Electrostatic Waves and Solitons in Electron-Positron Plasmas

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Contents

Preface

The work described in this thesis was carried out by the author from September 1993 to September 1998, in the Department of Physics, University of Natal, Durban, under the supervision of Professor M.A. Hellberg. These studies represent original work by the author and have not been submitted in any form to another university. Where use was made of the work of others, it has been duly acknowledged in the text.

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I would like to dedicate this thesis to my Mother, Wendy Dawn Gray, whom I miss and love.

For those of you who knew I had the tenacity to see it through, I thank you for your belief and encouragement.

In loving memory of my mother, Wendy Dawn.

Abstract

The magnetosphere of pulsars is thought to consist of an electron-positron plasma rotating in the pulsar magnetic field (Beskin, Gurevich & Istomin 1983; Lominadze, Melikidze & Pataraya 1984; Gurevich & Istomin 1985). A finite, and indeed large, longitudinal electric field exists outside the star, and may accelerate particles, stripped from the surface, to high energies (Goldreich & Julian 1969; Beskin 1993). These particles may leave the magnetosphere via open magnetic field lines at the poles of the pulsar. This depletion of particles causes a vacuum gap to arise, a double layer of substantial potential difference. The primary particles, extracted from the star's surface, are accelerated in the double layer, along the pulsar magnetic field lines, and so produce curvature radiation. The curvature photons, having travelled the distance of the double layer may produce electron-positron pairs above the vacuum gap. These first-generation secondary particles, although no longer accelerating, may synchroradiate, generating photons which may then produce further electron-positron pairs. These synchrophoton produced pairs will be at energies lower than curvature photon produced pairs, since synchrophoton energies are approximately an order of magnitude less than that of the parent curvature photon.

An attempt to model the electron-positron pulsar magnetosphere is made. A four component fluid electron-positron plasma is considered, consisting of a hot electron and positron species, at temperature T_h, and a cool electron and positron species at temperature T_c. The hot components represent the parent first-generation curvature-born pairs, and the cooler components represent the second-generation pairs, born of synchrophotons. The hot components are assumed to be highly mobile, and are thus described by a Boltzmann density distribution. The cool components are more sluggish and are thus described as adiabatic fluids. The model is symmetric in accordance with pair production mechanisms, so that both species of hot(cool) electrons and positrons have the same temperature $T_h(T_c)$, and number density $N_h(N_c)$.

In the interests of completeness, linear electrostatic waves in five different types of electron-positron plasmas are considered. The dispersion relations for electrostatic waves arising in these unmagnetized plasmas are derived. Single species electron-positron plasmas are investigated, considering the constituents to be: both Boltzmann distributed; both adiabatic fluids; and finally, one species of each type. Linear electrostatic acoustic waves in multi-component electron-positron plasmas are then considered, under the

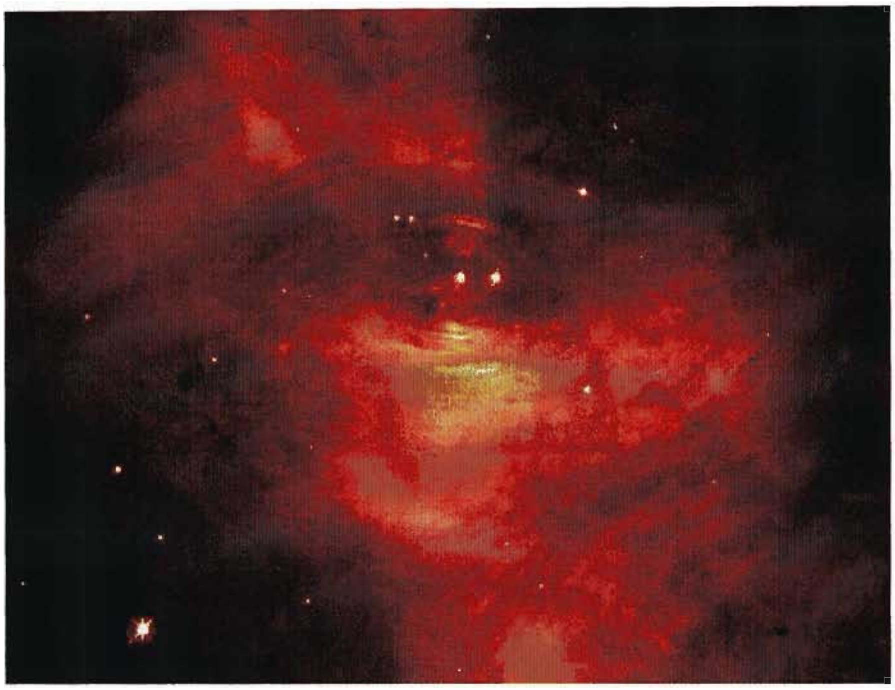
four component model and a three component model (Srinivas, Popel & Shukla 1996).

Small amplitude nonlinear electron-positron acoustic waves are investigated, under the four component electron-positron plasma model. Reductive perturbation techniques (Washimi & Taniuti 1966) and a derivation of the Korteweg-de Vries equation result in a zero nonlinear coefficient, and a purely dispersive governing wave equation. Higher order nonlinearity is included, leading to a modified Korteweg-de Vries equation (Watanabe 1984; Verheest 1988), which yields stationary soliton solutions with a sech dependence rather than the more familiar sech^2.

Arbitrary amplitude solitons are then considered via both numerical and analytical (Chatterjee & Roychoudhury 1995) analysis of the Sagdeev potential. The symmetric nature of the model leads to the existence of purely symmetrical compressive and rarefactive soliton solutions. Small and arbitrary amplitude soliton solutions are compared, and show good correlation.

Under the assumption of Boltzmann distributed hot particles, severe restrictions are imposed on the existence domains of arbitrary amplitude soliton solutions. The Boltzmann assumption places a stringent upper limit on the cool species number density, in order for the solutions to be physical.

An investigation is made of results obtained for an asymmetric electron-positron plasma (Pillay & Bharuthram 1992), consisting of cold electrons and positrons, and hot Boltzmann electrons and positrons at different temperatures T_{eh} and T_{ph}, and number density N_{eh} and N_{ph}. It is found that the assumption of Boltzmann particles again places restrictions on the acoustic soliton existence space, and that the results obtained may be physically invalid. Valid solutions are obtained numerically, within the boundaries of allowed cool species density values.



The Pulsar Powered Crab
Credit: J. Hester and P. Scowen (ASU), NASA
http://antwrp.gsfc.nasa.gov/apod/ap960531.html

In the image above, the pulsar is the left most of the two bright central stars.

Chapter 1

Introduction

The discovery of pulsars and the belief that pulsar magnetospheres consist of electron-positron plasma in a strong magnetic field (Beskin, Gurevich & Istomin 1983; Lominadze, Melikidze & Pataraya 1984; Gurevich & Istomin 1985), has lead to the investigation of the nature of electron-positron plasmas and collective processes that they exhibit. Pair plasmas are also thought to have been in predominance in the structure of the early universe. In response to the popularity of pair plasmas in plasma theory, electron-positron plasmas have been generated under laboratory conditions. We thus investigate the electrodynamics of the pulsar magnetosphere and the mechanisms of electron-positron pair generation. A review of the collective processes of pair plasmas follows with particular attention paid to electrostatic longitudinal waves. We are interested specifically in acoustic solitons, prompted by recent studies of nonlinear acoustic modes (Baboolal, Bharuthram & Hellberg 1988; 1989; 1990; Verheest 1988; Mace, Baboolal, Bharuthram & Hellberg 1991; Rice, Hellberg, Baboolal, Mace & Gray 1993; Verheest, Hellberg, Gray & Mace 1996).

1.1 Pulsars

The discovery of sources of pulsed cosmic radio emission by Hewish and his collaborators in 1967 (Hewish, Bell, Pilkington, Scott & Collins 1968), presented astrophysics with a new object to study: the pulsar. The pulsar was identified as a rotating neutron star (Gold 1968), the product of intense gravitational contraction of a collapsing star which has exhausted its stores of nuclear fuel. The gravitational forces are brought to equilibrium by the pressure of strongly compressed nuclear matter. Neutron stars are thus ultra-dense structures; their mass is of order of the solar mass, $M_{\odot} = 2 \times 10^{30}$ kg but their radius is only about 10–15 km (Shapiro & Teukolsky 1983).

The macroscopic density of the pulsar reaches nuclear values of $10^{17} - 10^{18}$ kg.m^{-3} (Beskin 1993) with surface magnetic fields of the order 10^{8} T. Such a large value for the magnetic field is associated with magnetic flux conservation during gravitational collapse. Considering the contraction of a normal star with a surface magnetic field of approximately 10^{-2} T, since the magnetic field lines are frozen in the star, the field strength will increase under contraction as $\rho^{\frac{2}{3}}$ (Beskin 1993), where ρ is the mean matter density. So for typical densities of the order 10^{17} kg.m^{-3} the magnetic field can attain a value of about 10^{8} T (Beskin 1993).

1.1.1 Pulsar magnetospheres

It was initially believed (Hoyle, Narlikar & Wheeler 1964; Pacini 1968) that the plasma density surrounding a rotating magnetic neutron star must be very low, considering the extremely large gravitational binding energies of electrons and protons at the surface of the star. This lead Pacini to assume a quasi-vacuum magnetosphere in his model of pulsar radiation. However, it is now widely accepted that a pulsar has a plasma magnetosphere (Goldreich & Julian 1969; Sturrock 1971).

We will describe the generation of plasma in the pulsar magnetosphere following Goldreich & Julian (1969). Consider a neutron star which, if nonrotating, would have a dipolar magnetic field, continuous at the stellar surface. Assume this star is rotating about an arbitrary rotational axis with an angular velocity Ω.

The conductivity of the neutron star material is extremely high and may be regarded as infinite. Thus, within the star, Ohm's law may be stated as

$$\vec{E_{int}} + \frac{1}{c}\left[(\vec{\Omega}\times\vec{r})\times\vec{B}\right] = 0, \tag{1.1}$$

where $\vec{B}$ is the magnetic field, $\vec{E_{int}}$ the electric field in the interior of the star, and $(\vec{\Omega} \times \vec{r})$ is the corotation velocity. Thus, because of rotation, there arises an electric field $\vec{E_{int}}$ caused by charge redistribution inside the pulsar, in order for (1.1) to be satisfied. The electric field at the surface of the star is

$$E \sim \frac{\Omega R}{c} B \tag{1.2}$$

where R is the distance from the centre of the star to the surface. $E(R)$ is found to be of the order $10^{10} - 10^{12}$ V.cm^{-1} (Beskin 1993).

The charge redistribution inside the neutron star generates an electric field not only inside the star. The vacuum electric field just outside the star, has a finite component parallel to the magnetic field. This parallel component $E_{\parallel}$ appears to be of the same order of magnitude as the internal E field (Mestel 1971). Thus, the dipole magnetic field results in a quadrupole electric field outside the star.

The electric force on a particle at the star's surface would thus exceed the gravitational force by many orders of magnitude, causing acceleration of surface particles along the magnetic field lines. The ratio of the gravitational force mg to the electric force qE, acting on an electron near the neutron star surface, is given by (Beskin 1993)

$$\epsilon = \frac{m_e g}{eE} \sim 10^{-13} - 10^{-15}. \tag{1.3}$$

Under these conditions particles will be stripped from the stellar surface by the large electric field.

1.1.2 Electron-positron plasma in the magnetosphere

It has been established that the pulsar's strong electric field exceeds the force of gravitational attraction, and extracts charged particles from the stellar surface. However, the extensive magnetic field at the pulsar surface prevents positive ions from being removed, as these ions form long molecular chains which are laterally attracted by fringe fields, forming a tightly bound, dense solid (Ruderman 1971). It is therefore unlikely that an electron-ion plasma will occur in the pulsar magnetosphere.

Consider an electron extracted from the surface, under the influence of the longitudinal electric field $E_{\parallel}$. Such a charged particle would be accelerated along the magnetic field outside the star. The particle's transverse motion (with respect to the magnetic field line) will be negligible (Kaplan, Tsytovich & Ter Haar 1972). If a particle possesses a noticeable momentum component perpendicular to the field, synchrotron emission will be large and this will rapidly make the transverse component $p_{\perp}$ smaller. The radiation 'fall down' time is of the order 10^{-19} s (Beskin 1993). Since the dipole magnetic field is curvilinear the particle acquires energy and begins to emit electromagnetic curvature radiation. Calculations (Beskin 1993) reveal very high particle energies: for electric fields of order $10^{10} - 10^{12}$ V.cm^{-1} and a curvature radius of the magnetic field of order 10^7 cm, accelerated particles obtain energies as high as $10^7 - 10^8$ MeV, with corresponding curvature photon energy of $10^6 - 10^7$ MeV.

Quantum electrodynamics dictates that an energetic photon propagating in a magnetic field will be converted into an electron-positron (EP) pair if certain conditions, pertaining to photon energy, magnetic field strength and photon momentum angle, are met (Berestetskii, Lifshits & Pitaevskii 1971). These conditions for pair production can be reduced to the requirement that the photon intersect a magnetic field line at a critical angle, which depends on the photon energy and the strength of the magnetic field.

The accelerated electrons radiate photons in the instantaneous direction of particle motion, which means initially the photons cannot convert to electron-positron pairs. However, since the magnetic field lines are curved, the photon's straight-line trajectory will eventually intersect a field line which satisfies the critical pair-production condition. The electron of the newly created pair will then be accelerated along the field line intersected by the photon in one direction, and the positron, in the opposite direction, according to the direction of the longitudinal electric field. The electron and positron may then in turn, produce curvature radiation, creating further EP pairs. In this way the magnetosphere is unstable to 'vacuum breakdown' resulting in an electron-positron plasma surrounding the pulsar. Such a mechanism for electron-positron pair generation was first suggested by Sturrock in 1971 (Sturrock 1971) and developed by Ruderman and Sutherland in 1975 (Ruderman & Sutherland 1975).

The pair plasma fills the magnetosphere and eventually screens the longitudinal $E_{\|}$ field. Because of the screening the plasma begins to corotate with the pulsar as a solid body.

1.1.3 The double layer or vacuum gap

Since the pulsar is rotating, the dipole magnetic field is deformed by the rotation of charge in the magnetosphere. This perturbation tends to lengthen the magnetic field lines orthogonally to the rotation axis, extending, and finally opening far field lines. These far field lines originate at the poles of the star, known as the polar caps. Thus the pulsar magnetosphere consists of two distinct regions: areas of closed or open field lines.

In the region of the polar caps the pair plasma no longer corotates with the pulsar, as particles stream out along the open field lines and are lost from the star. A region of vacuum builds up close to the star surface where plasma is absent, due to the escape of particles along the open B field lines. The screening effect, $E_{||} = 0$, is thus violated in this 'vacuum gap' and $E_{||}$ can attain values as high as $10^{13} - 10^{15}$ V (Beskin 1993).

The gap width h and the potential difference proportional to h^2 both increase with the outflow of charged particles from the magnetosphere, until the vacuum becomes unstable against the avalanche growth of EP pairs. This occurs when the potential drop reaches approximately $10^{13} - 10^{14}$ V (Ruderman & Sutherland 1975; Gurevich & Istomin 1985). The electron-positron creation provides the necessary charge to reduce the potential difference, and thus pair production is permanently maintained near the magnetic poles of the star.

It is also interesting to note that about 10^6 stray galactic photons fall on the polar caps every second (Kennel, Fujimura & Pellat 1979). These photons could produce EP pairs if they satisfy the Berestetskii condition (Berestetskii, Lifshits & Pitaevskii 1971).

The continuous production of pairs in the vacuum gap requires a gap width of about 10^2 m (Beskin 1993). For 'vacuum breakdown' it is necessary that charged particles accelerated in the gap produce curvature radiation and electron-positron pairs before reaching the gap boundary. The downward accelerated particle (sign depending on the electric field) in turn produces further pairs, and so on, until cascading occurs.

The vacuum gap is often referred to as the double layer since it has the properties of a plasma double layer: the sheathing effect that forms near the surface of a body in a plasma. As has previously been mentioned there is no electric field in the pair plasma above the vacuum gap because of screening. Thus the potential ϕ is zero there. However, the potential in the gap is finite and indeed large, so there must therefore exist a layer in the plasma where the potential varies.

1.1.4 Primary and secondary plasma

It is customary to refer to the plasma in the region of the magnetic poles in terms of primary and secondary particles. The primary particles are those produced near the surface of the neutron star (or ejected from it), which are then accelerated in the gap potential to extreme relativistic energies of order 10^7 MeV (Beskin 1993).

These primary particles, relinquishing almost all of their energy, produce curvature photons with energies greater than $2m_ec^2$. These curvature photons thus produce electron-positron pairs. These are the first-generation secondary plasma produced.

Above the gap boundary particles are no longer accelerated by the large potential difference in the gap, because of screening. However, further electron-positron generation occurs via synchrotron radiation. The first-generation secondary particles are created with a finite orthogonal momentum, therefore within a time τ of order 10^{-19} s (Beskin 1993) they radiate synchrophotons which are also capable of pair production. Pairs generated by the synchrophotons are referred to as second-generation secondary plasma. Usually particle creation stops after the second generation.

The energy of the secondary particles ranges from 10–30 MeV to 10^5 MeV (Beskin 1993). The range in energies is presumably due to the different pair-production mechanisms. Synchrophoton produced pairs will be at energies lower than curvature photon produced pairs, since synchrophoton energies are approximately an order of magnitude less than that of the parent curvature photon.

Thus we have two distinct pair-production processes. Firstly, EP pairs produced from curvature photons, and secondly, EP pairs produced from synchrophotons. An indication of the ratio of energies of the particles created from these two processes can be found using the range of secondary particle energies given by Beskin (Beskin 1993)

$$\frac{\varepsilon_{min}}{\varepsilon_{max}} \sim \frac{10^1}{10^5} \sim 10^{-4}. \tag{1.4}$$

1.2 Collective modes in nonrelativistic two component EP plasmas

In order to explain pulsar emission, studies in EP plasmas have mostly considered relativistic plasmas (Lominadze & Mikhailovskii 1979; Lominadze, Mikhailovskii & Sagdeev 1979; Suvorov & Chugunov 1980; Mikhailovskii 1980; Mamradze, Machabeli & Melikidze 1980; Chian & Kennel 1983; Yu, Shukla & Rao 1984; Mikhailovskii, Onishchenko & Tatarinov 1985; Yu & Rao 1985; Berezhiani, Skarka & Mahajan 1993; Mofiz & Mamun 1993; Shukla 1993; Verheest 1996a; Verheest & Lakhina 1996), since pair production involves high energy processes under astrophysical conditions. However, experimentalists have been successful in creating nonrelativistic electron-positron plasma in the laboratory. The nonrelativistic EP plasma may be produced when a relativistic electron beam impinges on a high-Z target, where positrons are produced copiously. The relativistic pair plasma is then trapped in a magnetic mirror and is expected to cool rapidly by radiation (Trivelpiece 1972).

1.2.1 Particle behaviour

The physics behind EP plasmas is unique, as it is highly symmetrical. Electrons and positrons both have the same dynamical properties, owing to their similar masses and electric charge magnitudes. Therefore their dynamical behaviour is the same, in contrast to electron-ion plasmas whose constituents exhibit different dynamical time scales. Because of such different relaxation time scales, electron-ion plasmas may exist as a two temperature plasma where the electrons and ions are both in thermal equilibrium, but at different temperatures, T_e and T_i, respectively. For an EP plasma however, the relaxation time scales are comparable, which leads to the conclusion that it is not possible to produce a two temperature EP plasma at equilibrium in a laboratory. When an electron plasma in thermal equilibrium, at temperature T_e, is mixed with a positron plasma at temperature T_p, thermal equilibrium for each component will not be attained until the whole EP plasma reaches a thermal equilibrium state (Iwamoto 1993).

In space plasmas however, for example the pulsar magnetosphere, there is the possibility of continuous streaming of high energy particles (electrons and positrons in opposite directions) within the background plasma. Under these conditions, thermal equilibrium between the species does not have time to occur, and so it may indeed be possible to sustain different species at different temperatures.

In the presence of a magnetic field, the electrons and positrons perform gyromotion at the same frequency $\mid \Omega_e \mid = \mid \Omega_p \mid$, in opposite directions. In contrast, electron-ion plasmas have $\mid \Omega_i \mid \ll \mid \Omega_e \mid$.

In addition to ordinary plasma processes, pair annihilation can take place in an EP plasma. However, under realistic conditions the pair plasma is well defined, in that its lifetime against pair annihilation is much larger than the characteristic time scales for collective oscillations (Iwamoto 1993).

1.2.2 Longitudinal and transverse modes with $B = 0$

For the case of a neutral EP plasma with particle equilibrium densities $n_{eo} = n_{po}$, and with no external magnetic field, the general longitudinal dielectric function may be obtained (Iwamoto 1993) from the combination of the Maxwell equations and the linearized Vlasov equation, as

$$\epsilon(k,\omega) = 1 + \sum_j \frac{k_j^2}{k^2} Z\left(\frac{\omega}{kv_j}\right), \tag{1.5}$$

where $Z(\xi)$ is the plasma dispersion function (Fried & Conte 1961); $v_j = \left(\frac{T_j}{m_j}\right)^{\frac{1}{2}}$ is the electron (positron) thermal speed, for $j = e(p)$; $k_j = \left(\frac{4\pi n_{jo} q_j^2}{T_j}\right)^{\frac{1}{2}}$ is the Debye wavenumber with n_{jo} the equilibrium number density, q_j the charge, m_j the mass for the particle species j; and T_j the temperature, where K the Boltzmann constant has been absorbed in the definition of T. In this particular case of a two species EP plasma in equilibrium, $T_e = T_p \equiv T$ and $m_e = m_p \equiv m$ so (1.5) reduces to

$$\epsilon(k,\omega) = 1 + \frac{k_{De}^2}{k^2} Z\left(\frac{\omega}{kv_j}\right), \tag{1.6}$$

where $k_{De}^2 \equiv k_e^2 + k_p^2$.

