MULTI-PARAMETER PERTURBATION ANALYSIS OF A SECOND GRADE FLUID FLOW PAST AN OSCILLATING INFINITE PLATE

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Abstract

In this dissertation we consider the two dimensional flow of an incompressible and electrically conducting second grade fluid past a vertical porous plate with constant suction. The flow is permeated by a uniform transverse magnetic field. The aim of this study is to use the multi-parameter perturbation technique to study the effects of Eckert numbers on the flow of a pulsatile second grade fluid along a vertical plate. We further aim to investigate the effects of other fluid and physical parameters such as the Prandtl numbers, Hartmann numbers, viscoelastic parameter, angular frequency and suction velocity on boundary layer velocity, temperature, skin friction and the rate of heat transfer.

Similarity transformations are used to reduce the governing partial differential equations to ordinary differential equations. We used perturbation methods to solve the coupled ordinary differential equations for zero Eckert number and the multi-parameter perturbation technique to solve the coupled ordinary differential equations for small viscoelastic parameters and Eckert numbers. It is found that increasing the

Eckert number or the viscoelastic parameter enhances the boundary layer velocity while reducing the temperature, the rate of heat transfer and the skin-friction. The results for the boundary layer velocity and the temperature are presented graphically and discussed. The results for the rate of heat transfer in terms of the Nusselt number and the skin friction are tabulated and discussed. A good agreement is found between these results and other published research. The comparison between the results for zero Eckert numbers and small Eckert numbers is also presented graphically and discussed.

Declaration

The work described in this dissertation was carried out under the supervision and direction of Professor P Sibanda, School of Mathematical Sciences, University of KwaZulu-Natal, Pietermaritzburg, from July 2007 to April 2009.

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institution of learning. The thesis is my original work except where due reference and credit is given.

Signature: ....

Date: ....

Dedication

To God almighty for his mercy that endures forever. To my wife, Umugiraneza Odette, and my son Byiringiro Néré Blaise, who throughout this work did not set their eyes on me. I am sincerely grateful to them for their understanding, patience and love.

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I gratefully acknowledge financial support from the government of Rwanda. I extend my acknowledgement to the administration of Kigali Institute of Education for allowing me study leave for this period of research. I am also indebted to a number of people who directly or indirectly made contributions to this work. In particular, thanks are due to E. Gakuba, G. Kabera, A. Nsabimana, the Reverend C . Hategekimana the Reverend W. Kisseka and M. Lilongwe for their moral support. I wish to thank my officemates; A. Basheer, T. T. Seretlo, F. Awad, J. Pedro and J. M. T. Ngnotchouye for creating a pleasant working environment. I also wish to thank staff at the Cecil Renaud library, especially Mr N. A. Gani for his hardwork providing access to overseas journals.

My parents, brothers and sisters have always been a pillar of strength, this is an opportunity to thank them for their sacrifices in ensuring my earlier studies.

Chapter 1

Introduction

Fluid mechanics is a branch of mechanics that is concerned with the properties and the behaviour of liquids and gases, Yuan (1967), Massey (1986), Spurk (1997). The study of fluid motions helps to shed light on many other aspects of applied sciences and engineering, for example, in bioengineering, fluid control systems, energy conversion systems, climatology and oceanography, Yuan (1967) and Allen and Ditsworth (1972). Fluids may be divided into two broad categories, namely inviscid and viscous fluids.

Inviscid or ideal fluids are fluids whose viscosity is negligible, such as water and air under normal conditions and viscous fluids are the thick fluids whose viscosity cannot be neglected. Examples of viscous fluids are oils, grease, paint and some food products. Viscosity relates applied stress in moving fluid to the resulting strain rate, Hughes and Brighton (1967). If this relationship is linear the fluid is termed Newtonian. Water, oil and gases are examples of Newtonian fluids. If the relation between the applied stress and the shear strain rate is nonlinear the fluid is non-Newtonian. Common examples of non-Newtonian fluids include solutions of various polymers, the emulsion of water in oil, toothpaste and quicksand. According to Hughes and Brighton (1967), non-Newtonian fluids may be classified into three groups as follow:

1. Time *independent* non-Newtonian fluids . These are fluids whose shear rate is a non-linear function of the shear stress. This group is further subdivided into dilatants, pseudoplastics and yield stress fluids such as Bingham plastics and Carson plastics. Bingham plastics are idealized materials which behave partly like solids and partly like fluids, White (1991). These fluids would not flow until a certain yield stress is exceeded. Thereafter the relationship between shear stress and shear rate is linear, see Figure 1.1. Pseudoplastic fluids are characterized by a decreasing slope (local viscosity) with increasing stress. Most non-Newtonian fluids belong to this category. If the apparent viscosity is increasing with increasing shear rate the fluid is known as a dilatant.

2. Time *dependent* non-Newtonian fluids are fluids whose behaviour may vary with time, Figure 1.2. If the shear strain is kept constant, the stress may vary and vice versa, Hughes and Brighton (1967). If the shear stress decreases with time as the fluid is sheared the fluid is known as thixotropic and the opposite effect is rheopectic. Many fluids in this category lose their rheopectic property at large shear rates to become thixotropic fluids, White (1991).

3. *Viscoelastic fluids*. These fluids are characterized by the fact that their shear stress is a non-linear function of the strain rate, for example, bitumen and flour dough. In the deformation of an elastic solid, in general, some energy of a viscoelastic fluid may be recoverable. This contrasts with perfect fluids in which all energy of deformation is dissipated. A viscoelastic fluid combines both elastic and viscous characteristics. These fluids have attracted many researchers because of their interesting properties and uses of such fluids in industrial applications. In this study, we are interested in the behaviour of a viscoelastic fluid.

Figures 1.1 and 1.2 depict the different behavioural characteristics of non-Newtonian fluids.

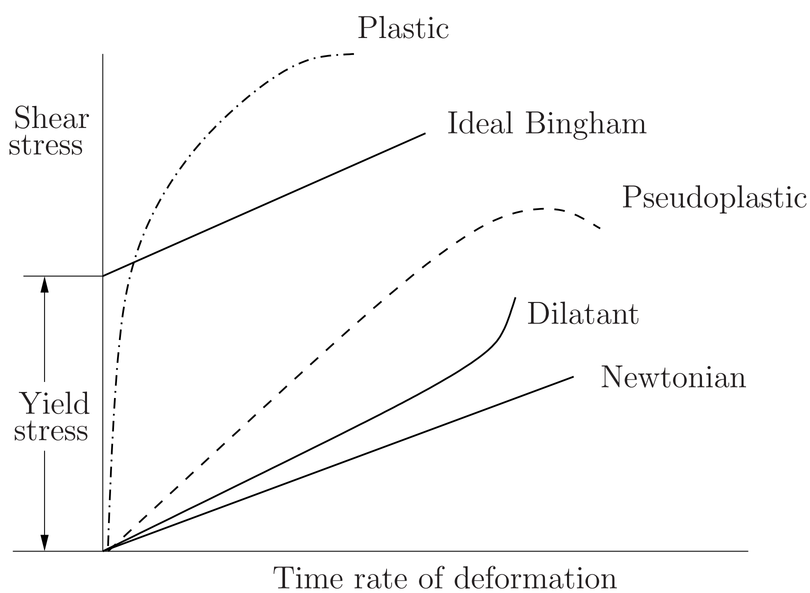

Figure 1.1: Stress- strain rate relationships in time-independent non-Newtonian fluids. Source: Massey (1986).

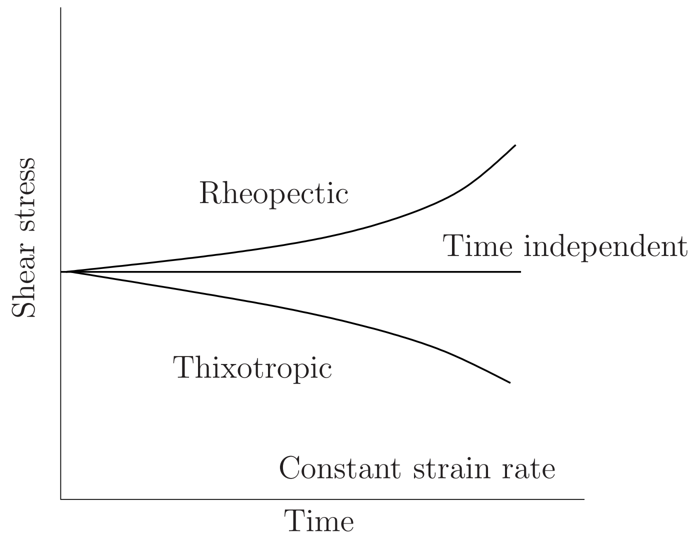

Figure 1.2: Stress- strain rate relationships in time-dependent non-Newtonian fluids. Source: Massey (1986).

1.1 Magnetohydrodynamics

Magnetohydrodynamics (MHD) is a branch of fluid dynamics that deals with the interaction between electrically conducting fluids and an electromagnetic field, Moreau (1990). Any electrically conducting material moving in a magnetic field generates an electromotive force that in turn induces its own magnetic field, Moreau (1990). The applied magnetic field has the following two principal effects:

1. it produces an induced magnetic field which has the effect of perturbing the original applied magnetic field

2. it creates an electromagnetic force due to the interaction of the current and the

magnetic field known as the Laplace force (also called a Lorentz force), Moreau (1990) . This force has the effect of perturbing the original motion, Moreau (1990), Schecliff (1965).

In general, the fluid velocity field $\mathbf{u}$ and the magnetic field (of strength B are coupled, Moreau (1990). The ratio of the induced magnetic field to the imposed magnetic field is known as the induced Reynolds number. In low Reynolds number flow, the induced magnetic field is usually ignored so that in MHD flow equations, the magnetic field strength B is replaced by the known magnetic field B_0, Schecliff (1965). Many devices have been made that rely on the application of MHD principles directly, or involve fluid-electromagnetic interactions, for example, in electron beam dynamics, electrical discharges and power generators, Hughes and Brighton (1967), Sattar and Alam (1995). There is a large body of literature on MHD flows, and some of the recent studies include, for example, Attar and Alam(1995), Hayat et al (2003), Vajravelu and Rollins (2004), Hsiao (2007), Murthy et al (2007) and Mustafa (2008). The magnetic field affects all electrically conducting fluids, for example, it increases or decreases the fluid velocity or temperature depending on the nature and quantum of the other fluid parameters, Anuar Bég et al (2005).

1.2 Heat and mass transfer

It is a fundamental law of physics that energy can neither be created nor destroyed, but can only be changed from one form to another. In thermodynamics energy is used to specify the state of a system. The science of thermodynamics involves the study of the relationships between various forms of energy and heat. The analysis of the rate of heat transfer in a system is known as the science of heat transfer. The transfer of heat is normally from a region of higher temperature to low temperature zones, see Özişik (1988).There are three modes of transferring heat from one medium to another, namely conduction, convection and radiation, Özişik (1988) and Welty et al (1984). In conduction, the heat is transferred from an object at higher temperature to an object at a lower temperature by kinematic motion in the case of fluids and by the movement of free electrons in solids. Convection is a mode of heat transfer which involves the energy exchange between a fluid and a solid due to temperature differences in the fluid and the solid surface. This mode of heat transfer is further subdivided into free or natural convection (which is a mode of heat transfer caused by density differences due to temperature gradients) and forced convection which may be defined as a mode of heat transfer in which the fluid motion is caused by external forces, for example by a pump, Colson and Richardison (1977). At temperatures higher than absolute zero all materials emit energy. This energy is called thermal radiation and is transferred in the form of electromagnetic radiation or photons, Bennet and Myers (1982). The energy transferred may be reflected, emitted or absorbed. The energy transferred by radiation is maximum when it is propagating in a vacuum, Özişik (1988). Heat and mass transfer play an important role in many engineering and industrial processes such as in heat exchangers, radiators, condensers and in the nuclear reactor core. In aerospace technology the temperature distribution and heat transfer characteristics are essential because of safety considerations. Studies have been done on heat and mass transfer by, among others, Sharma and Mathur (1995) who investigated the effects of different physical parameters such as the Prandtl and Eckert numbers on the steady laminar free convection flow of an electrically conducting fluid. They found that the temperature decreases with increasing Eckert and Prandtl numbers. They also found that the rate of heat transfer increases with increases in the strength of an applied magnetic field. Gokhale and Samman (2003) studied the effects of mass transfer on the transient free convection flow of a dissipative fluid including heat flux. They however found that the rate of heat transfer decreases with increasing Eckert numbers whereas the skin friction increases with increasing Eckert numbers. Soundalgekar et al (2004) studied the effects of viscous dissipative heat on transient free convection flow. They compared the temperature for zero and non-zero Eckert numbers. They found that the temperature in the presence of viscous dissipative heat is less than that at zero Eckert numbers. They also found that the rate of heat transfer decreases with increasing Eckert numbers whereas the skin friction increases with viscous dissipative heat. This is in conformity with the results of Gokhale and Samman (2003) on the rate of heat transfer and the skin friction. However in their research, Gokhale and Samman (2003) included the effects of heat flux. A related study by Soundalgekar et al (2004) included the effects of a constant suction velocity. Abel et al (2007) studied non-Newtonian viscoelastic boundary layer flows. They found that the temperature increases with Eckert numbers. Bateller (2007) extended the work of Abel et al (2007) by including radiative heat flux and elastic deformation. In addition to confirming the findings of Abel et al (2007), they also found that the temperature increases with increasing elastic deformation. Alam et al (2007) investigated the effects of heat generation and thermophoresis on steady laminar MHD with heat and mass transfer. They found that the temperature and the rate of heat transfer increase with increases in the applied magnetic field. In their study they did not however consider the effects of the Eckert number.

1.3 The Eckert number

In heat transfer problems, the Eckert number is one of a number of fundamental dimensionless parameters. It relates the flow's kinematic energy and enthalpy and characterizes the dissipation. The physical significance of the Eckert number is better understood when we analyze the role of this parameter in equations of motion and energy, Özişik (1988). After non-dimensionalizing the governing equations, the Eckert number enters as a multiplier of the viscous energy. If the Eckert number is too small, viscous energy generation is negligible. In this way the magnitude of the Eckert number serves as a criterion in the decision on whether viscous energy dissipation effects can be retained or neglected in the heat transfer analysis. Many researchers have investigated the effects of the Eckert number on the temperature, the boundary layer velocity, the rate of heat transfer and the skin friction. Soundalgekar et al (2004) studied the effect of the Eckert number on transient temperature, velocity, skin friction and the rate of heat transfer. They found that the temperature and skin friction of air increases with Eckert numbers whereas the Nusselt number decreases. Their study did not however include an applied magnetic field. Bateller (2007) investigated the effects of the Eckert number on viscous dissipation, work done due to deformation, internal heat generation and thermal radiation. He found that the temperature increases with increasing Eckert numbers. Mustafa et al (2008) analyzed the effect of flow parameters such as Prandtl numbers, the Hartmann number and the Eckert number on temperature, boundary layer velocity, skin friction and the rate of heat transfer. They found that the skin friction increases with increasing Hartmann numbers and magnetic Reynolds numbers. They also found that the rate of heat transfer increases with increasing Prandtl numbers and decreases with increasing Hartmann numbers, Eckert numbers and heat generation. In their study he did not consider the effect of the Eckert numbers on the skin friction. Hsiao (2007) studied MHD mixed convection of viscoelastic fluid over a streching sheet with ohmic dissipation. He found that temperature increases with increasing Hartmann numbers and decreases with increasing Prandtl numbers. He did not however analyze the effects of the Eckert number on the temperature, the velocity and the rate of heat transfer. After examining all these studies, one concludes that the effect of the Eckert number is dependent on the nature of the problem and the other parameters inherent in the system. The primary assumption in this thesis is that the Eckert number is very small, an assumption that is justified for low speed incompressible fluid flows, Sharma and Muthur (1995). Another justification for the low Eckert number assumption is that it has been found, for example in the case of the cooling of hot surfaces in gas flows that maximum heat transfer occurs when the Eckert number is 0.3, Gshwerndtner (2004).

1.4 The constitutive equations

A constitute equation is a mathematical statement of the mechanical behaviour of a group of materials, Astarita and Marrucci (1974). Constitutive equations are, in general, subdivided into three categories ; differential, integral and rate type, Trusdell and Noll (1965). A class of equations whose stress tensor is a function of differential kinematics at the moment of observation is called a constitutive equation of differential type. These are equations of fluid memory because higher order deformation tensors are involved. In the second category the stress is given by one or more integrals of deformation history whereas rate equations are those equations which have at least one time derivative of the stress tensor. Both differential and integral type constitutive equations are explicit in the stress tensor but rate equations are not explicit in the stress. The rate of change of the stress that appears in these equations gives the name to this category. Our study involves constitutive equations of differential type.

Non-Newtonian fluids, unlike Newtonian fluids present characteristics which cannot be described by the classical linear viscous model, Dunn and Rajagopal (1995). Rivlin and Eriksen (1955) and Truesdell and Nol (1965) devised the earliest method of classifying viscoelastic fluids. They presented a constitutive equation for a group of fluids that have come to be known as Rivlin-Eriksen fluids or fluids of a differential type, Vajravelu and Rollins (2004). The constitutive equations for these fluids present a complexity in momentum equations where the equations of motion are an order of magnitude higher than the Navier-Stokes equations, for more details see Rajagopal and Gupta (1984), Rajagopal and Kaloni (1989), Rajagopal (1995).

The Cauchy stress tensor $\mathbf{T}$ for a second grade fluid of differential type is defined by

$$\mathbf{T} = -p\mathbf{I} + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{1.1}$$

where μ is the viscosity, p is the pressure, α_1 and α_2 are normal stress moduli A_1 and A_2 are the first two Rivlin-Ericksen tensors defined as

$$A_1 = (grad\ v) + (grad\ v)^T, \quad A_2 = \frac{d}{dt}A_1 + A_1.grad\ v + (grad\ v)^T.A_1,$$

where d/dt is the material derivative and v is the velocity field. For thermodynamic reasons, the material parameter α_1 must be positive, Dunn and Rajagopal (1995). For a fluid of differential type modelled by equation (1.1) to be compatible with thermodynamics and to satisfy the Clausius-Duhem inequality, and the assumption that the specific Helmoltz free energy of the fluid be a minimum when the fluid is locally at rest, we require that

$$\mu \geq 0, \qquad \alpha_1 \geq 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{1.2}$$

For details see Vajravelu and Rollins (2004), Garg and Rajagopal (1991) and Dunn and Rajagopal (1995). The sign of α_1 and the stability or instability of the fluid motions have however been a subject of controversy and misunderstanding. A thorough discussion of such problems can be found in the critical review by Dunn and Rajagopal (1995).

1.5 Second grade fluids

Second grade fluids are non-Newtonian fluids whose stress tensor is given by equation (1.1) and the conditions (1.2). A second grade fluid belongs to a subclass of fluids of differential type, Dunn and Rajagopal (1995). Studies have been conducted on second grade fluids by, among others, Vajravelu and Rollins (2004), Vajravelu and Roper (1999) and Hayat et al(2003). Some of these studies, such as Vajravelu and Rollins (2004) and Dunn and Rajagopal (1995), considered heat transfer in second grade fluids and the others, for example, Hayat et al(2003), the flow of a second grade fluid without heat transfer.

In the case of heat transfer, most researchers are interested in investigating the effects of different physical parameters on the boundary layer velocity, temperature, skin friction and the rate of heat transfer. The study by Vajravelu and Roper (1999) investigated the effects of viscous dissipation and internal heat generation or absorption and work done due to deformation. In their study, they found that the boundary layer temperature increases with increasing viscoelastic parameters and decreases with increasing Prandtl numbers.

The viscous property of second grade fluids is due to the transport phenomenon of the fluid molecules whereas the elastic property is due to the chemical structure and configuration of polymer molecules, Hsiao (2007).

Baris (2003) studied the effect of elasticity on boundary layer velocity and temperature. He used a perturbation technique and showed that the elasticity of the fluid affects both the temperature and the velocity. It increases or reduces the velocity depending on the value of other flow parameters present. This was confirmed later by Vajravelu and Rollins (2004) in their study of the hydrodynamic flow of a second grade fluid over a stretching sheet. They found that the velocity increases with increasing viscoelastic parameters and decreases with increasing Hartmann numbers.

Murthy et al (2007) studied MHD unsteady free convective flow of Walter's fluid with constant suction and a heat sink. They investigated the effect of Hartmann numbers, Prandtl numbers and sink-strength on the mean velocity, mean temperature, heat transfer and mean skin-friction. They found that the mean velocity decreases with increasing Hartmann or Prandtl numbers. They also found that the mean temperature increases with increasing Eckert numbers. They did not however consider the effect of the viscoelasticity parameter and the angular frequency on either the velocity or the temperature field nor was the effect of the Eckert number on the velocity considered.

1.6 Perturbation methods

The perturbation technique is one of the essential tools of applied mathematics and theoretical physics used in the solution of linear and nonlinear differential equations. Perturbation and asymptotic methods consist of expanding the solution in an asymptotic series in terms of a small parameter. The classic perturbation method of Poincaré (1892) consists in expanding the solution in an asymptotic series of initial or boundary value problem using a single parameter or functions of such a parameter. When the series converges or is expected to converge the method is known as a perturbation method. If the series is diverging asymptotically, the method is called an asymptotic method, Zauderer (1989) and Holmes (1995). The perturbation method has been applied to different types of problems in the physical sciences, for example, the technique has been applied to problems in optics, the boundary layer theory of viscous flows, shock waves, reaction diffusion equations, celestial mechanics and non-linear oscillations. In this study we are particularly interested in the perturbation method. The perturbation method is used to approximate the solution to a differential equation analytically. This method may solve a problem that may otherwise be impossible to solve using numerical methods to obtain reasonable accuracy. The finite difference method is, for example, widely used in numerical analysis due to its efficiency, accuracy and stability but it may fail to solve stiff equations, Misra et al (2008). A good number of researchers have used the perturbation method, for example, Beard and Walters (1964) in their study of an elastico-viscous boundary layer. They suggested the use of the perturbation method when solving problems with insufficient boundary conditions. The perturbation method has the advantage of reducing the order of the equations and treating a singular perturbation problem as a regular perturbation problem, Beard and Walters (1964), Verma and Sharma (1988). In addition, it has been suggested by Simmonds and Mann (1986) that the approximate analytic solutions obtained using the perturbation method often reveal the important dependence of the exact solution on the flow or material parameters in manner that is not possible with a full numerical solution. These reasons provide sufficient motivation and justification for using perturbation methods to investigate the problem of second grade fluid flow over an oscillating infinite flat-plate in this research.

The classical perturbation method of Poincaré has been extended by Nowinski and

Ismail (1965) to many parameters. This generalization is called the multi-parameter perturbation technique. To apply this method the parameters must be independent of each another and must be of the same order. In addition, these parameters must describe different physical and fluid properties such as the material and dynamic properties and must be small so that their higher powers and products can be neglected, Nowinski and Ismail (1965). These methods use small parameters which, perhaps may be their main weakness since some behaviours of the solutions may not be observed. The multi-parameter perturbation technique has been used by many researchers such as Sahoo et al (2003) in their study of heat transfer in mercury and electrolytic solutions past an infinite porous plate with constant suction. Anuar-Bég et al (2005) used the multi-parameter perturbation technique to study the combined effect of periodic suction velocity and heat sinks in unsteady natural convection flow of a viscous fluid past an infinite vertical porous plate. A similar study by Murthy et al (2007) extended the work of Sahoo et al (2003) to include boundary layer heating and cooling.

A more recent addition to the list of perturbation methods is the concept of multi-scale perturbation. This method has the advantage of remedying the difficulties caused by the secular behaviour in a perturbation series, Zeytounian (2002) and Zauderer (1989). It has been used, for example by, Zhe-Wei (1994) in the stability analysis of plane couette flow and Nemirovskii (2005) in the analysis of the hydrodynamics of superfluid turbulence.

1.7 objectives

The main objective of this study is to use the multi-parameter perturbation technique to study the effects of Eckert numbers on the flow of a pulsatile second grade fluid along a vertical plate. We further aim;

- to investigate the effects of an applied magnetic field on the velocity and temperature fields.
- to investigate the effects of the Prandtl number on the velocity and temperature fields.
- to investigate the effects of the viscoelastic parameter on the velocity and temperature fields.
- to investigate the effects of the angular frequency on the velocity and temperature fields.
- to investigate the effects of suction velocity on the velocity and temperature fields.

The rest of this thesis is structured as follows. In Chapter 2 we formulate the problem mathematically. We use similarity transformations to reduce the governing partial differential equations to ordinary differential equations. We use perturbation methods to solve the coupled ordinary differential equations for zero Eckert numbers. We present the results graphically and discuss them. In Chapter 3 we use the multi-parameter perturbation technique to solve the coupled ordinary differential equations for small Eckert numbers. We compare the results of zero and small Eckert numbers. Finally we present our conclusions in Chapter 4.

Chapter 2

Flow of a second grade fluid

2.1 Mathematical formulation

In the present work we consider the two dimensional flow of a second grade fluid past an infinite flat plate placed along the plane $y' = 0$, the flow being confined to $y' > 0$ and obeying the constitutive equation (1.1).

