Iterative Approaches to Convex Feasibility Problems

by

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Declaration

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I, Paranjothi Pillay, hereby declare that the thesis entitled ITERATIVE APPROACHES TO CONVEX FEASIBILITY PROBLEMS is a result of my own investigation and research and that it has not been submitted in part or in full for any other degree or to any other university.

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Paranjothi Pillay

28/09/2001
....
Date

For Suresh and Jyestha.

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Abstract

Solutions to convex feasibility problems are generally found by iteratively constructing sequences that converge strongly or weakly to it. In this study, four types of iteration schemes are considered in an attempt to find a point in the intersection of some closed and convex sets.

The iteration scheme $x_{n+1} = (1 - \lambda_{n+1})y + \lambda_{n+1}T_{n+1}x_n$ is first considered for infinitely many nonexpansive maps $T_1, T_2, T_3, \ldots$ in a Hilbert space. A result of Shimizu and Takahashi [33] is generalized, and it is shown that the sequence of iterates converge to Py, where P is some projection. This is further generalized to a uniformly smooth Banach space having a weakly continuous duality map. Here the iterates converge to Qy, where Q is a sunny nonexpansive retraction. For this same iteration scheme, with finitely many maps $T_1, T_2, \ldots, T_N$, a complementary result to a result of Bauschke [2] is proved by introducing a new condition on the sequence of parameters (λ_n). The iterates converge to Py, where P is the projection onto the intersection of the fixed point sets of the T_is. Both this result and Bauschke's result [2] are then generalized to a uniformly smooth Banach space, and to a reflexive Banach space having a weakly continuous duality map and having Reich's property. Now the iterates converge to Qy, where Q is the unique sunny nonexpansive retraction onto the intersection of the fixed point sets of the T_is.

For a random map $r : \mathbb{N} \to \{1, 2, \ldots, N\}$, the iteration scheme $x_{n+1} = T_{r(n+1)}x_n$ is considered. In a finite dimensional Hilbert space with $T_{r(n)} = P_{r(n)}$, the iterates converge to a point in the intersection of the fixed point sets of the P_is. In an arbitrary Banach space, under certain conditions on the mappings, the iterates converge to a point in the intersection of the fixed point sets of the T_is.

For the scheme $x_{n+1} = (1-\lambda_{n+1})x_n + \lambda_{n+1}T_{r(n+1)}x_n$, in a finite dimensional Hilbert space the iterates converge to a point in the intersection of the fixed point sets of the

T_is, and in an infinite dimensional Hilbert space with the added assumption that the random map r is quasi-cyclic, then the iterates converge weakly to a point in the intersection of the fixed point sets of the T_is.

Lastly, the minimization of a convex function θ is considered over some closed and convex subset of a Hilbert space. For both the case where θ is a quadratic function and for the general case, first the unique fixed points of some maps T^{λ} are shown to converge to the unique minimizer of θ and then an algorithm is proposed that converges to this unique minimizer.

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Chapter 1

Introduction

Numerous problems, coming from disciplines as diverse as approximation theory, integral equations, signal and image processing, computerized tomography and control theory ([10], [11],[12], [21], [32], [35], [40], [41]) can be realised as a convex feasibility problem (CFP). A convex feasibility problem can be mathematically formulated as follows:

> Assume $\{C_i\}_{i=1}^N$ is a finite family of nonempty, closed and convex subsets of a Hilbert space H with $C := \bigcap_{i=1}^{N} C_i \neq \emptyset$. Find a point in C.

A CFP is usually formulated in a Hilbert space H and the fixed point sets of certain projections define the nonempty, closed and convex subsets C_i.

A typical application of the CFP is in image recovery. The problem of image recovery can be stated as follows:

> The original (unknown) image f is known *a priori* to belong to the intersection of N well-defined closed convex sets $C_1, \ldots, C_N$, in a Hilbert space H. Given only the metric projections P_i of H onto C_i $(i = 1, 2, \ldots, N)$, recover f by an iterative scheme.

Solutions to CFPs are generally found by iteratively constructing sequences that converge strongly or weakly to it. Different iteration schemes have been proposed by various authors. In this study the following schemes are considered: the scheme introduced by Halpern [19], the Mann [24] iteration scheme, the random iteration scheme and the scheme proposed by Deutsch and Yamada [13].

For N nonexpansive maps $T_1, T_2, \ldots, T_N$, the iteration scheme

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n \tag{1.1}$$

was introduced first by Halpern [19] in 1967 in which he considered the case where $y = 0$ and $N = 1$ (i.e. he considered only one map T). He showed that the conditions $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$ were necessary conditions for the convergence of the iterates to a fixed point of T. In 1977 Lions [23] considered the above scheme with the additional assumption $\lim_{n\to\infty} \dfrac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$ on the parameters, and obtained convergence of the iterates. However, Lions' conditions on the parameters did not include the natural choice of parameters, $\lambda_n = \dfrac{1}{n+1}$. Reich [30] in 1983 posed the following problem:

> In a Banach space, what conditions on the sequence of parameters (λ_n) will ensure convergence of the iterates?

Wittmann [38] in 1992, in the setting of a Hilbert space, obtained convergence of the iterates, where the parameters satisfied $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$, in addition to the two necessary conditions. Reich [31] in 1994 obtained strong convergence of the iterates where the underlying space was uniformly smooth and had a weakly continuous duality map. His result was proved for the case of a single map (i.e. $N = 1$), and the parameters satisfied the two necessary conditions for convergence in addition to the fact that they were increasing. Bauschke [2] in 1996, generalized Wittmann's result to finitely many maps, where $T_n := T_{n \bmod N}$. The additional condition on the parameters that he used was $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$. He also provided an algorithmic proof which has been used successfully, with modifications, by many authors ([13], [33], [39]). In 1997, Shioji and Takahashi [34] extended Wittmann's result to a Banach space. Chapters 3 and 4 of this thesis provide some answers to the problem posed by Reich [30], by introducing a new condition on the parameters, $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$, in the framework of both Hilbert and Banach spaces.

Mann [24], in 1953, introduced what is now known as the Mann iteration scheme:

$$x_{n+1} = (1 - \lambda_{n+1})x_n + \lambda_{n+1}Tx_n.$$

Mann showed that for a continuous selfmap T of a closed interval $[a, b]$ having one fixed point, convergence to this fixed point is obtained for the case of $\lambda_n = \frac{1}{n}$. Reich [28], in 1979, showed that in a uniformly convex Banach space having a Frechét differentiable norm and with T nonexpansive and having a fixed point, then weak convergence of the iterates is obtained under certain conditions on the parameters (λ_n). Tseng [36], in 1992, considered the scheme

$$x_{n+1} = (1 - \lambda_{n+1})x_n + \lambda_{n+1}T_{r(n+1)}x_n,$$

where $r : \mathbb{N} \to \{1, 2, \ldots N\}$ is a random map, and the maps $T_1, \ldots, T_N$ are used. He proved that in a finite dimensional Hilbert space, convergence of this scheme is obtained, resolving a conjecture posed by Bruck [26] in 1983, at least in the finite dimensional case. He also shows, in an infinite dimensional Hilbert space, under the quasi-cyclic order, the iterates converge weakly.

The random iteration scheme

$$x_{n+1} = T_{r(n+1)}x_n$$

has been considered by various authors ([3], [5], [7], [15], [16], [20]). Dye and Re ich [16] showed that in a Hilbert space with r quasi-periodic, the iterates converge weakly. In [15], they were able to extend this result to reflexive Banach spaces with a weakly continuous duality map. However, the result could only be proved if the pool of maps to be drawn from consisted of only two maps. In [14], Dye *et al* proved their result for Banach spaces that have Opial's property. However, in all of these results, only weak convergence of the iterates is obtained.

The scheme considered by Deutsch and Yamada [13] is defined by

$$x_{n+1} = Tx_n - \lambda_{n+1}\mu\theta'(Tx_n).$$

Deutsch and Yamada considered this scheme in the context of the minimization problem:

$$\text{Find } u^* \in C \text{ so that } \theta(u^*) = \min_{u\in C} \theta(u).$$

Under certain conditions they show that the above scheme converges to the unique minimizer of θ. This scheme generalizes the scheme introduced by Yamada *et al* [39] with $\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$. It also generalizes the scheme introduced by Halpern [19], Lions [23], Wittmann [38] and Bauschke [2].

For the remainder of this chapter a brief overview of the results that are obtained in this study is presented.

In chapter 2, definitions, notations and fundamental results are provided, and proofs of some of these results are included. The topics covered are results for real numbers, uniformly convex spaces, smooth Banach spaces, projections, duality maps, nonexpansive mappings and sunny nonexpansive retractions.

In chapter 3, all the results are set in a real Hilbert space. The iterative scheme

$$x_{n+1} = \lambda_{n+1}y + (1 - \lambda_{n+1})T_{n+1}x_n$$

is considered and convergence under different conditions is investigated. In Theorem 3.4 for infinitely many nonexpansive maps $T_1, T_2, T_3, \ldots$, under the assumptions that $\lim_{n\to\infty} \sup_{x \in \tilde{C}} \|T_n x - V_k(T_n x)\| = 0$, ($V_k$ nonexpansive, $k = 1, 2, \ldots, N$), $\bigcap_{i=1}^{\infty} Fix(T_i) \supseteq \bigcap_{k=1}^{N} Fix(V_k) \neq \emptyset$, $\lim_{n\to\infty} \lambda_n = 0$ and $\sum_{n=1}^{\infty} \lambda_n$, the iterates are shown to converge to Py, where P is a projection onto $\bigcap_{k=1}^{N} Fix(V_k)$. Theorem 3.4 generalizes a result of Shimizu and Takahashi ([33]; Theorem 1), which is included in Corollary 3.5. Theorem 3.7 is a complementary result to Theorem 3.1 of Bauschke [2] in which the condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$ is replaced by the new condition $\lim_{n\to\infty} \dfrac{\lambda_n}{\lambda_{n+N}} = 1$. As is shown in Example 3.6, neither condition is stronger than the other.

In chapter 4, the iteration scheme

$$x_{n+1} = \lambda_{n+1}y + (1 - \lambda_{n+1})T_{n+1}x_n$$

is again considered, but this time the underlying space is a Banach space. Theorem 4.1 extends Theorem 3.4 with the sunny nonexpansive retraction replacing the projection in a Hilbert space. In addition, it extends Theorem 1 of Shimizu and Takahashi [33] to Banach spaces. In Theorem 4.2, the condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$ provides an extension of Bauschke's Theorem 3.1 [2] to uniformly smooth Banach spaces, while the condition $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$ extends Theorem 3.7 to uniformly smooth Banach spaces. Theorem 4.4 is new and generalizes Theorem 3.1 of Bauschke [2] to a reflexive Banach space having a weakly continuous duality map and having Reich's property.

In chapter 5, the random iteration scheme

$$x_{n+1} = T_{r(n+1)}x_n$$

and the relaxed iteration scheme

$$x_{n+1} = (1 - \lambda_{n+1})x_n + \lambda_{n+1}T_{r(n+1)}x_n$$

are considered for finitely many maps $T_1, T_2, \ldots, T_N$. In Theorem 5.1, the random iteration scheme in a finite dimensional real Hilbert space is considered, where the mappings are projections, and convergence of the iterates is obtained. Tseng ([36], Theorem 1) showed the convergence of the relaxed iteration scheme in a finite dimensional Hilbert space, with the assumption that each map is chosen infinitely often. Theorem 5.2 is exactly Tseng's result, but an alternate proof is provided. Theorem 5.4 considers the relaxed iteration scheme in an infinite dimensional Hilbert space with the added assumption that r is quasi-cyclic, and obtains weak convergence of the iterates. This is the same as Theorem 2 of Tseng [36], but a variation of the proof is provided. In Theorem 5.5, the random iteration scheme is considered in a Banach space with certain conditions imposed on the mappings, and strong convergence of the iterates is obtained. Theorem 5.5 is a generalization of Theorem 5.1, but the proof is included because it is much simpler than the rather technical proof of Theorem 5.5.

In chapter 6, applications to minimization problems are considered. All the results hold in the framework of a Hilbert space. Firstly, we consider the scheme proposed by Yamada *et al* [39]. Theorem 6.4 is a more general result than Theorem 1 of [39], where it is shown that the unique fixed points of some mappings T^λ converge to the unique minimizer of a quadratic function θ. Theorem 6.5 is a generalization of Theorem 2 of [39], where we replace Lions' condition $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_{n+1}^2} = 0$ by our new, more general condition $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$, and finitely many mappings are considered. We then consider the iteration scheme proposed by Deutsch and Yamada [13]. In Theorem 6.13 it is shown the unique fixed points of T^λ converge to the unique minimizer of the problem. This result is new since Deutsch and Yamada [13] did not consider the behaviour of these fixed points. Theorem 6.14 is the main result of this chapter, where it is shown that the iterates as defined by Deutsch and Yamada [13] converge to the unique minimizer of the minimization problem. This result is complementary to a result of Deutsch and Yamada ([13]; Theorem 3.7), in which the condition $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$ is replaced by the condition $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

Chapter 2

Preliminaries

This chapter forms the basis for the rest of the thesis. Definitions, notations and some basic results are provided. The proofs for some results are included, and references are provided if they are not proved. Many of the results are very well-known but there are some that are not widely known.

For the remainder of the thesis we work either in a real Hilbert space or a real Banach space. A Banach space will be denoted by X with norm $\|\cdot\|$. A Hilbert space will be denoted by H with the inner product $\langle\cdot,\cdot\rangle$.

$\mathbb{R}$ and $\mathbb{N}$ will denote the sets of real numbers and natural numbers, respectively. $\tilde{\mathbb{R}}$ will denote the set of extended real numbers. If X and Y are Banach spaces then $L(X,Y)$ (resp. $B(X,Y)$) will denote the space of all linear (resp. bounded linear) operators from X to Y. $B(H)$ denotes the set of all bounded linear operators from H to H. The dual space of X, denoted by X^*, is the space of all bounded linear operators (or functionals) from X to $\mathbb{R}$.

In a real Banach space X, for $x \in X$ and $x^* \in X$, we sometimes write $\langle x, x^* \rangle$ for $x^*(x)$.

We write $x_n \longrightarrow x$ if the sequence (x_n) **converges** to x. We sometimes refer to this as strong convergence. A sequence (x_n) is said to **converge weakly** to a point $x \in X$ if

$$x^*(x_n) \longrightarrow x^*(x) \text{ for all } x^* \in X^*.$$

We write $x_n \rightharpoonup x$ if x_n converges weakly to x. Clearly, strong convergence implies weak convergence.
We define weak*-convergence in X^* as follows: $x_n^* \rightharpoonup^* x^*$ if $x_n^*(x) \longrightarrow x^*(x)$ for all $x \in X$.

Lemma 2.0.1 ([22]). *Let X be a Banach space and let $x_n \rightharpoonup x$ in X. Then (x_n) is bounded.*

Lemma 2.0.2. *Let X be a Banach space and let (x_λ) be a net in X. If every subsequence of (x_λ) has a subsequence that converges to x, then $x_\lambda \longrightarrow x$.*

Proof. If (x_λ) does not converge to x, then there exists $\epsilon > 0$ such that for all $\Lambda \in (0,1)$, there exists λ, $0 < \lambda \leq \Lambda$ such that $x_\lambda \notin N(x,\epsilon)$ ($N(x,\epsilon)$ is the ϵ-neighbourhood of x); i.e. $\|x_\lambda - x\| \geq \epsilon$.
Thus there exists λ_1, $0 < \lambda_1 \leq \frac{1}{2}$ such that $\|x_{\lambda_1} - x\| \geq \epsilon$.
Again there exists λ_2, $0 < \lambda_2 \leq \frac{1}{2}\lambda_1$ such that $\|x_{\lambda_2} - x\| \geq \epsilon$.
Continuing in this way, we can find a subsequence (x_{λ_n}) of (x_λ) such that $\|x_{\lambda_n} - x\| \geq \epsilon$ for all $n \geq 1$. This implies that (x_{λ_n}) has no subsequence that converges to x, contradicting the hypothesis. $\square$

Definition 2.0.3. Let C be a nonempty, closed and convex subset of a Banach space X. A sequence (x_n) is said to be **Fejér monotone with respect to C** if

$$\|x_{n+1} - c\| \leq \|x_n - c\| \text{ for all } c \in C.$$

The following result gives properties of a sequence that is Fejér monotone in a Hilbert space.

Theorem 2.0.4 ([4]; Theorem 2.16). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let (x_n) be a sequence in H that is Fejér monotone with respect to C. Then*

(i) *(x_n) is bounded.*

(ii) *(x_n) has at most one weak cluster point in C.*
Consequently, (x_n) converges weakly to some point in C iff all weak cluster points of (x_n) lie in C.

(iii) *The following are equivalent:*

1. *(x_n) converges in norm to some point in C.*
2. *(x_n) has norm cluster points, all lying in C.*
3. *(x_n) has norm cluster points, one lying in C.*
4. *$d(x_n, C) \longrightarrow 0$.*

We sometimes use the notation $\underline{\lim}\, a_n$ for $\liminf a_n$, and $\overline{\lim}\, a_n$ for $\limsup a_n$.

The following facts are well-known and can be found in most standard functional analysis texts. The results hold in a real Banach space.

Fact 2.0.5 (**[17]**). *A set A is relatively compact if and only if every sequence in A has a convergent subsequence.*

Fact 2.0.6 (**[17]**). *A set A is relatively weakly compact if and only if every sequence in A has a weakly convergent subsequence.*

The following fact is extremely important and is frequently used, often without reference.

Fact 2.0.7 (**[17]**). *The following are equivalent:*

(a) X is reflexive.

(b) Every bounded sequence in X has a weakly convergent subsequence.

(c) Every bounded set in X is relatively weakly compact.

Fact 2.0.8 (**[17]**). *If X is a reflexive space, then every closed, bounded and convex set in X is weakly compact.*

Definition 2.0.9. If D is a subset of X, then $\text{Int}(D)$ denotes the **interior** of D.

Definition 2.0.10. If X is a Banach space and $A \subseteq X$, then the **convex hull** of A is defined by

$$\text{conv}A = \bigcap \{K \subseteq X : K \supseteq A \text{ and } K \text{ is convex}\}.$$

This is clearly the smallest convex set that contains A.

Definition 2.0.11. A function $\phi : X \to \mathbb{R}$ is said to be **convex** if

$$\phi(tx + (1-t)y) \leq t\phi(x) + (1-t)\phi(y)$$

for all $x, y \in X$ and $t \in [0,1]$.

ϕ is called a **proper convex** function if it is convex and its domain is nonempty.

2.1 Results for Real Numbers

We now provide three results about real numbers that will be used later. Lemma 2.1.2 is especially useful.

Lemma 2.1.1 ([2]). *Let (λ_n) be a sequence in $[0,1)$ such that $\lim_{n\to\infty} \lambda_n = 0$. Then*

$$\sum_{n=1}^{\infty} \lambda_n = \infty \Leftrightarrow \prod_{n=1}^{\infty}(1-\lambda_n) = 0.$$

Proof. Since $\lim_{n\to\infty} \lambda_n = 0$, we may assume, without loss of generality, that $\lambda_n \leq \frac{1}{2}$ for all n. If $f(x) = \ln(1-x)$, then for $0 \leq x \leq \frac{1}{2}$, it is clear that $-2 \leq f'(x) \leq -1$. Thus $f'(x)+1 \leq 0$ and $f'(x)+2 \geq 0$. This means that $H_1(x) := f(x)+x$ is decreasing and $H_2(x) := f(x) + 2x$ is increasing in the region $0 \leq x \leq \frac{1}{2}$. But $H_1(0) = 0$ and $H_2(0) = 0$. Hence $H_1(x) \leq 0$ and $H_2(x) \geq 0$ in this region, showing that

$$-\lambda_n \geq \ln(1-\lambda_n) \geq -2\lambda_n$$

for all $\lambda_n \in [0, \frac{1}{2}]$. By taking sums

$$-\sum_{n=1}^{k} \lambda_n \geq \sum_{n=1}^{k} \ln(1-\lambda_n) \geq -2\sum_{n=1}^{k} \lambda_n;$$

i.e.

$$-\sum_{n=1}^{k}\lambda_n \geq \ln\prod_{n=1}^{k}(1-\lambda_n) \geq -2\sum_{n=1}^{k}\lambda_n.$$

Taking the limit as $k \longrightarrow \infty$ gives the result. $\square$

Lemma 2.1.2. *Let (λ_n) be a sequence in $[0,1)$ that satisfies $\lim\limits_{n\to\infty}\lambda_n = 0$ and $\sum\limits_{n=1}^{\infty}\lambda_n = \infty$. Let (a_n) be a sequence of nonnegative real numbers that satisfies any one of the following conditions:*

(a) For all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n\epsilon.$$

(b) $a_{n+1} \leq (1-\lambda_n)a_n + \mu_n$, $n \geq 0$ where $\mu_n > 0$ satisfies $\lim\limits_{n\to\infty}\dfrac{\mu_n}{\lambda_n} = 0$.

(c) $a_{n+1} \leq (1-\lambda_n)a_n + \lambda_n c_n$ where $\overline{\lim}\, c_n \leq 0$.

Then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. (a) For $n \geq N$,

$$\begin{aligned}
a_{n+1} &\leq (1-\lambda_n)a_n + \lambda_n\epsilon \\
&\leq (1-\lambda_n)[(1-\lambda_{n-1})a_{n-1} + \lambda_{n-1}\epsilon] + \lambda_n\epsilon \\
&= (1-\lambda_n)(1-\lambda_{n-1})a_{n-1} + \epsilon[1-(1-\lambda_n)(1-\lambda_{n-1})] \\
&\quad\vdots \\
&\leq \prod_{j=N}^{n}(1-\lambda_j)a_N + \epsilon\left[1-\prod_{j=N}^{n}(1-\lambda_j)\right].
\end{aligned}$$

Taking limits as $n \longrightarrow \infty$, and noting Lemma 2.1.1 , gives $\overline{\lim}\, a_n \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, $\overline{\lim}\, a_n = 0$. Thus $\lim\limits_{n\to\infty} a_n = 0$.

(b) If $\lim_{n\to\infty} \frac{\mu_n}{\lambda_n} = 0$, then for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\frac{\mu_n}{\lambda_n} < \epsilon$ for all $n \geq N$. Hence $\mu_n < \lambda_n \epsilon$ for all $n \geq N$. So for all $n \geq N$,

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \epsilon.$$

By (a), $\lim_{n\to\infty} a_n = 0$.

(c) Since $\overline{\lim}\, c_n \leq 0$, we have for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\sup\{c_n, c_{n+1}, \ldots\} < \epsilon$ for all $n \geq N$.

In particular $c_n < \epsilon$ for all $n \geq N$. Hence for all $n \geq N$,

$$a_{n+1} \leq (1 - \lambda_n)a_n + \lambda_n \epsilon.$$

By (a), $\lim_{n\to\infty} a_n = 0$. ☐

Lemma 2.1.3. *Let (a_n) and (b_n) be nonnegative sequences in $\mathbb{R}$, with $\sum a_n b_n < \infty$ and $\sum a_n = \infty$. Then $\underline{\lim}\, b_n = 0$.*

Proof. If $b = \underline{\lim}\, b_n > 0$, then there exists $N \in \mathbb{N}$ such that $b_n > \frac{b}{2}$ for $n \geq N$. Therefore

$$\sum_{n=N}^{\infty} a_n b_n \geq \frac{b}{2} \sum_{n=N}^{\infty} a_n = \infty.$$

☐

2.2 Uniformly Convex Banach Spaces

Definition 2.2.1. 1. A Banach space X is said to be **strictly convex** if the following implication holds for all $x, y \in X$:

$$\left.\begin{array}{l} ||x|| \leq 1 \\ ||y|| \leq 1 \\ ||x-y|| > 0 \end{array}\right\} \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| < 1.$$

2. A Banach space X is said to be **uniformly convex** if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that the following implication holds for all $x, y \in X$:

$$\left.\begin{array}{l} ||x|| \leq 1 \\ ||y|| \leq 1 \\ ||x-y|| \geq \epsilon \end{array}\right\} \quad \Rightarrow \quad \left\|\frac{x+y}{2}\right\| \leq \delta.$$

Clearly, a uniformly convex Banach space is strictly convex. In a finite-dimensional space, they are equivalent.

Definition 2.2.2. Let X be a Banach space. The **modulus of convexity** is the function $\delta_X : [0, 2] \to [0, 1]$ defined by

$$\delta_X(\epsilon) := \inf \left\{ 1 - \left\|\frac{x+y}{2}\right\| : ||x|| \leq 1, ||y|| \leq 1, ||x-y|| \geq \epsilon \right\}.$$

We will write $\delta(\epsilon)$ if it is understood that we are working in the space X.

The following result is clear from the definitions.

Theorem 2.2.3. *X is a uniformly convex Banach space if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0,2]$.*

Example 2.2.4. *A Hilbert space is uniformly convex.*

Let H be a Hilbert space. Let $\epsilon \in (0,2]$ and let $x, y \in H$ with $||x|| = 1, ||y|| = 1$ and $||x - y|| = \epsilon$. By the parallelogram law:

$$||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2),$$

we have

$$\begin{aligned} \left\|\frac{x+y}{2}\right\|^2 &= \frac{2}{4}(||x||^2 + ||y||^2) - \left\|\frac{x-y}{2}\right\|^2 \\ &= 1 - \left(\frac{\epsilon}{2}\right)^2, \end{aligned}$$

and so

$$\left\|\frac{x+y}{2}\right\| = \left(1 - \left(\frac{\epsilon}{2}\right)^2\right)^{\frac{1}{2}}.$$

Thus $\delta_H(\epsilon) = 1 - \left(1 - \left(\frac{\epsilon}{2}\right)^2\right)^{\frac{1}{2}} > 0$ since $\epsilon \in (0,2]$.
Therefore H is uniformly convex. $\diamondsuit$

It is also clear from the definition of the modulus of convexity that for $x, y \in X$ with $||x|| \leq 1, ||y|| \leq 1$ and $||x - y|| \geq \epsilon$,

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\epsilon).$$

We have a more general result than this.

Lemma 2.2.5. *Let X be a Banach space. Let $\epsilon \in (0,2]$ and $0 < r < 1$. Then for each $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x - y\| \geq \epsilon$,*

$$\|\lambda x + (1-\lambda)y\| \leq 1 - 2\min\{\lambda, 1-\lambda\}\,\delta(\epsilon).$$

Proof. *If* $\lambda = \frac{1}{2}$, then the result is clear. Suppose that $0 < \lambda < \frac{1}{2}$. Then

$$\begin{aligned}
\|\lambda x + (1-\lambda)y\| &= \|\lambda(x+y) + (1-2\lambda)y\| \\
&= \left\|2\lambda\left(\frac{x+y}{2}\right) + (1-2\lambda)y\right\| \\
&\leq 2\lambda\left\|\frac{x+y}{2}\right\| + (1-2\lambda) \\
&\leq 2\lambda(1-\delta(\epsilon)) + (1-2\lambda) \\
&= 1 - 2\lambda\delta(\epsilon).
\end{aligned}$$

If $\frac{1}{2} < \lambda < 1$, then $0 < 1 - \lambda < \frac{1}{2}$, and so

$$\|\lambda x + (1-\lambda)y\| \leq 1 - 2(1-\lambda)\delta(\epsilon).$$

Hence, in general,

$$\|\lambda x + (1-\lambda)y\| \leq 1 - 2\min\{\lambda, 1-\lambda\}\,\delta(\epsilon).$$

□

Theorem 2.2.6 ([8]; Theorem 2.9). *If X is a uniformly convex Banach space then X is reflexive.*

More information on uniform convexity can be found in [18], pages 6-11.

2.3 Smooth Banach Spaces

Definition 2.3.1. Let X and Y be Banach spaces and let $D \subseteq X$ be open. If $F : D \to Y$ and $x \in D$, then F is said to be **Gateaux-differentiable** (G-differentiable) at x if there exists $F'(x) \in L(X,Y)$ such that

$$\lim_{t \to \infty} \frac{F(x+ty) - F(x)}{t} = F'(x)y := \langle y, F'(x) \rangle \quad \text{for all} \quad y \in X.$$

Definition 2.3.2. Let X be a Banach space. $f : X \to \tilde{\mathbb{R}}$ is said to be **subdifferentiable** at a point $x \in X$ if there exists $x^* \in X^*$ such that

$$f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \text{for all} \quad y \in X. \tag{2.1}$$

x^* is called a **subgradient** of f at x.

The set of all subgradients of f at x is denoted by $\partial f(x)$; i.e.

