CONFORMAL MOTIONS
IN
BIANCHI I SPACETIME

by

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Abstract

In this thesis we study the physical properties of the manifold in general relativity that admits a conformal motion. The results obtained are general as the metric tensor field is not specified. We obtain the Lie derivative along a conformal Killing vector of the kinematical and dynamical quantities for the general energy–momentum tensor of neutral matter. Equations obtained previously are regained as special cases from our results. We also find the Lie derivative of the energy–momentum tensor for the electromagnetic field. In particular we comprehensively study conformal symmetries in the Bianchi I spacetime. The conformal Killing vector equation is integrated to obtain the general conformal Killing vector and the conformal factor subject to integrability conditions. These conditions place restrictions on the metric functions. A particular solution is exhibited which demonstrates that these conditions have a nonempty solution set. The solution obtained is a generalisation of the results of Moodley (1991) who considered locally rotationally symmetric spacetimes. The Killing vectors are regained as special cases of the conformal solution. There do not exist any proper special conformal Killing vectors in the Bianchi I spacetime. The homothetic vector is found for a nonvanishing constant conformal factor. We establish that the vacuum Kasner solution is the only Bianchi I spacetime that admits a homothetic vector. Furthermore we isolate a class of vectors from the solution which causes the Bianchi I model to degenerate into a spacetime of higher symmetry.

To my family and friends

for being pillars of support and encouragement.

Preface

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period January 1992 to December 1992. This thesis was completed under the excellent supervision of Dr. S. D. Maharaj.

This study represents original work by the author. It has not been submitted in any form to another University nor has it been published previously. Where use was made of the work of others it has been acknowledged duly in the text.

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Contents

1 Introduction

General relativity is a theory of gravity. It has wide applications in astrophysics and cosmology. In general relativity the gravitational field of a body is contained in the curvature of spacetime. The Riemann tensor describes the curvature of the spacetime manifold. Spacetime is taken to be a 4–dimensional, pseudo–Riemannian manifold possessing a symmetric, non–degenerate metric tensor field. The gravitational field is described by the Einstein tensor which is related to the curvature of spacetime via the Ricci tensor and the Ricci scalar. The matter content is represented by the symmetric energy–momentum tensor. The spacetime geometry is related to the matter content via the Einstein field equations. The field equations are a nonlinear coupled system of partial differential equations which satisfy conservation laws called the Bianchi identities.

There exist many solutions to the Einstein field equations in the literature. Exact solutions to the field equations are important because they facilitate the investigation of the physical properties of specific models. Although a large number of solutions are known today, many of these are not physical. A comprehensive list of exact solutions to the Einstein field equations is given by Kramer *et al* (1980). Exact solutions may be found in an *ad hoc* fashion by specifying one or more of the geometric and matter variables and solving the field equations to find the remaining variables. An alternative method of generating solutions to the field equations is to suppose that the gravitational field possesses a symmetry, e.g. a Killing vector. Such an assumption simplifies the field equations and often makes them easier to integrate. Most of the classical solutions, e.g. the Schwarzschild and the Robertson–Walker models, are spacetimes of high symmetry. Models with less symmetry are more difficult to investigate.

In recent attempts to obtain exact solutions a conformal symmetry requirement is imposed on the spacetime manifold, i.e. the manifold is invariant under the action of a group of conformal motions. A number of exact solutions have been found in various models with the assumption that the spacetime admits a conformal Killing vector. Perfect fluid spacetimes and anisotropic fluid spacetimes with a conformal Killing vector have been investigated by Herrera and Ponce de León (1985a, b, c), Herrera *et al* (1984), Maartens and Maharaj (1990), Maartens *et al* (1986), Mason and Maartens (1987), Mason and Tsamparlis (1985) and Saridakis and Tsamparlis (1991). Many of these solutions have been concerned with astrophysical applications. Spherically symmetric cosmological models, with vanishing shear, admitting a conformal Killing vector have been studied by Dyer *et al* (1987) and Maharaj *et al* (1991). A number of these authors have also extensively investigated the kinematical and dynamical properties of the solutions to the field equations with a conformal symmetry. In particular various spacetimes admitting an inheriting conformal Killing vector, a special conformal symmetry, have been analysed by Coley (1991) and Coley and Tupper (1989, 1990a, b, c, d). The analysis of conformal motions in general relativity is important. It is clear that explicitly finding conformal Killing vectors assists in the analysis of exact solutions. Conformal Killing vectors have been found in certain spacetimes: Minkowski spacetime (Choquet–Bruhat *et al* 1977), Robertson–Walker spacetimes (Maartens and Maharaj 1986), *pp*–wave spacetimes (Maartens and Maharaj 1991) and locally rotationally symmetric spacetimes (Moodley 1991).

In this thesis we consider the conformal geometry of the Bianchi I spacetime. The Bianchi spacetimes are spatially homogeneous and anisotropic and are often used in the study of anisotropic cosmological models. The Bianchi I spacetime is a generalisation of the corresponding locally rotationally symmetric spacetime, studied by Moodley (1991). The Bianchi models have only three Killing vectors in contrast to the Robertson–Walker models which have six. Consequently the integration of the conformal Killing vector equation is more complicated in the Bianchi I spacetime. We integrate the conformal Killing vector equation for the Bianchi I metric to obtain the conformal Killing vector and the conformal factor in general.

In chapter 2 we briefly discuss concepts in differential geometry necessary for this thesis. We begin with a description of manifolds. Coordinate transformations, vector fields and tensor fields are defined on the manifold. We introduce differentiation on manifolds: the covariant derivative and the Lie derivative are defined. The curvature of the spacetime manifold is described by the curvature tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor. We consider the energy–momentum tensor for neutral matter and charged matter. We then motivate the Einstein field equations with nonvanishing cosmological constant. The Lie bracket, Lie algebras and Lie groups are introduced briefly. We present the conformal Killing vector equation and the special cases of Killing, homothetic, special and nonspecial conformal Killing vectors are listed.

In chapter 3 we investigate the effect of the existence of a conformal symmetry on the Einstein field equations in general. We find the Lie derivative of the kinematical and dynamical quantities. The general energy–momentum tensor utilised is applicable to neutral matter. Equations of other authors are regained as special cases of our results. We also find the Lie derivative of the energy–momentum tensor of the electromagnetic field. We briefly review results obtained on conformal symmetries and consider other types of symmetries on the spacetime manifold.

In chapter 4 we discuss the spacetime geometry of the Bianchi I model. The Einstein field equations for a perfect fluid energy–momentum tensor are derived. We then integrate the conformal Killing vector equation for the Bianchi I metric to obtain the conformal Killing vector and the conformal factor, subject to integrability conditions. The integrability conditions place restrictions on the metric functions. We provide all the details of the integration process. In particular we consider a simplified class of conformal Killing vectors by setting certain functions of integration to zero; this ensures that the integrability conditions have a nonempty solution set. The conformal Killing vector obtained generalises results found previously on locally rotationally symmetric spacetimes. The special cases of Killing vectors, homothetic vectors and special conformal Killing vectors are considered.

The results obtained in this thesis are summarised in the conclusion. Some avenues for future work are pointed out. This work is a generalisation of results obtained by other authors. We believe that the results obtained in this thesis are original. We have not found any published work in the literature on the general solution of the conformal Killing vector equation in Bianchi I spacetimes.

2 Manifolds and Tensor Fields

2.1 Introduction

In this chapter we briefly review and discuss aspects of differential geometry, manifolds and tensor fields. We concentrate only on those aspects that are necessary for this thesis. We begin by heuristically introducing the 4–dimensional spacetime structure of a manifold which admits a Lorentzian metric in the neighbourhood of every point. The additional structure of an affine connection is also necessary. Spacetime is a 4–dimensional differentiable manifold endowed with a symmetric metric tensor field which describes the gravitational field. In addition to the manifold we also consider in §2.2 general coordinate transformations, tensor products and tensor fields as natural geometric objects on the manifold. For more comprehensive expositions on manifolds and related concepts the reader is referred to Bishop and Goldberg (1968), Choquet–Bruhat *et al* (1977), Hawking and Ellis (1973) and Wald (1984). The covariant derivative plays a significant role when considering the curvature of spacetime in general relativity. The Lie derivative is important in the study of symmetries in general relativity. The Lie derivative provides a coordinate independent description of symmetries. We consider the covariant derivative, the Lie derivative and their properties in §2.3. The additional structure of the connection and the related Christoffel symbols are also introduced in §2.3. The curvature tensor and the

Einstein field equations are discussed in §2.4. The curvature tensor is derived and the various identities which it satisfies are listed. The Ricci tensor, the Ricci scalar, the Einstein tensor and the energy–momentum tensor are also defined. The Einstein field equations for neutral matter are briefly introduced. We also present Maxwell's equations and the electromagnetic tensor for charged matter. The Einstein field equations are adapted to accommodate charged matter. We discuss Lie algebras and Lie groups and the relations between them in §2.5. We require that spacetime be invariant under a conformal Killing vector. The existence of a conformal symmetry imposes restrictions on the metric tensor and often leads to a simplification of the Einstein field equations. The existence of a conformal symmetry and its effect on the Einstein field equations are pursued in later chapters. The general conformal Killing vector is defined in §2.5 and the special cases of Killing, homothetic, special and nonspecial conformal Killing vectors are listed.

2.2 Manifolds and Tensor Fields

We may consider a manifold as a Hausdorff space that can be continuously parametrised. The number of independent parameters generates the dimension of the manifold, and the parameters are the coordinates of the manifold. Locally a manifold has the structure of Euclidean space in that it may be covered by coordinate neighbourhoods. It is important to note that the global structure of the manifold, however, may be very different from that of Euclidean space. For the purposes of general relativity we require the mathematical structure of a 4–dimensional differentiable manifold. Points in the 4–dimensional manifold are labelled by the real coordinates x^0, x^1, x^2, x^3. Here $x^0 = ct$ is the timelike coordinate (we will take the speed of light

$c = 1$) and x^1, x^2, x^3 are the spacelike coordinates. The manifold has to support a differentiable structure so that differentiation of functions, involving changes of coordinates in overlapping coordinate neighbourhoods, is permissible. For a rigorous definition of a differentiable manifold the reader is referred to Bishop and Goldberg (1968), Hawking and Ellis (1973), Misner *et al* (1973) and Wald (1984). Here we will present only those aspects of manifolds necessary for this thesis.

Suppose that M is a set of points and let $\{O_\alpha\}$ be a collection of open subsets in M. The function $\psi_\alpha : O_\alpha \longrightarrow \Re^4$ is bijective and maps the open set O_α to an open region of $\Re^4$. The purpose of each map ψ_α is to attach coordinates to points in O_α of the manifold. The map ψ_α together with the open subset O_α comprises the pair (ψ_α, O_α) called a chart. If P is a point in O_α then we sometimes call (ψ_α, O_α) a coordinate system about P. Consider the set of charts $\{(O_\alpha, \psi_\alpha)\}_{\alpha \in I}$ where I is some index set. For a well–defined manifold we require that the following conditions apply. The set $\{O_\alpha\}$ covers M so that each point of M is contained in at least one O_α. We also require that for all O_α there exists an O_β such that $O_\alpha \cap O_\beta \neq \emptyset$. This ensures that in the intersecting region the composite functions $\psi_\alpha \circ \psi_\beta^{-1}$ and $\psi_\beta \circ \psi_\alpha^{-1}$ are differentiable functions from $\Re^4$ to $\Re^4$. The inverse maps ψ_α^{-1} and ψ_β^{-1} are defined as ψ_α and ψ_β are injective. Further it is necessary that the collection $\{(O_\alpha, \psi_\alpha)\}_{\alpha \in I}$ is maximal so that any other chart (O_α, ψ_α) is contained in this set. This prevents the definition of new manifolds by the mere addition or deletion of a chart. The set $\{(O_\alpha, \psi_\alpha)\}_{\alpha \in I}$ satisfying the above conditions is called an atlas. The set M together with its atlas comprises a 4–dimensional differentiable manifold.

Consider the charts (ψ_α, O_α) and (ψ_β, O_β) with intersecting coordinate neighbourhoods. The maps $\psi_\alpha : O_\alpha \longrightarrow \Re^4$ and $\psi_\beta : O_\beta \longrightarrow \Re^4$ generate the coordinate systems $x^{a'}$ and x^a, respectively. These coordinates are related by the composite functions $\psi_\alpha \circ \psi_\beta^{-1} : \Re^4 \longrightarrow \Re^4$ and $\psi_\beta \circ \psi_\alpha^{-1} : \Re^4 \longrightarrow \Re^4$ because in the overlap $O_\alpha \cap O_\beta \neq \emptyset$. These composite functions imply the functional relationships

$$x^{a'} = x^{a'}(x^0, x^1, x^2, x^3)$$

and the inverse relationships

$$x^a = x^a(x^{0'}, x^{1'}, x^{2'}, x^{3'})$$

The functions $x^{a'}$ and x^a given above are both differentiable and injective. The Jacobians of the matrices

$$X_b^{a'} = \frac{\partial x^{a'}}{\partial x^b} \qquad \text{and} \qquad X_{a'}^b = \frac{\partial x^b}{\partial x^{a'}}$$

are nonvanishing in the overlapping region $O_\alpha \cap O_\beta \neq \emptyset$. It is also possible to establish the converse result. Suppose that we are given a chart (O_α, ψ_α) and the system of equations $x^{a'} = x^{a'}(x^0, x^1, x^2, x^3)$ with $|X_b^{a'}| \neq 0$ for some point $P \in O_\alpha$ with coordinates x^a. Then we can establish, utilising the inverse–function theorem, the existence of a coordinate system (O_β, ψ_β) about P whose coordinates are related to those of the previous chart (O_α, ψ_α) by $x^a = x^a(x^{0'}, x^{1'}, x^{2'}, x^{3'})$. It is sufficient for our purposes to require that the differentiability class of the manifold M is at least C^2 to ensure that operations which depend on the continuity of partial derivatives are valid.

We define a regularly parametrised smooth curve on the manifold M by the continuous functions $x^a : u \longrightarrow M$, where u is a real parameter. A curve with a given parametrisation yields a tangent vector at a point $P \in M$, and conversely the tangent vector is tangent to some curve through P. Let T_P represent the set of vectors tangent to a curve at the point P in M. The set of tangent vectors T_P generates a vector space at P. The dual tangent space T_P^* at P is defined by the real–valued function $T_P^* : T_P \longrightarrow \Re$. The dual space T_P^* satisfies the vector space axioms. We can then construct spaces $(T_s^r)_P$ of type (r,s) tensors at P by taking repeated tensor products of T_P and T_P^*:

$$T_s^r \equiv \underbrace{T \otimes T \otimes \cdots \otimes T}_{r \text{ times}} \otimes \underbrace{T^* \otimes T^* \otimes \cdots \otimes T^*}_{s \text{ times}}$$

so that we have

$$T_s^r : \underbrace{T^* \times T^* \times \cdots \times T^*}_{r \text{ times}} \times \underbrace{T \times T \times \cdots \times T}_{s \text{ times}} \longrightarrow \Re$$

(Bishop and Goldberg 1968, Misner *et al* 1973, Schutz 1980). It is easily established that the space $(T_s^r)_P$ of multilinear functionals is also a vector space at P. A type (r,s) tensor field on M is an assignment of a member of $(T_s^r)_P$ to each point $P \in M$. It is convenient to represent the set of all type (r,s) tensor fields on M by T_s^r. The quantity $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ represents the components of a (r,s) tensor field $\mathbf{T}$ in T_s^r. Under a change of coordinates the components $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ transform according to the rule

$$T^{a_1' a_2' \ldots a_r'}{}_{b_1' b_2' \ldots b_s'} = X^{a_1'}_{c_1} X^{a_2'}_{c_2} \ldots X^{a_r'}_{c_r} X^{d_1}_{b_1'} X^{d_2}_{b_2'} \ldots X^{d_s}_{b_s'} T^{c_1 c_2 \ldots c_r}{}_{d_1 d_2 \ldots d_s} \tag{2.1}$$

in the manifold M.

In order to discuss metrical properties we need to endow M with a metric tensor field $\mathbf{g}$ of rank two. In the case of an indefinite metric tensor field the manifold M is called a pseudo–Riemannian manifold (Misner *et al* 1973, Stephani 1990). Spacetime M is a Hausdorff, oriented, smooth 4–dimensional manifold endowed with a symmetric, non–degenerate tensor field $\mathbf{g}$ of signature $(- + + +)$. By definition the tensor field $\mathbf{g}$ satisfies (2.1). The metric tensor $\mathbf{g}$ is fundamental to the invariant definition of the length of a curve in M which is given by the integral

$$s = \int_{u_1}^{u_2} |g_{ab}\dot{x}^a \dot{x}^b|^{\frac{1}{2}} du$$

where $\dot{x}^a = dx^a/du$. This definition reduces to the infinitesimal line element or fundamental metric form

$$ds^2 = g_{ab}dx^a dx^b \tag{2.2}$$

where we have dropped the modulus sign. The line element (2.2) gives a measure of the infinitesimal interval between neighbouring points x^a and $x^a + dx^a$ in the manifold. We can construct a coordinate system about any point P in the spacetime of general relativity such that in the neighbourhood of P we have

$$g_{ab} \approx \eta_{ab} + \frac{1}{2}\left(\frac{\partial^2 g_{ab}}{\partial x^c \partial x^d}\right)x^c x^d \tag{2.3}$$

The metric tensor η_{ab} is that of special relativity. At any point P in spacetime there exists a coordinate system in which the metric tensor takes the Lorentzian form

$$[\eta_{ab}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the case of special relativity there exist global coordinate systems for which the metric tensor takes the above form. Such coordinate systems are called inertial or Cartesian. However in the 4–dimensional manifold of general relativity Cartesian coordinate systems occur only locally in the neighbourhood of a point. We distinguish between the two by saying that the spacetime of special relativity is flat, while that of general relativity is curved. The departure from flatness in general relativity is due to the nonvanishing of the second derivatives in (2.3). (In §2.4 we formally define the Riemann curvature tensor).

