Some Notions of Amenability of Banach Semigroup Algebras

by

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As the candidate's supervisor, I have approved this dissertation for submission.

Dr. O. T. Mewomo.

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Abstract

This master's dissertation deals with some notions of amenability of general Banach algebras and that of Banach semigroup algebras $\ell^1(S)$ in relation to the amenability of the semigroup S. A Banach algebra $\mathcal{A}$ is said to be amenable if every continuous derivation D from $\mathcal{A}$ into X' is inner for every Banach $\mathcal{A}$-bimodule X, where X' denotes the dual space of X. In this dissertation, we give an explicit proof of some characterizations, hereditary properties and some basic results in literature on *contractible, amenable, approximate amenable and pseudo-amenable* Banach algebras. In addition, we give a survey of results concerning the above mentioned notions of amenability on Banach semigroup algebras and also highlight some important structures of the semigroups. The last chapter of this dissertation is a catalogue of results that we obtained in the course of this research work.

Keywords : Contractible, amenable, approximately amenable, pseudo-amenable, semigroup, semigroup algebras.

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Dedication

This work is dedicated to God Almighty and to my beloved family.

Contents

Declaration

This dissertation, in its entirety or in part, has not been submitted to this or any other institution in support of an application for the award of a degree. It represents the author's own work and where the work of others has been used in the text, proper reference has been made.

Mebawondu Akindele Adebayo.

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Chapter 1

Introduction

1.1 Background

Amenability is an abstract mathematical concept which has its roots in measure theory. The notion of amenability started in the early twentieth century and the class of amenable groups was introduced and studied by John Von Neumann. He defined amenable groups as follows: A locally compact group G is amenable if there exists a left translation invariant mean on G. That is, if there is a linear functional $m : L^\infty(G) \to \mathbb{C}$ satisfying

$$m(1) = 1 = \|m\| \text{ and } m(l_x f) = m(f) \quad (x \in G, f \in L^\infty(G)).$$

He later used this concept to explain why the Banach-Tarski paradox holds for only dimension greater than or equal to three. Since then, amenability has become an important concept in abstract harmonic analysis. In 1972, B. E. Johnson showed that the amenability of a locally compact group G can be characterized in terms of the Hochschild cohomology of its group algebra $L^1(G)$. This result initiated the theory of amenable Banach algebras.

The concept of amenability for Banach algebras was introduced and studied by B. E. Johnson in [31]. According to Johnson 1972, a Banach algebra $\mathcal{A}$ is amenable if every continuous derivation $D : \mathcal{A} \to X'$ is inner for every Banach $\mathcal{A}$-bimodule X, where X' denotes the dual of X. In this definition, if we replace X' with X, we say that $\mathcal{A}$ is a *contractible* Banach algebra. Johnson in [31] and [32] was able to establish some intrinsic characterizations and hereditary properties of amenable Banach algebras. Since then, the concept of amenability has been a fruitful area of research in Banach algebras, operator algebras and abstract harmonic analysis. In the course of exploring the concept of amenability for Banach algebras, it was discovered that the above definition given by Johnson for amenable Banach algebras was too strong in the sense that, although it allows for nice theorems, it does not give room for enough and important examples.

For example, the founding result that was proved by Johnson in [31] was that, the group algebra $L^1(G)$ is amenable if and only if the locally compact group G is amenable. It was also established that, every finite group is amenable, furthermore, many results which holds for finite group were extended to amenable groups but these results never hold for larger class. In addition, it was conjectured in [29] that a contractible Banach algebra must be finite-dimensional. For this reason, several researchers in this area critically looked into the definition given by Johnson for amenable Banach algebras and were able to identify, relax some of the constraints in the definition and establish some generalizations and modifications of Johnson's definition of amenability for Banach algebras. In time past, some of the notable generalizations and modifications includes:

1. *weak amenability* introduced by *Bade, Curtis and Dales* in [3];
2. *essential and approximate amenability* introduced by *Ghahramani and Loy* in [18];
3. *operator amenability* introduced by *Ruan* in [46];
4. *Connes amenability* introduced by *Runde* in [47];
5. *character amenability* introduced by *Kaniuth* in [35] and *Sangani* in [42];
6. *approximate character amenability* introduced by *Mewomo and Okelo* in [41] and *Aghababa, Luo and Wu* in [1].

For more details on various notions of amenability in Banach algebras, see [39].

In this master's dissertation, we review and obtain some results on notions of amenability on general Banach algebras and that of Banach semigroup algebra $\ell^1(S)$ in relation to the amenability of the semigroup S.

1.2 Objectives

The main objectives of this study are to:

1. review some known results on some notions of amenability in general Banach algebras.
2. review some known results on amenability of Banach semigroup algebras in relation to the structures of the semigroups.
3. investigate some notions of amenability for general Banach algebras.
4. study some notions of amenability of the Banach semigroup algebras in relation to the semigroups.

1.3 Work Plan

This master's dissertation comprises of five chapters. In Chapter two, we lay down the basic background material that we shall need from Banach spaces, Banach algebras, Banach modules, semigroups and semigroup algebras. We also introduce some basic definitions and examples that will be useful in the course of our study. In chapter three, we introduce the definitions of some important notions of amenability that we shall consider in this study. In particular, we consider the notion of contractible, amenable, approximate amenable and pseudo-amenable Banach algebras. Furthermore, we give an explicit proof of the main characterization, hereditary properties and some interesting results of a general Banach algebra regarding these notions.

Chapter four is concerned with a general survey of results on the notion of amenable, approximate amenable and pseudo-amenable of Banach semigroup algebras $\ell^1(S)$. More so, some interesting results are presented. Chapter five is a catalogue of the results that we obtained in the course of the research work. The results in this chapter serve as our contribution to knowledge.

Chapter 2

Preliminaries

The purpose of this chapter is to introduce and develop basic concepts in the theory of Banach algebras, semigroups, semigroup algebras and the Gelfand theory of commutative Banach algebras. Other sections develop the basic theory of Banach modules and tensor product. We also recall some basic definitions, prove some results and give some examples that are relevant to our study. For more details on Banach spaces, Banach algebras, semigroups and semigroup algebras, see [2, 9, 11, 30].

2.1 Banach Space

Let X be a Banach space. The *dual* space of X is the space of continuous linear functionals on X, it is denoted by X'. That is $X' = \mathcal{B}(X, \mathbb{C}) = \{f : X \to \mathbb{C} \mid \text{ f is linear and continuous}\}$. Throughout this dissertation, for $x \in X$ and $f \in X'$, we write $\langle x, f\rangle := f(x)$. The higher duals of X are $X'' = (X')', X''' = (X'')', ..., X^n \ \ (n \in \mathbb{N})$, with $X^0 = X$ and X^n the n^{th} dual space of X. The canonical embedding of X into X'' is denoted by i and is the map

$$i : X \to X''$$

defined as

$$\langle f, i(x)\rangle = \langle x, f\rangle \quad (x \in X, f \in X').$$

The map i is an isometry and the space X is *reflexive* if i is onto. The image of X in X'' under i is denoted by $\widehat{X}$.

Theorem 2.1.1 (Hahn-Banach). *Let Y be a linear subspace of a normed space X. Then for each $f \in Y'$ there exists an extension $f' \in X'$ of f such that $\|f'\| = \|f\|$.*

The Hahn-Banach theorem guarantees that the canonical embedding i from X into its second dual X'' defined above is isometric. Let Y be a closed subspace of a normed space

X. The *annihilator* $Y^\perp$ of Y is defined as

$$Y^\perp = \{f \in X' \mid \langle y, f\rangle = 0 \ \forall \ y \in Y\}.$$

Also, we define ${}^\perp(Y^\perp) := \{x \in X \mid \langle x, f\rangle = 0 \ \forall f \in Y^\perp\}$.

Theorem 2.1.2 ([7]). *Let Y be a closed subspace of a normed space X. Then ${}^\perp(Y^\perp) = Y$.*

Definition 2.1.3. *Let X be a Banach space. The* weak topology *on X, denoted by $\sigma(X, X')$, is the topology generated by the family of seminorms $\{p_f \mid f \in X'\}$, where*

$$p_f(x) = |\langle x, f\rangle| \quad (x \in X).$$

The weak* topology *on X', denoted by $\sigma(X', X)$, is the topology generated by the family of seminorms $\{p_{i(x)} \mid x \in X\}$.*

A net $(x_\alpha) \subset X$ converges to $x \in X$ in $\sigma(X, X')$ if and only if

$$\langle x_\alpha, f\rangle \to \langle x, f\rangle \quad (f \in X')$$

and a net $(f_\alpha) \subset X'$ converges to $f \in X'$ in $\sigma(X', X)$ if and only if

$$\langle x, f_\alpha\rangle \to \langle x, f\rangle \quad (x \in X).$$

We now recall some theorems that are of great importance in the course of this study.

Theorem 2.1.4. *Let X be a Banach space.*

1. *(Goldstine). For each Φ in X'', there is a net $(x_\alpha) \subset X$ such that $\|x_\alpha\| \leq \|\Phi\|$ and $i(x_\alpha) \to \Phi$ in $\sigma(X'', X')$ with the limit taken in $\sigma(X'', X')$ on X''.*

2. *($Banach - Alaoglu$). The closed unit ball of X' is compact in $\sigma(X', X)$. Every bounded net in X' has a $\sigma(X', X)$- accumulation point and a $\sigma(X', X)$- convergent subnet.*

3. *($Mazur$). For each convex set $Y \subset X$, the closures of Y in $(X, \|\cdot\|)$ and $(X, \sigma(X, X'))$ are the same.*

2.2 Banach Algebra

In this section, we develop the theory of Banach algebras and since this dissertation is centred around the concept of Banach algebras, we begin by recalling the definition of an algebra.

Definition 2.2.1. *Let $\mathcal{A}$ be a vector space over a scalar field $\mathbb{F}$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$). Then $\mathcal{A}$ is an* algebra *if it also has an operation*

$$\times : \mathcal{A} \times \mathcal{A} \to \mathcal{A},$$

$$(x, y) \mapsto xy,$$

known as multiplication *or* product*, which satisfies the following axioms for all $x, y, z \in \mathcal{A}$ and every $\alpha \in \mathbb{F}$:*

1. $x(yz) = (xy)z;$
2. $(x + y)z = xz + yz;$
3. $x(y + z) = xy + xz;$
4. $(\alpha x)y = x(\alpha y) = \alpha(xy).$

Thus, an (associative) algebra is an algebraic structure that is both a ring and a vector space, where the addition of the ring is the same as the vector addition and multiplication by scalars relates to the ring multiplication by axiom (4) in the above definition.

Remark 2.2.2. *It is good to note that an algebra is characterized by the ring structure.*

Definition 2.2.3. *An algebra $\mathcal{A}$*

1. *is* commutative (abelian) *if its ring multiplication is commutative, that is*
$$xy = yx \quad (x, y \in \mathcal{A});$$
2. *has an* identity *element, say e, if $xe = ex = x$ for every x in $\mathcal{A}$.*

Definition 2.2.4. *Let $\mathcal{A}$ be an algebra. An* ideal *I of $\mathcal{A}$ is a subset of $\mathcal{A}$ such that:*

1. *I is a vector subspace of $\mathcal{A}$;*
2. *$\mathcal{A}I \subseteq I$ and $I\mathcal{A} \subseteq I$.*

Definition 2.2.5. *Let $\mathcal{A}$ be an algebra over a scalar field $\mathbb{F}$. An* algebra norm *on $\mathcal{A}$ is a mapping $\|\cdot\| : \mathcal{A} \to \mathbb{R}^+$ defined as $a \mapsto \|a\|$ such that:*

1. *$(\mathcal{A}; \|\cdot\|)$ is a normed space over $\mathbb{F}$;*
2. $\|xy\| \leq \|x\|\|y\| \quad (x, y \in \mathcal{A}).$

An algebra equipped with an algebra norm is called a *normed algebra*. A complete normed algebra is called a *Banach algebra*. A Banach algebra $\mathcal{A}$ is said to be *unital* if it has an identity, say 1 such that $\|1\| = 1$. Suppose that $\mathcal{A}$ is a Banach algebra; any non-empty subset of $\mathcal{A}$ which forms a Banach algebra under the induced norm of $\mathcal{A}$ is called a *subBanach algebra*.

Remark 2.2.6. *The algebraic multiplication $\times : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ defined as $(x, y) \mapsto xy$ is jointly continuous. The inequality $\|xy\| \leq \|x\|\|y\|$ makes the multiplication continuous. If $x_n \to x$ in $\mathcal{A}$ and $y_n \to y$ in $\mathcal{A}$ then $x_n y_n \to xy$ in $\mathcal{A}$.*

A Banach algebra $\mathcal{A}$ without a unit can always be embedded isometrically into a unital Banach algebra denoted by $\mathcal{A}^{\#}$ called the *unitization* of $\mathcal{A}$, such that $\mathcal{A}$ is closed ideal of $\mathcal{A}^{\#}$. Let $\mathcal{A}$ be a Banach algebra without a unit, we then consider $\mathcal{A}^{\#} = \mathcal{A} \odot \mathbb{C}$ with pointwise addition, scalar multiplication and multiplication defined as

$$(a, \alpha)(b, \beta) = (ab + a\beta + \alpha b, \alpha\beta)$$

and the norm $\|(a, \alpha)\|_{\mathcal{A}^{\#}} = \|a\|_{\mathcal{A}} + |\alpha| \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C})$. It is easy to show that $\mathcal{A}^{\#}$ is a Banach algebra. Indeed, for all $a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}$, we have

$$\begin{aligned}
\|(a, \alpha)(b, \beta)\| &= \|ab + a\beta + \alpha b, \alpha\beta\| = \|ab + a\beta + \alpha b\| + |\alpha\beta| \\
&\leq \|a\|\|b\| + \|a\||\beta| + |\alpha|\|b\| + |\alpha||\beta| \\
&= \|a\|\|b\| + |\alpha|\|b\| + \|a\||\beta| + |\alpha||\beta| \\
&= (\|a\| + |\alpha|)\|b\| + (\|a\| + |\alpha|)|\beta| \\
&= (\|a\| + |\alpha|)(\|b\| + \beta|) \\
&= \|(a, \alpha)\|\|(b, \beta)\|.
\end{aligned}$$

Clearly, $\mathcal{A}^{\#}$ is a unital Banach algebra with unit $(0, 1)$. More so, $\mathcal{A}^{\#}$ is commutative if, $\mathcal{A}$ is commutative.
Many natural occurring Banach algebras are not unital, but most of them possess sequence or net called *approximate identity* which behaves like a multiplicative identity in the limit. The concept of approximate identities was first studied explicitly by I. E. Segal, who proved that any norm closed self adjoint subalgebra of the algebra of bounded linear operators on a Hilbert space contains an approximate identity. Paul J. Cohen also proved that every element in an algebra with a suitable approximate identity can be factored. In an algebra with a unit, every element factors trivially ($a = a1 = 1a$).

Definition 2.2.7. *Let $\mathcal{A}$ be a normed algebra.*

1. *A* left approximate identity *for $\mathcal{A}$ is a net $(e_\alpha)_{\alpha \in D}$ in $\mathcal{A}$ such that $e_\alpha a$ converges in the norm to $a \in \mathcal{A}$. For every $a \in \mathcal{A}$, that is*

$$\lim_\alpha e_\alpha a = a, \quad (a \in \mathcal{A}).$$

2. *A* right approximate identity *for $\mathcal{A}$ is a net $(e_\alpha)_{\alpha\in D}$ in $\mathcal{A}$ such that ae_α converges in the norm to $a \in \mathcal{A}$. For every $a \in \mathcal{A}$, that is*

$$\lim_{\alpha} ae_\alpha = a, \quad (a \in \mathcal{A}).$$

3. *An* approximate identity *for $\mathcal{A}$ is a net $(e_\alpha)_{\alpha\in D}$ which is both left and right approximate identity.*

4. *A left or right approximate identity $(e_\alpha)_{\alpha\in D}$ is* bounded *by $M > 0$, if $\|e_\alpha\| \leq M$ for all $\alpha \in D$.*

5. *A left or right approximate identity $(e_\alpha)_{\alpha\in D}$ is* bounded *if it is bounded by some $M > 0$.*

6. *$\mathcal{A}$ is said to have a* bounded approximate identity *if it has a left and a right bounded approximate identity and it is* approximately unital *if it has a bounded approximate identity.*

7. *$\mathcal{A}$ has a* left approximate unit *if for all $a \in \mathcal{A}$ and $\epsilon > 0$ there exists $u \in \mathcal{A}$ (depending on a and ϵ) such that $\|a - ua\| < \epsilon$.*

8. *$\mathcal{A}$ has a* right approximate unit *if for all $a \in \mathcal{A}$ and $\epsilon > 0$ there exists $u \in \mathcal{A}$ (depending on a and ϵ) such that $\|a - au\| < \epsilon$.*

9. *$\mathcal{A}$ has a left or right approximate unit bounded by $M > 0$, if the element $u \in \mathcal{A}$ can be chosen such that $\|u\| \leq M$.*

Remark 2.2.8. *If the net is a sequence, the approximate identity is called a* sequential.

Theorem 2.2.9 ([34])**.** *Let $(e_\alpha)_{\alpha\in D}$ and $(f_\beta)_{\beta\in D'}$ be bounded left and right approximate identities for a normed algebra $\mathcal{A}$ respectively. Then the net*

$$(e_\alpha + f_\beta - e_\alpha f_\beta)_{(\alpha,\beta)\in D\times D'}$$

is a bounded approximate identity for $\mathcal{A}$.

Proof. Let $(e_\alpha)_{\alpha\in D}$ be a left approximate identity bounded by $M_1 > 0$ and let $(f_\beta)_{\beta\in D'}$ be a right approximate identity bounded by $M_2 > 0$. Since $(e_\alpha)_{\alpha\in D}$ and $(f_\beta)_{\beta\in D'}$ are left and right bounded approximate identity for $\mathcal{A}$ respectively. Let $\|e_\alpha a - a\| < \frac{\epsilon}{1+M_2}$ and $\|af_\beta - a\| < \frac{\epsilon}{1+M_1}$ for all $a \in \mathcal{A}$. Now let

$$(h_{\alpha\beta}) = (e_\alpha + f_\beta - e_\alpha f_\beta)_{(\alpha,\beta)\in D\times D'},$$

we then have that

$$\begin{aligned}
\|h_{\alpha\beta}a - a\| &= \|(e_\alpha + f_\beta - e_\alpha f_\beta)a - a\| = \|e_\alpha a + f_\beta a - e_\alpha f_\beta a - a\| \\
&= \|e_\alpha a - a + f_\beta a - e_\alpha f_\beta a\| = \|(e_\alpha a - a) - f_\beta(e_\alpha a - a)\| \\
&= \|(e_\alpha a - a)(1 - f_\beta)\| \\
&\leq \|e_\alpha a - a\|\|1 - f_\beta\| \\
&\leq \|e_\alpha a - a\|(1 + \|f_\beta\|) \\
&< \frac{\epsilon(1 + M_2)}{1 + M_2} = \epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
\|ah_{\alpha\beta} - a\| &= \|a(e_\alpha + f_\beta - e_\alpha f_\beta) - a\|\| = \|ae_\alpha + af_\beta - ae_\alpha f_\beta - a\| \\
&= \|ae_\alpha - ae_\alpha f_\beta + af_\beta - a\| = \|(a - af_\beta)e_\alpha - (a - af_\beta)\| \\
&= \|(e_\alpha - 1)(a - af_\beta)\| \\
&\leq \|(e_\alpha - 1)\|\|(a - af_\beta)\| \\
&= \|e_\alpha - 1\|\|af_\beta - a\| \\
&= (\|e_\alpha\| + 1)\|af_\beta - a\| \\
&< (M_1 + 1)\frac{\epsilon}{1 + M_1} = \epsilon.
\end{aligned}$$

Hence, $(h_{\alpha\beta})$ is an approximate identity for $\mathcal{A}$. For boundedness, observe that

$$\begin{aligned}
\|h_{\alpha\beta}\| &= \|(e_\alpha + f_\beta - e_\alpha f_\beta)\| \\
&\leq \|e_\alpha\| + \|f_\beta\| + \|e_\alpha\|\|f_\beta\| \\
&< M_1 + M_2 + M_1 M_2.
\end{aligned}$$

Let $M = M_1 + M_2 + M_1 M_2$. Therefore, $\|h_{\alpha\beta}\| < M$. Hence, $(h_{\alpha\beta})$ is bounded approximate identity for $\mathcal{A}$. □

Remark 2.2.10. *The above theorem guarantees that a Banach algebra with left and right (bounded) approximate identity has a (bounded) approximate identity.*

Theorem 2.2.11 ([18])**.** *Let $\mathcal{A}$ be a Banach algebra and suppose that $\mathcal{A}$ has a weak left (right) approximate identity. Then $\mathcal{A}$ has a left (right) approximate identity.*

Definition 2.2.12.

A topological space (X, τ) is said to be

1. compact *if every open cover of X has a finite sub-cover.*

2. Hausdorff *if for any distinct points $x_1, x_2 \in X$, there are disjoint open sets $G_1, G_2 \subseteq X$ with $x_1 \in G_1$ and $x_2 \in G_2$.*

3. locally compact *if every $x \in X$ has a compact neighbourhood.*

Definition 2.2.13. *A group G endowed with a topology is a topological group if both the multiplication*

$$G \times G \to G,\ (g, h) \mapsto gh$$

and the inversion

$$G \to G,\ g \mapsto g^{-1}$$

are continuous mapping. A locally compact group is a topological group whose topology is locally compact and Hausdorff

Every locally compact group G has a left Haar measure μ. We recall that a left invariant *Haar measure* on G is a Borel measure μ satisfying the following conditions:

1. $\mu(xU) = \mu(U)$ for every x in G and every measurable subset U of G;

2. $\mu(V) > 0$ for every non-empty open subset V of G;

3. $\mu(W) < \infty$ for every compact subset W of G.

Remark 2.2.14. *For example, the Lebesgue measure is an invariant Haar measure on the real numbers.*

There are several examples and classes of Banach algebras, which include the following: group algebra, Segal algebra, semigroup algebra, operator algebra, Fourier algebra, measure algebra and so on. Some of these examples are given below. For further details and examples see, [9] and [43].

Example 2.2.15. *1. (Group algebra). Let G be a locally compact group. We denote by $L^1(G)$ the group algebra of G. This is the Banach space*

$$\{f : G \to \mathbb{C}, f \textit{ is measurable} \mid \|f\|_1 = \int_G |f| d\mu < \infty\},$$

where μ denote left Haar measure on G and we equate functions that are equal almost everywhere with respect to μ. By defining a convolution multiplication on $L^1(G)$ by

$$(f * g)(t) = \int_G f(s) g(s^{-1}t) d\mu(s) \quad (f, g \in L^1(G), t \in G),$$

then $(L^1(G), *, \|\cdot\|_1)$ *is a Banach algebra. It is easy to check that* $\|f*g\|_1 \leq \|f\|_1\|g\|_1$. *Indeed, for all* $f, g \in L^1(G)$, *we have*

$$\begin{aligned}
\|f*g\|_1 &= \int_G |(f*g)(x)|d\mu = \int_G \left| \int_G (f(y)g(y^{-1}x) \right| d\mu(y)d\mu(x) \\
&\leq \int_G \int_G |(f(y)g(y^{-1}x)|d\mu(y)d\mu(x) = \int_G \int_G |(f(y)||g(y^{-1}x)|d\mu(x)d\mu(y) \\
&= \int_G |(f(y)| \Big(\int_G |g(y^{-1}x)|d\mu(x) \Big) d\mu(y) = \int_G |(f(y)| \Big(\int_G |g(z)|d\mu(z) \Big) d\mu(y) \\
&= \|f\|_1\|g\|_1.
\end{aligned}$$

$L^1(G)$ *is a commutative Banach algebra if and only if the group* G *is commutative. In the case where* G *is discrete, we write* $l^1(G)$ *for* $L^1(G)$.

2. *(Measure algebra). Let* G *be a locally compact group. We denote* $M(G)$ *for the space of all finite complex regular Borel measures on* G. $M(G)$ *can be identified with the dual of* $C_0(G)$, *the space of all continuous functions on* G *that vanish at infinity.* $M(G)$ *equipped with the total variation norm given by* $\|\mu\| = |\mu|(G)$ *for all* $\mu \in M(G)$, *this space is a Banach space and becomes a Banach algebra when the following convolution product is defined on it:*

$$(\mu * \upsilon)(f) = \int_G \Big(\int_G f(gh)d\mu(g) \Big) d\upsilon(h) \quad (\mu, \upsilon \in M(G), f \in C_0(G),\ g, h \in G).$$

According to Fubini's theorem, the order of integration does not matter. Also, we recall that a function f vanishes at infinity *if for all* $\varepsilon > 0$ *there exists a compact subset* K *of a locally compact Hausdorff space* X *such that,* $|f(x)| < \varepsilon$ *for each* $x \in X - K$.

3. *(Fourier algebra). Let* G *be a locally compact group and let* $p, q \in (1, \infty)$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *The space* $A_p(G)$ *consists of all functions* $f \in C_0(G)$ *such that* $(g_n)_{n\in\mathbb{N}} \in L^p(G)$ *and* $(h_n)_{n\in\mathbb{N}} \in L^q(G)$ *with* $\sum_{n=1}^{\infty} \|g_n\|_p\|h_n\|_q < \infty$ *and we define* $f(x) = \sum_{n=1}^{\infty}(g_n * h_n(x^{-1}))$. *Then*

$$\|f\|_A = \inf\Big\{ \sum_{n=1}^{\infty} \|g_n\|_p\|h_n\|_q \mid f = \sum_{n=1}^{\infty}(g_n * h_n) \Big\}.$$

The space $A_p(G)$ *together with the pointwise multiplication forms a Banach algebra. This algebra is called* Figa-Talamnce Herz. *It is called a* Fourier *algebra when* $p = 2$ *and it is denoted by* $A(G)$.

4. *(Segal algebra). Let* $\mathcal{A}$ *be a Banach algebra with norm* $\|\cdot\|_{\mathcal{A}}$ *and let* B *be a dense left ideal in* $\mathcal{A}$ *such that*

(a) B *is a Banach algebra with respect to some norm* $\|\cdot\|_B$;

(b) *there exists a constant* $K > 0$ *such that*

$$\|b\|_{\mathcal{A}} \leq K\|b\|_B \quad \forall\ b \in B;$$

(c) *there is a constant* $C > 0$ *such that*

$$\|ab\|_B \leq C\|a\|_{\mathcal{A}}\|b\|_B \quad \forall\ a, b \in B.$$

Then, recall from [40] that B *is called an* abstract Segal algebra *in* $\mathcal{A}$. *For* $\mathcal{A} = L^1(G)$, *we write* $S^1(G)$ *instead of* B, *with the following additional condition that* $S^1(G)$ *is closed under left translation;* $L_x f \in S^1(G)$ *for all* $x \in G$ *and* $f \in S^1(G)$, *where* $L_x f(y) = f(x^{-1}y)$ *for* $y \in G$. *Conditions (a) - (c) above on* $B = S^1(G)$ *are equivalent to the map*

$$(x, f) \mapsto L_x f : G \times S^1(G) \to S^1(G)$$

being continuous with $\|L_x f\|_{S^1(G)} = \|f\|_{S^1(G)}$ *for* $f \in S^1(G), s \in G$.

5. *(Operator algebra). Let* X *be a Banach space, the space* $\mathcal{B}(X)$ *denotes the set of all bounded linear operators on* X. $\mathcal{B}(X)$ *is a Banach space with the operator norm,*

$$\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}.$$

It becomes a Banach algebra with the composition product

$$(ST)(x) = (S \circ T)(x) = S(Tx), \quad (x \in X, S, T \in \mathcal{B}(X)).$$

Clearly, we have $\|ST\| \leq \|S\|\|T\|$. *Indeed, for all* $x \in X, S, T \in \mathcal{B}(X)$ *and* $\|x\| \leq 1$, *we have*

$$\|(ST)x\| = \|S(Tx)\| \leq \|S\|\|Tx\| \leq \|S\|\|T\|\|x\| \leq \|S\|\|T\|.$$

$\mathcal{B}(X)$ *is a non-commutative, unital Banach algebra. The unit is the* identity operator. *There are some closed ideals of* $\mathcal{B}(X)$ *that are also Banach algebras. For example* $\mathcal{K}(X)$, *the ideal of compact operators on* X, $\mathcal{A}(X)$, *the ideals of approximable operators on* X *and* $\mathcal{N}(X)$, *the ideal of nuclear operators on* X.

6. *(Function algebra). Let* (X, τ) *be a compact topological space. We denote by* $\mathcal{C}(X)$ *the set of all continuous complex valued functions on* X. $\mathcal{C}(X)$ *equipped with the norm*

$$\|f\| = \sup_{x \in X} |f(x)|, \quad (f \in \mathcal{C}(X), x \in X)$$

and pointwise addition and scalar multiplication is a Banach space and a Banach algebra with the pointwise product. Clearly,

$$\|fg\| = \sup_{x\in X} |f(x)g(x)| = \sup_{x\in X} |f(x)||g(x)| \leq \sup_{x\in X} |f(x)| \sup_{x\in X} |g(x)| = \|f\|\|g\|.$$

$\mathcal{C}(X)$ is a commutative Banach algebra with unit, where the unit is the constant function 1.

7. *(ℓ^1-Munn algebra). Let $\mathcal{A}$ be a unital Banach algebra, I and J be arbitrary index sets and P be a $J \times I$ non-zero matrix over $\mathcal{A}$ such that $\|P\|_\infty = \sup\{\|P_{ji} : i \in I, j \in J\|\} \leq 1$. Let $\mathcal{LM}(\mathcal{A}, P)$ be vector space of all $J \times I$ matrices A over $\mathcal{A}$ such that $\|A\|_1 = \sum_{i\in I, j\in J} \|A_{JI}\| < \infty$. Then, it is easy to check that $\mathcal{LM}(\mathcal{A}, P)$ with the product $A \circ B = APB, A, B \in \mathcal{LM}(\mathcal{A}, P)$ and the norm ℓ^1- norm $\|(a_{ij})\| = \sum_{i\in I, j\in J} \|a_{ij}\| < \infty$ is a Banach algebra that is called* ℓ^1-Munn algebra.

Definition 2.2.16. *Let $\mathcal{A}$ be a Banach algebra with unit* 1*. An element $a \in \mathcal{A}$ is* invertible, *if there exists $b \in \mathcal{A}$ such that $ab = ba = 1$ and we write $b = a^{-1}$.*

The set of invertible elements of $\mathcal{A}$ is denoted as $Inv(\mathcal{A})$. This set forms a group under the usual multiplication.

Remark 2.2.17. *Every invertible element of a Banach algebra has a unique inverse, if it exists.*

Example 2.2.18. 1. *Let X be a finite-dimensional Banach space then*

$$Inv(\mathcal{B}(X)) = \{T \in \mathcal{B}(X) \mid ker\ T = \{0\}\}.$$

2. *If X is a compact topological space then*

$$Inv(\mathcal{C}(X)) = \{f \in \mathcal{C}(X) \mid f(x) \neq 0\ \forall\ x \in X\}.$$

Theorem 2.2.19 ([2])**.** *Let $\mathcal{A}$ be a Banach algebra with unit 1. If $a \in \mathcal{A}$ with $\|a\| < 1$ then $(1 - a) \in Inv(\mathcal{A})$ and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n. \tag{2.1}$$

Proof. Let $a, a_0 \in \mathcal{A}$. Since $\|a^n\| \leq \|a\|^n$ for all $n \in \mathbb{N}$ and $\|a\| < 1$, the series 2.1 is absolutely convergent and so convergent by the completeness of $\mathcal{A}$. Let the series 2.1 converge to a_0 and let a_k be the kth partial sum of the series. Then, observe that

$$\begin{aligned}
\lim_{k\to\infty} \left(\sum_{n=0}^{k} a^n (1-a) \right) = \lim_{k\to\infty} \left((1-a) \sum_{n=0}^{k} a^n \right) &= \lim_{k\to\infty} [(1-a)(1+a+a^2+a^3+\cdots+a^k)] \\
&= \lim_{k\to\infty} (1-a+a+a^2-a^2+a^3-a^3+\cdots+a^{k+1}) \\
&= \lim_{k\to\infty} (1-a^{k+1}) = 1.
\end{aligned}$$

Therefore, we have that

$$\begin{aligned} & a_0(1-a) = (1-a)a_0 = 1 \\ \Rightarrow & a_0 = (1-a)^{-1} \\ \Rightarrow & (1-a) \in Inv(\mathcal{A}). \end{aligned}$$

□

Corollary 2.2.20 ([2]). *Let $\mathcal{A}$ be a unital Banach algebra. Then $Inv(\mathcal{A})$ is an open subset of $\mathcal{A}$.*

Proof. Let $a_0 \in Inv(\mathcal{A})$ and let $r_{a_0} = \frac{1}{\|a_0^{-1}\|} > 0$. We want to show that, the open ball $B(a_0, r_{a_0}) \subset Inv(\mathcal{A})$. Let $a \in B(a_0, r_{a_0})$, then $\|a_0 - a\| < r_{a_0}$. We then need to show that $a \in Inv(\mathcal{A})$. From the fact that

$$a = a_0 - a_0 + a = (a_0 - (a_0 - a))a_0^{-1}a_0 = (1 - (a_0 - a)a_0^{-1})a_0$$

and

$$\|(a_0 - a)a_0^{-1}\| \leq \|a_0 - a\|\|a_0^{-1}\| < r_{a_0}\|a_0^{-1}\| = 1,$$

it follows from Theorem 2.2.19 that $(1 - (a_0 - a)a_0^{-1})$ is invertible. Since a_0 is also in $Inv(\mathcal{A})$ and a is the product of two invertible elements, hence a is in $Inv(\mathcal{A})$. Therefore, $Inv(\mathcal{A})$ is open. □

Remark 2.2.21. *From Corollary 2.2.20, we have that, every element of $Inv(\mathcal{A})$ is contained in an open ball and this ball is contained in $Inv(\mathcal{A})$.*

Corollary 2.2.22 ([2]). *Let $\mathcal{A}$ be a Banach algebra with unit 1. Then the mapping*

$$\rho : Inv(\mathcal{A}) \to Inv(\mathcal{A}), \quad a \mapsto a^{-1}$$

is continuous.

Proof. Let $a, a_0 \in Inv(\mathcal{A})$ with $\|a - a_0\| < \frac{1}{2\|a^{-1}\|}$, using

$$a^{-1} - a_0^{-1} = a^{-1}(a_0 - a)a_0^{-1}$$

and

$$\begin{aligned} \|a_0^{-1}\| &= \|a^{-1} - a^{-1} + a_0^{-1}\| = \|a^{-1} - (a^{-1} - a_0^{-1})\| \\ &= \|a^{-1} - [a^{-1}(a_0 - a)a_0^{-1}]\| \leq \|a^{-1}\| + \|a^{-1}\|\|a_0 - a\|\|a_0^{-1}\| \\ &\leq \|a^{-1}\| + \|a^{-1}\|\frac{1}{2\|a^{-1}\|}\|a_0^{-1}\| = \|a^{-1}\| + \frac{1}{2}\|a_0^{-1}\|. \end{aligned}$$

It follows that $\|a_0^{-1}\| \leq 2\|a^{-1}\|$. By definition, the map ρ from $Inv(\mathcal{A})$ into $Inv(\mathcal{A})$ is said to be continuous at a point $a_0 \in Inv(\mathcal{A})$, if for every $\epsilon > 0$, there exists $\delta > 0$, such that if $a_0 \in Inv(\mathcal{A})$ satisfies $\|a - a_0\| < \delta$ then $\|\rho(a) - \rho(a_0)\| < \epsilon$. Suppose $a_0 \in Inv(\mathcal{A})$ satisfies $\|a - a_0\| < \delta$. Then

$$\begin{aligned}\|\rho(a) - \rho(a_0)\| &= \|a^{-1} - a_0^{-1}\| = \|a^{-1}(a_0 - a)a_0^{-1}\| \\ &\leq \|a^{-1}\|\|a_0 - a\|\|a_0^{-1}\| \\ &= 2\|a^{-1}\|^2\|a_0 - a\| < 2\|a^{-1}\|^2\delta < \epsilon,\end{aligned}$$

where $\delta = \frac{\epsilon}{4\|a^{-1}\|^2}$. Hence, continuity holds. □

Remark 2.2.23. *The map ρ from $Inv(\mathcal{A})$ into $Inv(\mathcal{A})$ above is a homeomorphism, since bijectivity holds from the fact that $(a^{-1})^{-1} = a$ and the continuity of ρ^{-1} follows from $\rho = \rho^{-1}$.*

Remark 2.2.24. *From the above theorem and corollaries, it is clear that the set of all invertible elements of a Banach algebra $\mathcal{A}$ denoted by $Inv(\mathcal{A})$ is a topological group.*

2.2.1 Spectrum of a Banach Algebra

In this section, all Banach algebras are assumed to have a unit.