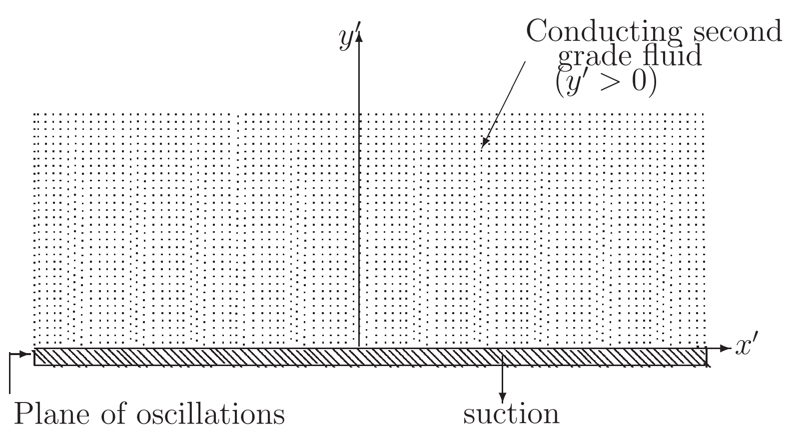

Figure 2.1: schematic diagram for the problem

Assuming that velocities are independent of x', the streamwise coordinate, the pressure is contant, Hayat et al (2003), and that the normal velocity v' is constant, the equations of motion are

$$\frac{\partial u'}{\partial x'} = 0, \tag{2.1}$$

$$\frac{\partial u'}{\partial t'} + v'\frac{\partial u'}{\partial y'} = \nu\frac{\partial^2 u'}{\partial y'^2} + \alpha^*\left[\frac{\partial^3 u'}{\partial y'^2\partial t'} + v'\frac{\partial^3 u'}{\partial y'^3}\right] - \frac{\sigma B_0^2 u'}{\rho} + g\beta(T' - T'_\infty), \tag{2.2}$$

$$\rho c_p\left[\frac{\partial T'}{\partial t'} + v'\frac{\partial T'}{\partial y'}\right] = k\frac{\partial^2 T'}{\partial y'^2} + \mu\left(\frac{\partial u'}{\partial y'}\right)^2, \tag{2.3}$$

where ν $(= \mu/\rho)$ is the coefficient of kinematic viscosity, B_0 is the uniform magnetic field strength along the y' axis, σ is the fluid electrical conductivity, c_p is the specific heat at constant pressure and k $(= \lambda_c/\rho c_p)$ is the thermal diffusivity where λ_c is the fluid thermal conductivity.

The appropriate boundary conditions for this problem are

$$u' = u_0\epsilon e^{i\omega' t'}, \quad v' = -v_0, \quad T' = T'_w + \epsilon\left(T'_w - T'_\infty\right)e^{i\omega t'}, \quad y' = 0,\ t > 0, \tag{2.4}$$

$$u' \to 0, \qquad T' \to T'_\infty \quad \text{as} \quad y' \to \infty, \tag{2.5}$$

where $\epsilon << 1$ is an amplitude parameter, v_0 is a constant suction parameter, u_0 is a constant velocity, ω' is the angular frequency, T'_w is the temperature of the surface and T'_∞ is the ambient fluid temperature. This model is an extension of the earlier study of Hayat et al (2003) to include heat transfer and constant suction. It is also an extension of the work by Soundalgekar et al (2004) to include the effects of an applied magnetic field and a second grade fluid. Murthy et al (2007) did a similar study on MHD unsteady free convective flow of Walter's fluid with constant suction and a heat sink. This study however differs from the work of Murthy et al (2007) in that their study included a heat sink as an additional term in the energy equations and the boundary conditions. To non-dimensionalize equations (2.1) - (2.3) we introduce the following dimensionless variables

$$\eta = \frac{y' v_0}{\nu}, \quad t = \frac{v_0^2 t'}{\nu}, \quad u = \frac{u'}{u_0}, \quad \theta = \frac{T' - T'_\infty}{T'_w - T'_\infty}. \tag{2.6}$$

These variables are similar to those used by Soundalgekar et al (2004), Sharma et al (1995), Hayat et al (2003), Sahoo et al (2003) and Murthy et al (2007). Substituting (2.6) into (2.1) - (2.3) gives

$$\frac{\partial u}{\partial t} - \frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} + \alpha \left(\frac{\partial^3 u}{\partial \eta^2 \partial t} - \frac{\partial^3 u}{\partial \eta^3} \right) - Mu + \frac{Gr\,\theta}{v_0}, \tag{2.7}$$

$$\frac{\partial \theta}{\partial t} - \frac{\partial \theta}{\partial \eta} = \frac{1}{P_r} \frac{\partial^2 \theta}{\partial \eta^2} + E \left(\frac{\partial u}{\partial \eta} \right)^2, \tag{2.8}$$

where the central parameters are the Eckert number E, the Prandtl number P_r which is the ratio of the momentum diffusivity to the heat diffusivity, Özişik (1988) and the Hartmann number M. These parameters are defined respectively by

$$E = \frac{u_0^2}{C_p \Delta T}, \quad P_r = \frac{\mu c_p}{k} \quad \text{and} \quad M = \frac{\sigma B_0^2 \nu}{\rho v_0^2}. \tag{2.9}$$

The other terms are defined by

$$\alpha = \frac{\alpha^* v_0^2}{\nu^2}, \quad u_0 = \left(\frac{g \beta \Delta T \nu}{v_0} \right)^{\frac{1}{2}}, \quad \omega = \frac{\nu \omega'}{v_0^2} \quad \text{and} \quad Gr = \frac{g \beta \nu \Delta T}{u_0 v_0}, \tag{2.10}$$

where α is the viscoelastic parameter and Gr is the Grashof number. The appropriate boundary conditions are:

$$u = u_0 \epsilon e^{i\omega t}, \quad \theta = 1 + \epsilon e^{i\omega t} \quad \text{at} \quad \eta = 0, \quad t > 0,$$

$$u = 0, \qquad \theta = 0 \quad \text{as} \quad \eta \to \infty,$$

where ω is a non-dimensional angular frequency. Various methods have been used previously to solve equations similar in form to equations (2.7) and (2.8). Numerical methods such as finite differences have been used by among others, Soundalgekar et al (2004), Saeid (2004) and Hsiao (2007). Many researchers, for example, Bateller (2007), Murthy et al (2007), used similarity transformations to reduce the partial differential equations to ordinary differential equations. In that case one may solve the resulting ordinary differential equations analytically, for example, using perturbation methods, Hayat et al (2003), multi-parameter perturbation techniques, Sahoo et al (2003), Anuar-Bég et al (2005) and Murthy et al (2007), Fourier series, Kelly (1965), Soundalgekar (1976), Messiha (1966) and the Laplace transform method, Hayat et al (2005). The equations have also been solved numerically using shooting methods, Alam et al (2007).

Following Bateller (2007) and others, we reduce the partial differential equations (2.7) and (2.8) to ordinary differential equations by assuming sinusoidally periodic solutions of the form

$$\begin{aligned} u(\eta, t) &= u_0 \left[f_1(\eta) + \epsilon f_2(\eta) e^{i\omega t} \right], & (2.11) \\ \theta(\eta, t) &= h_1(\eta) + \frac{\epsilon}{2} \left[h_2(\eta) + h_3(\eta) \right] e^{i\omega t}. & (2.12) \end{aligned}$$

Similar expansions have been used by a number of researchers such as Hayat et al (2003), Kelly (1965) , Sahoo et al (2003), Murthy et al (2007) and Anuar-Bég et al (2005). For convenience we have assumed here that $h_3 = 0(h_1)$ and u_0 as a constant. In the absence of clear means of obtaining additional boundary conditions, Beard and

Walter (1964) suggested a perturbation method to solve the problem. Substituting the expansions (2.11) and (2.12) into equations (2.7)- (2.8) gives the following zeroth order (in ϵ) differential equations

$$\alpha f_1''' - f_1'' - f_1' + M f_1 - \frac{1}{v_0} h_1 = 0, \tag{2.13}$$

$$\frac{1}{P_r} h_1'' + h_1' + E u_0^2 f_1'^2 = 0, \tag{2.14}$$

where G_r has been simplified by u_0 from equation (2.10). The boundary conditions are

$$f_1 = 0, \quad h_1 = 1 \quad \text{at} \quad \eta = 0, \tag{2.15}$$

$$f_1 = 0, \quad h_1 = 0 \quad \text{as} \quad \eta \to \infty. \tag{2.16}$$

The first order differential equations are

$$\alpha f_2''' - (1 + \alpha i w)\ f_2'' - f_2' + (M + i\omega)\ f_2 - \frac{1}{v_0}(h_2 + h_1) = 0, \tag{2.17}$$

$$\frac{1}{P_r} h_2'' + h_2' - i\omega h_2 + 2E u_0^2 f_1' f_2' = -\frac{1}{P_r} h_1'' - h_1' + i\omega h_1, \tag{2.18}$$

with appropriate boundary conditions

$$f_2 = 1 \quad \text{and} \quad h_2 = 1 \quad \text{at} \quad \eta = 0, \tag{2.19}$$

$$f_2 = 0 \quad \text{and} \quad h_2 = 0 \quad \text{as} \quad \eta \to \infty. \tag{2.20}$$

2.2 Zero Eckert number solutions

The Eckert number is a dimensionless parameter which characterizes dissipation and relates the kinetic energy to enthalpy. The dissipation may be large or small depending on the magnitude of the characteristic velocity of the flow, the specific heat capacity and the temperature differences. In this section, we solve the problem when the Eckert number is zero. Equation (2.14) reduces to the simple second order boundary value problem

$$h_1''(\eta) + P_r h_1'(\eta) = 0, \quad h_1(0) = 1, \quad h_1(\infty) = 0,$$

whose exact solution is

$$h_1(\eta) = e^{-P_r\,\eta}. \tag{2.21}$$

Similarly, equation (2.18) reduces to

$$h_2'' + P_r\, h_2' - iwP_r h_2 = -\frac{1}{P_r} h_1'' - h_1' + i\omega h_1, \quad h_2(0) = 1, \quad h_2(\infty) = 0,$$

with exact solution

$$h_2(\eta) = 2e^{-p_2\,\eta} - e^{-Pr\,\eta}, \tag{2.22}$$

where

$$p_1 = \frac{-P_r + \sqrt{P_r^2 + 4iwP_r}}{2} \quad \text{and} \quad p_2 = \frac{P_r + \sqrt{P_r^2 + 4iwP_r}}{2}.$$

Hence for zero Eckert numbers we find the boundary layer temperature to be an exponentially decreasing function of the Prandtl number, angular frequency and time of the form

$$\theta(\eta,\ t) = e^{-P_r\eta} + \epsilon e^{-p_2\,\eta + i\omega t}. \tag{2.23}$$

Now using the solution (2.21) in (2.13) we get

$$\alpha f_1'''(\eta) - f_1''(\eta) - f_1' + M f_1 = \frac{1}{v_0}\, e^{-P_r\eta}. \tag{2.24}$$

To solve this equation we assume perturbation solutions of the form

$$f_1 = f_{10} + \alpha f_{11} + 0(\alpha^2) \tag{2.25}$$

that are valid for sufficiently small α, see Beard and Walters (1964). Using the approximation (2.25) in equation (2.24) gives

$$\alpha\,(f_{10}''' + \alpha f_{11}''') - f_{10}'' - \alpha f_{11}'' - f_{10}' - \alpha f_{11}' + M(f_{10} + \alpha f_{11}) = \frac{1}{v_0}\, e^{-P_r\,\eta},$$

where the equation for f_{10} is

$$f_{10}'' + f_{10}' - M f_{10} = -\frac{1}{v_0} e^{-P_r\,\eta},$$

with general solution

$$f_{10}(\eta) = C_1 e^{\beta_1\eta} + C_2 e^{-\beta_2\eta} - \frac{e^{-P_r\eta}}{v_0 H},$$

where C_1 and C_2 are constants and,

$$\beta_1 = \frac{-1 + \sqrt{1+4M}}{2}, \quad \beta_2 = \frac{1 + \sqrt{1+4M}}{2} \quad \text{and} \quad H = P_r^2 - P_r - M.$$

Using the boundary conditions (2.15) and (2.16) gives

$$f_{10}(\eta) = \frac{1}{v_0 H}\left(e^{-\beta_2\,\eta} - e^{-Pr\,\eta}\right). \tag{2.26}$$

The equation for f_{11} is

$$\begin{aligned} f''_{11} + f'_{11} - M f_{11} &= f'''_{10} \\ &= \frac{1}{v_0 H}\left(-\beta_2^3 e^{-\beta_2 \eta} + P_r^3 e^{-P_r\,\eta}\right). \end{aligned} \tag{2.27}$$

The general solution to equation (2.27) is

$$f_{11}(\eta) = K_1 e^{\beta_1\,\eta} + K_2 e^{-\beta_2\,\eta} + \frac{\beta_2^3 \eta e^{-\beta_2\,\eta}}{v_0 H\sqrt{1+4M}} + \frac{P_r^3 e^{-P_r\,\eta}}{v_0 H^2}. \tag{2.28}$$

Using the boundary conditions (2.15) and (2.16) we get

$$K_1 = 0 \quad \text{and} \quad K_2 = \frac{-P_r^3}{v_0 H^2}.$$

Hence

$$f_{11}(\eta) = \left(\frac{\beta_2^3 \eta}{v_0 H\sqrt{1+4M}} - \frac{P_r^3}{v_0 H^2}\right) e^{-\beta_2\,\eta} + \frac{P_r^3 e^{-P_r \eta}}{v_0 H^2}. \tag{2.29}$$

Substituting (2.26) and (2.29) in (2.25) gives

$$\begin{aligned} f_1(\eta) &= \frac{1}{v_0 H}\left(e^{-\beta_2\,\eta} - e^{-P_r\,\eta}\right) \\ &+ \left(\frac{\alpha\beta_2^3\eta}{v_0 H\sqrt{1+4M}} - \frac{\alpha P_r^3}{(v_0 H^2)}\right) e^{-\beta_2\,\eta} \\ &+ \left(\frac{\alpha P_r^3}{v_0 H^2}\right) e^{-Pr\,\eta}. \end{aligned} \tag{2.30}$$

To find f_2 we now substitute equations (2.21) and (2.22) in (2.17) to get:

$$\alpha f'''_2 - f''_2 (1 + \alpha i\omega) - f'_2 + (M + i\omega) f_2 = \frac{2}{v_0} e^{-p_2\,\eta}. \tag{2.31}$$

Now assume a solution of the form

$$f_2 = f_{20} + \alpha f_{21} + O(\alpha^2). \tag{2.32}$$

This gives the two differential equations

$$0(\alpha^0): \quad f''_{20} + f'_{20} - (M + iw) f_{20} = \frac{-2}{v_0} e^{-p_2\,\eta}, \tag{2.33}$$

$$0(\alpha^1): \quad f''_{21} + f'_{21} - (M + i\omega) f_{21} = f'''_{20} - i\omega\, f''_{20}. \tag{2.34}$$

The auxiliary equation for (2.33) is

$$k^2 + k - (M + i\omega) = 0$$

with solutions $k_1 = \frac{1}{2}[-1 + \sqrt{1 + 4(M + i\omega)}]$ and $k_2 = \frac{1}{2}[1 + \sqrt{1 + 4(M + i\omega)}]$.

Therefore the solution for f_{20} is

$$f_{20}(\eta) = C_3 e^{k_1 \eta} + C_4 e^{-k_2\,\eta} - \frac{2}{v_0 H_1} e^{-p_2\,\eta}, \tag{2.35}$$

where $H_1 = p_2^2 - p_2 - (M + i\omega)$. The solution that satisfies the boundary conditions (2.19) and (2.20) gives

$$C_3 = 0 \quad \text{and} \quad \mathrm{C}_4 = \frac{\mathrm{v}_0 \mathrm{H}_1 + 2}{\mathrm{v}_0 \mathrm{H}_1}.$$

Hence

$$f_{20}(\eta) = \frac{1}{v_0 H_1} (v_0 H_1 + 2)\, e^{-k_2\,\eta} - \frac{2}{v_0 H_1} e^{-p_2\,\eta}. \tag{2.36}$$

Equation (2.34) can now be written as

$$\begin{aligned} f''_{21} + f'_{21} - (M + i\omega) f_{21} &= \frac{-k_2^2}{v_0 H_1}(v_0 H_1 + 2)(k_2 + i\omega) e^{k_2\,\eta} \\ &+ \frac{2p_2^2(p_2 + i\omega)}{v_0 H_1} e^{-p_2\,\eta}. \end{aligned}$$

The solution of the homogeneous part of the equation is

$$f_{21}(\eta) = C_5 e^{k_1 \eta} + C_6 e^{-k_2\, \eta} \tag{2.37}$$

and the particular solution is

$$f_{21p}(\eta) = \frac{k_2^2(v_0 H_1 + 2)(k_2 + i\omega)\eta}{v_0 H_1 \sqrt{1 + 4(M + i\omega)}} e^{-k_2 \eta} + \frac{2p_2^2(p_2 + i\omega)}{v_0 H_1^2} e^{-p_2\, \eta}. \tag{2.38}$$

Hence the general solution is

$$\begin{aligned} f_{21}(\eta) &= C_5 e^{k_1\, \eta} + C_6 e^{-k_2\, \eta} + \frac{k_2^2(v_0 H_1 + 2)(k_2 + i\omega)\eta}{v_0 H_1 \sqrt{1 + 4(M + i\omega)}} e^{-k_2\, \eta} \\ &+ \frac{2p_2^2(p_2 + i\omega)}{v_0 H_1^2} e^{-p_2\, \eta}, \end{aligned} \tag{2.39}$$

where

$$C_5 = 0 \quad \text{and} \quad \mathrm{C_6} = \frac{-2\mathrm{p}_2^2(\mathrm{p}_2 + \mathrm{i}\omega)}{\mathrm{v}_0 \mathrm{H}_1^2}.$$

Therefore

$$\begin{aligned} f_{21}(\eta) &= \left(\frac{-2p_2^2(p_2 + i\omega)}{v_0 H_1^2} + \frac{k_2^2(v_0 H_1 + 2)(k_2 + i\omega)\eta}{v_0 H_1 \sqrt{1 + 4(M + i\omega)}} \right) e^{-k_2\, \eta} \\ &+ \frac{2p_2^2(p_2 + i\omega)}{v_0 H_1^2} e^{-p_2\, \eta}. \end{aligned} \tag{2.40}$$

Substituting equations (2.36) and (2.40) in (2.32) finally gives

$$\begin{aligned} f_2(\eta) &= \frac{1}{v_0 H_1}(v_0 H_1 + 2)) e^{-k_2\, \eta} - \frac{2}{v_0 H_1} e^{-p_2\, \eta} \\ &+ \left(\alpha \left(\frac{-2p_2^2(p_2 + i\omega)}{v_0 H_1^2} \right) + \frac{\alpha k_2^2(v_0 H_1 + 2)(k_2 + i\omega)\eta}{v_0 H_1 \sqrt{1 + 4(M + i\omega)}} \right) e^{-k_2\, \eta} \\ &+ \frac{2\alpha p_2^2(p_2 + i\omega)}{v_0 H_1^2} e^{-p_2\, \eta}. \end{aligned} \tag{2.41}$$

For convenience the above equation can be written as

$$
\begin{aligned}
f_2(\eta) &= \left(\frac{v_0H_1+2}{v_0H_1}+\frac{\alpha(-2p_2^2(p_2+i\omega))}{v_0H_1^2}\right)e^{-k_2\,\eta} \\
&+ \left(\frac{\alpha k_2^2(v_0H_1+2)(k_2+i\omega)\eta}{v_0H_1\sqrt{1+4(M+i\omega)}}\right)e^{-k_2\,\eta} \\
&+ \left(\frac{2\alpha p_2^2(p_2+i\omega)}{v_0H_1^2}-\frac{2}{v_0H_1}\right)e^{-p_2\,\eta}. \qquad (2.42)
\end{aligned}
$$

Now, substituting (2.42) and (2.30) in (2.11) we find the boundary-layer fluid velocity to be

$$
\begin{aligned}
u(\eta,\ t) &= \frac{u_0}{v_0H}\left(1+\frac{\alpha\beta_2^3\eta}{\sqrt{1+4M}}-\frac{\alpha P_r^3}{P_r^2-P_r-M}\right)e^{-\beta_2\,\eta} \\
&+ \frac{u_0}{v_0H}\left(\frac{\alpha P_r^3}{P_r^2-P_r-M}-1\right)e^{-P_r\,\eta} \\
&+ u_0\epsilon e^{iwt-k_2\,\eta}\left(\frac{v_0H_1+2}{v_0H_1}+\frac{\alpha(-2p_2^2(p_2+i\omega))}{v_0H_1^2}\right) \qquad (2.43) \\
&+ \left(\frac{u_0k_2^2\alpha(v_0H_1+2)(k_2+i\omega)\eta}{v_0H_1\sqrt{1+4(M+i\omega)}}\right)\epsilon e^{i\omega t-k_2\,\eta} \\
&+ u_0\epsilon\left(\frac{2\alpha p_2^2(p_2+i\omega)}{v_0H_1^2}-\frac{2}{v_0H_1}\right)e^{i\omega t-p_2\,\eta}. \qquad (2.44)
\end{aligned}
$$

2.3 Results and Discussions

Figure 2.2 shows the variation of boundary layer velocity with the visco-elastic parameter. The velocity decreases with increasing α. The general trend in these results are similar to what was obtained by Hayat et al (2003). In Figure 2.2(b) we decreased M while keeping the same values of α as in Figure 2.2 (a). We observed that the smaller the Hartmann number the larger the velocity amplitude, confirming that a magnetic field may be used as a means to reduce the boundary layer velocity. This is explained by the fact that increasing α is, in fact, equivalent to increasing the suction velocity as can be seen from (2.10), implying that more fluid is being taken out of the system, thereby reducing the momentum boundary layer.

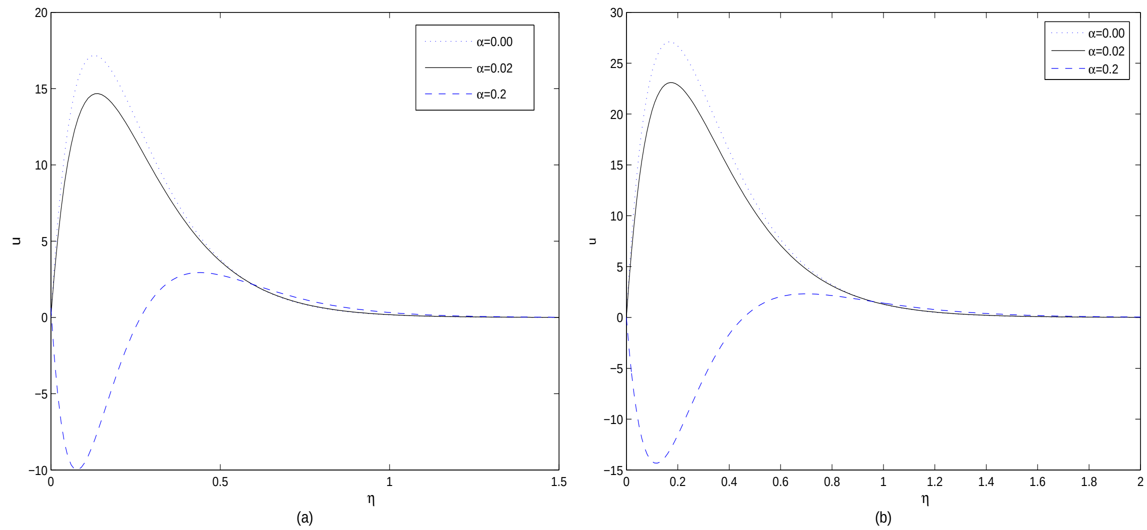

Figure 2.2: The velocity profile with $P_r = 7$, $v_0 = 0.01$, $\epsilon = 0.01$, $\omega = 1.5$, $u_0 = 0.01$, $t = \pi/2\omega$; (a) $M = 65$ and (b) $M = 20$.

Figure 2.3 shows the variation of the velocity profiles for different values of Hartmann numbers M. The velocity decreases with increasing Hartmann numbers. Similar results have been observed in earlier studies, for example, in Hayat et al (2003) and Vajravelu and Rolins (2004). The reduction in the boundary layer velocity with increasing magnetic force provides a means of controlling the flow in the desired direction. This control of the boundary layer flow is of practical significance, for example, in the control of flow around aircraft wings. Other methods that have been proposed for the purpose of artificially controlling the behaviour of boundary layer flow include streamlining the body shape, Hayat et al (2002), suction and the use of flexible walls as proposed by Kramer (1957). The application of MHD principles is another method for controlling the flow field by altering the structure of the boundary layer.