2.3 Differentiation on Manifolds

To define the covariant derivative on M we need to introduce additional structure on the manifold. The derivative operator ∇, sometimes called the covariant derivative operator, on the manifold M is a map which takes each smooth tensor field of type (r, s) to a smooth tensor field of type $(r, s+1)$. The components of the tensor field resulting from the action of ∇ on a type (r, s) tensor field $\mathbf{T}$ are denoted by $\nabla_c T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$. The derivative operator satisfies the following properties:

(i) ∇ is linear and Leibnitz,

(ii) ∇ commutes with contraction,

(iii) For $f \in \mathcal{F}$ and $\mathbf{V} \in T_P(M)$

$$\mathbf{V}(f) = V^a \nabla_a f$$

where $\mathcal{F} : M \longrightarrow \Re$ is the collection of C^∞ maps. This requirement is consistent with the notion of tangent vectors as directional derivatives on scalar fields,

(iv) ∇ is torsion free so that

$$\nabla_a \nabla_b f = \nabla_b \nabla_a f$$

for $f \in \mathcal{F}$.

Note that some authors treat property (iii) as a definition after first defining $\mathbf{V}(f)$ along a parametrised curve. The usual partial derivative operator ∂_a of the components $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ of the (r, s) tensor field $\mathbf{T}$ given by

$$\partial_c T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s} = \frac{\partial T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}}{\partial x^c}$$

$$= T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s, c}$$

where the comma denotes partial differentiation, satisfies the above conditions. The components $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s, c}$ in another coordinate system, however, do not transform like a tensor. Thus the partial derivative is coordinate dependent and is not naturally associated with the structure of the manifold.

Property (iii) implies that any two derivative operators ∇_a and $\tilde{\nabla}_a$ must agree in their action on scalar fields. However disagreement in the action of ∇_a and $\tilde{\nabla}_a$ on tensor fields of higher rank is possible. Consider the difference $\tilde{\nabla}_a(fW_b) - \nabla_a(fW_b)$, for some covariant vector field $\mathbf{W}$ and an arbitrary scalar field $f \in \mathcal{F}$. Using properties (i) and (iii) we obtain

$$\tilde{\nabla}_a(fW_b) - \nabla_a(fW_b) = f(\tilde{\nabla}_a W_b - \nabla_a W_b)$$

At a point $P \in M$, $\tilde{\nabla}_a W_b$ and $\nabla_a W_b$ each depend on changes in $\mathbf{W}$ as we move away from P. However the above equation shows that the difference $\tilde{\nabla}_a W_b - \nabla_a W_b$ depends only on the value of $\mathbf{W}$ at point P. Thus $\tilde{\nabla}_a - \nabla_a$ defines a map of covariant vectors at P to tensors of type (0, 2) at P. Since this map is linear by property (i), $(\tilde{\nabla}_a - \nabla_a)$ defines a quantity at P denoted by $C^c{}_{ab}$. Thus given any two derivative operators $\tilde{\nabla}_a$ and ∇_a there exists the object $C^c{}_{ab}$ such that

$$\nabla_a W_b = \tilde{\nabla}_a W_b - C^c{}_{ab} W_c \tag{2.4}$$

The torsion free restriction implies that $C^c{}_{ab}$ is symmetric in the indices a and b. The difference in action of ∇_a and $\tilde{\nabla}_a$ on vector fields and all higher rank tensor fields is determined by equation (2.4), and properties (i) and (iii). For example we have for a contravariant vector field $\mathbf{V}$

$$\nabla_a V^b = \tilde{\nabla}_a V^b + C^b{}_{ac} V^c$$

For the general formula for the action of ∇_a on an arbitrary tensor field given in terms of $\tilde{\nabla}_a$ and $C^b{}_{ac}$ see Wald (1984). The difference between the two derivative operators ∇_a and $\tilde{\nabla}_a$ is completely characterised by the object $C^b{}_{ac}$. Conversely, if $\tilde{\nabla}_a$ is a derivative operator and $C^b{}_{ac}$ is an arbitrary symmetric smooth quantity, it is easy to show that ∇_a will also be a derivative operator.

For our purposes we need to consider the case where $\tilde{\nabla}_a$ is the partial derivative operator ∂_a. In this case, the object $C^c{}_{ab}$ is replaced by $\Gamma^c{}_{ab}$ which is called a Christoffel symbol. For a contravariant vector field $\mathbf{T}$ of type $(1,0)$ we have from above

$$\nabla_a T^b = \partial_a T^b + \Gamma^b{}_{ac} T^c$$

or in a different notation

$$T^b{}_{;a} = T^b{}_{,a} + \Gamma^b{}_{ac} T^c \tag{2.5}$$

where the semicolon denotes covariant differentiation. As further examples we list the covariant derivatives of the $(0,1), (1,1), (0,2)$ and the $(2,0)$ tensor fields:

$$T_{a;b} = T_{a,b} - \Gamma^c{}_{ab} T_c \tag{2.6}$$

$$T^a{}_{b;c} = T^a{}_{b,c} + \Gamma^a{}_{dc} T^d{}_b - \Gamma^d{}_{bc} T^a{}_d \tag{2.7}$$

$$T_{ab;c} = T_{ab,c} - \Gamma^d{}_{ac} T_{db} - \Gamma^d{}_{bc} T_{ad} \tag{2.8}$$

$$T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^a{}_{dc} T^{db} + \Gamma^b{}_{dc} T^{ad} \tag{2.9}$$

The covariant derivative of a type (r,s) tensor field follows the pattern suggested by equations (2.5)–(2.9).

A vector $\mathbf{V}$ given at each point on a curve with a tangent vector $\mathbf{T}$ is said to be parallel transported as we move along the curve if

$$T^a \nabla_a V^b = 0$$

is satisfied. Note that we can also define the parallel transport of a tensor field of arbitrary rank. A vector at a point P on the curve uniquely defines a parallel transported vector everywhere else on the curve. The notion of parallel transport may be used to identify the tangent spaces $T_P(M)$ and $T_Q(M)$ of points P and Q if we are given a derivative operator and a curve connecting P and Q. The mathematical structure arising from the identification of tangent spaces of different points is called a connection. Conversely we may start with the definition of a connection, and then develop a derivative operator. It is possible to define many distinct derivative operators on the manifold. In particular the fundamental theorem of Riemannian geometry states that for the metric tensor field $\mathbf{g}$ there exists a unique derivative operator ∇_a satisfying

$$\nabla_a g_{bc} = 0$$

This is due to the requirement that the inner product of two vectors remains unchanged if we parallel transport them along any curve. Thus the metric tensor field $\mathbf{g}$ naturally determines the covariant derivative operator ∇_a. The connection of this derivative operator is called the metric connection. For the covariant derivative the metric connection is comprised of the Christoffel symbols (of the second kind) which are also called the connection coefficients. The Christoffel symbols are given in terms of the metric and its derivatives as follows

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad} \left(g_{cd,b} + g_{db,c} - g_{bc,d} \right) \tag{2.10}$$

In general the connection coefficients (2.10) do not transform tensorially.

In order to define the covariant derivative we needed to impose the additional structure of a connection on the manifold M. Other operators such as the exterior derivative and the Lie derivative are defined on a differentiable manifold without imposing additional structure on M. We do not consider the exterior derivative since it acts only on forms and is not relevant to this thesis. However we do consider the Lie derivative of a tensor field since it provides a coordinate independent description of a symmetry property in the manifold M.

We can define the vector field $\mathbf{X}$ to operate on the scalar field $f \in \mathcal{F}$ to give the C^∞ scalar field $\mathbf{X}(f)$. The Lie derivative with respect to $\mathbf{X}$ is an extension of this operation to an operator $\mathcal{L}_\mathbf{X}$ on all C^∞ tensor fields which preserves the tensor type. This derivative corresponds to the change determined by an observer in going from a point P, with coordinates x^a, in the direction of a vector field $\mathbf{X}$ to an infinitesimally neighbouring point Q, with coordinates $x^a + \varepsilon X^a$, and transporting the coordinate system from P to Q. Consider an infinitesimal coordinate transformation

$$x^{a'} = x^a - \varepsilon X^a$$

We obtain the Lie derivative of a type $(1,0)$ tensor field $\mathbf{T}$ by comparing the contravariant components T^a at a point P and at an infinitesimally neighbouring point Q. To first order in ε we can establish the difference

$$T^{a'}(Q) = X^{a'}_b T^b(x^c + \varepsilon X^c)$$

which implies

$$T^{a'}(Q) - T^a(P) = \varepsilon X^b T^a{}_{,b}(P) - \varepsilon X^a{}_{,b} T^b(P) \tag{2.11}$$

Then the Lie derivative of T^a in the direction of the vector field $\mathbf{X}$ is defined as the limiting value

$$\mathcal{L}_\mathbf{X} T^a = \lim_{\varepsilon \to 0} \frac{T^{a'}(Q) - T^a(P)}{\varepsilon} \tag{2.12}$$

From (2.11) and (2.12) we obtain the equivalent expression

$$\mathcal{L}_{\mathbf{X}} T^a = T^a{}_{,b} X^b - T^b X^a{}_{,b} \tag{2.13}$$

which is the Lie derivative of the $(1,0)$ tensor field $\mathbf{T}$. As examples we list the Lie derivative of $(0,1), (1,1), (0,2)$ and $(2,0)$ types of tensor fields :

$$\mathcal{L}_{\mathbf{X}} T_a = T_{a,b} X^b + T_b X^b{}_{,a} \tag{2.14}$$

$$\mathcal{L}_{\mathbf{X}} T^a{}_b = T^a{}_{b,c} X^c - T^c{}_b X^a{}_{,c} + T^a{}_c X^c{}_{,b} \tag{2.15}$$

$$\mathcal{L}_{\mathbf{X}} T_{ab} = T_{ab,c} X^c + T_{cb} X^c{}_{,a} + T_{ac} X^c{}_{,b} \tag{2.16}$$

$$\mathcal{L}_{\mathbf{X}} T^{ab} = T^{ab}{}_{,c} X^c - T^{cb} X^a{}_{,c} - T^{ac} X^b{}_{,c} \tag{2.17}$$

For an (r,s) tensor field the Lie derivative follows the pattern contained in (2.13)–(2.17). Note that we can introduce the Christoffel symbols (2.10) to replace partial derivatives with covariant derivatives in (2.13)–(2.17). This explicitly demonstrates that the Lie derivative of a tensor is also a tensorial quantity. The Lie derivative $\mathcal{L}_{\mathbf{X}}$ satisfies a number of useful properties which may be used to simplify calculations. We list these properties without proof (Stephani 1990, Wald 1984):

(i) $\mathcal{L}_{\mathbf{X}}$ preserves tensor type, i.e. $\mathcal{L}_{\mathbf{X}}\mathbf{T}$ is a tensor field of the same type as $\mathbf{T}$.

(ii) $\mathcal{L}_{\mathbf{X}}$ is linear and Leibnitz.

(iii) $\mathcal{L}_{\mathbf{X}}$ commutes with contraction.

(iv) $\mathcal{L}_{\mathbf{X}} f = \mathbf{X}(f)$ where $f \in \mathcal{F}$.

(v) $\mathcal{L}_{\mathbf{X}}$ commutes with the partial derivative.

Two further properties satisfied by the Lie derivative and related to the Lie bracket are given in §2.5.

In this section we have defined the covariant derivative and the Lie derivative. The Lie derivative arises naturally on the manifold. This derivative is introduced without defining further structure on the manifold. Note that in order to define the covariant derivative we have to impose the additional structure of a connection on the manifold M. The covariant derivative generates the curvature tensor on the manifold (see §2.4). The Lie derivative is important for the description of symmetries of gravitational fields (see §2.5) and also other physical fields (Burke 1985, Schutz 1980).

2.4 Curvature and the Field Equations

The notion of curvature arises from the path dependence of parallel transport of a covariant vector field $\mathbf{W}$. This path dependence is directly related to the non-commutativity of covariant derivatives of $\mathbf{W}$:

$$\begin{aligned} W_{a;bc} - W_{a;cb} &= \left(\Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec}\right) W_d \\ &= R^d{}_{abc} W_d \end{aligned}$$

where

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac}\Gamma^d{}_{eb} - \Gamma^e{}_{ab}\Gamma^d{}_{ec} \tag{2.18}$$

are the components of the Riemann curvature tensor $\mathbf{R}$. The tensor field $\mathbf{R}$ provides a measure of the curvature of a manifold. For $\mathbf{R} = \mathbf{0}$ we have flat spacetime and for

$\mathbf{R} \neq \mathbf{0}$ the spacetime is curved. The components $R^a{}_{bcd}$ satisfy the following identities

$$R_{abcd} = -R_{bacd} \tag{2.19}$$

$$R_{abcd} = -R_{abdc} \tag{2.20}$$

$$R_{abcd} = R_{cdab} \tag{2.21}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0 \tag{2.22}$$

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0 \tag{2.23}$$

The identities (2.19)–(2.23) assist in calculations that involve the curvature of the manifold and are important in the formulation of the Einstein field equations. The equation (2.23) is called the Bianchi identity. Upon contraction of the Riemann tensor (2.18) we obtain the Ricci tensor

$$R_{ab} = R^c{}_{acb} \tag{2.24}$$

The scalar curvature R is defined as the trace of the Ricci tensor

$$R = R^a{}_a \tag{2.25}$$

Contracting the Bianchi identity (2.23) and utilising the above identities for the Riemann tensor yields

$$\left(R^{ab} - \tfrac{1}{2}Rg^{ab}\right)_{;b} = 0$$

This has the equivalent form

$$G^{ab}{}_{;b} = 0 \tag{2.26}$$

if we define

$$G_{ab} = R_{ab} - \tfrac{1}{2} R g_{ab} \tag{2.27}$$

where the tensor $\mathbf{G}$ is called the Einstein tensor.

In general relativity the matter distribution is described by the energy–momentum tensor $\mathbf{T}$ which is given by

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} + q_a u_b + q_b u_a + \pi_{ab} \tag{2.28}$$

where the energy density μ, the isotropic pressure p, the energy flux vector q^a ($q^a u_a = 0$), and the trace free anisotropic pressure tensor π_{ab} are measured relative to the 4-velocity u^a. The 4-velocity $\mathbf{u}$ is timelike so that $u^a u_a = -1$. For perfect fluids the energy flux vector and the stress tensor vanish so that (2.28) becomes

$$T_{ab} = (\mu + p) u_a u_b + p g_{ab} \tag{2.29}$$

The energy–momentum tensor $\mathbf{T}$, given by (2.28), is coupled to the Einstein tensor $\mathbf{G}$, given by (2.27), via the Einstein field equations

$$G_{ab} + \Lambda g_{ab} = T_{ab} \tag{2.30}$$

where Λ is the cosmological constant. We are using units in which the coupling constant in (2.30) is unity. From equations (2.26) and (2.30) it follows that

$$T^{ab}{}_{;b} = 0 \tag{2.31}$$

which is a conservation law. The field equations constitute a system of ten nonlinear partial differential equations which determine the gravitational field. The equations (2.30) express the relationship between the curvature of the manifold structure and the matter distribution in spacetime. Equation (2.26) implies that not all of the field equations are independent. For further information on various categories of exact solutions to the Einstein field equations (2.30) the reader is referred to the comprehensive list contained in Kramer *et al* (1980).

