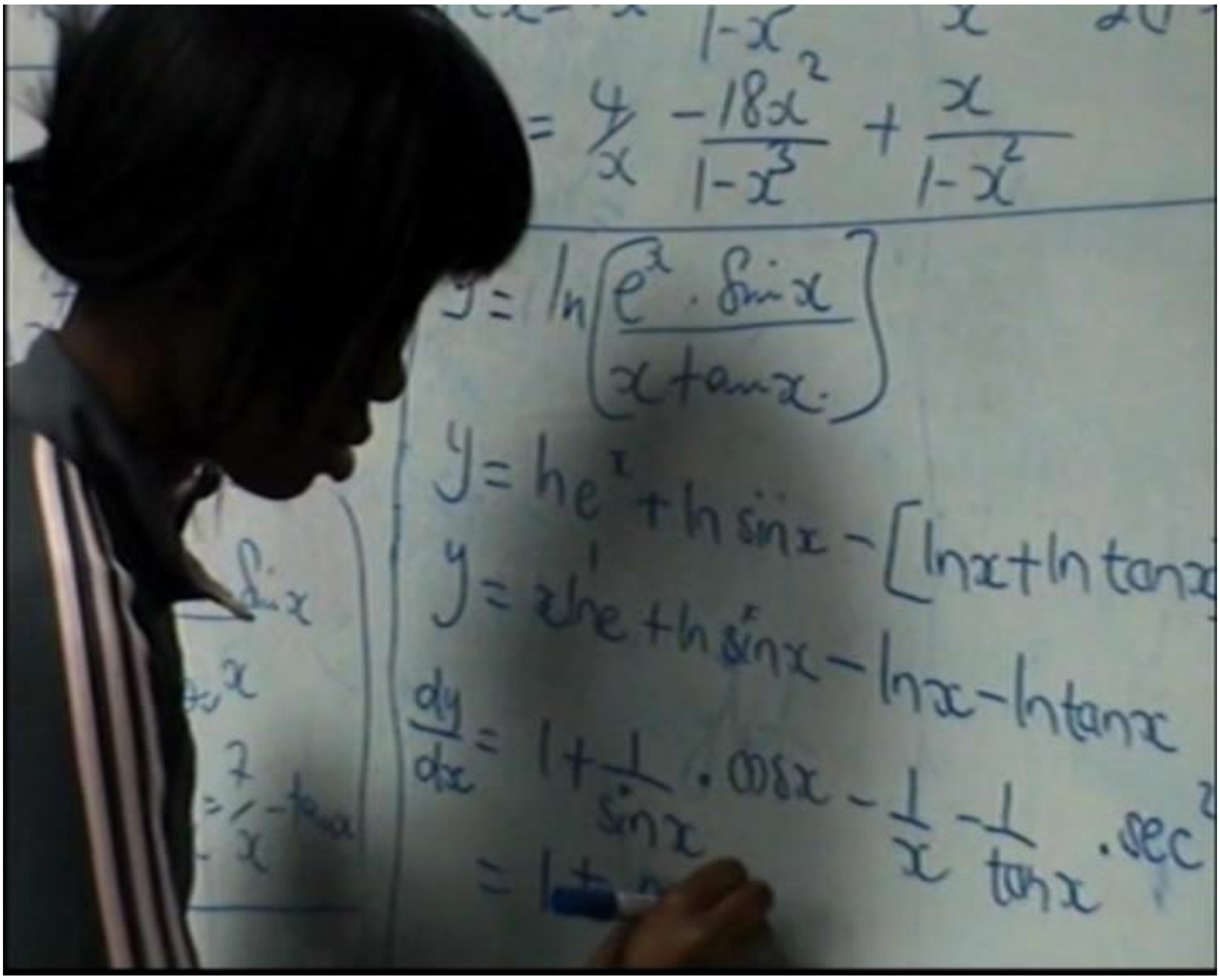


Figure 6.7: Presentation by Zonke

During the lesson the researcher socially interacted with the students to promote learning. Social interaction and use of Zulu language was used by the researcher to engage with the students in order to promote learning. Vygotsky's (1986) learning theory on scaffolding agrees with this. This theory asserts that learning is enhanced through social interaction between the student and the teacher. Vygotsky views the teacher as the one who is able to lift the students' achievement level, and he also claims that immediate assistance whilst supervising the students also lifts their performance.

The activities on differentiating (1) $y = \cot^3(x^5 + e^{\sqrt{x^2+3}})$ and (2) $y = e^{\tan^2(x^2+\sqrt{x^2+5})}$ were given as exercises to be done for homework away from class without the researcher's supervision. It was very interesting to mark Lutho's response to the first homework question displayed as follows in Extract 6.2.

$$y = \cot^3(x^5 + e^{\sqrt{x^2+3}})$$

$$y' = 3\cot^2(x^5 + e^{\sqrt{x^2+3}} \cdot \frac{d}{dx}(\cot(x^5 + e^{\sqrt{x^2+3}})$$

$$y' = 3\cot^2(x^5 + e^{\sqrt{x^2+3}} \cdot (-\mathrm{cosec}^2 \cancel{\cot}(x^5 + e^{\sqrt{x^2+5}}) \frac{d}{dx}(x^5 + e^{\sqrt{x^2+3}})$$

$$y' = 3\cot^2(x^5 + e^{\sqrt{x^2+3}}) \cdot -\mathrm{cosec}^2(x^5 + e^{\sqrt{x^2+5}}) \cdot [5x^4 + e^{\sqrt{x^2+3}} \cdot \frac{d}{dx}\sqrt{x^2+3}]$$

$$y' = 3\cot^2(x^5 + e^{\sqrt{x^2+3}}) \cdot -\mathrm{cosec}^2(x^5 + e^{\sqrt{x^2+3}}) \cdot [5x^4 + e^{\sqrt{x^2+3}} \cdot \frac{1}{2}(x^2+3) \cdot 2]$$

$$= -3\cot^2(x^5 + e^{\sqrt{x^2+3}})\,\mathrm{cosec}^2(x^5 + e^{\sqrt{x^2+3}}) \cdot 5x^4 + e^{\sqrt{x^2+3}} \cdot (x^2+3)$$

Extract 6.2 Lutho's response

When I asked Lutho to tell me how he arrived at his answer, he said, 'First I take care of the power, mhm……

Researcher: yes, go on…

Lutho: *Ngiya differentiyeyitha ipower kuqala, bese wonke ama functions engingekawenzi, ngiloko ngibhala u* $\frac{d}{dx}$ *for everything that I have not differentiated njalo, ngize ngiqede. (I differentiate with respect to the power first, and always write* $\frac{d}{dx}$ *for every function that I haven't differentiated until I finish).*

Angithi yiyona ichain rule leyo, mangabe ngizwe kahle? (Is that not the chain rule?)

Researcher: Oh well I am impressed except that the last part, (pointing at 2)…

Lutho: *Sorry mhem, iphutha lami,(my mistake), supposed to be 2x. (He said that grabbing his pen quickly and rectifying the mistake, putting 2x instead of 2.*

In step two of his solution he left out the second bracket but recaptured it back in the following steps. This was a common error made by other students regarding change in operations after differentiation. Also Lutho presented the derivative of $\sqrt{x^2+3}$ as $\frac{1}{2}(x^2+3)$ instead of $\frac{1}{2}(x^2+3)^{-\frac{1}{2}}$. The latter error led him to multiply $\frac{1}{2}\times 2$ and got (x^2+3). Lutho's solution is a guided notation using the *link* form technique of chain rule application and was used with caution by him initially. He demonstrated his mental construction of the chain rule as an object. At this point in his development he displayed the ability to reverse certain processes. He could trace back his steps of which functions were already and still to be differentiated. According to the Triad, Lutho operated in the Inter- stage with regards to chain rule application since he displayed construction of the underlying structure of the chain rule as an object through reflection on relationships between various processes from previous stages. This was done by putting $\frac{d}{dx}$ before functions still to be differentiated. The two errors are not associated with chain rule applications but ascertain that he did not verify his response. He seemed excited and sure about chain rule application.

6.4 Lesson two: Presentation and discussions

This was a follow up lesson with the same class the following day in a lecture hall at the University of Technology. First year engineering students (197) were assembled in one lecture hall of capacity three hundred students. Not all students attended the lecture, some were absent. All students had already been introduced to the chain rule and its applications during the previous lesson by different lecturers. Exercises were given to students to consolidate the understanding of the concept of the chain rule and to ascertain that all students in the group shared the same understanding of the concept and its applications. The exercises were done in class. Exercises ranged in their level of difficulty from the simple one to more complicated differentiation problem.

A part of the lecture hall where the students were seated in rows is indicated in Figure 6.8.

Figure 6.8: Students seated in a lecture hall

The main activities in the lecture included discussion of examples on differentiation in calculus, using the chain rule.

It was promising to learn that the students were able to pronounce correct answers, telling the researcher what to write on the chalkboard for most of the simple problems. Regarding the problem $\ln\frac{e^{4x}.\sin x}{x\tan x}$, I noticed that students were so keen to use the straight form technique of the chain rule. Zuko said, '*it will be one over everything*.'

Researcher: What is everything? And why are you saying one over?

Zuko: *Look mhem, You differentiate with respect to ln and it becomes one over that whole function, (pointing at* $\frac{e^{4x}.\sin x}{x\tan x}$)

Researcher: Then what?

Zuko: *You just multiply by eh, eh……..*

Researcher: What?

Zuko: *The derivative yale eseleyo ifunction (the remaining function), using product and quotient rule. This is what it will look like:* (he then wrote, see Extract 6.3):

$$y' \text{ of } y = \ln \frac{e^{4x}.\,\text{Sin}\,x}{x \tan x}$$

$$y' = \frac{1}{\frac{e^{4x}.\,\text{Sin}\,x}{x.\tan x}} \times \frac{d}{dx}\left(\frac{e^{4x}.\,\text{Sin}\,x}{x \tan x}\right)$$

$$y' = \frac{1}{\frac{e^{4x}.\,\text{Sin}\,x}{x.\tan x}} \times \left[\frac{4e^{4x}\sin x + e^{4x}\text{Cos}\,x)(x\tan x) - (e^{4x}\,\text{Sin}\,x)[(1)\tan x + x\sec^2 x]}{x^2\tan^2 x}\right]$$

Extract 6.3: Zuko's response

Researcher: Wow! What about interpreting the *ln* function first?

Zuko: *'Well even that can* work *but anything wrong with using the chain rule straight, ngoba mina angifuni kwenza amamistakes using ama ln laws egingekho sure ngawo.'* (I don't want to make mistakes by incorrectly using the *ln* laws because I am not sure of them).

Researcher: And eh…. Ar'nt you scared that this is too long?

Zuko*: No mhe…em, as long as I am sure that it's correct, because I differentiated everything.*

Zuko used the *link* form technique of the chain rule to differentiate at first. He then incorporated the *straight* form technique applying the product and quotient rule correctly. We could say that Zuko has a schema of the chain rule. He has internalized all the processes of differentiation as a procedure. He sees the procedure of differentiation as a whole and applies the chain rule freely. According to APOS, the process has been encapsulated into an object of the concept of the chain rule. He sees the chain rule as a total entity that can be acted upon by actions and processes and transformed as he applies the product and quotient rules. He looked at the given problem as an object where the product and the quotient rule could be exercised. He was able to jump back and forth between a collection of actions, processes, objects and other schemas of differentiation during the use of the chain rule. According to the Piaget's Triad, the student has moved to the Trans- stage of development. He showed evidence of reflecting on the explicit structure of the chain rule, being able to operate on the mental constructions which made up his collection, by using the chain rule proficiently. He had constructed the underlying structure of the chain rule and reflected on relationships between various objects from previous stages. So he had a chain rule schema.

During the lesson each student who had finished doing his/her problem was asked to read out his/her answer so that the researcher displayed it on the chalkboard (see Figure 6.9). This was done to ensure that all the students in class got the correct solution to each problem with discussions and explanations of how one arrived at his/her answer. It was emphasized that in the solutions that were read out by students, it was not about a correct or wrong answer, but rather how he/she arrived at the particular answer. All the solutions to the given problems were displayed on the chalkboard. This helped even other students to check with their own solutions regarding each problem. These can be viewed in the video clip captured in Figure 6.9.

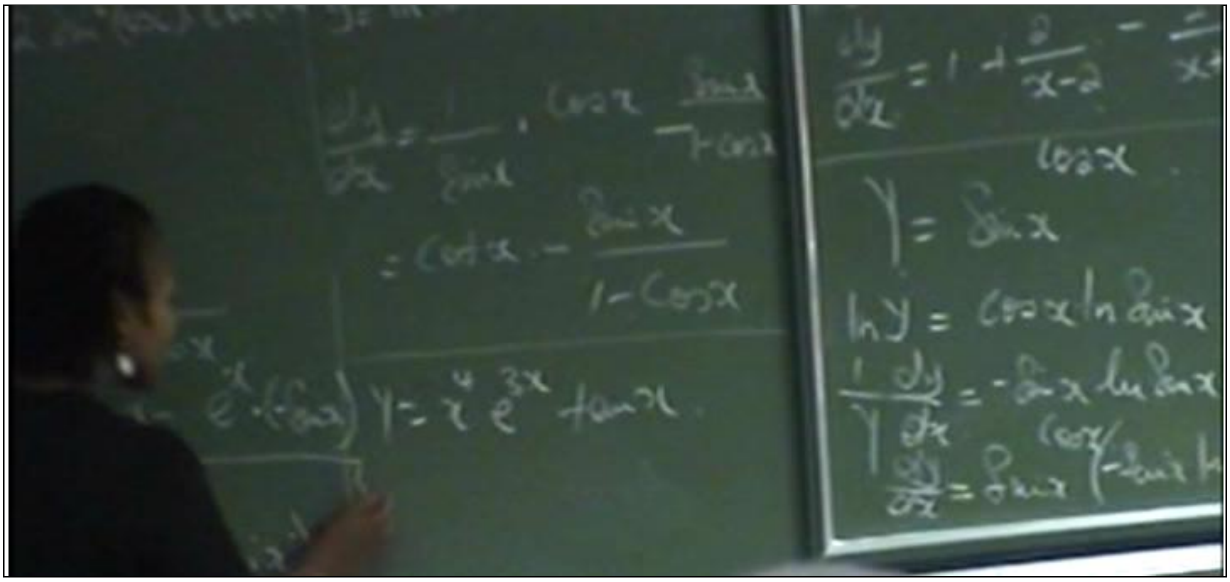


Figure 6.9: Presentation of students' responses in lesson two

The list of problems attempted by the students in class, are listed in Appendix B3.

6.5 Lesson three: Presentation and discussion

The third lesson involved integration by substitution using the chain rule. Various examples were demonstrated by the researcher with full explanations on the u-substitution (see Figure 6.10). Worksheets with a number of exercises (see Appendix B2) on the use of the chain rule were then issued to students. There was space provided below each task in the worksheet for students' responses.

This was done to reinforce the learning that took place in the three sequential lesson components. The aim was to provide students with opportunities to make applications of the chain rule they learnt and prepare them for the mathematics in which chain rule would be applied. This time students worked collaboratively. The activities were designed to foster the students' development of mental structures called for in the initial genetic decomposition. The genetic decomposition assumed the actions, processes, and objects that play a role in the construction of a mental schema for dealing with the chain rule.

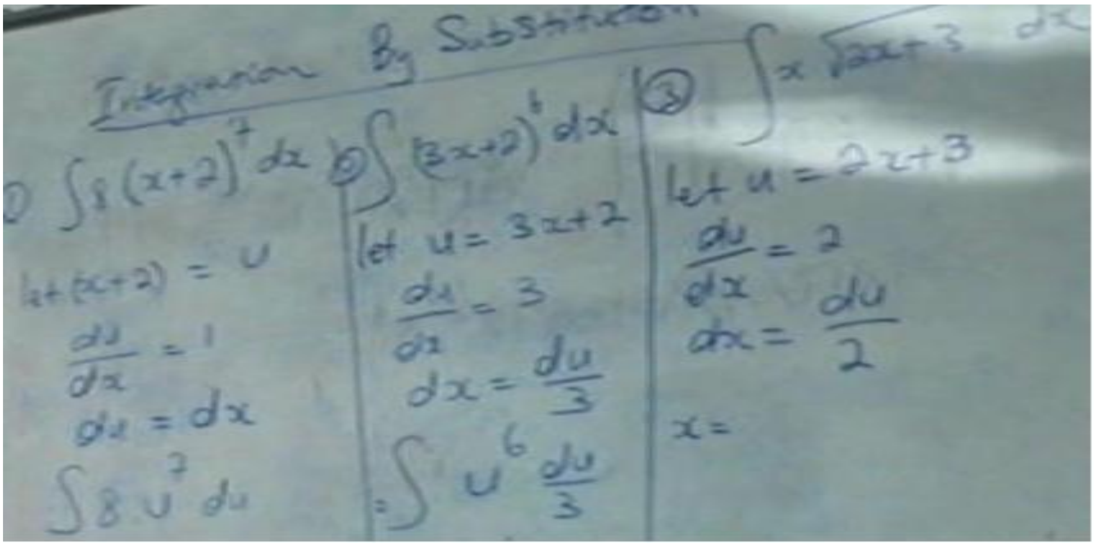


Figure 6.10: Discussions on Integration

Whilst working in groups students discussed their results and listened to explanations given by fellow students. The students worked collaboratively on mathematics tasks designed to help them use the mental structures that they had built during the two previous lessons. In some cases, students worked on a task as a group, whilst in other cases they worked as individuals and then compared notes, and then negotiated a group solution to the problem. They then reported their results in the class. During this process, the emphasis was on: (1) discussions, (2) reflection explanations by the researcher where appropriate, (3) completion of the tasks by the students, and (4) understanding the use and application of the chain rule.

As the researcher moved from group to group, she noticed that some students used a lead pencil to record their responses on the worksheet. They were trying to avoid mistakes and allow correction of an incorrect response without spoiling the worksheet. In some groups, after transcriptions of agreed responses, all the members of the group satisfied themselves that the submitted response was appropriate. They argued from time to time of the positions where brackets should be inserted. Even after submissions of completed worksheets, other students continued convincing and teaching the inquisitive students on how the chain rule works. This can be seen in the still photo in Figure 6.11.

Figure 6.11: Students explaining to each other in groups.

It was so interesting to watch the students refering back to their notes in their books before attempting the questions. Asked about this Zazi answered*: I remember a problem that you did for us, it looked like this one. So I want to compare and then differentiate this one.* Although Zazi is operating in the action stage, he needed to gain experience constructing actions similar and corresponding to differentiating using the chain rule. The experience of differentiation using the chain rule was built upon in subsequent activities like those in the worksheet, where he was asked to reconstruct familiar actions as general manipulations.

The researcher noticed that students in some groups would first copy a task in the worksheet onto their books. They would then work on it as individuals after which they compared their answers. Students argued and agreed upon certain responses. Individuals justified how they arrived at their responses. This way they taught each other and gave verbal descriptions of actions taken in their own words. They then repeated the actions many times with different tasks in their books and in the worksheet. Thus the worksheet helped the students interiorise the actions.

It was also noticed that most students in different groups were operating in the Intra- stage of the Triad. They had a collection of rules of differentiation with no recognition of relationships between them. Those students were helped by others who reflected on using the chain rule by applying actions to dynamic processes. The latter group had created an object of the chain rule. At the same time they applied actions on differentiation and as such the process of differentiating using the chain rule was encapsulated to form an object.

The worksheets were analyzed for meaning which is one of the mechanisms necessary for underststnding a concept. These included detecting (1) the connections made by students to other concepts, (2) calculations made using the chain rule, (3) the chain rule technique used, and (4) mental images on which the chain rule is based.

All the groups applied the chain rule to the first task $y = tan^2\,(3x + e^{\sqrt{x^2+1}})$ correctly using the *straight* form technique although only two out of twelve groups presented a solution with brackets, when they differentiated the composite function inside the brackets in the given task. One of the groups who left out the bracket then went on to detach the derivative 3 of 3*x* from the + sign. This 3 now multiplied the first two functions (see Extract 6.4).

Differentiate:

$y = \tan^2(3x + e^{\sqrt{x^2+1}})$

$= 2\tan(3x + e^{\sqrt{x^2+1}}).\sec^2(3x + e^{\sqrt{x^2+1}}).3 + e^{\sqrt{x^2+1}}.\frac{1}{2}(x^2+1)^{-\frac{1}{2}}.2x$

$= 2\tan(3x + e^{\sqrt{x^2+1}}).\sec^2(3x + e^{\sqrt{x^2+1}}).3 + e^{\sqrt{x^2+1}}.\frac{1}{2\sqrt{x^2+1}}.2x$

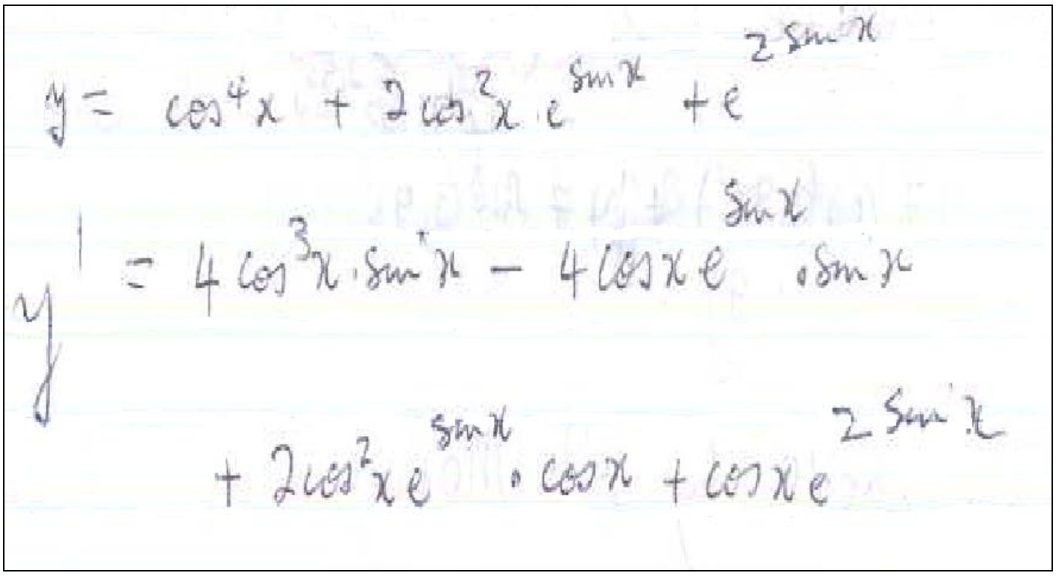
$= \cancel{6\tan}$
$= 6\tan(3x + e^{\sqrt{x^2+1}}).\sec^2(3x + e^{\sqrt{x^2+1}}) + e^{\sqrt{x^2+1}}.\frac{2x}{2\sqrt{x^2+1}}$

$= 6\tan(3x + e^{\sqrt{x^2+1}}).\sec^2(3x + e^{\sqrt{x^2+1}}) + e^{\sqrt{x^2+1}}.\frac{x}{\sqrt{x^2+1}}$

Extract 6.4: One group's presentation of task 1

This mistake was not detected by any of the other members of the same group. Those students struggled with the connection of previously learnt algebraic skills like use of brackets where appropriate and manipulation of algebraic terms in a function. The calculations presented after differentiating using the chain rule successfully were therefore not correct for seven out of twelve responses received. The mental images constructed by the seven groups in using the chain rule were incomplete. Although the actions were interiorized into processes, the processes were not encapsulated to objects. This could partly be attributed to previous knowledge of algebraic skills which were just actions and never interiorized. According to the Triad students in the said groups saw the chain rule as a procedure of differentiation which could not be connected or related to other processes applied to functions. Thus most students operated in the Intra- stage regarding task 1. This concurs with what Lakof & Nunez (1997) asserted that mathematics begins with direct human experience and ends there for some people. According to APOS, we observed that some students could only go as far as the action stage.

The second problem $y = (cos^2 x + e^{sin\, x})^2$ was presented correctly by nine out of twelve groups. Only one group avoided the use of the chain rule by squaring the given function and then differentiating. This was a brilliant idea but still required them to apply the chain rule on the individual terms, cos^4x, $2cos^2x\ e^{\sin x}$ and $e^{2\sin x}$. They then used *straight* form technique to differentiate. Those students were connecting the given function to a square of a binomial. Thus a part of understanding the concept of the chain rule is a mental process involving sorting out the given function, dealing with its composition, and connecting the two to find the derivative. They indicated a process construction of mental images since they transformed the given function to a trinomial which was operated on by repeating the actions of differentiation. Their work has been captured in Extract 6.5.



Extract 6.5 Chain rule application after squaring a binomial

The third task required students to differentiate $\sin(x+y) = e^{y^2+2x}$ implicitly using the chain rule. Five groups out of twelve groups introduced natural logarithms on both sides of the equation before differentiating. They explained that they connected the relationships of exponentials in the right hand side function with logarithms which would get rid of the exponent. In this way they ended up with simple expressions on both sides and thus allowed them to use the *straight* form technique of chain rule differentiation (see Extract 6.6).

