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The Characteristic Approach in Determining First Integrals of a Predator-Prey System

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The Characteristic Approach in Determining First Integrals of a Predator-Prey System

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As the candidate's supervisor, I have approved this dissertation for submission.

Signed: Dr Riven Narain November 2016

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Abstract

Predator-Prey systems are an intriguing symbiosis of living species that interplay during the fluctuations of birth, growth and death during any period. In the light of understanding the behavioural patterns of the species, models are constructed via differential equations. These differential equations can be solved through a variety of techniques. We focus on applying the characteristic method via the multiplier approach. The multiplier is applied to the differential equation. This leads to a first integral which can be used to obtain a solution for the system under certain initial conditions. We then look at the comparison of first integrals by using two different approaches for various biological models. The method of the Jacobi Last Multiplier is used to obtain a Lagrangian. The Lagrangian can be used via Noether's Theorem to obtain a first integral for the system.

Contents

Chapter 1

Mathematical Biology

1.1 Introduction

Mathematicians and biologists, including medical scientists, have a long history of working successfully together. Sophisticated mathematical approaches have been applied to the study of biology. There are also new techniques emerging to study the systems in Biology. The development of stochastic processes and statistical methods to solve a variety of population problems in Demography, Ecology, Genetics and Epidemiology as a few examples. Before Pythagoras, Aristotle, Fibonacci, Bernoulli, Euler, Fourier, Laplace, Gauss, Malthus, Riemann, Verhulst, Einstein, Thompson, von Neumann, Poincare, Thom and Keller are some names associated with both significant applications of mathematics to biological problems providing significant developments in mathematics.

The fast-growing discipline of Mathematical Biology is the application of mathematical modelling of biological systems to help analyse and understand biological processes in Nature. It is a highly interdisciplinary area that defies classification into the usual categories of mathematical research, although it has involved many areas of mathematics intertwined with interdisciplinary fields also used in the applications of biotechnology, biophysics, biochemistry, medicine and other areas of biology [10]. The value of mathematics in biology comes partly from applications of statistics and calculus to quantifying biological phenomena, but more importantly from the sophisticated point of view it can bring to complicated real-life systems by organizing information and identifying and studying emergent structures [10].

Mathematical models of biological systems, simple and complex, are used to explain biological processes through applied mathematical techniques. These systems can be expressed by deterministic or stochastic processes [6]. A mathematical model of a biological system is a set of equations that is derived from the biological reasoning which describes the system. The solution of these equations, by either analytical or numerical means, describes how the biological system behaves either over time or at equilibrium. There are many different types of equations and the type of behaviour that can occur is dependent upon both the model and the equations used. The type of model is often used to make assumptions about the system. The equations may also make assumptions about the nature of what may occur.

1.2 Biological Models

A Biological Model is a term that can be expressed in many different ways depending upon the context, which simply tells us that a Biological Model is or refers to a mathematical model of a biological system. There are many biological models that have been studied using various different mathematical approaches. Each system is unique and can be altered for experimental purposes which allow mathematicians to use their knowledge to submerge the biology of a system into a mathematical model which can explain how the system can work theoretically, even by taking environmental factors in consideration.

1.2.1 Exponential-Growth Model

In 1798, Thomas Robert Malthus, founder of the exponential model, wrote **An Essay on the Principle of Population** in which he describes how population growth occurs and is affected due to a certain factor. A biological system [6] describing a population of a certain species is reasonable to consider the increase or decrease, depending on the conditions, of the population number as time progresses. Consider a culture of bacteria. Depending upon the conditions we can expect the population of bacteria to increase or decrease. If we denote the population size as N, then it is natural to consider N as a function of time such that $N(t) = N$.

If b and d represent the average birth and death rates, respectively, of a certain species, then the difference of these two must give the average growth rate which is,

$$\frac{\dot{N}}{N} = b - d, \tag{1.1}$$

where $b - d$ as the *instrinsic growth rate*. Let $r = b - d$. It follows that (1.1) becomes,

$$\dot{N} = rN \tag{1.2}$$

which is a simple first-order ordinary differential equation that relates the intrinsic growth rate to the average growth rate. The solution to equation (1.2) is

$$N(t) = N_0 e^{rt}, \tag{1.3}$$

where N_0 is a constant. Biologically N_0 represents the initial size of the population at the start of the experiment. Below is a graph showing the growth of the population with $r > 0$.

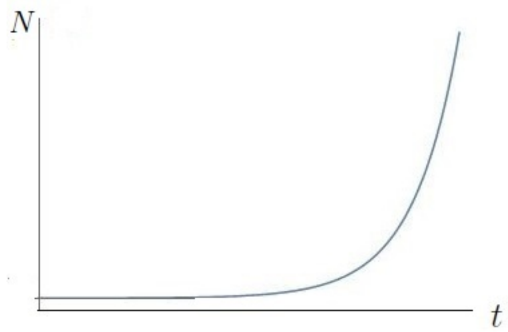

Figure 1.1: Population trajectory

We have successfully expressed the growth of a certain species as a mathematical equation and presented it as a graph in figure 1.1 [6] which describes the behaviour of this system. Equation (1.3) is an example of a simple mathematical model of a biological model commonly known as the exponential-growth model. It should be noted that equation (1.3) is an idealistic model as we have not taken into account factors such as immigration, emigration and the environment which include predators, diseases, quantity of food and other factors. Including these factors leads to a more realistic model. However, the cost of this is that the equations describing the biological system are, in general, more complex. Although the exponential growth model is an idealistic one, it provides an insight on how to build mathematical models that provide a better approximation to biological phenomena.

1.2.2 Logistic Growth Model

The logistic growth model, also known as the *Verhulst* model [6], is a modification of the exponential-growth model. A new parameter, K, is introduced into the equation which is called the *carrying capacity*. The significance of K is that it describes the maximum population that the containing environment can sustain. The introduction of K is biologically reasonable and makes the logistic growth model a more realistic model when compared to the exponential-growth model. This model is given by

$$\dot{N} = rN(1 - \frac{N}{K}), \tag{1.4}$$

where r is the intrinsic growth rate. Equation (1.4) can be easily solved to yield

$$N(t) = \frac{K}{1 + (\frac{N_0}{K} - 1)e^{-rt}}. \tag{1.5}$$

There are two considerations of r, where r can be negative or positive.

(i) Suppose that $r > 0$ such that the birth rate is higher than the death rate. The long-term behaviour ($t \to \infty$) predicts that

$$\lim_{t\to\infty} N(t) = K$$

meaning that in the case of a positive growth rate, the population grows to the maximum number that the environment can sustain. Figure 1.2 describes this behaviour of the system when $r > 0$.

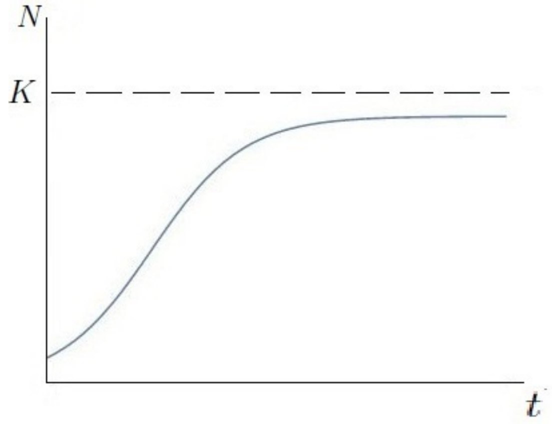

Figure 1.2: Logistic growth for $r > 0$

(ii) Suppose that $r < 0$, that is, the death rate is higher than the birth rate. The model predicts that in the long-term ($t \to \infty$)

$$\lim_{t\to\infty} N(t) = 0,$$

which means that the species eventually dies out. Figure 1.3 describes this behaviour of the system when $r < 0$.

Both cases predict behaviour that is realistic which is shown in each figure.

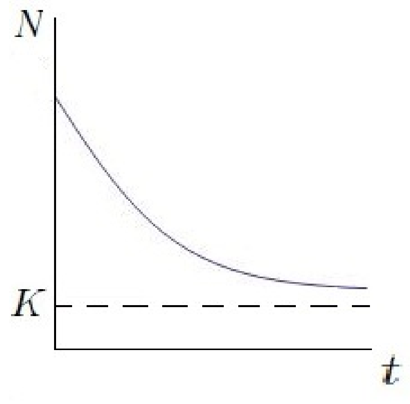

Figure 1.3: Logistic growth for $r < 0$

1.2.3 Chemostat Model

A chemostat model [6] is a biological system that is artificially created to observe the behavioural patterns of organisms under various induced conditions. The chemostat model received its name because the chemical environment of this model is static or at a steady state. This is due to environmental factors that are kept constant, such as fluid volume, pH levels, cell concentrations and other factors. One of the main benefits of this model is that it is a continuous process at which the bacterial growth can be maintained at a steady state. Chemostats have many applications and are very important because they are used in pharmaceuticals to produce different bacteria. In the food industry this has led to the production of fermented foods, manufacturing for the production of ethanol and antibiotics, as well as research purposes in different fields of study such as mathematical biology. This is a general chemostat model [6]. Nutrients (substrate) are pumped into the system for an organism (heterotroph) to grow over a certain period of time which leads to bacterial production that is used in industry today.

In the model $a_1(t)$ represents the substrate (nutrients) as a function of time and $a_2(t)$ represents the heterotroph (organism). The model is represented by a two-dimensional system of first-order nonlinear ordinary differential equations given by

$$\dot{a}_1 = D(F - a_1) - \frac{\mu_1}{\mu_2} a_1 a_2 \tag{1.6a}$$

$$\dot{a}_2 = \mu_1 a_1 a_2 - D a_2, \tag{1.6b}$$

where D is dilution rate, F is the inflow concentration of the substrate, μ_1 is the growth rate of the heterotroph and μ_2 is the yield of heterotrophs. Nondimensionalisation is the partial or full removal of quantities from an equation with suitable variables which simplifies problems in which measured quantities are involved. The Chemostat model given by (1.6a) and (1.6b) has variables that can be rescaled, thus removing the extra constants in the two-dimensional system. This can be done using the following ratios [6],

$$x(t) \equiv \tfrac{a_1}{F}, \quad y(t) \equiv \tfrac{a_2}{\mu_2 F}, \quad t \equiv DT,$$

which leads to a two-dimensional system with new variables,

$$\dot{x} = 1 - x - Axy \tag{1.7a}$$

$$\dot{y} = Axy - y, \tag{1.7b}$$

where $A = \frac{\mu_1 F}{D}$ is a constant.