The energy–momentum tensor $\mathbf{T}$ in equation (2.28) represents neutral matter only. To accommodate charged matter we need to supplement the right hand side of (2.30) with a term representing the electromagnetic field. The energy–momentum tensor $\mathbf{E}$ for the electromagnetic field is given by

$$E_{ab} = F_{ac}F_{b}{}^{c} - \tfrac{1}{4}g_{ab}F_{de}F^{de} \tag{2.32}$$

Here the components of the skew–symmetric electromagnetic field tensor $\mathbf{F}$ may be given in terms of a 4–potential $\mathbf{A}$:

$$F_{ab} = A_{b;a} - A_{a;b}$$

(Misner *et al* 1973, Stephani 1990). The electromagnetic field tensor $\mathbf{F}$ satisfies Maxwell's equations

$$F^{ab}{}_{;b} = J^{a} \tag{2.33}$$

$$F_{ab;c} + F_{bc;a} + F_{ca;b} = 0 \tag{2.34}$$

where $\mathbf{J}$ represents the current density. With the electromagnetic field tensor given by $F_{ab} = A_{b;a} - A_{a;b}$ we note that (2.34) is identically satisfied. The Maxwell equations (2.33)–(2.34) are the basic equations of the electromagnetic field in a curved space. For electrodynamics in a curved background we need to supplement these equations with the Lorentz equation. The Einstein field equations (2.30) have to be adapted to the form

$$G_{ab} + \Lambda g_{ab} = T_{ab} + E_{ab} \tag{2.35}$$

to incorporate the electromagnetic field. Maartens and Maharaj (1990) find solutions to the field equations (2.35), with a conformal symmetry, for a charged nonconducting imperfect fluid without energy flow for symmetric static fluid spheres.

2.5 Lie Algebras and Conformal Motions

In this section we only summarise those elements of the theory of groups of transformations necessary for this thesis. Transformations are maps of a manifold into itself and symmetries are those transformations that do not change the mathematical structure of the manifold. An important feature of the study of transformations is the infinitesimal transformation described by a vector field. Of particular interest is the infinitesimal transformation described by the Lie derivative (2.12). We introduce the concepts of Lie algebras and Lie groups and discuss the relation between them. Furthermore we define a conformal Killing vector which is important for later sections. For a more comprehensive treatment of Lie theory and its applications to physics the reader is referred to Dubrovin *et al* (1984, 1985).

A Lie algebra is a vector space upon which is defined a bilinear multiplication operation [,] which from any two C^∞ vectors $\mathbf{X}$ and $\mathbf{Y}$ produces another vector $[\mathbf{X},\mathbf{Y}]$ satisfying:

(i) $[\mathbf{X},\mathbf{Y}] = -[\mathbf{Y},\mathbf{X}]$

(ii) $[\mathbf{X},[\mathbf{Y},\mathbf{Z}]] + [\mathbf{Y},[\mathbf{Z},\mathbf{X}]] + [\mathbf{Z},[\mathbf{X},\mathbf{Y}]] = 0$

Property (i) shows that [,] is skew–symmetric and property (ii) is called the Jacobi identity. One such operation is the Lie bracket:

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}$$

where $\mathbf{XY}$ is the composition of the vectors $\mathbf{X}$ and $\mathbf{Y}$ on scalar fields. The Lie algebra is closed under the operation of the Lie bracket. The Lie bracket is associated with the Lie derivative by the following properties:

$$\mathcal{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$$

$$\mathcal{L}_{[\mathbf{X},\mathbf{Y}]} = \mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{Y}} - \mathcal{L}_{\mathbf{Y}}\mathcal{L}_{\mathbf{X}}$$

for all vector fields $\mathbf{X}, \mathbf{Y}$. Thus in addition to properties (i)–(v) listed in §2.3 the Lie derivative also satisfies the above properties.

An r–dimensional Lie group G_r is an r–dimensional differentiable manifold whose underlying set is a topological group. The group composition

$$G_r \times G_r \longrightarrow G_r$$

and group inverse

$$G_r \longrightarrow G_r$$

are smooth functions. Every Lie group defines a unique Lie algebra and conversely every Lie algebra defines a unique Lie group (Choquet–Bruhat *et al* 1977, Dubrovin *et al* 1984, 1985, Kramer *et al* 1980). For our purposes we use Lie groups to represent the symmetries of mathematical structures on manifolds. Each of the elements or generators of the Lie algebra of G_r represents an infinitesimal transformation. The relevance of Lie algebras and Lie groups to various classes of solutions to the Einstein field equations is comprehensively discussed by Kramer *et al* (1980).

Manifolds with structure may admit groups of transformations which preserve this structure. A conformal motion preserves the metric up to a factor. A conformal Killing vector $\mathbf{X}$ is defined by

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\psi g_{ab} \tag{2.36}$$

where $\psi = \psi(x^a)$ is the conformal factor and $\mathbf{g}$ is the metric tensor. We normally distinguish between four categories of symmetries admitted by the conformal Killing equation (2.36):

(i) $\mathbf{X}$ is a Killing vector when $\psi = 0$.

(ii) $\mathbf{X}$ is a homothetic Killing vector when $\psi_{,a} = 0 \neq \psi$.

(iii) $\mathbf{X}$ is a special conformal Killing vector when $\psi_{;ab} = 0$.

(iv) $\mathbf{X}$ is a nonspecial conformal Killing vector when $\psi_{;ab} \neq 0$.

The Killing vectors span a group of isometries, which may be utilised to characterise solutions of the Einstein field equations systematically and invariantly (Kramer *et al* 1980). Killing vectors generate constants of the motion along geodesics. Conformal Killing vectors generate constants of the motion along null geodesics for massless particles. Solutions to the Einstein field equations may be obtained by supposing that spacetime admits a group of conformal motions G_r of infinitesimal transformations. These symmetries impose restrictions on the metric functions and consequently obtaining solutions to the Einstein field equations is simplified (Castejón–Amenedo and Coley 1992, Coley and Tupper 1990a, b, c, Dyer *et al* 1987, Maharaj *et al* 1991, Van den Bergh 1988).

The set of all conformal Killing vectors generates a Lie algebra with basis

$\{\mathbf{X}_I\}$. The elements of the basis are related by

$$[\mathbf{X}_I, \mathbf{X}_J] = C^K{}_{IJ}\mathbf{X}_K \tag{2.37}$$

where the $C^K{}_{IJ}$ are the structure constants of the group. From the Jacobi identity and equation (2.37) we obtain the Lie identity

$$C^K{}_{LM}C^M{}_{IJ} + C^K{}_{IM}C^M{}_{JL} + C^K{}_{JM}C^M{}_{LI} = 0 \tag{2.38}$$

Any set of constants $C^K{}_{IJ}$ satisfying

$$C^K{}_{IJ} = C^K{}_{[IJ]}$$

and (2.38) are the structure constants of a group. The maximal order r of a group of conformal motions G_r for an n–dimensional manifold is given by

$$r = \tfrac{1}{2}(n+1)(n+2)$$

(Choquet–Bruhat *et al* 1977). The maximal dimensionality of the Lie algebra in a 4–dimensional spacetime is $r = 15$. The generators of the G_{15} of conformal Killing vectors for flat space are given by Choquet–Bruhat *et al* (1977), and Maartens and Maharaj (1986) give the fifteen generators for Robertson–Walker spacetimes. Also the conformal Killing vectors of *pp*–wave spacetimes were found recently by Maartens and Maharaj (1991).

3 The Lie Derivative and the Field Equations

3.1 Introduction

In this chapter we consider the kinematical and dynamical properties of spacetimes that admit a conformal motion. Our results apply in general to the Einstein field equations as we have not specified a particular form for the metric tensor field **g**. We consider the Lie symmetries in the general case of an imperfect fluid energy–momentum tensor for neutral matter. In addition we briefly consider the symmetries of the energy–momentum tensor of the electromagnetic field. In §3.2 the kinematical properties of the Lie derivative along a conformal Killing vector are derived. The results are applied to the unit fluid 4–velocity vector that generates the kinematical quantities. The Lie derivatives of the kinematical quantities along the conformal Killing vector are explicitly determined. In §3.3 we derive the Lie derivative of the energy flow vector. This result together with the kinematical results are applied to the energy–momentum tensor for neutral matter. We then calculate the Lie derivative of the Einstein field equations with nonzero cosmological constant. Consequently we obtain equations involving the Lie derivatives of the dynamical quantities. We regain results given previously as special cases of our solutions. The Lie derivative of the electromagnetic energy–momentum tensor is derived in §3.4 by using the general decomposition of vectors established earlier. Thus it is possible to extend the results established in §3.3 to charged matter in general. We briefly review results obtained on symmetry inheritance of conformal Killing vectors in §3.5. Other types of symmetry inheritance are also considered.

3.2 The Kinematical Quantities

Before establishing the results of the Lie derivative of the kinematic quantities we first prove a useful identity for the Lie derivative of a unit vector **X** along the conformal Killing vector $\boldsymbol{\xi}$. Recall that if $\boldsymbol{\xi}$ is a conformal Killing vector then (2.36) is satisfied:

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 2\psi g_{ab}$$

This implies the useful result

$$\mathcal{L}_{\boldsymbol{\xi}} g^{ab} = -2\psi g^{ab}$$

which is utilised in subsequent calculations. We can always write, for any **X**,

$$\mathcal{L}_{\boldsymbol{\xi}} X^a = \alpha X^a + Y^a \tag{3.1}$$

where **Y** is orthogonal to **X** ($\mathbf{X} \cdot \mathbf{Y} = \mathbf{0}$). We take **X** to be a unit vector. The quantity α is a scalar. As **X** is a unit vector we can write

$$X^a X_a = \varepsilon$$

where

$$\varepsilon = \begin{cases} +1 & \text{if } \mathbf{X} \text{ is spacelike} \\ \\ -1 & \text{if } \mathbf{X} \text{ is timelike} \end{cases}$$

Taking the Lie derivative of $X^a X_a = \varepsilon$ along the conformal Killing vector $\boldsymbol{\xi}$ yields

$$X^a \mathcal{L}_{\boldsymbol{\xi}} X_a + X_a \mathcal{L}_{\boldsymbol{\xi}} X^a = 0$$

$$X^a \mathcal{L}_{\boldsymbol{\xi}} (g_{ab} X^b) + X_a \mathcal{L}_{\boldsymbol{\xi}} X^a = 0$$

$$X^a \left(2\psi g_{ab} X^b + g_{ab} \mathcal{L}_{\boldsymbol{\xi}} X^b \right) + X_a \mathcal{L}_{\boldsymbol{\xi}} X^a = 0$$

$$\Rightarrow \quad X_a \mathcal{L}_{\boldsymbol{\xi}} X^a = -\psi\varepsilon$$

However contracting (3.1) with X_a also gives

$$X_a \mathcal{L}_{\boldsymbol{\xi}} X^a = \alpha\varepsilon$$

which together with the above result implies

$$\psi = -\alpha$$

Thus we have established the results

$$\mathcal{L}_{\boldsymbol{\xi}} X^a = -\psi X^a + Y^a \tag{3.2}$$

$$\mathcal{L}_{\boldsymbol{\xi}} X_a = \psi X_a + Y_a \tag{3.3}$$

for the Lie derivative of the unit vector $\mathbf{X}$ along the conformal Killing vector $\boldsymbol{\xi}$ where $X^a Y_a = 0$. Applying (3.2) and (3.3) to the fluid 4–velocity vector $\mathbf{u}$ we have that

$$\mathcal{L}_{\boldsymbol{\xi}} u^a = -\psi u^a + v^a \tag{3.4}$$

$$\mathcal{L}_{\boldsymbol{\xi}} u_a = \psi u_a + v_a \tag{3.5}$$

where $\mathbf{v}$ is a spacelike vector and $u^a v_a = 0$.

If $\mathbf{u}$ is the fluid 4–velocity vector then we can establish the kinematic result (Ellis 1973)

$$u_{a;b} = \sigma_{ab} + \tfrac{1}{3}\Theta h_{ab} + \omega_{ab} - \dot{u}_a u_b \tag{3.6}$$

where

$$\Theta = u^a{}_{;a} \tag{3.7}$$

is the rate of expansion,

$$h_{ab} = g_{ab} + u_a u_b \tag{3.8}$$

is the symmetric projection tensor ($h_{ab}u^b = 0$),

$$\sigma_{ab} = \tfrac{1}{2}(u_{a;c}h^c{}_b + u_{b;c}h^c{}_a) - \tfrac{1}{3}\Theta h_{ab} \tag{3.9}$$

is the symmetric shear tensor ($\sigma_{ab}u^a = 0 = \sigma^a{}_a$),

$$\omega_{ab} = h^c{}_a h^d{}_b u_{[c;d]} \tag{3.10}$$

is the skew–symmetric vorticity tensor ($\omega_{ab}u^b = 0$) and

$$\dot{u}_a = u_{a;b}u^b \tag{3.11}$$

is the acceleration vector ($\dot{u}_a u^a = 0$). The overhead dot denotes covariant differentiation along a fluid particle worldline. Square brackets denote skew–symmetrisation.

Taking the Lie derivative of the connection coefficients (2.10) along the conformal Killing vector $\boldsymbol{\xi}$ yields

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bc} &= \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\xi}}g^{ad}[g_{db,c} + g_{cd,b} - g_{bc,d}] \\
&\quad + \tfrac{1}{2}g^{ad}\mathcal{L}_{\boldsymbol{\xi}}[g_{db,c} + g_{cd,b} - g_{bc,d}] \\
&= \tfrac{1}{2}(-2\psi g^{ad})[g_{db,c} + g_{cd,b} - g_{bc,d}]
\end{aligned}$$

$$+\tfrac{1}{2}g^{ad}\left[\left(\mathcal{L}_{\boldsymbol{\xi}}g_{db}\right)_{,c}+\left(\mathcal{L}_{\boldsymbol{\xi}}g_{cd}\right)_{,b}-\left(\mathcal{L}_{\boldsymbol{\xi}}g_{bc}\right)_{,d}\right]$$

$$= g^{ad}\left[\psi_{,c}g_{db}+\psi_{,b}g_{cd}-\psi_{,d}g_{bc}\right]$$

$$= \psi_{,c}\delta^{a}{}_{b}+\psi_{,b}\delta^{a}{}_{c}-g_{bc}\psi^{,a} \tag{3.12}$$

We use (3.5) and (3.12) to establish the next result. The Lie derivative of (3.6) is given by

$$\mathcal{L}_{\boldsymbol{\xi}}u_{a;b} = \mathcal{L}_{\boldsymbol{\xi}}u_{a,b}-\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^{c}{}_{ab}\right)u_{c}-\Gamma^{c}{}_{ab}\left(\mathcal{L}_{\boldsymbol{\xi}}u_{c}\right)$$

$$= (\psi u_a+v_a)_{,b}-(\psi_{,b}\delta^{c}{}_{a}+\psi_{,a}\delta^{c}{}_{b}-g_{ab}\psi^{,c})u_c-\Gamma^{c}{}_{ab}(\psi u_c+v_c)$$

$$= \psi_{,b}u_a+\psi u_{a,b}+v_{a,b}-u_a\psi_{,b}-u_b\psi_{,a}$$
$$+\,g_{ab}u_c\psi^{,c}-\psi\Gamma^{c}{}_{ab}u_c-\Gamma^{c}{}_{ab}v_c$$

$$= \psi u_{a;b}+v_{a;b}+g_{ab}u_c\psi^{,c}-\psi_{,a}u_b \tag{3.13}$$

As the expansion scalar for the fluid velocity congruence is defined by (3.7) we have that

$$\mathcal{L}_{\boldsymbol{\xi}}\Theta = \left(\mathcal{L}_{\boldsymbol{\xi}}u^{a}\right)_{,a}+\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^{a}{}_{ba}\right)u^{b}+\Gamma^{a}{}_{ba}\left(\mathcal{L}_{\boldsymbol{\xi}}u^{b}\right)$$

$$= (-\psi u^a+v^a)_{,a}+(\psi_{,b}\delta^{a}{}_{a}+\psi_{,a}\delta^{a}{}_{b}-g_{ba}\psi^{,a})u^b+\Gamma^{a}{}_{ba}(-\psi u^b+v^b)$$

$$= -\psi_{,a}u^a-\psi u^{a}{}_{,a}+v^{a}{}_{,a}+4u^b\psi_{,b}+\psi_{,a}u^a$$
$$-\,u^a\psi_{,a}-\psi\Gamma^{a}{}_{ba}u^b+\Gamma^{a}{}_{ba}v^b$$

$$= \quad -\psi\Theta + v^a{}_{;a} + 3\psi_{,a}u^a \tag{3.14}$$

where we have utilised (3.4) and (3.12). The Lie derivative of the projection tensor (3.8) becomes

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} h_{ab} &= \mathcal{L}_{\boldsymbol{\xi}} g_{ab} + u_a \mathcal{L}_{\boldsymbol{\xi}} u_b + u_b \mathcal{L}_{\boldsymbol{\xi}} u_a \\
&= 2\psi g_{ab} + 2\psi u_a u_b + 2u_{(a}v_{b)} \\
&= 2\psi h_{ab} + 2u_{(a}v_{b)}
\end{aligned} \tag{3.15}$$

where we have used (3.5). The round brackets above denote symmetrisation. Note that since $u^a v_a = 0$ we have

$$\mathcal{L}_{\boldsymbol{\xi}} u^a = -\psi u^a \quad \Leftrightarrow \quad \mathcal{L}_{\boldsymbol{\xi}} h_{ab} = 2\psi h_{ab}$$

so that $\boldsymbol{\xi}$ is a conformal motion of the projection tensor h_{ab} if and only if $v^a = 0$. The Lie derivative of the shear tensor (3.9) along the conformal Killing vector $\boldsymbol{\xi}$ reduces to the following:

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} \sigma_{ab} &= \tfrac{1}{2}\mathcal{L}_{\boldsymbol{\xi}}(u_{a;c}h^c{}_b + u_{b;c}h^c{}_a) - \tfrac{1}{3}\mathcal{L}_{\boldsymbol{\xi}}(\Theta h_{ab}) \\
&= \psi\sigma_{ab} - \tfrac{1}{3}h_{ab}v^c{}_{;c} - \tfrac{2}{3}\Theta u_{(a}v_{b)} + \tfrac{1}{2}(v_{a;c}h^c{}_b + v_{b;c}h^c{}_a) \\
&\quad - \tfrac{1}{2}(\psi_{,a}h^c{}_b + \psi_{,b}h^c{}_a)u_c + g^{cd}(u_{a;c}u_{(d}v_{b)} + u_{b;c}u_{(d}v_{a)}) \\
&= \psi\sigma_{ab} - \tfrac{1}{3}h_{ab}v^c{}_{;c} - \tfrac{2}{3}\Theta u_{(a}v_{b)} \\
&\quad + \dot{v}_{(a}u_{b)} + v_{(a;b)} + \dot{u}_{(a}v_{b)} + u_{(a}u_{b);c}v^c
\end{aligned} \tag{3.16}$$

where we have used (3.13)–(3.15). Note that (3.16) does not explicitly depend on the derivatives of ψ. Also note that if $v_a = 0$ then

$$\mathcal{L}_{\boldsymbol{\xi}}\sigma_{ab} = \psi\sigma_{ab}$$

and $\boldsymbol{\xi}$ is a conformal motion of the shear tensor σ_{ab}. The Lie derivative of the acceleration vector (3.11) is given by

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\dot{u}_a &= \mathcal{L}_{\boldsymbol{\xi}}(u_{a;b}u^b) \\
&= u^b\mathcal{L}_{\boldsymbol{\xi}}u_{a;b} + u_{a;b}\mathcal{L}_{\boldsymbol{\xi}}u^b \\
&= u^b v_{a;b} + v^b u_{a;b} + \psi_{,a} + u_a\psi_{,c}u^c \qquad (3.17)
\end{aligned}$$

where we have utilised (3.4) and (3.13). On substituting (3.13)–(3.17) into (3.6) we obtain the Lie derivative of the vorticity tensor

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\omega_{ab} &= \mathcal{L}_{\boldsymbol{\xi}}[u_{a;b} - \sigma_{ab} - \tfrac{1}{3}\Theta h_{ab} + \dot{u}_a u_b] \\
&= \psi\omega_{ab} + v_{[a;b]} + \dot{u}_{[a}v_{b]} + \dot{v}_{[a}u_{b]} - u_{[a}u_{b];c}v^c \qquad (3.18)
\end{aligned}$$

which does not explicitly depend on the derivatives of ψ. Again note that if $v_a = 0$ then

$$\mathcal{L}_{\boldsymbol{\xi}}\omega_{ab} = \psi\omega_{ab}$$

and $\boldsymbol{\xi}$ is a conformal motion of the vorticity tensor ω_{ab}.

For easy reference we list the Lie derivative of the kinematical quantities (3.14), (3.16)–(3.18) together:

$$\mathcal{L}_{\boldsymbol{\xi}}\Theta = -\psi\Theta + v^a{}_{;a} + 3\psi_{,a}u^a$$

$$\mathcal{L}_{\boldsymbol{\xi}}\dot{u}_a = u^b v_{a;b} + v^b u_{a;b} + \psi_{,a} + u_a \psi_{,c} u^c$$

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\xi}}\sigma_{ab} &= \psi\sigma_{ab} - \tfrac{1}{3}h_{ab}v^c{}_{;c} - \tfrac{2}{3}\Theta u_{(a}v_{b)} \\ &\quad + \dot{v}_{(a}u_{b)} + v_{(a;b)} + \dot{u}_{(a}v_{b)} + u_{(a}u_{b);c}v^c\end{aligned}$$

$$\mathcal{L}_{\boldsymbol{\xi}}\omega_{ab} = \psi\omega_{ab} + v_{[a;b]} + \dot{u}_{[a}v_{b]} + \dot{v}_{[a}u_{b]} - u_{[a}u_{b];c}v^c$$

Special cases of the above properties have been listed by other authors depending on their applications. Herrera *et al* (1984) made the implicit assumption that $v^a = 0$ in (3.4) in their study of conformally invariant solutions to the Einstein field equations. Maartens *et al* (1986) give the kinematical quantities in a slightly different form; in particular the 4–acceleration vector $\dot{u}^a$ and the vorticity tensor ω^{ab} are given in terms of a scalar which reduces to an acceleration potential for an irrotational fluid.

3.3 The Dynamical Quantities

The energy flow vector q_a is not a unit vector. However by using an argument similar to that in §3.2 we can find the Lie derivative of q_a along $\boldsymbol{\xi}$. Let

$$q^a q_a = Q^2 \tag{3.19}$$

where Q is the magnitude of the energy flow. As before we express the Lie derivative of q_a as follows

$$\mathcal{L}_{\boldsymbol{\xi}} q^a = \alpha q^a + w^a$$

where $w^a q_a = 0$. Contracting this equation with q_a gives

$$q^a \mathcal{L}_{\boldsymbol{\xi}} q^a = \alpha Q^2$$

Taking the Lie derivative of (3.19) along the conformal Killing vector $\boldsymbol{\xi}$ yields

$$q^a \mathcal{L}_{\boldsymbol{\xi}} q_a + q_a \mathcal{L}_{\boldsymbol{\xi}} q^a = 2Q \mathcal{L}_{\boldsymbol{\xi}} Q$$

$$q^a \left(2\psi q_a + g_{ab} \mathcal{L}_{\boldsymbol{\xi}} q^b\right) + q_a \mathcal{L}_{\boldsymbol{\xi}} q^a = 2Q \mathcal{L}_{\boldsymbol{\xi}} Q$$

$$\Rightarrow \quad q_a \mathcal{L}_{\boldsymbol{\xi}} q^a = \left(-\psi + Q^{-1} \mathcal{L}_{\boldsymbol{\xi}} Q\right) Q^2$$

Comparing the right hand side of the above two equations gives

$$\alpha = -\psi + Q^{-1} \mathcal{L}_{\boldsymbol{\xi}} Q$$

Thus we have established the following results for the energy flow vector

$$\mathcal{L}_{\boldsymbol{\xi}} q^a = \left[-\psi + Q^{-1} \mathcal{L}_{\boldsymbol{\xi}} Q\right] q^a + w^a \tag{3.20}$$

$$\mathcal{L}_{\boldsymbol{\xi}} q_a = \left[\psi + Q^{-1} \mathcal{L}_{\boldsymbol{\xi}} Q\right] q_a + w_a \tag{3.21}$$

Note that the above results for q^a and q_a apply to any vector of variable magnitude Q. For a unit vector $\mathbf{q}$, (3.20)–(3.21) reduce to (3.2)–(3.3). Since $u^a q_a = 0$ it is convenient to define the scalar quantity

$$\Delta \equiv -u^a \mathcal{L}_{\boldsymbol{\xi}} q_a$$

$$= q^a \mathcal{L}_{\boldsymbol{\xi}} u_a$$

The above implies that

$$\Delta = -u^a w_a = v^a q_a$$

which we use later.

In order to establish the Lie derivative of the dynamical quantities it is useful to first find the Lie derivative of the Riemann tensor (2.18):

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} R^a{}_{bcd} &= \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bd}\right)_{,c} - \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bc}\right)_{,d} + \Gamma^a{}_{ce}\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^e{}_{bd}\right) \\
&+ \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{ce}\right)\Gamma^e{}_{bd} - \Gamma^a{}_{de}\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^e{}_{bc}\right) - \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{de}\right)\Gamma^e{}_{bc} \\
&= \left\{\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bd}\right)_{,c} + \Gamma^a{}_{ce}\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^e{}_{bd}\right) - \Gamma^e{}_{bc}\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{de}\right)\right\} \\
&- \left\{\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bc}\right)_{,d} + \Gamma^a{}_{de}\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^e{}_{bc}\right) - \Gamma^e{}_{bd}\left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{ce}\right)\right\} \\
&= \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bd}\right)_{;c} - \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^a{}_{bc}\right)_{;d} \qquad (3.22)
\end{aligned}$$

On setting $a = c$ in (3.22) we establish the Lie derivative of the Ricci tensor

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} R_{ab} &= \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^c{}_{ab}\right)_{;c} - \left(\mathcal{L}_{\boldsymbol{\xi}}\Gamma^c{}_{ac}\right)_{;b} \\
&= \psi_{;ac}\delta^c{}_b + \psi_{;bc}\delta^c{}_a - g_{ab}\psi^{;c}{}_{;c} \\
&\quad -(4\psi_{;ab} + \psi_{;cb}\delta^c{}_a - g_{ac}\psi^{;c}{}_{;b}) \\
&= -2\psi_{;ab} - g_{ab}\Box\psi \qquad (3.23)
\end{aligned}$$

where $\Box\psi = g^{ab}\psi_{;ab}$. Contracting the Lie derivative of the Ricci tensor in (3.23) gives the Lie derivative of the Ricci scalar

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} R &= \mathcal{L}_{\boldsymbol{\xi}}(g^{ab}R_{ab}) \\
&= (-2\psi g^{ab})R_{ab} + g^{ab}(-2\psi_{;ab} - g_{ab}\Box\psi)
\end{aligned}$$

$$= \quad -2\psi R - 6\Box\psi \tag{3.24}$$

Taking the Lie derivative of the Einstein tensor (2.27) and utilising (3.23) and (3.24) gives the expression

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} G_{ab} &= -2\psi_{;ab} - g_{ab}\Box\psi - \tfrac{1}{2}[(-2\psi R - 6\Box\psi)g_{ab} + R(2\psi g_{ab})] \\
&= 2g_{ab}\Box\psi - 2\psi_{;ab}
\end{aligned} \tag{3.25}$$

The results (3.22)–(3.25) give the Lie derivative of quantities related to the curvature of the manifold.

The Lie derivative along a conformal Killing vector $\boldsymbol{\xi}$ of the energy–momentum tensor (2.28) is given by

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} T_{ab} &= u_a u_b \mathcal{L}_{\boldsymbol{\xi}}\mu + h_{ab}\mathcal{L}_{\boldsymbol{\xi}} p + 2\psi(\mu u_a u_b + p h_{ab}) \\
&\quad + 2(\mu + p)u_{(a}v_{b)} + \mathcal{L}_{\boldsymbol{\xi}}\pi_{ab} \\
&\quad + 2\left(Q^{-1}\mathcal{L}_{\boldsymbol{\xi}} Q + 2\psi\right) u_{(a}q_{b)} + 2q_{(a}v_{b)} + 2u_{(a}w_{b)}
\end{aligned} \tag{3.26}$$

where we have used (3.5), (3.15) and (3.21). Then the Lie derivative of the Einstein field equations (2.30) becomes

$$\begin{aligned}
&u_a u_b \mathcal{L}_{\boldsymbol{\xi}}\mu + h_{ab}\mathcal{L}_{\boldsymbol{\xi}} p + 2\psi(\mu u_a u_b + p h_{ab}) + 2(\mu + p)u_{(a}v_{b)} \\
&+\mathcal{L}_{\boldsymbol{\xi}}\pi_{ab} + 2\left(Q^{-1}\mathcal{L}_{\boldsymbol{\xi}} Q + 2\psi\right) u_{(a}q_{b)} + 2q_{(a}v_{b)} + 2u_{(a}w_{b)} \\
&\qquad = 2(\Box\psi + \Lambda\psi)g_{ab} - 2\psi_{;ab}
\end{aligned} \tag{3.27}$$

on utilising (3.25) and (3.26). Contracting (3.27) with $u^a u^b$, h^{ab}, $u^a h^b{}_c$, $h^{ac}h^{bd} - \frac{1}{3}h^{ab}h^{cd}$, q^b, $q^a u^b$ and $q^a q^b$ in turn yields the following set of equations that involve the dynamical quantities:

$$u^a u^b \mathcal{L}_{\boldsymbol{\xi}} \pi_{ab} + \mathcal{L}_{\boldsymbol{\xi}} \mu + 2\psi\mu + 2\Delta = -2(\Box\psi + \Lambda\psi) - 2u^a u^b \psi_{;ab} \tag{3.28}$$

$$h^{ab} \mathcal{L}_{\boldsymbol{\xi}} \pi_{ab} + 3\mathcal{L}_{\boldsymbol{\xi}} p + 6\psi p + 2\Delta = 6(\Box\psi + \Lambda\psi) - 2h^{ab}\psi_{;ab} \tag{3.29}$$

$$u^a h^b{}_c \mathcal{L}_{\boldsymbol{\xi}} \pi_{ab} - (\mu + p)v_c - \left(Q^{-1}\mathcal{L}_{\boldsymbol{\xi}} Q + 2\psi\right) q_c - w_c + \Delta u_c = -2u^a h^b{}_c \psi_{;ab} \tag{3.30}$$

$$\left(h^{ac}h^{bd} - \tfrac{1}{3}h^{ab}h^{cd}\right)\left(\mathcal{L}_{\boldsymbol{\xi}} \pi_{ab} + \psi_{;ab}\right) + 2g^{ac}g^{bd}q_{(a}v_{b)} - \tfrac{2}{3}h^{cd}\Delta = 0 \tag{3.31}$$

$$\begin{aligned} q_a \left[\mathcal{L}_{\boldsymbol{\xi}} p + 2\psi p + \Delta\right] + u_a \left[\Delta(\mu + p) + Q\mathcal{L}_{\boldsymbol{\xi}} Q + 2\psi Q^2\right] \\ + q^b \mathcal{L}_{\xi} \pi_{ab} = 2(\Box\psi + \Lambda\psi)q_a - q^b\psi_{;ab} - Q^2 v_a \end{aligned} \tag{3.32}$$

$$q^a u^b \mathcal{L}_{\boldsymbol{\xi}} \pi_{ab} - \Delta(\mu + p) - \left(Q\mathcal{L}_{\boldsymbol{\xi}} Q + 2\psi Q^2\right) = -q^a u^b \psi_{;ab} \tag{3.33}$$

$$Q^2\left(\mathcal{L}_{\boldsymbol{\xi}} p + 2\psi p\right) + 2Q^2\Delta + q^a q^b \mathcal{L}_{\boldsymbol{\xi}} \pi_{ab} = 2\Delta(\Box\psi + \Lambda\psi) - q^a q^b \psi_{;ab} \tag{3.34}$$

Thus we have found the Lie derivative of the dynamical quantities in (3.28)–(3.34) in general without any assumptions.

Special cases of the above system arising from taking the Lie derivative of the Einstein field equations, have been considered by various authors. For cosmological purposes the most important special case of (3.28)–(3.34) is for a perfect fluid. By setting $q_a = 0 = \pi_{ab}$ in the above set of equations we obtain after some simplification:

$$\mathcal{L}_{\boldsymbol{\xi}}\mu = -2\psi(\mu + \Lambda) - 2\Box\psi - 2u^a u^b \psi_{;ab} \tag{3.35}$$

$$3\mathcal{L}_{\boldsymbol{\xi}}p = 4\Box\psi + 6\psi(\Lambda - p) - 2u^a u^b \psi_{;ab} \tag{3.36}$$

$$(\mu + p)v_c = 2u^a \psi_{;ac} + 2u^a u^b u_c \psi_{;ab} \tag{3.37}$$

$$\begin{aligned}\psi_{;}{}^{cd} + u^b u^d \psi_{;}{}^{c}{}_{b} + u^a u^c \psi_{;a}{}^{d} + \tfrac{2}{3} u^a u^b u^c u^d \psi_{;ab} \\ - \tfrac{1}{3} h^{cd}\Box\psi - \tfrac{1}{3} g^{cd} u^a u^b \psi_{;ab} = 0\end{aligned} \tag{3.38}$$

The subcase of (3.28)–(3.34) for a special conformal Killing vector ($\psi_{;ab} = 0$) with $\Lambda = 0$ was first considered by Herrera *et al* (1984). The case with $q_a = 0$ and an anisotropic stress tensor was found by Maartens *et al* (1986) generalising results obtained by Herrera *et al* (1984). Coley and Tupper (1989) imposed the restriction $q_a \neq 0$ and restricted the anisotropic stress tensor to satisfy the phenomenological equation $\pi_{ab} = -2\eta\sigma_{ab}$ where η is the bulk viscosity. In the case of Coley and Tupper (1989) the system (3.28)–(3.34) reduces to the following:

$$\mathcal{L}_{\boldsymbol{\xi}}\mu + 2\psi\mu + 2\Delta = -2(\Box\psi + \Lambda\psi) - 2u^a u^b \psi_{;ab} \tag{3.39}$$

$$3\mathcal{L}_{\boldsymbol{\xi}}p + 6\psi p + 2\Delta = 6(\Box\psi + \Lambda\psi) - 2h^{ab}\psi_{;ab} \tag{3.40}$$

$$2\eta\sigma_{cd}v^d - (\mu + p)v_c - \left(Q^{-1}\mathcal{L}_{\boldsymbol{\xi}}Q + 2\psi\right) q_c - w_c + \Delta u_c = -2u^a h^b{}_c \psi_{;ab} \tag{3.41}$$

$$\left(h^{ac}h^{bd} - \tfrac{1}{3}h^{ab}h^{cd}\right)\left(\mathcal{L}_{\boldsymbol{\xi}}\pi_{ab} + \psi_{;ab}\right) + 2g^{ac}g^{bd}q_{(a}v_{b)} - \tfrac{2}{3}h^{cd}\Delta = 0 \tag{3.42}$$

$$\begin{aligned} q_a\left[\mathcal{L}_{\boldsymbol{\xi}}p + 2\psi p + \Delta\right] + u_a\left[\Delta(\mu + p) + Q\mathcal{L}_{\boldsymbol{\xi}}Q + 2\psi Q^2\right] & \\ - 2q^b\mathcal{L}_{\boldsymbol{\xi}}(\eta\sigma_{ab}) = 2(\Box\psi + \Lambda\psi)q_a - q^b\psi_{;ab} - Q^2v_a & \end{aligned} \tag{3.43}$$

$$2\eta\sigma_{ab}q^a v^b - \Delta(\mu + p) - \left(Q\mathcal{L}_{\boldsymbol{\xi}}Q + 2\psi Q^2\right) = -q^a u^b \psi_{;ab} \tag{3.44}$$

$$Q^2\left(\mathcal{L}_{\boldsymbol{\xi}}p + 2\psi p\right) + 2Q^2\Delta + 2q^a q^b \mathcal{L}_{\boldsymbol{\xi}}(\eta\sigma_{ab}) = 2\Delta(\Box\psi + \Lambda\psi) - q^a q^b \psi_{;ab} \tag{3.45}$$

Our equations (3.28)–(3.34) unify results presented previously in a consistent notation. Also our equations generalise these results to the most general energy–momentum tensor for neutral matter.