Differentiate implicitly:

$$\sin(x+y) = e^{y^2+2x}$$

$$n\,\sin(x+y) = \ln e^{y^2+2x}$$

$$\ln \sin(x+y) = y^2 + 2x$$

$$\frac{\cos(x+y)}{\sin(x+y)}\left[1+\frac{dy}{dx}\right] = 2y.\ \frac{dy}{dx} + 2$$

$$\frac{\cot(x+y)}{\sin(x+y)}\left(1+\frac{dy}{dx}\right) = 2y \cdot \frac{dy}{dx} + 2$$

$$\cot(x+y)\,(1 + \frac{dy}{dx} = 2y\,\frac{dy}{dx} + 2$$

$$\cot(x+y) + \cot(x+y)\,\frac{dy}{dx} = 2y\,\frac{dy}{dx} + 2$$

$$[\cot(x+y) - 2y) = 2 - \cot(x+y)$$

$$\frac{dy}{dx} = \frac{2-\cot(x+y)}{\cot(x+y)-2y}$$

Extract 6.6: Differentiation using natural logarithms

Their calculations indicated a full understanding of the use of the chain rule. They operated in the Trans- stage of the triad since they could reflect on relationships between various objects from previous stages. They displayed coherence of understanding of differentiation rules and composition of functions. Three of the five groups presented responses of full construction of mental images of the chain rule and a connection between understanding of algebraic manipulations of the derivative and function composition. The other seven groups applied the chain rule directly using the *straight* form technique and then processed the resulting function to get the derivative. Two of the responses indicated a transition from an operational to a structural mode of thinking since they brought the concept of the chain rule into existence and used it with caution, and preferred it over other methods of differentiation (see Extract 6.7).

Differentiate implicitly:

$\sin(x+y) = e^{y^2+2x}$

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = e^{y^2+2x}\cdot 2y\cdot\frac{dy}{dx}+2$$

$$\cos(x+y) + \cos x + y\frac{dy}{dx} = 2ye^{y^2+2x}\frac{dy}{dx}+2$$

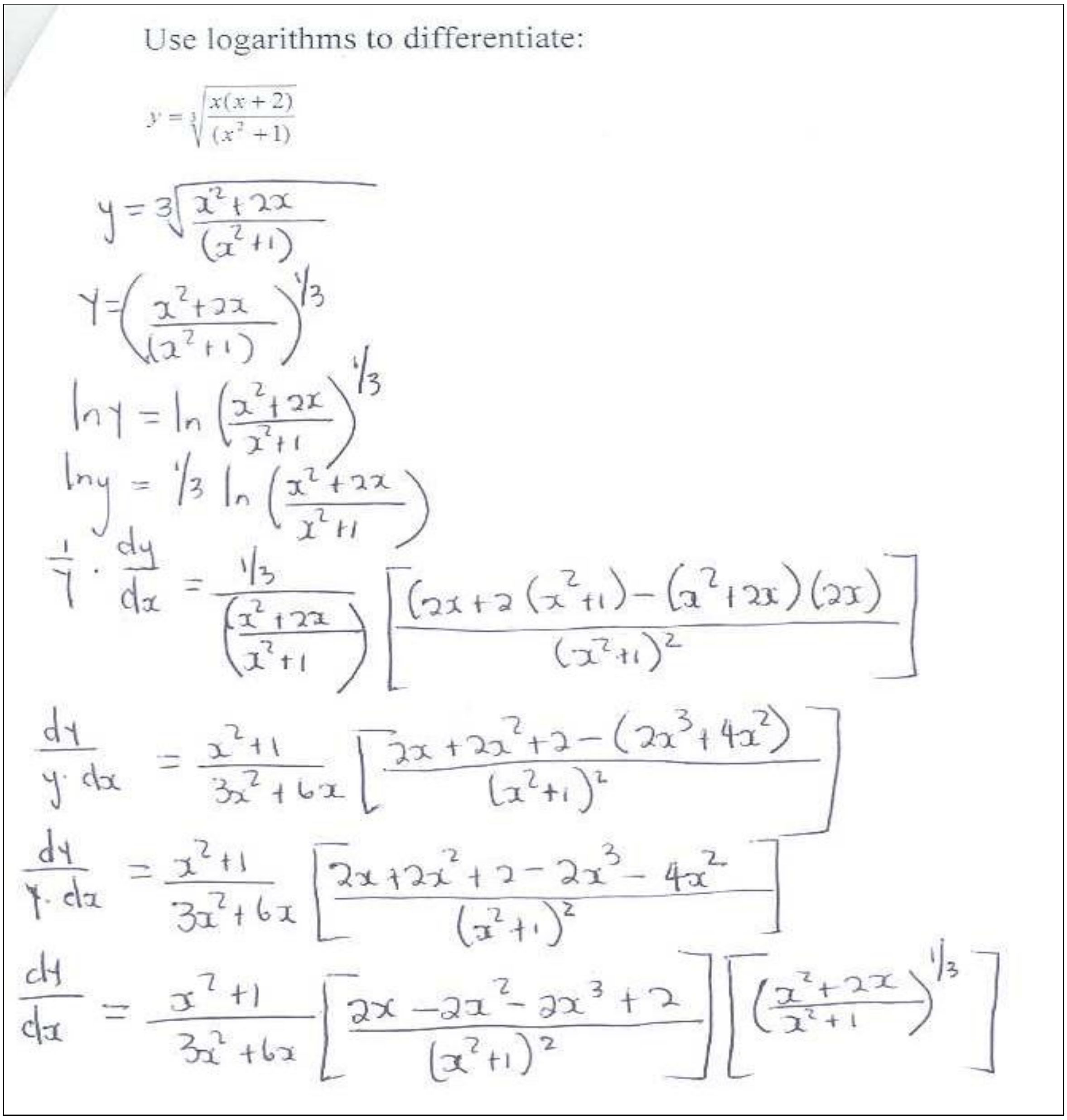
$$\cos(x+y)\frac{dy}{dx} - 2ye^{y^2+2x}\frac{dy}{dx} = 2-\cos(x+y)$$

$$\frac{dy}{dx}\left(\cos(x+y) - 2ye^{y^2+2x}\right. = 2 - \cos(x+y)$$

$$\frac{dy}{dx} = \frac{2-\cos(x+y)}{\cos(x+y)-2ye^{y^2+2x}}$$

Extract 6.7: Straight form technique used in differentiation

The last task involved differentiating $y=\sqrt[3]{\dfrac{x(x+2)}{(x^2+1)}}$ by applying the chain rule. Generally, one of two strategies was employed by students. The first form technique called for a specific connection between application of natural logarithms and differentiation.



Use logarithms to differentiate:

$$y = \sqrt[3]{\frac{x(x+2)}{(x^2+1)}}$$

$$y = 3\sqrt{\frac{x^2+2x}{(x^2+1)}}$$

$$Y = \left(\frac{x^2+2x}{(x^2+1)}\right)^{1/3}$$

$$\ln Y = \ln\left(\frac{x^2+2x}{x^2+1}\right)^{1/3}$$

$$\ln y = 1/3 \ln\left(\frac{x^2+2x}{x^2+1}\right)$$

$$\frac{1}{Y}\cdot\frac{dy}{dx} = \frac{1/3}{\left(\frac{x^2+2x}{x^2+1}\right)}\left[\frac{(2x+2(x^2+1)-(x^2+2x)(2x)}{(x^2+1)^2}\right]$$

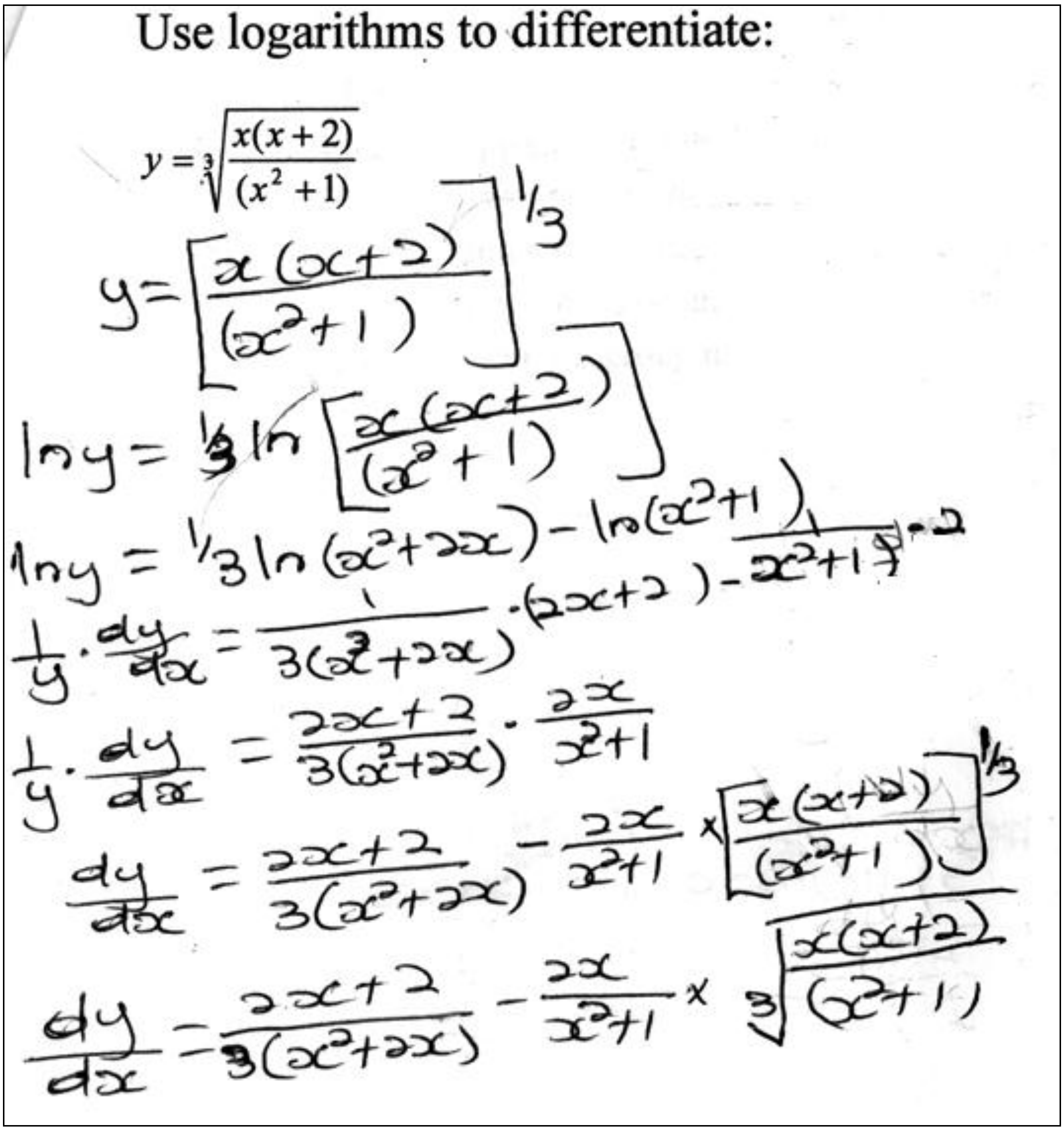
$$\frac{dy}{y\cdot dx} = \frac{x^2+1}{3x^2+6x}\left[\frac{2x+2x^2+2-(2x^3+4x^2)}{(x^2+1)^2}\right]$$

$$\frac{dy}{Y\cdot dx} = \frac{x^2+1}{3x^2+6x}\left[\frac{2x+2x^2+2-2x^3-4x^2}{(x^2+1)^2}\right]$$

$$\frac{dy}{dx} = \frac{x^2+1}{3x^2+6x}\left[\frac{2x-2x^2-2x^3+2}{(x^2+1)^2}\right]\left[\left(\frac{x^2+2x}{x^2+1}\right)^{1/3}\right]$$

Extract 6.8: Group 3's response on logarithmic differentiation

Only two groups displayed a coherent collection of the logarithmic rules and differentiation. Those groups were operating in the Trans- stage since they reflected on the explicit structure of the chain rule and were also able to operate on the mental constructions which made up their collection. Those students presented responses showing internal processes for manipulating logarithmic objects. Their schema enabled them to understand, organize, deal with and make sense out of application of the product rule, quotient, logarithmic rules and the chain rule. The other three groups could not apply logarithmic rules correctly and as such could not process the differentiation of the given task. This is illustrated in Extract 6.8 where students resolved the surd form of the function correctly and took natural logarithms both sides of the equation. The interpretation of logarithms was then incorrect since a bracket was left out in step three of the response. Thus the function differentiated was not the originally given one. Even in their process of differentiation some brackets were still left out when they should have been there. Also the derivative of the last term, $-\ln(x^2 + 1)$ in step four was recorded as $\frac{1}{x^2+1}.2$ instead of $\frac{1}{x^2+1}.2x$. In the next step the subtraction sign has been left out and then restored back again in the following one. The students in this group's actions indicated that they knew which steps to follow when differentiating. Their mental manipulations did not react to external cues of basic algebraic manipulations and as such transformation was not complete and their actions were not interiorized. Those students did not recognize the relationships between application of natural logarithms and algebraic manipulations resulting in multiplications when they were due and subtractions where appropriate. They perceived differentiation as a separate entities and even the rules applied were not remembered correctly. These were operating in the Intra- stage of the Triad. Their response is illustrated in Extract 6.9.



Use logarithms to differentiate:

$$y=\sqrt[3]{\frac{x(x+2)}{(x^2+1)}}$$

$$y=\left[\frac{x(x+2)}{(x^2+1)}\right]^{1/3}$$

$$\ln y=\tfrac{1}{3}\ln\left[\frac{x(x+2)}{(x^2+1)}\right]$$

$$\ln y=\tfrac{1}{3}\ln(x^2+2x)-\ln(x^2+1)$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{1}{3(x^2+2x)}\cdot(2x+2)-\frac{1}{x^2+1}\cdot 2$$

$$\frac{1}{y}\cdot\frac{dy}{dx}=\frac{2x+2}{3(x^2+2x)}\cdot\frac{2x}{x^2+1}$$

$$\frac{dy}{dx}=\frac{2x+2}{3(x^2+2x)}-\frac{2x}{x^2+1}\times\left[\frac{x(x+2)}{(x^2+1)}\right]^{1/3}$$

$$\frac{dy}{dx}=\frac{2x+2}{3(x^2+2x)}-\frac{2x}{x^2+1}\times\sqrt[3]{\frac{x(x+2)}{(x^2+1)}}$$

Extract 6.9: Incorrect application of chain rule in differentiation

The other group employed the *straight* form technique after converting the surd form to its exponential form. However, they did not then utilize the product and quotient rules appropriately. Their actions were not interiorized with regards to logarithms and this had an impact on applying the chain rule in the given task. Their mental images could not be related to the string of symbols forming the expression, since they could not interpret both the symbols and or manipulations.

Since calculations reflect the active part of mental constructions, the rules for these students were not perceived as entities on which actions could be made. Dubinsky (2010) asserts that in such cases the difficulty does not depend on the nature of the formal expressions, but rather in the loss of the connections between the expressions and the situation instructions.

6.6 Conclusion

The students' responses discussed above indicate that the instructional pedagogy should accommodate presentation of tasks that evoke rigorous deductive reasoning enabling the students to write and reflect on how they construct various mental images. A wide range of interactions between students themselves and between students and the researcher were discussed. In the next chapter I present analysis of interviews of selected subjects based on their responses to the written instrument.

CHAPTER 7

ANALYSIS OF WRITTEN RESPONSES AND INTERVIEWS

7.1 Introduction

In chapter six, video-recording, interactions between students, between students and the researcher, and instructional design were outlined. In this chapter, a transcription of the students' interviews on selected tasks based on their written responses is presented. The selected subjects were asked various questions in an effort to extract how they constructed various mental structures. It was of importance to detect whether they recognized and applied the chain rule in the context of the given tasks. This chapter reports the analysis of six interviews based on the descriptions of the Triad and mental constructions called upon in the initial genetic decomposition. The Triad mechanism is used to interpret the observations, and the data for each subject is described by a level in the Triad in conjunction with APOS.

7.2 The structure and analysis of the interviews

Semi-structured interviews of 30 minutes to 1 hour duration each were conducted by the researcher with each of the six participants selected from the 30 first year students. An interview schedule was prepared with the participants of the study to use their and the researcher's time appropriately. The purpose of the interview was explained each time to each selected subject prior to conducting the interview. The confidentiality of their names and the time needed was also specified. Fictitious names for the participants have been used to ascertain confidentiality and anonymity.

The interviews were audio-taped. It was sometimes required of the participants to write their answers on a worksheet besides expressing them verbally to explain their thinking even if the answers were incorrect. Some sub-questions related to the notions: (1) the definition of a function, (2) the chain rule, (3) composite function, (4) relationship between the chain rule and the composition of functions and (5) the structure of the integrand. These were asked at the convenient moment according to the way the interview was going. Probing questions were used to elicit information on the chain rule more fully. This question type was used extensively in this study because the researcher wanted to explore students' thinking processes. Misconceptions and difficulties that emerged during the interviews were analyzed with the view of establishing how students construct various structures to recognize and apply the chain rule in the context of calculus.

The different categories of: functions, composite functions, derivative and the structure of the integrand were searched, for the mental construction in relation to the proposed initial genetic decomposition in Chapter Three. It was also of importance to establish whether the students' category schema aligned with the genetic decomposition. The Triad stage of schema development in which the students were operating with respect to the chain rule and their identification of the reverse application of the chain rule in the substitution technique for integration were of significance to this study. Some of the questions in the interview were geared at finding out if students were just using memorized rules of special composite functions (such as power functions and trigonometric functions). Was the chain rule user-friendly as a way for finding derivatives of composite functions?

7.3 Analysis and discussion of written responses and interviews.

In addition to classroom observations when students did activities, class work exercises and tutorials on the chain rule applications, semi-structured and unstructured interviews were conducted with six subjects based on their responses to the different activities in the questionnaire. The objective of the interviews was to (1) get clarity on the written responses, (2) classify the chain rule schema development with respect to the Triad, (3) check for encapsulation of processes to objects or de-encapsulation of those processes, and (4) the development of mental constructions in relation to the proposed genetic decomposition.

It was requested that some students who struggled with explanations write down their answers during the interview to justify their mental constructions. In these interviews the students were asked to respond in an open-ended fashion to the following issues: (1) justifying their responses to particular questions in the research instrument, (2) the rule they used in differentiating, (3) stating the chain rule, and (4) identifying the students' preferential methods in integration. Some of the common questions asked in order to assess the student's conceptual understanding included: (1) State the chain rule? (2) Could you write down a mathematical general formula for the chain rule? The instrument whose reliability was validated that was used in the pilot study was administered to 30 first year students, who volunteered to partake in this of the study. The skills addressed by the questionnaire per item and the scoring code distribution in analysis, was illustrated in Table 5.1 (on page 86). The summary of the scores gained by each participant, in each category indicated higher scores than the scores on the Phase 1 of the study. Figure 7.1 represents this illustration.

It is evident from the graph above that the least category understood by students is composition of functions. A higher score of 3.0 is recorded than a previous evaluation with students who had already passed the course, which recorded a mean score of 2.7. The increase in scores would be associated with the instructional treatment offered in the three sequential lessons given on the chain rule. Also students did not make sense of tasks on functions as this section was not popular in examinations. Again we face the problem that students only study for marks and not concept understanding. The highest score of 4.7 was recorded for the derivative indicating an increased understanding of differentiation, using the chain rule. Once more the high scores indicate that students provided more correct responses for derivatives even though they could not make sense of composition or decomposition of functions.

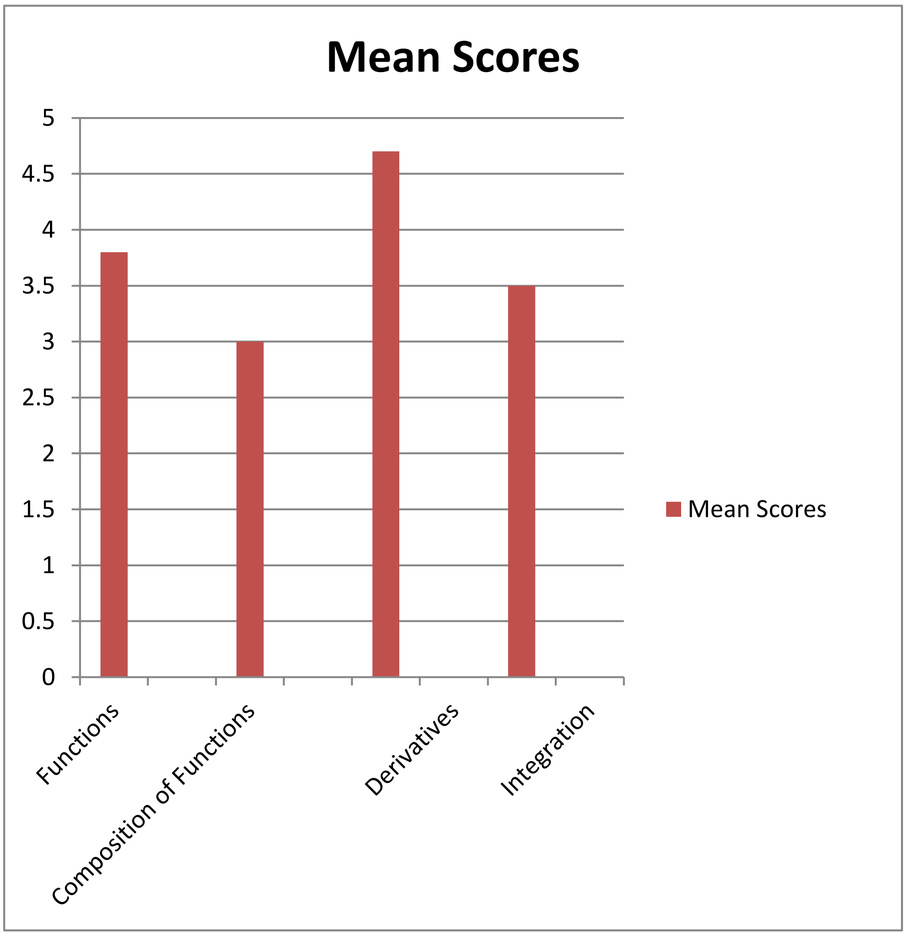


Figure 7.1: Mean scores for all categories

The scores for each category and coding used for displayed mental constructions were outlined in detail in Chapter Five in Tables 5.1 to 5.6. This coding was adopted in the main study since the instrument tested out to be valid and reliable.

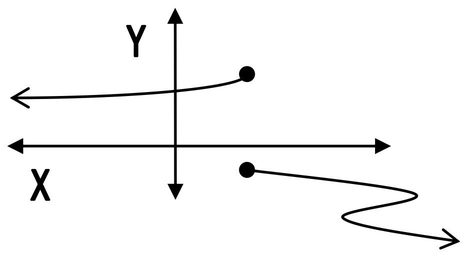
7.3.1 Category A: Functions

Two items in this category were used to find mental constructs displayed by students in the understanding of functions.

7.3.1.1 Item 1

The task for Item 1 appears below.

Is a student correct to identify the following as a function?



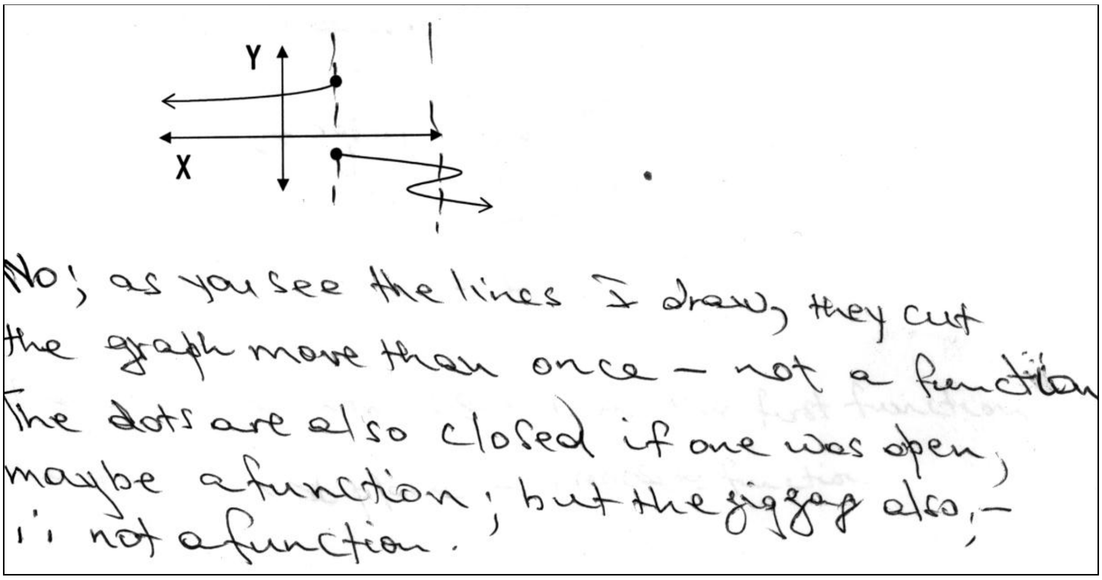
Explain.

About 60% of the 30 students gave correct responses to this Item. One of the responses indicated two vertical lines introduced on the graph. Extract 7.1 captures this response. An interview with her indicated:

Researcher: I noticed that you drew two lines in the given graph in question 1. Could you justify that?