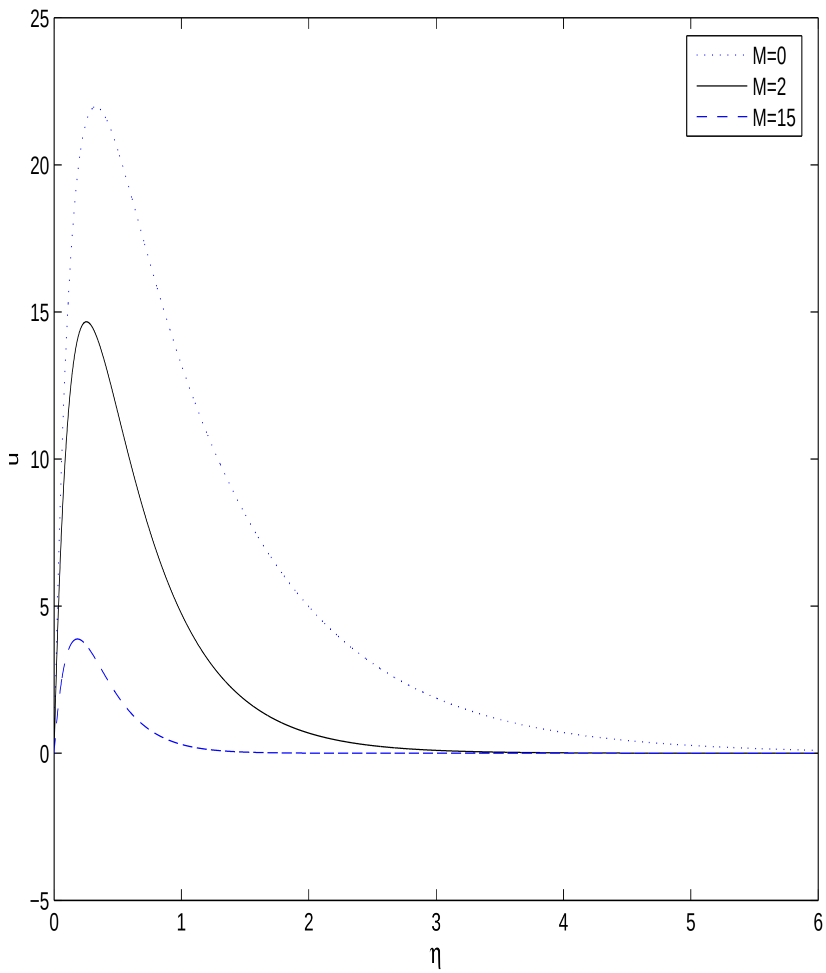

Figure 2.3: The velocity profile with $P_r = 7$, $v_0 = 0.01$, $\omega = 1.5$, $\epsilon = 0.01$, $t = \pi/2\omega$, $\alpha = 0.02$, $u_0 = 0.01$.

Figure 2.4 shows the effect of suction on the boundary layer velocity for different Hartmann numbers. The boundary layer velocity decreases with increasing suction. Increasing suction means that more fluid is taken out of the system, thereby reducing the momentum boundary layer. For Newtonian fluids (that is, $\alpha = 0$), when $M = 0$ we found the same results as Soundalgekar et al (2004) and Attia (2003). In Figure 2.4(b) we increased the Hartmann number from $M = 5$ to $M = 45$ while keeping the same suction values as in Figure 2.4 (a). Comparing Figure 2.4(a) and Figure 2.4(b), we found that the boundary layer velocity decrease faster in Figure 2.4(b) with large Hartmann numbers than in Figure 2.4 (a).

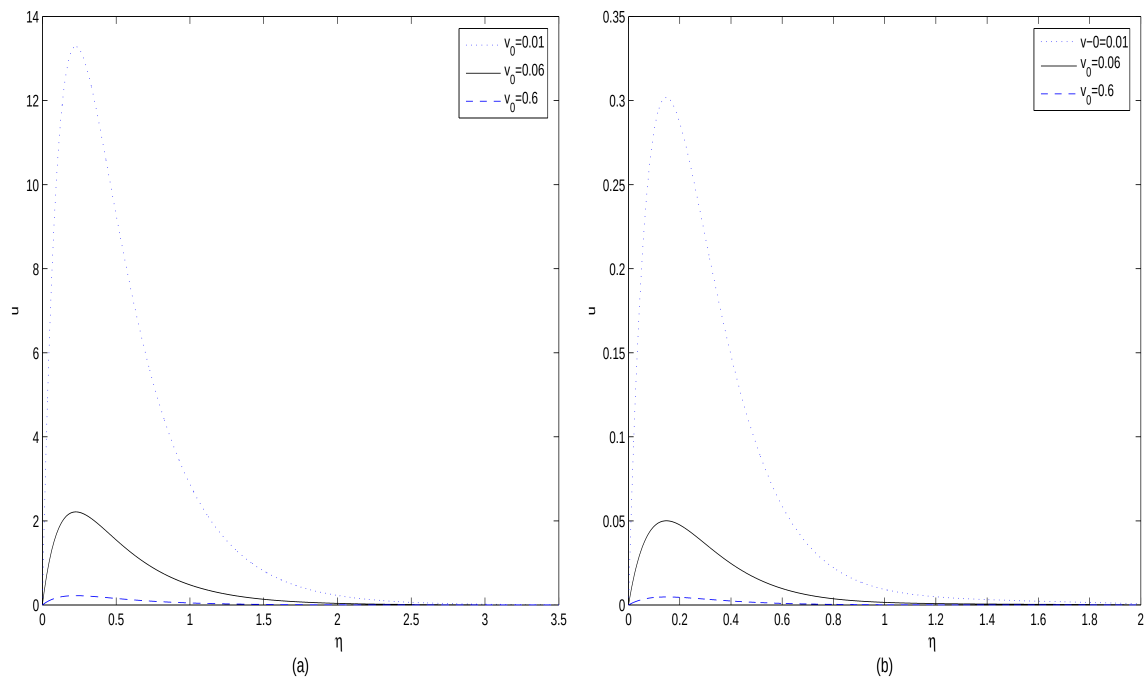

Figure 2.4: The velocity profile with $Pr = 7$, $\omega = 1.5$, $\epsilon = 0.01$, $\alpha = 0.02$, $t = \pi/2\omega$, $u0 = 0.01$; (a) $M = 5$ and (b) $M = 45$.

Figure 2.5 shows the variation of the velocity with Prandtl numbers. We observe from Figure 2.5 (a) that the velocity increases with increasing Prandtl numbers whereas in Figure 2.5 (b) it is decreasing with increasing Prandtl numbers. This difference is caused by the effect of the magnetic field. Increasing the Prandtl number is equivalent to increasing the kinematic viscosity, Cebeci and Bradshaw (1984)

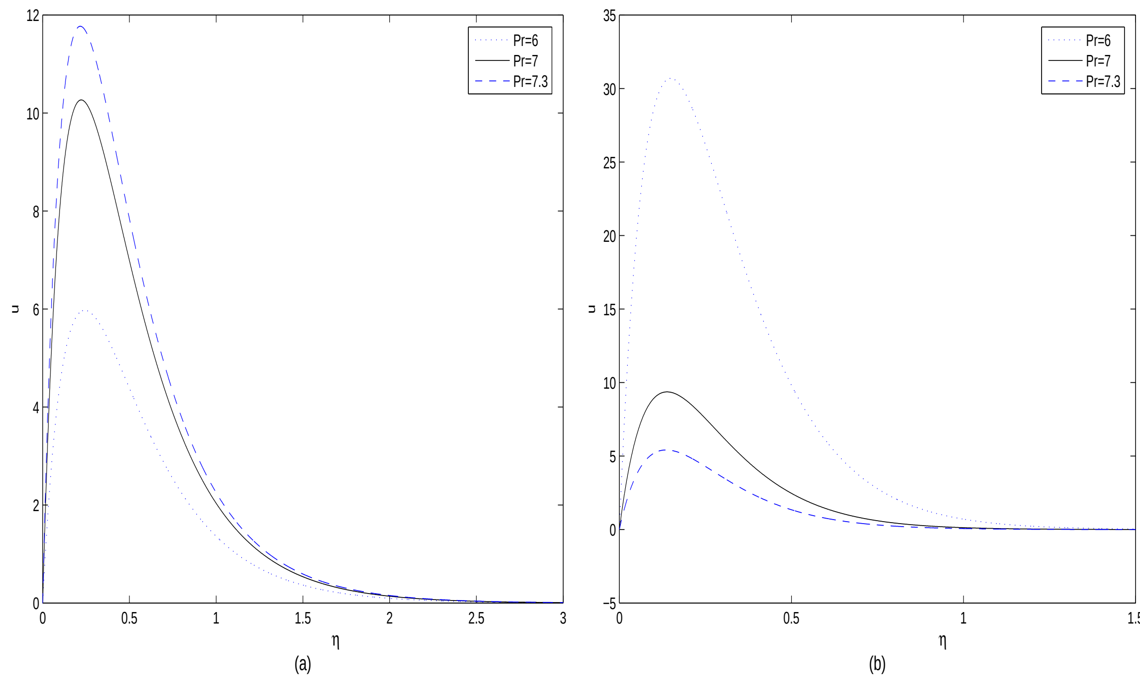

Figure 2.5: The velocity profile with $\omega = 1.5$, $\epsilon = 0.01$, $\alpha = 0.02$, $t = \pi/2\omega$, $u_0 = 0.01$; (a) $M = 5$ and (b) $M = 60$.

Figure 2.6 shows the variation of the velocity with angular frequency. The velocity does not depend on the angular frequency.

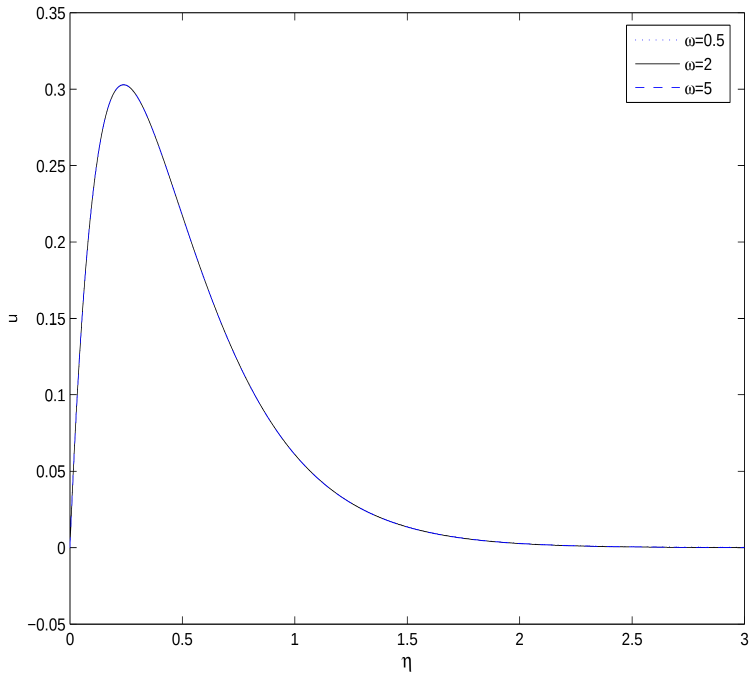

Figure 2.6: The velocity profile with $P_r = 7$, $v_0 = 0.01$, $\epsilon = 0.01$, $M = 10$, $\alpha = 0.08$, $t = \pi/2\omega$, $u_0 = 0.01$.

Figure 2.7 shows the variation of the temperature with the Prandtl number. The temperature decreases with increasing Prandtl numbers. This may be explained by the fact that increasing the Prandtl number is equivalent to reducing thermal diffusivity which characterizes the rate at which the heat is conducted, White (1999) and Cebeci and Bradshaw (1984). This is also consistent with the earlier findings by Sattar and Alam (1995), Vajravelu and Roper (1999), Massoudi (2001), Soundalgekar et al (2004) and Pantokratorras (2008).

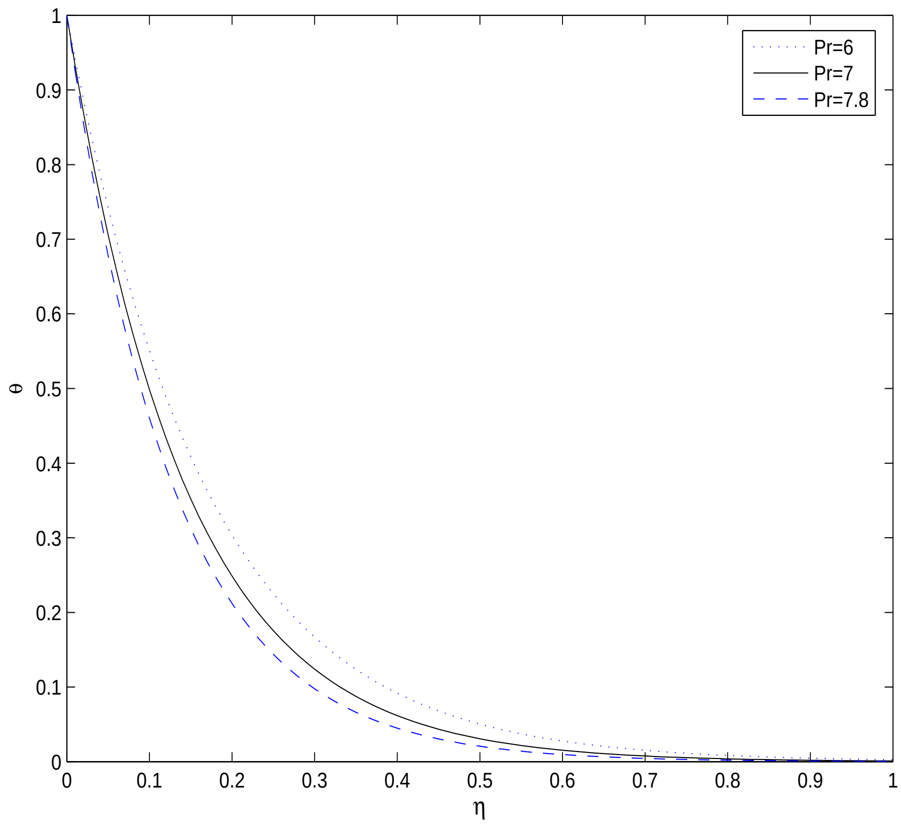

Figure 2.7: The temperature profile when $\epsilon = 0.1, \quad \omega = 1, \quad t = \pi/2\omega, \quad \alpha = 0.02,$ $u_0 = 0.01, \quad M = 5.$

Figure 2.8 shows the variation of the temperature profile with the angular frequency. The temperature increases only marginally with increasing angular frequencies. This is consistent with the results of Alam et al (2007).

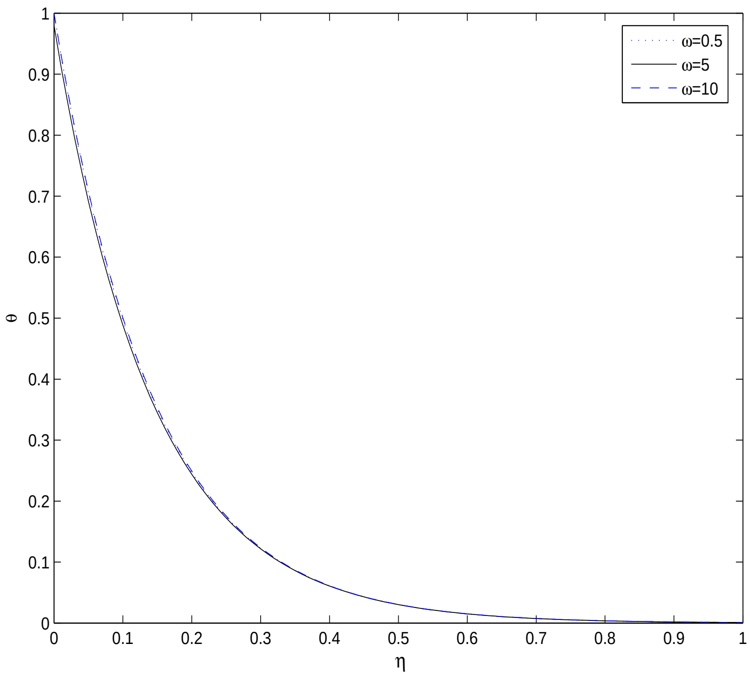

Figure 2.8: The temperature with $P_r = 7, \quad \epsilon = 0.01, \quad \alpha = 0.02, \quad t = \pi/2\omega, \quad u_0 = 0.01, \quad M = 5.$

2.4 Conclusion

In this chapter we have considered the solution of the equations of motion of a second grade fluid when the Eckert number is zero and the viscoelastic parameter is small. We have considered the effects of various physical parameters such as the Hartmann number, suction, Prandtl number, viscoelastic parameter and angular frequency on the properties of the fluid. Our investigations show that in the case of zero Eckert numbers;

- increasing the magnetic field reduces the boundary layer velocity. This is in line with the earlier finding by Hayat et al (2003) and Vajravelu and Rolins (2004)
- The boundary layer velocity increases or decreases with Prandtl numbers depending on the nature and quantum of the other physical parameters, this is in line with the earlier findings by Cebeci and Bradshaw (1984).
- The boundary layer velocity decreases with increasing suction velocity
- The velocity decreases with increasing visco-elastic parameter α. As α increases we obtained back flow.
- Both the temperature and velocity are independent of the angular frequency.
- The temperature decreases with increasing Prandtl numbers.

Chapter 3

Non-zero Eckert number solutions

3.1 Introduction

In this chapter we solve the problem defined by equations (2.13)-(2.20) when the Eckert number is small but non-zero. In this case there are four coupled third order differential equations. To solve these equations we use the multi-parameter perturbation technique. This technique is very useful when solving problems with several embedded small parameters. These parameters describe different physical and fluid properties such as the material and the dynamic properties. For the technique to be applicable the parameters must be of the same order but independent of each other, Nowinski and Ismail (1965). This technique has been used by many researchers such as Sahoo et al (2003) in his study of heat transfer in mercury and electrolytic solutions past an infinite porous plate with constant suction. Anuar-Bég et al (2005) used the multi-parameter perturbation technique to study the combined effects of periodic suction velocity and heat sinks in unsteady natural convection flow of a viscous fluid flow past an infinite vertical porous plate. A similar study by Murthy et al (2007) extended the work of Sahoo et al (2003) to include boundary layer heating and cooling. The parameters of primary interest and to which the multi-parameter perturbation technique is applied in this chapter are the Eckert number E and the visco-elastic parameter α. These parameters, in addition to ϵ, are assumed to be small so that any product of the three parameters may be neglected.

3.2 Perturbation analysis

In this section we solve equations (2.13)- (2.20) using the multi-parameter perturbation technique. We expand $f_1(\eta)$ and $f_2(\eta)$ in terms of E and α as follows:

$$f_1 = f_{100} + Ef_{101} + \alpha f_{111} + E\alpha f_{112} + 0(E^2), \tag{3.1}$$

$$f_2 = f_{200} + Ef_{201} + \alpha f_{211} + E\alpha f_{212} + 0(E^2), \tag{3.2}$$

$$h_1 = h_{10} + Eh_{11} + 0(E^2), \tag{3.3}$$

$$h_2 = h_{20} + Eh_{21} + 0(E^2), \tag{3.4}$$

where α, E and ϵ are small so that products of these parameters all tend to zero. The boundary conditions to be satisfied are:

$$f_{100} = f_{101} = f_{111} = f_{112} = 0, \quad \text{as} \quad \eta = 0, \tag{3.5}$$

$$f_{100} = f_{101} = f_{111} = f_{112} = 0, \quad \text{as} \quad \eta \to \infty, \tag{3.6}$$

$$f_{200} = 1, \quad f_{201} = f_{211} = f_{212} = 0, \quad \text{as} \quad \eta = 0, \tag{3.7}$$

$$f_{200} = f_{201} = f_{211} = f_{212} = 0, \quad \text{as} \quad \eta \to \infty, \tag{3.8}$$

$$h_{10} = 1, \quad h_{11} = 0, \quad h_{20} = 1, \quad h_{21} = 0, \qquad \text{as} \quad \eta = 0, \tag{3.9}$$

$$h_{10} = h_{11} = h_{20} = h_{21} = 0, \qquad \text{as} \quad \eta \to \infty. \tag{3.10}$$

Substituting the expansions (3.1)-(3.4) in equations (2.13), (2.14), (2.17) and (2.18) gives:

Zeroth order (E^0) :

$$f''_{100} + f'_{100} - Mf_{100} = -\frac{h_{10}}{v_0}, \tag{3.11}$$

$$f''_{200} + f'_{200} - (M + i\omega)f_{200} = -\left(\frac{h_{10} + h_{20}}{v_0}\right), \tag{3.12}$$

$$h''_{10} + Prh'_{10} = 0 \tag{3.13}$$

$$h''_{20} + Prh'_{20} - Pri\omega h_{20} = -h''_{10} - Prh'_{10} + i\omega h_{10}, \tag{3.14}$$

$$f''_{111} + f'_{111} - Mf_{111} = f'''_{100} \tag{3.15}$$

$$f''_{211} + f'_{211} - (M + i\omega)f_{211} = f'''_{200} - i\omega f''_{200}. \tag{3.16}$$

First order (E^1) :

$$f''_{101} + f'_{101} - Mf_{101} = -\frac{h_{11}}{v_0}, \tag{3.17}$$

$$f''_{201} + f'_{201} - (M + i\omega)f_{201} = -\left(\frac{h_{11} + h_{21}}{v_0}\right), \tag{3.18}$$

$$f''_{112} + f'_{112} - Mf_{112} = f'''_{101}, \tag{3.19}$$

$$f''_{212} + f'_{212} - (M + i\omega)f_{212} = f'''_{201} - i\omega f''_{201} \tag{3.20}$$

$$h''_{11} + P_r h'_{11} = -u_0^2 P_r \left(f'^2_{100} + 2\alpha f'_{100} f'_{111}\right), \tag{3.21}$$

$$h''_{21} + P_r h'_{21} - Pri\omega h_{21} = -h''_{11} - P_r h'_{11} + P_r i\omega h_{11} - 2u_0^2 P_r f'_{100} f'_{200}. \tag{3.22}$$

The solution to equation (3.13) is given by

$$h_{10}(\eta) = e^{-P_r\eta}. \tag{3.23}$$

Substituting equation (3.23) in (3.11) gives

$$f''_{100} + f'_{100} - Mf_{100} = -\frac{e^{-P_r\eta}}{v_0}, \tag{3.24}$$

with the particular solution

$$f_{100}(\eta) = \frac{1}{v_0 H}\left(e^{-\beta_2\eta} - e^{-P_r\eta}\right). \tag{3.25}$$

Now substituting equation (3.23) in (3.14) to get

$$h''_{20} + P_r h'_{20} - P_r i\omega h_{20} = P_r i\omega e^{-P_r\eta}. \tag{3.26}$$

Solving the above equation and using the boundary conditions (3.9) and (3.10) we get

$$h_{20}(\eta) = 2e^{-p_2\eta} - e^{-P_r\,\eta}. \tag{3.27}$$

Substituting equation (3.25) in (3.15) gives

$$f_{111}'' + f_{111}' - Mf_{111} = \frac{1}{v_0 H^2}\left(P_r^3 e^{-P_r\,\eta} - \beta_2^3 e^{-\beta_2\,\eta}\right). \tag{3.28}$$

The solution of the above equation is given by

$$f_{111}(\eta) = \frac{P_r^3 e^{-P_r\,\eta}}{v_0 H^2} + \left(\frac{-P_r^3}{v_0 H^2} + \frac{\beta_2^3 \eta}{v_0 H\sqrt{(1+4M)}}\right) e^{-\beta_2\,\eta}. \tag{3.29}$$

Substituting equations (3.23)and (3.27) in (3.12) give

$$f_{200}'' + f_{200}' - (M+i\omega)f_{200} = -\left(\frac{2e^{-p_2\eta}}{v_0}\right). \tag{3.30}$$

Solving the above equation and using the boundary conditions (3.7) and (3.8) gives

$$f_{200}(\eta) = \frac{1}{v_0 H_1}\left(v_0 H_1 + 2\right) e^{-k_2\,\eta} - \frac{2}{v_0 H_1} e^{-p_2\,\eta}, \tag{3.31}$$

where $H_1 = p_2^2 - p_2 - (M+i\omega)$ and $k_2 = \frac{1}{2}[1+\sqrt{1+4(M+i\omega)}]$.