Definition 2.2.25. *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$. The* spectrum *of a in $\mathcal{A}$ is defined as*

$$\sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda 1 - a \notin Inv(\mathcal{A})\}.$$

We denote the spectrum of $a \in \mathcal{A}$ by $\sigma(a)$ and we will write λ instead of $\lambda 1$ for $\lambda \in \mathbb{C}$. If $\mathcal{B}$ is a subBanach algebra of $\mathcal{A}$, then, we have that

$$\sigma_{\mathcal{A}}(a) \subseteq \sigma_{\mathcal{B}}(a) \quad (a \in \mathcal{B}).$$

Example 2.2.26. *1. If X is a finite-dimensional Banach space and $T \in \mathcal{B}(X)$, then*

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda \textit{ is an eigenvalue of } T\}.$$

Proof. By definition,

$$\sigma(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \notin Inv(\mathcal{A})\}$$

from Example 2.2.18 (1), we have that

$$Inv(\mathcal{B}(X))\sigma(T) = \{\lambda \in \mathbb{C} \mid \lambda \text{ is an eigenvalue of } T\}. = \{T \in \mathcal{B}(X) \mid ker\ T = \{0\}\}.$$

It then follows that

$$\begin{aligned}\sigma(T) &= \{\lambda \in \mathbb{C} \mid \lambda - T \notin Inv(\mathcal{B}(X))\}\\ &= \{\lambda \in \mathbb{C} \mid ker(\lambda - T) \neq \{0\}\}\\ &= \{\lambda \in \mathbb{C} \mid (\lambda - T)(x) = 0\} \qquad for\ some\ x \in X\\ &= \{\lambda \in \mathbb{C} \mid \lambda x - Tx = 0\}\\ &= \{\lambda \in \mathbb{C} \mid \lambda x = Tx\}.\end{aligned}$$

From the definition of eigenvalues and eigenvectors, we know that any x that satisfies $\lambda x = Tx$ is an eigenvector and λ is the corresponding eigenvalue. $\square$

2. *The $\sigma(\lambda) = \{\lambda\}$ for all $\lambda \in \mathbb{C}$.*

Definition 2.2.27. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$.*

1. *The* resolvent *of a in $\mathcal{A}$ is defined as*

$$\varrho_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} \mid \lambda - a \in InvA\}.$$

   That is $\varrho(a) = \mathbb{C} \backslash \sigma_{\mathcal{A}}(a)$.

2. *The map $R : \varrho(a) \to \mathcal{A}$ defined as $\lambda \mapsto (\lambda - a)^{-1}$ is called the* resolvent function *of a in $\mathcal{A}$.*

Definition 2.2.28. *A unital algebra $\mathcal{A}$ is said to be a division algebra if every non-zero element in $\mathcal{A}$ is invertible. That is $Inv(\mathcal{A}) = \mathcal{A} \setminus \{0\}$.*

Theorem 2.2.29 ([2])**.** *Let $\mathcal{A}$ be a unital Banach algebra. If $a \in \mathcal{A}$ then $\sigma(a)$ is a non-empty compact subset of $\mathbb{C}$ with $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$.*

Proof. For each $\lambda \in \mathbb{C}$ such that $|\lambda| > \|a\|$, we have $\lambda - a = \lambda(1 - \lambda^{-1}a)$. But $\|\lambda^{-1}a\| = |\lambda^{-1}|\|a\| < 1$ and so by Theorem 2.2.19, $\lambda - a \in Inv(\mathcal{A})$, which implies that $\lambda \notin \sigma(a)$. Hence, $\sigma(a) \subseteq \{\lambda \in \mathbb{C} : |\lambda| \leq \|a\|\}$. Also, $\sigma(a)$ is closed because it is the inverse image with respect to the continuous map $\lambda \mapsto \lambda - a$, $\mathbb{C} \to \mathcal{A}$, of the closed subset $\mathcal{A} \setminus Inv(\mathcal{A})$ of $\mathcal{A}$. Thus, $\sigma(a)$ is bounded and closed, so $\sigma(a)$ is a compact subset of $\mathbb{C}$.

Lastly, we need to show that $\sigma(a) \neq \emptyset$. If $\sigma_{\mathcal{A}}(a) = \emptyset$, then $\varrho(a) = \mathbb{C}$. Then we have

$$R : \mathbb{C} \to \mathcal{A}, \quad \lambda \mapsto (\lambda - a)^{-1}.$$

Corollary 2.2.22, shows that R is continuous, more so, R is analytic and bounded. Indeed, for all $\lambda_0 \in \mathbb{C}$, we have

$$\begin{aligned}
\lim_{\lambda \to \lambda_0} \left(\frac{R(\lambda) - R(\lambda_0)}{\lambda - \lambda_0} \right) &= \frac{(\lambda - a)^{-1} - (\lambda_0 - a)^{-1}}{\lambda - \lambda_0} \\
&= \frac{(\lambda - a)^{-1}[(\lambda_0 - a) - (\lambda - a)](\lambda_0 - a)^{-1}}{\lambda - \lambda_0} \\
&= \frac{(\lambda - a)^{-1}(\lambda_0 - \lambda)(\lambda_0 - a)^{-1}}{\lambda - \lambda_0} \\
&= \frac{-(\lambda - \lambda_0)(\lambda - a)^{-1}(\lambda_0 - a)^{-1}}{\lambda - \lambda_0} \\
&= -(\lambda - a)^{-1}(\lambda_0 - a)^{-1}. \\
\lim_{\lambda \to \lambda_0} \left(\frac{R(\lambda) - R(\lambda_0)}{\lambda - \lambda_0} \right) &= \lim_{\lambda \to \lambda_0} [-(\lambda - a)^{-1}(\lambda_0 - a)^{-1}] \\
&= -(\lambda_0 - a)^{-1}(\lambda_0 - a)^{-1} = -(\lambda_0 - a)^{-2}.
\end{aligned}$$

Since λ_0 is arbitrary, then R is analytic. Let us now show that R is bounded.

$$\lim_{|\lambda| \to \infty} R(\lambda) = \lim_{|\lambda| \to \infty} (\lambda - a)^{-1} = \lim_{|\lambda| \to \infty} \frac{(1 - a/\lambda)^{-1}}{\lambda} = \frac{(1)^{-1}}{\infty} = 0 \quad \forall\ \lambda.$$

Hence, R is bounded. Since we have show that R is analytic and bounded, then by Louville's theorem, R is a constant. Since, $R(\lambda) \to 0$ as $|\lambda| \to \infty$ for all $\lambda \in \mathbb{C}$. This is a contradiction, since $R(\lambda)$ is invertible. Hence, $\sigma_{\mathcal{A}}(a) \neq \emptyset$. □

Theorem 2.2.30 (Gelfand - Mazur)**.** *If $\mathcal{A}$ is a unital Banach algebra in which every non-zero element is invertible then $\mathcal{A} \cong \mathbb{C}$.*

Proof. Theorem 2.2.29 guarantees that $\sigma_{\mathcal{A}}(a) \neq \emptyset$. Let $a \in \mathcal{A}$, then there exists an element $\lambda \in \sigma_{\mathcal{A}}(a)$, such that $\lambda 1 - a \notin Inv(\mathcal{A})$. Since $\mathcal{A}$ is a division algebra, we have

$$\lambda 1 - a = 0, \quad \Rightarrow \lambda 1 = a.$$

This gives an isomorphism from $\mathcal{A}$ onto $\mathbb{C}$. Define $\phi : \mathcal{A} \to \mathbb{C}$ as $a \mapsto \lambda$. For isometric, we need to show that $\|\phi(a)\| = \|a\|$. We have that

$$\|a\| = \|\lambda 1\| = \|\lambda\| = \|\phi(a)\|.$$

Hence, the map is an isometry. □

Remark 2.2.31. *The above result shows that $\mathbb{C}$ is essentially the only unital Banach algebra which is also a field.*

Theorem 2.2.32 (Spectral mapping property for polynomials)**.** *Let $\mathcal{A}$ be a complex algebra with an identity, let $a \in \mathcal{A}$ and let p be a complex polynomial. Then*

$$\sigma(p(a)) = \{p(\lambda) \mid \lambda \in \sigma(a)\}.$$

Definition 2.2.33. *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$, the* spectral radius *of a is*

$$r_{\mathcal{A}}(a) = sup\{|\lambda| \mid \lambda \in \sigma_{\mathcal{A}}(a)\}.$$

Example 2.2.34. *Let X be a compact topological space and f in $\mathcal{A} := \mathcal{C}(X)$, we have*

$$r_{\mathcal{A}}(f) = \sup_{\lambda \in \sigma_{\mathcal{A}}(f)} |\lambda| = \sup_{\lambda \in f(X)} |\lambda| = \|f\|.$$

Theorem 2.2.35 (Spectral radius formula)**.** *Let $\mathcal{A}$ be a unital Banach algebra and let $a \in \mathcal{A}$, then*

$$r_{\mathcal{A}}(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}.$$

Corollary 2.2.36 ([34])**.** *If $\mathcal{A}$ is a unital Banach algebra and $\mathcal{B}$ is a closed unital subalgebra of $\mathcal{A}$ then $r_{\mathcal{A}}(b) = r_{\mathcal{B}}(b)$ for all b in $\mathcal{B}$.*

Proof. Using the spectral radius formula, for all $b \in \mathcal{B}$, we have

$$r_{\mathcal{A}}(b) = \lim_{n\to\infty} \|b\|^{\frac{1}{n}} = r_{\mathcal{B}}(b).$$

□

Remark 2.2.37. *Clearly, the above result guarantees that the spectral radius of $a \in \mathcal{A}$ does not change when computed in any subBanach algebra of $\mathcal{A}$, containing a.*

Remark 2.2.38. *It is easy to see that*

$$r_{\mathcal{A}}(\lambda a) = |\lambda| r_{\mathcal{A}}(a) \quad (\lambda \in \mathbb{C})$$

and by Theorem 2.2.29, we have $r_{\mathcal{A}}(a) \leq \|a\|$.

Theorem 2.2.39 ([34])**.** *Let $\mathcal{A}$ be a unital Banach algebra and let $a, b \in \mathcal{A}$ with $ab = ba$. Then*

$$r_{\mathcal{A}}(ab) \leq r_{\mathcal{A}}(a) r_{\mathcal{A}}(b).$$

Proof. Since $ab = ba$, we have $(ab)^n = a^n b^n \quad (n \in \mathbb{N})$. It then follows that

$$\begin{aligned} r_{\mathcal{A}}(ab) &= \inf_{n\in\mathbb{N}} \|(ab)^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n b^n\|^{1/n} \\ &\leq \inf_{n\in\mathbb{N}} \|a^n\|^{1/n} \inf_{n\in\mathbb{N}} \|b^n\|^{1/n} = r_{\mathcal{A}}(a) r_{\mathcal{A}}(b). \end{aligned}$$

□

2.2.2 Ideals, Quotients and Homomorphism of Banach Algebras

Definition 2.2.40. *Let $\mathcal{A}$ be a Banach algebra. An* ideal *I of $\mathcal{A}$ is a vector subspace I of $\mathcal{A}$ such that for all $x \in I$ and $a \in \mathcal{A}$, we have $ax \in I$ and $xa \in I$.*

It is well known that, if $\mathcal{A}$ is a Banach space and I a closed ideal of $\mathcal{A}$, the quotient space $\mathcal{A}/I$ is a Banach space with respect to the quotient norm $\|a + I\| = \inf_{x\in I} \|a + x\|$, for all $a \in \mathcal{A}$. If we define the product $(a + I)(b + I) = ab + I$ on $\mathcal{A}/I$ for all $a, b \in \mathcal{A}$. Then $\mathcal{A}/I$ becomes a Banach algebra. This result is shown in the next theorem.

Theorem 2.2.41 ([2])**.** *If I is a closed ideal of a Banach algebra $\mathcal{A}$, then $\mathcal{A}/I$ is a Banach algebra. If $\mathcal{A}$ is abelian so is $\mathcal{A}/I$.*

Proof. It is clear that the product $(a + I)(b + I) = ab + I$ is well defined for all $a, b \in \mathcal{A}$. Since $\mathcal{A}/I$ is a Banach space, it is suffices to show that $\|(a+I)(b+I)\| \leq \|a+I\|\|b+I\|$. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
\|(a + I)(b + I)\| = \|ab + I\| = \inf_{x\in I} \|ab + x\| &\leq \inf_{x,y\in I} \|ab + ay + xb + xy\| \\
&= \inf_{x,y\in I} \|(a + x)(b + y)\| \leq \inf_{x,y\in I} \|a + x\|\|b + y\| \\
&= \|a + I\|\|b + I\|.
\end{aligned}$$

Hence, $\mathcal{A}/I$ is a Banach algebra. Suppose $\mathcal{A}$ is abelian, then for all $a, b \in \mathcal{A}$, we have

$$(a + I)(b + I) = ab + I = ba + I = (b + I)(a + I).$$

Hence, $\mathcal{A}/I$ is abelian. □

Definition 2.2.42. *Let $\mathcal{A}$ be a Banach algebra.*

1. *An ideal I in $\mathcal{A}$ is* proper*, if I is not equal to $\mathcal{A}$.*

2. *An ideal I is a* maximal *ideal of $\mathcal{A}$, if I is a proper ideal such that I is not contained in any strictly larger proper ideal of $\mathcal{A}$.*

Lemma 2.2.43 ([2])**.** *Let $\mathcal{A}$ be a unital Banach algebra. If I is an ideal of $\mathcal{A}$, then I is a proper ideal if and only if $I \cap Inv(\mathcal{A}) = \emptyset$.*

Theorem 2.2.44 ([2])**.** *Let $\mathcal{A}$ be a unital Banach algebra.*

1. *If I is a proper ideal of $\mathcal{A}$ then the closure $\overline{I}$ is also a proper ideal of $\mathcal{A}$.*

2. *Any maximal ideal of $\mathcal{A}$ is closed.*

Definition 2.2.45. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. A* homomorphism *from $\mathcal{A}$ to $\mathcal{B}$ is a linear map $\theta : \mathcal{A} \to \mathcal{B}$ which is multiplicative in the sense that $\theta(ab) = \theta(a)\theta(b), \quad (a, b \in \mathcal{A})$. The* kernel *of such a homomorphism θ is the set*

$$ker\ \theta = \{a \in \mathcal{A} : \theta(a) = 0\}.$$

If $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras, we say that a homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ is unital if $\theta(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. A bijective homomorphism $\theta : \mathcal{A} \to \mathcal{B}$ is an *isomorphism.* If such an isomorphism exists then the Banach algebras $\mathcal{A}$ and $\mathcal{B}$ are *isomorphic.* The kernel of θ is a proper ideal if θ is not equal to 0. The quotient map $\pi : \mathcal{A} \to \mathcal{A}/I$ is surjective homomorphism with $ker\ \pi = I$. Suppose that $\theta : \mathcal{A} \to \mathcal{B}$ is a homomorphism of Banach algebras with $ker\ \theta = I$. Then there exists a unique embedding $\psi : \mathcal{A}/I \to \mathcal{B}$ such that $\theta = \psi \circ \pi$.

Remark 2.2.46. *If two "objects" are isomorphic, we say that they have the same structure. In the context of Banach algebras, we need more that isomorphism. Two Banach algebras have the same structure if they are isometrically isomorphic.*

Example 2.2.47. *Suppose that $\mathcal{A}$ is a non-unital Banach algebra then the map*

$$\theta : \mathcal{A} \to \mathcal{A}^{\#},\ \ a \mapsto (a, 0)$$

is an isometric homomorphism. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\theta(ab) = (ab, 0) = (a, 0)(b, 0) = \theta(a)\theta(b) \quad \text{and}$$
$$\|\theta(a)\|_{\mathcal{A}} = \|(a, 0)\|_{\mathcal{A}^{\#}} = \|a\|_{\mathcal{A}} + |0| = \|a\|_{\mathcal{A}}.$$

Hence, $\theta(\mathcal{A})$ is a Banach subalgebra of $\mathcal{A}^{\#}$ which is isometrically isomorphic to $\mathcal{A}$.

2.3 Gelfand Theory

In this section, we give a brief introduction to Gelfand's theory for commutative Banach algebras. For further details see [43].

Definition 2.3.1. *Let $\mathcal{A}$ be a Banach algebra. Then a* character *on $\mathcal{A}$ is a non-zero homomorphism $\mathcal{A} \to \mathbb{C}$; that is, a non-zero linear map $\varphi : \mathcal{A} \to \mathbb{C}$ which satisfies $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$.*

We denote the set of all character on $\mathcal{A}$ by $\Phi_{\mathcal{A}}$. Suppose that $1 \in \mathcal{A}$. Then $\varphi(1) = 1$ for every $\varphi \in \Phi_{\mathcal{A}}$ and so a character is a unital homomorphism.

Remark 2.3.2. *It is good to note that $\Phi_{\mathcal{A}}$ and $\Phi_{\mathcal{A}^{\#}}$ are related. Indeed, for every $\psi \in \Phi_{\mathcal{A}^{\#}},\ \psi(e) = 1$ and for every $\varphi \in \Phi_{\mathcal{A}}$, there exists a unique extension $\overline{\varphi} \in \Phi_{\mathcal{A}^{\#}}$ given as*

$$\overline{\varphi}(a + \lambda 1) = \varphi(a) + \lambda, \quad (a \in \mathcal{A}, \lambda \in \mathbb{C}).$$

Example 2.3.3. *1. Let $\mathcal{A} := \mathcal{C}(X)$ where X is a compact topological space. For any $x \in X$, the map $\varphi_x : \mathcal{A} \to \mathbb{C}$, $f \mapsto f(x)$ is a character on $\mathcal{A}$. Indeed, for all $f, g \in \mathcal{A}$, we have*

$$\varphi_x(fg) = (fg)(x) = f(x)g(x) = \varphi_x(f)\varphi_x(g).$$

2. Let $\mathcal{A} := A(\overline{\mathbb{D}})$, the disc algebra. For each $z \in \overline{\mathbb{D}}$, the map $\varphi_z : \mathcal{A} \to \mathbb{C}$, $f \mapsto f(z)$ is a character on $\mathcal{A}$. Indeed, for all $f, g \in \mathcal{A}$, we have

$$\varphi_z(fg) = (fg)(z) = f(z)g(z) = \varphi_z(f)\varphi_z(g).$$

Proposition 2.3.4 ([2])**.** *Any unital Banach algebra $\mathcal{A}$ possesses at least one character.*

Theorem 2.3.5 ([2])**.** *Let $\mathcal{A}$ be a Banach algebra and let $\varphi \in \Phi_{\mathcal{A}}$. Then φ is continuous and $\|\varphi\| \leq 1$. Suppose $\mathcal{A}$ is unital then $\|\varphi\| = 1$.*

In particular, $\Phi_{\mathcal{A}}$ is a subset of the closed unit ball of $\mathcal{A}'$.

Definition 2.3.6. *Let $\mathcal{A}$ be a Banach algebra and $\Phi_{\mathcal{A}}$ the set of all character on $\mathcal{A}$. We endow $\Phi_{\mathcal{A}}$ with the weakest topology with respect to which all the functions*

$$\Phi_{\mathcal{A}} \to \mathbb{C}, \quad \varphi \mapsto \varphi(a), \quad (a \in \mathcal{A})$$

are continuous. The topology on $\Phi_{\mathcal{A}}$ is called the Gelfand topology*.*

Remark 2.3.7. *From Theorem 2.3.5, we have that $\Phi_{\mathcal{A}}$ is contained in the unit ball of $\mathcal{A}'$. The Gelfand topology obviously coincides with the weak* topology of $\mathcal{A}'$ on $\Phi_{\mathcal{A}}$. This topology is sometimes called the weak* topology on $\Phi_{\mathcal{A}}$.*

Theorem 2.3.8 ([2])**.** *Let $\mathcal{A}$ be a unital Banach algebra. Then $\Phi_{\mathcal{A}}$ is a non-empty compact Hausdorff space in the Gelfand's topology.*

Lemma 2.3.9 ([2])**.** *Let $\mathcal{A}$ be a unital commutative Banach algebra.*

1. *If $\varphi \in \Phi_{\mathcal{A}}$, then ker φ is a maximal ideal of $\mathcal{A}$.*
2. *If J is a maximal ideal of $\mathcal{A}$, then the map $\mathbb{C} \to \mathcal{A}/J$, $\lambda \mapsto \lambda + J$ is an isometric isomorphism.*

Proof. 1. Since $\varphi \in \Phi_{\mathcal{A}}$, then φ is a non-zero homomorphism and so $I = ker\ \varphi$ is a proper ideal of $\mathcal{A}$. Suppose that J is another ideal of $\mathcal{A}$ such that $I \subsetneq J$ and let $a \in J \setminus I$. Then $\varphi(a) \neq 0$, so $b = \varphi(a)^{-1}a \in J$ and $\varphi(b) = 1$. Since $\varphi(1) = 1$, using Theorem 2.3.5, we then have that $1 - b \in I$ and it follows that $1 = b + 1 - b \in J$. By Lemma 2.2.43, $J = \mathcal{A}$. This shows that I is not contained in any strictly larger ideal of $\mathcal{A}$, hence I is a maximal ideal of $\mathcal{A}$.

2. Let J be a maximal ideal of $\mathcal{A}$. By Theorem 2.2.41, Theorem 2.2.44 (2) and the fact that if, $\mathcal{A}$ is a unital Banach algebra so is $\mathcal{A}/J$. The unit of $\mathcal{A}/J$ is of the form $1+J$. If $a+J$ is a non-zero element of $\mathcal{A}/J$ then $a \in \mathcal{A}\backslash J$. Let $K = \{ab+j : j \in J, b \in \mathcal{A}\}$. Since $\mathcal{A}$ is commutative and J is a maximal ideal, it is easy to see that K is an ideal of $\mathcal{A}$ and $J \subsetneq K$. Since J is a maximal ideal, $K = \mathcal{A}$. Then we have that $1 \in K$ and $ab + j = 1$, for some $b \in \mathcal{A}, j \in J$. Now observe that

$$(a + J)(b + J) = ab + J = ab + j + J = 1 + J.$$

Hence, $(a + J)$ is invertible. Then since every non-zero elements of $\mathcal{A}/J$ are invertible, applying Theorem 2.2.30, we have that $\mathcal{A}/J = 1_{\mathcal{A}/J} = 1 + J$. It is easy to see that the map $\mathbb{C} \to \mathcal{A}/J,\ \lambda \mapsto \lambda + J$ is an isometric homomorphism, since we have shown that it is surjective. Hence, the result hold.

□

Theorem 2.3.10 ([2]). *Let $\mathcal{A}$ be a unital commutative Banach algebra. The mapping $\varphi \mapsto ker\varphi$ is a bijection from $\Phi_{\mathcal{A}}$ onto the set of maximal ideals of $\mathcal{A}$.*

Lemma 2.3.11 ([2]). *Let $\mathcal{A}$ be a unital commutative Banach algebra and let $a \in \mathcal{A}$. Then the following are equivalent;*

1. *$a \notin Inv(\mathcal{A})$;*
2. *$a \in I$ for some proper ideal I of $\mathcal{A}$;*
3. *$a \in J$ for some maximal ideal J of $\mathcal{A}$.*

Corollary 2.3.12 ([2]). *Let $\mathcal{A}$ be a unital commutative Banach algebra and let $a \in \mathcal{A}$. Then the following holds.*

1. *$a \in Inv(\mathcal{A})$ if and only if $\varphi(a) \neq 0$ for all $\varphi \in \Phi_{\mathcal{A}}$.*
2. *$\sigma_{\mathcal{A}}(a) = \{\varphi(a) : \varphi \in \Phi_{\mathcal{A}}\}$.*
3. *$r_{\mathcal{A}}(a) = \sup_{\varphi \in \Phi_{\mathcal{A}}} |\varphi(a)|$*

2.3.1 Gelfand Representation

Definition 2.3.13. *Let $\mathcal{A}$ be a unital commutative Banach algebra. For $a \in \mathcal{A}$, the* Gelfand transform *of a is the mapping*

$$\widehat{a} : \Phi_{\mathcal{A}} \to \mathbb{C}, \quad \varphi \mapsto \varphi(a).$$

Example 2.3.14. *Let $\mathcal{A} = \mathcal{C}(X)$, where X is a compact Hausdorff space. If $f \in \mathcal{C}(X)$, then*

$$\widehat{f} : \Phi_{\mathcal{A}} \to \mathbb{C}, \quad \varphi_x \mapsto \varphi_x(f) = f(x).$$

Theorem 2.3.15 ([2])**.** *Let $\mathcal{A}$ be a unital commutative Banach algebra. For each $a \in \mathcal{A}$, the Gelfand transform $\widehat{a}$ is in $C(\Phi_{\mathcal{A}})$. Moreover, the mapping*

$$\Gamma_{\mathcal{A}} : \mathcal{A} \to C(\Phi_{\mathcal{A}}), \quad a \mapsto \widehat{a}$$

is a unital, norm-decreasing (and hence continuous) homomorphism and for each $a \in \mathcal{A}$ we have

$$\sigma(a) = \sigma_{C(\Phi_{\mathcal{A}})}(\widehat{a}) = \{\widehat{a}(\varphi) : \varphi \in \Phi_{\mathcal{A}}\} \quad \text{and} \quad r_{\mathcal{A}}(a) = \|\widehat{a}\|.$$

Proof. It is clear that $\widehat{a}$ is in $C(\Phi_{\mathcal{A}})$ from the Gelfand topology on $\Phi_{\mathcal{A}}$. It is also easy to see that Γ is a homomorphism. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\Gamma(ab) = \varphi(ab) = \varphi(a)\varphi(b) = \Gamma(a)\Gamma(b).$$

By Theorem 2.3.5, Γ is unital. The fact that $\sigma(a) = \sigma_{C(\Phi_{\mathcal{A}})}(\widehat{a})$ follows from Corollary 2.3.12 (2). Also, $r_{\mathcal{A}}(a) = \|\widehat{a}\| \leq \|a\|$ follows from Corollary 2.3.12 (3) and Remark 2.2.38, so Γ is linear and norm-decreasing, hence continuous. ☐

Definition 2.3.16. *If $\mathcal{A}$ is a unital commutative Banach algebra, then the unital homomorphism*

$$\Gamma : \mathcal{A} \to C(\Phi_{\mathcal{A}}), \quad a \mapsto \widehat{a}$$

is called the Gelfand representation of $\mathcal{A}$.

Remark 2.3.17. *In general, the Gelfand representation is neither injective or surjective.*

2.4 Banach Modules

In this section, we give a brief introduction to Banach modules. For details, see [9].

Definition 2.4.1. *Let $\mathcal{A}$ be an algebra over a scalar field $\mathbb{F}$.*

1. *By a* right $\mathcal{A}$-module*, we mean a vector space X over a scalar field $\mathbb{F}$ together with a map $\cdot : X \times \mathcal{A} \to X, (x, a) \mapsto x \cdot a$ which satisfies*
   (a) $(x + y) \cdot a = x \cdot a + y \cdot a \quad (x, y \in X, a \in \mathcal{A})$;
   (b) $x \cdot (a + b) = x \cdot a + x \cdot b \quad (x \in X, a, b \in \mathcal{A})$;

(c) $x \cdot (ab) = (x \cdot a) \cdot b \quad (x \in X, a, b \in \mathcal{A})$.

2. *By a* left $\mathcal{A}$-module*, we mean a vector space X over a scalar field $\mathbb{F}$ together with a map $\cdot : \mathcal{A} \times X \to X, (a, x) \mapsto a \cdot x$ which satisfies*

   (a) $(a + b) \cdot x = a \cdot x + b \cdot x \quad (x, \in X, a, b \in \mathcal{A})$;

   (b) $a \cdot (x + y) = a \cdot x + a \cdot y \quad (x, y \in X, a \in \mathcal{A})$;

   (c) $a \cdot (b \cdot x) = (ab) \cdot x \quad (x \in X, a, b \in \mathcal{A})$.

3. *By an* $\mathcal{A}$-bimodule*, we mean both left and right $\mathcal{A}$-modules and also satisfies*

$$(a \cdot x) \cdot b = a \cdot (x \cdot b), \quad (a, b \in \mathcal{A}, x \in X).$$

Definition 2.4.2. *Let $\mathcal{A}$ be a Banach algebra. A Banach space X which is*

1. *also a left $\mathcal{A}$-module, is called a* left Banach $\mathcal{A}$-module *if there exists a constant $M > 0$ such that $\|a \cdot x\| \leq M\|a\|\|x\|, (x \in X, a \in \mathcal{A})$;*

2. *also a right $\mathcal{A}$-module is called a* right Banach $\mathcal{A}$-module *if there exists a constant $M > 0$ such that $\|x \cdot a\| \leq M\|a\|\|x\|, (x \in X, a \in \mathcal{A})$;*

3. *both left and right Banach $\mathcal{A}$-module, is called a* Banach $\mathcal{A}$-bimodule.

Remark 2.4.3. *If we renorm X, we may take $M = 1$.*

Example 2.4.4.

1. *The Banach algebra $\mathcal{A}$ itself is a Banach $\mathcal{A}$-bimodule with the module operations taken as the Banach algebra multiplication operation.*

2. *Let X be a Banach $\mathcal{A}$-bimodule and let Y be a closed submodule of X, then the quotient bimodule X/Y is a Banach $\mathcal{A}$-bimodule with the module operations*

$$a \cdot (x + Y) = a \cdot x + Y \quad \text{and} \quad (x + Y) \cdot a = x \cdot a + Y \quad (a \in \mathcal{A}, x \in X).$$

3. *Let X be a Banach $\mathcal{A}$-bimodule and $\mathcal{A}$ a Banach algebra. The canonical way of making X' into a Banach $\mathcal{A}$-bimodule is by defining the left and right module operations as*

$$\langle x, a \cdot \psi \rangle = \langle x \cdot a, \psi \rangle \quad \text{and} \quad \langle x, \psi \cdot a \rangle = \langle a \cdot x, \psi \rangle \quad (x \in X, a \in \mathcal{A}, \psi \in X').$$

   It easy to check that

$$a \cdot (\psi \cdot b) = (a \cdot \psi) \cdot b,$$
$$\|a \cdot \psi\| \leq M\|a\|\|\psi\| \quad \text{and} \quad \|\psi \cdot a\| \leq M\|a\|\|\psi\|.$$

Indeed, for all $a, b \in \mathcal{A}, x \in X$ *and* $\psi \in X'$, *we have*

$$\begin{aligned}\langle x, (a \cdot \psi) \cdot b\rangle &= \langle b \cdot x, a \cdot \psi\rangle = \langle (b \cdot x) \cdot a, \psi\rangle = \langle b \cdot (x \cdot a), \psi\rangle \\ &= \langle x \cdot a, \psi \cdot b\rangle = \langle x, a \cdot (\psi \cdot b)\rangle.\end{aligned}$$

Also,

$$\begin{aligned}\|a \cdot \psi\| &= \sup\{|\langle x, a \cdot \psi| \mid \|x\| \le 1\} = \sup\{|\langle x \cdot a, \psi| \mid \|x\| \le 1\} \\ &\le \sup\{M\|x\|\|a\|\|\psi\| \mid \|x\| \le 1\} = M\|a\|\|\psi\|.\end{aligned}$$

Using similar approach, we have $\|\psi \cdot a\| \le M\|a\|\|\psi\|$.
In general, the nth dual space $X^{(n)}$ *of* X *are Banach* $\mathcal{A}$*-bimodules for* $n \in \mathbb{N}$ *with* $X^{(0)} := X$.

4. *Let* $\mathcal{A}$ *be a Banach algebra and* X *a Banach* $\mathcal{A}$*-bimodule. Then* $\mathcal{L}(\mathcal{A}, X)$ *is a Banach* $\mathcal{A}$*-bimodule with the module operations given as*

$$(a \cdot T)(b) = a \cdot (Tb) \quad \text{and} \quad (T \cdot a)(b) = T(ab) \quad (a, b \in \mathcal{A}, T \in \mathcal{L}(\mathcal{A}, X)).$$

2.5 Tensor Product

In this section, we give a brief introduction to tensor product. For details, see [43].

Definition 2.5.1. *Let* $\mathcal{A}, \mathcal{B}$ *and* $\mathcal{C}$ *be vector spaces. An* algebraic tensor product *of* $\mathcal{A}$ *and* $\mathcal{B}$ *is a pair* $(\mathcal{C}, \theta)$, *where* $\theta : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ *is a bilinear map called the* tensor map *such that if* $\mathcal{D}$ *is any vector space and for each bilinear map* $f : \mathcal{A} \times \mathcal{B} \to \mathcal{D}$, *there exists a unique* $g : \mathcal{C} \to \mathcal{D}$, *such that* $f = g \circ \theta$.

It is well known that for any two vector spaces say, $\mathcal{A}, \mathcal{B}$, the algebraic tensor product of $\mathcal{A}, \mathcal{B}$ always exists, unique up to isomorphism and inherits the property of the structure in which it is defined on. The tensor product $(\mathcal{C}, \theta)$ is denoted as $\mathcal{A} \otimes \mathcal{B}$, elements of $\mathcal{A} \otimes \mathcal{B}$ are called *tensors*. We also denote $\theta(a, b) = a \otimes b$, for all $a \in \mathcal{A}, b \in \mathcal{B}$ and elements of this form are called *elementary tensors*. An element, say $t \in \mathcal{A} \otimes \mathcal{B}$ is of the form

$$t = \sum_{i=1}^{m} a_i \otimes b_i \quad m \in \mathbb{N}, a_i \in \mathcal{A}, b_i \in \mathcal{B}. \tag{2.2}$$

Remark 2.5.2. *The representation for* t *in Equation (2.2) is not unique.*

On the tensor product $\mathcal{A} \otimes \mathcal{B}$, different types of norms can be defined. In this dissertation, we only consider the projective norm.

Definition 2.5.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be normed vector spaces, for $t \in \mathcal{A} \otimes \mathcal{B}$, we define* projective tensor product norm *as*

$$\|t\|_p = \inf \left\{ \sum_{i=1}^{m} \|a_i\| \|b_i\| < \infty \mid t = \sum_{i=1}^{m} a_i \otimes b_i, a_i \in \mathcal{A}, b_i \in \mathcal{B} \right\}.$$

Theorem 2.5.4 ([5])**.** *Given a bilinear mapping $\phi : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$, there exists a unique linear map $\psi : \mathcal{A} \otimes \mathcal{B} \to \mathcal{C}$, such that $\psi(a \otimes b) = \phi(a, b), \quad (a \in \mathcal{A}, b \in \mathcal{B})$.*

Remark 2.5.5. *By Proposition 2.5.4, we have that $(\mathcal{A} \otimes \mathcal{B})' \cong \mathcal{L}(\mathcal{A}, \mathcal{B}')$.*

Theorem 2.5.6 ([5])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be normed algebras over a scalar field $\mathbb{F}$.There exists a unique product on $\mathcal{A} \otimes \mathcal{B}$ with respect to which $\mathcal{A} \otimes \mathcal{B}$ is an algebra and*

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1 b_1 \otimes a_2 b_2 \quad (a_1, b_1 \in \mathcal{A}, a_2, b_2 \in \mathcal{B}).$$

Theorem 2.5.7 ([5])**.** *Let $\mathcal{A}, \mathcal{B}$ be normed algebras over a scalar field $\mathbb{F}$. Then the projective tensor norm on $\mathcal{A} \otimes \mathcal{B}$ is an algebra norm.*

Definition 2.5.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then their projective tensor product denoted by $\mathcal{A} \widehat{\otimes} \mathcal{B}$ is the completion of $\mathcal{A} \otimes \mathcal{B}$ with respect to the projective tensor norm.*

Now let $\mathcal{A}, \mathcal{B}$ be Banach algebras over a scalar field $\mathbb{F}$. Using Proposition 2.5.7, we may extend the product on $\mathcal{A} \otimes \mathcal{B}$ to $\mathcal{A} \widehat{\otimes} \mathcal{B}$ so that $\mathcal{A} \widehat{\otimes} \mathcal{B}$ becomes a Banach algebra.

