Credit Derivative Valuation and Parameter Estimation for CIR and Vasicek-type models

UNIVERSITY OF
KWAZULU-NATAL

ALMA PRELL BIMBABOU MABOULOU
212561780

This dissertation is submitted to the School of Mathematics, Statistics and Computer Science, College of Agriculture, Engineering and Science University of KwaZulu-Natal, Durban, in fulfilment of the requirements for the degree of Master in Science, under the supervision of Doctor K.Arunakirinathar.

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed: Dr K.Arunakirinathar December 2013.

Abstract

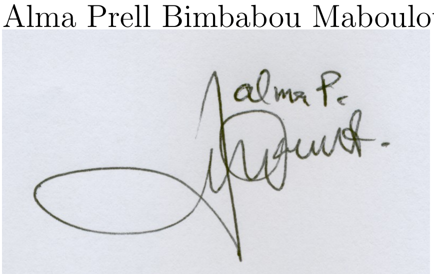
A credit default swap is a contract that ensures protection against losses occurring due to a default event of an certain entity. It is crucial to know how default should be modelled for valuation or estimating of credit derivatives. In this dissertation, we first review the structural approach for modelling credit risk. The model is an approach for assessing the credit risk of a firm by typifying the firms equity as a European call option on its assets, with the strike price (or exercise price) being the promised debt repayment at the maturity. The model can be used to determine the probability that the firm will default (default probability) and the Credit Spread.

We second concentrate on the valuation of credit derivatives, in particular the Credit Default Swap (CDS) when the hazard rate (or even of default) is modelled as the Vasicek-type model. The other objective is, by using South African credit spread data on defaultable bonds to estimate parameters on CIR and Vasicek-type Hazard rate models such as stochastic differential equation models of term structure. The parameters are estimated numerically by the Moment Method.

Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

Alma Prell Bimbabou Maboulou



Declaration 1 - Plagiarism

I, Alma Prell Bimbabou Maboulou, declare that

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Dedication

I dedicate this dissertation to my beloved mother
NZOUSSI Marie Jose
and to my wonderful father
Alphonse BIMBABOU.

Acknowledgments

First of all, I thank the goodness of God for delivering me the strength to finish my thesis. I wish to express my hearty gratitude to my supervisor, Doctor K.Arunakirinathar, for his guidance and presence during this project. My special thanks go to Doctor G.Amery who helped me with editorial suggestions. My special thanks also go to many lecturers who helped me throughout the postgraduate course at the African Institute for Mathematical Sciences (AIMS), especially Professor Bernd Schroers, Professor David Parkirson and Professor Robert Beezer. Rodrigue Massoukou, Roland Loufouma.M and Jefta Mvukaye.S are also thanked for their great advice and suggestions. Sincere gratitude goes to my parents, Alphonse Bimbabou and Nzoussi Marie Jose, who always believed in me and from whom I have always drawn inspiration. I also want to thank rest of my family for their support and prayers. Finally, special thanks go to AIMS community and the University of KwaZulu-Natal (UKZN) for the opportunities given.

Contents

List of notation

SDE	stochastic differential equation
ZCB	zero-coupon bond
r	risk free interest rate or yield on Treasury bond
r^d	yield of the bond
$B(.)$	bank account process
a.s.	almost surely
$\mathcal{N}(.)$	cumulative standard normal distribution function
$\mathbb{Q}$	risk neutral probability
PD	probability of default
T	maturity time
$\mathcal{F}$	filtration
$E(...)$	expectation value
$var(...)$	variance
PD_M	default probability given by the Merton model
CDs	Credit Derivatives
CDS	Credit Default Swap
CIR	Cox-Ingersoll-Ross
$B(t,T)$	Price at time t of a zero free coupon bond with maturity T
$p(\tau < t)$	survival probability in term of default intensity
$\gamma(t)$	hazard rate function
δ	recovery rate
$S(t,T)$	credit spread at time $t < T$

D_p price of the fixed side of CDS

$CC(t)$ cum-coupon amount of the underlying defaultable bond

D_R price of the recovery side of CDS

$D(t, T)$ the price of a zero-coupon defaultable bond with notional value 1 or price of a defaultable bond, which pays 1 at maturity if no default and δ at maturity T if the default has occurred before maturity

Chapter 1

Introduction

The major credit problems and important failures faced by Banks during Global Financial Crises, for example the recent financial crisis or credit crisis of 2007-2008 [?] and the failures of large prestigious institutions such as Lehman Brothers, Bear Sterns, Fannie Mae and Freddie Mac [?, ?], have highlighted the importance of modelling and providing for a Credit Risk quantifier. Credit Risk is the risk that a borrower (company, individual, sovereign government) will default on any type of debt by failing to meet its financial obligation. It emerged to be not just the traditional risk that lenders or ownership of the bonds or loans (example financial institutions) spare when lending out money, but also a financial contract traded or exchanged around the world.

The designation or development of new products, such as Credit Derivatives (CDs), by all investors and financial institutions to reduce or remove any Credit Risk arise from lenders or bondholders (example banks), and to allow banks to deliver more loans seemed to want a share in it. The most widely used product of CDs is the Credit Default Swap (CDS). The Credit Default Swap is a contract entered between two parties that provides a protection against losses occurring due to a default event of an certain entity. Since its introduction in the mid-1990s, the growth of the global market has been overwhelming: for example the market size for CDS almost doubled biannually from 1996 to 2004, and even quadrupled to over a peak notional outstanding amount of US $ 20 trillion during 2004-2006 [?].

The measurement or modelling of Credit Risk, however, provides its own set of challenges. There exist many ways of modelling Credit Risk with the implication that Banks can face a quandary of choosing the models. Historically, the most important models include Merton Models (Structural approach model and Reduced form approach [?, ?, ?], which main goal is to determine the probability that the firm will default), the Credit Rating Model (ratings are given by rating agencies like Moodys, Standard and Poors (S&P) and Fitch, and provide a measure of the relative creditworthiness of the entity [?, ?]), and the Financial Statement Analysis Model (the model provides the rating based on the financial statements of the borrowers, [?, ?]). We only study a particular class of credit risk models such as the Merton structural model (or asset values approach) through the thesis.

Merton's model is known to us, as a Structural Approach to Credit risk modelling because it reposes completely upon the capital structure of the firm (firm's asset value, equity and firm's debt) for modelling credit risk of the firm. Merton's model was the genesis for understanding the link between the market value of the firm's assets, the market value of the firm's debt, and the market value of firm equity. In the Merton's approach, the default event was modelled as the time when the firm's asset values drop below some default barriers. This assumption served to evaluate a pricing formula for corporate bonds. Black and Cox[?] extended Merton's framework in several directions, including some of the realities of corporate bonds, like safety covenants, debt subordinates, and the dividend payments. Longstaff and Schwartz [?], developed the Structural Approach for stochastic interest rate, and for the firm's total asset values and interest rate processes to be correlated (related). Zhou [?] initiated a jump-diffusion approach to the firm's asset value. Bryis and Vanne [?] extended this approach by evaluating the risk-free interest rates and firm's asset value with more general diffusion processes. Although considerable advances have been made within the Merton's framework, the predictability of default events has remained the weak assumption that actually limits their practical implementation. Therefore, the introduction of an assumption (an early default event) has led to considerable effort directed toward another class of credit risk analysis, such as the Reduced form Approach.

In the case of the Reduced form approach, the default time is assumed to be specified exogenously with the time being evaluated directly, and the relationship between the default and the firm's asset value is not explicitly evaluated. A characteristic of the reduced form approach is that the firm may default at any time, not only at maturity of the debt. This makes the default event unpredictable, and comes as a complete shock to the financial market. This is a threat that is more literal than the nature of predicted default even in structure approach. Jarrow and al.[**?**] studied the default event to be the time at which a company's credit rating enters in default state, specified by Markov process. Furthermore, the most current reduced form approaches are those that employ or involve hazard rate and stochastic intensities to model the default time.

The main point of this dissertation is to consider how the Credit Default Swap is modelled for estimating the value of CDS. There are different ways for modelling the Credit Derivatives, typically characterised by how they characterize the default event. Duffie and Signeton [**?**] suggested the approach in which the default is indicated or characterized by a hazard rate largely related to the distribution of default time. Their approach suggests that the defaultable claims can be priced the same way as the non default claims. Using the Duffie and Signeton framework, David and Mavroid [**?**] suppose the hazard rate is a Gaussian model with time-dependent deterministic drift to revise the valuation of the Credit Default Swap. Aonuma and Nakagawa [**?**] extend David and Mavroid work to model the hazard rate in form of affine type or quadratic Gaussian type term structure model, such as Vasick model and give the valuation formula for the credit default swap that contains basket type. In this thesis we also inspire for these previous models, to consider the default model which takes the hazard rate as principal factor.

This dissertation is structured into seven chapters, and we work under the following organisation:
In Chapter[2], we introduce and deal with some preliminary knowledge that is required for the remainder of this dissertation.We studied a survey of the most widely accepted single-factor models of the short-term interest rate with their solutions, among them, Vasicek and Cox-Ingersoll-Ross models (CIR).

Chapter[3] presents the brief summary of the main development within the Structural approach to modelling and valuation of Credit Risk. Specially, we review the classical Structural models, introduce by Merton [?] and we make its application to South Africa market data.

In Chapter[4], we follow the framework proposed by David and Mavroidis[?], and Aonuma and Nakagawa [?] to give the main techniques used to value credit default swap under the Vasicek-type hazard rate. These are done for the only one defaultable bond issued by the company. Here the explicit value of the credit default swap for both the fixed side, and the recovery side, are established in a quite general form.

In chapter[5], we discuss how to estimate hazard rate which follows the dynamic of the Vasicek-type model and a CIR-type model by using the relationship between credit spread and hazard rate, that, is necessary for switching the market credit spread data collecting directly from the market data into the CIR and the Vasicek-type Hazard rate data. We analyse the conditional survival probability for both the CIR and the Vasicek processes and give some results.

Chapter[6] treats the parameter estimation associated to the hazard rate models using the Generalized Moment Method. And we shall investigate 20 South African firm's debt terms, with different rating from AAA to BBB and different market credit spread for maturity one year, three years and five years to analyse and estimate the parameters in 2 models (Vasicek-type model and a CIR-type model).

In chapter[7] we summarize the dissertation, make our conclusions, and discuss potential future-work.

In the appendix, we present some basic results concerning the Merton model (the default probability and credit spread).

Chapter 2

Preliminaries

Here we introduce some terms useful in bond derivative pricing. In addition, we specify the filtered probability space and the fundamental concepts that will be needed in the following discussion. We also studied a survey of the most widely accepted single-factor models of the short-term interest rate with their solutions, among them, Vasicek and Cox-Ingersoll-Ross models.

2.1 Filtered space

Definition 2.1.1. ($\sigma-$algebra, Measurable sets and Measurable space.) Let Ω be a set, and 2^{Ω} symbolically represent its power set. Then a subset $\mathcal{F} \subset 2^{\Omega}$ is called a $\sigma-$algebra if it satisfies the following properties:

1. $\mathcal{F} \neq \emptyset$.
2. $\mathcal{F}$ is closed under complementation which means that, if $C \in \mathcal{F}$, then its complement $\Omega \diagdown C \in \mathcal{F}$.
3. $\mathcal{F}$ is closed under countable unions which means that, if $C_1, C_2, \ldots$ are in $\mathcal{F}$, then $C = C_1 \cup C_2 \cup C_3 \cup \cdots \in \mathcal{F}$.

The magma $(\Omega, \mathcal{F})$ is called a measurable space and we refer to the element of $\mathcal{F}$ as measurable sets or events. By applying De Morgan's laws to these properties, it follows that the $\sigma-$algebra is also closed under countable intersections.

Note that $\mathcal{F} = \{\emptyset, \Omega\}$ is a trial set.

Definition 2.1.2. (General filtration) Let us consider $0 \leq t \leq T$ the trading date, the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$ presents an information system of subsets and satisfies

$$\sigma(\{\Omega\}) = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_T \text{ and } \cup_{t=0}^{T} \mathcal{F}_t = \sigma(\Omega).$$

The algebra $\mathbf{F}$ gives all the information that is available at time t and we never lose any information and our knowledge increases with time t, as action is finite $t = 0, 1, \ldots, T$, $\mathcal{F}_T = \sigma(\Omega)$. The gradually available information is modelled by the triple $(\Omega, \mathcal{F}, \mathbb{P})$.

Throughout this thesis, if we consider the probability $\mathbb{P}$ to be a martingale measure (or risk neutral measure) then all the discounted asset price under the probability $\mathbb{P}$, are martingales.

Definition 2.1.3. (Probability space and filtered space) Let us fix a finite date T, and assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which models all states of the financial market,is endowed with some filtration $\mathbf{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$. Note that the collection $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ is referred as a filtered probability space.

Definition 2.1.4. (Continuity and completeness of filtered probability space.) Given the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if $B \subset A$, $\mathbb{P}(A) = 0 \Rightarrow B \in \mathcal{F}$, then the space is complete. When the probability space is complete and $\mathcal{F}_0$ contains all sets $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 0$, then the filtered probability space is complete.
The right limit of a filtration $\mathbf{F}$ is designed by

$$\mathcal{F}_{t+} = \bigcap_{t<s} \mathcal{F}_s,$$

and whenever we have

$$\mathcal{F}_{t+} = \mathcal{F}_t, \quad \forall t$$

then $\mathbf{F}$ is a right continuous.

The left limit of a filtration $\mathbf{F}$ is designed by

$$\mathcal{F}_{t-} = \sigma\left(\bigcup_{t>s} \mathcal{F}_s\right)$$

if

$$\mathcal{F}_{t-} = \mathcal{F}_t, \quad \forall t$$

then $\mathbf{F}$ is a left continuous.

2.2 Terms related to Bond Derivative

Let us describe the fundamental elements of Bond Derivative Pricing and define important definitions, that will be helpful through the whole of this dissertation.

Definition 2.2.1. (Maturity time). The maturity time T of the bond is the prescribed time in the future, and is also known as the expiry date of the bond. This date is also referred to as the life of the bond ends.

Though the maturity date is the expiration date, it must be specified at the time of opening the contract. The payment of the face value of bond is made at the maturity time. Therefore, is also known as redemption date.

Definition 2.2.2. (Bank account or Saving account). A bank account evolves deterministically as

$$B(t) = \exp\left(\int_0^t r_u du\right),$$

i.e.

$$\begin{cases} dB(t) = r(t)B(t)dt, \\ B(0) = 1, \end{cases}$$

where r is specified adapted stochastic process. The bank account can be viewed as a money market-account, and describes a bank with the stochastic short rate interest r. The obvious and most common use of the bank account is to be chosen as the numeraire to find the martingale price of the bond.

Definition 2.2.3. (Discount factor.) At given time $t < T$, the discount factor $B(t,T)$ is the value of one unit cash of payment at maturity T and is expressed as

$$B(t,T) = \frac{B(t)}{B(T)} = \exp\left(-\int_t^T r_u du\right). \tag{2.1}$$

Since, in order to have one unit of cash at time T the amount $1/B(T)$ has to be invested at the beginning. Then, the value of this initial investment constitutes $B(t)(1/B(T))$ at time $t > 0$.

Definition 2.2.4. (Bond). A bond is a financial security of debts, which matures at a precise date in the future T, refunds its face value at future date T and pays interest rate periodically in form of coupon payment. A zero coupon bond with maturity date T is a contract that guarantees to pay the bond holder δ dollars at time T . The face value δ is usually substituted by 1 for computational convenience. Denote the price of a bond with maturity date T at time t as $p(t,T)$ and $p(T,T) = 1$ for all t.
With visible prices of zero-coupon bonds in the market, one can define interest rates.

Definition 2.2.5. (Risk-free zero coupon bond). Let r_t be the short rate interest independent of W_t. Then the price of risk-free zero coupon bond at time $t \in [0,T]$ is given by

$$D(t,T) = E\left[\exp\left(-\int_t^T r_u du\right)\right] \tag{2.2}$$

This is actually the martingale price of bond at time t with payoff 1 dollars at maturity T, given the dynamic of short rate interest rate r_t we can deduce the bond value $D(t,T)$.
Observe that there is a close relationship between the price of the bond and the discount factor $B(t,T)$. Where W_t is the wiener process or standard Brownian motion defined in (**??**) below.

Definition 2.2.6. (Yield or spread). Give $p(t,T)$ the value of the bond. The yield to the maturity $r^d(t,T)$ of the bond $p(t,T)$ is the discounted value that make the current value of the cashflows equal to the value of the bond at time t, it is determine by the below formula

$$p(t,T) = e^{-r^d(t,T)(T-t)} \tag{2.3}$$

thereafter

$$r^d(t,T) = -\frac{\ln(p(t,T))}{T-t}. \tag{2.4}$$

The value $r^d(t,t)$ can be used to evaluate, or model the evolution of the short rate interest process r_t or hazard rate process of zero-coupon bonds through the maturity time T. The trajectory of the term structure yield can have the flat shape, decreasing or increasing depend on the value of the interest rate r_t, and other related parameters values.

Definition 2.2.7. (Premium). A premium is a kind of loan, which the ownership of the bond delivers to the bond's issuer.

This quantity establishes the value of the bond. Actually, the value of bond is giving by summing the principal and its premium. The principal of the bond is also termed the face value or the par value.

Definition 2.2.8. (Cum-coupon and ex-coupon). The cum-coupon bond is a contract that require the buyer of the bond to pay the seller the accrued interest rate on the bond, and ex-coupon is its opposite. United State's bonds are always cum-coupon bonds.

Definition 2.2.9. (Recovery rules). Let δ be the recovery value, if default does not occur before or at the maturity time T, then the claim is paid one monetary unit. Otherwise, depending on the market convention, either (1) the payment of δ monetary is made at the maturity time T, or (2) the payment of $\delta(\tau)$ monetary units is paid at time τ.

Definition 2.2.10. (Contingent claim). Contingent claim is defined as a contract that specifies X, the stochastic amount of money is to be redeemed at the maturity time T to the holder of contract.

Example 2.2.11. For the famous Black-Scholes model, an example of a contingent claim is that of a European Call option X on S (asset price) with the exercise price (strike price) K and the maturity date T. The pay off is given by

$$X = (S(T) - K)^+ \tag{2.5}$$

and gives the holder the right, but not the obligation to purchase one share of the stock at the prescribed price K, at the prescribed time in the future T. Where $S(T)$ denoted the asset price at maturity date T.

The section below introduces credit risk from a modelling perspective. We first explain credit risk and measure of credit risk.

2.3 Credit risk

Credit risk is gaining much attention and becoming an increasingly important subject for evaluation in the financial market (or industry). There are several types of risk that a smart investor should consider and pay careful attention to. The most important of all risks are Credit risk and market risk. Market risk, is also known as a systematic risk, is the risk of losses experienced by the investor due to factors that affect the overall performance of markets prices. Market risks are characterized by currency risk, equity risk, commodity risk, and interest rate risk. Credit risk and its measurement are discuss below.

Definition 2.3.1. (Credit risk). Credit Risk is defined as the risk that a borrower (company, individual, sovereign government) will default on any type of debt by failing to meet its financial commitment. In other word, is the chance of a loss due to inability of a counterpart (borrower) to fulfil its financial obligation. US treasure securities are assumed to be free of credit risk.

