Exploring learners' understanding of mathematical concepts necessary in the learning of grade 11 algebraic functions: the case of three schools in uMgungundlovu District

Nkosinathi Ndlovu

(211507215)

Supervisor: Ms Busisiwe Goba

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School of Education, Mathematics & Computer Science Education Cluster, University of KwaZulu-Natal, South Africa

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ABSTRACT

The purpose of this study was to explore learners' understanding of mathematical concepts in the learning of grade 11 algebraic functions in uMgungundlovu district, KwaZulu-Natal. In order to gain insights into learners' understanding of mathematical concepts in learning grade 11 algebraic functions, APOS theory was used as a theoretical lens to explore learners' level of understanding of functions. This study describes the mathematical concepts that are important in the learning of grade 11 algebraic functions. The CAPS document was used to analyse the mathematical concepts for functions to be learnt in grade 11. The data was gathered through written tasks and interviews of grade 11 learners in three schools in one district in KwaZulu-Natal. The research approach used for this study was the mixed method. Sixty grade 11 learners (twenty in each school) were purposively selected; however, this sample selection was conveniently done since learners were able to participate in the study after school. This study employed the interpretive paradigm and nine learners (three from each school) were interviewed during data collection.

Multiple methods were employed for data collection in this study. Qualitative data was organised using interview transcripts and quantitative data was organised using the APOS analytical framework. The findings of this study confirm that learners' level of understanding of algebraic functions at an object level is extremely poor.

PREFACE

The work described in this thesis was carried out in the School of Science, Mathematics and Technology Education, University of KwaZulu-Natal, from February 2018 to November 2019 under the supervision of Ms Busisiwe Goba. This study represents original work by the author and has not otherwise been submitted in any form for any degree or diploma to any tertiary institution. Where use has been made of the work of others, it is duly acknowledged in the text.

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ABBREVIATIONS

APOS- Action-Process-Object-Schema

CAPS - Curriculum and Assessment Policy Statement

MKO- More Knowledgeable Other

NDR- National Diagnostic Report

NSC- National Senior Certificate

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I feel highly favoured from the Lord almighty who has been with me throughout the entire journey. Even though this was not an easy journey, my passion was driven by the word of God from the book of Proverbs 3:5 which reads: “Trust in the Lord with all your heart and do not learn on your own understanding. In all your ways acknowledge him and He will make your paths straight”. I would then like to thank God for his presence in this journey and the strength He provided which made me not give-up. I would also like to extend my sincere words of gratitude to the following people for their massive role in this study:

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CHAPTER 1

COMPREHENDING MATHEMATICS ISSUES AND LEARNERS' PERCEPTION OF FUNCTIONS

1.1 Introduction

This chapter aims at discussing the research purpose. Firstly, a full description of the background and the objectives of the study are given. The problem statement, rationale and the definition of the concepts necessary for this study follow thereafter. Furthermore, this study was done in such a manner as to answer the research questions stated below. This chapter concludes with a summary of the overview of the thesis.

1.2 Background and the problem statement

Generally, South African Grade 12 learners perform poorly in mathematics. It is well known that learners struggle in mathematics paper 2. Therefore, if the learners' performance in mathematics could be improved then the learners could be expected to get better marks in mathematics paper 1. However, the National Diagnostic Report (NDR)[1] of learner performance in 2016, 2017 and 2018 acknowledges that learners can improve in their performance if they have a better understanding of functions. Functions make up 35% of paper 1 and in paper 2 they take the form of trigonometric graphs. In other words, if learners can improve on the understanding of functions, then the overall mathematics learner performance can be improved. Also, the diagnostic report points to learners not being able to respond to items on functions in exams because they lack basic mathematical concepts, which relate to functions. The Department of Basic Education (2017) succinctly summarises this point:

> The algebraic skills of the candidates are poor. Most candidates lacked fundamental mathematical competencies, which could have been acquired in lower grades. Whilst

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[1] In this document, there is information aimed at assisting teachers and department officials about how learners perform in National Senior Certificate (NSC) examinations in selected subjects including mathematics.

> calculations and performing well-known routine procedures form the basis of answering Mathematics questions, an in-depth understanding of definitions and concepts cannot be overlooked (Department of Basic Education, 2017, p153).

The above extract refers to the poor algebraic skills that candidates displayed in the 2017 National Senior Certificate (NSC) examinations and the lack of mathematical understanding of concepts. Since the study is exploring learners' understanding of mathematical concepts necessary for learning functions, thus their algebraic thinking and concept definitions are vital to their understanding of functions in mathematics.

Contemporary researchers argue that there is little evidence of learner experiences of functions and how they represent their understanding of the concepts in functions. Some researchers have explored learners' development and understanding of function concepts; however, these studies varied in focus in terms of the level of learners' schooling and the theoretical perspective used. According to Ayalon, Watson and Lerman (2017) learners possessing prior knowledge of the word "function" have a strong insight of functions as an object compared to learners without such prior knowledge. The use of the word "function" is not the only way to improve understanding of the concept, however, through visualization of diagrams displayed by educators in the classroom can improve the understanding of functions (Mudaly & Rampersad, 2010). Contrary to the use of the word and visualization of a function, learners' use of procedural knowledge to explain simple concepts showed that they had a weak understanding of functions (Mudaly & Rampersad, 2010). The studies done on learners' conception of functions that resulted in improved understanding of functions due to the use of both word and visual representation were reviewed. However, none of these studies investigated grade 11 learners' insight into mathematical concepts necessary for learning algebraic functions. Therefore, there is a need for this study in which learners' understanding of mathematical concepts to learn functions is explored. From the above-mentioned background, it is evident that there is still the need for mathematics educators to improve their skills when teaching mathematics (Rowland & Rutvhen, 2011). Even though educators can be more knowledgeable in the subject, however, it is significant for them to possess skills necessary in conveying such knowledge to learners. This implies that the presentation of mathematical concepts should be in the manner that learners can make a connection of such concepts with function types.

1.3 Rationale for the study

In three years of mathematics teaching experience, the researcher has realised that learners fail to respond correctly to questions involving assessing algebraic functions. What is not clear is a deep understanding of the main cause of this problem that students have with functions. The interest in gaining insight into learners' experiences with the learning of algebraic functions emanates from informal conversations held with grade 10 and 11 mathematics learners about their mathematics learning and performance. From these conversations with learners, it has become clear that there is a huge need for teaching functions in mathematics.

The learners' understanding of mathematical concepts necessary for learning grade 11 algebraic function is the main focus of this study. Also, a discussion is done on the learners' difficulties observed during each problem-solving activity. The need for conducting this study is due to a huge contribution made by "functions" throughout the entire mathematics curriculum (Gcasamba, 2014). In this curriculum, the ability to sketch and interpret graphs is one of the requirements in understanding the chapter of functions. The study by Ayalon, Watson and Lerman (2017) emphasised functions as crucial in mathematics, and the assortment of its interpretations and representations is also spreading on both pure and applied mathematics. While this is the case, the National Diagnostic Report (NDR) learners' poor answering of questions relating to functions due to numerous reasons:

> Most candidates could not state the range of the hyperbola. Candidates were unable to differentiate between $y\epsilon R$, $y \neq -1$. Some gave the answers in terms of x; however, these candidates confused the domain and range. Few candidates were able to determine the equation of the axis of symmetry. Some of those who were able to determine the equation of the axis of symmetry did not realise that the x-intercept of the axis of symmetry passes through B (Department of Basic Education, 2017, p. 159).

Concerning these above-mentioned quotations, there is a huge role that teachers must play in teaching learners such that they interpret functions conveniently. In addition, the characteristics and features of the graphs in functions are also significant for understanding the concepts that are involved with functions. In mathematics (paper 1)[2] the chances of learners passing the examination rely precisely on their knowledge and understanding of functions and algebra, thus these two sections are connected. For instance, we use algebraic skills in functions, such as interpreting the point of intersections, domain and range on the graphs. In paper 1, approximately 35% of the questions assess or are related to functions. This suggests that learners' performance in mathematics will be better if they master the function concept. Mathematics educators should insist on teaching algebraic skills and functions in previous grades so that learners can respond correctly and accurately in their matric exam papers.

Therefore, it is important to explore learners' experiences with functions to gain insight into how they interpret and conceptualise functions to find ways to improve their level of understanding of functions. Such learners' experiences about learning functions can help educators to formulate strategies for teaching functions in earlier grades so that learners become experts on functions in secondary education. The culture of learning functions in mathematics should be the one that promotes a clear understanding by learners. This should reduce the number of comments from the National Diagnostic Report about errors evident in mathematics paper 1. Even though numerous studies have explored the teaching and learning of functions, this study specifically explores learners' understanding of mathematical concepts necessary in learning functions. This includes how the learners interpret functions in connection with the prior knowledge necessary for learning functions. Thus, it is necessary for the conduction of this study that the numerous aspects that hinder a learner's ability to conceptualise functions are determined.

1.4 Purpose of the study

The purpose of this study is threefold. Firstly, to describe the mathematical concepts learners need in the learning of functions in mathematics. Secondly, to explore the manner in which learners use these mathematical concepts while learning functions in mathematics. Thirdly, to understand the reasons for learners using mathematical concepts

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[2] This paper assesses functions and graphs at various cognitive levels with the attention on process skills, critical thinking and scientific reasoning. In the National Senior Certificate examination, the entire paper is out of 150 marks and entails approximately 25% assessment on algebraic function.

while learning functions in the manner in which they do. In this study, algebraic reasoning is included as the underlying factor that influences the learning of functions. Concerning functions in mathematics, it appears that there is a great deal of literature reported on students' experiences of functions (Nielsen, 2015). However, there is a dearth in research about learners' interpretation of mathematics concepts while learning functions, therefore, this study aspires to provide insight into the learners' use of concepts in functions. In addition, it addresses key solutions that can help learners have an improved understanding of functions in mathematics.

1.5 Objectives of the study

The objectives of this study are:

i) To identify the mathematical concepts necessary for learning algebraic functions.

ii) To examine learners' understanding of mathematical concepts necessary for learning algebraic functions.

iii) To understand why learners understand these mathematical concepts for learning algebraic functions in the way they do.

1.6 Research questions

The research questions addressed by this study are threefold:

i) What are the mathematical concepts that are necessary for learning grade 11 algebraic functions?

ii) How are learners understanding these mathematical concepts when learning grade 11 algebraic functions?

iii) Why are learners understanding these mathematical concepts necessary for learning grade 11 algebraic functions in the way they do?

1.7 Definition of the concepts used in the study

In this study, the researcher has used various terms that could have different contextual meanings, or could be written differently. Consequently, the researcher elucidates the meanings that these terms have to convey in this thesis.

1.7.1 Function

Clements and Sarama (2008) defined a function as the correspondence that associates each element of a domain with each element of a range. Similarly, a function can be defined as the relationship consisting of one element of the range being associated with one element of the domain (Vinner, 1992). The introduction of algebraic and graphical representations of functions (Leinhardt, Zaslavsky, & Stein, 1990) is significant in mathematics learning. This concept 'function' is significant in this study since it draws the reader's attention to what functions in mathematics education are all about.

1.7.2 Learning in mathematics

Learning is the transformation brought by advancing a new skill, understanding scientific law and developing new attitude (Sequeira, 2012). According to Tabach and Nachlieli (2015), learning in mathematics education includes the ability to use mathematical keywords as accepted by mathematicians. Learning mathematics requires a learner to become more proficient in mathematics communication (Tabach & Nachlieli, 2015).

1.7.3 Mathematical concept

The learning and comprehending mathematical definitions are crucial for learning mathematical concepts; the power of definition of concepts is captured in the roles and features of the definitions (Thomson, 2015).

1.8 Structure of the thesis

This thesis is organised into the following six chapters:

Chapter 1 – Introduction: This chapter introduces the reader to the background of the study and highlights the significance of understanding learners' use of mathematical concepts in learning functions. This chapter also presents debates about learners' knowledge and understanding of the concepts embraced in functions. The rationale for conducting this study, the purpose, specific objectives, research questions, and relevant definitions for this study are provided.

Chapter 2 – Literature Review: This chapter presents literature that addresses learners' experiences in learning algebraic functions. The literature associated with the history of the function concept is also presented in this chapter. While this study focuses on learners' use of mathematical concepts in the learning of grade 11 functions; literature on learners' understanding of mathematical concepts is also provided. In addition, acknowledgement is made of the challenges faced by learners in the learning of mathematics which can have an impact on learners' understanding of the function concept. Furthermore, the literature that is discussed also presents the algebraic skills necessary for the learning of algebraic functions in mathematics.

Chapter 3 –Theoretical Framework: This chapter looks at the theory that is related to an exploration of learners' use of mathematical concepts in the learning of functions. To do this, the researcher discusses in detail the APOS theory and highlights how the framework is used in this current study.

Chapter 4 –Research Methodology: This chapter presents in detail the empirical process of this study. The selected research methodology for this study is also provided. In this chapter the researcher outlines the research design, population, sampling procedure, data collection instruments, data processing and analysis procedure, ethical considerations and issues of validity and trustworthiness. In addition, the limitations of this study are also included in this chapter.

Chapter 5 –Data presentation, Analysis and Discussion of the Findings: This chapter presents the raw data and analyses using tables. The reviewed literature and the suggested framework for the study are also discussed in this chapter.

Chapter 6 – Summary of the Findings, Conclusion and Recommendations: The chapter summarises the purpose of the study, and the methodology used, and presents the main findings, the conclusion and recommendations.

1.9 Conclusion

This chapter presents the introduction and background of the study. It outlines the rationale, purpose, objectives, research questions, and definitions of the concepts that guide this study. In addition, this chapter briefly highlights the structure of this thesis.

CHAPTER 2

THE ALGEBRAIC FUNCTIONS AND LEARNERS' EXPERIENCES: A REVIEW OF THE LITERATURE

2.1 Introduction

Recent debates for research in South African mathematics education are predominantly concerned with the teaching and learning standards of mathematics (Spaull, 2013). This study focuses on learners' understanding of mathematical concepts in learning grade 11 algebraic functions in uMgundundlovu District, KwaZulu-Natal. The 'function' definition and concepts involved around it are explored by numerous scholars in mathematics education (Bardini, Pierce, Vincent, & King, 2014; Chitsike, 2013; Breen, Larson, O'Shea, & Petterson, 2015). This chapter commences with a discussion of seminal contributions that have been made in understanding the history of the function concept. This is followed by a discussion of the learners' understanding of mathematical concepts and learners' understanding of the function concept. Also, the researcher then presented a comprehensive review of the challenges faced by learners in learning mathematical concepts. Furthermore, the researcher explored the outline of functions as given in the South African curriculum of mathematics education to construe how these functions are connected across all grades. I also realised the significance of understanding the algebraic thinking skills necessary for the learning of algebraic functions in this chapter.

2.2 The history of defining the function concept

In mathematics education, numerous branches of mathematics deal with functions directly or indirectly. Most of the mathematics curricular consider it important to study the properties of functions of one, two or n variables. Other fields of mathematics deal with concepts that constitute generalizations or outgrowths of the notions of functions, for example, algebra contemplates operations and relations. According to Kleiner (1989), functions should constitute a fundamental concept in secondary school mathematics. The South African mathematics curriculum still clearly emphasises the significance of functions (Denbel, 2015). Depending on the dominant mathematical viewpoint, there are many ways to consider the notion of functions (Kleiner, 1989). This section reviews some of the more prominent features of the history of the concept of function. The researcher examines the relation of functions to other sciences and discusses functions used in the study of real-world situations. In addition, the researcher considers the problems of certain pedagogical approaches with special attention given to the nature of the function concept underlying the activities of learners.

The current definition of a function and the teaching of its concepts are based on the great advancements in algebra and geometry that took place over centuries. The developments on function concepts began during Leibnitz's first introduction of the word function in a geometric context in 1673. In 1718, Bernoulli followed by proposing definitions of algebraic equations as formulae. One of the definitions reads; "a function is a quantity composed in any manner of a variable and including any constants (Kleiner, 1989)". Thereafter, Euler (1748) proposed a function in terms of an analytical expression: "A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constants quantities (Kleiner, 1989, p.283)".

The dependence between the variables in equations or formulae as representations was the tactic to functions that were considered at that time (Kleiner, 1989). The development of lasted for more than two centuries and represented a tug of war between the geometric and algebraic approaches (Kleiner, 1989). The developments of the function concept last for more than two centuries and represented a tug of war between the geometric and algebraic approaches (Kleiner, 1989). During the introduction of the new version of the definition of a function, it was discovered that the geometric definition falls short of expectations when it comes to the algebraic definition, thus that definition was rejected and a new version was formulated. Definitions of functions, therefore, evolved with each extending on the existing version until Bourbaki's (1970) set theory and abstract algebra that resulted in a set-theoretic definition. The definition reads:

> Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x (element) E, if there exists a unique y (element) F which is in the given relation with x. A function is given by an operation that associates every element x (element E) with the

element *y* (element F) given in relation with *x; y*. This function is said to be the value of the function at the element *x*, and is said to be determined by the given functional relation. Two equivalent functional relations determine the same function (Kleiner, 1989, p.299).

From the above discussion, it has been understood that the ‘function concept’ development has been cyclical and continuing over centuries. The psychological development of algebraic concepts in learners emanates from the historical development of the function. Thus, Nachlieli and Tabach (2012) pointed no reason that those who presently learn functions will struggle in similar to those mathematicians in the past. It is evident in mathematics education research that learners experienced difficulties predominantly when learning functions. These difficulties will be part of the discussion later in this chapter. For this study, South African learners are not immune to difficulties experienced by other learners elsewhere, and as discussed in chapter one, this is evident from the National Diagnostic Report (Department of Basic Education, 2017).

2.3 Learners’ understanding mathematical concepts

The definition of concepts in mathematics is essential for reaching an agreement about the nature and properties of mathematical objects. The attainment of educational goals in mathematics relies on learners’ understanding of mathematical concepts. The learning of mathematics should also embrace the ability to use acceptable mathematical keywords for each aspect of mathematics. Mathematics education comprises concepts that are numerical, algebraic, statistical, probabilistic, and analytical. The possession of these key concepts is paramount in the teaching and learning of mathematics (Watson, Jones, & Pratt, 2013), most especially in the early stages of development. Failure in teaching the concepts necessary for functions in prior grade results in learners failing to understand functions at the high school and tertiary level (Veloo, Krishnasamy, & Abdullah, 2015). In addition, besides the possession of the key mathematical concepts, reasoning about key concepts in mathematics is paramount since it promotes a better understanding of the concepts.

Previous studies have shown that learners often display reluctance in helping themselves with definitions while categorising mathematical concepts (Nachlieli & Tabach, 2012). Such reluctance can lead to learners failing to understand chapters of mathematics or even mathematics as a whole since many of these concepts are connected with information in other chapters. Learners' reluctance in using mathematical concepts to learn mathematics depends solely on the context in which they use these concepts (Mukono, 2015). For instance, it is pointless to teach mathematics using examples that are irrelevant to a learner's context. Instead, the commendable teaching strategy is the one that introduces mathematical concepts in terms of learners' exposure to the context. This can lead to learners' convenience in understanding the connections of prior learned concepts with newly learned mathematical concepts.

In mathematics learning, learners comprise numerous learning capabilities that can help in their successful understanding of mathematical concepts. For instance, some visual learners are mostly dominant in mathematics classrooms (Nel, N & Nel, M, 2013). Their preference is that of portraying information using diagrams, graphs and other illustrative methods (Fleming, 2015). This possibly justifies that most teachers should teach mathematical concepts using diagrams and graphs. A typical mathematics classroom also includes auditory learners whose preference is to listen to spoken information, and their best learning criteria is through group discussions and lesson presentations (Fleming, 2015; Juskeviciene & Kurilovas, 2014). The promotion of group discussion in mathematics classroom should accommodate these learners and help them in understanding mathematical concepts better. Some learners learn most effectively through reading and writing. Their preference is that of displaying mathematical instructions in words (Fleming, 2015; Juskeviciene & Kurilovas, 2014). From this discussion, it is evident that the presentation of mathematical concepts should not be limited to just one strategy. Making mathematical concepts understandable requires teachers to take steps to accommodate different learners' routines in which they understand such concepts best.

Learners should not simply learn about mathematical concepts but also think about the concepts. This is important since reasoning about such concepts is also significant in promoting a better understanding of mathematics. For example, the study by Welder (2012) posits that learners often omit to reason about an overall goal or the concept entailed in a problem. Instead, they look for an implied procedure inherent in the equations and directly apply it when trying to solve a problem (Welder, 2012). For instance, algebraic equations expressed in letter form are seen as representing a range of unspecified values and a systematic relationship is seen to occur between two sets of values (Welder, 2012). These equations are what often results in great confusion for learners (Welder, 2012). For learners to have a deeper understanding of these concepts in terms of content and application, practical work, and the use of manipulatives are extremely important in their learning.

The study by Thomson (2015) revealed learners' adversities in relating functional representation with its definition. For instance, their responses in explaining the domain and range were sketchy, meaning they could not express these concepts orally and in writing (Thomson, 2015). Similarly, Mpofu and Pournara (2016) found that learners' description of a hyperbola to be only visual and not by using literal words. Most learners portray hyperbolic graphs displaying asymptotes but talk as if there are no asymptotes (Mpofu & Pournara, 2016). These learners' perception of a hyperbola yields to the reasoning that function with a fraction displays a hyperbola whilst the one not expressed in standard form does not represent a hyperbola. For example, it can be convenient for learners to see that a particular graph is a hyperbola if they see the function with a fraction, but they cannot identify it as a hyperbola if the function is expressed in a different form. Therefore, teachers must adopt diverse pedagogies in enabling learners to present concepts of functions in different ways, as this will result in better mathematical reasoning.

This section attempts to provide a comprehensive review of learners' understanding of mathematical concepts. However, there is limited literature describing the relationship between a learner's understanding of the formal definitions in mathematics and their capability to classify mathematical objects to an extent that is consistent with these definitions (Nachlieli & Tabach, 2012). Thus, it is beyond the scope and primary purpose of this section to attempt to provide a comprehensive review of learners' understanding of definitions and their capability to classify mathematical objects.