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle y - x, x^* \rangle \quad \text{for} \quad \text{all} \quad y \in X\}.$$

The mapping $\partial f : X \to 2^{X^*}$ is called the subdifferential of f.

Theorem 2.3.3 ([8]; Corollary 2.7). *A proper convex function f is G-differentiable at $x \in \operatorname{Int} \mathrm{D}(f)$ if and only if it has a unique subgradient at x; in this case $\partial f(x) = F'(x)$.*

Definition 2.3.4. A Banach space X is said to be **smooth** if for every $x \neq 0$ in X, there is a unique $x^* \in X^*$ such that $\|x^*\| = 1$ and $\langle x, x^* \rangle = \|x\|$.

Theorem 2.3.5 ([8]; Theorem 3.5). *X is smooth if and only if the norm is G-differentiable on $X\backslash\{0\}$.*

Example 2.3.6. *A Hilbert space is smooth.*

Let $x \in X\backslash\{0\}$ and $y \in X$. Let $\phi(x) = \|x\|$. Then

$$\begin{aligned}
\lim_{t\to 0}\frac{\phi(x+ty)-\phi(x)}{t} &= \lim_{t\to 0}\frac{\|x+ty\|-\|x\|}{t} \\
&= \lim_{t\to 0}\frac{\langle x+ty, x+ty\rangle^{\frac{1}{2}}-\langle x,x\rangle^{\frac{1}{2}}}{t} \\
&= \lim_{t\to 0}\frac{\langle x+ty, x+ty\rangle-\langle x,x\rangle}{[\langle x+ty, x+ty\rangle^{\frac{1}{2}}+\langle x,x\rangle^{\frac{1}{2}}]t} \\
&= \lim_{t\to 0}\frac{\langle x,x\rangle+2t\langle x,y\rangle+t^2\langle y,y\rangle-\langle x,x\rangle}{[\langle x+ty, x+ty\rangle^{\frac{1}{2}}+\langle x,x\rangle^{\frac{1}{2}}]t} \\
&= \lim_{t\to 0}\frac{2\langle x,y\rangle+t\langle y,y\rangle}{\|x+ty\|+\|x\|} \\
&= \frac{2\langle x,y\rangle}{2\|x\|} \\
&= \frac{\langle x,y\rangle}{\|x\|}.
\end{aligned}$$

So $\phi'(x) = \dfrac{x}{\|x\|}$. ◊

Lemma 2.3.7. *Let X be a Hilbert space. If $\phi(x) = \frac{1}{2}\|x\|^2$, then $\phi'(x) = x$.*

Proof.

$$\begin{aligned}
\lim_{t\to 0}\frac{\phi(x+ty)-\phi(x)}{t} &= \lim_{t\to 0}\frac{\frac{1}{2}\|x+ty\|^2-\frac{1}{2}\|x\|^2}{t} \\
&= \lim_{t\to 0}\frac{\|x\|^2+2t\langle x,y\rangle+t^2\|y\|^2-\|x\|^2}{2t} \\
&= \lim_{t\to 0}\frac{2t\langle x,y\rangle+t^2\|y\|^2}{2t} \\
&= \langle y,x\rangle.
\end{aligned}$$

Hence $\phi'(x) = x$. □

Definition 2.3.8. A Banach space X is said to be **uniformly smooth** if

$$\lim_{t\to 0} \sup_{||x||=||y||=1} \frac{||x+ty||+||x-ty||-2}{2t} = 0.$$

Lemma 2.3.9 ([8]; Proposition 3.11). *If X is uniformly smooth, then X is smooth.*

Lemma 2.3.10 ([8]). *If X is uniformly smooth, then X is reflexive.*

2.4 Projections

In this section, we work only in a real Hilbert space H. We define projections and give the nice properties they have. The first result is standard and its proof can be found in most standard Functional Analysis texts. In particular, more information on the nearest point projections can be found in [18].

Theorem 2.4.1 ([22]). *Let H be a Hilbert space and let K be a nonempty, closed and convex subset of H. Then for $x \in X$, there exists a unique $y \in K$ such that*

$$\inf_{k\in K} ||x-k|| = ||x-y||.$$

Definition 2.4.2. Let H be a Hilbert space and let K be a nonempty, closed and convex subset of H. Then we define for any $x \in H$,

$$\mathrm{d}(x, K) := \inf_{k\in K} ||x-k||.$$

The unique $y \in K$, as obtained in Theorem 2.4.1 , will be denoted by $P_K x$.
Thus $||x - P_K x|| = \mathrm{d}(x, K)$, and $P_K : H \to K$ is a well-defined mapping.
P_K is called the **nearest point projection** (or simply the projection) of H onto K.

The following lemma gives a characterization of projections.

Lemma 2.4.3 ([17]; Lemma 12.1). *Let H be a Hilbert space and K a nonempty, closed and convex subset of H. Then for $x \in H$,*

(a) $\langle z - P_K x, P_K x - x\rangle \geq 0$ for all $z \in K$.

(b) if $\langle z - y, y - x\rangle \geq 0$ for each $z \in K$, then $y = P_K x$.

The following result shows that a projection is nonexpansive and firmly nonexpansive, as is defined in Section 2.6.

Lemma 2.4.4 ([17]). *Let H be a Hilbert space and K a nonempty, closed and convex subset of H. Then for any $x, y \in H$,*

$$\langle P_K x - P_K y, x - y\rangle \geq ||P_K x - P_K y||^2$$

and

$$||P_K x - P_K y|| \leq ||x - y||.$$

Proof. We will write $P = P_K$. Then by Lemma 2.4.3 ,

$$\langle Px - x, Py - Px\rangle \geq 0 \tag{2.2}$$

and

$$\langle Py - y, Px - Py\rangle \geq 0. \tag{2.3}$$

Equation 2.2 becomes,

$$\langle x - Px, Px - Py\rangle \geq 0 \tag{2.4}$$

and adding equations 2.3 and 2.4 gives,

$$\langle (x-y)-(Px-Py), Px-Py\rangle \geq 0.$$

Therefore

$$\langle x-y, Px-Py\rangle - ||Px-Py||^2 \geq 0,$$

and hence

$$\langle Px-Py, x-y\rangle \geq ||Px-Py||^2.$$

Also,

$$||Px-Py||^2 \leq \langle Px-Py, x-y\rangle \leq ||Px-Py||\,||x-y||$$

and so

$$||Px-Py|| \leq ||x-y||.$$

□

The following result also gives a characterization of projections.

Lemma 2.4.5. *Let H be a Hilbert space and K a nonempty, closed and convex subset of H. Then for any $x \in H$,*

(a) $||x-P_Kx||^2 \leq ||x-y||^2 - ||P_Kx-y||^2$ *for all* $y \in H$.

(b) if $||x-z||^2 \leq ||x-y||^2 - ||z-y||^2$ *for all* $y \in H$, *then* $z = P_Kx$.

Proof. Let $P = P_K$. Then

$$\begin{aligned}
||x - Px||^2 &= ||x-y||^2 + ||y - Px||^2 + 2\langle x - y, y - Px\rangle \\
&= ||x-y||^2 + ||y - Px||^2 + 2\langle x - Px, y - Px\rangle + 2\langle Px - y, y - Px\rangle \\
&= ||x-y||^2 + ||y - Px||^2 + 2\langle x - Px, y - Px\rangle - 2||y - Px||^2 \\
&= ||x-y||^2 - ||y - Px||^2 + 2\langle x - Px, y - Px\rangle,
\end{aligned}$$

and the result follows from Lemma 2.4.3. □

2.5 Duality Maps

The concepts of an inner product and projections in a Hilbert space provide us with all the nice inequalities that we use. Also, using the inner product, we are able to get an isomorphism from H to H^*, where each $x \in H$ is associated with an $x^* \in H^*$ satisfying the property

$$\langle x, x^* \rangle = ||x||^2 = ||x^*||^2$$

using Riesz's Representation Theorem.

In a Banach space X, the normalized duality map,

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2\},$$

which is multivalued, generalizes this isomorphism.

In what follows, X will denote a real Banach space. For any $x \in X$ and $x^* \in X^*$, by $\langle x, x^* \rangle$ we mean $x^*(x)$.

Definition 2.5.1. (i) A continuous, strictly increasing function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfies $\phi(0) = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$ is called a **weight/gauge** function.

(ii) A **duality mapping** of weight ϕ is a map $J_\phi : X \to 2^{X^*}$ defined by

$$J_\phi(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x^*\| \, \|x\|, \, \phi(\|x\|) = \|x^*\|\} \tag{2.5}$$

The Hahn-Banach Theorem ensures that $J_\phi(x)$ is nonempty for each $x \in X$.

If the weight function is defined by $\phi(t) = t$, then the corresponding duality map is called the **normalized duality map**. Hence the normalized duality map is given by

$$J(x) = \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{2.6}$$

We note that in a Hilbert space, the normalized duality map is the identity map.

Definition 2.5.2. If ϕ is a weight function, then define

$$\Phi(t) = \int_0^t \phi(s)ds.$$

Theorem 2.5.3 ([8]; Theorem 4.4). *If J_ϕ is a duality map of weight ϕ, then $J_\phi(x) = \partial\Phi(\|x\|)$.*

Hence, if J is the normalized duality map, i.e. $\phi(t) = t$, then $\Phi(t) = \frac{1}{2}t^2$, and $J(x) = \partial(\frac{1}{2}\|x\|^2)$.

Noting Theorem 2.5.3 , we obtain the following subdifferential inequality from equation (2.1) :

$$\Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, j_\phi(x) \rangle \quad \text{for any} \quad j_\phi(x) \in J_\phi(x). \tag{2.7}$$

This can clearly be rewritten as

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_\phi(x + y) \rangle \quad \text{for any} \quad j_\phi(x + y) \in J_\phi(x + y), \tag{2.8}$$

and the inequality is most often used in this form.

For the normalized duality map J, the subdifferential inequality (2.8) becomes

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle \quad \text{for any} \quad j(x + y) \in J(x + y). \tag{2.9}$$

Lemma 2.5.4. *If $\phi(t) = f(x + ty)$, then $\phi'(t) = \langle y, f'(x + ty) \rangle$.*

Proof.

$$\begin{aligned} \phi'(t) &= \lim_{k \to 0} \frac{\phi(t + k) - \phi(t)}{k} \\ &= \lim_{k \to 0} \frac{f(x + (t + k)y) - f(x + ty)}{k} \\ &= \lim_{k \to 0} \frac{f(x + ty + ky) - f(x + ty)}{k} \\ &= \langle y, f'(x + ty) \rangle. \end{aligned}$$

□

Lemma 2.5.5 ([8]; Corollary 2.7). *If f is a proper, convex function and is continuous at $x \in \operatorname{Int} \mathrm{D}(f)$, then $\partial f(x) = f'(x)$.*

Corollary 2.5.6. *If J_ϕ is single-valued, then we have the following identity:*

$$\Phi(\|x+h\|) - \Phi(\|x\|) = \int_0^1 \langle h, J_\phi(x+th)\rangle dt.$$

Proof. Let $J = J_\phi$. If $g(t) = f(x+th) = \Phi(\|x+th\|)$, then by Lemma 2.5.4

$$g'(t) = \langle h, f'(x+th)\rangle.$$

By Lemma 2.5.5 and Theorem 2.5.3 ,

$$f'(x+th) = \partial f(x+th) = \partial(\Phi(\|x+th\|) = J(x+th);$$

i.e.

$$\frac{d}{dt}\Phi(\|x+th\|) = g'(t) = \langle h, J(x+th)\rangle.$$

So by integrating sides from 0 to 1 we obtain

$$\Phi(\|x+h\|) - \Phi(\|x\|) = \int_0^1 \langle h, J(x+th)\rangle dt.$$

□

The following result gives another property of the function Φ.

Lemma 2.5.7. *For any $s \geq 0$ and any $t \in [0,1]$,*

$$\Phi(ts) \leq t\,\Phi(s).$$

Proof. Since Φ is convex, we have

$$\begin{aligned}\Phi(ts) &= \Phi(ts + (1-t)0)\\ &\leq t\Phi(s) + (1-t)\Phi(0)\\ &= t\Phi(s).\end{aligned}$$

□

We now look at some properties of the duality map.

Theorem 2.5.8 ([8]; Proposition 4.7). *Let J_ϕ be a duality map having weight ϕ. Then*

(a) $J_\phi(-x) = -J_\phi(x)$ *for* $x \in X$.

(b) $J_\phi(\lambda x) = \dfrac{\phi(\lambda\|x\|)}{\phi(\|x\|)} J_\phi(x)$, $x \in X, \lambda > 0$.

(c) If J_{ϕ_1} *is another duality map having weight* ϕ_1*, then*

$$\phi(\|x\|)J_{\phi_1}(x) = \phi_1(\|x\|)J_\phi(x), \ x \in X.$$

Proof. Write $J = J_\phi$ and $J_1 = J_{\phi_1}$.

(a) By Theorem 2.5.3 , and by definition of the subdifferential

$$\begin{aligned} J(x) &= \partial(\Phi(\|x\|)) \\ &= \{x^* \in X^* : \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y - x, x^* \rangle \quad \text{for all} \quad y \in X\}. \end{aligned}$$

Now

$$\begin{aligned} x^* \in J(-x) &\Leftrightarrow \Phi(\|y\|) - \Phi(\|x\|) \geq \langle y + x, x^* \rangle \quad \text{for all} \quad y \in X \\ &\Leftrightarrow \Phi(\|y\|) - \Phi(\|x\|) \geq \langle -y - x, -x^* \rangle \quad \text{for all} \quad -y \in X \\ &\Leftrightarrow -x^* \in J(x) \\ &\Leftrightarrow x^* \in -J(x). \end{aligned}$$

Hence $J(-x) = -J(x)$.

(b) Let $x^* \in J(x)$ and let $\alpha = \dfrac{\phi(\lambda\|x\|)}{\phi(\|x\|)}$. We will show that $\alpha x^* \in J(\lambda x)$. Indeed,

$$\begin{aligned} \langle \lambda x, \alpha x^* \rangle &= \lambda\alpha\langle x, x^* \rangle \\ &= \lambda\alpha\|x^*\| \, \|x\| \\ &= \|\alpha x^*\| \, \|\lambda x\| \end{aligned}$$

and

$$\begin{aligned}
\|\alpha x^*\| &= \alpha\|x^*\| \\
&= \frac{\phi(\lambda\|x\|)}{\phi(\|x\|)}\phi(\|x\|) \\
&= \phi(\lambda\|x\|) \\
&= \phi(\|\lambda x\|).
\end{aligned}$$

Thus $\alpha x^* \in J(\lambda x)$.

The converse can be similarly proved.

(c) Let $x^* \in J_1(x)$ and $\alpha = \dfrac{\phi(\|x\|)}{\phi_1(\|x\|)}$. We will show that $\alpha x^* \in J(x)$.

$$\begin{aligned}
\langle x, \alpha x^*\rangle &= \alpha\langle x, x^*\rangle \\
&= \alpha\|x^*\|\,\|x\| \\
&= \|\alpha x^*\|\,\|x\|
\end{aligned}$$

and

$$\begin{aligned}
\|\alpha x^*\| &= \alpha\|x^*\| \\
&= \alpha\phi_1(\|x\|) \\
&= \phi(\|x\|).
\end{aligned}$$

Hence $J_1(x) \subseteq \alpha J(x)$.

By symmetry, $J(x) \subseteq \dfrac{1}{\alpha}J_1(x)$, and so we obtain the result. □

Part (c) of the previous theorem shows that if any one duality map is single-valued, then all the duality maps must be single-valued.

Theorem 2.5.9 ([8]; Corollary 4.5). *A Banach space X is smooth if and only if every duality map on X is single-valued; in this case*

$$\langle y, J_\phi(x)\rangle = \frac{d}{dt}\Phi(\|x+ty\|)|_{t=0}, \quad \text{for all} \quad x, y \in X.$$

By the remark before the statement of this theorem, the single-valuedness of only one duality map will ensure that a Banach space is smooth.

Lemma 2.5.10. *If X is a uniformly smooth Banach space then the duality map J_ϕ is uniformly continuous on bounded sets.*

Definition 2.5.11. A Banach space X is said to have a **weakly continuous duality map** if there exists a weight ϕ such that J_ϕ is single-valued and weak-weak* sequentially continuous; i.e. if $x_n \rightharpoonup x$ in X, then $J(x_n) \rightharpoonup^* J(x)$ in X^*.

We note that Banach space that has a weakly continuous duality map is necessarily smooth. In a smooth Banach space, $\|J(x)\| = \|x\|$, where J is the normalized duality map.

Example 2.5.12 ([8]; Proposition 4.9; Corollary 4.11).

The duality map on the L^p-space,$1 < p < \infty$ corresponding to the weight $\phi(t) = t^{p-1}$ is given by

$$Jf = \|f\|^{p-1}\mathbf{sgn}f \quad , f \in L^p.$$

The duality map on the ℓ^p-space, $1 < p < \infty$, corresponding to the weight $\phi(t) = t^{p-1}$ is given by

$$Jx = (|x_k|^{p-1}\mathbf{sgn}x_k)_{k\in\mathbb{N}}, \quad x = (x_k)_k \in \ell^p.$$

$\diamondsuit$

2.6 Nonexpansive Mappings

Definition 2.6.1. Let X be a normed linear space and let C be a nonempty, closed and convex subset of X.

1. A mapping $T : X \to X$ is said to be a **contraction** if there exists a constant k, $0 < k < 1$, such that

$$||Tx - Ty|| \leq k||x - y|| \quad \text{for all} \quad x, y \in X.$$

2. If X is a Banach space, then a map $T : C \to C$ is said to be **nonexpansive** if

$$||Tx - Ty|| \leq ||x - y|| \quad \text{for all} \quad x, y \in C.$$

3. $x \in C$ is called a **fixed point** of $T : C \to C$ if $Tx = x$. The set of all fixed points of T is denoted by $Fix(T)$; i.e. $Fix(T) = \{x \in C : Tx = x\}$.

The next result is the famous Banach's Contraction Mapping Principle, the proof of which can be found in [17].

Theorem 2.6.2 ([17]; Theorem 2.1). *(Banach's Contraction Mapping Principle) Let X be a Banach space, C a closed subset of X and $T : C \to C$ be a contraction.*

Then T has a unique fixed point in C. Moreover, for each $x_0 \in C$, the sequence of iterates $(T^n x_0)$ converges to this fixed point.

The following result ensures the existence of the nearest point projection of H onto $Fix(T)$.

Theorem 2.6.3 ([17]; Lemma 3.4). *Let H be a Hilbert space and $C \subseteq H$ a nonempty, closed and convex set. If $T : C \to C$ is nonexpansive, then $Fix(T)$ is closed and convex.*

A firmly nonexpansive map can be defined on a Banach space, but all we need is a firmly nonexpansive map in a Hilbert space. The definition is greatly simplified in a Hilbert space, and hence this is the definition we provide.

Definition 2.6.4. Let H be a Hilbert space and let $C \subseteq H$ be a nonempty, closed and convex set. Then a mapping $T : C \to C$ is said to be **firmly nonexpansive** if $T = \frac{1}{2}(I + S)$ where $S : C \to C$ is a nonexpansive map.

The next result gives a characterization of firmly nonexpansive maps in Hilbert spaces. For the proof, see [17].

Theorem 2.6.5 ([17]; Theorem 12.1). *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T : C \to C$ be a mapping. Then the following conditions are equivalent:*

(a) T is firmly nonexpansive.

(b) $||Tx - Ty||^2 \leq \langle Tx - Ty, x - y \rangle$ for all $x, y \in C$.

(c) $2T - I$ *is nonexpansive.*

A firmly nonexpansive map is clearly nonexpansive. We include the following result about firmly nonexpansive maps.

Lemma 2.6.6 ([4]; Lemma 2.4(ii)). *If C is nonempty, closed and convex subset of a Hilbert space H and if $T : C \to C$ is firmly nonexpansive with $Fix(T) \neq \emptyset$, then for $x \in C$ and $f \in Fix(T)$,*

(i) $\langle Tx - f, x - Tx \rangle \geq 0$.

(ii) $||x - f||^2 - ||Rx - f||^2 \geq \alpha(2 - \alpha)||x - Tx||^2$ *where* $R := (1 - \alpha)I + \alpha T$, $\alpha \in (0, 2)$.

Theorem 2.6.7. *Let C be a nonempty, closed and convex subset of a Hilbert space H. Then the nearest point projection $P_C : H \to C$ is firmly nonexpansive.*

The above result follows from Lemma 2.4.4 .

The following result gives the fixed point set for a convex combination of nonexpansive maps, in a uniformly convex Banach space.

Lemma 2.6.8. *Let X be a uniformly convex Banach space, and let $C \subseteq X$ be a nonempty, closed and convex set. Let $T_i : C \to C$ be nonexpansive $(i = 1, 2, \ldots, N)$ such that $\bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ and let $V = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_N T_N$ where $\lambda_i > 0$ for $i = 1, 2, \ldots, N$ and $\sum_{i=1}^{N} \lambda_i = 1$. Then*

$$Fix(V) = \bigcap_{i=1}^{N} Fix(T_i).$$

Proof. We will prove this result by induction on N. It is clearly true for $N = 1$. We will now prove it for $N = 2$. It is clear that $Fix(V) \supseteq \bigcap_{i=1}^{2} Fix(T_i)$. Now let $p \in Fix(V)$. Then $\lambda_1 T_1(p) + \lambda_2 T_2(p) = p$. We need to show that $T_1(p) = p$ and $T_2(p) = p$. If $T_1(p) = p$, then

$$\lambda_2 T_2(p) = p - \lambda_1 T_1(p) = p - \lambda_1 p = (1 - \lambda_1)p = \lambda_2 p.$$

Hence $T_2(p) = p$ as well. So we may assume that $T_1(p) \neq p$ and $T_2(p) \neq p$. This also implies that $T_1(p) \neq T_2(p)$. For any $q \in Fix(T_1) \bigcap Fix(T_2)$, since $p \neq q$, we have

$$\begin{aligned}
\|p - q\| &= \|\lambda_1 T_1(p) + \lambda_2 T_2(p) - q\| \\
&= \|\lambda_1 (T_1(p) - q) + \lambda_2 (T_2(p) - q)\| \\
&= \|p - q\| \left\| \lambda_1 \left(\frac{T_1(p) - q}{\|p - q\|} \right) + \lambda_2 \left(\frac{T_2(p) - q}{\|p - q\|} \right) \right\|. \qquad (2.10)
\end{aligned}$$

By the nonexpansivity of T_1 and T_2 and the fact that $q \in Fix(T_1) \bigcap Fix(T_2)$,

$$\left\| \frac{T_1(p) - q}{\|p - q\|} \right\| \leq \frac{\|p - q\|}{\|p - q\|} = 1,$$

$$\left\| \frac{T_2(p) - q}{\|p - q\|} \right\| \leq \frac{\|p - q\|}{\|p - q\|} = 1,$$

and

$$\left\| \frac{T_1(p) - q}{\|p - q\|} - \frac{T_2(p) - q}{\|p - q\|} \right\| = \frac{1}{\|p - q\|} \|T_1(p) - T_2(p)\| = \epsilon$$

with $\epsilon > 0$ and $\epsilon = \frac{1}{||p-q||}||T_1(p) - T_2(p)|| \le \frac{1}{||p-q||}(||T_1(p) - q|| + ||T_2(p) - q||) \le 2$.
By Lemma 2.2.5, equation (2.10) becomes

$$||p - q|| \le ||p - q|| \, [1 - 2\min\{\lambda_1, \lambda_2\}\, \delta(\epsilon)] \tag{2.11}$$

and by uniform convexity, $\delta(\epsilon) > 0$ by Theorem 2.2.3 . Hence equation (2.11) becomes $||p - q|| < ||p - q||$, a contradiction. So $T_1(p) = p$ and $T_2(p) = p$. Hence $Fix(V) = Fix(T_1) \bigcap Fix(T_2)$. So the result is true for $N = 2$.

Assume now that the result is true for $N \ge 3$. Next we will show that the result is also true for $N+1$ and by induction it will be true for all N. Let $V = \sum_{i=1}^{N+1} \lambda_i T_i$ and put $T = \sum_{i=2}^{N+1} \frac{\lambda_i}{1-\lambda_1} T_i$. Then $V = \lambda_1 T_1 + (1 - \lambda_1)T$. Hence

$$\begin{aligned} Fix(V) &= Fix(T_1) \bigcap Fix(T) \\ &= Fix(T_1) \bigcap \left(\bigcap_{i=2}^{N+1} Fix(T_i) \right) \\ &= \bigcap_{i=1}^{N+1} Fix(T_i). \end{aligned}$$

□

2.7 Opial's Property and the Demiclosedness Principle

Definition 2.7.1. A Banach space X is said to have **Opial's Property** if whenever $x_n \rightharpoonup x$ and $y \neq x$, then $\overline{\lim}\, ||x_n - x|| < \overline{\lim}\, ||x_n - y||$.

Opial's property was introduced by Opial [25] in 1967, where he showed that

Hilbert spaces and ℓ^p spaces, $1 < p < \infty$, have this property.

Definition 2.7.2. A Banach space X is said to satisfy the **Demiclosedness Principle** if for any nonempty, closed and convex subset C of X and any nonexpansive map $T : C \to C$, $I - T$ is demiclosed; i.e. if $x_n \rightharpoonup x$ in C and $(I-T)(x_n) \longrightarrow y$, then $(I - T)(x) = y$.

The next result shows in which space the Demiclosedness Principle holds.

Theorem 2.7.3 ([17]; Theorem 10.4). *If X is a uniformly convex Banach space, then the Demiclosedness Principle is satisfied.*

A space having Opial's Property has the following nice property.

Theorem 2.7.4 ([17]). *Let X be a Banach space that has Opial's Property. Then X also satisfies the Demiclosedness Principle.*

Proof. Let (x_n) be a sequence in C such that $x_n \rightharpoonup x$ and $(I - T)(x_n) \longrightarrow y$. Define $T_y(z) = Tz + y$ for $z \in C$. Then for $z_1, z_2 \in C$,

$$\begin{aligned} ||T_y(z_1) - T_y(z_2)|| &= ||(T(z_1) + y) - (T(z_2) + y)|| \\ &= ||T(z_1) - T(z_2)|| \\ &\leq ||z_1 - z_2||. \end{aligned}$$

Hence T_y is nonexpansive with $(I - T_y)(x_n) \longrightarrow 0$.

We need to show that $T_y x = x$.

$$\begin{aligned} ||x_n - T_y x|| &\leq ||x_n - T_y x_n|| + ||T_y x_n - T_y x|| \\ &\leq ||(I - T_y)(x_n)|| + ||x_n - x||. \end{aligned}$$

Hence $\overline{\lim}\, \|x_n - T_y x\| \le \overline{\lim}\, \|x_n - x\|$.

By Opial's Property, $x = T_y x$ and hence $(I - T)(x) = y$. $\square$

The above proof can be found in [17], and even though their hypothesis includes X reflexive, no use of this fact is made in the proof.

As mentioned earlier, all Hilbert spaces and all ℓ^p spaces, $1 < p < \infty$, have Opial's Property. More generally, all spaces that have a weakly continuous duality map have Opial's Property. To prove this, we require the following lemma.