Differentiating equation (3.31) and substituting in equation (3.16) gives

$$f_{211}'' + f_{211}' - (M+i\omega)f_{211} = \frac{-k_2^2}{v_0 H_1}(v_0 H_1 + 2)(k_2 + i\omega)e^{-k_2\,\eta} + \frac{2p_2^2(p_2+i\omega)}{v_0 H_1} e^{-p_2\,\eta}. \tag{3.32}$$

The above equation has the solution

$$\begin{aligned} f_{211} \quad &= \quad \left(\frac{-2p_2^2(p_2+i\omega)}{v_0 H_1^2} + \frac{k_2^2(v_0 H_1 + 2)(k_2+i\omega)\eta}{v_0 H_1\sqrt{1+4(M+i\omega)}}\right) e^{-k_2\,\eta} \\ &+ \quad \frac{2p_2^2(p_2+i\omega)}{v_0 H_1^2} e^{-p_2\,\eta}. \end{aligned} \tag{3.33}$$

Now differentiating equations (3.25) and (3.29) and substituting in equation (3.21) gives

$$h_{11}'' + P_r h_{11}' = A_4 e^{-2P_r\,\eta} + A_5 e^{-(\beta_2+P_r)\,\eta} + A_6 e^{-2\beta_2\,\eta}, \tag{3.34}$$

where

$$B_1 = -2u_0^2 P_r \alpha \left(\frac{\beta_2^3 P_r}{v_0^2 H^2 \sqrt{(1+4M)}} + \frac{2\beta_2 P_r^4}{v_0^2 H^3} - \frac{P_r \beta_2^4 \eta}{v_0^2 H^2 \sqrt{(}1+4M)} \right),$$

$$B_2 = \frac{2u_0^2 P_r^6 \alpha}{v_0^2 H^3},$$

$$A_3 = 2u_0^2 Pr\alpha \left(\frac{\beta_2^4}{v_0^2 H^2 \sqrt{1+4M}} + \frac{\beta_2^2 P_r^3}{v_0^2 H^3} - \frac{\beta_2^5 \eta}{v_0^2 H^2 \sqrt{1+4M}} \right),$$

$$A_4 = B_2 - \frac{u_0^2 P_r^3}{v_0^2 H^2},$$

$$A_5 = B_1 + \frac{2u_0^2 P_r^2 \beta_2}{v_0^2 H^2},$$

$$A_6 = A_3 - \frac{u_0^2 P_r \beta_2^2}{v_0^2 H^2}.$$

The solution to equation (3.34) is given by

$$h_{11}(\eta) = K_5 e^{-P_r\,\eta} + \frac{A_4 e^{-2P_r\,\eta}}{2P_r^2} + \frac{A_5 e^{-(P_r+\beta_2)\eta}}{\beta_2(\beta_2+P_r)} + \frac{A_6 e^{-2\beta_2\,\eta}}{2\beta_2(2\beta_2-P_r)}, \tag{3.35}$$

and the boundary conditions (3.9)-(3.10) give

$$K_5 = -\frac{A_5}{\beta_2(\beta_2+P_r)} - \frac{A_4}{2P_r^2} - \frac{A_6}{2\beta_2(2\beta_2-P_r)}.$$

Equations (3.23) and (3.35) then give

$$h_1(\eta) = (1+EK_5)e^{-P_r\,\eta} + \frac{EA_4}{2P_r^2} E e^{-2P_r\,\eta} + \frac{EA_5\; e^{-(P_r+\beta_2)\eta}}{\beta_2(\beta_2+P_r)} + \frac{EA_6\; e^{-2\beta_2\,\eta}}{2\beta_2(2\beta_2-P_r)}. \tag{3.36}$$

Differentiating equations (3.25), (3.31) and (3.35) and substituting in (3.22) gives

$$\begin{aligned} h_{21}'' + P_r h_{21}' - P_r i\omega h_{21} &= P_r i\omega K_5\; e^{-P_r\eta} + A_7\; e^{-2P_r\eta} + A_8\; e^{-(\beta_2+P_r)\,\eta} \\ &+ A_9\; e^{-2\beta_2\,\eta} \qquad (3.37) \\ &+ A_{10}\; e^{-(P_r+p_2)\eta} + A_{11}\; e^{-(P_r+k_2)\,\eta} \\ &+ A_{12}\; e^{-(\beta_2+p_2)\,\eta} + A_{13}\; e^{-(\beta_2+k_2)\eta}, \end{aligned}$$

where

$$A_7 = A_4\left(\frac{i\omega}{2P_r}-1\right), \qquad A_8 = A_5\left(\frac{P_r i\omega}{\beta_2(\beta_2+P_r)}-1\right),$$
$$A_9 = A_6\left(\frac{P_r i\omega}{2\beta_2(2\beta_2-P_r)}-1\right), \qquad A_{10} = \frac{-4u_0^2Pr^2p_2}{v_0^2HH_1},$$
$$A_{11} = \frac{2u_0^2P_r^2k_2(v_0HH_1+2)}{v_0^2HH_1}, \qquad A_{12} = \frac{4u_0^2P_rp_2\beta_2}{v_0^2HH_1},$$
$$A_{13} = \frac{-2u_0^2P_rk_2\beta_2}{v_0^2HH_1}.$$

The solution to equation (3.37) is given by

$$\begin{aligned}
h_{21}(\eta) &= F_6\ e^{-p_2\ \eta} - K_5\ e^{-P_r\eta} + \frac{A_7}{P_r(2P_r-i\omega)}e^{-2P_r\ \eta} \\
&+ \frac{A_8}{\beta_2^2+P_r(\beta_2-i\omega)}e^{-(P_r+\beta_2)\ \eta} \\
&+ \frac{A_9}{4\beta_2^2-P_r(2\beta_2+i\omega)}e^{-2\beta_2\ \eta} + \frac{A_{10}}{p_2^2+P_r(p_2-i\omega)}e^{-(P_r+p_2);\eta} \\
&+ \frac{A_{11}}{k_2^2+P_r(k_2-i\omega)}e^{-(P_r+k_2)\ \eta} \\
&+ \frac{A_{12}}{(p_2+\beta_2)^2-P_r(p_2+\beta_2)-P_r\ i\omega)}e^{-(\beta_2+p_2)\ \eta} \\
&+ \frac{A_{13}}{(k_2+\beta_2)^2-P_r(k_2+\beta_2)-P_r\ i\omega)}e^{-(\beta_2+k_2)\ \eta},
\end{aligned} \tag{3.38}$$

where

$$\begin{aligned}
F_6 &= K_5 - \frac{A_7}{P_r(2P_r-i\omega)} - \frac{A_8}{\beta_2^2+P_r(\beta_2-i\omega)} - \frac{A_9}{4\beta_2^2-P_r(2\beta_2+i\omega)} - \frac{A_{10}}{\psi_1} \\
&- \frac{A_{11}}{k_2^2+P_r(k_2-i\omega)} - \frac{A_{12}}{\psi_2} - \frac{A_{13}}{\psi_3}
\end{aligned}$$

with $\psi_1 = p_2^2 + P_r(p_2 - i\omega)$, $\psi_2 = (\beta_2+p_2)^2 - Pr(\beta_2+p_2) - P_r\ i\omega$

and $\psi_3 = (\beta_2+k_2)^2 - P_r(\beta_2+k_2) - P_r\ i\omega$.

Substituting equations (3.27) and (3.38) in(3.4) we get

$$
\begin{aligned}
h_2(\eta) &= 2e^{-p_2\eta} - e^{-P_r\,\eta} + EF_6\; e^{-p_2\,\eta} - EK_5\; e^{-Pr\,\eta} \\
&+ \frac{EA_7}{Pr(2P_r - i\omega)} e^{-2P_r\,\eta} \\
&+ \frac{EA_8}{\beta_2^2 + Pr(\beta_2 - i\omega)} e^{-(P_r+\beta_2)\,\eta} \\
&+ \frac{EA_9}{4\beta_2^2 - P_r(2\beta_2 + i\omega)} e^{-2\beta_2\,\eta} + \frac{EA_{10}}{p_2^2 + P_r(p_2 - i\omega)} e^{-(P_r+p_2)\,\eta} \\
&+ \frac{EA_{11}}{k_2^2 + P_r(k_2 - i\omega)} e^{-(P_r+k_2)\,\eta} \\
&+ \frac{EA_{12}}{(p_2+\beta_2)^2 - P_r(p_2+\beta_2) - P_r\; i\omega} e^{-(\beta_2+p_2)\,\eta} \\
&+ \frac{EA_{13}}{(k_2+\beta_2)^2 - P_r(k_2+\beta_2) - P_r\; i\omega} e^{-(\beta_2+k_2)\,\eta}.
\end{aligned}
\tag{3.39}
$$

From equations (3.39) and (3.36) we get

$$
\begin{aligned}
\theta(\eta,t) &= (1+EK_5)e^{-P_r\,\eta} + \frac{EA_4}{2P_r^2} e^{-2P_r\,\eta} + \frac{EA_5\; e^{-(P_r+\beta_2)\,\eta}}{\beta_2(\beta_2+P_r)} \\
&+ \frac{EA_6\; e^{-2\beta_2\,\eta}}{2\beta_2(2\beta_2 - P_r)} + \frac{\epsilon}{2} e^{i\omega t}\left((1+EK_5)e^{-P_r\,\eta} + \frac{EA_4}{2P_r^2} e^{-2P_r\,\eta}\right) \\
&+ \epsilon e^{i\omega t} \frac{EA_5\; e^{-(P_r+\beta_2)\,\eta}}{2\beta_2(\beta_2+P_r)} \\
&+ \epsilon e^{i\omega t} \frac{EA_6\; e^{-2\beta_2\,\eta}}{4\beta_2(2\beta_2 - P_r)} \\
&+ \frac{\epsilon}{2} e^{i\omega t}\left(2e^{-p_2\eta} - e^{-P_r\,\eta} + EF_6\; e^{-p_2\,\eta} - EK_5\; e^{-P_r\,\eta}\right) \\
&+ \frac{\epsilon}{2} e^{i\omega t}\left(\frac{EA_7}{Pr(2P_r - i\omega)} e^{-2P_r\,\eta} + \frac{EA_8}{\beta_2^2 + P_r(\beta_2 - i\omega)} e^{-(P_r+\beta_2)\,\eta}\right) \\
&+ \frac{\epsilon}{2} e^{i\omega t}\left(\frac{EA_9}{4\beta_2^2 - P_r(2\beta_2 + i\omega)} e^{-2\beta_2\,\eta} + \frac{EA_{10}}{p_2^2 + P_r(p_2 - i\omega)} e^{-(P_r+p_2)\,\eta}\right) \\
&+ \frac{\epsilon}{2} e^{i\omega t}\left(\frac{EA_{11}}{k_2^2 + P_r(k_2 - i\omega)} e^{-(P_r+k_2)\eta}\right) \\
&+ \frac{\epsilon}{2} e^{i\omega t}\left(\frac{EA_{12}}{(p_2+\beta_2)^2 - P_r(p_2+\beta_2) - P_r\; i\omega} e^{-(\beta_2+p_2)\,\eta}\right) \\
&+ \frac{\epsilon}{2} e^{i\omega t}\left(\frac{EA_{13}}{((k_2+\beta_2)^2 - P_r(k_2+\beta_2) - P_r\; i\omega)} e^{-(\beta_2+k_2)\,\eta}\right).
\end{aligned}
\tag{3.40}
$$

Substituting h_{11} into (3.17) gives

$$f''_{101} + f'_{101} - Mf_{101} = -\frac{K_5}{v_0}e^{-P_r\,\eta} - \frac{A_4\ e^{-2P_r\,\eta}}{2v_0P_r^2} - \frac{A_5\ e^{-(P_r+\beta_2)\,\eta}}{v_0\beta_2(\beta_2+P_r)} - \frac{A_6\ e^{-2\beta_2\,\eta}}{2v_0\beta_2(2\beta_2-P_r)}. \tag{3.41}$$

The boundary conditions are

$$f_{101} = 0 \quad \text{as} \quad \eta \to 0 \quad \text{and} \quad f_{101} = 0 \quad \text{as} \quad \eta \to \infty. \tag{3.42}$$

The complementary solution to equation (3.41) is given by

$$f_{101}(\eta) = F_7\ e^{\beta_1\,\eta} + F_8\ e^{-\beta_2\,\eta}, \tag{3.43}$$

where F_7 and F_8 are constants to be determined. The particular solution to equation (3.41) is

$$\begin{aligned} f_{101p}(\eta) &= -\frac{K_5}{v_0H}e^{-P_r\,\eta} - \frac{A_4\ e^{-2P_r\,\eta}}{2v_0P_r^2(4Pr^2-2P_r-M)} \\ &- \frac{A_5\ e^{-(P_r+\beta_2)\,\eta}}{v_0\beta_2(\beta_2+P_r)((\beta_2+P_r)^2-(\beta_2+P_r)-M)} \\ &- \frac{A_6\ e^{-2\beta_2\,\eta}}{2v_0\beta_2(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)}. \end{aligned} \tag{3.44}$$

Finally, equations (3.43), (3.44) and the boundary conditions (3.5)and (3.6) gives

$$\begin{aligned} f_{101}(\eta) &= F_8\ e^{-\beta_2\,\eta} - \frac{K_5}{v_0H}e^{-P_r\,\eta} - \frac{A_4\ e^{-2P_r\,\eta}}{2v_0Pr^2(4P_r^2-2P_r-M)} \\ &- \frac{A_5\ e^{-(P_r+\beta_2)\,\eta}}{v_0\beta_2(\beta_2+P_r)((\beta_2+P_r)^2-(\beta_2+P_r)-M)} \\ &- \frac{A_6\ e^{-2\beta_2\,\eta}}{2v_0\beta_2(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)}, \end{aligned} \tag{3.45}$$

where

$$\begin{aligned}
F_8 &= \frac{K_5}{v_0 H} + \frac{A_4}{2v_0 Pr^2(4P_r^2 - 2Pr - M)} \\
&+ \frac{A_5}{v_0\beta_2(\beta_2 + P_r)((\beta_2 + P_r)^2 - (\beta_2 + P_r) - M)} \\
&+ \frac{A_6}{2v_0\beta_2(2\beta_2 - P_r)(4\beta_2^2 - 2\beta_2 - M)},
\end{aligned}$$

and $F_7 = 0$.

Differentiating equation (3.45) and substituting it in equation(3.19) gives

$$\begin{aligned}
f''_{112} + f'_{112} - Mf_{112} &= -\beta_2^3 F_8\ e^{-\beta_2\,\eta} + \frac{K_5 P_r^3}{v_0 H} e^{-P_r\,\eta} + F_9\ e^{-2P_r\,\eta} \\
&+ F_{10}\ e^{-(P_r+\beta_2)\,\eta} + F_{11}\ e^{-2\beta_2\,\eta}, \qquad (3.46)
\end{aligned}$$

where

$$F_9 = \frac{4P_r A_4}{v_0(4P_r^2 - 2P_r - M)} \qquad (3.47)$$

$$F_{10} = \frac{(P_r + \beta_2)^2 A_5}{v_0\beta_2((\beta_2 + P_r)^2 - (P_r + \beta_2) - M)} \qquad (3.48)$$

$$F_{11} = \frac{4\beta_2^2 A_6}{v_0(2\beta_2 - P_r)(4\beta_2^2 - 2\beta_2 - M)}. \qquad (3.49)$$

The appropriate boundary conditions are

$$f_{112} = 0 \quad \text{as} \quad \eta \to 0 \quad \text{and} \quad f_{112} = 0 \quad \text{as} \quad \eta \to \infty. \qquad (3.50)$$

The solution to equation (3.46) is thus

$$\begin{aligned}
f_{112}(\eta) &= F_{12}\ e^{-\beta_2\,\eta} + \frac{\beta_2^3 \eta F_8\ e^{-\beta_2\eta}}{\sqrt{1+4M}} + \frac{K_5 P_r^3}{v_0 H^2} e^{-P_r\,\eta} + \frac{F_9\ e^{-2P_r\,\eta}}{4P_r^2 - 2P_r - M} \qquad (3.51) \\
&+ \frac{F_{10}\ e^{-(P_r+\beta_2)\,\eta}}{(P_r + \beta_2)^2 - (P_r + \beta_2) - M} + \frac{F_{11}\ e^{-2\beta_2\,\eta}}{4\beta_2^2 - 2\beta_2 - M},
\end{aligned}$$

where

$$
\begin{aligned}
F_{12} &= -\frac{K_5 Pr^3}{v_0 H^2} - \frac{F_9}{4Pr^2 - 2P_r - M} - \frac{F_{10}}{(P_r+\beta_2)^2 - (P_r+\beta_2) - M} \\
&- \frac{F_{11}}{4\beta_2^2 - 2\beta_2 - M}. \qquad (3.52)
\end{aligned}
$$

Substituting equations (3.51), (3.45), (3.29) and (3.25) in (3.1) we get

$$
\begin{aligned}
f_1(\eta) &= \frac{1}{v_0 H}\left(e^{-\beta_2\,\eta} - e^{-P_r \eta}\right) + EF_8\, e^{-\beta_2\,\eta} \\
&-\frac{EK_5}{v_0 H} e^{-P_r\,\eta} - \frac{EA_4\, e^{-2P_r\,\eta}}{2v_0 P_r^2(4P_r^2 - 2P_r - M)} \\
&-\frac{EA_5\, e^{-(P_r+\beta_2)\,\eta}}{v_0\beta_2(\beta_2+P_r)((\beta_2+P_r)^2 - (\beta_2+P_r) - M)} \\
&-\frac{EA_6\, e^{-2\beta_2\,\eta}}{2v_0\beta_2(2\beta_2 - P_r)(4\beta_2^2 - 2\beta_2 - M)} \\
&+\alpha\frac{P_r^3\, e^{-P_r\,\eta}}{v_0 H^2} + \alpha\left(\frac{-P_r^3}{v_0 H^2} + \frac{\beta_2^3\,\eta}{v_0 H\sqrt{(1+4M)}}\right) e^{-\beta_2\,\eta} \\
&+E\alpha F_{12}\, e^{-\beta_2\,\eta} + \frac{E\alpha\beta_2^3 F_8 \eta e^{-\beta_2\,\eta}}{\sqrt{1+4M}} + \frac{\alpha E K_5 P_r^3}{v_0 H^2} e^{-P_r\,\eta} \\
&+\frac{E\alpha F_9\, e^{-2Pr\,\eta}}{4P_r^2 - 2Pr - M} \\
&+\frac{E\alpha F_{10}\, e^{-(P_r+\beta_2)\,\eta}}{(P_r+\beta_2)^2 - (P_r+\beta_2) - M} + \frac{E\alpha F_{11}\, e^{-2\beta_2\,\eta}}{4\beta_2^2 - 2\beta_2 - M}. \qquad (3.53)
\end{aligned}
$$

Substituting equations (3.38) and (3.35) into (3.18) give

$$
\begin{aligned}
f''_{201} + f'_{201} - (M + i\omega) f_{201} &= D_4\, e^{-2P_r\,\eta} + D_5\, e^{-(P_r+\beta_2)\,\eta} + D_6\, e^{-2\beta_2\,\eta} \\
&- \frac{F_6}{v_0} e^{-p_2\eta} \\
&+ D_7\, e^{-(P_r+k_2)\,\eta} + D_8\, e^{-(P_r+p_2)\,\eta} + D_9\, e^{-(\beta_2+p_2)\,\eta} \\
&+ D_{10}\, e^{-(\beta_2+k_2)\,\eta}, \qquad (3.54)
\end{aligned}
$$

where

$$
\begin{aligned}
D_4 &= -\frac{A_4}{2v_0P_r^2} - \frac{A_7}{v_0Pr(2P_r - i\omega)},\\
D_5 &= -\frac{A_5}{v_0\beta_2(\beta_2 + P)_r)} - \frac{A_8}{v_0(\beta_2^2 + P_r(\beta_2 - i\omega))},\\
D_6 &= -\frac{A_6}{2v_0\beta_2(2\beta_2 - P_r)} - \frac{A_9}{v_0(4\beta_2^2 - P_r(2\beta_2 + i\omega))},\\
D_7 &= -\frac{A_{11}}{v_0(k_2^2 + P_r(k_2 - i\omega))},\\
D_8 &= -\frac{A_{10}}{v_0(p_2^2 + P_r(p_2 - i\omega))},\\
D_9 &= -\frac{A_{12}}{v_0((\beta_2 + p_2)^2 - P_r(\beta_2 + p_2) - Pri\omega)},\\
D_{10} &= -\frac{A_{13}}{v_0((\beta_2 + k_2)^2 - P_r(\beta_2 + k_2) - P_ri\omega)}.
\end{aligned}
$$

The appropriate boundary conditions are

$$
f_{201} = 0 \quad \text{as} \quad \eta \to 0 \quad \text{and} \quad f_{201} = 0 \quad \text{as} \quad \eta \to \infty. \tag{3.55}
$$

The complementary solution to (3.54) is given by

$$
f_{(201)c}(\eta) = D_{11}\, e^{-k_2\,\eta} + D_{12}\, e^{k_1\,\eta}, \tag{3.56}
$$

and its particular solution is given by

$$
\begin{aligned}
f_{(201)p}(\eta) \quad = \quad & \frac{D_4}{4P_r^2 - 2P_r - (M + i\omega)} e^{-2P_r\,\eta} \\
& + \frac{D_5}{(P_r + \beta_2)^2 - (P_r + \beta_2) - (M + i\omega)} e^{-(P_r+\beta_2)\,\eta} \\
& + \frac{D_6}{4\beta_2^2 - 2\beta_2 - (M + \omega)} e^{2\beta_2\,\eta} - \frac{F_6}{v_0 H_1} e^{-p_2\,\eta} \\
& + \frac{D_7}{(P_r + k_2)^2 - (P_r + k_2) - (M + i\omega)} e^{-(P_r+k_2)\,\eta} \\
& + \frac{D_8}{(P_r + p_2)^2 - (P_r + p_2) - (M + i\omega)} e^{-(P_r+p_2)\,\eta} \\
& + \frac{D_9}{(p_2 + \beta_2)^2 - (p_2 + \beta_2) - (M + i\omega)} e^{-(p_2+\beta_2)\,\eta} \\
& + \frac{D_{10}}{(k_2 + \beta_2)^2 - (k_2 + \beta_2) - (M + i\omega)} e^{-(k_2+\beta_2)\,\eta}.
\end{aligned}
\tag{3.57}
$$

Equations (3.56) and (3.57) and the boundary conditions (3.55) give us the following solution:

$$
\begin{aligned}
f_{201}(\eta) \quad = \quad & D_{11}\, e^{-k_2\,\eta} + \frac{D_4}{4P_r^2 - 2P_r - (M + i\omega)} e^{-2Pr\,\eta} && (3.58) \\
& + \frac{D_5}{(P_r + \beta_2)^2 - (P_r + \beta_2) - (M + i\omega)} e^{-(P_r+\beta_2)\,\eta} \\
& + \frac{D_6}{4\beta_2^2 - 2\beta_2 - (M + \omega)} e^{-2\beta_2\,\eta} - \frac{F_7}{v_0 H_1} e^{-p_2\eta} \\
& + \frac{D_7}{(P_r + k_2)^2 - (P_r + k_2) - (M + i\omega)} e^{-(P_r+k_2)\,\eta} \\
& + \frac{D_8}{(P_r + p_2)^2 - (P_r + p_2) - (M + i\omega)} e^{-(P_r+p_2)\,\eta} \\
& + \frac{D_9}{(p_2 + \beta_2)^2 - (p_2 + \beta_2) - (M + i\omega)} e^{-(p_2+\beta-2)\,\eta} \\
& + \frac{D_{10}}{(k_2 + \beta_2)^2 - (k_2 + \beta_2) - (M + i\omega)} e^{-(k_2+\beta_2)\,\eta},
\end{aligned}
\tag{3.59}
$$

where

$$
\begin{aligned}
D_{11} &= -\frac{D_4}{4P_r^2-2P_r-(M+i\omega)}-\frac{D_5}{(P_r+\beta_2)^2-(P_r+\beta_2)-(M+i\omega)}\\
&- \frac{D_6}{4\beta_2^2-2\beta_2-(M+\omega)}+\frac{F_7}{v_0H_1}\\
&- \frac{D_7}{(P_r+k_2)^2-(P_r+k_2)-(M+i\omega)}\\
&- \frac{D_8}{(P_r+p_2)^2-(P_r+p_2)-(M+i\omega)}\\
&- \frac{D_9}{(p_2+\beta_2)^2-(p_2+\beta_2)-(M+i\omega)}\\
&- \frac{D_{10}}{(k_2+\beta_2)^2-(k_2+\beta_2)-(M+i\omega)},
\end{aligned}
$$

and $D_{12}=0$. Substituting equations (3.58), (3.33) and (3.31) in (3.2) gives

$$
\begin{aligned}
f_2(\eta) &= \frac{1}{v_0H_1}\left(v_0H_1+2\right)e^{-k_2\,\eta}-\frac{2}{v_0H_1}e^{-p_2\,\eta}\\
&+ED_{11}e^{-k_2\,\eta}+\frac{ED_4}{4P_r^2-2P_r-(M+i\omega)}e^{-2P_r\,\eta}\\
&+\frac{ED_5}{(P_r+\beta_2)^2-(P_r+\beta_2)-(M+i\omega)}e^{-(P_r+\beta_2)\,\eta}\\
&+\frac{ED_6}{4\beta_2^2-2\beta_2-(M+\omega)}e^{-2\beta_2\,\eta}-\frac{EF_7}{v_0H_1}e^{-p_2\,\eta}\\
&+\frac{ED_7}{(P_r+k_2)^2-(P_r+k_2)-(M+i\omega)}e^{-(P_r+k_2)\,\eta}\\
&+\frac{ED_8}{(P_r+p_2)^2-(P_r+p_2)-(M+i\omega)}e^{-(P_r+p_2)\eta}\\
&+\frac{ED_9}{(p_2+\beta_2)^2-(p_2+\beta_2)-(M+i\omega)}e^{-(p_2+\beta_2)\,\eta}\\
&+\frac{ED_{10}}{(k_2+\beta_2)^2-(k_2+\beta_2)-(M+i\omega)}e^{-(k_2+\beta_2)\,\eta}\\
&+\alpha\left(\frac{-2p_2^2(p_2+i\omega)}{v_0H_1^2}+\frac{k_2^2(v_0H_1+2)(k_2+i\omega)\eta}{v_0H_1\sqrt{1+4(M+i\omega)}}\right)e^{-k_2\eta}\\
&+\alpha\frac{2p_2^2(p_2+i\omega)}{v_0H_1^2}e^{-p_2\,\eta}. \qquad (3.60)
\end{aligned}
$$