We now are faced with the problem of the solution of a system of two first-order differential equations. The system can be merged into one second-order nonlinear ordinary differential equation by substituting one equation into the other. We can now try to solve the two-dimensional system by solving for x and $\dot{x}$ and substituting into equation (1.7b) which results in

$$x = \frac{\dot{y} + y}{Ay} \quad \text{and} \quad \dot{x} = \frac{1}{A}\left(\frac{\ddot{y}y + \dot{y}^2}{y^2}\right). \tag{1.8}$$

The substitution of (1.8) into (1.7a) forms the second-order nonlinear differential equation,

$$y\ddot{y} - \dot{y}^2 + (Ay^2 + y)\dot{y} + Ay^3 + y^2(1 - A) = 0. \tag{1.9}$$

1.2.4 Gompertz Model

The model used in this case [11] comprises

$$\dot{w}_1 = w_1\left(A\ln\frac{w_1}{m_1} + Bw_2\right) \tag{1.10a}$$

$$\dot{w}_2 = w_2\left(a\ln\frac{w_2}{m_2} + bw_1\right). \tag{1.10b}$$

Introducing a change of variables to simplify the system, we obtain

$$w_1 = m_1e^{x_1} \qquad w_2 = m_2e^{x_2},$$

which then yields the system to be

$$\dot{x}_1 = Ax_1 + m_2Be^{x_2} \tag{1.11}$$

$$\dot{x}_2 = ax_2 + m_1be^{x_1}. \tag{1.12}$$

Taking our system and using a change of variables, we can rearrange to get

$$x_2 = \ln\left(\frac{\dot{x}_1 - Ar_1}{Bm_2}\right) \tag{1.13}$$

which leads to a second-order differential equation given by

$$\ddot{x}_1 = \left[bm_1e^{x_1} + a\ln\left(\frac{\dot{x}_1 - Ax_1}{Bm_2}\right)\right](\dot{x}_1 - Ax_1) + A\dot{x}_1 \tag{1.14}$$

1.2.5 Lotka-Volterra Model

The Lotka-Volterra model [11] is the simplest model in which predator-prey interaction is taken into account. It is the most famous of all predator-prey models and has profound historical interest. It is also structurally unstable due to environmental factors not being taken into account.

$$\dot{w}_1 = w_1(a + bw_2) \tag{1.15a}$$

$$\dot{w}_2 = w_2(A + bw_1). \tag{1.15b}$$

For simplicity, we introduce a change of variables,

$$w_1 = e^{x_1} \quad \text{and} \quad w_2 = e^{x_2}. \tag{1.16}$$

The system becomes

$$\dot{x}_1 = a + be^{x_2} \tag{1.17a}$$

$$\dot{x}_2 = A + Be^{x_1}. \tag{1.17b}$$

If we then introduce a change of variables, we can rearrange one equation in our system to get

$$x_1 = \ln\left(\frac{\dot{x}_2 - A}{B}\right) \tag{1.18}$$

which leads to a second-order differential equation given by

$$\ddot{x}_2 = -\left(be^{x_2} + a\right)\left(A - \dot{x}_2\right). \tag{1.19}$$

For calculations in the latter part of this dissertation, we let $x_2 = x$. If the system is solved numerically, we see that the resulting graph given below shows the behaviour of

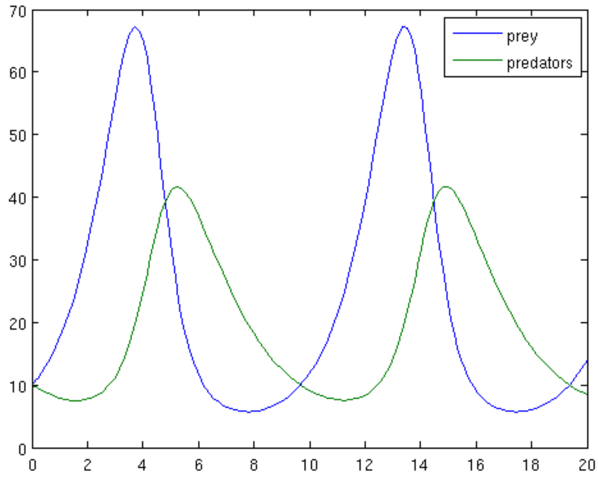

Figure 1.4: Predator-Prey projection

the system.

Figure 1.4 describes the behaviour of the system by two oscillating graphs for the predator and prey. As the prey increases, the predator also increases at a slower rate, after a certain period of time. Each species eventually reaches a peak and then starts to decrease because of the predation rate, but the predator will decrease at a lower rate, after a certain period of time, as compared to the prey. Thus the system becomes a continuous cycle between the two species.

1.2.6 Host-Parasite Model

This is another type of a predator-prey model [11] that describes an interaction between some host and some parasite. This model takes into account the nonlinear effects of the size of the host population on the growth rate of the parasite population which is governed by

$$\dot{u}_1 = (a - bu_2)u_1 \tag{1.20a}$$

$$\dot{u}_2 = \left(A - B\frac{u_2}{u_1}\right)u_2. \tag{1.20b}$$

For simplicity, we introduce a change of variables,

$$u_1 = x_1 e^{at} \quad , \quad u_2 = x_2 e^{At}.$$

Substitution of these variables back into our original system yields

$$\dot{x}_1 = -be^{At}x_1x_2 \tag{1.21a}$$

$$\dot{x}_2 = -\frac{Be^{At}{x_2}^2}{e^{at}x_1}. \tag{1.21b}$$

We can now remove one variable by transforming our system into an equivalent second-order differential equation. If we eliminate x_2, we get

$$x_2 = -\frac{\dot{x}_1}{be^{At}x_1} \tag{1.22}$$

which yields

$$\ddot{x}_1 = \frac{be^{at}x_1 + B}{be^{at}{x_1}^2}\dot{x}_1^2 + A\dot{x}_1. \tag{1.23}$$

1.3 Conclusion

In this Chapter we have investigated mathematical models of different biological systems. There has been a vast study and analysis performed on these models, which brings the reader to the models that we are interested to study. Our investigation leads to the behavioural study of the two-dimensional system, Lotka-Volterra Model, under certain conditions in Nature by finding first integrals for this system.

Chapter 2

First Integrals via Noether's Theorem

2.1 Introduction

Emmy Noether [5] developed a ground-breaking result in 1915, commonly known as Noether's Theorem. She proved that every differentiable symmetry of the Action of a physical system has a corresponding conservation law which was important at the time for Theoretical Physics and later the Calculus of Variations. It provided a generalisation of the formulations on constants of motion in Lagrangian and Hamiltonian Mechanics [17]. The original version only applied to ordinary differential equations which assumed that the Lagrangian only depends upon the first derivative, while later versions generalise the theorem to Lagrangians depending on the higher derivatives.

For Noether's theorem to be applied, the nature of the infinitesimal transformations and the invariance is to be considered. It is noted that the complete theorem requires two conditions just as the existence of a first integral of a differential equation does in Lie's Theory. The functional

$$A = \int_{x_0}^{x_1} L(x, y, \dot{y}) dx \tag{2.1}$$

in invariant, up to a gauge function, under an infinitesimal transformation generated by the operator

$$G = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \tag{2.2}$$

if

$$\int_{x_0}^{x_1} L\left(\underline{x}, \underline{y}, \underline{\dot{y}}\right) d\underline{x} - \int_{x_0}^{x_1} L\left(x, y, \dot{y}\right) dx = \epsilon\left(f(x_1) - f(x_0)\right), \tag{2.3}$$

where ϵ is the parameter of smallness and the infinitesimal transformation causes a zero end-point variation. Since

$$\underline{x} = x + \epsilon\xi, \tag{2.4a}$$

$$\underline{y} = y + \epsilon\eta, \tag{2.4b}$$

$$\underline{\dot{y}} = \dot{y} + \epsilon\left(\dot{\eta} - \dot{y}\dot{\xi}\right), \tag{2.4c}$$

the differential equation of equation (2.3) becomes

$$\xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial y} + \left(\dot{\eta} - \dot{y}\dot{\xi}\right)\frac{\partial L}{\partial \dot{y}} + L\dot{\xi} = \dot{f}. \tag{2.5}$$

This condition for G, in equation (2.2), is said to be a symmetry of equation (2.1). There also hasn't been any mention of a first integral thus far. We now analyse equation

(2.5) further. Collecting total derivatives on one side we have

$$\begin{aligned}
\frac{d}{dx}(f - L\xi) &= \xi\frac{\partial L}{\partial x} + \eta\frac{\partial L}{\partial y} + (\dot\eta - \dot y\dot\xi)\frac{\partial L}{\partial \dot y} - L\dot\xi && \text{(2.6a)}\\
&= \eta\frac{\partial L}{\partial y} + (\dot\eta - \dot y\dot\xi)\frac{\partial L}{\partial \dot y} - \epsilon\left(\dot y\frac{\partial L}{\partial y} + \ddot y\frac{\partial L}{\partial \dot y}\right) && \text{(2.6b)}\\
&= (\eta - \dot y\xi)\frac{\partial L}{\partial y} + (\dot\eta - \dot y\dot\xi - \ddot y\xi)\frac{\partial L}{\partial \dot y} && \text{(2.6c)}\\
\frac{d}{dx}\left(f - L\xi - (\eta - \dot y\xi)\frac{\partial L}{\partial \dot y}\right) &= (\eta - \dot y\xi)\frac{\partial L}{\partial y} - (\eta - \dot y\xi)\frac{d}{dx}\left(\frac{\partial L}{\partial \dot y}\right). && \text{(2.6d)}
\end{aligned}$$

If we now require that the Action Integral takes a stationary (not minimum) value, the Euler-Lagrange equation

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial \dot y}\right) = 0 \tag{2.7}$$

applies and we get

$$\frac{d}{dx}\left(f - L\xi - (\eta - \dot y\xi)\frac{\partial L}{\partial \dot y}\right) = 0. \tag{2.8}$$

Hence, by integration,

$$I = f - \left[L\xi + (\eta - \dot y\xi)\frac{\partial L}{\partial \dot y}\right] \tag{2.9}$$

is a *first integral*. The symmetry G exists independently of the requirement that the variation of the functional in equation (2.1) be zero. When the extra condition is added, the first integral exists.

In the derivation above, no mention has been made of the functional dependence of ξ and η. They may depend upon $\dot y$ as well as x and y. Then G becomes a generalised symmetry. When dependence upon $\dot y$ is allowed, the Noether symmetries need not be symmetries of the differential equation which arises from the variational principle. A Noether symmetry can only give rise to a single first integral via equation (2.9).

To illustrate we consider the free particle equation for which

$$L = \frac{1}{2}\dot{y}^2, \tag{2.10}$$

through the application of equation (2.5), we obtain the expression

$$(\dot{\eta} - \dot{y}\dot{\xi})\dot{y} + \frac{1}{2}\dot{y}^2\dot{\xi} = \dot{f}. \tag{2.11}$$

Suppose that G is a Noether point symmetry, then (2.11) gives the following determining equations separated by powers of $\dot{y}$,

$$\begin{aligned}
\dot{y}^3 &: \quad -\frac{1}{2}\frac{\partial \xi}{\partial y} = 0, && (2.12)\\
\dot{y}^2 &: \quad \frac{\partial \eta}{\partial y} - \frac{1}{2}\frac{\partial \xi}{\partial x} = 0, && (2.13)\\
\dot{y}^1 &: \quad \frac{\partial \eta}{\partial x} - \frac{\partial f}{\partial y} = 0, && (2.14)\\
\dot{y}^0 &: \quad \frac{\partial f}{\partial x} = 0. && (2.15)
\end{aligned}$$

Then integrating each equation yields the results,

$$\begin{aligned}
\xi &= a(x), && (2.16)\\
\eta &= \frac{1}{2}\dot{a}y + b(x), && (2.17)\\
f &= \frac{1}{4}\ddot{a}y^2 + \dot{b}y + c(x), && (2.18)\\
0 &= \frac{1}{4}\dddot{a}y^2 + \ddot{b}y + \dot{c}. && (2.19)
\end{aligned}$$

Hence the arbitrary functions are

$$\begin{aligned}
a &= A_0 + A_1 x + A_2 x^2, && (2.20)\\
b &= B_0 + B_1 x, && (2.21)\\
c &= C_0. && (2.22)
\end{aligned}$$

Since c is a constant, it is ignored. There are five Noether point symmetries and their associated first integrals are

$$G_1 = \frac{\partial}{\partial y} \qquad \Longrightarrow \qquad I_1 = -\dot{y} \tag{2.23}$$

$$G_2 = x\frac{\partial}{\partial y} \qquad \Longrightarrow \qquad I_2 = y - x\dot{y} \tag{2.24}$$

$$G_3 = \frac{\partial}{\partial x} \qquad \Longrightarrow \qquad I_3 = \frac{1}{2}\dot{y}^2 \tag{2.25}$$

$$G_4 = x\frac{\partial}{\partial x} + \frac{1}{2}y\frac{\partial}{\partial y} \qquad \Longrightarrow \qquad I_4 = -\frac{1}{2}\dot{y}(y - x\dot{y}) \tag{2.26}$$

$$G_5 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \qquad \Longrightarrow \qquad I_5 = \frac{1}{2}(y - x\dot{y})^2. \tag{2.27}$$

2.2 Jacobi Last Multiplier

The Jacobi Last Multiplier, found in [13] is a method that can be used to find a first integral to a system of differential equations or a differential equation that is second-order and higher. We start by looking at a partial differential equation

$$Af = \Sigma_{i=1}^{n} a_i(x_i, \ldots, x_n)\frac{\partial f}{\partial x_i} = 0 \tag{2.28}$$

which has a Lagrange system

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \ldots = \frac{x_n}{a_n}. \tag{2.29}$$

The Jacobi multiplier is then given by

$$\frac{\partial(f, w_1, w_2, \ldots, w_{n-1})}{\partial(x_1, x_2, \ldots, x_n)} = MAf, \tag{2.30}$$

where

$$\frac{\partial(f,w_1,w_2,\ldots,w_{n-1})}{\partial(x_1,x_2,\ldots,x_n)} = det\begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \\ \frac{\partial w_1}{\partial x_1} & & \frac{\partial w_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial w_{n-1}}{\partial x_1} & \cdots & \frac{\partial w_{n-1}}{\partial x_n} \end{bmatrix} = 0 \tag{2.31}$$

where $w_1,\ldots,w_{n-1}$ are $n-1$ solutions of (2.28) or first integrals of (2.29) independent of each other. M is a function of $(x_1,\ldots,x_n)$ and depends on the chosen $n-1$ solutions.