Lemma 2.7.5. *Let X be a Banach space that has a weakly continuous duality map J_ϕ. If $x_n \rightharpoonup x$, then for all $y \in X$, we have the following identity:*

$$\overline{\lim}\, \Phi(\|x_n - y\|) = \overline{\lim}\, \Phi(\|x_n - x\|) + \Phi(\|x - y\|).$$

Proof. First take $x = 0$. We have the following identity (Corollary 2.5.6):

$$\Phi(\|x_n + y\|) = \Phi(\|x_n\|) + \int_0^1 \langle y, J_\phi(x_n + ty)\rangle dt.$$

So

$$\begin{aligned}
\overline{\lim}\, \Phi(\|x_n + y\|) &= \overline{\lim} \left[\Phi(\|x_n\|) + \int_0^1 \langle y, J_\phi(ty)\rangle dt\right] \\
&= \overline{\lim}\, \Phi(\|x_n\|) + \int_0^1 \langle y, J_\phi(ty)\rangle dt \\
&= \overline{\lim}\, \Phi(\|x_n\|) + \int_0^1 \frac{1}{t} \langle ty, J_\phi(ty)\rangle dt \\
&= \overline{\lim}\, \Phi(\|x_n\|) + \int_0^1 \frac{1}{t} \|ty\|\, \phi(\|ty\|) dt \\
&= \overline{\lim}\, \Phi(\|x_n\|) + \int_0^1 \|y\|\, \phi(\|ty\|) dt.
\end{aligned}$$

If we let $s = \|ty\|$ in the integral, then

$$\begin{aligned}\int_0^1 \|y\|\,\phi(\|ty\|)dt &= \int_0^{\|y\|} \phi(s)ds \\ &= \Phi(\|y\|).\end{aligned}$$

For the general case, take $z_n = x_n - x \rightharpoonup 0$. So for any $z \in X$,

$$\overline{\lim}\,\Phi(\|z_n - z\|) = \overline{\lim}\,\Phi(\|z_n\|) + \Phi(\|z\|).$$

Let $y = x + z$. Then

$$\overline{\lim}\,\Phi(\|x_n - y\|) = \overline{\lim}\,\Phi(\|x_n - x\|) + \Phi(\|y - x\|).$$

□

Theorem 2.7.6. *Let X be a Banach space that has a weakly continuous duality map J_ϕ. Then X has Opial's Property.*

Proof. Suppose $x_n \rightharpoonup x$ and suppose that $y \neq x$. From Lemma 2.7.5 and the fact that Φ is strictly increasing,

$$\begin{aligned}\overline{\lim}\,\Phi(\|x_n - y\|) &= \overline{\lim}\,\Phi(\|x_n - x\|) + \Phi(\|y - x\|) \\ &> \overline{\lim}\,\Phi(\|x_n - x\|).\end{aligned}$$

Thus X has Opial's Property. □

2.8 Sunny Nonexpansive Retractions

Sunny nonexpansive retractions in Banach spaces together with duality maps, have characterizations which are analagous to projections in Hilbert spaces. We define the following concepts below.

Definition 2.8.1. Let X be a Banach space, C a nonempty, closed and convex subset of X, and K a nonempty subset of C. Let $P : C \to K$. P is said to be:

1. **sunny** if for each $x \in C$ we have
$$P(tx + (1-t)Px) = Px.$$
2. a **retraction** of C onto K if P is onto and
$$Px = x \quad \text{for all} \quad x \in K.$$
3. a **sunny nonexpansive retraction** if P is sunny, nonexpansive and a retraction of C onto K.

The following result gives a characterization of sunny nonexpansive retractions on a smooth Banach space.

Theorem 2.8.2 ([18], [27]). *Let X be a smooth Banach space and let C be a nonempty, closed and convex subset of X. Let $Q : X \to C$ be a retraction and let J be the normalized duality map on X. Then the following are equivalent:*

(a) *Q is sunny and nonexpansive.*

(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle \quad \text{for all} \quad x, y \in X.$

(c) $\langle x - Qx, J(y - Qx)\rangle \leq 0 \quad \text{for all} \quad x \in X \quad \text{and} \quad y \in C.$

Hence there is at most one sunny nonexpansive retraction on C.

Lemma 2.8.3. *Let X be a Banach space and C a nonempty, closed and convex subset of X. If $T : C \to C$ is a nonexpansive map and $z \in X$, then for each $0 < t < 1$, there exists a unique $z_t \in C$ such that $z_t = tz + (1-t)Tz_t$.*

Proof. Let $S(x) = tz + (1-t)T(x)$ for $x \in X$. Then

$$\begin{aligned} \|S(x_1) - S(x_2)\| &= \|(1-t)(T(x_1) - T(x_2))\| \\ &\leq (1-t)\|x_1 - x_2\|. \end{aligned}$$

Since $0 < 1 - t < 1$, S is a contraction, and so by Banach's Contraction Mapping Principle, S has a unique fixed point z_t. Thus $z_t = S(z_t) = tz + (1-t)Tz_t$. □

Reich [29] proved the following result.

Theorem 2.8.4 ([29]). *Let X be a uniformly smooth Banach space and C a nonempty, closed and convex subset of X. Let $T : C \to C$ be nonexpansive and let $z \in X$. For each $0 < t < 1$, there exists a unique $z_t \in X$ satisfying $z_t = tz + (1-t)Tz_t$ and $(z_t)_{0<t<1}$ converges to a fixed point of T as $t \longrightarrow 0^+$.*

Definition 2.8.5. A Banach space X is said to have **Reich's Property** if for any weakly compact and convex subset C of X, any nonexpansive mapping $T : C \to C$ and any $z \in C$, (z_t) (as obtained in Lemma 2.8.3) converges to a fixed point of T as $t \longrightarrow 0^+$.

Thus we have that every uniformly smooth Banach space has Reich's property. The following result gives a property for the normalized duality map which will be used in Theorem 2.8.8 .

Lemma 2.8.6. *Let X be a smooth Banach space, C a nonempty, closed and convex subset of X and $T : C \to C$ nonexpansive. If J is the normalized duality map on X, then*

$$\langle (I-T)(x) - (I-T)(y), J(x-y)\rangle \geq 0 \quad \text{for all} \quad x, y \in C.$$

Proof. Since X is smooth, J is single-valued. Then by definition of the normalized duality map

$$\begin{aligned} \langle (I-T)(x) - (I-T)(y), J(x-y)\rangle &= \langle x-y, J(x-y)\rangle - \langle Tx - Ty, J(x-y)\rangle \\ &= \|x-y\|^2 - \langle Tx - Ty, J(x-y)\rangle \\ &\geq \|x-y\|^2 - \|Tx-Ty\|\,\|J(x-y)\| \\ &\geq \|x-y\|^2 - \|x-y\|\,\|x-y\| \\ &= 0. \end{aligned}$$

□

The next lemma can be found in [8] and it gives a relationship between smooth spaces and strictly convex spaces.

Lemma 2.8.7 ([8]; Theorem 1.3, Corollary 1.4 and Corollary 1.5). *Let X be a Banach space.*

(a) If X^* is strictly convex, then X is smooth.

(b) If X is reflexive, then X is smooth if and only if X^* is strictly convex.

(c) Let X be a Banach space with X^* strictly convex. Then any duality map is norm-to-weak-continuous.*

The following result gives conditions for the existence of a sunny nonexpansive retraction.

Theorem 2.8.8. *Let X^* be a strictly convex Banach space, X have Reich's property, C be a nonempty, closed and convex subset of X and $T : C \to C$ a nonexpansive map with $Fix(T) \neq \emptyset$. Then $Q : C \to Fix(T)$ defined by*

$$Q(z) = \lim_{t \to 0^+} z_t$$

is a sunny nonexpansive retraction.

Proof. For any $z \in C$, there exists a unique $z_t \in C$ such that $z_t = tz + (1-t)Tz_t$ for $0 < t < 1$. By Riech's property, $\lim_{t \to 0} z_t$ exists. Define $Q(z) = \lim_{t \to 0} z_t$. Now since,

$$\begin{aligned} z &= \frac{1}{t} z_t - \frac{(1-t)}{t} T z_t \\ &= \frac{1}{t}(z_t - Tz_t) + Tz_t. \end{aligned}$$

So

$$\begin{aligned} z_t - z &= z_t - \frac{1}{t}(z_t - Tz_t) - Tz_t \\ &= (1 - \frac{1}{t})(z_t - Tz_t) \\ &= -\frac{(1-t)}{t}(I - T)(z_t). \end{aligned}$$

For all $f \in Fix(T)$, we have

$$\begin{aligned}
\langle z_t - z, J(z_t - f)\rangle &= -\frac{(1-t)}{t}\langle (I-T)(z_t), J(z_t - f)\rangle \\
&= -\frac{(1-t)}{t}\langle (I-T)(z_t) - (I-T)(f), J(z_t - f)\rangle \\
&\leq 0
\end{aligned}$$

by Lemma 2.8.6 . Taking limits as $t \longrightarrow 0$, and noting that J is norm-weak*-continuous by Lemma 2.8.7 , we have

$$\langle Q(z) - z, J(Q(z) - f)\rangle \leq 0.$$

Hence for any x and y in C,

$$\langle Q(x) - x, J(Q(x) - Q(y))\rangle \leq 0$$

and

$$\langle Q(y) - y, J(Q(y) - Q(x))\rangle \leq 0.$$

Adding these two inequalities and noting that J is odd, gives us

$$\langle Q(x) - x + y - Q(y), J(Q(x) - Q(y))\rangle \leq 0.$$

Therefore,

$$\begin{aligned}
\langle y - x, J(Q(x) - Q(y))\rangle &\leq -\langle Q(x) - Q(y), J(Q(x) - Q(y))\rangle \\
&= -\|Q(x) - Q(y)\|^2.
\end{aligned}$$

By Theorem 2.8.2 , Q is a sunny nonexpansive retraction. □

We are now ready to obtain the existence of a unique sunny nonexpansive retraction on certain spaces.

Theorem 2.8.9. *Let X be a uniformly smooth Banach space, and C a nonempty, closed and convex subset of X. Let $T : C \to C$ be a nonexpansive map with $Fix(T) \neq \emptyset$. Then there exists a unique sunny nonexpansive retraction $Q : C \to Fix(T)$.*

Proof. Since X is uniformly smooth, it must also be smooth and hence be Lemma 2.8.7 X^* is strictly convex. Existence now follows from Theorem 2.8.8 and uniqueness follows from Theorem 2.8.2 . □

The following lemma establishes conditions under which a space has Reich's property. Reich [27] proved this result for the normalized duality map. Here we extend the result to an arbitrary duality map, but we use the idea of the proof employed in [27].

Lemma 2.8.10. *Let X be a smooth Banach space having Opial's Property and having some duality map J_ϕ weakly sequentially continuous at 0. Then X has Reich's Property.*

Proof. Let C be weakly compact and convex and let $T : C \to C$ be nonexpansive. For $\lambda_n \in (0,1)$, let $z_n := z_{\lambda_n}$ be a subsequence of (z_t). Now

$$z_n = (1 - \lambda_n)z + \lambda_n T z_n.$$

Let $y \in Fix(T)$. Then

$$\begin{aligned} z_n - y &= (1 - \lambda_n)(z - y) + \lambda_n(Tz_n - y) \\ &= (1 - \lambda_n)(z - y) + \lambda_n(Tz_n - Ty). \end{aligned}$$

Therefore

$$\begin{aligned}
&\langle z_n - y, J_\phi(z_n - y)\rangle \\
= \;& (1-\lambda_n)\langle z - y, J_\phi(z_n - y)\rangle + \lambda_n \langle Tz_n - Ty, J_\phi(z_n - y)\rangle \\
\leq \;& (1-\lambda_n)\langle z - y, J_\phi(z_n - y)\rangle + \lambda_n \|z_n - y\| \, \|J_\phi(z_n - y)\|.
\end{aligned}$$

But $\langle z_n - y, J_\phi(z_n - y)\rangle = \|z_n - y\|\phi(\|z_n - y\|)$. So

$$\|z_n - y\|\phi(\|z_n - y\|) \leq (1-\lambda_n)\langle z - y, J_\phi(z_n - y)\rangle + \lambda_n\|z_n - y\|\phi(\|z_n - y\|).$$

Hence

$$(1-\lambda_n)\|z_n - y\|\phi(\|z_n - y\|) \leq (1-\lambda_n)\langle z - y, J_\phi(z_n - y)\rangle.$$

So for all $y \in Fix(T)$,

$$\|z_n - y\|\phi(\|z_n - y\|) \leq \langle z - y, J_\phi(z_n - y)\rangle.$$

Also

$$\begin{aligned}
&\langle z_n - z, J_\phi(y - z_n)\rangle \\
= \;& \langle z_n - y, J_\phi(y - z_n)\rangle + \langle y - z, J_\phi(y - z_n)\rangle \\
\geq \;& -\|z_n - y\|\phi(\|z_n - y\|) + \|z_n - y\|\phi(\|z_n - y\|) \\
= \;& 0.
\end{aligned}$$

So

$$\langle z_n - z, J_\phi(y - z_n)\rangle \geq 0.$$

Now

$$\begin{aligned}
\|z_n - y\| &\leq (1-\lambda_n)\|z - y\| + \lambda_n\|Tz_n - Ty\| \\
&\leq (1-\lambda_n)\|z - y\| + \lambda_n\|z_n - y\|.
\end{aligned}$$

So $(1-\lambda_n)||z_n - y|| \leq (1-\lambda_n)||z-y||$ and hence

$$||z_n - y|| \leq ||z-y||.$$

Therefore $||Tz_n - Ty|| \leq ||z_n - y|| \leq ||z-y||$ and so (Tz_n) is bounded. Since C is weakly compact, (Tz_n) has a weakly convergent subsequence, say $Tz_{n_k} \rightharpoonup u \in C$. If $\lambda_n \to 1$ as $n \to \infty$, then $z_n - Tz_n = (1-\lambda_n)z - (1-\lambda_n)Tz_n \to 0$. Hence as $k \to \infty$,

$$z_{n_k} \rightharpoonup u.$$

Opial's Property implies the Demiclosedness Principle, and so

$$u \in Fix(T).$$

Therefore as $k \to \infty$,

$$||z_{n_k} - u||\, \phi(||z_{n_k} - u||) \leq \langle z - u, J_\phi(z_{n_k} - u\rangle \to 0.$$

Since ϕ is strictly increasing, $z_{n_k} \to u$.

We will now show that every weakly convergent subsequence of (z_n) has the same limit.

Suppose that $z_{n_k} \rightharpoonup u$ and $z_{m_j} \rightharpoonup v$. Then by the previous proof , $u, v \in Fix(T)$ and $z_{n_k} \to u$ and $z_{m_j} \to v$. Now for all $y \in Fix(T)$,

$$||z_n - y||\, \phi(||z_n - y||) \leq \langle z - y, J_\phi(z_n - y)\rangle.$$

So

$$||z_{n_k} - y||\, \phi(||z_{n_k} - y||) \leq \langle z - y, J_\phi(z_{n_k} - y)\rangle$$

and

$$||z_{m_j} - y||\, \phi(||z_{m_j} - y||) \leq \langle z - y, J_\phi(z_{m_j} - y)\rangle.$$

Taking limits, we get

$$\|u - v\| \, \phi(\|u - v\|) \leq \langle z - v, J_\phi(u - v)\rangle \tag{2.12}$$

and

$$\|v - u\| \, \phi(\|v - u\|) \leq \langle z - u, J_\phi(v - u)\rangle \tag{2.13}$$

Adding equations 2.12 and 2.13 gives

$$\begin{aligned} 2\|u - v\| \, \phi(\|u - v\|) &\leq \langle u - v, J_\phi(u - v)\rangle \\ &= \|u - v\| \, \phi(\|u - v\|). \end{aligned}$$

So $\|u - v\| \, \phi(\|u - v\|) \leq 0$ and hence $u = v$.

Therefore every weakly convergent subsequence of (z_n) converges strongly to some unique limit Pz. Thus every subsequence of (z_t) has a subsequence converging to Pz. Hence by Lemma 2.0.2, (z_t) converges to Pz. □

Theorem 2.8.11. *Let X be a reflexive Banach space having a weakly continuous duality map J_ϕ. Let C be a nonempty, closed and convex subset of X and let $T : C \to C$ be nonexpansive with $Fix(T) \neq \emptyset$. Then there exists a unique sunny nonexpansive retraction $Q : C \to Fix(T)$.*

Proof. We note that since J_ϕ is weakly continuous, X must be smooth. Also by Lemma 2.8.10, X has Reich's Property. Then Theorem 2.8.8 ensures the existence of a sunny nonexpansive retraction defined by $Q(z) = \lim_{t \to 0^+} z_t$ where $z_t = tz + (1 - t)Tz_t$ for $0 < t < 1$.

For the proof of uniqueness, let $Q : C \to Fix(T)$ be any sunny nonexpansive retraction. Fix any $y \in C$, define a sequence (x_n) as follows: for $x_0 = y \in C$,

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T x_n, \qquad n \geq 0,$$

where $(\lambda_n) \subset (0,1)$ satisfies the following conditions:

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+1}} = 1.$$

Then it shown in Theorem 4.4 (chapter 4) that $x_n \longrightarrow Qy$ for the case of $N = 1$. But $Q : C \to Fix(T)$ was any arbitrary sunny nonexpansive map and $y \in C$ was arbitrary. Hence there can be only one sunny nonexpansive retraction from C onto $Fix(T)$. $\square$

Chapter 3

Iterative Methods in Hilbert Spaces

Throughout this chapter we work only in a real Hilbert space. We are concerned with the convergence properties of the following algorithm:

> Let H be a Hilbert space, C a nonempty, closed and convex subset of H and $T_n : C \to C$ a nonexpansive map for each $n = 1, 2, \dots$. For $x_0, y \in C$ define a sequence (x_n) in C by the iterative relationship
>
> $$x_{n+1} = \lambda_{n+1}\, y + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad n = 0, 1, 2, \dots$$
>
> where (λ_n) is a sequence of control parameters.

In this chapter, we generalize the results of Shimizu and Takahashi [33], and Bauschke [2]. The following equality is a generalization of the equality that is used in the proof of Lemma 1 in Shimizu and Takahashi [33]. Here the equality holds for any quantity that is defined as an average of some sort.

Theorem 3.1. *Let H be a Hilbert space and $\{x_\alpha\}_{\alpha\in I} \subseteq H$, for some index set I having K elements. If $y = \frac{1}{K}\sum_{\alpha\in I} x_\alpha$, then for any $v \in H$,*

$$||y-v||^2 = \frac{1}{K}\sum_{\alpha\in I}||x_\alpha - v||^2 - \frac{1}{K}\sum_{\alpha\in I}||x_\alpha - y||^2.$$

Proof.

$$\begin{aligned}
\frac{1}{K}\sum_{\alpha\in I}||x_\alpha - v||^2 - \frac{1}{K}\sum_{\alpha\in I}||x_\alpha - y||^2 &= \frac{1}{K}\sum_{\alpha\in I}[\,||x_\alpha - v||^2 - ||x_\alpha - y||^2] \\
&= \frac{1}{K}\sum_{\alpha\in I}[\,||x_\alpha - y||^2 + ||y-v||^2 \\
&\quad +2\langle x_\alpha - y, y-v\rangle - ||x_\alpha - y||^2] \\
&= \frac{1}{K}\sum_{\alpha\in I}[\,||y-v||^2 + 2\langle x_\alpha - y, y-v\rangle] \\
&= \frac{1}{K}K||y-v||^2 + \frac{2}{K}\sum_{\alpha\in I}\langle x_\alpha - y, y-v\rangle \\
&= ||y-v||^2 + 2\langle \frac{1}{K}\sum_{\alpha\in I}x_\alpha - \frac{1}{K}\sum_{\alpha\in I}y, y-v\rangle \\
&= ||y-v||^2 + 2\langle y - \frac{1}{K}Ky, y-v\rangle \\
&= ||y-v||^2 + 2\langle 0, y-v\rangle \\
&= ||y-v||^2.
\end{aligned}$$

□

The following lemma, whose proof we do not include, is due to Shimizu and Takahashi [33]. It is required in the proof of a later result.

Lemma 3.2 ([33]). *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H, and let $S : C \to C$ and $T : C \to C$ be nonexpansive maps such that*

$ST = TS$. For $x \in C$, put

$$T_n(x) = \frac{2}{n(n+1)} \sum_{k=0}^{n-1} \sum_{i+j=k} S^i T^j x.$$

Then

$$\lim_{n \to \infty} \sup_{x \in C} \|T_n x - S(T_n x)\| = 0$$

and

$$\lim_{n \to \infty} \sup_{x \in C} \|T_n x - T(T_n x)\| = 0.$$

Leading on from Lemma 3.2 we have the following result, for which we include the proof and this also gives an idea of the proof of Lemma 3.2.

Lemma 3.3. *Let C be a nonempty, bounded, closed and convex subset of a Hilbert space H and let $T : C \to C$ be nonexpansive. For each $x \in C$, if we define*

$$T_n(x) = \frac{1}{n} \sum_{j=0}^{n-1} T^j(x),$$

then

$$\lim_{n \to \infty} \sup_{x \in C} \|T_n x - T(T_n x)\| = 0.$$

Proof. For $x \in C$, let $y_n = T_n x$. Then we have by Lemma 3.1 and by the nonexpansivity of T that

$$\begin{aligned}
\|T_n x - T(T_n x)\|^2 &= \|y_n - Ty_n\|^2 \\
&= \frac{1}{n}\sum_{j=0}^{n-1} \|T^j(x) - Ty_n\|^2 - \frac{1}{n}\sum_{j=0}^{n-1} \|T^j(x) - y_n\|^2 \\
&= \frac{1}{n}\sum_{j=1}^{n-1} \|T^j(x) - Ty_n\|^2 + \frac{1}{n}\|x - Ty_n\|^2 - \frac{1}{n}\sum_{j=0}^{n-1} \|T^j(x) - y_n\|^2 \\
&\leq \frac{1}{n}\sum_{j=1}^{n-1} \|T^{j-1}(x) - y_n\|^2 + \frac{1}{n}\|x - Ty_n\|^2 \\
&\quad -\frac{1}{n}\sum_{j=0}^{n-1} \|T^j(x) - y_n\|^2 \\
&= \frac{1}{n}\sum_{j=0}^{n-2} \|T^j(x) - y_n\|^2 - \frac{1}{n}\sum_{j=0}^{n-1} \|T^j(x) - y_n\|^2 + \frac{1}{n}\|x - Ty_n\|^2 \\
&= \frac{1}{n}\|x - Ty_n\|^2 - \frac{1}{n}\|T^{n-1}(x) - y_n\|^2 \\
&\leq \frac{1}{n}\|x - Ty_n\|^2 \\
&\leq \frac{1}{n}(\operatorname{diam}(C))^2.
\end{aligned}$$

Thus $\sup_{x\in C} \|T_n x - T(T_n(x))\| \leq \frac{1}{\sqrt{n}} \operatorname{diam}(C)$, and so

$$\lim_{n\to\infty} \sup_{x\in C} \|T_n x - T(T_n(x))\| = 0.$$

□

The following result is a strong convergence result in a Hilbert space which generalizes Theorem 1 of Shimizu and Takahashi [33].

Theorem 3.4. *Let* $(\lambda_n) \subseteq (0,1)$ *satisfy* $\lim_{n\to\infty} \lambda_n = 0$ *and* $\sum_{n=1}^{\infty} \lambda_n = \infty$. *Let* C *be a nonempty, closed and convex subset of a Hilbert space* H *and let*

$T_n : C \to C$, $(n = 1, 2, 3, \ldots)$, *be nonexpansive mappings such that* $F := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. *Assume that*

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T_n x - V_k(T_n x)\| = 0 \qquad \text{for all } k = 1, 2, \ldots N,$$

where $\tilde{C}$ *is any bounded subset of* C *and* $V_k : C \to C$ *are nonexpansive mappings* $(k = 1, 2, \ldots, N)$ *with* $F \supseteq \bigcap_{k=1}^{N} Fix(V_k) \neq \emptyset$. *For* $x_0 \in C$ *and* $y \in C$ *define*

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad n \geq 0.$$

Then $x_n \to Py$ *where* P *is the projection from* C *onto* $\bigcap_{k=1}^{N} Fix(V_k)$.

Proof. We proceed with the following steps.

(1) $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and for all $f \in F$:

We use an inductive argument. The result is clearly true for $n = 0$. Suppose the result is true for n. Let $f \in F$. Then by the nonexpansivity of T_{n+1},

$$\begin{aligned}
\|x_{n+1} - f\| &= \|\lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n - f\| \\
&= \|\lambda_{n+1}(y - f) + (1 - \lambda_{n+1})(T_{n+1} x_n - f)\| \\
&\leq \lambda_{n+1}\|y - f\| + (1 - \lambda_{n+1})\|T_{n+1} x_n - f\| \\
&\leq \lambda_{n+1}\|y - f\| + (1 - \lambda_{n+1})\|x_n - f\| \\
&\leq \lambda_{n+1} \max\{\|x_0 - f\|, \|y - f\|\} + (1 - \lambda_{n+1}) \max\{\|x_0 - f\|, \|y - f\|\} \\
&= \max\{\|x_0 - f\|, \|y - f\|\}.
\end{aligned}$$

(2) (x_n) is bounded:

For all $n \geq 0$ and for any $f \in F$,

$$\begin{aligned}
\|x_n\| &\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|.
\end{aligned}$$

(3) $(T_{n+1}x_n)$ is bounded:

For all $n \geq 0$ and for any $f \in F$,

$$\begin{aligned} \|T_{n+1}x_n\| &\leq \|T_{n+1}x_n - f\| + \|f\| \\ &\leq \|x_n - f\| + \|f\| \\ &\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|. \end{aligned}$$

(4) $\overline{\lim}\,\langle T_{n+1}x_n - Py, y - Py\rangle \leq 0$:

By step (3), $\langle T_{n+1}x_n - Py, y - Py\rangle$ is bounded, hence $\overline{\lim}\,\langle T_{n+1}x_n - Py, y - Py\rangle$ exists. Thus we can find a subsequence (n_j) of (n) such that

$$\overline{\lim}\,\langle T_{n+1}x_n - Py, y - Py\rangle = \lim_j \langle T_{n_j+1}x_{n_j} - Py, y - Py\rangle$$

and

$$T_{n_j+1}x_{n_j} \rightharpoonup p \qquad \text{for some } p \in C.$$

By our assumption, we have for any $k = 1, 2, \ldots, N$ and for $\tilde{C} = \{x_n\}_{n\in\mathbb{N}}$,

$$\begin{aligned} 0 = \lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T_{n+1}x - V_k(T_{n+1}x)\| &\geq \lim_{n\to\infty} \|T_{n+1}x_n - V_k(T_{n+1}x_n)\| \\ &= \lim_{j\to\infty} \|T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j})\|. \end{aligned}$$

Thus

$$\lim_j \|T_{n_j+1}x_{n_j} - V_k(T_{n_j+1}x_{n_j})\| = 0 \qquad \text{for all } k = 1, 2, \ldots, N.$$

So $p \in Fix(V_k)$ for $k = 1, 2, \ldots, N$ by the Demiclosedness Principle; i.e. $p \in \bigcap_{k=1}^{N} Fix(V_k)$. It follows, by Lemma 2.4.3 that

$$\overline{\lim}\,\langle T_{n+1}x_n - Py, y - Py\rangle = \langle p - Py, y - Py\rangle \leq 0.$$

By step (4) and the fact that $\lambda_n \to 0$, we have for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that

$$\langle T_{n+1}x_n - Py, y - Py\rangle \leq \frac{\epsilon}{4}$$

and

$$\lambda_{n+1}\|y - Py\|^2 < \frac{\epsilon}{2}.$$

Hence we have for all $n \geq N$,

$$\begin{aligned}
\|x_{n+1} - Py\|^2 &= \|(1-\lambda_{n+1})(T_{n+1}x_n - Py) + \lambda_{n+1}(y - Py)\|^2 \\
&= (1-\lambda_{n+1})^2\|T_{n+1}x_n - Py\|^2 + {\lambda_{n+1}}^2\|y - Py\|^2 \\
&\quad +2\lambda_{n+1}(1-\lambda_{n+1})\langle T_{n+1}x_n - Py, y - Py\rangle \\
&\leq (1-\lambda_{n+1})\|x_n - Py\|^2 + {\lambda_{n+1}}^2\|y - Py\|^2 \\
&\quad +2\lambda_{n+1}(1-\lambda_{n+1})\langle T_{n+1}x_n - Py, y - Py\rangle \\
&\leq (1-\lambda_{n+1})\|x_n - Py\|^2 \\
&\quad +\lambda_{n+1}[\,\lambda_{n+1}\|y - Py\|^2 + 2\langle T_{n+1}x_n - Py, y - Py\rangle\,] \\
&< (1-\lambda_{n+1})\|x_n - Py\|^2 + \lambda_{n+1}\left(\frac{\epsilon}{2} + \frac{\epsilon}{2}\right) \\
&\leq (1-\lambda_{n+1})\|x_n - Py\|^2 + \lambda_{n+1}\epsilon.
\end{aligned}$$

By Lemma 2.1.2 , $x_n \to Py$. □

Theorem 1 of [33] now comes out as a corollary to Theorem 3.4, as is seen in Corollary 3.5(b).

Corollary 3.5. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $S : C \to C$ and $T : C \to C$ be nonexpansive mappings with $ST = TS$ and $Fix(T) \cap Fix(S) \neq \phi$. Suppose that $(\lambda_n) \subseteq (0, 1)$ satisfies*

$$\lim_{n\to\infty} \lambda_n = 0 \qquad \textit{and} \qquad \sum_{n=1}^{\infty} \lambda_n = \infty.$$

Suppose that T_n is defined as either

$$\text{(a) } T_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} T^j x, \qquad n \geq 1$$

or

$$\text{(b) } T_n(x) = \frac{2}{n(n+1)}\sum_{k=0}^{n-1}\sum_{i+j=k} S^i T^j(x), \qquad n \geq 1.$$

For $x_0 \in C$ and $y = x_0$, define

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad n \geq 0.$$

Then x_n converges to Py where P is the projection from C onto $Fix(S) \cap Fix(T)$ (i.e. $Py = Proj_{Fix(s)\cap Fix(T)}(y)$).