Solving $\epsilon(k,\omega) = 0$, Iwamoto (Iwamoto 1993) finds the dispersion relations for the longitudinal collective modes. A well defined mode exists where the phase velocity of the wave is much larger than the thermal velocity of the particles, so that $\frac{\omega}{k} \gg \left(\frac{T}{m}\right)^{\frac{1}{2}}$. In this region Iwamoto (Iwamoto 1993) finds

$$\omega(k) = \omega_p \left[1 + \frac{3k^2}{2k_{De}^2} + \cdots\right], \tag{1.7}$$

$$\gamma(k) = -\left(\frac{\pi}{8}\right)^{\frac{1}{2}} \omega_p \left(\frac{k_{De}}{k}\right)^3 \exp\left[-\frac{1}{2}\left(\frac{k_{De}}{k}\right)^2 - \frac{3}{2}\right], \tag{1.8}$$

(cf. Section 2.3.2), where $\omega_p^2 \equiv \sum_j \omega_{pj}^2$ with $\omega_{pj}^2 = \frac{4\pi n_{jo} q_j^2}{m}$, $j = e, p$. This is the Langmuir or plasma mode.

In the absence of a magnetic field, the transverse dielectric function for a single species EP plasma at temperature T is (Iwamoto 1993)

$$\epsilon_T(k,\omega) = 1 - \frac{\omega_p^2}{\omega^2}\left[1 - Z\left(\frac{\omega}{kv_j}\right)\right], \tag{1.9}$$

where $\omega_p^2 = \sum_j \omega_{pj}$ for $j = e, p$. Solving the equation

$$\epsilon_T(k,\omega) = \left(\frac{ck}{\omega}\right)^2,$$

Iwamoto (1993) finds the dispersion relation for a transverse mode of the form $\omega(k) + i\gamma(k)$ as

$$\omega(k)^2 = \omega_p^2 + c^2k^2, \tag{1.10}$$

$$\gamma(k) = 0. \tag{1.11}$$

The damping is absent since the phase velocity of the wave, obtained from (1.10) is always greater than the speed of light, so that no particles can be resonant with the wave (Iwamoto 1993).

Conspicuous by its absence is the acoustic branch which Iwamoto (Iwamoto 1993) does not find in his longitudinal mode analysis without an external magnetic field. This is worth closer attention. Since the acoustic mode is related to the ion motion within an electron-ion plasma, it would seem logical to conclude that any co-operative phenomena which originate from the mass difference between the electron and the ion will not appear in an EP plasma. This has lead to the conclusion by a number of authors (Sakai & Kawata 1980; Chian & Kennel 1983; Shukla, Rao, Yu & Tsintsadze 1986; Stewart & Laing 1992; Tsintsadze 1992) that the acoustic branch would thus be absent from electron-positron collective processes. This is certainly the case for a neutral ($n_e = n_p$) two component EP plasma in equilibrium with $T_e = T_p$, but not so for multi-species EP plasmas.

1.3 Multi-component pair plasmas

As we have seen, the magnetosphere of the pulsar near the poles is a highly dynamic and complicated system of electron-positron plasma. In attempts to model this region several authors (Bharuthram 1992; Pillay & Bharuthram 1992; Srinivas, Popel & Shukla 1996; Verheest, Hellberg, Gray & Mace 1996) have assumed that the plasma will consist of a number of components of electrons and positrons, rather than the much used two-component plasma model. In all such multi-component EP plasmas the acoustic mode is physical.

It is perhaps interesting to note that among the multi-component models there is some diversity. For example, electrostatic solitons in an EP plasma were investigated by Srinivas *et al.* (Srinivas, Popel & Shukla 1996) using a *three* component model consisting of a cold electron and positron component with temperatures $T_c = 0$, and a warm positron component, with temperature T_h.

In this case it would seem that Srinivas *et al.* are modelling the secondary plasma as a cold fluid, and then regarding a primary beam of fast positrons accelerated in the vacuum gap, passing through it. These may be the primary positrons after their energy has been reduced by curvature radiation. Because of their reduced energy, they have negligible further energy loss and so easily catch up and ultimately pass through the more slowly moving secondary pairs (Ruderman & Sutherland 1975). The cold components are then described by the equations of continuity and motion, and the warm positrons are described by a Boltzmann distribution. Closing the set of equations with the Poisson equation and requiring, for charge neutrality, that the number density of the hot and cold species of positrons together are equal to the cold electrons', Srinivas *et al.* conduct both linear and nonlinear analyses.

They obtain acoustic-like linear waves with a dispersion relation of the form

$$\omega^2 = \frac{c_a^2 k^2}{1 + k^2\lambda_d^2}, \tag{1.12}$$

with $c_a^2 = \frac{T_h}{m}\left(1 + 2\frac{n_{pco}}{n_{pho}}\right)$ and $\lambda_d = \left(\frac{T_h}{4\pi n_{pho} e^2}\right)^{\frac{1}{2}}$ (cf. Section 2.3.4). Srinivas *et al.* then investigate the existence of nonlinear acoustic soliton solutions under these conditions.

Bharuthram (1992) and Pillay & Bharuthram (1992) investigate electrostatic double layers and solitons respectively, under two differing *four* component EP plasma models. Pillay & Bharuthram (1992) assume Boltzmann hot electrons and positrons and cold electrons and positrons to model the pulsar EP plasma. This model inspired the one used for the results obtained in this thesis. However, Pillay *et al.* (Pillay & Bharuthram 1992) subscribe to a non-symmetric model in which the temperatures and number densities of each of the component species are not equal. The symmetric model used in this thesis seems to be more likely under the conditions of pair production, in which photons produce EP pairs with the same energy, and therefore temperature. Pillay *et al.* (Pillay & Bharuthram 1992) remarked on this asymmetry, stating that it was necessary in order to maintain double layers at all, since the existence of double layers requires some asymmetry in the motion or density of the particles in the plasma.

A symmetric model is more plausible, and could be sustained for longer than an asymmetric case, which can only last for a short time. However, considering the complex nature of the electron-positron plasma the Pillay/Bharuthram case could indeed model the region outward of the pulsar double layer vacuum gap. A complete review of the Pillay and Bharuthram model can be found in Chapter 5.

1.4 The four component symmetrical model - justification

We assume an unmagnetized, four component, electron-positron plasma to model the pulsar magnetosphere, with equal temperatures and densities for the electrons and positrons of each type. We have a hot species of electrons and positrons to model those secondary plasma particles (first-generation), born of curvature photons, with an average temperature T_h. And we have a cool species of electrons and positrons with a temperature T_c to model the secondary plasma particles (second-generation), born of synchrophotons. We are justified in assuming symmetry between the electrons and the positrons because we are dealing with two pair-production mechanisms which will yield equal numbers of electrons and positrons at two distinct temperatures. We have a ratio of energies and therefore temperatures from (1.4) which gives us a guideline as to the temperatures of our two temperature-component types.

The choice of an unmagnetized plasma may seem rather presumptuous in the light of such extreme pulsar magnetic fields. This assumption is valid if we consider the motion of the particles along the magnetic field lines. As has been stated previously (cf. Section 1.1.2), the motion of the particles is mainly longitudinal with respect to the magnetic field of the pulsar. Because of the strength of the magnetic field, any transverse momentum that the particles may have is quickly radiated away (Kaplan, Tsytovich & Ter Haar 1972). We may thus regard the motion of these particles as "beads on a wire". Since the particles travel along a magnetic field line of particular strength, in their rest frame they do not effectively "see" a magnetic field. Thus we may indeed assume $B = 0$ in our approximation.

1.5 Nonlinear acoustic solitons

We are particularly interested in nonlinear soliton wave structures. A soliton is a single pulse-like waveform where the plasma approaches the same asymptotic equilibrium state on either side. All physically relevant quantities are localized in space, thus the waveform is *solitary*. The amplitude, width and velocity of solitons are related, so that large amplitude solitons are faster (cf. equation 1.13) and narrower than their smaller amplitude counterparts. These waveforms retain their particle-like nature in collisions, and thus are referred to as *solitons* rather than solitary waves, which may be unstable during collisions.

The first recorded observation of such a structure is generally attributed to John Scott Russell in 1834 who observed a solitary wave in a canal near Edinburgh. Russell is reputed to have chased the single water pulse which he observed, down the canal on horseback, noting with incredulity its constant speed and amplitude.

Russell performed laboratory experiments, generating solitary waves by dropping a weight at one end of a wave channel. He was able to deduce empirically that the speed of the wave is obtained from

$$c^2 = f(h + a), \tag{1.13}$$

where h is the undisturbed depth of water and a is the amplitude of the wave (Russell 1844).

Boussinesq (1871) and Rayleigh (1876) followed the work of Russell, showing that the one dimensional water wave profile is given by

$$\zeta(x, t) = a\,\mathrm{sech}^2[\beta(x - ct)], \tag{1.14}$$

for β, a specific relation between water depth and wave amplitude. Groundbreaking analysis by Korteweg and de Vries (1895) produced a simple nonlinear wave equation which admits (1.14) as a solution. This Korteweg-de Vries (KdV) equation as it came to be known, is a nonlinear equation fundamental to small amplitude nonlinear plasma wave theory. This equation embodies in it an intricate balance between nonlinear and dispersive effects.

Washimi & Taniuti (1966) later derived a related KdV equation for ion-acoustic solitary waves in a plasma consisting of ions and electrons. In an unmagnetized plasma two electrostatic longitudinal modes can occur: acoustic and Langmuir oscillations. A detailed equilibrium between nonlinear and dispersive effects is possible in the acoustic wave mode, resulting in stable wave profiles that propagate unchanged with time. These are acoustic solitons. Both large and small amplitude ion acoustic solitons have been extensively investigated and are now well defined in plasma theory (see for example, Chen 1984).

The introduction of negative ions (Das & Tagare 1975) has lead to some interesting anomalies in the small amplitude nonlinear wave analysis, culminating in the derivation of a modified KdV equation (Watanabe 1984; Verheest 1988) with soliton solutions with a sech dependence rather than the well known sech^2 profile. As shall be revealed, this modified Korteweg-de Vries (mKdV) is significantly relevant to our work.

1.5.1 Outline of the following chapters

In Chapter 2 we outline the nonrelativistic fluid model of an electron-positron plasma which we will use to model the pulsar magnetosphere. The model consists of four components: a hot Boltzmann species of electrons and positrons, and a cool adiabatic species of the same. We then consider linear, longitudinal, electrostatic waves in *general* multi-species electron-positron plasmas, and derive the dispersion relations for these linear waves.

In Chapter 3 we discuss nonlinear processes in plasmas. We see that the inclusion of both dispersion and nonlinear effects in plasmas leads to the Korteweg de-Vries equation (Korteweg & de Vries 1895). We thus introduce *small amplitude* nonlinear electron-positron acoustic waves, and derive the Korteweg de-Vries (KdV) equation for our model, hoping for equilibrium between dispersive and nonlinear effects culminating in stable electron-positron soliton solutions. We find, however, that the resulting wave equation is purely dispersive with no nonlinear term. This leads to the investigation of the modified KdV (mKdV) equation (Verheest 1988) under our four component EP model. We obtain stationary solutions to the mKdV equation yielding soliton profiles of a sech dependence. We then consider arbitrary amplitude solitons following the method based on the work done on ion-acoustic solitons and double layers by Baboolal *et al.* (Baboolal, Bharuthram & Hellberg 1988, 1989, 1990) and on electron-acoustic solitons and double layers by Mace *et al.* (Mace & Hellberg 1993a; Mace, Baboolal, Bharuthram & Hellberg 1991; Mace, Hellberg, Bharuthram & Baboolal 1992). We obtain expressions for the electron and positron cool species density, in algebraic form via the equations of continuity and motion. We then introduce the Sagdeev potential (Sagdeev 1966), through integration of the Possion equation, yielding its algebraic form. Under the constraints imposed for soliton existence (Sagdeev 1966) we obtain limits for soliton Mach number for our electron-positron plasma model. We observe that the Mach number limit is dependent on cool species density, which suggests some restriction on the existence domain of EP solitons.

In Chapter 4 we obtain numerical and analytical solutions for both small and arbitrary amplitude solitons. We present graphical results in the form of soliton profiles with their corresponding Sagdeev potentials and soliton existence domains in cool species density–soliton amplitude space. These results are analysed and it is indeed found that the choice of model, that is, the assumption of Boltzmann hot species, places severe restrictions on the existence of these solutions.

In Chapter 5 we review the study of solitons in an asymmetric EP plasma conducted by Pillay & Bharuthram (1992). We discover that the results obtained in this work may be physically invalid in the light of restrictions imposed by the choice of model.

We conclude with a summary of the original results obtained in this thesis, reiterating the consequences of the Boltzmann restriction. We question the validity of the fluid model and suggest that further research into the kinetic theory of relativistic electron-positron plasmas with the inclusion of dust particles would be a worthwhile pursuit.

Chapter 2

Multi-species electron-positron plasmas

This chapter outlines a nonrelativistic fluid model of an electron-positron plasma in the pulsar magnetosphere, consisting of four components: hot Boltzmann species of electrons and positrons, and cool adiabatic species of the same. Linear, longitudinal, electrostatic waves in general multi-species electron-positron plasmas are then discussed, and the dispersion relations for these linear waves are derived. The method of linearization that we use follows that of Chen (1984).

2.1 Introduction

We begin by creating a simple model of the electron-positron plasma pulsar magnetosphere. We thus assume two species of electrons: a hot species at a temperature T_{eh}, to model the first-generation (curvature born) secondary electrons, and a cool species at a temperature T_{ec}, to model the second-generation (born of synchroradiation) secondary electrons. Similarly we have two species of positrons at temperatures T_{ph} and T_{pc}. The symmetry of electron-positron pair creation implies the production of equal densities of positrons and electrons at equal temperatures, at equilibrium. Defining electron and positron densities as n_{ec}, n_{eh}, n_{pc}, and n_{ph}, we thus have equal equilibrium densities for electrons and positrons, applying to both hot and cool species

$$n_{eho} = n_{pho} \equiv N_h,$$

and

$$n_{eco} = n_{pco} \equiv N_c,$$

where the subscript $_o$ refers to the equilibrium state, and we define N_h and N_c by the above equations. In addition, we have hot electrons and positrons at temperatures $T_{eh} = T_{ph} \equiv T_h$ and cool electrons and positrons at temperatures $T_{ec} = T_{pc} \equiv T_c$. The plasma is uniform and electrically neutral at rest (if we assume screening of the pulsar electric field), before it is disturbed. Thus the charge neutrality condition holds

$$n_{eo} = n_{po} = n_o \equiv N_o, \tag{2.1}$$

$$n_{eho} + n_{eco} = n_{pho} + n_{pco} = N_h + N_c \equiv N_o,$$

where N_o is the total equilibrium plasma density.

Since the particle motion is largely in the direction of the magnetic field, that is, the particles' transverse motion is negligible compared with their longitudinal motion (Kaplan, Tsytovich & Ter Haar 1972), we may make the assumption that $B = 0$ in our model. This assumption allows us to investigate the existence of electrostatic oscillations. The EP plasma in the magnetosphere will screen the pulsar's large E field, and so we may assume that at equilibrium $E_o = 0$, implying $\phi_o = 0$. It follows that the particles have no drift velocities. The fluid velocities are also zero at equilibrium

$$u_o = 0,$$

for all species of electrons and positrons, where u is the particle velocity and the subscript $_o$ refers to the equilibrium state. This implies that the plasma is stationary at equilibrium.

Finally, we impose the constraint that the electron-positron plasma is nonrelativistic. Although the primary particles may reach relativistic speeds in the double layer, we will consider a simpler nonrelativistic model.

2.2 Basic equations

Consider an infinite, collisionless, unmagnetized, one-dimensional electron-positron plasma consisting of four fluid components. The plasma is composed of hot species of electrons and positrons, and cool species of the same. The fluid equations describing each fluid component are the continuity equation, and the equation of motion

$$\frac{\partial n_{jk}}{\partial t} + \frac{\partial}{\partial x}(n_{jk}u_{jk}) = 0, \tag{2.2}$$

$$m_j n_{jk}\left[\frac{\partial u_{jk}}{\partial t} + u_{jk}\frac{\partial u_{jk}}{\partial x}\right] + \frac{\partial p_{jk}}{\partial x} = -q_j n_{jk}\frac{\partial \phi}{\partial x}. \tag{2.3}$$

This system of fluid equations is closed by Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi \sum_{k=h,c}\sum_{j=e,p} -q_j n_{jk}. \tag{2.4}$$

Here n_{jk} is the density of the particular species, u_{jk} the average velocity of the particles, p_{jk} the partial pressure, m_j the particle mass and q_j the particle charge. In this model m_j will be the electron mass so that $m_j \equiv m$, for $j = e, p$, and $q_j = Z_j e$ will be $-e$ and e for electrons and positrons, respectively.

The thermodynamic equation of state is

$$p_{jk} n_{jk}^{-\gamma_k} = C, \tag{2.5}$$

for C some constant, where $p_{jk} = n_{jk}T_k$ by definition. Here T_k is the temperature of the particular species (either T_c or T_h) and γ_k is the corresponding ratio of specific heats. The Boltzmann constant K is absorbed in the definition of T.

At infinity the following boundary conditions hold, to ensure electrical neutrality

$$n_{jk} \to n_{jko} \qquad p_{jk} \to p_{jko} \qquad \phi \to \phi_o \qquad \sum_k \sum_j q_j n_{jko} = 0. \tag{2.6}$$

Now (2.5) with (2.6) implies that

$$p_{jk} n_{jk}^{-\gamma_k} = p_{jko} n_{jko}^{-\gamma_k},$$

and with the definition $p_{jko} = n_{jko}T_k$ this gives

$$p_{jk} = \frac{n_{jk}^{\gamma_k}}{n_{jko}^{\gamma_k - 1}} T_k.$$

The fluid equation of momentum may thus be written

$$mn_{jk}\left[\frac{\partial u_{jk}}{\partial t}+u_{jk}\frac{\partial u_{jk}}{\partial x}\right]+\gamma_k\frac{n_{jk}^{\gamma_k-1}}{n_{jko}^{\gamma_k-1}}T_k\frac{\partial n_{jk}}{\partial x}=-Z_jen_{jk}\frac{\partial\phi}{\partial x},\qquad(2.7)$$

where $Z_j=(-1)(+1)$ for $j=(e)(p)$.

The hot components may be described by a Boltzmann distribution, effectively neglecting their inertia. The assumption of hot, 'massless' Boltzmann electrons and positrons is justified only if the thermal velocity of the hot components v_h, is far greater than the velocity of a waveform supported by the plasma, so that v_h is large enough that it may be considered as infinite when compared to the phase velocity of the waveform. Thus the hot components must be isothermal. The validity of this assumption will be considered in more detail later. The approximation that the hot particles be inertialess also requires that their thermal velocity is large in comparison with that of the cool fluids, which are then regarded as sluggish.

For isothermal fluids we have $\gamma_h = 1$, and since we assume that the hot electron and positron species are highly mobile we may take $m\to 0$, implying, from equation (2.7), that the pressure and electrostatic gradient forces must be closely in balance. Thus (2.7) may then be integrated (with $m\to 0$) to yield the Boltzmann relation for electrons and positrons

$$n_{eh}=N_h\exp(\frac{e\phi}{T_h}),\qquad(2.8)$$

and

$$n_{ph}=N_h\exp(-\frac{e\phi}{T_h}).\qquad(2.9)$$

The cool constituents are assumed to have thermal speeds which are significantly less than the speed of the waveform and are thus modelled as adiabatic fluids, described by the cool fluid continuity equation

$$\frac{\partial n_{jc}}{\partial t}+\frac{\partial}{\partial x}(n_{jc}u_{jc})=0,\qquad(2.10)$$

and equation (2.7) with $\gamma_c=3$

$$mn_{jc}\left[\frac{\partial u_{jc}}{\partial t}+u_{jc}\frac{\partial u_{jc}}{\partial x}\right]+3\frac{n_{jc}^2}{n_{jco}^2}T_c\frac{\partial n_{jc}}{\partial x}=-Z_jen_{jc}\frac{\partial\phi}{\partial x}.\qquad(2.11)$$

Equations (2.4), (2.8), (2.9), (2.10) and (2.11) may be expressed in terms of specific units so that they are dimensionless. Densities are normalized with respect to N_o the total equilibrium density value; temperatures with respect to T_h the hot species temperature; velocities by the hot electron thermal speed $v_h = \left(\frac{T_h}{m}\right)^{\frac{1}{2}}$; spatial length by the Debye length $\lambda_{Dh} = \left(\frac{T_h}{4\pi N_o e^2}\right)^{\frac{1}{2}}$ and time by $\omega_p^{-1} = \left(\frac{m}{4\pi N_o e^2}\right)^{\frac{1}{2}}$ the inverse electron plasma frequency, so that

$$\tilde{T} = \frac{T}{T_h}; \qquad \tilde{\phi} = \frac{e\phi}{T_h}; \qquad \tilde{x} = \frac{x}{\lambda_{Dh}}; \qquad \tilde{t} = t\omega_p; \qquad \tilde{u} = \frac{u}{v_h}, \tag{2.12}$$

where the tilde denotes the appropriate normalized variable. For convenience we shall henceforth omit the tilde and consider only normalized variables, unless otherwise specified.