3.4 The Electromagnetic Field Tensor

The results of §3.3 may be extended to include charged matter. In this section we find the Lie derivative along the conformal Killing vector $\boldsymbol{\xi}$ of the electromagnetic tensor (2.32). Since $F_{ab} = A_{b,a} - A_{a,b}$ we have that

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\xi}}F_{ca} &= \left(\mathcal{L}_{\boldsymbol{\xi}}A_a\right)_{,c} - \left(\mathcal{L}_{\boldsymbol{\xi}}A_c\right)_{,a} \\ \mathcal{L}_{\boldsymbol{\xi}}F^c{}_b &= -2\psi F^c{}_b + g^{ac}\left[\left(\mathcal{L}_{\boldsymbol{\xi}}A_b\right)_{,a} - \left(\mathcal{L}_{\boldsymbol{\xi}}A_a\right)_{,b}\right] \\ \mathcal{L}_{\boldsymbol{\xi}}F^{de} &= -4\psi F^{de} + g^{ad}g^{be}\mathcal{L}_{\boldsymbol{\xi}}F_{ab} \end{aligned}$$

Thus substituting the above into the Lie derivative of (2.32) we obtain

$$\begin{aligned}
\mathcal{L}_{\xi} E_{ab} &= \mathcal{L}_{\xi}\left(F_{ca}F^{c}{}_{b}\right) - \mathcal{L}_{\xi}\left(\tfrac{1}{4} g_{ab} F^{de} F_{de}\right) \\
&= F^{f}{}_{a} \mathcal{L}_{\xi} F_{fb} - 2\psi F_{ca} F^{c}{}_{b} + F^{c}{}_{b} \mathcal{L}_{\xi} F_{ca} \\
&\quad - \tfrac{1}{4}\left[-2\psi g_{ab} F^{de} F_{de} + g_{ab} F_{de} g^{gd} g^{he} \mathcal{L}_{\xi} F_{gh} + g_{ab} F^{de} \mathcal{L}_{\xi} F_{de}\right] \\
&= -2\psi E_{ab} + F^{c}{}_{a} \mathcal{L}_{\xi} F_{cb} + F^{c}{}_{b} \mathcal{L}_{\xi} F_{ca} - \tfrac{1}{4}\left[2 g_{ab} F^{de} \mathcal{L}_{\xi} F_{de}\right] \qquad (3.46)
\end{aligned}$$

To eliminate the Lie derivative of the electromagnetic tensor on the right hand side of (3.46) we have to find the Lie derivative of the 4–potential A_a. We use an argument similar to that in §3.2. Let

$$A^a A_a = A^2$$

where A is the magnitude of the 4–potential $\mathbf{A}$. We can express the Lie derivative of A_a as follows

$$\mathcal{L}_{\xi} A_a = \alpha A_a + B_a \qquad (3.47)$$

where $A^a B_a = 0$. We can then prove as before that

$$\alpha = -\psi + A^{-1} \mathcal{L}_{\xi} A \qquad (3.48)$$

Thus we obtain the quantities

$$\begin{aligned}
F^{c}{}_{b} \mathcal{L}_{\xi} F_{ca} &= g^{cf}(A_{b,f} - A_{f,b})\left[(\alpha A_a + B_a)_{,c} - (\alpha A_c + B_c)_{,a}\right] \\
&= 4 g^{cf} A_{[b,f]}\left(\alpha_{[,c} A_{a]} + \alpha A_{[a,c]} + B_{[a,c]}\right)
\end{aligned}$$

$$\begin{aligned}
F^{de} \mathcal{L}_{\xi} F_{de} &= g^{dg} g^{eh}(A_{h,g} - A_{g,h})\left[(\alpha A_e + B_e)_{,d} - (\alpha A_d + B_d)_{,e}\right] \\
&= 4 A^{[e,d]}\left(\alpha_{[,d} A_{e]} + \alpha A_{[e,d]} + B_{[e,d]}\right)
\end{aligned}$$

Substituting the above into (3.46) we obtain

$$\begin{aligned}\mathcal{L}_{\boldsymbol{\xi}} E_{ab} &= -2\psi E_{ab} + 4g^{cf} A_{[a,f]} \left(\alpha_{[,c} A_{b]} + \alpha A_{[b,c]} + B_{[b,c]}\right) \\ &\quad + 4g^{cf} A_{[b,f]} \left(\alpha_{[,c} A_{a]} + \alpha A_{[a,c]} + B_{[a,c]}\right) \\ &\quad - 2g_{ab} A^{[e,d]} \left(\alpha_{[,d} A_{e]} + \alpha A_{[e,d]} + B_{[e,d]}\right) \end{aligned} \tag{3.49}$$

where α is given by (3.48).

Thus we have found the Lie derivative of the electromagnetic energy–momentum tensor $\mathbf{E}$ along the conformal Killing vector $\boldsymbol{\xi}$ in general. We have utilised the decomposition (3.47) of $\mathcal{L}_{\boldsymbol{\xi}} A_a$ to express $\mathcal{L}_{\boldsymbol{\xi}} E_{ab}$ in the compact form (3.49). We have not seen this Lie property of the decomposition of the 4–potential $\mathbf{A}$ applied to the electromagnetic energy–momentum tensor previously. Maartens *et al* (1986), using a different notation, find the Lie derivative along $\boldsymbol{\xi}$ of $\mathbf{E}$ for the special case when the electric field vanishes. Using (3.49) we are now in a position to find the analogue of the system (3.28)–(3.34) for a charged fluid satisfying the Einstein field equations, and also to study its physical properties. As this falls outside the scope of this thesis it will be an area for future investigation.

3.5 Symmetry Inheritance

The idea that kinematical and dynamical quantities in general relativity inherit a symmetry property has been investigated by many authors. It is hoped that the imposition of a symmetry requirement will simplify the highly nonlinear Einstein field equations and will lead to new solutions. Also a symmetry property provides a mechanism to categorise invariantly solutions to the field equations in a systematic manner. In this section we briefly review symmetry inheritance properties of conformal Killing vectors in particular, and give some recent results on other types of symmetry inheritance.

A number of authors have investigated restrictions to the Einstein field equations by analysing the kinematic quantities with a conformal motion. If $v^a = 0$ then (3.4) gives

$$\mathcal{L}_{\boldsymbol{\xi}} u^a = -\psi u^a$$

and fluid flow lines are mapped conformally into fluid flow lines. Herrera *et al* (1984) imposed the condition $v^a = 0$ in their analysis of fluids without energy flow but with a preferred direction of anisotropy. They presented solutions to the Einstein field equations with a conformal symmetry for both isotropic and anisotropic pressures. Coley and Tupper (1989) introduced the notion of an inheriting conformal Killing vector if $\mathcal{L}_{\boldsymbol{\xi}} u^a = -\psi u^a$ is satisfied with $v^a = 0$. The properties of spacetimes admitting a special conformal Killing vector where the symmetries are inherited, were analysed by Coley and Tupper (1989). Maartens *et al* (1986), in their analysis of anisotropic fluids, provide a counter example that illustrates in general that fluid flow lines are not necessarily mapped conformally into fluid flow lines, i.e. $v^a \neq 0$. The condition of an inheriting conformal Killing vector is restrictive and Coley and Tupper (1990*a*) showed that even for a perfect fluid energy–momentum tensor conformal Killing vectors do not in general map fluid flow lines conformally. In addition they point out that Robertson–Walker spacetimes do not contain a special conformal Killing vector; this motivates the study of other forms of vectors, e.g. inheriting conformal Killing vectors. Furthermore Coley and Tupper (1990*b*) proved that orthogonal synchronous perfect fluid spacetimes, other than Robertson–Walker, admit no proper inheriting conformal Killing vector. (The term 'proper' means that the conformal Killing vector does not reduce to the special case of Killing vector, homothetic vector or special conformal Killing vector). Coley and Tupper (1990c) found all spherically symmetric spacetimes admitting a proper inheriting conformal Killing vector for a perfect fluid with a barotropic equation of state satisfying the energy conditions. With the assumption that the conformal Killing vector $\boldsymbol{\xi}$ is parallel to the fluid 4–velocity $\mathbf{u}$, Coley (1991) showed that any perfect fluid solution of the Einstein field equations, with a barotropic equation of state and satisfying $\mu + p \neq 0$, is locally a Robertson–Walker model.

The field equations for a symmetry vector orthogonal to the fluid 4–velocity vector were presented by Saridakis and Tsamparlis (1991). The results obtained were applied to a conformal Killing vector in particular. A conformal collineation is generated by an affine conformal vector $\boldsymbol{\xi}$ so that

$$\mathcal{L}_{\boldsymbol{\xi}} g_{ab} = 2\psi g_{ab} + H_{ab}$$

where H_{ab} is a symmetric Killing tensor. (The properties of a Killing tensor, which generalises a Killing vector, are discussed by Kramer *et al* 1980). The affine conformal vector is a generalisation of the conformal Killing vector. The kinematics and dynamics of conformal collineations for anisotropic fluids were analysed by Mason and Maartens (1987). Coley and Tupper (1990d) investigated the relationship between special affine conformal vectors and conformal Killing vectors. Katzin *et al* (1969) introduced a symmetry called a curvature collineation defined by a vector $\boldsymbol{\xi}$ satisfying

$$\mathcal{L}_{\boldsymbol{\xi}} R^{a}{}_{bcd} = 0$$

Collinson (1970a), Katzin and Levine (1970a, b, c, 1971, 1972) and McIntosh and Halford (1981) have investigated the properties of curvature collineations. Note that a special case of curvature collineations is a Ricci collineation defined by

$$\mathcal{L}_{\boldsymbol{\xi}} R_{ab} = 0$$

Ricci collineations have been studied by Collinson (1970*b*), Melfo *et al* (1992), Oliver and Davis (1977, 1979) and Tsamparlis and Mason (1990). Curvature collineations are unfortunately very restrictive. Katzin *et al* (1969) showed that a curvature collineation is also a conformal motion if and only if the conformal Killing vector is special. Consequently the definition of a curvature collineation should be adapted to allow for more general symmetry properties. Duggal (1992) generalised the concept of curvature collineations by introducing the notion of curvature inheritance defined by

$$\mathcal{L}_{\boldsymbol{\xi}} R^{a}{}_{bcd} = 2\alpha R^{a}{}_{bcd}$$

where $\alpha = \alpha(x^a)$ is a scalar function in an n–dimensional differentiable manifold. This allows for proper ($\alpha \neq 0$) curvature inheritance to reduce to nonspecial conformal Killing vectors. Note that the curvature inheritance is related not only to conformal Killing vectors but to projection collineations as well. The geometrical and physical properties, in particular an equation of state, of curvature inheritances were investigated by Duggal (1992). In particular he showed that a spacetime admitting a curvature inheritance and a conformal Killing vector is necessarily conformally flat.

4 The Bianchi I Spacetime

4.1 Introduction

In this chapter we investigate the conformal symmetries of the Bianchi I spacetime. This is an attempt to consider the conformal geometry of spacetimes with less symmetry than models studied in the past. As the Bianchi I spacetime is the simplest of the nine Bianchi models the method of solution will provide guidelines for the study of conformal motions in the other more complicated Bianchi models. The spacetime geometry of the Bianchi I model is reviewed in §4.2. The Einstein field equations for a perfect fluid energy–momentum tensor are derived. In §4.3 we briefly review results on conformal geometry in other spacetimes obtained by other authors. The system of equations governing the conformal geometry in Bianchi I spacetime is presented. This is a coupled system of first order partial differential equations. The conformal Killing vector equations are integrated in §4.4 to generate the general conformal Killing vector subject to integrability conditions. The existence of a conformal symmetry places restrictions on the gravitational potentials. We provide all the details of the integration process as the procedure is not obvious. We do not fully solve the integrability conditions. However, we do obtain a particular solution showing that these conditions have a nonempty solution set. The special cases of Killing vectors, homothetic vectors and special conformal Killing vectors are considered in §4.5. Two other cases are isolated from the solution and discussed in §4.6.

4.2 Spacetime Geometry and Field Equations

The Robertson–Walker spacetimes are the standard cosmological models describing a homogeneous and isotropic universe. In coordinates $(x^a) = (t, x, y, z)$ the Robertson–Walker line element is given by

$$ds^2 = -dt^2 + \frac{R^2(t)}{\left[1 + \frac{1}{4}k(x^2 + y^2 + z^2)\right]^2}\,(dx^2 + dy^2 + dz^2)$$

where $k = 0, 1, -1$ and $R(t)$ is the scale factor. The conformal Killing vectors of this line element, for each of the three cases $k = 0, 1, -1$, have been found by Maartens and Maharaj (1986). The Robertson–Walker spacetimes admit a maximal G_{15} Lie algebra of conformal Killing vectors as they are conformally flat. These spacetimes have a G_6 Lie algebra of Killing vectors. A generalisation of the $k = 0$ Robertson–Walker metric is the locally rotationally symmetric metric

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + dz^2]$$

This spacetime is called AIa in the MacCallum (1980) classification and it is of type VI_4 in the Petrov classification (1969). The conformal geometry of this locally rotationally symmetric metric was studied by Moodley (1991). This model has a G_4 Lie algebra of Killing vectors.

In this chapter we consider the spatially homogeneous and anisotropic Bianchi I spacetime described by the line element

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2 \tag{4.1}$$

This spacetime is a generalisation of the locally rotationally symmetric spacetime given above and is often used in the study of anisotropic models. The line element (4.1) admits a G_3 Lie algebra of Killing vectors. The study of the conformal Killing vectors of the Bianchi I metric is more complicated than the previous cases because it has the least symmetry; in contrast the Robertson–Walker spacetimes have six Killing vectors and the locally rotationally symmetric spacetime has four. The Abelian Lie algebra of Killing vectors of (4.1) is spanned by

$$\mathbf{X}_1 = \frac{\partial}{\partial x}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial y}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial z}$$

For a detailed analysis of the group structure and classification of Bianchi cosmologies see Ellis and MacCallum (1969) and Ryan and Shepley (1975).