Proof. Let $w \in Fix(S) \cap Fix(T)$, put $r = \|w - y\|$ and let $D = \{x \in C : \|x - w\| \leq r\}$. Now D is nonempty ($y, w \in D$), closed, bounded and convex, and it is S and T invariant. We may thus assume that S and T are mappings from D to D, and hence so is T_n (defined by either (a) or (b)).

By Lemma 3.2 and Lemma 3.3,

$$\lim_{n\to\infty} \sup_{x\in D} \|T_n x - V(T_n x)\| = 0.$$

where V is T if T_n is defined by (a) or V is either S or T if T_n is defined by (b). Theorem 3.4 implies that $x_n \longrightarrow Py$. □

We now turn our attention to a scenario developed in Bauschke [2]. In this paper Bauschke defined the following control conditions on the parameters (λ_n):

[B1] $\lim_{n\to\infty} \lambda_n = 0$.

[B2] $\sum_{n=1}^{\infty} \lambda_n = \infty$.

[B3] $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$.

We will replace [B3] by the condition:

[N3] $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

This condition is new and has not been used in the literature before.

Lions' condition,

[L3] $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$,

implies [N3], and [N3] includes the natural candidate of $\lambda_n = \dfrac{1}{n+1}$. Comparing [B3] and [N3] we find that no one condition is stronger than the other, as is demonstrated in the following example.

Example 3.6.

If (λ_n) is a decreasing sequence with $\lambda_n \longrightarrow 0$, then [B3] always holds for $N = 1$. Indeed,

$$\begin{aligned}
\sum_{k=1}^{n} |\lambda_k - \lambda_{k+1}| &= \sum_{k=1}^{n} (\lambda_k - \lambda_{k+1}) \\
&= (\lambda_1 - \lambda_2) + (\lambda_2 - \lambda_3) + \ldots + (\lambda_n - \lambda_{n+1}) \\
&= \lambda_1 - \lambda_{n+1} \longrightarrow \lambda_1.
\end{aligned}$$

So

$$\sum_{k=1}^{\infty} |\lambda_k - \lambda_{k+1}| < \infty.$$

[B3] $\not\Rightarrow$ [N3]: The sequence (λ_n) with $\lambda_n = \exp(-n^2)$ is a decreasing sequence and so [B3] holds for $N = 1$. Also $\lambda_n \to 0$. For $N = 1$,

$$\frac{\lambda_n}{\lambda_{n+1}} = \frac{\exp((n+1)^2)}{\exp(n^2)} = \exp(2n+1) \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty,$$

and hence [N3] is not satisfied.

[N3] $\not\Rightarrow$ [B3]: Let (λ_n) be the sequence defined by $\lambda_n = \begin{cases} \frac{1}{\sqrt{n}} & \text{if } n \text{ is odd} \\ \frac{1}{\sqrt{n}-1} & \text{if } n \text{ is even.} \end{cases}$.

Then $\lambda_n \to 0$ and

$$\frac{\lambda_n}{\lambda_{n+1}} = \begin{cases} \frac{\sqrt{n+1}-1}{\sqrt{n}} & \text{if n is odd} \\ \frac{\sqrt{n+1}}{\sqrt{n}-1} & \text{if n is even.} \end{cases}$$

which converges to 1 as $n \longrightarrow \infty$. Thus [N3] is satisfied for $N = 1$, but [B3] is not. Indeed, if n is odd, then

$$\begin{aligned}
|\lambda_n - \lambda_{n+1}| &= \left| \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}-1} \right| \\
&= \frac{|\sqrt{n+1} - 1 - \sqrt{n}|}{\sqrt{n}(\sqrt{n+1}-1)} \\
&= \frac{|\sqrt{n+1} - \sqrt{n} - 1|}{\sqrt{n}(\sqrt{n+1}-1)} \\
&= \frac{\left| \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} - 1 \right|}{\sqrt{n}(\sqrt{n+1}-1)} \\
&= \frac{1 - \frac{1}{\sqrt{n+1}+\sqrt{n}}}{\sqrt{n}(\sqrt{n+1}-1)} \\
&= O(\frac{1}{n})
\end{aligned}$$

and if n is even, then

$$\begin{aligned}
|\lambda_n - \lambda_{n+1}| &= \left|\frac{1}{\sqrt{n}-1} - \frac{1}{\sqrt{n+1}}\right| \\
&= \frac{|\sqrt{n+1} - \sqrt{n} + 1|}{\sqrt{n+1}(\sqrt{n}-1)} \\
&= \frac{\frac{1}{\sqrt{n+1}+\sqrt{n}} + 1}{\sqrt{n+1}(\sqrt{n}-1)} \\
&= O(\frac{1}{n})
\end{aligned}$$

where $u_n = O(\frac{1}{n})$ means that $\lim\limits_{n\to\infty} \frac{u_n}{\frac{1}{n}} \neq 0$. Thus $|\lambda_n - \lambda_{n+1}| = O(\frac{1}{n})$ and since $\sum\limits_{n=1}^{\infty} \frac{1}{n} = \infty$ we have $\sum\limits_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| = \infty$. ◊

We will now prove a complementary result to Theorem 3.1 of Bauschke [2], with condition [B3] replaced by condition [N3].

We consider N maps $T_1, T_2, \ldots, T_N$. For any $n \geq 1$,

$$T_n := T_{n \mod N},$$

where $n \mod N$ is defined as follows: if $n = kN + l,\quad 0 \leq l < N$, then

$$n \mod N := \begin{cases} l & \text{if } l \neq 0 \\ N & \text{if } l = 0. \end{cases}$$

Theorem 3.7. *Let C be a nonempty, closed and convex subset of a Hilbert space H and let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings of C into C with $F := \bigcap\limits_{i=1}^{N} Fix(T_i) \neq \emptyset$ and*

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_2) = \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Let $(\lambda_n) \subseteq (0,1)$ satisfy the following conditions:

[*N1*] $\lim_{n\to\infty} \lambda_n = 0.$

[*N2*] $\sum_{n=1}^{\infty} \lambda_n = \infty.$

[*N3*] $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1.$

Given points $x_0, y \in C$, the sequence $(x_n) \subseteq C$ is defined by

$$x_{n+1} = \lambda_{n+1} y + (1-\lambda_{n+1}) T_{n+1} x_n, \qquad n \geq 0.$$

Then $x_n \longrightarrow P_F y$ where P_F is the projection of C onto F.

Proof. We proceed with the following steps in the proof.

(1) $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and for all $f \in F$:

We prove this by induction. It is clearly true for $n = 0$.

Assume that $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$. Then

$$\begin{aligned}
\|x_{n+1} - f\| &= \|\lambda_{n+1}(y - f) + (1-\lambda_{n+1})(T_{n+1}x_n - f)\| \\
&\leq \lambda_{n+1}\|y - f\| + (1-\lambda_{n+1})\|T_{n+1}x_n - f\| \\
&\leq \lambda_{n+1}\|y - f\| + (1-\lambda_{n+1})\|x_n - f\| \\
&\leq \lambda_{n+1}\max\{\|x_0 - f\|, \|y - f\|\} + (1-\lambda_{n+1})\max\{\|x_0 - f\|, \|y - f\|\} \\
&= \max\{\|x_0 - f\|, \|y - f\|\}.
\end{aligned}$$

Hence by induction, $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and all $f \in F$.

(2) (x_n) is bounded:

For any $f \in F$ and for all $n \geq 0$,

$$\begin{aligned}
\|x_n\| &\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|.
\end{aligned}$$

(3) $(T_{n+1}x_n)$ is bounded:

For any $f \in F$ and for all $n \geq 0$,

$$\begin{aligned} \|T_{n+1}x_n\| &\leq \|T_{n+1}x_n - f\| + \|f\| \\ &\leq \|x_n - f\| + \|f\| \\ &\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|. \end{aligned}$$

Thus $(T_{n+1}x_n)$ is bounded.

(4) $x_{n+1} - T_{n+1}x_n \longrightarrow 0$:

$$\begin{aligned} \|x_{n+1} - T_{n+1}x_n\| &= \lambda_{n+1}\|y - T_{n+1}x_n\| \\ &\leq \lambda_{n+1}(\|y\| + \|T_{n+1}x_n\|) \\ &\leq \lambda_{n+1}(\|y\| + M), \qquad \text{for some } M. \end{aligned}$$

Since $\lambda_{n+1} \longrightarrow 0$,

$$x_{n+1} - T_{n+1}x_n \longrightarrow 0.$$

(5) $x_{n+N} - x_n \longrightarrow 0$:

By (2) and (3), there exists a constant $L > 0$ such that for all $n \geq 1$,

$$\|y - T_{n+1}x_n\| \leq L.$$

Since for all $n \geq 1, T_{n+N} = T_n$, we have

$$\begin{aligned} &\|x_{n+N} - x_n\| \\ =\ &\|(\lambda_{n+N} - \lambda_n)(y - T_{n+N}x_{n+N-1}) + (1 - \lambda_{n+N})(T_n x_{n+N-1} - T_n x_{n-1})\| \\ \leq\ &L|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| \\ =\ &(1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| + \lambda_{n+N}L\left\|1 - \frac{\lambda_n}{\lambda_{n+N}}\right\|. \end{aligned}$$

By [N3], we have $\lim_{n\to\infty} L \left\|1 - \frac{\lambda_n}{\lambda_{n+N}}\right\| = 0$ and so by Lemma 2.1.2 ,

$$x_{n+N} - x_n \longrightarrow 0.$$

(6) $x_n - T_{n+N} \cdots T_{n+1}x_n \longrightarrow 0$:

Noting (5), it suffices to show that

$$x_{n+N} - T_{n+N} \cdots T_{n+1}x_n \longrightarrow 0$$

By (4),

$$x_{n+N} - T_{n+N}x_{n+N-1} \longrightarrow 0.$$

Again by (4),

$$x_{n+N-1} - T_{n+N-1}x_{n+N-2} \longrightarrow 0.$$

Using the nonexpansivity of T_{n+N} we get

$$T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} \longrightarrow 0.$$

Using (4) again, we have

$$x_{n+N-2} - T_{n+N-2}x_{n+N-3} \longrightarrow 0,$$

and by the nonexpansivity of T_{n+N} and T_{n+N-1}, we get

$$T_{n+N}T_{n+N-1}x_{n+N-2} - T_{n+N}T_{n+N-1}T_{n+N-2}x_{n+N-3} \longrightarrow 0.$$

Continuing in this way we get

$$\vdots$$

$$T_{n+N}T_{n+N-1} \cdots T_{n+2}x_{n+1} - T_{n+N}T_{n+N-1} \cdots T_{n+1}x_n \longrightarrow 0.$$

Adding these N sequences yields

$$x_{n+N} - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0.$$

(7) $\overline{\lim}\, \langle T_{n+1}x_n - P_F y, y - P_F y\rangle \leq 0$:

By (3), $\langle T_{n+1}x_n - P_F y, y - P_F y\rangle$ is bounded, hence $\overline{\lim}\, \langle T_{n+1}x_n - P_F y, y - P_F y\rangle$ exists. We can find a subsequence (n_j) of (n) so that

$$\overline{\lim}\, \langle T_{n+1}x_n - P_F y, y - P_F y\rangle = \lim_{j\to\infty} \langle T_{n_j+1}x_{n_j} - P_F y, y - P_F y\rangle,$$

$$(n_j + 1) \mod N = i \qquad \text{for some } i \text{ and for all } j \geq 1$$

and

$$x_{n_j+1} \rightharpoonup x \qquad \text{for some } c \in C.$$

By (6)

$x_{n_j+1} - T_{i+N} \cdots T_{i+1} x_{n_j+1} = x_{n_j+1} - T_{n_j+N+1} \cdots T_{n_j+2} x_{n_j+1} \longrightarrow 0$. By the Demiclosedness Principle, $x \in Fix(T_{i+N} \cdots T_{i+1}) = F$. Thus, by (4) and Lemma 2.4.3 we obtain,

$$\begin{aligned}
\overline{\lim}\, \langle T_{n+1}x_n - P_F y, y - P_F y\rangle &= \lim_{j\to\infty} \langle T_{n_j+1}x_{n_j} - P_F y, y - P_F y\rangle \\
&= \lim_{j\to\infty} \langle T_{n_j+1}x_{n_j} - x_{n_j+1}, y - P_F y\rangle \\
&\quad + \lim_{j\to\infty} \langle x_{n_j+1} - P_F y, y - P_F y\rangle \\
&= 0 + \langle x - P_F y, y - P_F y\rangle \\
&\leq 0.
\end{aligned}$$

Fix any $\epsilon > 0$. By (7) and [N1] we can find $N_\epsilon \in \mathbb{N}$ so that for all $n \geq N_\epsilon$,

$$\langle T_{n+1}x_n - P_F y, y - P_F y\rangle < \frac{\epsilon}{4}$$

and

$$\lambda_{n+1}\|y - P_F y\|^2 < \frac{\epsilon}{2}.$$

Then for all $n \geq N_\epsilon$,

$$\begin{aligned}
\|x_{n+1} - P_F y\|^2 &= \|\lambda_{n+1}(y - P_F y) + (1-\lambda_{n+1})(T_{n+1}x_n - P_F y)\|^2 \\
&= \lambda_{n+1}^2\|y - P_F y\|^2 + (1-\lambda_{n+1})^2\|T_{n+1}x_n - P_F y\|^2 \\
&\quad +2\lambda_{n+1}(1-\lambda_{n+1})\langle y - P_F y, T_{n+1}x_n - P_F y\rangle \\
&\leq \lambda_{n+1}(\lambda_{n+1}\|y - P_F y\|^2) + (1-\lambda_{n+1})\|T_{n+1}x_n - P_F y\|^2 \\
&\quad +2\lambda_{n+1}\langle T_{n+1}x_n - P_F y, y - P_F y\rangle \\
&\leq \lambda_{n+1}\frac{\epsilon}{2} + (1-\lambda_{n+1})\|x_n - P_F y\|^2 + \lambda_{n+1}\frac{\epsilon}{2} \\
&= (1-\lambda_{n+1})\|x_n - P_F y\|^2 + \lambda_{n+1}\epsilon
\end{aligned}$$

By Lemma 2.1.2 , $x_n \longrightarrow P_F y$. □

Chapter 4

Iterative Methods in Banach Spaces

In this chapter we work only in a Banach spaces and we again consider the following iteration scheme

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad n \geq 0.$$

In a Hilbert space the concepts of a projection and the inner product exist, and they give us nice inequalities. We do not have these in a Banach space, but we do have the duality maps and the existence of sunny nonexpansive retractions in certain spaces that give us the inequalities that are analagous to those found in Hilbert spaces. For example, the nice property of projections (Lemma 2.4.3) also holds for sunny nonexpansive retractions (Theorem 2.8.2). Since the normalized duality map in a Hilbert space is the identity map, every projection is a sunny nonexpansive retraction. Therefore, in certain "nice" Banach spaces, we have convergence of the iterates.

Bauschke [2] in his proof of his main result provides an algorithmic proof to obtain his result. This technique has proved to be extremely useful. With appropriate modifications this algorithm has been effectively used by many authors to obtain convergence, for example in [33], [39] and [13]. We use this technique in Theorems 3.4 and 3.7, and in this chapter in Theorems 4.1, 4.2 and 4.4.

The following result generalizes Theorem 3.4 to Banach spaces, and hence it extends Theorem 1 of [33] to Banach spaces. It also complements the main result of [1] where they only consider the mapping $T_n = \frac{1}{n^2}\sum_{i,j=0}^{n-1} S^i T^j$ in a uniformly convex Banach space, and obtain weak convergence of the iterates, whereas Theorem 4.1 gets strong convergence. From Theorem 2.8.9, the existence of a unique sunny nonexpansive retraction is guaranteed in a uniformly smooth Banach space. This fact is used in the following two theorems.

Theorem 4.1. *Let X be a uniformly smooth Banach that has a weakly continuous duality map J. Let $(\lambda_n) \subseteq (0,1)$ satisfy*

[N1] $\lim_{n\to\infty} \lambda_n = 0.$

[N2] $\sum_{n=1}^{\infty} \lambda_n = \infty.$

Let C be a nonempty, closed and convex subset of X and let $T_n : C \to C$ be nonexpansive mappings $(n = 1, 2, 3, \ldots)$ such that $F := \bigcap_{n=1}^{\infty} Fix(T_n) \neq \emptyset$. Assume that

$$\lim_{n\to\infty} \sup_{x\in\tilde{C}} \|T_n(x) - T(T_n(x))\| = 0$$

where $T : C \to C$ is nonexpansive with $\emptyset \neq Fix(T) \subseteq F$ and $\tilde{C}$ is any bounded subset of C. For $x_0, y \in C$, define

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad n \geq 0.$$

Then $x_n \longrightarrow Qy$ where Q is the unique sunny nonexpansive retraction from C onto $Fix(T)$.

Proof. Since a uniformly smooth Banach space is reflexive and smooth (Lemmas 2.3.10 and 2.3.9), Theorems 2.8.9 and 2.8.8 defines the unique sunny retraction $Q : C \to Fix(T)$ by $Q(z) = \lim_{t \to 0^+} z_t$ where z_t $(0 < t < 1)$ is the unique fixed point of

$$S_t(x) = tz + (1-t)Tx, \qquad x \in C.$$

(1) $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and for all $f \in F$:
We use an inductive argument. The result is clearly true for $n = 0$. Suppose the result is true for n. Then by the nonexpansivity of T_{n+1},

$$\begin{aligned}
\|x_{n+1} - f\| &= \|\lambda_{n+1}y + (1-\lambda_{n+1})T_{n+1}x_n - f\| \\
&= \|\lambda_{n+1}(y - f) + (1-\lambda_{n+1})(T_{n+1}x_n - f)\| \\
&\leq \lambda_{n+1}\|y - f\| + (1-\lambda_{n+1})\|T_{n+1}x_n - f\| \\
&\leq \lambda_{n+1}\|y - f\| + (1-\lambda_{n+1})\|x_n - f\| \\
&\leq \lambda_{n+1}\max\{\|x_0 - f\|, \|y - f\|\} + (1-\lambda_{n+1})\max\{\|x_0 - f\|, \|y - f\|\} \\
&= \max\{\|x_0 - f\|, \|y - f\|\}.
\end{aligned}$$

(2) (x_n) is bounded:
For each $f \in F$ and for each $n \geq 0$

$$\begin{aligned}
\|x_n\| &\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|.
\end{aligned}$$

(3) $(T_{n+1}x_n)$ is bounded:
For any $f \in F$ and for all $n \geq 0$,

$$\begin{aligned}
\|T_{n+1}x_n\| &\leq \|T_{n+1}x_n - f\| + \|f\| \\
&\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0 - f\|, \|y - f\|\} + \|f\|.
\end{aligned}$$

(4) $||x_{n+1} - T_{n+1}x_n|| \longrightarrow 0$:

By (3) we can find an $M > 0$ so that

$$\begin{aligned} ||x_{n+1} - T_{n+1}x_n|| &= \lambda_{n+1}||y - T_{n+1}x_n|| \\ &\leq \lambda_{n+1}(||y|| + ||T_{n+1}x_n||) \\ &\leq \lambda_{n+1}M \\ &\longrightarrow 0 \end{aligned}$$

by [N1].

(5) $x_n - Tx_n \longrightarrow 0$:

For $\tilde{C} = \{x_n\}_{n\geq 1}$, we have

$$\begin{aligned} ||x_n - Tx_n|| &\leq ||x_n - T_nx_{n-1}|| + ||T_nx_{n-1} - T(T_nx_{n-1})|| + ||T(T_nx_{n-1}) - Tx_n|| \\ &\leq ||x_n - T_nx_{n-1}|| + ||T_nx_{n-1} - T(T_nx_{n-1})|| + ||T_nx_{n-1} - x_n|| \\ &= 2||x_n - T_nx_{n-1}|| + ||T_nx_{n-1} - T(T_nx_{n-1})|| \\ &\leq 2||x_n - T_nx_{n-1}|| + \sup_{x\in\tilde{C}} ||T_nx - T(T_nx)|| \end{aligned}$$

which converges to 0 by (4) and our hypothesis. Therefore

$$x_n - Tx_n \longrightarrow 0.$$

(6) $\overline{\lim}\, \langle y - Q(y), J(x_n - Q(y))\rangle \leq 0$:

We may assume that there exists subsequence (n_j) of (n) such that

$$x_{n_j} \rightharpoonup u$$

and

$$\overline{\lim}\, \langle y - Q(y), J(x_n - Q(y))\rangle = \lim_j \langle y - Q(y), J(x_{n_j} - Q(y))\rangle.$$

By (5), $x_{n_j} - Tx_{n_j} \longrightarrow 0$. The Demiclosedness Principle holds in a space that has a weakly continuous duality map by Theorems 2.7.6 and 2.7.4. Hence $u \in Fix(T)$. Therefore

$$\begin{aligned}
\overline{\lim}\, \langle y - Q(y), J(x_n - Q(y)) \rangle &= \lim_j \langle y - Q(y), J(x_{n_j} - Q(y)) \rangle \\
&= \langle y - Q(y), J(u - Q(y)) \rangle \\
&\leq 0
\end{aligned}$$

by Theorem 2.8.2 .

Using the subdifferential inequality (2.8)

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, J(x + y) \rangle$$

we get

$$\begin{aligned}
\Phi(\|x_n - Q(y)\|) &= \Phi(\|\lambda_n(y - Q(y)) + (1 - \lambda_n)\,(T_n x_{n-1} - Q(y))\|\,) \\
&\leq (1 - \lambda_n)\,\Phi(\|x_{n-1} - Q(y)\|) + \lambda_n \langle y - Q(y), J(x_n - Q(y)) \rangle.
\end{aligned}$$

Since $\overline{\lim}\, \langle y - Q(y), J(x_n - Q(y)) \rangle \leq 0$ by (6), Lemma 2.1.2 gives us $x_n \longrightarrow Q(y)$. $\square$

The next result is an extension of Bauschke's Theorem 3.1 [2] to Banach spaces, if we use the first assumption of condition [N3]. The second assumption of condition [N3] extends Theorem 3.7 to Banach spaces.

We will again define T_n as follows for any $n \geq 1$:

$$T_n := T_{n \mod N}$$

where $n \mod N$ is defined as follows: if $n = kN + \ell,\ 0 \leq \ell < N$, then

$$n \mod N = \begin{cases} \ell & \text{if } \ell \neq 0 \\ N & \text{if } \ell = 0. \end{cases}$$

Theorem 4.2. *Let X be a uniformly smooth Banach space and let C be a nonempty, closed and convex subset of X. Let $T_i : C \to C$, $(i = 1, 2, \ldots, N)$ be nonexpansive mappings and assume that $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Assume also that*

$$F = Fix(T_N T_{N-1} \cdots T_2 T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Let $(\lambda_n) \subseteq (0,1)$ satisfy the following conditions:

[*N1*] $\lim_{n\to\infty} \lambda_n$.

[*N2*] $\sum_{n=1}^{\infty} \lambda_n = \infty$.

[*N3*] $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty \quad or \quad \lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1.$

If the iterative process is defined by

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n, \quad y, x_0 \in C,$$

then $x_n \longrightarrow Q(y)$, where Q is the unique sunny nonexpansive retraction from C onto F.

Proof. We proceed with the following steps.

(1) $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and for all $f \in F$:

We use an inductive argument. The result is clearly true for $n = 0$. Suppose the result is true for n. Then by the nonexpansivity of T_{n+1},

$$\begin{aligned}
\|x_{n+1} - f\| &= \|\lambda_{n+1}y + (1-\lambda_{n+1})T_{n+1}x_n - f\| \\
&= \|\lambda_{n+1}(y-f) + (1-\lambda_{n+1})(T_{n+1}x_n - f)\| \\
&\leq \lambda_{n+1}\|y-f\| + (1-\lambda_{n+1})\|T_{n+1}x_n - f\| \\
&\leq \lambda_{n+1}\|y-f\| + (1-\lambda_{n+1})\|x_n - f\| \\
&\leq \lambda_{n+1}\max\{\|x_0 - f\|, \|y-f\|\} + (1-\lambda_{n+1})\max\{\|x_0-f\|, \|y-f\|\} \\
&= \max\{\|x_0 - f\|, \|y-f\|\}.
\end{aligned}$$

(2) (x_n) is bounded:

For each $f \in F$ and all $n \geq 0$,

$$\begin{aligned}
\|x_n\| &\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0 - f\|, \|y-f\|\} + \|f\|.
\end{aligned}$$

(3) $(T_{n+1}x_n)$ is bounded:

For all $n \geq 0$ and for each $f \in F$,

$$\begin{aligned}
\|T_{n+1}x_n\| &\leq \|T_{n+1}x_n - f\| + \|f\| \\
&\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0 - f\|, \|y-f\|\} + \|f\|.
\end{aligned}$$

(4) $x_{n+1} - T_{n+1}x_n \longrightarrow 0$: By (3), we can find $M > 0$ such that

$$\begin{aligned}
\|x_{n+1} - T_{n+1}x_n\| &= \lambda_{n+1}\|y - T_{n+1}x_n\| \\
&\leq \lambda_{n+1}(\|y\| + \|T_{n+1}x_n\|) \\
&\leq \lambda_{n+1}M.
\end{aligned}$$

By assumption [N1], $x_{n+1} - T_{n+1}x_n \longrightarrow 0$.

(5) $x_{n+N} - x_n \longrightarrow 0$:

By (2) and (3), we can find a constant M so that for all $n \geq 0$,

$$\|x_{n+N} - x_n\| \leq M$$

and

$$\|y - T_{n+1}x_n\| \leq M.$$

Noting that $T_{n+N} = T_n$ we have for $n \geq 0$,

$$\begin{aligned}
& \|x_{n+N} - x_n\| \\
= \;& \|(\lambda_{n+N} - \lambda_n)(y - T_n x_{n-1}) + (1 - \lambda_{n+N})(T_{n+N}x_{n+N-1} - T_n x_{n-1})\| \\
\leq \;& M|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|T_n x_{n+N-1} - T_n x_{n-1}\| \\
\leq \;& M|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| \qquad (4.1) \\
\leq \;& M|\lambda_{n+N} - \lambda_n| \\
& +(1 - \lambda_{n+N})[\, M|\lambda_{n+N-1} - \lambda_{n-1}| + (1 - \lambda_{n+N-1})\|x_{n+N-2} - x_{n-2}\|\,] \\
= \;& M[\,|\lambda_{n+N} - \lambda_n| + |\lambda_{n+N-1} - \lambda_{n-1}|\,] \\
& +(1 - \lambda_{n+N})(1 - \lambda_{n+N-1})\,\|x_{n+N-2} - x_{n-2}\| \\
\leq \;& \cdots \\
\leq \;& M\sum_{k=m}^{n} |\lambda_{k+N} - \lambda_k| + M\prod_{k=m}^{n}(1 - \lambda_{k+N}). \qquad (4.2)
\end{aligned}$$

Letting $n \to \infty$ in inequality 4.2 yields

$$\overline{\lim}\, \|x_{n+N} - x_n\| \leq M\sum_{k=m}^{\infty} |\lambda_{k+N} - \lambda_k| + M\prod_{k=m}^{\infty}(1 - \lambda_{k+N}). \qquad (4.3)$$

Condition [N2] implies that $\lim\limits_{m\to\infty} \prod\limits_{k=m}^{\infty}(1 - \lambda_{k+N}) = 0$ and the first assumption of [N3]

implies that $\lim_{m\to\infty}\sum_{k=m}^{\infty}|\lambda_{k+N}-\lambda_k|=0.$
If we let $m\to\infty$ in (4.3), we get $x_{n+N}-x_n\longrightarrow 0$ (by using the first assumption of [N3]).