Sindi: *Mahm engikwenzileyo lana, ngibone kunamajiko-jiko, nalana kunamagqudu avalekile,* (I was showing the zig-zag part and then the closed dots), *so I wanted to show that any of the two parts would lead the given graph not to be a function. That is why I drew the two lines.*

She indicated the closed dots and later explained that drawing the second line was to ascertain that the closed dots were in line. This for her implied that the graph was continuous which was incorrect. Not only that, Sindi knew that the zig-zag part of the graph did indicate a function. Sindi's actions of understanding of graphical representations had not been interiorized to a process She was mixing two issues, continuity of a function which was not applicable to the given graph and the definition of a function. This was evident from the complete response and the contradicting explanation given when interviewed, where she indicated continuity and vertical line test as necessary characteristics for a graph to be a function. Sindi was initially scored 5 since she showed all the mental constructions proposed in the genetic decomposition regarding the graphical representation of a function in the written response.



Extract 7.1: Sindi's response to Item 1

Sindi was one of 6 students who gave complete written responses. All the others displayed correct responses without reasons. This suggests that those students operated on the action stage of APOS. The response, No, could have just been a guess. That was confirmed by Nodi, asked to justify why he thought the graph was not a function. He indicated: *I just looked at it and decided it's not a function, was I correct?*

Even though Nodi indicated a correct response, he had no idea of what a function is. He was at the Intra- stage of the Triad. He had no idea of the relationships and properties that a graph should display for it to be a function. Figure 7.2 illustrates the number of students versus score performance explanations. It is revealed in the graph that many students displayed few mental constructions with no reasoning and explanations to justify the written response. Nodi's score fell in this range.

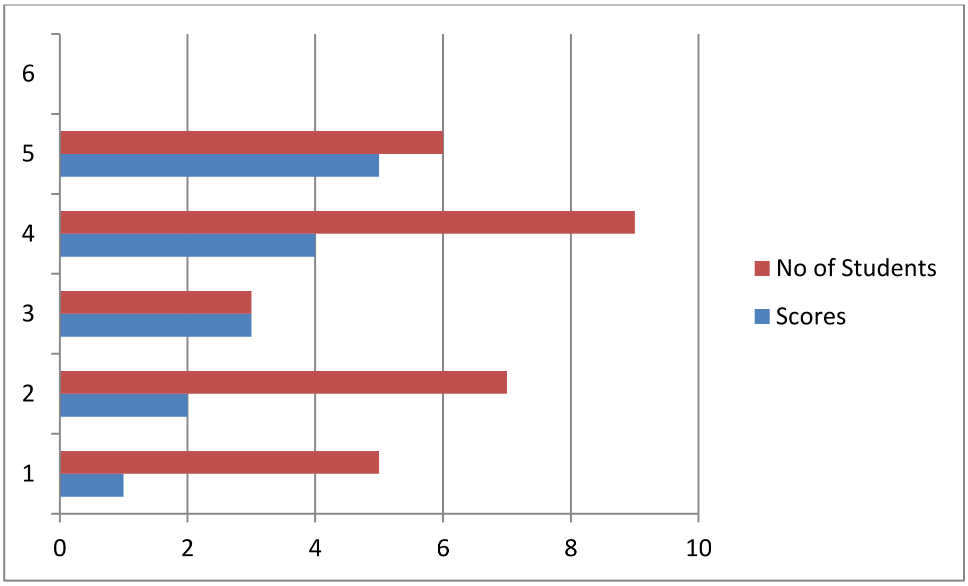


Figure 7.2: Bar graph illustrating student scores for item1

Zwayi gave only one reason on why the graph in Item 1 did not represent a function. He was scored 3, since he did not mention anything regarding the zigzag part of the graph. It was interesting to learn that when he was interviewed, he pointed to the zigzag part of the graph before giving an explanation about closed dots in the graph closed. Dubinsky (2000) proposed a set of mechanisms for constructing mathematical concepts. He asserted that these mechanisms involve mental steps such as interiorization to reinterpret an action as a process. Zwayi displayed a process understanding of a function when he wrote the activity. The encapsulation of this process to an object was incomplete. During the interview encapsulation and de-encapsulation were completed and Zwayi, displayed a better understanding of graphical representation of a function. This was not the case with Zandi. The interview transcription with her can be accessed in Appendix A. Zandi interpreted the given graph as two graphs since it was broken. The concept of continuity could not be evoked even by the questions asked during the interview. This was a new concept that had to be taught from scratch. Zandi was scored 1 since she showed no appropriate mental construction for graphs, at all.

A different case ensued with Popo. During the interview he explained and rectified some of his work. He had drawn a line to join the two dots in the diagram but labeled the line as horizontal. In an interview with him when he was asked to explain his use of vertical lines to draw conclusions, he responded:

Popo: *If the vertical line meets the graph more than once, then not a function.*

Researcher: Could you explain why you labeled 'horizontal line there?'(pointing at the label in the diagram).

Popo: *My mistake there mheem, angazi kwenzekeni kumina, (I don't know what was wrong with me) I wanted to write vertical line.*

Researcher: What about the zig-zag part?

Popo: *That part is obvious, not a function, I just wanted to be sure that the graph was continuous.*

Now Popo is coming up with the property of continuity of functions which was not mentioned or implied in her answer. Thus in terms of APOS analysis I suggest that looking at the graph and deciding with an indication of vertical line test fully understood by a student is a process. The action of determining continuity would have led to encapsulation of this process and the selection of the whole figure as an object function if the function was continuous.

7.3.1.2 Item 2

The statement for Item 2 follows.

A given correspondence associates 3 with each positive number, -3 with each negative number and 1 with 0. A student has marked afore - mentioned relationship as a function, is that correct, support your answer.

Only 5 students out of 30 gave a complete response indicating complete mathematical understanding of domain and range of the function. Those students were scored 5, also gave a complete response for Item 1. Figure 7.3 illustrates the number of students who attained the different scores. Twelve students could not make sense of whether a function was represented by a set of ordered pairs. A few students when interviewed indicated that this concept was not part of the examinable content in course 1, mathematics, so they did not think it was important. When Zwayi was interviewed further for mental constructions involved in Item 2, he displayed the same discourse as he had in Item 1. The discourse was noticed with interpretation of Item 2, where he had to explain a function using the domain and range. When he was interviewed he gave the following explanations:

Researcher: This second one, pointing at Item 2, is that a function?

Zwayi: *Yes, it's a function.*

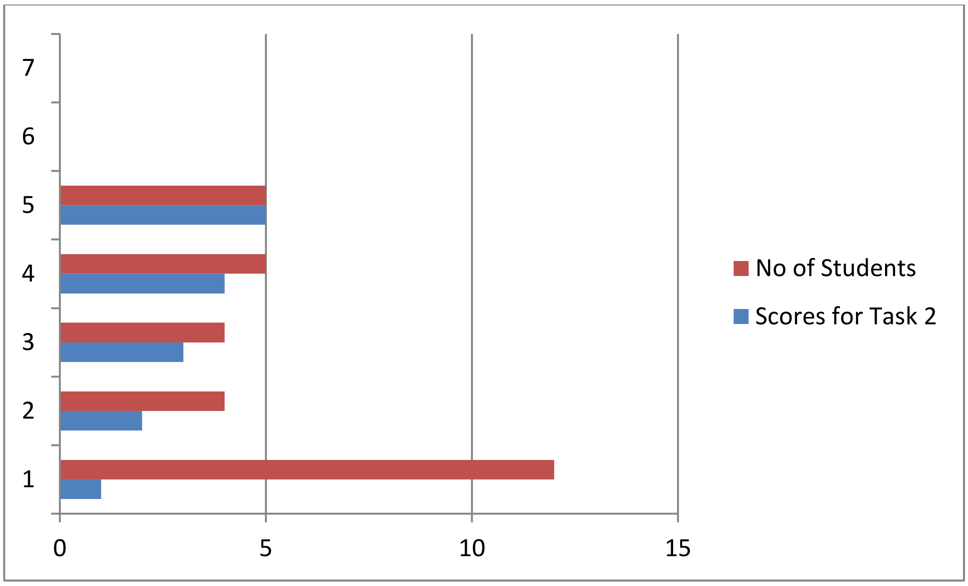


Figure 7.3: Bar graph showing number of students and scores for Item 2

Researcher: Why?

Zwayi: *Because the members of the domain are not repeated.*

Researcher: Which members form the domain?

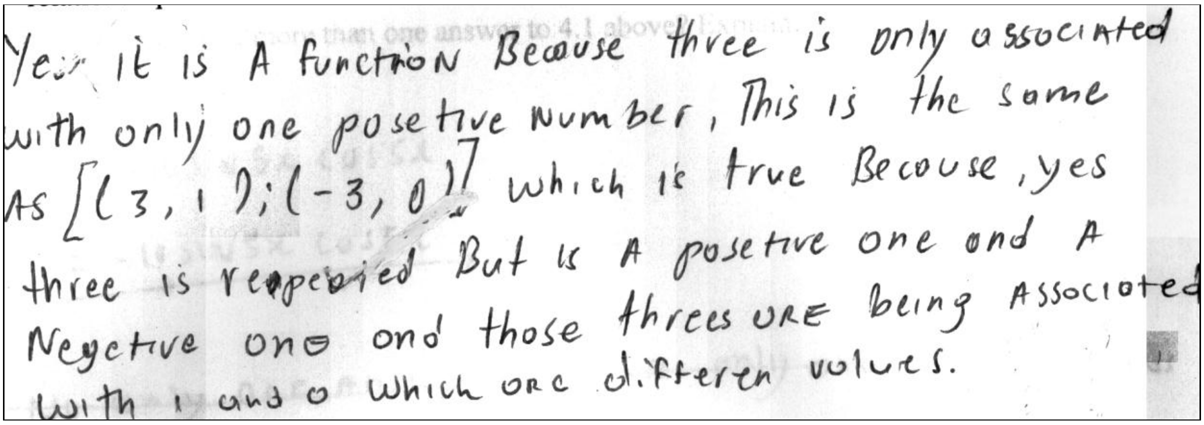
Zwayi: *The x- values*.

Researcher: Which ones are the x-values?

Zwayi: *We have (3; 1); (3;2)… Hayi akuyona ifunction. (No it's not a function) idomain iphindene. (domain entities have been repeated mapping with other entities for the range) 3….; 3….;.*

Zwayi went back to Item 2, read it again, listed a set of ordered pairs, and then made his decision based on his definition of a function regarding domain and range. Apparently, he knew this definition, but did not make sense of it regarding the task in the questionnaire. Sfard cited in Dubinsky (2000) explains that for an individual to understand a process as an object, nouns, new names or new symbols have to be introduced to refocus discourse. She claims that this new signifier will help students for example in going from a function conception of an input/output machine to functions as objects that can be acted upon. Zwayi has a process understanding of a function. He knows the descriptions of a function but cannot relate those descriptions to a given situation. According to the Triad, Zwayi is operating in the inter stage since he displayed the ability to begin to (mentally) collect all different cases regarding a function in his verbal explanations, and recognized that these were related. He shows evidence of having collected some or all the interpretations of a function without yet constructing the underlying structure of their relationships.

Popo (see Extract 7.2) knows that a definition of function in terms of ordered pairs requires non-repetition of the domain entity. He gave all the rules for an ordered pair relationship to be a function, but makes no meaning of them. This action is not interiorized into a process leading him to present a contradicting response. The action of mapping is not complete in his mind.


Yes it is A function Becouse three is only associated
with only one posetive number, This is the same
As [(3, 1); (-3, 0)] which is true Becouse, yes
three is repeated But is A posetive one ond A
Negetive one ond those threes are being Associated
with 1 and 0 which are differen values.

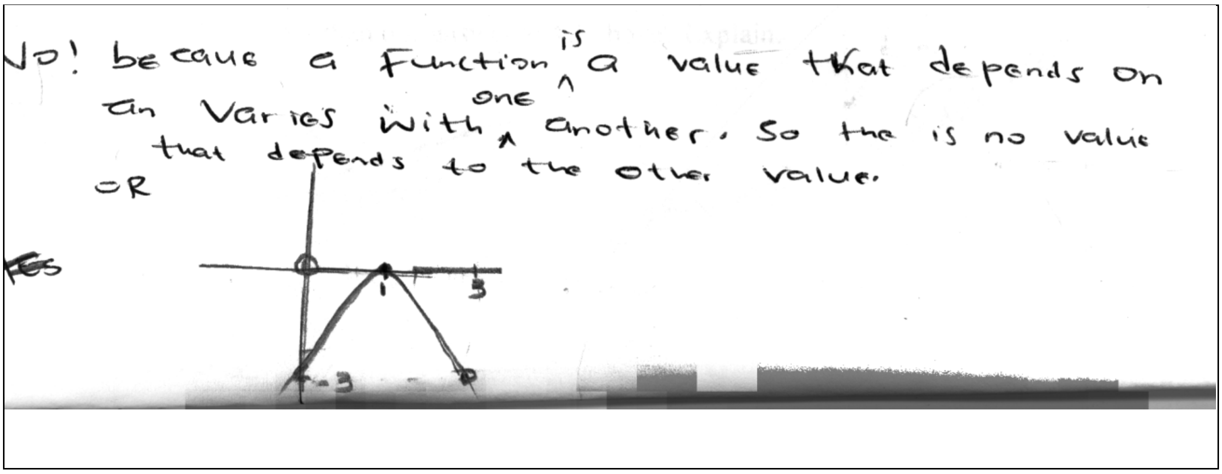
Extract 7.2: Popo’s response to item 2

The explanation to repetition in his case does not refer to domain entities, but to the 3 whether positive or negative. The researcher believes that the student failed to make sense of the given scenario in the question. Language might be one of the barriers to understanding for this student. He is said to operate in the intra stage with respect to the Triad. His response was coded one. A response that displayed a totality of encapsulation of a process is listed in Appendix C4. This response was scored 5 as it gave all the explanations, interpretations and contextual applications on the graphical representation of a function.

Zandi had seen the graph in item 1 as one graph. She was further interviewed to explain her response to item 2.

Researcher: Alright Zandi, can you now justify your answer to question 2.

Zandi read her response displayed as Extract 7.3, and said:



Extract 7.3: Zandi's response to item 2

Zandi*: Jah…..I see, kusho ukuthi bengithatha ngokuthi u 3 uhamba ne number ezingu 3, kodwa ngabuye ngabona ukuthi u 1 uhamba no 0, bese ngicabanga ukuthi sengathi kuthathw' inumber ezikwi* (I thought that 3 mapped to three other numbers, but noticed that 1 mapped to 0. So I thought the numbers belonged to) *turning point just like in this diagram.*

Researcher: In other words the graph that you have drawn here illustrates the description given in the task.

Zandi: *Yes.*

Zandi has an idea of a definition of a function regarding mapping of domain and range. The reasons and the sketch provided, rejects this claim. Thus according to APOS, the actions of mapping have not been interiorized into a process. Zandi can't relate any of the descriptions of mapping with graphical representations. For Zandi, these are two separate entities not related in any way.

Sindi on the other hand operated in the Trans- stage. Her explanations indicated that she had made all the mental constructions regarding a function definition as she did with graphical representations of functions. She had a collection of the objects and processes of a function, and had constructed the underlying structure of a function. She constructed the function schema through reflections on relationships of graphical representations and mappings, using domain and range. In an interview with her:

Researcher: I was also impressed by the response you gave in question 2. Would you want to explain how you came up with it?

Sindi*: I wasn't sure at first, which is why I wrote the ordered pairs. Ngicabangu'ukuthi iyaqhubeka ufake amanye amapositive numbers mamanegative numbers* (I think this is continuing, where you map with positive numbers and then with negative numbers). *What I am trying to say is that, if you repeat the x-coordinate, mapping it with other numbers, it won't be a function.*

The summary of the scores in this category is outlined in Figure 7.4.

Figure 7.4: Bar graph showing a summary of mean scores for items 1 & 2

It is evident from the graph that those students who gave correct responses for Item 1 did not have a complete mathematical understanding of functions, since they could interpret a function in terms of domain and range elements, (tested in Item 2).

7.3.2 Category B: Composition of Functions

7.3.2.1 Item 3

The task for Item 3 appears below.

Given two functions, $f(x)$, and $g(x)$ such that $f(x) = 4e^x$ and $g(x) = 3\sin x$. Find $(f \circ g)(\pi)$.

The results showed that most students experienced difficulty in dealing with the composition of functions (only 5 out of 30 displayed complete understanding of composition of functions). About 48% of the students were scored one for this item since they only evaluated $f(\pi)$ and $g(\pi)$ and stopped without further computations. Twelve out of thirty students were able to find $(f \circ g)(\pi)$.

The summary of the scores for this item is illustrated in Figure7.5

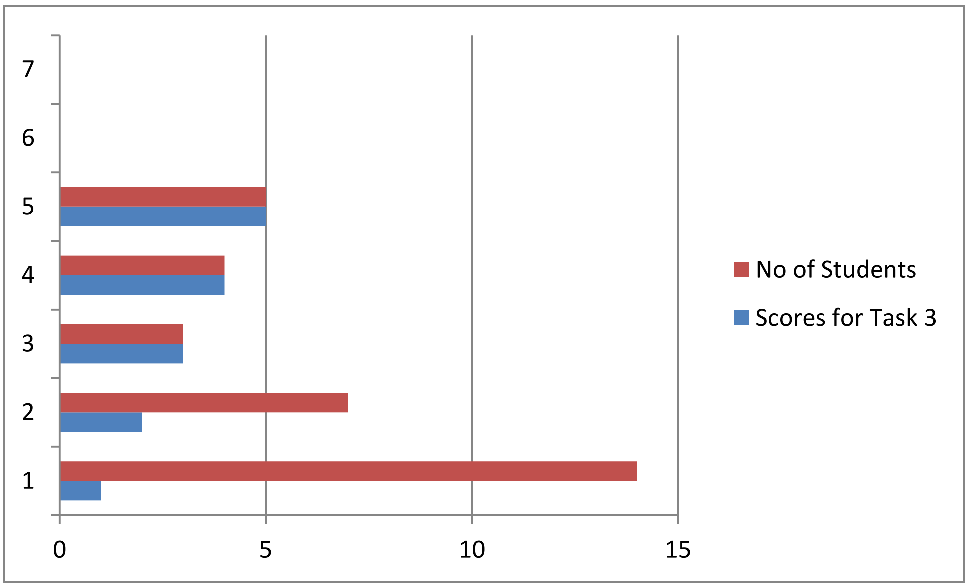


Figure 7.5: Bar graph showing scores for task 3

Sindi displayed complete understanding of this item and was scored 5. A conversation with her revealed:

Researcher: That is well said Sindi, could you explain how you did question 3?

Sindy: *I think yilaa ufike uxhume khona ifunction kwenye ebivele ikhona* (I think this is where you insert one function in another so as to get a composite function.) *So I inserted 3sin x in the place of x in the function f and then evaluated to get 4.*

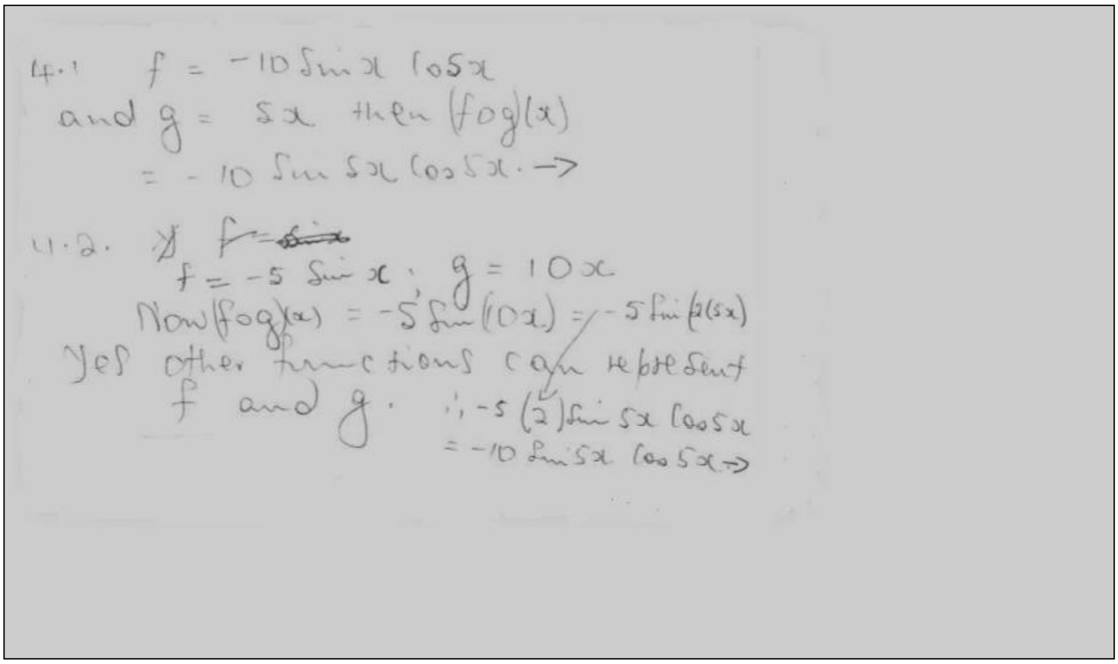
7.3.2.2 **Item 4**

The question for Item 4 appears below.

Given that $(f \circ g)(x) = -10\sin 5x \cos 5x$
4.1 Find functions f and g that satisfy this condition. 4.2 Is there more than one answer to part (a)? Explain.

This item required decomposition of a composite function and very few students (10%) displayed complete understanding of decomposition of the given composite function. 12 out of 30 students were scored 1 as some of them left the question blank and others displayed irrelevant constructions for Item 4. Nonetheless in a continued interview with Sindi who was scored 4 for this item and had displayed more than one answer, she explained:

Sindi: *I was not sure at first and ngaze ngacabang' ukuthi kukhona namanye amafunctions engingase ngiwafake*, (then I figured out later on and decided to include all relevant functions). Sindi's response for Item 4 is displayed in Extract 7.4.



4.1 $f = -10 \sin x \cos x$
and $g = 5x$ then $(f \circ g)(x)$
$= -10 \sin 5x \cos 5x \rightarrow$

4.2. $f = \cancel{\sin}$
$f = -5 \sin x$; $g = 10x$
Now $(f \circ g)(x) = -5 \sin(10x) = -5 \sin 2(5x)$
Yes other functions can represent f and g. $\therefore -5(2) \sin 5x \cos 5x$
$= -10 \sin 5x \cos 5x \rightarrow$

Extract 7.4: Sindi's response to item 4

Sindi had been scored 4 in item 4 since she just gave a correct response without explanations. She was amongst those students who had made all the mental constructions regarding decomposition of functions. After the full explanation she was scored 5. This suggests that she has an object conception of composition and decomposition of functions. She could reflect on operations applied to the composition process and was able to reverse it. She displayed awareness of the process as a totality and could construct transformations on the functions. The process had been encapsulated into an object.

Twelve students were scored 1 for this item. They operated in the action stage regarding decomposing the given function to *f* and *g* that satisfied the condition. They did not know which steps to take because they were restricted to the formula interpretation of the composed function. A summary of the scores in this category is illustrated in Figure 7.6.

Figure 7.6: Summary of scores for category B

7.3.3 Category C: Derivative

This category consisted of six items dealing with differentiation of functions, which follows:

Differentiate the following, with respect to x:

5.1 $y = -3\sin x + 2e^{\cos x} - 5e^{x}$

5.2 $y = \cos(2x-5)^3$

5.3 $f(x) = \sin^3(4x)$

5.4 $y = \sin^2(4x^2 + e^{\sqrt{2x - cose}})$

5.5 $f(x) = \ln[\cos(7x)]$

5.6 $y = (\cos ec^3 x + e^{\tan x})^2$

The results indicated high scores for this category. The category dealt with differentiation of functions and consisted of six items outlined above. The items ranged from simple tasks to overloaded composite functions requiring use of chain rule for differentiation. This category recorded a mean score of 4.7 higher than all the categories. This suggests that decomposition and composition of functions is not a pre-requisite for the understanding of the chain rule in differentiation. A summary of the scores for the different items in this category is displayed in Figure 7.7.

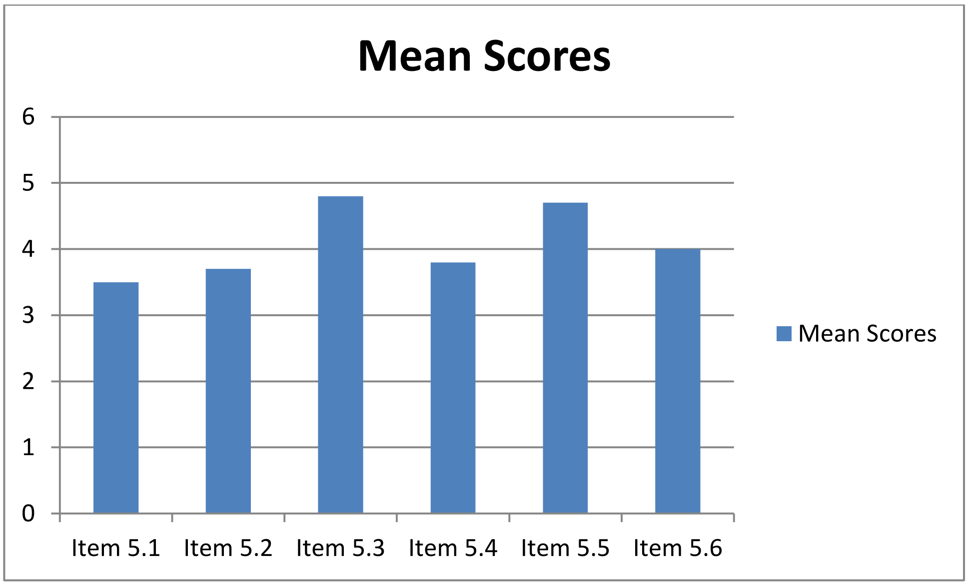


Figure 7.7: Summary of scores for Category C

7.3.3.1 Item 5.1

It is evident from Figure 7.8 that a higher number of scores was clustered around a mean score of 4. Although most of the students (19 out of 30) differentiated correctly, brackets and format of the response were neglected. Some students misrepresented the derivative of $\cos x$ *as* $\sin x$, they left out the negative sign. Those students were just differentiating as an action not taking care and without constructing a meaning into it. For them it's just using rules and knowing that the derivative of this function is just that. There were no processes coordinated. Those students were operating in the action stage since they saw the given function as a formula and the errors of signs left out where they should be, meant that those actions were not interiorized to processes.