The following example of a financial transaction elucidates the definition of the credit risk : A bank gives a loan of R1m to the company and they agree that the company refunds the loan two years from now on. The bank lends the money to the counter-party bower in the contract. From this time on, the bank deals with the credit risk of the transaction. Basically, the bower pays back the outstanding amounts of Rand 1m (plus coupon) to the lender. However, if the default happens or the debtor fails to meet its financial commitment, a procedure is started to recover moneys from the firms assets to refund the lenders. This is probably causing a great loss to the lender (or bank): in instance only 30% of the loan can be recovered causing a loss of Rand 0.7m. This exemplifies that credit risk in a financial transaction can cause large losses. Wherefore, lenders will carefully evaluate or measure this risk of a counter-party before entering into the agreement.

Definition 2.3.2. (Counterparty risk). Counterparty risk is traditionally thought of as credit risk between derivatives counterparties or is in one sense a specific form of credit risk. Hence, in the context of financial risk, it is merely a subset of a single risk type [?]. The counterparty risk has been considered by most market participants to be the key financial risk since the global credit crisis of 2007 onwards and the failures of large prestigious institutions such as Bear Sterns, Lehman Brothers, Fannie Mae and Freddie Mac.

Definition 2.3.3. (Credit spread). The credit spread is the excess return on a defaultable bond, (or the difference in yield between two bonds of similar maturity but different credit quality). The credit spread is influenced by the credit quality of the issuer and the maturity time of the bond, and is the recompense an investor received for undertaking the credit risk originally secured with the same security.

2.4 Credit Derivatives

Credit derivatives (CDs) are financial securities whose payoff depends on the occurrence of a credit event affecting an underlying financial entity or reference entity.

Credit derivatives are primarily used to:

1. express a positive or negative credit view on a single entity or a portfolio of entities, independent of any other exposures to the entity one might have.
2. reduce risk arising from ownership of bonds or loans.

Since its introduction in the mid-1990s, the growth of the global market has been overwhelming. The most widely used product of CDs is the Credit Default Swap (CDS). The CDS is defined in Chapter 4, Section (**??**).

2.5 Stochastic processes

This section will provide a very brief overview of stochastic calculus.

Definition 2.5.1. (Wiener process). The Wiener process or standard Brownian motion W_t, $t \geq 0$ is actually a stochastic process which satisfies the following conditions:

1. $W(0) = 0$,
2. the trajectory W_t is almost surely continuous,
3. the process W_t has independent increments, which mean that

$$\text{for} \quad 0 \leqslant s_1 \leqslant t_1 \leqslant s_2 \leqslant t_2, \text{then} \quad W_{t_1} - W_{s_1} \quad \text{and} \quad W_{t_2} - W_{s_2} \quad \text{are independent.}$$

4. For any $0 \leq s \leq t$, $W_t - W_s \sim \mathcal{N}(0, t - s)$,

where $\mathcal{N}(\mu, \sigma^2)$ characterises the normal distribution with μ and σ^2 expected value and variance respectively.

This definition allowed us to deduce the following basic properties related to Brownian motion. Note that Brownian motion is described by the Wiener process.

Property 2.5.2. *$E(W_t) = 0$ and $Var(W_t) = t$. Since its variance increases respect to the time t, and when t increases the trajectory of W_t moves away from the horizontal axis [**?**].*

Definition 2.5.3. (Predictable process.) Given a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. For any $t \geq 1$, if Z_t is $\mathcal{F}_{t-1}-$measurable and Z_0 is $\mathcal{F}_0-$measurable, then a stochastic process $Z = \{Z_t\}\,(t \in [0, T])$ is predictable. Note that $\mathcal{F}_0$ is assumed to be trivially measurable for all t. The similarity of predictable in continues time is less consider. For example, let $Z : \mathbb{R}_+ \times \Omega \longrightarrow \mathbb{R}$ be a continuous stochastic process, if the process is measurable with respect to the predictable σ-algebra, then it is predictable.

Definition 2.5.4. (Stopping time.) Let τ be a random variable defined on the filtered space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. If for all $t \geq 0$, τ is measurable i.e $\{\tau \leq t\} \in \mathcal{F}_t$ and taking values in $[0, \infty]$, then τ is a stopping time.

Definition 2.5.5. (Default indicator process and survival indicator.) Given the random default time τ, we characterize the right continuous jump process H, for any $t \in \mathbb{R}_+$, as following

$$H_t = 1_{\{\tau \leq t\}}, \tag{2.6}$$

this process is referred to as the jump or default indicator process and the process $1-H_t = 1_{\{\tau>t\}}$ is the survival indicator of the company. The process H jumps from 0 to 1 at the default time τ.

Definition 2.5.6. (Conditional expectation.) Let X be an integral random variable on a given a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, and $\mathcal{H} \subset \mathcal{F}$. The conditional expectation of X given $\mathcal{H}$ satisfies the following equation

$$\int_B E\left(X|\mathcal{H}\right) d\mathbb{P} = \int_B X d\mathbb{P}, \tag{2.7}$$

for any $B \in \mathcal{H}$.

Theorem 2.5.1. *Given a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$. Let Z and Y be random variables defined on a given space and, with sub-σ-algebras $\mathcal{H}$ and $\mathcal{G}$. We characterize below the results with respect to the conditional expectation values:*

1. $E(Z|\mathcal{H}_t) = E(Z)$, *if Z is independent of $\mathcal{F}_t$,*
2. $E(ZY|\mathcal{H}_t) = ZE(Y|\mathcal{H}_t)$, *given Z-$\mathcal{H}_t$ measurable,*
3. $E\left(E(Z|\mathcal{H}_t)\right) = E(Z)$,
4. $E\left(E(Z|\mathcal{H}_t)|\mathcal{G}_t\right) = E(Z|\mathcal{H}_t)$, *given $\mathcal{G}_t \subset \mathcal{H}_t$.*

Proof 2.5.7. *See [**?**].*

Definition 2.5.8. (Martingale and sub-Martingale.) Let a random X_t be an $\mathcal{F}-$ adapted process, then the process X_t is a martingale if

$$E\left(|X_t|\right) < \infty, \forall t \in [0, T] \tag{2.8}$$

and

$$\text{given} \quad t \in [0, T-1], \quad E\left(X_{t+1}/\mathcal{F}_t\right) = X_t. \tag{2.9}$$

This definition can be interpreted as: The best prediction for future values that could be constructed based on available information of a martingale process is its current value.

Notice that if we replace the equality **??** by $\geq$ or $\leq$, X_t is said sub-martingale and super-martingale respectively.

Definition 2.5.9. (Martingale measure.) Given $\mathbb{Q}$ and $\mathbb{P}$ on measure space $(\Omega, \mathcal{F})$, if

1. $\mathbb{Q}(w) \geq 0$, $\forall w \in \Omega$, and

2. $\mathbb{Q}(B) = 0$, iff $\mathbb{P}(B) = 0$, $\forall B \in \mathcal{F}$,

we said the probability measure $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on measure space $(\Omega, \mathcal{F})$, and $\mathbb{Q}$ is called a martingale measure for discounted process X_t^* if the below condition satisfies

$$E_{\mathbb{Q}}\left(X_{t+1}^*/\mathcal{F}_t\right) = X_t^*. \tag{2.10}$$

Note that a martingale measure is similar to the risk-neutral probability measure.

Definition 2.5.10. (Markov process.) Given a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ and $\{X_t\}_{t<T}$ defined as Markov, if for every n and $t < t < \cdots < t_n$ we have

$$\mathbb{P}(X_{t_n}|X_{t_{n-1}}, \ldots, X_{t_1}) = \mathbb{P}(X_{t_n}|X_{t_{n-1}}), \tag{2.11}$$

then we have a Markov process.

Theorem 2.5.2. *The Markov property for Itô processes. Let X_t^x be a time-homogeneous Itô process on it expressible in the form*

$$dX_t^x = \mu(X_t^x)dt + \sigma(X_t^x)dW_t, \quad X_0^x = x, \tag{2.12}$$

where μ and σ are Lipschitians and let $h : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a bounded Borel function. Therefore, for any t, $s \geq 0$

$$E(h(X_t^x)|\mathcal{F}_s) = E(h(X_t^y))|_{y=X_t^x}. \tag{2.13}$$

Proof 2.5.11. *See [**?**, **?**].*

The Markov's property is useful for calculating the price of a zero-coupon defaultable bond, the hazard rate and the conditional survival probability.

Lemma 2.5.12. *Suppose $\mathcal{Q}$ is a continuous martingale on the filtered space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, and let ϕ be a function that defines the process $\mathcal{Q} = \int_0^t \phi(s, w)dW_s$, then*

$$E\left[\left(\int_0^t \phi(s, w)dW_s\right)^2\right] = \int_0^t E\left[\phi(s, w)\right]^2 dW_s \tag{2.14}$$

We shall refer to this important lemma in the subsequent chapter(s).

2.6 Short-Term Rate Models (Single-factor Models)

In this section, we provide a survey of the most widely accepted single-factor models of the short-term interest rate. It is assumed throughout that the dynamics of r_t are specified under the martingale probability measure $\mathbb{Q}$, and the the underlying Brownian motion W is assumed to be one-dimensional. As a consequence, the risk premium does not appear explicitly in our formulas. In this sense, the models considered in this section are based on a single source of uncertainty, so that they belong to the class of single-factor models. we will consider various commonly used models. We will list their properties and make comparisons between them.

2.6.1 Vasicek model and its properties

Vasicek model defines the short rate process r_t by the affine stochastic differential equation (SDE):

$$dr_t = a(b - r_t)dt + \sigma dW_t, \quad r(0) = r_0 > 0, \tag{2.15}$$

where a, b and σ are strictly positive constants and W_t is a standard Wienner process. Note that one can write $r(t) = r_t$. The parameters a, b and σ are viewed as the mean reversion rate, the mean reversion level, and the volatility respectively. This SDE is known as the mean-reverting Ornstein-Uhlenbeck process.

The solution of the Vasicek process.

By applying Itô's lemma we can determine the solution of the stochastic differential equation (**??**). Let assume $X_t = r_t - b$. Therefore, the stochastic equation (**??**) can be written as

$$dX_t = dr_t = -aX_t dt + \sigma dW_t.$$

Substituting $Z_t = e^{at} X_t$, yields

$$\begin{aligned} dZ_t &= ae^{at} X_t dt + e^{at} dX_t \\ &= ae^{at} X_t dt + (-aX_t dt + \sigma dW_t)\, e^{at} \\ &= \sigma e^{at} dW_t. \end{aligned}$$

By applying the Itô's integral we obtain

$$Z_t = Z(0) + \int_0^t \sigma e^{as} dW_s. \tag{2.16}$$

Besides

$$Z_t = e^{at} X_t = e^{at} \left(r_t - b\right). \tag{2.17}$$

Clearly, from the previous equation (**??**) we see for $t = 0$, $Z(0) = r(0) - b$. Thereafter, the equation (**??**) maybe expressed as

$$e^{at} \left(r_t - b\right) = \left(r_0 - b\right) + \int_0^t \sigma e^{as} dW_s. \tag{2.18}$$

Then, we find the value of r_t as follows:

$$r_t = b + \left(r_0 - b\right) e^{-at} + e^{-at} \int_0^t \sigma e^{as} dW_s = b + \left(r_0 - b\right) e^{-at} + \sigma \int_0^t e^{-a(t-s)} dW_s. \tag{2.19}$$

This is the solution of the stochastic differential equation (SDE) that defines the Vasicek-type mode.

Lemma 2.6.1. *For any $t \geq s$, the unique solution to the stochastic differential equation (SDE) (**??**) is given by the formula:*

$$r_t = b + \left(r_s - b\right) e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u. \tag{2.20}$$

The conditional law of the short rate r_t with respect to the $\sigma-$field $\mathcal{F}_s$ is Gaussian.

- **The mean of the Vasicek-type model.**

  For any $t \geq s$, the mean is determined by evaluation of the expectation value of r_t given in equation (**??**), so

$$\begin{aligned} E\left(r_t | \mathcal{F}_s\right) &= E\left(b + \left(r_s - b\right) e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} dW_u | \mathcal{F}_s\right) \\ &= b + \left(r_s - b\right) e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)} E\left(dW_u\right), \end{aligned}$$

  since, for any function f, the Itô integral $\int_s^t f(u) dW_u$ is a random variable independent of the filtration $\mathcal{F}_s$, and applying the Brownian motion's properties, $E\left(dW_U\right) = 0$, we obtain

$$E\left(r_t | \mathcal{F}_s\right) = b + \left(r_s - b\right) e^{-a(t-s)}. \tag{2.21}$$

The limit of $E\left(r_t|\mathcal{F}_s\right)$, when t goes to infinity (i.e. $t-s\Longrightarrow\infty$),

$$\lim_{(t-s)\to\infty} E\left(r_t|\mathcal{F}_s\right)=b. \tag{2.22}$$

- **The variance of the Vasicek-type model.**

To evaluate the conditional variance of Vasicek process, we re-express equation (**??**) as

$$r_t-\left(b+\left(r_s-b\right)e^{-a(t-s)}\right)=\sigma\int_s^t e^{-a(t-u)}dW_u. \tag{2.23}$$

We clearly see that expression $b+\left(r_s-b\right)e^{-a(t-s)}$ on the left hand side of previous Equation (**??**) is equal to the conditional mean of Vasicek short rate model r_t. Hence, one may rewrite Equation (**??**) as follows:

$$r_t-\left(b+\left(r_s-b\right)e^{-a(t-s)}\right)=r_t-E\left(r_t|\mathcal{F}_s\right)=\sigma\int_s^t e^{-a(t-u)}dW_u. \tag{2.24}$$

It is well known that the variance is defined by

$$var\left(r_t|\mathcal{F}_s\right)=E\left[\left(r_t-E\left(r_t\right)\right)^2|\mathcal{F}_s\right]. \tag{2.25}$$

Substituting equation (**??**) into equation (**??**), the variance is

$$\begin{aligned}
var\left(r_t|\mathcal{F}_s\right)=&E\left[\left(\sigma\int_s^t e^{-a(t-u)}dW_u\right)^2|\mathcal{F}_s\right]\\
&=\sigma^2E\left[\left(\int_s^t e^{-a(t-u)}dW_u\right)^2\right]\\
&=\sigma^2E\left[\left(\int_s^t e^{-2a(t-u)}du\right)\right]\\
&=\frac{\sigma^2}{2a}E\left[e^{-2a(t-t)}-e^{-2a(t-s)}\right]\\
&=\frac{\sigma^2}{2a}\left[1-e^{-2a(t-s)}\right].
\end{aligned}$$

Taking the limit $(t-s)\longrightarrow\infty$, the conditional variance of the Vasicek-type model converges to

$$\lim_{(t-s)\longrightarrow\infty} var\left(r_t|\mathcal{F}_s\right)=\frac{\sigma^2}{2a}. \tag{2.26}$$

With regard to equation (??) and (??), the conditional expectation value tends to b when $(t-s)$ goes to infinity, the Vasicek model r_t is mean reverting, which can be interpreted as a long-term average and make sense economically. The drawback of the process is simply the serious defect of allowing the interest rate to take negative value with a positive probability.

It is obvious that Vasicek's model, as indeed any Gaussian approach, allows for negative values for the short term interest rate . This property is obviously incompatible with no-arbitrage in the presence of cash in the economy. This significant drawback of Gaussian models was explored or investigated by, among others, Dothan (1978). To overcome this shortcoming or imperfection of Vasicek's model, Cox et al (1985) proposed the modification of the short-term rate process r_t discussed in the following section.

2.6.2 Cox-Ingersoll-Ross (CIR) model (or square-root process)

This model was developed in 1985 by Cox et at. to eliminate any negative value of interest rate, by the following modification of the mean-reverting Ornstein-Uhlenbeck process, known as the square-root process and expressed by the SDE:

$$dr_t = a(b - r_t)dt + \sigma\sqrt{r_t}dW_t, \tag{2.27}$$

where a, b are strictly positive constants, and σ constant volatility and W_t is a standard Wiener process.

Regarding to the square-root in the diffusion coefficient, the CIR model yields positive values. Specifically, it can reach zero value, but it never takes negative value.

Lemma 2.6.2. *For any $s \leq t$, the unique solution to the stochastic differential equation (SDE)* (??) *is given by the formula:*

$$r_t = b + (r_s - b)\,e^{-a(t-s)} + \sigma \int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u. \tag{2.28}$$

with conditional expectation value

$$E\left(r_t|\mathcal{F}_s\right) = b + (r_s - b)\,e^{-a(t-s)}, \tag{2.29}$$

and conditional variance

$$var\left(r_t|\mathcal{F}_s\right)=\frac{\sigma^2 b}{2a}\left(1-e^{-a(t-s)}\right)^2+\frac{\sigma^2 r_s}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right). \tag{2.30}$$

When $(t-s)$ goes to infinity, the limits of $E\left(r_t|\mathcal{F}_s\right)$ and $var\left(r_t|\mathcal{F}_s\right)$ are given by:

$$\lim_{(t-s)\to\infty} E\left(r_t|\mathcal{F}_s\right)=b, \tag{2.31}$$

and

$$\lim_{(t-s)\longrightarrow\infty} var\left(r_t|\mathcal{F}_s\right)=\frac{\sigma^2 b}{2a}. \tag{2.32}$$

Proof 2.6.3. *In a way similar to the previous case, we prove that the CIR model has a unique solution, and has the property of positive interest rate.*
*Assuming the process $Z_t=r_te^{at}$ taking its derivative, and using equation (**??**), we obtain*

$$\begin{aligned} dZ_t &= e^{at}dr_t+ar_te^{at}dt+d<e^{at},r_t> \\ &= e^{at}\left(a(b-r_t)dt+\sigma\sqrt{r_t}dW_t\right)+ar_te^{at}dt \\ &= e^{at}\left(ab\ dt+\sigma\sqrt{r_t}dW_t\right), \end{aligned}$$

for all $t\geq s$. Taking the integral of previous equation yields

$$Z_t-Z_s=ab\int_s^t e^{au}du+\sigma\int_s^t e^{au}\sqrt{r_u}dW_u, \tag{2.33}$$

substituting the process $Z_t=r_te^{at}$, we obtain after rearranging

$$\begin{aligned} r_t &= r_se^{-a(t-s)}+abe^{-at}\int_s^t e^{au}du+\sigma e^{-at}\int_s^t e^{au}\sqrt{r_u}dW_u \\ &= r_se^{-a(t-s)}+b\left(1-e^{-a(t-s)}\right)+\sigma e^{-at}\int_s^t e^{au}\sqrt{r_u}dW_u \\ &= b+\left(r_s-b\right)e^{-a(t-s)}+\sigma\int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u. \end{aligned}$$