If we now use the second assumption of [N3], then inequality (4.1) is

$$\|x_{n+N}-x_n\|\leq M|\lambda_{n+N}-\lambda_n|+(1-\lambda_{n+N})\|x_{n+N-1}-x_{n-1}\|$$

with $\dfrac{|\lambda_{n+N}-\lambda_n|}{\lambda_{n+N}}\longrightarrow 0$. By Lemma 2.1.2 , $x_{n+N}-x_n\longrightarrow 0.$

(6) $x_n - T_{n+N}\cdots T_{n+1}x_n\longrightarrow 0.$

Noting (5), it is sufficient to show that

$$x_{n+N}-T_{n+N}\cdots T_{n+1}x_n\longrightarrow 0.$$

By (4),

$$x_{n+N}-T_{n+N}x_{n+N-1}\longrightarrow 0.$$

Again by (4),

$$x_{n+N-1}-T_{n+N-1}x_{n+N-2}\longrightarrow 0$$

and hence by the nonexpansivity of T_{n+N},

$$T_{n+N}x_{n+N-1}-T_{n+N}T_{n+N-1}x_{n+N-2}\longrightarrow 0.$$

Similarly,

$$\begin{aligned}
T_{n+N}T_{n+N-1}x_{n+N-2}-T_{n+N}T_{n+N-1}T_{n+N-2}x_{n+N-3} &\longrightarrow 0\\
&\vdots\\
T_{n+N}\cdots T_{n+2}x_{n+1}-T_{n+N}\cdots T_{n+1}x_n &\longrightarrow 0.
\end{aligned}$$

Adding these N sequences yields the desired result.

(7) $\overline{\lim}\,\langle y - Q(y), J(x_n - Q(y))\rangle \leq 0$:

We may choose a subsequence (n_j) of (n) so that

$$\overline{\lim}\,\langle y - Q(y), J(x_n - Q(y))\rangle = \lim_j \langle y - Q(y), J(x_{n_j} - Q(y))\rangle,$$

$$x_{n_j} \rightharpoonup \tilde{x}$$

and

$$T_{n_j} = T_i$$

for some $i \in \{1, 2, \ldots, N\}$ and for all $j \geq 1$.

Then $T_{n_j+N} \cdots T_{n_j+1} = T_{i+N} \cdots T_{i+1}$ for all $j \geq 1$.

Put $\tilde{S} = T_{i+N} \cdots T_{i+1}$. Then $Fix(\tilde{S}) = F$.

Since $\tilde{S}$ is nonexpansive, with $Fix(\tilde{S}) = F \neq \emptyset$, we have by Theorem 2.8.9 that there exists a unique sunny nonexpansive retraction from C onto F.

Now $Q(y) = \lim_{t\to 0} z_t$, where $z_t = ty + (1-t)\tilde{S}z_t$. If J denotes the normalized duality map, then by inequality (2.9)

$$\begin{aligned}
&\|z_t - x_{n_j}\|^2 \\
= \;& \|(1-t)(\tilde{S}z_t - x_{n_j}) + t(y - x_{n_j})\|^2 \\
\leq \;& (1-t)^2\|\tilde{S}z_t - x_{n_j}\|^2 + 2t\langle y - x_{n_j}, J(z_t - x_{n_j})\rangle \\
\leq \;& (1-t)^2\left(\|\tilde{S}z_t - \tilde{S}x_{n_j}\| + \|\tilde{S}x_{n_j} - x_{n_j}\|\right)^2 + 2t\langle y - z_t + z_t - x_{n_j}, J(z_t - x_{n_j})\rangle \\
= \;& (1-t)^2\left(\|\tilde{S}z_t - \tilde{S}x_{n_j}\| + \|\tilde{S}x_{n_j} - x_{n_j}\|\right)^2 \\
& +2t\left(\|z_t - x_{n_j}\|^2 + \langle y - z_t, J(z_t - x_{n_j})\rangle\right) \\
\leq \;& (1+t^2)\|z_t - x_{n_j}\|^2 + a_j + 2t\langle y - z_t, J(z_t - x_{n_j})\rangle,
\end{aligned}$$

where $a_j = (1-t)^2\left[2\|\tilde{S}x_{n_j} - x_{n_j}\|\,\|z_t - x_{n_j}\| + \|\tilde{S}x_{n_j} - x_{n_j}\|^2\right]$. Since $\|\tilde{S}x_{n_j} - x_{n_j}\| \longrightarrow 0$ by (6), and $\|z_t - x_{n_j}\|$ is bounded because $x_{n_j} \rightharpoonup \tilde{x}$ and $z_t \longrightarrow Q(y)$, we have $a_j \longrightarrow 0$ as $j \longrightarrow \infty$. Now

$$\|z_t - x_{n_j}\|^2 \leq (1+t^2)\|z_t - x_{n_j}\|^2 + a_j + 2t\langle y - z_t, J(z_t - x_{n_j})\rangle.$$

Therefore

$$\langle y - z_t, J(x_{n_j} - z_t)\rangle \leq \frac{t}{2}\|z_t - x_{n_j}\|^2 + \frac{1}{2t}a_j.$$

Taking lim sup as $j \longrightarrow \infty$ gives

$$\overline{\lim_j}\,\langle y - z_t, J(x_{n_j} - z_t)\rangle \leq c\frac{t}{2}$$

for some $c > 0$.

Now let $t \to 0$ to get

$$\overline{\lim_t}\,\overline{\lim_j}\,\langle y - z_t, J(x_{n_j} - z_t)\rangle \leq 0.$$

Now J is uniformly continuous on bounded sets. Hence we may swap the order of the limits to get

$$\begin{aligned} 0 &\geq \overline{\lim_j}\,\overline{\lim_t}\,\langle y - z_t, J(x_{n_j} - z_t)\rangle \\ &= \lim_j \langle y - Q(y), J(x_{n_j} - Q(y))\rangle \end{aligned}$$

proving (7).

Now by the subdifferential inequality (2.9)

$$\begin{aligned} &\|x_{n+1} - Q(y)\|^2 \\ = \;& \|(1-\lambda_{n+1})(T_{n+1}x_n - Q(y)) + \lambda_{n+1}(y - Q(y))\|^2 \\ = \;& (1-\lambda_{n+1})^2\|T_{n+1}x_n - Q(y)\|^2 + 2\lambda_{n+1}\langle y - Q(y), J(x_{n+1} - Q(y))\rangle \\ \leq \;& (1-\lambda_{n+1})\|x_n - Q(y)\|^2 + 2\lambda_{n+1}\langle y - Q(y), J(x_{n+1} - Q(y))\rangle \end{aligned}$$

By Lemma 2.1.2 , $x_n \longrightarrow Q(y)$. □

Remark 4.3. 1. The uniform smoothness assumption in Theorem 4.2 can be weakened to the assumption that the norm of X is uniformly Gateaux differentiable and each nonempty, closed and convex subset of X possesses the fixed point property for nonexpansive mappings.

2. Shioji and Takahashi [33] have proved the previous theorem with the above assumption but for a single map T. The proof of Theorem 4.2 shows that the use of a Banach limit in the proof of Shioji and Takahashi's theorem can be avoided.

3. Reich [31] obtained strong convergence of the iterates for the case $N = 1$, in the setting of a uniformly smooth Banach space having a weakly continuous duality map.

Under the conditions of Theorem 2.8.11, the existence of a unique nonexpansive retraction from C onto $Fix(T)$ is guaranteed for any nonexpansive mapping T from C to C. A space having a weakly continuous duality map satisfies the Demiclosedness Principle. Hence the following result is also an extension of Bauschke's Theorem 3.1 [2] to Banach spaces. This result is new and using the second of the two conditions in [N3], we even obtain a new result in Hilbert spaces (see Theorem 3.7).

Theorem 4.4. *Let X be a reflexive Banach space having a weakly continuous duality map J_ϕ. Let C be a nonempty, closed and convex subset of H and let $T_i : C \rightarrow$*

C, $(i = 1, 2, \ldots, N)$ be nonexpansive maps satisfying

$$F = Fix(T_N T_{N-1} \cdots T_2 T_1) = Fix(T_1 T_N \cdots T_3 T_2) = \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N),$$

where $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. *Let the sequence* $(\lambda_n) \subseteq (0, 1)$ *satisfy the following conditions:*

[*N1*] $\lim_{n\to\infty} \lambda_n = 0.$

[*N2*] $\sum_{n=1}^{\infty} \lambda_n = \infty.$

[*N3*] $\sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+N}| < \infty$ *or* $\lim_{n\to\infty} \dfrac{\lambda_n}{\lambda_{n+N}} = 1.$

Define x_n as follows:

$$x_{n+1} = \lambda_{n+1} y + (1 - \lambda_{n+1}) T_{n+1} x_n, \qquad n \geq 0, \quad x_0, y \in C.$$

Then $x_n \longrightarrow Q(y)$ where Q is the unique sunny nonexpansive retraction from C onto F.

Proof. Put $S = T_N \cdots T_1$. Then $Fix(S) = F = \bigcap_{i=1}^{N} Fix(T_i)$. By Theorem 2.8.11 there exists a unique sunny nonexpansive retraction $Q : C \to Fix(S)$.

We will show that $x_n \longrightarrow Q(y)$. Let $z = Q(y)$.

(1) $\|x_n - f\| \leq \max\{\|x_0 - f\|, \|y - f\|\}$ for all $n \geq 0$ and for all $f \in F$:

We use an inductive argument. The result is clearly true for $n = 0$. Suppose the result is true for n. Then by the nonexpansivity of T_{n+1},

$$\begin{aligned}
\|x_{n+1} - f\| &= \|\lambda_{n+1}y + (1-\lambda_{n+1})T_{n+1}x_n - f\| \\
&= \|\lambda_{n+1}(y-f) + (1-\lambda_{n+1})(T_{n+1}x_n - f)\| \\
&\leq \lambda_{n+1}\|y-f\| + (1-\lambda_{n+1})\|T_{n+1}x_n - f\| \\
&\leq \lambda_{n+1}\|y-f\| + (1-\lambda_{n+1})\|x_n - f\| \\
&\leq \lambda_{n+1}\max\{\|x_0-f\|, \|y-f\|\} + (1-\lambda_{n+1})\max\{\|x_0-f\|, \|y-f\|\} \\
&= \max\{\|x_0-f\|, \|y-f\|\}.
\end{aligned}$$

(2) (x_n) is bounded:

For each $f \in F$ and for all $n \geq 0$,

$$\begin{aligned}
\|x_n\| &\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0-f\|, \|y-f\|\} + \|f\|.
\end{aligned}$$

(3) $(T_{n+1}x_n)$ is bounded:

For each $f \in F$ and for all $n \geq 0$,

$$\begin{aligned}
\|T_{n+1}x_n\| &\leq \|T_{n+1}x_n - f\| + \|f\| \\
&\leq \|x_n - f\| + \|f\| \\
&\leq \max\{\|x_0-f\|, \|y-f\|\} + \|f\|.
\end{aligned}$$

(4) $x_{n+1} - T_{n+1}x_n \longrightarrow 0$: By (3) we can find $M > 0$ such that

$$\begin{aligned}
\|x_{n+1} - T_{n+1}x_n\| &= \lambda_{n+1}\|y - T_{n+1}x_n\| \\
&\leq \lambda_{n+1}(\|y\| + \|T_{n+1}x_n\|) \\
&\leq \lambda_{n+1}M.
\end{aligned}$$

By assumption [N1] $x_{n+1} - T_{n+1}x_n \longrightarrow 0$.

(5) $x_{n+N} - x_n \longrightarrow 0$:

By (2) and (3), we can find a constant M so that for all $n \geq 0$

$$\|x_{n+N} - x_n\| \leq M$$

and

$$\|y - T_{n+1}x_n\| \leq M.$$

Noting that $T_{n+N} = T_n$, we have for $n \geq 0$,

$$\begin{aligned}
& \|x_{n+N} - x_n\| \\
= \; & \|(\lambda_{n+N} - \lambda_n)(y - T_n x_{n-1}) + (1 - \lambda_{n+N})(T_{n+N}x_{n+N-1} - T_n x_{n-1})\| \\
\leq \; & M|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|T_n x_{n+N-1} - T_n x_{n-1}\| \\
\leq \; & M|\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N})\|x_{n+N-1} - x_{n-1}\| && (4.4) \\
\leq \; & M|\lambda_{n+N} - \lambda_n| \\
& +(1 - \lambda_{n+N})[\,M|\lambda_{n+N-1} - \lambda_{n-1}| + (1 - \lambda_{n+N-1})\|x_{n+N-2} - x_{n-2}\|\,] \\
= \; & M[|\lambda_{n+N} - \lambda_n| + |\lambda_{n+N-1} - \lambda_{n-1}|] \\
& +(1 - \lambda_{n+N})(1 - \lambda_{n+N-1})\|x_{n+N-2} - x_{n-2}\| \\
\leq \; & \cdots && (4.5) \\
\leq \; & M\sum_{k=m}^{n} |\lambda_{k+N} - \lambda_k| + M\prod_{k=m}^{n}(1 - \lambda_{k+N}). && (4.6)
\end{aligned}$$

Letting $n \to \infty$ in inequality (4.6) yields

$$\overline{\lim}\,\|x_{n+N} - x_n\| \leq M\sum_{k=m}^{\infty} |\lambda_{k+N} - \lambda_k| + M\prod_{k=m}^{\infty}(1 - \lambda_{k+N}). \tag{4.7}$$

Assumption [N2] implies that $\lim_{m\to\infty}\prod_{k=m}^{\infty}(1-\lambda_{k+N}) = 0$ and the first assumption of [N3] implies that $\lim_{m\to\infty}\sum_{k=m}^{\infty}|\lambda_{k+N}-\lambda_k| = 0$.

If we let $m \to \infty$ in inequality (4.7), we get $x_{n+N} - x_n \longrightarrow 0$ (by using the first assumption of [N3]).

If we now use the second assumption of [N3], then inequality (4.4) is

$$\|x_{n+N} - x_n\| \le M|\lambda_{n+N} - \lambda_n| + (1-\lambda_{n+N})\|x_{n+N-1} - x_{n-1}\|$$

with $\dfrac{|\lambda_{n+N}-\lambda_n|}{\lambda_{n+N}} \longrightarrow 0$. By Lemma 2.1.2 , $x_{n+N} - x_n \longrightarrow 0$.

(6) $x_n - T_{n+N}\cdots T_{n+1}x_n \longrightarrow 0$.

Noting (5) it is sufficient to show that

$$x_{n+N} - T_{n+N}\cdots T_{n+1}x_n \longrightarrow 0.$$

By (4),

$$x_{n+N} - T_{n+N}x_{n+N-1} \longrightarrow 0.$$

Again by (4),

$$x_{n+N-1} - T_{n+N-1}x_{n+N-2} \longrightarrow 0$$

and hence by the nonexpansivity of T_{n+N},

$$T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} \longrightarrow 0$$

Similarly,

$$\begin{aligned} T_{n+N}T_{n+N-1}x_{n+N-2} - T_{n+N}T_{n+N-1}T_{n+N-2}x_{n+N-3} &\longrightarrow 0 \\ &\vdots \\ T_{n+N}\cdots T_{n+2}x_{n+1} - T_{n+N}\cdots T_{n+1}x_n &\longrightarrow 0. \end{aligned}$$

Adding these N sequences yields the desired result.

(7) $\overline{\lim}\,\langle y - Q(y), J_\phi(x_n - Q(y))\rangle \le 0$:

Since $(\langle y - Q(y), J_\phi(x_n - Q(y))\rangle)$ is bounded and since X is reflexive, we can find a subsequence (n_j) of (n) such that

$$\overline{\lim}\,\langle y - Q(y), J_\phi(x_n - Q(y))\rangle = \lim_j \langle y - Q(y), J_\phi(x_{n_j} - Q(y))\rangle,$$

$$x_{n_j} \rightharpoonup \tilde{x},$$

and

$$T_{n_j} = T_i$$

for some $i \in \{1, 2, \ldots, N\}$ and for all j.

Then

$$\begin{aligned} \overline{\lim}\,\langle y - Q(y), J_\phi(x_n - Q(y))\rangle &= \lim_j \langle y - Q(y), J_\phi(x_{n_j} - Q(y))\rangle \\ &= \langle y - Q(y), J_\phi(\tilde{x} - Q(y))\rangle. \end{aligned}$$

By (6),

$$x_{n_j} - T_{i+N} \cdots T_i x_{n_j} = x_{n_j} - T_{n_j+N+1} \cdots T_{n_j+1} x_{n_j} \longrightarrow 0.$$

By the Demiclosedness Principle (from Theorem 2.7.6 and Theorem 2.7.4), $\tilde{x} \in Fix(T_{i+N} \cdots T_i) = Fix(S) = F$. By Theorem 2.8.2 ,

$$\langle y - Q(y), J(\tilde{x} - Q(y))\rangle \le 0$$

where J is the normalized duality map. But by Theorem 2.5.8 ,

$$J_\phi(x) = \frac{\phi(\|x\|)}{\|x\|} J(x), \text{ for } x \ne 0,$$

and so $\langle y - Q(y), J_\phi(\tilde{x} - Q(y))\rangle \le 0$, proving (7).

Now

$$\begin{aligned} x_{n+1} - Q(y) &= x_{n+1} - z \\ &= (1 - \lambda_{n+1})(T_{n+1}x_n - z) + \lambda_{n+1}(y - z). \end{aligned}$$

So by inequality (2.8) and by Lemma 2.5.7

$$\begin{aligned} \Phi(\|x_{n+1} - z\|) &\leq \Phi((1 - \lambda_{n+1})(T_{n+1}x_n - z)) + \lambda_{n+1}\langle x_0 - z, J_\phi(x_{n+1} - z)\rangle \\ &\leq (1 - \lambda_{n+1})\Phi(\|x_n - z\|) + \lambda_{n+1}\langle x_0 - z, J_\phi(x_{n+1} - z)\rangle. \end{aligned}$$

By step (7) and Lemma 2.1.2 , $\Phi(\|x_n - z\|) \longrightarrow 0$. Since Φ is increasing, $x_n - z \longrightarrow 0$. □

Remark 4.5. The weak continuity of the duality map in Theorem 4.4 makes the above proof much simpler than that of Theorem 4.2.

Chapter 5

Random Iterations

In this chapter the following iteration schemes are considered:

$$x_{n+1} = T_{r(n+1)}x_n \tag{5.1}$$

and

$$x_{n+1} = (1 - \lambda_{n+1})x_n + \lambda_{n+1}T_{r(n+1)}x_n. \tag{5.2}$$

The mappings are chosen in a random manner or in what is termed a quasi-cyclic manner. Scheme (5.1) is called the random iteration scheme and scheme (5.2) is called the relaxed iteration scheme with (λ_n) the relaxation parameters. In most cases, as in chapters 3 and 4, the mappings are chosen in a cyclic manner, i.e. every mapping is chosen at least once every M iterations, say, where M is bigger than or equal to the maximum number of mappings. In chapters 3 and 4, M is equal to the number of mappings. The quasi-cyclic order of choosing the mappings is a generalization of the cyclic order, where the cycle lengths are no longer fixed, but are allowed to increase at a slow rate.

In the following result we consider the random iteration scheme in a finite dimensional real Hilbert space, where the mappings are projections, and convergence of the iterates is obtained.

In both Theorems 5.1 and 5.2 we work in a finite dimensional Hilbert space, in which weak and strong convergence are equivalent.

Theorem 5.1. *Let H be a finite dimensional Hilbert space and let $C_1, C_2, \ldots, C_N$ be closed convex subsets of H such that $C := \bigcap_{i=1}^{N} C_i \neq \emptyset$. Let $P_i : H \to C_i$ be the nearest point projection for each $i = 1, 2, \ldots N$. For any $x_0 \in H$, define*

$$x_n = P_{r(n)} x_{n-1}, \qquad n \geq 1,$$

where $r : \mathbb{N} \to \{1, 2, \ldots, N\}$ is arbitrary, taking on each $j \in \{1, 2, \ldots, N\}$ infinitely often. Then (x_n) converges to a point in C.

Proof. Since $P_{r(n)}$ is nonexpansive, we have for each $x^* \in C$ and for all $n \geq 1$,

$$\begin{aligned} \|x_n - x^*\| &= \|P_{r(n)} x_{n-1} - x^*\| \\ &= \|P_{r(n)} x_{n-1} - P_{r(n)} x^*\| \\ &\leq \|x_{n-1} - x^*\|. \end{aligned} \tag{5.3}$$

Hence $\lim_{n \to \infty} \|x_n - x^*\|$ exists for each $x^* \in C$ and so (x_n) is bounded. For any projection P_K we have by Lemma 2.4.5 that for all $x \in H$ and all $y \in K$,

$$\|x - P_K x\|^2 \leq \|x - y\|^2 - \|P_K x - y\|^2.$$

Therefore, for any $x^* \in C$,

$$\begin{aligned} \|x_{n-1} - x_n\|^2 &= \|x_{n-1} - P_{r(n)} x_{n-1}\|^2 \\ &\leq \|x_{n-1} - x^*\|^2 - \|P_{r(n)} x_{n-1} - x^*\|^2 \\ &= \|x_{n-1} - x^*\|^2 - \|x_n - x^*\|^2. \end{aligned}$$

Since $\lim_{n\to\infty} \|x_n - x^*\|$ exists, $\lim_{n\to\infty} \|x_{n-1} - x_n\|^2 = 0$. Hence

$$\lim_{n\to\infty} \|x_{n-1} - x_n\| = 0. \tag{5.4}$$

Now there exists a subsequence (x_{n_j}) such that

$$x_{n_j} \rightharpoonup \overline{x} \qquad \text{for some } \overline{x} \in H$$

and for some $i \in \{1, 2, \ldots, N\}$,

$$r(n_j) = i \qquad \text{for all } j \geq 1.$$

But $\dim H < \infty$, so $x_{n_j} \longrightarrow \overline{x}$. Also

$$x_{n_j} = P_{r(n_j)} x_{n_j - 1} = P_i x_{n_j - 1}. \tag{5.5}$$

We will now show that $\overline{x} \in C$.

Since $\lim_{n\to\infty} \|x_{n-1} - x_n\| = 0$ and $x_{n_j} \longrightarrow \overline{x}$, we have

$$x_{n_j - 1} \longrightarrow \overline{x}. \tag{5.6}$$

Noting the continuity of P_i, we may take limits in equation (5.5) to get $\overline{x} = P_i \overline{x}$. Therefore $\overline{x} \in C_i$. If we assume that $\overline{x} \notin C$, then $\overline{x} \notin C_{k_0}$ for some $k_0 \in \{1, 2, \ldots, N\}$. By rearrangement, if necessary, we may assume that $\overline{x} \in C_1, C_2, \ldots, C_L$ and $\overline{x} \notin \; C_{L+1}, \ldots, C_N$, where $1 \leq L < N$.

Fix $k \geq 1$ and choose m_k to be the smallest integer τ, $\tau > n_k$, such that $r(\tau) > L$; i.e. $m_k = \min\{\tau > n_k : r(\tau) > L\}$. So $r(m_k) > L$ and $r(m_k - 1) \leq L$. For any j with $n_k \leq j < m_k$, we have $r(j) \leq L$, so that

$$\begin{aligned} \|x_j - \overline{x}\| &= \|P_{r(j)} x_{j-1} - \overline{x}\| \\ &= \|P_{r(j)} x_{j-1} - P_{r(j)} \overline{x}\| \\ &\leq \|x_{j-1} - \overline{x}\|, \end{aligned}$$

since $\overline{x} \in C_{r(j)}$ and by the nonexpansivity of $P_{r(j)}$.

In particular,

$$\begin{aligned} \|x_{m_k-1} - \overline{x}\| &\leq \|x_{m_k-2} - \overline{x}\| \\ &\leq \|x_{m_k-3} - \overline{x}\| \\ &\ \ \vdots \\ &\leq \|x_{n_k-1} - \overline{x}\| \\ &\longrightarrow 0. \end{aligned} \tag{5.7}$$

by (5.6).

Now $r(m_k) \in \{L+1, \ldots, N\}$ for all k. Hence we can choose a subsequence $(m_{k'})$ of (m_k) so that $r(m_{k'}) = j_0$ for some $j_0 \in \{L+1, \ldots, N\}$ and for all $k' \geq 1$. Then

$$x_{m_{k'}} = P_{r(m_{k'})} x_{m_{k'}-1} = P_{j_0} x_{m_{k'}-1}. \tag{5.8}$$

By (5.7), $x_{m_{k'}-1} \longrightarrow \overline{x}$ and by equation (5.4), $\|x_{m_{k'}-1} - x_{m_{k'}}\| \longrightarrow 0$. So $x_{m_{k'}} \longrightarrow \overline{x}$. Taking limits as $k' \to \infty$ in equation (5.8) yields $\overline{x} = P_{j_0}\overline{x}$, since P_{j_0} is continuous. Hence $\overline{x} \in C_{j_0}$ with $j_0 > L$, a contradiction. Hence we may conclude that $\overline{x} \in C$. By Fejér monotonicity of (x_n) with respect to C (inequality (5.3)), and since $x_{n_j} \longrightarrow \overline{x} \in C$, we have $x_n \longrightarrow \overline{x}$. □

The following result is a result of Tseng [36], but here we provide an alternate proof.

Theorem 5.2 ([36]). *Let H be a finite dimensional Hilbert space and let $T_i : H \to H$ be firmly nonexpansive mappings (i=1,2,...,N), with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. For any*

$x_0 \in H$ and any $\epsilon \in (0,2)$, define

$$x_{n+1} = (1-\lambda_{n+1})x_n + \lambda_{n+1}T_{r(n+1)}x_n, \qquad n \geq 0$$

where $\epsilon \leq \lambda_n \leq 2-\epsilon$ for all $n \geq 1$ and $r : \mathbb{N} \to \{1,2,\ldots,N\}$ is arbitrary, taking on each $j \in \{1,2,\ldots,N\}$ infinitely often. Then (x_n) converges to a point in F.

Proof. We can rewrite the process as

$$x_n = x_{n+1} + \lambda_{n+1}(x_n - T_{r(n+1)}x_n).$$

For each $x^* \in F$, and for all $n \geq 0$,

$$\begin{aligned}
\|x_n - x^*\|^2 &= \|x_{n+1} - x^* + \lambda_{n+1}(x_n - T_{r(n+1)}x_n)\|^2 \\
&= \|x_{n+1} - x^*\|^2 + \lambda_{n+1}^2\|x_n - T_{r(n+1)}x_n\|^2 \\
&\quad +2\lambda_{n+1}\langle x_{n+1} - x^*, x_n - T_{r(n+1)}x_n\rangle \\
&= \|x_{n+1} - x^*\|^2 + \lambda_{n+1}^2\|x_n - T_{r(n+1)}x_n\|^2 \\
&\quad +2\lambda_{n+1}\langle x_{n+1} - T_{r(n+1)}x_n, x_n - T_{r(n+1)}x_n\rangle \\
&\quad +2\lambda_{n+1}\langle T_{r(n+1)}x_n - x^*, x_n - T_{r(n+1)}x_n\rangle \\
&\geq \|x_{n+1} - x^*\|^2 + \lambda_{n+1}^2\|x_n - T_{r(n+1)}x_n\|^2 \\
&\quad +2\lambda_{n+1}(1-\lambda_{n+1})\|x_n - T_{r(n+1)}x_n\|^2 \\
&= \|x_{n+1} - x^*\|^2 + \lambda_{n+1}(2-\lambda_{n+1})\|x_n - T_{r(n+1)}x_n\|^2 \qquad (5.9) \\
&\geq \|x_{n+1} - x^*\|^2 + \epsilon^2\|x_n - T_{r(n+1)}x_n\|^2,
\end{aligned}$$

by Lemma 2.6.6 and by noting that $x_{n+1} - T_{r(n+1)}x_n = (1-\lambda_{n+1})(x_n - T_{r(n+1)}x_n)$. Hence

$$\epsilon^2\|x_n - T_{r(n+1)}x_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2. \qquad (5.10)$$

This implies that $(\|x_n - x^*\|)$ is a decreasing sequence and hence the limit exists. Thus by inequality (5.10),

$$\|x_n - T_{r(n+1)}x_n\| \longrightarrow 0. \tag{5.11}$$

By (5.11) we get

$$\|x_n - x_{n+1}\| = \lambda_{n+1}\|x_n - T_{r(n+1)}x_n\| \longrightarrow 0. \tag{5.12}$$

We also have (x_n) is bounded since $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Thus we may choose a subsequence (x_{n_j}) of (x_n) such that

$$x_{n_j} \longrightarrow \overline{x} \tag{5.13}$$

and

$$r(n_j + 1) = i_0$$

for some $i_0 \in \{1, 2, \ldots, N\}$ and for all $j \geq 1$.

By (5.11) we get $x_{n_j} - T_{i_0}x_{n_j} \longrightarrow 0$. By continuity of T_{i_0} and by (5.13) we also have

$$x_{n_j} - T_{i_0}x_{n_j} \longrightarrow \overline{x} - T_{i_0}\overline{x}.$$

Hence $\overline{x} \in Fix(T_{i_0})$.

If $\overline{x} \notin F$, then $\overline{x} \notin Fix(T_i)$ for some i. So we may assume (by rearrangement if necessary) that

$$\overline{x} \in Fix(T_i), \qquad 1 \leq i \leq L$$

and

$$\overline{x} \notin Fix(T_i), \qquad L + 1 \leq i \leq N,$$

where $1 \leq L < N$.