Properties of the Jacobi Last Multiplier [13] are:

(a) If one selects a different set of $n-1$ independent solutions $\eta_1,\ldots,\eta_{n-1}$ of equation (2.28), then the corresponding Jacobi Last Multiplier N is linked to M by the relationship:

$$N = M\frac{\partial(\eta_1,\ldots,\eta_{n-1})}{\partial(w_1,\ldots,w_{n-1})}. \tag{2.32}$$

(b) Given a non-singular transformation of variables

$$\tau: \quad (x_1,x_2,\ldots,x_n) \to (x_1',x_2',\ldots,x_n'), \tag{2.33}$$

then the last Multiplier M' of $A'F=0$ is given by:

$$M' = M\frac{\partial\left(x_1,x_2,\ldots,x_n\right)}{\partial\left(x_1',x_2',\ldots,x_n'\right)}, \tag{2.34}$$

where M obviously comes from the $n-1$ solutions of $AF=0$ which corresponds to those chosen for $A'F=0$ through the inverse transformation τ^{-1}.

(c) One can prove that each Multiplier M is a solution of the following linear partial differential equation:

$$\Sigma_{i=1}^{n}\frac{\partial(Ma_i)}{\partial x_i}=0, \tag{2.35}$$

vice versa every solution M of this equation is a Jacobi Multiplier.

(d) If one knows two Jacobi Last Multipliers M_1 and M_2 of equation (2.28), then their ratio is a solution ω of (2.28) or a first integral of (2.29). Naturally the ratio could be a constant. Vice versa the product of a Multiplier M_1 times any solution ω yields another last Multiplier $M_2 = M_1\omega$. In this case there exists a constant multiplier, then any Jacobi Last Multiplier is a first integral.

Another property of the Jacobi Last Multiplier is its relationship with the Lagrangian, $L = L(t, x, \dot{x})$, for any second-order equation

$$\ddot{x} = \phi(t, x, \dot{x}) \tag{2.36}$$

and

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} \tag{2.37}$$

where $M = M(t, x, \dot{x})$ satisfies the following equation

$$\frac{d}{dt}(\log M) + \frac{\partial\phi}{\partial\dot{x}} = 0. \tag{2.38}$$

Then equation (2.36) becomes the Euler-Lagrange equation

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}}\right) + \frac{\partial L}{\partial x} = 0. \tag{2.39}$$

The proof is given by taking the derivative of (2.39) by $\dot{x}$ and showing that this yields (2.38). If one knows a Jacobi Last Multiplier, then L can be obtained by a double integration,

$$L = \int \left(\int M d\dot{x} \right) d\dot{x} + l_1(t,x)\dot{x} + l_2(t,x), \tag{2.40}$$

where l_1 and l_2 are functions of t and x which have to satisfy a single partial differential equation related to (2.36). The functions l_1 and l_2 are related to the gauge function $F = F(t,x)$. We may assume,

$$\begin{aligned} l_1 &= \frac{\partial F}{\partial x}, & (2.41) \\ l_2 &= \frac{\partial F}{\partial x} + l_3(t,x) & (2.42) \end{aligned}$$

where l_3 has to satisfy the mentioned partial differential equation and F is arbitrary.

We can also show that a system of two first-order differential equations [13] can be used with small differences for this method, where

$$\begin{aligned} \dot{x_1} &= \phi_1(t,x_1,x_2), & (2.43) \\ \dot{x_2} &= \phi_2(t,x_1,x_2) & (2.44) \end{aligned}$$

always admits a Lagrangian of the form

$$L = X_1(t,x_1,x_2)\dot{x_1} + X_2(t,x_1,x_2)\dot{x_2} - V(t,x_1,x_2). \tag{2.45}$$

The key is a function W such that

$$W = -\frac{\partial X_1}{\partial x_2} = \frac{\partial X_2}{\partial x_1}, \tag{2.46}$$

is also the Hamiltonian for the system, which is used to yield the Lagrangian of the system, and thus, for a two-dimensional system we have

$$\frac{d}{dt}(\log W) + \frac{\partial \phi_1}{\partial x_1} + \frac{\partial \phi_2}{\partial x_2} = 0. \tag{2.47}$$

We can see that equation (2.47) is the same as equation (2.35) of the Jacobi Last Multiplier for the system. Once a Jacobi Last Multiplier $M(t, x_1, x_2)$ has been found, then a Lagrangian of the system can be obtained [13] by two integrations given by

$$L = \left(\int M dx_1\right) \dot{x_2} - \left(\int M dx_2\right) \dot{x_1} + g(t, x_1, x_2) + \frac{d}{dt} G(t, x_1, x_2), \tag{2.48}$$

where $g(t, x_1, x_2)$ satisfies two linear differential equations of first-order that can be integrated, and $G(t, x_1, x_2)$ is an arbitrary gauge function.

2.3 Lotka-Volterra Model

As the Jacobi Last Multiplier [11] for this system is a constant, the Lagrangian is

$$L_x = x_1 \dot{x_2} - x_2 \dot{x_1} + 2(-Be^{x_1}) + be^{x_2} - Ax_1 + ax_2 + \frac{d}{dt} G(t, x_1, x_2), \tag{2.49}$$

where G is an arbitrary gauge function. By using the transformation, we can find a Jacobi Last Multiplier for the original system by looking at the Jacobian in terms of the Jacobi Last Multiplier [11] for the transformation. Thus we have

$$M_w = M_x \frac{\partial(x_1, x_2)}{\partial(w_1, w_2)} = \begin{vmatrix} \frac{1}{w_1} & 0 \\ 0 & \frac{1}{w_2} \end{vmatrix} = \frac{1}{w_1 w_2}. \tag{2.50}$$

Therefore the Lagrangian, in terms of the original variables, is given by

$$L_w = \ln(w_1)\frac{\dot{w}_2}{w_2} - \ln(w_2)\frac{\dot{w}_1}{w_1} + 2(-A\ln(w_1) + a\ln(w_2) - Bw_1 + bw_2) + \frac{d}{dt}G(t, w_1, w_2). \tag{2.51}$$

This system can be solved using Noether's Theorem. We can do this by using the Lagrangian found in terms of the original system or we can transform the system into a second-order differential equation given by equation (1.19), using the change of variables, in which another Lagrangian can be found and a first integral can be produced from this differential equation. The Jacobi Last Multiplier for equation (1.19) is given by

$$\begin{aligned} \frac{d}{dt}(\ln M) + be^{x_2} + a &= 0, \\ \implies \quad \frac{d}{dt}(\ln M) + \dot{x_1} = 0. \end{aligned} \tag{2.52}$$

Therefore the Jacobi Last Multiplier, due to (1.18), is given by

$$M = e^{-x_1} = \frac{B}{\dot{x_2} - A}, \tag{2.53}$$

and the following Lagrangian is obtained

$$L = B\Big[(\dot{x_2} - A)\ln(A - \dot{x_2}) - \dot{x_2} + be^{x_2} + ax_2\Big] + \frac{d}{dt}F(t, x_2), \tag{2.54}$$

where $F(t, x_2)$ is the gauge function. We do not consider the gauge function using Noether's Theorem and equation (2.9). When we use the Lagrangian in equation (2.54) yields

$$I = f - \Big[\xi B\Big((\dot{x}_2 - A)\ln(A - \dot{x}_2) - \dot{x}_2 + be^{x_2} + ax_2\Big) + (\eta - \dot{x}_2\xi)\ln(A - \dot{x}_2)\Big], \tag{2.55}$$

where $\dot{y} = \dot{x}_2$ and, f, ξ and η are found by Noether's Theorem when taking the Lagrangian into account.

2.4 Host-parasite Model

A Jacobi Last Mulitplier [11] given by

$$M_{[u]} = \frac{e^{At}}{u_1 {u_2}^2}, \tag{2.56}$$

yields the Lagrangian

$$L_{[u]} = e^{At}\left[\ln(u_1)\frac{\dot{u}_2}{{u_2}^2} + \frac{\dot{u}_1}{u_1u_2} - 2\frac{a}{u_1} - 2\frac{B}{u_1} - \ln(u_1)\frac{A}{u_2} - 2b\ln(u_2)\right] + \frac{d}{dt}G(t,u_1,u_2). \tag{2.57}$$

A Jacobi Last Multiplier [11] of this system is then given by

$$M_{[x]} = M_{[u]}\frac{\partial(u_1,u_2)}{\partial(x_1,x_2)} = \frac{1}{x_1{x_2}^2} \tag{2.58}$$

and the corresponding Lagrangian is

$$L_{[x]} = \frac{\ln(x_1)\dot{x}_2}{\dot{x}_2^2} + \frac{\dot{x}_1}{x_1x_2} - 2e^{At}\frac{bx_1\ln(x_2)e^{at} + B}{x_1e^{at}} + \frac{d}{dt}G(t,x_1,x_2). \tag{2.59}$$

Through a Jacobi Last Multiplier for the differential equation, (1.23), a Lagrangian can be obtained which is

$$L_1 = be^{At}\ln(\dot{x}_1) - be^{At}\ln(x_1) + \frac{Be^{At}}{e^{at}x_1} + \frac{d}{dt}F(t,x_1). \tag{2.60}$$

From the previous model, the same would apply for this model, where we do not consider or take into account the gauge function. So by using Noether's Theorem and equation (2.9), the Lagrangian in equation (2.60) can be written for simplicity as

$$L = \left(be^{At}\right)\ln\frac{\dot{x}_1}{x_1} + \frac{Be^{t(A-a)}}{x_1}$$

and so

$$I = f - \left[\left(\xi be^{At}\right)\ln\frac{\dot{x}_1}{x_1} + \frac{Be^{t(A-a)}}{x_1} + (\eta - \dot{x}_1\xi)\,\frac{x_1be^{At}}{\dot{x}_1}\right], \tag{2.61}$$

where $\dot{y} = \dot{x}_1$ and, f, ξ and η are found by Noether's Theorem when taking the Lagrangian into account.