Remark 2.5.9. *$\mathcal{A} \widehat{\otimes} \mathcal{B}$ is a commutative and unital Banach algebra if and only if both $\mathcal{A}$ and $\mathcal{B}$ are commutative and unital.*

Definition 2.5.10. *Let $\mathcal{A}$ be a Banach algebra, then $\mathcal{A} \widehat{\otimes} \mathcal{A}$ becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as*

$$a \cdot (b \otimes c) = ab \otimes c \quad \textit{and} \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{A}).$$

We can make $(\mathcal{A} \widehat{\otimes} \mathcal{A})'$ and $(\mathcal{A} \widehat{\otimes} \mathcal{A})''$ into a Banach $\mathcal{A}$-bimodule in the canonical way. As in Remark 2.5.5, if we identify $(\mathcal{A} \widehat{\otimes} \mathcal{A})'$ with $\mathcal{L}(\mathcal{A}, \mathcal{A}')$, then the canonical Banach $\mathcal{A}$-bimodule operations on $(\mathcal{A} \widehat{\otimes} \mathcal{A})'$ takes the form in Example 2.4.4 (4). If $\mathcal{A}$ is a Banach algebra, $\mathcal{A}^{op}$ is also a Banach algebra with the product defined in reverse order, that is $a \times b = ba$. We can also make $\mathcal{A}$ into a left $\mathcal{A} \widehat{\otimes} \mathcal{A}^{op}$-module by the module operation defined by

$$(a \otimes b) \cdot c = acb \quad (a, b, c \in \mathcal{A}).$$

2.6 Semigroups and Semigroup Algebras

In this section, we shall recall some basic definitions, some properties of semigroups and semigroup algebras that will be relevant for our study.

2.6.1 Semigroups

Definition 2.6.1. *A* semigroup *is a non-empty set S together with an associative binary operation, denoted by*

$$S \times S \to S, \quad (s,t) \mapsto st \quad (s,t \in S).$$

A non-empty subset T of S is a *subsemigroup* if T is a semigroup under the induce binary operation of S. In general, we do not suppose that a semigroup have an identity. In the case where a semigroup S have an identity, we denote the identity of S by e_S. A semigroup S with an identity is said to be unital. If a semigroup S is non-unital, we can adjoin an identity to S and S becomes $S^{\#} := S \cup \{e\}$, where $S^{\#}$ is a semigroup with an identity adjoined. Then $S^{\#}$ is a semigroup containing S as a subsemigroup. The semigroup S is *abelian* if for all $s,t \in S$, we have $st = ts$. If a semigroup S with at least two elements contains an element 0 such that

$$0s = 0s = 0 \quad (s \in S),$$

we say that 0 is a *zero* of S and S is a semigroup with zero. Also, if S is a semigroup without a zero element, we can also adjoin the element 0 to S. A semigroup in which a zero is adjoined is denoted by S^o an $S^o := S \cup \{0\}$. Then S^o is semigroup containing S as a subsemigroup. A non-empty subset I of S is a *left (right) ideal* of S if, $SI \subset I$ ($IS \subset I$). A non-empty subset of S which is both left and right ideal is an ideal.

Definition 2.6.2. *Let S be a semigroup.*

1. *Let $s \in S$. An element $s^* \in S$ is called an* inverse *of s if*

$$ss^*s = s \quad \text{and} \quad s^*ss^* = s^*.$$

2. *An element $s \in S$ is called* regular *if there exists $t \in S$ such that $sts = s$.*

3. *An element $s \in S$ is called* completely regular *if there exists $t \in S$ such that $sts = s$ and $ts = st$.*

4. *S is called* regular *if each $s \in S$ is a regular element.*

5. *S is called* completely regular *if each $s \in S$ is a completely regular element.*

6. *S is called an* inverse semigroup *if S is regular and every element in S has a unique inverse.*

7. *An element $s \in S$ is* left cancellable *if $s = t$ whenever $vs = vt$, for $t, v \in S$.*

8. *An element $s \in S$ is* right cancellable *if $s = t$ whenever $sv = tv$ for $t, v \in S$.*

9. *An element $s \in S$ is* cancellable *if it is both left and right cancellable.*

10. *S is* cancellative *if each element is cancellable.*

11. *S is* left reversible *if for all $x, y \in S, xS \cap yS \neq \emptyset$.*

12. *S is* right reversible *if for all $x, y \in S, Sx \cap Sy \neq \emptyset$.*

13. *S is* reversible *if it is both left and right reversible.*

14. *An element $p \in S$ is called an* idempotent *if $p^2 = p$, the set of idempotents of S is denoted by $E(S)$.*

Remark 2.6.3. *An element $s \in S$ has an inverse if and only if is regular. Indeed, suppose that $s \in S$ has an inverse. By definition, it follows that s is regular. Conversely, suppose that $s \in S$ is regular, then there exists $t \in S$ such that $sts = s$. Set $u = tst$ and observe that*

$$sus = ststs = (sts)ts = sts = s$$

and

$$usu = tststst = t(sts)tst = tstst = t(sts)t = tst = u.$$

Remark 2.6.4. *If $s \in S$ has an inverse, then it is said to be regular and if not is called* singular. *We denote the inverse of an element s in an inverse semigroup as s^{-1}.*

Proposition 2.6.5 ([30])**.** *An inverse semigroup with a unique idempotent is a group.*

Proposition 2.6.6 ([30])**.** *A semigroup S is an inverse semigroup if and only if S is regular and the idempotent commutes.*

Definition 2.6.7. *A semigroup S is called* semilattice *if S is commutative and $E(S) = S$.*

Let S be an inverse semigroup. The canonical partial order defined on S and $E(S)$ coincides. The natural partial order on S is defined as

$$s \leq t \Leftrightarrow s = ss^{-1}t \quad (s, t \in S)$$

and that of $E(S)$ is defined as

$$p \leq q \Leftrightarrow p = pq = qp \quad (p, q \in E(S)).$$

An idempotent p is *maximal* if $p = q$ whenever $p \leq q$.

Definition 2.6.8. *Let P be a partially ordered set. For $p \in P$, we define $(p] = \{x : x \leq p\}$ and $[p) = \{x : p \leq x\}$. Then P is* locally finite *if $(p]$ is finite for each $p \in P$ and P is* locally C-finite *for some constant $C > 1$ if $|(p]| < C$ for each $p \in P$. A partially ordered set that is locally C-finite for some C is* uniformly locally finite.

Definition 2.6.9. *Let S be an inverse semigroup. Then S is* [locally finite / C-finite / uniformly locally finite] *respectively if the partial ordered set $(E(S), \leq)$ has the corresponding property.*

Proposition 2.6.10 ([44])**.** *Let S be an inverse semigroup. Suppose that $(E(S), \leq)$ is [uniformly] locally finite. Then $(S, \leq)$ is [uniformly] locally finite.*

Proposition 2.6.11 ([30])**.** *Let S be an inverse semigroup and let $s, t \in S$. Then $s\mathcal{D}t$ if and only if there exists $x \in S$ such that $s^{-1}s = xx^{-1}$ and $t^{-1}t = x^{-1}x$.*

Let S be an inverse semigroup, $p \in E(S)$ and $\{D_\lambda \mid \lambda \in \Lambda\}$ be the collection of all $\mathcal{D}$-class on S and $p_\lambda \in E(D_\lambda)$. The *maximal subgroup* of S at p_λ is denoted by G_{p_λ} and

$$G_{p_\lambda} = \{s \in S : ss^{-1} = s^{-1}s\}.$$

We recall from [38] the following definitions of semigroups.

Definition 2.6.12. *Let S be a semigroup.*

1. *S is called a* band *if $S = E(S)$.*
2. *S is called a* rectangular band *semigroup if it is a band semigroup and for each $x, y \in S, xyx = x$.*
3. *S is a* Clifford *semigroup if S is an inverse semigroup such that*

   $$ss^{-1} = s^{-1}s \quad (s \in S).$$

   For e in $E(S)$, let $G_e = \{s \in S : s^{-1}s = e\}$, we have that G_e is a group, $S = \bigcup_{e \in E(S)} G_e$ and $G_eG_f \subset G_{ef}$, where G_e's are the maximal subgroup of S. A Clifford semigroup is also known as a semilattice of groups.
4. *S with a zero element 0 is called* Brandt *semigroup if it satisfies the following axioms:*
   (a) *For each non-zero element s of S there corresponding unique elements t, u and s' in S such that*

   $$ts = s, \quad su = s, \quad s's = u.$$

   (b) *If $t, u \in E(S)$ and are non-zero, then $tSu \neq \{0\}$.*

   A Brandt semigroup S over a group G with index set J consist of all canonical $J \times J$ matrix units over $G \cup \{0\}$ and a zero matrix 0. It is an inverse semigroup over G with index set J given as

   $$S = \{(g)_{ij} \mid g \in G, i, j \in J\} \cup \{0\},$$

where $(g)_{ij}$ is the $J \times J$ matrix with (k,l)- entry equal to g if $(k,l) = (i,j)$ and 0 *if $(k,l) \neq (i,j)$ and multiplication defined as*

$$(g)_{ij}(h)_{kl} = \begin{cases} (gh)_{il} & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

5. *A* bicyclic *semigroup is a semigroup $S = \{p, q, e : pq = e\}$ generated by an identity element e and two more elements p and q such that $pq = e$.*

Definition 2.6.13. *Let G be a group, I and Λ be arbitrary non-empty sets and $G^o = G \cup \{0\}$ be a group with zero adjoined. A sandwich matrix $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix with entries being elements of G^o such that each row and column of P has at least one non-zero entry. The set $S = G \times I \times \Lambda$ with the composition*

$$(a, i, j) \circ (b, l, k) = (aP_{jl}b, i, k) \quad (a, i, j), (b, l, k) \in S$$

is a semigroup that we denote by $\mathcal{M}(G, I, \Lambda, P)$. Similarly if P is a $\Lambda \times I$ matrix over G^o, then $S = G \times I \times \Lambda \cup \{0\}$ is a semigroup under the following composition operation

$$(a, i, jg) \circ (b, l, k) = \begin{cases} (aP_{jl}b, i, k) & \text{if } P_{jl} \neq 0 \\ 0 & \text{if } jP_{jl} = 0. \end{cases}$$

$$(a, i, j) \circ 0 = 0 \circ (a, i, j) = 0 \circ 0 = 0.$$

This semigroup which is denoted by $\mathcal{M}^o(G, I, \Lambda, P)$ can also be describe in the following way. An $I \times \Lambda$ matrix A over G^o that has at most one non-zero entry $a = A(i,j)$ is called a Rees $I \times \Lambda$ matrix *over G^o and is denoted by $(a)_{ij}$. The set of all Rees $I \times \Lambda$ matrices over G^o form a semigroup under the binary operation $A \cdot B = APB$, which is called the* Rees matrix semigroup *over G^o.*

The above sandwich matrix P is regular if every row and column contains at least one entry in G and the semigroup $\mathcal{M}^o(G, I, \Lambda, P)$ is regular as a semigroup if and only if the sandwich matrix is regular.

Definition 2.6.14. *Let S be a semigroup. A* principal series *of ideals for S is a chain*

$$S = I_1 \supset I_2 \supset \cdots \supset I_{m-1} \supset I_m = K(S),$$

where $I_1, I_2, ...I_m$ are ideal in S and there is no ideals of S strictly between I_j and I_{j+1} for each $j \in \mathbb{N}_{m-1}$ and $K(S)$ is the minimum ideal *of S.*

Example 2.6.15. *1. Let S be an infinite set with the product given by*

$$st = t \quad (s, t \in S),$$

so that S is a right zero semigroup. *Then S is a semigroup which is right cancellative and each element of S is a left identity.*

2. Let S be an infinite set with the product given by

$$st = s \quad (s, t \in S),$$

so that S is a left zero semigroup. *Then S is a semigroup which is left cancellative and each element of S is a right identity.*

3. Let $S = \mathbb{Z}^2$, such that $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ with the binary operation

$$(t, u) \cdot (v, w) = (t + u, w) \quad (t, u, v, w \in \mathbb{Z}).$$

Then $(S, \cdot)$ is a non-abelian semigroup which is left cancellative.

4. Let S be an infinite semigroup and set $S_1 = S \times S$ as a set. Define

$$(a, x) \cdot (b, y) = (ab, ay) \quad (a, b, c, d \in S).$$

Then $(S_1, \cdot)$ is a semigroup. It is left cancellative whenever S is left cancellative.

5. Let $S = \mathbb{N}$, with the product

$$\wedge : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m \wedge n := \min\{m, n\}.$$

Then $(S, \wedge)$ is an abelian semigroup and 1 acts as a zero. Clearly $S = E(S)$, but S does not have an identity.

6. Let $S = \mathbb{N}$, with the product

$$\vee : \mathbb{N} \times \mathbb{N} \to \mathbb{N}, \quad (m, n) \mapsto m \vee n := \max\{m, n\}.$$

Then $(S, \vee)$ is an abelian semigroup with identity 1. *Clearly $S = E(S)$.*

2.6.2 Semigroup Algebras

Let S be a semigroup. We define $\ell^1(S) = \{f : S \to \mathbb{C} \mid \sum_{s \in S} |f(s)| < \infty\}$, with the norm $\|f\| = \sum_{s \in S} |f(s)|$. $\ell^1(S)$ is a Banach space and it becomes a Banach algebra when the following convolution product is defined on it:

$$(f * g)(t) = \sum \left\{ f(r)g(s) \mid r, s \in S, \ rs = t \right\} \quad (t \in S),$$

where we take $(f*g)(t) = 0$ where there are no elements $r, s \in S$ with $sr = t$. It can be easily shown that $\|f*g\| \leq \|f\|\|g\|$. Indeed, for all $f, g \in \ell^1(S)$, we have

$$\begin{aligned}\|f*g\| &= \sum_{t\in S} |(f*g)(t)| = \sum_{t\in S} |\sum_{sr=t} f(r)g(s)| \\ &\leq \sum_{t\in S}\sum_{sr=t} |f(r)\|g(s) \leq \sum_{(r,s)\in S\times S} |f(r)\|g(s)| \\ &= \sum_{r\in S}\sum_{s\in S} |f(r)\|g(s)| = \|f\|\|g\|.\end{aligned}$$

$(\ell^1(S), *)$ is a Banach algebra, called the *Banach semigroup algebra* on S. $\ell^1(S)$ is a *commutative* Banach algebra if and only if S is commutative. Every $f \in \ell^1(S)$ can be represented as

$$f = \sum_{s\in S} f(s)\delta_s$$

where δ_s is the Dirac measure,

$$\delta_s(t) = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{if } s \neq t. \end{cases}$$

For further discussion of this algebra, see [9] and [11].
It is well known that the semigroup algebra $\ell^1(S)$ may have an identity even if S does not. This fact was justified in [25], Proposition 2.1, where S is a finite semilattice. Furthermore, the necessary and sufficient condition for $\ell^1(S)$ to have a bounded approximate identity for inverse semigroup and other semigroups were given in [13, 28].

Let S and T be semigroups and let $\theta : S \to T$ be an epimorphism. Then there is an induced contractive epimormphism $\theta : \ell^1(S) \to \ell^1(T)$ defined by requiring that $\theta|_S$ takes specified values in $T \subset \ell^1(T)$. If S and T are isomorphic, then $\ell^1(S)$ and $\ell^1(T)$ are isometrically isomorphic. For each $\varphi \in \Phi_S$, the map

$$\sum_{s\in S} f(s)\delta_s \mapsto \sum_{s\in S} f(s)\varphi(s)$$

is a character on $\ell^1(S)$ and every character on $\ell^1(S)$ arises in this way. There is always one character on the Banach algebra $\ell^1(S)$, this is the augmentation character

$$\varphi_S : \ell^1(S) \to \mathbb{C}, \quad f \mapsto \sum_{s\in S} f(s).$$

Suppose that T is a subsemigroup of S. Then

$$\Phi_{\ell^1(T)} = \{\varphi_S|_{\ell^1(T)} \mid \varphi_S \in \Phi_S\}.$$

For a semigroup S, it is an established result that $\ell^1(S)\widehat{\otimes}\ell^1(S)$ is isometrically isomorphic to $\ell^1(S\times S)$ and so, we identify $(\ell^1(S)\widehat{\otimes}\ell^1(S))''$ with $(\ell^1(S\times S))''$. Using this identification, the bimodule operations are defined as follows: Let $M \in (\ell^\infty(S \times S))'$, $s \in S$, then for all $f \in \ell^\infty(S \times S)$,

$$Ms(f) = M(sf), \quad sM(f) = M(fs)$$

and

$$fs(u, v) = f(su, v), \quad sf(u, v) = f(u, vs).$$

More so, for all $s, t \in S$, we have $\delta_s * \delta_t = \delta_{st}$ and $\delta_t * \delta_s = \delta_{ts}$ and so $\ell^1(S)$ is a Banach $\ell^1(S)$-bimodule.

Remark 2.6.16. *If the semigroup S is a semilattice, then $\ell^1(S)$ is a commutative $\ell^1(S)$-module.*

Chapter 3

Notions of Amenability in Banach Algebras

In this chapter, we shall give the definitions of some important notions of amenability in Banach algebras that we shall study in this work. In particular, we shall give an explicit prove of some intrinsic characterizations, hereditary properties and some interesting results of *contractible, amenable, approximately amenable and pseudo-amenable* Banach algebras.

3.1 Basic Definitions

Definition 3.1.1. *Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A* derivation *from $\mathcal{A}$ into X is a bounded linear map $D : \mathcal{A} \to X$ such that*

$$D(ab) = a \cdot D(b) + D(b) \cdot a, \quad (a, b \in \mathcal{A}).$$

For example, let $x \in X$, then the linear map $\delta_x : \mathcal{A} \to X$ defined as

$$\delta_x(a) = a \cdot x - x \cdot a$$

is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
\delta_x(ab) = (ab) \cdot x - x \cdot (ab) &= (ab) \cdot x - a \cdot (x \cdot b) + a \cdot (x \cdot b) - x \cdot (ab) \\
&= a \cdot (b \cdot x) - a \cdot (x \cdot b) + (a \cdot x) \cdot b - (x \cdot a) \cdot b \\
&= a \cdot (b \cdot x - x \cdot b) + (a \cdot x - x \cdot a) \cdot b \\
&= a \cdot \delta_x(b) + \delta_x(a) \cdot b.
\end{aligned}$$

Hence, δ_x is a derivation. This type of derivation is called *inner* derivation and it is implemented by x. Note that inner derivations are automatically continuous linear maps and derivations which are not inner are called *outer* derivations. Let $\varphi \in \Phi_{\mathcal{A}}$. Then a *point derivation* at φ is a linear functional $d : \mathcal{A} \to \mathbb{C}$ such that

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b), \quad (a, b \in \mathcal{A}).$$

That is, d is a derivation into the bimodule $\mathbb{C}$, where $\mathbb{C}$ has the $\mathcal{A}$-bimodule operations

$$a \cdot z = z \cdot a = \varphi(a)z \quad (a \in \mathcal{A}, z \in \mathbb{C}).$$

Remark 3.1.2. *It is good to note that for any two elements $x, y \in X$, we have $\delta_{x+y}(a) = \delta_x(a) + \delta_y(a)$. Indeed, for all $a, b \in \mathcal{A}$, we have*

$$\begin{aligned} \delta_{x+y}(ab) &= (ab) \cdot (x + y) - (x + y) \cdot (ab) \\ &= (ab) \cdot x + (ab) \cdot y - [x \cdot (ab) + y \cdot (ab)] \\ &= (ab) \cdot x - x \cdot (ab) + (ab) \cdot y - y \cdot (ab) \\ &= \delta_x(ab) + \delta_y(ab). \end{aligned}$$

More so, any x, y in X can form the same inner derivation. If this occur, we have $\delta_x = \delta_y$ which implies that $\delta_x - \delta_y = \delta_{x-y} = 0$.

Let $\mathcal{Z}^1(\mathcal{A}, X)$ denote the space of all continuous derivations from $\mathcal{A}$ into X and let $\mathcal{B}^1(\mathcal{A}, X)$ denote the space of all continuous inner derivations from $\mathcal{A}$ into X. Then, the *first Hochschild Cohomology group* of $\mathcal{A}$ with coefficients in X is the quotient vector space

$$\mathcal{H}^1(\mathcal{A}, X) := \mathcal{Z}^1(\mathcal{A}, X)/\mathcal{B}^1(\mathcal{A}, X).$$

Clearly, $\mathcal{H}^1(\mathcal{A}, X) = \{0\}$ if and only if $\mathcal{Z}^1(\mathcal{A}, X) = \mathcal{B}^1(\mathcal{A}, X)$. A trivial case in which $\mathcal{H}^1(\mathcal{A}, X) = \{0\}$ is the case in which $\mathcal{A}$ acts trivially on right (left) of X. That is $X \cdot \mathcal{A} = 0$ ($\mathcal{A} \cdot X = 0$).

Definition 3.1.3. *Let $\mathcal{A}$ be a Banach algebra. Then:*

1. *$\mathcal{A}$ is* contractible *if, for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to X$ is inner;*

2. *$\mathcal{A}$ is* amenable *if, for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to X'$ is inner;*

3. *$\mathcal{A}$ is* weakly amenable *if every continuous derivation $D : \mathcal{A} \to \mathcal{A}'$ is inner;*

4. *a continuous derivation $D : \mathcal{A} \to X$ is said to be* approximately inner *if there exists a net (x_α) in X such that*

   $$D(a) = \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) = \lim_\alpha \delta_x(a) \quad (a \in \mathcal{A}),$$

   were the limit is taken in the norm of X;

5. $\mathcal{A}$ *is* approximately contractible *if, for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to X$ is approximately inner;*

6. $\mathcal{A}$ *is* approximately amenable *if, for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to X'$ is approximately inner;*

7. $\mathcal{A}$ *is* approximately weakly amenable *if every continuous derivation $D : \mathcal{A} \to \mathcal{A}'$ is approximately inner;*

8. $\mathcal{A}$ *is* boundedly approximately contractible *if, for each Banach $\mathcal{A}$-bimodule X and each continuous derivation $D : \mathcal{A} \to X$, there exists $M > 0$ and a net (x_α) in X such that $\|a \cdot x_\alpha - x_\alpha \cdot a\| \le M\|a\|$ for all a in $\mathcal{A}$ and each α and that $D(a) = \lim_\alpha (b \cdot x_\alpha - x_\alpha \cdot b), \ (b \in \mathcal{A})$;*

9. $\mathcal{A}$ *is* boundedly approximately amenable *if, for each Banach $\mathcal{A}$-bimodule X and each continuous derivation $D : \mathcal{A} \to X'$, there exists $M > 0$ and a net (x_α) in X' such that $\|a \cdot x_\alpha - x_\alpha \cdot a\| \le M\|a\|$ for all a in $\mathcal{A}$ and each α and that $D(a) = \lim_\alpha (b \cdot x_\alpha - x_\alpha \cdot b), \ (b \in \mathcal{A})$.*

Remark 3.1.4. *For the approximate notions, we use the qualifier* uniform *when that convergence of the net is uniform over the unit ball and similarly we use weak* when the convergence is in the appropriate weak* topology.*

Example 3.1.5. 1. *The set of complex number $\mathbb{C}$, is an amenable Banach algebra with the usual product and norm.*

Proof. Let 0 denote the trivial group and $\mathcal{A} := \ell^1(0)$ be an amenable Banach algebra. Define $\phi : \mathcal{A} \to \mathbb{C}$ as $\phi(f) = f(0)$. Clearly, ϕ is an isomorphism, more so, it is an isometry. Indeed, for all $f \in \mathcal{A}$, we have

$$|\phi(f)| = |f(0)| = \|f\|_1.$$

We then have that $\mathcal{A} \cong \mathbb{C}$ and since $\mathcal{A}$ is amenable, so is $\mathbb{C}$. □

2. *The Banach algebra $A(\mathbb{D})$ (disc algebra) is not amenable.*

Proof. Let $x \in \mathbb{D}$ and the module operations of $A(\mathbb{D})$ on $\mathbb{C}$ are given by

$$f \cdot z := f(x)z \text{ and } z \cdot f := f(x)z \quad (f \in A(\mathbb{D}), z \in \mathbb{C}).$$

Define $D : A(\mathbb{D}) \to \mathbb{C}$ as $f \mapsto f'(x) \quad (f'$ denote the derivative of $f)$. Clearly, D is a continuous derivation and every inner derivation at z is zero. Indeed, for all

$f, g \in A(\mathbb{D})$, we have

$$\begin{aligned} D(fg)(x) &= (fg)'(x) = (fg' + f'g)(x) \\ &= f(x)g'(x) + f'(x)g(x) \\ &= f(x)D(g) + D(f)g(x) \\ &= f \cdot D(g) + D(f) \cdot g. \end{aligned}$$

For inner derivation, we have

$$\delta_z(f) = f \cdot z - z \cdot f = f(x)z - f(x)z = 0.$$

Since all continuous derivation is not inner, hence $A(\mathbb{D})$ is not amenable. $\square$

Definition 3.1.6. *Let $\mathcal{A}$ be a Banach algebra.*

1. *The operator $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to \mathcal{A}$, defined by $(a \otimes b) \mapsto ab$ for all $a, b \in \mathcal{A}$ is called the* diagonal operator.
2. *A diagonal for $\mathcal{A}$ is an element $u \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that*

$$a \cdot u = u \cdot a \quad \textit{and} \quad a \cdot \pi(u) = a \quad (a \in \mathcal{A}).$$

It is well known that π is an $\mathcal{A}$-bimodule homomorphism with respect to the module operations on $\mathcal{A}\widehat{\otimes}\mathcal{A}$. For a diagonal $u = \sum_{j=1}^{n} a_j \otimes b_j$ in $\mathcal{A}\widehat{\otimes}\mathcal{A}$, we have

$$\pi(u) = \pi\left(\sum_{j=1}^{n} a_j \otimes b_j\right) = \sum_{j=1}^{n} \pi(a_j \otimes b_j) = \sum_{j=1}^{n} a_j b_j = e_{\mathcal{A}},$$

where $e_{\mathcal{A}}$ is the identity in $\mathcal{A}$. Also, $a \cdot u = u \cdot a$ implies that

$$\sum_{j=1}^{n} aa_j \otimes b_j = \sum_{j=1}^{n} a_j \otimes b_j a.$$

Suppose X is a Banach $\mathcal{A}$-bimodule and $\psi \in \mathcal{L}(\mathcal{A}, X)$. Then there exists a linear map

$$\begin{aligned} \Phi :&\mathcal{A}\widehat{\otimes}\mathcal{A} \to X, \\ \Phi(a \otimes b) &= a \cdot \psi(b) \quad (a, b \in \mathcal{A}). \end{aligned} \tag{3.1}$$

Since $a \cdot u = u \cdot a$ for $a \in \mathcal{A}$ and u a diagonal in $\mathcal{A}\widehat{\otimes}\mathcal{A}$, then

$$\Phi(a \cdot u) = \Phi(u \cdot a), \quad \Rightarrow \quad \Phi\left(\sum_{j=1}^{n} aa_j \otimes b_j\right) = \Phi\left(\sum_{j=1}^{n} a_j \otimes b_j a\right),$$

then

$$\sum_{j=1}^{n} aa_j\psi(b_j) = \sum_{j=1}^{n} a_j\psi(b_j a) \quad (a, b \in \mathcal{A}). \tag{3.2}$$

Lemma 3.1.7. *Let $\mathcal{A}$ be a Banach algebra, X a Banach $\mathcal{A}$-bimodule and the map Φ as defined in Equation (3.1). Then:*

1. *Φ is bounded;*
2. *$\Phi(a \otimes \lambda b) = \Phi(\lambda a \otimes b) \quad (a, b \in \mathcal{A}, \lambda \in \mathbb{C})$;*
3. *$a \cdot \Phi(b \otimes c) = \Phi(ab \otimes c) \quad (a, b, c \in \mathcal{A})$;*
4. *$a \cdot \Phi(u) = \Phi(a \cdot u) \quad (a \in \mathcal{A}, u \in \mathcal{A}\widehat{\otimes}\mathcal{A})$.*

Proof. 1. Let $a, b \in \mathcal{A}$ and X a Banach $\mathcal{A}$-bimodule. We then have

$$\|\Phi(a \otimes b)\| = \|a \cdot \psi(b)\| \leq K\|a\|\|\psi(b)\| \leq K\|a\|\|b\|\|\psi\|.$$

2. Let $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we then have

$$\Phi(a \otimes \lambda b) = a \cdot \psi(\lambda b) = a \cdot \lambda\psi(b) = \lambda a \cdot \psi(b) = \Phi(\lambda a \otimes b).$$

3. For all $a, b, c \in \mathcal{A}$, we then have

$$a \cdot \Phi(b \otimes c) = a \cdot b \cdot \psi(c) = (ab) \cdot \psi(c) = \Phi(ab \otimes c).$$

4. Let $a, b, c \in \mathcal{A}$ and $u \in \mathcal{A}\widehat{\otimes}\mathcal{A}$, where $u = \sum_{j=1}^{n} b_j \otimes c_j$, we then have

$$\begin{aligned} a \cdot \Phi(u) = a \cdot \Phi\left(\sum_{j}^{n} b_j \otimes c_j\right) &= a \cdot \sum_{j}^{n} b_j \cdot \psi(c_j) = \sum_{j}^{n} ab_j \cdot \psi(c_j) \\ &= \sum_{j}^{n} \Phi(ab_j \otimes c_j) = \Phi\left(\sum_{j}^{n} ab_j \otimes c_j\right) = \Phi(a \cdot u). \end{aligned}$$

□

Definition 3.1.8. *Let $\mathcal{A}$ be a Banach algebra.*

1. *An element M in $(\mathcal{A}\widehat{\otimes}\mathcal{A})''$ is called a* virtual diagonal *for $\mathcal{A}$ if*

$$a \cdot M = M \cdot a \quad \textit{and} \quad a \cdot \pi''(M) = a \quad (a \in \mathcal{A}).$$

2. *A net (M_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is called an* approximate diagonal *for $\mathcal{A}$ if*

$$a \cdot M_\alpha - M_\alpha \cdot a \to 0 \quad \textit{and} \quad a \cdot \pi(M_\alpha) \to a \quad (a \in \mathcal{A}).$$

3. *A* bounded approximate diagonal *for $\mathcal{A}$ is a bounded net (M_α) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that*

$$a \cdot M_\alpha - M_\alpha \cdot a \to 0 \quad \text{and} \quad a \cdot \pi(M_\alpha) \to a \quad (a \in \mathcal{A}).$$

Remark 3.1.9. *If $(M_\alpha) \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ is an approximate diagonal for $\mathcal{A}$, then $(\pi(M_\alpha))_\alpha$ is an approximate identity for $\mathcal{A}$. Since $(\pi(M_\alpha))_\alpha$ is a right approximate identity by definition, we then show that $(\pi(M_\alpha))_\alpha$ is a left approximate identity. Indeed, for all $a \in \mathcal{A}$, we have that*

$$\begin{aligned}
\|\pi(M_\alpha) \cdot a - a\| &= \|\pi(M_\alpha) \cdot a - a \cdot \pi(M_\alpha) + a \cdot \pi(M_\alpha) - a\| \\
&\leq \|\pi(M_\alpha) \cdot a - a \cdot \pi(M_\alpha)\| + \|a \cdot \pi(M_\alpha) - a\| \\
&= \|a \cdot \pi(M_\alpha) - \pi(M_\alpha) \cdot a\| + \|a \cdot \pi(M_\alpha) - a\| \\
&= \|a \cdot \pi(M_\alpha) - \pi(M_\alpha) \cdot a\| = \|\pi(a \cdot M_\alpha - M_\alpha \cdot a)\| \to 0.
\end{aligned}$$

Remark 3.1.10. *If $\mathcal{A}$ is a finite-dimensional Banach algebra, then any virtual diagonal for $\mathcal{A}$ is also a diagonal for $\mathcal{A}$.*

Definition 3.1.11. *Let $\mathcal{A}$ be a Banach algebra. Then:*

1. *$\mathcal{A}$ is* pseudo-amenable *if it possesses an (possible unbounded) approximate diagonal;*

2. *$\mathcal{A}$ is* pseudo-contractible *if it possesses a (possible unbounded) central approximate diagonal, that is an approximate diagonal (M_α) satisfying $a \cdot M_\alpha = M_\alpha \cdot a$ for all $a \in \mathcal{A}$ and all α.*

Definition 3.1.12. *A Banach $\mathcal{A}$-bimodule X is said to be* pseudo-unital or neo-unital *if for every $x \in X$ there exists $y \in X$ and $a, b \in \mathcal{A}$ such that $x = a \cdot y \cdot b$.*

3.2 Preliminary Results.

The following preliminary results are useful in establishing some results that will be proved in the course of this study.

Proposition 3.2.1 ([48])**.** *For a Banach algebra $\mathcal{A}$ with a bounded approximate identity, the following are equivalent:*

1. *$H^1(\mathcal{A}, X') = \{0\}$ for each Banach $\mathcal{A}$-bimodule X;*

2. *$H^1(\mathcal{A}, X') = \{0\}$ for each pseudo-unital Banach $\mathcal{A}$-bimodule X.*

Proposition 3.2.2 ([48])**.** *Let $\mathcal{A}$ be a Banach algebra with bounded approximate identity which is contained as a closed ideal in a Banach algebra $\mathcal{B}$. Let X be pseudo-unital Banach $\mathcal{A}$-bimodule, and let $D \in \mathcal{Z}^1(\mathcal{A}, X')$. Then X is a Banach $\mathcal{B}$-bimodule in a canonical fashion and there is a unique $\overline{D} \in \mathcal{Z}^1(\mathcal{B}, X')$ such that:*

1. $\overline{D}|_{\mathcal{A}} = D$;

2. $\overline{D}$ *is continuous with respect to the strict topology on* $\mathcal{B}$ *and the weak* topology in* X'.

Proposition 3.2.3 ([18]). *Suppose that* $\mathcal{A}$ *has a bounded approximate identity. Then* $\mathcal{A}$ *is approximately amenable if and only if every derivation into the dual of any neo-unital bimodule is approximately inner.*

Lemma 3.2.4 ([18]). *Let* $\mathcal{A}$ *be a unital Banach algebra with identity* e, X *an* $\mathcal{A}$*-bimodule,* $D : \mathcal{A} \to X'$ *a derivation. Then there exists* $D_1 : \mathcal{A} \to e \cdot X' \cdot e$ *and* $\eta \in X'$ *such that*

1. $\|\eta\| \leq 2C_x\|D\|$.

2. $D = D_1 + \delta_\eta$.

Proposition 3.2.5 ([31]). *Every finite-dimensional approximately amenable Banach algebra is amenable.*

3.3 Amenable and Contractible Banach Algebras.

We next give some hereditary properties and some characterizations of amenability and contractibility in Banach algebras.

3.3.1 Hereditary Properties of Amenable Banach Algebras.

Amenable Banach algebras have some nice hereditary, stability and useful properties. In this section, we are concerned with giving explicit proofs of some hereditary properties of amenable Banach algebras.

Proposition 3.3.1 ([31]). *Let* $\mathcal{A}$ *be a Banach algebra. If* $\mathcal{A}$ *is amenable and* $\mathcal{B}$ *is another Banach algebra such that* $\theta : \mathcal{A} \to \mathcal{B}$ *is a continuous homomorphism with dense range, then* $\mathcal{B}$ *is amenable. In particular,* $\mathcal{A}/I$ *is amenable for every closed ideal* I *of* $\mathcal{A}$.

Proof. Let X be a Banach $\mathcal{B}$-bimodule, X becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$x \cdot a = x \cdot \theta(a), \quad a \cdot x = \theta(a) \cdot x \quad (a \in \mathcal{A}, x \in X).$$

Let $\theta : \mathcal{A} \to \mathcal{B}$ be a homomorphism and suppose that $D : \mathcal{B} \to X'$ is a derivation. The map $\overline{D} := (D \circ \theta) : \mathcal{A} \to X'$ is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\overline{D}(ab) &= (D \circ \theta)(ab) = D(\theta(a)\theta(b)) = \theta(a) \cdot D(\theta(b)) + D(\theta(a)) \cdot \theta(b)\\ &= a \cdot (D \circ \theta)(b) + (D \circ \theta)(a) \cdot b = a \cdot \overline{D}(b) + \overline{D}(a) \cdot b. \end{aligned} \tag{3.3}$$

Hence, $\overline{D}$ is a derivation. Since $\mathcal{A}$ is amenable, there exists $x \in X'$ such that $\overline{D}(a) = a \cdot x - x \cdot a = \delta_x(a)$ for all $a \in \mathcal{A}$. By our hypothesis that $\theta : \mathcal{A} \to \mathcal{B}$ has a dense range, then for all $b \in \mathcal{B}$ there exists a sequence $(a_n) \subset \mathcal{A}$ such that $\lim_n(\theta(a_n)) = b$. It then follows that for every $b \in \mathcal{B}$, we have that

$$\begin{aligned} D(b) &= D(\lim_n \theta(a_n)) = \lim_n D(\theta(a_n)) = \lim_n((D \circ \theta)(a_n)) = \lim_n(\overline{D}(a_n)) \\ &= \lim_n(a_n \cdot x - x \cdot a_n) = \lim_n(\theta(a_n) \cdot x - x \cdot \theta(a_n)) = b \cdot x - x \cdot b = \delta_x(b), \end{aligned}$$

which implies that the derivation D from $\mathcal{B}$ into X' is inner. Hence, $\mathcal{B}$ is amenable.