- **The mean of the CIR-type model.**

  For any $t\geq s$, the mean is determined by evaluation of the expectation value of r_t given in previous equation , so

$$E\left(r_t|\mathcal{F}_s\right)=E\left(b+\left(r_s-b\right)e^{-a(t-s)}+\sigma\int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u|\mathcal{F}_s\right) \tag{2.34}$$

$$=b+\left(r_s-b\right)e^{-a(t-s)}+E\left(\sigma\int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u|\mathcal{F}_s\right), \tag{2.35}$$

and for any function f, the Itô integral $\int_s^t f(u)dW_u$ is a random variable independent of the filtration $\mathcal{F}_s$, and applying the Wienner process's property, $E\left(dW_U\right)=0$, we obtain

$$E\left(r_t|\mathcal{F}_s\right) = b + \left(r_s - b\right)e^{-a(t-s)}. \tag{2.36}$$

The limit of $E\left(r_t|\mathcal{F}_s\right)$, when t goes to infinity (i.e. $t - s \Longrightarrow \infty$),

$$\lim_{(t-s)\to\infty} E\left(r_t|\mathcal{F}_s\right) = b. \tag{2.37}$$

- **The Variance of the CIR-type model.**

*To evaluate the conditional variance of CIR process, we re-express equation (**??**) as*

$$r_t - \left(b + \left(r_s - b\right)e^{-a(t-s)}\right) = \sigma \int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u. \tag{2.38}$$

*We clearly see that, expression $b+b+\left(r_s - b\right)e^{-a(t-s)}$ on the left hand side of previous Equation (**??**) is equal to the conditional mean of CIR short rate model r_t. We may write Equation (**??**) as following*

$$r_t - \left(b + \left(r_s - b\right)e^{-a(t-s)}\right) = r_t - E\left(r_t|\mathcal{F}_s\right) = \sigma \int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u. \tag{2.39}$$

Again, the variance is defined by

$$var\left(r_t|\mathcal{F}_s\right) = E\left[\left(r_t - E\left(r_t\right)\right)^2|\mathcal{F}_s\right]. \tag{2.40}$$

*Substituting equation (**??**) into equation (**??**) and using lemma(**??**) , the variance is :*

$$
\begin{aligned}
var\left(r_t|\mathcal{F}_s\right) =& E\left[\left(\sigma\int_s^t e^{-a(t-u)}\sqrt{r_u}dW_u\right)^2|\mathcal{F}_s\right] \\
&= \sigma^2\int_s^t\left(e^{-2a(t-u)}E\left(r_u|\mathcal{F}_s\right)\right)du \\
&= \sigma^2\int_s^t\left(e^{-2a(t-u)}\left(b+\left(r_s-b\right)e^{-a(u-s)}\right)\right)du \\
&= \sigma^2 b\int_s^t e^{-2a(t-u)}du+\sigma^2\left(r_s-b\right)e^{-2at}\int_s^t e^{a(u+s)}du \\
&= \frac{b\sigma^2}{2a}\left(1-e^{-2a(t-s)}\right)+\frac{\sigma^2\left(r_s-b\right)}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right) \\
&= \frac{b\sigma^2}{2a}\left(1-e^{-2a(t-s)}\right)-\frac{b\sigma^2}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right)+\frac{\sigma^2 r_s}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right) \\
&= \frac{b\sigma^2}{2a}\left(1-e^{-2a(t-s)}-2e^{-a(t-s)}+2e^{-2a(t-s)}\right)+\frac{\sigma^2 r_s}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right) \\
&= \frac{b\sigma^2}{2a}\left(1-2e^{-a(t-s)}+e^{-2a(t-s)}\right)+\frac{\sigma^2 r_s}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right) \\
&= \frac{b\sigma^2}{2a}\left(1-e^{-a(t-s)}\right)^2+\frac{\sigma^2 r_s}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right).
\end{aligned}
$$

Taking the limit $(t-s)\longrightarrow\infty$, the conditional variance of the CIR-type model converges to

$$
\begin{aligned}
\lim_{(t-s)\longrightarrow\infty} var\left(r_t|\mathcal{F}_s\right) &= \lim_{(t-s)\longrightarrow\infty}\left[\frac{b\sigma^2}{2a}\left(1-e^{-a(t-s)}\right)^2+\frac{\sigma^2 r_s}{a}\left(e^{-a(t-s)}-e^{-2a(t-s)}\right)\right] \\
&= \frac{b\sigma^2}{2a}.
\end{aligned}
$$

Our future goal in this dissertation is to find the credit risk, the price of credit derivative swaps under Vasicek-type hazard model (refer in Chapter 4), the hazard rate function for the Vasicek and CIR models, and the survival probability under Vasicek-type hazard model and CIR-type model (these will be discussed in Chapter 5). To this end, we shall use the probability risk-neutral measure or martingale valuation formula directly. The following is the valuation of the bond price under CIR, which is helpful in determining the CIR mode type hazard rate in Section (**??**).

Bond price under the CIR model

Firstly, we consider the Laplace transform of the joint integral of the squared CIR process and squared root process

$$\left(\int_t^T r_u du, r_T\right) \quad \text{given} \quad r_t.$$

By using the representation of the Laplace transform obtained in [?].

Indeed, we observe

$$L(t,T,r,\lambda,\eta) = E\left[\exp\left(-\lambda\int_t^T r_u du - \eta r_T\right)|r_t = r\right] \tag{2.41}$$

A quick way to determine the value of the bond is to modify the equation (**??**), taking $\lambda = 1$ and $\eta = 0$, we come out with the CIR formula for the price of a zero-coupon bond (Cox et al, 1985) $L(t,T,r,\lambda,\eta) = B(t,T,r)$.

Theorem 2.6.1. *Suppose r_t is a solution of squared root process, the Laplace L is affine process and characterise by*

$$L(t,T,r,\lambda,\eta) = \exp\left(\Phi(t,T,\lambda,\eta) - \Psi(t,T,\lambda,\eta)\right) \tag{2.42}$$

where

$$\Phi(t,T,\lambda,\eta) = \frac{2ab}{\sigma^2}\ln\left(\frac{2h(\lambda)e^{(a+h(\lambda))(T-t)/2}}{2h(\lambda) + (a+\sigma^2\eta + h(\lambda))\left(e^{h(\lambda)(T-t)} - 1\right)}\right),$$

$$h(\lambda) = \sqrt{a^2 + 2\lambda\sigma^2}$$

and

$$\Psi(t,T,\lambda,\eta) = \frac{2\eta h(\lambda) + \eta\left(h(\lambda) - a\right)\left(e^{h(\lambda)(T-t)} - 1\right) + 2\lambda\left(e^{h(\lambda)(T-t)} - 1\right)}{2h(\lambda) + (a+\sigma^2\eta + h(\lambda))\left(e^{h(\lambda)(T-t)} - 1\right)}.$$

Let us take now $\lambda = 1$ and $\eta = 0$, we find the bond price maturing in T year, under CIR model as

$$D(t,T,r) = \exp\left(\Phi_*(t,T) - \Psi_*(t,T)r_t\right) \tag{2.43}$$

where

$$\Phi_*(t,T) = \frac{2ab}{\sigma^2} \ln\left(\frac{2he^{(a+h)(T-t)/2}}{2h + (a+h)\left(e^{h(T-t)} - 1\right)}\right),$$

$$\Psi_*(t,T) = \frac{2\left(e^{h(T-t)} - 1\right)}{2h + (a+h)\left(e^{h(T-t)} - 1\right)},$$

$$h = \sqrt{a^2 + 2\sigma^2}.$$

Furthermore we will refer to this process to find the hazard rate function and survival probability under the CIR-type model in Chapter 5.

2.7 Summary

In this chapter, we provide the basics on modelling credit risk and credit derivatives necessary to understand the remainder of this dissertation. We have specified the filtered probability space and the fundamental concepts that will be needed in the following discussion.

Furthermore, we studied a survey of the most widely accepted single-factor models of the short-term interest rate with their solutions. Among them, we focussed on Vasicek and Cox-Ingersoll-Ross models and determined that structural models are able to meet the research objectives or main scope of this study: prediction of the survival probability. These models are therefore further investigated in the sequel.

Chapter 3

Classical Structural Approach

3.1 Merton Approach

The structural approach for credit risk was first articulated by the famous Robert Merton (1974) in his paper on the valuation of corporate debt. Merton extends the existing framework that relates the asset value of the firm and its credit risk. Therefore, use of the Black-Scholes (1973) option pricing formulas to determine the defaultable bond price and firm's equity, and for evaluating credit risk of a firm. The Merton model is an important quantity to consider when predicting default. In this chapter, we will be presented a literature overview of Merton model and its application to South Africa market data. The present chapter is largely based on Bielecki and Rutkowski[**?**], and [**?**]; the interested reader may thus consult [**?**], [**?**] for more details.

3.1.1 Basic assumption and default conditions

The Merton model is a somewhat stylized structural approach that requires the following assumptions

- The assumption of a constant and flat risk-free interest rate r.
- The default happens only at maturity time T and not before. If, the firm's asset value

falls down to the default point (minimal levels of debt) before the debt's maturity but it is possible to recover and face the payment of the debt at time T, then the default will be avoided in the model.

- The rest of the assumptions Merton (1974) adopts are: the inexistence of bankruptcy costs, transaction costs, taxes or problems with indivisibilities of assets; continuous time trading; unrestricted borrowing and lending at a constant interest rate r; no restrictions on the short selling of the assets; and that the value of the firm is invariant under changes in its capital structure (Modigliani-Miller Theorem[1]).

- The dynamics of the firm's asset value V_t are given by the SDE:

$$dV_t = V_t \left((\mu_V - \kappa)dt + \sigma_V dW_t \right), \tag{3.1}$$

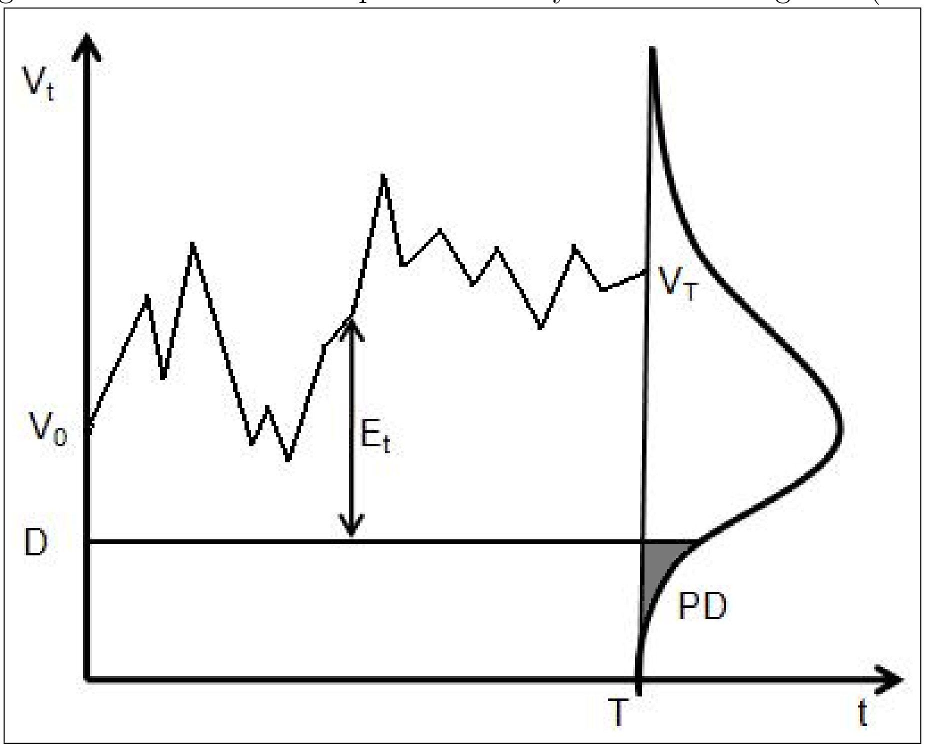
  where κ is the constant payout (dividend) ratio per unit time, μ_V is the expected return of the firm's assets per unit time, σ_V is the volatility of the firm's assets per unit time, and dW_t is a standard Brownian motion.

With regard to the Merton model (1974) not all of these assumptions are required to realize the model, but the assumptions are assumed for success of the model. The most important assumptions are that the firms asset value follows a diffusion process (the last assumption) and continuous time trading. Moreover, the model adopts a simplified of the firm's capital structure. The model does not distinguish amongst different types of debt and consist of only one zero-coupon bond (ZCB) that will become due at maturity date T. The equity of the firm represent the ordinary share. The equity and debt's face value are a contingent claim on the firm's assets, and the market value of the firm's asset V_t comprises of the value equity E_t and market value of total debt D_t, given by $V_t = E_t + D_t$.
The ZCB has a payoff L at maturity T (or face value of ZCB).
If, at maturity T, the firm's assets value exceeds the promised payment L (notional value), the

[1]Modigliani-Miller Theorem stipulates that, under a certain market price process (the classical random walk), in the absence of taxes, agency costs, bankruptcy costs, in an efficient market with asymmetric information, the value of a firm is independent of its capital structure (equity and debt).

bondholders will receive the full face value L and the equityholders (shareholders) then receive the residual asset value $V_T - L$. Otherwise, the firm defaults, the bondholders take control of the firm and receive the firm value V_T, and the shareholders receive nothing when the firm's value falls below the notional value L at maturity T. At default the equityholders never compensated the losses of the bondholders. This maybe viewed as: E_T cannot be negative.

Figure 3.1: Merton model representation by Duffie and Singleton (2003)



This figure (**??**) shows the dynamic in the Merton Approach. The evolution of the firm's asset value process is obtained by stimulating varies paths until maturity. As we can observe, the total value of the debt does not change over time and the firm's equity value fluctuates under the firm's assets value. Default happens only when the asset value of the firm falls below the default point (default barrier) at future time T so that $V_T < L$. The tinted zone of this figure (**??**) should be the expected default frequency or the probability of default.

3.1.2 Merton's formula (Security Pricing)

Referring to the assumptions and default conditions described above, we can derive the price of equity and debt in the Merton framework.

In accordance with the Black-Scholes option pricing theory, we will consider ourselves under risk neutral framework (probability measure $\mathbb{Q}$[2]). We follow Jin al. (2004) and Hull (2003) to derive the formulas.

The payment of equity and debt at maturity, can be represented as a European style written on the assets of firm with the strike price being the debt's face value L.

Specifically, a bit of algebra show that the payoff at maturity to the bondholders equals

$$D_T = \min(V_T, L) = L - \max(V_T, L) = L - (L - V_T)^+ . \tag{3.3}$$

The latter equality shows that the risky debt (payoff to the bondholders) is equivalent to a portfolio consisting of (a) a long term default-risk free bond paying L, and (b) a short term a European put option on the firm's assets with the strike price L and maturity T. This decomposition of the debt's payoff is illustrated in figure(**??**) below. [3] Note that the notation $(...)^+$ represents the pay off of an European Call option, see (**??**).

Once the payment of the debt is made at maturity the remaining assets value belongs to the equityholders. Actually, the payoff to the shareholders is similar to the payoff of a European

[2]Working under risk neutral measure, allows us to replace the expected return of the firm's asset value μ_V, by the risk free interest rate r to evaluate the firm's asset value as a diffusion process (and assuming payout zero e.i $\kappa = 0$):

$$dV_t = V_t(rdt + \sigma_V dW_t) \tag{3.2}$$

More generally, in risk-neutralized world, the expected return of asset is the risk-free rate r.

[3]The similar, but alternative, way of representing the payoff to the bondholders as an option position is given below. At maturity, it is easily verified that the payoff to the bondholders can also be expressed as

$$V_T - \max\{V_T - D, 0\} .$$

That means, the risky debt (payoff to the bondholders) is equivalent to a portfolio consisting of (a) a long position in assets of the firm, and (b) a short position of the call option on the firm's assets with exercise price L and maturity T. Obviously, both decompositions are similar by put-call parity, so either can be worked out for the problem of pricing bond.

call option written on the firm's assets with strike price L:

$$E_T = \max\{V_T - L, 0\}. \tag{3.4}$$

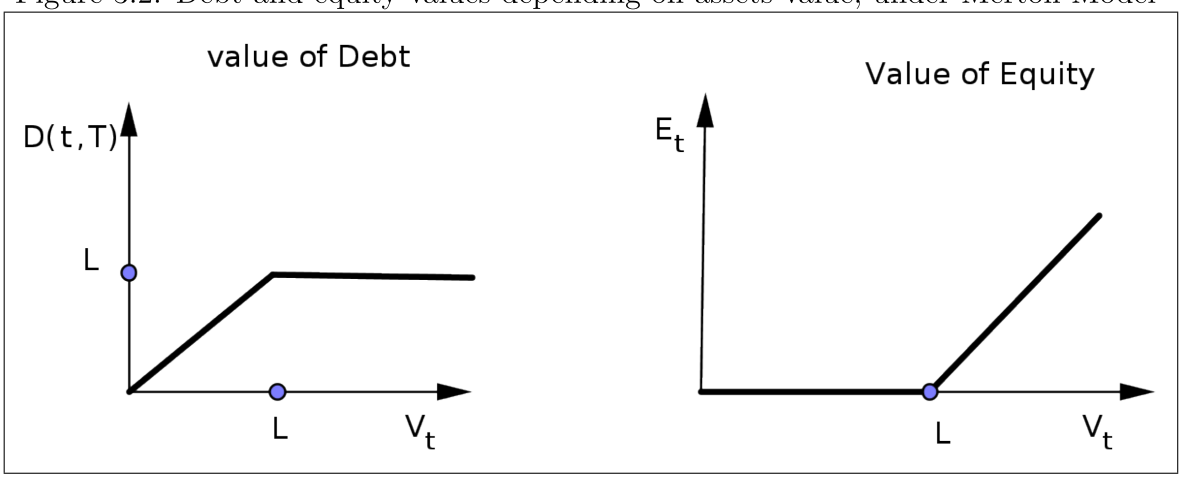
Thus, in figure(**??**), the line from 0 to upper part of graph represents the market value of the firm's debt as a function of the assets value e.i. $D(V_t)$. let $D(t,T)$ be the price at $t < T$ of the debt. Clearly, the value of firm's debt equals

$$D(V_t) = D(t,T) = LB(t,T) - (L - V_t)^+ .$$

The right graph illustrates the equityholders and the payoff of a European call option as function of the assets value $E(V_t)$ given by

$$E(V_t) = V_t - D(V_t) = V_t - LB(t,T) + (L - V_t)^+ = \max\{V_t - L, 0\}. \tag{3.5}$$

Figure 3.2: Debt and equity values depending on assets value, under Merton Model



The holders of the call option, will not exercise their option and will leave the firm to its creditors. Applying the Black and Scholes pricing formula, Merton derived a closed-form expression for its arbitrage price. For every $0 \leqslant t \leqslant T$ the value of the firm's debt and equity equal

$$D(t,T) = V_t\mathcal{N}(-d_1) + Le^{-r(T-t)}\mathcal{N}(d_2) \tag{3.6}$$

$$E_t = V_t\mathcal{N}(d_1) - Le^{-r(T-t)}\mathcal{N}(d_2) \tag{3.7}$$

where

- r is the risk-free interest rate,
- $\mathcal{N}(.)$ is the standard normal distribution function characterized by

$$\mathcal{N}(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} \exp\left(-\frac{x^2}{2}\right) dx,$$

and d_1 and d_2 are given by

$$d_1 = \frac{\ln(V_t/L) + (r + \frac{\sigma_V^2}{2})(T-t)}{\sigma_V \sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma_V \sqrt{T-t}. \tag{3.8}$$

However, we use risk neutral measure (the martingale approach) based on the Feynman Kac stochastic representation formula to find equation (**??**). More detail can be seen in Appendix A. One can also solve the PDE[4], with its boundary conditions directly to find (**??**), and the value of the firm's debt (**??**) can also be found in the same way.