2.5 Gompertz Model

Using the formula , we can obtain a Jacobi Last Multiplier [11] given by

$$\frac{d}{dt}(\ln M_x) = -(a+A) \quad \Longrightarrow \quad M_x = e^{-(a+A)t} \tag{2.62}$$

and from it the Lagrangian

$$L_x = e^{-(a+A)t}\Big[x_1\dot{x_2} - x_2\dot{x_1} - 2m_1 b e^{x_1} + 2m_2 B e^{x_2} + x_1 x_2(A-a)\Big] + \frac{d}{dt}G(t, x_1, x_2), \tag{2.63}$$

where G is an arbitrary gauge function. By using the transformation, we can find a Jacobi Last Multiplier for the original system by looking at the Jacobian between (w_1, w_2) and (x_1, x_2) in terms of the Jacobi Last Multiplier for the transformed system. Thus we have

$$M_w = M_x \frac{\partial(x_1, x_2)}{\partial(w_1, w_2)} = e^{-(a+A)t} \begin{vmatrix} \frac{1}{w_1} & 0 \\ 0 & \frac{1}{w_2} \end{vmatrix} = e^{-(a+A)t}\left(\frac{1}{w_1 w_2}\right). \tag{2.64}$$

Therefore the Lagrangian [11], in terms of the original variables, is given by

$$\begin{aligned} L_w &= e^{-(a+A)t}\Bigg[\ln\left(\frac{w_1^2}{w_2^2}\frac{\dot{w}_2}{\dot{w}_1}\right) + 2a\ln\left(\frac{w_2}{m_2}\right)\ln(w_1) + 2A\ln\left(\frac{w_1}{m_1}\right)\ln(w_2) \\ &+ 2Bw_2 - 2bw_1 - (A-a)\ln(w_1)\ln(w_2)\Bigg] + \frac{d}{dt}G(t, w_1, w_2). \end{aligned} \tag{2.65}$$

This system can be solved using Noether's Theorem. We can do this by using the Lagrangian found in terms of the original system or we can transform the system into a second-order differential equation given by (1.14), using the change of variables, in which another Lagrangian can be found. The Jacobi Last Multiplier for this equation (1.14), due to (1.13), is given by

$$M = e^{-(a+A)t}\left(\frac{1}{\dot{x_1} - Ax_1}\right) \tag{2.66}$$

and the following Lagrangian is obtained

$$\begin{aligned} L &= e^{-(a+A)t}\Big[(\dot{x_1} - Ax_1)\ln(\dot{x_1} - Ax_1) + m_1 b e^{x_1} - ax_1 \ln(Bm_2) - ax_1\Big] \\ &+ \frac{d}{dt}F(t, x_1), \end{aligned} \tag{2.67}$$

where $F(t, x_2)$ is the gauge function. The same analysis applies for this model, where we do not consider or take into account the gauge function. So by using Noether's Theorem and equation (2.9), the Lagrangian in equation (2.67) yields

$$\begin{aligned} I &= f - e^{-(a+A)t}\Bigg[\xi\Big[(\dot{x_1} - Ax_1)\ln(\dot{x_1} - Ax_1) + m_1 b e^{x_1} - ax_1 \ln(Bm_2) - ax_1\Big] \\ &+ (\eta - \dot{x}_1\xi)\left[\ln(\dot{x_1} - Ax_1) + 1\right]\Bigg], \end{aligned} \tag{2.68}$$

where $\dot{y} = \dot{x}_1$ and, f, ξ and η are found by Noether's Theorem when taking the Lagrangian into account.

2.6 Conclusion

In this Section we used Noether's Theorem in a general case to show how first integrals can be found. Each first integral is unique because it corresponds to the Noether symmetry found and can either be a reduction of the differential equation for the system or it can be a solution depending upon the nature of the first integral.

Chapter 3

Multipliers for the Lotka-Volterra Model

3.1 Introduction

Consider an rth-order system of partial differential equations of n independent variables $\underline{s} = (s_1, s_2, \ldots, s_n)$ and m dependent variables $u = (u_1, u_2, \ldots, u_m)$ viz.,

$$E(\underline{s}, u, u_{(1)}, \ldots, u_{(r)}) = 0, \qquad u = 1, \ldots, \tilde{m}, \tag{3.1}$$

where a locally analytic function $f(\underline{s}, u, u_1, \ldots, u_k)$ of a finite number of dependent variables $u, u_1, \ldots, u_k$ denote the collections of all first , second ,..., kth-order partial derivatives and s is a multivariable, that is

$$u_i^\alpha = D_i(u^\alpha), \qquad u_{ij}^\alpha = D_j D_i(u^\alpha), \ldots \tag{3.2}$$

respectively, with the total differentiation operator with respect to s^i given by,

$$D_i = \frac{\partial}{\partial s^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \dots \qquad i = 1, \dots, m. \tag{3.3}$$

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well known result that the Euler-Lagrange operator annihilates a total divergence. Firstly, if $(T^{s1}, T^{s1}, \dots)$ is a conserved vector corresponding to a conservation law, then

$$D_{s1}T^{s1} + D_{s1}T^{s1} + \dots = 0 \tag{3.4}$$

along the solutions of the differential equation $(E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0$.

Moreover, if there exists a nontrivial differential function Q, called a 'multiplier', such that

$$Q(\underline{s}, u, u_{(1)} \dots)E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = D_{s1}T^{s1} + D_{s1}T^{s1} + \dots, \tag{3.5}$$

for some (conserved) vector $(T^{s1}, T^{s1}, \dots)$, then

$$\frac{\delta}{\delta u} Q(\underline{s}, u, u_{(1)} \dots)E(\underline{s}, u, u_{(1)}, \dots, u_{(r)}) = 0, \tag{3.6}$$

where $\frac{\delta}{\delta u}$ is the Euler operator. Hence, one may determine the multipliers, using (3.6) and then construct the corresponding conserved vectors.

3.2 Multipliers of the Lotka-Volterra Model

We now find the various multipliers for the (1.19) which has been redefined as a second-order differential equation. In this section we employ the methods demonstrated using the equation (3.6) to obtain the multipliers of equation (1.19).

Case 1: We established a general multiplier of the form

$$\mu = R(x)[1 + \tan(S(x,t))]^{\frac{1}{2}} - T(\dot{x}), \tag{3.7}$$

where the functions R and S are $R(x) = \frac{e^{-\frac{1}{2}(C_3 A a + e^x b)}}{A}$, $T(\dot{x}) = \frac{C_1 \dot{x}}{A(A-\dot{x})}$ and $S(x,t) = \frac{(C_2+t)^2}{4}\left[-(e^x)^2 b^2 + 4be^x(A - \frac{a}{2})a^2\right]$. In the subcases that follow, we have selected constants that can vanish forming new multipliers.

Case 1.1 : We consider $C_1 = 1$ and $C_2 = C_3 = 0$. This simplifies the multiplier (3.7) for the equation (1.19) which results in a simplified form

$$\mu_1 = R_1(x)[1 + \tan(S_1(x,t))]^{\frac{1}{2}} - T(\dot{x}), \tag{3.8}$$

where the functions $R_1(x) = \frac{e^{-\frac{1}{2}(e^x b)}}{A}$ and $S_1(x,t) = \frac{t^2}{4}\left[-(e^x)^2 b^2 + 4be^x\left(A - \frac{a}{2}\right)a^2\right]$.

Case 1.2 : We consider $C_2 = 1$ and $C_1 = C_3 = 0$, this simplifies the multiplier in equation (3.7) for the equation (1.19) which results in a simplified form

$$\mu_2 = R_2\left[1 + \tan(S_2)\right]^{\frac{1}{2}}, \tag{3.9}$$

where the functions $R_2(x) = \frac{e^{-\frac{1}{2}e^x b}}{A}$ and $S_2(x,t) = \frac{(1+t)^2}{4}\left[-(e^x)^2 b^2 + 4be^x\left(A - \frac{a}{2}\right)a^2\right]$.

Case 1.3: We consider $C_3 = 1$ and $C_1 = C_2 = 0$. This simplifies the multiplier (3.7) for the equation (1.19) which results in a simplified form

$$\mu_3 = R_3\left[1 + \tan(S_3)\right]^{\frac{1}{2}}, \tag{3.10}$$

where the functions $R_3(x) = \frac{e^{-\frac{1}{2}(Aa+e^x b)}}{A}$ and $S_3(x,t) = \frac{t2}{4}\left[-(e^x)^2 b^2 + 4be^x(A - \frac{a}{2})a^2\right]$.

Case 2 : We consider a multiplier in which we look at different forms of the parameter a which gives us different multipliers and first integrals.

Case 2.1 : When a is unspecified, the multiplier for equation (1.19) is given by

$$\mu = \frac{C_1 \dot{x}}{(A - \dot{x})}. \tag{3.11}$$

If we consider $C_1 = 1$, the resulting multiplier of equation (3.11) is given by

$$\mu = \frac{\dot{x}}{(A - \dot{x})}. \tag{3.12}$$

Case 2.2 : When $a = -be^x - C_2$, the multiplier for equation (1.19) is given by

$$\mu = \frac{R(t)(A - \dot{x})\cos\left[S(x,t)\right] - C_1 \dot{x}}{A(A - \dot{x})}, \tag{3.13}$$

where $R(t) = \sqrt{(e^{C_4 A})^2 e^{C_2 t}}$, and $S(x,t) = \frac{1}{2}\sqrt{4Abe^x - C_2^2}(C_3 + t)$.

Case 2.2.1 : When we take $C_1 = 1$ and $C_2 = C_3 = C_4 = 0$, this simplifies our multiplier in equation (3.13) to

$$\mu_1 = \frac{1}{A}\cos(S_1) - \frac{\dot{x}}{A(A - \dot{x})}, \tag{3.14}$$

where $S_1(x,t) = \frac{t}{2}\left(4Abe^x\right)^{\frac{1}{2}}$.

Case 2.2.2 : If we now consider $C_2 = 1$ and $C_1 = C_3 = C_4 = 0$, our multiplier in equation (3.13) is simplified to

$$\mu_2 = R_2 \cos(S_2), \tag{3.15}$$

where $R_2(t) = \frac{e^{\frac{t}{2}}}{A}$ and $S_2(x,t) = \frac{t}{2} \left(4Abe^x - 1\right)^{\frac{1}{2}}$.

Case 2.2.3 : Now taking $C_3 = 1$ and $C_1 = C_2 = C_4 = 0$, this simplifies our multiplier in equation (3.13) to

$$\mu_3 = \frac{1}{A} \cos(S_3), \tag{3.16}$$

where $S_3(x,t) = \frac{(1+t)}{2} \left(4Abe^x\right)^{\frac{1}{2}}$.

Case 2.2.4 : Taking $C_4 = 1$ and $C_1 = C_2 = C_3 = 0$, this simplifies our multiplier in equation (3.13) which results in

$$\mu_4 = \frac{e^A}{A} \cos(S_4). \tag{3.17}$$

where $S_4(x,t) = \frac{t}{2} \left(4Abe^x\right)^{\frac{1}{2}}$.

Case 3: We now consider a multiplier which takes into account different forms of the parameter 'b'.

Case 3.1 : When $b = b$, then the multiplier for equation (1.19) is given by,

$$\mu = \frac{C_1 \dot{x}}{(A - \dot{x})}. \tag{3.18}$$

If we let $C_1 = 1$, then the resulting multiplier of equation (3.18) is,

$$\mu = \frac{\dot{x}}{(A - \dot{x})}. \tag{3.19}$$

Case 3.2 : When $b = e^{-x}(A - a - C_2)$, then the multiplier for equation (1.19) is given by

$$\mu = \frac{R(t)(A - \dot{x})\cos[S(t)] - C_1\dot{x}}{A(A - \dot{x})}, \tag{3.20}$$

where $R(t) = \sqrt{\frac{(e^{C_4A})^2 e^{C_2 t}}{e^{At}}}$, and $S(t) = \frac{1}{2}(C_3 + t)\sqrt{6A^2 - 4A(a + \frac{C_2}{2}) - C_2^2}$

Case 3.2.1 : Taking $C_1 = 1$ and $C_2 = C_3 = C_4 = 0$, this simplifies our multiplier in equation (3.20) which results in,

$$\mu_1 = R_1\cos(S_1) - \frac{\dot{x}}{A(A - \dot{x})}, \tag{3.21}$$

where $R_1(t) = \frac{1}{A}e^{\frac{-At}{2}}$ and $S_1(t) = \frac{t}{2}\left(6A^2 - 4Aa\right)^{\frac{1}{2}}$.

Case 3.2.2 : Taking $C_2 = 1$ and $C_1 = C_3 = C_4 = 0$, simplifies our multiplier in equation (3.20) to

$$\mu_2 = R_2\cos(S_2), \tag{3.22}$$

where $R_2(t) = \frac{e^{\frac{t}{2}(1-A)}}{A}$ and $S_2(t) = \frac{t}{2}\left(6A^2 - 4A\left(a + \frac{1}{2}\right) - 1\right)^{\frac{1}{2}}$.