In particular, let X be $\mathcal{A}/I$-bimodule in the canonical fashion, $D : \mathcal{A}/I \to X'$ a derivation and $\theta : \mathcal{A} \to \mathcal{A}/I$ a canonical surjective homomorphism. The map $d := (D \circ \theta) : \mathcal{A} \to X'$ is a derivation and since $\mathcal{A}$ amenable, there exists $x \in X'$ such that

$$d(a) = a \cdot x - x \cdot a = \delta_x(a) \quad (a \in \mathcal{A}).$$

Now, observe that

$$\begin{aligned} D(a + I) &= D(\theta(a)) = (D \circ \theta)(a) = d(a) \\ &= a \cdot x - x \cdot a = \theta(a) \cdot x - x \cdot \theta(a) \\ &= (a + I) \cdot x - x \cdot (a + I) \\ &= \delta_x(a + I) \quad (a \in \mathcal{A}). \end{aligned}$$

Hence, $\mathcal{A}/I$ is amenable. □

Proposition 3.3.2 ([31])**.** *Let $\mathcal{A}$ be a Banach algebra. If I is a closed ideal of $\mathcal{A}$ such that both I and $\mathcal{A}/I$ are amenable, then $\mathcal{A}$ is amenable.*

Proof. Suppose I and $\mathcal{A}/I$ are amenable and $D \in \mathcal{Z}^1(\mathcal{A}, X')$, where X is a Banach $\mathcal{A}$-bimodule. Since X can be identified with a Banach I-bimodule, we then have that $D|_I \in \mathcal{Z}^1(I, X')$. Using the amenability of I, then, there exists $x \in X'$ such that $D(a) = a \cdot x - x \cdot a = \delta_x(a)$ for all $a \in I$. Now, let $\overline{\delta_x}$ be the canonical extension of δ_x on $\mathcal{A}$. We then have that $D = \overline{\delta_x}$ which implies that $D - \overline{\delta_x} = 0$ on I. For all $a \in \mathcal{A}, b \in I$, we have that

$$0 = (D - \overline{\delta_x})(ab) = a \cdot (D - \overline{\delta_x})(b) + (D - \overline{\delta_x})(a) \cdot b.$$

It follows that

$$(D - \overline{\delta_x})(a) \cdot b = 0,$$

since $a \cdot (D - \overline{\delta_x})(b) = 0$ on I. Similarly, we have that

$$0 = (D - \overline{\delta_x})(ba) = b \cdot (D - \overline{\delta_x})(a) + (D - \overline{\delta_x})(b) \cdot a.$$

Also,

$$b \cdot (D - \overline{\delta_x})(a) = 0,$$

since $(D - \overline{\delta_x})(b) \cdot a = 0$ on I. Then for any $x \in X$, we have

$$\langle b \cdot x, (D - \overline{\delta_x})(a) \rangle = \langle x, (D - \overline{\delta_x})(a) \cdot b \rangle = 0$$

and

$$\langle x \cdot b, (D - \overline{\delta_x})(a) \rangle = \langle x, b \cdot (D - \overline{\delta_x})(a) \rangle = 0.$$

Let $X_I = \operatorname{span}\{a \cdot x + y \cdot b \mid a, b \in I, x, y \in X\}$. X_I is a closed linear span of $I \cdot X \cup X \cdot I$. The above equations implies that $D - \overline{\delta_x}$ maps $\mathcal{A}$ into $X_I^{\perp}$ and $D - \overline{\delta_x} \in \mathcal{Z}^1(\mathcal{A}, X_I^{\perp}) = \mathcal{Z}^1(\mathcal{A}, (X/X_I)')$. It is clear that, X/X_I becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$a \cdot (x + X_I) = a \cdot x + X_I, \quad (x + X_I) \cdot a = x \cdot a + X_I \quad (a \in \mathcal{A}, x \in X).$$

Also, X/X_I becomes an $\mathcal{A}/I$-bimodule with the module operation on X/X_I defined as

$$\begin{aligned}
&(x + X_I) \cdot (a + I) = (x + X_I) \cdot a = x \cdot a + X_I, \\
&(a + I) \cdot (x + X_I) = a \cdot (x + X_I) = a \cdot x + X_I \quad (a \in \mathcal{A}, x \in X).
\end{aligned}$$

We claim that $\widehat{D} \in \mathcal{Z}^1(\mathcal{A}/I, (X/X_I)')$, where $\widehat{D}$ is defined as $\widehat{D}(a + I) = (D - \overline{\delta_x})(a)$ for all $a \in \mathcal{A}$. The map $\widehat{D}$ from $\mathcal{A}/I$ into $(X/X_I)'$ is well defined and also continuous since $D - \overline{\delta_x} = 0$ on I. For all $(a + I), (b + I) \in \mathcal{A}/I$, we have

$$\begin{aligned}
\widehat{D}((a + I)(b + I)) &= \widehat{D}(ab + I) = (D - \overline{\delta_x})(ab) \\
&= a \cdot (D - \overline{\delta_x})(b) + (D - \overline{\delta_x})(a) \cdot b \\
&= (a + I) \cdot (D - \overline{\delta_x})(b) + (D - \overline{\delta_x})(a) \cdot (b + I) \\
&= (a + I) \cdot \widehat{D}(b + I) + \widehat{D}(a + I) \cdot (b + I).
\end{aligned}$$

Hence, our claim is justified. Since $\mathcal{A}/I$ is amenable, there exists $y \in (x/X_I)' = X_I^{\perp} \subset X'$ such that $\widehat{D} = \overline{\delta}_y$ where $\overline{\delta}_y$ is an inner derivation for $\mathcal{A}/I$. We then have that

$$\begin{aligned}
(D - \overline{\delta_x})(a) &= \widehat{D}(a + I) = \overline{\delta}_y(a + I) \\
&= (a + I) \cdot y - y \cdot (a + I) \\
&= a \cdot y + I - y \cdot a - I \\
&= a \cdot y - y \cdot a.
\end{aligned}$$

It then implies that $D - \overline{\delta_x} = \overline{\delta}_y$ and so, $D = \overline{\delta_x} + \overline{\delta}_y = \overline{\delta}_{x+y}$. Hence, $\mathcal{A}$ is amenable. $\square$

Proposition 3.3.3 ([48]). *Let $\mathcal{A}$ be an amenable Banach algebra. Then $\mathcal{A}$ has a bounded approximate identity.*

Proof. Let us take $X = \mathcal{A}$ and define the left and right bimodule module operations as

$$a \cdot x = ax, \quad x \cdot a = 0 \quad (a \in \mathcal{A}, x \in X).$$

We also make $X' = \mathcal{A}'$ into a Banach $\mathcal{A}$-bimodule in the usual way, that is

$$\langle x \cdot a, \alpha \rangle = \langle x, a \cdot \alpha \rangle = 0, \quad \langle a \cdot x, \alpha \rangle = \langle x, \alpha \cdot a \rangle \quad (a \in \mathcal{A}, x \in X, \alpha \in X').$$

We also make $X'' = \mathcal{A}''$ into a Banach $\mathcal{A}$-bimodule in the canonical fashion

$$\langle \alpha \cdot a, \Psi \rangle = \langle \alpha, a \cdot \Psi \rangle, \quad \langle \alpha, \Psi \cdot a \rangle = \langle a \cdot \alpha, \Psi \rangle = 0 \quad (a \in \mathcal{A}, \alpha \in X', \Psi \in X'').$$

The canonical embedding $D : \mathcal{A} \to X''$, defined as $D(a)(\alpha) = \alpha(a),\ a \in \mathcal{A}, \alpha \in X'$ is a derivation. Indeed, for all $a, b \in \mathcal{A}$ and $\alpha \in X'$, we have

$$\begin{aligned} \langle \alpha, D(ab) \rangle &= \langle ab, \alpha \rangle = \langle a \cdot b, \alpha \rangle = \langle b, \alpha \cdot a \rangle = \langle \alpha \cdot a, D(b) \rangle = \langle \alpha, a \cdot D(b) \rangle \\ &= \langle \alpha, a \cdot D(b) \rangle + \langle \alpha, D(a) \cdot b \rangle = \langle \alpha, a \cdot D(b) + D(a) \cdot b \rangle. \end{aligned} \tag{3.4}$$

The last equality in the Equation (3.4) holds because X'' has a right zero action and so $\langle \alpha, D(a) \cdot b \rangle = 0$. Since $\mathcal{A}$ is amenable, there exists $\Phi \in X''$ such that

$$D(a) = a \cdot \Phi - \Phi \cdot a = a \cdot \Phi \quad (a \in \mathcal{A}).$$

Then by Goldstine's theorem, there exists a bounded net $(e_\lambda)_{\lambda \in P} \in X$ such that $D(e_\lambda)$ converges to Φ in the weak* topology on X''. We then have that $\langle \alpha, D(e_\lambda) \rangle \to \langle \alpha, \Phi \rangle$ for all $\alpha \in X'$. It then follows that $\langle \alpha a, D(e_\lambda) \rangle \to \langle \alpha a, \Phi \rangle$ for all $\alpha \in X', a \in \mathcal{A}$, implying that $\langle a e_\lambda, \alpha \rangle \to \langle a, \alpha \rangle$. Thus (e_λ) is a bounded weak right approximate identify for $\mathcal{A}$ and so by Theorem 2.2.11, $\mathcal{A}$ has a bounded right approximate identity (e_λ). Doing the same thing all over again by defining the module operations as

$$a \cdot x = 0 \quad x \cdot a = xa \quad (a \in \mathcal{A}, x \in X),$$

we obtain a bounded left approximate identity. We now need to show that the left and right bounded approximate identity is a bounded approximate identity for $\mathcal{A}$.
Let $\{e_\lambda\}_{\lambda \in P}$ and $\{f_\beta\}_{\beta \in Q}$ be bounded left and right approximate identities for $\mathcal{A}$ respectively and suppose they are bounded by M_1 and M_2 respectively. We make $P \times Q$ into a directed set by defining a partial order $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq p_2$ and $q_1 \leq q_2$ for all $p_1, p_2 \in P, q_1, q_2 \in Q$. We then have that the net

$$(h_{\lambda\beta}) = (e_\lambda + f_\beta - e_\lambda f_\beta)_{(\lambda,\beta) \in P \times Q}$$

is a bounded approximate identity for $\mathcal{A}$, by Theorem 2.2.9. $\square$

Proposition 3.3.4 ([48]). *Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is amenable if and only if $\mathcal{A}^{\#}$ is amenable.*

Proof. Suppose that $\mathcal{A}^{\#}$ is amenable, let $D \in \mathcal{Z}^1(\mathcal{A}, X')$, where X is a Banach $\mathcal{A}$-bimodule. X becomes a Banach $\mathcal{A}^{\#}$-bimodule with the module operations defined as

$$(a, \alpha) \cdot x = \alpha x + a \cdot x, \quad x \cdot (a, \alpha) = \alpha x + x \cdot a \quad ((a, \alpha) \in \mathcal{A}^{\#}, x \in X). \tag{3.5}$$

Define $d : \mathcal{A}^{\#} \to X'$ as $d(a, \alpha) = D(a) \quad ((a, \alpha) \in \mathcal{A}^{\#})$. This map is well defined, linear and a derivation. For linearity, observe that

$$d((a, \alpha) + (b, \beta)) = d(a + b, \alpha + \beta) = D(a + b) = D(a) + D(b) = d(a, \alpha) + d(b, \beta), \text{ also}$$
$$d(\alpha(a, \beta)) = d(\alpha a, \alpha\beta) = D(\alpha a) = \alpha D(a) = \alpha d(a, \beta) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}).$$

For derivation, observe that

$$\begin{aligned} d((a, \alpha)(b, \beta)) &= d(ab + a\beta + \alpha b, \alpha\beta) = D(ab + a\beta + \alpha b) \\ &= D(ab) + D(a\beta) + D(\alpha b) \\ &= a \cdot D(b) + D(a) \cdot b + \beta D(a) + \alpha D(b) \\ &= \alpha D(b) + a \cdot D(b) + \beta D(a) + D(a) \cdot b \\ &\text{(using the module operations in Equation (3.5))} \\ &= (a, \alpha) \cdot D(b) + D(a) \cdot (b, \beta) \\ &= (a, \alpha) \cdot d(b, \beta) + d(a, \alpha) \cdot (b, \beta) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}). \end{aligned}$$

Hence, d is a derivation. Since $\mathcal{A}^{\#}$ is amenable, there exists $x \in X'$ such that $d(a, \alpha) = (a, \alpha) \cdot x - x \cdot (a, \alpha) = \delta_x(a, \alpha) \quad ((a, \alpha) \in \mathcal{A}^{\#})$. Indeed, for all $a \in \mathcal{A}, \alpha \in \mathbb{C}$, we have

$$D(a) = d(a, \alpha) = (a, \alpha) \cdot x - x \cdot (a, \alpha) = a \cdot x + \alpha x - x \cdot a - \alpha x = a \cdot x - x \cdot a = \delta_x(a).$$

Therefore, $\mathcal{A}$ is amenable.
Conversely, suppose that $\mathcal{A}$ is amenable, by Proposition 3.3.3, $\mathcal{A}$ has a bounded approximate identity. Also, by Proposition 3.2.1, we can take X to be pseudo-unital and since $\mathcal{A}$ is a closed ideal of $\mathcal{A}^{\#}$, then by Proposition 3.2.2, we have that $D \in \mathcal{Z}^1(\mathcal{A}, X')$ and there exists $\overline{D} \in \mathcal{Z}^1(\mathcal{A}^{\#}, X')$ such that $\overline{D}|_{\mathcal{A}} = D$. Hence, $\mathcal{A}^{\#}$ is amenable. □

Proposition 3.3.5 ([31]). *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is amenable and $\mathcal{B}$ is also an amenable Banach algebra, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is amenable.*

Proof. By Proposition 3.3.4, we may assume that $\mathcal{A}$ and $\mathcal{B}$ are unital Banach algebras with identities $e_{\mathcal{A}}$ and $e_{\mathcal{B}}$ respectively. Let X be a Banach $\mathcal{A}\widehat{\otimes}\mathcal{B}$-bimodule and suppose that $D : \mathcal{A}\widehat{\otimes}\mathcal{B} \to X'$ is a derivation. Since X can be identified with a Banach $\mathcal{A}$-bimodule with the module operations given as

$$a \cdot x = (a \otimes e_{\mathcal{B}}) \cdot x, \quad x \cdot a = x \cdot (a \otimes e_{\mathcal{A}}) \quad (a \in \mathcal{A}, x \in X).$$

Define $D_{\mathcal{A}} : \mathcal{A} \to X'$ as $D_{\mathcal{A}}(a) = D(a \otimes e_{\mathcal{B}})$ for all $a \in \mathcal{A}$. It is easy to check that $D_{\mathcal{A}}$ is a continuous derivation from $\mathcal{A}$ into X'. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned} D_{\mathcal{A}}(ab) &= D(ab \otimes e_{\mathcal{B}}) = D[(a \otimes e_{\mathcal{B}})(b \otimes e_{\mathcal{B}})] = (a \otimes e_{\mathcal{B}}) \cdot D(b \otimes e_{\mathcal{B}}) + D(a \otimes e_{\mathcal{B}}) \cdot (b \otimes e_{\mathcal{B}}) \\ &= a \cdot D(b \otimes e_{\mathcal{B}}) + D(a \otimes e_{\mathcal{B}}) \cdot b = a \cdot D_{\mathcal{A}}(b) + D_{\mathcal{A}}(a) \cdot b. \end{aligned}$$

Since $\mathcal{A}$ is amenable, there exists $x \in X'$ such that $D_{\mathcal{A}}(a) = a \cdot x - x \cdot a$, for all $a \in \mathcal{A}$. Let δ_x be the inner derivation from $\mathcal{A}\widehat{\otimes}\mathcal{B}$ into X'. Then, we have

$$\begin{aligned} D(a \otimes e_{\mathcal{B}}) &= D_{\mathcal{A}}(a) = a \cdot x - x \cdot a = (a \otimes e_{\mathcal{B}}) \cdot x - x \cdot (a \otimes e_{\mathcal{B}}) \\ &= \delta_x(a \otimes e_{\mathcal{B}}). \end{aligned}$$

And so, $D = \delta_x$, which implies that $D - \delta_x = 0$. Let $\overline{D} := D - \delta_x = 0$ on $\mathcal{A} \otimes e_{\mathcal{B}}$. Note that

$$(a \otimes b) = (a \otimes e_{\mathcal{B}})(e_{\mathcal{A}} \otimes b) = (e_{\mathcal{A}} \otimes b)(a \otimes e_{\mathcal{B}}) \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

This implies that $\mathcal{A} \otimes e_B$ and $e_{\mathcal{A}} \otimes \mathcal{B}$ commutes. Then for all $a \in \mathcal{A}, b \in \mathcal{B}$, we have

$$\begin{aligned} \overline{D}(a \otimes b) &= \overline{D}((a \otimes e_{\mathcal{B}})(e_{\mathcal{A}} \otimes b)) = (a \otimes e_{\mathcal{B}}) \cdot \overline{D}(e_{\mathcal{A}} \otimes b)) + \overline{D}(a \otimes e_{\mathcal{B}}) \cdot (e_{\mathcal{A}} \otimes b) \\ &= (a \otimes e_{\mathcal{B}}) \cdot \overline{D}(e_{\mathcal{A}} \otimes b), \end{aligned}$$

since $\overline{D}(a \otimes e_{\mathcal{B}}) \cdot (e_{\mathcal{A}} \otimes b) = 0$ on $\mathcal{A} \otimes e_{\mathcal{B}}$. Similarly,

$$\begin{aligned} \overline{D}(a \otimes b) &= \overline{D}((e_{\mathcal{A}} \otimes b)(a \otimes e_{\mathcal{B}})) = (e_{\mathcal{A}} \otimes b) \cdot \overline{D}(a \otimes e_{\mathcal{B}}) + \overline{D}(e_{\mathcal{A}} \otimes b) \cdot (a \otimes e_{\mathcal{B}}) \\ &= \overline{D}(e_{\mathcal{A}} \otimes b) \cdot (a \otimes e_{\mathcal{B}}), \end{aligned}$$

since $(e_{\mathcal{A}} \otimes b) \cdot \overline{D}(a \otimes e_{\mathcal{B}}) = 0$ on $\mathcal{A} \otimes e_{\mathcal{B}}$. This implies that $\overline{D}(a \otimes b) = \overline{D}(e_{\mathcal{A}} \otimes b) \cdot (a \otimes e_{\mathcal{B}}) = (a \otimes e_{\mathcal{B}}) \cdot \overline{D}(e_{\mathcal{A}} \otimes b)$, then $\overline{D}(e_{\mathcal{A}} \otimes b) \cdot (a \otimes e_{\mathcal{B}}) - (a \otimes e_{\mathcal{B}}) \cdot \overline{D}(e_{\mathcal{A}} \otimes b) = 0$ and taking the closure in weak* topology of X', then for $h \in \overline{\overline{D}(e_{\mathcal{A}} \otimes \mathcal{B})}$, $\delta_h(\mathcal{A} \otimes e_{\mathcal{B}}) = \{0\}$. Let C be the annihilator of $\overline{D}(e_{\mathcal{A}} \otimes \mathcal{B})$ in X. We make X a Banach $\mathcal{B}$-bimodule with the module operations defined as

$$b \cdot x = (e_{\mathcal{A}} \otimes b) \cdot x, \quad x \cdot b = x \cdot (e_{\mathcal{A}} \otimes b) \quad (b \in \mathcal{B}, x \in X).$$

Clearly, C is a Banach $\mathcal{B}$-bimodule of X. We then have that X/C is a Banach $\mathcal{B}$-bimodule with the module operations

$$(x + C) \cdot b = x \cdot b + C, \quad b \cdot (x + C) = b \cdot x + C \quad (b \in \mathcal{B}, x \in X).$$

By Corollary 1.9 of [7], we have

$$(X/C)' = C^{\perp} = ({}^{\perp}\overline{D}(e_{\mathcal{A}} \otimes \mathcal{B}))^{\perp} = \overline{\overline{D}(e_{\mathcal{A}} \otimes \mathcal{B})}$$

and $(X/C)' \subset X'$. Define $D_{\mathcal{B}} : \mathcal{B} \to (X/C)'$, by $D_{\mathcal{B}}(b) = \overline{D}(e_{\mathcal{A}} \otimes b)$ for all $b \in \mathcal{B}$. It is easy to check that $D_{\mathcal{B}}$ is a continuous derivation from $\mathcal{B}$ into $(X/C)'$. Indeed, for all $a, b \in \mathcal{B}$, we have

$$\begin{aligned} D_{\mathcal{B}}(ab) &= \overline{D}(e_{\mathcal{A}} \otimes ab) = \overline{D}((e_{\mathcal{A}} \otimes a)(e_{\mathcal{A}} \otimes b)) = (e_{\mathcal{A}} \otimes a) \cdot \overline{D}(e_{\mathcal{A}} \otimes b) + \overline{D}(e_{\mathcal{A}} \otimes a) \cdot (e_{\mathcal{A}} \otimes b) \\ &= a \cdot D_{\mathcal{B}}(b) + D_{\mathcal{B}}(a) \cdot b. \end{aligned}$$

Since $\mathcal{B}$ is amenable, there exist $y \in (X/C)'$ such that

$$D_{\mathcal{B}}(b) = b \cdot y - y \cdot b \quad (b \in \mathcal{B}).$$

Now, note that

$$\overline{D}(e_{\mathcal{A}} \otimes b) = D_{\mathcal{B}}(b) = b \cdot y - y \cdot b = (e_{\mathcal{A}} \otimes b) \cdot y - y \cdot (e_{\mathcal{A}} \otimes b) = \delta_y(e_{\mathcal{A}} \otimes b).$$

But we have that $\delta_h(\mathcal{A} \otimes e_{\mathcal{B}}) = \{0\}$, then such y must satisfy $\delta_y|_{\mathcal{A}\otimes e_{\mathcal{B}}} = 0$. Hence $\overline{D} - \delta_y = 0$ and $\overline{D} - \delta_y$ is a derivation of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ that vanishes on $\mathcal{A} \otimes e_{\mathcal{B}}$ and $e_{\mathcal{A}} \otimes \mathcal{B}$. Since $(\mathcal{A} \otimes e_{\mathcal{B}}) \cup (e_{\mathcal{A}} \otimes \mathcal{B})$ generates $\mathcal{A}\widehat{\otimes}\mathcal{B}$. It then follows that $\overline{D} = \delta_y$ vanishes on $\mathcal{A}\widehat{\otimes}\mathcal{B}$, then $D = \delta_x + \delta_y = \delta_{x+y}$. Hence, $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is amenable. □

Remark 3.3.6. *The converse of Proposition 3.3.5 is not true in general. However in [33] Proposition 3.5, B. E. Johnson proved that the amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ implies the amenability of $\mathcal{A}$ if the Banach algebra $\mathcal{B}$ is subjected to some conditions. In addition, F. Ghahramani and R. J. Loy in [23] also show that the amenability of $\mathcal{A}\widehat{\otimes}\mathcal{B}$ implies the amenability of $\mathcal{A}$ and $\mathcal{B}$ in the frame work or sense of semi-inner derivations.*

3.3.2 Characterization of Amenable Banach Algebras.

To determine if a Banach algebra is amenable or not via the definition given above is very difficult. There are however some interesting characterization of amenable Banach algebras. In this section, we shall consider the intrinsic characterization given by *B. E. Johnson* in [31], *Curtis* and *Loy* in [8] and *A. T. Lau* in [36]. An explicit proof of these characterization will be given.

We begin with the characterization given by B. E. Johnson in [31].

Theorem 3.3.7 ([31]). *Let $\mathcal{A}$ be a Banach algebra. Then the following are equivalent:*

1. *$\mathcal{A}$ is amenable;*
2. *$\mathcal{A}$ has an approximate diagonal;*
3. *$\mathcal{A}$ has a virtual diagonal.*

Proof. 1 $\Rightarrow$ 3. By Proposition 3.3.3, $\mathcal{A}$ has a bounded approximate identity. Let (e_α) be the bounded approximate identity for $\mathcal{A}$ and consider the bounded net $(e_\alpha \otimes e_\alpha)$ in $(\mathcal{A}\widehat{\otimes}\mathcal{A})''$. Let $E \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ be a w^*-accumulation point of $(e_\alpha \otimes e_\alpha)$. We define $D_E : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ by $D_E(a) = a \cdot E - E \cdot a$ for all $a \in \mathcal{A}, E \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''$. The map D_E from $\mathcal{A}$ into $(\mathcal{A}\widehat{\otimes}\mathcal{A})''$ is a derivation. Indeed, we have

$$\begin{aligned} D_E(ab) &= (ab) \cdot E - E \cdot (ab) = (ab) \cdot E - a \cdot (E \cdot b) + a \cdot (E \cdot b) - E \cdot (ab) \\ &= a \cdot (b \cdot E) - a \cdot (E \cdot b) + (a \cdot E) \cdot b - (E \cdot a) \cdot b \\ &= a \cdot (b \cdot E - E \cdot b) + (a \cdot E - E \cdot a) \cdot b \\ &= a \cdot D_E(b) + D_E(a) \cdot b \quad (a, b \in \mathcal{A}, E \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''). \end{aligned}$$

Hence, D_E is a derivation. Then for all $a \in \mathcal{A}$, we have

$$\begin{aligned} \pi''(D_E(a)) &= \pi''(a \cdot E - E \cdot a) = w^* - \lim_\alpha \pi''[a \cdot (e_\alpha \otimes e_\alpha) - (e_\alpha \otimes e_\alpha) \cdot a] \\ &= w - \lim_\alpha [\pi(ae_\alpha \otimes e_\alpha - e_\alpha \otimes e_\alpha a)] = w - \lim_\alpha [\pi(ae_\alpha \otimes e_\alpha) - \pi(e_\alpha \otimes e_\alpha a)] \\ &= w - \lim_\alpha (ae_\alpha^2 - e_\alpha^2 a) = w - \lim_\alpha (ae_\alpha^2) - [w - \lim_\alpha (e_\alpha^2 a)] = a - a = 0 \end{aligned} \tag{3.6}$$

The last equality in Equation (3.6) holds because (e_α^2) is also a bounded approximate identity for $\mathcal{A}$. Indeed, for all $a \in \mathcal{A}$, we have

$$\begin{aligned} \|e_\alpha^2 a - a\| &= \|e_\alpha^2 a - e_\alpha a + e_\alpha a - a\| = \|(e_\alpha + 1)(e_\alpha a - a)\| \\ &\leq (\|e_\alpha\| + 1)\|e_\alpha a - a\| \to 0. \end{aligned} \tag{3.7}$$

We use similar argument for $\|ae_\alpha^2 - a\|$. Since π is a bimodule homomorphism so is π'' and consequently, $ker\ \pi''$ is a Banach $\mathcal{A}$-bimodule. Also, since $\mathcal{A}$ has a bounded approximate identity, Cohen's factorization theorem (see [48]) implies that π is surjective and thus open. Consequently, $ker\ \pi'' \approxeq (ker\ \pi)''$, so that $ker\ \pi''$ is in fact a dual Banach $\mathcal{A}$-bimodule. Since $\mathcal{A}$ is amenable, there exists an $N \in ker\ \pi''$ such that $D_E = D_N$. Let take $M := E - N$, for all $a \in \mathcal{A}$, we have

$$\begin{aligned} a \cdot M - M \cdot a &= a \cdot (E - N) - (E - N) \cdot a \\ &= a \cdot E - a \cdot N - E \cdot a + N \cdot a \\ &= a \cdot E - E \cdot a - (a \cdot N - N \cdot a) \\ &= D_E(a) - D_N(a) = 0. \end{aligned}$$

Also, note that for all $a \in \mathcal{A}$, we have

$$\begin{aligned} a \cdot \pi''(M) &= a \cdot \pi''(E - N) = a \cdot (\pi''(E) - \pi''(N)) = a \cdot \pi''(E) \\ &= w^* - \lim_\alpha [a \cdot \pi''(e_\alpha \otimes e_\alpha)] = w - \lim_\alpha [a \cdot \pi(e_\alpha \otimes e_\alpha)] \\ &= w - \lim_\alpha ae_\alpha^2 = a. \end{aligned}$$

Hence, M is a virtual diagonal.
$3 \Rightarrow 2$. Suppose that M is a virtual diagonal for $\mathcal{A}$ and let (m_α) be a bounded net in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ with $M = w^* - \lim_\alpha \widehat{m}_\alpha$, where $\widehat{m}_\alpha \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''$. Then for all $a \in \mathcal{A}$, we have

$$w - \lim_\alpha [a \cdot m_\alpha - m_\alpha \cdot a] = w^* - \lim_\alpha [a \cdot \widehat{m}_\alpha - \widehat{m}_\alpha \cdot a] = a \cdot M - M \cdot a = 0.$$

Also, for all $a \in \mathcal{A}$, we have

$$w - \lim_\alpha a \cdot \pi(m_\alpha) = w^* - \lim_\alpha a \cdot \pi''(\widehat{m}_\alpha) = a \cdot \pi''(M) = a.$$

$2 \Rightarrow 1$. Let (m_α) be an approximate diagonal for $\mathcal{A}$. By remark 3.1.9, $(\pi(m_\alpha))_\alpha$ is a bounded approximate identity for $\mathcal{A}$. Let X be a Banach $\mathcal{A}$-bimodule. We need to show that, for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \to X'$ is inner. By Proposition 3.2.1, there is no loss of generality if we suppose that X is pseudo-unital. Let $D : \mathcal{A} \to X'$ be a derivation and let $m_\alpha = \sum_{j=1}^{\infty} a_j^{(\alpha)} \otimes b_j^{(\alpha)}$ be with $\sum_{j=1}^{\infty} \|a_j^{(\alpha)}\|\|b_j^{(\alpha)}\| \leq \infty$. Then $\left(\sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \right)_\alpha$ is a bounded net in X', without loss of generality we may suppose that λ is the $w^* - \lim$ of $\left(\sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \right)_\alpha$. Then, for all $a \in \mathcal{A}$ and $x \in X$, we have

$$\begin{aligned}
\langle x, a \cdot \lambda \rangle &= \lim_\alpha \left\langle x, a \cdot \sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \right\rangle = \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \right\rangle, \\
&\text{(since } D \text{ is linear, using Equation (3.2))} \\
&= \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \right\rangle = \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot [D(b_j^{(\alpha)} a)] \right\rangle \\
&= \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot [b_j^{(\alpha)} \cdot D(a) + D(b_j^{(\alpha)}) \cdot a] \right\rangle \\
&= \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a_j^{(\alpha)} b_j^{(\alpha)} \cdot D(a) + \sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \cdot a \right\rangle \\
&= \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a_j^{(\alpha)} b_j^{(\alpha)} \cdot D(a) \right\rangle + \lim_\alpha \left\langle x, \sum_{j=1}^{\infty} a_j^{(\alpha)} \cdot D(b_j^{(\alpha)}) \cdot a \right\rangle \\
&= \lim_\alpha \left\langle x \cdot \sum_{j=1}^{\infty} a_j^{(\alpha)} b_j^{(\alpha)}, D(a) \right\rangle + \langle x, \lambda \cdot a \rangle \\
&= \lim_\alpha \langle x \cdot e_\alpha, D(a) \rangle + \langle x, \lambda \cdot a \rangle \\
&= \langle x, D(a) \rangle + \langle x, \lambda \cdot a \rangle
\end{aligned}$$

It follows that $D(a) = \delta_\lambda(a)$ for all $a \in \mathcal{A}$. Hence, $\mathcal{A}$ is amenable. $\square$

Let $\mathcal{A}$ be a Banach algebra and suppose that X, Y and Z are Banach $\mathcal{A}$-bimodule and $f : X \to Y, g : Y \to Z$ are Banach $\mathcal{A}$-module homomorphism. Then the sequence

$$\Sigma : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

is *short exact* if f is one-to-one, g is onto and $im(f) = ker(g)$. The exact sequence Σ is *admissible* if there is a bounded linear map $F : Y \to X$ such that $Ff = I_X$. The exact sequence Σ *splits* if there is a Banach $\mathcal{A}$-module homomorphism $F : Y \to X$ such that $Ff = I_X$. Curtis and Loy relate the concept of amenability of Banach algebra to the splitting of Π' and in terms of the splitting of an admissible sequence

$$\begin{aligned} \Sigma : 0 \to X' \xrightarrow{f} Y \xrightarrow{g} Z \to 0 \text{ where} \\ \Pi : 0 \to K \xrightarrow{i} \mathcal{A}\widehat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0 \text{ and its dual} \\ \Pi' : 0 \to \mathcal{A} \xrightarrow{\pi'} (\mathcal{A}\widehat{\otimes}\mathcal{A})' \xrightarrow{i'} K' \to 0. \end{aligned}$$

We now state and prove the characterization given by Curtis and Loy in [8].

Theorem 3.3.8 ([8])**.** *The Banach algebra $\mathcal{A}$ is amenable if and only if*

1. *$\mathcal{A}$ has a bounded approximate identity and*
2. *the exact sequence Π' of $\mathcal{A}$-bimodules splits.*

Proof. Suppose that $\mathcal{A}$ is amenable, then by Proposition 3.3.3, $\mathcal{A}$ has a bounded approximate identity and also by Theorem 3.3.7, $\mathcal{A}$ has a virtual diagonal. Let M be the virtual diagonal for $\mathcal{A}$, for $f \in (\mathcal{A}\widehat{\otimes}\mathcal{A})'$ define $\langle a, \theta f\rangle = \langle f \cdot a, M\rangle$ for all $a \in \mathcal{A}$. We claim that $\theta\pi' = I$ and that θ is an $\mathcal{A}$-bimodule homomorphism. Indeed, for all $a \in \mathcal{A}$ and $\lambda \in \mathcal{A}'$, we have

$$\begin{aligned} \langle a, \theta\pi'\lambda\rangle &= \langle(\pi'\lambda) \cdot a, M\rangle = \langle\pi'(\lambda \cdot a), M\rangle \\ &= \langle\lambda \cdot a, \pi''(M)\rangle = \langle a, \pi''(M) \cdot \lambda\rangle \\ &= \langle a, \lambda\rangle. \end{aligned}$$

Lastly, let us show that θ is an $\mathcal{A}$-bimodule homomorphism. For all $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \langle a, \theta(f \cdot b)\rangle &= \langle(b \cdot f) \cdot a, M\rangle = \langle b \cdot (f \cdot a), M\rangle \\ &= \langle f \cdot a, M \cdot b\rangle = \langle f \cdot a, b \cdot M\rangle \\ &= \langle(f \cdot a) \cdot b, M\rangle = \langle f \cdot (ab), M\rangle \\ &= \langle ab, \theta(f)\rangle = \langle a, b \cdot \theta(f)\rangle. \end{aligned}$$

Similarly, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned} \langle a, \theta(f \cdot b)\rangle &= \langle(f \cdot b) \cdot a, M\rangle = \langle b \cdot f \cdot (ba), M\rangle \\ &= \langle ba, \theta(f)\rangle = \langle a, \theta(f) \cdot b\rangle. \end{aligned}$$

Conversely, suppose that $\mathcal{A}$ has a bounded approximate identity (e_α), θ an $\mathcal{A}$-bimodule homomorphism with $\theta\pi' = I$ and supposing that the net $(e_\alpha \otimes e_\alpha)$ converging *weak** to $u \in (\mathcal{A}\widehat{\otimes}\mathcal{A})''$. Let us take $M = \theta'\pi''u$. We claim that M is a virtual diagonal for $\mathcal{A}$. Indeed, for all $a \in \mathcal{A}$, $f \in (\mathcal{A}\widehat{\otimes}\mathcal{A})'$, we have

$$\begin{aligned}
\langle f, a \cdot M\rangle &= \langle f, a \cdot (\theta'(\pi''u))\rangle = \langle f \cdot a, \theta'(\pi''u)\rangle = \langle \theta(f \cdot a), \pi''(u)\rangle \\
&= \langle \pi'(\theta(f \cdot a)), u\rangle = \lim_\alpha \langle \pi'(\theta(f \cdot a)), (e_\alpha \otimes e_\alpha)\rangle \\
&= \lim_\alpha \langle \theta(f \cdot a), \pi(e_\alpha \otimes e_\alpha)\rangle = \lim_\alpha \langle \theta(f) \cdot a, e_\alpha^2\rangle \\
&= \lim_\alpha \langle \theta(f), a e_\alpha^2\rangle = \langle \theta(f), a\rangle \quad \text{(the last equality follows from Equation (3.7))} \\
&= \lim_\alpha \langle \theta(f), e_\alpha^2 a\rangle = \lim_\alpha \langle a \cdot \theta(f), e_\alpha^2\rangle \\
&= \lim_\alpha \langle \theta(a \cdot f), \pi(e_\alpha \otimes e_\alpha)\rangle = \lim_\alpha \langle \pi'\theta(a \cdot f), (e_\alpha \otimes e_\alpha)\rangle \\
&= \langle \pi'\theta(a \cdot f), u\rangle = \langle \theta(a \cdot f), \pi''u\rangle = \langle a \cdot f, \theta'\pi''u\rangle \\
&= \langle f, (\theta'\pi''u) \cdot a\rangle = \langle f, M \cdot a\rangle.
\end{aligned}$$

Lastly, we have

$$\begin{aligned}
\langle f, \pi''(M) \cdot a\rangle &= \langle a \cdot f, \pi''(M)\rangle = \langle \pi'(a \cdot f), M\rangle = \langle \pi'(a \cdot f), \theta'\pi''u\rangle \\
&= \langle \theta\pi'(a \cdot f), \pi''u\rangle = \langle (a \cdot f), \pi''u\rangle \quad (\text{since } \theta\pi' = I) \\
&= \langle \pi'(a \cdot f), u\rangle = \lim_\alpha \langle \pi'(a \cdot f), (e_\alpha \otimes e_\alpha)\rangle \\
&= \lim_\alpha \langle a \cdot f, \pi(e_\alpha \otimes e_\alpha)\rangle = \lim_\alpha \langle f, e_\alpha^2 a\rangle \\
&= \langle f, a\rangle.
\end{aligned}$$

Hence, M is a virtual diagonal. Then, by Theorem 3.3.7, $\mathcal{A}$ is amenable. □

Theorem 3.3.9 ([8])**.** *Let $\mathcal{A}$ be an amenable Banach algebra and let*

$$\Sigma : 0 \to X' \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an admissible short exact sequence of left or right $\mathcal{A}$-module with X' a dual Banach $\mathcal{A}$-module. Then Σ splits.