Equations (2.8), (2.9), (2.10), (2.11) and (2.4) thus become

$$n_{eh} = N_h \exp(\phi), \tag{2.13}$$

$$n_{ph} = N_h \exp(-\phi), \tag{2.14}$$

$$\frac{\partial n_{jc}}{\partial t} + \frac{\partial}{\partial x}(n_{jc}u_{jc}) = 0, \tag{2.15}$$

$$\left[\frac{\partial u_{jc}}{\partial t} + u_{jc}\frac{\partial u_{jc}}{\partial x}\right] + 3T_c\frac{n_{jc}}{N_c^2}\frac{\partial n_{jc}}{\partial x} = -Z_j\frac{\partial \phi}{\partial x}, \tag{2.16}$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = \sum_{j=e,p}\sum_{k=c,h} -Z_j n_{jk}. \tag{2.17}$$

2.3 Linear waves

Any periodic motion of a fluid can be represented as a superposition of sinusoidal oscillations with different frequencies ω and wavenumbers k. A simple wave is any one of these components, and travels at a phase velocity $v_\phi = \frac{\omega}{k}$. The concept of dispersive waves is defined by a relationship between ω and k of the form $\omega = \omega(k)$, which is known as the dispersion relation. If ω depends on k we find that components of different wavenumber propagate at different frequencies.

Let us consider linear waves in electron-positron plasmas. Linear wave analysis requires the perturbation values from equilibrium of the velocities, densities and electrostatic potential to be small, so that any quadratic and higher order terms can be ignored.

We will consider five different models of electron-positron plasma for completeness.

2.3.1 Hot Boltzmann electrons and positrons

Perhaps the simplest model of an electron-positron plasma is the single species pair plasma, containing one species each of electrons and positrons at equal temperatures. If we assume the temperature of the electrons and positrons to be hot, say T_h, so that their thermal velocity $v_h = \left(\frac{T_h}{m}\right)^{\frac{1}{2}}$ is much greater than the speed of a waveform supported by the plasma, we may model them as isothermal fluids. In this case the particles would be highly mobile, and we may ignore their inertia. The densities of both particles would then be given by the Boltzmann relations

$$n_e = n_{eo} \exp(\phi),$$

$$n_p = n_{po} \exp(-\phi),$$

or more generally

$$n_j = n_{jo} \exp(-Z_j \phi). \tag{2.18}$$

The system of density equations is closed by Poisson's equation

$$\frac{\partial^2 \phi}{\partial x^2} = \sum_j -Z_j n_j, \tag{2.19}$$

and the charge neutrality condition (since we assume an electrically neutral plasma at equilibrium)

$$\sum_j Z_j n_{jo} = 0, \tag{2.20}$$

where the subscript $_j$ refers to either electrons or positrons, and the subscript $_o$ denotes the equilibrium state. The above equations are normalized with respect to (2.12), remembering (2.1).

We obtain linear waves by the process of linearization of the system of equations describing the electron-positron plasma (Chen 1984). Assuming that the amplitude of the electrostatic oscillation is small we may neglect any higher order terms and express the variables in terms of their equilibrium values and a perturbation, thus

$$n_j = n_{jo} + n_{j1} \qquad \phi = \phi_o + \phi_1. \tag{2.21}$$

Since we assume an electrically neutral, uniform, stationary plasma at equilibrium, we have the condition that $\phi_o = 0$.

The exponential term in equation (2.18) may be expanded so that

$$n_j = n_{jo}(1 - Z_j\phi + \cdots). \tag{2.22}$$

Substituting (2.21) into (2.22) and ignoring all higher order terms gives

$$\begin{aligned} (n_{jo} + n_{j1}) &= n_{jo} - Z_j n_{jo}\phi_1 \\ \Rightarrow n_{j1} &= -Z_j n_{jo}\phi_1. \end{aligned} \tag{2.23}$$

Equation (2.19) with (2.21) becomes

$$\begin{aligned} \tfrac{\partial^2\phi_1}{\partial x^2} &= (n_{eo} - n_{po}) + n_{e1} - n_{p1} \\ &= n_{e1} - n_{p1} \\ &= n_{eo}\phi_1 + n_{po}\phi_1, \end{aligned} \tag{2.24}$$

with the charge neutrality condition (2.20) and equation (2.23).

The perturbations from equilibrium are assumed to be sinusoidal, that is, we can consider a Fourier mode (ω, k) such that

$$\begin{aligned} n_{j1} &= \tilde{n_{j1}} \exp[i(kx - wt)] \\ \phi_1 &= \tilde{\phi_1} \exp[i(kx - wt)], \end{aligned} \tag{2.25}$$

where the variables with tilde are the amplitudes of the sinusoidal variations. We will omit the tilde for simplicity. Substituting (2.25) into equation (2.24) we obtain

$$\begin{aligned} -k^2\phi_1 &= (n_{eo} + n_{po})\phi_1 \\ \Rightarrow -k^2 &= n_{eo} + n_{po}. \end{aligned} \tag{2.26}$$

The charge neutrality condition implies that the electrons and the positrons have equal densities at equilibrium, that is, in unnormalized terms

$$n_{eo} = n_{po} = n_o,$$

(in normalized terms this reduces to $n_{eo} = n_{po} = 1$ since densities are normalized with respect to the equilibrium density value n_o). Thus equation (2.26) reduces to

$$-k^2 = 2. \tag{2.27}$$

Equation (2.27) should represent the dispersion relation for the waves which are supported by this electron-positron plasma. However, ω is indeterminate in this equation, signifying an absence of waves in this single species, isothermal electron-positron plasma. For an oscillation, one requires a restoring force, and thus inertia. In this model the particles are assumed to be isothermal, and thus no inertia term is retained, thus we cannot expect any waveform.

The assumption of isothermal fluids limits us to the investigation of low phase velocity waves, so that $v_\phi < v_h$. The dispersion relation above, precludes the existence of such oscillations. Any waves occurring would thus be high phase velocity waves with $v_\phi > v_{th}$. We thus consider a cool two species model consisting of particles with temperature T_c, so that we may investigate waveforms with phase velocities greater than the thermal speed $\left(\frac{T_c}{m}\right)^{\frac{1}{2}}$, of these particles.

2.3.2 Cool adiabatic electrons and positrons

Consider such a single species electron-positron plasma model: again both particles are at the same temperature T_c, but in this case they are assumed to be cool, and adiabatic, so that their thermal velocity $v_c = \left(\frac{T_c}{m}\right)^{\frac{1}{2}}$ is much smaller than the velocity of a waveform supported by the plasma. Both electrons and positrons are then described generally by the equation of continuity, equation of motion and Poisson's equation

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j u_j) = 0,$$

$$m_j n_j \left[\frac{\partial u_j}{\partial t} + u_j \frac{\partial u_j}{\partial x}\right] + 3\frac{n_j^2}{n_{jo}^2} T_c \frac{\partial n_j}{\partial x} = -Z_j e n_j \frac{\partial \phi}{\partial x},$$

and

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi \sum_j -q_j n_j.$$

We follow the usual normalization as outlined earlier, but expressed in terms of the cool temperature T_c, so that the above equations become

$$\frac{\partial n_j}{\partial t} + \frac{\partial}{\partial x}(n_j u_j) = 0, \tag{2.28}$$

$$\frac{\partial u_j}{\partial t} + u_j\frac{\partial u_j}{\partial x} + 3\frac{n_j}{n_{jo}^2}\frac{\partial n_j}{\partial x} = -Z_j\frac{\partial \phi}{\partial x}, \tag{2.29}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \sum_j -Z_j n_j. \tag{2.30}$$

Linearizing in the same manner as previously shown for isothermal electrons and positrons, we have

$$n_j = n_{jo} + n_{j1} \qquad \phi = \phi_1 \qquad u_j = u_{jo} + u_{j1}. \tag{2.31}$$

We consider a stationary plasma, so that $u_{jo} = 0$, that is, at equilibrium the electron-positron fluids have no flow velocities. Expanding with respect to these linear variables, neglecting quadratic and higher order terms, and assuming sinusoidal, small amplitude perturbations, equations (2.28), (2.29) and (2.30) become

$$\begin{aligned} -i\omega n_{j1} + ikn_{jo}u_{j1} &= 0 \\ \Rightarrow u_{j1} &= \frac{\omega}{k}\frac{n_{j1}}{n_{jo}}, \end{aligned} \tag{2.32}$$

$$-i\omega u_{j1} + ik3\frac{n_{j1}}{n_{jo}} = -ikZ_j\phi_1, \tag{2.33}$$

$$\begin{aligned} -k^2\phi_1 &= (n_{eo} - n_{po}) + n_{e1} - n_{p1} \\ &= n_{e1} - n_{p1}. \end{aligned} \tag{2.34}$$

Substituting (2.32) into (2.33) gives

$$\frac{n_{j1}}{n_{jo}} = \frac{Z_j\phi_1}{\frac{\omega^2}{k^2} - 3}. \tag{2.35}$$

Using equation (2.35) in equation (2.34) gives

$$k^2\phi_1 = \frac{\phi_1}{\frac{\omega^2}{k^2} - 3}\left(n_{eo} + n_{po}\right),$$

which reduces to

$$k^2 = \frac{2}{\frac{\omega^2}{k^2} - 3},$$

where $n_{eo} = n_{po} = 1$ is a consequence of the charge neutrality condition

$$\sum_j Z_j n_{jo} = 0,$$

assuming an electrically neutral plasma at equilibrium, and normalization with respect to n_o.

This gives a dispersion relation of

$$\omega^2 = 2 + 3k^2, \tag{2.36}$$

which has the form as that of an electron plasma wave (Chen 1984), (bearing in mind that equation (2.36) is normalized). This shows that the medium may support a plasma-like wave, with phase velocity $v_\phi = \frac{\omega}{k}$ greater than the particle thermal velocity.

Figure 2.1 shows the dispersion curve for this electron-positron plasma wave. The slope at any point on the curve gives the group velocity of the wave, $v_g = \frac{\partial\omega}{\partial k}$. This is clearly always less than $\sqrt{3}$, although at very large k values the slope of the ω, k plot tends to this value.

It should be noted that, for $\frac{3}{2}k^2 \ll 1$, the dispersion relation (2.36) has the same form as that of Iwamoto (c.f. equation (1.7)). To recognise that, it is necessary to reconsider the definitions of k_{De} and ω_p which are used in (1.7). Iwamoto defines $k_{De}^2 \equiv k_e^2 + k_p^2$ with $k_j = \left(\frac{4\pi n_{jo} q_j^2}{T_j}\right)^{\frac{1}{2}}$ and $\omega_p^2 \equiv \sum_j \omega_{pj}^2$ with $\omega_{pj}^2 = \frac{4\pi n_{jo} q_j^2}{m}$, $j = e, p$. He also assumes a neutral ($n_{eo} = n_{po}$) EP plasma with $T_e = T_p \equiv T$. We note that $k_{De} = \sqrt{2}k_e$ and $\omega_p = \sqrt{2}\omega_{pe}$. Thus (1.7) becomes

$$\omega = \sqrt{2}\omega_p \left[1 + \frac{3}{4}\frac{k^2}{k_e^2}\right], \tag{2.37}$$

which is simply (2.36) under binomial series expansion in the range $\frac{3}{2}k^2 \ll 1$.

2.3.3 Boltzmann electrons and adiabatic positrons

The next model we shall consider is an electron-positron plasma with hot electrons at temperature T_h and cool positrons at temperature T_c, where $T_h \gg T_c$. We assume equal electron and positron densities at equilibrium so that

$$n_{eo} = n_{po} = n_o.$$

Watanabe & Taniuti (1977), employing a fluid treatment of a plasma consisting of ions, cool and hot electrons, showed that if the temperature of the hot electrons greatly exceeds that of the cooler, then the equations admit wave solutions whose phase velocity satisfies and $v_h \gg v_\phi \gg v_c$, and are therefore

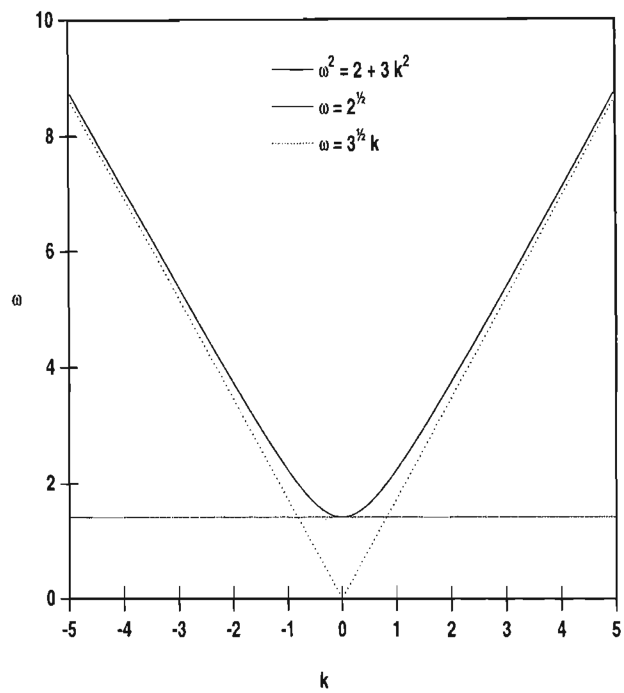


Figure 2.1: A plot of the dispersion relation, equation (2.36) $\omega^2 = 2 + 3k^2$, for an electron-positron plasma of cool adiabatic electrons and positrons, showing a plasma wave mode.

weakly Landau damped. In the same way, we assume that the hot species thermal velocity v_h is greater than the phase velocity, $v_\phi = \frac{\omega}{k}$, of a waveform supported by the plasma, and the cool species' thermal velocity is assumed to be less than this phase velocity. We may then model the hot electrons as isothermal with a Boltzmann density distribution function

$$n_e = n_{eo} \exp(\phi). \tag{2.38}$$

The cool positrons are considered to be an adiabatic fluid with describing equations (continuity and motion)

$$\frac{\partial n_p}{\partial t} + \frac{\partial}{\partial x}(n_p u_{pc}) = 0, \tag{2.39}$$

$$\frac{\partial u_p}{\partial t} + u_p \frac{\partial u_p}{\partial x} + 3T_c \frac{n_p}{n_{po}^2} \frac{\partial n_p}{\partial x} = -\frac{\partial \phi}{\partial x}. \tag{2.40}$$

These equations are closed by the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = \sum_j -Z_j n_j, \tag{2.41}$$

and the charge neutrality condition $\sum_j Z_j n_{jo} = 0$, such that $n_{eo} - n_{po} = 0$. The above equations are normalized with respect to (2.12), bearing in mind (2.1), so that all references to temperature T_c suggest a temperature ratio $\frac{T_c}{T_h}$. Equation (2.38) may be expanded to

$$n_e = n_{eo} \left(1 + \phi + \cdots\right). \tag{2.42}$$

Considering small amplitude electrostatic waves, we may linearize equations (2.42), (2.39), (2.40) and (2.41) with sinusoidal variables, to obtain the following equations relating ω and k

$$n_{e1} = n_{eo}\phi_1, \tag{2.43}$$

$$-i\omega n_{p1} + ikn_{po}u_{pc1} = 0 \qquad \Rightarrow u_{p1} = \frac{\omega}{k}\frac{n_{p1}}{n_{po}}, \tag{2.44}$$

$$-i\omega u_{p1} + ik3T_c \frac{n_{p1}}{n_{po}} = -ik\phi_1, \tag{2.45}$$

$$\begin{aligned} -k^2 &= (n_{eo} - n_{po}) + n_{e1} - n_{p1} \\ &= n_{e1} - n_{p1}. \end{aligned} \tag{2.46}$$

Substituting (2.43), (2.44) and (2.45) into (2.46) gives

$$-k^2 = n_{eo} - \frac{n_{po}}{\frac{\omega^2}{k^2} - 3T_c}. \tag{2.47}$$

This yields the dispersion relation

$$\omega^2 = \left(\frac{\frac{n_{po}}{n_{eo}}}{1 + \frac{k^2}{n_{eo}}} + 3T_c \right) k^2. \tag{2.48}$$

In the limit of small k equation (2.47) becomes

$$\begin{aligned} \omega &= \left(\tfrac{n_{po}}{n_{eo}} + 3T_c\right)^{\frac{1}{2}} k \\ &= Vk, \end{aligned} \tag{2.49}$$

where V is the normalized wave sound speed $\frac{v_s}{v_h}$. Equation (2.49) is of the form $\omega = v_s k$, where v_s is the sound speed of the wave, and is thus an acoustic-type wave, with $v_s = v_h \left(\frac{n_{po}}{n_{eo}} + 3T_c\right)^{\frac{1}{2}}$. This is clearly analogous to the usual electron-acoustic wave (Mace & Hellberg 1993a).

Figure 2.2 shows a plot of the dispersion relation equation (2.48), for a hot electron, cool positron plasma. It can be seen from the plot that at very large wavelengths ($k \to 0$) all the harmonics travel at the same speed V, the normalized sound speed.

It should be noted, however, that because of the symmetry of the pair production process, this type of plasma would not be an appropriate model for a naturally forming electron-positron plasma, unless there was some streaming effect or heating effect that differentiates between electrons and positrons.

2.3.4 A three component model

Next we consider the three component model (Srinivas, Popel & Shukla 1996) mentioned in Section 1.3. Consider an electron-positron plasma consisting of cold inertial electron and positron fluids (with $T_c = 0$), with a component of energetic positrons at T_h, to model the region of EP plasma above the vacuum gap through which positrons pass, having been accelerated to high energies in the double layer.

At equilibrium $\phi_o = 0$ and charge neutrality dictates that

$$n_{eo} = n_{eco} = n_{po} = n_{pho} + n_{pco} = n_o. \tag{2.50}$$

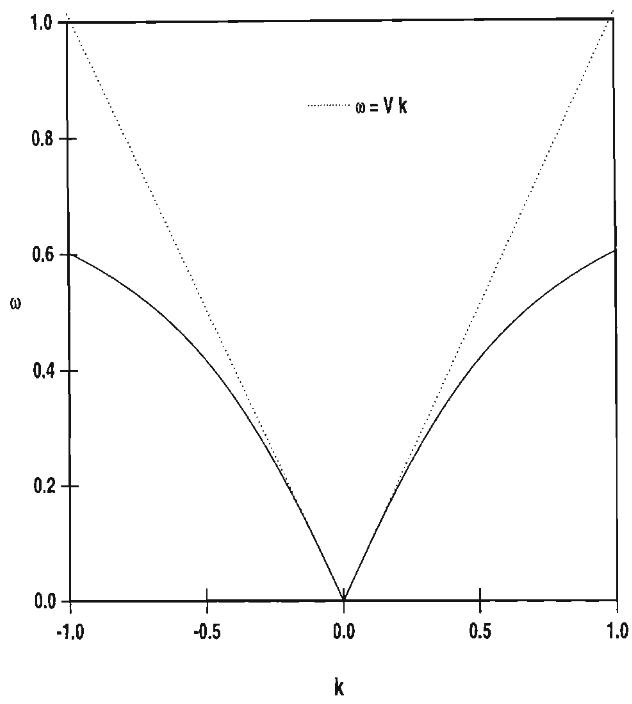


Figure 2.2: A plot of the dispersion curve, equation (2.48) (solid curve), showing an acoustic mode supported by an electron-positron plasma of hot Boltzmann electrons and cool adiabatic positrons. The dashed curve shows the limit of small k, equation (2.49). Plots are obtained using $n_{po} = n_{eo} = 0.5$, and $\frac{T_c}{T_h} = 0.01$.

The hot positron species may be considered to be Boltzmann distributed, if the energies of the positrons are sufficiently high. Thus the density of the hot positrons is given by

$$n_{ph} = n_{pho} \exp(-\phi). \tag{2.51}$$

The cold electron and positron fluids are described by the fluid equations of continuity and motion (neglecting the pressure term)

$$\frac{\partial n_{jc}}{\partial t} + \frac{\partial}{\partial x}(n_{jc}u_{jc}) = 0, \tag{2.52}$$

$$\frac{\partial u_{jc}}{\partial t} + u_{pc}\frac{\partial u_{jc}}{\partial x} = -Z_j\frac{\partial \phi}{\partial x}. \tag{2.53}$$

This set of equations is closed by the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = n_{ec} - (n_{ph} + n_{pc}). \tag{2.54}$$

The above equations are normalized with respect to (2.12), remembering (2.1). Equation (2.51) may be expanded as

$$n_{ph} = n_{pho}\left(1 - \phi \cdots\right). \tag{2.55}$$

Considering small amplitude electrostatic waves, we may linearize equations (2.55), (2.52), (2.53) and (2.54) with sinusoidal variables, to obtain the following equations relating ω and k

$$n_{ph1} = -n_{pho}\phi_1, \tag{2.56}$$

$$-i\omega n_{jc1} + ikn_{jco}u_{jc1} = 0, \tag{2.57}$$

$$-i\omega u_{jc1} = -Z_j ik\phi_1, \tag{2.58}$$

and

$$-k^2\phi_1 = n_{ec1} - n_{pc1} - n_{ph1}. \tag{2.59}$$

Equations (2.57) and (2.58) give

$$n_{jc1} = Z_j\frac{k^2}{\omega^2}n_{jco}\phi_1. \tag{2.60}$$

Substituting (2.56) and (2.60) into (2.59) we obtain

$$k^2 = n_{eco}\frac{k^2}{\omega^2} + n_{pco}\frac{k^2}{\omega^2} - n_{pho},$$

which may be rearranged to yield the dispersion relation

$$\omega^2 = \left(\frac{n_{eco} + n_{pco}}{k^2 + n_{pho}}\right) k^2,$$

or

$$\omega^2 = \left(\frac{1 + 2\frac{n_{pco}}{n_{pho}}}{1 + \frac{k^2}{n_{pho}}}\right) k^2, \tag{2.61}$$

bearing in mind that $n_{eco} = n_o = n_{pco} + n_{pho}$.