Therefore the boundary layer velocity is given by

$$
\begin{aligned}
u(\eta,t) \;=\;& \frac{u_0}{v_0 H}\left(e^{-\beta_2\,\eta}-e^{-P_r\,\eta}\right)+u_0 E F_8\, e^{-\beta_2\,\eta} \\
&-\frac{u_0 E K_5}{v_0 H}e^{-P_r\,\eta}-\frac{u_0 E A_4\, e^{-2P_r\,\eta}}{2v_0 P_r^2(4P_r^2-2P_r-M)} \\
&-\frac{u_0 E A_5\, e^{-(P_r+\beta_2)\,\eta}}{v_0\beta_2(\beta_2+P_r)((\beta_2+P_r)^2-(\beta_2+P_r)-M)} \\
&-\frac{u_0 E A_6\, e^{-2\beta_2\,\eta}}{2v_0\beta_2(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)} \\
&+\alpha u_0\frac{P_r^3 e^{-P_r\,\eta}}{v_0 H^2}+\alpha u_0\left(\frac{-P_r^3}{v_0H^2}+\frac{\beta_2^3\eta}{v_0 H\sqrt{(1+4M)}}\right)e^{-\beta_2\,\eta} \qquad (3.61)\\
&+\left(E\alpha u_0 F_{12}+E\alpha u_0\frac{\beta_2^3 F_8\eta}{\sqrt{1+4M}}\right)e^{-\beta_2\,\eta}+\frac{E u_0\alpha K_5 P_r^3}{v_0H^2}e^{-P_r\,\eta} \\
&+\frac{u_0 E\alpha F_9\, e^{-2P_r\,\eta}}{4P_r^2-2P_r-M} \\
&+\frac{E\alpha u_0 F_{10}\, e^{-(P_r+\beta_2)\eta}}{(P_r+\beta_2)^2-(P_r+\beta_2)-M}+\frac{E u_0\alpha F_{11}\, e^{-2\beta_2\,\eta}}{4\beta_2^2-2\beta_2-M} \\
&+\frac{\epsilon u_0 e^{(i\omega t-k_2\,\eta)}}{v_0 H_1}(v_0H_1+2)-\frac{2\epsilon u_0 e^{(i\omega t-p_2\,\eta)}}{v_0H_1} \\
&+\epsilon E D_{11}\, e^{(i\omega t-k_2\,\eta)}+\frac{D_4\epsilon u_0 e^{(i\omega t-2P_r\,\eta)}}{4P_r^2-2P_r-(M+i\omega)} \\
&+\frac{E D_5 u_0\epsilon e^{(i\omega t-(P_r+\beta_2)\,\eta)}}{(P_r+\beta_2)^2-(P_r+\beta_2)-(M+i\omega)} \\
&+\frac{E D_6\epsilon u_0 e^{(i\omega t-2\beta_2\,\eta)}}{4\beta_2^2-2\beta_2-(M+\omega)}-\frac{\epsilon u_0 E F_7\, e^{(i\omega t-p_2\,\eta)}}{v_0H_1} \\
&+\frac{E D_7\epsilon u_0 e^{(i\omega t-(P_r+k_2)\,\eta)}}{(P_r+k_2)^2-(P_r+k_2)-(M+i\omega)} \\
&+\frac{E D_8 u_0\epsilon e^{(i\omega t-(P_r+p_2)\,\eta)}}{(P_r+p_2)^2-(P_r+p_2)-(M+i\omega)} \\
&+\frac{E D_9 u_0\epsilon e^{(i\omega t-(p_2+\beta_2)\,\eta)}}{(p_2+\beta_2)^2-(p_2+\beta_2)-(M+i\omega)} \\
&+K_1\, e^{(i\omega t-(k_2+\beta_2)\,\eta)}+K_2\, e^{(i\omega t-k_2\,\eta)} \\
&+K_3\, e^{(i\omega t-p_2\,\eta)},
\end{aligned}
$$

where

$$
\begin{aligned}
K_1 &= \frac{ED_{10}u_0\epsilon}{(k_2+\beta_2)^2-(k_2+\beta_2)-(M+i\omega)}, \\
K_2 &= \left(\frac{-2p_2^2(p_2+i\omega)}{v_0H_1^2}+\frac{k_2^2(v_0H_1+2)(k_2+i\omega)\eta}{v_0H_1\sqrt{1+4(M+i\omega)}}\right)\alpha u_0\epsilon, \\
K_3 &= \left(\frac{2p_2^2(p_2+i\omega)}{v_0H_1^2}\right)\alpha u_0\epsilon.
\end{aligned}
$$

3.2.1 Results and Discussions

Figure 3.1 shows the variation of the temperature profile with the Hartmann number M. The temperature increases with increasing M in both Figures 3.1 (a) and 3.1 (b) implying that an applied magnetic field can be used as a means of heating a fluid. This result agrees with published results such as Sahoo et al (2003), Anuar Bég et al (2005) and Alam et al (2007). In Figure 3.1 (b) the Eckert number is increased from $E = 0.001$ to $E = 0.01$ while maintaining the same values of the Hartmann number as in Figure 3.1 (a). The temperature becomes negative for small values of M implying a rapid cooling of the fluid with increasing Eckert numbers. This suggests that, perhaps a combination of moderate Eckert numbers and small Hartmann numbers could be used as a means of reducing the boundary layer temperature.

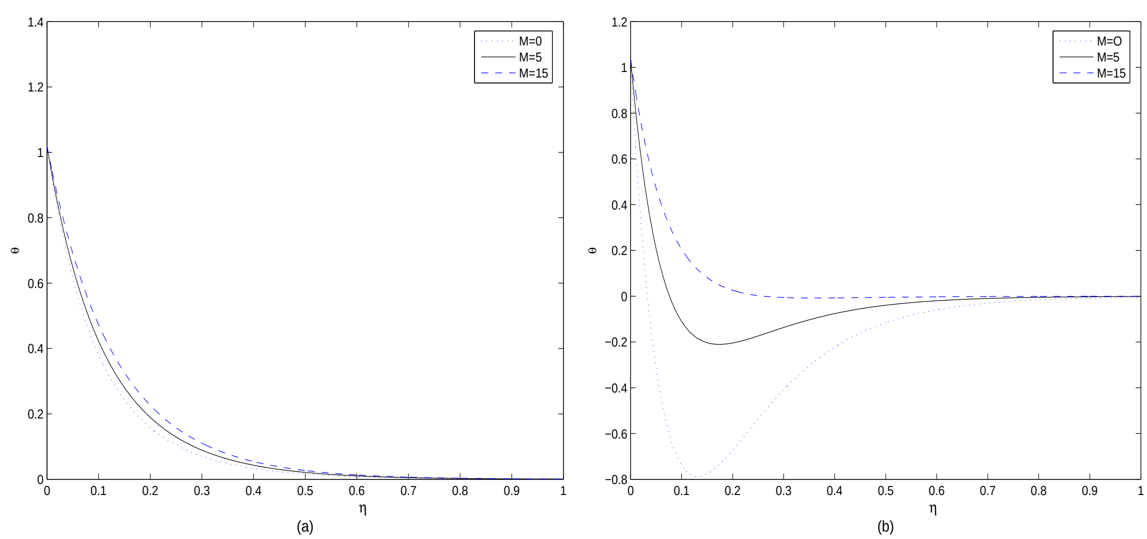

Figure 3.1: The temperature profile with $P_r = 7$, $v_0 = 0.9$, $\epsilon = 0.001$, $\alpha = 0.001$, $t = \pi/2\omega$, $v_0 = 0.9$. $\omega = 2.5$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$

In Figure 3.2 the temperature decreases with increasing viscoelastic parameter α, Hsiao (2007) and Bataller (2007). When the viscoelastic parameter is zero which implies essentially a Newtonian fluid, the temperature is higher than when $\alpha \neq 0$. This means that a Newtonian fluid tends to heat-up much faster than a non-Newtonian fluid. In Figure 3.2 (b), we used $E = 0.01$ instead of $E = 0.001$ and kept the variation of the viscoelastic parameter as in Figure 3.2 (a). The temperature still decreases with the increasing viscoelastic parameter but at a much quicker rate than in Figure 3.2 (a), and becomes negative for larger viscoelastic parameterS.

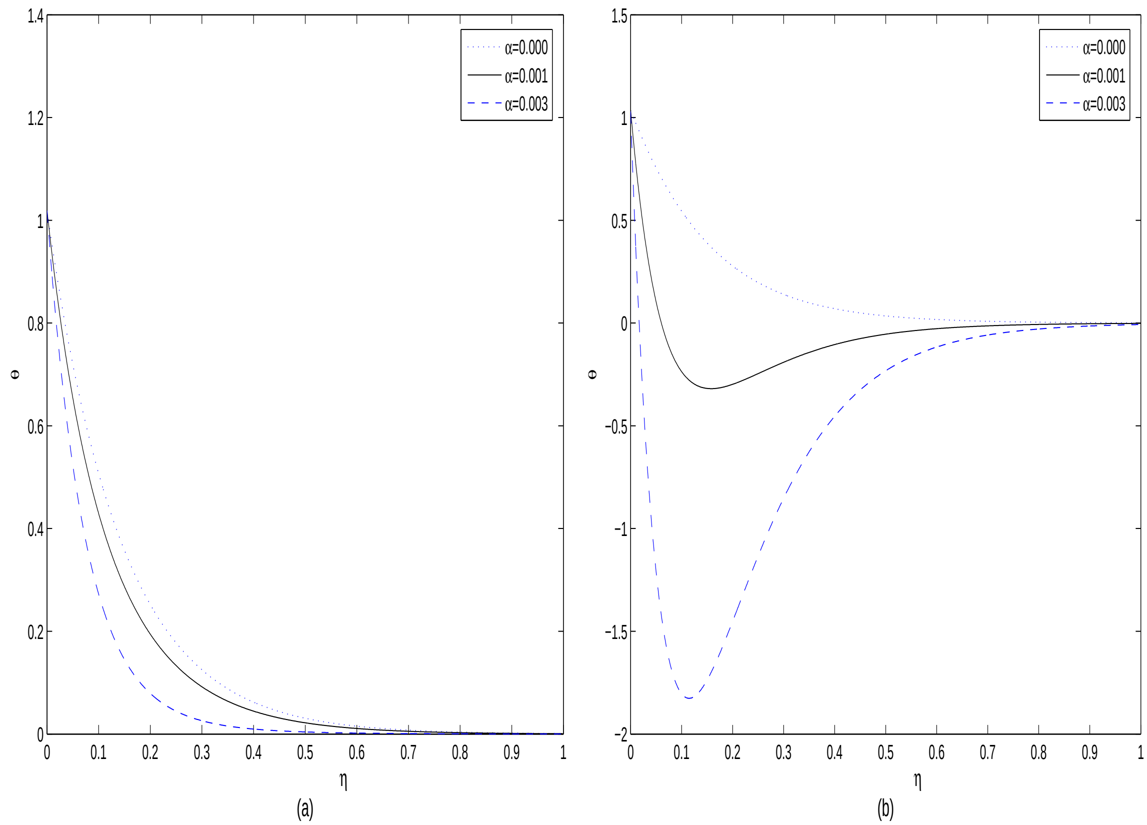

Figure 3.2: The temperature profile with $P_r = 7$, $\epsilon = 0.001$, $t = \pi/2\omega$, $M = 6$, $v_0 = 0.9$. $\omega = 2.5$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$

Figure 3.3 shows the effect of suction on the temperature . The temperature increases with increasing suction, Sahoo et al (2003) and Soundalgekar et al (2004). In Figure 3.3 (b) we increased the Eckert number from $E = 0.001$ to $E = 0.01$ while keeping the same values of suction as in Figure 3.3 (a). The temperature still increases with increasing suction. In addition we observe negative temperatures in Figure 3.3 (b). These differences are due to the effect of the Eckert number. This again suggests the possibility that a combination of suction and the Eckert number could be used as a means of controlling the boundary layer temperature.

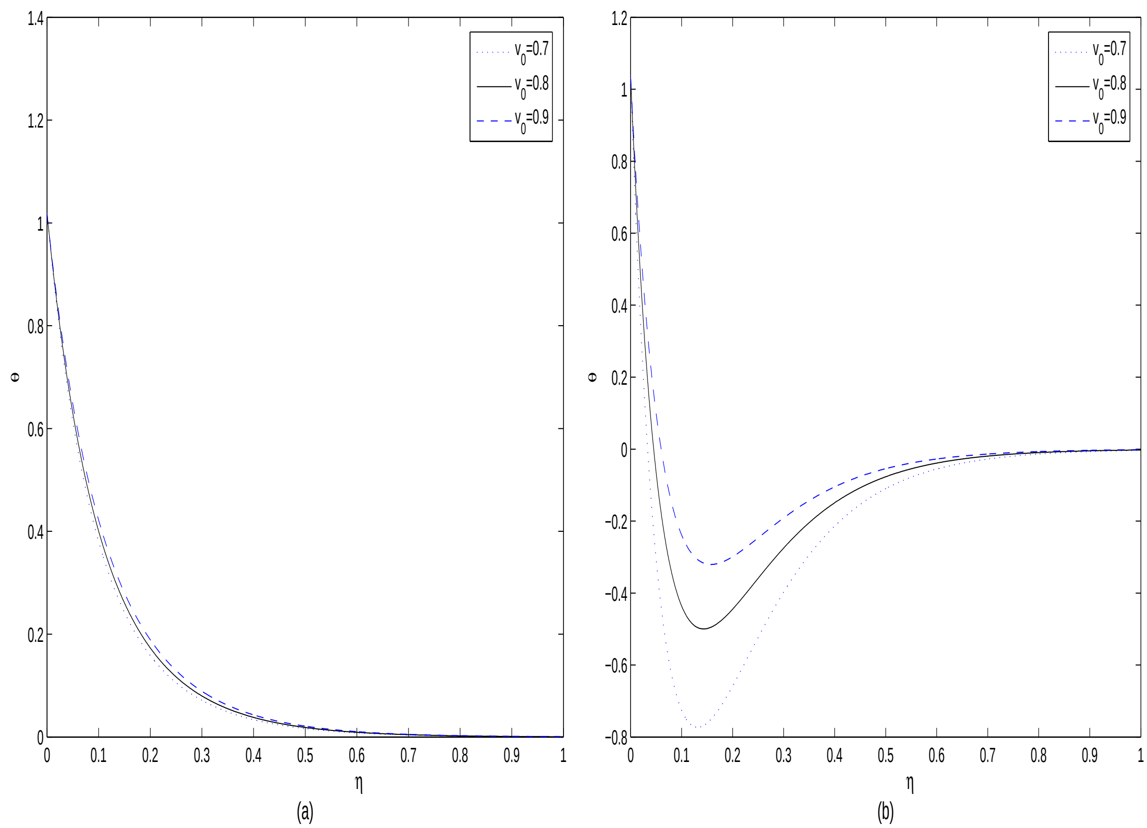

Figure 3.3: The temperature profile with $P_r = 7$,$M = 6$, $\epsilon = 0.001$, $\alpha = 0.001$, $\omega = 2.5$, $t = \pi/2\omega$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$.

Figure 3.4 shows the variation of the temperature profile with the angular frequency. In Figure 3.4 (a), for small Eckert numbers, the temperature increases slightly with increasing ω. These results agree with the earlier findings, for example by Alam (2007). In Figure 3.4 (b) where the Eckert number is larger, the temperature dips below zero before increasing to zero far from the plate.

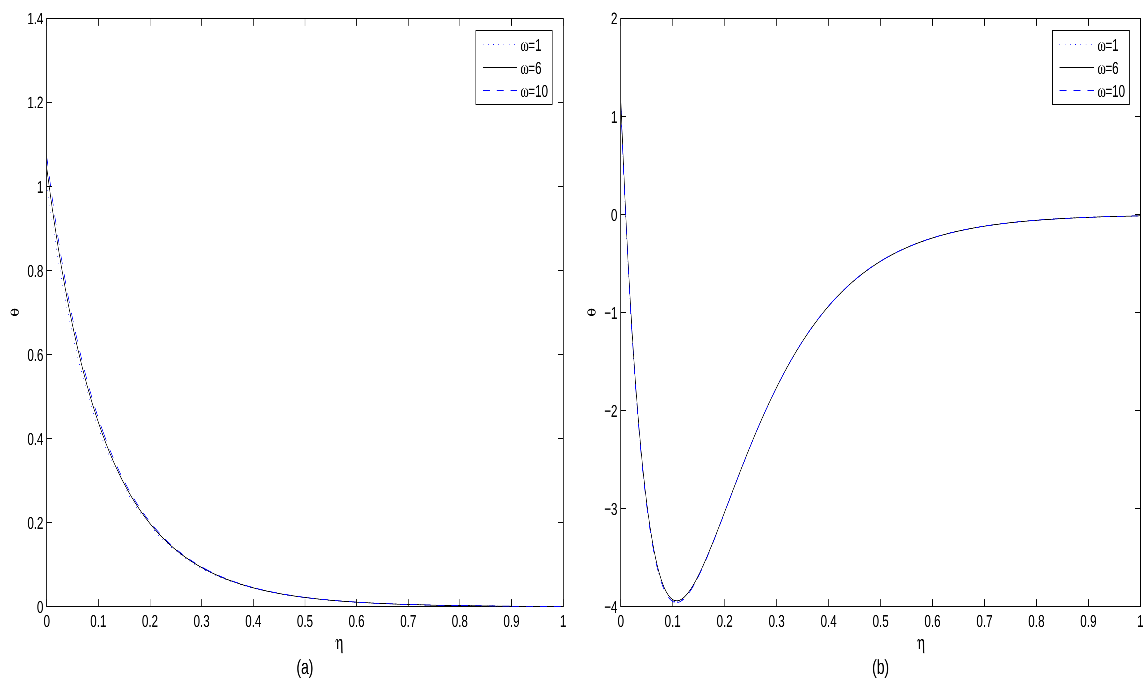

Figure 3.4: The temperature profile with $P_r = 7$, $M = 6$, $\epsilon = 0.001$, $\alpha = 0.001$, $v_0 = 0.9$, $t = \pi/2\omega$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.06$.

Figure 3.5 shows the variation of the temperature profiles with Eckert numbers. The temperature decreases with increasing Eckert numbers, Gschwendtner (2004). In addition, for small Eckert numbers the boundary layer temperature dips below zero near the plate surface. This means that the Eckert number tends to reduce the boundary layer temperature. This may be explained by the fact that increasing the Eckert number is equivalent to decreasing the specific heat capacity, which is a measure of the energy needed to increase the temperature.

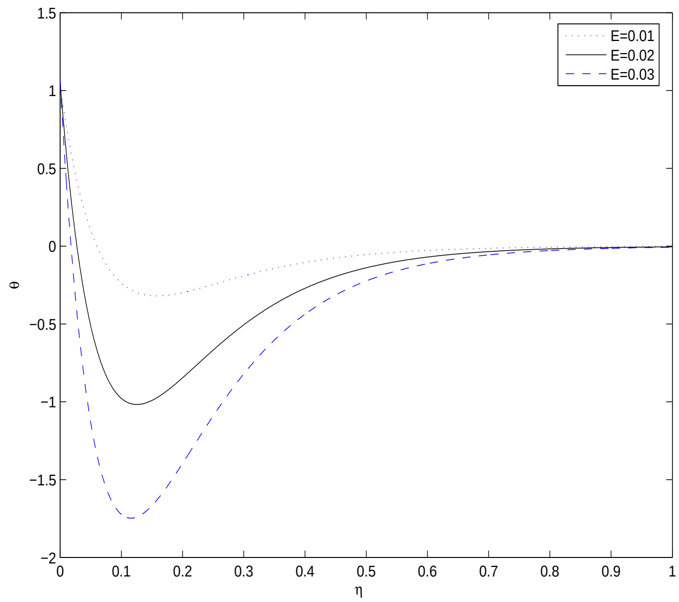

Figure 3.5: The temperature profile with $P_r = 7, \quad M = 6, \quad \epsilon = 0.001, \quad v_0 = 0.9,$ $\alpha = 0.001, \quad t = \pi/2\omega, \quad \omega = 2.5, \quad u_0 = 0.001.$

In Figure 3.6 we observe that the temperature decreases with increasing Prandtl numbers. This may be explained by the fact that increasing the Prandtl number is equivalent to reducing thermal diffusivity which characterizes the rate at which the heat is conducted, White (1999) and Cebeci and Bradshaw (1984). This result agrees well with the earlier work, for example, Sattar and Alam (1995), Vajravelu and Roper (1999), Soundalgekar et al (2004), Bataller (2007) and Pantokratorras (2008). In Figure 3.6 (b), larger Eckert numbers lead to very rapid decreases in the temperature to below zero near the plate surface. This is consistent with the fact that the thermal boundary layer thickness decreases with increasing Prandtl number, Cebeci and Bradshaw (1984).This result is in good agreement with Soundalgekar (2004) and Attia (2003).

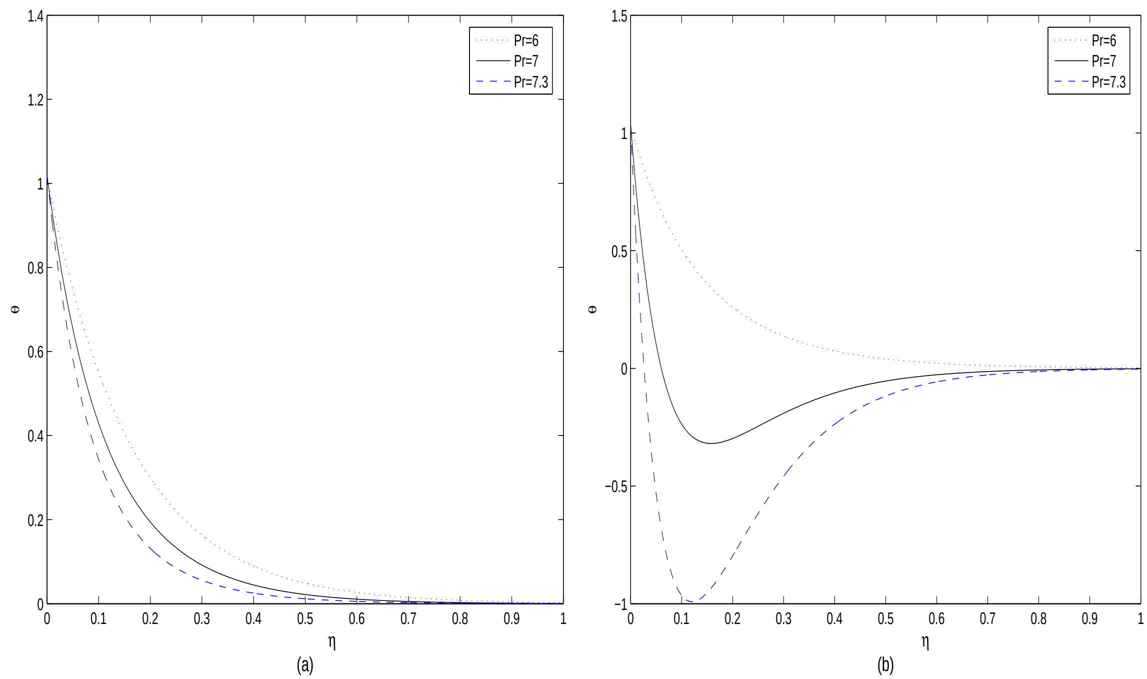

Figure 3.6: The temperature profile with $M = 6$, $\epsilon = 0.001$, $\alpha = 0.001$, $\omega = 2.5$, $t = \pi/2\omega$, $v_0 = 0.9$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$.

Figure 3.7 shows the variation of boundary layer velocity with suction. As expected the boundary layer velocity decreases with increasing suction. Indeed when $\alpha = 0$, we found the same results as Soundalgekar et al (2004). Increasing suction means that more fluid is taken out of the system, thereby reducing the momentum boundary layer.