2.4 Learners' understanding of the function concept

Several factors affect learners' learning and performance in mathematics. One of these factors includes learners' attitudes towards the subject, teachers' pedagogical practices and school environment (Mazana, Montero, & Casmir, 2019). In mathematics, learners can portray a positive or negative attitude while learning concepts for each topic. In a study by Mazana, Montero and Casmir (2019), initially learners displayed a positive attitude towards mathematics; however, such attitude faded as they shifted towards higher levels of education. This occurs due to teacher-related factors such as difficulties in teaching the subject, which have a negative influence on their teaching (Ajai & Iyekekpolor, 2016; Kravitz, 2013). Thus, having a negative attitude in mathematics learning can be one of the reasons for failure in using correct concepts for learning mathematics.

The learning context for learners' interaction with mathematics becomes more significant in emphasising learners' experiences (Mata, Monteiro, & Piexoto, 2012). In the study by Fraser and Kahle (2007), learners' learning at home, school and with a peer group is accounted for a significant amount of variance. Learners' exposure to the context corresponds with their attitude in learning the subject. In other words, the exposure of the context where learners interact contributes to the understanding of mathematical concepts. It is, therefore, the role of teachers to influence learning such that learners' exposure to the context contributes to having a positive attitude in mathematics learning (Ajai & Iyekekpolor, 2016).

There are many challenges learners face in mathematics learning. One of these challenges includes making connections with concepts, manipulating information, stating mathematical sentences and determining applicable formulae (Tambychik, 2010). The lack of these information skills in mathematics learning results in difficulties with problem-solving. From the researcher's personal experience in teaching functions, it has become clear that learners fail to solve problems in functions, and cannot apply suitable formulae correctly. For instance, factorising quadratic functions is a huge difficulty for many learners. Even though they can use the quadratic formula to determine the factors, they cannot substitute into a correct formula.

2.5 Challenges faced by learners in learning mathematical concepts

The curriculum in mathematics education views the 'function concept' as a unifying theme (Steele, Hillen, & Smith, 2013) consisting of tables, symbolic equations and verbatim as multiple representations of the function concept (Chitsike, 2013). It entails using distinct representations where each concept representation offers information about a specific aspect of the concept without its complete description (Gagatsi & Shiakalli, 2004). The understanding of the function concept in mathematics education is paramount for learners and is a major goal secondary curriculum together with a collegiate curriculum (Lin & Cooney, 2016). It enables learners to use various representations and to translate important features from one form to another (Lin & Cooney, 2016).

Numerous scholars have recognised the role that connections of representations play in functions and problem-solving (Gagatsi & Shiakalli, 2004; Monoyiou & Gagatsis, 2008). Connecting representations of functions with problem-solving promote learners' problem-solving abilities (Gagatsi & Shiakalli, 2004). This means that learners with a more powerful understanding of relationships between various kinds of representations are more successful in problem-solving than other learners. However, this depends on the learners' insight in defining the function concept since it is crucial in the mathematics curriculum.

Clements and Sarama (2008) defined a function as a correspondence associating each element of *x* with each element of *y*. In addition, the symbolic representation $y = f(x)$ characterises a function comprising a single variable *x* that produces a mapping from *x*-values to *y*-values (Clements & Sarama, 2008). Similarly, Chitsike (2013) defined a function as a relationship consisting of one element of the domain being associated with one unique element of the range. This definition refers to a domain as a set of *x*-values and the ranges refer to a set of *y*-values (Chitsike, 2013). The definitions mentioned cannot be the only ones to consider in defining a function concept and a function can also be a many-to-one relationship.

In mathematics education, the ability of learners to define and make sense of a function is significant. Defining and making sense of a function entails the possession of ideas, covariation reasoning[3], and mapping (Breen, Larson, O'Shea, & Petterson, 2015). Knowing the change and variation of quantities are also important for the functional thinking ability of learners (Wilkie, 2016). In the study by Thompson and Carlson (2016), it is suggested that learners should gain covariation and quantitative reasoning in their mathematics lessons. These reasoning types are significant for their real lives and advanced mathematical understanding (Thompson & Carlson, 2016).

[3] Saldanha and Thompson described covariation understanding of a function as "holding in mind a sustained image of two quantities' values (magnitude) simultaneously" (Saldanha & Thompson, 1998).

Concerning the above definitions of functions, this study explores learners' understanding of mathematical concepts necessary for learning grade 11 algebraic functions. Therefore, learners understand the function concept in numerous perspectives including algebraic and geometric concept images of a function (Breen, Larson, O'Shea, & Petterson, 2015).

However, research on the function concept has revealed instructional difficulties related to the learning of functions in mathematics (Viirman, 2014). In other words, learners' conception of the function concept displayed inconsistencies both within and between the conceptions and definitions (Viirman, 2014). The most occurrence of these difficulties is through the use of textual, algebraic and graphic representations involving daily life situations in problem-solving (Okur, 2013). Learners struggle with the multivariable function concept, including identifying domain and range (Kabael, 2011; Martinez-Planell & Trigueros, 2012), drawing correct graphs (Dorko, 2016) and working with free variables (Dorko, 2017). These difficulties may hinder learners from reasoning correctly about multivariable functions in other settings such as physics, statistics, and engineering (Dorko, 2017). Thus, research regarding learners' thinking about the definition and representations of functions can assist in informing instruction at the secondary and college education levels.

Currently, studies persist in identifying learners' challenges in defining a function and solving function problems (Panaourna, Michael-Chrysanthou, & Philippou, 2016). Research also indicates representational obstacles learners experience when dealing with functions (Bardini, Pierce, Vincent, & King, 2014). They cannot appropriately define or recognise if a graph displays a function or rule that done a function (Bardini, Pierce, Vincent, & King, 2014). In addition, they could not relate between function graphs and a table of values (Bardini, Pierce, Vincent, & King, 2014). Thus, it is paramount to familiarise learners with defining the 'function concept' with its application in solving function problems. There is a huge role that teachers need to play in understanding the image created in learners' minds about the specific concept as internal representations of the concept (Panaoura, Michael-Chrysanthou, Gagatsis, Elia, & Philippou, 2017).

There are numerous types of functions and their concepts included in the South African schools' mathematics curriculum. The learning of some of these functions starts at the primary school level, while others start at the high school level. As mentioned above, these functions include the linear, quadratic, exponential, hyperbolic and trigonometric functions (Department of Basic Education, 2011). It is, therefore, paramount to give the reader a precise understanding of these function types and how learners perceive them.

2.5.1 Linear Function

The learners' initial experiences with functions typically embrace the study of linear relationships, before constructing more advanced functional relationships (Nagle & Moore-Russo, 2013). These linear relationships involve linear functions and comprise variables appearing in the first degree only (Webster, 2016). Linear functions are commonly sighted with a rule, $y = mx + c$ and require that learners cannot only develop an understanding of the variables x and y, but also understand the meaning of m and c. The learning of linear functions in the mathematics curriculum requires learners to perceive the role played by parameters m and c in the graph of the function (Pierce, Stacey, & Bardini, 2010). The value of m denotes the gradient or the slope of a function and c denotes the y-intercept of a function.

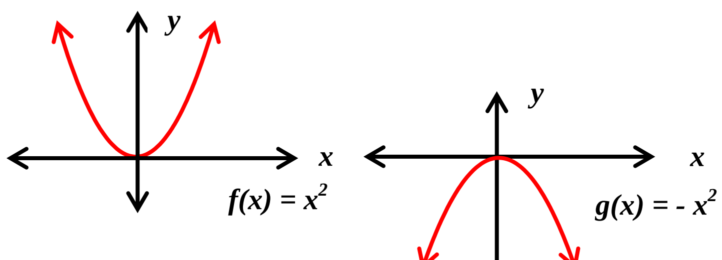
Numerous scholars have noted that learners' understanding of the parameters of linear functions is a problem. For instance, according to Pierce, Stacey and Bardini (2010), learners often neglect the parameter c from verbal and symbolic descriptions and treat it as if it is not_an essential part of the linear function. In addition, even college undergraduate students lack the insight of the slope and equations of linear functions (Mielicki & Wisely, 2016) which are essential for advanced algebraic thinking. This lack of knowledge began during learners' earliest experiences with graphing linear functions. It has been discovered that most learners perceive the connection from the equation to the graph (Soots & Shafer, 2018), however, they fail to construct equations from a given graph (Knuth, 2000). This inability of learners to recognise algebraic and graphical representations hinders their ability to truly understand mathematics (Bayazit & Aksoy, 2010). Therefore, the introduction of linear functional relationships in earlier grades should not only focus on learners' ability to sketch or plot linear graphs. A deep understanding of the standard form $y = mx + c$ together with its parameters is crucial.

2.5.2 Quadratic Function

A quadratic function is one of the basic polynomial functions consisting of a degree greater than one (Burns-Childers & Vidakovic, 2018). It describes the connection between amongst two variables, where an independent variable has exactly one dependent variable (Bansilal & Ubah, 2018). This function type is one of the most significant ideas for learners to learn about in school mathematics (Nielsen, 2015; Parent, 2015; Benning & Agyei, 2016) since it plays an important role in calculus courses (Burns-Childers & Vidakovic, 2018). In addition, the concepts of quadratic functions are extremely significant in higher mathematics, especially when dealing with higher polynomial functions (Suzame, 2015).

A good understanding of quadratic functions enables learners to explore numerous function types including cubic, trigonometric, exponential and logarithmic functions, leading to real-life uses of this concept (Bansilal & Ubah, 2018). In the South African curriculum, quadratic functions form a part of the grade 10 mathematics introduction and are explained further in grade 11 (Department of Basic Education, 2011). The standard form of the quadratic function $f(x) = ax^2 + bx + c$ is where a, b, and c are numbers that are not equal to zero (Nielsen, 2015). The graph of a quadratic function $f(x) = ax^2 + bx + c$ is called a parabola (Yeo, Seng, Ye, & Chew, 2013) and can either concave up or down as shown in figure 2.1 below.

There are numerous forms of expressing quadratic functions in the South African curriculum. Firstly, the grade 10 curriculum only introduces quadratic functions in the form $f(x) = ax^2 + q$. Secondly, the grade 11 curriculum further displays quadratic functions in factorised or intercept form and vertex form. The factorised or intercept representation of a quadratic function is given by the equation $f(x) = a(x - x_1)(x - x_2)$. The values x_1 and x_2 in this equation denotes the roots of the corresponding quadratic function (Bansilal & Ubah, 2018). There is also a quadratic function written in a standard form given by $f(x) = ax^2 + bx + c$, where the value of c represents the y-intercept of the equation (Bansilal & Ubah, 2018). In grade 11, learners are introduced to the quadratic function $f(x) = a(x - p)^2 + q$ that is written in vertex form with p and q representing the coordinates of the turning point (Parent, 2015). The following figure represents the shape of a quadratic function:



Since quadratic functions can be represented in different forms, numerous scholars in mathematics education (Parent, 2015; Nielsen, 2015; Endang, Nanang, Sufyani, & Ruli, 2018) have explored learners' challenges in learning quadratic functions. For a complete understanding of quadratic functions, learners need to relate quadratic equations with functions. While this should be the case, learners are inept when solving quadratic equations for quadratic functions (Didis, Bas, & Erbas, 2011). The conception of solving quadratic equations is paramount in learning quadratic equations. However, learners' concept image of quadratic equations is limited and dominated by ideas concerning factorising (Kabar, 2018). In other words, learners lack prerequisite knowledge such as knowing the degree of a polynomial, variable, and equal sign (Kabar, 2018). Thus, this reduces learners' competences in learning quadratic functions and other polynomials with a degree greater than two.

As mentioned above, quadratic functions can be represented in three forms. However, amongst these forms research suggested that the learners' preferences of learning quadratic functions lie in standard[4] form rather than vertex[5] form or factored[6] form (Nielsen, 2015). The interpretation of the vertex form and factored form of quadratic functions are assessed in the National Senior Certificate examinations, and thus it is paramount that learners know these forms. In summary, the focus on pedagogy for teaching quadratic functions is necessary to

[4] The standard form of a quadratic function is written as $f(x) = ax^2 + bx + c$

[5] The vertex form of a quadratic function is written as $f(x) = a(x - p)^2 + q$, where $(p; q)$ represents the coordinates of the stationary points of a function.

[6] The factored form is written as $f(x) = (x - x_1)(x - x_2)$, with x_1 and x_2 denoting the x-intercepts of the function.

address issues concerning learners' thinking and difficulties in mathematics (Celik & Guzel, 2017).

2.5.3 Hyperbolic Function

In South African mathematics curriculum, hyperbolic function is considered as the only rational functions present in the National Senior Certificate examination among all other function types (Department of Basic Education, 2018). While this is the case, the National Diagnostic Report displayed challenges faced by learners in interpreting this function type (Department of Basic Education, 2018). This challenge resulted in a 27% national average score on questions involving the hyperbola (Department of Basic Education, 2018). Therefore, the exploration of learners' understanding of mathematical concepts necessary for learning function is essential.

The study by Mpofu and Pournara (2016) focused on three ways that learners displayed the representation of the hyperbola, namely: formula (equation), graph, and table. It was found that the majority of learners sketched the graph of a hyperbola displaying a vertical asymptote yet talked as if there was no vertical asymptote (Mpofu & Pournara, 2016). Similarly, Mpofu and Pournara (2018), elucidated on a learners' tendency of sketching graphs showing two asymptotes whilst talking as if there were only one asymptote. In other words, it is a huge challenge for learners to see the axes of the Cartesian plane as asymptotes (Mpofu & Pournara, 2018). In addition, these learners failed to display asymptotic behaviour on the table of values yet they drew graphs with asymptotes (Mpofu & Pournara, 2018). This demonstrates that a teacher's pedagogy in introducing the hyperbola should be done to enable learners to display hyperbolic function in numerous transformations (tables, graphs and equations). In most cases, the point-by-point plotting is the most commonly used strategy for introducing the concept of function (Mpofu & Pournara, 2018). However, the selection of the points emphasising the horizontal and vertical asymptotes is vital in the case of hyperbolic function (Mpofu & Pournara, 2018).

2.5.4 *Exponential and Logarithmic Function*

The function type consisting of the relationship between variables represented by $f(x) = a.b^x + q$ is an exponential function (Mousel, 2006; Webber, 2002). It is also included in the grade 11 mathematics curriculum and requires a basic understanding of exponential laws. The exponential functions are connected to logarithmic functions, and therefore should not be taught in isolation (Makgakga & Sepeng, 2013). However, looking at the South African mathematics curriculum shows that exponential functions are first introduced in grade 10 and then later dealt with deeply in grade 11, yet these concepts are taught separately from logarithmic functions. The logarithmic function as the inverse of an exponential function is only introduced in grade 12 whilst there is the need for learners to study and interpret the relationship between these function types in earlier grades (Makgakga & Sepeng, 2013).

The current literature on learners' understanding of exponential functions is extremely scant. It has been previously explored that learners appear to encounter adversities in understanding exponential functions and they struggle to express the exponential equations as $y = a^x$ (Makgakga & Sepeng, 2013). This is supported by the study of Webber (2002) which demonstrated that even college undergraduate students struggled to explain what a function such as $f(x) = a^x$ meant, and the reason for a function such as $f(x) = \left(\frac{1}{2}\right)^x$ to be considered as a decreasing function. The lack of understanding exponential functions contributes to learners' challenges with interpreting this function type, more especially in grade 12. The learners' difficulties in understanding exponential functions are due to teachers' difficulty with using covariation as a tool for building an understanding of these functions (Strom, 2007). Hence, the designed pedagogy for teaching exponential functions in grade 10 should outline the relations of exponential and logarithmic functions. This will result in learners being able to reason why a logarithmic function is an inverse of an exponential function.

2.6 Functions in the South African curriculum

The mathematics curriculum in South Africa comprises functions as the central topic, which contributes about 50% of the marks in the National Senior Certificate (NSC) examinations. The learning of functions includes transformations on points in the Cartesian plane, intersections, algebraic operations on numbers and combinations of pairs of sets (Denbel, 2015). While this is the case, the South African curriculum encourages a shift from traditional ways of teaching and learning to the use of more interactive approaches (Department of Basic Education, 2011). Thus, learners must be encouraged to take the study of functions seriously and to know and understand its connection with other chapters in mathematics. In addition, their knowledge construction in learning functions should be a priority, and teaching should reconstruct meaning, where learners interpret what they see based on what they already know (Department of Basic Education, 2011).

The mathematics curriculum for secondary schools in South Africa entails function topics from grades 8 to 12. However, the curriculum does not introduce all the function types across all grades. For instance, grades 8 and 9 explores only the concepts involved around the linear function, and grades 10 and 11 entails quadratic, hyperbolic, and exponential functions. In grade 12, logarithmic and cubic functions are additional function types that are examined.

This section discusses the outline of functions in the Curriculum Assessment and Policy Statement (CAPS), which indicates four learning outcomes that learners need to achieve in learning functions in grades 10 to 12. In mathematics, functions are also included in trigonometry; however, the researcher’s focus in this study is precisely limited to algebraic functions and their characteristics. Most importantly how grade 11 learners use mathematical concepts to learn these functions. This study focuses precisely on learners’ understanding of mathematical concepts necessary in learning grade 11 algebraic functions. There are four types of algebraic functions in the grade 11 mathematics curriculum namely, linear functions, quadratic functions, hyperbolic functions, and exponential functions.

With these function types mentioned above, grade 11 learners are anticipated to explore the concepts involved around them as stipulated in the curriculum (Department of Basic Education, 2011). The curriculum also emphasises the pointwise approach, such as performing operations like sketching graphs and manipulating algebraic expressions (Department of Basic Education, 2011). The interpretation of global features of representations, for example, investigating the properties of graphs, is another emphasis of the curriculum. While analysing the mathematics curriculum document (Department of Basic Education, 2011), I have realised that it is only in grade 12 that the formal definition of a function is explored.

2.7 Algebraic knowledge necessary for the learning of functions

In mathematics, the learning of numerous types of functions requires the use of algebraic thinking skills. These thinking skills include reasoning, using notations, and calculations of unknown variables and numbers (Radford, 2014). The possession of algebraic thinking results in an ability to think and solve problems logically (Baltaci & Yildiz, 2015). The focus of this study is on learners' use of mathematical concepts in learning grade 11 algebraic functions. The algebraic thinking revolves around functions, and it is thus paramount to foster it through primary and secondary education (Chimoni & Pitta-Pantazi, 2015). Engaging elementary learners in rich, age-appropriate tasks can improve their foundational understanding of core mathematical concepts, including variables and functions (Blanton, et al., 2015).

2.8 Conclusion

This chapter explored the literature related to function concepts and learners' learning of algebraic functions. The chapter commenced by exploring the historical developments of the function concept. There were inferences made by numerous scholars that functions should constitute a fundamental concept in secondary school mathematics (Kleiner, 1989) since most recent curricula clearly emphasise the significance of functions. The researcher also explored the literature about learners' understanding of mathematical concepts. From the reviewed literature, it seems that learners' understanding of mathematical concepts strongly depends on their ability to use simpler terms in defining the concept as well as their use of representations.

To comprehend the nature of learners' learning of algebraic functions in South Africa, more attention was paid to the type of research that South African mathematics researchers conducted. It appears that there is very little literature about learners' learning of algebraic functions in South Africa, particularly in rural areas. Therefore, this study contributes to the knowledge about learners' perceptions of algebraic functions in rural mathematics education research. This can help researchers to understand what rural learners experience about the learning of algebraic functions. The literature in this study also points out that the meaning of the function concept requires the relevant experience of earlier concepts (Ayalon, Watson, & Lerman, 2017). Besides that, the visualisation of diagrams displaying the concept "function" and its representations play a huge role in mathematics classrooms (Mudaly & Rampersad, 2010). This chapter has illustrated the great need for mathematics research more especially in a rural context where there is a dearth of existing literature pertaining to learners’ understanding of functions.

CHAPTER 3

RECONCEPTUALISING MATHEMATICAL CONCEPTS IN FUNCTIONS: THE THEORETICAL FRAMEWORK

3.1 Introduction

The learners' understanding of mathematical concepts necessary in learning grade 11 algebraic functions is what this study intends to explore. In the previous chapter, the researcher introduced the literature informing this study. Therefore, this chapter discussed in detail a theoretical framework within which this study is located. The beginning of this chapter is that of the researcher's beliefs about learning mathematics education which emanates from several theories that intersect with the theory underpinning this study. Subsequently, the researcher elucidates the significance of the APOS theory that this study used. According to (Eisenberg, 1992), a theoretical framework guides the researcher with a formal theory constructed with the use of a recognised, clear explanation of certain phenomena and relationships. This formally constructed theory shapes the thinking about the nature of the study and planning of the study process. Thus, a researcher cannot begin what he or she intends to do without a theoretical framework as it is a lens through which the study is viewed. Before relating the APOS theory with functions in this study, there is the need to provide the reader with an outline of the researcher's theoretical understanding of mathematics learning.

3.2 Researcher's theoretical understanding of learning mathematics

In this study, the researcher used the APOS theory. Further, the researcher explained the concepts this framework comprises to the study. Before engaging the reader in understanding the framework for this study, the researcher will first elucidate his theoretical understanding of learning mathematics. These theories about mathematics learning inform the researcher's engagement in this study.

In the learning process, the constructivist theory guides knowledge construction. This theory emanates from the arguments of the scientists Piaget, Dewey and Vygotsky and has an important influence on mathematics learning theories and reasoning (Hanna & Jahnke, 1996). In addition, the constructivist theory is also significant and relevant to the current South African curriculum (Department of Basic Education, 2011). Since functions are significant in mathematics (Dubinsky & Harel, 1992), the researcher believes that learners can dependently construct knowledge. This is because mathematics learning by an individual is not direct; rather Vygotsky (1978) declared that contributes to a child's development. Even though learners can help construct their knowledge (Piaget & Garcia, 1989), their thinking and problem solving can best be helped with the assistance of an educator (Vygotsky, 1978). Subsequently, this study explores learners' learning of mathematics, thus a "More Knowledgeable Other" (MKO) by Vygotsky (1978) plays a vital role. Also, without MKO, learners will struggle in their articulations of what they are doing and why, which results in learners losing motivation when trying to solve a problem.

The constructivist perspective posits that the transforming way of thinking entails cognitive structures through prior separate structures into general and powerful structures (Brodie, 2010). In grade 11 algebraic functions, there are previously learned concepts that are required for learners to generate new powerful concepts. Therefore, the constructivist theory considers an active process of sorting or constructing knowledge together as part of learning (Cobern, 1995). In addition, an individual's mind is the unit of analysis in constructivist theory (Brodie, 2010), and social collaboration is important as it constrains individual learning (Hatano, 1996). Piaget inferred that in learning, constructivism leads to "assimilation" and "accommodation". Assimilation encompasses taking in new information and fitting it into the pre-existing schema (Piaget, 1964), ensuring that this provides a picture of what is learned in the later stage (Bruner, 1960). In this case, what is newly absorbed loses most of its original meaning and acquires new meaning due to the pre-existing schema (Mokolo, 2017). Accommodation happens when a completely new schema forms due to newly acquired information not matching with the pre-existing schema (Hatano, 1996; Sarwadi & Shahril, 2014). The learning of functions in grade 11 requires that there be the pre-existing schema of functions obtained from previous grades (e.g. Grades 9 and 10). Thus, assimilation forms part of the new schema of grade 11 functions.