Again for small k^2, this is an acoustic mode with sound speed $v_s = v_h\left(1 + 2\frac{n_{eco}}{n_{pco}}\right)^{\frac{1}{2}}$. This may seem to differ from the original dispersion relation obtained by Srinivas *et al.* (Srinivas, Popel & Shukla 1996)

$$\omega^2 = \frac{c_a^2 k^2}{1 + k^2\lambda_d^2}, \tag{2.62}$$

with $c_a^2 = \frac{T_h}{m}\left(1 + 2\frac{n_{pco}}{n_{pho}}\right)$, and $\lambda_d = \left(\frac{T_h}{4\pi n_{pho} e^2}\right)^{\frac{1}{2}}$. However the disparity occurs simply because equation (2.61) is normalized. In unnormalized terms (cf. equation (2.12) and (2.1)), (2.61) becomes

$$\frac{\omega^2}{\omega_p^2} = \frac{\left(1 + 2\frac{n_{pco}}{n_{pho}}\right)}{1 + \frac{k^2\lambda_{Dh}^2}{\frac{n_{pho}}{n_o}}} k^2\lambda_{Dh}^2,$$

which reduces directly to (2.62), since $\omega_p^2\lambda_{Dh}^2 = \frac{T_h}{m} = v_h^2$.

2.3.5 The symmetric four component model

Finally we consider linear waves in the four component electron-positron plasma described in Section 2.2. Again it is assumed (Watanabe & Taniuti 1977) that the hot species thermal velocity v_h is very much greater than the phase velocity, $v_\phi = \frac{\omega}{k}$, of a waveform supported by the plasma, and the cool species thermal velocity is very much less than this phase velocity, that is,

$T_h \gg T_c$ and $v_h \gg v_\phi \gg v_c$. This assumption is justified in light of the large energy difference between the first and second generation EP plasma in the magnetosphere (cf. energy relation equation (1.4) in Section 1.1.4.)

Consider the fluid equations for the hot and cool species

$$n_{jh} = N_h \exp(-Z_j\phi), \tag{2.63}$$

$$\frac{\partial u_{jc}}{\partial t} + u_{jc}\frac{\partial u_{jc}}{\partial x} + 3T_c\frac{n_{jc}}{N_c^2}\frac{\partial n_{jc}}{\partial x} = -Z_j\frac{\partial \phi}{\partial x}, \tag{2.64}$$

$$\frac{\partial n_{jc}}{\partial t} + \frac{\partial}{\partial x}(n_{jc}u_{jc}) = 0, \tag{2.65}$$

with $j = e, p$ and $Z_j = \frac{q}{e}$; and Poisson's equation is

$$\frac{\partial^2 \phi}{\partial x^2} = (n_{eh} + n_{ec}) - (n_{ph} + n_{pc}). \tag{2.66}$$

Assuming that the amplitude of oscillation is small, we may use the process of linearization (Chen 1984) in which higher order amplitude factors are neglected. All variables are expressed in terms of their equilibrium value, and a perturbation, denoted by a subscript $_1$

$$n_{jk} = n_{jko} + n_{jk1}, \quad u_{jk} = u_{jko} + u_{jk1}, \quad \phi = \phi_o + \phi_1. \tag{2.67}$$

We assume an electrically neutral, stationary, uniform plasma at equilibrium before perturbation, thus

$$u_{jko} = \phi_o = 0, \quad \tfrac{\partial n_{jko}}{\partial x} = \tfrac{\partial n_{jko}}{\partial t} = 0. \tag{2.68}$$

The exponential term in (2.63) may be expanded so that

$$n_{jh} = N_h(1 - Z_j\phi + \cdots),$$

and with (2.67) and (2.68) this becomes

$$\begin{aligned} &(N_h + n_{jh1}) = N_h - Z_jN_h\phi_1 \\ &\Rightarrow n_{jh1} = -Z_jN_h\phi_1. \end{aligned} \tag{2.69}$$

Expanding (2.64), (2.65) and (2.66) with (2.67) and neglecting quadratic and higher order terms gives

$$\frac{\partial u_{jc1}}{\partial t} + \frac{3T_c}{N_c}\frac{\partial n_{jc1}}{\partial x} = -Z_j\frac{\partial \phi_1}{\partial x}, \tag{2.70}$$

$$\frac{\partial n_{jc1}}{\partial t} + N_c \frac{\partial u_{jc1}}{\partial x} = 0, \tag{2.71}$$

and

$$\frac{\partial^2 \phi_1}{\partial x^2} = (N_h + n_{eh1}) + (N_c + n_{ec1}) - (N_h + n_{ph1}) - (N_c + n_{pc1}). \tag{2.72}$$

Thus

$$\frac{\partial^2 \phi_1}{\partial x^2} = (n_{eh1} + n_{ec1}) - (n_{ph1} + n_{pc1}). \tag{2.73}$$

The perturbations are assumed to be sinusoidal, that is we consider a Fourier mode (ω, k)

$$\begin{aligned} u_{jc1} &= \tilde{u_{jc1}} \exp[i(kx - \omega t)], \\ n_{jc1} &= \tilde{n_{jc1}} \exp[i(kx - \omega t)], \\ \phi_1 &= \tilde{\phi_1} \exp[i(kx - \omega t)], \end{aligned} \tag{2.74}$$

where as usual the tilde denotes the amplitude, and is omitted for simplicity.

Substituting (2.74) into equations (2.70)–(2.73) yields

$$-i\omega u_{jc1} + ik3T_c \frac{n_{jc1}}{N_c} = -Z_j ik\phi_1, \tag{2.75}$$

$$-i\omega n_{jc1} + ikN_c u_{jc1} = 0 \quad \Rightarrow u_{jc1} = \frac{\omega}{k}\frac{n_{jc1}}{N_c}, \tag{2.76}$$

$$-k^2\phi_1 = (n_{eh1} + n_{ec1}) - (n_{ph1} + n_{pc1}). \tag{2.77}$$

Further substitution of (2.76) into (2.75) gives

$$\frac{n_{jc1}}{N_c} = \frac{Z_j \phi_1}{\frac{\omega^2}{k^2} - 3T_c}. \tag{2.78}$$

Finally substituting (2.78) and (2.69) into (2.77) gives

$$-k^2\phi_1 = N_h\phi_1 - \frac{N_c\phi_1}{\frac{\omega^2}{k^2} - 3T_c} + N_h\phi_1 - \frac{N_c\phi_1}{\frac{\omega^2}{k^2} - 3T_c},$$

that is,

$$k^2 = \frac{2N_c}{\frac{\omega^2}{k^2} - 3T_c} - 2N_h, \tag{2.79}$$

where $k = k'\lambda_{Dh}$ is the normalized wavenumber and $\omega = \frac{\omega'}{\omega_p}$ the normalized frequency, with primes referring to unnormalized variables. Rearranging (2.79) we obtain the expression

$$\omega^2 = \left(\frac{2N_c}{k^2 + 2N_h} + 3T_c\right) k^2. \tag{2.80}$$

We may rearrange equation (2.80) such that

$$\omega^2 = k^2 \left(\frac{N_c}{N_h} \left(1 + \frac{k^2}{2N_h} \right)^{-1} + 3T_c \right).$$

For k small so that $k^2 < 2N_h$ binomial expansion yields

$$\omega^2 = k^2 \left[\frac{N_c}{N_h} \left(1 - \frac{k^2}{2N_h} \right) + 3T_c \right], \tag{2.81}$$

$$\omega^2 = \left(\frac{N_c}{N_h} + 3T_c \right) k^2 - \frac{N_c}{2N_h^2} k^4. \tag{2.82}$$

In the very long wavelength limit for $k \to 0$ and with $V = \frac{v_s}{v_h}$ as the normalized sound speed, this equation reduces to

$$\omega = Vk,$$

where

$$V = \left(\frac{N_c}{N_h} + 3T_c \right)^{\frac{1}{2}}. \tag{2.83}$$

Equation (2.82) is in the form of the general acoustic dispersion relation (Chen 1984),

$$\omega^2 = v_s k^2 - \gamma k^4,$$

for γ some constant, and v_s, the sound speed of the acoustic wave, $\frac{\omega}{k} = v_s$ in the long wavelength limit $k \to 0$. This form of the dispersion relation is directly related to the nonlinear Korteweg-de Vries equation which will be discussed in Chapter 3.

Figure 2.3 shows a plot of ω versus k for equation (2.80), for $N_c = 0.2, N_h = 0.8$ and $T_c = 10^{-4}$ (cf. energy relation equation (1.4), recalling that T_c is in fact the ratio $\frac{T_c}{T_h}$ in unnormalized terms). It can be seen, as in the case of single hot and cool species, that at long wavelengths ($k \to 0$), all the harmonics travel at the normalized sound speed V.

The condition $k^2 < 2N_h$ which allows the binomial expansion in equation (2.81) is inherent in the requirement that ω be real, for small T_c. For a general acoustic dispersion relation

$$\omega^2 = ak^2 - bk^4,$$

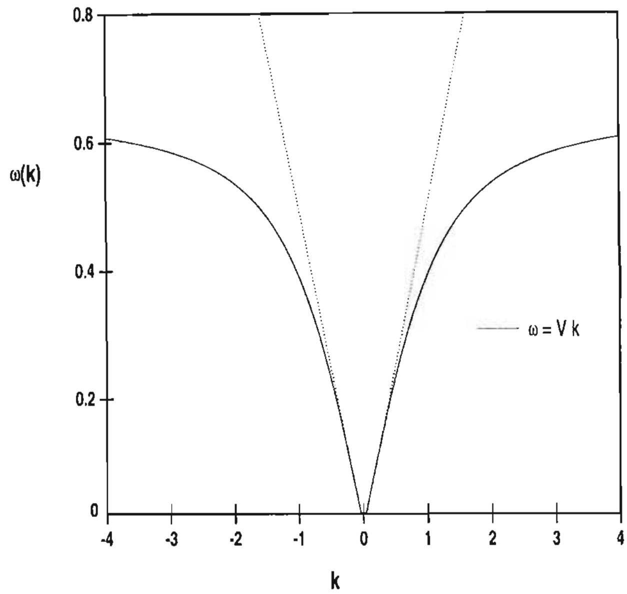


Figure 2.3: A plot of the dispersion relation for linear waveforms supported by a four component electron-positron plasma of hot Boltzmann electrons and positrons and cool adiabatic electrons and positrons, c.f. equation (2.80) (solid curve), with $N_c = 0.2, N_h = 0.8$ and $T_c = 10^{-4}$. The dotted curve shows that in the limit as $k \to 0$ the dispersion relation reduces to $\omega = Vk$, with V given by (2.83).

$k < \left(\frac{a}{b}\right)^{\frac{1}{2}}$ for real ω. Specifically for equation (2.82)

$$k^2 < 2N_h + 6T_c\frac{N_h^2}{N_c}.$$

In the limit as T_c tends to zero, we have $k^2 < 2N_h$, that is, a small k limit.

It is evident that the dispersion relation (2.80) is dependent on the cool electron-positron equilibrium density, N_c, ($N_h = 1 - N_c$). An interesting anomaly occurs if we assume $N_c = 0$, that is, that there are no cool adiabatic particles. Equation (2.80) then reduces to $\omega = \sqrt{3T_c}k$, where ω depends on the temperature of the non-existent cool particles! To resolve this apparent contradiction it is necessary to return to the initial form of the dispersion relation as derived from the linearization process, equation (2.79)

$$k^2 = \frac{2N_c}{\frac{\omega^2}{k^2} - 3T_c} - 2N_h.$$

When N_c is zero, this becomes $k^2 = -2N_h$ (actually $k^2 = -2$ since we would have $N_h = n_{pho} = n_{eho} = n_o$), implying ω undefined. This is just the case of a two component hot plasma, of electrons and positrons, discussed earlier. It was noted that such low phase velocity waves will not occur due to the lack of inertia of the hot, isothermal, Boltzmann particles. Thus when there are no cool particles we get a breakdown of wave formation in the plasma.

On the other hand, if there are no hot particles, that is, the hot equilibrium number density N_h is zero, ($N_c = 1$), equation (2.80) becomes

$$\omega^2 = 2 + 3T_c k^2, \tag{2.84}$$

which is a plasma-like wave similar to equation (2.36) for the case of cool adiabatic electrons and positrons. Since there are no hot particles, normalization with respect to T_h is invalid. Normalizing with respect to T_c gives

$$\omega^2 = 2 + 3k^2,$$

which is simply equation (2.36). In the absence of hot particles the four component model reduces to the two species cool, adiabatic case.

Let us consider the case of N_c or N_h tending to zero, that is, there are still very small quantities of hot or cool particles. Consider equation (2.80) with N_c tending to zero, assuming the existence of a small number of cool

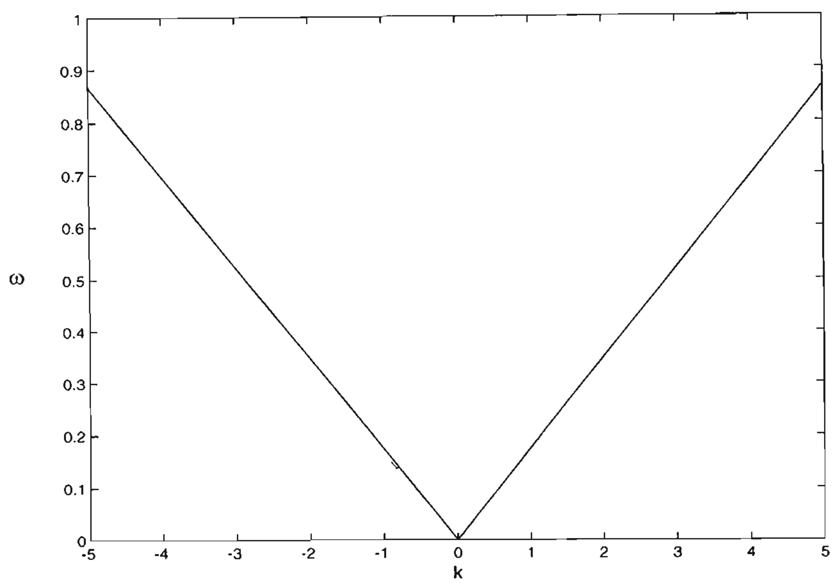


Figure 2.4: As the number density of the cool particles N_c, tends to zero the plasma supports a purely acoustic mode and (2.80) reduces to $\omega = \sqrt{(3T_c)}k$. Specific values of $T_c = 10^{-2}$ and $N_c = 10^{-3}$ have been used in (2.80).

particles. If we take N_c to be of order 10^{-3} with $T_c = 10^{-2}$, we obtain Figure 2.4.

This shows that as the number density of the cool particles tends to zero, the plasma supports a purely acoustic wave with a sound speed dependent on the cool species temperature, $v_s = \sqrt{3T_c}v_h$.

As N_h tends to zero, the plasma begins to support a plasma-like wave. Figures 2.5 and 2.6 show a plot of the dispersion relation with $N_h = 10^{-3}$, $T_c = 10^{-2}$ and $N_h = 10^{-6}$, $T_c = 10^{-2}$ respectively. We still have an acoustic-like wave at small k, however the wave tends to be plasma-like at large k. The smaller N_h becomes the smaller the range of k that supports the acoustic mode.

This analysis is not valid in light of the limits imposed by the model. We have assumed (Watanabe & Taniuti 1977)

$$v_h \gg v_\phi \gg v_c$$

so that the hot species thermal velocity be very much greater than the phase velocity of a waveform, which in turn, must be very much greater than the cool species thermal velocity. In the limit as $k \to 0$ this becomes

$$v_h \gg v_s \gg v_c,$$

or for the upper bound in normalized terms,

$$1 \gg V.$$

Since equation (2.83) gives

$$V = \left(\frac{N_c}{N_h} + 3T_c\right)^{\frac{1}{2}},$$

in the limit as T_c tends to zero we require

$$\left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} \ll 1.$$

This implies that

$$N_c^{\frac{1}{2}} \ll (1 - N_c)^{\frac{1}{2}}.$$

Figure 2.7 shows a plot of $F(N_c) = N_c^{\frac{1}{2}}$ and $F(N_c) = (1 - N_c)^{\frac{1}{2}}$ versus N_c. For $N_c^{\frac{1}{2}} \ll (1 - N_c)^{\frac{1}{2}}$ to be satisfied, $N_c \ll 0.5$. Thus N_h has a lower

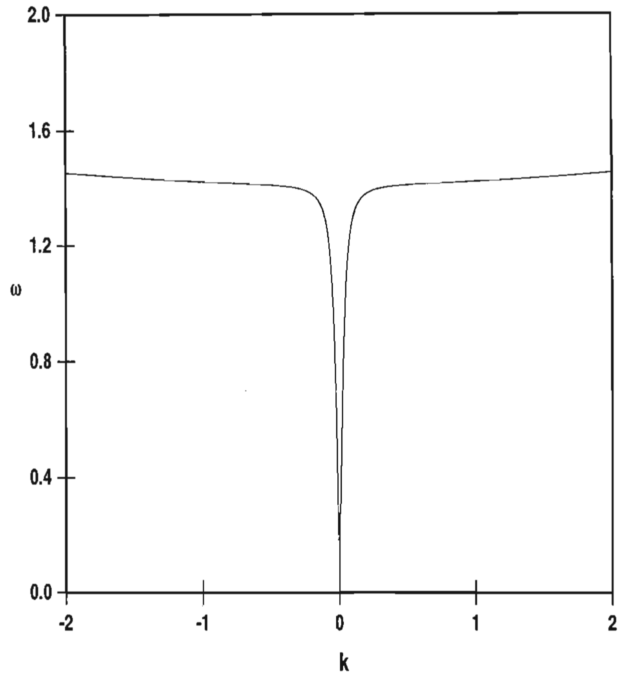


Figure 2.5: A plot of the dispersion relation (2.80) for $N_h = 10^{-3}$ at $T_c = 10^{-2}$, supporting an acoustic-like mode only for very small k.

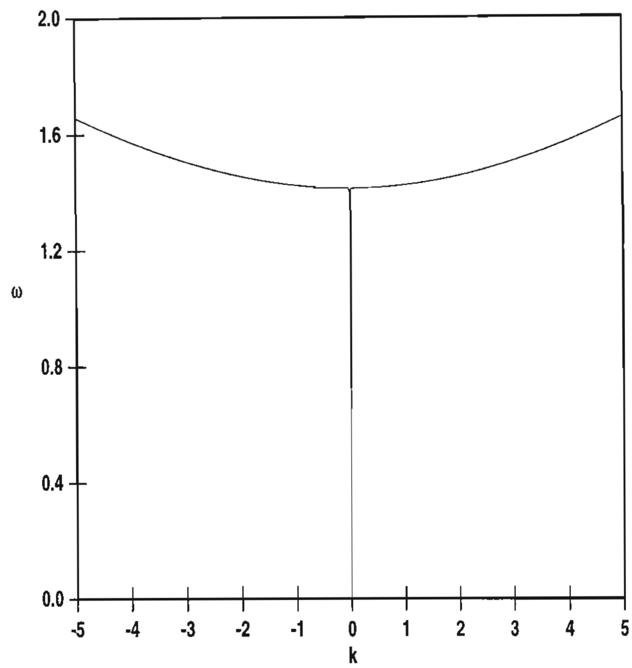


Figure 2.6: A plot of the dispersion relation (2.80) for $N_h = 10^{-6}$ at $T_c = 10^{-2}$. As the number density of the hot particles N_h decreases, so does the range of k that supports an acoustic type mode.

limit which is at most 0.5. Hence letting $N_h \to 0$ (as discussed above) leads to the breakdown of the assumption that the phase velocity is much smaller than the thermal velocity of the hot particles, and thus is invalid under the limits of the model.

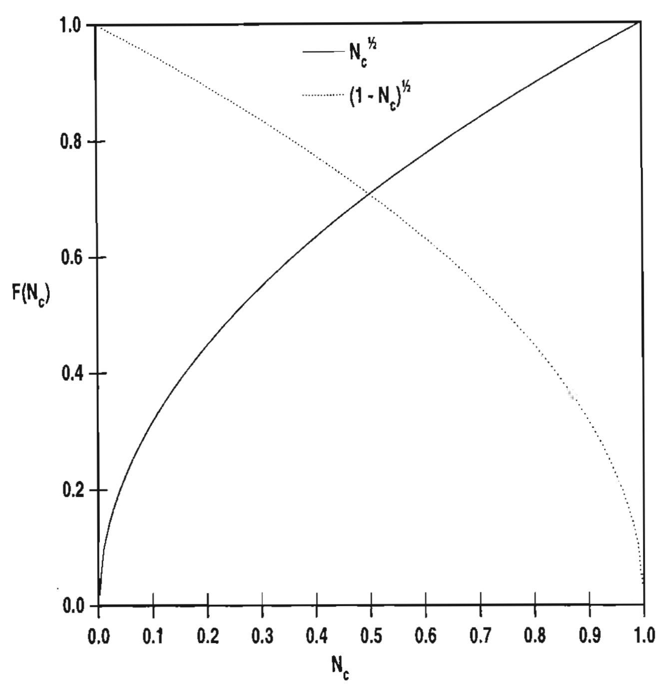


Figure 2.7: A plot of $F(N_c) = N_c^{\frac{1}{2}}$ (solid curve) and $F(N_c) = (1 - N_c)^{\frac{1}{2}}$ (dotted curve) versus N_c, showing that for $N_c^{\frac{1}{2}} \ll (1 - N_c)^{\frac{1}{2}}$, $N_c \ll 0.5$.

Chapter 3

Nonlinear electron-positron acoustic waves

3.1 Introduction

Most wave processes in nonlinear dispersive media can be represented as a competition between nonlinear and dispersive effects in the evolution of an initial disturbance. Nonlinearity, the dependence of the behaviour of a wavepacket on its amplitude, results in the generation of harmonics with greater wavenumbers, revealing itself in a steepening of the wavepacket, ultimately leading to 'wavebreaking': the collapse or breakdown of the wave.