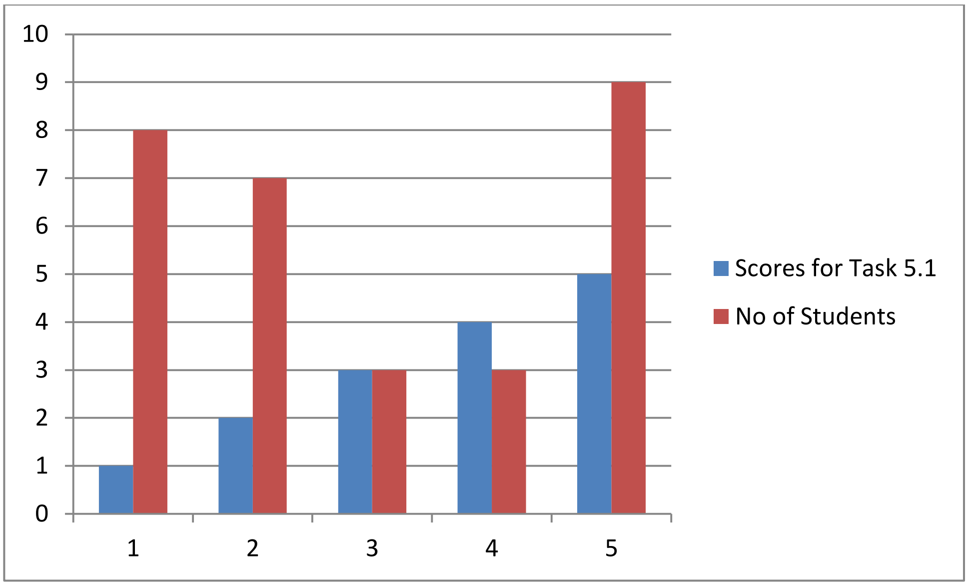


Figure 7.8: Scores for item 5.1

Responses to Item 5.1 revealed a mean score of 3.1 being the lowest of all the other tasks in this category. Most students missed the correct signs, recording the derivative of $-3\sin x$ as $3\cos x$ instead of $-3\cos x$. Most interestingly, the term, $2e^{\cos x}$ which involved application of the chain rule was differentiated correctly by 70% of the students.

7.3.3.2 Items 5.2 to 5.6

These items dealt with evaluation of the derivative using the chain rule. Scores were allocated specifically for correct application of chain rule, or trying other means to avoid it, for example expanding the given function and then differentiating. Item 5.2 had least number of correct responses while Item 5.5 enjoyed the highest number of correct presentations. In a continuation with Zwayi:

Researcher: Let us now check how you did Item 5.2. Can you explain?

Zwayi: *Oh yes,*

Researcher: Which rule are you using there?

Zwayi: *Chain rule?*

Researcher: Can you state the chain rule?

Zwayi*: mh….You find the derivative of cos first and write it as – sin this thing (pointing at (2x -5)^2) bese ungena phakathi udifferentiate the power ye angle and then the angle. (you then proceed inside the bracket and differentiate with respect to the power of the angle and then the angle itself.*

Researcher: Yes, I understand how you came up with your answer to Item 5.2. My question is can you state the chain rule.

Zwayi: *You mean I must write it down. Kungeko sibalo, kungekho lutho.* (When there is no calculation just general)

Researcher: Given $f(g(x))$. Find the derivative

He took his pen preparing to write, then shook his head and said:

Zwayi: '*Iyangihlula ke mayigeneral, kodwa nje makuyesinye isibalo ngiyayishaya.' (I have problems with the general form, but if given another problem in context, I can gladly apply it.)*

Researcher: Now back to 5.2, would it make any difference if you were given $y = \cos^2(2x - 5)$ to differentiate?

Zwayi: *Oh yes, in this case it's the function that is squared, so I would start with the power, to get 2 cos(2x -5) and then multiply by the derivative of the function cos and then times derivative of angle 2x -5.*

Zwayi could not state the chain rule. He was nonetheless able to apply it with reasonable explanations and displayed full understanding of the concept. He had been scored 5 as per explanation he gave on how he arrived at his answer. The reflection on operations applied to differentiation, realization of transformations, and being able to construct the transformations proved that Zwayi had an object conception of the chain rule. The totality of the concept was complete though he couldn't state the rule. This concurs with Naidoo (2007), who asserted that the correct answers given by students do not necessarily prove that students understand the concept. Buhle gave response displayed in Extract 7.5, and tried to justify the method he used differentiating 5.2 as opposed to 5.3.

5.2 $y = \cos(2x-5)^3$

Let $u = 2x - 5$ $\therefore$ $y = \cos u^3$

Now $v = u^3$ and $y = \cos v$

Then $\frac{du}{dx} = 2$; $\frac{dy}{dv} = -\sin v$;

$\frac{dv}{du} = 3u^2$. Now

$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$ (chain rule

$= -\sin v \cdot 3u^2 \cdot 2 = -\sin u^3 \cdot 3(2x-5)^2 \cdot 2$

$= -6\sin(2x-5)^3 \cdot (2x-5)^2 \rightarrow$

Extract 7.5: Buhle’s response to 5.2

Buhle: *Lapha ku 5.2, the angle is cubed, so I thought maybe I could make a mistake if I do straight differentiation. So I decided to do substitution using u and use the chain rule. The following problem looks more, simpler to me, because in 5.3 it’s the function, that is cubed.*

Researcher: By function what do you mean?

Buhle*: I am referring to the sin function in 5.3, kanti (but) in 5.2 it’s the angle, 2x…mh….2x – 5 that is cubed.*

Researcher: In 5.4 you wrote, ‘I make sure that I differentiate all the functions one by one until I finish.’ What do you mean by that?

Buhle*: Phela mhem lapho ngisho ukuthi kufanele uqale la ekuqaleni, kukhona ipower, then ubese uya kwi function lena ye trig, mawuqeda lapho bese uya kumabrackets uqhubeke noku differentiatha, uloko umultiplaya. Konke nje, kanye, kanye, sengathi uhlubula amalayers* (Look ma’hm, I mean that you should start from the beginning with the power, then differentiate the trigonometric function. After you finish, get into the brackets and go on differentiating and multiplying each time. You do this one by one as if you are peeling layers.)

Buhle used *Leibniz* form technique of differentiation in Item 5.2. In his explanation he links the composition and decomposition of functions to differentiation. He is clear of the function affected by the power rule and demarcates between the Item 5.2, where the angle is squared compared to Item 5.3 where the trigonometric function is cubed. In 5.3 to 5.6 he used *straight* form technique of chain rule differentiation and application of the chain rule without u-substitution. He ran back and forth applying differentiation freely and correctly. The process of differentiation has been encapsulated to an object derivative. He displayed coherence of understanding of a collection of derivative rules and understanding of a composition of function. According to the Triad he is operating in the Trans- stage since he recognized various forms of the chain rule as linked. Buhle scored 5 in all the items involving use of the chain rule, composition and decomposition of functions. Some of his work is illustrated in Appendix D2. The summary of scores for Item 5.2 is given in Figure 7.9. Only 5 students could not make the relevant mental constructions while 12 out of 30 students displayed complete understanding showing all the aspects of mental constructions proposed in the genetic decomposition.

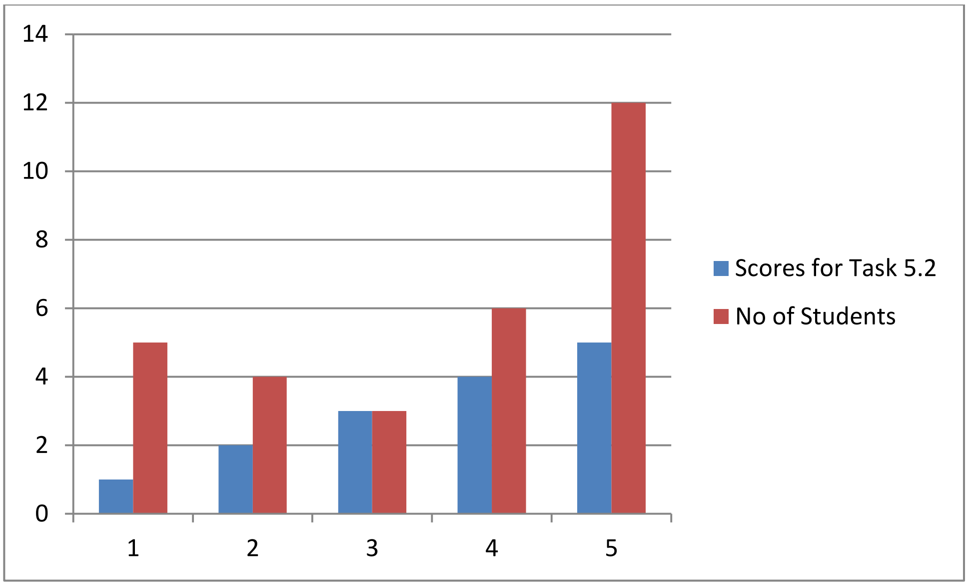


Figure 7.9: Scores allocation for item 5.2.

Almost all students gave a correct response for Item 5.3. This is illustrated by Figure 7.10 where 17 out of 30 students gave a complete response. No students mixed composition of functions or attempted to avoid the chain rule by expanding or rearranging the terms. Only 1 student displayed a response with no evidence of considering the chain rule. We could say that a collection of pre-schema for the chain rule was formed since the collection of elements in the chain rule was identified in their responses. With such a task we cannot conclude that the underlying structure of the chain rule was formed. Most of these students operated in the Inter- stage of the Triad since there was evidence of students' ability to (mentally) collect and relate the different cases of the chain rule. Some of the students during interviews, and further questioning explained the connection between the general statement of the chain rule and its applicability to the task.

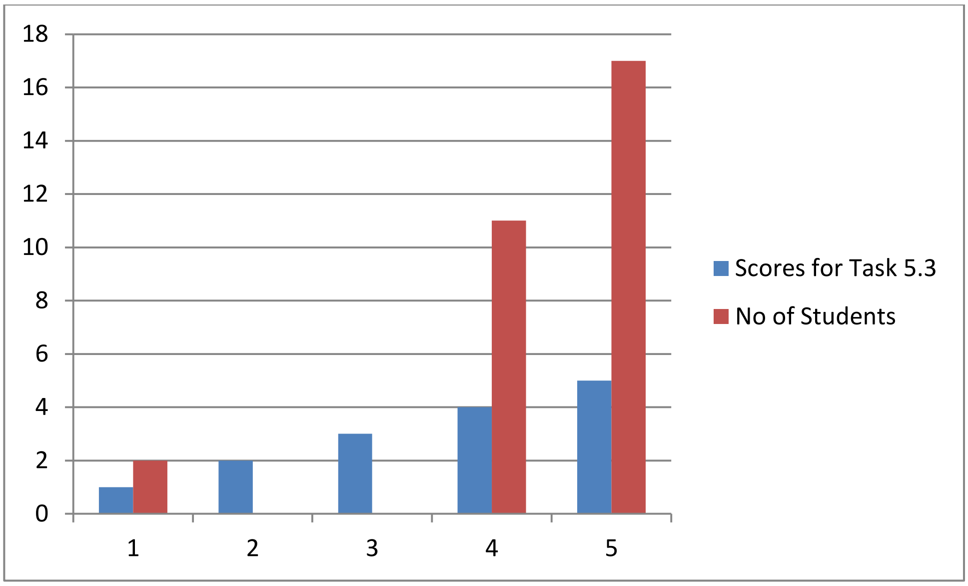


Figure 7.10 Scores for Item 5.3

Looking at the responses displayed by first year students the researcher noticed that the complexity of the function to be differentiated from 5.4 to 5.6 was indirectly proportional to the scores gained by students. Item 5.5 recorded the highest complete responses displaying all mental constructions made regarding the chain rule. The graphs displaying the scores of these performances can be found in Appendix D3. In an interview with Dube on how he managed to give correct responses on every item in differentiation, the following transpired:

Researcher: Looking at Item 5.4, I can see that you did it correctly, together with 5.5 and 5.6, using the chain rule in differentiation. What would you advise a person with problems of applying the chain rule in 5.4?

Dube: *Akabheke ipower le kuqala, athole iderivative yayo, then the function sin, Bese engena ngaphakathi,* (let the person start with the power, and find the derivative of the sin function, after which he/she must get inside the bracket*) differentiate one by one and multiply by each derivative until aqede (he/she finishes).*

Researcher: Maybe. Why is it called the chain rule?

Dube: *I can say it's because you find the each derivative uloko u(time)zile ugcin'usuthola ichain lamadervatives in a product* (and always multiply each of the derivative to get a chain of them).

Unlike 4 students interviewed who acknowledged learning the chain rule but could not state or express it although they applied it correctly, Dube knew the chain rule. He (1) understood all the procedures involved in doing calculations involving the chain rule, (2) was able to access the chain rule when he needed it, (3) was paying conscious attention to algorithms and techniques used in dealing with the chain rule and as such reflect on it and (4) jumped back and forth from *Leibniz* form technique by using the u-substitution and differentiating function by function without substitution. Some of his work is displayed in Extract 7.6.

$$y = \sin^2(4x^2 + e^{\sqrt{2x+\cos e}}$$

$$\frac{dy}{dx} = 2\sin(4x^2 + e^{\sqrt{2x+\cos e}}) \cdot \frac{d}{dx}\sin(4x^2 + e^{\sqrt{2x-\cos e}})$$

$$= 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \frac{d}{dx}(4x^2 + e^{\sqrt{2x-\cos e}})$$

$$= 2\sin(4x^2 + e^{\sqrt{2x \mp \cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x \mp \cos e}} \cdot [8x + e^{\sqrt{2x-\cos e}} \cdot \frac{1}{2}(2x - \cos e)^{-\frac{1}{2}} \cdot 2 \cdot -0]$$

Extract 7.6: Dube's response to item 5.4

Researcher: I want you to be explicit in explaining your understanding of the chain rule. What does it say? Can you write it down maybe?

Dube: *Angikhon' ukuyichaza kodwa ngiyazi ukuyisebenzisa. (*I am unable to explain it but I can use it)

Researcher: If for example I give you, $y = f(g(x))$, to differentiate, what would be the derivative of *y*?

Dube: *It would be, let me write it down, f prime g, then g prime, and we multiply. (He wrote:* $f'g(x) \times (g'(x))$.

In responding to Item 5.4, Dube used the link form technique of the chain rule for differentiation. Dube also explained that he differentiates the first function with respect to the power and writes $\frac{d}{dx}$ for every function that has not been differentiated yet. This helped him with knowing functions already differentiated and those still to be differentiated. This concise explanation convinced the researcher that he has a schema of the chain rule. He used previously constructed schemas of functions and derivatives to express the concept of the chain rule. He also recognized the given functions in Items 5.2 to 5.6 as composition of two or more functions, took their derivatives separately and then multiplied them. Part of this is the explanation he gave when he was asked on how he would help other students to deal with the chain rule. According to the Triad, Dube operated in the Trans- stage since he used the chain rule proficiently.

7.3.4 Category D: The structure of the integrand

7.3.4.1 Items 6.1 and 6.2

Evaluate:

6.1 $\int 2x\sqrt{1+x^2}\,dx$

6.2 $\int (3x+2)^6\,dx$

In this category, the chain rule was embedded in the structure of the integrand. The scores for mental construction made were allocated according to Table 5.7. This category aimed at reinforcing the use of the chain rule in expressions where it was embedded in the structure of the integrand. The table of standard integrals was for this reason not issued to the students. The summary of the scores in this category is illustrated by Figure7.11.

Figure: 7.11: Summary of scores for Category D

Dube used the u-substitution with the chain rule to respond to Items 6.1 and 6.2. He argued during the interview that they normally dealt with tasks in integration by referring to a table of standard integrals, which was not issued for this study. About 75% of the responses indicated the use of chain rule via *Leibniz* form technique of differentiation, reversing the process and then integrating. More complete responses were reflected for Item 6.2 than 6.1. All students who made relevant mental constructions for Item 6.1 also gave correct responses for 6.2. Most students operated in the Inter- stage in this category. This is encapsulation implied by Piaget's notion of reflective abstraction where the construction of the chain rule extends to another level and includes new forms of the process being built, drawing from previous ones to form an object. 75% of those students were able to think of the interiorized action a reverse and indicated evidence of constructing a new process in reverse.

7.4 Conclusion

Students were prompted through guiding questions to collect sets of tightly associated elements of the chain rule to examine their (1) knowledge structures, (2 internal representations, (3) how these were coordinated, and (4) the mental models they formed. The interviews conducted with the different subjects on various selected items to clarify some responses presented on the written instrument, helped the students in constructing and reconstructing mental objects. It was evident from their explanations that when the questions were revisited they had to think deeply about the functions and application of the chain rule. By the cycle of APOS theory students were led to reflection and reconstruction of operations important in differentiation of composite functions.

This chapter gave explanations of how students in the first year engineering course constructed concepts of functions and their derivatives using the chain rule. Analysis presented in graphs and students' written work extracts served to explore the conceptual understanding of these concepts using APOS and the stages on which mental constructs on the chain rule were made with regards to the Triad mechanism. The next chapter concludes the study by discussing the findings, researcher's thoughts, recommendations and limitations of the study.

CHAPTER 8

CONCLUSIONS AND RECOMMENDATIONS

8.1 Introduction

This chapter presents (1) the findings and conclusions of this study, (2) recommendations, (3) limitations of this study, and (4) themes for further research.

8.2 Findings and Conclusions

The findings and conclusions on the main research question addressing how students constructed various structures to recognize and apply the chain rule to functions in the context of calculus, which answer the sub-questions indicated in Chapter 1, are presented in a structured discussion under the following headings: (1) students understanding of the function concept, (2) students understanding of function composition, (3) students' understanding of the derivative (4) students' difficulties in using the chain rule, (5) students' schema alignment with the genetic decomposition of the chain rule, (6) the triad stage of schema development in which students are operating with respect to the chain rule, (7) students' identification of the reverse application of the chain rule in the substitution technique for integration, and (8) possible modifications of the proposed genetic decomposition.

8.2.1 Students' understanding of the function concept

The mean scores for the understanding of functions revealed a score of 0.8 higher than that of composition of functions. During interviews with the chosen subjects, it was evident that the 60% correct written responses received for the Item 1 on functions did not reveal students' understanding of a function. Also, a lower percentage (16,7)% correct responses was recorded for the Item 2 which was based on the concept of domain and range of a function. These findings indicate that most students displayed inadequate understanding of the function conception. When analyzing their responses it was evident that they only could manipulate and quantify functions since they emphasized numbers. They mostly operated in the process stage of APOS since they could make general arguments about functions, imagining transformations and performing actions without external stimuli. This understanding is insufficient to help students to deal with higher order calculus requiring understanding of the concept of function.

8.2.2 Students' understanding of function composition

Students in this category could not go beyond plugging in or substituting the one function for a variable in the other function. The mean scores revealed an action stage conception of composition and decomposition of functions. Some students interpreted composition as multiplication where they multiplied functions as objects. In this case $f((g(x))$ was interpreted as, $(f(x).g(x)$. Students not only multiply numbers but also multiplied entities such as functions. The process of substitution was treated as an entity which then became the objects of other actions and procedures of multiplication. This misconception was also evident in the pilot study.

There were a few misunderstandings hidden in the notation in function composition and the use of the '$\circ$' notation. Results showed little evidence (33%), where students treated $(f \circ g)(x)$ as a binary operation on two functions f and g, resulting in a new function $(f \circ g)$. These functions were the objects acted upon as a result of replaced processes. The processes were encapsulated to objects. Some students who operated in the Inter- stage of the Triad where they showed evidence of understanding composition of functions, operated in the Intra- stage regarding decomposition. They knew the rules of substitution to form the composite function but were unable to reverse the actions. Although instructional design accommodated intervention on stressing the understanding of the concepts of composition and decomposition of functions after the pilot study, the little difference of 0,3 in mean scores still indicates a lack of understanding of these concepts. The results indicated that 67% of student operated in the Intra-stage. They had a collection of rules for substituting in algebraic expressions in various situations but had no recognition of identifying the explicit functions that made the composite. Students who are in the Intra- stage of composition of functions schema development were those who saw the various rules of composition and decomposition as not related. Those were students who were skilled at algebraic manipulations, easily able to assimilate rules and procedures in a cognitive structure that consisted of a list of unconnected actions, processes and objects to produce correct answers. The actions had not been interiorized to processes. This was evident from the answers and responses given during interviews by students who left blank responses for those items indicating they knew nothing about decomposing a function. Success in composition and decomposition of functions is not a necessary requirement for successful application of chain rule. This is supported in the data illustrated in figure 7.1.

8.2.3 Students' understanding of the derivative

The highest mean score (4.7) recorded for this category indicated that most students displayed an object understanding of differentiation. The first item addressed finding the derivative of $y = -3\sin x + 2e^{\cos x} - 5e^{x}$. This item recorded the least number of correct responses in this category with a mean score of 3.5 despite the fact that only the middle term called for differentiation using the chain rule. A common error where students recorded the derivative of *cos x* correctly as – *sin x* but left out the brackets to end up with $y = -3\cos x + 2e^{\cos x} - \sin x - 5e^{x}$ was observed. Such students' actions of differentiation are detached from the basic algebraic operational signs. The multiplication sign left out indicates the absence of links between actions and procedures. Knowing the derivative of a particular function is not an indication of conceptual understanding since the relationships constructed internally were not connected to existing ideas. This understanding should also involve the knowledge and application of mathematical ideas and procedures related to basic arithmetic facts.

About a quarter of the students, (8 out of 30) were scored 1, indicating a guessed response and neglecting the application of the chain rule when differentiating the middle term $2e^{\cos x}$. Those subjects were operating in the Intra- stage of the Triad since they could not realize that the middle term was a composite function. Their responses displayed a mixture of not using appropriate rules when differentiating and making compound errors with rules used. According to APOS, they were reacting to external cues that give precise details on what steps to take. For example, most students interpreted the idea of differentiating *cos x* to get *sin x* which is incorrect since the negative sign has been left out, and indicates a void in their understanding of derivatives of trigonometric functions. During interviews, students would recall and want to rectify omissions of signs and brackets that they left out in their written responses. This could be as a result of differentiation from first principles not forming part of examinable topics in first year mathematics. On the contrary, 9 out of 30 students' responses displayed a schema understanding of differentiation regarding this item. Their responses indicated a collection of cognitive objects, their connections and internal processes for manipulating the processes of differentiation and operation signs correctly.

Items 5.2 to 5.6 dealt with differentiation using the chain rule. The findings are discussed and classified according to the techniques followed by the students in applying the chain rule. Most students, who used the straight form technique in differentiating, $y = \cos(2x-5)^3$, treated the trigonometric function as being cubed instead of the angle $(2x - 5)$. They indicated an action understanding of the given function where the existing schema of powers and a function of an angle were not taken to account. It was only during interviews with some of them that they took note of this. The students in the Pilot study who indicated and associated using the chain rule as peeling an onion, could state the chain rule, but displayed an incorrect response using the *straight* form technique to solve Item 5.2 as indicated in chapter 5 of this study. The 12 out of 30 subjects who used either the *Leibniz* form technique or the *link* form technique displayed correct responses. Also they indicated a totality of understanding of the chain rule with regard to this item.

Item 5.3 with a mean score of 4.8 recorded the highest in this category. Only one student used the *Leibniz* form technique to differentiate, $f(x) = \sin^3(4x)$. The other students used the *straight* form technique correctly. Mistakes with two students' responses occurred as they left out the square function after differentiation. Their responses were displayed as $f'(x) = 4.3\sin\ (4x)$ instead of $f'(x) = 12\sin^2(4x)$. Also the different subjects interviewed indicated that they used the chain rule to differentiate. Even those students, who used the *Leibniz* form and *link* form techniques for Item 5.2, used the *straight* form technique to deal with Item 5.3. Evidence of schema understanding of this item was indicated by 57% of the subjects in interviews and in written responses while 36% had the processes of differentiation encapsulated as objects. Those students could reflect on operations applied to differentiation, realized transformations and were able to construct such transformations in totality.