The belief entailed in the constructivist theory is that of learners' construction of knowledge through the learning process. During knowledge construction, learners should work on concepts until they can process them to form objects, which helps them to develop schemas. The development of these schemas will help them to accommodate or assimilate new schemas into existing ones they already know. In addition, the application of mental structures is required to make sense of a concept (Piaget & Garcia, 1989).

The situated theory is the theory that views learning participation as a community of practice (Lave & Wenger, 1991). In other words, the learning process in this theory is defined as an increase in an individual's participation in practice. This theory deals with how learners use mathematical tools while participating in the community of practice. In addition, making connections and generalizing ideas are significant in situated theory. However, the researcher discusses these ideas in conversation, rather than seen as structures in the mind (Brodie, 2010). According to Brodie (2010), the classroom allows affordances and limited interactional situations that impact on learners' lives beyond the classroom.

In cognitive psychology and mathematics education, Kilpatrick, Swarfford and Findell (2001) adopted a complete view of successful mathematics learning. They chose strands of mathematical proficiency to achieve all aspects of expertise, competence and mathematical knowledge as well. These strands of mathematical proficiency are divided into five aspects namely: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and procedural disposition (Kilpatrick et al., 2001). In light of these strands of mathematical proficiency, it is paramount that learners apply them effectively in their learning. In addition, the Kilpatrick's strands of mathematical concepts seemed to be similar to Polya's (2014) theory of mathematical problem-solving. Theory by Polya (2014) established three steps to be considered in problem-solving namely; understanding the problem, strategic planning and reflection. In relation to the problem of this study, the researcher expects that a learner holds an ability to analyse the mathematical problem to gain insight into it. Through the attainment of this insight, the plan for problem-solving is carried out where a student uses various ideas that can lead to a solution until the correct idea is found. At a later time, Polya (2014) elucidated on the necessity to modify correct ideas. In light of the above literature, the researcher anticipates that learners will apply such theories in their learning of functions. However, it is possible that they do not follow the strands of mathematical proficiency of Kilpatrick, Swafford and Findell as well as, or Polya's steps of problem-solving.

3.3 The APOS theoretical model

There is a bulk of research focusing on the idea that a learners' view of functions has a great influence on their learning of functions. The definition of these views or conceptions of functions (Breidenbach, Dubinsky, Hawks, & Nicholas, 1992) is what eventually developed into APOS (Action-Process-Object-Schema) theory. However, Eduard Dubinsky is the actual founder of APOS theory, which emanates from Piaget's major idea of mental construction known as 'reflective abstraction' to post-secondary mathematics.

Thereafter, Dubinsky developed ideas that led to the APOS theory in 1984 during the proceedings of a conference in Helsinki and Finland (Dubinsky, 1984). In these proceedings, he distinguished an individual's thought about functions as a "Process" and as an "Object", and further explained how one applies "Actions" to mental "Objects". Before I further elucidate about Dubinsky's APOS theory, it is significant to understand Piaget's definition of reflective abstraction since it forms the basis of this theory. Reflective abstraction refers to the reconstruction and reorganisation of content and operations from a lower cognitive level or stage to a higher cognitive level (Piaget, 1974). Piaget asserted that:

> An action conception is a transformation of a mathematical object by individuals according to an explicit algorithm that is conceived as externally driven. Through individuals' reflection on their actions, subsequently, they can interiorize them into a process. Each step of a transformation may be described or reflected upon without actually performing it. When a person reflects on actions applied to a particular process, he or she becomes aware of the process as a whole, or encapsulate it. A mathematical schema is considered as a collection of action, process and object conceptions, and other previously constructed schemas, which are synthesised to form mathematical structures utilised in problem situations (Trigueros & Martinez-Planel, 2010, p.146).

Numerous scholars have used APOS theory globally to understand students' construction of knowledge in sections of mathematics. Some of these scholars (Steward

& Thomas, 2009; Parranguez & Oktac, 2010) have used APOS theory to scrutinize learners' mental construction in the learning of linear algebra. In addition, others have used this theory to describe learners' understanding of algebraic functions (Martinez-Planell & Gaismas, 2012; Mahir, 2010; Thompson, 1994). Similarly, this study seeks to investigate learners' understanding of mathematical concepts in the learning of grade 11 algebraic functions. It is paramount to inform the reader that this researcher is not the first South African researcher to use APOS theory while trying to understand learners' construction of mathematical concepts. Ndlovu (2014), Bansilal (2015) and Maharaj (2010) are prominent South African scholars in mathematics education who used APOS theory in their previous research studies. The APOS theory is a model for a detail description of learning mathematical concepts and how learners can mentally construct their understanding of mathematical concepts (Arnon, Dubinsky, RoaFuentes, Wellerr, Cottril, Oktac & Trigueros, 2014). It postulates that mathematical knowledge comprises a learner's tendency to deal with perceived mathematical concepts to solve the problems and make sense of the situations (Dubinsky & McDonald, 2008). Mathematics education requires learners to build and use certain mental structures (or constructions), which APOS theory referred to as stages in the learning of mathematical concepts (Arnon, et al., 2014). These mental structures arise through reflective abstraction (discussed above), which in APOS theory entails interiorization, encapsulation, coordination, reversal, de-encapsulation, and thematization. From a cognitive viewpoint, an elucidation of constructions of mental structures and mechanisms needed to learn a certain mathematical concept is referred to as genetic decomposition (Arnon, et al., 2014). A genetic decomposition may comprise an explanation of how these structures are related and arranged into a larger mental structure known as schema (Arnon, et al., 2014). The mental structures in this theory link the components of action, process, object and schema. The following is a summary of the vital components of the APOS theory:

3.3.1 Action

This component of APOS theory is external such that each step of knowledge transformation needs to be performed, guided by external instructions (Arnon, et al., 2014). The action level in this theory is necessary for performing each step of transformation clearly and considers algebraic functions as static. This means that a learner's understanding of function concepts at an action level is tied to a specific rule, formula or computation (Oehrtman, Carlson & Thompson, 2008). For example, at the action level, the learners need to understand the rule or computation such as 'division by zero', solving quadratic and exponential equations before they can understand the hyperbolic, quadratic and exponential functions. Moore and Carlson (2012) argue that it is unlikely a learner working at this level will be able to solve a situational problem that involves a function without the provision of a formula. That is, the learner at the action level cannot determine the equation of a function given some coordinates without being given the general formula for that particular function.

3.3.2 Process

At the process level, a learner begins to reflect upon the action, which he or she is performing. A learner can also 'turn back, express or even reverse the step of transformation on previously learned objects without actually performing those steps (Dubinsky, 1991). For learners to be at the process level they should have understood the concepts or rules required at "action level" through the process of *interiorization*. For instance, the learner's ability to perceive the asymptotes of a hyperbolic function entirely depends on the interiorized prior concept of "division by zero" respectively. The interiorization of this concept will also lead a learner to be able to draw these functions respectively.

3.3.3 Object

An individual's awareness of a "process level" as a totality yields to comprehending that manipulations can act in totality (Weller, Arnon & Dubinsky, 2009). In addition, they can also construct such manipulations of which we regard them as having 'encapsulated' the process into an object (Weller, Arnon & Dubinsky, 2009). The construction of these manipulations could embrace discerning an operation that takes two functions on the same set of axes. Similarly, when a learner is given two functions $f(x)$ and $g(x)$, at the object level the learner can give the composition, $f(g(x))$. Simultaneously, if given composite function $f(g(x))$ and $f(x)$, the learner can identify $g(x)$ thus showing that he or she can de-encapsulate the concept of function (see figure

3.1 on the next page). This is supported by Eisenberg (1992) arguing that teachers need to assist learners to apply reverse-path-development thus encapsulating functions.

The literature also implies that if a learners' process level of functions is sufficient, then that learner holds an object of functions. Further, the literature suggests that the learner's understanding of functions at the object level can result in an understanding of other branches in mathematics such as, for example, calculus concepts (Eisenberg,1992; Weller, Arnon & Dubinsky, 2009).

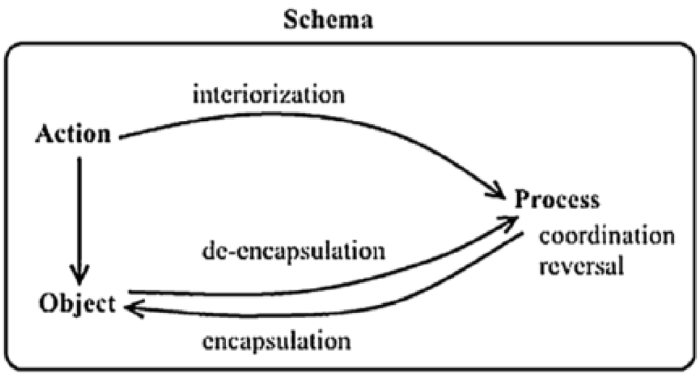


Figure 3.1: Illustration of mental structures of APOS theory

Adapted from Dubinsky and Harel (1992)

3.3.4 Schema

A chapter in mathematics involves several actions, process and objects required to be arranged and connected into intelligible frameworks, called schemas (Weller, Arnon & Dubinsky, 2009). The schema offers a learner with a technique of deciding when given a specific mathematical situation, where it applies. Learners with a well-constructed function schema perceive when there is a need for a particular view of function to be used to solve a problem. They know the relationship between various views and possess the flexibility to shift between them fluidly. Therefore, in this study, if learners can make connections of other thinkable objects for functions, they would have gone through the thematization process of functions. In this study, it is the researcher intends to explore learners' understanding of mathematical concepts necessary in the learning of grade 11 algebraic functions. Therefore, the data collection instrument (task) was created using the mental constructs (action, process and object) that learners need to have a schema for functions.

3.4 The genetic decomposition for functions

In this section, I propose the genetic decomposition for functions (see figure 3.2 on the next page). In understanding learners' development of knowledge of grade 11 algebraic functions, I have used Dubinsky's APOS theory (Dubinsky, 1991). The prosed genetic decomposition for functions has the mathematical concepts such as division by zero, solving quadratic and exponential equations at the action level. When the learners interiorise these mathematical concepts, rules or computations they can understand functions at a process level. At the process level, the learners will be able to determine range and domains of all functions, determine the asymptotes for hyperbolic functions and find intercepts for all functions. When learners can encapsulate the aforementioned mathematical concepts for functions they will then move to the object level.

At the object level, the learners can be able to solve complicated problems for functions such as shifts and a variety of functions in one set of axes. When learners can make connections of different objects for functions they would have gone through the process of thematization for functions (see figure 3.2 on the next page). This study proposed a genetic decomposition for functions comparable to Breidenbach, Dubinsky, Hawks and Nichols (1992), which investigates learners' interpretation of functions in general. The success or failure of learners in using mathematical concepts to understand algebraic functions can be identified with mental constructions that the learner has acquired (Hartati, 2014). For a further articulation of the genetic decomposition of functions for this study, I have used an example of a grade 11 learner's understanding of a concept used in functions.

In figure 3.2 on the next page, the learners' understanding of the 'asymptote' at the process level depends on his or her ability to conceive the effect of a 'zero-divisor' in a 'quotient'. This conception of 'division by zero' at the action level results in an 'undefined quotient (∞)' which is the concept learned externally to assist grade 11 learners' understanding of functions. Hence, learning of the asymptotic behaviour of a hyperbola in this study is based on learners' interiorization of 'division by zero' at the action level. In doing so, the learner uses his or her image of the action without necessarily having to perform each step explicitly (Arnon, et al., 2014). Then we say a learner has interiorised an action concept into a process.

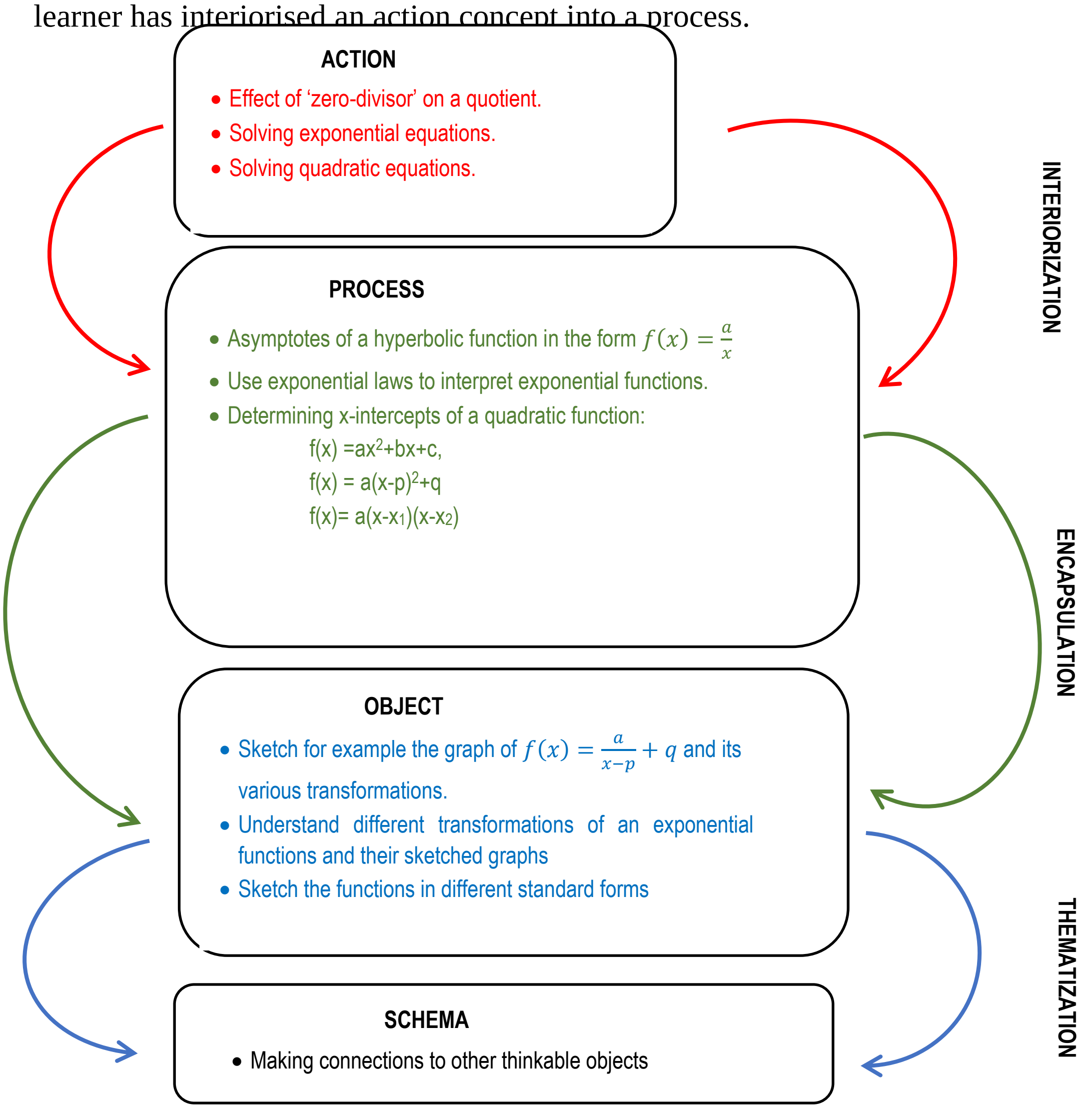


Figure 3.2: A proposed genetic decomposition for functions
Adapted from Brijlal (2019)

After a process level, learners can construct their object level understanding of mathematical concepts. Here, learners need to realise a process as a totality and to transform it (Weller, Arnon & Dubinsky, 2009). For instance, the conception of factorising quadratic equations leads to interiorization of the concept into determining the *x*-intercept of a quadratic function. This knowledge of determining the *x*-intercepts of a quadratic function can then be encapsulated into a cognitive object where one can interpret the *x*-intercepts of a parabolic graph.

Once the learner has a collection of actions, processes and objects, he or she then possesses the schema of mathematical concepts. The Action, Process, Object and

Schema conception can be designed in the task item. Below are examples of items in the task used to collect data about the quadratic function that connects to Action, Process and Object conception:

1. This item in the task evaluates whether learners understand the mathematical concept at the *action level*, which later aids them to understand the parabola.

Solve for *x* in the following equations: a) $x^2 - x - 30 = 0$ b) $\left(x - \frac{3}{2}\right)(2x + 5) = 0$

Solving the quadratic equation in 1 (a) above can be done by using a factor method or a quadratic formula. If a learner can only solve for *x* in 1 (a), but he or she cannot solve for *x* in 1 (b) then the learner does not have complete knowledge of the mathematical concept (factorisation) necessary to understand parabola at the process level.

2. This item of the task evaluates learners' understanding of the parabola at the *process level*.

Given a function $f: y + 4 = (x - 5)^2$. Determine the x-intercepts of *f*.

The learner's ability to transpose 4 to the other side to make $y = (x - 5)^2 - 4$, designates the "action level" understanding. Furthermore, if a learner gives a correct answer by factorising the standard form of the equation, then we say he or she has interiorized the concept 'factorisation' into the process of finding the *x*-intercept.

3. The item in the task evaluates whether learners understand the *object level* of a parabolic function.

Sketch the graph of *f*, showing ALL the intercepts with axes and the turning points.

If learners can answer this question perfectly, then we say they are at the object level and have developed a schema of the parabola. The examples of items in the task given above depict problems that can be used to observe the level of learners’ understanding of mathematical concepts from an APOS perspective. This study used a task comprising of items that represented an action, process, object, and schema conception.

3.5 Conclusion

The APOS theory, as explained in this chapter, is a suitable theoretical lens for exploring learners’ understanding of mathematical concepts necessary for learning grade 11 functions. Given this, the theory allows for a more detailed understanding of the formation of the genetic decomposition for functions. The theory also suggests that a detailed insight of children’s learning of functions depends on external experience embraced about function. In addition, the APOS theory posits a hierarchal understanding of learning and development in mathematics since we consider the development of learners once they reach the schema level of this theory.

CHAPTER 4

AN EXCURSION TO THE FIELD: RESEARCH DESIGN

4.1 Introduction

The previous chapter reflected on APOS theory as a comprehensive framework for reconceptualising mathematical concepts necessary in understanding functions. The use of the theory in developing a detailed analysis of learners' experiences in learning functions in grade 11 was explained. Then, this chapter focuses on the presentation, discussion and explanation of the constructs of the methodology the researcher employed in addressing the purpose of the study. In Bertram and Christiansen (2014) perspective, research is the systematic enquiry aiming at discovering and interpreting information about the subject under study.

This chapter explained the in-depth process and methods undertaken to collect, organize, and analyse data gathered through tasks and semi-structured interviews. The use of data collection processes is extremely significant in realizing the purpose and objectives of the study (Bertram & Christiansen, 2014). For the reader to be acquainted with the context of the study, the researcher outlined briefly the research design and paradigm. In addition, the researcher explains and justifies techniques for selecting participants, collecting and analysing data for this study. Furthermore, a detailed ethical consideration, trustworthiness and validity are provided in this chapter.

4.2 Research design

The research design is the researcher's strategy of systematically collecting and analysing data to respond to the research questions (Bertram & Christiansen, 2014). Similarly, Babbie and Mouton (2007) defined the research design as the plan that the researcher configures in addressing the research objectives and questions of the study. The research questions entailed in Chapter 1 of this study are there to guide the research design of this chapter (Bertram & Christiansen, 2014). Therefore, the research design comprises the methods used by the researcher to gather data and present the findings. Briefly, the research design is the blueprint that guides the entire research study.

The research design suitable to explore learners' understanding of functions in this study was a case study. The case study was defined by numerous scholars in research, including Robert Yin whose is the founder of case studies in research. He defined a case study as "an empirical inquiry that investigates a contemporary phenomenon in depth and within its real-life context" (Yin, 2014). Additionally, Hancock and Algozzine (2015) defined a case study as an intensive investigation based on a single unit. These definitions are also similar to Baxter and Jack's (2008) definition that a case study is a qualitative enquiry of intensive, holistic and analysis of a single entity, phenomenon, event, process or social group. These definitions evoked the researcher's interest to define a case study in a specific event, not only as of the method of enquiry (Niewenhuis, 2010).

Scholars use certain research designs for certain reasons. The reasons for using the case study as the design for this study are threefold: Firstly, to give insight to a certain instance by providing a thick, rich description of the case and enlightening how this relates in a broader context (Peter & Vaughm, 2011). Secondly, to generate a problem or issue within a limited and focused setting (Peter & Vaughm, 2011). Lastly, to create theoretical insights, in testing and developing the theory regarding the case (Peter & Vaughm, 2011). In addition to the above-mentioned reasons, the researcher used the case study to generate insight into APOS theory concerning each case. These reasons correspond with the researchers' reasons for using case studies.

Yin (2014) mentioned that case studied are preferable in responding to questions of "how" and "why"; where the researcher focuses on the current phenomenon and having a little control over it. Therefore, the following research questions numbered (ii) and (iii) yields to a researcher proposing a case study as a suitable design for this study:

i) What are the mathematical concepts that are necessary for learning grade 11 algebraic functions?
ii) How are learners understanding these mathematical concepts when learning grade 11 algebraic functions?
iii) Why are learners understanding these mathematical concepts necessary for learning grade 11 algebraic functions in the way they do?

These questions prompted the researcher to gain insight into learners' understanding of mathematical concepts necessary in the learning of grade 11 algebraic functions. Therefore, the researcher focused on a case study done in each of three schools as being suitable for this study. Below the researcher describes ontological and epistemological underpinnings in order to discuss the paradigmatic disposition in this study.

4.2.1 Research paradigm

In research, there is an emerging consensus informally presenting a paradigmatic framework for a study. Epistemological, ontological, and methodological stances are three dimensions that can structure such a framework (Cohen, Manion & Morrison, 2013). In this section, the researcher provides a brief definition of the term "paradigm" from numerous scholars, then describes ontological and epistemological stances in this study and paradigmatic stance as well. The reason for providing such discussions is that the researcher wants to present the study concerning the chosen paradigm, design, and methodology. The researcher believed that there is a strong relationship between ontological, epistemological stances and the methodology that has been chosen for this study. Furthermore, the view by Sike (2004) is that the "researcher's personality" informs the selection of the research methodology.