Dispersion, the dependence of the phase velocity, $\frac{\omega}{k}$, of a component wave, on its wavenumber k, results in the spreading of the wavepacket because of phase mixing of the harmonics in time. Dispersion causes each new harmonic generated by nonlinearity to travel at a different speed. This phase mixing suppresses the infinite steepening of an initial disturbance which would otherwise result in a nonlinear dispersionless medium.

A detailed equilibrium between nonlinear and dispersive effects is possible in the acoustic wave mode, resulting in stable wave profiles that propagate unchanged with time. These are acoustic solitons. It would appear perhaps that such a balance is so critical that the formation of such profiles would be rare, however the balance leads to stability, so that any initial finite disturbance in a nonlinear dispersive medium will generally form an ordered train of solitons, in the limit as $t \to \infty$.

3.2 Nonlinearity, dispersion and dissipation

As an introduction to the physics behind the Korteweg de Vries equation we include a review of nonlinear, dispersive and dissipative systems as laid out in Drazin & Johnson (1989). It is common practice to develop the concepts of wave propagation from the simplest, although idealized, model for one dimensional wave motion

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \tag{3.1}$$

where $u(x,t)$ is the amplitude of the wave and c is a positive constant. This equation has a simple, well known general solution, expressed in terms of characteristic variables $(x \pm ct)$ as

$$u(x,t) = f(x-ct) + g(x+ct), \tag{3.2}$$

where f and g are arbitrary functions, determined by the initial conditions $u(x,0)$ and $\frac{\partial u(x,0)}{\partial t}$. The solution (3.2) referred to as d'Alembert's solution, describes two distinct waves; one moving to the left and one moving to the right, both at speed c. These waves do not interact with themselves nor with each other, a consequence of (3.1) being linear, and furthermore, they do not change their shape as they propagate. To be more specific, we may restrict ourselves to waves which propagate only in one direction, that is, solutions of

$$u_t + u_x = 0, \tag{3.3}$$

with $c = 1$, and where short hand notation for partial derivatives has been used. The general solution of (3.3) is

$$u(x,t) = f(x-t),$$

where f is an arbitrary function as before.

When wave equations are derived from general governing equations, certain simplifying assumptions are made, for example, that solutions do not interact with themselves or each other, and that they do not change their shape as they propagate, as mentioned above. In such extreme cases we may derive equations (3.1) or (3.3). However, if the assumptions are less extreme we may obtain equations which retain more of the physical detail, for example: dispersion, dissipation or nonlinearity.

The concept of dispersive waves is usually defined by a relationship between ω and k of the form

$$\omega = \omega(k),$$

which is known as the dispersion relation. Consider the equation

$$u_t + u_x + u_{xxx} = 0. \tag{3.4}$$

This is the simplest linear dispersive wave equation. Assuming a simple harmonic solution of the form

$$u(x,t) = e^{i(kx-\omega t)}, \tag{3.5}$$

it may be seen that $u(x,t)$ is a solution only if

$$\omega = k - k^3. \tag{3.6}$$

This is the dispersion relation, which determines $\omega(k)$ for given k. This implies that

$$kx - \omega t = k[x - (1-k^2)t],$$

which describes a wave propagating at the velocity

$$\frac{\omega}{k} = 1 - k^2.$$

Thus a single wave profile which may be represented by the sum of just two components of different wavenumber, each like solution (3.5), will change its shape as time evolves by virtue of the different velocities of the two components. To extend this interpretation to a general wavepacket we integrate over all k to yield

$$u(x,t) = \int_{-\infty}^{\infty} A(k) e^{i(kx-\omega t)} dk,$$

for some given $A(k)$, the Fourier transform of $u(x,0)$. The overall effect is to produce a wave profile which changes its shape as it moves. Since components of different wavenumber propagate at different frequencies, the profile will spread out or disperse.

In dispersive systems, the dispersion relation may give complex ω for real k. This occurs for even derivatives of u with respect to x, for example

$$u_t + u_x - u_{xx} = 0. \tag{3.7}$$

With equation (3.5) this gives the dispersion relation

$$\omega = k - ik^2.$$

So

$$kx - \omega t = kx - (k - ik^2)t,$$

and

$$u(x,t) = e^{-k^2 t} e^{ik(x-t)},$$

is a solution of equation (3.7). This describes a wave which propagates at a speed of unity for all k, but which also decays exponentially for any real k, as $t \to \infty$. This decay is known as dissipation. Thus we could have an equation of both even and odd derivatives, describing both a dispersive and dissipative medium. In this case the harmonic wave solution would exhibit both dispersive and dissipative properties. Not only would the phase velocity of the component wave depend on its wavelength (or wavenumber), but so would its effective amplitude, which would be attenuated with time if $\mathcal{I}(\omega) < 0$. If $\mathcal{I}(\omega) > 0$, the effective amplitude of the component wave will grow without bound with time, defining an instability.

If $\omega = \omega(k)$ is a real function of a real k then neither dissipation nor instability occurs. This pure dispersion is the case for the system defined in Chapter 2: nonlinear electron-positron acoustic waves in an unmagnetized plasma, modelled under fluid theory. This is not to say that our EP acoustic waves will not be damped, but rather that fluid theory fails to give an accurate description of the damping mechanism. Under a kinetic analysis of the four component EP plasma one would indeed obtain an imaginary component in the dispersion relation, however this is beyond the scope of this thesis.

Most wave equations like (3.1) and (3.3) are valid only for sufficiently small amplitudes. If some account is taken of the amplitude, for a better approximation to the true physical nature of wave motion, we may obtain the nonlinear wave equation

$$u_t + (1 + u)u_x = 0. \tag{3.8}$$

This equation embodies the simplest kind of nonlinearity, uu_x. The general solution of (3.8) is

$$u(x,t) = f[x - (1 + u)t], \tag{3.9}$$

where f is an arbitrary function. Now, given a wave profile $u(x,0) = f(x)$ we must solve (3.9) for u. We obtain a single valued solution for u only for a finite time, thereafter the solution will be non-unique. This multi-valued solution can be described as a wave which has 'broken'. As time increases the wave steepens, until its front becomes vertical. Thereafter the solution is triple valued. Thus the solution must change its shape as it propagates. What is occurring is the generation of higher order harmonics causing wave steepening and finally wave breaking.

By making suitable assumptions in a given physical problem we may obtain an equation containing both nonlinear and dispersive terms. The simplest equation embodying nonlinearity and dispersion is the Korteweg-de Vries equation (Korteweg & de Vries 1895),

$$u_t + auu_x + bu_{xxx} = 0.$$

3.3 Small amplitude nonlinear EP acoustic waves

Among wide classes of general dispersive weakly nonlinear systems, when the linearized dispersion relation has the form

$$\omega = ak + bk^3 + \cdots$$

for small values of k, where a and b are real constants, then the original system of nonlinear equations can often be reduced, in the small wavenumber limit, to the Korteweg-de Vries equation, or in special cases to the modified Korteweg-de Vries equation, if one considers small, finite amplitude waves (Jeffery & Kakutani 1972).

3.3.1 The Korteweg-de Vries equation

The theory of Korteweg and de Vries (1895) has had a decisive influence on the development of nonlinear wave theory. Korteweg and de Vries proposed that the wave equations of a physical system may be reduced to a much simpler form, while still retaining the essential features of the nonlinear phenomenon. It is only necessary to tackle nonlinear and dispersive terms with the same degree of accuracy.

Classical ion-electron plasmas propagate longitudinal, plane waves at or near a characteristic sound speed v_s (acoustic type waves). It can be seen through the linear wave analysis of the various models of EP plasma in Chapter 2, that given a medium consisting of species of slow and fast moving particles [1] we obtain acoustic-like waves with a dispersion relation given in the form of

$$w(k) = Vk, \tag{3.10}$$

in the limit as $k \to 0$, where $V = \frac{v_s}{v_h}$ represents the normalized electron-positron sound speed. This indicates that at sufficiently long wavelengths the waves all propagate at the same phase and group velocity. Equation (3.10) has the form of the general dispersion relation for all linear, acoustic-type waves in isotropic media, in the limit as $k \to 0$.

Thus linear, long-wavelength acoustic modes propagate with little or no dispersion and the group velocity $\frac{\partial \omega}{\partial k}$ is equal to the phase velocity, $\frac{\omega}{k}$ such that

$$\frac{\partial \omega}{\partial k} = v_s = \frac{\omega}{k}.$$

Dispersion will, however, exist for k other than very small. We note that equation (2.82) gives the general linear dispersion relation for the four component electron-positron plasma model to be

$$\omega^2 = \left(\frac{N_c}{N_h} + 3T_c\right) k^2 - \frac{N_c}{2N_h^2} k^4,$$

for $k^2 < 2N_h$. In the limit as $T_c \to 0$ this becomes

$$\omega^2 = \left(\frac{N_c}{N_h}\right) k^2 - \left(\frac{N_c}{N_h}\right) \frac{1}{2N_h} k^4,$$

or in unnormalized terms,

$$\omega^2 = \left(\frac{N_c}{N_h}\right) \omega_p^2 \lambda_{Dh}^2 k^2 \left[1 - \left(\frac{N_o}{2N_h}\right) \lambda_{Dh}^2 k^2\right]. \tag{3.11}$$

[1] In order to obtain acoustic-like waves it is required that the particles in the medium exhibit non-symmetric motion. Analogous to the ion acoustic oscillation, we must have an ion-like "heavy" species, and an electron-like "light" species. In the case of multi-species electron-positron plasmas, both electrons and positrons have the same mass. Thus it is necessary to assume one species, at least, to be hot and thus Boltzmann distributed, effectively inertialess, and one species, at least, to be cool and adiabatic, and thus "massive".

This equation has the form of the general small k dispersion relation for acoustic waves (Chen 1984)

$$\omega^2 = v_s^2 k^2 - \gamma^2 k^4, \tag{3.12}$$

with $v_s^2 = \left(\frac{N_c}{N_h}\right) \omega_p^2 \lambda_{Dh}^2 = \left(\frac{N_c}{N_h}\right) v_h^2$ and $\gamma^2 = \frac{N_o}{2N_h} \left(\frac{N_c}{N_h}\right) \lambda_{Dh}^2 v_h^2$.

Equation (3.12) reduces to

$$\omega = v_s k - (\frac{\gamma^2}{2v_s}) k^3 + ...$$

for $\frac{\gamma^2}{v_s^2} k^2 < 1$. Specifically, equation (3.11), in the binomial limit $(\frac{N_o}{2N_h} k^2 \lambda_{Dh}^2 < 1)$, is

$$\omega = \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} v_h k - \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} \left(\frac{N_o}{4N_h}\right) v_h \lambda_{Dh}^2 k^3.$$

This suggests the following differential equation for $u(x, t)$

$$\frac{\partial u}{\partial t} + v_s \frac{\partial u}{\partial x} + \frac{\gamma^2}{2v_s} \frac{\partial^3 u}{\partial x^3} = 0,$$

which has the form of (3.4), a linear, purely dispersive wave equation, leading to unchecked spreading of the initial profile $A(k) = u(x, 0)$.

If we replace the second term by the more general convective derivative, introducing a nonlinear term in $u(x, t)$, we may combat the dispersion. That is, in general form,

$$a_1 \frac{\partial u}{\partial t} + a_2 u \frac{\partial u}{\partial x} + a_3 \frac{\partial^3 u}{\partial x^3} = 0.$$

This is known as the Korteweg-de Vries (KdV) equation, including dispersion and nonlinearity, allowing us to hope for a stationary, pulse-like solution to exist, if the nonlinearity, which leads to wave steepening, can just counteract the dispersion. Once steepening of waves occurs, the higher order derivative (dispersive) term begins to play a role (Jeffrey & Kakutani 1972), acting as a check or balancing effect against the nonlinear steepening of waves, resulting in an eventual steady state.

3.3.2 Derivation of the KdV equation

We will assume weak nonlinearity, so that the amplitude of the electrostatic potential is small, $|\ \phi\ | < 1$, and that the perturbations in the densities and velocities are all small compared to unity. Employing reductive perturbation techniques based on that of Washimi & Taniuti (1966), we may derive the evolutionary equations describing weakly nonlinear electron-positron acoustic waves from the system of equations (2.63–2.66) describing the four component fluid model.

Coordinate stretching

We follow the method of Leroy (1989) whereby a dispersive parameter $\beta^2 = k^2\lambda_{Dh}^2$, and an amplitude parameter ϵ is introduced. Under the restrictions of weakly nonlinear, long wavelength waves, we consider small amplitude waves, so that the perturbations from equilibrium are small in comparison to unity, taking into account only the first order $O(\epsilon)$ deviation from linearity, and weak dispersion, such that $\beta^2 \ll 1$, so that both nonlinear and dispersive effects are of the same order of magnitude, that is $\beta^2 = O(\epsilon)$ (Leroy 1989).

We use a similar technique to that of Mace (1991), in order to recover the specific coordinate stretchings required in the reductive perturbation technique for the case of a four component electron-positron plasma. We may write the linear dispersion relation (2.80) in the limit as $T_c \to 0$ as

$$\omega = \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} \left(1 + \frac{k^2}{2N_h}\right)^{-\frac{1}{2}} k. \tag{3.13}$$

In the binomial limit $\frac{k^2}{2N_h} < 1$ this reduces to

$$\omega = \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} \left(k - \frac{k^3}{4N_h}\right).$$

If we consider plane waves, propagating from left to right in the x direction, so that the phase argument is $kx - \omega(k)t$, then

$$\begin{aligned} kx - \omega t &= kx - \left(\tfrac{N_c}{N_h}\right)^{\frac{1}{2}} \left(k - \tfrac{k^3}{4N_h}\right) t \\ &= k\left[x - \left(\tfrac{N_c}{N_h}\right)^{\frac{1}{2}} t\right] + \left(\tfrac{N_c}{N_h}\right)^{\frac{1}{2}} \tfrac{1}{4N_h} k^3 t. \end{aligned} \tag{3.14}$$

In unnormalized terms the above equation becomes

$$kx - \omega t = k\lambda_{Dh}\left[\frac{x}{\lambda_{Dh}} - \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} t\omega_p\right] + \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} \frac{N_o}{4N_h} k^3\lambda_{Dh}^3 t\omega_p, \tag{3.15}$$

where $\lambda_{Dh} = \left(\frac{T_h}{4\pi N_o e^2}\right)^{\frac{1}{2}}$, and $\omega_p = \left(\frac{4\pi N_o e^2}{m}\right)^{\frac{1}{2}}$. Recalling that $V = \frac{v_s}{v_h} = \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}}$, in the limit as $T_c \to 0$, we have

$$kx - \omega t = \beta\left[\frac{x}{\lambda_{Dh}} - Vt\omega_p\right] + \beta^3 \frac{N_o}{4N_h} Vt\omega_p,$$

which implies the following coordinate stretchings

$$\xi = \beta(x - Vt), \quad \tau = \beta^3 Vt. \tag{3.16}$$

The parameter β, a measure of the wave dispersion is considered small, $\beta^2 \ll 1$ (Leroy 1989).

Now, the densities, velocities and electrostatic potential may be expanded in terms of an equilibrium value, and a nonlinear perturbation. Higher order amplitude factors are written in terms of a specific variable ϵ, defining their 'strength'. If we choose ϵ so that $\beta^2 \sim O(\epsilon)$, we imply a relation between the nonlinearity and the dispersion of the wavepacket, linking the opposing effects so that a balance is attained. We may then write (3.16) as

$$\xi = \epsilon^{\frac{1}{2}}(x - Vt), \quad \tau = \epsilon^{\frac{3}{2}} Vt, \tag{3.17}$$

where ξ and τ are referred to as slow variables since it requires a large change in x and t in order to change ξ and τ appreciably ($\epsilon \ll 1$). This coordinate stretching follows that of Verheest (1988).

Basic equations

An infinite, collisionless, unmagnetized electron-positron plasma with four fluid components is described by the following one-dimensional, normalized equations. The hot components are described by Maxwell-Boltzmann distributions, their densities given by

$$n_{eh} = N_h \exp(\phi), \qquad n_{ph} = N_h \exp(-\phi), \tag{3.18}$$

which may be written in terms of an exponential power series

$$\begin{aligned} n_{eh} &= N_h\left(1 + \phi + \tfrac{1}{2}\phi^2 + \cdots\right) \\ n_{ph} &= N_h\left(1 - \phi + \tfrac{1}{2}\phi^2 - \cdots\right). \end{aligned} \tag{3.19}$$

The cool fluid components are described by the fluid equations of continuity and motion, with $\gamma_c = 3$,

$$\frac{\partial n_{jc}}{\partial t} + \frac{\partial}{\partial x}\left(n_{jc}u_{jc}\right) = 0, \tag{3.20}$$

$$\frac{\partial u_{jc}}{\partial t} + u_{jc}\frac{\partial u_{jc}}{\partial x} + 3T_c\frac{n_{jc}}{N_c^2}\frac{\partial n_{jc}}{\partial x} = -Z_j\frac{\partial \phi}{\partial x}, \tag{3.21}$$

where $Z_j = \frac{q_j}{e}$. The system of equations is closed by the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} = n_{eh} - n_{ph} + \sum_j -Z_j n_{jc}. \tag{3.22}$$

These equations are normalized as stated in (2.12).

The expanded form of the densities, velocities and electrostatic potential is given by

$$\begin{array}{rcl} n_{jc} & = & N_c + \epsilon n_{jc1} + \epsilon^2 n_{jc2} + \cdots \\ u_{jc} & = & \epsilon u_{jc1} + \epsilon^2 u_{jc2} + \cdots \\ \phi & = & \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots \end{array} \tag{3.23}$$

noting that $\phi_o = u_{jco} = 0$ and $n_{jco} = N_c$ at equilibrium. The deviations of the macroscopic quantities from their equilibrium values are all at most of order ϵ, satisfying the requirement that perturbations are small compared to unity.

The reductive perturbation technique

We follow the reductive perturbation technique (Washimi & Tanuiti 1966), (Baboolal, Bharuthram & Hellberg 1989) and employ the spatial and temporal stretched variables (3.17) (Verheest 1988)

$$\xi = \epsilon^{\frac{1}{2}}\left(x - Vt\right), \qquad \tau = \epsilon^{\frac{3}{2}}Vt,$$

implying the following transformations

$$\frac{\partial}{\partial x} = \epsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi}; \qquad \frac{\partial}{\partial t} = -\epsilon^{\frac{1}{2}}V\frac{\partial}{\partial \xi} + \epsilon^{\frac{3}{2}}V\frac{\partial}{\partial \tau},$$

to equations (3.20), (3.21) and (3.22) to obtain

$$-\epsilon^{\frac{1}{2}}V\frac{\partial n_{jc}}{\partial \xi} + \epsilon^{\frac{3}{2}}V\frac{\partial n_{jc}}{\partial \tau} + \epsilon^{\frac{1}{2}}\frac{\partial}{\partial \xi}\left(n_{jc}u_{jc}\right) = 0, \tag{3.24}$$

$$-\epsilon^{\frac{1}{2}}V\frac{\partial u_{jc}}{\partial \xi}+\epsilon^{\frac{3}{2}}V\frac{\partial u_{jc}}{\partial \tau}+u_{jc}\epsilon^{\frac{1}{2}}\frac{\partial u_{jc}}{\partial \xi}+\frac{3T_c}{N_c^2}\epsilon^{\frac{1}{2}}n_{jc}\frac{\partial n_{jc}}{\partial \xi}=-Z_j\epsilon^{\frac{1}{2}}\frac{\partial \phi}{\partial \xi}, \quad (3.25)$$

$$\epsilon\frac{\partial^2\phi}{\partial\xi^2}=2N_h\left(\phi+\frac{1}{3!}\phi^3+\cdots\right)-\sum_j Z_j n_{jc}, \quad (3.26)$$

where we have expanded the hot particle densities in terms of (3.19) in the transformed Poisson equation.