The nonvanishing connection coefficients (2.10) for the line element (4.1) are given by

$$\Gamma^0{}_{11} = A\dot{A} \qquad \Gamma^1{}_{01} = \frac{\dot{A}}{A}$$

$$\Gamma^0{}_{22} = B\dot{B} \qquad \Gamma^2{}_{02} = \frac{\dot{B}}{B}$$

$$\Gamma^0{}_{33} = C\dot{C} \qquad \Gamma^3{}_{03} = \frac{\dot{C}}{C}$$

where the dots denote differentiation with respect to the timelike coordinate t. With the help of the above connection coefficients we calculate the components of the Ricci tensor (2.24):

$$R_{00} = -\left(\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}\right) \tag{4.2}$$

$$R_{11} = A^2\left[\frac{\ddot{A}}{A}+\frac{\dot{A}}{A}\left(\frac{\dot{B}}{B}+\frac{\dot{C}}{C}\right)\right] \tag{4.3}$$

$$R_{22} = B^2\left[\frac{\ddot{B}}{B}+\frac{\dot{B}}{B}\left(\frac{\dot{A}}{A}+\frac{\dot{C}}{C}\right)\right] \tag{4.4}$$

$$R_{33} = C^2\left[\frac{\ddot{C}}{C}+\frac{\dot{C}}{C}\left(\frac{\dot{A}}{A}+\frac{\dot{B}}{B}\right)\right] \tag{4.5}$$

$$R_{ab} = 0, \quad a \neq b$$

The components (4.2)–(4.5) of the Ricci tensor together with the definition (2.25) yield the Ricci scalar

$$R = 2\left\{\frac{\ddot{A}}{A}+\frac{\ddot{B}}{B}+\frac{\ddot{C}}{C}+\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC}\right\} \tag{4.6}$$

The components of the Einstein tensor (2.27) for the Bianchi I spacetime (4.1) then become

$$G_{00} = \frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC} \tag{4.7}$$

$$G_{11} = -A^2\left[\frac{\ddot{B}}{B}+\frac{\dot{B}\dot{C}}{BC}+\frac{\ddot{C}}{C}\right] \tag{4.8}$$

$$G_{22} = -B^2\left[\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{C}}{AC}+\frac{\ddot{C}}{C}\right] \tag{4.9}$$

$$G_{33} = -C^2\left[\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{B}}{AB}+\frac{\ddot{B}}{B}\right] \tag{4.10}$$

$$G_{ab} = 0, \quad a \neq b$$

With the assistance of (4.2)–(4.6), the Einstein field equations (2.30) with vanishing cosmological constant, are equivalent to the system

$$\frac{\dot{A}\dot{B}}{AB}+\frac{\dot{A}\dot{C}}{AC}+\frac{\dot{B}\dot{C}}{BC} = \mu \tag{4.11}$$

$$\frac{\ddot{B}}{B}+\frac{\dot{B}\dot{C}}{BC}+\frac{\ddot{C}}{C} = -p \tag{4.12}$$

$$\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{C}}{AC}+\frac{\ddot{C}}{C} = -p \tag{4.13}$$

$$\frac{\ddot{A}}{A}+\frac{\dot{A}\dot{B}}{AB}+\frac{\ddot{B}}{B} = -p \tag{4.14}$$

for the Einstein tensor components (4.7)–(4.10), and the perfect fluid energy–momentum tensor (2.29).

4.3 Conformal Killing Vector Equation

The Einstein field equations (4.11)–(4.14) are highly nonlinear and it is difficult to find exact solutions in general. However the exact vacuum solution ($\mu = 0 = p$) has been found. This solution is called the Kasner solution. Also the solution for dust ($\mu \neq 0 = p$) and some other solutions with an equation of state are known (Kramer *et al* 1980). In an attempt to simplify the field equations we impose a symmetry requirement on the spacetime manifold, namely a conformal Killing vector symmetry. In many cases this assumption simplifies the field equations and leads to new solutions. Even if no new solutions are found this approach leads to a deeper understanding of the spacetime geometry concerned. The G_{15} Lie algebra of conformal motions in Minkowski spacetime is given by Choquet–Bruhat *et al* (1977). Maartens and Maharaj (1986) have found the fifteen conformal Killing vectors in the homogeneous and isotropic Robertson–Walker spacetimes for all three cases of the spatial geometry: $k = 0, 1, -1$. The conformal geometry of certain anisotropic locally rotationally symmetric spacetimes have been considered by Moodley (1991). In this section we investigate the conformal geometry of an example of the anisotropic Bianchi models, the Bianchi I spacetime. This spacetime is homogeneous but anisotropic and generalises the results of Moodley (1991). Also note that Maartens and Maharaj (1991) have found the conformal Killing vectors in the *pp*–wave spacetimes, the plane fronted gravitational waves with parallel rays, and have related their results to the Einstein–Maxwell and the Einstein–Klein–Gordon field equations. The *pp*–wave spacetimes admit only one Killing symmetry which is a null vector. Solutions to the Einstein field equations with a conformal symmetry have been investigated by Dyer *et al* (1987) and Maharaj *et al* (1991) for applications in cosmology. For astrophysical models with a conformal symmetry see Herrera and Ponce de León (1985a, b, c), Herrera *et al* (1984) and Maartens and Maharaj (1990).

The conformal Killing equation (2.36) for the line element (4.1) reduces to the following system of ten equations:

$$X^0_t = \psi \tag{4.15}$$

$$A^2 X^1_t - X^0_x = 0 \tag{4.16}$$

$$B^2 X^2_t - X^0_y = 0 \tag{4.17}$$

$$C^2 X^3_t - X^0_z = 0 \tag{4.18}$$

$$\dot{A} X^0 + A X^1_x = A\psi \tag{4.19}$$

$$B^2 X^2_x + A^2 X^1_y = 0 \tag{4.20}$$

$$C^2 X^3_x + A^2 X^1_z = 0 \tag{4.21}$$

$$\dot{B} X^0 + B X^2_y = B\psi \tag{4.22}$$

$$C^2 X^3_y + B^2 X^2_z = 0 \tag{4.23}$$

$$\dot{C} X^0 + C X^3_z = C\psi \tag{4.24}$$

In the above system the subscipts t, x, y and z denote partial differentiation. The system (4.15)–(4.24) is a coupled system of first order partial differential equations. We need to solve this system to obtain the conformal Killing vector $\mathbf{X} = (X^0, X^1, X^2, X^3)$ and the conformal factor ψ in terms of the metric functions $A(t), B(t)$ and $C(t)$. The solution obtained will be subject to integrability conditions. The integration of (4.15)–(4.24) is performed in the next section. As the system (4.15)–(4.24) is a coupled system of equations we attempt to find differential equations involving only one component of $\mathbf{X}$ to simplify the integration process.

4.4 Solution of the Conformal Equation

On subtracting the derivative of (4.20) with respect to z from the derivative of (4.21) with respect to y, we obtain $B^2X^2_{zx} - C^2X^3_{yx} = 0$. This result, together with the derivative of (4.23) with respect to x, yields the identities

$$X^2_{zx} = 0 \qquad\qquad X^3_{yx} = 0$$

Also, subtracting the derivative of (4.23) with respect to x from the derivative of (4.20) with respect to z, gives $A^2X^1_{yz} - C^2X^3_{yx} = 0$. This result and the derivative of (4.21) with respect to y yields the identities

$$X^1_{yz} = 0 \qquad\qquad X^3_{xy} = 0$$

Thus in general we have established

$$X^1_{yz} = 0 \tag{4.25}$$

$$X^2_{xz} = 0 \tag{4.26}$$

$$X^3_{xy} = 0 \tag{4.27}$$

for the components X^1, X^2, X^3.

On subtracting the derivative of (4.18) with respect to y from the derivative of (4.17) with respect to z, we obtain

$$B^2X^2_{zt} - C^2X^3_{yt} = 0 \tag{4.28}$$

From equation (4.23) we obtain the result

$$B^2X^2_{zt} + B^2\left(\frac{C^2}{B^2}\right)^{\cdot}X^3_y + C^2X^3_{yt} = 0 \tag{4.29}$$

Subtracting (4.28) from (4.29) yields the following partial differential equation in the component X^3:

$$B^2\left(\frac{C^2}{B^2}\right)^{\cdot} X^3_y + 2C^2 X^3_{yt} = 0$$

Integrating the above with respect to t we obtain the first derivative

$$X^3_y = \frac{B}{C}\tilde{f}(x, y, z)$$

where $\tilde{f}(x, y, z)$ is a function of integration. However identity (4.27) implies that X^3_y is independent of x. Thus we can write

$$X^3_y = \frac{B}{C}\mathcal{F}(y, z) \tag{4.30}$$

where $\tilde{f} = \mathcal{F}$. From equations (4.30) and (4.23) we have

$$X^2_z = -\frac{C}{B}\mathcal{F}(y, z) \tag{4.31}$$

With the forms given in (4.30)–(4.31) we note that the conformal Killing vector equation (4.23) is identically satisfied.

We now subtract the derivative of (4.17) with respect to x from the derivative of (4.16) with respect to y to obtain

$$A^2 X^1_{yt} - B^2 X^2_{xt} = 0 \tag{4.32}$$

From equation (4.20) we have the result

$$B^2 X^2_{xt} + B^2\left(\frac{A^2}{B^2}\right)^{\cdot} X^1_y + A^2 X^1_{yt} = 0 \tag{4.33}$$

Equations (4.32)–(4.33) give the following partial differential equation in the component X^1:

$$B^2\left(\frac{A^2}{B^2}\right)^{\cdot} X^1_y + 2A^2 X^1_{yt} = 0$$

We integrate the above to obtain

$$X^1_y = \frac{B}{A}\tilde{g}(x,y,z)$$

where $\tilde{g}(x,y,z)$ is a function of integration. However, identity (4.25) implies that X^1_y is independent of z. Hence we can write

$$X^1_y = \frac{B}{A}\mathcal{G}(x,y) \tag{4.34}$$

where $\tilde{g} = \mathcal{G}$. Substituting (4.34) into (4.20) yields

$$X^2_x = -\frac{A}{B}\mathcal{G}(x,y) \tag{4.35}$$

With the results (4.34)–(4.35) we observe that the conformal Killing vector equation (4.20) is identically satisfied.

Now we subtract the derivative of (4.18) with respect to x from the derivative of (4.16) with respect to z to obtain

$$A^2X^1_{zt} - C^2X^3_{xt} = 0 \tag{4.36}$$

From equation (4.21) we obtain the partial differential equation

$$C^2X^3_{xt} + C^2\left(\frac{A^2}{C^2}\right)^{\cdot} X^1_z + A^2X^1_{zt} = 0 \tag{4.37}$$

Addition of equations (4.36) and (4.37) gives a partial differential equation in the component X^1:

$$C^2\left(\frac{A^2}{C^2}\right)^{\cdot} X^1_z + 2A^2X^1_{zt} = 0$$

The above equation can be integrated to obtain

$$X^1_z = \frac{C}{A}\tilde{h}(x,y,z)$$

where $\tilde{h}(x,y,z)$ is a function of integration. However identity (4.25) implies that X^1_z is independent of y. Therefore we can write

$$X^1_z = \frac{C}{A}\mathcal{H}(x,z) \tag{4.38}$$

where $\tilde{h} = \mathcal{H}$. Substituting (4.38) into (4.21) gives the result

$$X^3_x = -\frac{A}{C}\mathcal{H}(x,z) \tag{4.39}$$

Note that the conformal Killing vector equation (4.21) is identically satisfied if equations (4.38)–(4.39) hold.

Integrating (4.34) with respect to y gives the component

$$X^1 = \frac{B}{A}\mathcal{G}^y(x,y) + \tilde{\alpha}(t,x,z)$$

where we have set

$$\mathcal{G}^y(x,y) = \int \mathcal{G}(x,y)dy$$

and $\tilde{\alpha}(t,x,z)$ is a function of integration. Henceforth we will use the notation where the superscripts x,y,z denote integration. Substituting this form of X^1 into equation (4.38) implies that

$$\tilde{\alpha}_z = \frac{C}{A}\mathcal{H}(x,z)$$

Upon integration this partial differential equation yields

$$\tilde{\alpha} = \frac{C}{A}\mathcal{H}^z(x,z) + \alpha(t,x)$$

where we let

$$\mathcal{H}^z(x,z) = \int \mathcal{H}(x,z)dz$$

and $\alpha(t,x)$ is a function of integration. Thus we have established that the component X^1 is given by

$$X^1 = \frac{B}{A}\mathcal{G}^y(x,y) + \frac{C}{A}\mathcal{H}^z(x,z) + \alpha(t,x)$$

Similarly, integrating (4.35) with respect to x yields the component

$$X^2 = -\frac{A}{B}\mathcal{G}^x(x,y) + \tilde{\beta}(t,y,z)$$

where we have let

$$\mathcal{G}^x(x,y) = \int \mathcal{G}(x,y)dx$$

and $\tilde{\beta}(t,y,z)$ is a function of integration. Substituting the above form of X^2 into equation (4.31) we obtain

$$\tilde{\beta}_z = -\frac{C}{B}\mathcal{F}(y,z)$$

We integrate this partial differential equation to get

$$\tilde{\beta} = -\frac{C}{B}\mathcal{F}^z(y,z) + \beta(t,y)$$

where we have defined

$$\mathcal{F}^z(y,z) = \int \mathcal{F}(y,z)dz$$

and $\beta(t,y)$ is a function of integration. Thus we have the result that the component X^2 is given by

$$X^2 = -\frac{A}{B}\mathcal{G}^x(x,y) - \frac{C}{B}\mathcal{F}^z(y,z) + \beta(t,y)$$

Also integrating (4.39) with respect to x we obtain the component

$$X^3 = -\frac{A}{C}\mathcal{H}^x(x,z) + \tilde{\gamma}(t,y,z)$$

where we have set

$$\mathcal{H}^x(x,z) = \int \mathcal{H}(x,z)dx$$

and $\tilde{\gamma}(t,y,z)$ is a function of integration. On substituting this form of X^3 into equation (4.30) we obtain the partial differential equation

$$\tilde{\gamma}_y = \frac{B}{C}\mathcal{F}(y,z)$$

Integrating this equation gives

$$\tilde{\gamma} = \frac{B}{C}\mathcal{F}^y(y,z) + \gamma(t,z)$$

where we have defined

$$\mathcal{F}^y(y,z) = \int \mathcal{F}(y,z)dy$$

and $\gamma(t,z)$ is a function of integration. Hence we have established that

$$X^3 = -\frac{A}{C}\mathcal{H}^x(x,z) + \frac{B}{C}\mathcal{F}^y(y,z) + \gamma(t,z)$$

for the component X^3.

It is convenient to collect the results obtained thus far. We have established that

$$X^1 = \frac{B}{A}\mathcal{G}^y(x,y) + \frac{C}{A}\mathcal{H}^z(x,z) + \alpha(t,x) \tag{4.40}$$

$$X^2 = -\frac{A}{B}\mathcal{G}^x(x,y) - \frac{C}{B}\mathcal{F}^z(y,z) + \beta(t,y) \tag{4.41}$$

$$X^3 = -\frac{A}{C}\mathcal{H}^x(x,z) + \frac{B}{C}\mathcal{F}^y(y,z) + \gamma(t,z) \tag{4.42}$$

for the components X^1, X^2, X^3. With (4.40)–(4.42) we find that the conformal Killing vector equations (4.20)–(4.21) and (4.23) are identically satisfied. It remains to solve the other equations of the system (4.15)–(4.24) to obtain X^0 and the conformal factor ψ. We first obtain a form for the timelike component X^0 to supplement the spacelike components X^1, X^2, X^3.

On substituting (4.40) into (4.16) we obtain

$$X^0_x = A^2\left(\frac{B}{A}\right)^{\cdot}\mathcal{G}^y(x,y) + A^2\left(\frac{C}{A}\right)^{\cdot}\mathcal{H}^z(x,z) + A^2\alpha_t(t,x) \tag{4.43}$$

Similarly equations (4.41) and (4.17) give the differential equation

$$X^0_y = -B^2\left(\frac{A}{B}\right)^{\cdot}\mathcal{G}^x(x,y) - B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^z(y,z) + B^2\beta_t(t,y) \tag{4.44}$$

and substituting (4.42) into (4.18) yields

$$X^0_z = -C^2\left(\frac{A}{C}\right)^{\cdot}\mathcal{H}^x(x,z) + C^2\left(\frac{B}{C}\right)^{\cdot}\mathcal{F}^y(y,z) + C^2\gamma_t(t,z) \tag{4.45}$$

The system (4.43)–(4.45) may be solved to obtain the component X^0. Integrating (4.43) with respect to x gives

$$X^0 = A^2\left(\frac{B}{A}\right)^{\cdot}\mathcal{G}^{yx}(x,y) + A^2\left(\frac{C}{A}\right)^{\cdot}\mathcal{H}^{zx}(x,z) + A^2\alpha^x_t(t,x) + \tilde{\tilde{\rho}}\,(t,y,z) \tag{4.46}$$

where $\tilde{\tilde{\rho}}\,(t,y,z)$ is a function of integration and we have set

$$\begin{aligned}
\mathcal{G}^{yx}(x,y) &= \int \mathcal{G}^y(x,y)dx \\
\mathcal{H}^{zx}(x,z) &= \int \mathcal{H}^z(x,z)dx \\
\alpha^x_t(t,x) &= \int \alpha_t(t,x)dx
\end{aligned}$$

Substituting (4.46) into (4.44) we obtain

$$\tilde{\tilde{\rho}}_y(t,y,z) = -\left[B^2\left(\frac{A}{B}\right)^{\cdot} + A^2\left(\frac{B}{A}\right)^{\cdot}\right]\mathcal{G}^x(x,y) - B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^z(y,z) + B^2\beta_t(t,y)$$

$$\Rightarrow \quad \tilde{\tilde{\rho}}_y(t,y,z) = -B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^z(y,z) + B^2\beta_t(t,y)$$

Integrating the above with respect to y we obtain

$$\tilde{\tilde{\rho}}\,(t,y,z) = -B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^{zy}(y,z) + B^2\beta^y_t(t,y) + \tilde{\rho}(t,z)$$

where we let

$$\mathcal{F}^{zy}(y,z) = \int \mathcal{F}^{z}(y,z)dy$$

$$\beta_t^y(t,y) = \int \beta_t(t,y)dy$$

and $\tilde{\rho}(t,z)$ is a function of integration. On substutiting $\tilde{\tilde{\rho}}(t,y,z)$ in (4.46) we obtain the component

$$\begin{aligned} X^0 &= A^2\left(\frac{B}{A}\right)^{\cdot}\mathcal{G}^{yx}(x,y) + A^2\left(\frac{C}{A}\right)^{\cdot}\mathcal{H}^{zx}(x,z) + A^2\alpha_t^x(t,x) \\ &- B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^{zy}(y,z) + B^2\beta_t^y(t,y) + \tilde{\rho}(t,z) \end{aligned} \tag{4.47}$$

Now substituting (4.47) in (4.45) we obtain

$$\begin{aligned} \tilde{\rho}_z(t,z) &= -\mathcal{H}^x(x,z)\left[C^2\left(\frac{A}{C}\right)^{\cdot} + A^2\left(\frac{C}{A}\right)^{\cdot}\right] \\ &+ \mathcal{F}^y(y,z)\left[C^2\left(\frac{B}{C}\right)^{\cdot} + B^2\left(\frac{C}{B}\right)^{\cdot}\right] + C^2\gamma_t(t,z) \end{aligned}$$

$$\Rightarrow \quad \tilde{\rho}_z(t,z) = C^2\gamma_t(t,z)$$

Integrating the equation above gives

$$\tilde{\rho}(t,z) = C^2\gamma_t^z(t,z) + \rho(t)$$

where we set

$$\gamma_t^z(t,z) = \int \gamma_t(t,z)dz$$

and the function $\rho(t)$ results from the integration process.