According to Creswell (2007), paradigms refer to sets of assumptions, values and beliefs about essential characteristics of reality which give rise to a certain worldview. Additionally, Creswell (2013) defined a paradigm as a set of basic beliefs representing a worldview that designates the researcher, the nature of the world, the individual's location on it and the range of possible relationships to that world and its components. Therefore, according to the above-mentioned definitions, a research paradigm entails a general worldview guiding a researcher's interpretation of reality. Sike (2004) further alluded that a researcher's paradigmatic disposition informs knowledge, truth, and meanings, which shape the involvement with research participants to understand a particular phenomenon.

Ontology refers to the assumptions concerning the worldview or nature of the social phenomena being investigated (Cohen, Manion & Morrison, 2013). This links with Crotty's (2003) view that ontology refers to "the study of being" and focuses on the kind of world under investigation, with the nature of existence and the structure of reality. In this regard, the researcher's ontological assumptions are concerned with the question "What is the nature of reality?". "Is the reality of an objective nature, or the result of individual cognition?" (Cohen et.al, 2013). "Is it a given out there in the world, or is it created by one's mind?" (Cohen et.al, 2013).

Mathematics learning in this study is viewed based on the APOS theory through which learners construct knowledge of new schema with understanding prior learned concepts. The researcher strongly believes that learning experiences and attitudes towards learning mathematics in the same school are very different from social construction. The ontology in the research focuses on "the nature of reality", while the focus of epistemology is based on knowledge, its nature and forms of inquiry (Cohen et. al, 2013). Similarly, Crotty (2003) defined epistemology as "a way of understanding and explaining how we know what we know". To be more precise, the phenomenon in which the researcher seeks to gain insight determines the assumptions about the nature of knowledge. According to Cohen, Manion and Morrison (2013), knowledge is "hard, objective and tangible", thus this is how the researcher interpreted the way learners use mathematical concepts in understanding algebraic functions in the test the researcher gave them. The epistemological stance used in this study is social constructivism since knowledge is socially constructed and learners are part of the construction (Cohen et. al, 2013).

Considering the discussion of these ontological and epistemological positions mentioned above, the underpinning philosophical assumptions situate this study in an interpretive paradigm. The emphasis on the relationship between the goal, the exploration, and the path taken to reach the goal is one of the strengths of the interpretive paradigm (Mouton, 2012). Within this paradigm "the researcher can understand the subjective world of the human experience" (Cohen, Manion & Morrison, 2013). Also, the interpretive paradigm is a tool with which to uncover the fundamental set of beliefs, which guide the action of a person (Creswell, 2013). An epistemological stance of interpretive philosophy indicates the construction of knowledge by describing people's action, beliefs, values, understanding and construction of meaning (Henning, Van Rensburg & Smith, 2004). In this way, the researcher seeks to explore whether learners find the meaning of mathematical concepts for learning functions, and in particular how they construct their knowledge of concepts related to functions.

The use of the interpretive paradigm in this study seeks to guide how grade 11 learners respond to questions related to algebraic functions. As an educator, the researcher intends to share feelings and experiences, as well as to provide interpretations of grade 11 learners' actions when they respond to algebraic functions (Ndemuweda, 2011). Having stated the ontological, epistemological, and paradigmatic stance, in the next section the researcher will present the research approach for this study.

4.2.2 Research approach

The research divides research approaches into three categories namely: qualitative, quantitative and mixed-method. In this study, the researcher uses a qualitative approach to understand grade 11 learners' understanding of mathematical concepts necessary in the learning of algebraic functions. This approach requires a researcher to collect and analyse qualitative findings and draw inferences using these sorts of data in a single study (Tashakkori & Creswell, 2007). Even though this study uses qualitative approach at large, however, the task given to learners was analysed in Chapter 5 using tables and bar graphs. Within a unit study, a quantitative analysis examines multidimensional problems to inform the qualitative analysis and to triangulate the findings within sources of evidence (DeCuir-Gunby & Schutz, 2017). The results embraced in this study emanated from a qualitative approach, where learners responded to a task based on functions. Thereafter, the researcher included interviews to clearly understand learners' knowledge of mathematical concepts necessary for learning grade 11 algebraic functions. The latter results are dominant and demonstrate that this study falls within the interpretive paradigm (Ernest, 1998).

4.3 Research sampling

In an interpretive qualitative study, the selection of study samples according to numerous authors (Creswell, 2007), is based on the supposition that they are knowledgeable about the subject under study. This denotes, according to Babbie and Mouton (2001) that the researcher "speaks of respondents as people who provide information about themselves, allowing the researcher to construct a composite picture". In addition, when the researcher selects participants of the study, he or she precisely consider selecting participants who are knowledgeable and informative about the phenomena under scrutiny (McMillan & Schumacher, 2010). Thus, precisely grade 11 mathematics learners were selected since they are the ones who are informative about the study phenomenon. With these sampling steps in mind, the researcher has realized Mouton's (2012) definition of research sampling which is a technique the researcher employs in the selection of the sample that is suitable for the study.

In addressing the research questions, the considerable sampling procedure for this study involves determining the location or size of the study (Creswell, Clark & Plano, 2018). In doing so, the researcher considered the number of participants as significant in answering the research questions (Creswell, Clark & Plano, 2018). This study used a purposive and convenience sampling as the criteria for selecting participants since the research is not aimed at generalisations about the wider population (Cohen et. al, 2007). The selected participants have an experience of the central phenomenon being explored (Creswell, Clark & Plano, 2018). The researcher believed that grade 11 learners were most relevant and most informative (McMillan & Schumacher, 2010) about their experiences of learning algebraic functions since they selected this subject in grade 11. Since the data was collected both qualitatively and quantitatively, therefore the provision of the reader with a picture of the sample size chose for this study is essential. Creswell, Clark and Plano (2018) mentioned that a sample size needs to be sizeable enough to meet the requirements of the planned statistical tests and provide a good estimate for the parameters of the population. Hence, the sample size in this study was manageable enough to understand and interpret the findings.

The sample size comprises sixty grade 11 learners (twenty learners in each of the three schools). The researcher has found that each school comprises above twenty grade 11 learners who took mathematics, which is convenient for the researcher for selecting the needed sample. During the selection of the sample, the researcher expected that the sample size might not be the exact sixty responses required for this study. This was due to realising that learners hold the authority to refrain from being part of the study at any time. Therefore, the number of participants entirely relied on learners' interest to participate in the study.

Creswell (2003) declared that for phenomenological studies the sample size required ranges from 5-25 informative individuals for the study under scrutiny. Therefore, using twenty learners from each school in this case study met this criterion for phenomenological studies. The researcher selected these learners to maximise the variety of information obtained during the process of quantitative data analysis and to possess manageable data. The three schools in which this case study research was conducted are situated in the uMgungundlovu District in KwaZulu-Natal.

4.4 The distinctive characteristics of three schools

The three schools[7] within which this research was conducted are located in the uMgungundlovu district and are separated by two distinct circuits. The pseudonyms for these schools are Lanfield Secondary school, Lulove High school, and Maxiz High school. Lanfield Secondary school is in the semi-urban area called Edendale, under the uMsunduzi Municipality, whilst Lulove High school and Maxiz High school are in the rural Impendle area under the Impendle Municipality. While conducting this study, it was realised that the infrastructural conditions in these schools are not of the same quality (see images below). There are three space norms that each school in the country requires to provide education that is of high quality to its learners (Department of Basic Education, 2009). These space norms are core educational spaces, administrative spaces, and support educational spaces. Before the researcher presents the existing nature of the three schools, they draw from the Department of Basic Education (2009) specifications of the schools' space norms. Therefore, entering each school the focused was also on finding out about the presence and quality of the above-mentioned space types in addition to collecting data for the study.

It is difficult for learners to learn properly due to overcrowded classrooms in the three schools of this study. During informal conversations with teachers in each school, the researcher has found that they teach classrooms comprising more than fifty learners each. At that point, they cannot give learners individual attention and assistance since there is no space to move around in the classroom. The classrooms in both Lulove and Maxiz high schools are in poor condition and have broken window glasses, extremely old chalkboards, and poor floor space with holes in the floor. Taking into consideration that the area, Impendle, is characterised by extremely cold weather conditions throughout the year and

[7] For the protection of the identities of the three schools, I have used pseudonyms. Thus, throughout this research, the schools are referred to as Lulove Secondary School, Maxiz High School and Lanfield Secondary School.

that some classes have broken windows, thus learners cannot learn effectively. This is evidenced by the broken windows at Lulove high school (see figure 4.2, image 5 below) where cold air enters the classroom causing learners to feel cold. In Lanfield Secondary School, the classroom walls are in good condition, but the furniture (e.g. desk) is not (see image 3). This results in learners being unable to sit comfortably during their classroom lessons.

Image 1: Science Laboratory donated by Hulamin

Image 2: Library with packed furniture

Image 3: Classroom furniture

Image 4: Classroom roof with ceiling board and electricity

Image 5: Classroom whiteboard

Image 6: Mathematics Classroom with charts pasted on the notice

***Figure 4.1**:* Lanfield Secondary School

The ‘Science laboratory at Lanfield (image 1 in figure 4.1) is filled with the science kits for practical investigations in Physics, Chemistry, Life Sciences, and Technology. This laboratory was donated by Hulamin Company; however, it is not yet functioning. The Lanfield high school is also in possession of a library (image 2 in figure 4.1), however, other shelves are running short of books and the library floor is filled with desks packed together. Thus, it is not convenient for learners to use the school library while it is in such a condition. Lanfield Secondary School seems to be at an advantage as it comprises classrooms with electricity (image 4 in figure 4.1), a whiteboard (image 5 in figure 4.1), and Science and Mathematics posters pasted on classroom walls (image 6 in figure 4.1). This school also displayed the context that is distinct as compared to Lulove and Maxiz High schools in figure 4.2 and figure 4.3 below. It was very useful to understand the role that the context plays in shaping learners' learning of mathematics which is the major concern of this study.

Image 1: Lulove's light bulbs connected across the roof planks

Image 2: Lulove's classroom furniture

Image 5: Broken windows of Lulove high school

Image 3: Lulove's classroom walls with charts

Image 4: Lulove's classroom

Figure 4.2: Lulove High School

At Lulove high school, the classroom roofs are not constructed with ceiling boards (image 1 of figure 4.2), and the electric wires and light bulbs are connected across the timbers. During an informal conversation with a teacher in this school, I have discovered that the corrugated iron has holes, which allows water to leak through and into the classroom during the rainy season. At times lessons cannot proceed as normal due to classroom leaks which can damage learning resources (e.g. books). In addition, the grade 12 extra tuition programme is not offered during winter season due to extremely cold weather conditions.

Lulove high school has new classroom furniture (e.g. desks), however, there is a scarcity of this furniture and learners sit in threes which cause them to be uncomfortable, more especially when they write (image 2 of figure 4.2). The classroom charts posted on the walls (image 3 of figure 4.2) are written by hand and they are invisible unless you get closer to them. Extremely old chalkboards are still in existence at Lulove high school (image 4 of figure 4.3) of which some cannot produce visible writing by teachers during lessons.

Image 1: Maxiz high school classroom furniture

Image 2: Maxiz high school classroom chalkboard

Image 3: Maxiz high school toilets

Image 4: Maxiz high school garden

Figure 4.3: Maxiz high school

Since Lulove high school and Maxiz high school are located in a rural context, however, the features of these schools are slightly different. In common, both schools have electricity (image 1 and 2 of figure 4.3), which sometimes also affects the schools' functionality during winter seasons due to load shedding. In addition, the schools have old chalkboards in classrooms (image 2 of figure 4.3) as compared to whiteboards at Lanfield Secondary school (image 5 of figure 4.3). The classroom furniture for both Lulove and Lanfield high school is new and similar in design; however, Maxiz high school has furniture that accommodates only one learner. In other words, during classroom lessons at Maxiz high school learners can sit comfortably without sharing furniture. Image 3 of figure 4.3 portrays staff members and learners pit toilets which are non-flushing and are too far away from school buildings. These toilets are surrounded by bush behind which poses safety risks in a sense that snakes, for instance, can make such spaces their habitat. The school garden at Maxiz high school appeared to be taken care of, but with only cabbage as the vegetable being grown in the garden.

The above-mentioned features of the three schools illustrate the importance of context in understanding challenges faced by rural learners as compared to semi-urban learners with respect towards learning in general. In addition, it provides insight to a researcher about the learning of mathematics within different educational environments since they largely produce and equip the participants within each context. In this study, grade 11 learners form part of the sample, thus twenty learners were selected in each of the three schools.

4.5 Research instruments

In this study, the researcher used the CAPS document, task, and interviews to gather information about learners' use of the mathematical concept in their learning of grade 11 algebraic functions. The use of the CAPS document was to determine the mathematical concepts that grade 11 learners need in the learning of algebraic functions. Thereafter, the use of a task was to find mathematical concepts learners possess related to functions. Since the lens used in this study is APOS theory, thus, the task also played a role in identifying learners' possession of mental structures (action, process and object) in their learning of algebraic functions. In the task, the researcher has at his disposal a powerful method, which yields numerical data collection other than the verbal kind (Cohen, Manion & Morrison, 2007). Then, numerous issues need to be borne in mind for the consideration of the task. These issues include intelligence, achievement, personality, attitude or social adjustment which are aspects being tested by the researcher (Cohen, Manion & Morrison, 2007).

In this study, the task will provide the researcher with numerical data on how participants use mathematical concepts to learn algebraic functions. In addition, the researcher took notice of learners' knowledge of algebraic functions in this study and had learners give a verbal description of why they possess such knowledge. This was achieved through the marking of the task by the researcher, which then assisted in in identifying the patterns that could have evolved in the responses of each sub-question. Since the researcher knows the list for the number of grade 11 mathematics learners in each school, therefore, a maximum of twenty learners was used as participants in responding to the task.

Another instrument used in this study was interviews, which were defined by Bertram and Christiansen (2014) as conversations between the researcher and the respondent. The interviews in this study were extremely important in triangulating data and gaining descriptive data from the participants. In addition, the researcher wanted to know how learners generate their knowledge towards learning functions and wanted to understand the challenges they experienced on other concepts entailed in functions. Since this study took place in three schools, therefore, three learners were interviewed from each school, which made up nine interviewed learners total in this study. The conduction of these interviews was based on learners' performance in the task and APOS mental structures they possessed. All of these interviews were audio-recorded and analysed with permission from participants. The researcher believes that audio recording the interviews will provide time to transcribe audios and analyse them thoroughly rather than writing the interviews, which can lead to some of the verbal information from participants being lost.

4.6 Data organisation and analysis

According to Marshall and Rossman (1999), data organisation and analysis refers to the process through which the researcher brings order, structure, as well as creating meaning to the data collected. In this study, the researcher based the enquiry on the assumptions that diverse data brings a complete conception of the research problem rather than the use of one type of data. Both quantitative and qualitative data were collected and analysed separately. They were compared to find out if the findings confirm or disconfirm each other. The following figure on the next page displays each type of instrument used in answering the three stated research questions. The CAPS document was the first instrument used to analyse the pre-existing mathematical concepts that grade 11 learners need to know for the learning of algebraic functions.

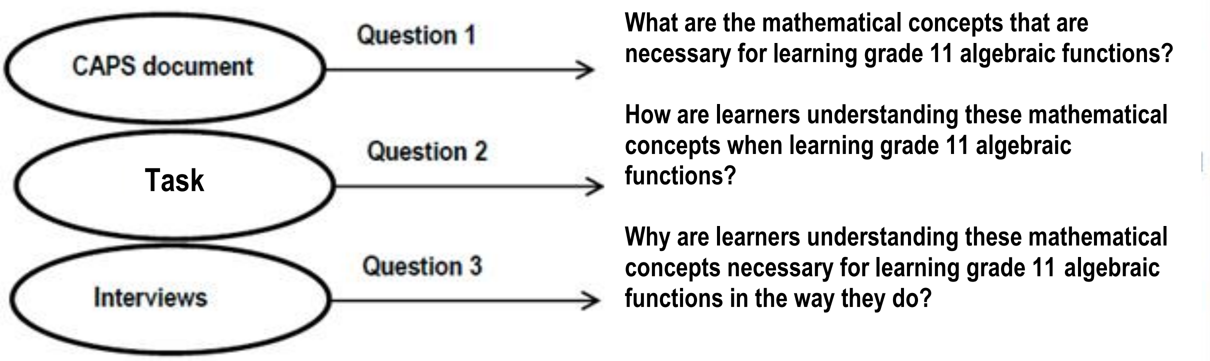


Figure 4.4: Illustration of data collection and analysis

This document analysis assisted the researcher to set an appropriate task for this study. The selection of participants in this study is based on Opie's (2004) idea that "the population should be defined with the objectives of the study in mind". Therefore, all sixty participants in this study made up of the twenty participants in each of the three chosen schools were given a task based on grade 11 functions. The researcher did this after hours of teaching and learning using their normal classroom. In addition, refreshments were provided which attracted them to fully participate in this study. Figure 4.5 below is the picture showing grade 11 learners of three different schools responding to a task:

Image 1: Lanfield Secondary School

Image 2: Lulove High School

Image 3: Maxiz High School

Figure 4.5: Learners of three participating schools writing a task

The task given to learners was comprised of three questions to be answered based on algebraic functions. All the scripts of the task were investigated and analysed in detail according to the APOS analytical framework (see Appendix G). These learners' scripts were analysed for good performance, adequate performance and poor performance.

Lastly, based on learners' responses from the task, the researcher then conducted one on one interviews for forty-five minutes each to interpret learners' experiences or viewpoints about concepts learned in algebraic functions. Since one cannot tell fully about learners' thoughts whilst responding to a task, hence the significance of interviews is that of getting their thinking from the "horses' mouth". During interviews, the researcher used an audio recorder to capture the learners' utterances. In addition, learners were permitted to write their responses down if they wished to. The researcher then transcribed the interviews verbatim by constructing a table of conversation between the researcher and participants in Microsoft Word (see Appendix I). Since the research approach of the current study was a mixed-method, it involved data that had to be analysed quantitatively and qualitatively. The recorded audios from the interviews were then transcribed to get a thick description of learners' understanding of the mathematical concepts in learning functions.

4.7 Ethical considerations

The interaction with participants and other beings in the research fraternity raises ethical issues (Babbie & Mounton, 2007). According to Durrheim and Wassenaar (2004), ethical principles are extremely significant in terms of autonomy, non-maleficence, and beneficence. In terms of the study's autonomy, the researcher is required to obtain consent from every participant who is part of the study (Durrheim & Wassenaar, 2004). In addition, the researcher must also understand that the participants are not forced to engage in the study, meaning they can withdraw at any time.

The researcher gave consent forms to learners for declaring their willingness to form part of the study. This was done before a task was written and before the condition of the interviews with them. The language in which these consent forms were written was understandable to the learners (see Appendix D). During data presentation and discussion, participants' identities were protected (pseudonyms were used) and they were informed that the data collected was to be used for the research purpose only.

The non-maleficence of participants is kept in this study, meaning that no harm will occur to participants during this study (Bertram & Christiansen, 2014). In addition, there is no form of physical, emotional or either social harm that will happen to participants. Furthermore, the participants were assured of the confidentiality of the information supplied to them. In other words, the researcher confirmed their identities and the schools in which they studied. The researcher's permission to carry out this study in selected schools was granted by the KwaZulu-Natal Department of Basic Education (see Appendix A). In addition to this, the researcher also wrote a letter to school principals requesting to use grade 11 mathematics learners for the data collection process (see Appendix B). The University of KwaZulu-Natal Ethics Committee also gave the researcher's authority to collect data for this study (see Appendix E). The data collected in this study was used for only the purpose of this study and will be stored in the Research Office for five years. The parents of learners who participated in this study were given consent forms to sign, indicating their willingness for their children to participate in the study (see Appendix D). Grade 11 learners were also given the informed consent forms (see Appendix D).

4.7.1 Piloting of instruments

In conducting any research study, the researcher is required to outline a strategy to follow (Dikko, 2016). The chosen research instruments for collecting data in this study went through the test of validity to confirm good measures for these instruments (Dikko, 2016). According to Sekaran (2003), a researcher can achieve the reliability of a measure through the consistency of the instrument.

This study employs two phases of piloting. The first phase was pre-piloting done at the beginning of this study to see whether grade 11 learners could talk about functions. I gave grade 11 learners a task based on the curriculum in which they tried to respond to curriculum questions and then we discussed their solutions. During the discussion of task solutions, the researcher intended to see whether they were at ease speaking English and were able to express their thinking on functions. It was evident during the interaction that learners were able to communicate. Through the scrutiny of their scripts, I was able to identify some errors they made and then develop the task based on the literature on functions as discussed in the chapter on the literature review. The results from the first phase of piloting did inform the questions used in the second phase (pilot of the main study).

In this study, the researcher did the piloting of the instruments (test and interviews) with a grade 11 class that was different from the one intended for the study (Creswell & Plano Clark, 2018). This was done to indicate to me the task's suitability for the study in terms of clarity in the instruction, structure, and content of the questions. The execution of the pilot in this study also gave an opportunity as a researcher to practice the administration protocols (Creswell & Plano Clark, 2018). In addition, it aided in deciding on the removal of any items that seemed to be irrelevant data for the study (Creswell & Plano Clark, 2018).

4.8 Ensuring quality of the findings: trustworthiness

To maximise the trustworthiness of the study, it is paramount to consider certain issues. The researcher must ensure that the study is valid, reliable, and is generalisable (Loh, 2013). In research, validity is a significant factor, and a research study without validity is meaningless (Cohen, Manion, & Morrison, 2011). Numerous authors (Babbie & Mounton, 2007; Loh, 2013) elucidate on four strategies for the establishment of validity and trustworthiness in qualitative research as invented by Lincoln and Guba (1989), namely; credibility, transferability, dependability, and confirmability.