The responses by first year students to more complex functions for example, $y = (\cos ec^3 x + e^{\tan x})^2$, indicated that the complexity of the function differentiated was indirectly proportional to the scores gained by students. The more complex the item differentiated, the higher the scores, to the extent that, Item 5.5 recorded the highest complete responses displaying all mental constructions made regarding the chain rule. The students interviewed also indicated full understanding of the chain rule and being able to state it and use it. A number of students used the *straight* form technique in differentiating these tasks. In this manner differentiation of each function in the composite function was accomplished. Students either operated in the Inter- or Trans stages of the Triad. Those students in the Inter- stage showed evidence of having collected some or all the differentiation rules in a group and perhaps provided the general statement of the chain rule without yet constructing the underlying structure of the relationships. Minor errors such as dropping (−) signs or arithmetic errors; or applied the chain rule but making an error with the derivative rule were common. Such students during interviews, and further questioning explained the connection between their general statements of the chain rule and its applicability. About 50% of the subjects were able to reflect on the explicit structure of the chain rule and were capable of operating on the mental constructions which made up their collection. Without stating the chain rule those students were able to use it proficiently. They operated at the Trans stage of Triad.

8.2.4 Students' difficulties in using the chain rule

The main difficulties noticed in using the chain rule in this study were not connected to differentiation. Data analyzed disputed the fact that to understand the chain rule, one needs decomposition of function schema. The fact that students differentiated correctly using the chain rule rejects this hypothesis. Students struggled with manipulation of algebraic terms. This involved leaving brackets where they were required and minor errors of inserting a – sign instead of a + sign. For example, in differentiating $y = (\cos ec^3 x + e^{\tan x})^2$, responses like, $\frac{dy}{dx} = 2(\cos ec^3 x + e^{\tan x} + 3\cos ec^2 x + e^{\tan x}.\sec^2 x,$ without closing the first bracket after the first e^{tanx} , and just continuing with addition instead of multiplication or opening another bracket before 3 $cosec^2x$. Nonetheless, high scores were recorded for differentiation against the lowest scores attained for decomposition of functions. In this study it was found that students who had an inadequate understanding of composition of functions, performed well in the application of the chain rule. This confirms studies by Clark et al, (1997); Cottrill, (1999), and Hassani, (1998) who acknowledged that students' difficulties include the inability to apply the chain rule to functions and also with composing and decomposing functions. It cannot be disputed that students have shown significant gaps between their conceptual understanding of the major ideas of calculus and their ability to perform procedures based on these ideas. Some of the students' difficulties (about 60%) could be attributed to their difficulties with prerequisite knowledge of composite function or the derivative function notion. This finding is consistent with Capistran's (2005) result in which he stated that most students do not like or understand the *Leibniz* notation. This finding already overlooks the framework of three worlds of mathematics in which $\frac{dy}{dx}$ is regarded as a process and a concept, and therefore is a procept. The procedural actions of application of the chain rule in the proceptual world did not depend on finding of individual derivatives of the complex composite functions from the embodied world. The derivatives were found but not proved in the formal world because of mistakes done either in the form of signs or algebraic manipulations like leaving out the brackets when required.

8.2.5 Students' schema alignment with the genetic decomposition of the chain rule

To have a schema of the chain rule one has to master the use of multiplication and brackets used in function composition. These results were similar to the results of Webster (1978) in that students who are successful solvers of routine chain rule problems are not necessarily successful solvers of non-routine problems. Some of them knew the chain rule but could not access it when it was needed and only remembered during interviews the correct answers they could have provided. Also it was revealed from the interviews with the students that they only concern themselves with concepts that are examined. Students though recognize and apply the chain rule to different situations. They were able to differentiate some functions by simply applying rules which they memorized, and in some cases recalled incorrectly. Those students were skilled at algebraic manipulations, easily able to assimilate rules and procedures in a cognitive structure that consists of a list of unconnected actions, processes and objects to produce correct answers. Such students performed mechanically without displaying understanding.

A schema for the chain rule is an individual's collection of actions, processes, objects as well as other concepts that are perceived to be linked to the chain rule. These concepts include students' understanding of functions, composition of functions, derivatives and use of the chain rule. Results revealed that students who operated in the Intra- stage with regard to understanding of functions and their compositions, operated in the Trans- stage with regard to differentiation. They displayed coherence of understanding of a collection of derivative rules even though they did not understand composition of functions. Also, results revealed most difficulties were related to symbolic or structural difficulties. Obviously, the difficulties could not be attributed to only having the structural or conceptual prerequisite knowledge, but also relating them. This could account for the difference in Triad stages in which a particular student is operating.

If a student having the formula of the chain rule cannot (1) associate the derivatives of specific composite functions with the formula, (2) make the relations, (3) take specific composite function derivatives as unrelated by using memorized rules, he is in the Intra- stage of the Triad. When the student can collect the various derivative rules in a group and might provide the general statement of the chain rule, but he or she has not yet constructed the underlying structure of the relationships of the functions, he operates in the Inter- stage of the Triad. One of the participants explained during interviews that he differentiated the first function with respect to the power and wrote $\frac{d}{dx}$ for every function that had not been differentiated yet. This helped him with knowing functions already differentiated and those still to be differentiated. He used the *link* form technique in differentiating the given loaded function. In Item 5.2 he had correctly used the *Leibniz* form technique. The concise explanation convinced the researcher that he had a schema of the chain rule. He used previously constructed schemas of functions and derivatives to define the chain rule. He also recognized the given functions in 5.2 to 5.6 as composition of two or more functions, took their derivatives separately and then multiplied them. Part of this is the explanation he gave when he was asked on how he would help other students to deal with the chain rule. According to the Triad, such student operated in the Trans- stage since he used the chain rule proficiently. That student was also able to choose the appropriate form technique to use per task given and his work displayed correct application of the chain rule in each case.

8.2.6 The triad stage of schema development in which students are operating with respect to the chain rule

Written responses indicated that over 60% of students had a process understanding of the chain rule. Their actions of differentiation had been interiorized and transformed explicitly from memory and followed step-by-step instructions using the *straight* form technique to differentiate composite functions. The students who were confined to the straight form technique often made errors associated with basic arithmetic operations. Those students were operating in the Intra- stage of the Triad since they used a collection of rules for differentiation but failed to apply knowledge acquired from basic algebraic manipulations and observations of correct signs where needed. Those students showed evidence of constructing very little while others constructed bits and pieces of the concept of chain rule.

Students who used the *straight* form technique together with the link and *Leibniz* form techniques based on the type of composition of the function indicated a schema understanding of the chain rule. They were aware of the processes of differentiation and integration in totality and realized that transformations could act on it. In integration they preferred to use the *Leibniz* form technique while they used either the *link* or *straight* form techniques in differentiation of composite functions. Those students operated in the Trans- stage of the Triad. They mastered the multiplication and brackets used in function composition. Those students displayed all the constructions proposed in the genetic decomposition. They showed evidence of understanding and possessing a schema for the chain rule.

8.2.7 Students' identification of the reverse application of the chain rule in the substitution technique for integration

About75% of the responses displayed the use of the chain rule via the *Leibniz* form technique of differentiation, reversing the process and then integrating. All students who made relevant mental constructions for Item 6.1 also gave correct responses for Item 6.2. Most students operated in the Inter- stage in this category. The subjects interviewed indicated the inconvenience they suffered as they were not issued with standard integral tables. They cited this as the reason that forced them to use the u-substitution and hence the *Leibniz* form technique. The scores in this category were lower than those in the derivative category. No evidence was found of the use of the *link* and the *straight* form techniques in integration. The written responses indicated that where existing schemas were applied such that after finding the derivatives of the various functions in a composition, they now had to be multiplied to put the chain rule into application. Examples include, $\int (3x+2)^6 dx$, where some students represented $3x+2$ as u such that we have $\int u^6 du$. The derivative, $\frac{du}{dx} = 3$ was found where $\frac{du}{3} = dx$. This allows for the replacement of the composite function, and multiplication by the newly formed derivative and as such reversing integration using the *Leibniz* form technique. New processes were constructed by means of reversing the existing chain rule schema.

8.2.8 Possible modifications of the proposed genetic decomposition

The initial genetic decomposition proposed for the chain rule suggested amongst other things that for a chain rule schema, a student needed to have schemas of functions, composition of functions, derivatives and use of chain rule. Results show that decomposing a composite function is not a pre-requisite for applying the chain rule. Thus the proposed genetic decomposition has been modified as:

For a student to have his or her function schema

(i) He or she had developed a process or object conception of a function and

(ii) Has developed an action or process conception of a composition of functions.

For a derivative schema,

(iii) He or she had developed an object conception of differentiation

(iv) The student then uses the previously constructed schemas of functions, composition of functions and derivative to define the chain rule. In this process, the student recognized a given function as the composition of two functions, took his derivatives separately and then multiplied them.

(v) The student recognized and applied the chain rule to specific situations using either the *straight, link or Leibniz* form techniques. The student could think of an interiorised process of differentiation in reverse to construct a new process by reversing the existing one

8.3 Recommendations

To present these in a structured manner, they have been documented under the following sub-headings: (1) sampling, (2) re-visiting content for first year calculus, (3) form techniques for the chain rule learning in calculus, and (4) use of APOS in exploring other concepts.

8.3.1 Sampling

A sample of 30 volunteering first year University of Technology civil engineering students were participants in this study. This implies that the data used in the analysis have been based purely on those students' written responses from the questionnaire administered, video recordings and interview schedules with six of those subjects. In addition data were also collected from tutorial worksheets on the chain rule presented collaboratively. While this has been adequate in using the APOS approach in exploring conceptual understanding displayed by first year, University of Technology students in learning the chain rule in calculus, for future research it is recommended that broader research that includes more first year students within the university be conducted. Samples can also be drawn for this research from electrical, chemical and mechanical engineering first year students in the university. Alternatively, this study may be extended to any first year engineering student of any university. All first year engineering students not only must they learn the chain rule, but ought to be brought to a schema level of understanding to enable its successful application in calculus.

8.3.2 Revisiting content for first year calculus

All first year students should be taught the concept of functions and their behaviour explicitly and these should be examinable. They should be guided through instructional design to a level of understanding functions that is past the process stage. Students should realize that some actions can be carried out resulting in some transformation on a function. The encapsulations of those processes would then result in students operating in the object stage of this concept. It is then a collection of all the actions on functions, processes, objects together with other schemas and their relationships will help the students to process the composition of functions. Processes like continuity, limits, mappings, graphical representations, composite functions and their decomposition as a totality should be a firm background knowledge on which first year students build their understanding of the chain rule. It is therefore recommended that the content for first year engineering mathematics accommodate this.

8.3.3 Form techniques for the chain rule learning in calculus

Also from the findings in this study, it is recommended that students be taught all three form techniques of chain rule differentiation identified in this research. This is because students exposed to all techniques indicated full understanding on how to use the chain rule. The *straight* form technique which involved straight application of the chain rule was easily used by students who had a schema of the chain rule. This involved a collection of actions on algebraic manipulations and use of multiplication and correct signs where necessary. Students who struggle with signs and use of brackets would be advised to stick to the *link* form technique where differentiation starts from the innermost function and moves outwards. Students struggling with the chain rule can always in any given composite function, use the *Leibniz* form technique where they substitute for various functions, differentiate and multiply.

8.3.4 Use of APOS in exploring other concepts

While APOS was used to explore the conceptual understanding of the chain rule in this study resulting in a genetic decomposition of the concept, it is recommended that for future research, researchers explore the conceptual understanding of other mathematical concepts, for example, product and quotient rule in calculus, exponential and natural logarithm rule, and understanding of algebraic functions. This would result in instructional treatment that would guide students to make mental constructions relevant to those concepts and as such improve their understanding of mathematics. This would result in the genetic decomposition of many other concepts in mathematics. Time spent learning for understanding is another factor. Students with deeper understanding of a concept are more likely to be able to transfer that concept to other situations.

The most important contribution that we obtain from the APOS analysis, in this study and in many others, is an increased understanding of mental mechanisms made by students as they dealt with the chain rule in calculus. This involves interiorization of actions to processes which are then transformed and encapsulated as objects. Based on the proposed genetic decompositions, these processes should then point us to effective pedagogy. It is therefore suggested that APOS be used to explore the understanding of other concepts taking into consideration the prerequisite concepts on which the understanding of the concept under scrutiny is based. Pedagogical strategies based on the analysis in this study and focusing at helping students to interiorize actions repeated without end, to reflect on seeing the chain rule in differentiation as a totality.

8.4 Limitations

This was a small-scale study. Only aspects directly related to the sample of first year engineering students participating and their responses to the written instrument, interviews and video-recordings in their mathematics classroom were investigated. Needless to say, there were other issues that could have been investigated however those were not within the scope of this doctoral study.

Also, since this study has been conducted in a University of Technology with only one group of first year engineering students that was purposely chosen because the researcher was their lecturer, the situations with other first year groups may differ. This may be the case because variables differ from one discipline to the other.

Also, in order to gain access to do research with the participants and gain their consent to proceed, the study was registered first with University of Technology's research department. The institution was informed the intentions, details and processes that would be followed in conducting this research. The consent forms were then read to each participant and explained in detail in terms of what they were expected to do for the study. This might have influenced the way they presented their responses on the questionnaires and worksheets. The responses could have been structured in the way they thought the researcher was anticipating. However, the researcher has addressed this possible bias by interviewing selected subject based on written responses and observations done when students interacted with the worksheet collaboratively.

The students who took part in the study were volunteers. The researcher had no choice but to work with responses that they gave. The researcher chose subjects for interviews from the volunteered sample. The students were not chosen according to performance in class or any other criterion.

In addition the tutorial sessions took place in laboratories. Those venues did not have instruments and space for instructional treatment records to be documented for students. This generated some concern for students about copying from the whiteboard before it was erased instead of interacting with the content presented. During Lesson Two, for example, the lecture in which the students were accommodated did not allow for the researcher to observe each student's work due to the seating arrangement in the lecture hall. Students had to call out their answers to the given activities.

Finally, on interrogating each video recording, it may have been beneficial to use two video cameras where one would focus on the lesson presented and the other on the learners. This would have captured the learners' responses together with their expressions especially in situations that were considered critical moments in each lesson. This would have captured relevant learner responses when the learners argued for correctness of their responses against each other as they exchanged their books.

8.5 Themes for further research

This study had a conceptual understanding focus. In the APOS exploration of the understanding of the concept of the chain rule in calculus, it was found that the following, amongst others, influence the teaching and learning of the chain rule: (1) understanding of the basic manipulation of algebraic symbols and operations by students, (2) accommodation of multiple ways of function representation in instructional design to enable students to make connections and have deeper understanding of the concept of function, (3) selecting activities to inform teaching strategies in the use of APOS paradigm for object stage understanding of the chain rule, (4) relating of the three worlds of mathematics where students have the general statement of the chain rule , relationship between the formulas of the chain rule in the function and the *Leibniz* form technique in the symbolic world, (5) formulation of the schema of the chain rule to be applied to a wider range of contexts in calculus, and (6) augmentation of the chain rule by instructional design based on the modified genetic decomposition for better understanding of the chain rule concept with skills. These have informed the following possible themes for further research:

(1) Continuation of research including all first years within the university on their understanding of the chain rule. Explore how we can bring first year students to a schema level of understanding of the chain rule to enable successful application of calculus.

(2) Investigations on the effect of use of multiple ways of function representations in the understanding of the chain rule by first year students. Explore whether this would result in students attaining a deeper understanding of the chain rule or other concepts in calculus.

(3) The effect of using the modified genetic decomposition in the teaching of the chain rule. Explore whether instructional design based on this genetic decomposition can help students to operate in the Trans- stage of the Triad?

(4) The use of APOS in exploring the other first year mathematical concepts to enhance the development of the genetic decomposition of various concepts, in a South African context.

REFERENCES

1. Akkoç, H., & Tall, D. (2002). The Simplicity, Complexity and Complication of the Function Concept. In A. D. Cockburn & E. Nardi (Eds.) *Proceedings of the 26th Conference of the International Group for the Psychology of Mathematics Education,* Norwich. Vol. 2, 25-32.

2. Akkoc, H., & Tall, D. (2003). 'The function concept: Comprehension and Complication', *Proceedings of the Day Conference of British Society Research on learning of Mathematics,* Sheffield Hallam University, UK (2), 25-29.

3. Akkoç, H., & Tall, D. (2005). A Mismatch between Curriculum Design and Student Learning: The Case of the Function Concept. In D. Hewitt & A. Noyes (Eds.) *Proceedings of the 6th British Congress of Mathematics Education,* Warwick. 1-8.

4. Altrichter, H., Feldman, A., Posch, P., & Somekh, B. (2008). Teachers investigate their work: *An introduction to action research across the professions.* Routledge. (2nd edition).

5. Artigue, M. (1998), L'évolution des problématiques en didactique de l'analyse, *Recherches en Didactique des Mathématiques,* 18.(2), 231-262.

6. Anderson, J.R. (1995). Cognitive Psychology and it's Implications. 4th Edition, W.H. Freeman.

7. Anderson J, Austin K, Barnard T and Jagger J,(2001) 'Do Third-Year Mathematics Undergraduates Know What They Are Supposed To Know', Inter J. Math. Ed. in Sci. and Tech., 29 3, 401-420.

8. Angen, M.J. (2000). Evaluating interpretive inquiry: Reviewing the validity debate and opening the dialogue. *Qualitative Health Research*. 10(3) p. 378-395.

9. Anghileri, J. (2006). Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education,* 9, 33 – 52.

10. Asiala, M., Brown, A., DeVries, D.J., Dubinsky, E., Matthews, D., & Thomas, K. (1996). A framework for Research and Development in Undergraduate Mathematics Education, 2, 1-32.

11. Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *Journal of Mathematical Behavior,* 16, 399–431.

12. Asiala, M., Brown, A., Devries, D.J., Dubinsky E., Mathews, D., & Thomas, K. (2004). A Framework for Research and Curriculum Development in Undergraduate Mathematics Education Research in Collegiate Mathematics Education II, *Issues in Mathematics Education* (CBMS), American Mathematical Society.

13. Baker, B., Cooley, L., Trigueros, M. The schema triad, a calculus example, submitted for publication.

14. Baker, W., & Czarnocha, B. (2002). Written meta-cognition and procedural knowledge. *Proceedings of the 2^{nd} International conference on the teaching of mathematics at the undergraduate level,* Hersonissos, Crete.

15. Berth, E. W., & Piaget, J. (1966). Mathematical Epistemology and Psychology (W. Mays, trans.), Dodtrecht: Reidel.

16. Bowie, L. (2000). A learning theory approach to students' errors in a calculus course. *Pythagoras*, 52, 2-14.

17. Bowie, L., (2003). A learning theory approach to students' errors in a calculus course, in *Proceedings of the Ninth National Congress of the Association of Mathematics Education:* Vol. 1. UCT Baxter Theatre Complex and Rustenburg Girls' Junior school.

18. Bransford, J., Brown, A., & Cocking, R. (2000). How people learn: Brain, mind, experience, and school. Washington, DC: National Academy Press.
19. Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process of conception of function. *Educational studies in Mathematics,* 23, 247-285.
20. Brijlall, D.; Maharaj, A. & Jojo, Z.M.M. (2006). The development of geometrical concepts through design activities during a Technology education class. *African Journal of Research in SMT Education,* Volume 10(1), 37-45.
21. Brijlall, D. & Maharaj, A. (2008). Applying APOS theory as a theoretical framework for collaborative learning in teacher education.
22. Brijlall, D. & Maharaj, A. (2009a). Using an inductive approach for definition making: Monotonocity and Boundedness of sequences. *Pythagoras*, 70, 68-79.
23. Brijlall, D. & Maharaj, A. (2009 b). An APOS analysis of students' construction of the concept of continuity of a single-valued function. 7th Southern Delta Conference on the 'Teaching and learning of undergraduate mathematics and statistics' *Proceedings of Gordon's Bay Delta'09, 30th Nov.- 4th Dec*: Villa Via Hotel, Gordon's Bay. South Africa.
24. Brijlall, D. & Maharajh, N. (2010). Exploring a theoretical framework to understand student learning of concepts in differential calculus. *Proceedings of the 16th Annual Congress of the Association of Mathematics Education of South Africa (pp81-98).* Durban: AMESA.
25. Brush, P.A., & Saye, J. (2001). The use of embedded scaffolds with hypermedia-supported student-centered learning. *Journal of Educational Multimedia and Hypermedia, 10 (1), 333 – 356.*
26. Burke, M., Erickson, D., Lott, J. W., & Obert, M. (2001). *Navigating Through Algebra in Grades 9-12*. Reston, VA: National Council of Teachers of Mathematics.
27. Byers, V. (1980). What does it mean to understand mathematics? *International Journal of Mathematical Education in Science and Technology,* 11(1), 1–10.

28. Byrnes, J. & Wasik, B. (1991). The role of conceptual knowledge in mathematical procedural learning. *Developmental Psychology,* 27(5), 777-786.
29. Cangelosi, J.S. (1996). Teaching Mathematics in Secondary and Middle School*:* An alternative approach, Second edition. New Jersey: Prentice Hill.
30. Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. *Research in Collegiate Mathematics Education,* 3, 114–162.
31. Carlson, M. & Jacobs, S. (2000). A General Framework for Describing Covariational Reasoning: Drawing from Piaget's Work. Presentation at the Fifth Annual Conference of the MAA *Special Interest Group on Research in Undergraduate Mathematics Education*.
32. Carlson, M., Jacobs, S., Coe, E., Larson, S., & Hsu, E. (2002). Applying covariational reasoning while modeling dynamic events: A framework and a study. Journal for *Research in Mathematics Education*, 33(2), 352-378.
33. Carpenter, T. P., & Lehrer, R. (1999). Teaching and Learning Mathematics with understanding. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 19-32). Mahwah, N.J.: Lawrence Erlbaum Associates.
34. Capistran, R. W. (2005). Concepts of the chain rule for first term calculus: A comparison across students, instructors, and professors. Unpublished doctoral dissertation, the University of Minnesota.
35. Clark, J. M., Cordero, F., Cottrill, J., Czarnocha, B., DeVries, D. J., St. John, D., Tolias, T., & Vidakovic, D. (1997). Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior*, 16, 345–364.
36. Claxton, C.S. and Murrell, P.H. 1987. Learning Styles*: Implications for improving Educational Practices.* ASHE-ERIC Higher education Report No. 4. Washington D.C. ERIC Clearing House on Higher Education.
37. Cohen, L., Manion, L., & Morrison, K. (2000). Research Methods in Education. Routledge Falmer: London.

38. Cohen, L. & Manion, L. (2000). Research Methods in Education (4th ed). London: Routledge.
39. Confrey, J. (1994). Six Approaches to Transformation of Function Using Multi-Representational Software. In J. Pedro da Ponte & J. Filipe Matos (eds) *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education,* University of Lisbon, Portugal, Vol. 2, 217-224.
40. Cooley, L., Trigueros, M., & Baker, B. (2007) Schema Thermatization: A framework and Example. *Journal for Research in Mathematics Education*. Vol. 38 US.
41. Cottrill, J. (1999) Students' understanding of the concept of chain rule in first year calculus and the relation to their understanding of composition of functions, Ph.D. thesis, Purdue University.
42. Creswell, JW. (1998). Qualitative Inquiry and Research Design Choosing Among Five Traditions. Thousand Oaks, CA: Sage Publications.
43. de Vos, A.S. (2002). Qualitative data analysis and Interpretation. In A.S. de Vos (Ed.), *Research at grassroots* (pp 339-355). Pretoria: Van Schaik Publishers.
44. de Vos, A.S., Fouche', C.B. & Venter, L.(2002). Quantitative data analysis and Interpretation. In A.S. de Vos (Ed.), *Research at grassroots* (pp 222-246). Pretoria: Van Schaik Publishers.
45. De Vries, M.J. (2006). Teaching and learning the nature of Technology. *Proceeding of the 14th Annual meeting of the Southern African Association for research in Mathematics, Science and Technology Education (SAARMSTE),* Pretoria, pp 1-13.
46. Denzin, N. (2006). Sociological Methods: A Sourcebook. Aldine Transaction. ISBN 9780-202308401. (5th edition).
47. Dreyer, J.J. (1985). Modern Mathematics for South African schools. Juta & Co, LTD: Kenwyn.

48. Dubinsky, E. (1991). Reflective Abstraction in Advanced Mathematical Thinking. In David O. Tall (Ed.), *Advanced Mathematical Thinking* (pp. 95–123). Kluwer: Dordrecht.
49. Dubinsky, E., (1991a). ‘Reflective Abstraction in advanced Mathematical Thinking’, in: Tall, D., ed, *Advanced Mathematical Thinking*, The Netherlands: Kluwer.
50. Dubinsky, E., (1991b). ‘Constructive aspects of Reflective Abstraction in Advanced Mathematics’’, in: Steffe , L., ed, *Epistemological Foundation of Mathematical Experience,* New York: Springer Verlag.
51. Dubinsky, E., (2000). Using a theory of learning in college mathematics courses [Electronic version]. *Teaching and Learning Undergraduate Mathematics, 12.*
52. Dubinsky, E. (2010). *In Proceedings of the eighteenth annual meeting of the SAARMSTE*: School of Science, Mathematics and Technology Education, University of KwaZulu Natal.
53. Dubinsky, E., & Harel, G. (1992). The Nature of the Process Conception of Function. In Harel, G. and Dubinsky, E. (Eds), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 85–106). Washington, D.C.: Mathematical Association of America.
54. Dubinsky, E. and McDonald M.A. (2001), APOS: A constructivist theory of learning in Undergraduate Mathematics Education Research in the *Teaching and learning of Mathematics at University level:* An ICMI study. Kluwer Academic Publishers: Netherlands.
55. Dubinsky, E., Weller, K., McDonald, M.A. & Brown, N. (2005). Some historical issues and paradoxes regarding the concept of infinity: An APOS analysis: Part 2. *Educational Studies in Mathematics*. Vol. 60, pp 253- 266.
56. Duffin, J.M., & Simpson, A.P. (2000). A search for understanding. *Journal of Mathematical Behavior,* 18(4), 415–427. *Education sector development plan*. (1992). Maseru: Ministry of Education.