Case 3.2.3 : If we consider $C_3 = 1$ and $C_1 = C_2 = C_4 = 0$, our multiplier in equation (3.20) simplifies to

$$\mu_3 = R_3\cos(S_3). \tag{3.23}$$

where $R_3(t) = \frac{e^{\frac{-At}{2}}}{A}$ and $S_3(t) = \frac{(t+1)}{2}\left(6A^2 - 4Aa\right)^{\frac{1}{2}}$.

Case 3.2.4 : Now taking $C_4 = 1$ and $C_1 = C_2 = C_3 = 0$, simplifies our multiplier in equation (3.20) to

$$\mu_4 = R_4 \cos(S_4). \tag{3.24}$$

where $R_4(t) = \frac{1}{A}\left(e^{A(2-t)}\right)^{\frac{1}{2}}$ and $S_4(t) = \frac{t}{2}\left(6A^2 - 4Aa\right)^{\frac{1}{2}}$.

Case 3.3 : When $b = 0$, the multiplier for equation (1.19) is given by

$$\mu = C_1 + C_2 e^{-at} + F_1[-(A - \dot{x})e^{-2at}]. \tag{3.25}$$

Case 3.3.1 : If we consider $C_1 = 1$ and $C_2 = F_1 = 0$, our multiplier in equation (3.25) simplifies to,

$$\mu_1 = 1. \tag{3.26}$$

Case 3.3.2 : Taking $C_2 = 1$ and $C_1 = F_1 = 0$, simplifies our multiplier in equation (3.25) to

$$\mu_2 = e^{-at}. \tag{3.27}$$

3.3 Conclusion

In this section we have explored different types of multipliers for specific parameters. Each multiplier is unique and corresponds to a specific first integral which is discussed in the next chapter.

Chapter 4

First integrals for the Lotka-Volterra Model

4.1 Introduction

The dynamical equations in biological systems involve second-order differential equations. First integrals are the result of integrating one time, to reduce the second-order equations to first-order differential equations. First integrals were introduced in physics problems which dealt with laws of motion and later on used in many other fields of mathematics. Any quantities which are not changing with time are called "first integrals." There are many ways in which first integrals can be found. In this chapter we look at two different types which include the use of Noether's Theorem and the Multiplier Method.

Since we are able to reduce our equations from a second-order to a first-order differential equation, it becomes much easier to solve the system. Many biological systems can be quite complicated to solve directly, thus by the use of first integrals, we are able to obtain exact solutions of our systems. These solutions would allow us to make physical conclusions regarding the behaviour of our systems. One of the ways in which first integrals can be found, is through the use of symmetry analysis [12]. However, in general, one must take into account the notion of symmetries. Suppose that

$$E(x, y, \dot{y}, \ddot{y}) = 0 \tag{4.1}$$

has a symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}. \tag{4.2}$$

A function $f(x, y, \dot{y})$ also possesses the symmetry G if

$$\begin{aligned} G^{[1]}f &= 0 && (4.3) \\ \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + (\dot{\eta} - \dot{y}\dot{\xi})\frac{\partial f}{\partial \dot{y}} &= 0. && (4.4) \end{aligned}$$

This has an associated Lagrange's system

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{d\dot{y}}{\dot{\eta} - \dot{y}\dot{\xi}}. \tag{4.5}$$

The two characteristics are, say u and v so that

$$f(x, y, \dot{y}) = g(u, v). \tag{4.6}$$

Now f is a first integral of the differential equation $E = 0$ if

$$\begin{aligned} \frac{df}{dx}_{|E=0} &= 0 \\ \implies \dot{u}\frac{\partial g}{\partial u} + \dot{v}\frac{\partial g}{\partial v} &= 0 && (4.7) \end{aligned}$$

which has the associated Lagrange's system given by

$$\frac{du}{\dot{u}} = \frac{dv}{\dot{v}}. \tag{4.8}$$

The system has the one characteristic, w, and so $f(x, y, \dot{y}) = h(w)$, where h is an arbitrary function of its argument. Usually h is taken to be the identity, but this is not necessary. In general a scalar ordinary differential equation of the n^{th} order

$$E(x, y, \dot{y}, \dots, y^{(n)}) = 0 \tag{4.9}$$

has a first integral

$$I = f(x, y, \dot{y}, \dots, y^{(n-1)}) \tag{4.10}$$

associated with the symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{4.11}$$

if

$$G^{[n-1]}f = 0 \tag{4.12}$$

and

$$\frac{df}{dx}_{|E=0} \tag{4.13}$$

Consider the equation

$$\ddot{y} = 0. \tag{4.14}$$

If we integrate this equation once, we get

$$\dot{y} = I_1, \tag{4.15}$$

where I_1 is a constant. We say that

$$I_1 = \dot{y} \tag{4.16}$$

is a first integral of equation (4.14). In Physics one talks of a constant of the motion or a conservation law. The main requirement for a nontrivial function, I, to be a first integral of an ordinary differential equation is that

$$I' = 0 \tag{4.17}$$

when the equation is taken into account. One can usually find an infinite number of first integrals for a given equation (especially because any function of a first integral is itself a first integral) but there are only a finite number of independent first integrals. In the case of equation (4.14) we can multiply by x and integrate to obtain another first integral given by

$$I_2 = y - x\dot{y} \tag{4.18}$$

which is independent of equation (4.16). Since both I_1 and I_2 are independent, one can write down the general solution to equation (4.14) by eliminating $\dot{y}$ from each expression. This leads to

$$y = I_2 + xI_1 \tag{4.19}$$

which we recognise as the general solution of equation (4.14). It is clear that, in simple cases, one can calculate the first integrals by inspection. The technique is presented more clearly through the given example.

Example : We consider a differential equation given by

$$2(x-1)y\ddot{y} + 2(1-2x)y\dot{y} + (1-4x+3x^2)\dot{y}^2, \tag{4.20}$$

where our multiplier is $\mu = \frac{e^{2x}}{\dot{y}^2}$. By applying equation (4.20) to (**??**), we obtain

$$\frac{e^{2x}}{\dot{y}^2}\left[2(x-1)y\ddot{y} + 2(1-2x)y\dot{y} + (1-4x+3x^2)\dot{y}^2\right] = \frac{\partial I}{\partial x} + \dot{y}\frac{\partial I}{\partial y} + \ddot{y}\frac{\partial I}{\partial \dot{y}}. \tag{4.21}$$

By equating coefficients of $\ddot{y}$,

$$\frac{\partial I}{\partial \dot{y}} = \frac{e^{2x}}{\dot{y}^2}\left[2(x-1)y\right]. \tag{4.22}$$

Integrating equation (4.22) results in,

$$I = -\frac{2}{\dot{y}}e^{2x}y(x-1) + \psi(x,y). \tag{4.23}$$

By differentiating and equating corresponding coefficients we have

$$\frac{\partial I}{\partial y} = -\frac{2}{\dot{y}}e^{2x}(x-1) + \psi_y(x,y), \tag{4.24}$$

$$\frac{\partial I}{\partial x} = -\frac{4}{\dot{y}}e^{2x}y(x-1) - \frac{2}{\dot{y}}ye^{2x} + \psi_x(x,y), \tag{4.25}$$

which results in

$$2e^{2x}(1-2x)\frac{y}{\dot{y}} + e^{2x}(1-4x+3x^2) = \frac{\partial I}{\partial x} + \dot{y}\frac{\partial I}{\partial y}, \tag{4.26}$$

$$\begin{aligned} 2e^{2x}(1-2x)y + e^{2x}\dot{y}(1-4x+3x^2) &= -4e^{2x}y(x-1) + 2ye^{2x} + \psi_x \\ &+ \dot{y}\left[-2e^{2x}(x-1) + \dot{y}\psi_y\right]. \end{aligned} \tag{4.27}$$

If we now separate by powers of $\dot{y}$,

$$\dot{y}^2 \quad : \quad \psi_y = 0 \quad \Longrightarrow \quad \psi(x), \tag{4.28}$$

$$\dot{y} \quad : \quad e^{2x}(1-4x+3x^2) + 2e^{2x}(x-1) = \psi_x, \tag{4.29}$$

$$\dot{y}^0 \quad : \quad 2e^{2x}(1-2x)y = -4e^{2x}y(x-1) - 2ye^{2x}. \tag{4.30}$$

Thus $\psi(x,y)$ is given by

$$\psi = \int e^{2x}(3x^2 - 2x - 1)dx. \tag{4.31}$$

Integrating this by parts yields,

$$I_1 = \frac{1}{2}e^{2x}(3x^2 - 2x - 1) - \frac{1}{2}\int e^{2x}(6x-2)dx, \tag{4.32}$$

integrating the second half of I_1 by parts, we get

$$I_2 = \frac{1}{4}e^{2x}(6x-2) - \int \frac{6}{4}e^{2x}dx = \frac{1}{4}e^{2x}(6x-2) - \frac{3}{4}e^{2x} + G. \tag{4.33}$$

and thus, I_1 results in

$$I_1 = \frac{1}{2}e^{2x}(3x^2 - 2x - 1) - \frac{1}{4}e^{2x}(6x-2) + \frac{3}{4}e^{2x} + G, \tag{4.34}$$

where G is the integration constant.

4.2 First integrals of the Lotka-Volterra Model

We now find the various multipliers for equation (1.19) which is the Lotka-Volterra model redefined as a second-order differential equation. In this section we employ the methods of characteristics using the equation (3.6) to obtain the first integrals of equation (1.19).

Case 1.1 : To determine the first integrals we apply the equation (3.8) to (3.6) to obtain the relation,

$$\left[R_1\left[1+\tan(S_1)\right]^{\frac{1}{2}} - \frac{\dot{x}}{A(A-\dot{x})}\right]\left[\ddot{x} + (be^x + a)(A - \dot{x})\right] \;=\; \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.35}$$

Equating coefficients of $\ddot{x}$ results in,

$$\frac{\partial I}{\partial \dot{x}} = R_1\left[1+\tan(S_1)\right]^{\frac{1}{2}} - \frac{\dot{x}}{A(A-\dot{x})}. \tag{4.36}$$

Integrating equation (4.36) leads to the generalised form of the first integral,

$$I = \dot{x}R_1\left[1+\tan(S_1)\right]^{\frac{1}{2}} + \frac{1}{A}\left[\dot{x} + A\ln(\dot{x}-A)\right] + \psi(x,t). \tag{4.37}$$

In order to solve for the general functions in (4.37), we first differentiate then equate like terms in the corresponding coefficients resulting with the following equations,

$$\begin{aligned}
\frac{\partial I}{\partial x} &= \dot{x}\left[R_{1x}\left[1+\tan(S_1)\right]^{\frac{1}{2}} + \frac{R_1}{2}\left[1+\tan(S_1)\right]^{-\frac{1}{2}}\sec^2(S_1)\cdot S_{1x}\right] + \psi_x, &(4.38)\\
\frac{\partial I}{\partial t} &= \frac{\dot{x}R_1}{2}\left[1+\tan(S_1)\right]^{-\frac{1}{2}}\sec^2(S_1)S_{1t} + \psi_t, &(4.39)
\end{aligned}$$

where the subscript x or t refers to partial differentiation with respect to the variable. Substituting the derivatives of $I(x,t)$ into the next part of the equation, we separate by powers of $\dot{x}$,

$$\begin{aligned}
\dot{x}^2 &: \left[R_{1x}\left[1+\tan(S_1)\right]^{\frac{1}{2}} + \frac{R_1}{2}\left[1+\tan(S_1)\right]^{-\frac{1}{2}}\sec^2(S_1)S_{1x}\right] = \frac{be^x+a}{A(A-\dot{x})}, &(4.40)\\
\dot{x} &: -\left(be^x+a\right)R_1\left[1+\tan(S_1)\right]^{\frac{1}{2}} - \frac{be^x+a}{A(A-\dot{x})} = \frac{R_1}{2}\left[1+\tan(S_1)\right]^{-\frac{1}{2}}\sec^2(S_1)S_{1t}\\
&\qquad +\psi_x. &(4.41)
\end{aligned}$$