4.8.1 Credibility

It is extremely important for any empirical research that the researcher makes sure that the researcher ensures the findings as a true reflection of the participants' information. Concerning this, the researcher's assurance located in the truth of the research findings gives the credibility of the research (Anney, 2014). There various distinct strategies used to establish the credibility of the study which includes field experience, triangulation, member checking, time sampling, interview technique and structural coherence (Anney, 2014). In this study, the researcher collected data for an entire month in three schools in the uMgungundlovu district. This also includes two days that were spent in each school to gain a comprehensive understanding of the nature of the schools with the culture of learning. Spending much time in each school enabled the researcher to gain insight into the nature of learning mathematics in each school. In addition, it gave insight into the factors that can shape learners' experiences of learning and understanding mathematics. For instance, a mathematics classroom in one school has technology resources for teaching and learning mathematics, such as geometers sketchpad, GeoGebra, and autograph.

The interview technique that was followed is another aspect that increased the credibility of this study. As mentioned earlier, a semi-structured interview technique was used, which allowed for open conversation with the learners about their understanding of mathematical concepts in algebraic functions. Furthermore, member checking is another to increase the credibility of this study. The researcher has provided the supervisor with both the data and the information provided by learners to check if they correlate. The verification of data was a priority in this study, for instance, the researcher went back to schools to verify the data before transcribing the interviews. The intention of doing this was to check whether what the learners said or interpretations that were made about the subject under scrutiny matched what they meant during conversations done in the interviews.

4.8.2 Dependability

The evaluation of the findings, clarification and recommendations of the study supported by the data received from participants is referred to as the dependability of the study (Korstjens & Moser, 2018). The qualitative research determines its philosophical and epistemological position by both the problem and the predisposition of the researcher in terms of the classification of data (Moon & Blackman, 2014). Thus, the researcher needs to report on the steps taken to manage and reflect on the effects of their philosophical or experiential preferences (Moon, Brewer, Januchowski-Hartey, Adams, & Blackman, 2016). This is to ensure that the study findings are based on the experiences and preferences of the research participants rather than those of the researcher (Moon et al., 2016).

To address issues of dependability in this study, the use of peer debriefing is important (Anney, 2014). During the research process, the supervisor was there to provide guidance which included scrutinising "the data, findings, interpretations and recommendations" (Korstjens & Moser, 2018) of the study to determine whether or not they correlated. In addition, she evaluated whether the interpretations of participants' responses, conclusions, and recommendations can be supported by the participants' information gained during the test and from the conversation with them during interviews. There are two significant criteria for an enquiry audit for the master's research and supervision at the University of Kwa-Zulu Natal. Firstly, during supervision sessions, the supervisor frequently challenged the researcher to critically think about each component of this study, to ensure that no information was treated as irrelevant. Secondly, before writing this thesis, the requirement was a proposal presentation and assessment. The supervisor organized cohort sessions where students had to present research proposals in preparation for presentation to the academic panel of lecturers who are specializing in the field of research. In the academic panel, the lecturers read the proposal and challenged and provided academic advice on how to refine the focus of the study. An examination is the last form of audit; this is done to ensure that all the ideas presented in this study are academically sound and coherent.

4.9 Conclusion

Entirely, this chapter presented a detailed research design, paradigm and an approach considered in this study. The researcher also recognized the sampling methods and the research tools that are in line with this study. Furthermore, there are different phases entailed by this mixed-method study through which data was collected. Finally, there was a discussion on the validity of the research instruments, the limitations of the study and ethical considerations.

CHAPTER 5

DATA PRESENTATION, ANALYSIS AND DISCUSSION OF THE FINDINGS

5.1 Introduction

This chapter presents and analyses findings intending to provide a picture of participants' experiences with algebraic functions. This is in line with the purpose of this study which was to describe mathematical concepts that learners need in the learning of grade 11 functions. The researcher has realised the need to embrace the CAPS document analysis in this chapter. Thereafter, each participant completed a task that was based on algebraic functions. The participants' interviews based on the task was analysed thoroughly to triangulate the data.

Combining qualitative and quantitative methods for data collection in this study was done to fully address the research question. The qualitative approach for the integration of both types of data allowed the researcher to investigate which commonalities emerged. The quantitative data were presented and analysed through the use of the APOS analytical framework (see Appendix G) in conjunction with participants' excerpts of the task. This visual representation of data allowed the researcher to undertake a convenient analysis and interpretation of quantitative data. The analysis of each question in this study is comprised of the mean percentage for each question item. From the analysis of these questions, the researcher has developed the formula for calculating the mean percentage of learners with errors in using the concept investigated in each question. The formula is:

$$\text{Question} = \frac{\text{Number of errors}}{60} \times 100$$

The analysis drawn from the interviews strengthened the links between the errors made and the APOS theory levels that the learners possessed. In some instances, the interviews revealed the opposite of what was expected to be the case. The intention of quantifying the data in this study was to give an overview of the learners' responses to the test, thereby answering the following questions:

i) What are the mathematical concepts that are necessary for learning grade 11 algebraic functions?

ii) How are learners understanding these mathematical concepts when learning grade 11 algebraic functions?

However, the interviews were also intended to answer the third question: iii) Why are learners understanding these mathematical concepts necessary for learning grade 11 algebraic functions in the way they do?

Before the researcher presents and analyses the findings for this study, it is extremely important to provide the reader with a brief discussion of general performance in mathematics by participants of this study. During the data collection process in each school, the researcher firstly discussed with the grade 11 mathematics educators about their learners' performance in mathematics. This was done to gain insight into the types of learners that would be used for data collection processes. Also, this enabled further adjustment of the research instruments such that it would provide precise data that would measure the phenomenon of this research and give a response to the research questions. It was found that each grade 11 class for each school comprised learners who performed better, fairly and those whose performance is bad. This gave a picture of expectations about learners' performance on the test, which was one of the research instruments used in this study. In this chapter, 5.2, the researcher presents and analyse the findings of this study. The analysis of the findings in this study is compared to the relevant literature in order to see what the commonalities or differences are between this study and past studies.

5.2 PRESENTATION AND ANALYSIS OF LEARNERS' MENTAL CONSTRUCTIONS

5.2.1 The CAPS document presentation and analysis

The Curriculum and Assessment Policy Statement (CAPS) is the curriculum statement that is currently used by the Department of Basic Education (DBE), which was amended from the National Curriculum Statement (NCS). This policy statement is not as advanced as a new curriculum since it still follows the same procedure used in the NCS Grades R-12 (Pinnock, 2011). In addition, it does not follow a certain procedure or give suggestions on how teachers should interpret it based on their pedagogy and educational needs. The teachers are freely able to decide on their teaching strategies, activities, tests, assignments, projects, etc. (Hoadley, 2012).

One of the research objectives of this study was to identify the mathematical concepts necessary for learning algebraic functions. Thus, it was felt that the analysis of both Senior Phase and FET CAPS documents was significant in this study. This will provide the reader with clear insight into previously learned mathematical concepts, which could assist learners to understand grade 11 algebraic functions.

The following figure 5.1 represents the layout of the topics to be covered on algebraic functions in FET phase. It is evident from figure 5.1 below that in grade 10 learners explore function types, namely, linear, some quadratic polynomial functions, exponential functions, and some rational functions. However, in grade 11 the content learned in grade 10 is extended to the relationships between variables in terms of numerical, graphical, verbal, and symbolic representation of functions. In addition, the function types explored in grade 10 are further explored in grade 11. In order for learners to understand grade 11 functions, there are necessary mathematical concepts covered in the Senior Phase mathematics CAPS documents. These mathematics concepts are highlighted in figure 5.2 on the next page.

1. FUNCTIONS					
	Grade 10		**Grade 11**		**Grade 12**
	Work with relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and some quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.		Extend Grade 10 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear and quadratic polynomial functions, exponential functions, some rational functions and trigonometric functions.		Introduce a more formal definition of a function and extend Grade 11 work on the relationships between variables in terms of numerical, graphical, verbal and symbolic representations of functions and convert flexibly between these representations (tables, graphs, words and formulae). Include linear, quadratic and some cubic polynomial functions, exponential and logarithmic functions, and some rational functions.
	Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effect of the parameter which results in a vertical shift and that which results in a vertical stretch and /or a reflection about the *x* axis.		Generate as many graphs as necessary, initially by means of point-by-point plotting, supported by available technology, to make and test conjectures and hence generalise the effects of the parameter which results in a horizontal shift and that which results in a horizontal stretch and/or reflection about the *y* axis.		The inverses of prescribed functions and be aware of the fact that, in the case of many-to-one functions, the domain has to be restricted if the inverse is to be a function.
	Problem solving and graph work involving the prescribed functions.		Problem solving and graph work involving the prescribed functions. Average gradient between two points.		Problem solving and graph work involving the prescribed functions (including the logarithmic function).

Figure 5.1: Illustration of functions in FET phase mathematics

CONTENT	GRADE 7	GRADE 8	GRADE 9
1.1 **Whole numbers**	**Properties of whole numbers** • Revise the following done in Grade 6: - recognize and use the commutative; associative; distributive properties of whole numbers - recognize and use 0 in terms of its additive property (identity element for addition) - recognize and use 1 in terms of its multiplicative property (identity element for multiplication)	**Properties of whole numbers** • Revise: - The commutative; associative; distributive properties of whole numbers - 0 in terms of its additive property (identity element for addition) - 1 in terms of its multiplicative property (identity element for multiplication) • Recognize the division property of 0, whereby any number divided by 0 is undefined **Calculations using whole numbers** • Revise calculations using all four operations on whole numbers, estimating and using calculators where appropriate	**Properties of numbers** • Describe the real number system by recognising, defining and distinguishing properties of: - natural numbers - whole numbers - integers - rational numbers - irrational numbers **Calculations using whole numbers** • Revise calculations using all four operations on whole numbers, estimating and using calculators where appropriate
1.2 **Exponents**	**Mental calculations** • Determine squares to at least 12^2 and their square roots • Determine cubes to at least 6^3 and cube roots **Comparing and representing numbers in exponential form** • Compare and represent whole numbers in exponential form: $a^b = a \times a \times a \times ...$ for b number of factors **Calculations using numbers in exponential form** • Recognize and use the appropriate laws of operations with numbers involving exponents and square and cube roots • Perform calculations involving all four operations using numbers in exponential form, limited to exponents up to 5, and square and cube roots **Solving problems** • Solve problems in contexts involving numbers in exponential form.	**Mental calculations** • Revise: - Squares to at least 12^2 and their square roots - Cubes to at least 6^3 and their cube roots **Comparing and representing numbers in exponential form** • Revise compare and represent whole numbers in exponential form • Compare and represent integers in exponential form • Compare and represent numbers in scientific notation, limited to positive exponents **Calculations using numbers in exponential form** • Establish general laws of exponents, limited to: - natural number exponents - $a^m \times a^n = a^{m+n}$ - $a^m \div a^n = a^{m-n}$, if $m>n$ - $(a^m)^n = a^{m \times n}$ - $(a \times t)^n = a^n \times t^n$ - $a^0 = 1$ • Recognize and use the appropriate laws of operations using numbers involving exponents and square and cube roots • Perform calculations involving all four operations with numbers that involve the squares, cubes, square roots and cube roots of integers • Calculate the squares, cubes, square roots and cube roots of rational numbers **Solving problems** • Solve problems in contexts involving numbers in exponential form	**Comparing and representing numbers in exponential form** • Revise compare and represent integers in exponential form - compare and represent numbers in scientific notation • Extend scientific notation to include negative exponents **Calculations using numbers in exponential form** • Revise the following general laws of exponents: - $a^m \times a^n = a^{m+n}$ - $a^m \div a^n = a^{m-n}$, if $m>n$ - $(a^m)^n = a^{m \times n}$ - $(a \times t)^n = a^n \times t^n$ - $a^0 = 1$ • Extend the general laws of exponents to include: - integer exponents - $a^{-m} = \frac{1}{a^m}$ • Perform calculations involving all four operations using numbers in exponential form, using the laws of exponents **Solving problems** • Solve problems in contexts involving numbers in exponential form, including scientific notation
2.3 **Algebraic expressions**		• Determine the squares, cubes, square roots and cube roots of single algebraic terms or like algebraic terms • Determine the numerical value of algebraic expressions by substitution	- Determine the squares, cubes, square roots and cube roots of single algebraic terms or like algebraic terms - Determine the numerical value of algebraic expressions by substitution • Extend the above algebraic manipulations to include: - Multiply integers and monomials by polynomials - Divide polynomials by integers or monomials - The product of two binomials - The square of a binomial **Factorize algebraic expressions** • Factorize algebraic expressions that involve: - common factors - difference of two squares - trinomials of the form: ♦ $x^2 + bx + c$ ♦ $ax^2 + bx + c$, where a is a common factor. • Simplify algebraic expressions that involve the above factorisation processes. • Simplify algebraic fractions using factorisation.
		• Use substitution in equations to generate tables of ordered pairs • Extend solving equations to include: - using additive and multiplicative inverses - using laws of exponents	- use substitution in equations to generate tables of ordered pairs • Extend solving equations to include: - using factorisation - equations of the form: a product of factors = 0

Figure 5.2: Concepts taught at Senior Phase mathematics necessary for learning functions

Figure 5.2 shows the mathematics concepts taught at the Senior Phase level which are necessary for the learning of grade 11 functions. These concepts are highlighted in red in figure 5.1 and are: "Recognize the division property of 0, whereby any number divided by 0 is undefined" (Department of Basic Education, p.13), the general law of exponents and factorisation of algebraic equations. By the time the learners reach grade 11 mathematics, they should have mastered the aforementioned concepts necessary in the learning of functions.

5.2.2 Presentation and Analysis of the task and interviews

This study bases its analysis of the findings on sixty individual participants (as discussed in Chapter 4) who wrote the task consisting of three sections, namely Section A, B and C. Section A of the task comprised questions related to hyperbolic functions. The mathematical concept that is under scrutiny in this section is the perception of the relationship between *"divisor"*, *"dividend"* and "*quotient"*. These concepts are extremely important in understanding the asymptotic behaviour of the function (Mpofu & Pournara, 2016) (See Appendix F).

In section B, the researcher intends to determine whether or not learners can correctly display their basic knowledge of exponential equations in Question 2.1.1 and 2.1.2, which links with learners' process levels of determining the equation of p(x) in Question 2.1.2. The learners' ability to determine the equation of p(x) in this section results in their understanding of the function as an object in Question 2.1.3. Section C of the test has quadratic functions, where *"factorisation"* of a quadratic equation is a mathematical concept investigated in this section. Learners' inability to factorise a quadratic equation will hinder their ability to determine the *x*-intercept of a quadratic function (Nielsen, 2015).

To present and analyse the mathematical concepts investigated in this study, I used the APOS theory[8] discussed in Chapter 3 and the related literature of this research. According to (Dubinsky & Harel, 1992), thinking about a mathematical concept can be a

[8] The question items of the test consist of the APOS theory (Dubinsky& Harel, 1992), which will be analysed using the APOS Analytical Framework. The questions of the test are set comprising questions that assess action level, process level and object level learners possess.

"Process" or an "Object". Sfard (2008) asserts further that a process view of a mathematical concept is operational, thus the object view is structural. The learners' ability to view a mathematical concept both as a process and as an object is crucial for a full understanding of mathematics (Sfard, 1991). In this learning of mathematics, learners can show the skills of using mathematical concepts by whether they can identify the correct options and use the correct procedures in responding to task items. For instance, responding to Question 1.1 does not necessarily demand the demonstration of a routine-driven mathematics procedure (Sfard, 2008, p.182). However, it requires basic understanding of the concepts of *"dividend" "divisor"* and *"quotient"*, which learners learned from previous grades. The following diagram represents the APOS theory concepts comprised in each sub-section of the task:

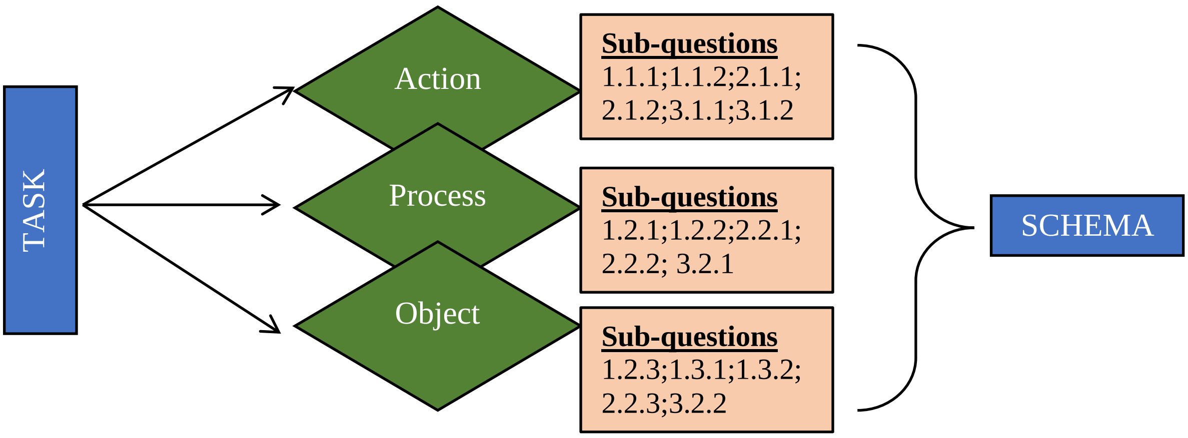


***Figure 5.3*:** Representation of APOS on questions items of the task

The diagram above shows the distribution of the framework of APOS concepts on sub-questions of the task. In this study, I constructed the task comprising the mathematical constructions for APOS theory on the sub-questions of the test. In addition, the use of mathematical concepts in learning algebraic functions in grade 11 is the phenomenon that was being investigated in this study. These mathematical concepts are *'factorisation'*, the relationship between the *'dividend, 'divisor' and quotient'* and *'solving exponential equations'*. These concepts play a huge role in the learning of algebraic functions in mathematics. The "dividend-divisor" relationship with the quotient is the basic concept needed to enable the learner to understand the asymptotic behaviour of the hyperbola. The concept of "factorisation" results in learners' knowledge of being able to find the *x*-

intercepts of a parabola. The knowledge of “solving exponential equations” contributes to understanding the domain and range of a function.

In order to present the findings, analysis and discussion for each of the three questions in a reader friendly format, the researcher has used the following sub-headings to describe the type of the questions used:

- The action level presentation and analysis
- The process level presentation and analysis
- The object level presentation and analysis
- Learners’ schema of functions

In each of these sub-headings, the researcher has provided the reader with the relevant task items excerpts, and question analysis. In order to analyse learners’ use of mathematical concepts in learning functions, the researcher has constructed an analytical framework displaying the performance of all learners who participated in the study (See Appendix G). The question analysis includes the actual number and percentage of learners who responded to the question. In addition, the researcher maintained the protection of learners’ identities by labelling each learner using L1 to L60 (See Appendix G). Furthermore, learners’ responses were classified as correct, incorrect, or no attempt.

5.2.2.1 The action level presentation and analysis

Table 5.1 on the next page shows learners’ responses to the items in the task at the action level. The data in table 5.1 shows the learners’ responses according to their schools using the schools’ pseudonyms (Lanfield Secondary, Lulove High and Maxiz High). There were twenty participating learners in each school. The task comprised of six sub-questions included for assessing learners’ *action level* of understanding algebraic functions. The researcher calculated the percentage of responses among learners who wrote the task in each school. A second level analysis of the items set at the action level is done in the subsequent page for items 1.1, 1.2 and 3.1. The data shown in table 5.1 followed by a bar graph in figure 5.3 represents the learners responses on the questions assessing their knowledge at action level. These quantitative data are analysed further in this study.

Table 5.1

Learner responses to sub-questions related to Action level understanding

SCHOOL	**No. of Learners**	**RESPONSES**	**ACTION (16)**					
		Sub-questions	**1.1.1**	**1.1.2**	**2.1.1**	**2.1.2**	**3.1.1**	**3.1.2**
		Responses						
Lanfield Secondary school	**20**	CORRECT	17	11	12	8	18	13
		INCORRECT	1	6	3	5	2	6
		NO ATTEMPT	2	3	5	7	0	1
Lulove High school	**20**	CORRECT	11	9	17	10	20	16
		INCORRECT	7	9	1	7	0	4
		NO ATTEMPT	2	2	2	3	0	0
Maxiz High school	**20**	CORRECT	14	11	14	11	18	13
		INCORRECT	5	8	3	5	1	7
		NO ATTEMPT	1	1	3	4	1	0
TOTAL		CORRECT	42	31	43	29	56	42
		INCORRECT	13	23	7	17	3	17
		NO ATTEMPT	5	6	10	14	1	1

The following bar graph below is the representation of the total number of learners' responses on action-level questions in all three respective schools:

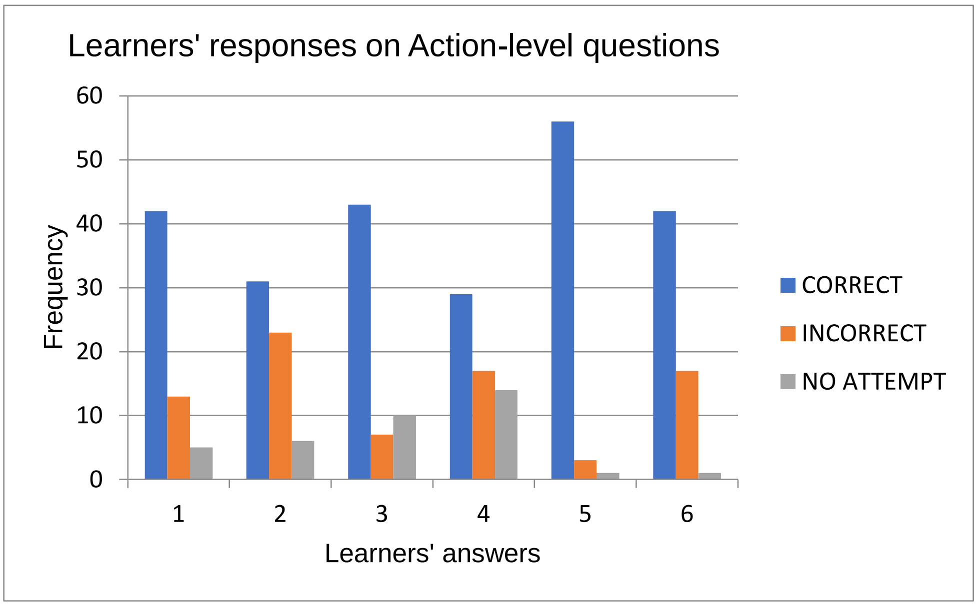


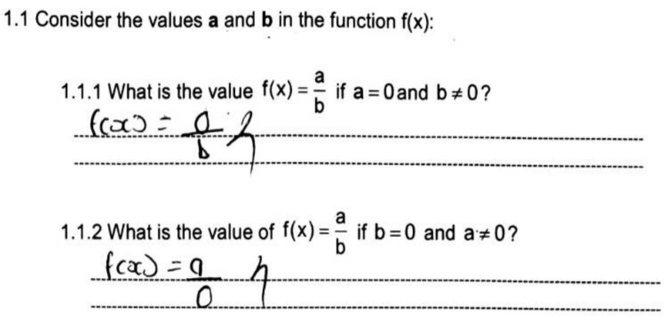
Figure 5.4: Bar graph illustrating learners' responses at action level

<u>**Question 1.1: The dividend, divisor and quotient relationship**</u>

The Questions 1.1.1 and 1.1.2 are based on investigating learners' knowledge of the relationship between the *'dividend'*, *'divisor'* and the *'quotient'*. In actual fact, the researcher intends to investigate learners' understanding of the effect of "zero dividend" on a "quotient" and "zero divisor" on a "quotient". This mathematical concept is extremely important in understanding the 'asymptotes' of a hyperbola.