57. Elia, I., & Spyrou, P. (2006). How students conceive a function: A triarchic conceptual-semiotic model of understanding of a complex concept. *The Montana Mathematics Enthusiast*, ISSN 1551-3440, Vol. 3, no2, pp.256-272.

58. Engelbregcht, J., Harding, A. & Potgieter, M. (2010). Undergraduate students' performance and confidence in procedural and conceptual mathematics. University of Pretoria:

59. Ernest, P. (1991). The Philosophy of Mathematics Education. London: The Falmer Press.

60. Evangelidou, A., Spyrou, P., Elia, I., & Gagatsis, A. (2004). 'University students' conceptions of function', in M. Johnsen Hoines and A. Berit Fuglestad (eds), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, Bergen University College, Norway, pp. 351-358.

61. Felder, R.M., (1996). Matters of Style. ASEE Prism, 6(4): 18-23 December.

62. Frey, J.H. (1994). Interviewing: the art of science. In N.K. Denzin, & Y.S. Lincoln, eds. Handbook of qualitative research, 361-77.

63. Frid, S.D. (2004). A constructivist' look at the nature and role of calculus students' use of mathematical symbols, technical and everyday language. Retrieved April 2, 2006.

64. Fouche', C.B. & Delport, C.S.L. (2002). Introduction to the research process. In A.S. de Vos (Ed.), *Research at grassroots* (pp77-92). Pretoria: Van Schaik Publishers.

65. Gagnon, G., & Collay, M. (2001). Designing for learning: Six elements in constructivist classrooms. Thousand Oaks, CA: Corwin Press.

66. Gordon, S.P. (2005). Discovering the chain rule graphically. *Mathematics and Computer Education*. 39(3), 195-197.

67. Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity and flexibility: a "proceptual" view of simple arithmetic. *Journal for Research in Mathematics Education*, 25 (2), 116–140.

68. Gray, E. M., and Tall, D. O. (2001). Relationships between embodied objects and symbolic procepts: an explanatory theory of success and failure in mathematics. In M. van den Heuval (Ed.), *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, p. 65-72). Utrecht, Netherlands.

69. Gray, E.M., & Tall, D.O. (2002). 'Relationships between embodied objects and symbolic procepts'. In Marja van den Heuvel-Panhuizen (Ed.) *Proceedings of the 25th Conference of PME,* 3, 65–72. Utrecht, The Netherlands.

70. Habineza, F. (2010). Developing first year mathematics student teachers' understanding of the concept of definite integrals and their link through the fundamental theorem of calculus: An action research. Unpublished doctoral dissertation, University of KwaZulu-Natal, Pietermaritzburg, South Africa.

71. Harel, G. & Kaput, J. (1991). 'The Role of Conceptual Entities and Their Symbols in Building Advanced Mathematical Concepts', in Tall, D., ed., *Advanced Mathematical Thinking,* The Netherlands: Kluwer.

72. Hapasaalo T. (2004). On instrumental genesis within Procedural and conceptual thinking. University of Joensuu, Finland.

73. Hassani, S. (1998). Calculus students' knowledge of the composition of functions and the chain rule. Unpublished doctoral dissertation, Illinois State University, Normal.

74. Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A.Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Mcmillan.

75. Hiebert, J., & Lefevre, P., 1986: 'Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis'. In Hiebert (Ed.) *Conceptual and*

procedural Knowledge: The Case for Mathematics, (pp. 1–27). Hillsdale. N.J.: Erlbaum.

76. Horvath, A. (2007). Looking at calculus students' understanding from the inside-out: The relationship between the chain rule and function composition. Michigan State University.
77. Jojo, Z.M.M., Brijlall, D., & Maharaj, A. (2010). Reflective abstraction and Mathematics Education: The genetic decomposition of the chain rule. *Proceedings of the eighteenth Annual meeting of the Southern African Association for research in Mathematics, Science and Technology Education, 1, 63-7.* Durban: SAARMSTE.
78. Jojo, Z.M.M., Brijlall, D., & Maharaj, A. (2011). The reliability of a research instrument used to measure mental constructs in the learning of the chain rule in calculus. *Proceedings of the 16th Annual Congress of the Association of Mathematics Education of South Africa (pp 336-349).* Johannesburg: AMESA.
79. Kaplan, W., (1984). 'Derivatives and differentials of composite functions' and 'the general chain rule' in advanced calculus. 3rd Ed. Reading. MA: ADISON Wesley, pp101-105 and 106-110.
80. Kendal, M. (2001). Teaching and learning introductory differential calculus with a computer algebra system. Phd Thesis: Department of Science and Mathematics Education, University of Melbourne.
81. Kieren, T.E. (1990). Understanding for teaching for understanding, *The Alberta Journal of Educational Research*, 36 (3), 191-201.
82. Kieran, C. (1992). The Learning and Teaching of School Algebra. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390-419). New York: Macmillan.
83. Krefting, L. (1991). Rigor in qualitative research: The assessment of trustworthiness. *The American Journal of Occupational Therapy*, 45(3), 214-222.

84. Kvale, S. (1996). Interviews: An introduction to qualitative research interviewing. London: Sage Publications.
85. Lankshear, & Knobel (2004)
86. Likwambe, B. & Christiansen, I.M. (2008). A Case Study of the development of In- service Teachers' Concept Images of the Derivative. *Proceedings of Association for Mathematics Education of South Africa.* Pythagoras: No68, December pp22-31.
87. Lutzer, C.V. (2003). Using motion to teach chain rule and u-substitution. PRIMUS, 13(1), 47-54.
88. Maharaj, A. (2010). An APOS Analysis of Student's Understanding of the Concept of a Limit of a Function. *Pythagoras,* 71, 41-51.
89. Maharajh, N., Brijlall, D., & Govender, N. (2008). Prer-service teachers' conception of continuity of single-valued functions. *Proceedings of the 16th Annual Conference of the Southern African Association for Research in Mathematiics, Science and Technology Education.* Maseru: SAARMSTE.
90. Malone, J.A. & Taylor, P.C. (1992). Constructivist Interpretations of Teaching and Learning Mathematics. *Handbook of Research on Mathematics Teaching and Learning.*
91. Mathews, John H. (1989). Teaching differentiation with muMATH. *Journal of Computers in Mathematics and Science Teaching,* 8(3), 50-53.
92. Mccosker, N. & Diezman, C.,(2009). Scaffolding students' thinking in mathematical investigations. APMC, 14 (3), 27 – 32.
93. Monk, S., & Nemirowsky, R. (1994). The case of Dan: Student construction of a functional situation through visual attributes, *Research on Collegiate Mathematics Education,* 1, 139-168.
94. Mwamwenda, T.S. (1989). An Educational Psychology: An African Perspective. Butterworths Professional Publishers (Pty) Ltd; Durban.
95. Naidoo, K. (2007). First year students' understanding of elementary concepts in differential calculus in a computer laboratory teaching environment. *Journal of College Teaching & Learning,* (pp55-66), Vol. 4, No9.

96. Naidoo, K., & Naidoo, R. (2007). First year students' understanding of Elementary Concepts in Differential Calculus in a Computer Laboratory Teaching Environment in the *Journal of College and Teaching & learning.* Vol.4, N0 9.
97. Nesamvuni, PT 2009, Communicative interactions in desegregated South African classrooms, MEd dissertation, University of Pretoria, Pretoria, viewed *yymmdd* < http://upetd.up.ac.za/thesis/available/etd-05102010-161421/>
98. Novotna, J., & Hoch, M. (2008). How structure sense for algebraic expressions or equations is related to structure sense for abstract algebra. *Mathematics Education Research Journal*, 20(2), 93-104.
99. O'Donoghue, T., Punch K. (2003). Qualitative Educational Research in Action: Doing and Reflecting. Routledge. p.78.
100. Oehrtman, M.C. (2002). Collapsing dimensions, Physical limitations, and other student metaphors for limit concepts: An Instrumentalist Investigation into Calculus students' spontaneous reasoning. PhD Dissertation, University of Texas, Austin, Texas.
101. Olsen, B.E. (2008). Classroom observation tips, in Student Teaching in Special Education.
102. Orton, A., (1983). Students' Understanding of Differentiation, *Educational Studies in Mathematics*, 14, 235-250.
103. Pascual, J. (2004). Reflections on working memory: Are the two models complementary?
Journal of Experimental Child Psychology, 77, 138-154.
104. Piaget, J. (1964). Development and learning. *Journal of Research in Science Teaching*, 2, 176-180.
105. Piaget, J. (1972). *The Principles of Genetic Epistemology* (W. Mays trans.) London: Routledge & Kegan Paul.
106. Piaget, J. (1975). The Equilibration of cognitive structures. Cambridge: Harvard University Press.

107. Piaget, J. (1980). Adaptation and intelligence: Organic selection and Photocopy. Chicago: University of Chicago Press.

108. Piaget, J. (1985): The Equilibrium of Cognitive Structures. Harvard University Press, Cambridge, Massachusette: USA.

109. Piaget, J., & Garcia, R. (1983). Psychogenèse et Histoire des Sciences. Paris: Flammarion.

110. Piaget, J., & Garcia, R. (1989). Psychogenesis and the history of science (H. Feider, Trans.). New York: Columbia University Press. (Original work published 1983).

111. Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterize it and how can we represent it? *Educational Studies in Mathematics, 26,* 165-190.

112. Romberg, T. A. (1992). Problematic features of the school mathematics curriculum. In P. W.

Jackson (Ed.), *Handbook of research on curriculum* (pp. 749-788). New York: Macmillan.

113. Rubin, H., & Rubin, I. (1995). Qualitative interviewing: The art of hearing data. Thousand Oaks, C.A: Sage.

114. Schutz, A. (1962). Collect Papers, Volume 1, The Hague, Martinus Nijhoff. See in particular: "Commonsense and scientific interpretations of human action" pp. 3-47; "Concept and theory formation in the social sciences" pp. 48-66; "On multiple realities" pp. 207-259.

115. Selden, J., Selden, A., & Mason, A. (1994). Even good calculus students can't solve non-routine problems. In J. J. Kaput & E. Dubinsky (Eds.), *Research issues in undergraduate mathematics learning*, MAA Notes 33 (pp. 19–26). Washington, DC: Mathematical Association of America.

116. Sewell, A. (2002). Constructivism and student misconceptions. Why every teacher needs to know about them. *Australian science teacher's journal, 4*8(4), 24-31.

117. Sfard, A. (1987). 'Two conceptions of Mathematical notions: operational and structural', PME11, Montreal.

118. Sfard, A. (1989). Transition from Operational to Structural Conception: The notion of function revisited, *Proceedings of PME XIII*, Paris, 151-158.

119. Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22, 1-36.

120. Sfard, A. (1992). Operational Origins of Mathematical Objects and the Quandary of Reification –The Case of Function. In C. Harel & E. Dubinsky (Eds.), *The Concept of Function. MAA Notes* 25, 59-84.

121. Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification - the case of algebra. *Educational Studies in Mathematics*, 26, 191-228.

122. Slavit, D. (1997). An alternate route to reification of function, *Educational Studies in Mathematics* 33, 259-281.

123. Smith, D.A., & Moore, L.C. (1991). Project Calc: An Integrated Laboratory Course. In C. Leinbach et al. (Eds.), The Laboratory Approach to Teaching Calculus, MAA Notes, 81-92, Washington DC: MAA.

124. South Australian Certificate of Education [SACE] Board of South Australia. (2009). Curriculum statement 2009. Retrieved from http://www.ssabsa.sa.edu.au/docs/cs-2009/2mme-cs-2009.pdf

125. Star, J. R. (2000). *On the Relationship between Knowing and Doing in Procedural Learning.* Paper presented at the Fourth International Conference of the Learning Sciences.

126. Swanson,T. (2006). Pre-calculus: A study of functions and their applications, Hope College: Holland.

127. Tall, D. (1991). 'Intuition and Rigour: The Role of Visualization in Calculus', in Zimmermann, W. & Cunningham, S. *Visualization in teaching and learning Mathematics.* MAA Notes series, USA: Mathematical Association of America.

128. Tall, D. (1992). The transition to advanced mathematical thinking: Functions, limits, infinity, and proof. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 495–511). New York: Macmillan.

129. Tall, D. (1996). Functions and calculus. In A.J. Bishop, K. Clements, C. Keitel, J. Kilpatrick & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 289–311). Dordrecht: Kluwer Academic Publishers.

130. Tall, D. (1999). Reflections on APOS Theory in Elementary and Advanced Mathematical Thinking. In O. Zaslavsky (Eds.) *Proceedings of the 23rd Conference of PME*, Haifa, Israel, 1, p111-118.

131. Tall, D., McGowen, M., & DeMarois, P. (2000). The function Machine as a cognitive root for the function concept. University of Warwick: Conventry CV47AL, UK.

132. Tall, D. (2004). Thinking through three worlds of mathematics. In M. Johnsen Høines, & A.B. Fuglestad (Eds.), *Proceedings of the 28:th conference of the International Group for the Psychology of Mathematics Education* (pp. 281-288). Norway: Bergen University College.

133. Tall, D. O. (1993). Real Mathematics, Rational Computers and Complex People. In Lum, L. (ed.), *Proceedings of the Fifth Annual International Conference on Technology in College Mathematics Teaching*, 243–258.

134. Tall, D. O. (1997) Functions and Calculus. In A. J. Bishop et al (Eds.), *International Handbook of Mathematics Education,* 289–325, Dordrecht: Kluwer.

135. Tall, D. O. (2007). Embodiment, symbolism and formalism in undergraduate mathematics education. *Electronic proceedings of the 10th Conference of the Special Interest Group of the Mathematical Association of America on Research in Undergraduate Mathematics Education.* San Diego, CA. Retrieved from http://sigmaa.maa.org/rume/crume2007/ eproc.html

136. Tall, D. O. (2008). The transition to formal thinking in mathematics. Mathematics Education Research Journal, 20(2), 5-24.

137. Thompson, P. W. (1994), 'Students, Functions, and the Undergraduate Curriculum', in E. Dubinsky, A. Schoenfeld, & J. Kaput (eds), *Research in Collegiate Mathematics Education*, *I, CBMS Issues in Mathematics Education*, 4, 21–44.

138. Thoo, J. B. (1995). Composition and the chain rule using arrow diagrams. PRIMUS, 5(3), 291-295.

139. Trochim, W.M. (2002). Concept mapping as an alternative approach for the analysis of open-ended survey responses. Organizational Research Methods, 5(4): 307-356.

140. Uygur, T., & Ozdas, A. (2005) Misconceptions and difficulties with the chain rule. In *The mathematics Education into the 21st century Project,* University of Teknologi Malaysia, Johor Bahru: Malaysia.

141. Uygur,T., & Ozdas,A. (2007). The effect of arrow diagrams on achievement in applying the chain rule. Primus. 17(2), 131-147.

142. Uygur, T. (2010). Cognitive development of applying the chain rule through the three worlds.

Australian Senior Mathematics JournalPublisher: The Australian Association of Mathematics Teachers, Inc. Audience: Academic. Vol. 24(2), pp 89-103.

143. Vinner S. & Dreyfus T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*. 20(4), 354-366.
144. Vygotsky, (1986). Thought and language. Cambridge, MA: Havard MIT Pres.
145. Webster, R. J. (1978). The effects of emphasizing composition and decomposition of various types of composite functions on the attainment of chain rule application skills in calculus. Unpublished doctoral dissertation, Florida State University, Tallahassee.
146. Weller, K., Clark, J., Dubinsky, E., Loch, S., McDonald, M. and Merkovsky, R.: 2003,

    'Student performance and attitudes in courses based on APOS Theory and the ACE Teaching Cycle', in A. Selden, E. Dubinsky, G. Harel, and F. Hitt (eds.), *Research in Collegiate Mathematics Education V, American Mathematical Society, Providence*, pp. 97–131.
147. Weyer, S.R. (2010). APOS theory as a conceptualization for understanding mathematics learning. pp 9-15. Http://ripon.edu/macs/summation.
148. Wiggins, G. (1993). Assessing student performance. *Exploring the limits of testing*. San Francisco: Josey-Bass.
149. Wiggins, G., & McTighe, J. (1998). Understanding by design. Alexandria, VA: Association of Supervision and Curriculum Development.
150. Woods, P. (2006). Qualitative research as an art form, albeit governed by scientific principles. London: Routledge.
151. Zandieh, M. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld & J. Kaput (Eds.), *Research in Collegiate Mathematics Education. IV* (Vol. 8, pp. 103-127). Washington DC.American Mathematical Society.

152. Zingiswa, M.M.J., Brijlall, D., & Maharaj, A. (2005). Mathematical Concepts embedded in Mathematics Education. *Proceedings of the 1st Africa Regional Congress of the International Commission of Mathematical Instruction*. Witwatersrand (on CD).

RESEARCH OFFICE (GOVAN MBEKI CENTRE)
WESTVILLE CAMPUS
TELEPHONE NO.: 031 – 2603587
EMAIL : sshrec@ukzn.ac.za

UNIVERSITY OF
KWAZULU-NATAL

18 FEBRUARY 2010

MRS. Z M M JOJO (201506651)
SCHOOL OF SOCIAL SCIENCE, MATHS AND TECH ED.

Dear Mrs. Jojo

PROTOCOL REFERENCE NUMBER: HSS/0862/09D
PROJECT TITLE: "AN EXPLORATION OF THE CONCEPTUAL UNDERSTANDING OF THE CHAIN RULE IN DIFFERENTIAL CALCULUS BY FIRST YEAR ENGINEERING STUDENTS"

EXPEDITED APPROVAL

This letter serves to notify you that your application in connection with the above has been granted full approval through an expedited review process.

Any alteration s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Methods must be reviewed and approved through the amendment /modification prior to its implementation. Please quote the above reference number for all queries relating to this study.

PLEASE NOTE: Research data should be securely stored in the school/department for a period of 5 years

Best wishes for the successful completion of your research protocol.

Yours faithfully

PROFESSOR STEVEN COLLINGS (CHAIR)
SOCIAL SCIENCES & HUMANITIES RESEARCH ETHICS COMMITTEE

cc. Supervisor (Dr. D Brijlall)
cc. Dr. A Maharaj
cc. Ms. T Khumalo/Mrs. R Govender

APPENDIX A2

Letter of Consent

To: Participant(s) and Lecturer/ Tutor

Research Project:
Year: 2009

Mrs Z.M.M. JOJO (PhD Student) is conducting a study through the **School of Education, Mathematics Education at the University of KwaZulu-Natal** under the supervision of **Dr Deonarain Brijlall** and **Dr A. Maharaj**. The proposed research looks towards 'an exploration of the conceptual understanding of the chain rule in calculus by first year engineering students'. In particular it is aimed at using the exploration of the conceptual understanding of the chain rule in calculus displayed by first year students.

Lecturers, students and tutors are requested to assist through participating in this research project as it would be of benefit to education practitioners and interested educationalists and/or mathematics lectures. However, participation is completely voluntary and has no impact or bearing on evaluation or assessment of the learner in any studies or course while at the university. Participants may be asked to take part in the post-course surveys and open-ended interviews after the worksheets have been completed. These interviews will be recorded as the time progresses, but initially no recording will take place. All participants will be noted on transcripts and data collections by a *pseudonym* (i.e. fictitious name). The identities of the interviewees will be kept strictly confidential. All data will be stored in a secure password protected server where authentication will be required to access such data.

Participants may revoke from the study at any time by advising the researcher of this intention. Participants may review and comment on any parts of the dissertation that represents this research before publication.

_________________________ (*Researcher's Signature*) ______________ (*Date*)

DECLARATION

I,____(*Participant's NAME*)
(*Signature*)

______________________ (*Date*)

☐ Agree.
N.B. Tick ONE
☐ Disagree.

To participate in the research being conducted by …Z.M.M Jojo concerning '*An exploration of the conceptual understanding of the chain rule in calculus by first year Engineering students*'.

APPENDIX A 3

TO WHOM IT MAY CONCERN

18 November 2011

This thesis, entitled *An APOS Exploration of Conceptual Understanding of the Chain Rule in Calculus by First Year Engineering Students*, has been edited to ensure technically accurate and contextually appropriate use of language.

Sincerely

CM ISRAEL
BA Hons (UDW) MA (UND) MA (US) PhD (UNH)
Language Editor

APPENDIX A4

Turnitin Originality Report
Jojo thesis by Monica Jojo
From thesis (Maths 410)

- Processed on 08-Dec-2011 11:44 AM CAT
- ID: 219810532
- Word Count: 52392

Similarity Index
27%
Similarity by Source
Internet Sources:
24%
Publications:
14%
Student Papers:
8%

APPENDIX B1

Activity worksheet 1

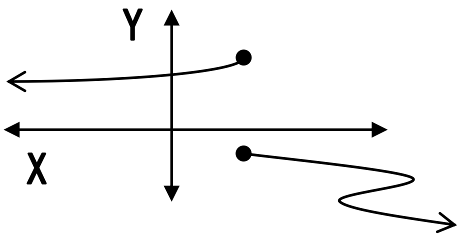
Student Number……………………………

Instructions:

- The following questions are designed to explore your range of understanding on a number of concepts in differential calculus. Please answer all questions to the best of your ability. Show all your working.
- For each question show in detail how you obtained your answer. Attempt all questions. If you like to write additional notes explaining how you got your answer, feel free to use the back of the page.
- Please do not write your name in any of the pages of this worksheet.

Question 1

Is a student correct to identify the following as a function? Explain.


Y
X

Question 2

A given correspondence associates 3 with each positive number, -3 with each negative number and 1 with 0. A student has identified the afore-mentioned relationship as a function. Is that correct? Explain.

Question3

$f(x) = 4e^{x}$ and $g(x) = 3\sin x$, Find $(f \circ g)(\pi)$

Question 4

Given that $(f \circ g)(x) = -10\sin 5x \cos 5x$

4.1 Find functions f and g that satisfy this condition.

4.2 Is there more than one answer to 4.1 above? Explain.

Question 5

Differentiate the following with respect to *x:*

5.1 $y = -3\sin x + 2e^{\cos x} - 5e^{x}$

5.2 $y = \cos(2x-5)^3$

5.3 $f(x) = \sin^3(4x)$

5.4 $y = \sin^2(4x^2 + e^{\sqrt{2x - \cos e}})$

5.5 $f(x) = \ln\left[\cos(7x)\right]$

5.6 $y = (\cos ec^3 x + e^{\tan x})^2$

Question 6

Determine the following:

6.1 $\int 2x\sqrt{1+x^2}\,dx$

6.2 Evaluate: $\int (3x+2)^6 \, dx$.

APPENDIX B2

ACTIVITY WORKSHEET

Group Number:……………….

INSTRUCTIONS:

- The following questions are designed to explore your range of understanding on the chain rule. Please answer all questions to the best of your ability.
- For each question show in detail how you obtained your answer.
- Do not write your names on any of the pages.
- One student from each group may be chosen to present the negotiated answers on the chalkboard.

Question 1

Differentiate:

$$y = \tan^2(3x + e^{\sqrt{x^2+1}})$$

1.2 $y = (\cos^2 x + e^{\sin x})^2$

Question 2

Differentiate implicitly:

$$\sin(x+y) = e^{y^2+2x}$$

Question 3

Use logarithms to differentiate:

$$y = \sqrt[3]{\frac{x(x+2)}{(x^2+1)}}$$

APPENDIX B3

EXERCISES

The list of problems for chain rule differentiation included:

1. $y = \tan^2(3x + e^{\sqrt{x^2+1}})$
2. $y = (x-3)^3$
3. $f(x) = \sin(\tan 3x)$
4. $y = \cos ec(x + \cos x)$
5. $y = \sin^3(4x)$
6. $y = \ln(x^2.\cos x)$
7. $f(x) = x^{2x}$
8. $f(x) = \ln \dfrac{x^5(1-x^2)^6}{x(\sqrt{1-x^3}}$
9. $y = \cot^3(x^5 + e^{\sqrt{x^2+3}}$
10. $y = e^{\tan^2(x^2+\sqrt{x^2+5}}$
11. $y = \sin^2(3x + e^{\sqrt{x^2+1}}$
12. $y = \ln\left[\dfrac{e^x.\sin x}{x \tan x}\right]$
13. $y = x^4 e^{3x} \tan x$
14. $y = \sin^{\cos x}$

APPENDIX C

1. Interviews with Zwayi

Researcher: You indicated no for task 1, why do you think the graph is not a function?

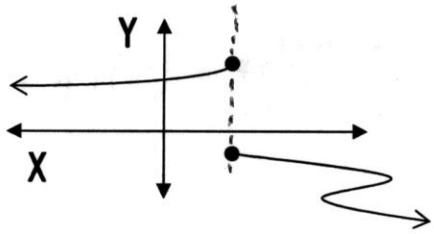
Zwayi: *Because uma ubek' istraight line la (pointing at the zig-zag part of the graph in task 1) sizotatsha igraph kathathu. (If you draw a straight line here, it will touch the graph three times)*

Researcher: So what?

Zwayi: *The vertical line test requires that a line drawn parallel to the y-axis should cut the graph only once for a function. This one is not a function.*

Researcher: Is that all?

Is a student correct to identify the following as a function? Explain.