3.1.3 Measurement of the risk neutral probability and objective probability of default

The figure (**??**) above illustrated that the probability of default in the Merton model is given by the probability that the firm's assets value drops below the promised debt payment L at maturity:

$$PD = P_r(V_T \leq L). \tag{3.10}$$

Therefore, at time $t = 0$, in the Merton framework under risk neutral measure, the default probability is simply expressed by

$$PD_M = \mathcal{N}(-d_2) = \mathcal{N}\left(-\frac{\ln(\frac{V_0}{L}) + (r - \frac{1}{2}\sigma_V^2)T}{\sigma\sqrt{T}}\right). \tag{3.11}$$

[4]The payoff to the shareholders (or equity's value) is a European call option in Merton framework, and expressed as (**??**). As usual in contingent claim modelling, the equity's value process follows the PDE :

$$\frac{\partial E_t}{\partial t} + rV_t \frac{\partial E_t}{\partial V_t} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 E_t}{\partial V_t^2} - rE_t = 0, \tag{3.9}$$

subjected to the following boundaries conditions: $E_T(V_T) = \max(V_T - L, 0), \quad E(0) = 0, \quad \frac{E(V)}{V} \leq 1.$

The probabilities of the Merton formula do not represent actually, the probability of sitting above or below the face value at maturity. Since the firm's asset is risky, it does not drift at risk free rate.

If, we change the risk free interest rate r, in equation (**??**) to the expected return on the firm's asset value μ_V, we get the probability of default of the firm under an objective probability measure

$$PD = \mathcal{N}\left(-\frac{\ln(\frac{V_0}{L}) + (\mu_V - \frac{1}{2}\sigma_V^2)T}{\sigma\sqrt{T}}\right). \tag{3.12}$$

As shown in Deliandes and Geske [2003], risk neutral probabilities serve as an upper bound to objective default probabilities. Despite the different distributions of the firm's asset value under risk neutral and objective measure, both have the similar volatility terms (i.e variance). Therefore, the objective distributions must basically have a mean or expected return greater than risk-free rate, in other word, the drift is generally higher than the risk free interest rate, it proves that the risk free neutral distribution allows a higher default probability.

Deliandes and Geske [2003] also demonstrates that risk neutral default probabilities have the equivalent sensitivities as objective default probabilities [**?**].

3.1.4 Term structure of Credit spreads (CS) under Merton Approach

For simplicity of notation, suppose the quantity $\Pi_t = \frac{V_t}{Le^{-r(T-t)}}$ is viewed as a proxy of the asset to debt ratio $V_t/D(t,T)$.

Then, Merton's debt value (**??**) becomes

$$D(t,T) = Le^{-r(T-t)}\left(\Pi_t\mathcal{N}(-d_1) + \mathcal{N}(d_2)\right), \tag{3.13}$$

so the corporate debt is a risk bond, and thus should be modelled at a credit spread (risk premium). Let $S(t,T)$ be denoted the continuously compounded credit spread at time $t < T$, while $Le^{-r(T-t)}$ represents the current (or present) value of the face value of firm's debt. [5]

[5]The inequality $D(t,T) < Le^{-r(T-t)}$ can be easily checked and it similar to the property of positivity of credit spread.

Observe that the yield to maturity, $r^d(t,T)$, of a corporate ZCB in continuous-time is implicitly expressed as

$$D(t,T) = Le^{-r^d(t,T)(T-t)}.$$

From this equality, and substituting $D(t,T)$ with expression (**??**), it follows :

$$r^d(t,T) = -\frac{\ln\left(\Pi_t e^{-r(T-t)}\mathcal{N}(-d_1) + \mathcal{N}(d_2)e^{-r(T-t)}\right)}{T-t}.$$

At $t < T$, the credit spread is the excess return on a defaultable bond (or the difference in yield between two bonds of similar maturity but different credit quality) $S(t,T) = r^d(t,T) - r(t,T)$.[6] Using the last equality, the credit spread in Merton's model is expressing as

$$S(t,T) = -\frac{\ln\left(\Pi_t \mathcal{N}(-d_1) + \mathcal{N}(d_2)\right)}{T-t} > 0. \tag{3.14}$$

This shows that, credit spread is a function of asset to debt ratio, time to maturity and the volatility of the firm's asset, and well agrees with the theory that risk bonds have the higher expected return than the risk-free interest rate (i.e, yields on Treasury bonds with matching notional value are lower than the yields on corporate bonds).

Therefore, let us analyse the behaviour of the credit spread in the Merton framework. When t belongs to $[0,T]$, the credit spread goes either to 0 or to infinity, according to whether we have default or not ($V_T < L$ or $V_T > T$). For this propose, we define the forward short spread at time T, as :

$$Fss_T = \lim_{t\to T} S(t,T),$$

and observe that

$$Fss_T(\theta) = \begin{cases} \infty, & \text{if} \quad \theta \in \{V_T < L\}, \\ 0, & \text{if} \quad \theta \in \{V_T > L\}. \end{cases}$$

[6]Here $r(t,T)$ is a yield continuously compound at time t on the future time T of the Treasure ZCB, and is a constant, equal to the risk free interest rate r. We assume

$$B(t,T) = e^{-r(t,T)(T-t)} = e^{-r(T-t)}.$$

(see appendix for a more detailed of credit spread)

While the volatility of the firm's asset can be obtained from historical data (or be modelled as in implied volatility), the market value of the firms asset V_t is not directly observable (Hull, 2000). On the other hand, E_t is easily observed in the marketplace, and we can estimate σ_E. We know that, according to our assumptions, the processes for the equity can be given by the following stochastic differential equation

$$dE_t = \mu_E E dt + \sigma_E dW_t, \tag{3.15}$$

where μ_E represent the drift in the equity and σ_E its volatility or diffusion term.
Given the market value of the firms asset V_t comprises of the value equity E_t and market value of total debt D_t , as $V_t = E_t + D_t$, by applying Itos lemma, we can also represent the value of equity E_t as:

$$dE_t = \left(\frac{\partial E_t}{\partial t} + \mu_V V_t \frac{\partial E_t}{\partial V_t} + \frac{1}{2}\sigma_V^2 V_t^2 \frac{\partial^2 E_t}{\partial V_t^2} \right) dt + \sigma_V V \frac{\partial E_t}{\partial V_t} dW_t. \tag{3.16}$$

By comparing the diffusion terms in the equity value process in (??) and (??) we obtain the following relationship:

$$\sigma_E = \sigma_V \left(\frac{V}{E}\right) \frac{\partial E_t}{\partial V_t} = \sigma_V \left(\frac{V}{E}\right) \mathcal{N}(d_1), \tag{3.17}$$

where $\mathcal{N}(d_1)$ is viewed as the hedge ratio or the " delta " in standard option terminology. This equation enable us to determine σ_V and V_t in terms of known values E_t, σ_E and L. Substituting d_1 and d_2 given by (??) into equation (??), and substituting d_1 from (??) into (??), yields the following system of 2 nonlinear equations after rearranging terms:

$$V_t \mathcal{N}\left(\frac{\ln(V_t/L) + (r + \frac{\sigma_V^2}{2})(T-t)}{\sigma_V \sqrt{T-t}} \right) - Le^{-r(T-t)} \mathcal{N}\left(\frac{\ln(V_t/L) + (r + \frac{\sigma_V^2}{2})(T-t)}{\sigma_V \sqrt{T-t}} - \sigma_V \sqrt{T-t} \right) = E_t, \tag{3.18}$$

$$\sigma_V V_t \mathcal{N}\left(\frac{\ln(V_t/L) + (r + \frac{\sigma_V^2}{2})(T-t)}{\sigma_V \sqrt{T-t}} \right) - \sigma_E E = 0. \tag{3.19}$$

Basically, if the values r, L, T, E_t and σ_E are given then, we can find the unknown variable σ_V and V_t by solving the system [7].

[7] This system of 2 nonlinear equations and 2 unknown variables basically, can be solved by applying a standard Newton Raphson algorithm, see [?].

3.1.5 Application of the Merton Approach to South African Bonds

In this section, we use the data given by [?] to find the credit spread (basis points abbreviated by bp) under Merton approach, we investigate a range of 20 South African firms, with different ratings ranging from AAA to BBB. We utilize the current debt of the firm and measure the debt financing amount (or the market value of the total debt) at different maturity times. The debt term of five years, three years and one year were considered and shown in below Table (??). Clearly, the credit spread increases with regard to the volatility of the firm's asset value and debt's face value L (formulated in term of the leverage ratio or the inverse of the proxy of the asset to debt ratio $d = \frac{1}{\pi_t}$). We find that, in the Banking sector the credit spread calculated varies from 6 bp to 297 bp, and in other sectors the credit spread calculated varies from 3 bp to 85 bp for a debt term of 5 years maturity. Considering a debt term of maturity 3 years, the credit spreads oscillates between 2 bp and 12 bp for companies rated AA. We notice that for the year's data, the credit spreads in the South African market in general were fluctuating between 30 and 60 Bps. Therefore, the result is in accord with the prediction.

Table 3.1: Credit spread for Merton's approach using various South African debt terms

company	Rating	Volatility of the firm's asset (%)	Leverage ratio (%)	1 year CS (Bp)	3 years CS (Bp)	5 year CS (Bp)
1	AAA	32	25	0.00	7	28
2*	AA^{+}	19	37	0.00	1	6
3	AA	27	24	0.00	2	11
4*	AA	5	84	0.2	7	18
5*	AA	4	87	0.2	7	18
6*	AA	7	82	0.4	12	29
7*	AA	3	91	0.2	6	15
8	AA^{-}	17	41	0.00	0	3
9	AA^{-}	20	46	0.00	5	19
10	A^{+}	21	37	0.00	2	10
11	A^{+}	29	38	0.6	33	85
12*	A^{+}	3	93	0.4	7	17
13*	A^{+}	7	84	0.7	15	36
14*	A^{-}	4	89	0.0	8	21
15	A^{-}	24	36	0.0	6	24
16	A^{-}	41	13	16	3	18
17*	BBB^{+}	30	49	1.6	160	297
18*	BBB	15	69	1.6	34	77
19*	BBB	18	67	7.4	84	169
20*	BBB	22	60	9.8	108	211

An asterisk indicates that the firm is in the banking sector.

3.2 Summary

This chapter presented a literature overview of the structural approach for modelling credit risk. We started with the first structural model of Merton (1974) and identified its shortcomings from a comparison between the measurement of the risk neutral probability and objective probability of default. The purpose of this study is to implement a credit risk model that can be used to value CDS contracts, estimate market CDS term structures and to perform analysis on several credit risk measures. Based on the analysis in this chapter and the evaluation of the structure of Credit spreads under the Merton approach in Section (**??**), we investigated a range of 20 South African firms, with different ratings from AAA to BBB and found the CS using the Merton model.

We further use those credit spreads calculated on South Africa Bonds from different maturities such as one year, three years and five years as data to estimate the parameters associated with Vasicek and CIR processes in the numerical aanalysis part. Chapter 4 further determines the credit default swap under a Vasicek-type hazard rate, and in Chapter 5, we estimate hazard rates type Vasicek model and CIR models, and analyse the results.

Chapter 4

Pricing Credit Default Swap under Vasicek-type Hazard rate

The primary motivation for using credit derivatives is to reduce risk arising from bondholders or owners of the loans. The credit default swap (CDS) is the cornerstone of the credit derivatives market, and is actually the most widely used credit derivative product. In this chapter we will follow [**?**] and the framework proposed by David and Mavroidis. We outline the main techniques used to value credit default swaps under a Vasicek-type hazard rate. We consider the procedure for only one bond issued by a firm or where there is only one reference bond in the market. These tools build on the framework of arbitrage-free opportunity pricing which was discussed in the previous chapter .

4.0.1 Framework

Given the filtered space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ and a standard Brownian motion W on space, we let $\mathcal{G} = \{\mathcal{G}_t\}(t \in [0, T])$ be a two dimensional filtration Brownian motion and denote by τ a non-negative random default time. We assume the 2-dimensional $(\Omega, \mathcal{G}_t, \mathbb{P})$-Brownian motion (W, W^*), that means, $\{\mathcal{G}_t\}(t \in [0, T])$ satisfies the ‘ regular conditions’ of completeness and right-continue. Also $\mathcal{G}_t$ is the smallest filtration which made W and W^* adapted (refer back to Preliminaries chapter).

If we suppose that we are given an auxiliary reference filtration $\sigma\{\tau \wedge t\}$ such that $\mathcal{F}_t = \bigcap_{t<s} (\mathcal{G}_s \vee \sigma\{\tau \wedge s\})$, then the all possible information available at time t is captured by the filtration $\mathcal{F}_t$, it is a right continuous τ and also is an $\mathcal{F}_t$-stopping time. In the above $\tau \wedge s$ is an abbreviation for $\min\{t, s\}$.

The following important lemma are introduced for further evaluation of credit default swaps.

Lemma 4.0.1. *Assume that the hazard rate process γ_t is a non-negative $\mathcal{G}_t$ progressively measurable. Then the process*

$$M_t = H_t - \int_0^t \gamma_s 1_{\{\tau > s\}} ds \tag{4.1}$$

is a Martingale on $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$.

Proof 4.0.2. *See [?, ?].*

Lemma 4.0.3. *Assume that, for $0 \leq t < T$, X be a $\mathcal{F}_T-$measurable and $\mathbb{Q}-$integrable random variable, then we have*

$$E_{\mathbb{Q}} \left[X 1_{\{\tau > t\}} | \mathcal{G}_t \right] = 1_{\{\tau > t\}} E_{\mathbb{Q}} \left[X \exp \left(- \int_t^T \gamma(u) du \right) | \mathcal{F}_t \right]. \tag{4.2}$$

Proof 4.0.4. *See Nakagawa[?], Duffie et al.[?] and Kusuoka[?].*

Corollary 4.0.5. *For any bounded $\mathcal{G}_{[0,T]}$-predictable process Z*

$$E \left[Z_\tau 1_{\{\tau \leq T\}} \right] = E \left[\int_0^T Z_t \gamma(t) \exp \left(- \int_0^t \gamma(u) du \right) dt \right] \tag{4.3}$$

[?].

Proof 4.0.6. *See Nakagawa[?], Duffie et al.[?] and Kusuoka[?] .*

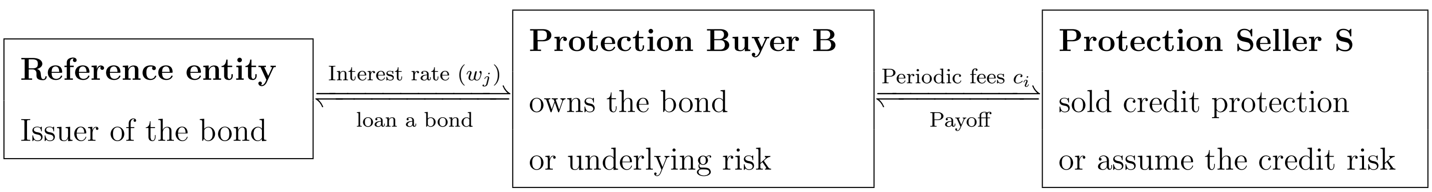
4.1 Definition of CDS and pricing CDS

We follow the framework of David and Mavroidis, [?] and Hidetoshi and al.[?] to value the CDS, starting first, by defining the credit default swaps, specifying the rule of the default swap and pricing CDS.

4.1.1 Definition of CDS

A credit default swap is an agreement designed between two parties that provides a protection or assurance against losses occurring due to a default event of an certain entity.

One party agreed to buy protection called protection buyer B (e.g a firm) and provides a regular payment c_i $(i = 1, 2, \dots, n)$ until the credit event occurs or at maturity of the contract (at expected time $t_i < \cdots < t_n \leqslant T$), the other is the seller of the protection S. Typically banks or insurance companies will assume the credit risk and deliver the difference between the notional value and some recovered value δ [1] from the bond issuer for the owner of the bond B, if the credit event of the bond issuer happens before the maturity date T. [2] The risky bond that the buyer B holds permits a fixed coupon w_j $(j = 1, 2, \dots)$ at each adopted time s_j $(j = 1, 2, \dots,$ $0 \leqslant s_1 < s_2 < \dots)$ except when the default even occurs.



A credit default swap agreement includes a fixed premium leg or fixed side and a recovery side (or contingent default leg).

- The fixed side corresponds to the series of payments made by the buyer B of the CDS-contract to the protection seller S of the contract up to the maturity time, unless a bankruptcy event or other credit event perturbs the contingent payment on a CDS.
- The recovery side corresponds to the net payment delivered by the counterparty protection seller S to the protection buyer in case of such default event happens.

[1] In this case, we assume the recovered value from the bond issuer is specified by $\delta\times$ (the instant pre-default bond's face value), where $0 < \delta < 1$ defined as the constant.

[2] T is the expiry date of the contract. Whether the default of the bond issuer happens before T, then the contract is strictly stopped.

The main goal of valuation of CDS is to obtain equilibrium premium (or regular payment) c_i's paid periodically by the reference holder, which is followed from the equality of the value between fixed premium leg and contingent default leg. Consider the risk-free interest rate r be independent of all factors related to credit risk, alike default time and the hazard rates. This assumption implies that we can value default swap in term of the default-free zero coupon bond's prices as follows.

4.1.2 The price of the fixed side

Since all payments are evaluated at the starting level of the contract and no payment is made after any default event occurs, the actual value of the fixed side D_p is basically defined by

$$D_p = E\left[\sum_{i=1}^{n} c_i \exp\left(-\int_0^{t_i} r_u du\right) 1_{\{\tau>t_i\}}\right], \tag{4.4}$$

where r is the short rate interest and $E(.)$ is the expectation value under the risk neutral measure $\mathbb{Q}$. It follow

$$\begin{aligned} D_p &= \sum_{i=1}^{n} c_i E\left[\exp\left(-\int_0^{t_i} r_u du\right) 1_{\{\tau>t_i\}}\right] \\ &= \sum_{i=1}^{n} c_i E\left[E\left[\exp\left(-\int_0^{t_i} r_u du\right) 1_{\{\tau>t_i\}}\right] |\mathcal{F}_t\right] \\ &= \sum_{i=1}^{n} c_i B(0,t_i) E\left[\exp\left(-\int_0^{t_i} \gamma_u du\right)\right] \\ &= \sum_{i=1}^{n} c_i B(0,t_i) p(\tau > t_i), \end{aligned}$$

using the previous work we get

$$D_p = \sum_{i=1}^{n} c_i B(0,t_i) \exp\left[\frac{\gamma_0}{a}\left(e^{-at_i}-1\right) - \frac{1}{a}\left(e^{-at_i}-1\right)\left(b + \frac{\sigma^2}{4a^2}\left(e^{-at_i}-3\right)\right) - t_i\left(b - \frac{\sigma^2}{2a^2}\right)\right] \tag{4.5}$$

is an explicit value of the fixed side. The next section discuss about the price of recovery side.