Thus we have established that (4.47) can be written as

$$X^0 = A^2\left(\frac{B}{A}\right)^{\cdot}\mathcal{G}^{yx}(x,y) + A^2\left(\frac{C}{A}\right)^{\cdot}\mathcal{H}^{zx}(x,z) + A^2\alpha_t^x(t,x)$$

$$- B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^{zy}(y,z) + B^2\beta_t^y(t,y) + C^2\gamma_t^z(t,z) + \rho(t) \tag{4.48}$$

where we have used the above form of $\tilde{\rho}(t,z)$. Thus we have found the timelike component X^0 of $\mathbf{X}$. With X^0 given by (4.48) we obtain the conformal factor from equation (4.15):

$$\begin{aligned}\psi &= \left[A^2\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G}^{yx}(x,y) + \left[A^2\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H}^{zx}(x,z) - \left[B^2\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F}^{zy}(y,z)\\ &+ \left[A^2\alpha_t^x(t,x) + B^2\beta_t^y(t,y) + C^2\gamma_t^z(t,z) + \rho(t)\right]_t \end{aligned} \tag{4.49}$$

Note that with the above forms of X^0 and ψ the conformal Killing vector equations (4.15)–(4.18), (4.20)–(4.21) and (4.23) are satisfied. It remains to integrate the equations (4.19), (4.22) and (4.24). These conformal Killing vector equations will generate integrability conditions that will govern the existence of a conformal Killing vector in the Bianchi I spacetime.

Substituting the timelike component X^0, given by (4.48), and the conformal factor ψ, given by (4.49), into the remaining conformal Killing vector equations (4.19), (4.22) and (4.24) we obtain after simplification:

$$\begin{aligned}&\frac{B}{A}\left\{\mathcal{G}_x^y(x,y) - \frac{A^2}{B}\left[A\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G}^{yx}(x,y)\right\}\\ &+ \frac{C}{A}\left\{\mathcal{H}_x^z(x,z) - \frac{A^2}{C}\left[A\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H}^{zx}(x,z)\right\}\\ &+ A\left[\frac{B^2}{A}\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F}^{zy}(y,z) + \alpha_x(t,x) - A[A\alpha_t^x(t,x)]_t - A\left(\frac{\rho(t)}{A}\right)^{\cdot}\\ &+ \frac{\dot{A}}{A}\left[B^2\beta_t^y(t,y) + C^2\gamma_t^z(t,z)\right] - \left[B^2\beta_t^y(t,y) + C^2\gamma_t^z(t,z)\right]_t = 0\end{aligned} \tag{4.50}$$

$$- \frac{A}{B}\left\{\mathcal{G}_y^x(x,y) + \frac{B^2}{A}\left[\frac{A^2}{B}\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G}^{yx}(x,y)\right\}$$

$$
\begin{aligned}
&- \frac{C}{B}\left\{\mathcal{F}^z_y(y,z) - \frac{B^2}{C}\left[B\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F}^{zy}(y,z)\right\} \\
&- B\left[\frac{A^2}{B}\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H}^{zx}(x,z) + \beta_y(t,y) - B[B\beta^y_t(t,y)]_t - B\left(\frac{\rho(t)}{B}\right)^{\cdot} \\
&+ \frac{\dot{B}}{B}\left[A^2\alpha^x_t(t,x) + C^2\gamma^z_t(t,z)\right] - \left[A^2\alpha^x_t(t,x) + C^2\gamma^z_t(t,z)\right]_t = 0
\end{aligned} \tag{4.51}
$$

$$
\begin{aligned}
&- \frac{A}{C}\left\{\mathcal{H}^x_z(x,z) + \frac{C^2}{A}\left[\frac{A^2}{C}\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H}^{zx}(x,z)\right\} \\
&+ \frac{B}{C}\left\{\mathcal{F}^y_z(y,z) + \frac{C^2}{B}\left[\frac{B^2}{C}\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F}^{zy}(y,z)\right\} \\
&- C\left[\frac{A^2}{C}\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G}^{yx}(x,y) + \gamma_z(t,z) - C[C\gamma^z_t(t,z)]_t - C\left(\frac{\rho(t)}{C}\right)^{\cdot} \\
&+ \frac{\dot{C}}{C}\left[A^2\alpha^x_t(t,x) + B^2\beta^y_t(t,y)\right] - \left[A^2\alpha^x_t(t,x) + B^2\beta^y_t(t,y)\right]_t = 0
\end{aligned} \tag{4.52}
$$

At this point we observe that we have, in fact, generated the general solution of the conformal Killing vector equations (4.15)–(4.24). The timelike component X^0 is given by (4.48), the spatial components X^1, X^2, X^3 are given by (4.40)–(4.42) and the conformal factor ψ is given by (4.49). This solution is subject to the three integrability conditions (4.50)–(4.52). The integrability conditions involve the functions of integration $\mathcal{F}, \mathcal{G}, \mathcal{H}, \alpha, \beta, \gamma$ and ρ and the metric functions $A(t), B(t)$ and $C(t)$. Ideally we would want to obtain the functional forms of $\mathcal{F}, \mathcal{G}, \mathcal{H}, \alpha, \beta, \gamma$ and ρ explicitly or equations governing their behaviour. Due to the complexity of the integrability conditions (4.50)–(4.52) we have not achieved this yet. However, we do find a particular solution to equations (4.50)–(4.52), illustrating that the integrability conditions have a nonempty solution set. In the remainder of this section we investigate the integrability conditions in detail and obtain the aforementioned particular solution.

In the given form the equations (4.50)–(4.52) are difficult to analyse as they are integral equations. We obtain differential equations from the integral equations which govern the behaviour of $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$. Differentiating equations (4.50)–(4.52) with respect to x and y we obtain the following system of equations in the integration function $\mathcal{G}(x,y)$:

$$\mathcal{G}_{xx} - \frac{A^2}{B}\left[A\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G} = 0 \tag{4.53}$$

$$\mathcal{G}_{yy} + \frac{B^2}{A}\left[\frac{A^2}{B}\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G} = 0 \tag{4.54}$$

$$\left[\frac{A^2}{C}\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G} = 0 \tag{4.55}$$

Similarly, differentiating equations (4.50)–(4.52) with respect to y and z we obtain the following set of equations for the function $\mathcal{F}(y,z)$:

$$\left[\frac{B^2}{A}\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F} = 0 \tag{4.56}$$

$$\mathcal{F}_{yy} - \frac{B^2}{C}\left[B\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F} = 0 \tag{4.57}$$

$$\mathcal{F}_{zz} + \frac{C^2}{B}\left[\frac{B^2}{C}\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F} = 0 \tag{4.58}$$

On differentiating equations (4.50)–(4.52) with respect to x and z we obtain the following system of equations for the function of integration $\mathcal{H}(x,z)$:

$$\mathcal{H}_{xx} - \frac{A^2}{C}\left[A\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H} = 0 \tag{4.59}$$

$$\left[\frac{A^2}{B}\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H} = 0 \tag{4.60}$$

$$\mathcal{H}_{zz} + \frac{C^2}{A}\left[\frac{A^2}{C}\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H} \;=\; 0 \tag{4.61}$$

The differential equations (4.53)–(4.61) can be integrated to obtain the functional dependence of $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$. If a function of integration vanishes then the corresponding system of three equations is identically satisfied. Conditions are placed on the metric functions $A(t), B(t)$ and $C(t)$ if the function of integration is nonvanishing.

With the help of the conditions (4.53)–(4.61) we find that the integrability conditions (4.50)–(4.52) reduce to the simpler form:

$$\begin{aligned}
&\frac{\dot{A}}{A}\left[B^2\beta^y_t(t,y) + C^2\gamma^z_t(t,z)\right] - \left[B^2\beta^y_t(t,y) + C^2\gamma^z_t(t,z)\right]_t \\
&- A[A\alpha^x_t(t,x)]_t - A\left(\frac{\rho(t)}{A}\right)^{\cdot} + \alpha_x(t,x) = 0
\end{aligned} \tag{4.62}$$

$$\begin{aligned}
&\frac{\dot{B}}{B}\left[A^2\alpha^x_t(t,x) + C^2\gamma^z_t(t,z)\right] - \left[A^2\alpha^x_t(t,x) + C^2\gamma^z_t(t,z)\right]_t \\
&- B[B\beta^y_t(t,y)]_t - B\left(\frac{\rho(t)}{B}\right)^{\cdot} + \beta_y(t,y) = 0
\end{aligned} \tag{4.63}$$

$$\begin{aligned}
&\frac{\dot{C}}{C}\left[A^2\alpha^x_t(t,x) + B^2\beta^y_t(t,y)\right] - \left[A^2\alpha^x_t(t,x) + B^2\beta^y_t(t,y)\right]_t \\
&- C[C\gamma^z_t(t,z)]_t - C\left(\frac{\rho(t)}{C}\right)^{\cdot} + \gamma_z(t,z) = 0
\end{aligned} \tag{4.64}$$

The functions of integration arising from equations (4.53)–(4.61), involving $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$, have been set to zero for convenience. In future work we intend to consider the general case with nonvanishing functions of integration.

Differentiating (4.62)–(4.64) with respect to x, y and z in turn gives systems of equations in the functions of integration α, β and γ. For the function $\alpha(t, x)$ we obtain the following system of equations

$$\alpha_{xx} - A[A\alpha_t]_t = 0 \tag{4.65}$$

$$\left[A^2\alpha_t\right]_t - \frac{\dot{B}}{B}\left[A^2\alpha_t\right] = 0 \tag{4.66}$$

$$\left[A^2\alpha_t\right]_t - \frac{\dot{C}}{C}\left[A^2\alpha_t\right] = 0 \tag{4.67}$$

For the function $\beta(t, y)$ we have

$$\beta_{yy} - B[B\beta_t]_t = 0 \tag{4.68}$$

$$\left[B^2\beta_t\right]_t - \frac{\dot{A}}{A}\left[B^2\beta_t\right] = 0 \tag{4.69}$$

$$\left[B^2\beta_t\right]_t - \frac{\dot{C}}{C}\left[B^2\beta_t\right] = 0 \tag{4.70}$$

Finally for the function $\gamma(t, z)$ we obtain the system

$$\gamma_{zz} - C[C\gamma_t]_t = 0 \tag{4.71}$$

$$\left[C^2\gamma_t\right]_t - \frac{\dot{B}}{B}\left[C^2\gamma_t\right] = 0 \tag{4.72}$$

$$\left[C^2\gamma_t\right]_t - \frac{\dot{A}}{A}\left[C^2\gamma_t\right] = 0 \tag{4.73}$$

We can integrate the above systems to obtain $\alpha(t, x), \beta(t, y)$ and $\gamma(t, z)$.

The solution of equations (4.66)–(4.67) is of the form $\alpha_t = 0$. Then (4.65) implies $\alpha_{xx} = 0$. Thus the solution to the system (4.65)–(4.67) is given by

$$\alpha = \alpha_1 x + \alpha_2 \tag{4.74}$$

where α_1 and α_2 are constants. The solution of equations (4.69)–(4.70) is given by $\beta_t = 0$. Equation (4.68) implies $\beta_{yy} = 0$ so that the system (4.68)–(4.70) has the solution

$$\beta = \beta_1 y + \beta_2 \tag{4.75}$$

where β_1 and β_2 are constants. The solution of equations (4.72)–(4.73) is of the form $\gamma_t = 0$. Then (4.71) implies $\gamma_{zz} = 0$. Hence the system (4.71)–(4.73) has the solution

$$\gamma = \gamma_1 z + \gamma_2 \tag{4.76}$$

where γ_1 and γ_2 are constants. Note that the solutions (4.74)–(4.76) are valid only if the functions $A(t), B(t)$ and $C(t)$ are not proportional to each other.

With the forms given by (4.74)–(4.76) we find that integrability conditions (4.62)–(4.64) become

$$A\left(\frac{\rho}{A}\right)^{\cdot} = \alpha_1 \tag{4.77}$$

$$B\left(\frac{\rho}{B}\right)^{\cdot} = \beta_1 \tag{4.78}$$

$$C\left(\frac{\rho}{C}\right)^{\cdot} = \gamma_1 \tag{4.79}$$

which are essentially the reduced forms of the conformal Killing vector equations (4.19), (4.22) and (4.24). Therefore with the restrictions (4.53)–(4.61) and the requirement that functions arising from the integration of these equations vanish, we find that the integrability conditions (4.50)–(4.52) are satisfied. Consequently all the conformal Killing vector equations (4.15)–(4.24) have been integrated. We note that by introducing the transformation

$$R(t) = \int \rho^{-1} dt$$

we obtain the following solution to the system (4.77)–(4.79):

$$A(t) = A_0\left[\dot{R}\exp(\alpha_1 R)\right]^{-1}$$

$$B(t) = B_0\left[\dot{R}\exp(\beta_1 R)\right]^{-1}$$

$$C(t) = C_0\left[\dot{R}\exp(\gamma_1 R)\right]^{-1}$$

where A_0, B_0 and C_0 are constants.

Thus the coupled system of equations (4.15)–(4.24) has the solution (4.40)–(4.42), (4.48)–(4.49) subject to the conditions (4.53)–(4.61), (4.74)–(4.79). Collecting the various results for easy reference we have the solution

$$\begin{aligned} X^0 &= A^2\left(\frac{B}{A}\right)^{\cdot}\mathcal{G}^{yx}(x,y) + A^2\left(\frac{C}{A}\right)^{\cdot}\mathcal{H}^{zx}(x,z) \\ &\quad - B^2\left(\frac{C}{B}\right)^{\cdot}\mathcal{F}^{zy}(y,z) + \rho(t) \end{aligned} \tag{4.80}$$

$$X^1 = \frac{B}{A}\mathcal{G}^y(x,y) + \frac{C}{A}\mathcal{H}^z(x,z) + \alpha_1 x + \alpha_2 \tag{4.81}$$

$$X^2 = -\frac{A}{B}\mathcal{G}^x(x,y) - \frac{C}{B}\mathcal{F}^z(y,z) + \beta_1 y + \beta_2 \tag{4.82}$$

$$X^3 = -\frac{A}{C}\mathcal{H}^x(x,z) + \frac{B}{C}\mathcal{F}^y(y,z) + \gamma_1 z + \gamma_2 \tag{4.83}$$

$$\begin{aligned}\psi &= \left[A^2\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G}^{yx}(x,y) + \left[A^2\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H}^{zx}(x,z) \\ &- \left[B^2\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F}^{zy}(y,z) + \dot{\rho}(t)\end{aligned} \tag{4.84}$$

subject to the following integrability conditions

$$\mathcal{F}_{yy} - \frac{B^2}{C}\left[B\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F} = 0 \tag{4.85}$$

$$\mathcal{F}_{zz} + \frac{C^2}{B}\left[\frac{B^2}{C}\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F} = 0 \tag{4.86}$$

$$\left[\frac{B^2}{A}\left(\frac{C}{B}\right)^{\cdot}\right]^{\cdot}\mathcal{F} = 0 \tag{4.87}$$

$$\mathcal{G}_{xx} - \frac{A^2}{B}\left[A\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G} = 0 \tag{4.88}$$

$$\mathcal{G}_{yy} + \frac{B^2}{A}\left[\frac{A^2}{B}\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G} = 0 \tag{4.89}$$

$$\left[\frac{A^2}{C}\left(\frac{B}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{G} = 0 \tag{4.90}$$

$$\mathcal{H}_{xx} - \frac{A^2}{C}\left[A\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H} = 0 \tag{4.91}$$

$$\mathcal{H}_{zz} + \frac{C^2}{A}\left[\frac{A^2}{C}\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H} = 0 \tag{4.92}$$

$$\left[\frac{A^2}{B}\left(\frac{C}{A}\right)^{\cdot}\right]^{\cdot}\mathcal{H} = 0 \tag{4.93}$$

$$\alpha_1 = A\left(\frac{\rho}{A}\right)^{\cdot} \tag{4.94}$$

$$\beta_1 = B\left(\frac{\rho}{B}\right)^{\cdot} \tag{4.95}$$

$$\gamma_1 = C\left(\frac{\rho}{C}\right)^{\cdot} \tag{4.96}$$

Hence we have found the conformal Killing vector in the Bianchi I spacetime subject to twelve integrability conditions. This solution holds when the functions of integration arising from (4.53)–(4.61) vanish. The conformal Killing vector (4.80)–(4.83) with conformal factor (4.84), subject to the integrability conditions (4.85)–(4.96), generalises the locally rotationally symmetric results of Moodley (1991). The existence of a conformal symmetry places restrictions on the metric functions $A(t), B(t)$ and $C(t)$. It would be interesting to determine the effect of the conformal Killing symmetry obtained on the Einstein field equations; this is an area of ongoing research.