No, when we are using the ~~horizon~~ vertical line ~~tester~~, the line must touch the two graph in the same cartesian plain where the graphs are ploted.

Zwayi: *Also, both dots are closed, so this graph is continuous, the line will cut twice, all in all this is not a function.*

Researcher: Closed dots do not indicate continuity.

2. Interview with Zandi

Researcher: Your answer to task 1, was yes it is a function. Is it?

Zandi: *Yes ma ubheke lena igraph (yes if you are looking at this one) pointing at the upper part of the given graph in item 1.*

Researcher: How many graphs do we have in item 1?

Zandi: *Two*

Researcher: Do you know anything about the concept of continuity?

Zandi: *Yes*

Researcher: When a graph is discontinuous at a point, does it split to two graphs?

Zandi*: No, ok….there is one graph, now I see?*

Researcher: What do you see?

Zandi: *That this is not a function*.

Researcher: Oh! Why are you changing now?

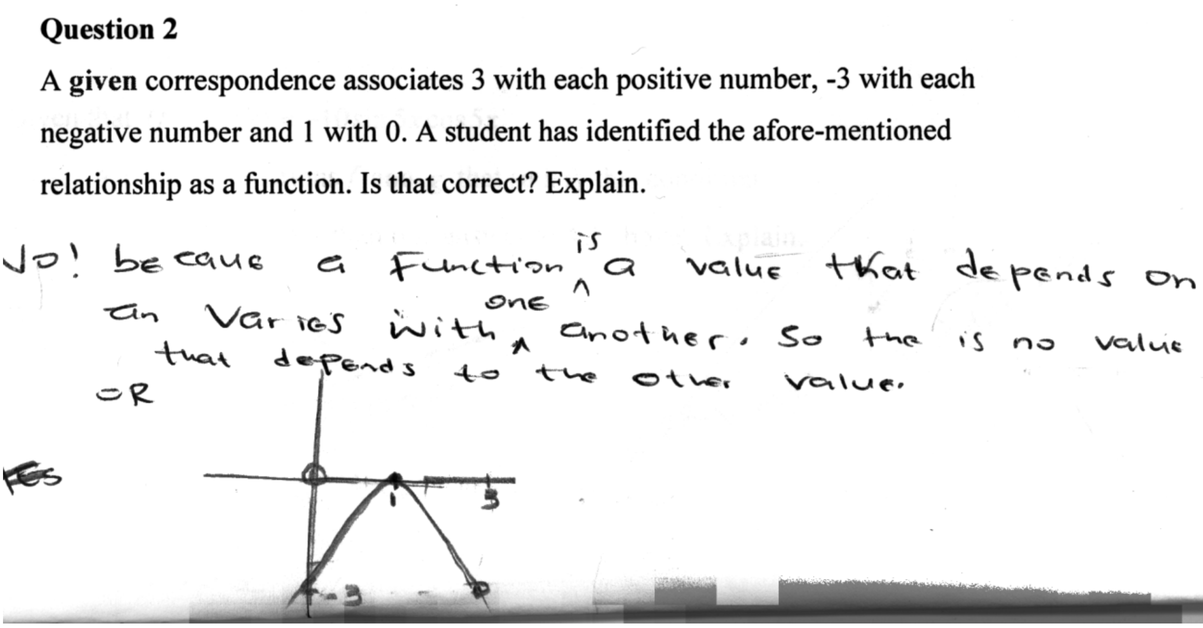
Zandi: *I had just looked at the structure and decided without thinking*.

Researcher: How is that possible?

Zandi*: Jaa, sometimes…….mh*

Researcher: Alright Zandi, can you now justify your answer to question 2. Zandi read her response again and said:

Zandi: *Jah…..I see, kusho ukuthi bengithatha ngokuthi u 3 uhamba ne number ezingu 3, kodwa ngabuye ngabona ukuthi u 1 uhamba no 0, bese ngicabanga ukuthi sengathi kuthathw' inmber ezikwi (I thought that 3 mapped to three other numbers, but noticed that 1 mapped to 0. So I thought the numbers belonged to) turning point just like in this diagram.*



Question 2

A **given** correspondence associates 3 with each positive number, -3 with each negative number and 1 with 0. A student has identified the afore-mentioned relationship as a function. Is that correct? Explain.

No! becaue a Function is a value that depends on an Varies with one another. So the is no value that depends to the other value.

OR

YES

Researcher: In other words the graph that you have drawn here illustrates the description given in the task.

Zandi: *Yes.*

Researcher: Ok, that's interesting, I have never thought of it like that.

Researcher: in question 4, you are wrote that f(x) = -10x and g(x) is that. What are the other f's and g's that can satisfy this condition? Are you in a position to think about any other functions that we can use. Please, identify them.

Zandi's answer to item was given by:

4.1 $f(x) = -10x$

$g(x) = \sin 5x \cos 5x$

4.2. yes. because you can still change g of to hav
a minuse sign from f(x).

Zandi: *You know what mhem, I know they are there, but I can't think of them now.*

Researcher: Ok then, let's jump to item 5.2.

Zandi: *I started with finding a derivative for the cosine function, followed by the derivative of the angle, then inside the brackets after which I multiplied everything.*

Researcher: There you are, just working it out. Were you perhaps using any rule in your differentiation?

Zandi: *I cannot say I was using product rule because there is only one function here, I was just differentiating*.

Researcher: You were not using any rule, even though in your explanation you are saying you started with that, followed by that….eh….

Zandi: *I used the chain rule.*

Researcher: Can you state the chain rule?

Zandi: *Yes*

Researcher: What does it say?

Zandi: *You just differentiate right through*.

Researcher: What do you mean by right through?

Zandi*: I mean its not like the product rule where you differentiate one function, live the other one, add and take the derivative of another one, you just differentiate okokoko (right through) and multiply*

Researcher: I want you to be explicit in explaining your understanding of the chain rule. What does it say? Can you write it down maybe?

Zandi*: Angikhon' ukuyichaza kodwa ngiyazi ukuyisebenzisa. (I am unable to explain it but I can use it)*

Researcher: If for example I give you $y = f(g(x))$, to differentiate, what would be the derivative of y?

Zandi: *It would be, let me write it down, f prime g, then g prime, and we multiply. (She wrote:* $f'g(x) \times g'(x)$

Researcher: Excellent, this is correct.

Zandi: *Jaaa mhem, ngikutshelile ukuthi ndiyayazi lento. (yes maa'm, I told you that I know this thing.*

Researcher: How different is the problem in 5.3 as compared to the one in 5.2/

Zandi: *In 5.3 the power affects the angle only, but in 5.2 the power is for the trigonometric function.*

Reaseacher: Can you now explain me how you differentiated item 5.5. Tell me the rule you used and how you used it.

Zandi: *Chain rule, ngiqale nga(differentiater) ipower ka sin, (I first differentiated with respect to the power of the sin function), then the sin itself, then got to the bracket, there I differentiate everything and just keep on multiplying.*

Researcher: How would you advise other students to do when differentiating using the chain rule?

Zandi: *Aqale abheke ukuthi usebenzisa yiphi irule (Decide on which rule to use first), they must also decide on what kind of a function is given.*

Researcher: Then?

Zandi: *When differentiating abheke (must look)….mhm…..Mhem konje oosin noo cos sibabiza ngokuthi ngama trig functions? (ma'hm, by the way do we call sin and cos trig functions?)*

Researcher: Yes

Zandi: *Identify if there is a power, start with differentiating with respect to the power, function, unpack everything contained in the function and differentiate all functions step by step, and multiplying by each result.*

Researcher: In question 6, did you use the chain rule to do your solutions?

Zandi: *I do not think I understand much of the process of integrating, I just refer to standard integrals.*

Researcher: Thank you very much for your time.

3. Interview with Nodi

Researcher: Your response presented for the first question does not indicate a reason, why do you think the graph is not a function?

Nodi: *I just looked at it and saw that it was not a function.*

Researcher: Ok, then can we jump to question 5, I need some explanations to your work.

Researcher: Can you tell me how you obtained your answers to question 5.4.

Nodi: *Ja mhem, I know I did not do my best.*

Researcher: Its not about that, but I am asking you because you did not explain in the worksheet how you arrived at your answers.

Nodi*: Ok then mhem*.

Researcher: For example, in 5.4 which rule did you use?

Nodi: *Chain rule*

Reseacher: How do you know that you should use the chain rule?

Nodi: *Let us say for the product rule you have a function multiplying another function. But now when you have a function kwenye ifunction (in another function), you use the chain rule.*

Researcher: All right then, explain how you used the chain rule in 5.4.

Nodi: *Start with the power, then the function, then go inside and just multiply each result you get, no plus as in product rule.*

Researcher: Can you state the chain rule

Nodi: *Ja, you just find f' and then g' then u (time)ze (you multiply).*

Researcher: What exactly do you mean?

Nodi: *That kwi chain rule you just differentiate one way uloko (and always) u multiplier by each derivative.*

Researcher: Explain how you arrive at your answer for 5.2.

Nodi: *I think you use the product rule here.*

Researcher: What would be the second function?

Nodi: *You have cos as a function and then* $(2x -5)^3$ *as another one.*

Researcher: Which one is the angle for cos?

Nodi: *Oh I see now, my mistake, I should have used the chain rule. We are differentiating a composite function.*

Reseacher: How come?

Nodi: *I should have differentiated the cosine function first. Then, I multiply by the derivative of the angle.*

Researcher: Let us visit your answer to ln[cos(7x)]. Which rule should be used for differentiation here?

Nodi: *I should have used the product rule*.

Researcher: Why? What are the function being multiplied?

Nodi*: ln and cos 7x.*

Researcher: ln of what?

Nodi: *No mhem, I am confused, let me admit, the chain rule is the one to be used.*

Researcher: I was surprised, because you differentiated correctly in the worksheet using the chain rule correctly.

Researcher: Ok then thank you for now.

For Nodi, it was more of a learning curve as he used the interview session also to pose questions of concern to him. For example, he asked at one stage, ‘Is it wrong for me to change the given function, $\sin^3 4x$ and write it as $\sin^2 4x$ multiplied by sin4x and differentiate using the product rule? This concurs with Horvath (2007) findings that students replaced function composition with function multiplication for functions that they had experienced in precalculus, but had not yet encountered in calculus (e.g., exponential, logarithm, and inverse trigonometric).

4. Research Interview with Popo

Researcher: There are a few questions that I would like to ask from you so as to get clarity on some of the answers given in your worksheet. Is that ok with you?

Popo: *Yes, no problem ma'hm*.

Researcher: Could you explain how you did task

Popo: *If the vertical line meets the graph more than once, then not a function.*

Researcher: Could you explain why you labeled 'horizontal line there?'(pointing at the label in the diagram).

Popo: *My mistake there mheem, angazi kwenzekeni kumina, (I don't know what was wrong with me) I wanted to write vertical line.*

Researcher: What about the zig-zag part?

Popo: *That part is obvious, not a function, I just wanted to be sure that the graph was continuous.*

Reseacher: Could you please clarify the main difference between questions 5.2 and 5.3.

Popo: *Kahle kahle, (truly speaking), other than it being a cos or sin functions, in 5.2 icube yeye angle ayiwukhavi u cos kanti ngala ucube ngowe function* *sine, (it's the angle that is expanded and in the case of 5.3), and we could write it as sin 4x multiplied by sin 4x and again by sin 4x.*

Researcher: I noticed that in almost all the subsections of item 5, you have indicated that you would use the chain rule. Can you state the chain rule? I am giving you f(g(x)) to differentiate.

Popo: *y = ……., may I write it down,*

He wrote:

Researcher: Did you use the chain rule to do questions 6.1 and 6.2?

Popo: *Oh yes!*

Researcher: It looks like you are comfortable with the chain rule and its applications. How would you assist a student who is struggling to do question 5.4?

Popo: *Find the derivative yayo yonke into nga one, nga one, nga one. Ngiqond'ukuthini ngalokho (of everything one by one. What do I mean by that)? Start with the power, uqede ngayo (and finish with it), then find the derivative of the sin function and continue differentiating each function in the bracket until you finish. Remember to multiply by each derivative.*

Researcher: Thank you Popo for your time and explanations.

5. Interview with Dube

Researcher: Dube , do you know what is a composite function?

Dube: *Ish……let us say a loaded function*

Researcher: How loaded?

Dube: *Like one function f, say, f = 2a and a = 3x + 5. Now you take a as stipulated and insert it in f, so f will now be composite.*

Researcher: Let us talk about the chain rule, can you state it.

Dube: *Haaawu, ngeke kodwa ma unginika isibalo (No I can't, but if you give me a problem) I can use it.*

Researcher: Can you explain how you can help another student who has problems in using the chain rule. Use question 5.6 for your explanation.

Dube: *Ngingathi, (I can say), start with the power rule for the square outside, then inside the bracket start with the cosec and differentiate everything in a line.*

Researcher: What do you mean by, in a line?

Dube: *I mean each function has to be differentiated one by one and multiply the result each time.*

Researcher: Let us look at the interesting way in which you did integration in 5.6 and 5.7. Can you explain?

Dube: *There I thought I must let one of the functions be u. I then differentiated to get* $\frac{du}{dx}$*. I then expressed du in terms of dx so that I could go back and substitute. I then worked everything out.*

Researcher: Would you say you used the chain rule in the process?

Dube: *Yes, that is exactly why I chose to use substitution by u. I wanted to use the chain rule.*

Researcher: But you said earlier on that you could not state it.

Dube: *Jaa that's correct, I assumed that there would never be question requiring the statement of the chain rule. Kodwa ke I think if you are given f(g(x)). Say for example you are given* $f(x) = 2x^2$ *and* $g(x) = 3x$*, then u fog(x) is* $2(3x)^2$*. Then we now use the chain rule.*

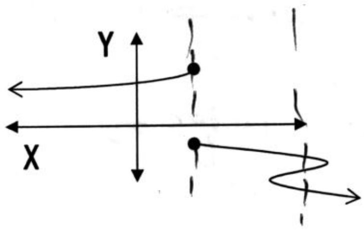
5. Interview with Sindi

Researcher: I noticed that you drew two lines in the given graph in question 1. Could you justify that?

Sindi: *Mahm engikwenzileyo lana, ngibone kunamajiko-jiko, nalana kunamagqudu avalekile, (I was showing the zig-zag part and then the closed dots), so I wanted to show that any of the two parts would lead the given graph not to be a function. That is why I drew the two lines. Sindy's response is displayed below.*

Question 1

Is a student correct to identify the following as a function? Explain.



No; as you see the lines I draw, they cut the graph more than once – not a function The dots are also closed if one was open, maybe a function; but the zigzag also, is not a function.

Researcher: I was also impressed by the response you gave in question 2. Would you want to explain how you came up with it?

Sindi: *I wasn't sure at first, which is why I wrote the ordered pairs. Ngicabangu'ukuthi iyaqhubeka ufake amanye amapositive numbers mamanegative numbers (I think this is continuing, where you map with positive numbers and then with negative numbers). What I am trying to say is that, if you repeat the x-coordinate, mapping it with other numbers, it won't be a function.*

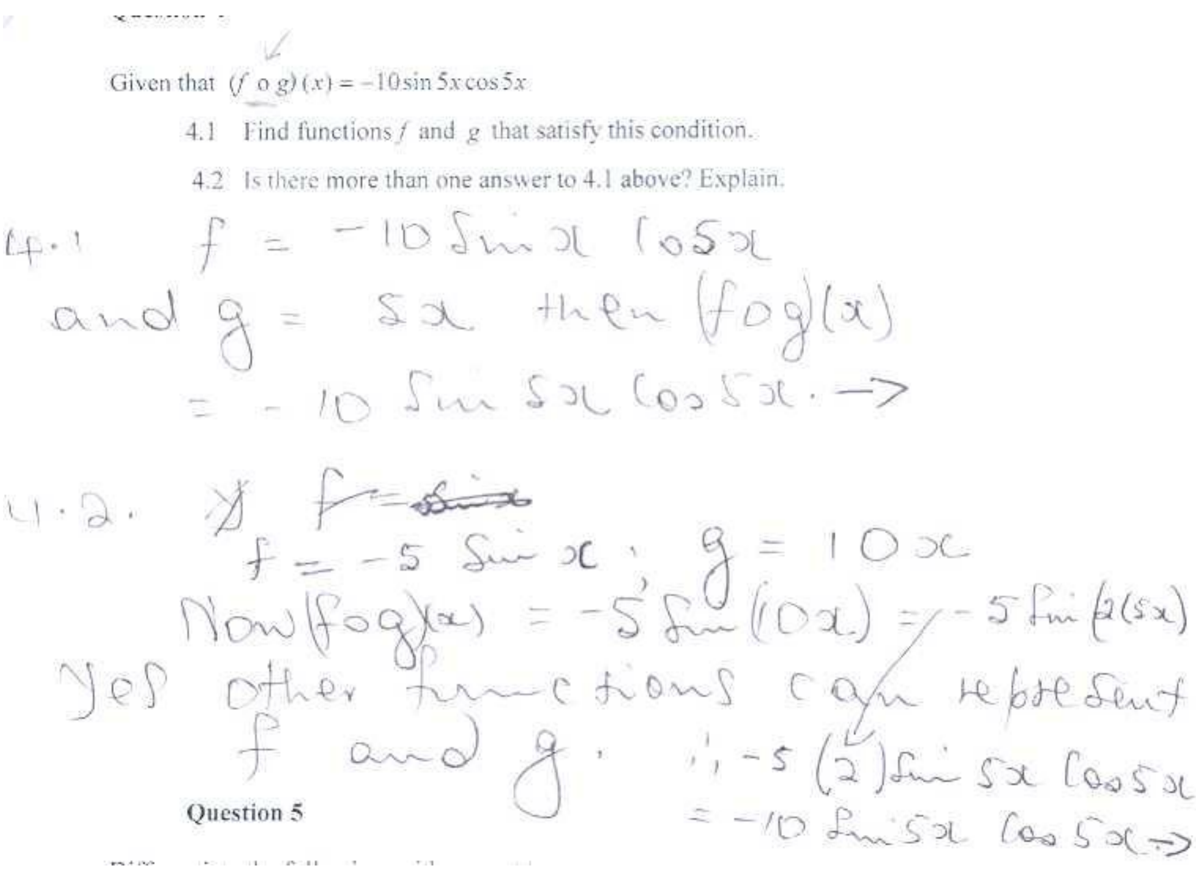
Researcher: That is well said Sindy, the explanation to question 3.

Sindi: *I think yilaa ufike uxhume khona ifunction kwenye ebivele ikhona (I think this is where you insert one function in another so as to get a composite function. So I inserted 3sin x in the place of x in the function f and then evaluated to get 4.*

Researcher: I have noticed that you listed a lot of functions in task 4. Could you explain.

Sindi: *I was not sure at first and ngaze ngacabang' ukuthi kukhona namanye amafunctions engingase ngiwafake, (then I figured out later on and decided to include all relevant functions).*

Sindi's response for question 4 is displayed below.



Given that $(f \circ g)(x) = -10\sin 5x\cos 5x$

4.1 Find functions f and g that satisfy this condition.

4.2 Is there more than one answer to 4.1 above? Explain.

4.1 $f = -10\sin x\cos 5x$ and $g = 5x$ then $(f \circ g)(x) = -10\sin 5x\cos 5x$ →

4.2. ~~f = sin~~

$f = -5\sin x$, $g = 10x$

Now $(f \circ g)(x) = -5\sin(10x) = -5\sin 2(5x)$

Yes other functions can represent f and g.

$= -5(2)\sin 5x\cos 5x$

$= -10\sin 5x\cos 5x$ →

Question 5

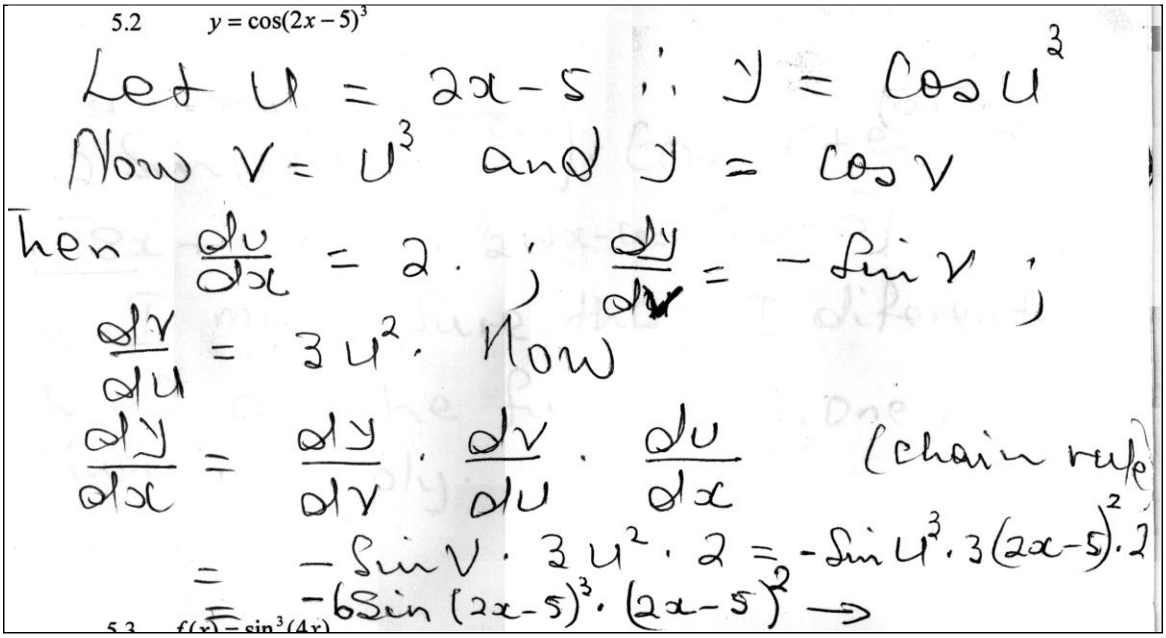
Researcher: Ok fine! Can we now go to question 5. I have no queries in your answers but I am very much interested in question 5.2 , the method you used which differs from the one you used in 5.3. Can you explain?

Sindi: *Lapha ku 5.2, the angle is cubed, so I thought maybe I could make a mistake if I do straight differentiation. So I decided to do substitution using u and use the chain rule. The following problem looks more, simpler to me, because in 5.3 it's the function, that is cubed.*

Researcher: By function what do you mean?

Sindi: I am referring to the sin function in 5.3, kanti (but) in 5.2 it's the angle, 2x…mh….2x – 5 that is cubed.

Her work is captured in the extract below.



5.2 $y = \cos(2x-5)^3$

Let $u = 2x-5$; $y = \cos u^3$

Now $v = u^3$ and $y = \cos v$

Then $\frac{du}{dx} = 2$; $\frac{dy}{dv} = -\sin v$;

$\frac{dv}{du} = 3u^2$. Now

$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$ (chain rule

$= -\sin v \cdot 3u^2 \cdot 2 = -\sin u^3 \cdot 3(2x-5)^2 \cdot 2$

$= -6\sin(2x-5)^3 \cdot (2x-5)^2 \rightarrow$

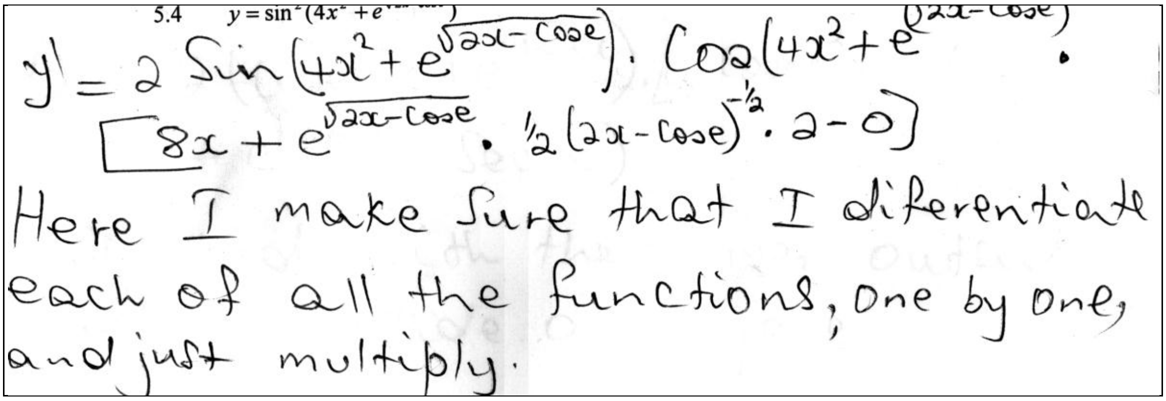
5.3 $f(x) = \sin^3(4x)$

Researcher: In 5.4 you wrote, 'I make sure that I differentiate all the functions one by one until I finish.' What do you mean by that?

Sindi: *Phela mhem lapho ngisho ukuthi kufanele uqale la ekuqaleni, kukhona ipower, then ubese uya kwi function lena ye trig, mawuqeda lapho bese uya* *kumabrackets uqhubeke noku differentiatha, uloko umultiplaya. Konke nje, kanye, kanye, sengathi uhlubula amalayers* (Look ma'hm, I mean that you should start from the beginning with the power, then differentiate the trigonometric function. After you finish, get into the brackets and go on differentiating and multiplying each time. You do this one by one as if you are peeling layers.)

Researcher: Ok then let's go to question six. I notice that you also used substitution. Why?