Proof. Suppose that $\mathcal{A}$ is amenable and Σ is a sequence of left $\mathcal{A}$-modules. Since Σ is admissible, there exists $G \in \mathcal{L}(Z, Y)$ such that $gG = I$ on Z. Define $D : \mathcal{A} \to \mathcal{L}(Z, Y)$ as $D(a) = a \cdot G - G \cdot a$ for all $a \in \mathcal{A}$. Clearly, D is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
D(ab) &= (ab) \cdot G - G \cdot (ab) = a \cdot (b \cdot G) - a \cdot (G \cdot b) + (a \cdot G) \cdot b - (G \cdot a) \cdot b \\
&= a \cdot (b \cdot G - G \cdot b) + (a \cdot G - G \cdot a) \cdot b = a \cdot D(b) + D(a) \cdot b.
\end{aligned}$$

Hence, D is a derivation. More so, for any $z \in Z$ and that $gG = I$, we have

$$g((D(a))(z)) = g(a \cdot G - G \cdot a)(z) = (a \cdot gG - gG \cdot a)(z) = az - az = 0.$$

Therefore $D(\mathcal{A}) \subset \mathcal{L}(Z, ker\ g) = \mathcal{L}(Z, imf)$. Clearly, $(f^{-1} \circ D) : \mathcal{A} \to \mathcal{L}(Z, X') = (Z \otimes X)'$ is a derivation. Since $\mathcal{A}$ is amenable, $f^{-1} \circ D$ is inner and so there exists $Q \in \mathcal{L}(Z, X')$ such that

$$D(a) = a \cdot G - G \cdot a = a \cdot fQ - fQ \cdot a \quad (a \in \mathcal{A}).$$

If $G_1 = G - fQ$, then $a \cdot G_1 = G_1 \cdot a$ and G_1 is a left $\mathcal{A}$-module homomorphism from Z to Y. Furthermore,

$$gG_1(z) = gG(z) - gfQ(z) = gG(z) - 0 = z,$$

since $gG = I$ and $imfQ \subset kerg$. Therefore G_1 is a right inverse for g and thus, the sequence Σ splits. $\square$

Definition 3.3.10. *If $\mathcal{A}$ is a Banach algebra and X is a Banach $\mathcal{A}$-bimodule, we write*

$$Z(\mathcal{A}, X') = \bigcap_{a \in \mathcal{A}} \{f \in X' \mid a \cdot f = f \cdot a\}.$$

Then $Z(\mathcal{A}, X')$ is a closed subspace of X' which is invariant under each bounded linear operator from X' into X' commuting with the action of $\mathcal{A}$.

Lastly, we state and prove the characterization given by A. T. Lau in [36].

Theorem 3.3.11 ([36])**.** *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*

1. *$\mathcal{A}$ is amenable.*
2. *For any Banach $\mathcal{A}$-bimodule X and any Banach $\mathcal{A}$-submodule Y of X, each linear functional in $Z(\mathcal{A}, Y')$ has an extension to a linear functional in $Z(\mathcal{A}, X')$.*
3. *For any Banach $\mathcal{A}$-bimodule, there exists a bounded projection from X' onto $Z(\mathcal{A}, X')$ which commutes with any weak* continuous bounded linear operator from X' into X' commuting with the action of $\mathcal{A}$ on X'.*

Proof. $1 \Rightarrow 2$. Since X is a Banach $\mathcal{A}$-bimodule, then the quotient Banach space X/Y becomes a Banach $\mathcal{A}$-bimodule with the canonical module operations given as

$$a \cdot (x + Y) = a \cdot x + Y, \quad (x + Y) \cdot a = x \cdot a + Y \quad (a \in \mathcal{A}, x \in X).$$

Let $f \in Z(\mathcal{A}, Y')$ and $\widehat{f} \in X'$ be any extension of f to X. If $a \in \mathcal{A}$, then $a \cdot \widehat{f} - \widehat{f} \cdot a \in Y^{\perp}$. It is known that $Q : Y^{\perp} \to (X/Y)'$ is an $\mathcal{A}$-module isometry and surjection mapping.

Define $D : \mathcal{A} \to (X/Y)'$ as $D(a) = Q(a \cdot \widehat{f} - \widehat{f} \cdot a)$ for all $a \in \mathcal{A}$. It is easy to see that the map D from $\mathcal{A}$ into $(X/Y)'$ is a bounded derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned} D(ab) &= Q((ab) \cdot \widehat{f} - \widehat{f} \cdot (ab)) = Q((ab) \cdot \widehat{f} - a \cdot \widehat{f} \cdot b + a \cdot \widehat{f} \cdot b - \widehat{f} \cdot (ab)) \\ &= Q(a \cdot (b \cdot \widehat{f}) - a \cdot (\widehat{f} \cdot b) + (a \cdot \widehat{f}) \cdot b - (\widehat{f} \cdot a) \cdot b) \\ &= Q(a \cdot (b \cdot \widehat{f} - \widehat{f} \cdot b) + (a \cdot \widehat{f} - \widehat{f} \cdot a) \cdot b) \\ &= a \cdot Q(b \cdot \widehat{f} - \widehat{f} \cdot b) + Q(a \cdot \widehat{f} - \widehat{f} \cdot a) \cdot b \\ &= a \cdot D(b) + D(a) \cdot b. \end{aligned}$$

Hence, D is a derivation. Since $\mathcal{A}$ is amenable, there exists $x \in (X/Y)'$ such that $D(a) = a \cdot x - x \cdot a = \delta_x(a)$ for all $a \in \mathcal{A}$. Then there exists $h \in Y^{\perp}$ (Q is surjective) such that $D(a) = a \cdot Q(h) - Q(h) \cdot a$, for all $a \in \mathcal{A}$. Let $g = \widehat{f} - h$ and for all $a \in \mathcal{A}$ and $y \in Y$, we have

$$\langle y, a \cdot (\widehat{f} - h) - (\widehat{f} - h) \cdot a \rangle = \langle y, a \cdot \widehat{f} - \widehat{f} \cdot a \rangle - \langle y, a \cdot h - h \cdot a \rangle = 0.$$

This implies that $g \in Z(\mathcal{A}, X')$ and g extends f.
$2 \Rightarrow 3$. The projective tensor product $X' \widehat{\otimes} X$ becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$(f \otimes x) \cdot a = f \otimes x \cdot a, \quad a \cdot (f \otimes x) = f \otimes a \cdot x \quad (a \in \mathcal{A}, x \in X, f \in X').$$

We define the sets H and K as follows

$$H := \overline{lin}\{T'(f) \otimes x - f \otimes T(x) \mid T \in \mathbb{B}(X), x \in X, f \in X'\}$$

and

$$K := \overline{lin}\{f \otimes x \mid f \in Z(\mathcal{A}, X'), x \in X\}.$$

By the definitions of H and K, it is clear that both H and K are Banach $\mathcal{A}$-submodule of $(X' \widehat{\otimes} X)$. Therefore, Y/H is also Banach $\mathcal{A}$-submodule of $(X' \widehat{\otimes} X)/H$. Let $\phi \in (X' \widehat{\otimes} X)'$ satisfying $\langle f \otimes x, \phi \rangle = f(x) \quad (x \in X, f \in X')$. Clearly, $\phi \in H^{\perp}$. Indeed, for all $x \in X, f \in X'$, observe that

$$\begin{aligned} \langle T'(f) \otimes x - f \otimes T(x), \phi \rangle &= \langle T'(f) \otimes x, \phi \rangle - \langle f \otimes T(x), \phi \rangle \\ &= \langle x, T'(f) \rangle - \langle T(x), f \rangle \\ &= \langle T(x), f \rangle - \langle T(x), f \rangle = 0. \end{aligned}$$

Hence, $\phi \in H^{\perp}$. Now, we have $\Phi \in (X' \widehat{\otimes} X/H)'$ such that $\Phi(y + H) = \phi(y)$ for all

$y \in (X'\widehat{\otimes}X)$. We need to check that $a \cdot \Phi = \Phi \cdot a$ is in $(Y/H)'$. For all $a \in \mathcal{A}$, we have

$$\begin{aligned}\langle f \otimes x + H, a \cdot \Phi\rangle - \langle f \otimes x + H, \Phi \cdot a\rangle &= \langle (f \otimes x + H) \cdot a, \Phi\rangle - \langle a \cdot (f \otimes x + H), \Phi\rangle \\ &= \langle f \otimes x \cdot a + H, \Phi\rangle - \langle f \otimes a \cdot x + H, \Phi\rangle \\ &= \langle f \otimes x \cdot a, \phi\rangle - \langle f \otimes a \cdot x, \phi\rangle \\ &= \langle x \cdot a, f\rangle - \langle a \cdot x, f\rangle = \langle x, a \cdot f\rangle - \langle x, f \cdot a\rangle \\ &= \langle x, a \cdot f - f \cdot a\rangle = 0.\end{aligned}$$

Since ϕ is in $H^{\perp}$, it follows that $a \cdot \Phi = \Phi \cdot a$ for all $a \in \mathcal{A}$ and $\Phi \in (Y/H)'$. By hypothesis (2), there exists an extension $\widehat{\Phi}$ of Φ such that $\widehat{\Phi} \in (X/H)'$ and $a \cdot \widehat{\Phi} = \widehat{\Phi} \cdot a$. Define $\langle x, P(f)\rangle = \langle f \otimes x + H, \widehat{\Phi}\rangle$ for all $x \in X, f \in X'$. We claim that the projection P is bounded and commutes with every $T' \in \mathbb{B}(X')$. Indeed, for all $x \in X$ and $f \in X'$, we have

$$\begin{aligned}\langle x, (P \circ P)(f)\rangle = \langle x, P(P(f))\rangle &= \langle P(f)(x), P\rangle = \langle ((\langle x, P(f)\rangle), P\rangle \\ &= \langle ((\langle f \otimes x + H, \widehat{\Phi}\rangle), P\rangle = \langle ((\langle f \otimes x, \phi\rangle), P\rangle = \langle f(x), P\rangle \\ &= \langle x, P(f)\rangle.\end{aligned}$$

Hence P is a bounded projection from X' onto $Z(\mathcal{A}, X')$. Also, observe that

$$\begin{aligned}\langle x, P(T'(f))\rangle &= \langle T'(f) \otimes x + H, \widehat{\Phi}\rangle = \langle T'(f) \otimes x, \phi\rangle \\ &= \langle x, T'(f)\rangle = \langle T(x), f\rangle = \langle f \otimes T(x), \phi\rangle \\ &= \langle f \otimes T(x) + H, \widehat{\Phi}\rangle \\ &= \langle T(x), P(f)\rangle = \langle x, T'(P(f))\rangle.\end{aligned}$$

Hence, P commutes with every $T' \in \mathbb{B}(X')$.
$3 \Rightarrow 1$. Without loss of generality, we may suppose that $\mathcal{A}$ is unital. Set $X := \mathcal{A}\widehat{\otimes}\mathcal{A}$, X becomes a Banach $\mathcal{A}$-bimodule in the canonical fashion. Let $\mathcal{F} = \{L_a, R_a : a \in \mathcal{A}\}$ be a family of bounded linear operators from X into X such that

$$L_a(b \otimes c) = b \otimes ac, \quad R_a(b \otimes c) = ba \otimes c \quad (a, b, c \in \mathcal{A}).$$

Each operator in $\mathcal{F}$ commutes with the actions of $\mathcal{A}$ on X. To see this, for all $a, b, c, d \in \mathcal{A}$, we have

$$b \cdot L_a(c \otimes d) = b \cdot (c \otimes ad) = bc \otimes ad = bL_a(c \otimes d)$$

and

$$b \cdot R_a(c \otimes d) = b \cdot (ca \otimes d) = bca \otimes d = bR_a(c \otimes d).$$

By hypothesis (3), there exists a bounded projection from X' onto $Z(\mathcal{A}, X')$, such that P commutes with T' ($PT' = T'P$) for all $T \in \mathcal{F}$. Let $q : X' \to X'$ be defined as $\langle a \otimes b, q(f)\rangle = \langle b \otimes a, f\rangle \quad (a, b \in \mathcal{A}, f \in X')$. From the way q is defined, observe that

$$\begin{aligned}\langle c \otimes d, q(x' \cdot a)\rangle &= \langle d \otimes c, x' \cdot a\rangle = \langle ad \otimes c, x'\rangle = \langle c \otimes ad, q(x')\rangle \\ &= \langle L_a(c \otimes d), q(x')\rangle = \langle (c \otimes d), L_a^{'} q(x')\rangle.\end{aligned}$$

Also, we have

$$\begin{aligned}\langle c \otimes d, q(a \cdot x')\rangle &= \langle d \otimes c, a \cdot x'\rangle = \langle d \otimes ca, x'\rangle = \langle ca \otimes d, q(x')\rangle \\ &= \langle R_a(c \otimes d), q(x')\rangle = \langle (c \otimes d), R_a^{'} q(x')\rangle,\end{aligned}$$

for all $a, b, c, d \in \mathcal{A}$ and $x' \in X'$. Then by our assumption in (3) we have that

$$PR_a^{'} = R_a^{'} P, \quad PL_a^{'} = L_a^{'} P.$$

Let $M = q'(P'(e \otimes e))$, where e is the identity of $\mathcal{A}$. We claim that M is a virtual diagonal for $\mathcal{A}$. Indeed, for all $a \in \mathcal{A}, x' \in X'$, we have

$$\begin{aligned}\langle x', M \cdot a\rangle &= \langle a \cdot x', M\rangle = \langle a \cdot x', q'(P'(e \otimes e))\rangle = \langle q(a \cdot x'), P'(e \otimes e)\rangle \\ &= \langle R_a^{'} q(x'), P'(e \otimes e)\rangle = \langle PR_a^{'} q(x'), (e \otimes e)\rangle \\ &= \langle R_a^{'} Pq(x'), (e \otimes e)\rangle = \langle Pq(x'), R_a(e \otimes e)\rangle \\ &= \langle Pq(x'), a \otimes e\rangle = \langle Pq(x'), a \cdot (e \otimes e)\rangle \\ &= \langle Pq(x') \cdot a, (e \otimes e)\rangle = \langle x' \cdot a, q'P'(e \otimes e)\rangle \\ &= \langle x', a \cdot q'P'(e \otimes e)\rangle = \langle x', a \cdot M\rangle.\end{aligned}$$

Similarly, for all $a \in \mathcal{A}, x' \in X'$, we have

$$\begin{aligned}\langle x', a \cdot M\rangle &= \langle x' \cdot a, M\rangle = \langle x' \cdot a, q'(P'(e \otimes e))\rangle = \langle q(x' \cdot a), P'(e \otimes e)\rangle \\ &= \langle L_a^{'} q(x'), P'(e \otimes e)\rangle = \langle PL_a^{'} q(x'), (e \otimes e)\rangle \\ &= \langle L_a^{'} Pq(x'), (e \otimes e)\rangle = \langle Pq(x'), L_a(e \otimes e)\rangle \\ &= \langle Pq(x'), e \otimes a\rangle = \langle Pq(x'), (e \otimes e) \cdot a\rangle \\ &= \langle x', q'P'(e \otimes e) \cdot a\rangle = \langle x', M \cdot a\rangle.\end{aligned}$$

Hence, $a \cdot M = M \cdot a$. Also, we have

$$\begin{aligned}\langle f, \pi''(M) \cdot a\rangle &= \langle a \cdot f, \pi''(M)\rangle = \langle \pi'(a \cdot f), M\rangle = \langle \pi'(a \cdot f), q'(P'(e \otimes e))\rangle \\ &= \langle P(q(\pi'(a \cdot f))), (e \otimes e)\rangle = \langle q(\pi'(a \cdot f)), (e \otimes e)\rangle \\ &= \langle \pi'(a \cdot f), (e \otimes e)\rangle = \langle a \cdot f, \pi(e \otimes e)\rangle \\ &= \langle f, \pi(e \otimes e) \cdot a\rangle = \langle f, a\rangle \quad (a \in \mathcal{A}, f \in \mathcal{A}').\end{aligned}$$

Then, by Theorem 3.3.7, $\mathcal{A}$ is amenable. $\square$

3.3.3 Hereditary Properties of Contractible Banach Algebras.

In this section, we are concerned with hereditary properties of contractible Banach algebras. The hereditary properties for contractible Banach algebras are as nice as the hereditary properties for amenable Banach algebras.

Proposition 3.3.12 ([48])**.** *Let $\mathcal{A}$ be a Banach algebra.*

1. *If $\mathcal{A}$ is contractible and $\mathcal{B}$ another Banach algebra such that $\theta : \mathcal{A} \to \mathcal{B}$ is a continuous homomorphism with dense range, then $\mathcal{B}$ is contractible.*

2. *If I is a closed ideal of $\mathcal{A}$ such that both I and $\mathcal{A}/I$ are contractible, then $\mathcal{A}$ is contractible.*

3. *If $\mathcal{A}$ is contractible and $\mathcal{B}$ is also contractible, then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is contractible.*

Remark 3.3.13. *To avoid repetition, the proofs will be omitted. The proofs of the above hereditary properties are similar to that of amenable Banach algebras. In most cases, we just need to replace X' with X.*

Remark 3.3.14. *All contractible Banach algebras are amenable. Hence all results that are true for contractible Banach algebras are true for amenable Banach algebras, but the converse is not true in general. Both amenability notions are stable under tensor product.*

3.3.4 Characterization of Contractible Banach Algebras.

In this section, we give the well known characterization for contractible Banach algebras. That is all contractible Banach algebras are unital.

Theorem 3.3.15 ([48])**.** *Let $\mathcal{A}$ a Banach algbera. Then the following are equivalent:*

1. *$\mathcal{A}$ is contractible.*

2. *$\mathcal{A}$ is unital and possesses a diagonal.*

Proof. $1 \Rightarrow 2$. We need to show that $\mathcal{A}$ has a unit and possess a diagonal. Firstly, we show that $\mathcal{A}$ has a unit. In doing this, we will show that $\mathcal{A}$ has both left and right unit. Set $X = \mathcal{A}$, with the module operations defined as

$$a \cdot x = ax, \quad x \cdot a = 0 \quad (a \in \mathcal{A}, x \in X).$$

The identity map $I : \mathcal{A} \to X$ is a derivation from $\mathcal{A}$ into X. Indeed, for all $a, b \in \mathcal{A}$, we have

$$I(ab) = ab,$$

on the other hand, we have

$$a \cdot I(b) + I(a) \cdot b = a \cdot b + 0 = ab.$$

Since $\mathcal{A}$ is contractible, there exists $x \in X$ such that $I(a) = a \cdot x - x \cdot a = ax - 0 = ax$, for all $a \in \mathcal{A}$, which implies that $a = ax$. Then, x is the right unit for $\mathcal{A}$. Using similar argument, but with the module operations defined as $a \cdot y = 0$ and $y \cdot a = ya \quad (a \in \mathcal{A}, y \in X)$. Since $\mathcal{A}$ is a contractible, there exists $y \in X$ such that $I(a) = a \cdot y - y \cdot a = 0 - ya = -ya$, which implies that $a = -ya$. Then, $-y$ is the left unit for $\mathcal{A}$. It it well known that if the unit of a Banach algebra exists such unit is unique. Therefore, x and $-y$ coincide. Hence $\mathcal{A}$ has a unit.
Secondly, let us show that $\mathcal{A}$ has a diagonal element $u \in \mathcal{A}\widehat{\otimes}\mathcal{A}$. To achieve this, let us set $X := ker\ \pi$. Define $D : \mathcal{A} \to ker\ \pi$, as $D(a) = a \otimes e - e \otimes a$. We claim that this map is a bounded derivation. Indeed, for all $a \in \mathcal{A}$, we have

$$\begin{aligned} \|D(a)\| &= \|a \otimes e - e \otimes a\| \leq \|a \otimes e\| + \|e \otimes a\| \\ &= \|a\|\|e\| + \|e\|\|a\| = 2\|a\|\|e\| \quad (a, e \in \mathcal{A}). \end{aligned} \tag{3.8}$$

Hence, D is bounded. For the derivation, note that

$$\begin{aligned} D(ab) &= ab \otimes e - e \otimes ab = ab \otimes e - a \otimes b + a \otimes b - e \otimes ab \\ &= a \cdot (b \otimes e - e \otimes b) + (a \otimes e - e \otimes a) \cdot b \\ &= a \cdot D(b) + D(a) \cdot b \quad (a, b, e \in \mathcal{A}). \end{aligned} \tag{3.9}$$

Hence, D is a derivation. Now, since $\mathcal{A}$ is contractible, there exists $x \in X = ker\ \pi$ such that

$$\begin{aligned} & D(a) = a \cdot x - x \cdot a \\ \Rightarrow & a \otimes e - e \otimes a = a \cdot x - x \cdot a \\ \Rightarrow & a \otimes e - a \cdot x = e \otimes a - x \cdot a \\ \Rightarrow & a \cdot (e \otimes e - e \cdot x) = (e \otimes e - e \cdot x) \cdot a. \end{aligned}$$

Comparing $a \cdot (e \otimes e - e \cdot x) = (e \otimes e - e \cdot x) \cdot a$ with $a \cdot u = u \cdot a$, it implies $u = e \otimes e - e \cdot x$. We need to affirm that $e \otimes e - e \cdot x$ is a diagonal for $\mathcal{A}$. In doing this, we need to check if, $a \cdot \pi(e \otimes e - e \cdot x) = a$, since $a \cdot (e \otimes e - e \cdot x) = (e \otimes e - e \cdot x) \cdot a$. Indeed, for all $a \in \mathcal{A}$, we have that

$$\begin{aligned} a \cdot \pi(e \otimes e - e \cdot x) &= a \cdot (\pi(e \otimes e) - \pi(x)) = a \cdot (e - 0) \quad (\text{since } x \in X = ker\ \pi) \\ &= a \cdot e = a. \end{aligned}$$

It follows that, $\pi(e \otimes e - e \cdot x) = e$. Therefore, $e \otimes e - e \cdot x$ is a diagonal for $\mathcal{A}$.

$2 \Rightarrow 1$. Let X be a Banach $\mathcal{A}$-bimodule and suppose that $u = \sum_j a_j \otimes b_j \in \mathcal{A} \otimes \mathcal{A}$ is a diagonal for $\mathcal{A}$ and let $\pi(u) = e \in \mathcal{A}$ be a unit on $\mathcal{A}$. Let $D : \mathcal{A} \to X$ be a continuous derivation. We need to show that D inner. Since $\mathcal{A}$ is unital, we have

$$D(a) = D(ea) = e \cdot D(a) + D(e) \cdot a \quad (a \in \mathcal{A}). \tag{3.10}$$

Observe that

$$\begin{aligned}
e \cdot D(a) = \pi(u) \cdot D(a) = \pi\left(\sum_j a_j \otimes b_j\right) \cdot D(a) &= \sum_j a_j b_j \cdot D(a) \\
&= \sum_j a_j \cdot (D(b_j a) - D(b_j) \cdot a) = \sum_j a_j \cdot D(b_j a) - \sum_j a_j \cdot D(b_j) \cdot a \\
&\text{(using Equation 3.1)} \\
&= \sum_j \Phi(a_j \otimes b_j a) - \sum_j \Phi(a_j \otimes b_j) \cdot a \\
&= \Phi\left(\sum_j a_j \otimes b_j a\right) - \Phi\left(\sum_j a_j \otimes b_j\right) \cdot a \\
&= \Phi(u \cdot a) - \Phi(u) \cdot a = \Phi(a \cdot u) - \Phi(u) \cdot a \quad \text{(since u is a diagonal)} \\
&= a \cdot \Phi(u) - \Phi(u) \cdot a \quad \text{(using Lemma 3.1.7 (4))} \\
\text{let } \Phi(u) = x \\
&= a \cdot x - x \cdot a.
\end{aligned} \tag{3.11}$$

More so, for all $a \in \mathcal{A}$, we have

$$e \cdot D(e) \cdot a = e \cdot (D(ea) - e \cdot D(a)) = e \cdot D(a) - e \cdot D(a) = 0,$$

and so, we have

$$\begin{aligned}
D(e) \cdot a &= a \cdot D(e) - a \cdot D(e) - 0 + D(e) \cdot a \\
&= a \cdot D(e) - a \cdot D(e) - e \cdot D(e) \cdot a + D(e) \cdot a \\
&= a \cdot (e \cdot D(e) - D(e)) - (e \cdot D(e) - D(e)) \cdot a \\
\text{take } e \cdot D(e) - D(e) = y \\
&= a \cdot y - y \cdot a.
\end{aligned} \tag{3.12}$$

Now substituting Equations (3.11) and (3.3.4) in Equation (3.10), we have

$$\begin{aligned}
D(a) &= a \cdot x - x \cdot a + a \cdot y - y \cdot a \\
&= a \cdot (x + y) - (x + y) \cdot a = a \cdot z - z \cdot a \quad (x + y = z).
\end{aligned}$$

Thus, D is inner and so $\mathcal{A}$ is contractible. $\square$

Contractible Banach algebras are not much studied like other notions of amenability due to the lack of non-trivial examples. Over the years the only known examples of contractible Banach algebras are finite dimensional Banach algebras and semisimple Banach algebras. So far, the well known contractible Banach algebras are the direct sum of finite full matrix algebras.

3.3.5 Some Basic Results

In this section, we study some basic results on amenable Banach algebras.

Theorem 3.3.16 ([31]). *Let $\mathcal{A}$ be an amenable Banach algebra and let I be a closed ideal in $\mathcal{A}$. Then I is amenable if and only if I contains a bounded approximate identity.*

Proof. Suppose that I contains a bounded approximate identity. By Proposition 3.2.1, in order to show that I is amenable, it is suffices to show that $\mathcal{H}^1(I, X') = \{0\}$ for any pseudo-unital Banach I-bimodule X. Let X be pseudo-unital Banach I-bimodule and let $D : I \to X'$ be a derivation. By Proposition 3.2.2, we make X into a Banach $\mathcal{A}$-bimodule in the canonical fashion such that D has a unique $\overline{D} \in \mathcal{Z}(\mathcal{A}, X')$. Since $\mathcal{A}$ is amenable, there exists $x \in X'$, such that $\overline{D}(a) = a \cdot x - x \cdot a = \delta_x(a)$ for all $a \in \mathcal{A}$. Then by the restriction of $\overline{D} = D|_I$, we therefore have that D is also inner. Hence I is amenable.
Conversely, suppose that I is amenable, then by Proposition 3.3.3, I has a bounded approximate identity. □

Theorem 3.3.17 ([31]). *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let X be a Banach $\mathcal{A}$-bimodule such that $\mathcal{A}$ acts trivially on one side. Then $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$.*

Proof. Let us take the left module operation to be trivial. That is $\mathcal{A} \cdot X = \{0\}$, which implies that $X' \cdot \mathcal{A} = \{0\}$. Let $D : \mathcal{A} \to X'$ be a continuous derivation from $\mathcal{A}$ into X'. Then for all $a, b \in \mathcal{A}$, we have

$$D(ab) = a \cdot D(b) + D(a) \cdot b = a \cdot D(b).$$

Now, suppose (e_α) is a bounded approximate identity for $\mathcal{A}$. We then have that $(D(e_\alpha))_\alpha$ is a bounded net in X', then by Banach-Alaoglu's theorem, there exists a w^*- accumulation point y of $D(e_\alpha)$. It is known that any subnet of (e_α) is also a bounded approximate identity for $\mathcal{A}$, therefore, we take $w^* - \lim_\alpha D(e_\alpha) = y$. It then follows that for all $a \in \mathcal{A}$ and $x \in X$ we have

$$\begin{aligned}\langle x, D(a)\rangle &= \lim_\alpha \langle x, D(ae_\alpha)\rangle = \lim_\alpha \langle x, a \cdot D(e_\alpha) + D(a) \cdot e_\alpha\rangle \\ &= \lim_\alpha \langle x, a \cdot D(e_\alpha)\rangle = \lim_\alpha \langle x \cdot a, D(e_\alpha)\rangle = \langle x \cdot a, y\rangle \\ &= \langle x, a \cdot y\rangle = \langle x, a \cdot y - y \cdot a\rangle\end{aligned}$$

The last equality holds because $y \cdot a = 0$. Hence the derivation D from $\mathcal{A}$ into X' is inner and so, $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$. $\square$

Proposition 3.3.18. *If $\mathcal{A}$ is an amenable Banach algebra, then $\mathcal{A}\widehat{\otimes}e$ and $e\widehat{\otimes}\mathcal{A}$ are amenable.*

Proof. Let $\mathcal{A}$ be an amenable Banach algebra, X a Banach $\mathcal{A}$-bimodule and suppose that $D : \mathcal{A} \to X'$ is a derivation. Since X can be identified with a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$a \cdot x = (a \otimes e) \cdot x, \quad x \cdot a = x \cdot (a \otimes e) \quad (a \in \mathcal{A}, x \in X).$$

We define $d : \mathcal{A}\widehat{\otimes}e \to X'$ by $d(a \otimes e) = D(a)$ for all $a \in \mathcal{A}$. The map d from $\mathcal{A}\widehat{\otimes}e$ into X' is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned} d((a \otimes e)(b \otimes e)) &= d(ab \otimes e) = D(ab) = a \cdot D(b) + D(a) \cdot b \\ &= (a \otimes e) \cdot d((b \otimes e)) + d((a \otimes e)) \cdot (b \otimes e) \end{aligned}$$

Hence, d is a derivation. Since $\mathcal{A}$ is amenable, there exists $x \in X'$, such that $D(a) = a \cdot x - x \cdot a = \delta_x(a)$ for all $a \in \mathcal{A}$. Now, observe that

$$d(a \otimes e) = D(a) = a \cdot x - x \cdot a = (a \otimes e) \cdot x - x \cdot (a \otimes e) = \delta_x(a \otimes e).$$

Therefore every continuous derivation $d : \mathcal{A}\widehat{\otimes}e \to X'$ is inner. Hence, $\mathcal{A} \otimes e$ is amenable. We use similar argument to show that $e \otimes \mathcal{A}$ is amenable. $\square$

Proposition 3.3.19. *If $\mathcal{A}$ is amenable Banach algebra and suppose that I is a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/I\widehat{\otimes}\mathcal{A}/I$ is amenable.*

Proof. Since $\mathcal{A}$ is amenable, by Proposition 3.3.2, $\mathcal{A}/I$ is amenable and so $\mathcal{A}/I\widehat{\otimes}\mathcal{A}/I$ is amenable by Proposition 3.3.5. $\square$

Proposition 3.3.20 ([31])**.** *Let $\mathcal{A}$ be Banach algebra, J a closed ideal of $\mathcal{A}$ with bounded approximate identity. If J is amenable, then $\mathcal{A}$ is amenable.*

Proof. By Proposition 3.2.1, we can take X to be a pseudo-unital Banach $\mathcal{A}$-bimodule. It then suffices to show that $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$ for all pseudo-unital Banach $\mathcal{A}$-bimodule. Let $D : J \to X'$ be a derivation, then, by Proposition 3.2.2, there exists $\overline{D} : \mathcal{A} \to X'$ a unique derivation such that $\overline{D}|_J = D$. Using Proposition 3.2.2 and the amenability of J, we have that $\mathcal{H}^1(\mathcal{A}, X') = \{0\}$. $\square$

3.4 Approximate and Pseudo-amenable Banach Algebras.

The notions of approximate and pseudo amenability were introduced by F. Ghahramani and R. J. Loy in [18]. They conceived the idea of coming up with an amenability notion that does not have bounded approximate identity, which was known for the amenability notion that was introduced by B. E. Johnson in [31]. The corresponding class of Banach algebras is larger than that of the amenable Banach algebras. After the concept of approximate amenability was introduced, all examples of Banach algebras that were studied in this regard in [[6], [10], [19], [22]] have bounded approximate identity. It then became an open question whether approximately amenable Banach algebras must have a bounded approximate identity. In a positive direction, the authors in [6] proved that every boundedly approximately contractible Banach algebras have bounded approximate identity. In the same manner one might think the same holds for boundedly approximately amenable Banach algebras, but this is false. In [21], F. Ghahramani and C. J. Read gave examples of bounded approximate amenable Banach algebras without bounded approximate identity and used these Banach algebras to answer some of the open questions in this area.

In this section, we give some characterizations, hereditary properties and some results in literature on approximate and pseudo-amenable Banach algebras.

3.4.1 Hereditary Properties of Approximately Amenable Banach Algebras.

In this section, we give a detailed proof of some hereditary properties of approximately amenable Banach algebras.