4.5 Special Cases

In this section we consider special cases arising from the conformal Killing vector solution (4.80)–(4.84). We can obtain the Killing vector of the spacetime (4.1) from the conformal Killing vector solution (4.80)–(4.84) such that $\psi = 0$. The following are the restrictions on the functions of integration in order to obtain a Killing vector:

$$\mathcal{F} = 0$$

$$\mathcal{G} = 0$$

$$\mathcal{H} = 0$$

$$\rho = 0$$

With the above values the integrability conditions (4.85)–(4.96) are identically satisfied. The components of the Killing vector $\mathbf{X}$ are given by

$$X^0 = 0$$

$$X^1 = \alpha_2$$

$$X^2 = \beta_2$$

$$X^3 = \gamma_2$$

so that the Killing vector can be written as

$$\mathbf{X} = \alpha_2 \frac{\partial}{\partial x} + \beta_2 \frac{\partial}{\partial y} + \gamma_2 \frac{\partial}{\partial z}$$

Clearly the Lie algebra of Killing vectors $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ of the Bianchi I spacetime may be regained from our Killing vector $\mathbf{X}$.

The homothetic vector of the spacetime (4.1) is obtained as a special case of the solution of (4.80)–(4.84) such that $\psi_{,a} = 0 \neq \psi$. For a homothetic vector the functions of integration satisfy:

$$\mathcal{F} = 0$$

$$\mathcal{G} = 0$$

$$\mathcal{H} = 0$$

$$\rho = \psi t + \theta$$

where θ is a constant. The integrability conditions (4.85)–(4.93) are identically satisfied, and equations (4.94)-(4.96) restrict the metric functions:

$$\frac{\dot{A}}{A} = \frac{\psi - \alpha_1}{\psi t + \theta}$$

$$\frac{\dot{B}}{B} = \frac{\psi - \beta_1}{\psi t + \theta}$$

$$\frac{\dot{C}}{C} = \frac{\psi - \gamma_1}{\psi t + \theta}$$

Therefore the existence of a homothetic vector places the following restrictions on the gravitational field:

$$A = \theta_1(\psi t + \theta)^{(\psi - \alpha_1)/\psi}$$

$$B = \theta_2(\psi t + \theta)^{(\psi - \beta_1)/\psi}$$

$$C = \theta_3(\psi t + \theta)^{(\psi - \gamma_1)/\psi}$$

where θ_1, θ_2 and θ_3 are constants resulting from the integration process. The components of the conformal vector $\mathbf{X}$ become

$$
\begin{aligned}
X^0 &= \psi t + \theta \\
X^1 &= \alpha_1 x + \alpha_2 \\
X^2 &= \beta_1 y + \beta_2 \\
X^3 &= \gamma_1 z + \gamma_2
\end{aligned}
$$

so that we can write

$$\mathbf{X} = (\psi t + \theta)\frac{\partial}{\partial t} + (\alpha_1 x + \alpha_2)\frac{\partial}{\partial x} + (\beta_1 y + \beta_2)\frac{\partial}{\partial y} + (\gamma_1 z + \gamma_2)\frac{\partial}{\partial z}$$

for the homothetic vector. The nonvanishing conformal factor is given by

$$\psi = \dot{\rho}$$

where ψ is a constant.

The homothetic Killing vector places restrictions on the metric functions $A(t), B(t)$ and $C(t)$ unlike the case of the Killing vector. Note that the line element (4.1) becomes

$$ds^2 = -dt^2 + \theta_1(\psi t + \theta)^{2p_1} dx^2 + \theta_2(\psi t + \theta)^{2p_2} dy^2 + \theta_3(\psi t + \theta)^{2p_3} dz^2$$

where we have set

$$p_1 = \frac{\psi - \alpha_1}{\psi} \qquad p_2 = \frac{\psi - \beta_1}{\psi} \qquad p_3 = \frac{\psi - \gamma_1}{\psi}$$

for the homothetic vector $\mathbf{X}$. This line element satisfies the vacuum Einstein field equations (4.11)–(4.14) if

$$p_1 + p_2 + p_3 = 1$$

$$p_1^2 + p_2^2 + p_3^2 = 1$$

Consequently we have generated the Kasner solution. We have established that the only homothetic vector $\mathbf{X}$ admitted by the Bianchi I spacetime is contained in the familiar Kasner solution.

In order to obtain a special conformal Killing vector of the spacetime (4.1) from the conformal Killing vector solution (4.80)–(4.84) we require that $\psi_{;ab} = 0 \neq \psi_{,a}$. Then the equations $\psi_{,12} = 0, \psi_{,13} = 0, \psi_{,23} = 0$ place the following restrictions on the functions of integration in order to obtain a special conformal Killing vector:

$$\mathcal{F} = 0$$

$$\mathcal{G} = 0$$

$$\mathcal{H} = 0$$

$$\dot{\rho} = \psi$$

These equations imply that ψ is a function of time. However in addition we have $\psi_{xx} = \dot{A}A\psi_t$ which implies that ψ is a constant. Thus the special conformal Killing vector reduces to a homothetic vector. Therefore the Bianchi I spacetime does not admit a proper special conformal Killing vector.

4.6 Some Other Cases

It is difficult to analyse the conformal Killing vector solution in the form given by equations (4.80)–(4.84). In an attempt to simplify the solution we place restrictions on the functions of integration $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$. We discuss two such restrictions below.

<u>CASE I</u>: $\mathcal{F} = \mathcal{G} = \mathcal{H} = 0$

In this case equations (4.80)–(4.84) reduce to

$$X^0 = \rho(t)$$

$$X^1 = \alpha_1 x + \alpha_2$$

$$X^2 = \beta_1 y + \beta_2$$

$$X^3 = \gamma_1 z + \gamma_2$$

$$\psi = \dot{\rho}(t)$$

The integrability conditions (4.85)–(4.93) are identically satisfied. The remaining conditions (4.94)–(4.96) yield

$$A(t) = A_0\Big[\dot{R}\exp\left(\alpha_1 R\right)\Big]^{-1}$$

$$B(t) = B_0\Big[\dot{R}\exp\left(\beta_1 R\right)\Big]^{-1}$$

$$C(t) = C_0\Big[\dot{R}\exp\left(\gamma_1 R\right)\Big]^{-1}$$

as indicated previously. Note that since α_1, β_1 and γ_1 are unequal in general the metric functions $A(t), B(t)$ and $C(t)$ are not proportional to each other.

Setting $\rho = \alpha_1 = \beta_1 = \gamma_1 = 0$ in the above solution we regain the Killing vector. With the conditions $\alpha_1 = \beta_1 = \gamma_1 = 0$ and requiring that ρ is linear in t we obtain the homothetic vector. We may also obtain our previous result on special conformal Killing vectors from the above solution by requiring that $\psi_{;ab} = 0 \neq \psi_{,a}$.

<u>CASE II</u>: $\mathcal{F} \neq 0, \mathcal{G} \neq 0, \mathcal{H} \neq 0$

We can show that for this case the conformal Killing vector solution becomes

$$X^0 = \mathcal{J}_1 C \mathcal{G}^{yx}(x,y) + \mathcal{J}_3 B \mathcal{H}^{zx}(x,z) - \mathcal{J}_2 A \mathcal{F}^{zy}(y,z) + \rho(t)$$

$$X^1 = \frac{B}{A}\mathcal{G}^{y}(x,y) + \frac{C}{A}\mathcal{H}^{z}(x,z) + \alpha_1 x + \alpha_2$$

$$X^2 = -\frac{A}{B}\mathcal{G}^{x}(x,y) - \frac{C}{B}\mathcal{F}^{z}(y,z) + \beta_1 y + \beta_2$$

$$X^3 = -\frac{A}{C}\mathcal{H}^{x}(x,z) + \frac{B}{C}\mathcal{F}^{y}(y,z) + \gamma_1 z + \gamma_2$$

$$\psi = \mathcal{J}_1 \dot{C} \mathcal{G}^{yx}(x,y) + \mathcal{J}_3 \dot{B} \mathcal{H}^{zx}(x,z) - \mathcal{J}_2 \dot{A} \mathcal{F}^{zy}(y,z) + \dot{\rho}(t)$$

where $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ are constants. The functions of integration $\mathcal{F} \neq 0, \mathcal{G} \neq 0$ and $\mathcal{H} \neq 0$ take a simple form in this class of solutions. These may be compactly expressed as follows

$$\begin{aligned}\mathcal{F} &= a_1 \exp\left\{y\sqrt{-\mathcal{J}_1\mathcal{J}_2} + iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\} + b_1 \exp\left\{y\sqrt{-\mathcal{J}_1\mathcal{J}_2} - iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\} \\ &\quad + c_1 \exp\left\{-y\sqrt{-\mathcal{J}_1\mathcal{J}_2} + iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\} + d_1 \exp\left\{-y\sqrt{-\mathcal{J}_1\mathcal{J}_2} - iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\}\end{aligned}$$

$$\begin{aligned}\mathcal{G} &= a_2 \exp\left\{x\sqrt{\mathcal{J}_1\mathcal{J}_3} + iy\sqrt{\mathcal{J}_1\mathcal{J}_2}\right\} + b_2 \exp\left\{x\sqrt{\mathcal{J}_1\mathcal{J}_3} - iy\sqrt{\mathcal{J}_1\mathcal{J}_2}\right\} \\ &\quad + c_2 \exp\left\{-x\sqrt{\mathcal{J}_1\mathcal{J}_3} + iy\sqrt{\mathcal{J}_1\mathcal{J}_2}\right\} + d_2 \exp\left\{-x\sqrt{\mathcal{J}_1\mathcal{J}_3} - iy\sqrt{\mathcal{J}_1\mathcal{J}_2}\right\}\end{aligned}$$

$$\begin{aligned}\mathcal{H} &= a_3 \exp\left\{x\sqrt{\mathcal{J}_1\mathcal{J}_3} + iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\} + b_3 \exp\left\{x\sqrt{\mathcal{J}_1\mathcal{J}_3} - iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\} \\ &\quad + c_3 \exp\left\{-x\sqrt{\mathcal{J}_1\mathcal{J}_3} + iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\} + d_3 \exp\left\{-x\sqrt{\mathcal{J}_1\mathcal{J}_3} - iz\sqrt{-\mathcal{J}_2\mathcal{J}_3}\right\}\end{aligned}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3$ are constants. Note that we have used a complex notation to represent our solutions. The above solutions may be equivalently expressed in terms of elementary functions. The constants $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ are specified by

$$\mathcal{J}_1 = \frac{A^2}{C}\left(\frac{B}{A}\right)^{\cdot}$$

$$\mathcal{J}_2 = \frac{B^2}{A}\left(\frac{C}{B}\right)^{\cdot}$$

$$\mathcal{J}_3 = \frac{A^2}{B}\left(\frac{C}{A}\right)^{\cdot}$$

and as for the conformal Killing vector solution we require

$$\alpha_1 = A\left(\frac{\rho}{A}\right)^{\cdot}$$

$$\beta_1 = B\left(\frac{\rho}{B}\right)^{\cdot}$$

$$\gamma_1 = C\left(\frac{\rho}{C}\right)^{\cdot}$$

which reduces to

$$
\begin{aligned}
A(t) &= A_0\Big[\dot{R}\exp\left(\alpha_1 R\right)\Big]^{-1} \\
B(t) &= B_0\Big[\dot{R}\exp\left(\beta_1 R\right)\Big]^{-1} \\
C(t) &= C_0\Big[\dot{R}\exp\left(\gamma_1 R\right)\Big]^{-1}
\end{aligned}
$$

where $R(t)$ was defined previously. Substituting these particular forms of the metric functions into the equations defining the constants $\mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$ we find that $\alpha_1 = \beta_1 = \gamma_1 = 0$ which implies that $\mathcal{J}_1 = \mathcal{J}_2 = \mathcal{J}_3 = 0$. This in turn implies that the metric functions are proportional to one another and that $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are constants. Thus for this case our Bianchi I model has degenerated into a spacetime of higher symmetry viz. the Robertson–Walker model.

5 Conclusion

The study of conformal symmetries and their relationship to exact solutions of the Einstein field equations has generated much interest recently. In this thesis we studied the kinematical and dynamical properties of spacetimes that admit a conformal symmetry in general relativity. We analysed the Einstein field equations, for neutral matter and charged matter, with a conformal symmetry. In particular we found explicitly the conformal symmetries in the spatially homogeneous and anisotropic Bianchi I spacetime which is often utilised as an anisotropic cosmological model. The equations obtained were general and we regained the work of other authors as special cases.

In chapter 2 we briefly introduced aspects of differential geometry necessary for later sections. We discussed the concept of a differentiable manifold. Vector fields and tensor fields were then introduced on the manifold. The covariant and Lie derivatives were defined and their properties discussed. Curvature was introduced via the Riemann tensor, and the Einstein field equations for both neutral and charged matter were motivated. We considered those aspects of Lie theory necessary for the development of conformal symmetries in later chapters. Lie algebras and Lie groups form the basis of the study of conformal symmetries. Finally we defined the conformal Killing vector which imposed the condition of a conformal symmetry on the manifold.

We investigated the effect of a conformal symmetry on the Einstein field equations in chapter 3. The results obtained were general because we did not specify a particular form of the metric tensor field. We considered the Lie symmetry in the general case of an imperfect energy–momentum tensor for neutral matter. The Lie derivatives of the kinematical quantities along a conformal Killing vector were derived. Then we found the Lie derivative of the general energy–momentum tensor. The Lie derivatives of the dynamical quantities were then found by considering the Einstein field equations with nonzero cosmological constant. Results found previously were regained as special cases of our solution. We obtained the Lie derivative of the electromagnetic energy–momentum tensor, thereby extending our results to charged matter. We also reviewed previously published results on conformal Killing vectors and other symmetries.

In chapter 4 we analysed the conformal geometry of the Bianchi I spacetime. We studied the spacetime geometry and derived the Einstein field equations with vanishing cosmological constant. The conformal Killing vector equation was explicitly solved to obtain the conformal Killing vector and the conformal factor, the solution being subject to integrability conditions that relate the functions of integration to the metric functions. We showed that these conditions have a nonempty solution set by obtaining a particular solution when certain functions of integration vanish. The results obtained generalise the conformal Killing vectors of the locally rotationally symmetric model studied by Moodley (1991). The conformal solution found contains the three Killing vectors of Bianchi I spacetime. We obtained the homothetic vector as a particular case from the conformal solution. From the integrability conditions we found the functional dependence of the metric functions and established that the only homothetic vector admitted by the Bianchi I spacetime is contained in the vacuum Kasner solution. The Bianchi I spacetime does not admit a proper special conformal Killing vector. We also considered some other cases by placing restrictions on the three functions of integration.

To summarise, we studied the effect of the existence of a conformal symmetry on the Einstein field equations and obtained the conformal Killing vectors in the anisotropic Bianchi I spacetime. The equations for neutral matter obtained in chapter 3 should be extended by finding an analogue of the system (3.28)–(3.34) for a charged fluid satisfying the Einstein field equations. It would be interesting to study the kinematics and dynamics of this general case. Also the integrability conditions, obtained in chapter 4, should be solved in general. In addition the physical properties of the Bianchi I spacetime admitting the conformal Killing vector should be studied further. This would involve an analysis of the nonlinear Einstein field equations in this spacetime. An analysis of other symmetries in the Bianchi I spacetime and other Bianchi models should also be pursued. However this would not be a simple matter as the equations involved are extremely complicated.

We believe that this thesis represents the first attempt to solve the conformal Killing vector equation in the Bianchi I spacetime. We hope that we have demonstrated that the study of symmetries is a fertile area of research and that further investigation of symmetries in the Bianchi I spacetime and other models should be pursued.

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