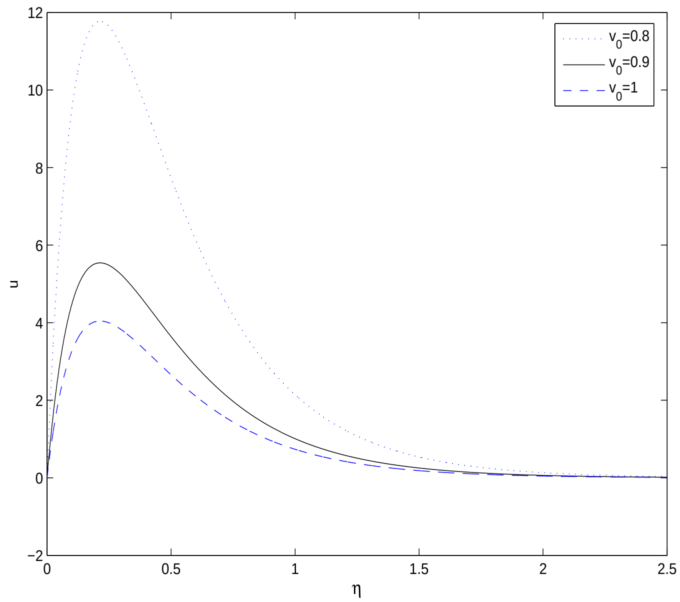

Figure 3.7: The velocity profile with $M = 5, \quad P_r = 7, \quad E = 0.001, \quad \epsilon = 0.001,$ $\alpha = 0.01, \quad \omega = 2.5, \quad t = \pi/2\omega, \quad u_0 = 0.001.$

Figure 3.8 shows the variation of the velocity profile with Eckert numbers. The boundary layer velocity increases with increasing E. Increasing the Eckert number is the equivalent to increasing the square of the characteristic velocity of the flow.

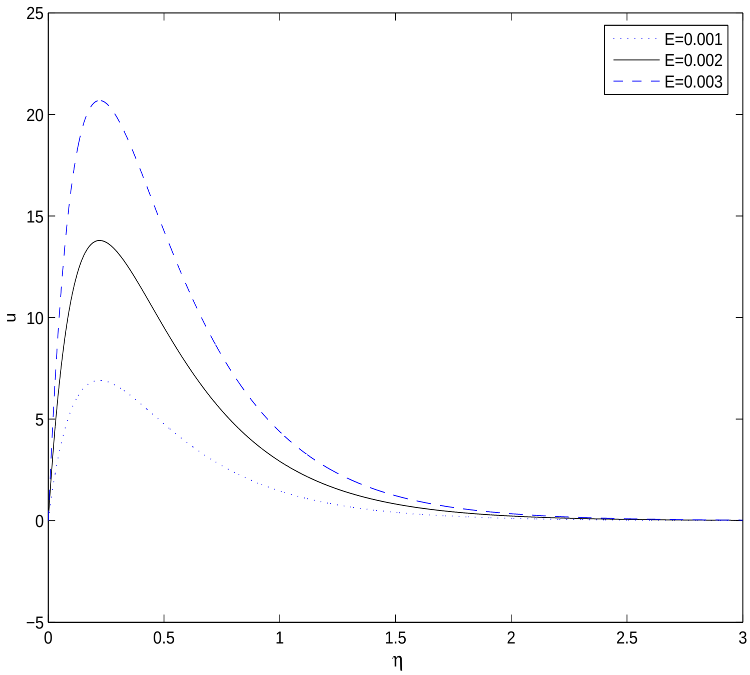

Figure 3.8: The velocity profile with $M = 4$, $P_r = 7$, $\epsilon = 0.001$, $\alpha = 0.01$, $v_0 = 0.9$, $t = \pi/2\omega$, $\omega = 2.5$, $u_0 = 0.001$.

Figure 3.9 shows the variation of boundary layer velocity profile with Hartmann numbers M. The velocity decreases with increasing Hartmann numbers for all Eckert numbers Hayat et al (2003), Vajravelu and Rolins (2004) and Anuar-Bég et al (2005). This is explained by the fact that increasing Hartmann numbers is the same as increasing the Lorentz force that opposes the flow. The reduction in the boundary layer velocity with increasing magnet force provides a means of controlling the flow in the desired direction. In Figure 3.9 (b) the Eckert number is increased while maintaining the same variation of Hartmann numbers as in Figure 3.9 (a). The larger the Eckert number, the larger the amplitude of the velocity. This suggests that a combination of suitable Hartmann numbers and desired values of Eckert numbers could possibly provide a means of controlling and reducing the boundary layer velocity.

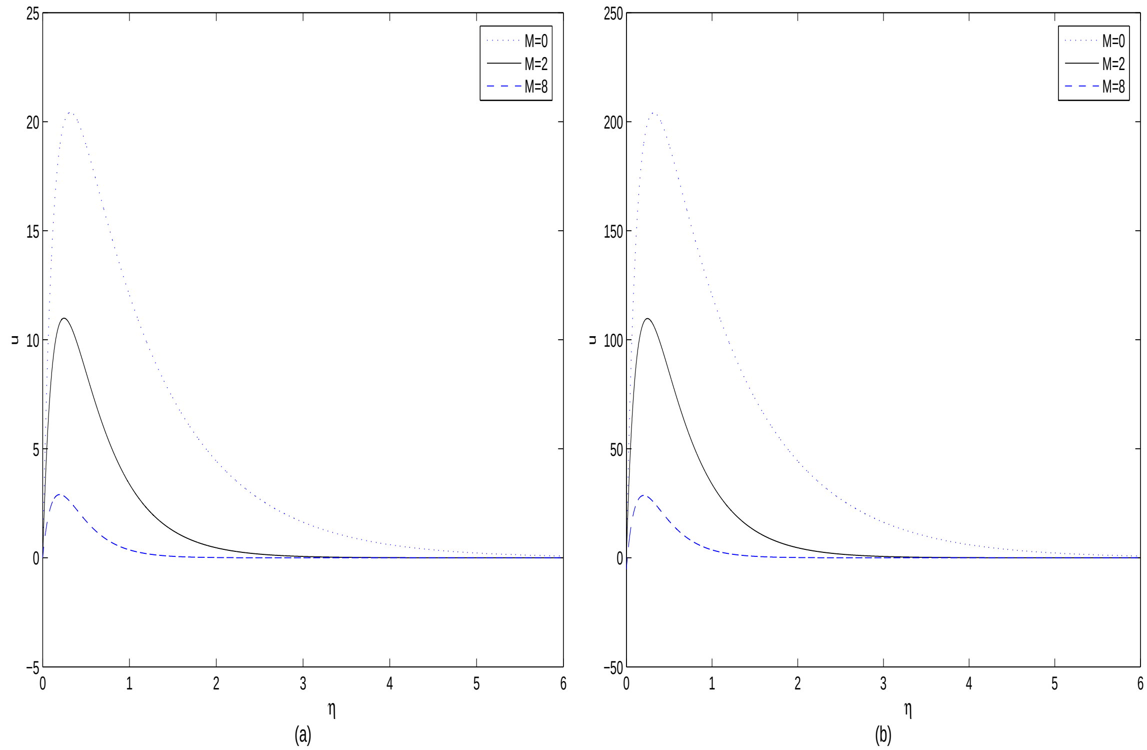

Figure 3.9: The velocity profile with $P_r = 7$, $\epsilon = 0.001$, $\alpha = 0.01$, $v_0 = 0.9$, $t = \pi/2\omega$, $\omega = 2.5$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$.

Figure 3.10 shows the variation of the velocity profile with the angular frequency. The velocity does not depend on the angular frequency. In Figure 3.10 (b) we increased Eckert number from $E = 0.001$ to $E = 0.01$ while maintaining the same values of ω in Figure 3.10 (a). The larger the Eckert number, the larger the amplitude of the velocity.

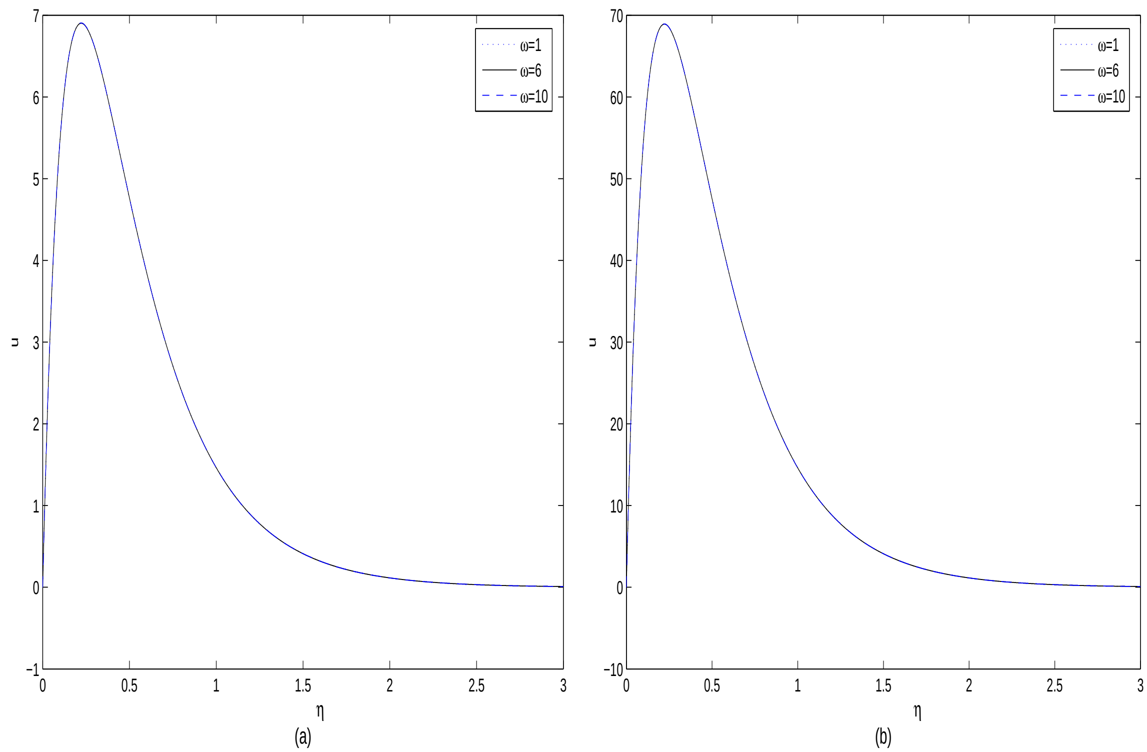

Figure 3.10: The velocity profile with $M = 4$, $P_r = 7$, $\epsilon = 0.001$, $\alpha = 0.01$, $v_0 = 0.9$, $t = \pi/2\omega$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$.

Figure 3.11 shows the variation of the boundary layer velocity with the viscoelastic parameter α. The boundary layer velocity increases with increasing α. In Figure 3.11 (b) the boundary layer velocity is increasing with increasing α faster than in Figure3.11 (a). This result agrees well with Alam et al (2007). This means that for large values of the Eckert number, the effect of the Eckert number is much more significant than that of α . When there is no energy changes (i. e $\theta = 0$), this result agrees with Hayat et al (2003).

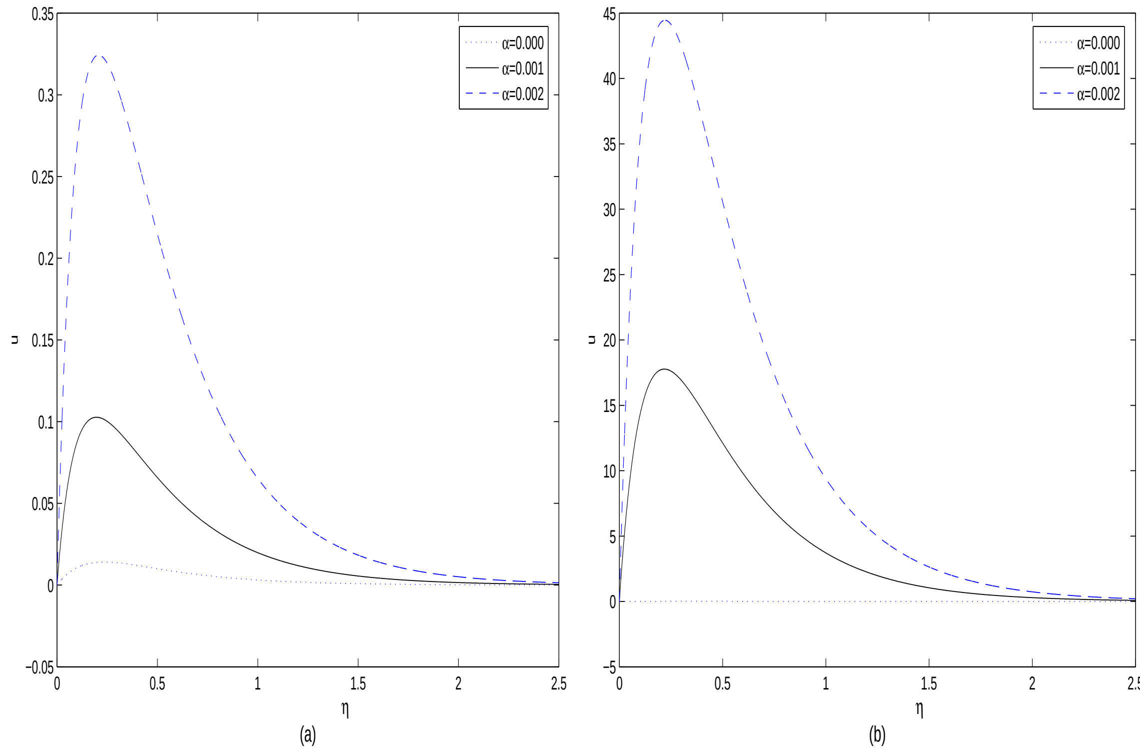

Figure 3.11: The velocity profile with $M = 4$, $\epsilon = 0.001$, $P_r = 7$, $v_0 = 0.9$, $t = \pi/2\omega$, $\omega = 2.5$, $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$.

Figure 3.12 shows the variation of the boundary layer velocity with Prandtl numbers. The velocity increases with increasing Prandtl numbers for all Eckert numbers. The Prandtl number is essential in heat transfer, it controls the relative thickness of the momentum and thermal boundary layers, Cebeci and Bradshaw (1984). Increasing Prandtl numbers is equivalent to increasing the momentum diffusivity. The velocity amplitude is however higher in Figure 3.12(b) than in Figure 3.12(a). This result agrees well with Sahoo et al (2003) and Attia (2003).

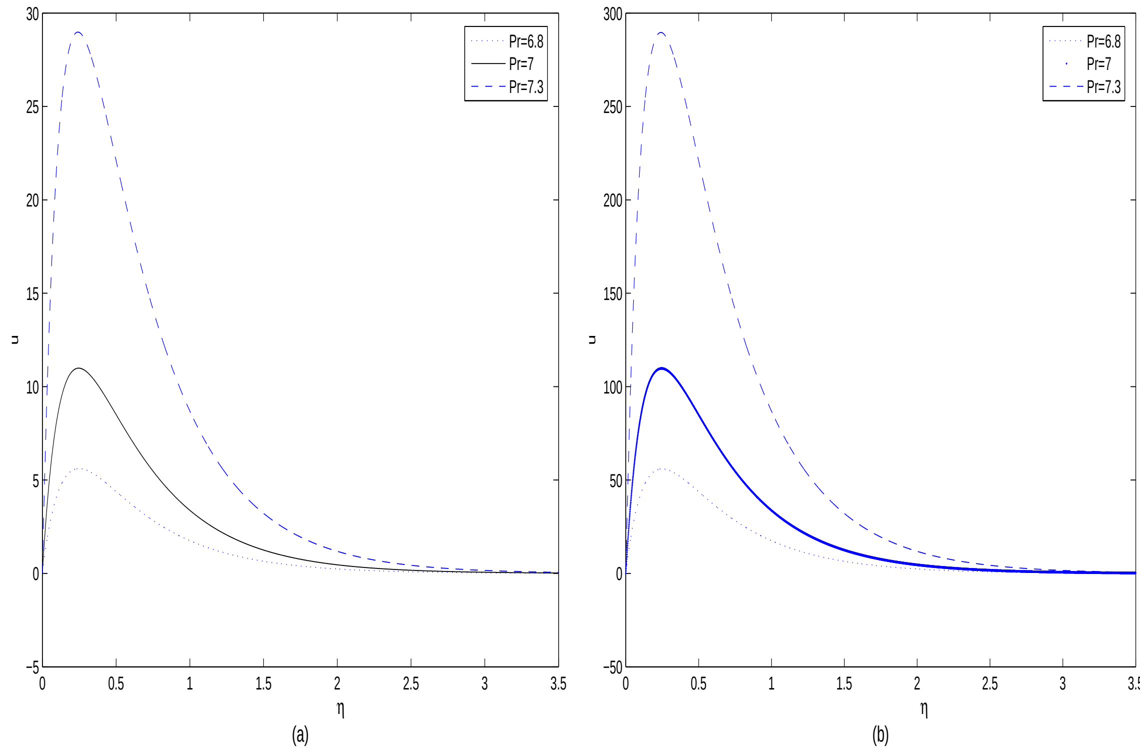

Figure 3.12: The velocity profile with $M = 2$, $\epsilon = 0.001$, $v_0 = 0.9$, $\omega = 2.5$, $t = \pi/2\omega$, $\alpha = 0.1$ $u_0 = 0.001$; (a) $E = 0.001$ and (b) $E = 0.01$.

3.3 Comparison of results when $E = 0$ and $E \neq 0$

In this section we compare the results of Chapters 2 and 3 for zero and small but non-zero Eckert numbers. Figure 3.13 shows the comparison of the variation of the temperature with the angular frequency. In Figure 3.13(a) the Eckert number is zero while in Figure 3.13 (b) the Eckert number is small but non-zero. In the absence of the Eckert number, the temperature decreases asymptotically with the angular frequency. For non-zero Eckert numbers, the temperature still does not depend on the angular frequency but there is now a rapid cooling of the fluid resulting in negative temperatures.

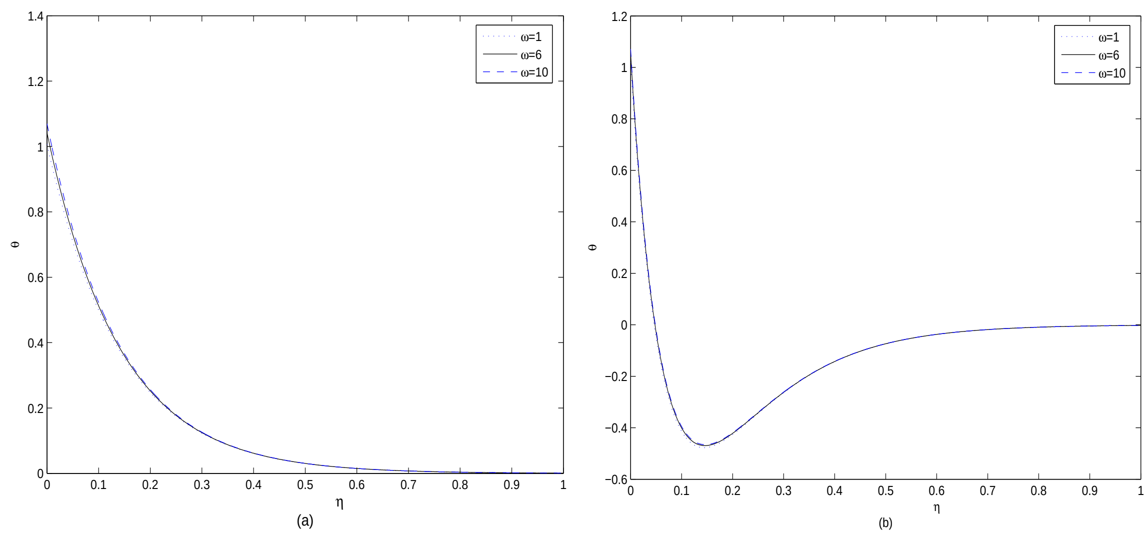

Figure 3.13: The temperature profile with $P_r = 7, \quad v_0 = 0.9, \quad \epsilon = 0.001, \quad \alpha = 0.001, \quad t = \pi/2\omega, \quad v_0 = 0.9. \quad M = 6;$ (a) $E = 0$ and (b) $E = 0.06$

Figure 3.14 shows a comparison of the variation of the temperature with the Prandtl number in the absence and presence of the Eckert number. In all these cases the temperature decreases with increasing Prandtl numbers. In Figure 3.14 (b) the boundary layer temperature dips below zero near the plate surface.

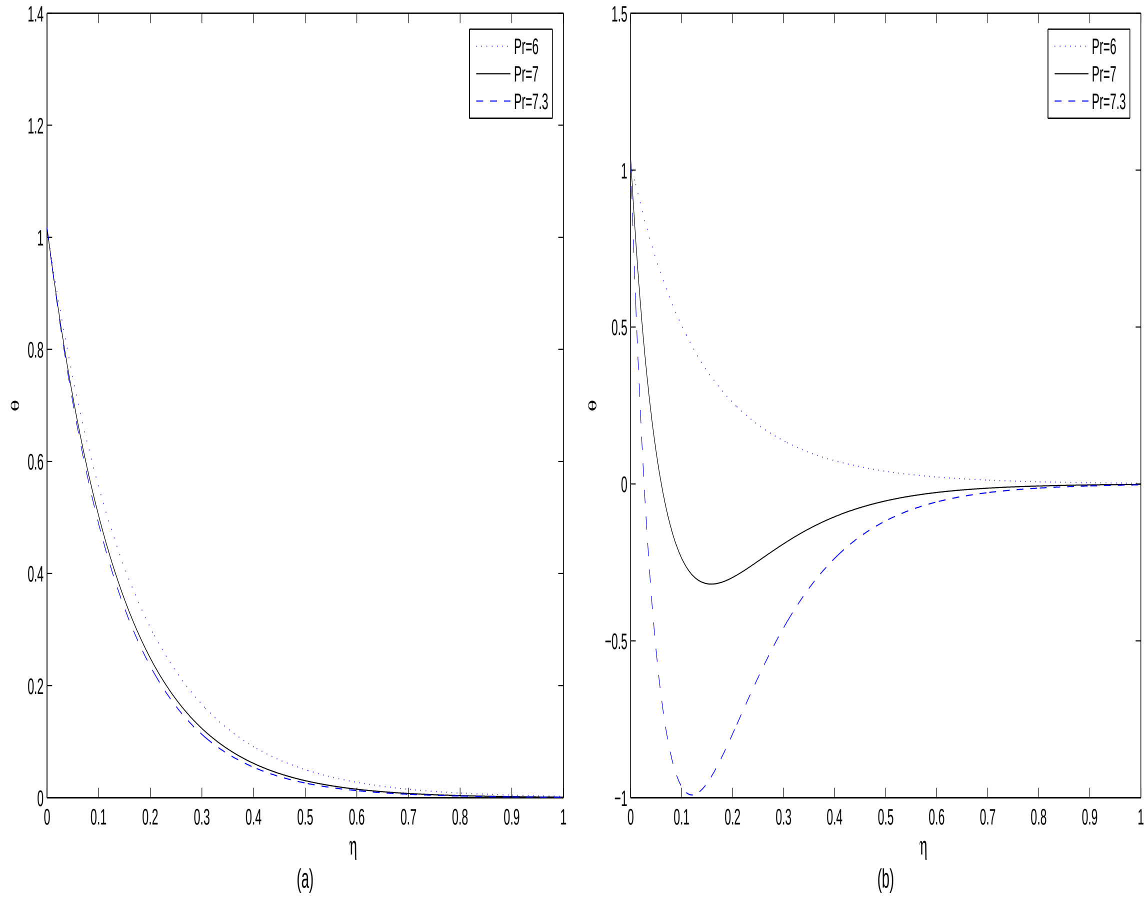

Figure 3.14: The temperature profile with $M = 6$, $v_0 = 0.9$, $\epsilon = 0.001$, $\alpha = 0.001$, $t = \pi/2\omega$, $v_0 = 0.9$. $\omega = 2.5$; (a) $E = 0$ and (b) $E = 0.001$

Figure 3.15 shows the variation of the boundary layer velocity with the Prandtl numbers for zero and small Eckert numbers respectively. The boundary layer velocity increases with increasing Prandtl numbers. In Figure 3.15 (b) the Eckert number is small but different to zero whereas in Figure 3.15 (a) the Eckert number is zero. The amplitude of the velocity is much higher in Figure 3.15 (b) than in Figure 3.15 (a). This means that the larger the Eckert number, the higher the amplitude of the velocity.