Fix $k \in \mathbb{N}$ and choose m_k as follows:

$$m_k = \min\{\tau > n_k : r(\tau) > L\}.$$

Thus $r(m_k) > L$ and $r(m_k - 1) \leq L$. For any i with $n_k \leq i < m_k$ we have $r(i) \leq L$, so that $\|x_i - \overline{x}\| \leq \|x_{i-1} - \overline{x}\|$ noting that $\overline{x} \in Fix(T_{r(i)})$ and using the argument used to obtain inequality (5.10). Hence

$$\|x_{m_k-1} - \overline{x}\| \leq \|x_{m_k-2} - \overline{x}\| \leq \cdots \leq \|x_{n_k-1} - \overline{x}\|.$$

By (5.13), $x_{n_k} \longrightarrow \overline{x}$ and by (5.12), $\|x_{n-1} - x_n\| \longrightarrow 0$. So $x_{n_k-1} \longrightarrow \overline{x}$. Therefore $x_{m_k-1} \longrightarrow \overline{x}$.

By passing to a subsequence $(m_{k'})$ we have $r(m_{k'}) = j_0$ for some $j_0 \in \{L+1, \ldots, N\}$. But $\|x_{m_{k'}-1} - T_{r(m_{k'})}x_{m_{k'}-1}\| \longrightarrow 0$ by (5.11) and

$$\|x_{m_{k'}-1} - T_{r(m_{k'})}x_{m_{k'}-1}\| = \|x_{m_{k'}-1} - T_{j_0}x_{m_{k'}-1}\| \longrightarrow \|\overline{x} - T_{j_0}\overline{x}\|.$$

So $\overline{x} = T_{j_0}\overline{x}$, i.e. $\overline{x} \in Fix(T_{j_0})$. This is a contradiction since $j_0 > L$. Hence $\overline{x} \in F$. Since $x_{n_j} \longrightarrow \overline{x}$ by (5.13) and (x_n) is Fejér monotone with respect to F (from inequality (5.10)) , we must have $x_n \longrightarrow \overline{x}$. $\square$

It still remains an open question whether Theorems 5.1 and 5.2 can be extended to infinite dimensional Hilbert spaces. In this connection, see Theorem 2 of [15].

Definition 5.3. A map $r : \mathbb{N} \to \{1, 2, \ldots, N\}$ is said to be **quasi-cyclic** if there is an increasing sequence (τ_k) satisfying

(a) $\tau_1 = 1$,

(b) $\tau_{k+1} - \tau_k \geq N$ for all $k \geq 1$,

(c) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\tau_{k+1} - \tau_k} = \infty$,

(d) $\{1, 2, \ldots, N\} \subseteq \{r(\tau_k), r(\tau_k + 1), \ldots, r(\tau_{k+1} - 1)\}$ for all $k \geq 1$.

Condition (b) says that the length of each cycle is at least N (where N is the number of mappings), condition (c) ensures that the lengths of the cycles do not increase too fast and condition (d) says that every mapping is chosen at least once in every cycle (i.e. between the τ_k^{th} and the $(\tau_{k+1} - 1)^{th}$ iterate for all $k \geq 1$). This control was introduced by Tseng and Bertsekas ([37]) in 1987.

In the following theorem we obtain weak convergence in a Hilbert space that is not necessarily finite dimensional. This is Theorem 2 of Tseng ([36]), but an alternate proof is provided.

Theorem 5.4. *Let H be a Hilbert space and let $\epsilon \in (0, 2)$ be arbitrary. For $i = 1, 2, \ldots, N$, let $T_i : H \to H$ be firmly nonexpansive with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$ and let $r : \mathbb{N} \to \{1, 2, \ldots, N\}$ be quasi-cyclic. For $\epsilon \leq \lambda_n \leq 2 - \epsilon$ and for $x_0 \in H$, define*

$$x_{n+1} = (1 - \lambda_{n+1})x_n + \lambda_{n+1} T_{r(n+1)} x_n, \; n \geq 0.$$

Then (x_n) converges weakly to a point in F.

Proof. There exists an increasing sequence (τ_k) satisfying conditions (a)-(d) of Definition 5.3 .

Let $\sigma_k = \sum_{i=\tau_k}^{\tau_{k+1}-1} ||x_{i+1} - x_i||$, $\quad k \geq 1$. By the Cauchy-Schwartz inequality,

$$\sigma_k{}^2 \leq \sum_{i=\tau_k}^{\tau_{k+1}-1} ||x_{i+1} - x_i||^2(\tau_{k+1} - \tau_k). \tag{5.14}$$

In the proof of Theorem 5.2 we obtained for each $x^* \in F$, inequality (5.10)

$$\epsilon^2||x_n - T_{r(n+1)}x_n||^2 \leq ||x_n - x^*||^2 - ||x_{n+1} - x^*||^2$$

and so by (5.12)

$$\begin{aligned} ||x_n - x_{n+1}||^2 &= \lambda_{n+1}^2||x_n - T_{r(n+1)}x_n||^2 \\ &\leq \frac{(2-\epsilon)^2}{\epsilon^2}[\,||x_n - x^*||^2 - ||x_{n+1} - x^*||^2\,]. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} ||x_n - x_{n+1}||^2 < \infty$ and $\lim_{n\to\infty} ||x_n - x_{n+1}|| = 0$. Inequallity (5.14) yields

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{\tau_{k+1} - \tau_k}\sigma_k{}^2 &\leq \sum_{k=1}^{\infty}\sum_{i=\tau_k}^{\tau_{k+1}-1} ||x_{i+1} - x_i||^2 \\ &= \sum_{n=1}^{\infty} ||x_{n+1} - x_n||^2 \\ &< \infty. \end{aligned}$$

But $\sum_{k=1}^{\infty} \frac{1}{\tau_{k+1} - \tau_k} = \infty$. Hence $\underline{\lim}\, \sigma_k^2 = 0$ by Lemma 2.1.3 , and so $\underline{\lim}\, \sigma_k = 0$.
Therefore we can find a subsequence (k') such that $\lim_{k'} \sigma_{k'} = 0$.
$(x_{\tau_{k'}})$ is bounded as is shown in the proof of Theorem 5.2. Hence it has a weakly convergent subsequence, which we may assume to be itself. So $x_{\tau_{k'}} \rightharpoonup \overline{x}$ for some $\overline{x} \in H$.

To simplify the notation, we will drop the "prime" in the remainder of the proof and write k for k'.

Fix $i \in \{1, 2, \ldots, N\}$. Then $i \in \{r(\tau_k), r(\tau_k + 1), \ldots, r(\tau_{k+1} - 1)\}$ for all $k \geq 1$. So for each $k \geq 1$, we can find ρ_k with $\rho_k + 1 \in \{\tau_k, \tau_k + 1, \ldots, \tau_{k+1} - 1\}$ so that $r(\rho_k + 1) = i$. Now

$$\begin{aligned} \|x_{\rho_k} - T_{r(\rho_k+1)} x_{\rho_k}\| &= \frac{1}{\lambda_{\rho_k+1}} \|x_{\rho_k+1} - x_{\rho_k}\| \\ &\leq \frac{1}{\epsilon} \|x_{\rho_k+1} - x_{\rho_k}\| && (5.15) \\ &\longrightarrow 0. && (5.16) \end{aligned}$$

So

$$\begin{aligned} \|x_{\rho_k+1} - T_{r(\rho_k+1)} x_{\rho_k}\| &\leq \|x_{\rho_k+1} - x_{\rho_k}\| + \|x_{\rho_k} - T_{r(\rho_k+1)} x_{\rho_k}\| \\ &\leq (1 + \frac{1}{\epsilon}) \|x_{\rho_k+1} - x_{\rho_k}\|. \end{aligned}$$

Therefore

$$\begin{aligned} \|x_{\tau_k} - T_{r(\rho_k+1)} x_{\rho_k}\| &= \|x_{\tau_k} - T_i x_{\rho_k}\| \\ &\leq \sum_{j=\tau_k}^{\rho_k} \|x_j - x_{j+1}\| + \|x_{\rho_k+1} - T_i x_{\rho_k}\| \\ &\leq \sum_{j=\tau_k}^{\tau_{k+1}-1} \|x_j - x_{j+1}\| + \|x_{\rho_k+1} - T_i x_{\rho_k}\| \\ &\leq \sigma_k + (1 + \frac{1}{\epsilon}) \|x_{\rho_k+1} - x_{\rho_k}\| \longrightarrow 0, \end{aligned}$$

i.e. $x_{\tau_k} - T_i x_{\rho_k} \longrightarrow 0$ as $k \longrightarrow \infty$. But $x_{\tau_k} \rightharpoonup \overline{x}$. So $T_i x_{\rho_k} \rightharpoonup \overline{x}$ and by (5.16), $x_{\rho_k} \rightharpoonup \overline{x}$. Again by (5.16), $(I - T_i)(x_{\rho_k}) \longrightarrow 0$. By the Demiclosedness Principle, $T_i \overline{x} = \overline{x}$; i.e. $\overline{x} \in Fix(T_i)$. Since i was arbitrary, $\overline{x} \in \bigcap_{i=1}^{N} Fix(T_i) = F$.

Now assume that (x_n) has another weak cluster point $\tilde{x}$. Then there exists a subsequence (x_{m_j}) such that $x_{m_j} \rightharpoonup \tilde{x}$. Suppose also that $x_{n_k} \rightharpoonup \overline{x}$, for some subsequence (x_{n_k}). By repeating the above argument, we can show that $\tilde{x} \in F$. From the proof of Theorem 5.2 , $\lim_{n\to\infty} \|x_n - x^*\|$ exists for each $x^* \in F$. Hence $\lim_{n\to\infty} \|x_n - \overline{x}\|$ and $\lim_{n\to\infty} \|x_n - \tilde{x}\|$ both exist. If $\tilde{x} \neq \overline{x}$, then by Opial's property,

$$\begin{aligned}
\lim_n \|x_n - \tilde{x}\| &= \lim_j \|x_{m_j} - \tilde{x}\| \\
&= \overline{\lim}\|x_{m_j} - \tilde{x}\| \\
&< \overline{\lim}\|x_{m_j} - \overline{x}\| \\
&= \lim_j \|x_{m_j} - \overline{x}\| \\
&= \lim_n \|x_n - \overline{x}\| \\
&= \lim_k \|x_{n_k} - \overline{x}\| \\
&= \overline{\lim}\|x_{n_k} - \overline{x}\| \\
&< \overline{\lim}\|x_{n_k} - \tilde{x}\| \\
&= \lim_k \|x_{n_k} - \tilde{x}\| \\
&= \lim_n \|x_n - \tilde{x}\|,
\end{aligned}$$

which is a contradiction. Thus (x_n) has a unique weak cluster point in F. Now (x_n) is Fejér monotone with respect to F (inequality (5.10) in the proof of Theorem 5.2), so by Theorem 2.0.4 (ii), (x_n) converges weakly to some point in F. Hence $x_n \rightharpoonup \overline{x}$. □

For a finite collection of nonexpansive maps $T_i : C \to C$, $(i = 1, 2, \ldots, N)$, we consider the following iteration scheme:

$$x_n = T_{r(n)} x_{n-1}.$$

This boils down to looking at the products of the form

$$T_{r(n)}T_{r(n-1)}\cdots T_{r(1)},$$

and we investigate the convergence of the above iterates.

Dye and Reich [16] showed that in a Hilbert space with r quasi-periodic, as they called it, the iterates converge weakly. In [15], they were able to extend this result to reflexive Banach spaces with a weakly sequentially continuous duality map. However, the result could only be proved if the pool of maps to be drawn from consisted of only two maps. In [14], Dye *et al* proved the result for Banach spaces that have Opial's property. However, in all of these results only weak convergence of the iterates is obtained. Our next result obtains strong convergence of the iterates in a Banach space that has certain conditions imposed on the fixed point sets.

Theorem 5.5. *Let X be a Banach space and let C be a nonempty, closed and convex subset of X. Let $T_i : C \to C$, $(i = 1, 2, \ldots, N)$ be nonexpansive with $F_i := Fix(T_i)$ and $F := \bigcap_{i=1}^{N} F_i \neq \emptyset$. Let $r : \mathbb{N} \to \{1, 2, \ldots, N\}$ be a map that takes on each $i \in \{1, 2, \ldots, N\}$ infinitely often. Assume also that :*

(i) there exists $c \in F$ such that for $i \in \{1, 2, \ldots, N\}$, and for all sequences $(u_n) \subseteq C$,

$$\|u_n - c\| - \|T_i u_n - c\| \longrightarrow 0 \quad \text{implies} \quad d(u_n, F_i) \longrightarrow 0.$$

(ii) for all sequences $(u_n) \subseteq X$,

$$\max_{1\le i\le N} d(u_n, F_i) \longrightarrow 0 \quad \text{implies} \quad d(u_n, F) \longrightarrow 0.$$

For $x_0 \in C$, if

$$x_n = T_{r(n)}x_{n-1}, \qquad n \ge 1,$$

then (x_n) *converges to a point in* F.

Proof. Since each $i \in \{1, 2, \ldots, N\}$ is taken infinitely often, there exists an increasing sequence $(n_k), n_k \longrightarrow \infty$ as $k \to \infty$, such that for each k,

$$\{r(n_k+1), r(n_k+2), \ldots, r(n_{k+1})\} \supseteq \{1, 2, \ldots, N\}.$$

Put $W_k = T_{r(n_{k+1})} \cdots T_{r(n_k+1)}$. Then $x_{n_{k+1}} = W_k x_{n_k}$. By nonexpansivity, we have for each $f \in F$,

$$\begin{aligned} \|x_{n+1} - f\| &= \|T_{r(n+1)}x_n - T_{r(n+1)}f\| \\ &\leq \|x_n - f\| \end{aligned}$$

Therefore (x_n) is Fejér monotone with respect to F. In particular, $c \in F$, so $(\|x_n - c\|)$ is decreasing and hence $\lim_n \|x_n - c\|$ exists. So

$$\|x_{n_k} - c\| - \|W_k x_{n_k} - c\| = \|x_{n_k} - c\| - \|x_{n_{k+1}} - c\| \longrightarrow 0. \tag{5.17}$$

Put $V_{k,j} = T_{r(n_k+j)} \cdots T_{r(n_k+1)}, \quad k \geq 1, j = 1, 2, \ldots, n_{k+1} - n_k$.
Then $(\|V_{k,j}x_{n_k} - c\|)$ is decreasing in j by nonexpansivity and the fact that $c \in F$, and for $j = 1, 2, \ldots, n_{k+1} - n_k$,

$$\|x_{n_{k+1}} - c\| \leq \|V_{k,j}x_{n_k} - c\| \leq \|x_{n_k} - c\|. \tag{5.18}$$

So for $j = 1, 2, \ldots, n_{k+1} - n_k$,

$$\|x_{n_k} - c\| - \|V_{k,j}x_{n_k} - c\| \leq \|x_{n_k} - c\| - \|x_{n_{k+1}} - c\| \longrightarrow 0. \tag{5.19}$$

Claim : For $j = 1, 2, \ldots, n_{k+1} - n_k$, $\|x_{n_k} - V_{k,j}x_{n_k}\| \longrightarrow 0$ as $k \longrightarrow \infty$.
We will prove this claim by induction.

$j = 1$: By Fejér monotonicity, (x_{n_k}) is bounded and hence $(V_{k,j}x_{n_k})$ is also bounded. So $(\|x_{n_k} - V_{k,1}x_{n_k}\|)$ is bounded. Hence there exists a subsequence $(n_{k'})$ such that

$$\overline{\lim}\, \|x_{n_k} - V_{k,1}x_{n_k}\| = \lim_{k'} \|x_{n_{k'}} - V_{k',1}x_{n_{k'}}\| \tag{5.20}$$

and

$$r(n_{k'} + 1) = i$$

for some $i \in \{1, 2, \ldots, N\}$. Noting that $V_{k',1} = T_{r(n_{k'}+1)}$, and by (5.19),

$$\|x_{n_{k'}} - c\| - \|T_i x_{n_{k'}} - c\| \longrightarrow 0.$$

By assumption (i), $d(x_{n_{k'}}, F_i) \longrightarrow 0$ as $k' \longrightarrow 0$. Now

$$\begin{aligned} \|x_{n_{k'}} - V_{k',1}x_{n_{k'}}\| &\leq \|x_{n_{k'}} - c_i\| + \|V_{k',1}x_{n_{k'}} - c_i\| \\ &\leq 2\|x_{n_{k'}} - c_i\|, \quad \text{for all} \quad c_i \in F_i. \end{aligned}$$

Hence

$$\|x_{n_{k'}} - V_{k',1}x_{n_{k'}}\| \leq 2\,\mathrm{d}(x_{n_{k'}}, F_i).$$

So by equation (5.20),

$$\begin{aligned} \overline{\lim}\, \|x_{n_k} - V_{k,1}x_{n_k}\| &= \lim_{k'} \|x_{n_{k'}} - V_{k',1}x_{n_{k'}}\| \\ &\leq \lim_{k'} 2\,\mathrm{d}(x_{n_{k'}}, F_i) \\ &= 0. \end{aligned}$$

Therefore $\lim\limits_{k\to\infty} \|x_{n_k} - V_{k,1}x_{n_k}\| = 0$. Hence the claim is true for $j = 1$.

We will now assume that $\|x_{n_k} - V_{k,j-1}x_{n_k}\| \longrightarrow 0$ as $k \to \infty$, and we will prove that for $1 \leq j \leq n_{k+1} - n_k$, $\|x_{n_k} - V_{k,j}x_{n_k}\| \longrightarrow 0$ as $k \to \infty$.

Since $(\|x_{n_k} - V_{k,j}x_{n_k}\|)$ is bounded, we can find a subsequence $(n_{k'})$ such that

$$\overline{\lim}\|x_{n_k} - V_{k,j}x_{n_k}\| = \lim_{k'} \|x_{n_{k'}} - V_{k',j}x_{n_{k'}}\| \tag{5.21}$$

and

$$r(n_{k'} + j) = i^*$$

for some $i \in \{1, 2, \ldots, N\}$ and for all $k' \geq 1$.

So from equation (5.21) and the induction assumption,

$$\begin{aligned}
\overline{\lim}\|x_{n_k} - V_{k,j}x_{n_k}\| &= \lim_{k'} \|x_{n_{k'}} - T_{r(n_{k'}+j)}V_{k',j-1}x_{n_{k'}}\| \\
&\leq \lim_{k'} \|x_{n_{k'}} - V_{k',j-1}x_{n_{k'}}\| \\
&\quad + \lim_{k'} \|T_iV_{k',j-1}x_{n_{k'}} - V_{k',j-1}x_{n_{k'}}\| \qquad (5.22) \\
&= \lim_{k'} \|T_iV_{k',j-1}x_{n_{k'}} - V_{k',j-1}x_{n_{k'}}\|. \qquad (5.23)
\end{aligned}$$

Now for any $c_i \in F_i$,

$$\begin{aligned}
\|T_iV_{k',j-1}x_{n_{k'}} - V_{k',j-1}x_{n_{k'}}\| &\leq \|T_iV_{k',j-1}x_{n_{k'}} - c_i\| + \|c_i - V_{k',j-1}x_{n_{k'}}\| \\
&\leq 2\|V_{k',j-1}x_{n_{k'}} - c_i\|.
\end{aligned}$$

Hence $\|T_iV_{k',j-1}x_{n_{k'}} - V_{k',j-1}x_{n_{k'}}\| \leq 2\,\mathrm{d}(V_{k',j-1}x_{n_{k'}}, F_i)$.

So from inequality (5.23), we have

$$\overline{\lim}\,\|x_{n_k} - V_{k,j}x_{n_k}\| \leq \lim_{k'} 2\,\mathrm{d}(V_{k',j-1}x_{n'_k}, F_i).$$

Now

$$\begin{aligned}
\|x_{n_k+1} - c\| &\leq \|T_{r(n_k+j)}V_{k,j-1}x_{n_k} - c\| \\
&\leq \|V_{k,j-1}x_{n_k} - c\| \\
&\leq \|x_{n_k} - c\|.
\end{aligned}$$

Therefore

$$\begin{aligned} \|V_{k,j-1}x_{n_k} - c\| - \|T_i V_{k,j-1}x_{n_k} - c\| &\leq \|x_{n_k} - c\| - \|x_{n_k+1} - c\| \\ &\longrightarrow 0, \end{aligned}$$

by (5.17).

By assumption (i), $\mathrm{d}(V_{k',j-1}x_{n_{k'}}, F_i) \longrightarrow 0$. So $\overline{\lim}\,\|x_{n_k} - V_{k,j}x_{n_k}\| = 0$ and hence $\lim_k \|x_{n_k} - V_{k,j}x_{n_k}\| = 0$. Thus the claim is proved; i.e. for $j = 1, 2, \ldots, n_{k+1} - n_k$,

$$\|x_{n_k} - V_{k,j}x_{n_k}\| \longrightarrow 0. \tag{5.24}$$

Now $\{r(n_k+1), r(n_k+2), \ldots, r(n_{k+1})\} \supseteq \{1, 2, \ldots, N\}$ for each k.

Fix $k \in \mathbb{N}$. Then for each $i \in \{1, 2, \ldots, N\}$, there exists $\rho_k \in \{n_k + 1, \ldots, n_{k+1}\}$ such that $r(\rho_k) = i$.

We can decompose $W_k = T_{r(n_{k+1})} \cdots T_{r(n_k+1)}$ as follows:

$$W_k = U_k T_i V_k$$

where $U_k = T_{r(n_{k+1})} \cdots T_{r(\rho_k+1)}$ and $V_k = T_{r(\rho_k-1)} \cdots T_{r(n_k+1)}$.

Then by nonexpansivity,

$$\begin{aligned} \|x_{n_{k+1}} - c\| &= \|W_k x_{n_k} - c\| \\ &= \|U_k T_i V_k x_{n_k} - c\| \\ &\leq \|T_i V_k x_{n_k} - c\| \\ &\leq \|V_k x_{n_k} - c\| \\ &\leq \|x_{n_k} - c\|. \end{aligned}$$

Since $\lim_n \|x_n - c\|$ exists,

$$\|V_k x_{n_k} - c\| - \|T_i V_k x_{n_k} - c\| \leq \|x_{n_k} - c\| - \|x_{n_{k+1}} - c\| \longrightarrow 0.$$

By assumption (i), $d(V_k x_{n_k}, F_i) \longrightarrow 0$. Again, by nonexpansivity of T_i

$$\begin{aligned} \|V_k x_{n_k} - T_i V_k x_{n_k}\| &\leq \|V_k x_{n_k} - c'\| + \|T_i V_k x_{n_k} - c'\| \\ &= 2\|V_k x_{n_k} - c'\| \end{aligned}$$

for all $c' \in F_i$.

Hence

$$\|V_k x_{n_k} - T_i V_k x_{n_k}\| \leq 2\,\mathrm{d}(V_k x_{n_k}, F_i) \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty. \tag{5.25}$$

Now $V_k = T_{r(\rho_k - 1)} \cdots T_{r(n_k)} = V_{k,j}$ for some $j \geq 1$. So

$$\begin{aligned} \|x_{n_k} - T_i x_{n_k}\| &\leq \|x_{n_k} - V_k x_{n_k}\| + \|V_k x_{n_k} - T_i V_k x_{n_k}\| + \|T_i V_k x_{n_k} - T_i x_{n_k}\| \\ &\leq 2\|x_{n_k} - V_k x_{n_k}\| + \|V_k x_{n_k} - T_i V_k x_{n_k}\| \\ &\longrightarrow 0 \end{aligned}$$

by (5.24) and (5.25).

Thus $\|x_{n_k} - c\| - \|T_i x_{n_k} - c\| \leq \|x_{n_k} - T_i x_{n_k}\| \longrightarrow 0$. By assumption (i), $\mathrm{d}(x_{n_k}, F_i) \longrightarrow 0$. Since i was arbitrary, we have by assumption (ii) that $\mathrm{d}(x_{n_k}, F) \longrightarrow 0$. So for each $\epsilon > 0$, there exists k so that $d(x_{n_k}, F) < \epsilon$.

In particular, for any $k \geq 1$, there exists $\tilde{k}$ so that $\mathrm{d}(x_{n_{\tilde{k}}}, F) < \dfrac{1}{2^k}$. Hence there exists $p_k \in F$ such that

$$\|x_{n_{\tilde{k}}} - p_k\| < \frac{1}{2^k}.$$

Since (x_n) is Fejér monotone with respect to F,

$$\begin{aligned} \|p_{k+1} - p_k\| &\leq \|p_{k+1} - x_{n_{\widetilde{k+1}}}\| + \|x_{n_{\widetilde{k+1}}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \|x_{n_{\tilde{k}}} - p_k\| \\ &< \frac{1}{2^{k+1}} + \frac{1}{2^k}. \end{aligned}$$

Thus (p_k) is a Cauchy sequence in F and hence converges to a point p in F since $F = \bigcap_{i=1}^{N} Fix(T_i)$ is closed. Therefore $p \in F$.

Now

$$\begin{aligned} \|x_{n_{\bar{k}}} - p\| &\leq \|x_{n_{\bar{k}}} - p_k\| + \|p_k - p\| \\ &< \frac{1}{2^k} + \|p_k - p\| \longrightarrow 0. \end{aligned}$$

Thus $x_{n_{\bar{k}}} \longrightarrow p$.

By Fejér monotonicity, the entire sequence must converge to p; i.e. $x_n \longrightarrow p$. □

Remark 5.6. (a) For any relaxed projection on a Hilbert space, i.e.

$$R := (1-\lambda)I + \lambda P_K, \qquad \lambda \in (0,2),$$

we have for any $c \in K$ and for any $x \in H$

$$\begin{aligned} \|x-c\|^2 - \|Rx - c\|^2 &\geq \lambda(2-\lambda)\|x - P_K x\|^2 \\ &= \lambda(2-\lambda)\, \mathrm{d}(x,K)^2 \end{aligned}$$

by Lemma 2.6.6 . Hence any projection on a Hilbert space satisfies assumption (i) of Theorem 5.5.

(b) Combettes [9] defines a family of sets (S_i) that satisfies condition (ii) as being boundedly regular. Condition (ii) of Theorem 5.5 will be satisfied if $\dim H < \infty$.

(c) By remarks (a) and (b) above, Theorem 5.1 is a special case of Theorem 5.5, although the proof of Theorem 5.1 is much simpler than the rather technical proof of Theorem 5.5 .

Chapter 6

Applications to Minimization Problems

In this chapter we work only in a real Hilbert space, and the aim is to minimize some objective function θ. The minimization problem can be stated simply as follows:

$$[\mathrm{P}] \qquad \text{Find } u^* \in F \text{ such that } \theta(u^*) = \inf_{u \in F} \theta(u).$$

Deutsch and Yamada [13] consider this problem for θ that satisfies certain conditions, and firstly show the existence and uniqueness of the solution to the above problem, and then propose an algorithm that converges to this unique minimizer. This algorithm extends the results of Yamada *et al* [39], where they consider the minimization problem only for a quadratic function θ. The algorithm of Deutsch and Yamada [13] also extends results by Halpern [19], Wittmann [38] and Bauschke [2], where the unique minimizer is the projection of some point y onto a set F.

In this chapter, we first consider the algorithm proposed by Yamada *et al* [39]. We obtain a generalization of their Theorem 1, which shows the convergence of some unique fixed points to the unique minimizer, by making fewer assumptions. We then obtain a generalization of their Theorem 2 and a complementary result to Theorem

3. We next consider the algorithm of Deutsch and Yamada [13] and obtain a complementary result to their main result. In addition, we also show the convergence of some fixed points to the unique minimizer. This has not been considered in [13].

For the rest of this chapter H denotes a real Hilbert space. The details about the problem to be solved are outlined below.

Let $T_i : H \to H$ $(i = 1, 2, \ldots, N)$ be nonexpansive mappings with $F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$. Let $\Delta := \bigcup_{i=1}^{N} \text{co}(T_i(H))$ and let the function $\theta : H \to \tilde{\mathbb{R}}$ be twice differentiable on some open set $U \supseteq \Delta$. Suppose $\theta'' : U \to \mathcal{B}(H)$ satisfies the properties that $\theta''(x)$ is self-adjoint for all $x \in \Delta$, and there exist scalars $M \geq m > 0$ such that

$$m\|v\|^2 \leq \langle \theta''(x)v, v\rangle \leq M\|v\|^2 \qquad \text{for all} \quad x \in \Delta \text{ and } v \in H. \tag{6.1}$$

For an arbitrary fixed μ with $0 < \mu < \dfrac{2}{M}$, let

$$\Psi(x) := \mu\theta(x) - \frac{1}{2}\|x\|^2 \text{ for all } x \in H \tag{6.2}$$

and let

$$T^{\lambda}(x) = Tx - \lambda\mu\theta'(Tx) \quad \text{for all} \quad x \in H \quad \text{and all} \quad \lambda \in [0,1]. \tag{6.3}$$

The iteration process that we consider is defined by

$$x_{n+1} = T_{n+1}^{\lambda_{n+1}} x_n \qquad \text{for any } x_0 \in H \tag{6.4}$$

where (λ_n) is a sequence of parameters that satisfies certain conditions.