Integrating with respect to x,

$$\begin{aligned}
\psi &= -\int\left[\left(be^x+a\right)R_1\left[1+\tan(S_1)\right]^{\frac{1}{2}} + \frac{R_1}{2}\left[1+\tan(S_1)\right]^{-\frac{1}{2}}\sec^2(S_1)S_{1t}\right.\\
&+ \left.\frac{be^x+a}{A(A-\dot{x})}\right]\partial x + F(t). &(4.42)
\end{aligned}$$

Equating coefficients by $\dot{x}^0$,

$$
\begin{aligned}
A\ R_1 & \quad [1+\tan(S_1)]^{\frac{1}{2}}\,(be^x+a) = \psi_t, && (4.43)\\
\Longrightarrow & \quad F(t) = AR_1\,[1+\tan(S_1)]^{\frac{1}{2}}\,(be^x+a), && (4.44)\\
+ & \quad \left(\int\left[(be^x+a)\,R_1\,[1+\tan(S_1)]^{\frac{1}{2}} + \frac{R_1}{2}\,[1+\tan(S_1)]^{-\frac{1}{2}}\sec^2(S_1)S_{1t}\right.\right. \\
+ & \quad \left.\left.\frac{be^x+a}{A(A-\dot{x})}\right]\partial x\right) + G. && (4.45)
\end{aligned}
$$

Thus, the first integral is given by

$$
I = \dot{x}R_1\,[1+\tan(S_1)]^{\frac{1}{2}} + \frac{1}{A}\,[\dot{x}+A\ln(\dot{x}-A)] + AR_1(be^x+a)\int[1+\tan(S_1)]^{\frac{1}{2}}\,\partial t + G. \tag{4.46}
$$

where G is the constant of integration.

Case 1.2: To determine the first integral we apply the equation (3.9) to (3.6) to obtain the relation,

$$
R_2\,[1+\tan(S_2)]^{\frac{1}{2}}\,[\ddot{x}+(be^x+a)(A-\dot{x})] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.47}
$$

Equating coefficients of $\ddot{x}$,

$$
\frac{\partial I}{\partial \dot{x}} = R_2\,[1+\tan(S_2)]^{\frac{1}{2}}\,. \tag{4.48}
$$

Integrating equation (4.48) results in,

$$
I = \dot{x}R_2\,[1+\tan(S_2)]^{\frac{1}{2}} + \psi(x,t). \tag{4.49}
$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$
\begin{aligned}
\frac{\partial I}{\partial x} &= \dot{x}\left[R_{2x}\,[1+\tan(S_2)]^{\frac{1}{2}} + \frac{R_2}{2}\,[1+\tan(S_2)]^{-\frac{1}{2}}\sec^2(S_2)S_{2x}\right] + \psi_x && (4.50)\\
\frac{\partial I}{\partial t} &= \frac{\dot{x}R_2}{2}\,[1+\tan(S_2)]^{-\frac{1}{2}}\sec^2(S_2)S_{2t} + \psi_t. && (4.51)
\end{aligned}
$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we separate by powers of $\dot{x}$,

$$\begin{aligned}
\dot{x}^2 \quad &: \quad \left[R_{2x}\left[1+\tan(S_2)\right]^{\frac{1}{2}}+\frac{R_2}{2}\left[1+\tan(S_2)\right]^{-\frac{1}{2}}\sec^2(S_2)S_{2x}\right]=0, && (4.52)\\
\dot{x} \quad &: \quad -\left(be^x+a\right)R_2\left[1+\tan(S_2)\right]^{\frac{1}{2}}=\frac{R_2}{2}\left[1+\tan(S_2)\right]^{-\frac{1}{2}}\sec^2(S_2)S_{2t}\\
&\qquad\qquad +\psi_x. && (4.53)
\end{aligned}$$

Integrating with respect to x we obtain,

$$\begin{aligned}
\psi &= -\int\left[\left(be^x+a\right)R_2\left[1+\tan(S_2)\right]^{\frac{1}{2}}+\frac{R_2}{2}\left[1+\tan(S_2)\right]^{-\frac{1}{2}}\sec^2(S_2)S_{2t}\right]\partial x\\
&+ F(t). && (4.54)
\end{aligned}$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned}
AR_2 \quad & \left[1+\tan(S_2)\right]^{\frac{1}{2}}\left(be^x+a\right)=\psi_t,\\
\Longrightarrow \quad & F(t)=AR_2\left[1+\tan(S_2)\right]^{\frac{1}{2}}\left(be^x+a\right)+\left(\int\left[\left(be^x+a\right)R_2\left[1+\tan(S_2)\right]^{\frac{1}{2}}\right.\right.\\
+ \quad & \left.\left.\frac{R_2}{2}\left[1+\tan(S_2)\right]^{-\frac{1}{2}}\sec^2(S_2)S_{2t}\right]\partial x\right)+G. && (4.55)
\end{aligned}$$

Thus, the first integral is given by

$$I=\dot{x}R_2\left[1+\tan(S_2)\right]^{\frac{1}{2}}+AR_2(be^x+a)\int\left[1+\tan(S_2)\right]^{\frac{1}{2}}\partial t+G. \qquad (4.56)$$

where G is the constant of integration.

Case 1.3: To determine the first integral we apply the equation (3.10) to (3.6) to obtain the relation,

$$R_3\left[1+\tan(S_3)\right]^{\frac{1}{2}}\left[\ddot{x}+(be^x+a)(A-\dot{x})\right]=\frac{\partial I}{\partial t}+\dot{x}\frac{\partial I}{\partial x}+\ddot{x}\frac{\partial I}{\partial \dot{x}}. \qquad (4.57)$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = R_3 \left[1 + \tan(S_3)\right]^{\frac{1}{2}}. \tag{4.58}$$

Integrating equation (4.58) results in,

$$I = \dot{x} R_3 \left[1 + \tan(S_3)\right]^{\frac{1}{2}} + \psi(x,t). \tag{4.59}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned} \frac{\partial I}{\partial x} &= \dot{x}\left[R_{3x}\left[1 + \tan(S_3)\right]^{\frac{1}{2}} + \frac{R_3}{2}\left[1 + \tan(S_3)\right]^{-\frac{1}{2}} \sec^2(S_3) S_{3x}\right] + \psi_x, & (4.60)\\ \frac{\partial I}{\partial t} &= \frac{\dot{x} R_3}{2}\left[1 + \tan(S_3)\right]^{-\frac{1}{2}} \sec^2(S_3) S_{3t} + \psi_t. & (4.61) \end{aligned}$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we separate by powers of $\dot{x}$,

$$\begin{aligned} \dot{x}^2 \quad &: \quad \left[R_{3x}\left[1 + \tan(S_3)\right]^{\frac{1}{2}} + \frac{R_3}{2}\left[1 + \tan(S_3)\right]^{-\frac{1}{2}} \sec^2(S_3) S_{3x}\right] = 0, & (4.62)\\ \dot{x} \quad &: \quad -\left(be^x + a\right) R_3 \left[1 + \tan(S_3)\right]^{\frac{1}{2}} = \frac{R_3}{2}\left[1 + \tan(S_3)\right]^{-\frac{1}{2}} \sec^2(S_3) S_{3t} \\ &\qquad + \psi_x. & (4.63) \end{aligned}$$

Integrating with respect to x we obtain,

$$\begin{aligned} \psi &= -\int \left[\left(be^x + a\right) R_3 \left[1 + \tan(S_3)\right]^{\frac{1}{2}} + \frac{R_3}{2}\left[1 + \tan(S_3)\right]^{-\frac{1}{2}} \sec^2(S_3) S_{3t}\right] \partial x \\ &+ F(t). & (4.64) \end{aligned}$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned} A\ R_3 \quad & \left[1 + \tan(S_3)\right]^{\frac{1}{2}} \left(be^x + a\right) = \psi_t, \\ \implies \quad & F(t) = AR\left[1 + \tan(S_3)\right]^{\frac{1}{2}} \left(be^x + a\right) + \left(\int \left[\left(be^x + a\right) R_3 \left[1 + \tan(S_3)\right]^{\frac{1}{2}}\right.\right. \\ + \quad & \left.\left.\frac{R_3}{2}\left[1 + \tan(S_3)\right]^{-\frac{1}{2}} \sec^2(S_3) S_{3t}\right] \partial x\right) + G. & (4.65) \end{aligned}$$

Thus, the first integral is given by

$$I = \dot{x}R_3\left[1+\tan(S_3)\right]^{\frac{1}{2}} + AR_3(be^{x}+a)\int\left[1+\tan(S_3)\right]^{\frac{1}{2}}\partial t + G. \tag{4.66}$$

where G is the constant of integration.

Case 2.1: To determine the first integral for this case, we apply equation (3.12) to (3.6) to obtain the relation,

$$\frac{\dot{x}}{A-\dot{x}}\Big[\ddot{x} + (be^{x}+a)(A-\dot{x})\Big] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.67}$$

By equating coefficients of $\ddot{x}$, we get

$$\frac{\partial I}{\partial \dot{x}} = \frac{\dot{x}}{A-\dot{x}} \tag{4.68}$$

Integrating equation (4.68) yields,

$$I = -\frac{\dot{x}}{A} - \ln(\dot{x}-A) + \psi(x,t). \tag{4.69}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\frac{\partial I}{\partial x} = \psi_x, \tag{4.70}$$

$$\frac{\partial I}{\partial t} = \psi_t. \tag{4.71}$$

Substituting the derivatives of $I(x,t)$, we have

$$\dot{x}(be^{at}+a) = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x}, \tag{4.72}$$

$$\dot{x}(be^{at}+a) = \psi_t + \dot{x}\psi_x, \tag{4.73}$$

and equating by powers of $\dot{x}$ we get,

$$\dot{x} \;:\; be^{x}+a = \psi_x, \tag{4.74}$$

$$\dot{x}^0 \;:\; 0 = \psi_t. \tag{4.75}$$

Integrating $\dot{x}^0$, yields

$$\begin{aligned} \psi &= C(x), & (4.76) \\ \psi_x &= \dot{C}(x), \\ \Longrightarrow \dot{C}(x) &= be^x + a, \\ \Longrightarrow C(x) &= be^x + ax + G. & (4.77) \end{aligned}$$

Therefore, our first integral becomes

$$I = -\frac{\dot{x}}{A} - \ln(\dot{x} - A) + be^x + ax + G. \tag{4.78}$$

where G is the constant of integration.