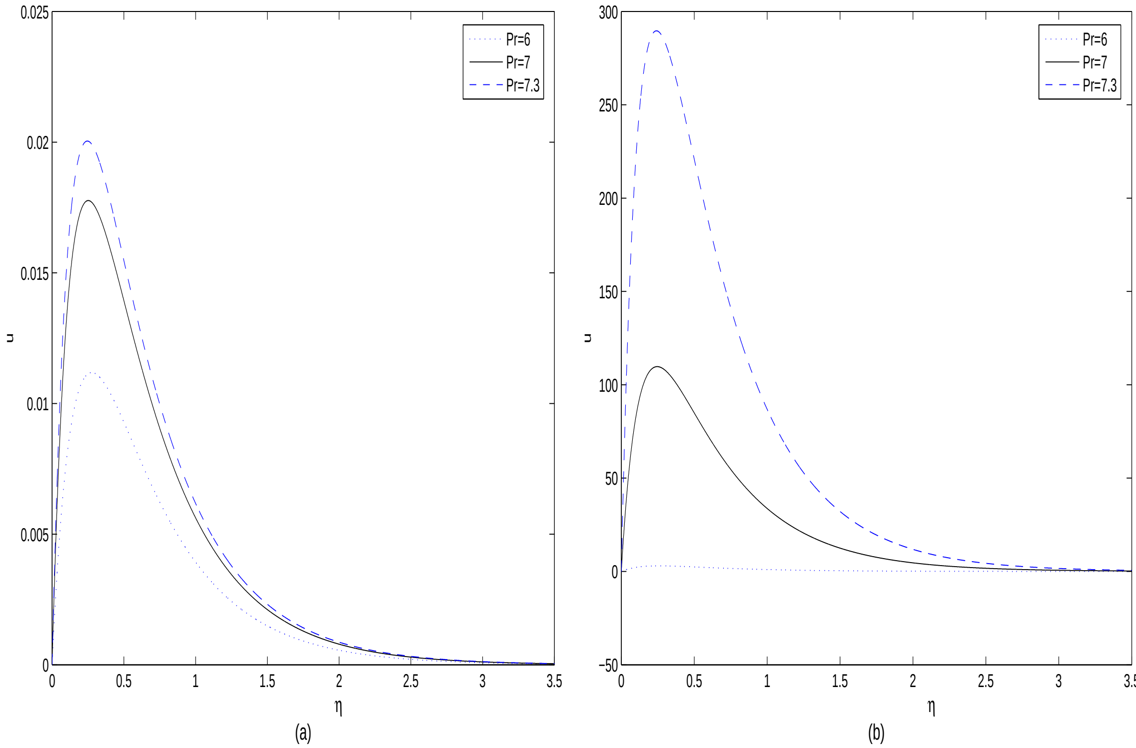

Figure 3.15: The velocity profile with $M = 2$, $v_0 = 0.9$, $\epsilon = 0.001$, $\alpha = 0.01$, $t = \pi/2\omega$, $v_0 = 0.9$. $\omega = 2.5$; (a) $E = 0$ and (b) $E = 0.01$

Figure 3.16 shows the comparison of the variation of the velocity profiles with the viscoelastic parameter in the absence and presence of Eckert numbers. The velocity decreases with increasing viscoelastic parameter in Figure 3.16 (a) and increases with the increasing viscoelastic parameter in Figure 3.16 (b) depending on the value of the Eckert number.

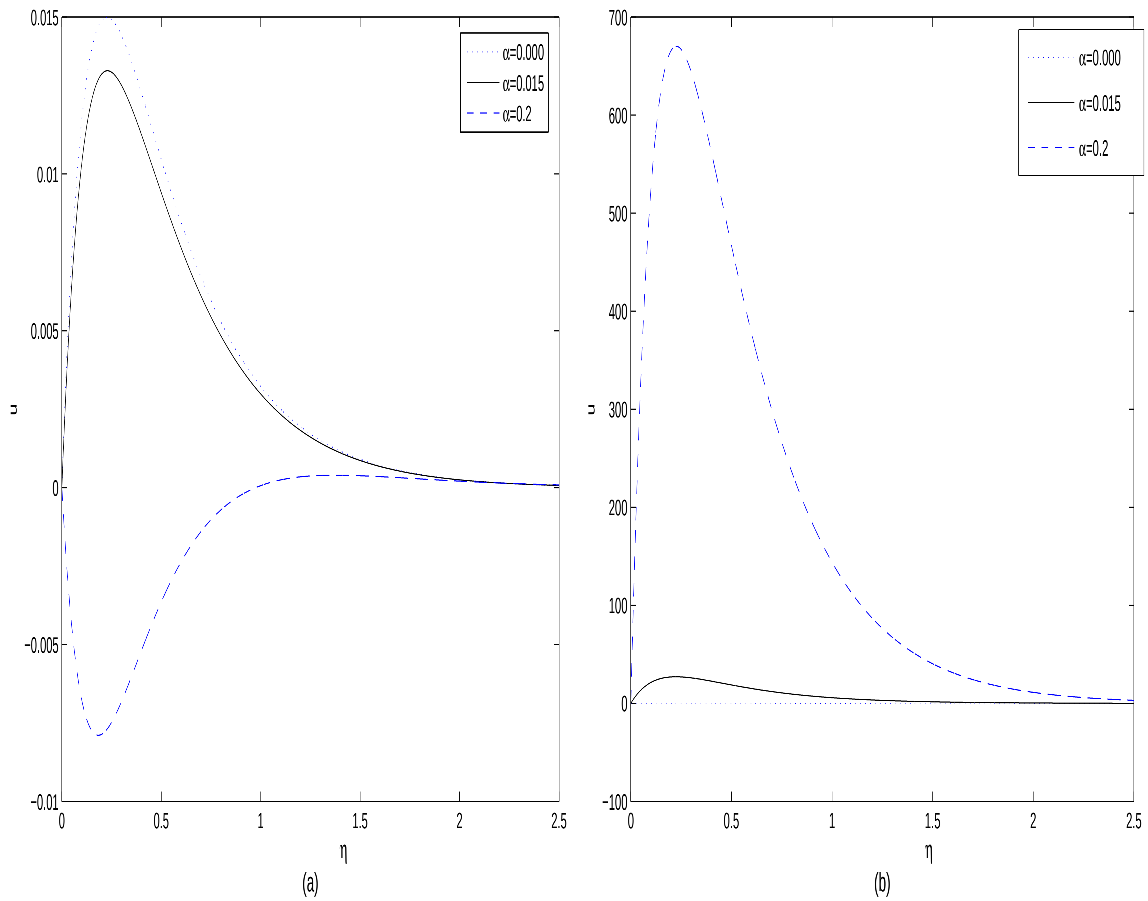

Figure 3.16: The velocity profile with $P_r = 7$, $v_0 = 0.9$, $\epsilon = 0.001$, $M = 4$, $t = \pi/2\omega$, $v_0 = 0.9$. $\omega = 1.5$; (a) $E = 0$ and (b) $E = 0.001$

Figure 3.17 shows the comparison of variation of boundary layer velocity with angular frequency in absence and presence of Eckert number.The velocity does not depend on the angular frequency. In Figure 3.17 (b) the amplitude of the velocity is much higher than in Figure 3.17 (a).

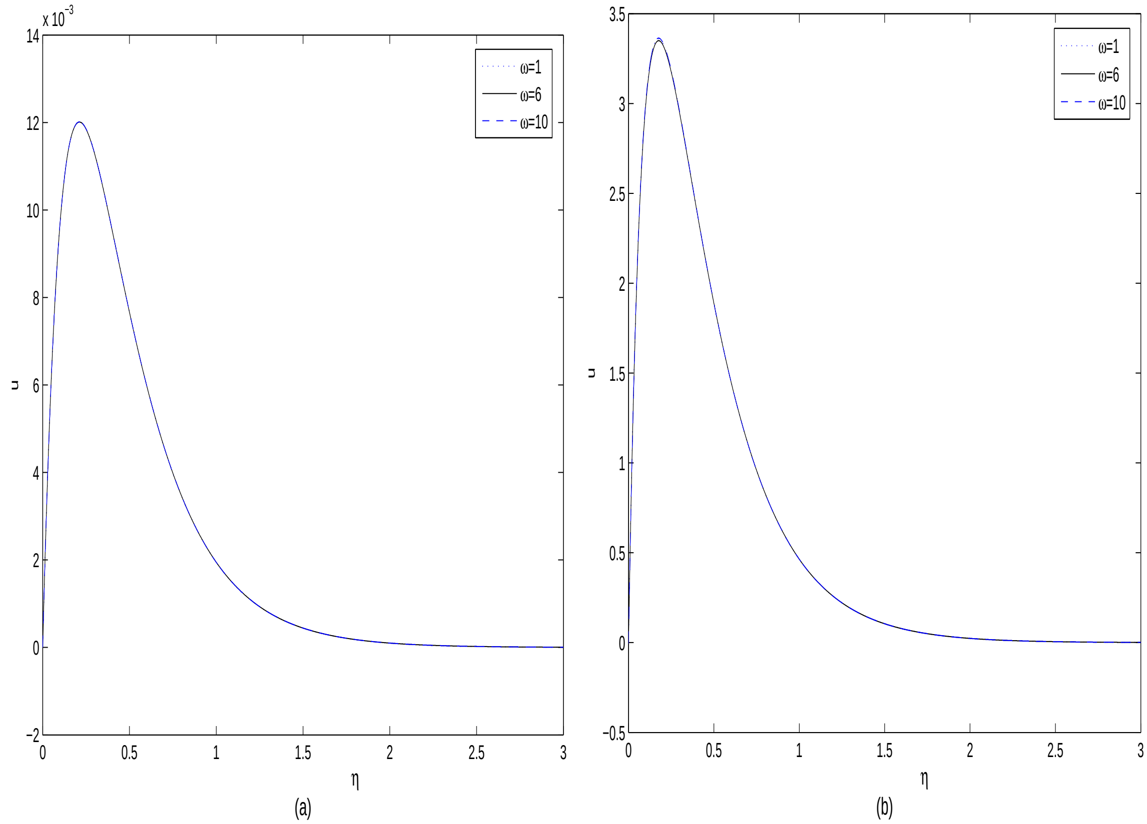

Figure 3.17: The velocity profile with $P_r = 7$, $v_0 = 0.9$, $\epsilon = 0.001$, $\alpha = 0.01$, $t = \pi/2\omega$, $M = 6$; (a) $E = 0$ and (b) $E = 0.06$

Figure 3.18 shows the comparison of variation of boundary layer velocity with suction for zero and non-zero Eckert numbers. The velocity is decreasing with increasing suction. In Figure 3.18 (b) the Eckert number is small but different to zero whereas in Figure 3.18 (a) the Eckert number is zero. The amplitude of the velocity is much higher in Figure 3.18 (b) than in Figure 3.18 (a). This suggests that combining the suction velocity and moderated Eckert number may serve as a means of controlling the boundary layer velocity.

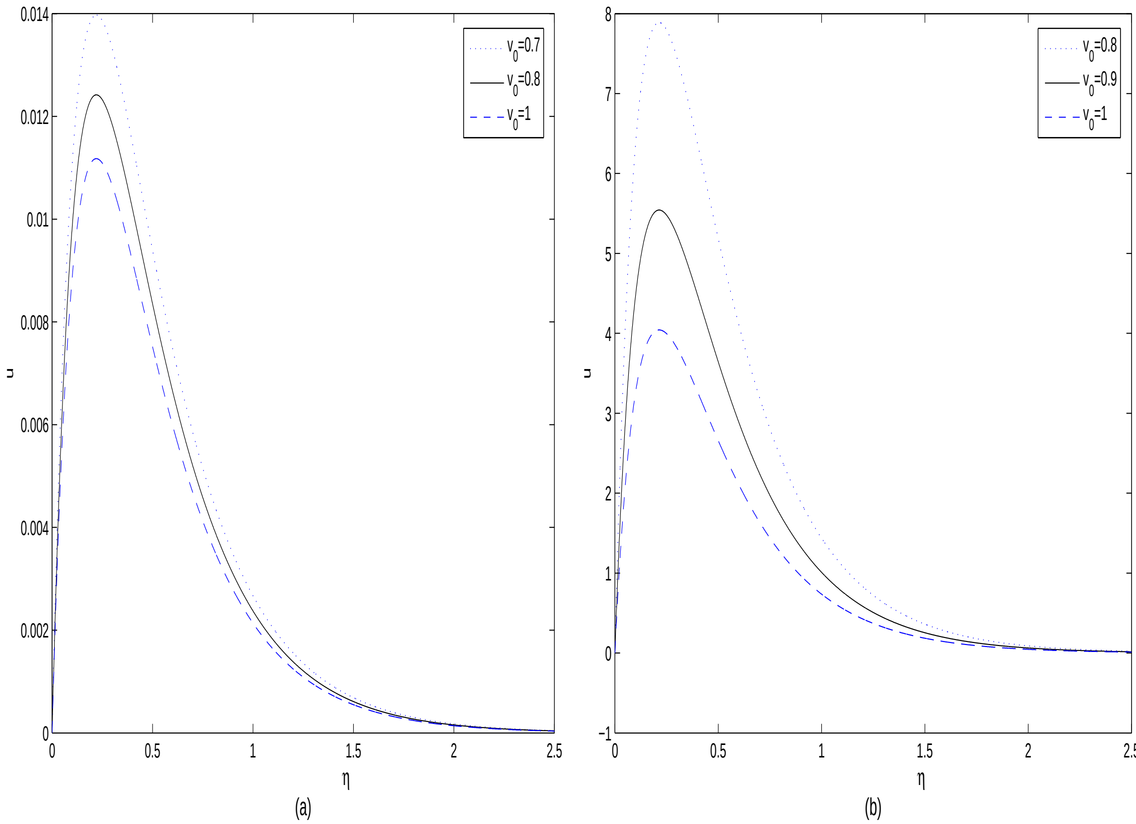

Figure 3.18: The velocity profile with $P_r = 7, \quad \omega = 2.5, \quad \epsilon = 0.001, \quad \alpha = 0.01,$ $t = \pi/2\omega, \quad M = 5; \quad$ (a) $\quad E = 0 \quad$ and (b) $\quad E = 0.01$

Figure 3.19 shows the comparison of variation of boundary layer velocity with Hartmann numbers. The velocity is decreasing with increasing Hartmann numbers. When we compare Figure 3.19 (b) and Figure 3.19 (a), we found that the velocity decreases faster in Figure 3.19 (a) than in Figure 3.19 (b).

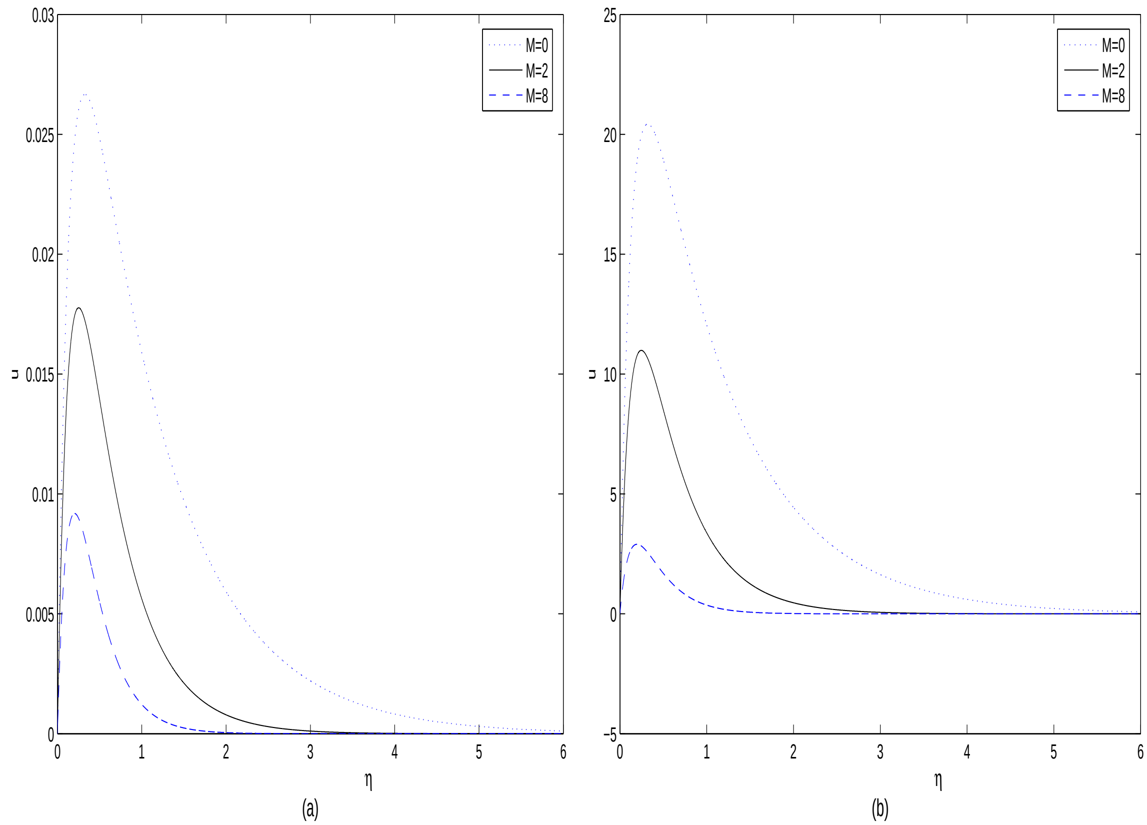

Figure 3.19: The velocity profile with $P_r = 7$, $\omega = 2.5$, $\epsilon = 0.001$, $\alpha = 0.01$, $t = \pi/2\omega$, $\omega = 1.5$; (a) $E = 0$ and (b) $E = 0.001$

From the above comparison, the physical effect of the Eckert number in our study is to reduce heat transfer and to enhance the boundary layer velocity.

3.4 Skin-friction and rate of heat transfer

The skin-friction coefficient or shear stress at the wall is defined as the force exerted by the fluid on the surface over which it flows. Physically, the skin-friction is the resistive drag force caused by shear in the boundary layer,Vajravelu and Mohapatra (1990). Boundary layer control is very important in many disciplines such as in aeronautical engineering where they use artificial methods such as boundary layer suction to reduce the drag and to improve the vehicle power, Hayat et al (2002). Among the recent studies Sahoo et al (2003) investigated the effects of an applied magnetic field and sink strength on the skin-friction. They found that the skin-friction decreases with the increase in the applied magnetic field and sink strength for both mercury and an electrolytic solution. Soundalgekar et al (2004) studied the effects of viscous dissipative heat on transient free convection flow. They investigated the effects of Eckert numbers and suction on the skin-friction. They found that the skin-friction increases with increasing Eckert numbers in the case of air and decreases with increasing Eckert number in the case of water. Murthy et al (2007) extended the study of Sahoo et al (2003) to memory fluids. They found that the skin-friction for both mercury and electrolytic solutions decreases with an increase in the applied magnetic field and the sink strength. They did not however consider the effect of the viscoelastic parameter, the Eckert number, the angular frequency and suction on the skin-friction.

Misra and Shit (2009) studied biomagnetic viscoelastic fluid flow over a stretching sheet. In their study they investigated the effect of fluid viscoelasticity and ferromagnetic interaction on skin-friction. They found that the skin-friction increases with increasing fluid viscoelasticity but they also found in the case of small viscoelasticity that the skin-friction increases with Prandtl numbers.

Mathematically, the skin friction is defined as:

$$\tau = -\mu \left(\frac{\partial u}{\partial \eta} \right)_{\eta=0} . \tag{3.62}$$

Equation (3.62) reduces to:

$$\tau = -u_0 \left(f_1'(0) + \epsilon e^{i\omega t} \; f_2'(0) \right), \tag{3.63}$$

and differentiating equations (3.53) and (3.60) and substituting into (3.63) gives:

$$\tau = -\left(\xi_1 + \xi_2 + \xi_3\right) - u_0 \epsilon e^{i\omega t} \left(\chi_1 + \chi_2 + \chi_3\right) \tag{3.64}$$

where

$$
\begin{aligned}
\xi_1 \;\; = \;\; & u_0\left(\frac{1}{v_0 H}(P_r-\beta_2)-E\beta_2 F_8\right) \\
& +\frac{Eu_0}{v_0 H}\left(-\frac{2\ u_0^2 P_r^2\alpha\beta_2^3}{v_0^2 H^2\sqrt{1+4M}(\beta_2+P_r)}+\frac{P_r u_0^2\alpha\beta_2^4}{v_0^2 H^2\sqrt{1+4M}(2\beta_2-P_r)}\right) \\
& +u_0\left(\frac{2\ u_0^2 P_r^2\alpha\beta_2^3}{(\beta_2+P_r)((\beta_2+P_r)^2-(\beta_2+P_r)-M)v_0^3 H^2\sqrt{1+4M}}+\frac{EK_5 P_r}{v_0 H}\right) \\
& +u_0\left(-\frac{u_0^2 P_r\alpha\beta_2^4}{(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)v_0^3 H^2\sqrt{1+4M}}\right) \\
& +u_0\left(\frac{2\ u_0^2 P_r^2\alpha\beta_2^3}{v_0^3 H^3\sqrt{1+4M}(\beta_2+P_r)}-\frac{u_0^2 P_r\alpha\beta_2^4}{v_0^3 H^3\sqrt{1+4M}(2\beta_2-P_r)}\right) \\
& +u_0\left(\frac{EA_4}{v_0 P_r(4P_r^2-2Pr-M)}+\frac{EA_5}{v_0\beta_2((P_r+\beta_2)^2-(P_r+\beta_2)-M)}\right) \\
& +u_0\left(-\frac{2\ u_0^2 P_r^2\alpha\beta_2^3}{(\beta_2+P_r)((\beta_2+P_r)^2-(\beta_2+P_r)-M)v_0^3 H^2\sqrt{1+4M}}\right) \\
& +\left(\frac{EA_6}{v_0(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)}\right), \\
\xi_2 \;\; = \;\; & +u_0\left(\frac{u_0^2 P_r\alpha\beta_2^4}{(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)v_0^3 H^2\sqrt{1+4M}}-\frac{\alpha P_r^4}{v_0 H^2}\right) \\
& +u_0\left(\frac{\alpha\beta_2 P_r^3}{v_0 H^2}+\frac{\alpha\beta_2^3}{v_0 H\sqrt{1+4M}}\right) \\
& +u_0\left(-E\beta_2\alpha F_{12}-\frac{E\alpha P_r^3}{v_0 H^2}\left(-\frac{2\ u_0^2 P_r^2\alpha\beta_2^3}{v_0^2 H^2\sqrt{1+4M}(\beta_2+P_r)}+\frac{2P_r u_0^2\alpha\beta_2^4}{v_0^2 H^2\sqrt{1+4M}(2\beta_2-P_r)}\right)\right) \\
& +u_0\left(-\frac{2E(P_r+\beta_2)^2 u_0^2 P_r^2\alpha^2\beta_2^3}{((\beta_2+P_r)^2-(\beta_2+P_r)-M)^2 v_0^3 H^2\sqrt{1+4M}}\right) \\
& +u_0\left(\frac{8E\alpha^2 u_0^2 P_r\beta_2^7}{v_0^3 H^2\sqrt{1+4M}(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)^2}\right) \\
& +u_0\left(\frac{E\alpha\beta_2^3\ F_8}{\sqrt{1+4M}}-\frac{\alpha EK_5\ P_r^4}{v_0 H^2}-\frac{2E\alpha\ P_r\ F_9}{4P_r^2-2P_r-M}\right)
\end{aligned}
$$

$$
\begin{aligned}
\xi_3 &= +u_0\left(\frac{EP_r^3\alpha}{v_0H^2}\left(-\frac{2\,u_0^2P_r^2\alpha\beta_2^3}{v_0^2H^2\sqrt{1+4M}(\beta_2+P_r)}+\frac{2P_ru_0^2\alpha\beta_2^4}{v_0^2H^2\sqrt{1+4M}(2\beta_2-Pr)}\right)\right)\\
&\quad +u_0\left(-\frac{(Pr+\beta_2)E\alpha F_{10}}{(P_r+\beta_2)^2-(P_r+\beta_2)-M}\right)\\
&\quad +\frac{2E(P_r+\beta_2)^2u_0^3P_r^2\alpha^2\beta_2^3}{v_0^3H^2\sqrt{1+4M}((P_r+\beta_2)^2-(P_r+\beta_2)-M)^2}\\
&\quad +u_0\left(-\frac{2E\alpha\ F_{11}}{4\beta_2^2-2\beta_2-M}-\frac{8P_r\ \alpha^2u_0^2\beta_2^7E}{v_0^3H^2\sqrt{1+4M}(2\beta_2-P_r)(4\beta_2^2-2\beta_2-M)^2}\right),
\end{aligned}
$$