This process generalizes the iteration scheme proposed by Bauschke [2] where he considered the case of $\theta(x) := \frac{1}{2}\|x - y\|^2$ for some $y \in H$, and the iteration scheme proposed by Yamada *et al* [39] where they considered the quadratic function $\theta(x) := \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$ for all $x \in H$.

We show in Examples 6.16 and 6.17 that the iteration schemes proposed by Bauschke and Yamada *et al*, are indeed special cases of the scheme defined in equation (6.4).

We first provide the following definition.

Definition 6.1. If H is a Hilbert space, then a bounded, self-adjoint operator $A \in \mathcal{B}(H)$ is said to be **strongly positive** if there exists $\alpha > 0$ such that

$$\langle Ax, x\rangle \geq \alpha\|x\|^2 \qquad \text{for all} \quad x \in H.$$

We will first consider the algorithm proposed by Yamada *et al* [39]:

Let $A \in \mathcal{B}(H)$ be self adjoint and strongly positive, and $b \in H$. Define a quadratic function $\theta : H \to \mathbb{R}$ as follows:

$$\theta(x) := \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle. \tag{6.5}$$

We note the following result.

Lemma 6.2 ([22])**.** *Let H be a Hilbert space and $A : H \to H$ a linear operator. If A is a bounded, self-adjoint operator, then*

$$\|A\| = \sup_{\|x\|=1} |\langle Ax, x\rangle|.$$

Since A is strongly positive, there exists $\alpha > 0$ such that $\langle Ax, x\rangle \geq \alpha\|x\|^2$. This is the α that is referred to in the following two results.

The following fact characterizes the minimizer of a quadratic function. It can be found in [42].

Lemma 6.3 ([39]; 25E; 25.23). *Let H be a Hilbert space, $b \in H$ and $A \in \mathcal{B}(H)$ be a self-adjoint and strongly positive operator. Let θ be defined by equation (6.5). Then for a nonempty, closed and convex subset C of H,*

(a) there exists a unique a minimizer u^* of θ over C; i.e. $\theta(u^*) = \min_{x \in C} \theta(x)$.

(b) $u^* \in C$ is a minimizer of θ over C if and only if u^* satisfies $\langle Au^* - b, x - u^* \rangle \geq 0$ for all $x \in C$.

The following result exhibits a sequence that converges to the unique minimizer described above. It is a generalization of a result by Browder [6] and it is proved by Yamada, *et al* ([39]; Theorem 1). They make the assumption that $\|I - A\| < 1$, but in our proof we avoid use of this assumption. In the statement of parts (a) and (b), we make an additional assumption that $\lambda \leq \frac{1}{\|A\|}$, but this is not restrictive, since in part (c) we take $\lambda \longrightarrow 0$ and so λ can be chosen as small as possible. In the proof of part(b), Yamada *et al* [39] make use of a projection, but our proof for part (b) avoids the use of a projection, and hence the result will be valid in any Banach space.

Theorem 6.4. *Let H be a Hilbert space and $T : H \to H$ nonexpansive with $Fix(T) \neq \emptyset$. Suppose that u^* is the unique minimizer of the quadratic function θ over $Fix(T)$ where θ is defined by equation (6.5) . Let*

$$C_f := \left\{ x \in H : \|x - f\| \leq \frac{\|b - Af\|}{\alpha} \right\} \quad \text{for all} \quad f \in Fix(T)$$

and

$$T^\lambda(x) = (I - \lambda A)Tx + \lambda b \quad \text{for all} \quad \lambda \in [0,1] \quad \text{and} \quad x \in H.$$

Then

(a) $T^\lambda(C_f) \subseteq C_f$ *for all* $f \in Fix(T)$ *and* $\lambda \in [0,1]$, *with* $\lambda < \dfrac{1}{\|A\|}$. *In particular,* $T(C_f) \subseteq C_f$ *for all* $f \in Fix(T)$.

(b) $T^\lambda : H \to H$ *is a contraction with its unique fixed point* $\xi_\lambda \in \bigcap_{f \in Fix(T)} C_f$ *for all* $\lambda \in (0,1]$, *with* $\lambda < \dfrac{1}{\|A\|}$.

(c) $\lim_{\lambda \to 0} \xi_\lambda = u^*$.

Proof. The existence of the unique minimizer is guaranteed by Lemma 6.3.

(a) For $x \in H$,

$$\begin{aligned} \alpha\|x\|^2 &\leq \langle Ax, x\rangle \\ &\leq \|A\|\,\|x\|^2. \end{aligned}$$

Hence

$$\|A\| \geq \alpha. \tag{6.6}$$

Also, for $x \in H,\ x \neq 0$,

$$\begin{aligned} \langle (I - \lambda A)x, x\rangle &= \|x\|^2 - \lambda\langle Ax, x\rangle \\ &\geq \|x\|^2 - \lambda\|A\|\,\|x\|^2 \\ &= (1 - \lambda\|A\|)\|x\|^2 \\ &> 0, \end{aligned}$$

since $\lambda < \dfrac{1}{||A||}$.

Hence, by Lemma 6.2 , for $x, y \in H$,

$$\begin{aligned}
||T^{\lambda}x - T^{\lambda}y|| &= ||(I - \lambda A)(Tx - Ty)|| \\
&\leq ||I - \lambda A|| \, ||Tx - Ty|| \\
&\leq ||I - \lambda A|| \, ||x - y|| \\
&= \left(\sup_{||u||=1} \langle (I - \lambda A)u, u \rangle \right) ||x - y|| \\
&= \sup_{||u||=1} (||u||^2 - \lambda \langle Au, u \rangle) ||x - y|| \\
&\leq \sup_{||u||=1} (||u||^2 - \lambda\alpha ||u||^2) ||x - y|| \\
&= (1 - \lambda\alpha) ||x - y||. \qquad (6.7)
\end{aligned}$$

Let $f \in Fix(T)$. Then

$$\begin{aligned}
T^{\lambda}f &= (I - \lambda A)Tf + \lambda b \\
&= (I - \lambda A)f + \lambda b,
\end{aligned}$$

and so

$$T^{\lambda}f - f = -\lambda(Af - b). \qquad (6.8)$$

Therefore for $x \in C_f$, we have

$$\begin{aligned}
||T^{\lambda}x - f|| &\leq ||T^{\lambda}x - T^{\lambda}f|| + ||T^{\lambda}f - f|| \\
&\leq (1 - \lambda\alpha) \, ||x - f|| + \lambda ||Af - b|| \\
&\leq (1 - \lambda\alpha) \frac{||b - Af||}{\alpha} + \lambda\alpha \frac{||Af - b||}{\alpha} \\
&= \frac{||b - Af||}{\alpha},
\end{aligned}$$

proving that $T^{\lambda}(C_f) \subseteq C_f$.

In particular, if $\lambda = 0$, then $T(C_f) \subseteq C_f$.

(b) Since $\lambda, \alpha > 0$, then $\lambda\alpha > 0$. Also from inequality (6.6) and our assumption on λ,

$$\lambda\alpha < \frac{1}{||A||}||A|| = 1.$$

So inequality (6.7) implies that T^λ is a contraction with constant $1-\lambda\alpha$ for $\lambda \in (0,1]$ and $\lambda < \frac{1}{||A||}$. By Banach's contraction mapping principle (Theorem 2.6.2), T^λ has a unique fixed point, ξ_λ.

For any $f \in Fix(T)$, we need to show that $\xi_\lambda \in C_f$. Now

$$\begin{aligned} ||\xi_\lambda - f|| &= ||T^\lambda \xi_\lambda - f|| \\ &\leq ||T^\lambda \xi_\lambda - T^\lambda f|| + ||T^\lambda f - f|| \\ &\leq (1-\lambda\alpha)||\xi_\lambda - f|| + \lambda ||Af - b|| \end{aligned}$$

by inequality (6.7) and equation (6.8). So

$$||\xi_\lambda - f|| \leq \frac{\lambda ||Af - b||}{\lambda\alpha} = \frac{||Af-b||}{\alpha}.$$

Thus $\xi_\lambda \in C_f$ for each $f \in Fix(T)$. Hence $\xi_\lambda \in \bigcap_{f \in Fix(T)} C_f$.

(c) By Lemma 2.0.2 , it suffices to show that every subsequence (ξ_n) of (ξ_λ) has a subsequence that converges to u^*, where $\xi_n := \xi_{\lambda_n}$, and $\lambda_n \longrightarrow 0$.

Now C_f is actually the closed ball centered at f with radius $\frac{||b - Af||}{\alpha}$. Hence C_f is weakly compact by Fact 2.0.8 . Since (ξ_n) is bounded, there exists a subsequence (ξ_{n_j}) that converges weakly to a point $v \in C_f$ for all $f \in Fix(T)$. We will write $v_j = \xi_{n_j}$ to simplify the notation. Hence $v_j \rightharpoonup v$.

Claim 1: T(v)=v. Since $(T(v_j))$ is bounded, there exists $r > 0$ such that $||T(v_j)|| < r$ for all $j \geq 1$. Now $T^{n_j} v_j := T^{\lambda_{n_j}} v_j = v_j$ and $\lim_{n\to\infty} \lambda_n = 0$ give us

$$\begin{aligned} \|T(v_j) - v_j\| &= \lambda_{n_j}\|AT(v_j) - b\| \\ &\leq \lambda_{n_j}(\|A\|\,\|T(v_j)\| + \|b\|) \\ &\leq \lambda_{n_j}(\|A\|r + \|b\|) \\ &\longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty. \end{aligned} \tag{6.9}$$

By the Demiclosedness Principle, $Tv = v$, as was claimed.

Claim 2:

$$\|v_j - u^*\|^2 \leq \frac{1}{\alpha}\langle b - Au^*, v_j - v\rangle. \tag{6.10}$$

Since $v_j = (I - \lambda_{n_j}A)Tv_j + \lambda_{n_j}b$ and $Tu^* = u^*$, we can write

$$v_j - u^* = (I - \lambda_{n_j}A)(Tv_j - Tu^*) + \lambda_{n_j}(b - Au^*).$$

In the proof of inequality (6.7), we have shown that for those λ with $\lambda < \dfrac{1}{\|A\|}$, $\|I - \lambda A\| \leq (1 - \lambda\alpha)$. We may choose j large enough so that $\lambda_{n_j} < \dfrac{1}{\|A\|}$. Then it follows that

$$\begin{aligned} \|v_j - u^*\| &= \langle v_j - u^*, v_j - u^*\rangle \\ &= \langle (I - \lambda_{n_j}A)(Tv_j - Tu^*) + \lambda_{n_j}(b - Au^*), v_j - u^*\rangle \\ &\leq \|I - \lambda_{n_j}A\|\,\|Tv_j - Tu^*\|\,\|v_j - u^*\| + \lambda_{n_j}\langle b - Au^*, v_j - u^*\rangle \\ &\leq (1 - \lambda_{n_j}\alpha)\|v_j - u^*\|^2 + \lambda_{n_j}\langle b - Au^*, v_j - u^*\rangle. \end{aligned}$$

Hence $\|v_j - u^*\|^2 \leq \dfrac{1}{\alpha}\langle b - Au^*, v_j - u^*\rangle$, which proves claim 2.

Since $v_j \rightharpoonup v \in Fix(T)$ we have by Lemma 6.3 that

$$\overline{\lim}\,\|v_j - u^*\|^2 \leq \frac{1}{\alpha}\langle b - Au^*, v - u^*\rangle \leq 0.$$

Thus $v_j \longrightarrow u^*$ as $j \longrightarrow \infty$. $\square$

Again, for finitely many maps $T_1, T_2, \ldots, T_N$, and for any $n \geq 1$, we will define T_n by $T_n := T_{\text{ mod } N}$.

The following theorem is an extension of Theorem 2 of [39] with $N = 1$, since a sequence of parameters (λ_n) with $\lim_{n\to\infty} \lambda_n = 0$ that satisfies Lions' condition $\lim_{n\to\infty} \frac{\lambda_n - \lambda_{n+1}}{\lambda_{n+1}^2} = 0$ will also satisfy [N3].

We also note that in Theorem 2 of [39] we can leave out the condition that $\|I - A\| < 1$. Example 6.17 shows that Theorem 6.14 is a generalization of Theorem 1 of [39] if $\|I - A\| < 1$.

Theorem 6.5. *Let H be a Hilbert space, $T_i : H \to H$ $(i = 1, 2, \ldots, N)$ are nonexpansive maps with*

$$F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$$

and

$$F = Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_2) = \cdots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_2).$$

Assume that a sequence $(\lambda_n) \subseteq (0, 1]$ satisfies

[N1] $\lim_{n\to\infty} \lambda_n = 0.$

[N2] $\sum_{n=1}^{\infty} \lambda_n = \infty.$

[N3] $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1.$

Then for any $x_0 \in H$, the sequence generated by

$$x_{n+1} = \lambda_{n+1} b + (I - \lambda_{n+1} A) T_{n+1} x_n, \quad n \geq 0$$

converges to u^*, the unique minimizer of the quadratic function θ, defined in equation (6.5) over F.

Proof. The existence of $u^* \in F$ is guaranteed by Lemma 6.3 .

Case 1. We will first assume that $x_0 \in C_{u^*} = \left\{ x \in H : \|x - u^*\| \leq \dfrac{\|b - Au^*\|}{\alpha} \right\}$. The general case will be reduced to this case. The proof follows the following steps.

(1) (x_n) and $(T_n x_{n-1})$ are bounded:

If $T^{\lambda_n} x = \lambda_n b + (I - \lambda_n A) T_n x$, then $x_n = T^{\lambda_n} x_{n-1}$. From Theorem 6.4 (a), $(x_n) \subseteq C_{u^*}$ and $(T_n x_{n-1}) \subseteq C_{u^*}$. So, (x_n) and $(T_n x_{n-1})$ are bounded.

(2) $x_{n+1} - T_{n+1} x_n \longrightarrow 0$:

$$\begin{aligned} \|x_{n+1} - T_{n+1} x_n\| &= \|\lambda_{n+1} b + (I - \lambda_{n+1} A) T_{n+1} x_n - T_{n+1} x_n\| \\ &= \lambda_{n+1} \|b - A T_{n+1} x_n\| \\ &\leq \lambda_{n+1} \left[\|b\| + \|A\| \, \|T_{n+1} x_n\| \right], \end{aligned}$$

since $A \in \mathcal{B}(H)$. $(T_{n+1} x_n)$ bounded, and $\lambda_n \longrightarrow 0$ imply that $x_{n+1} - T_{n+1} x_n \longrightarrow 0$.

(3) $x_{n+1} - x_n \longrightarrow 0$:

Choose n large enough so that $\lambda_{n+N} < \dfrac{1}{\|A\|}$.

Since $A \in \mathcal{B}(H)$ and $(T_n x_{n-1})$ is bounded, we can find $L > 0$ such that

$$\|b - A T_n x_{n-1}\| \leq L\alpha.$$

We also have $T_{n+N} = T_n$. Therefore by the proof of inequality (6.7) of Theorem 6.4 ,

$$\begin{aligned}
\|x_{n+N} - x_n\| &= \|\lambda_{n+N} b + (I - \lambda_{n+N} A) T_{n+N} x_{n+N-1} - [\lambda_n b + (I - \lambda_n A) T_n x_{n-1}]\| \\
&= \|(\lambda_{n+N} - \lambda_n)(b - A T_n x_{n-1}) + (I - \lambda_{n+N} A)(T_n x_{n+N-1} - T_n x_{n-1})\| \\
&\leq L\alpha |\lambda_{n+N} - \lambda_n| + \|I - \lambda_{n+N} A\| \, \|x_{n+N-1} - x_{n-1}\| \\
&\leq L\alpha |\lambda_{n+N} - \lambda_n| + (1 - \lambda_{n+N}\alpha) \|x_{n+N-1} - x_{n-1}\|
\end{aligned}$$

Now $\lim\limits_{n\to\infty} \dfrac{L\alpha|\lambda_{n+N} - \lambda_n|}{\lambda_{n+N}\alpha} = 0$ by [N3]. So by Lemma 2.1.2 , $x_{n+N} - x_n \longrightarrow 0$.

(4) $x_n - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0$:

In view of (3), it suffices to show that

$$x_{n+N} - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0.$$

By (2),

$$x_{n+N} - T_{n+N} x_{n+N-1} \longrightarrow 0$$

and

$$x_{n+N-1} - T_{n+N-1} x_{n+N-2} \longrightarrow 0.$$

T_{n+N} nonexpansive, implies that

$$T_{n+N} x_{n+N-1} - T_{n+N} T_{n+N-1} x_{n+N-2} \longrightarrow 0.$$

Similarly,

$$T_{n+N} T_{n+N-1} x_{n+N-2} - T_{n+N} T_{n+N-1} T_{n+N-2} x_{n+N-3} \longrightarrow 0$$

$$\vdots$$

$$T_{n+N} \cdots T_{n+2} x_{n+1} - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0.$$

Adding these N sequences yields

$$x_{n+N} - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0.$$

(5) $\overline{\lim}\, \langle T_{n+1}x_n - u^*, b - Au^* \rangle \leq 0$:

$(\langle T_{n+1}x_n - u^*, b - Au^* \rangle)$ is bounded since $(T_{n+1}x_n)$ is bounded. Also, C_{u^*} is a closed ball centered at u^*, hence it is weakly compact by Fact 2.0.8. Thus we can find a subsequence (n_j) such that

$$\overline{\lim}\, \langle T_{n+1}x_n - u^*, b - Au^* \rangle = \lim_j \langle T_{n_j+1}x_{n_j} - u^*, b - Au^* \rangle,$$

$$T_{n_j+1} = T_i \qquad \text{for some } i \in \{1, 2, \ldots, N\}$$

and

$$x_{n_j} \rightharpoonup \hat{x} \in C_{u^*}.$$

By (4) we get

$$x_{n_j} - T_{i+N-1} \cdots T_i x_{n_j} = x_{n_j} - T_{n_j+N} \cdots T_{n_j+1} x_{n_j} \longrightarrow 0.$$

By the Demiclosedness Principle , $\hat{x} \in Fix(T_{i+N} \cdots T_{i+1}) = F$. Therefore

$$\begin{aligned}
\overline{\lim}\, \langle T_{n+1}x_n - u^*, b - Au^* \rangle &= \lim_j \langle T_{n_j+1}x_{n_j} - u^*, b - Au^* \rangle \\
&= \lim_j \langle T_i x_{n_j} - u^*, b - Au^* \rangle \\
&= \langle T_i \hat{x} - u^*, b - Au^* \rangle \\
&= \langle \hat{x} - u^*, b - Au^* \rangle \\
&\leq 0
\end{aligned}$$

by Lemma 6.3 (b).

(6) $x_n \longrightarrow u^*$:

Since $(T_{n+1}x_n)$ is bounded, $A \in \mathcal{B}(H)$ and $\lambda_n \longrightarrow 0$, for any $\epsilon > 0$, we can find $M \in \mathbb{N}$ such that for all $n \geq M$ we have $\lambda_{n+1} < \dfrac{1}{||A||}$ and,

$$2\lambda_{n+1}|\langle b - Au^*, AT_{n+1}x_n - Au^*\rangle| < \frac{\epsilon\alpha}{3},$$

$$\lambda_{n+1}||b - Au^*||^2 < \frac{\epsilon\alpha}{3},$$

and

$$2\langle b - Au^*, T_{n+1}x_n - u^*\rangle < \frac{\epsilon\alpha}{3}.$$

Now

$$\begin{aligned}
||x_{n+1} - u^*||^2 &= ||\lambda_{n+1}(b - Au^*) + (I - \lambda_{n+1}A)(T_{n+1}x_n - u^*)||^2 \\
&= \lambda_{n+1}^2||b - Au^*||^2 + 2\lambda_{n+1}\langle b - Au^*, (I - \lambda_{n+1}A)(T_{n+1}x_n - u^*)\rangle \\
&\quad +||(I - \lambda_{n+1}A)(T_{n+1}x_n - u^*)||^2 \\
&\leq \lambda_{n+1}^2||b - Au^*||^2 + 2\lambda_{n+1}\langle b - Au^*, T_{n+1}x_n - u^*\rangle \\
&\quad -2\lambda_{n+1}^2\langle b - Au^*, AT_{n+1}x_n - Au^*\rangle + ||I - \lambda_{n+1}A||^2||T_{n+1}x_n - u^*||^2 \\
&\leq \lambda_{n+1}\frac{\epsilon\alpha}{3} + \lambda_{n+1}\frac{\epsilon\alpha}{3} + \lambda_{n+1}\frac{\epsilon\alpha}{3} + (1 - \lambda_{n+1}\alpha)||x_n - u^*||^2 \\
&= \epsilon\lambda_{n+1}\alpha + (1 - \lambda_{n+1}\alpha)||x_n - u^*||^2.
\end{aligned}$$

By Lemma 2.1.2 , $x_n \longrightarrow u^*$.

Case 2. We now consider the general case where x_0 is an arbitrary element of H. Let (x_n) be generated by x_0 and let (s_n) be generated by starting at $s_0 \in C_{u^*}$. Then by Case 1, $s_n \longrightarrow s_0$. Therefore it suffices to show that $||x_n - s_n|| \longrightarrow 0$. Since $\lambda_n \longrightarrow 0$ we may assume that there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $\lambda_n < \dfrac{1}{||A||}$.

Now

$$x_n = \lambda_n b + (I - \lambda_n A)T_n x_{n-1}$$

and

$$s_n = \lambda_n b + (I - \lambda_n A)T_n s_{n-1}.$$

Therefore it follows that for $n \geq N$

$$\begin{aligned} \|x_n - s_n\| &= \|(I - \lambda_n A)(T_n x_{n-1} - T_n s_{n-1})\| \\ &\leq \|I - \lambda_n A\| \, \|x_{n-1} - s_{n-1}\| \\ &\leq (1 - \lambda_n \alpha) \, \|x_{n-1} - s_{n-1}\| \\ &\vdots \\ &\leq \prod_{k=N}^{n} (1 - \lambda_k \alpha) \, \|x_{N-1} - s_{N-1}\| \longrightarrow 0 \end{aligned}$$

by [N2] and Lemma 2.1.2 . Hence $\|x_n - s_n\| \longrightarrow 0$. □

For $A = I$ we obtain the iteration

$$x_{n+1} = \lambda_{n+1} b + (1 - \lambda_{n+1})T_{n+1} x_n.$$

Hence Theorem 6.5 becomes an extension of Theorem 3.7 . By Theorem 3.7 , $x_n \longrightarrow P_F b$ where P_F is the projection of H onto F. In this case, therefore, the unique minimizer of the minimization problem [P], is $P_F b$.

For the remainder of this chapter, the iteration scheme proposed by Deutsch and Yamada [13] is considered and it is shown to converge to the solution of the optimization problem. The main result of their paper, Theorem 3.7, shows the convergence of the iterates to the unique minimizer of some function θ over F. We provide a complementary result to their main result where we replace Bauschke's condition

[B3] $\sum_{n=1}^{\infty} |\lambda_{n+N} - \lambda_n| < \infty$

on the parameters by our new condition

[N3] $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1.$

We first define those functions that will be considered in the problem [P].

Definition 6.6. Let S be a subset of a Hilbert space H, and let a function $\theta : H \to \tilde{\mathbb{R}}$ be twice differentiable on some open set $U \supseteq S$. Then $\theta'' : U \to \mathcal{B}(H)$ is said to be **uniformly strongly positive and uniformly bounded** (or USPUB) over S if $\theta''(x)$ is self-adjoint for all $x \in S$, and there exist scalars $M \geq m > 0$ such that

$$m\|v\|^2 \leq \langle \theta''(x)v, v\rangle \leq M\|v\|^2 \text{ for all } x \in S \text{ and } v \in H.$$

Definition 6.7. Let $f : H \to \tilde{\mathbb{R}}$. Then f is said to be **lower semicontinuous** at $x_0 \in H$ if

$$f(x_0) = \liminf_{x\to x_0} f(x) = \sup_{V\in N_{x_0}} \inf_{v\in V} f(x),$$

where N_{x_0} is the set of all neighbourhoods of the point x_0.

The following result was proved by Deutsch and Yamada [13] and it gives a characterization of a minimizer of a convex function.

Lemma 6.8 ([13]; Lemma 2.1). *(Characterization of minimizers of a convex differentiable function.) Let $\theta : H \to \tilde{\mathbb{R}}$ be lower semicontinuous over H, convex over a nonempty, closed and convex subset C of H, and differentiable over some open set $U \supseteq C$. Then u^* is a minimizer of θ over C if and only if $\langle \theta'(u^*), x - u^*\rangle \geq 0$ for all $x \in C$.*

The following result guarantees the existence of the unique minimizer and its proof can be found in [13].

Theorem 6.9 ([13]; Theorem 3.4). *(Existence and Uniqueness of Optimal Solutions) Let C be a nonempty, closed and convex subset of a Hilbert space H, and let $U \subseteq H$ be an open subset containing C. Assume that $\theta : H \to \tilde{\mathbb{R}}$ is twice differentiable on U, and there exists some $m > 0$ such that*

$$m\|v\|^2 \leq \langle \theta''(x)v, v\rangle \quad \text{for all} \quad x \in C \quad \text{and} \quad v \in H.$$

Then there exists a unique point $u^* \in C$ such that

$$\theta(u^*) = \inf_{u \in C} \theta(u).$$

The following three results are needed for the proof of the main result, the first of which is proved in [13]. The second result is well-known.

Lemma 6.10 ([13]). *Let S be a subset of a Hilbert space H, and let $\theta : H \to \tilde{\mathbb{R}}$ be twice differentiable on some open set $U \supseteq S$. Suppose that $\theta'' : U \to \mathcal{B}(H)$ satisfies the condition USPUB over S. Then for each $0 < \mu < \dfrac{2}{M}$, the function $\Psi = \Psi_\mu : H \to \mathbb{R}$ defined by*

$$\Psi(x) := \mu\theta(x) - \frac{1}{2}\|x\|^2$$

is twice differentiable and $\Psi'' : U \to \mathcal{B}(H)$ satisfies

$$|\langle \Psi''(x)v, v\rangle| \leq L\|v\|^2 \qquad \text{for all} \quad x \in S \quad \text{and} \quad v \in H.$$

In particular,

$$\|\Psi'(x) - \Psi'(y)\| \leq L\|x - y\| \qquad \text{for all } x, y \in S, \tag{6.11}$$

where $L := \max\{|\mu m - 1|, |\mu M - 1|\} < 1$.

Lemma 6.11. *Let H be a Hilbert space. If $h(x) = \frac{1}{2}||x||^2$, then $h'(x) = x$.*

Proof. For all $v \in V$,

$$\begin{aligned}\langle h'(x), v\rangle &= \lim_{t\to 0}\frac{h(x+tv)-h(x)}{t}\\ &= \lim_{t\to 0}\frac{\frac{1}{2}||x+tv||^2-\frac{1}{2}||x||^2}{t}\\ &= \lim_{t\to 0}\frac{||x||^2+t^2||v||^2+2t\langle x,v\rangle-||x||^2}{2t}\\ &= \lim_{t\to 0}\frac{t}{2}||v||+\langle x,v\rangle\\ &= \langle x,v\rangle.\end{aligned}$$

Hence $h'(x) = x$. ☐

Lemma 6.12 ([13]). *Let $T : H \to H$ be a nonexpansive mapping with $Fix(T) \neq \emptyset$. Suppose that a function $\theta : H \to \tilde{\mathbb{R}}$ is twice differentiable on some open set $U \supseteq$ co$(T(H))$, and $\theta'' : U \to \mathcal{B}(H)$ satisfies the condition USPUB over co$(T(H))$. For an arbitrarily fixed μ with $0 < \mu < \dfrac{2}{M}$, let*

$$\Psi(x) := \mu\theta(x) - \frac{1}{2}||x||^2 \quad \textit{for all} \quad x \in H,$$

$$\begin{aligned}T^\lambda(x) &:= T(x) - \lambda\mu\theta'(T(x))\\ &= (1-\lambda)T(x) - \lambda\Psi'(T(x)) \quad \textit{for all} \quad \lambda\in[0,1] \quad \textit{and} \quad x \in H\end{aligned}$$

and for $f \in Fix(T)$, let

$$C_f := \left\{x \in H : ||x - f|| \leq \frac{||f + \Psi'(f)||}{1-L}\right\},$$

where $L := \max\{|\mu m - 1|, |\mu M - 1|\} < 1$. Then

$$||T^\lambda(x) - T^\lambda(y)|| \leq [1 - \lambda(1-L)]||x - y|| \qquad \textit{for all } x, y \in H \tag{6.12}$$

and for all $f \in Fix(T)$,

$$T^{\lambda}(C_f) \subseteq C_f \qquad \textit{for all } \lambda \in [0,1].$$

We are able to deduce from inequality (6.12) that T^{λ} is a contraction for all $\lambda \in (0,1]$, and so by Banach's Contraction Mapping Principle (Theorem 2.6.2), T^{λ} has a unique fixed point u_{λ}. Before we show the convergence of the iterates defined by equation (6.4) we will first show the convergence of these fixed points to the solution of [P]. This result is new as the behaviour of these fixed points was not studied by Deutsch and Yamada [13].