Expressing (3.24),(3.25) and (3.26) in terms of the expanded quantities (3.23) we obtain

$$\begin{aligned}
&-\epsilon^{\frac{1}{2}}V\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\xi}+\cdots\right]+\epsilon^{\frac{3}{2}}V\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\tau}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\tau}+\cdots\right]\\
&+\epsilon^{\frac{1}{2}}\left(N_c+\epsilon n_{jc1}+\epsilon^2 n_{jc2}+\cdots\right)\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\xi}+\cdots\right]\\
&+\epsilon^{\frac{1}{2}}\left(\epsilon u_{jc1}+\epsilon^2 u_{jc2}+\cdots\right)\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\xi}+\cdots\right]\\
&=0,
\end{aligned} \quad (3.27)$$

$$\begin{aligned}
&-\epsilon^{\frac{1}{2}}V\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\xi}+\cdots\right]+\epsilon^{\frac{3}{2}}V\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\tau}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\tau}+\cdots\right]\\
&+\epsilon^{\frac{1}{2}}\left(\epsilon u_{jc1}+\epsilon^2 u_{jc2}+\cdots\right)\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\xi}+\cdots\right]\\
&+\tfrac{3T_c}{N_c^2}\epsilon^{\frac{1}{2}}\left(N_c+\epsilon n_{jc1}+\epsilon^2 n_{jc2}+\cdots\right)\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\xi}+\cdots\right]\\
&=-Z_j\epsilon^{\frac{1}{2}}\left[\epsilon\tfrac{\partial\phi_1}{\partial\xi}+\epsilon^2\tfrac{\partial\phi_2}{\partial\xi}+\cdots\right],
\end{aligned} \quad (3.28)$$

$$\begin{aligned}
&\epsilon\left[\epsilon\tfrac{\partial^2\phi_1}{\partial\xi^2}+\epsilon^2\tfrac{\partial^2\phi_2}{\partial\xi^2}+\cdots\right]=2N_h\left(\epsilon\phi_1+\epsilon^2\phi_2+\cdots\right)\\
&+\tfrac{2N_h}{3!}\left(\epsilon^3\phi_1^3+3\epsilon^4\phi_1^2\phi_2+3\epsilon^5\phi_1\phi_2^2+\epsilon^6\phi_2^3+\cdots\right)\\
&-\textstyle\sum_j Z_j\left(N_c+\epsilon n_{jc1}+\epsilon^2 n_{jc2}+\cdots\right).
\end{aligned} \quad (3.29)$$

Solving order by order in ϵ we have from (3.29)

$$O(\epsilon^0) \quad \textstyle\sum_j Z_j N_c = 0, \quad (3.30)$$

$$O(\epsilon^1) \quad 2N_h\phi_1-\textstyle\sum_j Z_j n_{jc1} = 0, \quad (3.31)$$

$$O(\epsilon^2) \quad \tfrac{\partial^2\phi_1}{\partial\xi^2} = 2N_h\phi_2-\textstyle\sum_j Z_j n_{jc2}. \quad (3.32)$$

From (3.27) we have

$$O\left(\epsilon^{\frac{3}{2}}\right) \quad -V\frac{\partial n_{jc1}}{\partial\xi}+N_c\frac{\partial u_{jc1}}{\partial\xi}=0, \quad (3.33)$$

$$O\left(\epsilon^{\frac{5}{2}}\right) \quad -V\frac{\partial n_{jc2}}{\partial\xi}+V\frac{\partial n_{jc1}}{\partial\tau}+N_c\frac{\partial u_{jc2}}{\partial\xi}+n_{jc1}\frac{\partial u_{jc1}}{\partial\xi}+u_{jc1}\frac{\partial n_{jc1}}{\partial\xi}=0, \quad (3.34)$$

and from (3.28)

$$O\left(\epsilon^{\frac{3}{2}}\right) \quad -V\frac{\partial u_{jc1}}{\partial \xi} + \frac{3T_c}{N_c}\frac{\partial n_{jc1}}{\partial \xi} = -Z_j\frac{\partial \phi_1}{\partial \xi}, \tag{3.35}$$

$$O\left(\epsilon^{\frac{5}{2}}\right) \quad -V\frac{\partial u_{jc2}}{\partial \xi}+V\frac{\partial u_{jc1}}{\partial \tau}+u_{jc1}\frac{\partial u_{jc1}}{\partial \xi}+\frac{3T_c}{N_c}\frac{\partial n_{jc2}}{\partial \xi}+\frac{3T_c}{N_c^2}n_{jc1}\frac{\partial n_{jc1}}{\partial \xi} = -Z_j\frac{\partial \phi_2}{\partial \xi}. \tag{3.36}$$

At $O(\epsilon^0)$, equation (3.30) is simply the charge neutrality condition for a neutral EP plasma. At $O(\epsilon^{\frac{3}{2}})$, integration of equations (3.33) and (3.35) yields the relations

$$Vn_{jc1} = N_c u_{jc1},$$

and

$$-Vu_{jc1} + \frac{3T_c}{N_c}n_{jc1} = -Z_j\phi_1,$$

which may be combined to yield expressions for n_{jc1} and u_{jc1} in terms of ϕ_1 only

$$n_{jc1} = \frac{Z_j N_c}{V^2 - 3T_c}\phi_1, \tag{3.37}$$

$$u_{jc1} = \frac{Z_j V}{V^2 - 3T_c}\phi_1. \tag{3.38}$$

Substituting (3.37) into $O(\epsilon)$ equation (3.31) we have

$$\left[2N_h - \sum_j \frac{Z_j^2 N_c}{V^2 - 3T_c}\right]\phi_1 = 0, \tag{3.39}$$

which, for $\phi_1 \neq 0$, gives the $O(\epsilon)$ dispersion relation in the limit as $k \to 0$, for $V = \frac{\omega}{k}$,

$$V^2 = \frac{N_c}{N_h} + 3T_c.$$

Consider the $O(\epsilon^{\frac{5}{2}})$ equations (3.34) and (3.36). Rearranging (3.34) we have

$$N_c\frac{\partial u_{jc2}}{\partial \xi} = V\frac{\partial n_{jc2}}{\partial \xi} - V\frac{\partial n_{jc1}}{\partial \tau} - n_{jc1}\frac{\partial u_{jc1}}{\partial \xi} - u_{jc1}\frac{\partial n_{jc1}}{\partial \xi}. \tag{3.40}$$

Multiplying the $O(\epsilon^{\frac{5}{2}})$ equation (3.36) by N_c and rearranging we have

$$3T_c\frac{\partial n_{jc2}}{\partial \xi} = N_cV\frac{\partial u_{jc2}}{\partial \xi} - N_cV\frac{\partial u_{jc1}}{\partial \tau} - N_cu_{jc1}\frac{\partial u_{jc1}}{\partial \xi} - \frac{3T_c}{N_c}n_{jc1}\frac{\partial n_{jc1}}{\partial \xi} - Z_jN_c\frac{\partial \phi_2}{\partial \xi}. \tag{3.41}$$

Substituting (3.40) into (3.41) gives

$$\begin{aligned}&(V^2 - 3T_c)\tfrac{\partial n_{jc2}}{\partial \xi} = N_cV\tfrac{\partial u_{jc1}}{\partial \tau} + V^2\tfrac{\partial n_{jc1}}{\partial \tau} + V\tfrac{\partial}{\partial \xi}\left(n_{jc1}u_{jc1}\right)\\ &+N_cu_{jc1}\tfrac{\partial u_{jc1}}{\partial \xi} + \tfrac{3T_c}{N_c}n_{jc1}\tfrac{\partial n_{jc1}}{\partial \xi} + Z_jN_c\tfrac{\partial \phi_2}{\partial \xi}.\end{aligned} \tag{3.42}$$

Using (3.37) and (3.38) in (3.42) gives

$$\tfrac{\partial n_{jc2}}{\partial \xi} = 2Z_j\tfrac{V^2N_c}{(V^2-3T_c)^2}\tfrac{\partial \phi_1}{\partial \tau} + 3Z_j^2\tfrac{N_c(V^2+T_c)}{(V^2-3T_c)^3}\phi_1\tfrac{\partial \phi_1}{\partial \xi} + \textstyle\sum_j Z_j\tfrac{N_c}{V^2-3T_c}\tfrac{\partial \phi_2}{\partial \xi}. \tag{3.43}$$

Considering the $O(\epsilon^2)$ equation (3.32), we obtain the derivative with respect to ξ,

$$\frac{\partial^3 \phi_1}{\partial \xi^3} = 2N_h\frac{\partial \phi_2}{\partial \xi} - \sum_j Z_j\frac{\partial n_{jc2}}{\partial \xi}. \tag{3.44}$$

Substituting (3.43) into (3.44) gives

$$\begin{aligned}&\tfrac{\partial^3 \phi_1}{\partial \xi^3} = \left[2N_h - \textstyle\sum_j Z_j^2\tfrac{N_c}{V^2-3T_c}\right]\tfrac{\partial \phi_2}{\partial \xi} - \left[2\textstyle\sum_j Z_j^2\tfrac{V^2N_c}{(V^2-3T_c)^2}\right]\tfrac{\partial \phi_1}{\partial \tau}\\ &-\left[3\textstyle\sum_j Z_j^3\tfrac{N_c\left(V^2+T_c\right)}{(V^2-3T_c)^3}\right]\phi_1\tfrac{\partial \phi_1}{\partial \xi}.\end{aligned} \tag{3.45}$$

Note that the coefficient of the $\frac{\partial \phi_2}{\partial \xi}$ term is the same as in equation (3.39), which, for $\phi_1 \neq 0$, is zero. Thus equation (3.45) reduces to

$$2\sum_j Z_j^2\frac{V^2N_c}{(V^2-3T_c)^2}\frac{\partial \phi_1}{\partial \tau} + 3\sum_j Z_j^3\frac{N_c\left(V^2+T_c\right)}{(V^2-3T_c)^3}\phi_1\frac{\partial \phi_1}{\partial \xi} + \frac{\partial^3 \phi_1}{\partial \xi^3} = 0,$$

or

$$\frac{\partial \phi_1}{\partial \tau} + a\phi_1\frac{\partial \phi_1}{\partial \xi} + b\frac{\partial^3 \phi_1}{\partial \xi^3} = 0,$$

which is the Korteweg-de Vries equation, with $a = \frac{B}{A}$ and $b = \frac{1}{A}$, where

$$A = 2\sum_j Z_j^2\frac{V^2N_c}{(V^2-3T_c)^2} \qquad B = 3\sum_j Z_j^3\frac{N_c\left(V^2+T_c\right)}{(V^2-3T_c)^3}.$$

The sum in B reduces to

$$3\frac{N_c(V^2+T_c)}{(V^2-3T_c)^3}-3\frac{N_c(V^2+T_c)}{(V^2-3T_c)^3}=0,$$

which implies $a=0$ and the KdV equation thus reduces to

$$\frac{\partial\phi_1}{\partial\tau}+b\frac{\partial^3\phi_1}{\partial\xi^3}=0,$$

which is purely dispersive, with no nonlinear term.

The reason for this loss of the nonlinear term is that, using the ordering $\beta^2=O(\epsilon)$, the symmetry in mass and antisymmetry in charge between the species leads to a balance between the terms forming B.

This brings to mind the study of the Korteweg-de Vries equation for nonlinear ion-acoustic waves in a plasma containing contaminating negative ions, derived by Das & Tagare (1975). Das *et al.* discovered that an increase in the concentration of the negative ions r, causes the coefficient of the nonlinear term $\left(\phi\frac{\partial\phi}{\partial x}\right)$ to decrease, and eventually become negative at a critical value $r=r_c$. As a result, the KdV equation predicts the existence of positive (compressive) solitons when $r<r_c$, and negative (rarefactive) solitons when $r>r_c$. When $r=r_c$ the nonlinear term vanishes, so that higher order nonlinearity must be included, which results in a modified KdV equation (Watanabe 1984). The modified KdV equation has soliton solutions (Verheest 1988) which allow for the existence of both negative and positive solitons in the plasma.

3.3.3 The modified KdV equation

We thus need to consider a modified Korteweg-de Vries equation (Watanabe 1984) with a different stretching, that allows for a higher degree of nonlinearity, in order to obtain an evolutionary equation containing both quadratic and cubic nonlinear terms on an equal footing. We follow Verheest (1988), and employ a reductive perturbation technique with the following coordinate stretchings

$$\xi=\epsilon(x-Vt)\qquad\tau=\epsilon^3Vt.$$

In this case the nonlinear parameter $\beta^2=k^2\lambda_D^2=O(\epsilon^2)$. The stretchings above, allow for the incorporation of even higher wavenumber harmonics in the wavepacket, $k\lambda_{Dh}\sim\epsilon$, as opposed to the KdV stretchings which have β^2

of order ϵ. The larger wavenumbers admit stronger wave dispersion, which for balance implies a greater degree of nonlinearity (Baboolal, Bharuthram & Hellberg 1988).

The densities, velocities and electrostatic potential may then be expanded in terms of the amplitude parameter ϵ so that

$$\begin{aligned} n_{jc} &= N_c + \epsilon n_{jc1} + \epsilon^2 n_{jc2} + \epsilon^3 n_{jc3} + \ldots \\ u_{jc} &= \epsilon u_{jc1} + \epsilon^2 u_{jc2} + \epsilon^3 u_{jc3} + \ldots \\ \phi &= \epsilon\phi_1 + \epsilon^2\phi_2 + \epsilon^3\phi_3 + \ldots \end{aligned} \tag{3.46}$$

which in this case are expanded to $O(\epsilon^3)$.

Proceeding in the usual manner (Baboolal, Bharuthram & Hellberg 1988), (cf. Appendix A for a complete derivation) we obtain to $O(\epsilon^2)$

$$n_{jc1} = \tfrac{Z_j N_c}{V^2 - 3T_c}\phi_1, \quad u_{jc1} = \tfrac{Z_j V}{V^2 - 3T_c}\phi_1, \tag{3.47}$$

where V, the normalized sound speed, satisfies the long-wavelength linear dispersion relation

$$2N_h - \sum_j \frac{Z_j^2 N_c}{V^2 - 3T_c} = 0. \tag{3.48}$$

These equations (3.47) and (3.48) are expressions of the first order nonlinear perturbations in u and n, and the dispersion relation obtained previously in the derivation of the Korteweg-de Vries equation.

To $O(\epsilon^3)$ we obtain

$$n_{jc2} = \frac{Z_j N_c}{V^2 - 3T_c}\phi_2 + \frac{3Z_j^2 N_c (V^2 + T_c)}{2(V^2 - 3T_c)^3}\phi_1^2 \tag{3.49}$$

$$u_{jc2} = \frac{Z_j V}{V^2 - 3T_c}\phi_2 + \frac{Z_j^2 V (V^2 + 9T_c)}{2(V^2 - 3T_c)^3}\phi_1^2. \tag{3.50}$$

At $O(\epsilon^4)$ we obtain from the continuity equation

$$V\frac{\partial n_{jc1}}{\partial \tau} - V\frac{\partial n_{jc3}}{\partial \xi} + N_c\frac{\partial u_{jc3}}{\partial \xi} + \frac{\partial}{\partial \xi}(n_{jc1}u_{jc2}) + \frac{\partial}{\partial \xi}(n_{jc2}u_{jc1}) = 0, \tag{3.51}$$

and from the equation of momentum

$$V\frac{\partial u_{jc1}}{\partial \tau} - V\frac{\partial u_{jc3}}{\partial \xi} + \frac{\partial}{\partial \xi}(u_{jc1}u_{jc2}) + \frac{3T_c}{N_c}\frac{\partial n_{jc3}}{\partial \xi} + \frac{3T_c}{N_c^2}\frac{\partial}{\partial \xi}(n_{jc1}n_{jc2}) = -Z_j\frac{\partial \phi_3}{\partial \xi}. \tag{3.52}$$

Eliminating u_{jc3} from (3.51), and using (3.47), (3.49) and (3.50) yields an equation for $\frac{\partial n_{jc3}}{\partial \xi}$, which when substituted into the $O(\epsilon^3)$ Poisson equation, after partial differentiation with respect to ξ (increasing the order to $O(\epsilon^4)$ (Mace & Hellberg 1993a)), gives the modified Korteweg-de Vries equation

$$\frac{\partial \phi_1}{\partial \tau} + B\frac{\partial \phi_1^3}{\partial \xi} + A\frac{\partial^3 \phi_1}{\partial \xi^3} = 0,$$

where $A = \frac{1}{a}$

$$a = 2\sum_j \frac{Z_j^2 N_c V^2}{\left(V^2 - 3T_c\right)^2}$$

and $B = \frac{b}{a}$, where

$$b = \sum_j \frac{Z_j^4 N_c \left(5V^4 + 30T_c V^2 + 9T_c^2\right)}{2\left(V^2 - 3T_c\right)^5} - \frac{N_h}{3}.$$

This mKdV equation, first derived by Watanabe (1984), has a cubic nonlinearity, rather than the familiar quadratic nonlinearity of the KdV equation.

Expanding the summations and noting that

$$V = \left(\frac{N_c}{N_h} + 3T_c\right)^{\frac{1}{2}},$$

from (3.48), the coefficients may be written as

$$A = \frac{\left(\frac{N_c}{N_h}\right)^2}{4N_c\left(\frac{N_c}{N_h} + 3T_c\right)}, \tag{3.53}$$

$$B = -\frac{\left(\frac{N_c}{N_h}\right)^4 - 15\left(\frac{N_c}{N_h}\right)^2 - 180\frac{N_c}{N_h}T_c - 432T_c^2}{12\left(\frac{N_c}{N_h} + 3T_c\right)\left(\frac{N_c}{N_h}\right)^3}. \tag{3.54}$$

We note that these coefficients bare a striking resemblance to those obtained by Mace & Hellberg (1993a) in their paper on the existence of stationary electron acoustic double layers. The model they were considering is similar to ours, in that it includes hot and cool electrons and cool ions. Thus the similarity in coefficients is not unexpected.

3.3.4 Stationary solutions to the mKdV equation

In order to obtain small amplitude soliton solutions to the mKdV equation

$$\frac{\partial \phi}{\partial \tau} + B\frac{\partial \phi^3}{\partial \xi} + A\frac{\partial^3 \phi}{\partial \xi^3} = 0,$$

we follow Mace (1991) and define a stationary frame so that

$$\phi = \phi(\nu) = \phi(\xi - U\tau),$$

where U is some velocity. These solutions must satisfy the following boundary conditions

$$\phi, \quad \frac{d\phi}{d\nu}, \quad \frac{d^2\phi}{d\nu^2} \to 0, \quad \nu \to \infty.$$

to obtain a satisfactory soliton profile. Note, we have suppressed the subscript in ϕ_1 for convenience. We may write the mKdV equation in the form

$$-U\frac{d\phi}{d\nu} + B\frac{d\phi^3}{d\nu} + A\frac{d^3\phi}{d\nu^3} = 0. \tag{3.55}$$

Equation (3.55) may then be integrated twice, using the boundary conditions, to yield

$$\left(\frac{d\phi}{d\nu}\right)^2 = \frac{U}{A}\phi^2 - \frac{B}{2A}\phi^4. \tag{3.56}$$

Taking the square root of both sides of equation (3.56), and manipulating gives an expression for ν

$$\nu + c = \pm\left(\frac{A}{U}\right)^{\frac{1}{2}} \int \frac{d\phi}{\phi\left(1 - \frac{B}{2U}\phi^2\right)^{\frac{1}{2}}}. \tag{3.57}$$

for c some arbitrary constant.

With the substitution $\theta = \frac{1}{\phi}$ equation (3.57) may be cast into integrable form

$$\nu + c = \pm\left(\frac{A}{U}\right)^{\frac{1}{2}} \int \frac{d\theta}{\left(\theta^2 - \frac{B}{2U}\right)^{\frac{1}{2}}}, \tag{3.58}$$

which is of the standard form (Spiegel 1974)

$$\int \frac{dx}{\sqrt{x^2 - \alpha^2}} = \cosh^{-1}\left(\frac{x}{\alpha}\right).$$

Thus equation (3.58) yields the solution

$$\nu + c = \pm \left(\frac{A}{U}\right)^{\frac{1}{2}} \cosh^{-1}\left[\left(\frac{2U}{B}\right)^{\frac{1}{2}} \frac{1}{\phi}\right]. \tag{3.59}$$

Equation (3.59) may be rearranged to obtain an expression for ϕ_1, the soliton profile, in terms of ξ and τ (remembering that we have suppressed the subscript in the derivation above)

$$\phi_1 = \left(\frac{2U}{B}\right)^{\frac{1}{2}} \operatorname{sech}\left[\pm\left(\frac{U}{A}\right)^{\frac{1}{2}} (\xi - U\tau + c)\right]. \tag{3.60}$$

3.4 Arbitrary amplitude theory

Nonlinear structures, in particular solitons, travelling in the x direction with dimensionless velocity μ are considered. We follow the analysis of Baboolal *et al.* (1988), so that equations (2.15), (2.16) and (2.17), are transformed to a frame stationary with respect to the soliton structures, through $s = x - \mu t$, with $\frac{\partial}{\partial x} = \frac{d}{ds}$; $\frac{\partial}{\partial t} = -\mu\frac{d}{ds}$ so that all variables depend only on s. These equations then become

$$-\mu\frac{dn_{jc}}{ds} + \frac{d}{ds}(n_{jc}u_{jc}) = 0 \tag{3.61}$$

$$-\mu\frac{du_{jc}}{ds} + u_{jc}\frac{du_{jc}}{ds} + 3T_c\frac{n_{jc}}{N_c^2}\frac{dn_{jc}}{ds} = -Z_j\frac{d\phi}{ds} \tag{3.62}$$

$$\frac{d^2\phi}{ds^2} = \sum_j\sum_k -Z_j n_{jk} \quad j = e, p \quad k = c, h. \tag{3.63}$$

Equations (3.61) and (3.62) can then be integrated exactly with the boundary conditions

$$\mid s \mid \to \infty: \quad \phi, \frac{d\phi}{ds} \to 0, \quad n_{jc} \to N_c, \quad u_{jc} \to 0 \tag{3.64}$$

assuming that the plasma is undisturbed at $\mid s \mid \to \infty$. These conditions ensure that the solutions of the set of equations (3.61)–(3.63) are localized in space, excluding infinite wavetrains.

Integrating the continuity equation (3.61) with (3.64) gives

$$u_{jc} = \mu\left(1 - \frac{N_c}{n_{jc}}\right), \tag{3.65}$$

whence

$$\frac{du_{jc}}{ds} = \frac{\mu N_c}{n_{jc}^2}\frac{dn_{jc}}{ds}. \tag{3.66}$$

Substitution of (3.65) and (3.66) into (3.62) yields

$$-\mu^2\frac{N_c^2}{n_{jc}^3}\frac{dn_{jc}}{ds} + \frac{3T_c}{N_c^2}n_{jc}\frac{dn_{jc}}{ds} = -Z_j\frac{d\phi}{ds}, \tag{3.67}$$

and integration with (3.64) allows us to obtain an expression for n_{jc} from

$$\mu^2\left(\frac{N_c^2}{n_{jc}^2} - 1\right) + 3T_c\left(\frac{n_{jc}^2}{N_c^2} - 1\right) = -Z_j 2\phi, \tag{3.68}$$

which can be solved as a quadratic equation in n_{jc}^2 yielding

$$n_{ec}^2 = \frac{N_c^2}{6T_c}\left[(\mu^2 + 2\phi + 3T_c) \pm \sqrt{(\mu^2 + 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right], \tag{3.69}$$

$$n_{pc}^2 = \frac{N_c^2}{6T_c}\left[(\mu^2 - 2\phi + 3T_c) \pm \sqrt{(\mu^2 - 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right]. \tag{3.70}$$

Equations (3.69) and (3.70) are similar to those obtained by Mace, Baboolal, Bharuthram and Hellberg 1991) in their investigation of electron acoustic solitons in a two electron component plasma. The $\pm$ signs refer to the two possible roots for n_{jc}^2. The positive root is discarded as spurious, in order that the equations (3.69) and (3.70) reduce to the correct form (Mace *et al.* 1991) in the limit at $T_c \to 0$ (cf. Appendix B).