Case 2.2.1: To determine the first integral we apply equation (3.14) to (3.6) which results in,

$$\left[\frac{1}{A}\cos(S_1) - \frac{\dot{x}_1}{A(A - \dot{x}_1)}\right][\ddot{x} + (be^x + a)(A - \dot{x})] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.79}$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = \frac{1}{A}\cos(S_1) - \frac{\dot{x}_1}{A(A - \dot{x}_1)} \tag{4.80}$$

Integrating equation (4.80) yields,

$$I = \frac{\dot{x}}{A}\cos(S_1) - \frac{1}{A}[\dot{x} + A\ln(\dot{x} - A)] + \psi(x, t). \tag{4.81}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned} \frac{\partial I}{\partial x} &= -\frac{\dot{x}}{A}\sin(S_1)S_{1x} + \psi_x, & (4.82) \\ \frac{\partial I}{\partial t} &= -\frac{\dot{x}}{A}\sin(S_1)S_{1t} + \psi_t. & (4.83) \end{aligned}$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we separate by powers of $\dot{x}$,

$$\begin{aligned}
\dot{x}^2 \quad &: \quad \frac{(be^x+a)}{A(A-\dot{x}_1)} = -\frac{1}{A}\sin(S)S_{1x}, && (4.84)\\
\dot{x} \quad &: \quad -\frac{(be^x+a)}{A}\cos(S_1) - \frac{(be^x+a)}{(A-\dot{x})} = -\frac{1}{A}\sin(S_1)S_{1t} + \psi_x. && (4.85)
\end{aligned}$$

Integrating with respect to x yields,

$$\begin{aligned}
\psi &= \int\left[-(be^x+a)\left(\frac{\cos(S_1)}{A} + \frac{1}{A-\dot{x}}\right) + \frac{1}{A}\sin(S_1)S_{1t}\right]\partial x + F(t), && (4.86)\\
\psi_t &= \frac{\partial}{\partial t}\left(\int\left[-(be^x+a)\left(\frac{\cos(S_1)}{A} + \frac{1}{A-\dot{x}}\right) + \frac{1}{A}\sin(S_1)S_{1t}\right]\partial x\right) + \dot{F} && (4.87)
\end{aligned}$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned}
(be^x+a)\cos(S_1) &= \psi_t, && (4.88)\\
\implies F(t) &= -\int\left[-(be^x+a)\left(\frac{\cos(S_1)}{A} + \frac{1}{A-\dot{x}}\right) + \frac{1}{A}\sin(S_1)S_{1t}\right]\partial x \\
&+ (be^x+a)\int\cos(S_1)\partial t + G. && (4.89)
\end{aligned}$$

Thus, the first integral is given by

$$I = \frac{\dot{x}}{A}\cos(S_1) + \frac{1}{A}\left[\dot{x} + A\ln(\dot{x}-A)\right] + (be^x+a)\int\cos(S_1)\partial t + G. \tag{4.90}$$

where G is the constant of integration.

Case 2.2.2: To acquire the first integral for this case, we apply equation (3.15) to (3.6) which yields,

$$R_2\cos(S_2)\left[\ddot{x} + (be^x+a)(A-\dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.91}$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = R_2 \cos(S_2). \tag{4.92}$$

Integrating equation (4.92) results in,

$$I = \dot{x} R_2 \cos(S_2) + \psi(x,t). \tag{4.93}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\frac{\partial I}{\partial x} = -\dot{x} R_2 \sin(S_2) S_{2x} + \psi_x, \tag{4.94}$$

$$\frac{\partial I}{\partial t} = \dot{x}\left[\dot{R}_2 \cos(S_2) - R_2 \sin(S_2) S_{2t}\right] + \psi_t. \tag{4.95}$$

Substituting the derivatives of $I(x,t)$, we separate by powers of $\dot{x}$,

$$\dot{x}^2 \quad : \quad -R_2 \sin(S_2) S_{2x} = 0, \tag{4.96}$$

$$\dot{x} \quad : \quad -(be^x + a) R_2 \cos(S_2) = \dot{R}_2 \cos(S_2) - R_2 \sin(S_2) S_{2t} + \psi_x. \tag{4.97}$$

Integrating with respect to x,

$$\begin{aligned} \psi &= -\int \left[(be^x + a) R_2 \cos(S_2) + \left(\dot{R}_2 \cos(S_2) - R_2 \sin(S_2) S_{2t}\right)\right] \partial x \\ &+ F(t). \end{aligned} \tag{4.98}$$

Equating coefficients by $\dot{x}^0$,

$$A(be^x + a) R_2 \cos(S_2) = \psi_t, \tag{4.99}$$

$$\begin{aligned} \Longrightarrow F(t) &= \int \left[(be^x + a) R_2 \cos(S_2) + \left(\dot{R}_2 \cos(S_2) - R_2 \sin(S_2) S_{2t}\right)\right] \partial x \\ &+ A(be^x + a) \int R_2 \cos(S_2) \partial t + G. \end{aligned} \tag{4.100}$$

Thus, the first integral is given by

$$I = \dot{x} R_2 \cos(S_2) + A(be^x + a) \int R_2 \cos(S_2) \partial t + G. \tag{4.101}$$

where G is the constant of integration.

Case 2.2.3 : In order to obtain the first integral, we apply equation (3.16) to (3.6) which gives us,

$$\frac{1}{A}\cos(S_3)\left[\ddot{x}+(be^x+a)(A-\dot{x})\right]=\frac{\partial I}{\partial t}+\dot{x}\frac{\partial I}{\partial x}+\ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.102}$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}}=\frac{1}{A}\cos(S_3). \tag{4.103}$$

$$I=\frac{\dot{x}}{A}\cos(S_3)+\psi(x,t). \tag{4.104}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\frac{\partial I}{\partial x} = -\frac{\dot{x}}{A}\sin(S_3)S_{3x}+\psi_x, \tag{4.105}$$

$$\frac{\partial I}{\partial t} = -\frac{\dot{x}}{A}\sin(S_3)S_{3t}+\psi_t. \tag{4.106}$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we separate by powers of $\dot{x}$,

$$\dot{x}^2 \quad : \quad -\frac{1}{A}\sin(S_3)S_{3x}=0, \tag{4.107}$$

$$\dot{x} \quad : \quad -\frac{(be^x+a)}{A}\cos(S_3)=-\frac{1}{A}\sin(S_3)S_{3t}+\psi_x. \tag{4.108}$$

Integrating with respect to x,

$$\psi = \frac{1}{A}\int\left[\frac{1}{A}\sin(S_3)S_{3t}-\frac{(be^x+a)}{A}\cos(S_3)\right]\partial x+F(t), \tag{4.109}$$

$$\psi_t = \frac{\partial}{\partial t}\left(\frac{1}{A}\int\left[\frac{1}{A}\sin(S_3)S_{3t}-\frac{(be^x+a)}{A}\cos(S_3)\right]\partial x\right)+\dot{F} \tag{4.110}$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned}
(be^x + a)\cos(S_3) &= \psi_t, && (4.111)\\
\implies F(t) &= -\frac{1}{A}\int\left[\frac{1}{A}\sin(S_3)S_{3t} - \frac{(be^x+a)}{A}\cos(S_3)\right]\partial x \\
&+ (be^x+a)\int \cos(S_3)\partial t + G. && (4.112)
\end{aligned}$$

Thus, the first integral is given by

$$I = \frac{\dot{x}}{A}\cos(S_3) + (be^x + a)\int \cos(S_3)\partial t + G. \quad (4.113)$$

where G is the constant of integration.

Case 2.2.4: To determine the first integral we apply equation (3.17) to (3.6) which yields,

$$\frac{e^A}{A}\cos(S_4)\left[\ddot{x} + (be^x + a)(A - \dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \quad (4.114)$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = \frac{e^A}{A}\cos(S_4). \quad (4.115)$$

Integrating equation (4.115) results in,

$$I = \frac{\dot{x}e^A}{A}\cos(S_4) + \psi(x,t). \quad (4.116)$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned}
\frac{\partial I}{\partial x} &= -\frac{\dot{x}e^{A+x}}{A}\sin(S_4)S_{4x} + \psi_x, && (4.117)\\
\frac{\partial I}{\partial t} &= -\frac{\dot{x}e^A}{A}\sin(S_4)S_{4t} + \psi_t. && (4.118)
\end{aligned}$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we separate by powers of $\dot{x}$,

$$\begin{aligned}
\dot{x}^2 &: \quad -\frac{e^{A+x}}{A}\sin(S_4)S_{4x} = 0, && (4.119)\\
\dot{x} &: \quad -\frac{(be^x+a)e^A}{A}\cos(S_4) = -\frac{e^A}{A}\sin(S_4)S_{4t} + \psi_x. && (4.120)
\end{aligned}$$

Integrating with respect to x,

$$\begin{aligned}
\psi &= \frac{e^A}{A}\int\left[\frac{e^A}{A}\sin(S_4)S_{4t} - \frac{(be^x+a)e^A}{A}\cos(S_4)\right]\partial x + F(t), && (4.121)\\
\psi_t &= \frac{\partial}{\partial t}\left(\frac{e^A}{A}\int\left[\frac{e^A}{A}\sin(S_4)S_{4t} - \frac{(be^x+a)e^A}{A}\cos(S_4)\right]\partial x\right) + \dot{F}. && (4.122)
\end{aligned}$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned}
e^A(be^x+a)\cos(S_4) &= \psi_t, && (4.123)\\
\Longrightarrow F(t) &= -\frac{e^A}{A}\int\left[\frac{e^A}{A}\sin(S_4)S_{4t} - \frac{(be^x+a)e^A}{A}\cos(S_4)\right]\partial x \\
&+ e^A(be^x+a)\int\cos(S_4)\partial t + G. && (4.124)
\end{aligned}$$

Thus, the first integral is given by

$$I = \frac{\dot{x}e^A}{A}\cos(S_4) + e^A(be^x+a)\int\cos(S_4)\partial t + G. \quad (4.125)$$

where G is the constant of integration.

Case 3.1: To obtain the first integral, we apply equation (3.19) to (3.6) which yields,

$$\frac{\dot{x}}{A-\dot{x}}\left[\ddot{x} + (be^x+a)(A-\dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \quad (4.126)$$

By equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = \frac{\dot{x}}{A-\dot{x}} \quad (4.127)$$

Integrating equation (4.127) results in,

$$I = -\frac{\dot{x}}{A} - \ln(\dot{x} - A) + \psi(x,t). \tag{4.128}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned} \frac{\partial I}{\partial x} &= \psi_x, &(4.129)\\ \frac{\partial I}{\partial t} &= \psi_t. &(4.130) \end{aligned}$$

Substituting the derivatives of $I(x,t)$, we have

$$\begin{aligned} \dot{x}(be^{at} + a) &= \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x}, &(4.131)\\ \dot{x}(be^{at} + a) &= \psi_t + \dot{x}\psi_x. &(4.132) \end{aligned}$$

So by equating coefficients we get,

$$\begin{aligned} \dot{x} &: \; be^{x} + a = \psi_x, &(4.133)\\ \dot{x}^0 &: \; 0 = \psi_t. &(4.134) \end{aligned}$$

Integrating $\dot{x}^0$, yields

$$\begin{aligned} \psi &= C(x), &(4.135)\\ \psi_x &= \dot{C}(x), &\\ \implies \dot{C}(x) &= be^{x} + a, &\\ \implies C(x) &= be^{x} + ax + G. &(4.136) \end{aligned}$$

Therefore our first integral becomes

$$I = -\frac{\dot{x}}{A} - \ln(\dot{x} - A) + be^{x} + ax + G. \tag{4.137}$$

where G is the constant of integration.

Case 3.2.1 : To determine the first integral we apply equation (3.21) to (3.6) which results in,

$$\left[R_1\cos(S_1) - \frac{\dot{x}_1}{A(A-\dot{x}_1)}\right][\ddot{x} + (be^x + a)(A-\dot{x})] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.138}$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = R_1\cos(S_1) - \frac{\dot{x}_1}{A(A-\dot{x}_1)}. \tag{4.139}$$

Integrating equation (4.139) yields,

$$I = \dot{x}R_1\cos(S_1) - \frac{1}{A}\left[\dot{x} + A\ln(\dot{x} - A)\right] + \psi(x,t). \tag{4.140}$$

Then differentiating and equating like terms in the corresponding coefficients leads to

$$\frac{\partial I}{\partial x} = \psi_x, \tag{4.141}$$

$$\frac{\partial I}{\partial t} = \dot{x}\left[\dot{R}_1\cos(S_1) - R_1\sin(S_1)S_{1t}\right] + \psi_t. \tag{4.142}$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we can then separate by powers of $\dot{x}$,

$$\dot{x}^2 \quad : \quad \frac{(be^x + a)}{A(A-\dot{x})} = 0 \tag{4.143}$$

$$\dot{x} \quad : \quad -R_1(be^x + a)\cos(S_1) - \frac{(be^x + a)}{(A-\dot{x})} = \dot{R}_1\cos(S_1) - R_1\sin(S_1)S_{1t} + \psi_x. \tag{4.144}$$

Integrating with respect to x we obtain,

$$\psi = -\int\left[R_1(be^x + a)\cos(S_1) + \frac{(be^x + a)}{(A-\dot{x})} + \left(\dot{R}_1\cos(S_1) - R_1\sin(S_1)S_{1t}\right)\right]\partial x + F(t). \tag{4.145}$$

Equating coefficients by $\dot{x}^0$,

$$
\begin{aligned}
A \quad & (be^x + a)R_1\cos(S_1) = \psi_t, \\
\implies \quad & F(t) = \int \left[R_1(be^x + a)\cos(S_1) + \frac{(be^x + a)}{(A - \dot{x})} + \left(\dot{R}_1\cos(S_1) - R_1\sin(S_1)S_{1t} \right) \right] \partial x \\
+ \quad & A(be^x + a) \int R\cos(S)\partial t + G. \qquad (4.146)
\end{aligned}
$$

Therefore, the first integral is given by

$$I = \dot{x}R_1\cos(S_1) - \frac{1}{A}\left[\dot{x} + A\ln(\dot{x} - A)\right] + A(be^x + a)\int R_1\cos(S_1)\partial t + G. \qquad (4.147)$$

where G is the constant of integration.