Proposition 3.4.1 ([18]). *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is approximately amenable and $\mathcal{B}$ is another Banach algebra such that $\theta : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism, then $\mathcal{B}$ is approximately amenable.*

Proof. Let X be a Banach $\mathcal{B}$-bimodule, X becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$x \cdot a = x \cdot \theta(a), \quad a \cdot x = \theta(a) \cdot x \quad (a \in \mathcal{A}, x \in X).$$

Let $\theta : \mathcal{A} \to \mathcal{B}$ be an epimorphism and suppose that $D : \mathcal{B} \to X'$ is a derivation. The map $\overline{D} := (D \circ \theta) : \mathcal{A} \to X'$ is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have that

$$\begin{aligned}\overline{D}(ab) &= (D \circ \theta)(ab) = D(\theta(a)\theta(b)) = \theta(a) \cdot D(\theta(b)) + D(\theta(a)) \cdot \theta(b) \\ &= a \cdot (D \circ \theta)(b) + (D \circ \theta)(a) \cdot b = a \cdot \overline{D}(b) + \overline{D}(a) \cdot b \qquad (a, b \in \mathcal{A}).\end{aligned}$$

Hence, $\overline{D}$ is a derivation. Since $\mathcal{A}$ is approximately amenable, there exists exists a net $(x_\alpha) \subset X'$ such that

$$\overline{D}(a) = \lim_\alpha [a \cdot x_\alpha - x_\alpha \cdot a] = \lim_\alpha \delta_{x_\alpha}(a) \quad (a \in \mathcal{A}).$$

By our hypothesis that, $\theta : \mathcal{A} \to \mathcal{B}$ is an epimorphism, that is for all $b \in \mathcal{B}$, there exists $a \in \mathcal{A}$ such that $\theta(a) = b$. Then for every $b \in \mathcal{B}$, we have that

$$\begin{aligned} D(b) = D(\theta(a)) = Do\theta(a) = \overline{D}(a) &= \lim_\alpha [a \cdot x_\alpha - x_\alpha \cdot a] \\ = \lim_\alpha [\theta(a) \cdot x_\alpha - x_\alpha \cdot \theta(a)] &= \lim_\alpha [b \cdot x_\alpha - x_\alpha \cdot b] = \lim_\alpha \delta_{x_\alpha}(b), \end{aligned}$$

which implies that the derivation D from $\mathcal{B}$ into X' is approximately inner. Hence, $\mathcal{B}$ is approximately amenable. $\square$

Remark 3.4.2. *We remark that this argument does not extend to the closure of a homomorphic image as that of amenable Banach algebras.*

Corollary 3.4.3 ([18])**.** *Suppose $\mathcal{A}$ is approximately amenable and I a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/I$ is approximately amenable.*

Proof. Let X be $\mathcal{A}/I$-bimodule in the canonical fashion, $D : \mathcal{A}/I \to X'$ a derivation and $\theta : \mathcal{A} \to \mathcal{A}/I$ a canonical surjective homomorphism. X becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$x \cdot a = x \cdot \theta(a), \quad a \cdot x = \theta(a) \cdot x \quad (a \in \mathcal{A}, x \in X).$$

The map $d := (D \circ \theta) : \mathcal{A} \to X'$ is a derivation and since $\mathcal{A}$ is approximately amenable, there exists a net $(x_\alpha) \subset X'$ such that

$$d(a) = \lim_\alpha [a \cdot x_\alpha - x_\alpha \cdot a] = \lim_\alpha \delta_{x_\alpha}(a) \quad (a \in \mathcal{A}).$$

Now, observe that

$$\begin{aligned} D(a + I) &= D(\theta(a)) = (D \circ \theta)(a) = d(a) \\ &= \lim_\alpha [a \cdot x_\alpha - x_\alpha \cdot a] = \lim_\alpha [\theta(a) \cdot x_\alpha - x_\alpha \cdot \theta(a)] \\ &= \lim_\alpha [(a + I) \cdot x_\alpha - x_\alpha \cdot (a + I)] \\ &= \delta_{x_\alpha}(a + I) \quad (a \in \mathcal{A}). \end{aligned}$$

Hence, $\mathcal{A}/I$ is approximately amenable. $\square$

Proposition 3.4.4 ([18])**.** *If $\mathcal{A}$ is approximately amenable and has a bounded approximate identity and $\mathcal{B}$ is an amenable Banach algebra. Then $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately amenable.*

Proof. The proof follows from Proposition 3.3.5. $\square$

Lemma 3.4.5. *Suppose that $\mathcal{A}$ be a unital Banach algebra with identity e, X a Banach $\mathcal{A}$-bimodule, with the following module operations*

$$a \cdot x = ax, \quad x \cdot a = xa \qquad (a \in \mathcal{A}, x \in X).$$

If $D : \mathcal{A} \to X'$ is a derivation, then $D(e) = 0$.

Proof. Let $D : \mathcal{A} \to X'$ be a derivation. We have that $D(e) = D(ee) = e \cdot D(e) + D(e) \cdot e$. It then follows that

$$\langle e \cdot x, D(e) \cdot e\rangle = \langle x, D(e) \cdot e \cdot e\rangle = \langle x, D(e) \cdot e\rangle = \langle x, D(e)\rangle.$$

Then, $D(e) \cdot e = D(e)$, it follows that $e \cdot D(e) = 0$ and since $e \neq 0$, it implies that $D(e) = 0$. $\square$

Proposition 3.4.6 ([18])**.** *Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is approximately amenable if and only if $\mathcal{A}^{\#}$ is approximately amenable.*

Proof. Suppose that $\mathcal{A}^{\#}$ is approximately amenable and $D \in \mathcal{Z}^1(\mathcal{A}, X')$, where X is a Banach $\mathcal{A}$-bimodule. X becomes a Banach $\mathcal{A}^{\#}$-bimodule with the module operations defined as

$$(a, \alpha) \cdot x = \alpha x + a \cdot x \quad x \cdot (a, \alpha) = \alpha x + x \cdot a \quad ((a, \alpha) \in \mathcal{A}^{\#}, x \in X). \tag{3.13}$$

Define $d : \mathcal{A}^{\#} \to X'$, as $d(a, \alpha) = D(a) \quad ((a, \alpha) \in \mathcal{A}^{\#})$. This map is well defined, linear and a derivation. For linearity, observe that

$$d((a, \alpha) + (b, \beta)) = d(a + b, \alpha + \beta) = D(a + b) = D(a) + D(b) = d(a, \alpha) + d(b, \beta), \text{ also}$$
$$d(\alpha(a, \beta)) = d(\alpha a, \alpha\beta) = D(\alpha a) = \alpha D(a) = \alpha d(a, \beta) \qquad \forall a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}.$$

For derivation, note that

$$\begin{aligned} d((a, \alpha)(b, \beta)) &= d(ab + a\beta + \alpha b, \alpha\beta) = D(ab + a\beta + \alpha b) \\ &= D(ab) + D(a\beta) + D(\alpha b) \\ &= a \cdot D(b) + D(a) \cdot b + \beta D(a) + \alpha D(b) \\ &= \alpha D(b) + a \cdot D(b) + \beta D(a) + D(a) \cdot b \\ &\text{(using the module operations in Equation (3.13))} \\ &= (a, \alpha) \cdot D(b) + D(a) \cdot (b, \beta) \\ &= (a, \alpha) \cdot d(b, \beta) + d(a, \alpha) \cdot (b, \beta) \quad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}). \end{aligned}$$

Hence, d is a derivation. Since $\mathcal{A}^\#$ is approximately amenable, there exists $(x_\beta) \subset X'$ such that

$$d(a,\alpha) = \lim_\beta[(a,\alpha)\cdot x_\beta - x_\beta\cdot(a,\alpha)] = \lim_\beta \delta_{x_\beta}(a,\alpha) \quad ((a,\alpha)\in\mathcal{A}^\#).$$

Now, note that

$$\begin{aligned} D(a) = d(a,\alpha) &= \lim_\beta[(a,\alpha)\cdot x_\beta - x_\beta\cdot(a,\alpha)] \\ &= \lim_\beta[\alpha x_\beta + a\cdot x_\beta - \alpha x_\beta - x_\beta\cdot a] = \lim_\beta[a\cdot x_\beta - x_\beta\cdot a] \\ &= \lim_\beta \delta_{x_\beta}(a) \quad (a\in\mathcal{A}). \end{aligned}$$

Therefore, $\mathcal{A}$ is approximately amenable.
Conversely, suppose that $\mathcal{A}$ is approximately amenable. Let X be a Banach $\mathcal{A}^\#$-bimodule and $D : \mathcal{A}^\# \to X'$ a derivation. By Lemma 3.2.4, there exists $\eta \in X'$ and $D_1 : \mathcal{A}^\# \to e\cdot X'\cdot e$, such that $D = D_1 + \delta_\eta$. Set $d := D_1|_\mathcal{A} : \mathcal{A} \to e\cdot X'\cdot e$. Clearly, d is a derivation. Since $\mathcal{A}$ is approximately amenable, there exist a net $(x_\beta) \subset X'$ such that

$$d(a) = \lim_\beta[a\cdot x_\beta - x_\beta\cdot a] = \lim_\beta \delta_{x_\beta}(a) \quad (a\in\mathcal{A}).$$

Hence

$$D_1(a) = \lim_\beta[a\cdot x_\beta - x_\beta\cdot a] \quad (a\in\mathcal{A}).$$

Since $e\cdot X'\cdot e$ is unital, then by Lemma 3.4.5, $D(e) = 0$ and for each $(a+\alpha)\in\mathcal{A}^\#$, we have that

$$\begin{aligned} D_1(a+\alpha) = D_1(a) + D_1(\alpha) = D_1(a) + D_1(\alpha e) &= D_1(a) + \alpha D_1(e) = D_1(a) \\ &= \lim_\beta[a\cdot x_\beta - x_\beta\cdot a] = \lim_\beta[a\cdot x_\beta + \alpha x_\beta - \alpha x_\beta - x_\beta\cdot a] \\ &= \lim_\beta[(a+\alpha)\cdot x_\beta - x_\beta\cdot(a+\alpha)]. \end{aligned}$$

Hence, D_1 is approximately inner and so $\mathcal{A}^\#$ is approximately amenable. □

Proposition 3.4.7 ([18]). *Suppose that $\mathcal{A}$ is approximately amenable. Then $\mathcal{A}$ has left and right approximate identities. In particular $\mathcal{A}^2$ is dense in $\mathcal{A}$.*

Proof. Let $\mathcal{A}$ be an approximately amenable Banach algebra and set $X = \mathcal{A}$, X becomes a Banach $\mathcal{A}$-bimodule with the left and right module operations given as

$$a\cdot x = ax, \quad x\cdot a = 0 \quad (a\in\mathcal{A}, x\in X).$$

We also make $X' = \mathcal{A}'$ into a Banach $\mathcal{A}$-bimodule in the usual way

$$\langle x \cdot a, \alpha \rangle = \langle x, a \cdot \alpha \rangle = 0, \quad \langle a \cdot x, \alpha \rangle = \langle x, \alpha \cdot a \rangle \quad (a \in \mathcal{A}, x \in X, \alpha \in X').$$

We also make $X'' = \mathcal{A}''$ into a Banach $\mathcal{A}$-bimodule in the canonical fashion

$$\langle \alpha \cdot a, \Psi \rangle = \langle \alpha, a \cdot \Psi \rangle, \quad \langle \alpha, \Psi \cdot a \rangle = \langle a \cdot \alpha, \Psi \rangle = 0 \quad (a \in \mathcal{A}, \alpha \in X', \Psi \in X'').$$

The natural injection $D : \mathcal{A} \to X''$, defined as $D(a)(\alpha) = \alpha(a),\ a \in \mathcal{A}, \alpha \in X'$ is a derivation. Indeed, for all $a, b \in \mathcal{A}, \alpha \in X'$, we have

$$\begin{aligned} \langle \alpha, D(ab) \rangle &= \langle ab, \alpha \rangle = \langle a \cdot b, \alpha \rangle = \langle b, \alpha \cdot a \rangle = \langle \alpha \cdot a, D(b) \rangle \\ &= \langle \alpha, a \cdot D(b) \rangle + \langle \alpha, D(a) \cdot b \rangle = \langle \alpha, a \cdot D(b) + D(a) \cdot b \rangle \end{aligned}$$

The last equality in the equation above holds because $D(a) \cdot b = 0 = \langle \alpha, D(a) \cdot b \rangle = 0$. Then there exists a net $(e_\beta) \subset X''$ such that $a \cdot e_\beta \to \alpha(a)$ for each $a \in \mathcal{A}$. Take a finite set $F \subset \mathcal{A}, \Psi \subset \mathcal{A}'$ and $\epsilon > 0$. Let $H = \{\phi \cdot a \mid a \in F, \phi \in \Psi\}, K = max\{\|\psi\|, \|\phi\| \mid \psi \in H, \phi \in \Psi\}$. Then there is a $\beta = \beta_{(F,\Psi,\epsilon)}$ such that $\|a \cdot e_\beta - \alpha(a)\| < \frac{\epsilon}{2K}$ for any $a \in F$. By Goldstine's theorem, there is $b_\beta \in \mathcal{A}$ such that

$$|\langle b_\beta, \psi \rangle - \langle \psi, e_\beta \rangle| < \frac{\epsilon}{2} \quad (\psi \in H).$$

Therefore, for all $a \in F, \phi \in \Psi$, we have that

$$\begin{aligned} |\langle ab_\beta, \phi \rangle - \langle a, \phi \rangle| &= |\langle ab_\beta, \phi \rangle - \langle \phi, a \cdot e_\beta + a \cdot e_\beta - \alpha(a) \rangle| \\ &\le |\langle ab_\beta, \phi \rangle - \langle \phi, a \cdot e_\beta \rangle| + |\langle \phi, a \cdot e_\beta - \alpha(a) \rangle| \\ &= \ |\langle b_\beta, \phi \cdot a \rangle - \langle \phi \cdot a, e_\beta \rangle| + K\frac{\epsilon}{2K} \\ &\le \epsilon. \end{aligned}$$

Hence $(b_\beta)_{(F,\Psi,\epsilon)}$ is a weak right approximate identity for $\mathcal{A}$. Then by Theorem 2.2.11, $(b_\beta)_{(F,\Psi,\epsilon)}$ is an approximate identity. We use similar argument for left. □

Proposition 3.4.8 ([18])**.** *Let $\mathcal{A}$ be a Banach algebra. If I is a closed ideal of $\mathcal{A}$ such that I is amenable and $\mathcal{A}/I$ is approximately amenable, then $\mathcal{A}$ is approximately amenable.*

Proof. The proof follows from Proposition 3.3.2. □

3.4.2 Characterization of Approximately Amenable Banach Algebras.

In this section, we give an explicit proof of some characterizations of approximately amenable Banach algebras.

Theorem 3.4.9 ([18]). *Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is approximately amenable if and only if either of the following equivalent conditions hold:*

1. *there is a net $(M_v) \subset (\mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#})''$ such that for each $a \in \mathcal{A}^{\#}$,*
$$a \cdot M_v - M_v \cdot a \to 0 \quad \textit{and} \quad \pi''(M_v) \to e;$$

2. *there is a net $(M_v') \subset (\mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#})''$ such that for each $a \in \mathcal{A}^{\#}$,*
$$a \cdot M_v' - M_v' \cdot a \to 0 \quad \textit{and} \quad \pi''(M_v') = e \textit{ for every } v.$$

Proof. Clearly, (2) implies (1).
Suppose $\mathcal{A}$ is approximately amenable, so by Proposition 3.4.6, $\mathcal{A}^{\#}$ is approximately amenable. $\mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#}$ is a Banach $\mathcal{A}^{\#}$-bimodule with the module operations defined as

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in \mathcal{A}^{\#}).$$

Let $u = e \otimes e$ and define $D_u : \mathcal{A}^{\#} \to ker\ \pi''$ as $D_u(a) = a \cdot u - u \cdot a$ for all $a \in \mathcal{A}^{\#}$. It is easy to check that D_u is a derivation. Indeed, we have

$$\begin{aligned} D_u(ab) &= (ab) \cdot u - u \cdot (ab) = a \cdot (b \cdot u) - a \cdot (u \cdot b) + (a \cdot u) \cdot b - (u \cdot a) \cdot b \\ &= a \cdot (b \cdot u - u \cdot b) + (a \cdot u - u \cdot a) \cdot b = a \cdot D_u(b) + D_u(a) \cdot b \quad (a, b \in \mathcal{A}^{\#}). \end{aligned}$$

Hence, D_u is a derivation. Since $\mathcal{A}^{\#}$ is approximately amenable, there exists a net $(e_\alpha) \subset ker\ \pi''$ such that

$$D_u(a) = \lim_\alpha [a \cdot e_\alpha - e_\alpha \cdot a] = \lim_\alpha \delta_{e_\alpha}(a) \quad (a \in \mathcal{A}^{\#}).$$

Set $M_v' = u - e_\alpha$. Then for all $a \in \mathcal{A}$, we have

$$\begin{aligned} a \cdot M_v' - M_v' \cdot a &= a \cdot (u - e_\alpha) - (u - e_\alpha) \cdot a = a \cdot u - a \cdot e_\alpha - u \cdot a + e_\alpha \cdot a \\ &= a \cdot u - u \cdot a - (a \cdot e_\alpha + e_\alpha \cdot a) \to 0. \end{aligned}$$

Also, for all v

$$\pi''(M_v') = \pi''(u - e_\alpha) = \pi''(u) - \pi''(e_\alpha) = \pi''(u) = e.$$

Thus (2) holds.
Suppose (1) holds. Then by Proposition 3.4.6, it suffices to show that $\mathcal{A}^{\#}$ is approximately amenable. Let $D : \mathcal{A}^{\#} \to X'$ be a derivation. By Propositions 3.4.7 and 3.2.3, we may take X to be neo-unital. So for each v set $f_v(x) = M_v(\mu_x)$, where $a, b \in \mathcal{A}^{\#}, x \in X, \mu_x(a \otimes b) = \langle x, aD(b)\rangle$. Then, with $(M_\beta) \subset \mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#}$ converging weak* to M_v and noting that for $m \in \mathcal{A}^{\#}\widehat{\otimes}\mathcal{A}^{\#}, \langle m, \mu_{a\cdot x - x\cdot a}\rangle = \langle m, a\cdot\mu_x - \mu_x\cdot a\rangle + \langle x, \pi(m)D(a)\rangle$. Since X is neo-unital, so $X = X \cdot \mathcal{A}^{\#}$, so for each $a \in \mathcal{A}^{\#}$ and $x \in X$, we have that

$$\begin{aligned}
\langle a\cdot x - x\cdot a, f_v\rangle = \langle \mu_{a\cdot x - x\cdot a}, M_v\rangle &= \lim_\beta \langle \mu_{a\cdot x - x\cdot a}, M_\beta\rangle \\
&= \langle a\cdot\mu_x - \mu_x\cdot a, M_v\rangle + \lim_\beta \langle x, \pi(M_\beta)D(a)\rangle \\
&= \langle \mu_x, M_v\cdot a - M_v\cdot a\rangle + \langle x, \pi''(M_v)D(a)\rangle.
\end{aligned}$$

So

$$\begin{aligned}
&\|\langle x, a\cdot f_v - f_v\cdot a\rangle - \langle x, D(a)\rangle\| \\
&= \|\langle \mu_x, a\cdot M_v - M_v\cdot a\rangle + \langle x, \pi''(M_v)D(a) - \langle x, D(a)\rangle\| \\
&= \|\langle \mu_x, a\cdot M_v - M_v\cdot a\rangle + \langle x, \pi''(M_v)D(a) - D(a)\rangle\| \\
&= \|\langle \mu_x, a\cdot M_v - M_v\cdot a\rangle + \langle x, (\pi''(M_v) - e)D(a)\rangle\| \\
&\le \|\mu_x\|\cdot\|a\cdot M_v - M_v\cdot a\| + \|x\|\cdot\|\pi''(M_v) - e\|\cdot\|D(a)\| \\
&= \|D\|\cdot\|x\|\cdot\|a\cdot M_v - M_v\cdot a\| + \|x\|\cdot\|\pi''(M_v) - e\|\cdot\|D(a)\|,
\end{aligned}$$

then $D(a) = \lim_v[a\cdot f_v - f_v\cdot a]$. It follows that $\mathcal{A}^{\#}$ is approximately amenable and so is $\mathcal{A}$ by Proposition 3.4.6. Then the equivalence holds. □

Corollary 3.4.10 ([18])**.** *Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is approximately amenable if and only if there are nets $(M_v'') \subset (\mathcal{A}\widehat{\otimes}\mathcal{A})'', (F_v), (G_v) \subset \mathcal{A}''$, such that for each $a \in \mathcal{A}$,*

1. $a\cdot M_v'' - M_v''\cdot a + F_v \otimes a - a \otimes G_v \to 0$
2. $a\cdot F_v \to a, G_v\cdot a \to a$ *and*
3. $\pi''(M_v'')\cdot a - F_v\cdot a - G_v\cdot a \to 0.$

Proof. Suppose $\mathcal{A}$ is approximately amenable. Using the net (M_v) given in Theorem 3.4.9 (1) and write $M_v = M_v'' - F_v\otimes e - e\otimes G_v + c_v e\otimes e$, where $(M_v'') \subset (\mathcal{A}\widehat{\otimes}\mathcal{A})'', (F_v), (G_v) \subset \mathcal{A}''$ and $(c_v) \subset \mathbb{C}$. Applying π'' on M_v we have

$$\begin{aligned}
\pi''(M_v) &= \pi''(M_v'' - F_v\otimes e - e\otimes G_v + c_v e\otimes e) \\
&= \pi''(M_v'') - \pi''(F_v\otimes e) - \pi''(e\otimes G_v) + \pi''(c_v e\otimes e) \\
&= \pi''(M_v'') - F_v - G_v + c_v e.
\end{aligned}$$

By Theorem 3.4.9, $\pi''(M_v) \to e$, then $\pi''(M_v) \cdot a \to e \cdot a$. Since

$$\pi''(M_v^{''}) - F_v - G_v + c_v e \to e,$$

it follows that $c_v \to 1$ and for any $a \in \mathcal{A}$, we have

$$\pi''(M_v^{''}) \cdot a - F_v \cdot a - G_v \cdot a + e \cdot a \to e \cdot a.$$

It implies that

$$\pi''(M_v^{''}) \cdot a - F_v \cdot a - G_v \cdot a \to 0.$$

So we have (3). By Theorem 3.4.9 (1), for $a \in \mathcal{A}^{\#}, a \cdot M_v - M_v \cdot a \to 0$. Since $M_v = M_v^{''} - F_v \otimes e - e \otimes G_v + e \otimes e$, we then have that

$$\begin{aligned}
&a \cdot (M_v^{''} - F_v \otimes e - e \otimes G_v + e \otimes e) - (M_v^{''} - F_v \otimes e - e \otimes G_v + e \otimes e) \cdot a \to 0 \\
\Rightarrow\ &a \cdot M_v^{''} - a \cdot F_v \otimes e - a \otimes G_v + a \otimes e - M_v^{''} \cdot a + F_v \otimes a + e \otimes G_v \cdot a - e \otimes a \to 0.
\end{aligned}$$

We must then have that

$$a \cdot M_v^{''} - M_v^{''} \cdot a + F_v \otimes a - a \otimes G_v \to 0, a \cdot F_v \to a \quad \text{and} \quad G_v \cdot a \to a.$$

Hence we have (1) and (2).
Conversely, let $c_v \to 1, a \cdot F_v \to a, G_v \cdot a \to a$ and $M_v = M_v^{''} - F_v \otimes e - e \otimes G_v + e \otimes e$. We then have that

$$\begin{aligned}
a \cdot M_v - M_v \cdot a &= a \cdot M_v^{''} - a \cdot F_v \otimes e - a \otimes G_v + a \otimes e - M_v^{''} \cdot a + F_v \otimes a + e \otimes G_v \cdot a - e \otimes a \\
&= a \cdot M_v^{''} - a \otimes e - a \otimes G_v + a \otimes e - M_v^{''} \cdot a + F_v \otimes a + e \otimes a - e \otimes a \\
&= a \cdot M_v^{''} - a \otimes G_v - M_v^{''} \cdot a + F_v \otimes a \to 0.
\end{aligned}$$

Hence, $a \cdot M_v - M_v \cdot a \to 0$, for all $a \in \mathcal{A}^{\#}$. Also, we have

$$\begin{aligned}
\pi''(M_v) \cdot a &= \pi''(M_v^{''} - F_v \otimes e - e \otimes G_v + e \otimes e) \cdot a \\
&= (\pi''(M_v^{''}) - F_v - G_v + e) \cdot a \\
&= \pi''(M_v^{''}) \cdot a - F_v \cdot a - G_v \cdot a + e \cdot a \\
&= 0 + a \to a.
\end{aligned}$$

and so, $\pi''(M_v) \to e$. Then by Proposition 3.4.6, $\mathcal{A}$ is approximately amenable. $\square$

Theorem 3.4.11 ([36]). *Let $\mathcal{A}$ be a Banach algebra. The following are equivalent:*

1. *$\mathcal{A}$ is approximately amenable.*

2. *For a Banach $\mathcal{A}$-bimodule X and a Banach $\mathcal{A}$-submodule Y of X, if $f \in Z(\mathcal{A}, Y')$, then there exists a net $(g_\alpha)_{\alpha\in\Gamma} \subset X'$ of extensions of f such that $\lim_\alpha(a\cdot g_\alpha - g_\alpha \cdot a) =$ 0.*

3. *For any Banach $\mathcal{A}$-bimodule X, there exists a net $(P_\alpha)_{\alpha\in\Gamma}$, $P_\alpha : X' \to X'$ each P_α is a continuous operator and $P_\alpha|_{Z(\mathcal{A},X')} = Id_{Z(\mathcal{A},X')}$. Also P_α commutes with every weak*-weak* continuous bounded linear operator from X' into X' commuting with the action of $\mathcal{A}$ on X' and $\lim_\alpha(a \cdot P_\alpha(f) - P_\alpha(f) \cdot a) = 0$ $(f \in X')$.*

Proof. $1 \Rightarrow 2$. Since X is a Banach $\mathcal{A}$-bimodule, the quotient Banach space X/Y becomes a Banach $\mathcal{A}$-bimodule with the canonical module operations given as

$$a \cdot (x + Y) = a \cdot x + Y, \quad (x + Y) \cdot a = x \cdot a + Y \quad (a \in \mathcal{A}, x \in X).$$

Let $f \in Z(\mathcal{A}, Y')$ and $\widehat{f} \in X'$ be any extension of f to X. Define $D_1 : \mathcal{A} \to (X/Y)'$ as $D_1(a) = a \cdot \widehat{f} - \widehat{f} \cdot a$ $(a \in \mathcal{A})$. Clearly, D_1 is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned} D_1(ab) &= (ab) \cdot \widehat{f} - \widehat{f} \cdot (ab) = (ab) \cdot \widehat{f} - a \cdot \widehat{f} \cdot b + a \cdot \widehat{f} \cdot b - \widehat{f} \cdot (ab) \\ &= a \cdot (b \cdot \widehat{f}) - a \cdot (\widehat{f} \cdot b) + (a \cdot \widehat{f}) \cdot b - (\widehat{f} \cdot a) \cdot b \\ &= a \cdot (b \cdot \widehat{f} - \widehat{f} \cdot b) + (a \cdot \widehat{f} - \widehat{f} \cdot a) \cdot b \\ &= a \cdot D(b) + D(a) \cdot b. \end{aligned}$$

Hence, D is a derivation. Since $\mathcal{A}$ is approximately amenable, there exists a net $(h_\alpha) \subset (X/Y)'$ such that $D_1(a) = \lim_\alpha(a \cdot h_\alpha - h_\alpha \cdot a)$. Then, we have that

$$\begin{aligned} D_1(a) &= \lim_\alpha (a \cdot h_\alpha - h_\alpha \cdot a) \\ a \cdot \widehat{f} - \widehat{f} \cdot a &= \lim_\alpha (a \cdot h_\alpha - h_\alpha \cdot a) \\ \lim_\alpha (a \cdot h_\alpha - a \cdot \widehat{f} - h_\alpha \cdot a + \widehat{f} \cdot a) &= 0 \\ \lim_\alpha (a \cdot (h_\alpha - \widehat{f}) - (h_\alpha - \widehat{f}) \cdot a) &= 0 \\ \lim_\alpha (a \cdot g_\alpha - g_\alpha \cdot a) &= 0, \end{aligned}$$

where $g_\alpha = h_\alpha - \widehat{f}$. This implies that g_α is an extension of f.

$2 \Rightarrow 3$. The projective tensor product $X'\widehat{\otimes}X$ becomes a Banach $\mathcal{A}$-bimodule with the module operations defined as

$$(f \otimes x) \cdot a = f \otimes x \cdot a, \quad a \cdot (f \otimes x) = f \otimes a \cdot x \quad (a \in \mathcal{A}, x \in X, f \in X').$$

Let us define the sets H and K as follows

$$H := \overline{lin}\{T'(f) \otimes x - f \otimes T(x) \mid T \in \mathbb{B}(X), x \in X, f \in X'\}$$

and

$$K := \overline{lin}\{f \otimes x \mid f \in Z(\mathcal{A}, X'), x \in X\}.$$

It is clear from the definitions of H and K that both H and K are Banach $\mathcal{A}$-submodule of $(X'\widehat{\otimes}X)$. Hence, Y/H is also Banach $\mathcal{A}$-submodule of $(X'\widehat{\otimes}X)/H$. Let $\phi \in (X'\widehat{\otimes}X)'$ satisfying $\langle f \otimes x, \phi \rangle = f(x) \quad (x \in X, f \in X')$. Clearly, $\phi \in H^{\perp}$. Indeed, for all $x \in X, f \in X'$, observe that

$$\begin{aligned}\langle T'(f) \otimes x - f \otimes T(x), \phi \rangle &= \langle T'(f) \otimes x, \phi \rangle - \langle f \otimes T(x), \phi \rangle \\ &= \langle x, T'(f) \rangle - \langle T(x), f \rangle \\ &= \langle T(x), f \rangle - \langle T(x), f \rangle = 0.\end{aligned}$$

Hence, $\phi \in H^{\perp}$. We have $\Phi \in (X'\widehat{\otimes}X/H)'$ such that $\Phi(y+H) = \phi(y)$ for all $y \in (X'\widehat{\otimes}X)$. Now if $f \in Z(\mathcal{A}, X'), x \in X$, we have

$$\begin{aligned}\langle f \otimes x + H, a \cdot \Phi \rangle &= \langle (f \otimes x + H) \cdot a, \Phi \rangle = \langle f \otimes x \cdot a + H, \Phi \rangle \\ &= \langle f \otimes x \cdot a, \phi \rangle = \langle x \cdot a, f \rangle = \langle a \cdot x, f \rangle \\ &= \langle f \otimes a \cdot x, \phi \rangle = \langle f \otimes a \cdot x + H, \Phi \rangle \\ &= \langle (f \otimes x + H) \cdot a, \Phi \rangle = \langle f \otimes x + H, \Phi \cdot a \rangle.\end{aligned}$$

By hypothesis (2), there exists a net $\widehat{\Phi}_\alpha \subset ((X'\widehat{\otimes}X)/H)'$ which is an extension of Φ such that $\lim_\alpha (a \cdot \widehat{\Phi}_\alpha - \widehat{\Phi}_\alpha \cdot a) = 0$. Let us define $\langle x, P_\alpha(f) \rangle = \langle f \otimes x + H, \widehat{\Phi}_\alpha \rangle$ for all $x \in X, f \in X'$. If $f \in Z(\mathcal{A}, X')$, then it follows that

$$\langle x, P_\alpha(f) \rangle = \langle f \otimes x + H, \widehat{\Phi}_\alpha \rangle = \langle f \otimes x + H, \Phi \rangle = \langle f \otimes x, \phi \rangle = \langle x, f \rangle.$$

Hence $P_\alpha|_{Z(\mathcal{A},X')} = Id_{Z(\mathcal{A},X')}$. Let $T : X' \to X'$ be a weak*-weak* continuous and bounded operator which commutes with the action of $\mathcal{A}$ on X'. Then, since T is bounded continuous operator, we have $T = S'$ for S in $\mathbb{B}(X)$. Indeed, for all $f \in X', a \in \mathcal{A}$ and $x \in X$, we have

$$\begin{aligned}\langle S(a \cdot x), f \rangle &= \langle a \cdot x, S'(f) \rangle = \langle a \cdot x, T(f) \rangle \\ &= \langle x, T(f) \cdot a \rangle = \langle x, T(f \cdot a) \rangle \\ &= \langle x, S'(f \cdot a) \rangle = \langle S(x), f \cdot a \rangle \\ &= \langle a \cdot S(x), f \rangle.\end{aligned}$$

Thus, we have $S(a \cdot x) = a \cdot S(x)$, in similar way we can show that $S(x \cdot a) = S(x) \cdot a$. Hence, we have that $S \in \mathbb{B}(X)$. We then have that

$$\langle x, P_\alpha(T(f)) \rangle = \langle T(f) \otimes x + H, \widehat{\Phi}_\alpha \rangle = \langle S'(f) \otimes x + H, \widehat{\Phi}_\alpha \rangle$$

and since, $S \in \mathbb{B}(X)$, we have

$$S'(f) \otimes x + H - f \otimes S(x) \in H$$

and so

$$\begin{aligned}
\langle x, P_\alpha(T(f)) \rangle &= \langle T(f) \otimes x + H, \widehat{\Phi}_\alpha \rangle = \langle S'(f) \otimes x + H, \widehat{\Phi}_\alpha \rangle \\
&= \langle f \otimes S(x) + H, \widehat{\Phi}_\alpha \rangle = \langle S(x), P_\alpha(f) \rangle \\
&= \langle x, S'(P_\alpha(f)) \rangle = \langle x, T(P_\alpha(f)) \rangle.
\end{aligned}$$

Hence for all α, we have $P_\alpha T = T P_\alpha$. More so, we have

$$\begin{aligned}
\langle x, a \cdot P_\alpha(f) - P_\alpha(f) \cdot a \rangle &= \langle x, a \cdot P_\alpha(f) \rangle - \langle x, P_\alpha(f) \cdot a \rangle \\
&= \langle x \cdot a, P_\alpha(f) \rangle - \langle a \cdot x, P_\alpha(f) \rangle \\
&= \langle x \cdot a - a \cdot x, P_\alpha(f) \rangle = \langle f \otimes (x \cdot a - a \cdot x) + H, \widehat{\Phi}_\alpha \rangle \\
&= \langle f \otimes x \cdot a + H - f \otimes a \cdot x + H, \widehat{\Phi}_\alpha \rangle \\
&= \langle (f \otimes x + H) \cdot a, \widehat{\Phi}_\alpha \rangle - \langle a \cdot (f \otimes x + H), \widehat{\Phi}_\alpha \rangle \\
&= \langle f \otimes x + H, a \cdot \widehat{\Phi}_\alpha \rangle - \langle f \otimes x + H, \widehat{\Phi}_\alpha \cdot a \rangle \\
&= \langle f \otimes x + H, a \cdot \widehat{\Phi}_\alpha - \widehat{\Phi}_\alpha \cdot a \rangle.
\end{aligned}$$

Then

$$\begin{aligned}
|\langle x, a \cdot P_\alpha(f) - P_\alpha(f) \cdot a \rangle| &= |\langle f \otimes x + H, a \cdot \widehat{\Phi}_\alpha - \widehat{\Phi}_\alpha \cdot a \rangle| \\
&\leq \|f\| \|x\| \|a \cdot \widehat{\Phi}_\alpha - \widehat{\Phi}_\alpha \cdot a\|,
\end{aligned}$$

so

$$\|a \cdot P_\alpha(f) - P_\alpha(f) \cdot a\| \leq \|a \cdot \widehat{\Phi}_\alpha - \widehat{\Phi}_\alpha \cdot a\| \|f\|.$$

Since $\lim_\alpha (a \cdot \widehat{\Phi}_\alpha - \widehat{\Phi}_\alpha \cdot a) = 0$, we have

$$\lim_\alpha (a \cdot P_\alpha(f) - P_\alpha(f) \cdot a) = 0.$$

Hence (3) holds.