$$
\begin{aligned}
\chi_1 &= -\frac{2EP_r^2u_0^2\alpha\beta_2^3}{[(\beta_2+P_r)^2-(P_r+\beta2)-(M+i\omega)](\beta_2+P_r)v_0^3H^2\sqrt{1+4M}}\\
&\quad -\frac{2EP_r^2u_0^2\alpha\beta_2^4}{\Gamma_2}\left(\frac{P_ri\omega}{\beta_2(\beta_2+P_r)}-1\right)\\
&\quad -\frac{2\beta_2\ E\ D_6}{4\beta_2^2-2\beta_2-(M+i\omega)}+\frac{EP_ru_0^2\alpha\beta_2^4}{\Gamma_3}\\
&\quad +\frac{Ep_2F_6}{v_0H_1}+\frac{2EPru_0^2\alpha\beta_2^5}{\Gamma_4}\left(\frac{P_ri\omega}{2\beta_2(2\beta_2-P_r)}-1\right)\\
\chi_2 &= -\frac{k_2}{v_0H_1}(v_0H_1+2)+\frac{2p_2}{v_0H_1}-k_2E\ D_{11}\\
&\quad +\frac{2EP_r^2u_0^2\alpha\beta_2^3}{((\beta_2+P_r)^2-(P_r+\beta_2)-(M+i\omega))v_0^3H^2\sqrt{1+4M}(\beta_2+P_r)}\\
&\quad +\frac{2EP_r^2u_0^2\alpha\beta_2^4}{\Gamma_1}\left(\frac{P_ri\omega}{\beta_2(\beta_2-P_r)}-1\right)\\
&\quad -\frac{EP_ru_0^2\alpha\beta_2^4}{v_0^3H^2\sqrt{1+4M}(2\beta_2-P_r)(4\beta_2^2-2\beta_2-(M+i\omega))}\\
&\quad -\left(\frac{EP_ru_0^2\alpha\beta_2^5}{v_0^3H^2\sqrt{1+4M}(4\beta_2^2-P_r(2\beta_2+i\omega))(4\beta_2^2-2\beta_2-(M+i\omega))}\right)\left(\frac{P_ri\omega}{2\beta_2(2\beta_2-P_r)}-1\right)\\
&\quad -\frac{2EP_r^2u_0^2\alpha\beta_2^3}{v_0^3H^2H_1\sqrt{1+4M}(\beta_2+P_r)}+\frac{2EP_ru_0^2\alpha\beta_2^4}{v_0^3H^2H_1\sqrt{1+4M}(2\beta_2-P_r)}\\
&\quad -\left(\frac{2EP_r^2u_0^2\alpha\beta_2^4}{v_0^3H^2H_1\sqrt{1+4M}(\beta_2^2+P_r(\beta_2-i\omega))}\right)\left(\frac{P_ri\omega}{\beta_2(\beta_2+P_r)}-1\right)\\
&\quad +\left(\frac{2EP_r^2u_0^2\alpha\beta_2^5}{v_0^3H^2H_1\sqrt{1+4M}(4\beta_2^2-P_r(2\beta_2+i\omega))}\right)\left(\frac{P_ri\omega}{2\beta_2(2\beta_2+P_r)}-1\right)\\
&\quad -\left(\frac{2P_r\ E\ D_4}{4Pr^2-2P_r-M}+\frac{(\beta_2+P_r)E\ D_5}{(P_r+\beta_2)^2-(P_r+\beta_2)-(M+i\omega)}\right),
\end{aligned}
$$

$$
\begin{aligned}
\chi_3 \;=\;& \frac{2E\, u_0^2 P_r^2 \alpha \beta_2^3}{v_0^3 H^2 H_1 \sqrt{1+4M}(\beta_2+P_r)} \\
& - \frac{E P_r u_0^2 \alpha \beta_2^4}{v_0^3 H^2 H_1 \sqrt{1+4M}(2\beta_2 - Pr)} \\
& + \frac{2EP_r^2 u_0^2 \alpha \beta_2^4}{v_0^3 H^2 H_1 \sqrt{1+4M}(\beta_2^2 + P_r(\beta_2 - i\omega))} \left(\frac{P_r i\omega}{\beta_2(\beta_2+P_r)} - 1 \right) \\
& - \frac{2EPr u_0^2 \alpha \beta_2^5}{v_0^3 H^2 H_1 \sqrt{1+4M}(4\beta_2^2 - P_r(2\beta_2 + i\omega))} \left(\frac{P_r i\omega}{2\beta_2(2\beta_2 - Pr)} - 1 \right) \\
& - \frac{ED_7(P_r+k_2)}{(P_r+k_2)^2 - (P_r+k_2) - (M+i\omega)} \\
& - \frac{ED_8(P_r+p_2)}{(P_r+p_2)^2 - (P_r+p_2) - (M+i\omega)} \\
& - \frac{ED_9(p_2+\beta_2)}{(\beta_2+p_2)^2 - (\beta_2+p_2) - (M+i\omega)} \\
& - \frac{ED_{10}(\beta_2+k_2)}{(\beta_2+k_2)^2 - (\beta_2+k_2) - (M+i\omega)} \\
& + \frac{2k_2 \alpha p_2^2 (p_2+i\omega)}{v_0 H_1^2} \\
& + \frac{\alpha k_2^2 (v_0 H_1 + 2)(k_2 + i\omega)}{v_0 H_1 \sqrt{1+4(M+i\omega)}} - \frac{2\alpha p_3^2 (p_2+i\omega)}{v_0 H_1^2}.
\end{aligned}
$$

Here

$$\Gamma_1 = \left[(\beta_2+P_r)^2 - (P_r+\beta_2) - (M+i\omega)\right]\left[\beta_2^2 + P_r(\beta_2 - i\omega)\right] v_0^3 H^2 \sqrt{1+4M},$$

$$\Gamma_2 = \left[(\beta_2+P_r)^2 - (Pr+\beta_2) - (M+i\omega)\right]\left[\beta_2^2 + P_r(\beta_2 - i\omega)\right] v_0^3 H^2 \sqrt{1+4M},$$

$$\Gamma_3 = v_0^3 H^2 \sqrt{1+4M}\left(4\beta_2^2 - 2\beta_2 - (M+i\omega)\right)(2\beta_2 - P_r),$$

$$\Gamma_4 = v_0^3 H^2 \sqrt{1+4M}\left(4\beta_2^2 - 2\beta_2 - (M+i\omega)\right)\left(4\beta_2^2 - P_r(2\beta_2 + i\omega)\right).$$

The variation of the skin friction with the various parameter values is shown in tables 3.1 and 3.2 for zero and small Eckert numbers respectively. The Prandtl number used corespond to water (7) and air (0.72).

Table 3.1: The skin friction for zero Ekert number

E	P_r	M	v_0	ω	α	τ
0.000	7	6	0.9	2.5	0.001	1.7975
0.000	0.72	6	0.9	2.5	0.001	−0.0149
0.000	1	6	0.9	2.5	0.001	−0.0125
0.000	7	6	0.9	2.5	0.001	1.7975
0.000	8	6	0.9	2.5	0.001	6.7889
0.000	7	6	0.9	2.5	0.001	1.7975
0.000	7	2	0.9	2.5	0.001	2.8086
0.000	7	6	0.9	2.5	0.001	1.7975
0.000	7	6	0.5	2.5	0.001	3.2410
0.000	7	6	0.9	2.5	0.001	1.7975
0.000	7	6	0.9	6	0.001	1.8666
0.000	7	6	0.9	2.5	0.001	1.7975
0.000	7	6	0.9	2.5	0.005	9.6543

Table 3.1 shows that the skin-friction coefficient in the case of zero Eckert numbers increases with increasing Prandtl numbers, angular frequency and viscoelastic parameter whereas it decreases with increases in Hartmann numbers and suction. These

findings are in agreement with the earlier results in Alam et al (2007) and Mustafa et al (2008).

Table 3.2: The skin friction for small Eckert number

E	P_r	M	v_0	ω	α	τ
0.005	7	6	0.9	2.5	0.001	-0.1751
0.009	7	6	0.9	2.5	0.001	-1.7686
0.005	0.72	6	0.9	2.5	0.001	-0.0149
0.005	1	6	0.9	2.5	0.001	-0.0125
0.005	7	6	0.9	2.5	0.001	-0.1751
0.005	8	6	0.9	2.5	0.001	-64.4977
0.005	7	6	0.9	2.5	0.001	-0.1751
0.005	7	2	0.9	2.5	0.001	-1.9990
0.005	7	6	0.9	2.5	0.001	-0.1751
0.005	7	6	0.5	2.5	0.001	-8.2636
0.005	7	6	0.9	2.5	0.001	-0.1751
0.005	7	6	0.9	6	0.001	-0.1141
0.005	7	6	0.9	2.5	0.001	-0.1751
0.005	7	6	0.9	2.5	0.005	-57.2924

Table 3.2 shows that the skin-friction coefficient in the case of small Eckert numbers increases with increase in Prandtl numbers in the case of (air), suction and Hartmann numbers, Soundalgekar et al (2004), whereas it is decreasing with increases in the Eckert number numbers, Prandtl numbers (in the case of water) and viscoelastic parameter, Mustafa et al (2008), Sahoo et al (2003), Ramana et al (2007), Alam et al (2007) and Murthy et al (2007). Increasing the skin-friction in any system generally reduces the velocity of the flows.

The rate of heat transfer is given in terms of the Nusselt number, Nu. It is an important dimensionless parameter that is generally expressed as the ratio of convective to conductive heat transfer across the fluid layer, Özişik (1988). Some of the recent studies that have considered heat transfer in boundary layer flows include those of Sahoo et al (2003) who studied MHD unsteady free convection flow past an infinite vertical plate with a constant suction. They investigated the effects of an applied magnetic field and sink strength on the rate of heat transfer. They found that the rate of heat transfer decreases with the increase in the applied magnetic field and sink strength for both mercury and electrolytic solutions. Soundalgekar et al (2004) studied the effects of viscous dissipative heat on transient free convection flow. They investigated the effects of Eckert numbers and suction on the rate of heat transfer. They found that the rate of heat transfer decreases with increases in the Eckert numbers whereas it increases with increasing suction. Murthy et al (2007) extended the study of Sahoo et al (2003) to memory fluids. They found that the heat transfer for mercury and electrolytic solution decreases with the increase in applied magnetic field. They did not however consider the effect of the viscoelastic parameter, the Eckert number, the angular frequency and suction on the Nusselt number. Misra and Shit (2009) studied biomagnetic viscoelastic fluid flow over a stretching sheet. In their study they investigated the effect of fluid viscoelasticity and ferromagnetic interaction on the rate of heat transfer.

They found that the rate of heat transfer decreases with increasing fluid viscoelasticity.

In mathematical form, the Nusselt number is defined as:

$$\begin{aligned} Nu &= -\left(\frac{\partial \theta}{\partial \eta}\right)_{\eta=0} \qquad (3.65) \\ &= -h_1'(0) - \left(h_2'(0) + h_1'(0)\right)\frac{\epsilon}{2}e^{i\omega t} \end{aligned}$$

Differentiating equations (3.39) and (3.36) and substituting into (3.66) gives:

$$
\begin{aligned}
Nu \;=\; & -\left(-P_r - EK_5\, P_r - \frac{2E\; u_0^2 P_r^2 \alpha \beta_2^3}{(\beta_2+P_r)v_0^2 H^2\sqrt{1+4M}}\right) \\
& -\left(\frac{EP_r u_0^2 \alpha \beta_2^4}{(2\beta_2-P_r)v_0^2 H^2\sqrt{1+4M}} - \frac{E\;A_4}{P_r} - \frac{EA_5}{\beta_2} - \frac{EA_6}{2\beta_2-P_r}\right) \\
& -\left(\frac{2E\; u_0^2 P_r^2 \alpha \beta_2^3}{(\beta_2+P_r)v_0^2 H^2\sqrt{1+4M}} - \frac{Eu_0^2 P_r \alpha \beta_2^4}{v_0^2 H^2 (2\beta_2-P_r)\sqrt{1+4M}}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(-\frac{2E\; u_0^2 P_r^2 \alpha \beta_2^3}{(\beta_2+P_r)v_0^2 H^2\sqrt{1+4M}} + \frac{EP_r u_0^2 \alpha \beta_2^4}{(2\beta_2-P_r)v_0^2 H^2\sqrt{1+4M}}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(\frac{2E\; u_0^2 P_r^2 \alpha \beta_2^3}{(\beta_2+P_r)v_0^2 H^2\sqrt{1+4M}}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(-\frac{EP_r u_0^2 \alpha \beta_2^4}{(2\beta_2-P_r)v_0^2 H^2\sqrt{1+4M}}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(-\frac{E\;A_4}{P_r} - \frac{EA_5}{\beta_2} - \frac{EA_6}{2\beta 2 -_r}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(-2p_2 - Ep_2F_6 + \frac{EP_r u_0^2 \alpha \beta_2^4}{(2\beta_2-P_r)v_0^2 H^2\sqrt{1+4M}}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(-\frac{2E\; u_0^2 P_r^2 \alpha \beta_2^3}{(\beta_2+P_r)v_0^2 H^2\sqrt{1+4M}}\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\Upsilon_1\left(\frac{P_r i\omega}{\beta_2(\beta_2+P_r)} - 1\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(\Upsilon_2\left(\frac{P_r i\omega}{2\beta_2(2\beta_2-P_r)} - 1\right)\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(\Upsilon_3+\Upsilon_4+\Upsilon_5+\Upsilon_6\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(\Upsilon_7\left(\frac{P_r i\omega}{\beta_2(\beta_2-i\omega)} - 1\right) + \Upsilon_8\right) \\
& -\epsilon e^{i\omega t}\left(\Upsilon_9\left(\frac{P_r i\omega}{2\beta_2(2\beta_2+P_r)} - 1\right)\right) \\
& -\frac{\epsilon}{2}e^{i\omega t}\left(\Upsilon_{10}+\Upsilon_{11}+\Upsilon_{12}+\Upsilon_{13}\right) \qquad (3.66)
\end{aligned}
$$

where

$$\Upsilon_1 = -\frac{2EP_r^2u_0^2\alpha\beta_2^4}{(\beta_2^2+P_r(\beta_2-i\omega))v_0^2H^2\sqrt{1+4M}},$$

$$\Upsilon_2 = \frac{2E\ P_r\ u_0^2\alpha\beta_2^5}{(4\beta_2^2-P_r(2\beta_2+i\omega))v_0^2H^2\sqrt{1+4M}},$$

$$\Upsilon_3 = \frac{2E\ u_0^2P_r^2\alpha\beta_2^3}{(\beta_2+P_r)v_0^2H^2\sqrt{1+4M}}, \qquad \Upsilon_4 = -\frac{EP_ru_0^2\alpha\beta_2^4}{(2\beta_2-P_r)v_0^2H^2\sqrt{1+4M}},$$

$$\Upsilon_5 = -\frac{2EA_7}{P_r(2P_r-i\omega)}, \qquad \Upsilon_6 = -\frac{(P_r+\beta_2)EA_8}{\beta_2^2+P_r(\beta_2-i\omega)},$$

$$\Upsilon_7 = \frac{2u_0^2\alpha P_r^2\beta_2^4}{v_0^2H^2\sqrt{1+4M}(\beta_2^2+P_r(\beta_2-i\omega))},$$

$$\Upsilon_8 = -\frac{2\beta_2EA_9}{4\beta_2^2-P_r(2\beta_2+i\omega)}, \qquad \Upsilon_9 = -\frac{2E\ P_r^2\ u_0^2\alpha\beta_2^4}{(\beta_2^2-P_r(2\beta_2-i\omega))v_0^2H^2\sqrt{1+4M}},$$

$$\Upsilon_{10} = -\frac{(P_r+p_2)EA_{10}}{p_2^2+Pr(p_2-i\omega)}, \qquad \Upsilon_{11} = -\frac{(P_r+k_2)EA_{11}}{p_2^2+P_r(k_2-i\omega)},$$

$$\Upsilon_{12} = -\frac{(\beta_2+p_2)EA_{12}}{(p_2+\beta_2)^2-P_r(p_2+\beta_2)-P_ri\omega},$$

$$\Upsilon_{13} = -\frac{(\beta_2+k_2)EA_{13}}{(k_2+\beta_2)^2-Pr(k_2+\beta_2)-Pri\omega}.$$

The rate of heat transfer for various parameter is given in tables 3.3 and 3.4.

Table 3.3: The rate of heat transfer for zero Ekert number

E	P_r	M	v_0	ω	α	Nu
0.000	7	6	0.9	2.5	0.001	6.9979
0.000	0.72	6	0.9	2.5	0.001	0.7191
0.000	1	6	0.9	2.5	0.001	0.9989
0.000	7	6	0.9	2.5	0.001	6.9979
0.000	8	6	0.9	2.5	0.001	7.9978
0.000	7	6	0.9	2.5	0.001	6.9979
0.000	7	2	0.9	2.5	0.001	6.9979
0.000	7	6	0.9	2.5	0.001	6.9979
0.000	7	6	0.5	2.5	0.001	6.9979
0.000	7	6	0.9	2.5	0.001	6.9979
0.000	7	6	0.9	6	0.001	6.9960
0.000	7	6	0.9	2.5	0.001	6.9979
0.000	7	6	0.9	2.5	0.005	6.9979

Table 3.3 shows that the Nusselt numbers at the surface in case of zero Eckert numbers decreases with increasing Prandtl numbers and is slightly decreasing with increasing

the angular frequency. In this case the Nusselt number is independent of Hartmann numbers, suction and the visco-elastic parameter.

Table 3.4: The rate of heat transfer for small Ekert number

E	P_r	M	v_0	ω	α	Nu
0.005	7	6	0.9	2.5	0.001	−2.7500
0.009	7	6	0.9	2.5	0.001	−10.5482
0.005	0.72	6	0.9	2.5	0.001	0.7191
0.005	1	6	0.9	2.5	0.001	0.9990
0.005	7	6	0.9	2.5	0.001	−2.7500
0.005	8	6	0.9	2.5	0.001	−83.5260
0.005	7	6	0.9	2.5	0.001	−2.7500
0.005	7	2	0.9	2.5	0.001	−6.6049
0.005	7	6	0.9	2.5	0.001	−2.7500
0.005	7	6	0.5	2.5	0.001	−24.5852
0.005	7	6	0.9	2.5	0.001	−2.7500
0.005	7	6	0.9	6	0.001	−2.7409
0.005	7	6	0.9	2.5	0.001	−2.7500
0.005	7	6	0.9	2.5	0.005	−43.3847

Table 3.4 shows that the Nusselt number in the case of small Eckert numbers decreases with increases in Prandtl numbers (water), the Eckert number, viscoelastic parameter. It however increases with increases in Hartmann numbers, suction, angular frequency and the Prandtl number (air). These results are also in agreement with the earlier findings by Soundalgekar et al (2004), Mustafa et al (2008), Sharma et al (1995) and Alam et al (2007). The Eckert number, Prandtl number (in the case of water) for example, and the viscoelastic parameter reduces heat transfer whereas in the case of air the Prandtl number, Hartmann numbers and suction velocity enhance heat transfer.

3.5 Conclusion

In this chapter we have considered the effects of small but non-zero Eckert numbers, small viscoelastic parameters, angular frequency, Hartmann number, suction parameter and the Prandtl number on the flow of a second grade viscoelastic fluid. The multi-parameter perturbation technique was used to solve the problem and our investigations show that;

- The temperature increases with Hartmann numbers.
- The temperature decreases with the viscoelastic parameter
- The temperature increases with suction.

- The temperature increases slightly with the angular frequency.
- The temperature decreases with increasing Eckert numbers.
- Increasing the Prandtl number reduces the boundary layer temperature.
- The boundary layer velocity decreases with increasing suction.
- The boundary layer velocity increases with Eckert numbers.
- The boundary layer velocity increases with increasing viscoelastic parameters.
- Increasing the Prandtl number increases the boundary layer velocity.
- The boundary layer velocity is independent of the angular frequency.
- Increasing the magnetic field reduces the boundary layer velocity.
- The boundary layer velocity decreases with increasing viscoelastic parameter for zero Eckert number whereas it increases with increasing viscoelastic parameter in presence of small Eckert number.
- The boundary layer velocity increases with increasing Prandtl number for zero and small Eckert number. The amplitude of the velocity is higher in the presence of small Eckert number.
- The temperature decreases with increasing Prandtl number for zero and small Eckert number. In presence of small Eckert number the boundary layer temperature dips below zero near the plate surface.

Chapter 4

Conclusion

We have considered the two dimensional flow of an incompressible and electrically conducting second grade fluid past an infinite porous plate with constant suction. The flow is permeated by a uniform transverse magnetic field. The aim of this study was to use the multi-parameter perturbation technique to study the effects of the Eckert number on the flow of a second grade fluid. We further aimed to investigate the effects of other fluid and physical parameters such as the Prandtl numbers, Hartmann numbers, the viscoelastic parameter, angular frequency and suction velocity on the boundary layer velocity, temperature, skin friction and the rate of heat transfer. Similarity transformations were used to reduce the governing partial differential equations to ordinary differential equations. We used perturbation methods to solve the coupled ordinary differential equations for zero Eckert numbers and the multi-parameter perturbation technique to solve the coupled ordinary differential equations for small Eckert numbers.

We analyzed the effect of the fluid parameters such as the Hartmann number, suction velocity, angular frequency, Prandtl number and the viscoelastic parameter on the boundary layer velocity and temperature. In the absence of the Eckert numbers we found that:

- increasing the magnetic field reduces the boundary layer velocity. This is in line with the earlier finding by, for example, Hayat et al (2003) and Vajravelu and Rolins (2004)

- The boundary layer velocity increases or decreases with Prandtl numbers depending on the quantum of the physical parameters. This is in line with the findings by Cebeci and Bradshaw (1984).

- The boundary layer velocity decreases with increasing suction velocity

- The velocity decreases with increases in the viscoelastic parameter α, so that large values of α lead to back flow.

- The temperature and velocity are both independent of the angular frequency.

- The temperature decreases with increasing Prandtl numbers.

In Chapter 3 we solved the coupled ordinary differential equations for small Eckert numbers. We found that:

- The temperature increases with Hartmann numbers.

- The temperature decreases with the viscoelastic parameter
- The temperature increases with suction.
- The temperature increases slightly with the angular frequency.
- The temperature decreases with increasing Eckert numbers.
- Increasing the Prandtl number reduces the boundary layer temperature.
- The boundary layer velocity decreases with increasing suction.
- The boundary layer velocity increases with Eckert numbers.
- The boundary layer velocity increases with increasing viscoelastic parameters.
- Increasing the Prandtl number reduces the boundary layer velocity.
- The Nusselt number and skin-friction decrease with increasing Eckert number or viscoelastic parameter.

In addition, we compared the results of zero and small Eckert numbers. We found that for small Eckert numbers the temperature decreases faster than for zero Eckert numbers. Small Eckert numbers enhance the boundary layer velocity. The temperature increases with increases in the applied magnetic field. This may be useful in medical applications such as in cancer therapy where this effect may help to arrest the cancerous growth of tumors, Misra et al (2008).

Our results agree or disagree with the earlier published works depending on the nature of parameters in the system, for example, when the energy equation is excluded our results agree with those of Hayat et al (2003) on the effects of Hartmann numbers, frequency and viscoelastic parameters on the boundary layer velocity in the case of constant suction. If $\alpha = 0$, our results agree with those of Soundalgekar et al (2004). Our results also agree with the results of Murthy et al (2007) on the effects of an applied magnetic field on the velocity but differ from their results on the effects of Eckert numbers. These differences may arise from the boundary conditions and the additional term in the energy equations. In our case we found that the applied magnetic field induces a resistive force to the flow whose consequence is to reduce the velocity and enhance the temperature. In addition, our results differ from the results of Vajravelu and Roper (1999) on the effects of viscoelastic parameter. These differences may arise from the additional term in the energy equations and in our case we included the applied magnetic field which is not in their study.

The perturbation and multi-parameter perturbation methods produce approximate analytic solutions that can be interpreted mathematically and physically for various parameters. Numerically evaluated solutions may not provide the same intuitive feel for the effects of these parameters. These methods give reasonable accuracy, Holmes (1995). Unfortunately these methods use small parameters which, perhaps may be their main weakness since some behaviours of the solutions may not be observed.

In future, it may be worthwhile to solve this problem using a combination of numerical methods for large Eckert and viscoelastic parameters and Fourier or Laplace transforms.

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Appendix

List of Symbols

$A_1,\ A_2$	First two Rivlin-Ericksen tensors
B_0	Uniform magnetic field strength
C_p	Specific heat at constant pressure
E	Eckert number, $E = u_0^2/C_p\Delta T$
g	Gravitational acceleration
Gr	Grashof number, $Gr = g\beta\nu\Delta T/u_0v_0$
k	Thermal diffusivity, $k = \lambda_c/\rho c_p$
M	Hartmann number, $M = \sigma B_0^2\nu/\rho v_0^2$
Nu	Nusselt number
P	Pressure
P_r	Prandtl number, $P_r = \mu c_p/k$

t	Dimensionless time
t'	time
T'	Fluid temperature
T'_w	Temperature of the surface
T'_∞	Ambient fluid temperature
u_0	Constant velocity
u'	Velocity component parallel to the plate
u	Non-dimensional velocity
v_0	Constant suction
v'	Constant normal velocity
x'	Cartesian coordinate
y'	Normal cartesian coordinate

Greek Symbols

α	Viscoelastic parameter, $\alpha = \alpha^* v_0^2/\nu^2$
$\alpha_1,\ \alpha_2$	Normal stress moduli
α^*	Material parameter of Second grade fluid
β	Coefficient of volume expansion
ϵ	Amplitude parameter
η	Similarity variable
θ	Dimensionless temperature
λ_c	Fluid thermal conductivity
μ	Coefficient of dynamic viscosity
ν	Coefficient of kinematic viscosity
ρ	Fluid density
σ	Fluid electrical conductivity
τ	Skin-friction
ω'	Angular frequency
ω	non-dimensional angular frequency