3.4.1 The generalized Sagdeev potential

Continuing along the lines of Baboolal *et al.* (1988), Poisson's equation (3.63) with (2.13), (2.14) and (3.69), (3.70) may be written as

$$\frac{d^2\phi}{ds^2} = N(\phi,\mu) = N_h e^{\phi} + n_{ec}(\phi,\mu) - N_h e^{(-\phi)} - n_{pc}(\phi,\mu) \equiv -\frac{\partial V(\phi,\mu)}{\partial\phi}, \tag{3.71}$$

which defines the Sagdeev pseudo-potential $V(\phi,\mu)$ (Sagdeev 1966), with n_{ec} and n_{pc} found from the appropriate roots of (3.69) and (3.70) respectively.

Writing the LHS of expression (3.71) as $\frac{d}{ds}\left[\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2\right]$ one can integrate both sides yielding

$$\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 + V(\phi,\mu) = 0, \tag{3.72}$$

with $V(\phi,\mu)$ given by

$$V(\phi,\mu) = 2N_h(1-\cosh\phi) - \int_0^\phi n_{ec}(\phi',\mu)d\phi' + \int_0^\phi n_{pc}(\phi',\mu)d\phi'. \tag{3.73}$$

Equation (3.72) represents the equation of motion of a particle of unit mass moving with velocity $\frac{d\phi}{ds}$ in a potential well $V(\phi,\mu)$ with pseudo-spatial co-ordinate ϕ.

For the existence of soliton potential structures one requires (Sagdeev 1966) that

$$\begin{array}{lll} (i) & V(\phi,\mu) = \frac{\partial V(\phi,\mu)}{\partial\phi} = 0 & \text{at} \quad \phi = 0 \\ (ii) & V(\phi,\mu) = 0 & \text{at} \quad \phi = \phi_0 \\ (iii) & V(\phi,\mu) < 0 & \text{for} \quad 0 < |\phi| < |\phi_0| \end{array} \tag{3.74}$$

where ϕ_0 is the soliton amplitude. Considering the classical mechanical analogy, conditions (3.74)(i),(ii) and (iii), ensure that the pseudo-particle with coordinate ϕ transits just once from $\phi = 0$ to $\phi = \phi_0$ and back, coming to rest at $\phi = 0$.

It can be seen that (3.74)(i) implies the charge neutrality condition at $\phi = 0$, that is when $s \to \pm\infty$. Expanding condition (i) gives

$$\begin{aligned} \frac{\partial V(\phi,\mu)}{\partial\phi}\Big|_{\phi=0} &= -N(\phi,\mu)\Big|_{\phi=0} \\ &= (n_{pc}+n_{ph})\Big|_{\phi=0} - (n_{ec}+n_{eh})\Big|_{\phi=0}. \end{aligned} \tag{3.75}$$

Now under the boundary conditions as $s \to \pm\infty$, $n_j \to n_{jo}$ so (3.75) becomes

$$\frac{\partial V(\phi,\mu)}{\partial\phi} = n_{po} - n_{eo} = 0. \tag{3.76}$$

3.4.2 Limits imposed on Mach number

Solutions for solitons exist only for a specific range of Mach numbers. We define the soliton speed relative to the thermal speed (via normalization) as μ, so that

$$\mu = \frac{v}{v_h}. \tag{3.77}$$

A lower limit for μ is given by the condition that

$$\frac{\partial^2 V(\phi,\mu)}{\partial \phi^2} < 0 \quad \text{at} \quad \phi = 0, \tag{3.78}$$

that is, there exists a stationary rest point at $\phi = 0$ (Baboolal, Bharuthram & Hellberg 1990). Since

$$\frac{\partial^2 V(\phi,\mu)}{\partial \phi^2} = -\frac{\partial N(\phi,\mu)}{\partial \phi}, \tag{3.79}$$

equation (3.78) becomes (cf. Appendix C)

$$\frac{N_c}{\mu^2 - 3T_c} < N_h,$$

yielding a lower limit for μ as

$$\mu > \left(\frac{N_c}{N_h} + 3T_c\right)^{\frac{1}{2}}. \tag{3.80}$$

Here the RHS of the equation is simply the normalized sound speed of the wave, V (cf. equation (2.83)) calculated from the linear dispersion relation, in the long wavelength limit as $k \to 0$. Thus the soliton speed is always greater than the sound speed v_s of the wave. We follow convention (Chen 1984) and define a Mach number so that

$$M = \frac{\mu}{V} > 1. \tag{3.81}$$

A further, important restriction is imposed by the requirement that the densities of the cool and hot species of electrons and positrons

$$n_{jc}^2 = \frac{N_c^2}{6T_c}\left[\left(\mu^2 - Z_j 2\phi + 3T_c\right) - \sqrt{(\mu^2 - Z_j 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right],$$

be real, that is, the terms under the square root sign must be greater than or equal to zero. It can be seen that the requirement of real densities imposes restrictions on ϕ such that

$$\left(\mu^2 - Z_j 2\phi + 3T_c\right)^2 - 12\mu^2 T_c \geq 0, \tag{3.82}$$

that is,

$$\phi_{min} = -\frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 \leq \phi \leq \frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 = \phi_{max}. \tag{3.83}$$

In the limit as $T_c \to 0$ this reduces to the condition

$$-\frac{\mu^2}{2} \leq \phi \leq \frac{\mu^2}{2}, \tag{3.84}$$

also found for ion acoustic solitons (Chen 1984).

The upper limit for M is imposed by the condition that the function $V(\phi, \mu)$ must cross the ϕ axis for some $\phi > \phi_0$ (Chen 1984), so that the pseudo-particle is reflected. That is

$$V(\phi, \mu) > 0, \quad \mid \phi \mid > \mid \phi_0 \mid, \tag{3.85}$$

for compressive and rarefactive solitons.

Since the integrals of the cool electron and positron density are complicated we will consider the cold case where $T_c \to 0$ to simplify matters. Since we require $V(\phi, \mu) > 0$ at some $\phi > \phi_0$ then $V(\phi_{max})$ must be positive and thus substituting $\phi_{max} = \frac{\mu^2}{2}$ for ϕ, (3.85) becomes (cf. Appendix D)

$$2N_h(\cosh\frac{\mu^2}{2} - 1) < N_c\mu^2(2 - \sqrt{2}). \tag{3.86}$$

Solving graphically for

$$M = \frac{\mu}{\left(\frac{N_c}{N_h}\right)^{\frac{1}{2}}},$$

using the value $N_c = 0.2$, gives $M < 1.53$. This corresponds to the critical Mach number of $M < 1.6$ (Chen 1984) for ion-acoustic solitons, and thus we would expect a similar cut-off to exist in general.

Figure 3.1 shows the intercept of the functions defined by the left and right hand sides of equation (3.86) for $N_c = 0.2$.

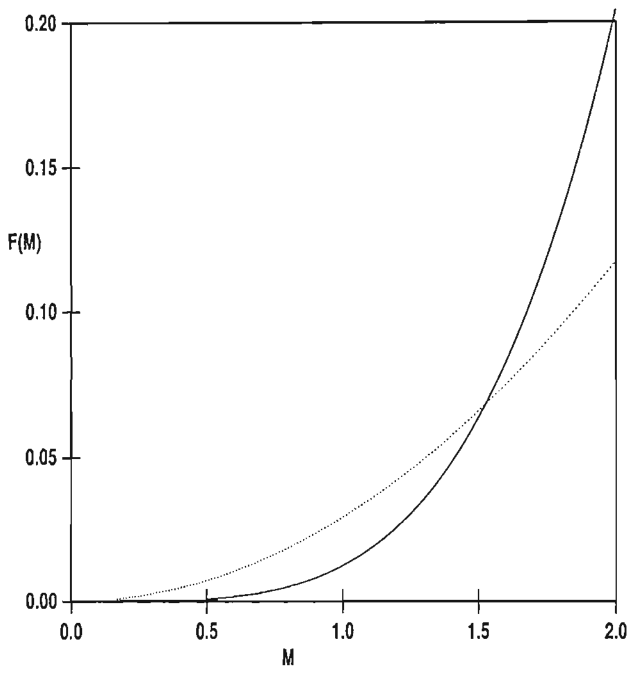


Figure 3.1: Plot showing the intercept of the functions defined by the left (solid curve) and right (dotted curve) hand sides of equation (3.86) for $N_c = 0.2$.

Solving

$$2N_h(\cosh\frac{M^2V^2}{2} - 1) - N_cM^2V^2(2 - \sqrt{2}) = 0, \tag{3.87}$$

where $V = \left(\frac{N_c}{N_h}\right)^{\frac{1}{2}}$ is the normalized sound speed in the limit as $T_c \to 0$, for M over the range of N_c values gives us the Mach number cut-off at a particular value of N_c. Figure 3.2 shows the Mach number cut-off for the range of cool species density N_c.

These graphs indicate that the upper Mach number limit depends on the cool species equilibrium number density, N_c. As N_c increases, the upper Mach number limit decreases. This suggests that there exists a specific range for N_c that is valid for the existence of significant Mach number solitons. This valid N_c range will be discussed in greater detail in the following chapter.

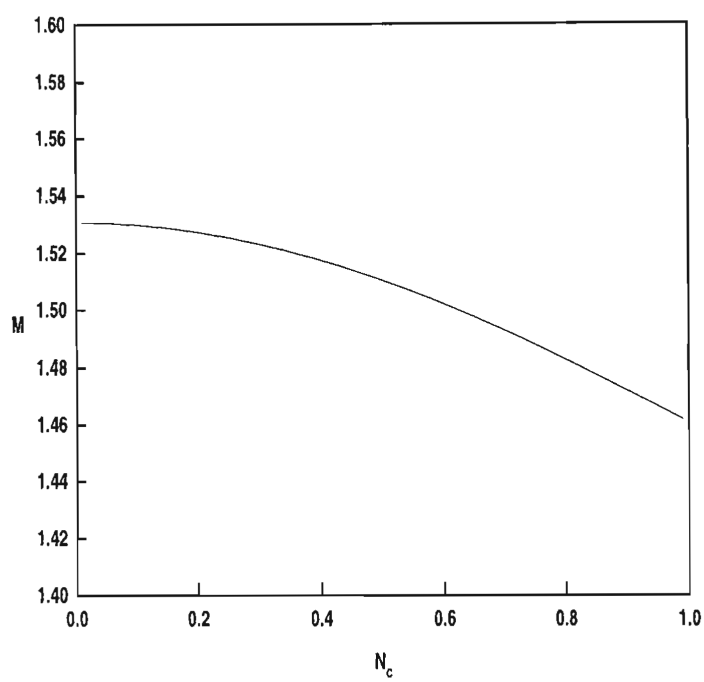


Figure 3.2: Mach number cut-off for the range of cool species density N_c, found by solving equation (3.87).

Chapter 4

Results and Discussion

In this chapter we obtain solutions for both small and arbitrary amplitude solitons, discussed in Chapter 3. These results are analysed graphically, with emphasis on the existence domains for both rarefactive and compressive solitons in cool species density and soliton amplitude space. Analysis of the limits imposed by the choice of model indicates severe restrictions on the existence of these solutions.

4.1 Small amplitude soliton solutions

The solution to the modified Korteweg-de Vries equation (3.60) is given by

$$\phi_1 = \left(\frac{2U}{B}\right)^{\frac{1}{2}} \operatorname{sech}\left[\pm\left(\frac{U}{A}\right)^{\frac{1}{2}}(\xi - U\tau + c)\right],$$

where U is the soliton speed, in terms of the variables ξ and τ, and A and B are given by equations (3.53) and (3.54).

Now, in terms of the original variables

$$\xi = \epsilon(x - Vt), \qquad \tau = \epsilon^3 Vt,$$

and recalling that $\phi = \epsilon\phi_1$, cf. equation (3.23) we obtain an expression for ϕ, the soliton profile

$$\phi = \left(\frac{2\epsilon^2 U}{B}\right)^{\frac{1}{2}} \operatorname{sech}\left[\pm\left(\frac{\epsilon^2 U}{A}\right)^{\frac{1}{2}}(x - V(1 + U\epsilon^2)t + c)\right]. \qquad (4.1)$$

We have considered arbitrary amplitude solitons in the stationary frame $s = x - \mu t$, thus we note that in the above equation

$$V(1 + U\epsilon^2) = \mu,$$

and with reference to equation (3.81) $M = \frac{\mu}{V}$,

$$M = (1 + U\epsilon^2). \tag{4.2}$$

Since the amplitude and speed of solitons are related, we may choose the amplitude expansion parameter ϵ, to be the Mach number excess, so that

$$\epsilon^2 \equiv \Delta M = M - 1.$$

Substituting this definition of ϵ into equation (4.2) shows that the soliton speed U, defined in terms of the stretched variables ξ and τ, is equal to unity in terms of the normalized units used to derive the mKdV equation (Chen 1984). Thus the modified Korteweg-de Vries equation admits a soliton solution, whose profile is given as a function of Mach number $M = \frac{\mu}{V}$, cool and hot species number density N_c and N_h, and temperature ratio $\frac{T_c}{T_h}$, by

$$\phi = \pm \left(\frac{2(M-1)}{B}\right)^{\frac{1}{2}} \operatorname{sech}\left[\pm\left(\frac{M-1}{A}\right)^{\frac{1}{2}} (x - \frac{M}{V}t)\right], \tag{4.3}$$

where A and B are defined in equations (3.53) and (3.54). Since sech(x) is even in x we may omit the $\pm$ in the square brackets of the above equation.

4.2 Small amplitude results

By simple substitution we may solve

$$\phi = \pm \left(\frac{2(M-1)}{B}\right)^{\frac{1}{2}} \operatorname{sech}\left[\left(\frac{M-1}{A}\right)^{\frac{1}{2}} s\right] \tag{4.4}$$

for s, to obtain soliton amplitude profiles for specific values of Mach number, temperature ratio and cool species number density. Figure 4.1 shows a typical mKdV compressive soliton profile.

We see that equation (4.4) takes the form of a solitary wave pulse, in which the harmonic generating effects of nonlinearity are balanced by the

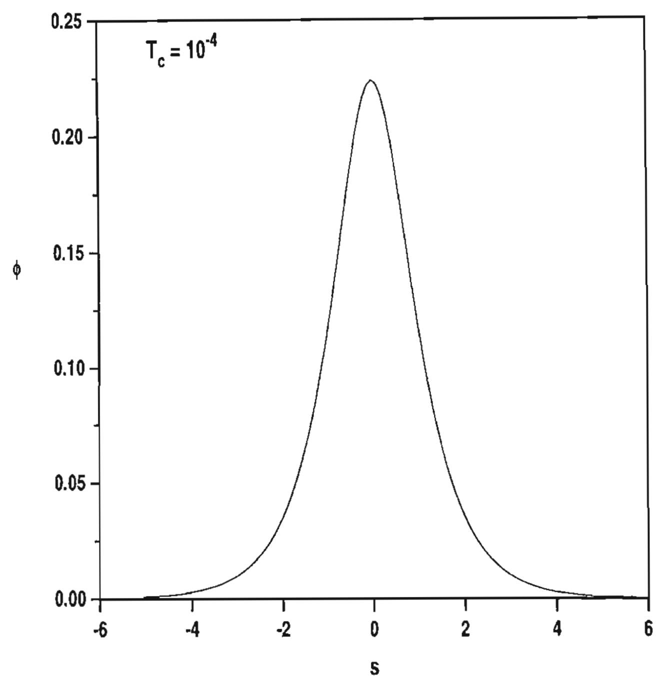


Figure 4.1: A typical mKdV compressive soliton profile. Parameters are $T_c = 10^{-4}, N_c = 0.2$ and $M = 1.5$.

phase mixing of the Fourier components caused by wave dispersion. Solitons with both positive and negative electrostatic potentials will exist since the amplitude ϕ_0 is the square root of a constant which can be positive or negative. The negative root gives rarefactive solitons, and the positive root, compressive solitons. There will be exact symmetry since

$$| \phi_0 |=| \left(\frac{2(M-1)}{B}\right)^{\frac{1}{2}} |$$

will have the same value for both roots, for the same values of M, N_c and T_c.

The amplitude of the soliton

$$\phi_0 = \left(\frac{2(M-1)}{B}\right)^{\frac{1}{2}}, \tag{4.5}$$

is proportional to the square root of the Mach number excess

$$\Delta M = M - 1, \tag{4.6}$$

and the soliton half-width

$$L = \left(\frac{A}{M-1}\right)^{\frac{1}{2}}, \tag{4.7}$$

is proportional to $\Delta M^{-\frac{1}{2}}$. This means that larger amplitude solitons occurring for higher Mach numbers, are narrower, and travel faster than their smaller counterparts. This is a general characteristic of soliton solutions of the KdV and mKdV equations. The Mach number M can thus be used to specify the energy of a soliton. The larger the energy, the larger the speed and amplitude, and the narrower the width. Figure 4.2 shows this relation between soliton speed, amplitude and width.

The occurrence of solitons will depend on whether the initial disturbance to the equilibrium plasma has enough energy, and the necessary phases (i.e. harmonics), if not, a nonlinear wave will appear. If the initial disturbance has the energy of several solitons, and the necessary phases, an N-soliton solution can be generated. Thus a localized initial disturbance, with energy and harmonics necessary for the formation of soliton structures, that evolves according to the modified Korteweg de Vries equation, usually emerges as a finite number of solitons arranged in order of increasing height, and speed (Chen 1984).

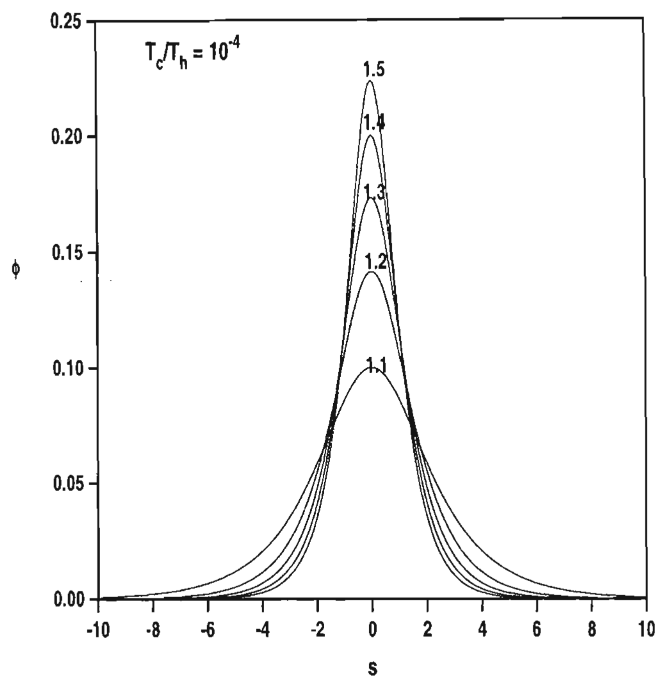


Figure 4.2: shows the relation between soliton amplitude, speed and width. The numbering on the curves refers to Mach number, M, and other parameters are $T_c = 10^{-4}, N_c = 0.2$. It is evident that solitons with larger Mach numbers have larger amplitudes, and are narrower.

It is also interesting to note (cf. Figure 4.3) that a decrease in the ratio of the temperatures $\frac{T_c}{T_h}$, results in an increase in soliton amplitude for a specific value of N_c and M.

We may also plot $\phi_0 = \left| \left(\frac{2(M-1)}{B}\right)^{\frac{1}{2}} \right|$, where

$$B = -\frac{\left(\frac{N_c}{N_h}\right)^4 - 15\left(\frac{N_c}{N_h}\right)^2 - 180\frac{N_c}{N_h}T_c - 432T_c^2}{12\left(\frac{N_c}{N_h} + 3T_c\right)\left(\frac{N_c}{N_h}\right)^3},$$

over a range of N_c values, to obtain existence domains for mKdV solitons in the space of amplitude ϕ_0, and cool species number density N_c. Figure 4.4 and 4.5 show existence domains for T_c values 10^{-2} and 10^{-4}.

Regarding equation (4.5), we note that in order for ϕ_0 to be real, $\frac{2(M-1)}{B} > 0$. Since the Mach number M is greater than unity, cf. equation (3.81), this implies that $B > 0$. Considering the more simplistic case, in the limit as $T_c \to 0$, B reduces to

$$B = -\frac{\left(\frac{N_c}{N_h}\right)^4 - 15\left(\frac{N_c}{N_h}\right)^2}{12\left(\frac{N_c}{N_h}\right)^4}. \tag{4.8}$$

Recalling that $N_h = 1 - N_c$, the condition $B > 0$ implies an upper N_c cut-off of $N_c < 0.79$. Thus, in the limit as $T_c \to 0$, a cool species number density greater than 0.79 results in an imaginary soliton amplitude.

The cut-off at $N_c = 0.79$ for real soliton amplitude can be seen clearly in Figures 4.4 and 4.5. It can also be noted from these figures that the mKdV theory allows for solitons with Mach numbers greatly in excess of the previously obtained cut-off value (cf. Section 3.4.2) $M < 1.53$. At such large Mach numbers the soliton amplitude ϕ_0 tends asymptotically to infinity for shorter and shorter N_c ranges. This is inappropriate since for mKdV solutions we require small amplitudes.