4.1.3 The price of the contingent default leg (recovery side)

We define $CC(t)$ be the cum-coupon amount of the underlying defaultable bond and its value is given by

$$CC(t) = E\left[\sum_{s_i \geq t} w_i \exp\left(-\int_t^{s_i} (r_u + (1-\delta)\gamma_u)du\right) |\mathcal{F}_t\right].$$

It is supposed that the fixed premium leg to the contract can recover the specific amount $\delta CC(\tau)$, when the default event occurs to the issuer of the bond, at default time τ.

Then, the price of the recovery side has been defined by Hidetoshi Nakagawa [**?**] as follow

$$\begin{aligned} D_R &= E\left[\exp\left(-\int_0^\tau r_u du\right)(1 - \delta CC(\tau))\, 1_{\{\tau \leqslant T\}}\right] \\ &= E\left[\exp\left(-\int_0^\tau r_u du\right) 1_{\{\tau \leqslant T\}}\right] - \delta E\left[\exp\left(-\int_0^\tau r_u du\right) CC(\tau) 1_{\{\tau \leqslant T\}}\right]. \end{aligned}$$

This equation can be evaluated separately as

$$\begin{aligned} E\left[\exp\left(-\int_0^\tau r_u du\right) 1_{\{\tau \leqslant T\}}\right] &= E\left[E\left[\exp\left(-\int_0^\tau r_u du\right) 1_{\{\tau \leqslant T\}}\right] |\mathcal{F}_t\right] \\ &= E\left[B(0,\tau) 1_{\{\tau \leqslant T\}}\right] \\ &= E\left[\int_0^T B(0,t)\gamma_t \exp\left(-\int_0^t \gamma_u du\right) dt\right] \\ &= \int_0^T B(0,t) E\left[\gamma_t \exp\left(-\int_0^t \gamma_u du\right)\right] dt. \end{aligned}$$

Since corollary (**??**), lemma (**??**) and using hazard rate mean reversion we get

$$\begin{aligned} E\left[\exp\left(-\int_0^\tau r_u du\right) 1_{\{\tau \leqslant T\}}\right] &= \int_0^T B(0,t) E\left[\gamma_t \left(bt + (\gamma_0 - b)\frac{1-e^{-at}}{a} + \frac{\sigma}{a}\int_0^t \left(1 - e^{a(u-t)}\right) dW_u\right)\right] dt \\ &= \int_0^T B(0,t)\gamma_t \left(bt + (\gamma_0 - b)\frac{1-e^{-at}}{a}\right) dt. \end{aligned}$$

And for second term with application of corollary (**??**)

$$
\begin{aligned}
&\delta E\left[\exp\left(-\int_0^\tau r_u du\right) CC(\tau)1_{\{\tau\leqslant T\}}\right]\\
=&\delta E\left[\int_0^T \exp\left(-\int_0^t r_u du\right) CC(t)\gamma_t \exp\left(-\int_0^t \gamma_u du\right) dt\right]\\
=&\delta E\left[\int_0^T \exp\left(-\int_0^t r_u du\right)\left(E\left[\sum_{s_i\geq t} w_i \exp\left(-\int_t^{s_i}(r_u+(1-\delta)\gamma_u)du\right)|\mathcal{F}_t\right]\right)\gamma_t \exp\left(-\int_0^t \gamma_u du\right) dt\right]\\
=&\delta \sum_{s_i\geq t} w_i E\left[\int_0^T \exp\left(-\int_0^t r_u du\right)\left(E\left[\exp\left(-\int_t^{s_i}(r_u+(1-\delta)\gamma_u)du\right)|\mathcal{F}_t\right]\right)\gamma_t \exp\left(-\int_0^t \gamma_u du\right) dt\right]\\
=&\delta \sum_{s_i\geq t} w_i E\left[\int_0^T \exp\left(-\int_0^t r_u du\right)\left(\exp\left(-\int_t^{s_i}(r_u+(1-\delta)\gamma_u)du\right)\right)\gamma_t \exp\left(-\int_0^t \gamma_u du\right) dt\right]\\
=&\delta \sum_{s_i\geq t} w_i E\left[\int_0^T \gamma_t \exp\left(-\int_0^{s_i} r_u du\right)\exp\left(-(1-\delta)\int_t^{s_i}\gamma_u)du\right)\exp\left(-\int_0^t \gamma_u du\right) dt\right]\\
=&\delta \sum_{s_i\geq t} w_i B(0,t_i) E\left[\int_0^T \gamma_t \exp\left(-\int_0^t \gamma_u du\right)\exp\left(-(1-\delta)\int_t^{s_i}\gamma_u)du\right) dt\right]\\
=&\delta \sum_{s_i\geq t} w_i B(0,t_i) E\left[\int_0^T \gamma_t \exp\left(-\int_0^t \gamma_u du\right) E\left[\exp\left(-(1-\delta)\int_t^{s_i}\gamma_u)du\right)|\mathcal{G}_t\right] dt\right].
\end{aligned}
$$

From the above the term

$$
E\left[\exp\left(-(1-\delta)\int_t^{s_i}\gamma_u)du\right)|\mathcal{G}_t\right] = \exp\left[\frac{1}{a}(1-\delta)\left(e^{-a(s_i-t)}-1\right)(\gamma(t)-b)-\left(e^{-a(s_i-t)}-1\right)\psi_i(t)\right], \tag{4.6}
$$

where

$$
\psi_i(t) = \frac{\frac{(1-\delta)}{a}(s_i-t)\left[b-\frac{\sigma^2}{2a}(1-\delta)\right]}{e^{-a(s_i-t)}-1} + \frac{\sigma^2}{2a^3}(1-\delta)^2\left(e^{-a(s_i-t)}-3\right).
$$

So

$$
\begin{aligned}
&E\left[\exp\left(-\int_0^\tau r_u du\right) CC(\tau)1_{\{\tau\leqslant T\}}\right]\\
=&\sum_{s_i\geq t} w_i B(0,t_i)\int_0^T \left(bt+(\gamma_0-b)\frac{1-e^{-at}}{a}\right)\gamma_t \exp\left[\frac{(1-\delta)}{a}\left(e^{-a(s_i-t)}-1\right)(\gamma_t-b)-\left(e^{-a(s_i-t)}-1\right)\psi_i(t)\right] dt.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
D_R = &\int_0^T B(0,t)\gamma_t \left(bt + (\gamma_0 - b)\frac{1-e^{-at}}{a} \right) dt \\
-\delta \sum_{s_i \geq t} w_i B(0,t_i) &\int_0^T \left(bt + (\gamma_0 - b)\frac{1-e^{-at}}{a} \right) \gamma_t \exp\left[\frac{(1-\delta)}{a} \left(e^{-a(s_i-t)} - 1\right)(\gamma_t - b) - \left(e^{-a(s_i-t)} - 1\right)\psi_i \right] dt.
\end{aligned}
$$

This is the price of the recovery side or contingent default leg, given the value of the default free zero coupon bond $B(0,t)$ its explicit value can be derived.

4.2 Summary

This chapter explained a credit default swap and determined the explicit value of the fixed side and recovery side of the credit default swap in a quite general form under hazard rates distributed by the Vasicek process, that is, it contains the type with counter party risk (or basket type). The next chapter estimates the hazard rate for Vasicek models and CIR models using the market data of credit spreads.

Chapter 5

Estimating Hazard rate process and Defautable Zero coupon bonds

In this chapter, we discuss how to estimate hazard rate type Vasicek model and CIR model. Using the relationship between credit spread and hazard rate, it is possible to convert the market credit spread data (collected directly from the market data) into CIR and Vasicek type Hazard rate data. We also analyse the conditional survival probability for both processes. Our first analysis, thus, assumes the risk-free interest rate r to be independent of all the hazard rates. Therefore the occurrence of default is not correlated with bond prices. This assumption implies that the level of default is cause by some factors affecting the issuer, not the level of risk-free interest rate.

5.1 Bond Valuation under stochastic hazard rate

Let assume that the reference bond is represented by the defaultable zero coupon bond with the single payoff 1 (notional value) and maturity time T.
This leads to the following description :

let τ be the default time, for an hazard process with stochastic intensity γ; then the price of a zero-coupon defaultable bond with notional value 1 is given by

$$D(t,T) = B(t,T).E\left[\exp\left(-\int_t^T (1-\delta)\gamma(s)ds\right)\right], \tag{5.1}$$

where $t \in [0,T]$, and δ is the recovery rate. Similarly, at time t the credit spread, viewed as the difference between the default adjusted interest rate and the risk-free interest rate is given by $(1-\delta)\gamma(t)$ [**?**].

We are now in a position to define the relationship between the credit spread and hazard rate process. This is useful in converting the credit spread data given from market data into the hazard rate.

For $t < T$, the credit spread process $S(t,T)$ for the bond with maturity T satisfies the relationship :

$$\exp\left(-S(t,T)(T-t)\right) = \frac{D(t,T)}{B(t,T)} \tag{5.2}$$

$$= E\left[\exp\left(-\int_t^T (1-\delta)\gamma(s)ds\right)\right]. \tag{5.3}$$

Note that it is impossible to estimate the recovery rate δ and the hazard rate γ separately from the credit spread: knowing δ (given by another technique), we may determine the parameters of the hazard rate γ. We therefore need to determine the distribution of the random variable $-\int_t^T (1-\delta)\gamma(s)ds$. To manage this, we need to know how the hazard rate process is distributed. For example :

Case of a constant hazard rate, $\gamma(t) = \gamma_0 > 0$.
Since $\exp\left(-S(t,T)(T-t)\right) = \exp\left(-(1-\delta)\gamma_0(T-t)\right)$ then

$$S(t,T) = (1-\delta)\gamma_0. \tag{5.4}$$

We calculate the estimators using this important data. The following section treats the hazard rate process distributed by the Vasicek model.

5.1.1 The Vasicek model type hazard rate

We define the hazard rate process γ by means of the affine stochastic differential equation (SDE)

$$d\gamma(t) = a(b - \gamma(t))dt + \sigma dW_t, \quad \gamma(0) = \gamma_0 > 0, \tag{5.5}$$

where a, b and σ are strictly positive constants, W_t is a standard Wienner process. The parameters a, b and σ are viewed as the mean reversion rate (or the mean reverting velocity), the mean reversion level, and the volatility respectively. This SDE is known as a mean-reverting Ornstein-Uhlenbeck process (discussed in Section (**??**)) and its solution $\gamma(t)$ is given by

$$\gamma(t) = b + (\gamma_0 - b)e^{-at} + \sigma e^{-at} \int_0^t e^{as} dW_s. \tag{5.6}$$

From equation (**??**) we see that; given γ_0, the possible values of γ are being normally distributed. The reason is that the distribution of dW_s is normal with mean zero and variance one. Therefore, the integral itself is normally distributed and, since the remaining part of the equation other than the stochastic part is deterministic, the current distribution of the stochastic part does not change except its mean and variance. Therefore, $\gamma(t)$ is normally distributed with mean and variance expressed respectively by

$$E^{\mathbb{Q}}(\gamma(t)) = b + (\gamma_0 - b)e^{-at}, \tag{5.7}$$

$$var^{\mathbb{Q}}(\gamma(t)) = \sigma^2 e^{-2at} \left(\int_0^t e^{as} dW_s \right)^2 = \sigma^2 e^{-2at} \int_0^t e^{2as} ds = \frac{\sigma^2}{2a}(1 - e^{-2at}). \tag{5.8}$$

Remark. According to equation (**??**), γ is mean reverting, since the expectation value tends to b, for t goes to infinity, which can be viewed as a long-term average value.

Given the relation of credit spread

$$\exp\left(-S(t,T)(T-t)\right) = E\left[\exp\left(-\int_t^T (1-\delta)\gamma(s)ds\right)\right], \tag{5.9}$$

we can now derive the hazard rate by computing the expectation. Using the Markov property for Itô processes (Theorem (**??**)) this expression can be represented as

$$E_{\mathbb{Q}}\left(\exp\left(-\int_0^{T-t} (1-\delta)\gamma(s)^y ds\right)\right)|_{y=\gamma(t)}. \tag{5.10}$$

Hence, we first evaluate $\left(-\int_0^{T-t}(1-\delta)\gamma(s)^y ds\right)|_{y=\gamma(t)}$:

$$\begin{aligned}\left(\int_0^{T-t}\gamma(s)^y ds\right)|_{y=\gamma(t)} &= b(T-t)+(\gamma(t)-b)\frac{1-e^{-a(T-t)}}{a}+\sigma\int_0^{T-t}\int_0^s e^{a(u-s)}dW_u ds\\ &= b(T-t)+(\gamma(t)-b)\frac{1-e^{-a(T-t)}}{a}+\sigma\int_0^{T-t}\left(\int_0^s e^{a(u-s)}ds\right)dW_u\\ &= b(T-t)+(\gamma(t)-b)\frac{1-e^{-a(T-t)}}{a}+\frac{\sigma}{a}\int_0^{T-t}\left(1-e^{a[u-(T-t)]}\right)dW_u.\end{aligned}$$

As $\left(\int_0^{T-t}\gamma(s)^y ds\right)|_{y=\gamma(t)}$ is clearly a Gaussian variable, we can now easily determine the expectation

$$\begin{aligned}&E_{\mathbb{Q}}\left(\exp\left(-\int_0^{T-t}(1-\delta)\gamma(s)^y ds\right)\right)|_{y=\gamma(t)}\\ &=\exp\left[E_{\mathbb{Q}}\left(-\int_0^{T-t}(1-\delta)\gamma(s)^y ds\right)|_{y=\gamma(t)}+\frac{1}{2}var^{\mathbb{Q}}\left(-\int_0^{T-t}(1-\delta)\gamma(s)^y ds|_{y=\gamma(t)}\right)\right],\end{aligned}$$

where

$$\begin{aligned}E_{\mathbb{Q}}\left(-\int_0^{T-t}(1-\delta)\gamma(s)^y ds\right)|_{y=\gamma(t)} &= E\left(b(T-t)+(\gamma(t)-b)\frac{1-e^{-a(T-t)}}{a}+\frac{\sigma}{a}\int_0^{T-t}\left(1-e^{a[u-(T-t)]}\right)dW_u\right)\\ &= b(T-t)+(\gamma(t)-b)\frac{1-e^{-a(T-t)}}{a}+\frac{\sigma}{a}E\left(\int_0^{T-t}\left(1-e^{a[u-(T-t)]}\right)dW_u\right)\\ &= -(1-\delta)\left(b(T-t)+(\gamma(t)-b)\frac{1-e^{-a(T-t)}}{a}\right),\end{aligned}$$

and

$$\begin{aligned}var^{\mathbb{Q}}\left(-\int_0^{T-t}(1-\delta)\gamma(s)^y ds|_{y=\gamma(t)}\right) &= E_{\mathbb{Q}}\left[\left(-\int_0^{T-t}(1-\delta)\gamma(s)ds\right)-E_{\mathbb{Q}}\left(-\int_0^{T-t}(1-\delta)\gamma(s)ds\right)\right]^2|_{y=\gamma(t)}\\ &= E_{\mathbb{Q}}\left[\left(-(1-\delta)\frac{\sigma}{a}\int_0^{T-t}\left(1-e^{a[u-(T-t)]}\right)dW_u\right)\right]^2|_{y=\gamma(t)}\\ &= \frac{\sigma^2}{a^2}(1-\delta)^2\int_0^{T-t}\left(1-e^{a[u-(T-t)]}\right)^2 du\\ &= \frac{\sigma^2}{a^2}(1-\delta)^2(T-t)+\frac{\sigma^2}{a^2}(1-\delta)^2\frac{e^{-a(T-t)}-1}{2a}.\end{aligned}$$

Hence the relation (**??**) becomes

$$\exp\left(-S(t,T)(T-t)\right)=\exp\left[\frac{1}{a}(1-\delta)\left(e^{-a(T-t)}-1\right)(\gamma(t)-b)-\left(e^{-a(T-t)}-1\right)\psi(t)\right].$$

Equivalently, the hazard rate is given by

$$\gamma(t) = \frac{a}{(1-\delta)} \left(\frac{-S(t,T)(T-t)}{e^{-a(T-t)}-1} - \psi(t) \right) + b, \tag{5.11}$$

where

$$\psi(t) = \frac{\frac{(1-\delta)}{a}(T-t)\left[b - \frac{\sigma^2}{2a}(1-\delta)\right]}{e^{-a(T-t)}-1} + \frac{\sigma^2}{2a^3}(1-\delta)^2\left(e^{-a(T-t)}-3\right).$$