Generally, learners from all three schools responded correctly to question 1.1.1. Seventy percent (42) of the learners answered the question item 1.1.1 correctly. In addition, 22% of the learners in all three schools did not answer question 1.1.1 correctly. Furthermore, there are 8% of the learners who did not attempt to respond to this question item. The findings also revealed that 52% of learners from three schools answered question item 1.1.2 correctly, whilst 12% of the learners from three schools gave incorrect answers for question item 1.1.2.

Looking at the percentage of learners who displayed incorrect answers to question 1.1.1 and 1.1.2, these results demonstrate that these learners are unable to conceptualise the relationship between 'dividend', 'divisor' and 'quotient'. In other words, they cannot recognise the effect of a 'zero-dividend' on a quotient and the 'zero-divisor' on a quotient. Through this, the responses of these learners to questions related to hyperbolic functions, especially the asymptotic behaviour and the function properties were also incorrect. In fact, they did not possess a complete understanding of the asymptotic behaviour of a hyperbolic function. In this study, the researcher has used the excerpts of learner 43 (L43) in figure 13 below to represent learners' who experienced adversities in answering question 1.1.1 and 1.1.2 at the action level:


1.1 Consider the values **a** and **b** in the function f(x):

1.1.1 What is the value $f(x)=\frac{a}{b}$ if $a=0$ and $b\neq 0$?

$f(x) = \frac{0}{b}$

1.1.2 What is the value of $f(x)=\frac{a}{b}$ if $b=0$ and $a\neq 0$?

$f(x) = \frac{a}{0}$

Figure 5.5: Learner (L43) response to Question 1.1.1 and Question 1.1.2

In the above figure, there are two inadequacies in the written responses of learner (L43): (1) failure to understand the question and (2) lack of knowledge of the relationship between the 'dividend', 'divisor' and 'quotient'. Even though learner (L43) portrayed the correct substitution into the functions, however, she did not give the correct answers after substitutions. During the interview, learner (L43) pointed out that she thought it was only the substitution that the questions required, however, she knew that the questions were function related. This shows the learners' lack of knowledge of the concept investigated in these questions. With regard to incorrect responses to Question 1.1.1 and 1.1.2 by learner (L43), the following emerged during the interview:

Researcher: How did you acquire knowledge/skills in responding to Question 1.1.1 and 1.1.2?

Learner (L43): I had no clue on where to start, I knew that the question was asking about functions but I did not know how to respond to it. I only thought of substitution into the given functions.

During conversation between the researcher and the learner (L43), the researcher has articulated to the learner that zero divided by any integer is zero, and any integer divided by zero is undefined. However, learner (L43) responded: "I couldn't think about that but I knew that the question was related to a function. The most confusion I came across with are letters *a* and *b* in a fraction". This indicates that learner (L43) does not know this rule since she could not apply it even if the fraction is in algebraic form. This made it a challenge for learner (L43) to encapsulate the action level understanding into understanding 'asymptotes' at the process level. The following excerpt is of the learner who wrote the incorrect answer to Question 1.2.1:

Table 5.2:

Example of transcript showing a learners' lack of knowledge of equation of asymptotes

Researcher	Looking at your response to Question 1.2.1, why did you work out the solution to this question the way you did? Explain.
Learner (L43)	In this question, I failed to recall how to determine the asymptotes which could have helped me to attain an accurate graph in Question 1.2.2. I had a difficulty in spotting that the asymptotes are both zero since p and q values were not given.

The requirement in question 1.2.1 was to grow a new schema to accommodate a completely new concept “asymptotes” without discarding the simple concepts of “dividend”, “divisor” and “quotient” relationships. These simple concepts are the subset of understanding asymptotic behaviour of a hyperbolic function. Since this is the case, learner (L43) in table 5.2 above displayed adversities with using schema of mathematical concepts to understand asymptotic behaviour of the hyperbola. This learner cannot assimilate these prior concepts with the symbolic and graphical representation of asymptotes in a hyperbola. This reveals the study by Thompson (2015) that learners cannot relate the definition of function concepts to its representations orally and in writing. The learners’ possession of mathematical concepts is paramount in their early stages of development (Watson, Jones, & Pratt, 2013) as it will aid them in using these concepts in higher levels

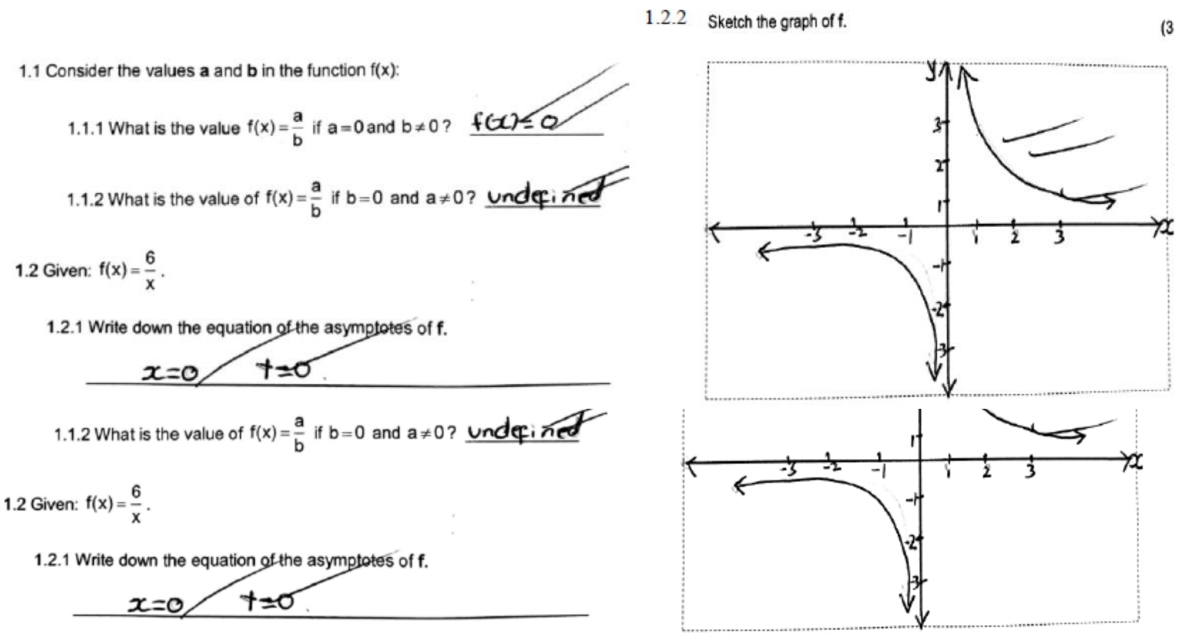


Figure 5.6: Learner 6 (L6) responses to questions 1.1.1, 1.1.2, 1.2.1 and 1.2.2

Based on concepts investigated in Question 1.1.1 and 1.1.2, learner (L6) portrayed the correct answers to these questions. His answers to these questions showed that he completely understands the relationship between the ‘dividend’, ‘divisor’ and ‘quotient’. However, during interviews he firstly encountered a challenge in answering a question since it comprises letters *a* and *b*.

Researcher: How did you acquire knowledge/skills in responding to Question 1.1.1 and 1.1.2?

Learner (L6): These questions (1.1.1 and 1.1.2) were a bit tricky since they have letters *a* and *b* as part of the fraction. I knew that if the numerator of the fraction is zero then the answer will be zero, but if the denominator is zero there will be no solution of the answer.

From Learner (L6) interview excerpt above, it appears that he is able to realise previously learnt schemas of knowledge. In this test, learner (L6) was able to encapsulate this knowledge into a process of understanding the symbolic and graphical representation of asymptotes in question 1.2.1 and 1.2.1. Clearly, it is not simply the concepts “dividend”, “divisor” and “quotient” relationship which are investigated in this study but the learners’ perceptions and ability for solving exponential equations are also investigated in question 2.1.1 and 2.1.2.

Question 2.1: Solving exponential equations

In the task, question 2.1.1 is an exponential equation which requires basic understanding of exponential laws; however, question 2.1.2 requires a deep understanding of exponential laws (see Appendix F). The learners’ knowledge of properties of an exponential function in order to determine its equation solely depends on the learners’ ability to solve basic exponential equations. From the results displayed in table 5.1 above, 72% of learners solved exponential equation in question 2.1.1 successfully whereas 28% of learners made errors in answering question 2.1.2. Looking at these results it is clear that the majority of learners in the study can solve basic exponential equations. However, the exponential equation that requires a deeper procedure cannot be solved successfully in question 2.1.2. Figure 5.7 below displays learner (L27) answers to question 2.1.1 and 2.1.2:

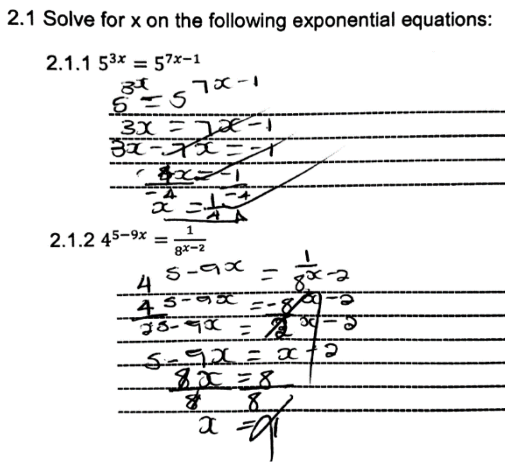


Figure 5.7: Illustration of learner’s answers in solving exponential equations

Learner (L27) above was able to solve an exponential equation in question 2.1.1 successfully. However, his answer to question 2.1.2 was incorrect due to the inability to apply the relevant laws of exponents. This algebraic knowledge is extremely important in exponential functions as it promotes the ability to determine the equation of the function (Makgakga & Sepeng, 2013). Below is an excerpt from the interview showing the learners understanding of question 2.1.1 and 2.1.2:

Table 5.3:

Learner's ability to solve exponential equations in Question 2.1.1 and 2.1.2

Researcher	**Do you think the exponential equations in Question 2.1.1 and 2.1.2 were correctly solved? Explain**
Learner (L27)	The exponential equation in Question 2.1.1 was correctly solved because I got the correct value of *x* by applying the correct law of exponents. The one in Question 2.1.2 was incorrect because of the sign I wrote on the final answer which was negative instead of being positive.

It appears above that the learner (L27) can solve an exponential equation in question 2.1.1, however, the learner's answer to question 2.1.2 was incorrect. Learners' difficulties with solving exponential equations have been documented by numerous scholars (Kothari, 2012; Pitta-Pantazi, Christou, & Zachariades, 2007) who found that learners struggle with applying the laws of exponents when solving exponential equations. In figure 5.7 above, it is clear that the learner (L27) does not clearly understand the procedure to use in solving the exponential equation provided in question 2.1.2. This is consistent with the findings of Khothari (2012) who posited that learners still struggle with mastering the exponential laws applicable for solving exponential equations. The reluctance by 28% of learners who displayed incorrect answers in question 2.1.1 and 12% of learners who wrote incorrect answers in question 2.1.2 led to challenges in answering questions assessing exponential functions. From the findings of this study, it became evident that learner (L6) encompassed a complete understanding of exponential equations. This learner can interiorise this understanding and accommodate it into determining the equation of *p(x)* in question 2.2.2 and the sketched graph in question 2.2.3. In other words, he encapsulates the basic schema of understanding exponential equations into a process of understanding exponential functions:

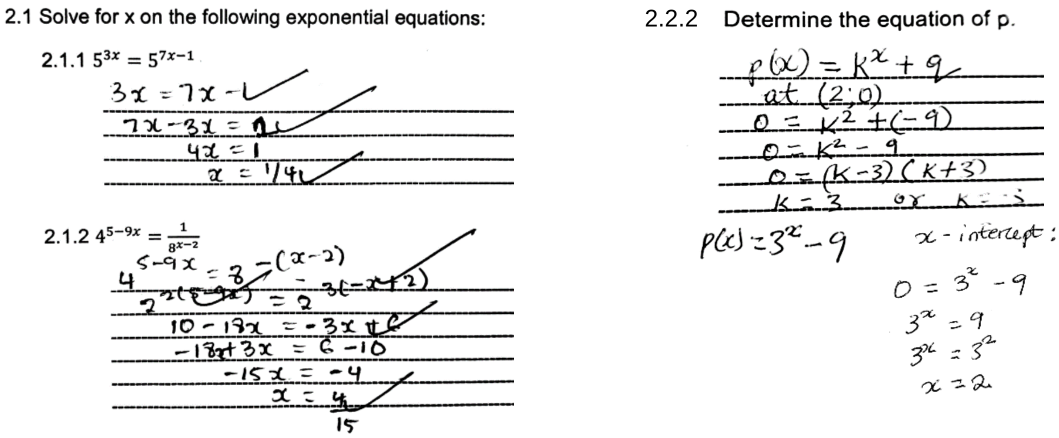


Figure 5.8: Illustration of learner's (L6) understanding of exponential equations at the process level

Learner (L6) holds a complete understanding of the use of exponential laws to solve the given equations. In addition, this learner interiorised this action level understanding into a process of determining the *x*-intercept of an exponential graph *p(x)* even thought it was unnecessary because the *x*-intercept was given. Furthermore, the process level understanding of this learner led him to sketch a correct graph of *p(x)* in the excerpt below:

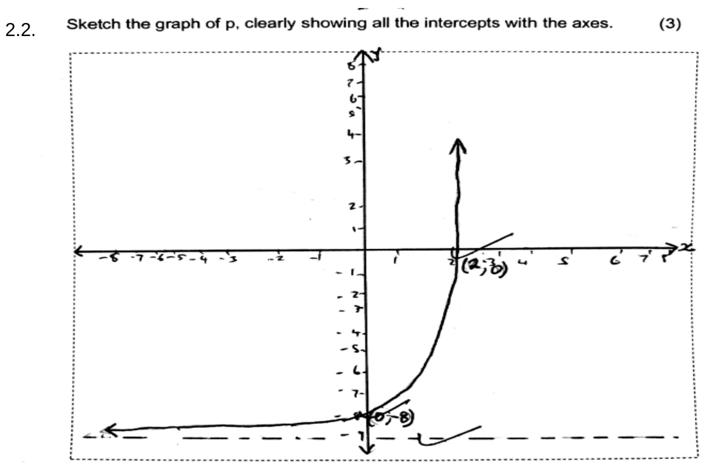


Figure 5.9: A sketch of an exponential function drawn by L6

This illustration of the sketched graph in figure 5.9 above depicts learner's (L6) understanding of the requirement of the entire question 2.2 since the intercepts and asymptotes are correctly shown. The learners' understanding of exponential equations will be analysed in more detail later in this study.

<u>Question 3.1: factorising quadratic equations</u>

Question 3.1 of the task was comprised of two sub-questions 3.1.1 and 3.1.2 which assessed learners' knowledge of factorising quadratic equations (See Appendix F). The findings of this study showed that 93% of learners clearly possess a good understanding of factorising quadratic equations. However, 7% of learners cannot factorise the quadratic equations. Amongst these learners, the incorrect answers to this question demonstrated three errors that they made. The first one is associated with the learners finding the incorrect quadratic factors. The second one is associated with the learners inserting incorrect signs in brackets and the other, with the incorrect application of a quadratic formula. Below is figure 5.10 showing a learner's difficulties in responding to question 3.1.1 and 3.1.2:

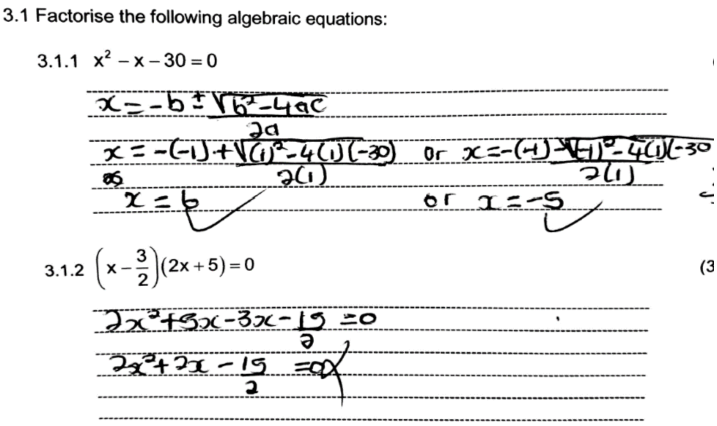


Figure 5.10: Learner's responses on solving quadratic equations

As shown in the figure above, the quadratic equation in Question 3.1.1 is given in a standard form $ax^2 + bx + c = 0$, however, the learner solved the equation using the quadratic formula. From figure 5.10 above, I can deduce that this learner can only solve quadratic equations in its standard form. This is because she did not realise that Question 3.1.2 is given in a different transformation. Looking at Question 3.1.2 in figure 5.10 above, extra hard work and calculations have been done by the learner instead of equating the factors to zero then finding the values of the unknown. This learner's notion is that the problem still needs to be worked on some more. Failing to realise that the equation is given in factors form, instead the learner multiplied out the brackets correctly which was unnecessary. This reveals Didis, Bas and Erbas (2011) findings that learners become aware of the zero-product property, but cannot apply it suitably when the equation is written in different transformations. In figure 5.10 above, the learner did not read Question 3.1.2 with understanding since she did not find the correct unknown values required by this question. Reading through the question to understand what it requires is one of the challenges that learners have to overcome. Learners in this study displayed incompetence when solving quadratic equations for quadratic functions (Didis, Bas, & Erbas, 2011). One of the more significant findings that emerge from this study is that most of the learners had in action conception of the effect that a 'zero divisor' has on the quotient. In addition, they could solve quadratic equations with procedures correctly to the mathematical object, but could not attain the next conception yet. Based on the APOS analytical framework (see Appendix G), the results of this study displayed that learners are still committing errors when solving exponential equations.

5.2.2.2 The process level presentation and analysis

As discussed in chapter 3 of this study, the process level of APOS theory is informed by learners' possession of the process level. In other words, the learner begins to "reflect upon the action or even reverse the steps" of a transformation on previously learned objects without performing those steps (Dubinsky, 1991). The focus of this study is on learners' understanding of mathematical concepts in learning algebraic functions. Thus, there are five question items indicated in table 5.4 below which seek to investigate the learners' encapsulation of the action level in the process. Table 5.4 shows the percentages of learners' answers of the five questions that assessed learners' knowledge of functions at the process level. Question 1.2.1 required a learner to be able to encapsulate the schema of the zero divisor into understanding the asymptotic equations of $f(x) = \frac{6}{x}$ (see Appendix F). In addition, Question 1.2.2 required a sketch graph be done showing all the intercepts and the asymptotes. The findings at large revealed 50% of learners who displayed the knowledge of "asymptotes" of a hyperbola at the process level in question 1.2.1 and 1.2.2.