Case 3.2.2 : To determine the first integral of this case, we apply equation (3.22) to (3.6)

$$R_2\cos(S_2)\left[\ddot{x} + (be^x + a)(A - \dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \qquad (4.148)$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = R_2\cos(S_2). \qquad (4.149)$$

Integrating equation (4.149) results in,

$$I = \dot{x}R_2\cos(S_2) + \psi(x,t). \qquad (4.150)$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$
\begin{aligned}
\frac{\partial I}{\partial x} &= \psi_x, && (4.151) \\
\frac{\partial I}{\partial t} &= \dot{x}\left[\dot{R}_2\cos(S_2) - R_2\sin(S_2)S_{2t}\right] + \psi_t. && (4.152)
\end{aligned}
$$

Substituting the derivatives of $I(x,t)$, we then separate by powers of $\dot{x}$,

$$\dot{x} \quad : \quad -R_2(be^x + a)\cos(S_2) = \dot{R}_2\cos(S_2) - R_2\sin(S_2)S_{2t} + \psi_x. \qquad (4.153)$$

Integrating with respect to x,

$$\psi = -R_2(be^x + ax)\cos(S_2) - x\left[\dot{R}_2\cos(S_2) - R_2\sin(S_2)S_{2t}\right] + F(t). \quad (4.154)$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned} e^A(be^x + a)\cos(S_2) &= \psi_t, && (4.155)\\ \implies F(t) &= R(be^x + ax)\cos(S_2) + x\left[\dot{R}_2\cos(S_2) - R_2\sin(S_2)S_{2t}\right] \\ &+ A(be^x + a)\int R_2\cos(S_2)\partial t + G. && (4.156)\end{aligned}$$

Thus, the first integral is given by

$$I = \dot{x}R_2\cos(S_2) + A(be^x + a)\int R_2\cos(S_2)\partial t + G. \quad (4.157)$$

where G is the constant of integration.

Case 3.2.3 : To determine the first integral, we apply equation (3.23) to (3.6) which results in,

$$R_3\cos(S_3)\left[\ddot{x} + (be^x + a)(A - \dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \quad (4.158)$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = R_3\cos(S_3). \quad (4.159)$$

Integrating equation (4.159) yields,

$$I = \dot{x}R_3\cos(S_3) + \psi(x,t). \quad (4.160)$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned} \frac{\partial I}{\partial x} &= \psi_x, && (4.161)\\ \frac{\partial I}{\partial t} &= \dot{x}\left[\dot{R}_3\cos(S_3) - R_3\sin(S_3)S_{3t}\right] + \psi_t. && (4.162)\end{aligned}$$

Substituting the derivatives of $I(x,t)$, we can separate by powers of $\dot{x}$,

$$\dot{x} \quad : \quad -R(be^x + a)\cos(S_3) = \dot{R}_3\cos(S_3) - R_3\sin(S_3)S_{3t} + \psi_x. \tag{4.163}$$

Integrating with respect to x,

$$\psi \;=\; -R_3(be^x + ax)\cos(S_3) - x\left[\dot{R}_3\cos(S_3) - R_3\sin(S_3)S_{3t}\right] + F(t). \tag{4.164}$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned} e^A(be^x + a)\cos(S_3) &= \psi_t, & (4.165)\\ \implies F(t) &= R_3(be^x + ax)\cos(S_3) + x\left[\dot{R}_3\cos(S_3) - R_3\sin(S_3)S_{3t}\right] \\ &+ A(be^x + a)\int R_3\cos(S_3)\partial t + G. & (4.166)\end{aligned}$$

Thus, the first integral is given by

$$I = \dot{x}R_3\cos(S_3) + A(be^x + a)\int R_3\cos(S_3)\partial t + G. \tag{4.167}$$

where G is the constant of integration.

Case 3.2.4 : To determine the first integral, we can apply (3.24) to (3.6) which yields,

$$R_4\cos(S_4)\left[\ddot{x} + (be^x + a)(A - \dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.168}$$

Equating coefficients of $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = R_4\cos(S_4). \tag{4.169}$$

Integrating equation (4.169) results in,

$$I = \dot{x}R_4\cos(S_4) + \psi(x,t). \tag{4.170}$$

The differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned}\frac{\partial I}{\partial x} &= \psi_x, && (4.171)\\ \frac{\partial I}{\partial t} &= \dot{x}\left[\dot{R}_4\cos(S_4) - R_4\sin(S_4)S_{4t}\right] + \psi_t. && (4.172)\end{aligned}$$

Substituting the derivatives of $I(x,t)$ into the next part of the equation, we can separate by powers of $\dot{x}$,

$$\dot{x} \quad : \quad -R_4(be^x + a)\cos(S_4) = \dot{R}_4\cos(S_4) - R_4\sin(S_4)S_{4t} + \psi_x. \qquad (4.173)$$

Integrating with respect to x,

$$\psi = -R_4(be^x + ax)\cos(S_4) - x\left[\dot{R}_4\cos(S_4) - R_4\sin(S_4)S_{4t}\right] + F(t). \qquad (4.174)$$

Equating coefficients by $\dot{x}^0$,

$$\begin{aligned} e^A(be^x + a)\cos(S_4) &= \psi_t, \\ \implies F(t) &= R_4(be^x + ax)\cos(S_4) + x\left[\dot{R}_4\cos(S_4) - R_4\sin(S_4)S_{4t}\right] \\ &+ A(be^x + a)\int R_4\cos(S_4)\partial t + G. && (4.175)\end{aligned}$$

Thus, the first integral is given by

$$I = \dot{x}R_4\cos(S_4) + A(be^x + a)\int R_4\cos(S_4)\partial t + G. \qquad (4.176)$$

where G is the constant of integration.

Case 3.3.1 : To determine the first integral for this case, we apply equation (3.26) to (3.6) which results in,

$$[\ddot{x} + (be^x + a)(A - \dot{x})] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \qquad (4.177)$$

By equating coefficients $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = 1. \tag{4.178}$$

Integrating equation (4.178) yields,

$$I = \dot{x} + \psi(x,t). \tag{4.179}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\frac{\partial I}{\partial x} = \psi_x, \tag{4.180}$$

$$\frac{\partial I}{\partial t} = \psi_t. \tag{4.181}$$

Substituting the derivatives of $I(x,t)$, we can then separate by powers of $\dot{x}$,

$$\begin{aligned}
\dot{x} : -(be^x + a) &= \psi_x, && (4.182\text{a})\\
\implies \psi &= -(be^x + ax) + C(t), && (4.182\text{b})\\
\psi_t &= \dot{C}(t).
\end{aligned}$$

$$\begin{aligned}
\dot{x}^0 : A(be^x + a) &= \psi_t, && (4.182\text{c})\\
A(be^x + a) &= \dot{C}(t),\\
\implies C(t) &= At(be^x + a) + G. && (4.182\text{d})
\end{aligned}$$

Therefore, our first integral becomes

$$I = \dot{x} - (be^x + a)(At - 1) + G. \tag{4.183}$$

where G is the constant of integration.

Case 3.3.2 : To determine the first integral, we apply equation (3.27) to (3.6) which results in,

$$e^{-at}\left[\ddot{x} + (be^x + a)(A - \dot{x})\right] = \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} + \ddot{x}\frac{\partial I}{\partial \dot{x}}. \tag{4.184}$$

By equating coefficients $\ddot{x}$,

$$\frac{\partial I}{\partial \dot{x}} = e^{-at}. \tag{4.185}$$

Integrating equation (4.185) yields,

$$I = e^{-at}\dot{x} + \psi(x,t). \tag{4.186}$$

Then differentiating and equating like terms in the corresponding coefficients leads to,

$$\begin{aligned} \frac{\partial I}{\partial x} &= \psi_x, & (4.187)\\ \frac{\partial I}{\partial t} &= -ae^{-at}\dot{x} + \psi_t. & (4.188) \end{aligned}$$

Substituting the derivatives of $I(x,t)$, we have

$$\begin{aligned} e^{-at}(be^x + a)(A - \dot{x}) &= \frac{\partial I}{\partial t} + \dot{x}\frac{\partial I}{\partial x} & (4.189)\\ e^{-at}(be^x + a)(A - \dot{x}) &= -ae^{-at}\dot{x} + \psi_t + \dot{x}\psi_x. & (4.190) \end{aligned}$$

We can now separate by powers of $\dot{x}$,

$$\begin{aligned} \dot{x} \ : \ -e^{-at}(be^x + a) &= -ae^{-at} + \psi_x, & (4.191a)\\ \psi_x &= -e^{-at}be^x, & \\ \psi &= -be^{x-at} + C(t), & (4.191b)\\ \psi_t &= abe^{x-at} + \dot{C}(t). & \end{aligned}$$

$$\begin{aligned} \dot{x}^0 \ : \ e^{-at}(be^x + a)A &= \psi_t, & (4.191c)\\ e^{-at}(be^x + a)A &= abe^{x-at} + \dot{C}(t), & \\ \dot{C}(t) &= e^{-at}(be^x - abe^x), & \\ \implies C(t) &= e^{-at}be^x - \frac{e^{-at}}{a}\left(be^x + a\right)A + G. & (4.191d) \end{aligned}$$

Therefore, our first integral becomes

$$I = e^{-at}\dot{x} - \frac{e^{-at}}{a}\left(be^{x} + a\right)A + G. \tag{4.192}$$

where G is the constant of integration.

4.3 Conclusion

In this chapter, we investigated one biological model via the Method of Multipliers/Characteristics. From our findings, we can conclude that each case can give a more precise exact solution due to the initial conditions for the system as well the multiplier that is used. This is a very effective method of finding exact solutions without the use of a Lagrangian, but the resulting first integral, although solvable, can be very highly nonlinear in some cases which could make it more challenging in finding an exact solution for the system.

Chapter 5

Conclusion

In this dissertation, two methods that lead to first integrals can show us how unique a solution can be. This is because of the initial conditions that are considered for each approach. Each method allows us to find closed-form solutions of a differential equation that arises from the system. Both methods are constructive approaches to First Integrals and Conservation Laws. As far as the authors knowledge this is the first time such study has been undertaken in a Biological framework , therefore producing new results in literature.

We firstly looked at Noether's theorem which requires a Lagrangian of the system. Using Noether symmetries, we are able to find as many first integrals as there are Noether symmetries. Even though we would find closed-form solutions to the system, the solutions may be very vague or may not adhere to some initial conditions or environmental factors that impact the system physically. Note it is possible to find the structure of a general first integral even if we do not attain the Noether symmetries of the system.

In the next method we applied the Characteristic approach associated with the uniqueness of the multiplier. This is a very simplistic and elegant method of solving differential equations via a reduction of a first integral, without the use of a Lagrangian. We were able to find multipliers for general cases as well as for environmental factors. The approach lead to obtaining some linear and nonlinear first integrals. With more introspection these solutions found could possibly give us a better understanding as to how the system would behave under varied conditions.

There are many methods that can be used or tested on a Biological system, but the characteristic method can be considered due its effectiveness in handling more than environmental factors, thus giving us a better outcome regarding the behaviour of the system.

Chapter 6

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