Figure 5.1: hazard rate function $\gamma(t)$ with $\sigma = 0.01$

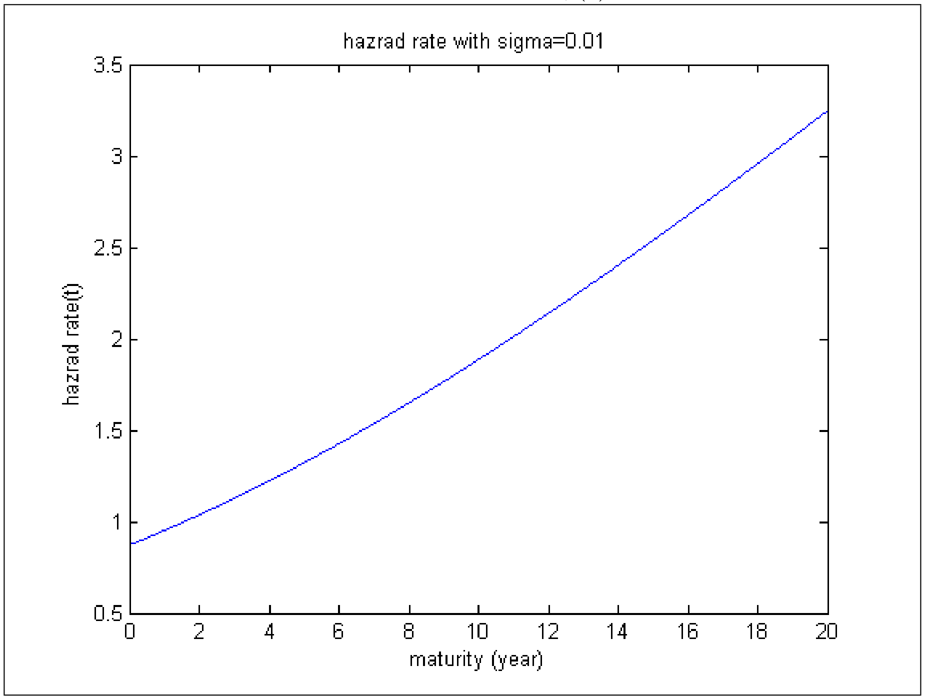


Figure 5.2: hazard rate function $\gamma(t)$ with $\sigma = 0.1$

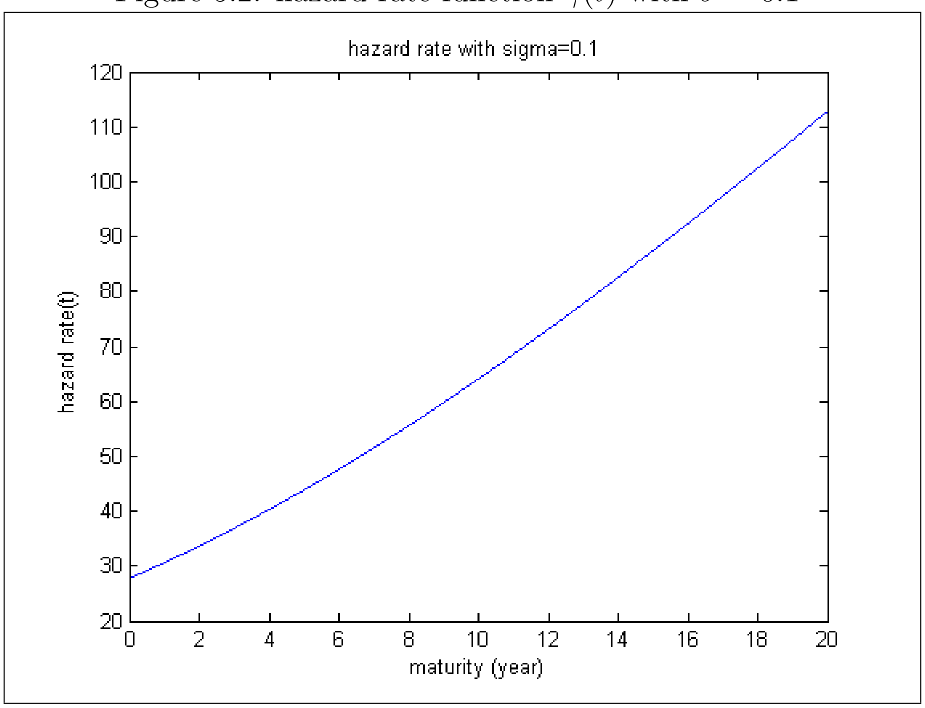


Figure 5.3: hazard rate function $\gamma(t)$ with $\sigma = 0.2$

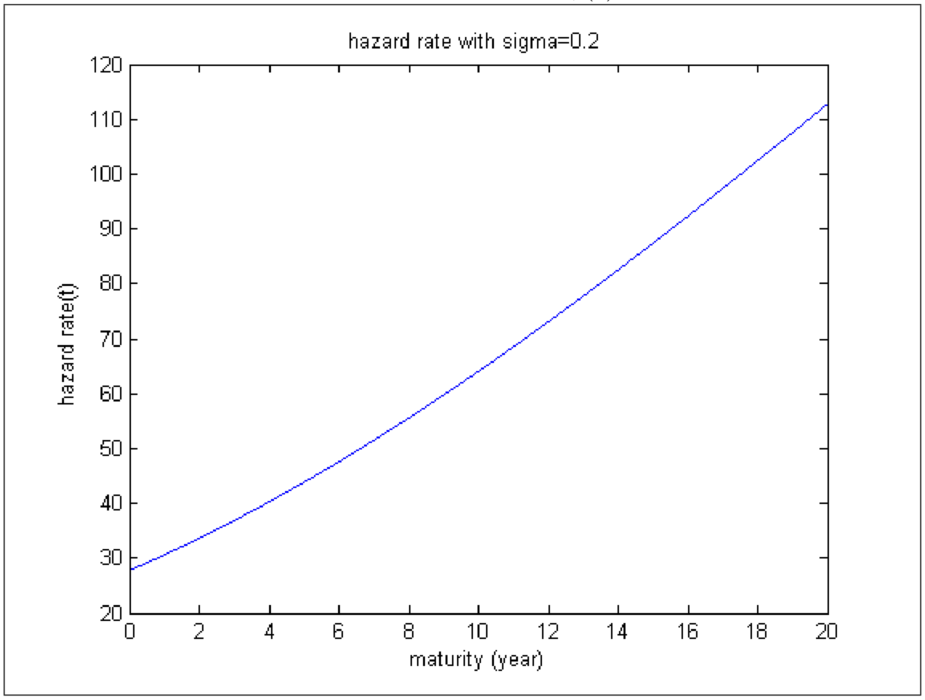


Figure 5.4: hazard rate function $\gamma(t)$ with $\sigma = 0.5$

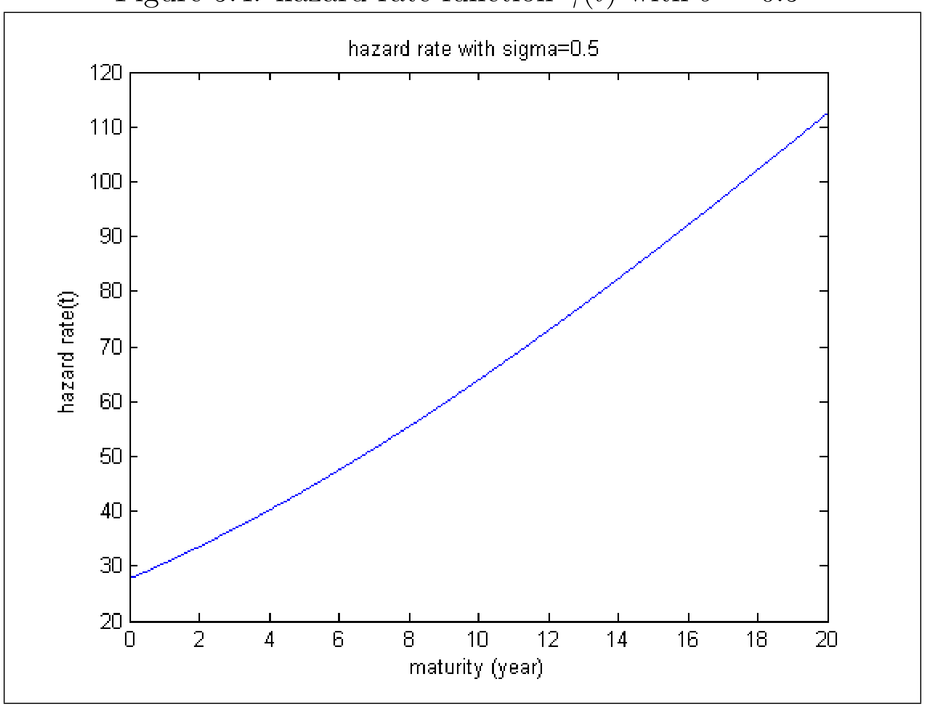
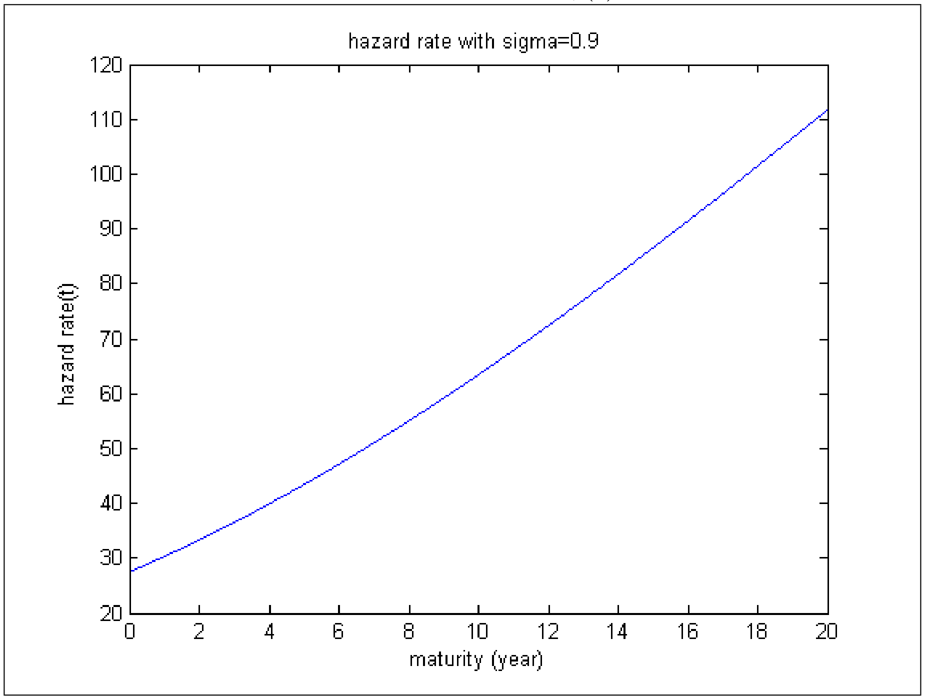


Figure 5.5: hazard rate function $\gamma(t)$ with $\sigma = 0.9$



These figures (??), (??), (??), (??) and (??) illustrate the hazard rate function of time, distributed as the Vasicek model of twenty years from now on, given the market price of credit spread. We took the volatility to be $\sigma = 0.01$, $\sigma = 0.1$, $\sigma = 0.2$, $\sigma = 0.5$ and $\sigma = 0.9$ respectively. Such curves as in figures(??), (??), (??), (??) and (??) are upward sloping curves. Those upward sloping curves show that hazard rate functions are increasing over time. Intuitively, this means that the probability of defaulting in any period (conditional on not having defaulted until then) increases as time goes on. Upward sloping curves mean that the market is implying not only that firms are more likely to default with every year that goes by, but also that the likelihood in each year is ever increasing. Credit risk is therefore getting increasingly worse for every year into the future.

5.1.2 The CIR model type hazard rate

Here the hazard rate process γ follows a stochastic differential equation (SDE) of the form

$$d\gamma(t) = a(b - \gamma(t))dt + \sigma\sqrt{\gamma_t}dW_t, \quad \gamma(0) = \gamma_0 > 0, \tag{5.12}$$

where a, b and σ are strictly positive constants and W_t is a standard Wienner process. The solution $\gamma(t)$ is obtained by equation (**??**). The principal advantage of the CIR-type model compared to the Vasicek-type model is that the solution $\gamma(t)$ of the SDE is guaranteed to remain positive (non-negative). Dissimilar to the Vasicek process, however, the squared root process is not Gaussian and is an affine process. Therefore, it is, considerably more bother to study.

The quick way to determine $E\left[\exp\left(-\int_t^T(1-\delta)\gamma(s)ds\right)\right]$ is to suitably transform it as

$$E\left[\exp\left(-\int_t^T (1-\delta)\gamma(s)ds\right)\right] = e^{(1-\delta)}E\left[\exp\left(-\int_t^T \gamma(s)ds\right)\right].$$

The second right hand expression is contained in the formula of the bond price (for $\delta = 0$), which is the expectation of the exponential of minus the integral of the short term process. Analogously to the previous work of the CIR model and Bond price under CIR in chapter [1] section (**??**), the CIR formula for the price of a zero-coupon bond is

$$E\left[\exp\left(-\int_t^T \gamma(s)ds\right)\right] = \exp\left(\Phi_*(t,T) - \Psi_*(t,T)\gamma_t\right)$$

where

$$\Phi_*(t,T) = \frac{2ab}{\sigma^2}\ln\left(\frac{2he^{(a+h)(T-t)/2}}{2h + (a+h)\left(e^{h(T-t)}-1\right)}\right),$$

$$\Psi_*(t,T) = \frac{2\left(e^{h(T-t)}-1\right)}{2h + (a+h)\left(e^{h(T-t)}-1\right)},$$

$$h = \sqrt{a^2 + 2\sigma^2}.$$

This new process immediately yields

$$E\left[\exp\left(-\int_t^T (1-\delta)\gamma(s)ds\right)\right] = \exp\left[(1-\delta)\,\Phi_*(t,T) - (1-\delta)\,\Psi_*(t,T)\gamma_t\right].$$

Given the relation of credit spread (**??**), we obtain

$$\exp\left(-S(t,T)(T-t)\right) = E\left[\exp\left(-\int_t^T (1-\delta)\gamma(s)ds\right)\right] \tag{5.13}$$

$$= \exp\left[(1-\delta)\,\Phi_*(t,T) - (1-\delta)\,\Psi_*(t,T)\gamma_t\right]. \tag{5.14}$$

Equivalently, the hazard rate in a CIR-type model is given by

$$\gamma(t) = \frac{1}{(1-\delta)\,\Psi_*(t,T)} \left[(1-\delta)\,\Phi_*(t,T) + S(t,T)(T-t)\right], \tag{5.15}$$

where

$$\Phi_*(t,T) = \frac{2ab}{\sigma^2} \ln\left(\frac{2he^{(a+h)(T-t)/2}}{2h + (a+h)\left(e^{h(T-t)}-1\right)}\right),$$

$$\Psi_*(t,T) = \frac{2\left(e^{h(T-t)}-1\right)}{2h + (a+h)\left(e^{h(T-t)}-1\right)},$$

$$h = \sqrt{a^2 + 2\sigma^2}.$$

Figure 5.6: Hazard rate function $\gamma(t)$ with $\sigma = 10\%$

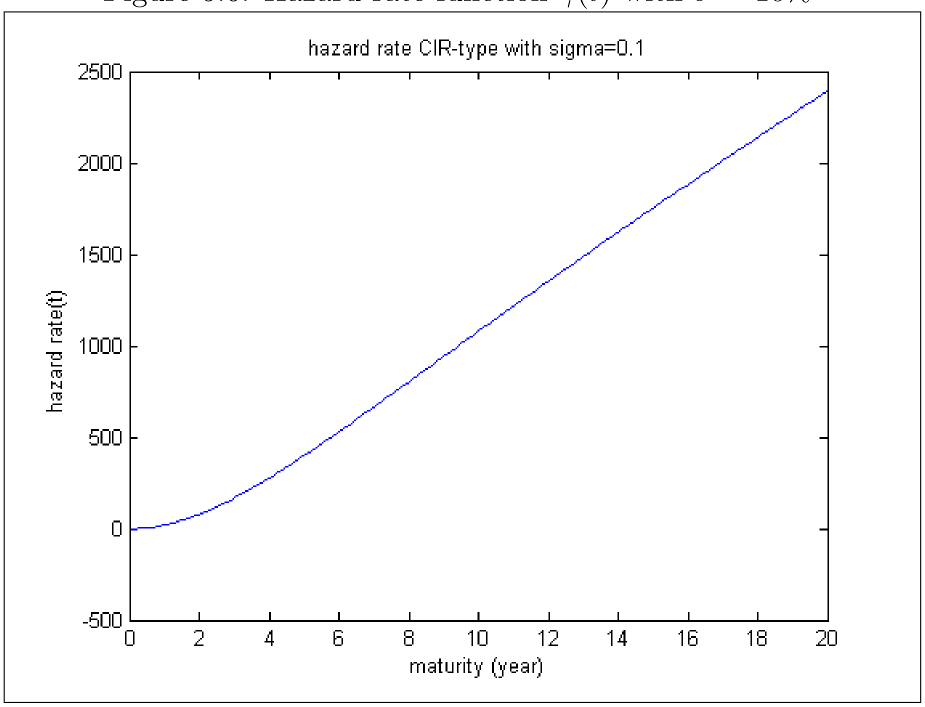


Figure 5.7: Hazard rate function $\gamma(t)$ with $\sigma = 20\%$

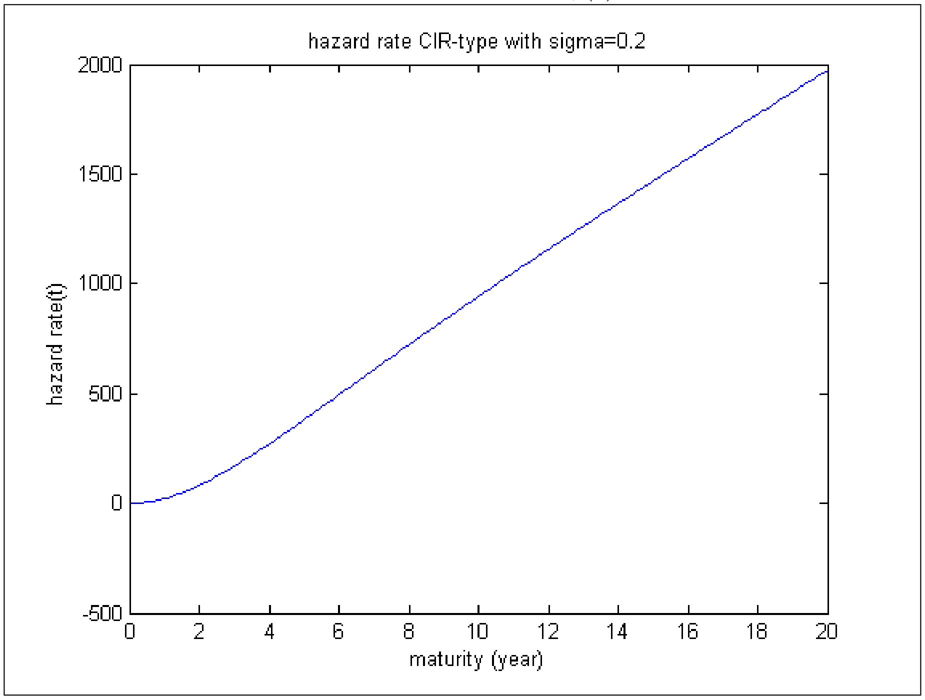


Figure 5.8: Hazard rate function $\gamma(t)$ with $\sigma = 50\%$

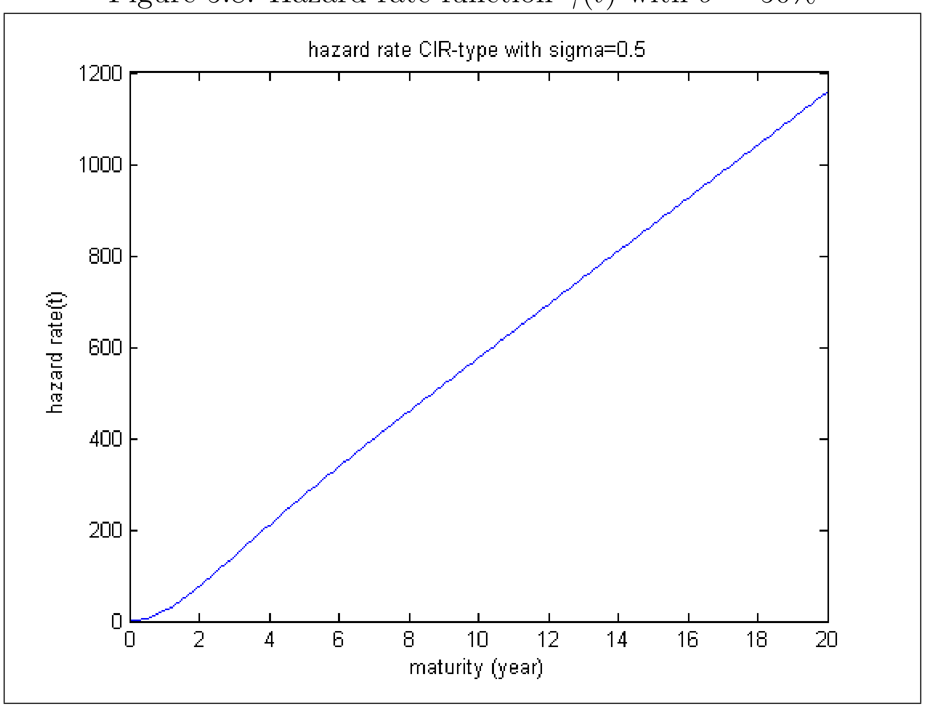
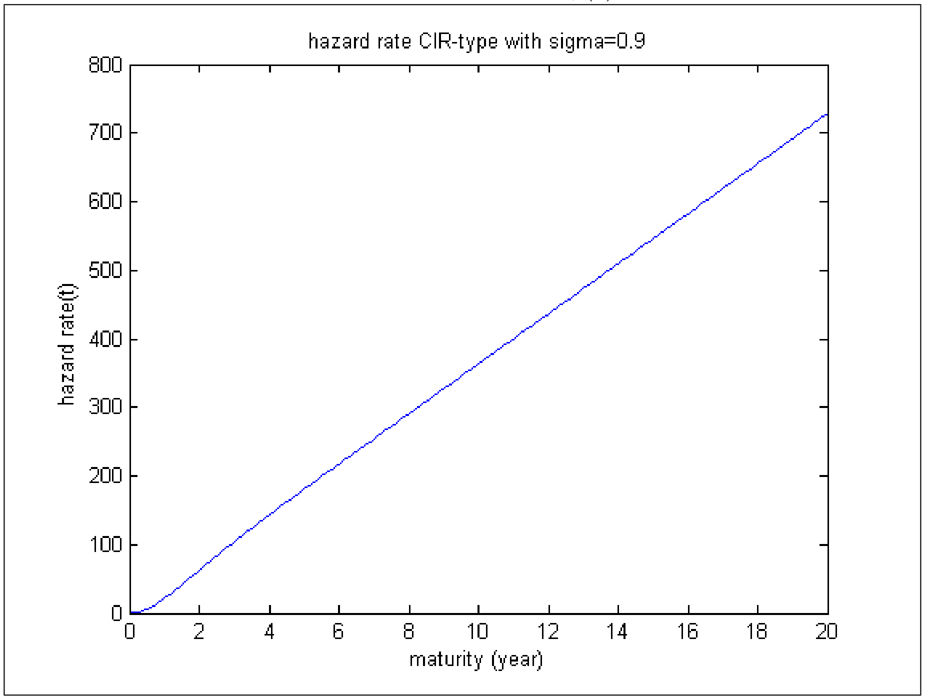


Figure 5.9: Hazard rate function $\gamma(t)$ with $\sigma = 90\%$



These figures (??), (??), (??) and (??) illustrate the hazard rate function of time of twenty years from now on, and distributed as the CIR-type model; given the market price of credit spread. We took the volatility to be $\sigma = 0.1$, $\sigma = 0.2$, $\sigma = 0.5$ and $\sigma = 0.9$ respectively. These figures monotonic increasing behaviour is subject to the same interpretation as for the Vasicek-type hazard rate (see, section (??)).