In contrast to this, the findings also revealed 28% of learners with incorrect equations of asymptotes in question 1.2.1 and incorrect graphical representation of asymptotes in question 1.2.2. The following table represents the learners' use of action level to understand the learners' process level while learning functions:

Table 5.4:
Learner responses to sub-questions related to process level

SCHOOL	No. of Learners	RESPONSES	PROCESS (13)				
		Sub-questions	**1.2.1**	**1.2.2**	**2.2.1**	**2.2.2**	**3.2.1**
		Responses					
Lanfield Secondary school	**20**	CORRECT	8	9	6	5	13
		INCORRECT	7	9	6	4	6
		NO ATTEMPT	5	2	8	11	1
Lulove High school	**20**	CORRECT	11	11	6	8	11
		INCORRECT	6	7	9	5	8
		NO ATTEMPT	3	2	5	7	1
Maxiz High school	**20**	CORRECT	11	10	4	5	13
		INCORRECT	4	6	5	3	5
		NO ATTEMPT	5	4	11	12	2
TOTAL		CORRECT	30	30	16	18	37
		INCORRECT	17	22	20	12	19
		NO ATTEMPT	13	8	24	30	4

The following bar graph below is the representation of the total number of learners' responses on process-level questions in all three respective schools:

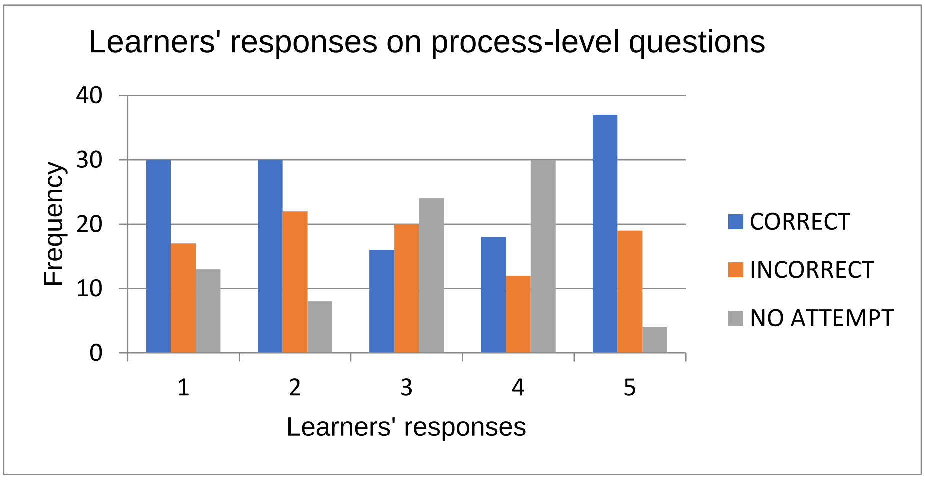


Figure 5.11: Bar graph illustrating learners' responses on process-level questions

These learners did not possess a good understanding of asymptotes in symbolic and graphical representations. Below is figure 5.12, illustrating a learner's incorrect equations of asymptotes in Question 1.2.1 and incorrect sketch in Question 1.2.2:

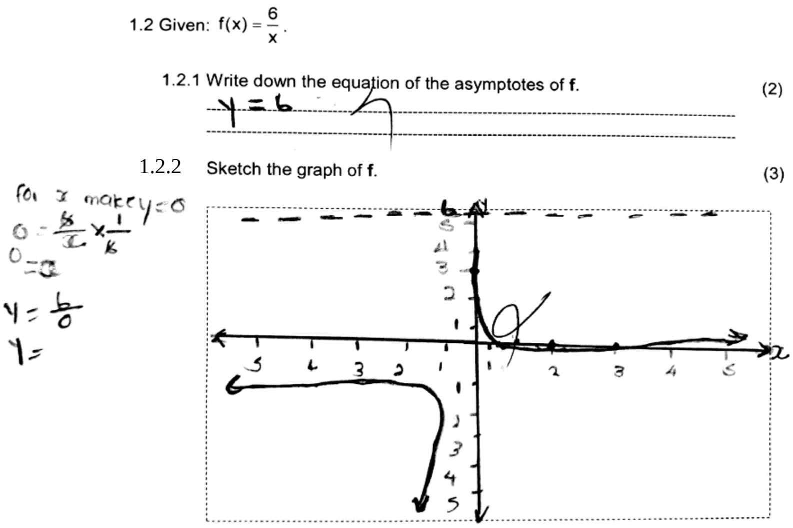


Figure 5.12: Learners' response to questions 1.2.1 and 1.2.2

Figure 5.12 clearly shows a learner's wrong interpretation of the hyperbolic function given in Question 1.2. This learner did not write the correct equations of the asymptotes of *f* in Question 1.2.1. In fact, he interpreted 6 as the horizontal asymptote of *f*, whereas it denotes the value of *a* in the standard form of a hyperbola. This wrong interpretation was also portrayed in the sketched graph with the horizontal asymptote at y=6 and the graph is cutting the x-and y-axis. Although the shape of the graph of *f* is correct, this does not mean learners understand the behaviour of the asymptotes on a hyperbola. This indicates a massive challenge experienced by learners in learning mathematical concepts in functions, particularly when it concerns a hyperbola. It is clear that learners cannot link prior concepts of ‘dividend’, ‘divisor’, ‘quotient’ relationships with asymptotes in a hyperbolic function. In fact, learner (L43) in this study was unable to assimilate the knowledge of the equations of the asymptotes of the hyperbola learnt in grade 10 with the one learnt in grade 11. Basically, she did not encapsulate this knowledge of equations of asymptotes in Questions 1.2.1 and 1.2.2 towards newly learnt mathematical information in Question 1.2.3. In contrast to the above-mentioned errors, Mpofu and Pournara (2016) posit that the majority of learners can sketch the hyperbolic graphs with correct asymptotes, whilst talking as if there are no asymptotes. Therefore, the learners’ lack of understanding of hyperbolic functions, especially ‘asymptotes’ is still a huge concern in mathematics education (Department of Basic Education, 2017).

According to Weller, Arnon and Dubinsky (2009), an individual’s understanding of a process as totality will rely on realising that manipulations can act as a totality which will lead them to construct such manipulations. Thus, looking at figure 5.12 above it is clear that they cannot encapsulate the concept into a process level. This learner also failed to represent the asymptotes of the function $f(x) = \frac{-3}{x+2} + 1$ at the object level in question 1.2.3.

<u>Question 2.2.1 and 2.2.2: the domain and the equation of the exponential function</u>

In these questions, learners were supposed to use the given characteristics of an exponential function and interpret the graph accordingly. However, there were adversities found with 33% of the learners who could not interpret the given characteristics of a function *p(x)* to determine its domain in question 2.2.1. Furthermore, 20% of learners could not determine the equation of *p(x)*. In general, some learners’ notion of interpreting the domain conveniently is to sketch the function (Swars, Stinson, & Lemons-Smith, 2009). However, the domain of *p(x)* in the task required learners to interpret the given instruction of the function and use the imagination of a function before sketching it. Thus, it was a challenge for learners to do so. This reveals prior findings (Swars, Stinson, & Lemons-Smith, 2009) that learners have difficulties when asked to find the domain for functions given in tabular or algebraic form.

The requirement of learners in question 2.2.2 of the task was to use the given characteristics of *p(x)* to determine its equation. The findings indicate that only 30% can correctly sketch the graph of *p(x)*. Then, 20% of learners could not answer this question and 50% of learners did not even try to answer the question. The question was presented in a transformation that hindered the learners' ability to respond correctly to it. This is evident from the interview with learner (L43) which is as follows:

Researcher: How did you respond to Question 2.2.2? Why?

Learner (L43): It was difficult for me to understand this question because of the manner in which it appeared. I respond incorrectly to this question because I was unfamiliar with revising question related to the manner in which Question 2.2.2 was presented.

With the above transcript, it becomes clear that creativity in equipping learners with the function concept is crucial. Learner (L43) made an error on question 2.2.2 due to her confusion with the mathematical concepts, rules, and procedures needed to attempt this question (Makonye & Nhlankla, 2014). This resulted in the learner (L43) not having the correct knowledge to determine the equation of the exponential function in question 2.2.2.

Question 3.2.1: the *x*-intercepts of a quadratic function

The ability to determine the *x*-intercepts of a quadratic function solely depends on the ability to factorise a quadratic equation. When learning quadratic functions, learners place their focus on three objects: quadratic equations, equations defining quadratic functions, and trinomial expressions. Question 3.2.1 of the test presents the equation $y + 4 = (x - 5)^2$ where learners are expected to determine the *x*-intercept of this function (see Appendix F). Looking at the results in table 5.2 above, 62% of learners possessed a complete understanding of the *x*-intercepts of a quadratic function and they determined it successfully. In determining the *x*-intercepts in question 3.2.1, some of these learners used the factor method and others used the quadratic formula. In contrast to this, 32% of learners lacked this knowledge. This percentage (32%) comprised learners who managed to determine the correct factors, however failed to insert the correct signs for factors. The inability of learners to factorise quadratic equations in this study hindered their ability to determine the *x*-intercepts of the quadratic function in question 3.2.1 of the task (see Appendix F). This was also shown by previous studies that some learners cannot define and solve the equations of quadratics (Nielsen, 2015). From the findings of this study, the researcher has identified persistent errors that learners made while learning algebraic functions.

Some learners portrayed the lack of basic knowledge of the properties of a parabola and its shape although this concept was introduced in grade 10. Instead of maintaining the shape of the function as concave up, they decided to draw the function as concave down. Furthermore, they did not indicate the coordinates of the turning point of this function. Figure 5.13 below displays an excerpt of a learner who could not determine the correct *x*-intercepts of the function which resulted to an incorrectly sketched graph in question 3.2.2:

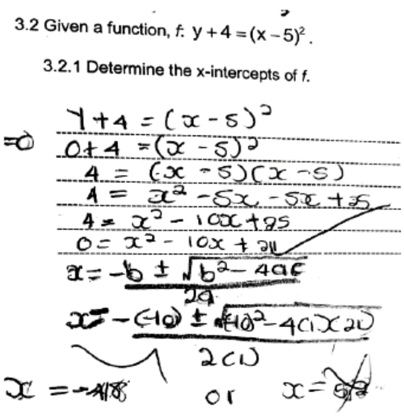


Figure 5.13: Illustration of the incorrect *x*-intercepts found

The factorisation of a quadratic equation requires that learners consider all the three terms of a quadratic trinomial at the same time. For instance, the factors of the first term and the last term in the equation added together or sometime subtracted must give the middle term. The learner in figure 5.13 simplified the equation correctly and arrived at an exact standard form, however, she failed to determine the correct *x*-intercepts even though she substituted correctly into the quadratic formula. This learner got the concept of factorising completely messed up, which resulted in an incorrect *x*-intercept of the quadratic function being calculated. This challenge that learners have is evident from the study by Guner (2017) which asserted that learners' errors in solving quadratic equations were due to their weaknesses in mastering the rules of quadratic equations and algebraic simplification.

Most learners who used the quadratic formula to factorise quadratic equations in questions 3.1.1 and 3.1.2 failed to determine the *x*-intercepts of the equation of a function *f(x)* in question 3.2.1. In general, this study revealed that the use of the quadratic formula did not encourage learners to construct flexible meanings in algebra. According to Kotsopoulos (2007), a flexible understanding would allow adjusting the solution method to the type of quadratic equation and would require using different types of quadratic equations, not only the standard form.

Interestingly, a positive correlation was found between the conception of the effect of the 'zero divisor' on the quotient and the 'asymptotes'. The learners' possession of mathematical conception in question 1.1.1 and 1.1.2 at the action level resulted in interiorization of such conceptions into the process level of understanding the asymptote. Similarly, learners' complete knowledge of factorising quadratic equations led to a complete insight of the procedure to determine the *x*-intercept of a quadratic function. Contrary, learners' adversities with solving exponential equations in question 2.1.2 impacted on their ability to interpret an exponential function in question 2.1.

5.2.2.3 Object level presentation and analysis

The results displayed in Table 5.5 below are of learners' responses for sub-questions related to the object level. The participants who successfully responded to these questions are holding the schema of algebraic functions that are learnt in grade 11 mathematics. In other words, they are cognisant of a process as a totality and realise the construction of manipulations (Weller, Arnon & Dubinsky, 2009). Table 5.5 presents the quantitative results of learners' responses of task sub-questions at the object level.

The four questions in table 5.5 are based on the use of mathematical concepts necessary for sketching the graphs of functions. An exception is question 1.3.1 which requires understanding the given properties of a hyperbolic function in order to determine its equation (see Appendix F). The question analysis in table 5.5, using the totals for three schools, indicates that 34 (57%) learners conceptualised the properties of a hyperbolic function required to sketch the function in question 1.2.3.

Table 5.5:
Learners' responses to sub-questions related to object level

SCHOOL	No. of Learners	RESPONSES Sub-questions Responses	OBJECT (21) 1.2.3	1.3.1	1.3.2	2.2.3	3.2.2
Lanfield Secondary school	20	CORRECT	9	4	4	4	9
		INCORRECT	8	3	1	2	2
		NO ATTEMPT	3	13	15	14	9
Lulove High school	20	CORRECT	10	5	5	6	9
		INCORRECT	8	3	1	3	6
		NO ATTEMPT	2	12	14	11	5
Maxiz High school	20	CORRECT	15	6	6	5	10
		INCORRECT	4	4	4	3	4
		NO ATTEMPT	1	9	10	12	6
TOTAL		CORRECT	34	16	15	15	28
		INCORRECT	20	10	6	8	12
		NO ATTEMPT	6	34	39	37	20

The following bar graph below is the representation of the total number of learners' responses on object-level questions:

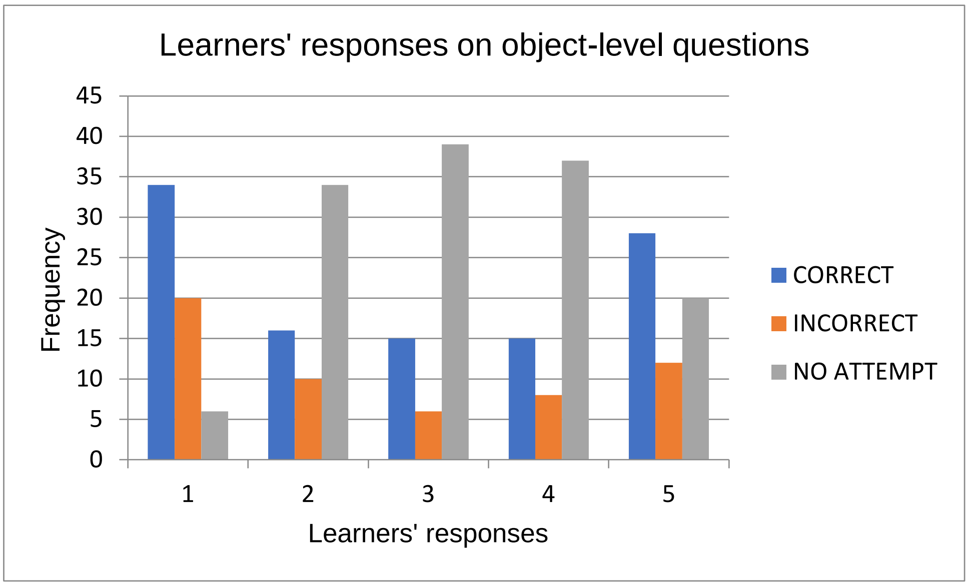


Figure 5.14: Bar graph showing learners' responses on object-level questions

In the APOS theoretical framework, this means that those learners' mental constructions were at best at the action level. However, 33% of learners had no idea regarding sketching a correct graph of the function provided in question 1.2.3. A possible reason 20 learners failed to answer this question is that they did not fully understand the concepts of hyperbolic functions. Figure 5.15 below provides evidence of learners' lack of knowledge of a hyperbolic function in question 1.2.3:

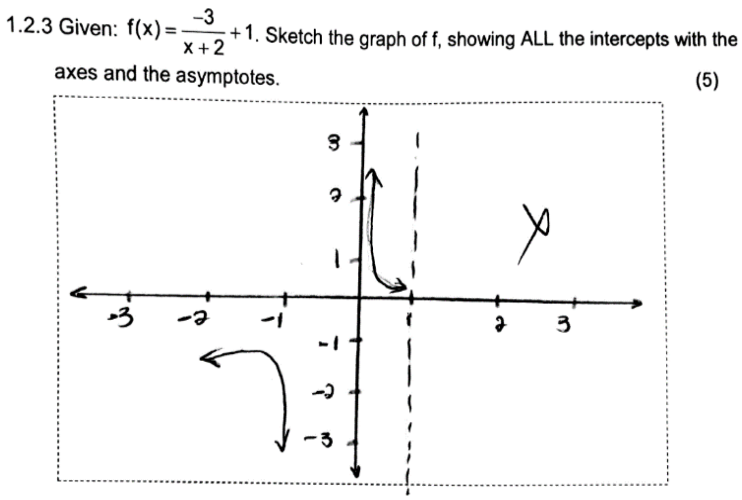


Figure 5.15: Illustration of incorrect hyperbolic function in 1.2.3

Sketching the graph of a hyperbolic function in question 1.2.3 presented three types of errors. The first one is associated with drawing the vertical and horizontal asymptotes of the function. This can be clarified by articulating that from $f(x)\frac{a}{x-p}+q$**,** the value of p denotes the vertical asymptotes whilst q denotes the horizontal asymptotes. The second error is associated with the shape of a function and a possible explanation for this could be that for $a > 0$, the function lies on the first and third quadrant. However, for $a < 0$, the function lies on the second and third quadrant. The last error is associated with the *x* and *y*-intercept. Looking at Learner (L53) in figure 5.15 above it can be surmised that the learner lacks the algebraic knowledge of determining the intercepts of the function. These intercepts are incorrectly displayed on the graph of *f(x)* in figure 5.15 above. This was due to challenges these learners faced in defining the *x* and *y* intercepts, the effects of the parameters on the graph a hyperbola and its interpretation (Moalosi, 2015). This is supported by previous research studies that posited that reasoning ability using notations and calculations of unknown and numbers is a significant part of algebra required for working with and understanding functions (Radford, 2014). The incorrect intercepts displayed that a learner failed to work with free variable *x* and *y* to determine the unknown values. This led to a learner sketching the graph of a function with the incorrect shape. This finding is supported by Dorko's (2016) study which showed that learners struggle with drawing correct graphs and working with free variables.

Question 1.3.1 is another transformation of a hyperbolic function where the requirement is to determine the equation using the given properties of a function (see Appendix F). The findings suggest that a total of 10 (exactly 17%) learners had no idea of what the question required. In addition, a total of 34 (57%) learners did not even attempt to answer the question. The following interview transcript in table 5.6 below displays the learners' lack of knowledge in answering question 1.3.1 and 1.3.2.

There is consistency in terms of learners' errors portrayed in their written responses as well as in the answers they gave in the interviews. The probing during interviews played a role in assisting the participants in reflecting deeply on their answers. This led to learner self-correction and misconception resolution. For instance, table 5.6 above displays learner's (L43) realisation of what he could have done in responding to the questions correctly. This learner often omits reasoning about the overall concepts entailed in question 1.3.1 and 1.3.2 due to his reluctance in reading the instructions clearly since he realised later what errors he had made. This result was similar to what was found by (Welder, 2012) and (Nachlieli & Tabach, 2012). In addition, this also supports the findings by Veloo et al. (2015) that major reasons for errors made in functions were a lack of understanding, procedures being forgotten and negligence in transcribing information from the question. In table 5.6, it is clear that the learner could not answer the question because of negligence and carelessness.

Table 5.6:

Example of excerpts showing a learners' lack of knowledge of mathematical concepts

Researcher	**Comment on your response to Question 1.3.1 and 1.3.2**
Learner (L43)	I failed to respond as of what the question needed for both 1.3.1 and 1.3.2. But after I did my remedial work, I understood that for sketching the graph I needed to clearly understand the characteristics which were described in the instruction. With understanding these instructions, I would have been able to know and sketch the type of a function required in Question 1.3.2.

In question 3.2.2 learners were asked to sketch the graph of a function *f*, showing all the intercepts with the axis. The findings of the task were that 47% of learners had complete understanding of the graph of a quadratic function. Contrary, 20% of learners did not sketch the correct graph of a quadratic function. Amongst these learners, 8% of them gave incomplete sketched graphs of the function. During the scrutiny of incomplete answers, the researcher was able to determine that nearly all of these learners factorised the quadratic equation representing the *x*-intercepts of the graph. However, they failed to graphically display these intercepts in a sketched quadratic function in question 3.2.2. This made the researcher realize that these learners had difficulties using parameters of the quadratic function and displaying these parameters correctly as an object.

Based on responses in figure 5.16 on the next page, the learner determined the correct *x*-intercept of *f* in Question 3.2.1 but sketched a completely incorrect graph of *f* in Question 3.2.2. The learner could successfully determine the *x*-intercept of *f* but using the quadratic formula, but could not present it in a function as an object. The learners in this study were most familiar with the *x*-intercepts of a quadratic function. However, they found it difficult to encapsulate this knowledge into an object. In figure 5.13 on the next page, the learner's concept images, reasoning, and difficulties indicated that she did not understand the meaning of a quadratic function to be sketched in question 3.2.2.

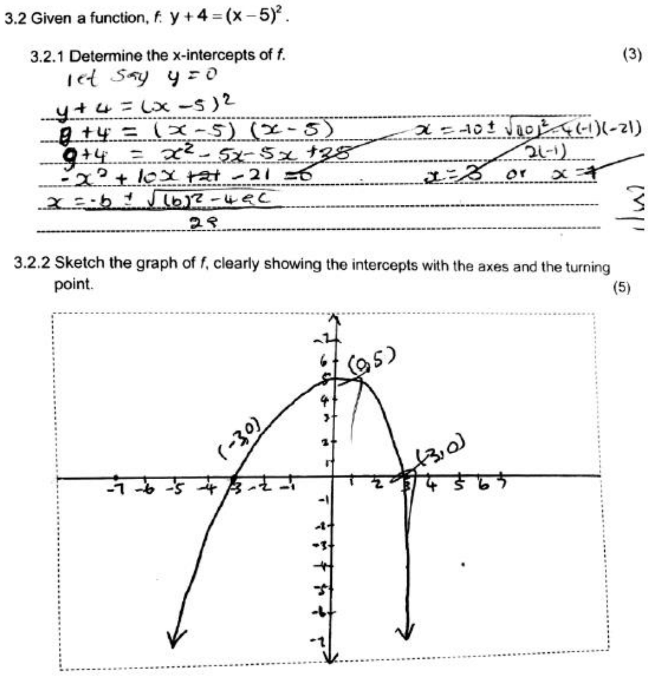


Figure 5.16: Learner's incorrect sketched graph of a quadratic function

The learner's low level of achievement in question 3.2.2 was caused by the lack of understanding of the graphing concepts involved in quadratic function (Hoon, Singh, & Halim, 2018). In order to sketch the graph of a quadratic function, it is important to understand and relate the concept of symmetry in determining the maximum or minimum points (Celik & Guzel, 2017). Therefore, the intercepts with the axis, minimum and maximum points, axis of symmetry, and the shape of the graph are parameters of the quadratic function.

Looking at the response in question 3.2.2 above, the learner's issue appeared to be struggles in correctly interpreting the parameters of a quadratic function (Ellis & Grinstead, 2008). The representation of quadratic functions on mathematical axes or determining the equation whose graph is given both develop the learners' reasoning power and enlarges upon their mathematical interpretation skills (Memnum, Aydin, Dinc, Coban, & Sevindik, 2015).Therefore, the learner's possession of knowledge about the parameters of quadratic function in question 3.2.2 could have aided her in representing the correct graph of this function. This study has found that generally learners' schema of the asymptotes of a hyperbolic function was encapsulated into the object level in question 1.2.3. In addition, their conceptions of quadratic equations contributed positively towards successfully sketching the graph of a quadratic function. However, the majority of learners cannot sketch exponential functions due to their lack of schema for solving exponential equations.

5.2.2.4 The schema of understanding algebraic functions

This section presents the learners' complete understanding of grade 11 algebraic functions. These learners are said to possess cognitive structures and use them to construct knowledge through action, process, object and prior schemas which are linked by general principles. Based on APOS theory, learners' construction of knowledge of the concepts of functions was investigated through identifying the relevant initial genetic decomposition. From the findings of this study, it is apparent that few learners possessed the schema of algebraic functions in grade 11 mathematics. These learners failed to encapsulate the mathematical concepts into understanding functions as objects. While this is the case, it is clear from this study that the majority of learners' schema is strong particularly on hyperbolic functions. Table 5.3 above indicates that 57% of learners possess schema of hyperbolic functions. These learners understood the concept 'asymptotes' at the action, process and object-level.