Theorem 6.13. *Let* u_{λ} *be the unique fixed point of* T^{λ} *for* $\lambda \in (0,1]$ *as is defined in equation (6.3). Then* $\lim\limits_{\lambda \to 0} u_{\lambda}$ *exists and solves the minimization problem [P].*

Proof. Let $u^* \in F(T)$ be the unique solution of the minimization problem. Its existence is guaranteed by Theorem 6.9 . Since u_{λ} is the fixed point of T^{λ}, we have

$$u_{\lambda} = Tu_{\lambda} - \lambda\mu\theta'(Tu_{\lambda}) = (1-\lambda)Tu_{\lambda} - \lambda\Psi'(Tu_{\lambda}).$$

Hence

$$(1-\lambda)(u_{\lambda} - Tu_{\lambda}) + \lambda(u_{\lambda} + \Psi'(Tu_{\lambda})) = 0.$$

It is evident that

$$(1-\lambda)(u^* - Tu^*) + \lambda(u^* + \Psi'(Tx^*)) = \lambda(u^* + \Psi'(Tu^*)).$$

Subtract these two equalities to get

$$\begin{aligned} & (1-\lambda)[(u_{\lambda} - u^*) - (Tu_{\lambda} - Tu^*)] + \lambda[u_{\lambda} - u^* + \Psi'(Tu_{\lambda}) - \Psi'(Tu^*)] \\ = \quad & -\lambda(u^* + \Psi'(Tu^*)). \end{aligned}$$

Noting that $u^* + \Psi'(Tu^*) = \mu\theta'(Tu^*)$ and using $u_\lambda - u^*$ to make the the inner product we obtain,

$$\begin{aligned} & (1-\lambda)\langle (u_\lambda - u^*) - (Tu_\lambda - Tu^*), u_\lambda - u^* \rangle \\ & +\lambda[||u_\lambda - u^*||^2 + \langle \Psi'(Tu_\lambda - \Psi'(Tu^*), u_\lambda - u^* \rangle] \\ = \quad & -\lambda\mu\langle \theta'(u^*), u_\lambda - u^* \rangle. \end{aligned} \tag{6.13}$$

Now

$$\begin{aligned} \langle (u_\lambda - u^*) - (Tu_\lambda - Tu^*), u_\lambda - u^* \rangle &= ||u_\lambda - u^*||^2 - \langle Tu_\lambda - Tu^*, u_\lambda - u^* \rangle \\ &\geq ||u_\lambda - u^*||^2 - ||Tu_\lambda - Tu^*||\,||u_\lambda - u^*|| \\ &\geq ||u_\lambda - u^*||^2 - ||u_\lambda - u^*||^2 \\ &= 0, \end{aligned} \tag{6.14}$$

and by Lemma 6.10 ,

$$\begin{aligned} |\langle \Psi'(Tu_\lambda) - \Psi'(Tu^*), u_\lambda - u^* \rangle| &\leq ||\Psi'(Tu_\lambda) - \Psi'(Tu^*)||\,||u_\lambda - u^*|| \\ &\leq L||Tu_\lambda - Tu^*||\,||u_\lambda - u^*|| \\ &\leq L||u_\lambda - u^*||^2. \end{aligned} \tag{6.15}$$

Thus we obtain from equation (6.13), and inequalities (6.14) and (6.15), that

$$-\lambda\mu\langle \theta'(u^*), u_\lambda - u^* \rangle \geq \lambda\,[\,||u_\lambda - u^*||^2 - L||u_\lambda - u^*||^2\,]$$

and we may deduce that

$$||u_\lambda - u^*||^2 \leq -\frac{\mu}{1-L}\langle \theta'(u^*), u_\lambda - u^* \rangle. \tag{6.16}$$

Since $(\theta'(Tu_\lambda))$ is bounded,

$$||u_\lambda - Tu_\lambda|| = \lambda\mu||\theta'(Tu_\lambda)|| \longrightarrow 0 \qquad \text{as } \lambda \longrightarrow 0.$$

Hence we have by the Demiclosedness Principle that every weak cluster point of (u_λ) is a fixed point of T.

Take any weak cluster point $\tilde{x}$ of (u_λ); i.e. $u_n := u_{\lambda_n} \rightharpoonup \tilde{x}$. Therefore $\tilde{x} \in Fix(T)$ and hence $\langle \theta'(u^*), \tilde{u} - u^* \rangle \geq 0$ by Lemma 6.8. Therefore by inequality (6.16), we have

$$\begin{aligned} \overline{\lim} \|u_n - u^*\|^2 &\leq -\frac{\mu}{1-L}\langle \theta'(u^*), \tilde{x} - u^* \rangle \\ &\leq 0. \end{aligned}$$

This implies that $u_n \longrightarrow u^*$, and hence $\tilde{x} = u^*$. So u^* is the only weak and norm cluster point of (u_λ) and therefore $u_\lambda \longrightarrow u^*$. □

We will now prove the main result of this section.

Theorem 6.14. *Let $T_i : H \to H$ $(i = 1, 2, \ldots, N)$ be nonexpansive mappings with*

$$F := \bigcap_{i=1}^{N} Fix(T_i) \neq \emptyset$$

and

$$F := Fix(T_N \cdots T_1) = Fix(T_1 T_N \cdots T_2) = \ldots = Fix(T_{N-1} T_{N-2} \cdots T_1 T_N).$$

Let $\Delta := \bigcup_{i=1}^{N} \mathrm{co}(T_i(H))$ and suppose that a function $\theta : H \longrightarrow \tilde{\mathbb{R}}$ is twice differentiable on some open set $U \supseteq \Delta$, and $\theta'' : U \longrightarrow \mathcal{B}(H)$ satisfies the condition USPUB over Δ. Assume that (λ_n) is a sequence of parameters in $(0, 1]$ that satisfies

[*N1*] $\lim_{n\to\infty} \lambda_n = 0$.

[*N2*] $\sum_{n\geq 1} \lambda_n = \infty$.

[*N3*] $\lim_{n\to\infty} \frac{\lambda_n}{\lambda_{n+N}} = 1$.

Then for any arbitrary fixed μ, $0 < \mu < \dfrac{2}{M}$, *and any* $x_0 \in H$, *the sequence* (x_n) *generated by*

$$x_{n+1} = T_{n+1}^{\lambda_{n+1}} x_n := T_{n+1}x_n - \lambda_{n+1}\mu\theta'(T_{n+1}x_n)$$

converges to the unique minimizer u^* *of the function* θ *over* F.

Proof. Case 1. We will assume that

$$x_0 \in C_{u^*} := \left\{ x \in H : \|x - u^*\| \leq \frac{\|u^* + \Psi'(u^*)\|}{1 - L} \right\}$$

where $L := \max\{|\mu m - 1|, |\mu M - 1|\} < 1$. The general case will be reduced to this case. We follow the following steps.

(1) (x_n) and (Tx_n) are bounded:

By Lemma 6.12 , $T^\lambda(C_{u^*}) \subseteq C_{u^*}$ for all $\lambda \in [0, 1]$. So $x_{n+1} \in C_{u^*}$ and $T_{n+1}x_n \in C_{u^*}$ (using $\lambda = 0$). Hence (x_n) and $(T_n x_{n-1})$ are bounded.

(2) $x_{n+1} - T_{n+1}x_n \longrightarrow 0$:

Define $\Psi(x) := \mu\theta(x) - \frac{1}{2}\|x\|^2$. By Lemma 6.10 , the nonexpansivity of T_{n+1} and the fact that $x_n \in C_{u^*}$, we have for all $n \geq 0$,

$$\begin{aligned} \|\Psi'(T_{n+1}x_n) - \Psi'(u^*)\| &\leq L\|T_{n+1}x_n - u^*\| \\ &\leq L\|x_n - u^*\| \\ &\leq \frac{L}{1 - L}\|u^* + \Psi'(u^*)\| \end{aligned}$$

This implies that $(\Psi'(T_{n+1}x_n))$ and hence $(\theta'(T_{n+1}x_n))$ are bounded. Noting that $\Psi'(x) = \mu\theta'(x) - x$, we have

$$\|x_{n+1} - T_{n+1}x_n\| = \lambda_{n+1}\|\mu\theta'(T_{n+1}x_n)\|.$$

Since $\lambda_n \longrightarrow 0$, we have $x_{n+1} - T_{n+1}x_n \longrightarrow 0$.

(3) $x_{n+N} - x_n \longrightarrow 0$:

By Lemma 6.12 , the definition of the iterates and the fact that $T_{n+N} = T_n$, we have

$$\begin{aligned}
& \|x_{n+N} - x_n\| \\
= \; & \|T_{n+N}^{\lambda_{n+N}} x_{n+N-1} - T_n^{\lambda_n} x_{n-1}\| \\
\leq \; & \|T_{n+N}^{\lambda_{n+N}} x_{n+N-1} - T_{n+N}^{\lambda_{n+N}} x_{n-1}\| + \|T_{n+N}^{\lambda_{n+N}} x_{n-1} - T_n^{\lambda_n} x_{n-1}\| \\
\leq \; & [1 - \lambda_{n+N}(1-L)] \, \|x_{n+N-1} - x_{n-1}\| \\
& + \|T_{n+N}x_{n-1} - \lambda_{n+N}\mu\theta'(T_{n+N}x_{n-1}) - T_n x_{n-1} + \lambda_n \mu\theta'(T_n x_{n-1})\| \\
= \; & [1 - (1-L)\lambda_{n+N}] \, \|x_{n+N-1} - x_{n-1}\| + |\lambda_{n+N} - \lambda_n| \, \|\mu\theta'(T_n x_{n-1})\| \\
= \; & [1 - (1-L)\lambda_{n+N}] \, \|x_{n+N-1} - x_{n-1}\| + |\lambda_{n+N} - \lambda_n| \, \|T_n x_{n-1} + \Psi'(T_n x_{n-1})\|.
\end{aligned}$$

Since $(\Psi'(T_n x_{n-1}))$ and $(T_n x_{n-1})$ are bounded, there exists $c > 0$ such that

$$\|T_n x_{n-1} + \Psi'(T_n x_{n-1})\| \leq c(1-L) \qquad \text{for all } n \geq 1.$$

Hence

$$\|x_{n+N} - x_n\| \leq [1 - (1-L)\lambda_{n+N}] \, \|x_{n+N-1} - x_{n-1}\| + c(1-L)|\lambda_{n+N} - \lambda_n|.$$

By [N3], $\lim_{n\to\infty} c(1-L)\dfrac{|\lambda_{n+N} - \lambda_n|}{\lambda_{n+N}} = 0$, and so $x_{n+N} - x_n \longrightarrow 0$ by Lemma 2.1.2 .

(4) $x_n - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0$:

In view of (3), it suffices to show that

$$x_{n+N} - T_{n+N} \cdots T_{n+1} x_n \longrightarrow 0.$$

By (2),

$$x_{n+N} - T_{n+N} x_{n+N-1} \longrightarrow 0$$

and

$$x_{n+N-1} - T_{n+N-1} x_{n+N-2} \longrightarrow 0.$$

T_{n+N} nonexpansive implies that

$$T_{n+N}x_{n+N-1} - T_{n+N}T_{n+N-1}x_{n+N-2} \longrightarrow 0.$$

Similarly,

$$\begin{aligned} T_{n+N}T_{n+N-1}x_{n+N-2} - T_{n+N}T_{n+N-1}T_{n+N-2}x_{n+N-3} &\longrightarrow 0 \\ &\vdots \\ T_{n+N}T_{n+N-1}\cdots T_{n+2}x_{n+1} - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n &\longrightarrow 0. \end{aligned}$$

Adding these N sequences yields

$$x_{n+N} - T_{n+N}T_{n+N-1}\cdots T_{n+1}x_n \longrightarrow 0.$$

(5) $\overline{\lim}\,\langle T_{n+1}x_n - u^*, -\theta'(u^*)\rangle \le 0$:

Since (x_n) and $(T_{n+1}x_n)$ are bounded, we can find a subsequence (n_j) such that

$$\overline{\lim}\,\langle T_{n+1}x_n - u^*, -\theta'(u^*)\rangle = \lim_j \langle Tx_{n_j+1}x_{n_j} - u^*, -\theta'(u^*)\rangle,$$

$$T_{n_j+1} = T_i \qquad \text{for some} \quad i \in \{1, 2, \ldots, N\} \text{ and for all } j \ge 1,$$

and

$$x_{n_j} \rightharpoonup \hat{x} \qquad \text{for some } \hat{x} \in C_{u^*}.$$

By (4) we obtain

$$x_{n_j} - T_{i+N}\cdots T_i x_{n_j} = x_{n_j} - T_{n_j+N}\cdots T_{n_j+1}x_{n_j} \longrightarrow 0.$$

By the Demiclosedness Principle, $\hat{x} \in Fix(T_{i+N} \cdots T_{i+1}) = F$. Therefore

$$\begin{aligned}
\overline{\lim}\,\langle T_{n+1}x_n - u^*, -\theta'(u^*)\rangle &= \lim_{j\to\infty}\langle T_{n_j+1}x_{n_j} - u^*, -\theta'(u^*)\rangle \\
&= \lim_{j\to\infty}\langle T_i x_{n_j} - u^*, -\theta'(u^*)\rangle \\
&= \langle T_i\hat{x} - u^*, -\theta'(u^*)\rangle \\
&= \langle \hat{x} - u^*, -\theta'(u^*)\rangle \\
&\leq 0
\end{aligned}$$

by Lemma 6.8 .

(6) $x_n \longrightarrow u^*$:

Noting that $T_{n+1}^{\lambda_{n+1}}u^* = T_{n+1}u^* - \lambda_{n+1}\mu\theta'(T_{n+1}u^*) = u^* - \lambda_{n+1}\mu\theta'(u^*)$,

$$\begin{aligned}
& \|x_{n+1} - u^*\|^2 \\
= \; & \|T_{n+1}^{\lambda_{n+1}}x_n - u^*\|^2 \\
= \; & \|T_{n+1}^{\lambda_{n+1}}x_n - T_{n+1}^{\lambda_{n+1}}u^* - \lambda_{n+1}\mu\theta'(u^*)\|^2 \\
= \; & \|T_{n+1}^{\lambda_{n+1}}x_n - T_{n+1}^{\lambda_{n+1}}u^*\|^2 \\
& +2\mu\lambda_{n+1}\langle T_{n+1}^{\lambda_{n+1}}x_n - T_{n+1}^{\lambda_{n+1}}u^*, -\theta'(u^*)\rangle + \lambda_{n+1}^2\mu^2\|\theta'(u^*)\|^2 \\
= \; & \|T_{n+1}^{\lambda_{n+1}}x_n - T_{n+1}^{\lambda_{n+1}}u^*\|^2 + \lambda_{n+1}^2\mu^2\|\theta'(u^*)\|^2 \\
& +2\mu\lambda_{n+1}\langle T_{n+1}x_n - \lambda_{n+1}\mu\theta'(T_{n+1}x_n) - u^* + \lambda_{n+1}\mu\theta'(u^*), -\theta'(u^*)\rangle \\
= \; & \|T_{n+1}^{\lambda_{n+1}}x_n - T_{n+1}^{\lambda_{n+1}}u^*\|^2 + \lambda_{n+1}^2\mu^2\|\theta'(u^*)\|^2 + 2\mu\lambda_{n+1}\langle T_{n+1}x_n - u^*, -\theta'(u^*)\rangle \\
& -2\mu^2\lambda_{n+1}^2\langle \theta'(T_{n+1}x_n) - \theta'(u^*), \theta'(u^*)\rangle \\
= \; & \|T_{n+1}^{\lambda_{n+1}}x_n - T_{n+1}^{\lambda_{n+1}}u^*\|^2 + 2\mu\lambda_{n+1}\langle T_{n+1}x_n - u^*, -\theta'(u^*)\rangle \\
& +\mu^2\lambda_{n+1}^2\,[\,\|\theta'(u^*)\|^2 - 2\langle\theta'(T_{n+1}x_n) - \theta'(u^*), \theta'(u^*)\rangle\,].
\end{aligned}$$

By the boundedness of $(\theta'(T_{n+1}(x_n)))$, by (5) and the fact that $\lambda_n \longrightarrow 0$, for any

$\epsilon > 0$, we can find $M \in \mathbb{N}$ such that for all $n \geq M$,

$$\begin{aligned} 2\mu\langle T_{n+1}x_n - u^*, -\theta'(u^*)\rangle &< \frac{\epsilon}{2}(1-L), \\ \mu^2\lambda_{n+1}\,[\,||\theta'(u^*)||^2 - 2\langle\theta'(T_{n+1}x_n) - \theta'(u^*), \theta'(u^*)\rangle\,] &< \frac{\epsilon}{2}(1-L). \end{aligned}$$

Using these and Lemma 6.12 , we have for all $n \geq M$,

$$\begin{aligned} ||x_{n+1} - u^*||^2 &\leq [1-\lambda_{n+1}(1-L)]^2||x_n - u^*||^2 + \lambda_{n+1}\frac{\epsilon}{2}(1-L) + \lambda_{n+1}\frac{\epsilon}{2}(1-L) \\ &\leq [1-\lambda_{n+1}(1-L)]\,||x_n - u^*||^2 + \lambda_{n+1}(1-L)\epsilon. \end{aligned}$$

By Lemma 2.1.2 , $x_n \longrightarrow u^*$. This completes the proof for Case 1.

Case 2. We now consider the general case where x_0 is an arbitrary element of H (x_0 may not belong to C_{u^*}). Let (x_n) be generated by x_0 and (s_n) be generated by $s_0 \in C_{u^*}$. Then by Case 1, $s_n \longrightarrow u^*$. Therefore, it suffices to show that $||x_n - s_n|| \longrightarrow 0$.

Now by Lemma 6.12 , we have

$$\begin{aligned} ||x_n - s_n|| &= ||T_n^{\lambda_n}x_{n-1} - T_n^{\lambda_n}s_{n-1}|| \\ &\leq [1-\lambda_n(1-L)]\,||x_{n-1} - s_{n-1}|| \\ &\quad\vdots \\ &\leq \prod_{k=1}^{n}[1-\lambda_k(1-L)]\,||x_0 - s_0|| \\ &\longrightarrow 0 \end{aligned}$$

by [N2] and Lemma 2.1.1 .

Hence $||x_n - s_n|| \longrightarrow 0$, proving the general case. $\square$

Even though the next result follows as a Corollary to Theorem 6.14 , we nevertheless include the proof, because of its simplicity. We replace [N3] in Theorem 6.14 by [L3] $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_{n+1}^2} = 0$, and clearly [L3] implies [N3] if $\lim_{n\to\infty} \lambda_n = 0$.

Corollary 6.15. *Let H be a Hilbert space. Assume that $T : H \to H$ is nonexpansive with $Fix(T) \neq \emptyset$. Assume that a sequence $(\lambda_n) \subseteq (0,1]$ satisfies*

[*N1*] $\lim_{n\to\infty} \lambda_n = 0$.

[*N2*] $\sum_{n=1}^{\infty} \lambda_n = \infty$.

[*N3*] $\lim_{n\to\infty} \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_{n+1}^2} = 0$.

If (x_n) is defined as follows

$$x_{n+1} = T^{\lambda_{n+1}} x_n := Tx_n - \lambda_{n+1}\mu\theta'(Tx_n), \quad x_0 \in H, \quad n \geq 0,$$

then (x_n) converges to u^*, the solution of the problem [P].

Proof. If $u_n := u_{\lambda_n}$ is the unique fixed point of T^{λ_n}, then Theorem 6.13 shows that $u_n \longrightarrow u^*$. Therefore, it suffices to show that $x_n - u_n \longrightarrow 0$ as $n \longrightarrow \infty$.
We have, by Lemma 6.12 , that

$$\begin{aligned} \|x_n - u_n\| &= \|T^{\lambda_n}(x_{n-1}) - T^{\lambda_n}(u_n)\| \\ &\leq [1 - \lambda_n(1-L)]\,\|x_{n-1} - u_n\| \\ &\leq [1 - \lambda_n(1-L)]\,\|x_{n-1} - u_{n-1}\| \\ &\quad +[1 - \lambda_n(1-L)]\,\|u_{n-1} - u_n\| \end{aligned} \tag{6.17}$$

Now by Lemma 6.10 ,

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \|(1-\lambda_n)Tu_n - \lambda_n\Psi'(Tu_n) - (1-\lambda_{n-1})Tu_{n-1} + \lambda_{n-1}\Psi'(Tu_{n-1})\| \\
&= \|(1-\lambda_n)(Tu_n - Tu_{n-1}) - (\lambda_n - \lambda_{n-1})Tu_{n-1} \\
&\quad -(\lambda_n - \lambda_{n-1})\Psi'(Tu_{n-1}) + \lambda_n(\Psi'(Tu_{n-1}) - \Psi'(Tu_n))\| \\
&\leq (1-\lambda_n)\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\,\|Tu_{n-1} + \Psi'(Tu_{n-1})\| \\
&\quad +\lambda_n L\|Tu_n - Tu_{n-1}\| \\
&\leq [1-\lambda_n(1-L)]\,\|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\,\|Tu_{n-1} + \Psi'(Tu_{n-1})\|.
\end{aligned}$$

Hence it follows that

$$\begin{aligned}
\|u_n - u_{n-1}\| &\leq \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n(1-L)}\|Tu_n + \Psi'(Tu_{n-1})\| \\
&\leq \frac{c|\lambda_n - \lambda_{n-1}|}{\lambda_n(1-L)}
\end{aligned}$$

since (Tu_{n-1}) and $(\Psi'(T(u_{n-1})))$ are bounded. By inequality (6.17),

$$\begin{aligned}
\|x_n - u_n\| &\leq [1-\lambda_n(1-L)]\,\|x_{n-1} - u_{n-1}\| + [1-\lambda_n(1-L)]\,\frac{c|\lambda_n - \lambda_{n-1}|}{\lambda_n(1-L)} \\
&= [1-\lambda_n(1-L)]\,\|x_{n-1} - u_{n-1}\| \\
&\quad +\lambda_n(1-L)\,\frac{c[1-\lambda_n(1-L)]}{(1-L)^2}\,\frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n^2} \\
&= [1-\lambda_n(1-L)]\,\|x_{n-1} - u_{n-1}\| + \lambda_n(1-L)\delta_n
\end{aligned}$$

where $\delta_n = \dfrac{c[1-\lambda_n(1-L)]}{(1-L)^2}\,\dfrac{|\lambda_n - \lambda_{n-1}|}{\lambda_n^2} \longrightarrow 0$ as $n \longrightarrow \infty$, by [L1] and [L3]. By Lemma 2.1.2 , $x_n - u_n \longrightarrow 0$. □

We will now show in Example 6.16 that the θ defined by $\theta(x) := \frac{1}{2}\|x-y\|^2$ is a special case of Theorem 6.14 and turns out to be the scheme proposed in Theorem 3.7. We will also show in Example 6.17 that Theorem 2 of [39] with θ defined by $\theta(x) := \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$ is a special case of Theorem 6.14.

Example 6.16. *For $\theta(x) = \frac{1}{2}||x-y||^2$, Theorem 6.14 is an extension of Theorem 3.7.*

$$\begin{aligned}
\langle \theta'(x), v\rangle &= \lim_{t\to 0} \frac{\theta(x+vt)-\theta(x)}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}||x+vt-y||^2 - \frac{1}{2}||x-y||^2}{t} \\
&= \lim_{t\to 0} \frac{||x-y||^2 + t^2||v||^2 + 2t\langle x-y, v\rangle - ||x-y||^2}{2t} \\
&= \lim_{t\to 0} \frac{t}{2}||v|| + \langle x-y, v\rangle \\
&= \langle x-y, v\rangle.
\end{aligned}$$

Thus $\theta'(x) = x - y$. Also,

$$\begin{aligned}
\langle \theta''(x), v\rangle &= \lim_{t\to 0} \frac{\theta'(x+vt)-\theta'(x)}{t} \\
&= \lim_{t\to 0} \frac{(x+vt-y)-(x-y)}{t} \\
&= v.
\end{aligned}$$

So $\theta'' = I$. Now $\langle \theta''(x)v, v\rangle = \langle v, v\rangle = ||v||^2$. So $M = 1$. Choose $\mu = 1 < \frac{2}{M}$. Also

$$\begin{aligned}
x_{n+1} &= T_{n+1}^{\lambda_{n+1}} x_n := T_{n+1}x_n - \lambda_{n+1}\mu\theta'(T_{n+1}x_n) \\
&= T_{n+1}x_n - \lambda_{n+1}(T_{n+1}x_n - y) \\
&= \lambda_{n+1}y + (1-\lambda_{n+1})T_{n+1}x_n,
\end{aligned}$$

which is the iteration scheme of Theorem 3.7, and hence converges to $P_F y$.

By Theorem 6.14, (x_n) converges to u^*, the unique minimizer of problem [P]. Therefore $P_F y = u^*$. This can be verified by the following:

$$\langle P_F y - y, u - P_F y\rangle \geq 0 \quad \text{for all} \quad u \in F$$

by Lemma 2.4.3, and by the Lemma 6.8, $P_F y = u^*$. $\Diamond$

Example 6.17. *For $\theta(x) = \frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle$ where $\mathcal{A} \in \mathcal{B}(H)$ is self-adjoint and strongly positive, and $\|I - A\| \leq 1$, Theorem 6.14 is an extension of Theorem 3 of [39].*

Since A is linear,

$$\begin{aligned}
\langle \theta'(x), v\rangle &= \lim_{t\to 0} \frac{\theta(x+vt) - \theta(x)}{t} \\
&= \lim_{t\to 0} \frac{\left[\,\frac{1}{2}\langle A(x+vt), x+vt\rangle - \langle b, x+vt\rangle\,\right] - \left[\,\frac{1}{2}\langle Ax, x\rangle - \langle b, x\rangle\,\right]}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}[\,\langle Ax, vt\rangle + \langle A(vt), x\rangle + \langle A(vt), vt\rangle\,] - \langle b, vt\rangle}{t} \\
&= \lim_{t\to 0} \frac{\frac{1}{2}[\,t\langle Ax, v\rangle + t\langle Av, x\rangle + t^2\langle A(vt), vt\rangle\,] - t\langle b, v\rangle}{t} \\
&= \frac{1}{2}\,[2\langle Ax, v\rangle] - \langle b, v\rangle \\
&= \langle Ax, v\rangle - \langle b, v\rangle \\
&= \langle Ax - b, v\rangle.
\end{aligned}$$

Thus $\theta'(x) = Ax - b$,

and so $\theta''(x) = A$.

Now A is strongly positive, so there exists $\alpha > 0$ such that $\langle Ax, x\rangle \geq \alpha\|x\|^2$ for all $x \in H$. Therefore

$$\alpha\|v\|^2 \leq \langle Av, v\rangle \leq \|A\|\,\|v\|^2.$$

Choose $M = \|A\| < 2$ and choose $\mu = 1 < \dfrac{2}{M}$. Then

$$\begin{aligned}
x_{n+1} &= T_{n+1}^{\lambda_{n+1}} x_n := T_{n+1}x_n - \lambda_{n+1}\mu\theta'(T_{n+1}x_n) \\
&= T_{n+1}x_n - \lambda_{n+1}(AT_{n+1}x_n - b) \\
&= \lambda_{n+1}b + (I - \lambda_{n+1}A)T_{n+1}x_n,
\end{aligned}$$

which is the iteration scheme of [39]. $\diamondsuit$

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