The survival probability (see section (??) below) can be calculated now if the law of hazard rate (that is, each parameter of the model) is given. We typically model survival probabilities by making them a function of a hazard rate.

5.2 A model for default

In this section, we define the joint survival probability in terms of default intensity processes under the assumption of conditional independence. The Probability of Survival $p(\tau > t)$ is the probability of not having defaulted in the period. We analyse the two different cases when the default intensity or hazard rate follow a Vasicek process or a CIR process.

Let τ be a default time associated with the filtration $\mathcal{F}_t$ of the reference issuer. Then the stochastic process $\gamma(t)$ is called an intensity process for τ. Duffie (1998) defines the conditional survival probability $p(\tau > t)$ of τ as follow :

$$p(\tau > t) = E\left[\exp\left(-\int_0^t \gamma(s)ds\right)\right]. \tag{5.16}$$

The material below investigates the survival probability under the Vasicek-type hazard rate and the CIR-type hazard rate.

5.2.1 The conditional survival probability, under Vasicek-type hazard rate

Similar reasoning to that used to evaluate equations (**??**) and (**??**), maybe used to express $p(\tau > t)$ as

$$p(\tau > t) = \exp\left[\frac{\gamma_0}{a}\left(e^{-at} - 1\right) - \frac{1}{a}\left(e^{-at} - 1\right)\left(b + \frac{\sigma^2}{4a^2}\left(e^{-at} - 3\right)\right) - t\left(b - \frac{\sigma^2}{2a^2}\right)\right]. \tag{5.17}$$

We note that, if using the Vasicek approach, the distribution of the default time may be not monotonously decreasing [**?**, **?**]. Equivalently, the survival probability may be over one.

Figure 5.10: Survival probability with $\sigma = 0.1\%$

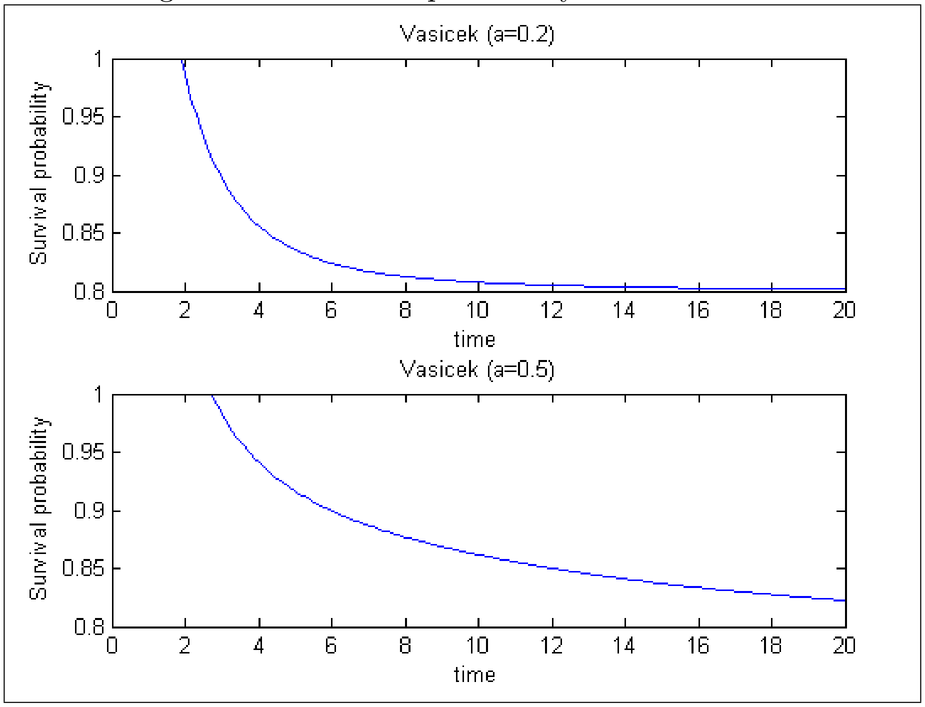


Figure 5.11: Survival probability with $\sigma = 1\%$

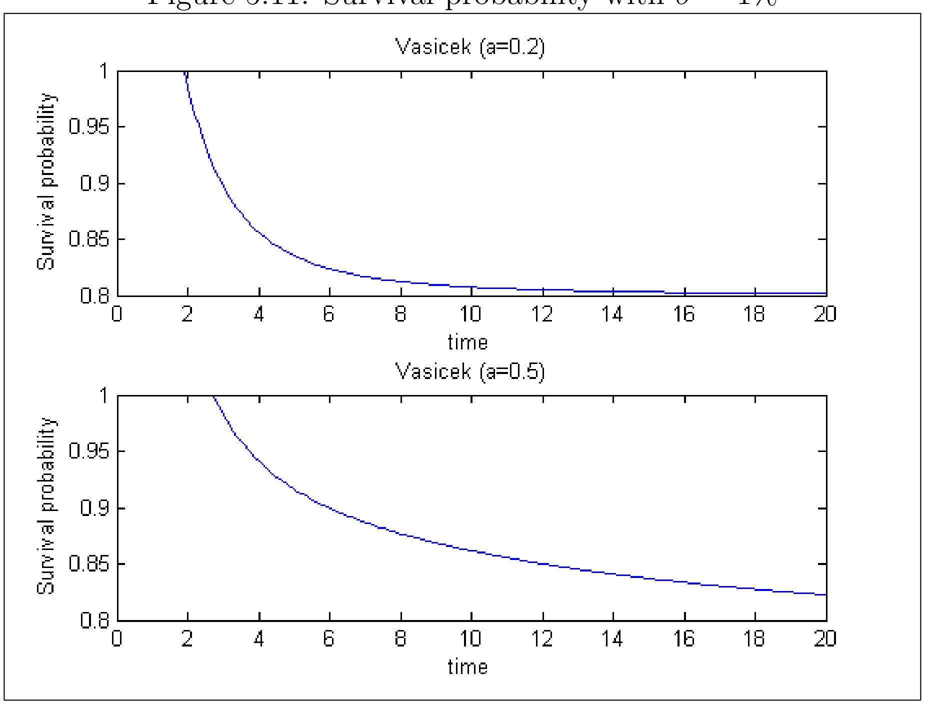
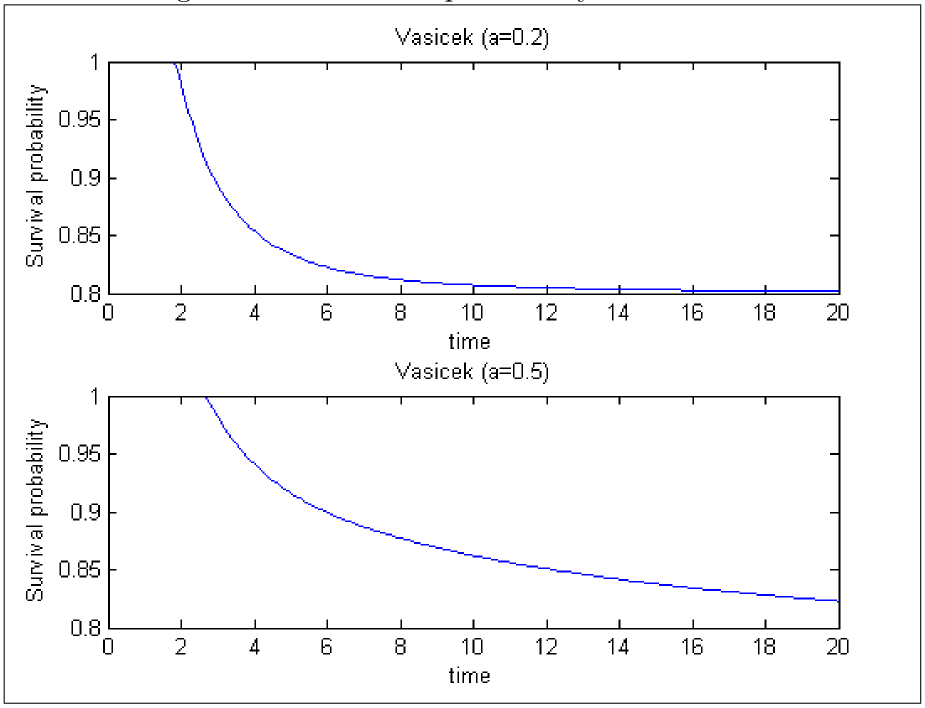


Figure 5.12: Survival probability with $\sigma = 5\%$



Figures (**??**), (**??**) and (**??**) illustrate the survival probability of the firm at time of twenty years from now on. We took the volatility to be $\sigma = 0.1\%$, $\sigma = 1\%$ and $\sigma = 5\%$ respectively, and the annually hazard rate at initial time is $\gamma_0 = 1\%$. In each figure, the two curves illustrate respectively the Vasicek-type process with mean-reversion rates $a = 0.2$ and $a = 0.5$.

We observe, at early times from the Vasicek-type model that the misgivings that the survival probability exceeds one is getting less for the smaller mean-reverting velocity (mean-reverting speed) Vasicek-type than the higher mean-reverting velocity Vasicek-type model. Analysing that the survival probability exhibits patterns that might correspond to the market's expectation in the issuer's ability to meet its debt obligation in early period [**?**]. It is essential to set a desirable value of the velocity (or speed) cautiously since the higher mean-reversion rate removes of the impact (or influence) of the volatility. Thereafter, it is concluded that Vasicek-type model with the high velocity is improved compared to the Vasicek-type with the smaller velocity (or speed).

5.2.2 The conditional survival probability, under CIR-type model hazard rate

Analogously to the previous discussion about the CIR model (see section (**??**)) and the CIR formula for the price of a zero-coupon bond (section (**??**)), we have

$$E\left[\exp\left(-\int_t^T \gamma(s)ds\right)\right] = \exp\left(\Phi_*(t,T) - \Psi_*(t,T)\gamma_t\right), \tag{5.18}$$

where

$$\Phi_*(t,T) = \frac{2ab}{\sigma^2}\ln\left(\frac{2he^{(a+h)(T-t)/2}}{2h+(a+h)\left(e^{h(T-t)}-1\right)}\right),$$

$$\Psi_*(t,T) = \frac{2\left(e^{h(T-t)}-1\right)}{2h+(a+h)\left(e^{h(T-t)}-1\right)},$$

$$h = \sqrt{a^2+2\sigma^2}.$$

Therefore we get

$$p(\tau > t) = \exp\left(\Phi_*(t) - \Psi_*(t)\gamma_0\right), \tag{5.19}$$

read

$$\Phi_*(t) = \frac{2ab}{\sigma^2}\ln\left(\frac{2he^{(a+h)t/2}}{2h+(a+h)\left(e^{ht}-1\right)}\right),$$

$$\Psi_*(t) = \frac{2\left(e^{ht}-1\right)}{2h+(a+h)\left(e^{ht}-1\right)},$$

$$h = \sqrt{a^2+2\sigma^2}.$$

Figure 5.13: Survival probability with $\sigma = 0.1\%$

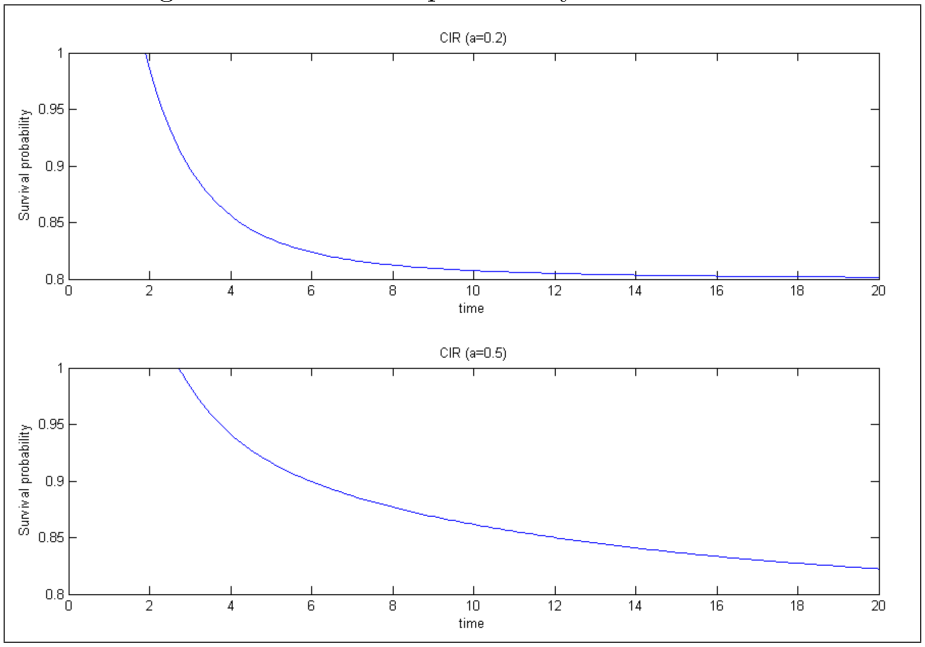


Figure 5.14: Survival probability with $\sigma = 1\%$

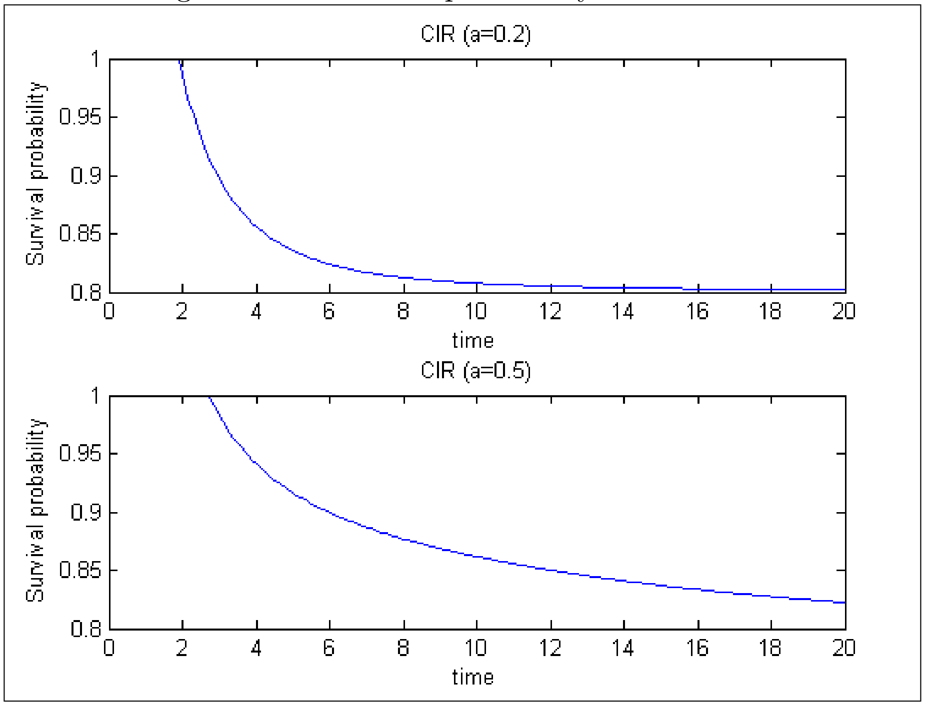
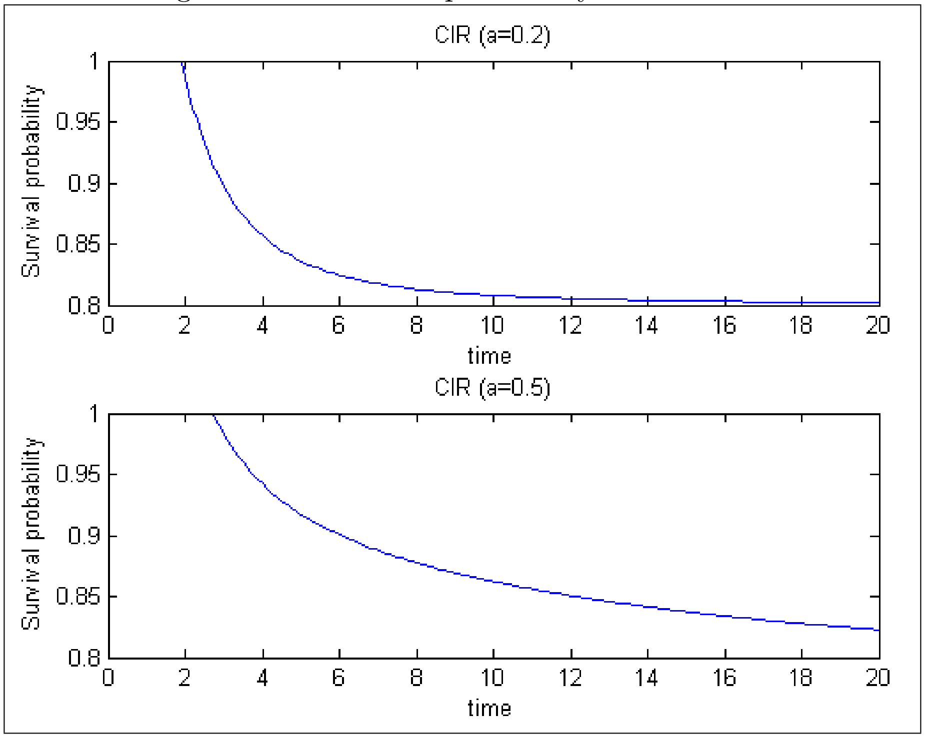


Figure 5.15: Survival probability with $\sigma = 5\%$



Figures (??), (??) and (??) illustrate the probability of the firm's survival in time of twenty years from now on. We took the volatility to be $\sigma = 0.1\%$, $\sigma = 1\%$ and $\sigma = 5\%$ respectively, and the annually hazard rate at initial time is $\gamma_0 = 1\%$.