The understanding of hyperbolic functions cannot be successful if the concept 'asymptotes' is not clearly understood. The understanding of hyperbolic functions as object requires that a learner is able to make relationships with prior assimilated schemas. For instance, in question 1.1.2, the learner (L6) interiorised the effect of a 'zero-divisor' in a 'quotient' as a process of understanding the 'asymptotes'. These concepts were then encapsulated into an object generally represented by a function equation $f(x) = \frac{-3}{x+2} + 1$ in question 1.2.3. Not only the hyperbolic function comprises the 'asymptote', and in fact, an exponential function and tan graph in trigonometry also entails 'asymptotes'. Thus, the lack of understanding of the 'asymptotes' by these learners can also hinder their ability to successfully understand these function types.

5.3 Conclusion

It is evident from the analysis of the findings from this study that the majority of learners cannot comprehend functions at an object level. This was seen in all function types under investigation, in which each function type had a high percentage of learners who didn't understand it at an object level. This analysis aimed at identifying common errors made by learners with hyperbolic, exponential and quadratic functions. Some of the errors learners made were on the application of algebra across the function types. The aim of this study was to investigate learners' understanding of mathematical concepts necessary in learning grade 11 functions. The question remains: how can learners use these mathematical concepts when learning grade 11 algebraic functions? In answering this question, the researcher gave the learners the task on algebraic functions in terms of the APOS theory, and later interviewed them to get an in-depth understanding of their experiences with functions.

CHAPTER 6

CONCLUSION AND RECOMMENDATIONS

6.1 Introduction

The overriding purpose of this study was to explore learners' understanding of mathematical concepts in the learning of grade 11 algebraic functions. Additionally, the study sought to understand the reasons resulting in learners using mathematical concepts while learning functions in the manner in which they do. Taking into consideration the driving aspiration of this study, the research question for this study are:

i) What are the mathematical concepts that are necessary for learning grade 11 algebraic functions?
ii) How are learners understanding these mathematical concepts when learning grade 11 algebraic functions?
iii) Why are learners understanding these mathematical concepts necessary for learning grade 11 algebraic functions in the way they do?

In chapter 1 and 2, the researcher highlighted the dearth of research in learners' understanding of mathematical concepts in learning grade 11 algebraic functions. The general literature in this study, specifically in the context of South Africa is inconclusive on several important questions within learners' understanding of grade 11 algebraic functions. Even to date, the scarcity of learners' understanding of functions within the South African context has not been able to clearly explain the manner in which learners learn algebraic functions.

This motivated the conceptualisation of the current study to gain insight into learners' use of mathematical concepts to understand grade 11 algebraic functions. This study used the APOS theory (Dubinsky, 1991) as the theoretical framework. The theory allowed me to observe the learners' level of understanding mathematical concepts for learning algebraic functions. Throughout the analysis of the quantitative findings of this study, an APOS analytical framework was construct which aided the researcher to explore learners' level of understanding mathematical concepts in learning grade 11 algebraic functions.

This concluding chapter begins with presenting a summary of the findings relating it to the research questions stated in Chapter 1. The study went further in providing a discussion on its significance, linking this research with contentious debates on learners' learning of functions, predominantly in a South African context. The limitations accompanying the study and the study's implications are discussed. Finally, the chapter concludes by making recommendations for future research, and stressing directions for future research in order to understand learners' experiences with algebraic functions.

6.2 Summary of the main findings

Looking at the CAPS document, the researcher has found that the learner's knowledge of the types of algebraic functions was built in grade 10 based on the inputs and output relationships that were learnt in grades 7, 8 and 9. The algebraic functions were extended in grade 11 and were based on grade 10 functions. In addition, learners should know both the horizontal and vertical shifts of the graphs of functions by grade 11.

The learner's conception of "division by zero" is one of the major findings emerged from this study. The learners understood the relationship between these concepts at an action level; however, they cannot link this concept with understanding 'asymptotes' of a hyperbola at an object level. Secondly, there was a strong correlation found between the conception of the 'asymptotes' and the action level conception of the effect of the 'zero-divisor' on the quotient. Most learners were able to interiorise the conception at the action level into understanding the asymptotes. However, learners encountered adversities with solving exponential equations, which then hindered their ability to interpret an exponential function in question 2.2.3. Thirdly, most learners were able to encapsulate the concept 'asymptotes' into an object level. In addition, learners' conceptions of quadratic equations contributed positively in sketching the graph of a quadratic equation.

6.3 Significance and implications of the study

In the introductory remarks in chapter one, the difficulties learners often have during assessments with algebraic function is noted as reported in the National Diagnostic Report (Department of Basic Education, 2017). These difficulties were associated with graphic representations, interpretations using function features, and algebraic calculations for functions. This research study has tried to articulate the underlying reasons for these adversities from an APOS perspective.

The significances of this study have contributed to a dearth of research on learners' understanding of algebraic functions, more especially in the South African context. In addition, this study is significant because it explored learners' understanding of mathematical concepts in learning functions using a mixed method approach. Many preceding research studies on functions fell into either the quantitative or qualitative research approach. Therefore, this study offers suggestive evidence for employing a mixed method approach to investigate learners' understanding of functions. The contribution to the current knowledge made by this study provides insight for teachers and curriculum developers about learners understanding of functions. Through this insight, it will be the teachers' responsibilities to vary their pedagogy so that it will promote the development of learners' schema of functions.

6.4 Limitations of the study

Before a discussion can take place on the limitations of the study, it is paramount to remind the reader about the focus of the study. This study focused on the exploration of the learners' use of mathematical concepts in the learning of algebraic functions in grade 11 at three schools in KwaZulu-Natal. This means that the researcher did not consider other schools within the province nor did the researcher engage with other grade 11 mathematics learners from other schools. Thus, the small size and arbitrary nature of the sample precluded drawing generalizable inferences about the manner in which grade 11 learners use mathematical concepts in the learning of algebraic functions. A second limitation was that only one test of each type with its equation wording was used as a criterion measure. Therefore, the findings in this study cannot be applicable to other contexts with the similar characteristics. In other words, the findings of this study are applicable to three schools in KwaZulu-Natal.

6.5 Recommendations for future research

This research did not embrace any analysis of learners' understanding of functions in mathematics. Thus, it would be interesting to explore the learners' understanding of mathematical concepts through discussion among themselves. The next step in better understanding learners' use of concepts in learning functions could be to conduct a similar study in a larger context, with diverse samples, embracing learners from numerous school settings. Another additional area for future research would be to explore learners' use of algebraic procedures in answering function-related questions in secondary school mathematics.

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APPENDICES

APPENDIX A

KwaZulu-Natal Department of Education approval letter

education

Department:
Education
PROVINCE OF KWAZULU-NATAL

Enquiries: Phindile Duma   Tel: 033 392 1063   Ref.:2/4/8/1492

Mr N. Ndlovu

Private Bag X524
Impendle
3227

Dear Mr Ndlovu

PERMISSION TO CONDUCT RESEARCH IN THE KZN DoE INSTITUTIONS

Your application to conduct research entitled: **"EXPLORING LEARNER'S USE OF MATHEMATICAL CONCEPTS IN THE LEARNING OF FUNCTIONS IN GRADE 11 MATHEMATICS"**, in the KwaZulu-Natal Department of Education Institutions has been approved. The conditions of the approval are as follows:

1. The researcher will make all the arrangements concerning the research and interviews.
2. The researcher must ensure that Educator and learning programmes are not interrupted.
3. Interviews are not conducted during the time of writing examinations in schools.
4. Learners, Educators, Schools and Institutions are not identifiable in any way from the results of the research.
5. A copy of this letter is submitted to District Managers, Principals and Heads of Institutions where the Intended research and interviews are to be conducted.
6. The period of investigation is limited to the period from 28 March 2018 to 09 July 2020.
7. Your research and interviews will be limited to the schools you have proposed and approved by the Head of Department. Please note that Principals, Educators, Departmental Officials and Learners are under no obligation to participate or assist you in your investigation.
8. Should you wish to extend the period of your survey at the school(s), please contact Miss Phindile Duma at the contact numbers below.
9. Upon completion of the research, a brief summary of the findings, recommendations or a full report/dissertation/thesis must be submitted to the research office of the Department. Please address it to The Office of the HOD, Private Bag X9137, Pietermaritzburg, 3200.
10. Please note that your research and interviews will be limited to schools and institutions in KwaZulu-Natal Department of Education.

1. Nyonithwele Secondary 2. Luthando High School 3.Mantomela High School

Nzama

Dr. EV Nzama
Head of Department: Education
Date: 27 March 2018

KWAZULU-NATAL DEPARTMENT OF EDUCATION
Postal Address: Private Bag X9137 • Pietermaritzburg • 3200 • Republic of South Africa
Physical Address: 247 Burger Street • Anton Lembede Building • Pietermaritzburg • 3201

...Championing Quality Education - Creating and Securing a Brighter Future

APPENDIX B

Information sheet for school principals

School of Education
College of Humanities
University of KwaZulu-Natal
Edgewood Campus

Manfield Secondary School
Private Bag x 503
Plessislaer
3216

Dear Sir/Madam

REQUEST FOR PERMISSION TO CONDUCT RESEARCH

My name is Nkosinathi Ndlovu, I am a Master's degree (Mathematics Education) candidate studying at the University of KwaZulu-Natal, Edgewood campus, South Africa. I am interested in exploring grade 11 learners' use of mathematical concepts in the learning of algebraic functions. I would like to use your school as one of the research sites, and your grade 11 learners as participants. This letter intends to request your permission. Should permission be granted, the interviews with the learners will be scheduled for dates and times that are convenient for them. Care will be taken that not disruption is caused during such interviews.

Please note that:

*Your learners' confidentiality is guaranteed as their inputs will not be attributed to them in person, but reported only as a population member opinion.

* The interviews may last for about 45 minutes to 1 hour.

* Any information given by your learners cannot be used against them, and the collected data will be used for purposes of this research only.

* Data will be stored in secure storage and destroyed after 5 years.

* Your learners have a choice to participate, not to participate or stop participating in the research. They will not be penalized for taking such action.

* Your learners' involvement is purely for academic purposes only, and there are no financial benefits involved. In addition, you are assured that details of the school and the participants will be kept confidential, and your identity will never be disclosed to anyone.

I can be contacted at:
Email: 211507215@stu.ukzn.ac.za
Cell: 0818732987

My Supervisor is Dr. Goba who is located at the School of Education, Edgewood campus of the University of Kwa-Zulu Natal.
Contact details: Gobab@ukzn.ac.za Phone number: +27 73 848 3377

You may also contact the Research Office through:
Ms Ximba (HSSREC Research Office)
Tel: 031 260 3587
Email: ximbap@ukzn.ac.za

Thanking you in advance
Mr N.E Ndlovu

School of Education
College of Humanities
University of KwaZulu-Natal
Edgewood Campus

LuLove High School
Private Bag x 524
Impendle
3227

School of Education
College of Humanities
University of KwaZulu-
Natal
Edgewood Campus

Maxiz High School
Private Bag x 524
Impendle
3227

APPENDIX C

Information sheet for parents and consent forms

PARENT'S CONSENT FORM

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the study titled: **Exploring learners' understanding of mathematical concepts in the learning of Grade 11 algebraic functions.**

I,____

Permission to be audiotaped

Mark the block with a cross

I agree that my child can be audiotaped during interviews. YES: ☐ / NO: ☐

I know that the audiotapes will be used for this project only. YES: ☐ NO: ☐

Informed Consent

I understand that:

- my child's name and information will be kept confidential and safe and that his or her name and the name of my child's school will not be revealed.
- my child does not have to answer every question and can withdraw from the study at any time.
- I can ask for my child not to be audiotaped during interviews.
- all the data collected during this study will be destroyed within five years after completion of the project.

Signature:________________ Date:_______________

APPENDIX D

Information sheet for learners and consent forms

LEARNER'S CONSENT FORM

Please fill in and return the reply slip below indicating your willingness to participate in the study titled: **Exploring learners' understanding of mathematical concepts in the learning of Grade 11 algebraic functions.**

I,______________________________________

Permission to be audiotaped

Mark the block with a cross

I allow the researcher the permission to audiotape my conversation during interviews. **YES**: ☐ / **NO**: ☐

I know that the audiotapes will be used for the purpose of this project only. **YES**: ☐ / **NO**: ☐

Informed Consent

I understand that:

- My name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I can withdraw in participating in this study at any time.
- I can ask the researcher not to audiotape my conversation during interviews.
- All the data collected during this study will be destroyed within five years after completion of the project.

Signature:_________________________ Date:_________________

APPENDIX E

Ethics Clearance letter

UNIVERSITY OF KWAZULU-NATAL
INYUVESI YAKWAZULU-NATALI

05 February

Mr Nkosinathi Ndlovu (211507215)
School of Education
Edgewood Campus

Dear Mr Ndlovu,

Protocol reference number: HSS/2066/018M
Project title: Exploring learners' understanding of Mathematical Concepts in the learning of Grade 11 Algebraic Functions: The case of three schools

Full Approval – Expedited Application

In response to your application received on 09 November 2018, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and FULL APPROVAL for the protocol has been granted.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment/modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

Naidoo
....
Dr Shamila Naidoo (Deputy Chair)

/ms

Cc Supervisor: Ms Busisiwe Goba
Cc Academic Leader Research: Dr SB Khoza
Cc School Administrator: Ms Sheryl Jeenarain

Humanities & Social Sciences Research Ethics Committee
Dr Rosemary Sibanda (Chair)
Westville Campus, Govan Mbeki Building
Postal Address: Private Bag X54001, Durban 4000
Telephone: +27 (0) 31 260 3587/8350/4557 **Facsimile:** +27 (0) 31 260 4609 **Email:** ximbap@ukzn.ac.za / snymanm@ukzn.ac.za / mohunp@ukzn.ac.za
Website: www.ukzn.ac.za

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APPENDIX F

The Task

MATHEMATICS: FUNCTIONS

SECTION A

QUESTION 1

1.1 Consider the function $\mathrm{f(x)} = \frac{\mathrm{a}}{\mathrm{b}}$:

1.1.1 What is the value of $(\mathrm{x}) = \frac{\mathrm{a}}{\mathrm{b}}$ if $a = 0$ and $b \neq 0$?

1.1.2 What is the value of $(\mathrm{x}) = \frac{\mathrm{a}}{\mathrm{b}}$ if $a \neq 0$ and $b = 0$?

1.2 Given: $f(x) = \frac{6}{x}$.

1.2.1 Write down the equation of the asymptotes of *f*.

1.2.2 Sketch the graph of *f*.

1.2.3 Given: $f(x) = \frac{-3}{x+2} + 1$. Sketch the graph of f, showing ALL the intercepts with the axes and the asymptotes.

1.3 A function, h, is described with the following characteristics:

- The equation of the vertical asymptote is x = 0
- The range of *h* is $(-\infty; 3) \cup (3; \infty)$
- The x-intercept of h is (2;0)

1.3.1 Determine the equation of h.

1.3.2 Sketch the graph of h, clearly showing ALL the intercepts with the axes and the asymptotes.

SECTION B

QUESTION 2

2.1 Solve for x in the following exponential equations:

2.1.1 $5^{3x} = 5^{7x-1}$

2.1.2 $5^{4-9x} = \frac{1}{8^{x-2}}$

2.2 The function $p(x) = k^x + q$ is described by the following properties:

- $k > 0\,; k \neq 0$
- x-intercept at (2;0)
- The horizontal asymptote is y = -9

2.2.1 Write down the domain of p.

2.2.2 Determine the equation of p.

2.2.3 Sketch the graph of p, clearly showing all the intercepts with the axes.

SECTION C

QUESTION 3

3.1 Factorise the following algebraic equations:

3.1.1 $x^2 - x - 30 = 0$

3.1.2 $\left(x - \frac{3}{2}\right)(2x + 5) = 0$

3.2 Given a function, f. $y + 4 = (x - 5)^2$.

3.2.1 Determine the x-intercept of f.

3.2.2 Sketch the graph of f, clearly showing the intercepts with the axes and the turning point.

APPENDIX G

APOS analytical framework

School	Learner	ACTION (16)						PROCESS (13)					OBJECT (21)					SCHEMA
		1.1.1 (2)	1.1.2 (2)	2.1.1 (3)	2.1.2 (4)	3.1.1 (2)	3.1.2 (3)	1.2.1 (2)	1.2.2 (3)	2.2.1 (2)	2.2.2 (3)	3.2.1 (3)	1.2.3 (5)	1.3.1 (4)	1.3.2 (4)	2.2.3 (3)	3.2.2 (5)	

School	Learner	ACTION (16)						PROCESS (13)					OBJECT (21)					SCHEMA
		1.1.1 (2)	1.1.2 (2)	2.1.1 (3)	2.1.2 (4)	3.1.1 (2)	3.1.2 (3)	1.2.1 (2)	1.2.2 (3)	2.2.1 (2)	2.2.2 (3)	3.2.1 (3)	1.2.3 (5)	1.3.1 (4)	1.3.2 (4)	2.2.3 (3)	3.2.2 (5)	
LuLove High School	L32	2	2	1	0	2	3	2	3	2	3	3	4	0	0	0	0	NO
	L33	0	0	3	0	2	3	0	0	0	0	3	3	-	-	-	0	NO
	L34	2	2	3	4	2	2	2	2	2	3	3	5	4	4	3	4	YES
	L35	2	2	3	4	2	3	1	3	2	3	3	5	4	4	3	2	YES
	L36	2	0	3	4	2	3	2	3	2	3	3	5	3	4	3	4	YES
	L37	0	2	3	4	2	1	0	3	-	-	1	4	-	-	-	-	NO
	L38	0	0	3	0	2	3	0	0	0	0	1	1	-	-	-	0	NO
	L39	2	0	1	0	2	0	0	3	0	0	3	0	-	-	0	0	NO
	L40	0	0	3	0	1	0	0	0	0	-	0	0	-	-	-	-	NO
	L41	0	0	3	4	2	3	-	-	-	-	3	0	-	-	-	2	NO
	L42	0	0	3	4	2	3	-	-	-	-	3	2	0	-	-	2	NO
	L43	2	2	3	0	0	0	1	-	-	-	3	5	4	2	-	4	YES

APPENDIX H

Interview Schedule

1. How many types of algebraic expressions do you know?

2. In which grade did you start learning about these algebraic functions?

3. Did you find learning about these algebraic functions easy or difficult? Explain.

4. How did you acquire the knowledge and/or skills for responding to QUESTION 1.1?

5. Looking at your response to QUESTION 1.2.1, why did you work out the solutions to this question? Explain.

6. What mathematical concepts assisted you to sketch the graph in QUESTION 1.2.2 and QUESTION 1.2.3? Explain.

7. Comment on your response to QUESTION 1.3.1 and 1.3.2.

8. How did you determine the domain in QUESTION 2.2.1? Why?

9. What mathematical procedure did you use to determine the equation of p(x) in QUESTION 2.2.2? Why?

10. Is the graph in QUESTION 2.2.3 sketched correctly? Explain

9. Do you think the algebraic equations in QUESTION 3.1 were factorized correctly? Why?

10. In QUESTION 3.2.1 you were required to determine the x-intercepts of *f*. Briefly explain the skills you have employed in responding to that question.

11. Is the graph in QUESTION 3.2.2 sketched correctly? Explain

APPENDIX I

Interview transcripts

LEARNER (L27)

Researcher	Learner
How many types of algebraic functions do you know? Name these functions.	There are 4 types of functions I know. These are linear, hyperbola, parabola and exponential functions.
In which grade did you started learning about these function types?	In grade 10
Did you find learning about these functions easy or difficult? Explain.	At first, I found everything about functions very difficult and this was caused by the poor algebra background I had. I knew nothing about solving exponential problems and interpreting graphs was devastating for me.
How did you acquire skills/knowledge to respond to Question 1.1?	I was given the numerator that it is equal to zero in 1.1.1, so zero divided by any number gives zero. But, any number divided by zero in 1.1.2 is undefined.
Looking at your response to Question 1.2.1, why did you work out the solutions to this question the way you did? Explain	I had difficulty in spotting that the asymptotes are both equal to zero that is why I had the incorrect solutions.
What mathematical concepts assisted you to sketch the graph in Question 1.2.2 and Question 1.2.3? Explain	Everything was given in the statement, all I had to do was to find the x-intercepts by letting y equal to 0 and find the y-intercepts by letting x equal to 0. Also, I spotted that the asymptotes are $x= -2$ and $y=1$.
Comment on your respond to Question 1.3.1 and 1.3.2	I can't comment because I did not understand the questions, and that is why I had incorrect answers.
Do you think the exponential equations in Question 2.1.1 and 2.1.2 were correctly solved? Explain.	Yes, the problems were a bit easy, all I had to do was to use exponential laws in solving these equations.
Why did you refrained from determining the domain in Question 2.2.1	I did not determine the domain because I have forgotten the method to determine it.
What mathematical procedure did you use to determine the equation of p(x) in Question 2.2.2? Why?	I was confused in this question
What confused you in this question?	It was the manner in which the question appeared. I was not familiar with it.
Do you think the algebraic equations in Question 3.1 were factorised correctly? Why?	Yes, I used the quadratic formula and substituted correctly in the quadratic formula and got the solutions.
How can you be so sure that the factorised equations have correct solutions?	I have substituted in the original equations by the values I got to check whether they give zero, then indeed I they give zero, that is why my answers are correct.
Is the graph in Question 2.2.3 sketched correctly? Explain.	No because I failed to determine the correct equation of p(x).

Learner (L43)

Researcher	**Learner**
How many types of functions do you know? Give name of these function types	There are 4 types, namely: Linear, Parabolic, hyperbolic and exponential function
In which grade did you start learning about these types of functions?	I began to learn about them in grade 10

	the test I was able to realize what was expected to be done.
Do you think the exponential equations in Question 2.1.1 and 2.1.2 were correctly solved? Explain	Yes, I was able to use the required laws of exponents.
How did you determine the domain in Question 2.2.1? Why?	I failed to respond to that question because I didn’t know where to start
Is the graph in Question 2.2.3 sketched correctly? Explain	No, I was unable to sketch the graph because I failed to determine its equation in Question 2.2.2
Do you think the algebraic equations in Question 3.1 were factorized correctly?	Yes, because I knew what to do as I was taught in grade 10

CERTIFICATE OF ENGLISH EDITING

This certificate confirms that the thesis entitled below has been edited by an English expert with a Ph.D. who has professional review and editing experience. The following errors were checked for and corrected: grammar, punctuation, spelling, word choice, sentence structure, and clarity.

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(A.A.S., A.A.S., B.Sc., B.Sc. (Hons), M.Sc., Ph.D.)

APPENDIX K

Turnitin Certificate

10/19/2019 Turnitin - Class Portfolio

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