Sindi: *Mhem you did not give us the standard integral table, but I knew that if I use the chain rule, the other form, I will not go wrong, So I did substitution to be sure that I am correct.*


5.4 $y = \sin^2(4x^2 + e^{\dots})$

$y' = 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot [8x + e^{\sqrt{2x-\cos e}} \cdot \frac{1}{2}(2x-\cos e)^{-\frac{1}{2}} \cdot 2 - 0]$

Here I make sure that I diferentiate each of all the functions, one by one, and just multiply.

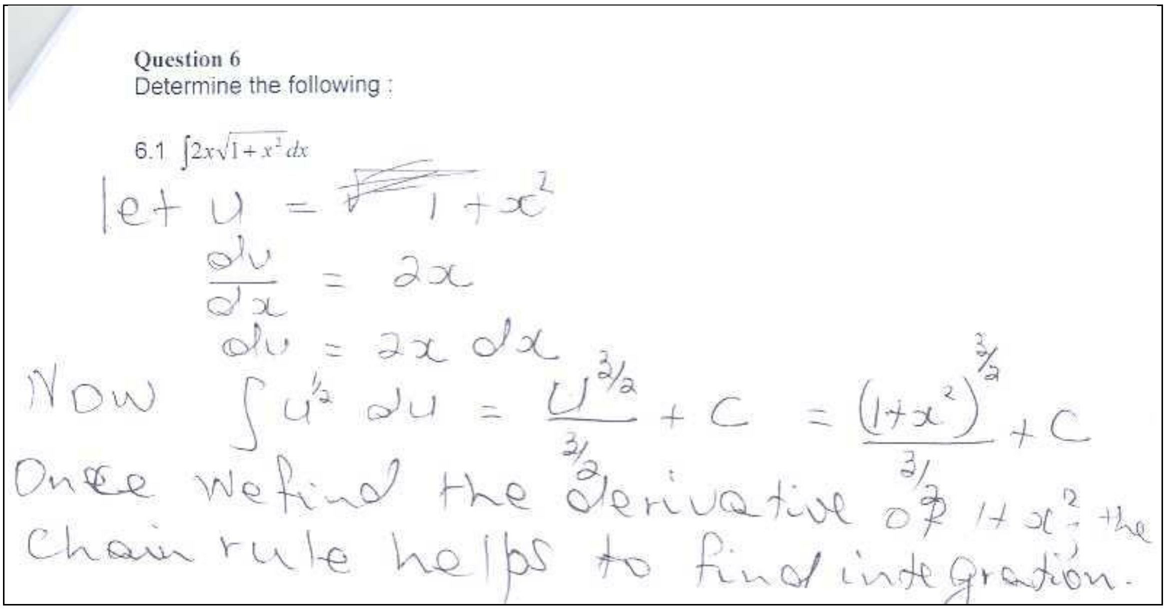
Researcher: You have been mentioning the chain rule right true. Do you know it or can you state the chain rule?

Sindi: *Eish mhe…m, Ja I think I know it.*

Researcher: What is it? What does it say?

Sindi: *Maybe if I write it down.*

This is what she wrote.



Question 6

Determine the following :

6.1 $\int 2x\sqrt{1+x^2}\,dx$

let $u = \cancel{\sqrt{}}\ 1 + x^2$

$\frac{du}{dx} = 2x$

$du = 2x\,dx$

Now $\int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{(1+x^2)^{3/2}}{3/2} + C$

Once we find the derivative of $1+x^2$; the chain rule helps to find integration.

Researcher: Thank you for your time.

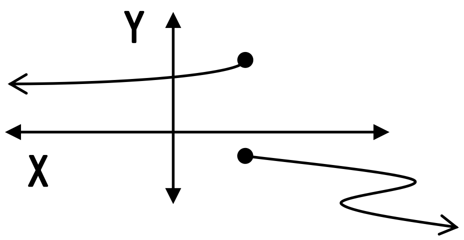
APPENDIX D1

EXTRACTS FOR THE PILOT STUDY

Students who participated in the pilot study were named as S1 up to S23.

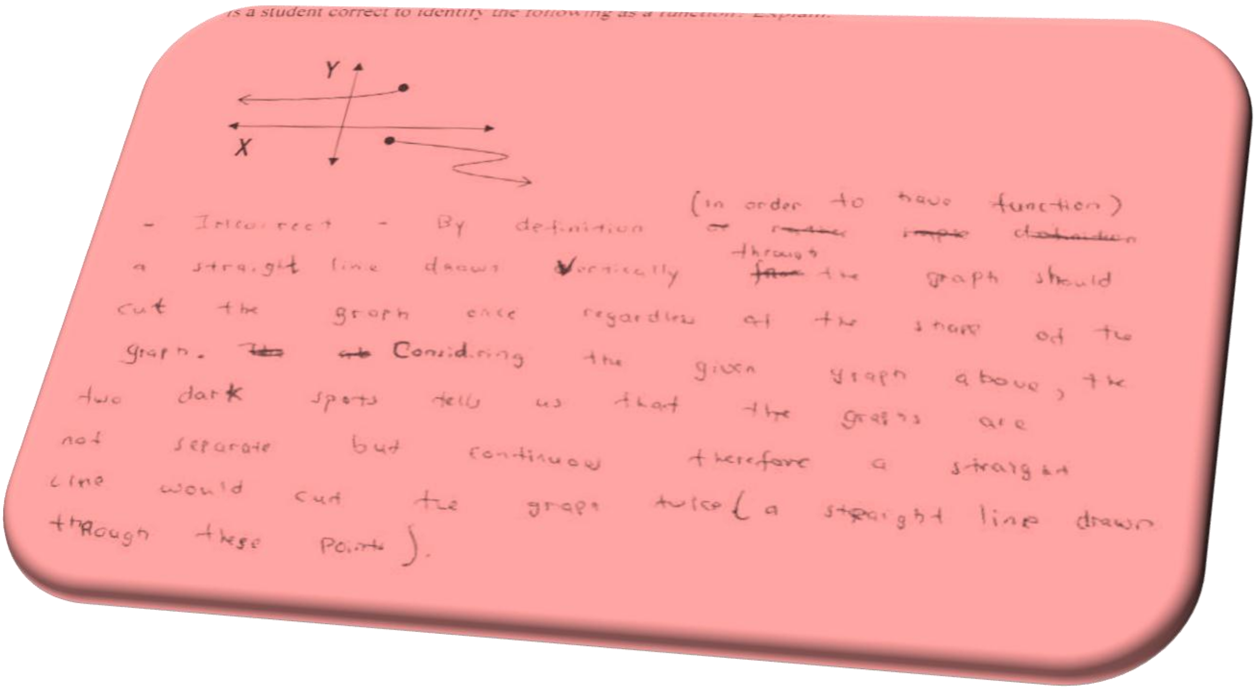
3.9 Responses for Item 1

1. Is the student correct to mark the following as a function? Explain.



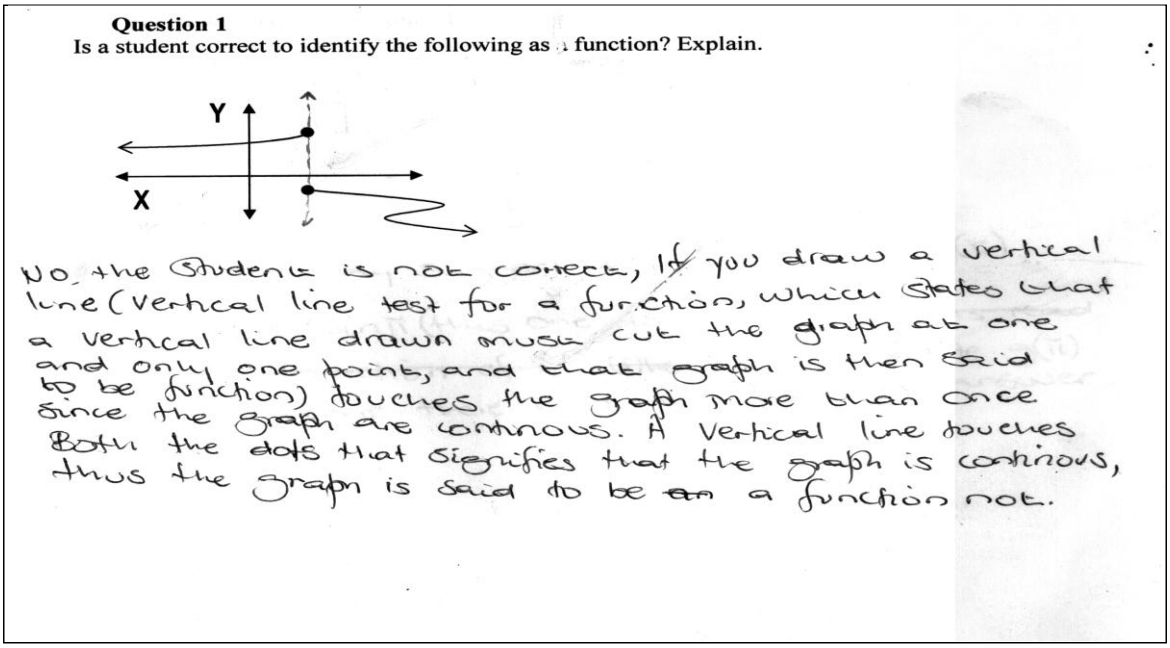
Some of the students' responses were:

S17, for example wrote,


Is a student correct to identify the following as a function? Explain.

- Incorrect - By definition (in order to have function) ~~of~~ ~~[illegible]~~ ~~[illegible]~~ ~~[illegible]~~ a straight line drawn Vertically through ~~from~~ the graph should cut the graph once regardless of the shape of the graph. ~~[illegible]~~ ~~[illegible]~~ Considering the given graph above, the two dark spots tells us that the graphs are not separate but continuous therefore a straight line would cut the graph twice (a straight line drawn through these points).

S15 on the other hand wrote:


Question 1
Is a student correct to identify the following as a function? Explain.

No the Student is not correct, If you draw a vertical line (Vertical line test for a function, which States that a vertical line drawn must cut the graph at one and only one point, and that graph is then said to be function) touches the graph more than once Since the graph are continous. A Vertical line touches Both the dots that signifies that the graph is continous, thus the graph is said to be ~~an~~ a function not.

4.13 Response to item 4

Given that $(f \circ g)(x) = -10 \sin 5x \cos 5x$

4.1 Find functions f and g that satisfy this condition.

4.2 Is there more than one answer to 4.1 above? Explain.

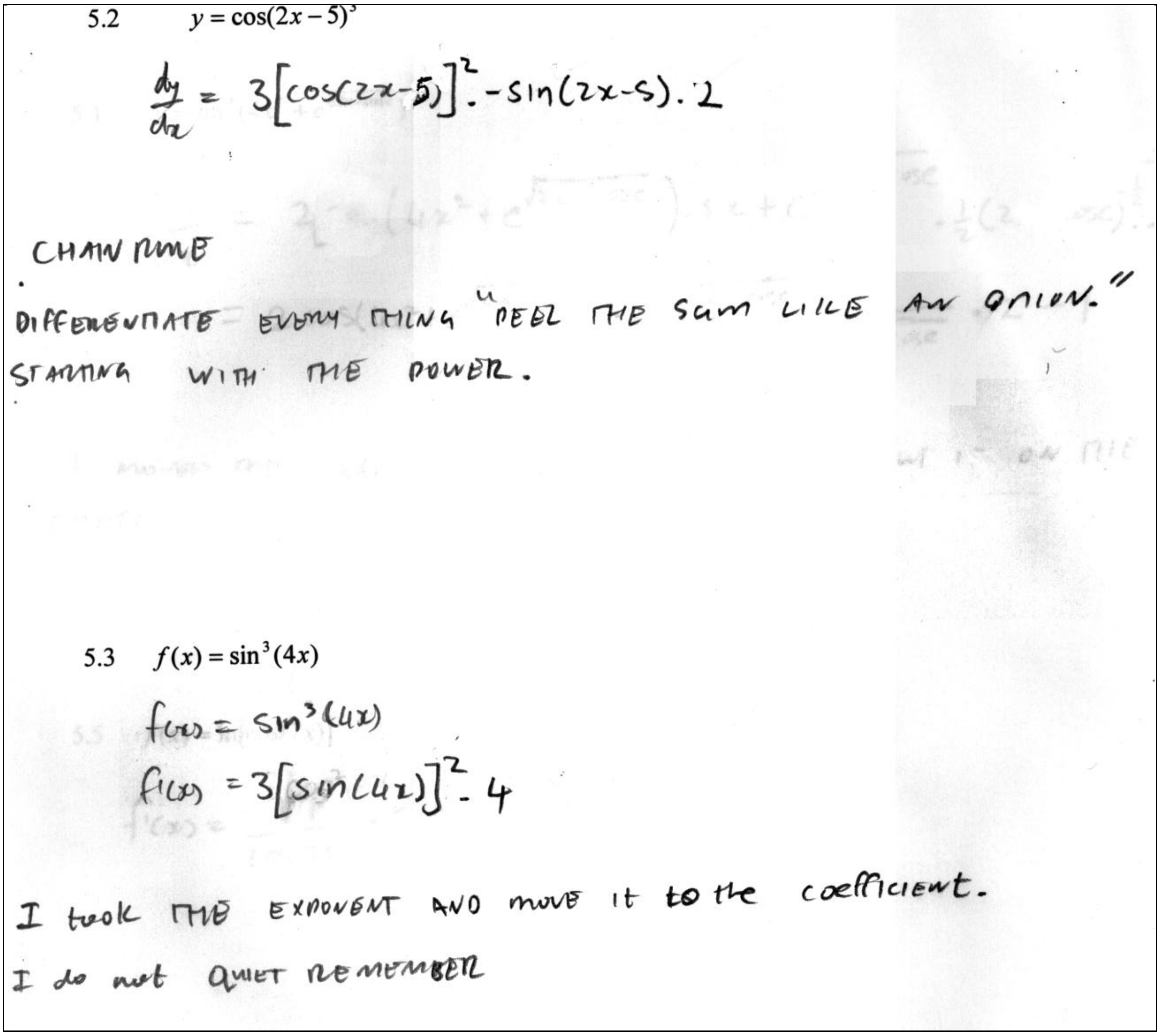
4.1. $f(x) = \cancel{\sin x \cos x}$ $\cos x \sin x$

$f(x) = 5x^2$

4.2. yes

→ If you can continue from the $(f \circ g)x$ and find the $(f' \circ g')x$ and find another solution in which it will be $(f' \circ g')x = -50 \cos 5x - 5 \sin 5x$, using the product Rule.

PS16 (Mzi') wrote an explanation of how he got to his answers.


5.2 $y = \cos(2x-5)^3$

$$\frac{dy}{dx} = 3[\cos(2x-5)]^2 \cdot -\sin(2x-5) \cdot 2$$

CHAIN RULE

DIFFERENTIATE EVERY THING "PEEL THE SUM LIKE AN ONION."

STARTING WITH THE POWER.

5.3 $f(x) = \sin^3(4x)$

$$f(x) = \sin^3(4x)$$

$$f(x) = 3[\sin(4x)]^2 \cdot 4$$

I took THE EXPONENT AND move it to the coefficient.

I do not QUIET REMEMBER

When Mzi was asked what he meant in question 5.2 when he wrote differentiation has to be like peeling an onion, he said:

Mzi: *To me differentiating such a function that is so loaded with many other functions is like peeling an onion, taking it layer by layer until you get to the inner one.*

Researcher: What do you mean?

Mzi*: In 5.6 for example, you can start off with the power outside, differentiate with respect to it. Jaa…., You have to imagine everything inside the bracket as one function, effect the power first and then come inside the bracket.*

Researcher: Then what?

Mzi: *Oh Ma….am, you see now, all you do is to attend to each function, find its derivative and keep on multiplying with every result until you finish.*

Researcher: Is there some rule that guided you in your differentiation?

Mzi: *Ja, the chain rule.*

Researcher: What does it say?

Mzi: *do you want me to put it in symbols?*

Researcher: Yes if you can.

Mzi*: I cannot be able to put it in symbols. But I can show you another way of doing the same problem.*

Researcher: How?

Mzi: *Where you substitute all the functions inside with symbols u, v. w. etc and then find the different derivatives, after which you multiply each result you get, it works just like the chain rule, but it's too long, I don't like it.*

Researcher: Ok then let's not do it, thanks for your time

S13 presented the use of the chain rule as

5.4 $y = \sin^2(4x^2 + e^{\sqrt{2x - \cos e}})$

$\frac{dy}{dx} = 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot [8x + e^{\sqrt{2x-\cos e}} \cdot \frac{1}{2}(2x - \cos e)^{-\frac{1}{2}} \cdot \cancel{2} - 0]$

$\frac{dy}{dx} = 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \left(8x + \frac{e^{\sqrt{2x-\cos e}}}{\sqrt{2x-\cos e}}\right)$

5.5 $f(x) = \ln[\cos(7x)]$

$f'(x) = \frac{1}{\cos(7x)} \cdot -7\sin 7x$

$= \frac{-7\sin 7x}{\cos 7x}$

$= -7\tan 7x$

APPENDIX D2

EXTRACTS FROM THE MAIN STUDY

Differentiate :-

① $y = \cot^3\left(x^5 + e^{\sqrt{x^2+3}}\right)$

$y' = 3\cot^2\left(x^5 + e^{\sqrt{x^2+3}} \cdot \frac{d}{dx}\left(\cot\left(x^5 + e^{\sqrt{x^2+3}}\right)\right.\right.$

$y' = 3\cot^2\left(x^5 + e^{\sqrt{x^2+3}} \cdot \left(-\mathrm{cosec}^2 \cancel{\cot}\left(x^5 + e^{\sqrt{x^2+5}}\right)\frac{d}{dx}\left(x^5 + e^{\sqrt{x^2+3}}\right)\right.\right.$

$y' = 3\cot^2\left(x^5 + e^{\sqrt{x^2+3}}\right) \cdot -\mathrm{cosec}^2\left(x^5 + e^{\sqrt{x^2+5}}\right) \cdot \left[5x^4 + e^{\sqrt{x^2+3}} \frac{d}{dx}\sqrt{x^2+3}\right]$

$y' = 3\cot^2\left(x^5 + e^{\sqrt{x^2+3}}\right) \cdot -\mathrm{cosec}^2\left(x^5 + e^{\sqrt{x^2+3}}\right) \cdot \left[5x^4 + e^{\sqrt{x^2+3}} \cdot \frac{1}{2}(x^2+3) \cdot 2\right]$

$= -3\cot^2\left(x^5 + e^{\sqrt{x^2+3}}\right)\mathrm{cosec}^2\left(x^5 + e^{\sqrt{x^2+3}}\right) \cdot 5x^4 + e^{\sqrt{x^2+3}} \cdot (x^2+3)$

$y = \sin^3(4x)$

$y_1 = 3\sin^2(4x) \frac{d}{dx}\sin(4x)$

$y_1 = 3\sin^2 4x \cos 4x \frac{d}{dx} 4x$

$y_1 = 3\sin^2(4x)\cos(4x) \cdot 4$

$= 12\sin^2(4x)\cos 4(x)$

① find $\frac{dy}{dx}$ of $y = \ln(x^2 \cdot \cos x)$

let $u = x^2 \cdot \cos x$

$$\frac{du}{dx} = 2x\cos x + x^2(-\sin x)$$

$$\frac{dy}{du} = \frac{1}{\ln u}$$

now $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

$$= \frac{1}{\ln u} \cdot (2x\cos x + x^2(-\sin x))$$

$$= \frac{1}{\ln(x^2\cos x)} \cdot 2x\cos x - x^2\sin x$$

$$= \frac{2x\cos x - x^2\sin x}{\ln(x^2\cos x)}$$

y' of $y = \ln \frac{e^{4x} \cdot \sin x}{x \tan x}$

$$y' = \frac{1}{\frac{e^{4x} \cdot \sin x}{x \cdot \tan x}} \times \frac{d}{dx}\left(\frac{e^{4x} \cdot \sin x}{x \tan x}\right)$$

$$y' = \frac{1}{\frac{e^{4x} \cdot \sin x}{x \cdot \tan x}} \times \frac{\left[4e^{4x}\sin x + e^{4x}\cos x)(x\tan x)\right] - \left(e^{4x}\sin x\right)\left[(1)\tan x + x\sec^2 x\right]}{x^2\tan^2 x}$$

$$y = \sin^2(4x^2 + e^{\sqrt{2x+\cos e}}$$

$$\frac{dy}{dx} = 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \frac{d}{dx} \sin(4x^2 + e^{\sqrt{2x-\cos e}})$$

$$= 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \frac{d}{dx}(4x^2 + e^{\sqrt{2x-\cos e}})$$

$$= 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot [8x + e^{\sqrt{2x-\cos e}} \cdot \frac{1}{2}(2x - \cos e)^{-\frac{1}{2}} \cdot 2 \cdot -0]$$

5.4 $\quad y = \sin^2(4x^2 + e^{\sqrt{2x-\cos e}})$

$$\cancel{y' = 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{\cos e}})(16x + e^{\sqrt{2x}} e^{\cos} \cdot \tfrac{1}{2}(}$$

$$y' = 2\sin(4x^2 + e^{\sqrt{2x-\cos e}}) \cdot \cos(4x^2 + e^{\sqrt{2x-\cos e}})(16x + e^{\sqrt{2x-\cos e}})(\tfrac{1}{2}(2x - \cos e)^{-\frac{1}{2}} \cdot 2 \cdot \text{sine}$$

I made the exponent becoms the base in sin ①
got the dirrivative of sin which is coss
then the rest is history

Question3

find

If $f(x) = 4e^{x}$ and $g(x) = 3\sin x$. Find $(f \circ g)(\pi)$.

$(f \circ g)(\pi)$

$$f \circ g(\pi) = 4e^{3\sin\pi}$$
$$= 4e^{0}$$
$$= 4 \rightarrow$$

I put the value of function $g(x)$ in $f(x)$ where I see x, so it becomes $4e^{3\sin\pi}$. I then evaluate to get 4. This is a function in another function.

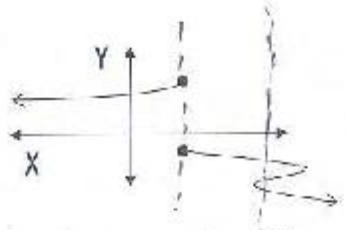
Question 2

A given correspondence associates 3 with each positive number, -3 with each negative number and 1 with 0. A student has identified the afore-mentioned relationship as a function. Is that correct? Explain.

$(3;1);(3;2);(3;3)$ --- and $(-3;-1)$;
$(-3;-2)$ --- ; Not a function; each member of domain repeated.
One to many correspondence!

Question 1

Is a student correct to identify the following as a function? Explain.



No, a vertical line test done at any of the two places indicated above, shows that the line meets the graph wice, so not a function.

6.1 $\int 2x\sqrt{1+x^2}\,dx$

$f'(x) = x^2+1$

$f'(x) = 2x$

$\dfrac{(1+x^2)^{3/2}}{3/2} + C$

$\frac{2}{3}\sqrt{(1+x^2)^3} + C$

The equation has been express in this format $\int f(x)\, f'(x)\, dx$ So the function shall ~~to~~ have it ~~apple~~ derivative. Than I will ~~will the~~ derive what is under the cube root and get it derivative. If we are integrating we add 1 on the exponet of the function

6.2 Evaluate: $\int (3x+2)^6\,dx$.

$\frac{1}{3}\int 3(3x+2)^6\,dx$

$f(x) = 3x+2$

$f'(x) = 3$

$\frac{1}{3}\dfrac{(3x+2)^7}{7} + C$

$\dfrac{(3x+2)^7}{21} + C$

The ~~ext~~ equation can be express like $\int 1.(3x+2)^6\,dx$ so the Should be the derivative of what is inside the bracket, than I will look for a derivative of what is inside the bracket and te derivating will be 3, so the equation to be in a standard form we need to multiply by 3 and divide by $\frac{1}{3}$

Questions

$f(x) = 4e^x$ and $g(x) = 3\sin x$, Find $(f \circ g)(\pi)$

$f \circ g(x) = 4 \cdot e^{3\sin x}$

$f \circ g(\pi) = 4e^{3\sin\pi}$

$= 4e^0$

$= 4$

The function of a function is when we put the fuction to another fuction. This time we want to put $g(x)$ to $f(x)$, so where we see x in a ~~fx~~ equation of $f(x)$ we will put $g(x)$.

$f \circ g(x) = \cancel{3\sin}$

$= 4e^{\sin x}$

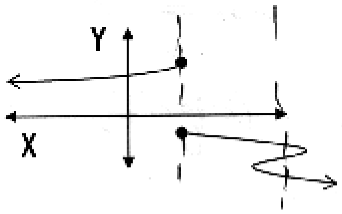
$f \circ g(\pi) = 4e^{\sin\pi}$

$= 4e^0$

$= 4$

Question 1

Is a student correct to identify the following as a function? Explain.



No; as you see the lines I draw, they cut the graph more than once – not a function. The dots are also closed if one was open, maybe a function; but the zigzag also, ∴ not a function.

Question3

$f(x) = 4e^x$ and $g(x) = 3\sin x$, Find $(f \circ g)(\pi)$

$(f \circ g) = 4e^{3\sin x}$

$= 4e^{3\sin(\pi)}$

$= 4e^{3(0)}$

$= 4(1)$

$= \underline{4}$

APPENDIX D3

GROUP DISCUSSIONS IN THE MAIN STUDY