$3 \Rightarrow 1$. Without loss of generality, we may suppose that $\mathcal{A}$ is unital. Set $X := \mathcal{A}\widehat{\otimes}\mathcal{A}$, X becomes a Banach $\mathcal{A}$-bimodule in the canonical fashion. Let $\mathcal{F} = \{L_a, R_a : a \in \mathcal{A}\}$ be a family of bounded linear operators from X into X defined as

$$L_a(b \otimes c) = b \otimes ac \quad R_a(b \otimes c) = ba \otimes c \quad (a, b, c \in \mathcal{A}).$$

Each operator in $\mathcal{F}$ commutes with the actions of $\mathcal{A}$ on X. Indeed, for all $a, b, c, d \in \mathcal{A}$, we have

$$b \cdot L_a(c \otimes d) = b \cdot (c \otimes ad) = bc \otimes ad = bL_a(c \otimes d)$$

also,

$$b \cdot R_a(c \otimes d) = b \cdot (ca \otimes d) = bca \otimes d = bR_a(c \otimes d).$$

Clearly, the operator R_a, L_a in $\mathcal{F}$ commutes with the actions of $\mathcal{A}$ on X. Now, suppose the net P_α has the properties mentioned in (3). Let $q : X' \to X'$ be defined by $\langle a \otimes b, q(f)\rangle = \langle b \otimes a, f\rangle \quad (a, b \in \mathcal{A}, f \in X')$. From the way q is defined, we have

$$\begin{aligned}\langle c \otimes d, q(a \cdot x')\rangle &= \langle d \otimes c, a \cdot x'\rangle = \langle d \otimes ca, x'\rangle = \langle ca \otimes d, q(x')\rangle \\ &= \langle R_a(c \otimes d), q(x')\rangle = \langle (c \otimes d), R_a^{'} q(x')\rangle\end{aligned}$$

and

$$\begin{aligned}\langle c \otimes d, q(x' \cdot a)\rangle &= \langle d \otimes c, x' \cdot a\rangle = \langle ad \otimes c, x'\rangle = \langle c \otimes ad, q(x')\rangle \\ &= \langle L_a(c \otimes d), q(x')\rangle = \langle (c \otimes d), L_a^{'} q(x')\rangle,\end{aligned}$$

for all $a, b, c, d \in \mathcal{A}$ and $x' \in X'$. It then follows from our hypothesis in (3) that

$$P_\alpha R_a^{'} = R_a^{'} P_\alpha, P_\alpha L_a^{'} = L_a^{'} P_\alpha.$$

Let $M_\alpha = q'(P_\alpha^{'}(e \otimes e))$, where e is the identity of $\mathcal{A}$. Now observe that for all $a \in \mathcal{A}, x' \in X'$, we have

$$\begin{aligned}\langle x', M_\alpha \cdot a\rangle &= \langle a \cdot x', M_\alpha\rangle = \langle a \cdot x', q'(P_\alpha^{'}(e \otimes e))\rangle = \langle q(a \cdot x'), P_\alpha^{'}(e \otimes e)\rangle \\ &= \langle R_a^{'} q(x'), P_\alpha^{'}(e \otimes e)\rangle = \langle P_\alpha R_a^{'} q(x'), (e \otimes e)\rangle \\ &= \langle R_a^{'} P_\alpha q(x'), (e \otimes e)\rangle = \langle P_\alpha q(x'), R_a(e \otimes e)\rangle \\ &= \langle P_\alpha q(x'), a \otimes e\rangle = \langle P_\alpha q(x'), a \cdot (e \otimes e)\rangle \\ &= \langle P_\alpha q(x') \cdot a, (e \otimes e)\rangle.\end{aligned}$$

Similarly, for all $a \in \mathcal{A}, x' \in X'$, we have

$$\begin{aligned}\langle x', a \cdot M_\alpha \rangle &= \langle x' \cdot a, M_\alpha \rangle = \langle x' \cdot a, q'(P'_\alpha(e \otimes e)) \rangle = \langle q(x' \cdot a), P'_\alpha(e \otimes e) \rangle \\ &= \langle L'_a q(x'), P'_\alpha(e \otimes e) \rangle = \langle P_\alpha L'_a q(x'), (e \otimes e) \rangle \\ &= \langle L'_a P_\alpha q(x'), (e \otimes e) \rangle = \langle P_\alpha q(x'), L_a(e \otimes e) \rangle \\ &= \langle P_\alpha q(x'), e \otimes a \rangle = \langle P_\alpha q(x'), (e \otimes e) \cdot a \rangle \\ &= \langle a \cdot P_\alpha q(x'), (e \otimes e) \rangle.\end{aligned}$$

By our assumption, we then have that

$$\lim_\alpha (\langle x', a \cdot M_\alpha - M_\alpha \cdot a \rangle) = \lim_\alpha (\langle a \cdot P_\alpha q(x') - P_\alpha q(x') \cdot a, e \otimes e \rangle) = 0,$$

so $wk* - \lim_\alpha (a \cdot M_\alpha - M_\alpha \cdot a) = 0$. It is easy to see that $q(\pi'(x')) \in Z(\mathcal{A}, X')$. Indeed, for all $a, b, c \in \mathcal{A}, x' \in \mathcal{A}'$ we have that

$$\begin{aligned}\langle b \otimes c, a \cdot q(\pi'(x')) \rangle &= \langle b \otimes ca, q(\pi'(x')) \rangle = \langle ca \otimes b, \pi'(x') \rangle \\ &= \langle \pi(ca \otimes b), x' \rangle = \langle cab, x' \rangle = \langle \pi(c \otimes ab), x' \rangle \\ &= \langle c \otimes ab, \pi'(x') \rangle = \langle ab \otimes c, q(\pi'(x')) \rangle \\ &= \langle a \cdot (b \otimes c), q(\pi'(x')) \rangle = \langle b \otimes c, q(\pi'(x')) \cdot a \rangle.\end{aligned}$$

Hence, $q(\pi'(x')) \in Z(\mathcal{A}, X')$. Also, we have

$$\begin{aligned}\langle f, \pi''(M_\alpha) \cdot a \rangle &= \langle a \cdot f, \pi''(M_\alpha) \rangle = \langle \pi'(a \cdot f), M \rangle = \langle \pi'(a \cdot f), q'(P'_\alpha(e \otimes e)) \rangle \\ &= \langle P_\alpha(q(\pi'(a \cdot f))), (e \otimes e) \rangle = \langle q(\pi'(a \cdot f)), (e \otimes e) \rangle \\ &= \langle \pi'(a \cdot f), (e \otimes e) \rangle = \langle a \cdot f, \pi(e \otimes e) \rangle \\ &= \langle f, \pi(e \otimes e) \cdot a \rangle = \langle f, a \rangle \quad (a \in \mathcal{A}, f \in \mathcal{A}').\end{aligned}$$

It implies that

$$\langle f, \pi''(M_\alpha) \rangle = \langle f, e \rangle$$

Then, by Theorem 3.4.9, $\mathcal{A}$ is approximately amenable. □

Theorem 3.4.12 ([18]). *Suppose $\mathcal{A}$ is approximately amenable and let*

$$\Sigma : 0 \to X' \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an admissible short exact sequence of left $\mathcal{A}$-module. Then Σ approximately splits. That is, there is a net $G_v : Z \to Y$ of right inverse maps to g such that

$$\lim_v (a \cdot G_v - G_v \cdot a) = 0 \quad (a \in \mathcal{A})$$

and a net $F_v : Y \to X'$ of left inverse maps to f such that

$$\lim_v (a \cdot F_v - F_v \cdot a) = 0 \quad (a \in \mathcal{A}).$$

Proof. Suppose that $\mathcal{A}$ is approximately amenable and that Σ is a sequence of left $\mathcal{A}$-modules. Since Σ is admissible, there exists $G \in \mathcal{L}(Z, Y)$ such that $gG = I$ on Z. Define $D : \mathcal{A} \to \mathcal{L}(Z, Y)$ as $D(a) = a \cdot G - G \cdot a$ for all $a \in \mathcal{A}$. It is easy to check that D is a derivation. Indeed, for all $a, b \in \mathcal{A}$, we have that

$$\begin{aligned} D(ab) &= (ab) \cdot G - G \cdot (ab) = a \cdot (b \cdot G)a \cdot (G \cdot b) + (a \cdot G) \cdot b - (a \cdot G) \cdot b \\ &= a \cdot (b \cdot G - G \cdot b) + (a \cdot G - G \cdot a) \cdot b = a \cdot D(b) + D(a) \cdot b. \end{aligned}$$

Hence, D is a derivation. Then, for any $z \in Z$ and $gG = I$, we have

$$g((D(a))(z)) = g(a \cdot G - G \cdot a)(z) = az - az = 0.$$

Therefore, $D(\mathcal{A}) \subset \mathcal{L}(Z, ker\ g) = \mathcal{L}(Z, im\ f)$. Clearly $f^{-1} \circ D : \mathcal{A} \to (Z, X') = (Z \otimes X)'$ is a derivation. Since $\mathcal{A}$ is approximately amenable and $f^{-1} \circ D$ is approximately inner, then there exists a net $(Q_\alpha) \subset \mathcal{L}(Z, X')$ such that

$$D(a) = a \cdot G - G \cdot a = \lim_\alpha [a \cdot fQ_\alpha - fQ_\alpha \cdot a].$$

If $G_v = G - fQ_\alpha$, then $a \cdot G_v = G_v \cdot a$ and G_v is a left $\mathcal{A}$-module homomorphism form Z to Y. Furthermore, $gG_v(z) = gG(z) - gfQ_\alpha(z) = gG(z) = z$. Since $gG = I$ and $im(fQ_\alpha) \subset ker\ g$. Therefore G_v is a right inverse map to g and $\lim_v [a \cdot G_v - G_v \cdot a] = 0$. Similar argument holds for F_v. $\square$

3.4.3 Hereditary Properties of Pseudo-amenable Banach Algebras.

In this section, we give the following hereditary properties for pseudo-amenable Banach algebras.

Proposition 3.4.13 ([24])**.** *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ is pseudo-amenable, $\mathcal{B}$ another Banach algebra, and $\theta : \mathcal{A} \to \mathcal{B}$ is a continuous epimorphism, then $\mathcal{B}$ is pseudo-amenable.*

Proof. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a continuous epimorphism and suppose that $\mathcal{A}$ is pseudo-amenable. The map $\theta \otimes \theta : \mathcal{A} \widehat{\otimes} \mathcal{A} \to \mathcal{B} \widehat{\otimes} \mathcal{B}$ defined as $(\theta \otimes \theta)(a \otimes b) = \theta(a) \otimes \theta(b)$ for all $a, b \in \mathcal{A}$, takes any approximate diagonal for $\mathcal{A}$ to an approximate diagonal for $\mathcal{B}$. $\square$

3.4.4 Some Basic Results.

In this section, we study some basic results for approximately amenable and pseudo-amenable Banach algebras.

Proposition 3.4.14 ([22]). *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are approximately amenable Banach algebras. Then, for any neo-unital $(\mathcal{A}\oplus\mathcal{B})$-bimodule X, continuous derivations from $\mathcal{A}\oplus\mathcal{B}$ into X' are weak* approximately inner.*

Proof. Let $D:\mathcal{A}\oplus\mathcal{B}\to X'$ be a continuous derivation. Then D induces (continuous) derivation $d:\mathcal{A}\to X'$, defined as $d(a)=D(a,0),\ (a\in\mathcal{A})$ and $d_1:\mathcal{B}\to X'$, defined as $d(b)=D(0,b),\ (b\in\mathcal{B})$. Note that, for all $a\in\mathcal{A}, b\in\mathcal{B}$, we have

$$\begin{aligned} d(a)+d_1(b) &= D(a,0)+D(0,b) \\ &= D((a,0)+(0,b)) = D(a,b) \quad \text{(derivations are linear map).} \end{aligned} \tag{3.14}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are approximately amenable, there are nets $(x_\alpha),(y_\alpha)\subset X'$ such that

$$d(a)=\lim_\alpha[(a,0)\cdot x_\alpha - x_\alpha\cdot(a,0)] \quad (a\in\mathcal{A}), \tag{3.15}$$

$$d(b)=\lim_\alpha[(0,b)\cdot y_\alpha - y_\alpha\cdot(0,b)] \quad (b\in\mathcal{B}). \tag{3.16}$$

By Proposition 3.4.7, $\mathcal{A}$ and $\mathcal{B}$ possesses an approximate identity. Let η_α,ι_α be an approximate identity of $\mathcal{A}$ and $\mathcal{B}$ respectively. We then have

$$(a,0)=\lim_\alpha(a,b)(\eta_\alpha,0)=\lim_\alpha(\eta_\alpha,0)(a,b), \tag{3.17}$$

$$(0,b)=\lim_\alpha(a,b)(0,\iota_\alpha)=\lim_\alpha(0,\iota_\alpha)(a,b). \tag{3.18}$$

Using Equations (3.15), (3.16), (3.17) and (3.18). We have that

$$D(a,b)=d(a)+d_1(b)=\lim_\beta[(a,b)\cdot\gamma_\beta-\zeta_\beta\cdot(a,b)], \quad (a\in\mathcal{A}, b\in\mathcal{B}).$$

Since D is a derivation (γ_β) and (ζ_β) in the above equation satisfy

$$(a,b)\cdot(\gamma_\beta-\zeta_\beta)\cdot(c,d)\xrightarrow{\beta}0, \quad (a\in\mathcal{A}, b\in\mathcal{B}).$$

So, we have

$$D(a,b)\cdot(c,d)=\lim_\beta[(a,b)\cdot\Phi_\beta-\Phi_\beta\cdot(a,b)]\cdot(c,d), \quad (a,c\in\mathcal{A}, b,d\in\mathcal{B}).$$

Since X is neo-unital $(\mathcal{A}\oplus\mathcal{B})$-bimodule, this implies that $D(a,b)=\text{weak}^*-\lim_\beta[(a,b)\cdot\Phi_\beta-\Phi_\beta\cdot(a,b)]$, for all $a\in\mathcal{A}, b\in\mathcal{B}$. Hence, D is weak* approximately inner. $\square$

Proposition 3.4.15 ([22]). *If $\mathcal{A}\oplus\mathcal{A}$ is approximately amenable, then $\mathcal{A}$ has an approximate identity.*

Proof. We make $X = \mathcal{A}$ and $(\mathcal{A} \oplus \mathcal{A})$-bimodule by defining the module operation as

$$(a, b) \cdot x = a \cdot x, \quad x \cdot (a, b) = xb \qquad (a, b, \in \mathcal{A}, x \in X).$$

Define $D : \mathcal{A} \oplus \mathcal{A} \to X$, as $(a, b) \mapsto a - b \quad (a, b \in \mathcal{A})$. It is easy to see that D is a derivation. Indeed, for all $a, b, c, d \in \mathcal{A}$, we have

$$\begin{aligned} D((a, b)(c, d)) &= D(ac, bd) = ac - bd \\ &= ac - ad + ad - bd \\ &= a(c - d) + (a - b)d \\ &= aD(c, d) + D(a, b)d \\ &= (a, b) \cdot D(c, d) + D(a, b) \cdot (c, d) \quad \text{(using the module operations).} \end{aligned} \tag{3.19}$$

Hence D is a derivation. Since $\mathcal{A} \oplus \mathcal{A}$ is approximately amenable, there exists a net $(x_\alpha) \subset X$ for which

$$D(a, b) = \lim_\alpha [(a, b) \cdot x_\alpha - x_\alpha \cdot (a, b)] = \lim_\alpha [a \cdot x_\alpha - x_\alpha b], \quad ((a, b) \in \mathcal{A} \oplus \mathcal{A}).$$

Now, observe that

$$\begin{aligned} a - b = D(a, b) &= \lim_\alpha \delta_{x_\alpha}(a, b) = \lim_\alpha [(a, b) \cdot x_\alpha - x_\alpha \cdot (a, b)] = \lim_\alpha [a x_\alpha - x_\alpha b] \\ &= \lim_\alpha a x_\alpha - \lim_\alpha x_\alpha b. \end{aligned}$$

In particular, we have that $a = \lim_\alpha a x_\alpha$ and $b = \lim_\alpha x_\alpha b$ for all $a, b \in \mathcal{A}$. Hence, (x_α) is an approximate identity. $\square$

Suppose that $\mathcal{A}$ is an approximately amenable Banach algebra. In particular, $\mathcal{A}$ has one sided approximate identities. Considering the topology τ determine by the seminorms $b \mapsto \|ab\| \quad (a \in \mathcal{A})$.

Proposition 3.4.16 ([22]). *Suppose that $\mathcal{A}$ is approximately amenable and that τ is stronger than the weak topology on $\mathcal{A}$. Then $\mathcal{A}$ has an approximate identity.*

Proof. We make $X = \mathcal{A}$ and $\mathcal{A} \oplus \mathcal{A}$-bimodule by defining the module operation as

$$(a, b) \cdot x = ax, \quad x \cdot (a, b) = xb \qquad (a, b, \in \mathcal{A}, x \in X).$$

The map $D : \mathcal{A} \oplus \mathcal{A} \to X, (a, b) \mapsto a - b \ \forall\ a, b \in \mathcal{A}$ is a derivation. By Proposition 3.4.14, any derivation $D : \mathcal{A} \oplus \mathcal{A} \to X$, there is a net (x_α) in X such that

$$\begin{aligned} D(a, b) \cdot (c, d) &= \lim_\alpha [(a, b) \cdot x_\alpha - x_\alpha \cdot (a, b)] \cdot (c, d) \\ D(a, b)d &= \lim_\alpha [a \cdot x_\alpha \cdot d - x_\alpha \cdot bd] \\ (a - b)d &= \lim_\alpha [a \cdot x_\alpha - x_\alpha \cdot b]d. \end{aligned}$$

Then by our assumption on τ,

$$a - b = \lim_{\alpha}[ax_\alpha - x_\alpha b].$$

In particular, we have that $a = weak - \lim_\alpha a \cdot x_\alpha$, $b = weak - \lim_\alpha x_\alpha \cdot b$ for all $a, b \in \mathcal{A}$. Thus (x_α) is a weak approximate identity for $\mathcal{A}$. Then by Theorem 2.2.11, (x_α) is an approximate identity for $\mathcal{A}$. □

Proposition 3.4.17 ([23]). *Suppose that $\mathcal{A}^\#\widehat{\otimes}\mathcal{B}^\#$ is approximately amenable. Then $\mathcal{A}, \mathcal{B}$ and $\mathcal{A} \oplus \mathcal{B}$ are approximately amenable.*

Proof. The Banach algebra $\mathcal{A}^\#$ admit a non-zero character φ. Define $\Psi : \mathcal{A}^\#\widehat{\otimes}\mathcal{B}^\# \to \mathcal{B}^\#$ as $(a \otimes b) \mapsto \varphi(a)b$, $(a \in \mathcal{A}^\#, b \in \mathcal{B}^\#)$. It is easy to check that Ψ an epimorphism. Indeed, we have that

$$\begin{aligned}\Psi((a \otimes b)(c \otimes d)) &= \Psi(ac \otimes bd) = \varphi(ac)bd = \varphi(a)\varphi(c)bd = \varphi(a)b\varphi(c)d \\ &= \Psi(a \otimes b)\Psi(c \otimes d), \quad (a, c \in \mathcal{A}, b, d \in \mathcal{B}).\end{aligned}$$

Thus, the map is a homomorphism. Since φ is a non-zero character, Ψ is surjective. Hence Ψ is an epimorphism. By Proposition 3.4.1, $\mathcal{B}^\#$ is approximately amenable, also, by Proposition 3.4.6, $\mathcal{B}$ is approximately amenable. We use similar argument for $\mathcal{A}$. We have the decomposition into closed subalgebras,

$$\mathcal{A}^\#\widehat{\otimes}\mathcal{B}^\# = (1_\mathcal{A} \otimes 1_\mathcal{B}) + (1_\mathcal{A} \otimes \mathcal{B}) + (\mathcal{A} \otimes 1_\mathcal{B}) + (\mathcal{A}\widehat{\otimes}\mathcal{B}).$$

Thus $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is a closed ideal, since $\mathcal{A}^\#\widehat{\otimes}\mathcal{B}^\#$ is approximately amenable, the quotient algebra

$$(1_\mathcal{A} \otimes 1_\mathcal{B}) + (1_\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes 1_\mathcal{B}) \cong \big((1_\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes 1_\mathcal{B})\big)^\#$$

is also approximately amenable. We now define a map

$$\Upsilon : (1_\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{A} \otimes 1_\mathcal{B}) \to \mathcal{A} \oplus \mathcal{B},\ ((1_\mathcal{A} \otimes b)(a \otimes 1)) \mapsto (a, b) \ \ (a \in \mathcal{A}, b \in \mathcal{B}).$$

The map Υ is well defined and isometric surjective algebra isomorphism. Then by Proposition 3.4.1, $\mathcal{A} \oplus \mathcal{B}$ is approximately amenable. □

Proposition 3.4.18 ([42]). *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. Then, $\mathcal{A} \times \mathcal{B}$ is approximately amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are amenable.*

Proof. Suppose $\mathcal{A} \times \mathcal{B}$ is approximately amenable. Define a map

$$\Phi : \mathcal{A} \times \mathcal{B} \to \mathcal{B},\ (a, b) \mapsto b.$$

Clearly, Φ is a continuous surjective homomorphism. Since $\mathcal{A} \times \mathcal{B}$ is approximately amenable, then by Proposition 3.4.1, $\mathcal{B}$ is approximately amenable. We use similarly argument for $\mathcal{A}$.
Conversely, suppose both $\mathcal{A}$ and $\mathcal{B}$ are approximately amenable. We define $C := \{(0, b) : b \in \mathcal{B}\}$. It is easy to see that $\mathcal{B} \cong C$ and that $(\mathcal{A} \times \mathcal{B})/\mathcal{B} = \mathcal{A}$, so by Proposition 3.4.8, $\mathcal{A} \times \mathcal{B}$ is approximately amenable. □

Proposition 3.4.19 ([24]). *Let $\mathcal{A}$ be pseudo-amenable Banach algebra and let J be a closed left/right ideal of $\mathcal{A}$. Then J has a left/right approximate identity if it is bounded approximately complemented in $\mathcal{A}$.*

Proof. Suppose that $\mathcal{A}$ is pseudo-amenable and J is a left ideal that is bounded approximately complemented in $\mathcal{A}$. Let $P_\alpha : \mathcal{A} \to J, \alpha \in \Gamma$, be a net such that $\|P_\alpha\| \leq M$, for some constant $M > 0$ and $P_\alpha(a) \xrightarrow{\alpha} a$ for each $a \in J$. Define $\Phi_\alpha : \mathcal{A}\widehat{\otimes}\mathcal{A} \to J$ by $\Phi_\alpha(a \otimes b) = aP_\alpha(b)$, for all $a, b \in \mathcal{A}$. Using similar argument as in Lemma 3.1.7 (1), Φ_α is a bounded linear operator. Then $\|\Phi_\alpha\| \leq M$. Given a finite set $F \subset J$ and $\epsilon > 0$, since $\mathcal{A}$ is pseudo-amenable, there exists an element $m \in \mathcal{A}\widehat{\otimes}\mathcal{A}$ such that

$$\|fm - mf\| < \frac{\epsilon}{3M} \text{ and } \|\pi(m)f - f\| < \frac{\epsilon}{3} \quad (f \in F).$$

Assume that $m = \sum_{j=1}^n a_j \otimes b_j$ and let $\mu_\alpha = \Phi_\alpha(m) = \sum_{j=1}^n a_j P_\alpha(b_j)$. It is clear that $\mu_\alpha \in J$. Furthermore, there is α such that

$$\|P_\alpha(b_j f) - b_j f\| < \frac{\epsilon}{3\sum_{i=1}^n \|a_i\|} \quad 1 \leq j \leq n \quad (f \in F).$$

Then for α and all $f \in F$, we have

$$\begin{aligned}
\|f\mu_\alpha - f\| &= \|f\Phi_\alpha(m) - f\| = \|f(\sum_{j=1}^n a_j P_\alpha b_j) - f\| = \|\sum_{j=1}^n f a_j P_\alpha b_j - f\| \\
&= \|\sum_{j=1}^n f a_j P_\alpha b_j - \sum_{j=1}^n a_j P_\alpha(b_j f) + \sum_{j=1}^n a_j P_\alpha(b_j f) - \sum_{j=1}^n a_j b_j f + \sum_{j=1}^n a_j b_j f - f\| \\
&= \|\Phi_\alpha(fm) - \Phi_\alpha(mf) + \sum_{j=1}^n a_j[P_\alpha(b_j f) - b_j f] + \sum_{j=1}^n a_j b_j f - f\| \\
&= \|\Phi_\alpha(fm) - \Phi_\alpha(mf) + \sum_{j=1}^n a_j[P_\alpha(b_j f) - b_j f] + \pi(m)f - f\|
\end{aligned}$$

$$\leq \|\Phi_\alpha(fm) - \Phi_\alpha(mf)\| + \sum_{j=1}^{n} \|a_j\| \|P_\alpha(b_j f) - b_j f\| + \|\pi(m)f - f\|$$
$$= \|\Phi_\alpha(fm - mf)\| + \sum_{j=1}^{n} \|a_j\| \|P_\alpha(b_j f) - b_j f\| + \|\pi(m)f - f\|$$
$$\leq \|\Phi_\alpha\| \|fm - mf\| + \sum_{j=1}^{n} \|a_j\| \|P_\alpha(b_j f) - b_j f\| + \|\pi(m)f - f\|$$
$$\leq M\frac{\epsilon}{3M} + \sum_{j=1}^{n} \|a_j\| \frac{\epsilon}{3\sum_{1=1}^{n} \|a_i\|} + \frac{\epsilon}{3} = \epsilon.$$

This shows that J has a right approximate identity. The proof in the case that J is a right ideal is similar. $\square$

Proposition 3.4.20 ([24]). *Let $\mathcal{A}$ be a pseudo-amenable Banach algebra and let J be a two sided closed ideal of $\mathcal{A}$. If J has an approximate identity (t_i) such that the associated left and right multiplication operators $L_i : a \mapsto t_i a$ and $R_i : a \mapsto a t_i$ from $\mathcal{A}$ to J are uniformly bounded, then J is pseudo-amenable.*

Proof. With the condition on (t_i), there is a constant $M > 0$ such that $\|t_i f\| \leq M\|f\|$ and $\|f t_i\| \leq M\|f\|$ for all t_i and all $f \in \mathcal{A}$. Then $\|t_i m\| \leq M\|m\|$ and $\|m t_i\| \leq M\|m\|$ for all t_i and $m \in \mathcal{A}\widehat{\otimes}\mathcal{A}$. Let $(M_\alpha) \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ be an approximate diagonal for $\mathcal{A}$. Given $\epsilon > 0$ and a finite set $F \subset J$, choose α such that $\|fM_\alpha - M_\alpha f\|M^2 \leq \frac{\epsilon}{2}$ and $\|\pi(M_\alpha)f - f\|M \leq \frac{\epsilon}{2}$ for $f \in F$. Also, choose i such that $\|f t_i - t_i f\|M\|M_\alpha\| \leq \frac{\epsilon}{4}$, $\|t_i f - f\| \leq \frac{\epsilon}{4}$ and $\|\pi(M_\alpha)(t_i f - f)\|M \leq \frac{\epsilon}{4}$ for all $f \in F$. We claim that the subnet of $(t_i M_\alpha t_i) \subset J\widehat{\otimes}J$ is an approximate diagonal for J. Indeed, for all $f \in F$, we have

$$\begin{aligned}
\|f t_i M_\alpha t_i - t_i M_\alpha t_i f\| &= \|f t_i M_\alpha t_i - t_i f M_\alpha t_i + t_i f M_\alpha t_i - t_i M_\alpha f t_i + t_i M_\alpha f t_i - t_i M_\alpha t_i f\| \\
&= \|(f t_i - t_i f) M_\alpha t_i + t_i (f M_\alpha - M_\alpha f) t_i + t_i M_\alpha (f t_i - t_i f)\| \\
&\leq \|(f t_i - t_i f) M_\alpha t_i\| + \|t_i (f M_\alpha - M_\alpha f) t_i\| + \|t_i M_\alpha (f t_i - t_i f)\| \\
&\leq \|f t_i - t_i f\| \|M_\alpha t_i\| + \|t_i\| \|f M_\alpha - M_\alpha f\| \|t_i\| + \|t_i M_\alpha\| \|f t_i - t_i f\| \\
&\leq \|f t_i - t_i f\| M \|M_\alpha\| + M\|f M_\alpha - M_\alpha f\| M + M\|M_\alpha\| \|f t_i - t_i f\| \\
&= 2\|f t_i - t_i f\| M \|M_\alpha\| + \|f M_\alpha - M_\alpha f\| M^2 \\
&\leq 2\frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Also, we have

$$\begin{aligned}
\|\pi(t_iM_\alpha t_i)f - f\| &= \|\pi(t_iM_\alpha t_i)f - t_i\pi(M_\alpha)f + t_i\pi(M_\alpha)f - t_if + t_if - f\| \\
&= \|t_i\pi(M_\alpha)t_if - t_i\pi(M_\alpha)f + t_i\pi(M_\alpha)f - t_if + t_if - f\| \\
&= \|t_i\pi(M_\alpha)[t_if - f] + t_i[\pi(M_\alpha)f - f] + t_if - f\| \\
&\leq \|t_i\pi(M_\alpha)[t_if - f]\| + \|t_i[\pi(M_\alpha)f - f]\| + \|t_if - f\| \\
&\leq \|t_i\|\|\pi(M_\alpha)[t_if - f]\| + \|t_i\|\|\pi(M_\alpha f) - f]\| + \|t_if - f\| \\
&\leq \|\pi(M_\alpha)[t_if - f]\|M + \|\pi(M_\alpha)f - f]\|M + \|t_if - f\| \\
&\leq \frac{\epsilon}{4} + \frac{\epsilon}{2} + \frac{\epsilon}{4} = \epsilon.
\end{aligned}$$

Hence our claim is justified. Hence J is pseudo-amenable. $\square$

Proposition 3.4.21 ([24])**.** *Let $\mathcal{A}$ be a Banach algebra having a central approximate identity. If $\mathcal{A}$ is approximately amenable, then it is pseudo-amenable.*

Proof. Let (e_α) be a central approximate identity for $\mathcal{A}$. Given $\epsilon > 0$ and a finite set $F \subset \mathcal{A}$, choose $e_{\alpha 1}, e_{\alpha 2} \in (e_\alpha)$ such that

$$\|e_{\alpha 1}a - a\| < \frac{\epsilon}{2}, \quad \|e_{\alpha 2}e_{\alpha 1}a - a\| < \frac{\epsilon}{2} \quad (a \in F).$$

Let $X = ker\pi$. Define $D : \mathcal{A} \to ker\pi$ by $D(a) = ae_{\alpha 1} \otimes e_{\alpha 2} - e_{\alpha 1} \otimes e_{\alpha 2}a$ for all $a \in \mathcal{A}$. Clearly, D is a bounded derivation. Indeed, for all $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
D(ab) &= abe_{\alpha 1} \otimes e_{\alpha 2} - e_{\alpha 1} \otimes e_{\alpha 2}ab = abe_{\alpha 1} \otimes e_{\alpha 2} - ae_{\alpha 1} \otimes e_{\alpha 2}b + ae_{\alpha 1} \otimes e_{\alpha 2}b - e_{\alpha 1} \otimes e_{\alpha 2}ab \\
&= a \cdot (be_{\alpha 1} \otimes e_{\alpha 2} - e_{\alpha 1} \otimes e_{\alpha 2}b) + (ae_{\alpha 1} \otimes e_{\alpha 2} - e_{\alpha 1} \otimes e_{\alpha 2}a) \cdot b \\
&= a \cdot D(b) + D(a) \cdot a.
\end{aligned}$$

Also,

$$\begin{aligned}
\|D(a)\| &= \|ae_{\alpha 1} \otimes e_{\alpha 2} - e_{\alpha 1} \otimes e_{\alpha 2}a\| \leq \|ae_{\alpha 1} \otimes e_{\alpha 2}\| + \|e_{\alpha 1} \otimes e_{\alpha 2}a\| \\
&= \|ae_{\alpha 1}\|\|e_{\alpha 2}\| + \|e_{\alpha 1}\|\|e_{\alpha 2}a\| \leq \|a\|\|e_{\alpha 1}\|\|e_{\alpha 2}\| + \|e_{\alpha 1}\|\|e_{\alpha 2}\|\|a\| \\
&= 2\|a\|\|e_{\alpha 1}\|\|e_{\alpha 2}\|.
\end{aligned}$$

Hence D is a bounded derivation. Since $\mathcal{A}$ is approximately amenable, it is approximately contractible, so there exist $u = u(e_{\alpha 1}, e_{\alpha 2}, \epsilon, F) \in X$ for which $\|D(a) - (a \cdot u - u \cdot a)\| < \epsilon, \quad (a \in F)$. Take $M = e_{\alpha 1} \otimes e_{\alpha 2} - u$. Then for all $a \in \mathcal{A}$, we have

$$\begin{aligned}
\|a \cdot M - M \cdot a\| &= \|a \cdot (e_{\alpha 1} \otimes e_{\alpha 2} - u) - (e_{\alpha 1} \otimes e_{\alpha 2} - u) \cdot a\| \\
&= \|ae_{\alpha 1} \otimes e_{\alpha 2} - a \cdot u - e_{\alpha 1} \otimes e_{\alpha 2}a + u \cdot a\| \\
&= \|ae_{\alpha 1} \otimes e_{\alpha 2} - e_{\alpha 1} \otimes e_{\alpha 2}a - (a \cdot u - u \cdot a)\| \\
&= \|D(a) - (a \cdot u - u \cdot a)\| < \epsilon,
\end{aligned}$$

also,

$$\begin{aligned}\|\pi(M)\cdot a-a\| &= \|\pi(e_{\alpha 1}\otimes e_{\alpha 2}-u)\cdot a-a\| = \|[\pi(e_{\alpha 1}\otimes e_{\alpha 2})-\pi(u)]\cdot a-a\| \\ &\text{(since u} \in X = ker\ \pi) \\ &= \|e_{\alpha 1}e_{\alpha 2}a-a\| = \|e_{\alpha 1}e_{\alpha 2}a-e_{\alpha 1}a+e_{\alpha 1}a-a\| \\ &\leq \|e_{\alpha 1}e_{\alpha 2}a-e_{\alpha 1}a\|+\|e_{\alpha 1}a-a\| \leq \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon, \quad (a\in F).\end{aligned}$$

This implies that

$$a\cdot M - M\cdot a \to 0, \quad \pi(M)\cdot a - a \to 0,$$

so therefore, $\mathcal{A}$ has an approximate diagonal. Hence, $\mathcal{A}$ is pseudo-amenable. $\square$

Proposition 3.4.22 ([24])**.** *Suppose that $\mathcal{A}$ is pseudo-amenable. Let X be a Banach $\mathcal{A}$-bimodule such that each approximate identity of $\mathcal{A}$ is also a one sided (i.e left or right) approximate identity for X. Then:*

1. *every continuous derivation $D : \mathcal{A} \to X$ is approximately inner;*

2. *every continuous derivation $D : \mathcal{A} \to X'$ is weak* approximately inner.*

Proof. Suppose that $(M_\alpha) \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ is an approximate diagonal for $\mathcal{A}$ and $(\pi(M_\alpha))$ is a right approximate identity for X. Let $M_\alpha = \sum_j a_j^{(\alpha)} \otimes b_j^{(\alpha)}$ and suppose that D is a continuous derivation from $\mathcal{A}$ into X. Since $\mathcal{A}$ is pseudo-amenable, it follows that $a\cdot M_\alpha \to M_\alpha \cdot a$ and $\pi(M_\alpha)\cdot a \to a$.

1. Observe that

$$\begin{aligned} a\cdot M_\alpha &\to M_\alpha \cdot a \\ \Rightarrow a\cdot\left(\sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}\right) &\to \left(\sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}\right)\cdot a \\ \Rightarrow \sum_j aa_j^{(\alpha)}\otimes b_j^{(\alpha)} &\to \sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}a. \end{aligned}$$

It follows that,

$$\sum_j D(aa_j^{(\alpha)})\cdot b_j^{(\alpha)} \to \sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)}a$$
$$\Rightarrow \sum_j a\cdot D(a_j^{(\alpha)})\cdot b_j^{(\alpha)} + \sum_j (D(a)\cdot a_j^{(\alpha)})\cdot b_j^{(\alpha)} \to \sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)}a$$
$$\Rightarrow a\cdot \sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)} + D(a)\cdot \sum_j a_j^{(\alpha)}b_j^{(\alpha)} \to \left(\sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)}\right)\cdot a$$
$$\Rightarrow D(a)\cdot \sum_j a_j^{(\alpha)}b_j^{(\alpha)} \to \left(\sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)}\right)\cdot a - a\cdot \sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)}$$
$$\text{set } x_\alpha = -\sum_j D(a_j^{(\alpha)})\cdot b_j^{(\alpha)}$$
$$\Rightarrow D(a)\cdot \pi(M_\alpha) \to a{\cdot}x_\alpha - x_\alpha\cdot a$$

Since $(\pi(M_\alpha))$ is a right approximate identity for X, we have that $D(a)\pi(M_\alpha) \to D(a)$.

$$D(a) \to a\cdot x_\alpha - x_\alpha\cdot a$$
$$D(a) = \lim_\alpha (a\cdot x_\alpha - x_\alpha\cdot a) \quad (a\in\mathcal{A}).$$

We use similar argument for the left.

2. Also, note that

$$a\cdot M_\alpha \to M_\alpha\cdot a$$
$$\Rightarrow a\cdot (\sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}) \to (\sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)})\cdot a$$
$$\Rightarrow \sum_j aa_j^{(\alpha)}\otimes b_j^{(\alpha)} \to \sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}a,$$

it then follows that,

$$\Rightarrow \sum_j aa_j^{(\alpha)}\cdot D(b_j^{(\alpha)}) \to \sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)}a)$$
$$\Rightarrow a\cdot\left(\sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})\right) \to \sum_j a_j^{(\alpha)}\cdot b_j^{(\alpha)}\cdot D(a) + \sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})\cdot a$$
$$\Rightarrow \sum_j a_j^{(\alpha)}\cdot (b_j^{(\alpha)}\cdot D(a)) \to -\sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})\cdot a + a\cdot (\sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})$$
$$\text{set } x_\alpha = \sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})$$
$$\Rightarrow \pi(M_\alpha)\cdot D(a) \to a\cdot x_\alpha - x_\alpha\cdot a$$

By our assumption, we have that $\pi(M_\alpha)D(a) \xrightarrow{w^*} D(a)$.

$$\begin{aligned}
&D(a) \to a \cdot x_\alpha - x_\alpha \cdot a \\
&D(a) = \text{weak* } \lim_\alpha (a \cdot x_\alpha - x_\alpha \cdot a) \quad (a \in \mathcal{A}).
\end{aligned}$$

We use similar argument for the left. $\square$

Chapter 4

Some Notions of Amenability in Banach Semigroup Algebras

A lot of investigative study into the concept of amenable, approximate amenable and pseudo-amenable Banach semigroup algebras $\ell^1(S)$ have been carried out by different researchers. In this chapter, a survey of results on the notions of amenable, approximate amenable and pseudo-amenable of Banach semigroup algebras $\ell^1(S)$ are presented.