These decreasing curves do imply a declining probability of survival over time (and therefore an increasing probability of default), as shown in exhibit(??), (??) and (??). Also means that we have an increasing hazard rate for each period as shown in figure (??), (??), (??) and (??). Similar conclusions can also be drawn with regard to the Vasicek-type model discussed in section (??).

5.3 Summary

We have seen in this chapter how one may model the hazard rate. Firstly, the hazard rate process was modelled as the Vasicek diffusion process, and the conditional survival probability is determined under this assumption. Secondly, we viewed the hazard rate as a CIR process, and analysed the conditional survival probability when the hazard rate follows a CIR model. By analysing the survival probability function under the two models with different values of mean-reverting speeds and volatility, we find that the higher mean-reversion rate removes the effect of the volatility and basically, the firm's remain stable at early times. The next chapter treats the parameter estimation associated to the hazard rate models using the Generalized Moment Method.

Chapter 6

Analysis and Numerical Results

6.1 Introduction

In this chapter we shall estimate parameters associated with hazard rate models using the Moment Method. We shall investigate 20 South African firm's debt terms, with different rating from AAA to BBB and different market credit spread for maturity one year, three years and five years to analyse and estimate the parameters for each of the 2 models described previously.

6.2 Parameter Estimation with Vasicek and CIR models

We recall from the previous chapter that the parameter of hazard rate process can be estimated from historical market data of credit spread. In the special cases such as :
-Vasicek model,
- and CIR model
(except for the recovery rate δ) we have to secure the estimator for a, b and σ. Meaning that we need to have at least four different credit spread $S(t,T)$ from market data of the firm. Nevertheless, it is not easy to get all those data since only a limited number of bonds are issued by each company and the amount traded in the marketplace is not enough to evaluate all the parameters. Thus, we cannot simplify the task by supposing some exogenous parameters

and finding the remaining ones implicitly. Hence, we consider the procedure of estimating parameters when only one defaultable bond issued by the company. As before we shall treat each of the Vasicek type model and the CIR type model below.

6.3 Parameter estimation

There exist different methods of estimating the parameters, including the implied volatility which is used in option valuation. Typically they are characterized or evaluated from the historical data. Though we consider the CIR and Vasicek processes in particular, we note that a similar procedure is possible for other models. We follow [?] to estimate the parameters, such as recovery rate δ and the long-term mean b using the South Africa data:

- Characteristic of the long-term mean hazard rate process b.

  Following the assumption that the long-term mean hazard rate process is similar for the same category of industry and the rating of the same class, we consider the estimation of b as the mean value of the probability of default on every category of industry and rating collected from rating agencies such as Standard & Poor, Moody's and Fitch.

- Rule of estimating recovery rate δ.

  Moody Agency's database includes detailed bond prices information after default, the historical market price of the bond for 30 days (one month) after the firm experienced the default event. This is viewed as the recovery rate from the default bond. Basically, the average of debt differs from issuers to issuers (or from firm to firm), in this discussion we assume that the recovery rate of the same class of financial rating and the same class of industry are shared and regard as the recovery rate δ of the firm. That is the mean value computed from Moody's data of recovery rate every class of industry and rating [?].

  Furthermore, the recovery rate can also be determined by different methods such as Ordinary Least Squares (OLS)[?], Multiple Additive Regression Trees (MART)[?], Classification And Regression Trees (CART)[?] and Waterfall model[?]. These listed methods estimate the recovery rate in default by using their capital structure and some econometric factors at the time of default. Moody's examined the determinants of recovery rates of defaulted corporate bonds and loans and proved that the recovery rates are strongly affected by many factors such as type of default event (eg. Bankruptcy, failure to pay and restructuring), the tangibility of its assets, the amount of the debt and macroeconomic factors [?].

Due to the difficulty of estimating the volatility σ and mean-reverting speed a separately from market data of credit spread using only one bond issuer (or reference bond), we restrict ourself to the limit distribution of the CIR hazard rate-type model and the Vasicek hazard rate-type model, and attempt to use the moment method to find those parameters. The moment method is a generic method or the most preferred numerical technique of estimating parameters in statistical model due to its less requirements of information.

6.3.1 Moment Method

Consider a set of observations of hazard rate $\gamma(t)$, which is obtained from historical market data calculated from the formula (??) and (??), respectively (for a hazard rate of Vasicek or CIR type models) for a certain period of time. The market data can be collected daily or at the end of the month.

From the discussion about hazard rate models in Chapter 5, we recall that the elements of γ are the speed of mean reversion a, the long mean rate b and the volatility σ.

In the case of the CIR model, CIR hazard rate $\gamma(t)$ has the following limits when $t \longrightarrow \infty$,

$$\begin{cases} \lim_{t\to\infty} E\left(\gamma(t)\right) = b \\ \\ \lim_{t\longrightarrow\infty} var\left(\gamma(t)\right) = \frac{\sigma^2 b}{2a} \end{cases} \Longrightarrow \gamma(t) \sim \mathcal{N}\left(a, \frac{\sigma^2 b}{2a}\right) \tag{6.1}$$

In the case of the Vasicek model, Vasicek hazard rate $\gamma(t)$ has the following limits when $t \longrightarrow \infty$,

$$\begin{cases} \lim_{t\to\infty} E\left(\gamma(t)\right) = b \\ \\ \lim_{t\longrightarrow\infty} var\left(\gamma(t)\right) = \frac{\sigma^2 b}{2a} \end{cases} \Longrightarrow \gamma(t) \sim \mathcal{N}\left(a, \frac{\sigma^2}{2a}\right) \tag{6.2}$$

We assume that the relations (**??**) and (**??**) are satisfied in general for any value of t. In order to use the moment method technique, we consider as data the value of hazard rates $\gamma(t)$ at n points $t_1, t_2, \ldots, t_n$ analogue as a vector :

$$t = (t_1, t_2, \ldots, t_n),$$

obtained by historical market data calculated from the formula (**??**) and (**??**). Using (**??**) and substituting equation (**??**) for the CIR model, we have

$$b = E(\gamma(t)) = \frac{1}{n}\sum_{k=1}^{n}\gamma(t_k). \tag{6.3}$$

Using (**??**) and the formula (**??**), and substituting Equation (**??**) for the CIR model, we have

$$\begin{aligned}\frac{\sigma^2 b}{2a} = var(\gamma(t)) &= E\left[(\gamma_t - E(\gamma_t))\right]^2 \\ &= E\left[\gamma^2(t)\right] - \left[E(\gamma_t)\right]^2 \\ &= \frac{1}{n}\sum_{k=1}^{n}\gamma^2(t_k) - b^2.\end{aligned}$$

We have, therefore,

$$\frac{1}{n}\sum_{k=1}^{n}\gamma^2(t_k) = b^2 + \frac{\sigma^2 b}{2a}. \tag{6.4}$$

Having the estimator parameters δ and b, we have only to obtain the parameters values σ and a which obey the equations (**??**) and (**??**) simultaneously. This may be done by solving, for example, the following the system of two equations with two unknowns a and σ :

$$\begin{cases} \frac{1}{n}\sum_{k=1}^{n}\gamma(t_k) = b \\ \\ \frac{1}{n}\sum_{k=1}^{n}\gamma^2(t_k) - \frac{\sigma^2 b}{2a} = b^2. \end{cases} \tag{6.5}$$

This system is for the CIR-type hazard rate process. For the Vasicek-type hazard one may similarly obtain the system :

$$\begin{cases} \frac{1}{n}\sum_{k=1}^{n}\gamma(t_k) = b \\ \\ \frac{1}{n}\sum_{k=1}^{n}\gamma^2(t_k) - \frac{\sigma^2}{2a} = b^2. \end{cases} \tag{6.6}$$

Note that both equations in system (??) and (??) are nonlinear equations. The solution of the system a and σ can be obtained by solving the systems of nonlinear simultaneous equations. (Typically, this is very difficult). Assume that

$$\begin{cases} \chi_1 = \frac{1}{n}\sum_{k=1}^{n}\gamma(t_k) - b = 0 \\ \\ \chi_2 = \frac{1}{n}\sum_{k=1}^{n}\gamma^2(t_k) - b^2 - \frac{\sigma^2 b}{2a} = 0, \end{cases} \tag{6.7}$$

this systems of two nonlinear simultaneous equations can be solved on Matlab.

This problem can also be formulated as the optimization problem means that we will seek to minimize χ_1^2 and χ_2^2 subject to $\sigma \geq 0$ and $a > 0$. This optimization problem is the problem of making the best possible choice of σ and a that can minimize the objective function χ_1^2 and χ_2^2 simultaneously.

Because of the rarity of data, we use data from [?]. We use matlab to find the solution to the system of non-linear equations (??), with different initial guess values of a and σ. These solutions are given in Tables **??** and **??** for Vasicek and CIR hazard rate-types respectively.

Table **??** shows the results for estimation of mean reversion a and volatility σ, using various debt terms for 20 South African firms, with different rating from AAA to BBB and different market credit spreads : for maturity in one year, three years, and five years. These are done when hazard rates are distributed using a Vasicek model. The results show that the mean reversion a increases when volatility decreases, and decreases when volatility increases for firms from banking sector and non-banking sector. We notice that volatilities found are quit similar to those for South African firms' market data given in [?] or (Table **??**, Chapter [4]).

Table 6.1: Hazard rate parameters for Vasicek-type model

company	Rating	Volatility of σ (%)	mean reversion a (%)	1 year CS (Bp)	3 years CS (Bp)	5 year CS (Bp)
1	AAA	03.170	03.49	0.00	7	28
2*	AA^+	94.03	07.49	0.00	1	6
3	AA	26.98	0.1018	0.00	2	11
4*	AA	5.31	10.30	0.2	7	18
5*	AA	4.212	04.48	0.2	7	18
6*	AA	7.153	0.0349	0.4	12	29
7*	AA	3.31	1.2945	0.2	6	15
8	AA^-	17.39	10.24	0.00	0	3
9	AA^-	20.75	04.25	0.00	5	19
10	A^+	21.18	05.85	0.00	2	10
11	A^+	29.12	0.0205	0.6	33	85
12*	A^+	3.13	04.61	0.4	7	17
13*	A^+	1.33	1.5287	0.7	15	36
14*	A^-	41.3	0.0412	0.0	8	21
15	A^-	23.60	03.77	0.0	6	24
16	A^-	41.37	04.97	16	3	18
17*	BBB^+	30.02	01.11	1.6	160	297
18*	BBB	15.07	02.18	1.6	34	77
19*	BBB	18.0	01.49	7.4	84	169
20*	BBB	22.02	01.33	9.8	108	211

An asterisk indicates that the firm is in the banking sector.

The results shown in Table **??** are the estimation of volatility σ and mean reversion a, when hazard rates are distributed using a CIR model. Those are estimated by using various debt terms for 20 South African firms, with different rating from AAA to BBB and different market credit spreads for maturity in one year, three years, and five years. the result show that the

Table 6.2: Hazard rate parameters for CIR-type model

company	Rating	Volatility of σ (%)	mean reversion a (%)	1 year CS (Bp)	3 years CS (Bp)	5 year CS (Bp)
1	AAA	32.10	03.29	0.00	7	28
2*	AA^+	19.03	01.09	0.00	1	6
3	AA	26.98	0.1018	0.00	2	11
4*	AA	5.31	10.30	0.2	7	18
5*	AA	4.212	04.48	0.2	7	18
6*	AA	7.153	0.0349	0.4	12	29
7*	AA	3.31	1.2945	0.2	6	15
8	AA^-	17.39	10.24	0.00	0	3
9	AA^-	20.15	04.25	0.00	5	19
10	A^+	21.18	05.85	0.00	2	10
11	A^+	29.02	0.0205	0.6	33	85
12*	A^+	3.13	04.61	0.4	7	17
13*	A^+	7.33	1.5287	0.7	15	36
14*	A^-	4.13	0.0412	0.0	8	21
15	A^-	23.60	03.77	0.0	6	24
16	A^-	41.37	04.97	16	3	18
17*	BBB^+	30.02	01.11	1.6	160	297
18*	BBB	15.07	02.18	1.6	34	77
19*	BBB	18.0	01.49	7.4	84	169
20*	BBB	22.02	01.33	9.8	108	211

An asterisk indicates that the firm is in the banking sector.

mean reversion a increases when volatility decreases, and decreases when volatility increases for firms from banking sector and non-banking sector. We notice that, the results are quit similar to those for the Vasicek-type model, and that the volatilities found are generally similar to volatilities of South African firm's market data given in [**?**] or (Table **??**, Chapter [4]).

6.4 Summary

This chapter provided the estimation of parameters associated to the hazard rate models using the Generalized Moment Method. We investigated 20 South African firms debt terms, with different ratings from AAA to BBB and different market credit spreads for maturity one year, three years and five years to analyse the results and estimate the parameters for each of the 2 models, such as, a Vasicek- type model and a CIR-type model. The next chapter summarizes the dissertation, makes a conclusion, and discusses potential future-work.

Chapter 7

Conclusion

The main research objective of this dissertation, was to determine the explicit value of fixed side and recovery side credit default swaps in a quite general form that contains counter party risk, under the hazard rates distributed as the Vasicek process that :

- can be used to value the single credit default swap,
- and can be used to reduce risk arising from bondholders (ownership of bonds) or lenders (ownership of loans).

The other objective was, to calculate the conditional survival probability in terms of default intensity processes (or hazard rates), and to estimate parameters associated with hazard rates Vasicek and CIR models using the relationship between credit spread and hazard rate necessary to switch market credit spread data collected directly from market into Vasicek and CIR type hazard rates. In particular, we modelled hazard rate as Vasicek and CIR type processes since they have some significant properties: mean-reverting and Gaussian, and affine respectively. We suggested one implementation procedure when the available market data are not sufficiently rich and provided some simulation results.

We gave a review of the one-factor short rate models focussing on two models: Vasicek and CIR, possessing different properties. We also gave a literature review of credit risk modelling and noted that structural credit risk models are able to meet these research objectives of the dissertation. In this approach the firm's asset value follows a diffusion process and continuous time trading, and default happens only when the asset value of the firm falls below the default point (default barrier) before maturity time T. In this manner, the structural approach elucidates an economic explanation for the default process. Moreover, it can be performed to derive default probabilities and credit spreads. We investigated a range of 20 South African firms, with different ratings: from AAA to BBB, and with different maturity dates one year, three years and five years. We found that the result is in accord with the prediction regarding the low credit spreads found using the Merton model made by L Smit, B Swart and F van Niekerk [**?**] and others researches.(Kim, Ramaswamy and Sundaresan 1993, Jarrow and Turnbull 1995, and Shimko 1999).

However, the work done in this dissertation has mainly been on the study of credit default swap under single factor hazard rate models, and the estimation of parameters for single factor hazard rate models and the conditional survival probability. These are done under the original risk-neutral or martingale measure. Moreover, as in real world the market is often incomplete, that is existence of many martingale measures or the risk-neutral probability is not unique.(Refer to Asset Pricing theorem). In future research work, we propose the following main topics : we should execute calculation under real world probability (or objective probability) and consider hazard rates in multi-factor models. We will investigate a case, when there is a correlation between a defaultable bond and risk-free interest rate (or correlation between the risk-free interest rate r and all the hazard rates), and consider when defaultable bonds issued by different firms.

There are many more interesting topics to study and thereby develop our models, such as modified CIR and Vasicek models (making the long term mean level time-dependent). However, the suggested future research indications are the first most significant steps to do the approaches more practical and to truly meet the research objectives of this dissertation.

Appendix

In the Merton framework under risk neutral measure, the default probability is simply expressed by

$$PD_M = \mathcal{N}(-d_2). \tag{7.1}$$

To obtain this probability, more information about the distribution of firm's asset value V has to be known. However, using the assumption that the asset value of the firm follows a diffusion process under risk neutral $\mathbb{Q}$ (equation (**??**)), we obtain the firm's asset value at maturity

$$V_T = V_0 e^{(r-\frac{1}{2}\sigma_V^2)T+\sigma\sqrt{T}W_T}.$$

We can get information about probability distribution of $\ln V_T$

$$\ln V_T \sim \mathcal{N}\left(\ln V_0 + (r - \frac{1}{2}\sigma_V^2)T, \sigma\sqrt{T}W_T\right).$$

The probability (**??**) is expressed as

$$\begin{aligned}
PD_M =& P_r(V_T \leq L) \\
=& P_r\left(\ln V_T \leq \ln L\right) \\
=& P_r\left(\ln V_0 + (r - \frac{1}{2}\sigma_V^2)T + \sigma\sqrt{T}W_T \leq \ln L\right) \\
=& P_r\left(\sigma\sqrt{T}W_T \leq \ln L - \ln V_0 - (r - \frac{1}{2}\sigma_V^2)T\right) \\
=& P_r\left(W_T \leq -\frac{\ln(\frac{V_0}{L}) + (r - \frac{1}{2}\sigma_V^2)T}{\sigma\sqrt{T}}\right) \\
=& P_r\left(W_T \leq -d_2\right) \\
=& \mathcal{N}(-d_2)
\end{aligned}$$

That is the default probability given by the Merton model. The following section discusses the credit spread under Merton model.

7.0.1 Credit spread in Merton Model

An important element of a defaultable bond is the difference between its yield and the yield of a similar default free bond, i.e, the credit spread. It is defined as

$$S(t,T) = r^d(t,T) - r(t,T) \tag{7.2}$$

where $r^d(t,T)$ and $r(t,T)$ are calculated through

$$B(t,T) = e^{-r(t,T)} \Longrightarrow r(t,T) = -\frac{\ln B(t,T)}{T-t} \tag{7.3}$$

$$D(t,T) = Le^{-r(t,T)} \Longrightarrow r^d(t,T) = -\frac{\ln D(t,T)/L}{T-t}. \tag{7.4}$$

Substituting (**??**) and (**??**) into (**??**) yields

$$S(t,T) = -\frac{\ln\left[D(t,T)/LB(t,T)\right]}{T-t}. \tag{7.5}$$

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