4.1 Amenability of $\ell^1(\mathbf{S})$

We remarked in the introduction that the notion of amenability was first studied for some classes of amenable groups by Von Neumann in 1904. After which, the concept of amenability was extended and studied for semigroups by M. M. Day. We recall that a discrete semigroup S is *left amenable* if the space $\ell^\infty(S)$ admits a functional m called *mean* such that $m(1) = \|m\| = 1$ and the mean is left invariant. That is $m(\ell_x f) = m(f)$, where $(\ell_x f)(y) = f(xy),\ (x, y \in S, f \in \ell^\infty(S))$. Similarly for right amenable. If S is both left and right amenable, then it is amenable.

For a locally compact group G. It is an established result that, G is amenable if and only if $L^1(G)$ is amenable, but this is not generally true for a semigroup S. For example, the bicyclic semigroup S is amenable, but $\ell^1(S)$ is not amenable, see [13]. The amenability of $\ell^1(S)$ as a Banach algebra is some how complicated. We shall present some partial results regarding amenability of $\ell^1(S)$ as a Banach algebra and also give the result that really determines exactly when $\ell^1(S)$ is amenable as a Banach algebra.

Theorem 4.1.1 ([13]). *Let S be an inverse semigroup with $E(S)$ finite. Then $\ell^1(S)$ is amenable if and only if each maximal subgroup of S is amenable.*

Theorem 4.1.2. *Let S be a semigroup.*

1. *Suppose that $\ell^1(S)$ is an amenable Banach algebra. Then:*

    (a) *S is an amenable semigroup;*

    (b) *S is left and right reversible;*

    (c) *S has only finitely many idempotents and each ideal I in S is regular and in particular $I^2 = I$, S has a minimal idempotents;*

    (d) *$\ell^1(S)$ has an identity, $K(S)$ exists and is an amenable group;*

    (e) *$\ell^1(S)$ is a semisimple algebra.*

2. *Suppose that S is unital and left or right cancellative. Then $\ell^1(S)$ is amenable if and only if S is an amenable group.*

3. *Suppose S is abelian. Then $\ell^1(S)$ is amenable if and only if S is a finite semilattice of amenable groups.*

Proof. 1. (a) See [13], Lemma 3.

    (b) See [50], Lemma 1.

    (c) See [14], Theorem 2.

    (d) See [11], Corollary 10.6.

    (e) See [15], Theorem 5.11.

2. See [28], Theorem 2.3.

3. See [27], Theorem 2.7.

□

From the above results, the condition for the amenability of $\ell^1(S)$ imposes strong algebraic constraints on the semigroup S. More so, $\ell^1(S)$ seems amenable if and only if S is constructed out of an amenable group. We now proceed to give the result which tells exactly when $\ell^1(S)$ is amenable.

Theorem 4.1.3 ([11])**.** *Let S be a semigroup. Then the Banach algebra $\ell^1(S)$ is amenable if and only if the minimum ideal $K(S)$ exists, $K(S)$ is an amenable group and S has a principal series $S = I_1 \supset I_2 \supset I_3 \supset \cdots \supset I_{m-1} \supset I_m = K(S)$ such that each quotients I_j/I_{j+1}, is a regular Rees matrix semigroup of the form $\mathcal{M}^0(G, P, n)$, where $n \in \mathbb{N}, G$ is an amenable group and the sandwich matrix P is invertible in $\mathcal{M}(\ell^1(G))$.*

Theorem 4.1.4 ([13])**.** *Let S be the Brandt semigroup over a group G with finite index set I. Then $\ell^1(S)$ is amenable if and only if G is amenable*

The above result does not hold if the indexing set is infinite.

Theorem 4.1.5 ([13])**.** *Let S be the Brandt semigroup with an infinite index set over an arbitrary group. Then $\ell^1(S)$ is not amenable.*

4.2 Approximate and Pseudo Amenability of $\ell^1(S)$

For a locally compact group G, the characterizations of approximate amenability and pseudo-amenability of $L^1(G)$ in terms of the amenability of G are well known results in the literature. But for a semigroup S, the characterizations of approximate amenability and pseudo-amenability of $\ell^1(S)$ in terms of the amenability of the semigroup S are not known in general. There are only partial results established on this in the literature. For example, it was established in [18, 24] that, for a locally compact group $G, L^1(G)$ is approximately (pseudo) amenable if and only if G is amenable. This is never true for a semigroup S. For instance, the bicyclic semigroup S is amenable, but $\ell^1(S)$ is never approximately amenable, see [26]. In [4], M. L. Bami and H. Samea investigated approximate amenability of the discrete semigroup algebras $\ell^1(S)$ for left cancellative semigroups. They established that, if $\ell^1(S)$ is approximately amenable as a Banach algebra, then the semigroup S is left amenable. The converse is false, but it was shown that if S is a finite semigroup and $\ell^1(S)$ is approximately amenable, then S is amenable. In [16] and [17], the authors considered the pseudo amenability of semigroups for certain classes of inverse, Brandt, band and cancellative semigroups. It was established in [16] that, for an inverse semigroup S with uniformly locally finite idempotent set S, the semigroup algebra $\ell^1(S)$ is pseudo-amenable if and only if each maximal subgroup of S is amenable.

In this section, we shall give some partial results regarding the characterizations of approximate amenability and pseudo amenability of $\ell^1(S)$ in terms of the amenability of the semigroup S.

Theorem 4.2.1. *Let S be a semigroup.*

1. *Suppose that the semigroup algebra $\ell^1(S)$ is approximately amenable. Then S is regular and amenable.*

2. *Suppose that $\ell^1(S)$ is approximately amenable and that S is right cancellative. Then S is an amenable group and so $\ell^1(S)$ is amenable.*

3. *Suppose $\ell^1(S)$ is approximately amenable and $E(S)$ is finite. Then $\ell^1(S)$ has identity.*

Proof.

1. See [22], Theorem 9.2.

2. See [13], Theorem 2.3.

3. See [12], Corollary 2.2.2.

□

Theorem 4.2.2 ([12])**.** *Let S be a semigroup such that $E(S)$ is finite and let T be an ideal in S. Suppose that $\ell^1(S)$ is approximately amenable. Then $\ell^1(T)$ is approximately amenable.*

Remark 4.2.3. *The above results gives an hereditary property of approximately amenable Banach semigroup algebras.*

Theorem 4.2.4 ([12])**.** *Let S be a semigroup such that $E(S)$ is finite. Then $\ell^1(S)$ is approximately amenable if and only if it is amenable.*

Theorem 4.2.5 ([49])**.** *Let S be the Brandt semigroup over the group G with index set I. Then the following are equivalent;*

1. *$\ell^1(S)$ is amenable;*
2. *$\ell^1(S)$ is approximately amenable;*
3. *I is finite and G is amenable.*

Theorem 4.2.6 ([45])**.** *Let S be a uniformly locally finite inverse semigroup. Then, the following are equivalent;*

1. *$\ell^1(S)$ is approximately amenable;*
2. *$E(S)$ is finite and each maximal subgroup of S is amenable;*
3. *$\ell^1(S)$ is amenable;*
4. *$\ell^1(S)$ is boundedly approximate contractible;*
5. *$\ell^1(S)$ is boundedly approximate amenable.*

We now give some partial results for pseudo amenability of $\ell^1(S)$.

Theorem 4.2.7 ([16])**.** *Let S be an inverse semigroup. Suppose $\ell^1(S)$ is pseudo-amenable, then S is an amenable group.*

Theorem 4.2.8 ([16])**.** *Let S be a band semigroup. Suppose $\ell^1(S)$ is pseudo-amenable, then S is semilattice and so amenable.*

Theorem 4.2.9 ([16])**.** *Let S be a uniformly locally finite band semigroup. Then $\ell^1(S)$ is pseudo-amenable if and only if S is semilattice.*

Theorem 4.2.10 ([17])**.** *Let $S = \cup_{p\in E(S)} G_p$ be the Clifford semigroup such that $E(S)$ is uniformly locally finite. Then $\ell^1(S)$ is pseudo-amenable if and only if G_p is amenable for every $p \in E(S)$.*

Theorem 4.2.11 ([16])**.** *Let S be a left cancellative semigroup. Then the following are equivalent:*

1. *$\ell^1(S)$ is pseudo-amenable;*

2. *S is an amenable group;*

3. *$\ell^1(S)$ is amenable.*

Definition 4.2.12. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is called* biflat *if there exists a bounded $\mathcal{A}$-bimodule homomorphism $\rho : \mathcal{A} \to (\mathcal{A}\widehat{\otimes}\mathcal{A})''$ such that $\pi'' \circ \rho = k_{\mathcal{A}}$, where $k_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}''$ is the natural embedding of $\mathcal{A}$ into its second dual.*

Theorem 4.2.13 ([17])**.** *Let S be an inverse semigroup such that $(E(S), \leq)$ is uniformly locally finite. Then the following are equivalent:*

1. *$\ell^1(S)$ is pseudo-amenable.*

2. *Each maximal subgroup of S is amenable.*

3. *$\ell^1(S)$ is biflat.*

4.3 Some Basic Results

In this section, we give some interesting result on the amenability notions of semigroup algebras $\ell^1(S)$.

Proposition 4.3.1. *Let S and T be semigroups and suppose $\ell^1(S)$ and $\ell^1(T)$ are semigroup algebras. If $\ell^1(S)$ and $\ell^1(T)$ are amenable, then $S \times T$ is amenable.*

Proof. Since $\ell^1(S)$ and $\ell^1(T)$ are amenable, then by Proposition 3.3.5, $\ell^1(S)\widehat{\otimes}\ell^1(T)$ is amenable. Since $\ell^1(S)\widehat{\otimes}\ell^1(T)$ can be identified with $\ell^1(S \times T)$, it follows that $\ell^1(S \times T)$ is amenable. Then by Theorem 4.1.2 (1a), $S \times T$ is amenable. $\square$

Proposition 4.3.2. *Let S be a semigroup and T a closed ideal of S. If $\ell^1(S)$ is amenable, then $\ell^1(S/T)$ and S/T are amenable.*

Proof. Since $\ell^1(S)$ is amenable and T a closed ideal of S. It is known that $\ell^1(T)$ is a closed ideal of $\ell^1(S)$, then by Proposition 3.3.1, $\ell^1(S)/\ell^1(T)$ is amenable. Since $\ell^1(S)/\ell^1(T)$ can be identified with $\ell^1(S/T)$, it follows that $\ell^1(S/T)$ is amenable. Then by Theorem 4.1.2 (1a), S/T is also amenable. $\square$

Proposition 4.3.3. *Let S and T be semigroups and suppose that $\theta : S \to T$ is an isomorphism. If $\ell^1(S)$ is amenable, then $\ell^1(T)$ is amenable.*

Proof. Since S and T are semigroups and $\theta : S \to T$ is an isomorphism. It is known that the isomorphism between S and T can be extended to their algebras and that $\ell^1(S)$ is isometrically isomorphic to $\ell^1(T)$. So if $\ell^1(S)$ is amenable so is $\ell^1(T)$. $\square$

Proposition 4.3.4. *Let S and T be semigroups and suppose that $\theta : S \to T$ is an epimorphism. If $\ell^1(S)$ is approximately amenable, then $\ell^1(T)$ is approximately amenable.*

Proof. Since S and T are semigroups and $\theta : S \to T$ is an epimorphism. It is known that the epimorphism between S and T can be extended to their algebras and since $\ell^1(S)$ is approximately amenable, then by Proposition 3.4.1 $\ell^1(T)$ is approximately amenable. $\square$

Remark 4.3.5. *The above result also hold for pseudo-amenability.*

Definition 4.3.6. *Let $\mathcal{A}$ be a Banach algebra and let I be a non-empty set. We denote $\mathbb{M}_I(\mathcal{A})$, the set of $I \times I$ matrices (a_{ij}) with entries in $\mathcal{A}$ such that*

$$\|(a_{ij})\| = \sum_{i,j\in I} \|a_{ij}\| < \infty.$$

Then $\mathbb{M}_I(\mathcal{A})$ with the usual matrix multiplication is a Banach algebra that belongs to the class of ℓ^1-Munn algebras. It is clear that the map $\theta : \mathbb{M}_I(\mathcal{A}) \to \mathcal{A}\widehat{\otimes}\mathbb{M}_I(\mathbb{C})$ defined as

$$\theta((a_{ij})) = \sum_{i,j\in I} a_{ij} \otimes E_{ij} \quad ((a_{ij}) \in \mathbb{M}_I(\mathcal{A})),$$

is an isometric isomorphism of Banach algebras, where E_{ij} are matrix units in $\mathbb{M}_I(\mathbb{C})$.

Proposition 4.3.7 ([12])**.** *Let G be a group and let $n \in \mathbb{N}$. Then*

1. *$\mathcal{M}_n(\ell^1(G))$ is approximately amenable if and only if it is amenable.*
2. *$\mathcal{M}_n(\ell^1(G))$ is pseudo-amenable if and only if it is amenable.*

Proof.
1. Suppose $\mathcal{M}_n(\ell^1(G))$ is approximately amenable. By Proposition 1.6.7 (ii) in [12], $\ell^1(G)$ is approximately amenable. Using Theorem 3.2 in [22], the algebra $\ell^1(G)$ is approximately amenable if and only if G is amenable and this holds if and only if $\ell^1(G)$ is amenable. We then have that the algebra $\ell^1(G)$ is amenable. From [11], Theorem 2.7 (i) $\mathcal{M}_n(\ell^1(G))$ is amenable.
2. Suppose $\mathcal{M}_n(\ell^1(G))$ is pseudo-amenable. By Corollary 3.3 in [17], $\ell^1(G)$ is pseudo-amenable. Using Theorem 4.1 in [24], the algebra $\ell^1(G)$ is pseudo-amenable if and only if G is amenable and this holds if and only if $\ell^1(G)$ is amenable. We then have that the algebra $\ell^1(G)$ is amenable. From [11], Theorem 2.7 (i) $\mathcal{M}_n(\ell^1(G))$ is amenable.

$\square$

Corollary 4.3.8. *Let G be a group and let $n \in \mathbb{N}$. If*

1. *$\mathcal{M}_n(\ell^1(G))$ is approximately amenable, then $\ell^1(G)$ is pseudo-amenable.*
2. *$\mathcal{M}_n(\ell^1(G))$ is pseudo-amenable, then $\ell^1(G)$ is approximately amenable.*

Chapter 5

Collection of Results

The results in this chapter serves as our contribution to knowledge.

5.1 Results on General Banach Algebras

Proposition 5.1.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and suppose that $\mathcal{A}$ and $\mathcal{B}$ admit non-zero character. Then $\mathcal{A}$, $\mathcal{B}$ are approximately amenable if $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately amenable.*

Proof. Suppose $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is approximately amenable and that $\mathcal{A}$ and $\mathcal{B}$ admit non-zero character. Let

$$\psi : \mathcal{A} \to \mathbb{C}$$

and

$$\phi : \mathcal{B} \to \mathbb{C}$$

be non-zero characters on $\mathcal{A}$ and $\mathcal{B}$ respectively. Define $\Upsilon_{\mathcal{B}} : \mathcal{A}\widehat{\otimes}\mathcal{B} \to \mathcal{B}$, as $(a \otimes b) \mapsto \psi(a)b,\ \forall\ a \in \mathcal{A}, b \in \mathcal{B}$. This map is well defined and continuous. It is easy to check that Υ is an epimorphism. Indeed, for all $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$, observe that

$$\begin{aligned}\Upsilon_{\mathcal{B}}((a \otimes b)(c \otimes d)) &= \Upsilon_{\mathcal{B}}(ac \otimes bd) = \psi(ac)bd = \psi(a)\psi(c)bd \\ &= \psi(a)b\psi(c)d = \Upsilon_{\mathcal{B}}((a \otimes b)\Upsilon_{\mathcal{B}}(c \otimes d).\end{aligned}$$

Hence, the map is a homomorphism. And since ψ is a non-zero character, $\Upsilon_{\mathcal{B}}$ is surjective. Therefore, $\Upsilon_{\mathcal{B}}$ is an epimorphism. Hence, the result holds from Proposition 3.4.1. We use similar argument for $\mathcal{A}$. □

Proposition 5.1.2. *Suppose $\mathcal{A}^{\#}\widehat{\otimes}\mathcal{B}^{\#}$ is finite-dimensional approximately amenable Banach algebra. Then $\mathcal{A}$ and $\mathcal{B}$ are amenable.*

Proof. Since $\mathcal{A}^{\#}\widehat{\otimes}\mathcal{B}^{\#}$ is approximately amenable, then by Theorem 2.7 of [23], $\mathcal{A}$ and $\mathcal{B}$ are approximately approximately amenable. Thus, the amenability of $\mathcal{A}$ and $\mathcal{B}$ follows from the fact that they are finite-dimensional and approximately amenable, see Proposition 3.2.5. $\square$

The next result is a slight modification of Proposition 3.4.22, but the approach and method of prove is completely different from that of Proposition 3.4.22 and thus we claim that the next result is new.

Proposition 5.1.3. *Suppose that $\mathcal{A}$ is pseudo-contractible Banach algebra and let X be a Banach $\mathcal{A}$-bimodule such that an approximate identity for $\mathcal{A}$ is also an approximate identity for X. Then every continuous derivation $D : \mathcal{A} \to X$ is approximately inner.*

Proof. Since $\mathcal{A}$ is a pseudo-contractible. Then $\mathcal{A}$ has a central approximate diagonal, say $(M_\alpha) \subset \mathcal{A}\widehat{\otimes}\mathcal{A}$ which is an approximate diagonal with $M_\alpha \cdot a = a \cdot M_\alpha$ for all $a \in \mathcal{A}$ and all α. Let $D : \mathcal{A} \to X$ be a continuous derivation. Observe that

$$\begin{aligned}
\lim_\alpha[\pi(M_\alpha)\cdot D(a)] &= \lim_\alpha\left[\sum_j \pi(a_j^{(\alpha)}\otimes b_j^{(\alpha)})\cdot D(a)\right] = \lim_\alpha\left[\sum_j a_j^{(\alpha)} b_j^{(\alpha)}\cdot D(a)\right] \\
&= \lim_\alpha\left[\sum_j a_j^{(\alpha)}\cdot\Big(D(b_j^{(\alpha)}a) - D(b_j^{(\alpha)})\cdot a\Big)\right] \\
&= \lim_\alpha\left[\sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)}a) - \sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})\cdot a\right],
\end{aligned}$$

where $\Phi : \mathcal{A}\widehat{\otimes}\mathcal{A} \to X$ is defined by $\Phi(a\otimes b) = a\cdot D(a)$. Using Equation (3.1), with $\psi = D$, we have

$$\begin{aligned}
\lim_\alpha\left[\sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)}a) - \sum_j a_j^{(\alpha)}\cdot D(b_j^{(\alpha)})\cdot a\right] &= \lim_\alpha\Bigg[\Phi\Big(\sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}a\Big) \\
&\quad - \Phi\Big(\sum_j a_j^{(\alpha)}\otimes b_j^{(\alpha)}\Big)\cdot a\Bigg] \\
&= \lim_\alpha\Big[\Phi(M_\alpha\cdot a) - \Phi(M_\alpha)\cdot a\Big].
\end{aligned}$$

Since $\mathcal{A}$ have a central approximate diagonal and also using Lemma 3.1.7 (4). It follows that

$$\begin{aligned}
\lim_\alpha\Big[\Phi(M_\alpha\cdot a) - \Phi(M_\alpha)\cdot a\Big] &= \lim_\alpha\Big[\Phi(a\cdot M_\alpha) - \Phi(M_\alpha)\cdot a\Big] \\
&= \lim_\alpha\Big[a\cdot\Phi(M_\alpha) - \Phi(M_\alpha)\cdot a\Big] \\
&= \lim_\alpha\Big[a\cdot x_\alpha - x_\alpha\cdot a\Big],
\end{aligned}$$

where $(\Phi(M_\alpha)) = (x_\alpha) \subset X$. Since $(\pi(M_\alpha))$ is an approximate identity for X, then $\pi(M_\alpha) \cdot D(a) = D(a)$, which implies that

$$D(a) = \lim_\alpha [a \cdot x_\alpha - x_\alpha \cdot a].$$

□

Proposition 5.1.4. *Let $\mathcal{A}$ be a unital Banach algebra and I a non-empty set. If $\mathbb{M}_I(\mathcal{A})$ is approximately amenable, then $\mathbb{M}_I(\mathcal{A})$ is pseudo-amenable amenable.*

Proof. Suppose $\mathbb{M}_I(\mathcal{A})$ is approximately amenable. Then by Theorem 5.2 of [45], $\mathcal{A}$ is approximately amenable and I is finite. Now, using Theorem 2.2 of [16], $\mathbb{M}_I(\mathcal{A})$ have a central approximate identity, since I is finite. Then, by Proposition 3.4.21, $\mathbb{M}_I(\mathcal{A})$ is pseudo-amenable. □

5.2 Results on Semigroups

We begin with some definitions that will be needed in proving the results in this section. We recall that a semigroup S is called a *band semigroup* if $S = E(S)$, where $E(S)$ is the set of idempotents of S. We say that S is a *rectangular band* semigroup if it is a band semigroup and for each $x, y \in S, xyx = x$. Let S be a semigroup and S^1 denote the unitization of S. We define an equivalence relation on S by

$$a\tau b \Leftrightarrow S^1 a S^1 = S^1 b S^1 \quad (a, b \in S).$$

If S is a regular semigroup, we have

$$a\tau b \;\Leftrightarrow\; SaS = SbS \quad (a, b \in S),$$

see [30].
If S is a band semigroup. Then by Theorem 4.4.1 of [30], S is a semilattice of rectangular band semigroup. Indeed, $S = \cup_{\alpha \in Y} S_\alpha$ where $Y = S/\tau$ and for each $\alpha = [s] \in Y, S_\alpha = [s]$.

Definition 5.2.1. *Let $\mathcal{A}$ be a Banach algebra, Λ be a semillatice and $\{\mathcal{A}_\alpha \mid \alpha \in \Lambda\}$ be a collection of closed subalgebras of $\mathcal{A}$. Suppose $\mathcal{A}$ is an ℓ^1-direct sum of $\mathcal{A}_\alpha'$s as a Banach space such that*

$$\mathcal{A}_\alpha \mathcal{A}_\beta \subseteq \mathcal{A}_{\alpha\beta} \quad (\alpha, \beta \in \Lambda).$$

Then $\mathcal{A}$ is called ℓ^1-graded of $\mathcal{A}_\alpha'$s over the semilattice Λ and denoted by $\mathcal{A} = \ell^1 - \oplus_{\alpha\in\Lambda}\mathcal{A}_\alpha$.

Proposition 5.2.2. *Let S be a band semigroup and suppose that L and R are left and right zero semigroups. Suppose that $\ell^1(S)$ is approximately amenable and that every approximate identity of a complemented ideal of $\ell^1(S)$ is bounded, then $\ell^1(L)\widehat{\otimes}\ell^1(R), \ell^1(L)$ and $\ell^1(R)$ are approximately amenable.*

Proof. By the above argument, suppose that $S = \cup_{\alpha\in\Lambda} S_\alpha$ is a semilattice of rectangular band semigroup. Indeed, we have

$$\mathcal{S}_\alpha \mathcal{S}_\beta \subseteq \mathcal{S}_{\alpha\beta} \quad (\alpha, \beta \in \Lambda).$$

It then follows that $\ell^1(S)$ is ℓ^1-graded of $\ell^1(S_\alpha)'s$ over semilattice Λ. Now, let $\alpha_1 \in \Lambda$. It is known that $\ell^1 - \oplus_{\alpha\leq\alpha_1} \ell^1(S_\alpha)$ is a closed complemented ideal of $\ell^1(S)$. By Corollary 2.4 of [18], $\ell^1 - \oplus_{\alpha\leq\alpha_1} \ell^1(S_\alpha)$ has a left and right approximate identity. Then by our assumption the approximate identity of $\ell^1 - \oplus_{\alpha\leq\alpha_1} \ell^1(S_\alpha)$ is bounded. Then using Corollary 2.3 of [18], $\ell^1 - \oplus_{\alpha\leq\alpha_1} \ell^1(S_\alpha)$ is approximately amenable. Furthermore, we have that $\ell^1(S_{\alpha_1})$ is a homomorphic image of $\ell^1 - \oplus_{\alpha\leq\alpha_1} \ell^1(S_\alpha)$, then $\ell^1(S_{\alpha_1})$ is approximately amenable. By Theorem 1.1.3 of [30], S_{α_1} is isomorphic to $L \times R$. The isomorphism between S_{α_1} and $L \times R$ can be extended to their algebras. That is

$$\ell^1(S_{\alpha_1}) \cong \ell^1(L \times R) \cong \ell^1(L)\widehat{\otimes}\ell^1(R).$$

Hence, $\ell^1(L \times R) \cong \ell^1(L)\widehat{\otimes}\ell^1(R)$ is approximately amenable. Since we can define a continuous epimorphism from $\ell^1(L)\widehat{\otimes}\ell^1(R)$ into $\ell^1(L)$ and $\ell^1(R)$, see [16]. Hence $\ell^1(L)$ and $\ell^1(R)$ are approximately amenable by Proposition 3.4.1. □

Proposition 5.2.3. *Let S be a band semigroup and suppose that L and R are left and right zero semigroups. Suppose that $\ell^1(S)$ is pseudo-amenable and that every approximate identity of a complemented ideal of $\ell^1(S)$ is bounded, then $\ell^1(L)\widehat{\otimes}\ell^1(R), \ell^1(L)$ and $\ell^1(R)$ are pseudo-amenable.*

Proof. The proof is similar to the one given above. It follows from similar argument given in the above proof by replacing approximate amenability with pseudo-amenability. □

Remark 5.2.4. *In the above proposition if, $\ell^1(S)$ is uniformly approximately amenable. The result hold without the boundedness condition.*

Proposition 5.2.5. *Let S be a uniformly locally finite inverse semigroup. If $\ell^1(S)$ is approximately amenable, then each maximal subgroup of S is amenable and each D-class has finitely many idempotent element.*

Proof. Suppose that $\ell^1(S)$ is approximately amenable. By Theorem 2.18 of [44], we have

$$\ell^1(S) \cong \ell^1 \oplus \{M_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\},$$

as a Banach algebras. Then for $\lambda \in \Lambda$, $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is a homomorphic image of $\ell^1(S)$ an so is approximately amenable. It is known that G_{p_λ} being a maximal subgroup of S, it is a group under the semigroup operation and so G_{p_λ} been a group, $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is approximately amenable if and only if it is amenable. Then $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ been amenable, it follows that $E(D_\lambda)$ is finite and $\ell^1(G_{p_\lambda})$ is amenable which implies that G_{p_λ} is amenable. Hence, the result holds. □

Proposition 5.2.6. *Let S be a uniformly locally finite inverse semigroup with $|E(D_\lambda)| = 1$. Then $\ell^1(S)$ is pseudo-amenable if and only if $E(S)$ is finite and each maximal subgroup of S is amenable.*

Proof. Suppose that $\ell^1(S)$ is pseudo-amenable. By Theorem 2.1.8 of [44], we have

$$\ell^1(S) \cong \ell^1 \oplus \{M_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) : \lambda \in \Lambda\},$$

as a Banach algebras. Then for $\lambda \in \Lambda, M_{E(D_\lambda)}(\ell^1(G_{p_\lambda}))$ is a homomorphic image of $\ell^1(S)$ an so is pseudo-amenable. Using the condition that $|E(D_\lambda)| = 1$, we have that $M_{E(D_\lambda)}(\ell^1(G_{p_\lambda})) \cong \ell^1(G_{p_\lambda})$ and so is pseudo-amenable. Since every unital pseudo-amenable Banach algebra is approximately amenable, see [24], Theorem 3.1. Then $E(S)$ is finite and each maximal subgroup of S is amenable from Theorem 4.2.6.
The converse follows from Thorem 4.2.13. □

Bibliography

[1] H. P. Aghababa, L. Y. Shi and Y. J. Wu, *Generalized notions of character amenability,* Acta Mathematica Sinica, English Series 29, No. 7 (2013), 1329-1350.

[2] G. R. Allan, *Introduction to Banach spaces and algebras,* No. 20, Oxford University Press, 2011.

[3] W. G. Bade, P. C. Curtis and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras,* Proceedings of the London Mathematical Society 3, No. 2 (1987), 359-377.

[4] M. L. Bami and H. Samea, *Approximate amenability of certain semigroup algebras,* In Semigroup Forum, Vol. 71, No. 2, pp. 312-322. Springer-Verlag, 2005.

[5] F. F. Bonsall and J. Duncan, *Complete normed algebras,* Vol. 80, Springer Science and Business Media, 2012.

[6] Y. Choi, F. Ghahramani and Y. Zhang, *Approximate and pseudo-amenability of various classes of Banach algebras,* Journal of Functional Analysis 256, No. 10 (2009), 3158-3191.

[7] J. B. Conway, *A course in functional analysis,* Vol. 96, Springer Science and Business Media, 2013.

[8] P. C. Curtis and R. J. Loy, *The structure of amenable Banach algebras,* Journal of the London Mathematical Society 2, No. 1 (1989), 89-104.

[9] H. G. Dales, *Banach algebras and automatic continuity,* Clarendon Press, 2000.

[10] H. G. Dales, R. J. Loy and Y. Zang, *Approximate amenability for Banach sequence algebras,* Studia Math 177, No. 1 (2006), 81-96.

[11] H. G. Dales, A. T. Lau and D. Strauss, *Banach algebras on semigroups and on their compactifications,* American Mathematical Soc., 2010.

[12] H. G. Dales and R. J. Loy, *Approximate amenability of semigroup algebras and Segal algebras,* Institute of Mathematics, Polish Academy of Sciences, 2010.

[13] J. Duncan and I. Namioka, *Amenability of inverse semigroups and their semigroup algebras,* Proceedings of the Royal Society of Edinburgh, Section A Mathematics 80, No. 3-4 (1978), 309-321.

[14] J. Duncan and A. L. T. Paterson, *Amenability for discrete convolution semigroup algebras,* Mathematica Scandinavica (1990), 141-146.

[15] G. H. Esslamzadeh, *Banach algebra structure and amenability of a class of matrix algebras with applications,* Journal of functional analysis 161, No. 2 (1999), 364-383.

[16] M. Essmaili, M. Rostami and A. R. Medghalchi, *Pseudo-contractibility and pseudo-amenability of semigroup algebras,* Archiv der Mathematik 97, No. 2 (2011), 167-177.

[17] M. Essmaili, M. Rostami and A. Pourabbas, *Pseudo-amenability of certain semigroup algebras,* In Semigroup Forum, Vol. 82, No. 3, pp. 478-484, Springer-Verlag, 2011.

[18] F. Ghahramani and R. J. Loy, *Generalized notions of amenability,* Journal of Functional Analysis 208, No. 1 (2004), 229-260.

[19] F. Ghahramani and R. Stokke, *Approximate and pseudo-amenability of the Fourier algebra,* Indiana University mathematics journal 56, No. 2 (2007), 909-930.

[20] F. Ghahramani, E. Samei and Y. Zhang, *Generalized amenability properties of the Beurling algebras,* Journal of the Australian Mathematical Society 89, No. 03 (2010), 359-376.

[21] F. Ghahramani and C. J. Read, *Approximate identities in approximate amenability,* Journal of Functional Analysis 262, No. 9 (2012), 3929-3945.

[22] F. Ghahramani, R. J. Loy and Y. Zhang, *Generalized notions of amenability, II,* Journal of Functional analysis 254, No. 7 (2008), 1776-1810.

[23] F. Ghahramani and R. J. Loy, *Approximate amenability of tensor products of Banach algebras,* arXiv preprint arXiv, 1604.08681 (2016).

[24] F. Ghahramani and Y. Zhang, *Pseudo-amenable and pseudo-contractible Banach algebras,* In Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 142, No. 01, pp. 111-123, Cambridge University Press, 2007.

[25] M. H. Ghandehari and N. Spronk, *Amenability constants for semilattice algebras,* In Semigroup Forum, Vol. 79, No. 2, pp. 279-297, Springer-Verlag, 2009.

[26] F. Gheorghe and Y. Zhang, *A note on the approximate amenability of semigroup algebras,* In Semigroup Forum, Vol. 79, No. 2, pp. 349-354, Springer-Verlag, 2009.

[27] N. Grønbæk, *Amenability of discrete convolution algebras, the commutative case,* Pacific Journal of Mathematics 143, No. 2 (1990), 243-249.

[28] N. Grønbæk, *Amenability of weighted discrete convolution algebras on cancellative semigroups,* Proceedings of the Royal Society of Edinburgh, Section A Mathematics 110, No. 3-4 (1988), 351-360.

[29] Y. A. Helemskii, *Banach and Locally Convex Algebras, General Theory, Representations, Homology,* (1993).

[30] J. M. Howie, *Fundamentals of semigroup theory,* London Mathematical Society Monographs, Vol. 12, Clarendon Press, Oxford, 1995.

[31] B. E. Johnson, *Cohomology in Banach algebras,* Vol. 127, American Mathematical Soc., 1972.

[32] B. E. Johnson, *Approximate diagonals and cohomology of certain annihilator Banach algebras,* American Journal of Mathematics 94, No. 3 (1972), 685-698.

[33] B. E. Johnson, *Symmetric amenability and the non-existence of Lie and Jordan derivations,* In Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 120, No. 03, pp. 455-473, Cambridge University Press, 1996.

[34] E. Kaniuth, *A course in commutative Banach algebras,* Vol. 246, Springer Science and Business Media, 2008.

[35] E. Kaniuth, A. T. Lau and J. Pym, *On φ-amenability of Banach algebras,* In Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 144, No. 01, pp. 85-96, Cambridge University Press, 2008.

[36] A. T. Lau, *Characterizations of amenable Banach algebras,* Proceedings of the American Mathematical Society 70, No. 2 (1978), 156-160.

[37] A. T. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups,* Fund. Math. 118, No. 3 (1983), 161-175.

[38] O. T. Mewomo, *Notions of amenability on semigroup algebras,* Journal of Semigroup Theory and Applications, (2013), Article-ID.

[39] O. T. Mewomo, *Various notions of amenability in Banach algebras,* Expositiones Mathematicae 29, No. 3 (2011), 283-299.

[40] O. T. Mewomo, S. M. Maepa and M. O. Uwala, *On character and approximate character amenability of various Segal algebras,* Acta Mathematica Academiae Paedagogicae Nyíregyháziensis 31, No. 2 (2015), 273-283.

[41] O. T. Mewomo and N. B. Okelo, *On approximate character amenability of Banach algebras,* J. Nig. Math. Soc 32 (2013), 303-315.

[42] M. S. Monfared, *Character amenability of Banach algebras,* In Mathematical Proceedings of the Cambridge Philosophical Society, Vol. 144, No. 03, pp. 697-706, Cambridge University Press, 2008.

[43] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras,* Volume 2,*-Algebras, Vol. 2. Cambridge University Press, 2001.

[44] P. Ramsden, *Biflatness of semigroup algebras,* In Semigroup Forum, Vol. 79, No. 3, pp. 515-530, Springer-Verlag, 2009.

[45] M. A. Rostami and M. Essmaili, *Approximate amenability of certain semigroup algebras,* Acta Mathematica Scientia 33, No. 2 (2013), 565-577.

[46] Z. Ruan, *The operator amenability of* $A(G)$, American Journal of Mathematics 117, No. 6 (1995), 1449-1474.

[47] V. Runde, *Connes-amenability and normal, virtual diagonals for measure algebras I,* Journal of the London Mathematical Society 67, No. 03 (2003), 643-656.

[48] V. Runde, *Lectures on amenability,* No. 1774, Springer Science and Business Media, 2002.

[49] M. M. Sadr and A. Pourabbas, *Approximate amenability of Banach category algebras with application to semigroup algebras,* In Semigroup Forum, Vol. 79, No. 1, pp. 55-64, Springer-Verlag, 2009.

[50] C. Wilde and L. Argabright, *Invariant means and factor-semigroups,* Proceedings of the American Mathematical Society 18, No. 2 (1967), 226-228.