In the mKdV derivation of soliton profiles to first degree nonlinearity, the electrostatic potential is of order ϵ, $\phi = \epsilon\phi_1$. Similarly $k\lambda_{Dh}$ is of order ϵ, so $\phi \sim k\lambda_{Dh}$. However $k^2\lambda_{Dh}^2 \ll 1$, so this implies that $\phi \ll 1$. Since ϕ_0 is the amplitude, and thus the maximum value of ϕ, we have $\phi_0 \ll 1$ by the above deduction. Figures 4.6 and 4.7 show the existence domains for T_c values 10^{-2} and 10^{-4} with an upper cut-off of $\phi_0 = 1$, an overestimate. Even with this imposed upper ϕ_0 limit, these figures suggest that very large Mach number solitons exist, in excess of the previously obtained limit ($M < 1.53$).

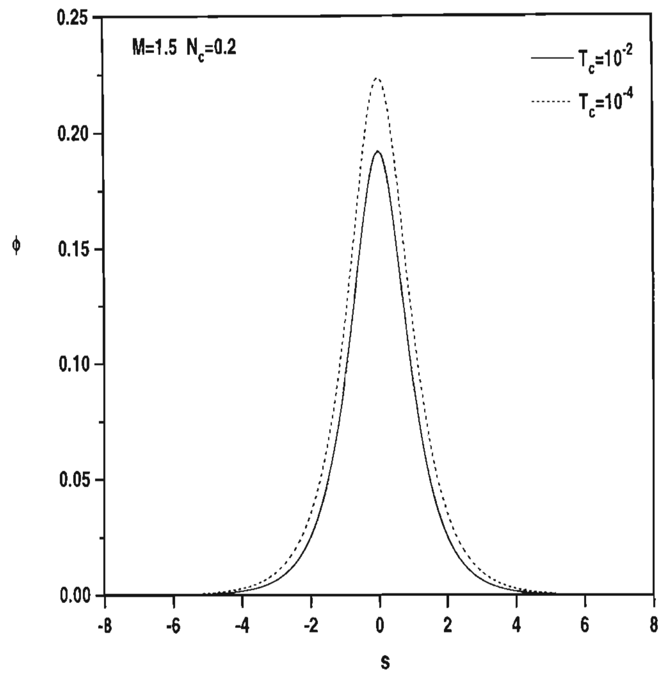


Figure 4.3: For specific values of Mach number and cool species number density, as we decrease the ratio of $\frac{T_c}{T_h}$, the amplitude of the soliton solution increases. Other parameters are $N_c = 0.2$ and $M = 1.5$.

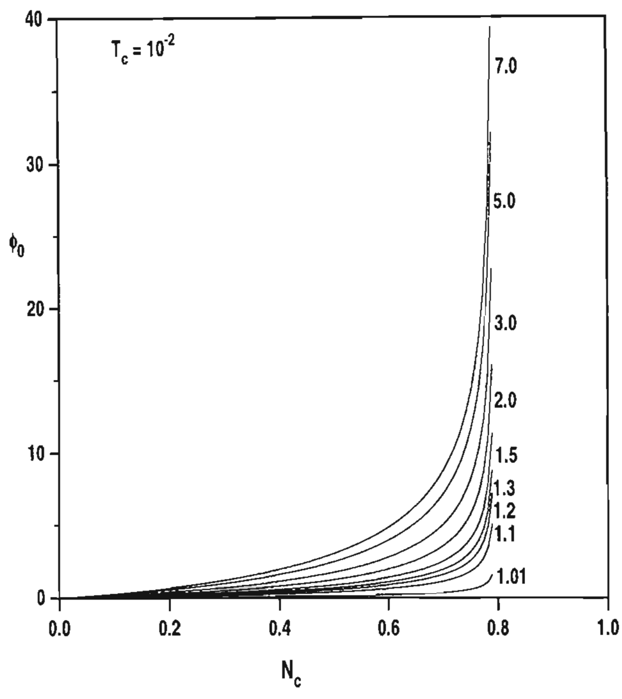


Figure 4.4: mKdV soliton existence domains for temperature ratio $\frac{T_c}{T_h} = 10^{-2}$, in amplitude ϕ_0 - N_c cool species number density space. The numbering on the curves refers to soliton Mach number.

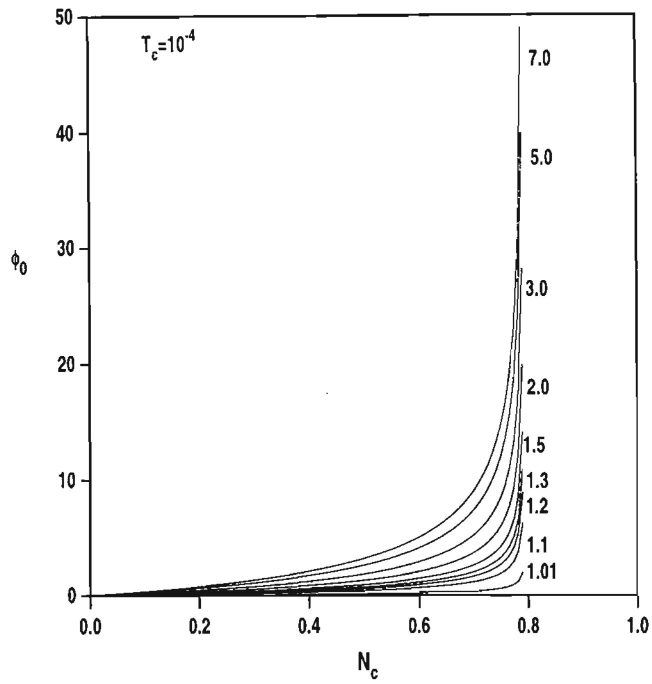


Figure 4.5: mKdV soliton existence domains for temperature ratio $\frac{T_c}{T_h} = 10^{-4}$, in amplitude $\phi_0 - N_c$ cool species number density space. The numbering on the curves refers to soliton Mach number.

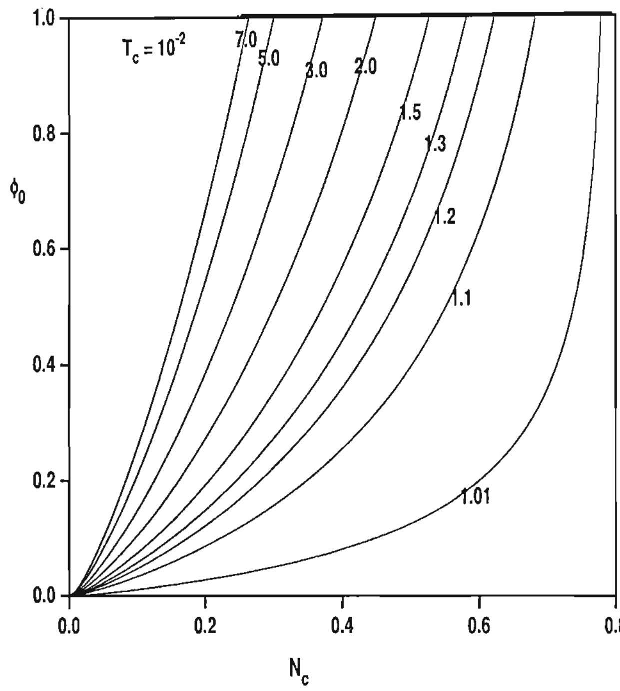


Figure 4.6: mKdV soliton existence domains for temperature ratio $\frac{T_c}{T_h} = 10^{-2}$, with the restriction $\phi_0 < 1$ imposed. The numbering on the curves refers to soliton Mach number.

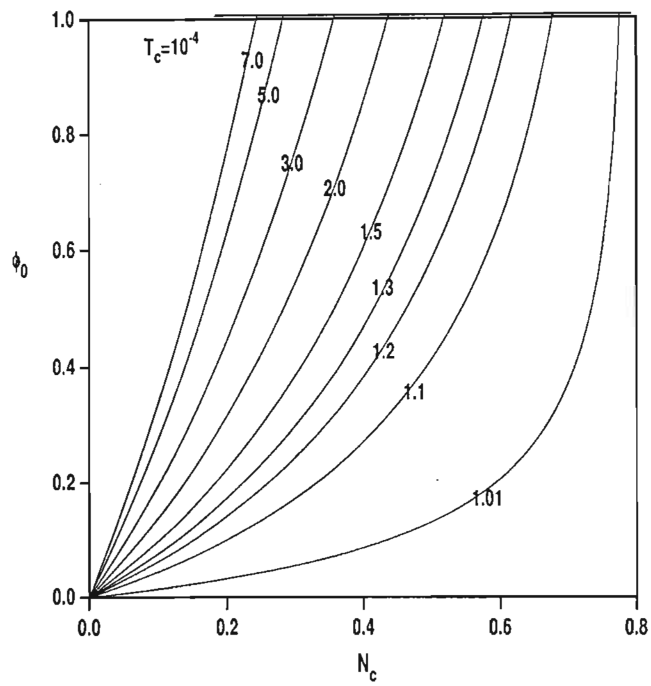


Figure 4.7: mKdV soliton existence domains for temperature ratio $\frac{T_c}{T_h} = 10^{-4}$, with the restriction $\phi_0 < 1$ imposed. The numbering on the curves refers to soliton Mach number.

A more likely cut-off is provided by equation (4.6) which states that $\Delta M = M - 1 = \epsilon^2$. Thus the Mach number excess is of order ϵ^2, and since $\epsilon \ll 1$, $M - 1 \ll 1$. This leads us to expect that solutions showing the existence of solitons with Mach number in excess of $M = 1.6$ are non-physical, since $M \ll 2$.

4.3 Numerical solutions

It was initially presumed that the Sagdeev potential equation

$$V(\phi, \mu) = 2N_h\,(1 - \cosh\phi) - \int_0^{\phi} n_{ec}(\phi', \mu)d\phi' + \int n_{pc}(\phi', \mu)d\phi',$$

could not be solved analytically for non-zero temperatures, since it appeared as if

$$n_{ec} = \frac{N_c}{\sqrt{6T_c}}\left[(\mu^2 + 2\phi + 3T_c) \pm \sqrt{(\mu^2 + 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right]^{\frac{1}{2}}, \qquad (4.9)$$

$$n_{pc} = \frac{N_c}{\sqrt{6T_c}}\left[(\mu^2 - 2\phi + 3T_c) \pm \sqrt{(\mu^2 - 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right]^{\frac{1}{2}}, \qquad (4.10)$$

could not be integrated simply. Thus the Sagdeev potential was constructed by numerical integration of $n_{ec}(\phi, \mu)$ and $n_{pc}(\phi, \mu)$, using Simpson's rule. The following method used follows that found in 'Numerical Recipes for C' (Press 1988).

We consider the integral of the function $f(x)$

$$I = \int_a^b f(x)dx.$$

The interval $[a, b]$ is subdivided into a finite number of equally spaced subintervals. These intervals may be denoted by $x_0 = a; x_1; x_2; \cdots; x_n = b$ such that

$$x_i = x_0 + ih \qquad i = 0; 1; \cdots; n$$

where h is a constant step size. Now the function $f(x)$ has known values at the x_i's

$$f(x_i) = f_i.$$

Simpson's method of quadrature involves the addition of these values of the integrand, under the conditions of the formula

$$\int_{x_0}^{x_n} f(x)dx = h[\frac{1}{3}f_1 + \frac{4}{3}f_2 + \frac{2}{3}f_3 + \cdots + \frac{2}{3}f_{n-2} + \frac{4}{3}f_{n-1} + \frac{1}{3}f_n] + O(\frac{1}{n^4})$$

where $O(\frac{1}{n^4})$ is the error term which signifies the difference between the estimate and the true answer, and is of the order $\frac{1}{n^4}$. For the interval $[0, \phi]$ subdivided into twenty equally spaced sub-intervals, the error in the integral of cool electron density, and cool positron density is of the order 10^{-6}.

The initial value ϕ_0 was then obtained by solving

$$V(\phi_0) = 0,$$

using Brent's Rootfinding method, for specific variables M, T_c, N_c. In this way existence regions for arbitrary amplitude solitons in the space of soliton amplitude and cool species density were explored.

Brent's method solves the equation $f(x) = 0$ numerically for x. The algorithm proceeds by iteration: starting at an approximate solution it will improve on this trial solution until a specified convergence criterion is satisfied. The initial 'guessed' solution is found by substitution of ϕ into the Sagdeev potential $V(\phi)$ in steps beginning at $\phi = \phi_{max}$, the maximum value of ϕ for which n_{ec} and n_{pc} are real; until $V(\phi_i)$ becomes negative at some ϕ_i. The algorithm then uses this value of ϕ_i and ϕ_{max} as brackets, as long as $V(\phi_i) < 0$ and $V(\phi_{max}) > 0$.

According to the Intermediate Value Theorem if $V(\phi_i)$ and $V(\phi_{max})$ have opposite signs, then at least one root must lie between ϕ_i and ϕ_{max}. The algorithm finds the root by use of the Bisection method. Since it is known that between $V(\phi_i)$ and $V(\phi_{max})$ the function passes through zero, because it changes sign, the Bisection method evaluates the function at the midpoint between ϕ_i and ϕ_{max}. It then uses this midpoint to replace whichever bracket has the same sign. Thus after each iteration the range between the boundary points is halved, and this continues until the two limits converge to within a certain tolerance, that is until the interval becomes smaller than this tolerance. The tolerance specifies the accuracy of the solution: for example convergence to within 10^{-6}, if the root is unity suggests a reasonable accuracy, however if the root is 10^{-10} and the tolerance is 10^{-6}, that is the algorithm halts when the interval is of order 10^{-6}, the root cannot be found with any degree of accuracy. A reasonable tolerance is $\frac{\epsilon}{2}(\mid x_1 \mid + \mid x_2 \mid)$ where ϵ is the machine precision and x_1 and x_2 the initial brackets.

The soliton profile was obtained by rearranging (3.72) to give

$$\frac{d\phi}{ds} = \pm\sqrt{-2V(\phi,\mu)},$$

and using the Runga-Kutta method to yield a potential profile of ϕ against s, together with the assumption of $\phi(s=0) = \phi_0$, that is maxima or minima occur at $s = 0$. The Runga-Kutta method solves the differential equation (in this case)

$$\frac{d\phi}{ds} = F(s,\phi) = \pm[-2V(\phi,\mu)]^{\frac{1}{2}}, \qquad \phi = \phi(s),$$

for a specific value of μ, with the boundary condition that $\phi(s=0) = \phi_0$, using the Euler formula, that is

$$\phi_{n+1} = \phi_n + hF(s_n, \phi_n).$$

This advances a solution from ϕ_n to $\phi_{n+1} \equiv \phi_n + h$ through an interval h. The fourth order Runga-Kutta formula is as follows

$$\begin{aligned} k_1 &= hF(s_n; \phi_n) \\ k_2 &= hF(s_n + \tfrac{h}{2}; \phi_n + \tfrac{k_1}{2}) \\ k_3 &= hF(s_n + \tfrac{h}{2}; \phi_n + \tfrac{k_2}{2}) \\ k_4 &= hF(s_n + h; \phi_n + k_3) \end{aligned}$$

$$\phi_{n+1} = \phi_n + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]. \tag{4.11}$$

The method thus generates ϕ_{n+1} using $s_n = nh$ in (4.11) for $n = 0; 1; \cdots; k$, where k is a suitable number of values for s.

4.4 Analytical solutions

It was subsequently discovered (Chatterjee & Roychoudhury 1995) that the exact pseudopotential $V(\phi,\mu)$ in equation (3.73) can be obtained in the case of non-zero temperature in analytic form. Equation (3.73) gives the Sagdeev potential as

$$V(\phi,\mu) = 2N_h(1 - \cosh\ \phi) - \int_0^{\phi} n_{ec}(\phi',\mu)d\phi' + \int_0^{\phi} n_{pc}(\phi',\mu)d\phi'$$

where

$$n_{jc}(\phi,\mu) = \frac{N_c}{\sqrt{6T_c}}\left[(\mu^2 - Z_j2\phi + 3T_c) - \sqrt{(\mu^2 - Z_j2\phi + 3T_c)^2 - 12\mu^2 T_c}\right]^{\frac{1}{2}},$$

for $Z_j = \frac{q_j}{e}$ for $j = e, p$.

Following Chatterjee *et al.* (1995) we introduce two new variables

$$\Theta_j = \cosh^{-1}\left[\frac{\mu^2 - Z_j2\phi + 3T_c}{\sqrt{12\mu^2 T_c}}\right] = \cosh^{-1}(\chi_j),$$

or

$$\cosh\Theta_j = \chi_j = \frac{\mu^2 - Z_j2\phi + 3T_c}{\sqrt{12\mu^2 T_c}}. \tag{4.12}$$

Differentiating both sides of equation (4.12) yields

$$\sinh\Theta_j\frac{d\Theta_j}{d\phi} = \frac{-2Z_j}{\sqrt{12\mu^2 T_c}},$$

and thus

$$d\phi = -\frac{\sqrt{12\mu^2 T_c}}{2Z_j}\sinh\Theta_j d\Theta.$$

Expressing n_{jc} in terms of Θ using (4.12) gives

$$\begin{aligned}
&n_{jc}(\Theta_j,\mu) = \tfrac{N_c}{\sqrt{6T_c}}\left[\sqrt{12\mu^2 T_c}\cosh\Theta_j - \sqrt{12\mu^2 T_c\cosh^2\Theta_j - 12\mu^2 T_c}\right]^{\frac{1}{2}}\\
&= \tfrac{(12\mu^2 T_c)^{\frac{1}{4}}N_c}{(6T_c)^{\frac{1}{2}}}\left[\cosh\Theta_j - \sqrt{\cosh^2\Theta_j - 1}\right]^{\frac{1}{2}}\\
&= \left(\tfrac{\mu^2}{3T_c}\right)^{\frac{1}{4}} N_c\left[\cosh\Theta_j - \sinh\Theta_j\right]^{\frac{1}{2}},
\end{aligned}$$

that is,

$$\begin{aligned}
n_{jc}(\Theta_j,\mu) &= \left(\tfrac{\mu^2}{3T_c}\right)^{\frac{1}{4}} N_c\left[\tfrac{e^{\Theta_j}+e^{-\Theta_j}}{2} - \tfrac{e^{\Theta_j}-e^{-\Theta_j}}{2}\right]^{\frac{1}{2}}\\
&= \left(\tfrac{\mu^2}{3T_c}\right)^{\frac{1}{4}} N_c\exp\left(-\tfrac{\Theta_j}{2}\right).
\end{aligned}$$

Thus the integral of n_{jc} in terms of Θ_j is as follows

$$\begin{aligned}
\textstyle\int_0^\phi n_{jc}(\phi')d\phi' &= -\left(\tfrac{\mu^2}{3T_c}\right)^{\frac{1}{4}}\tfrac{(12\mu^2 T_c)^{\frac{1}{2}}}{-2Z_j}\textstyle\int_{\Theta_j(0)}^{\Theta_j(\phi)}\exp\left(-\tfrac{\Theta_j}{2}\right)\sinh\Theta_j d\Theta_j\\
&= -(3\mu^6 T_c)^{\frac{1}{4}}\tfrac{N_c}{Z_j}\textstyle\int_{\Theta_j(0)}^{\Theta_j(\phi)} e^{\left(\frac{-\Theta_j}{2}\right)}\tfrac{(e^{\Theta_j}-e^{-\Theta_j})}{2}d\Theta_j\\
&= -(3\mu^6 T_c)^{\frac{1}{4}}\tfrac{N_c}{Z_j}\textstyle\int_{\Theta_j(0)}^{\Theta_j(\phi)}\tfrac{1}{2}\left(e^{\left(\frac{\Theta_j}{2}\right)} - e^{\left(-\frac{3\Theta_j}{2}\right)}\right)d\Theta_j.
\end{aligned}$$

Thus

$$\int_0^{\phi} n_{jc}(\phi')d\phi' = -\frac{N_c}{Z_j}(3\mu^6 T_c)^{\frac{1}{4}}\left[\exp\left(\frac{\Theta_j}{2}\right) + \frac{1}{3}\exp\left(-\frac{3\Theta_j}{2}\right)\right]_{\Theta_j(0)}^{\Theta_j(\phi)}.$$

Recall equation (4.12): $\Theta_j(\phi) = \cosh^{-1}\chi_j$. Now it is known (Spiegel 1974) that

$$\cosh^{-1}\chi_j = \ln(\chi_j + \sqrt{\chi_j^2 - 1}),$$

for $\chi_j > 1$. So

$$\exp\ \Theta_j = \chi_j + \sqrt{\chi^2 - 1},$$

and

$$\int_0^{\phi} n_{jc}(\phi')d\phi' =$$
$$-\tfrac{N_c}{Z_j}(3\mu^6 T_c)^{\frac{1}{4}}\left[\left(\chi_j + \sqrt{\chi_j^2 - 1}\right)^{\frac{1}{2}} + \tfrac{1}{3}\left(\chi_j + \sqrt{\chi_j^2 - 1}\right)^{-\frac{3}{2}}\right]_0^{\phi}.$$

Thus

$$\int_0^{\phi} n_{jc}(\phi)d\phi' = -\tfrac{N_c}{Z_j}(3\mu^6 T_c)^{\frac{1}{4}}[\left(\chi_j + \sqrt{\chi_j^2 - 1}\right)^{\frac{1}{2}} + \tfrac{1}{3}\left(\chi_j + \sqrt{\chi_j^2 - 1}\right)^{-\frac{3}{2}}$$
$$- \left(\chi_0 + \sqrt{\chi_0^2 - 1}\right)^{\frac{1}{2}} - \tfrac{1}{3}\left(\chi_0 + \sqrt{\chi_0^2 - 1}\right)^{-\frac{3}{2}}], \tag{4.13}$$

where

$$\chi_o = \frac{\mu^2 + 3T_c}{\sqrt{12\mu^2 T_c}}.$$

We may thus write the Sagdeev potential in closed form (Chatterjee *et al.* 1995) as

$$V(\phi, \mu) = 2N_h(1 - \cosh\ \phi) - \int_0^{\phi} n_{ec}(\phi', \mu)d\phi' + \int_0^{\phi} n_{pc}(\phi', \mu)d\phi', \tag{4.14}$$

with the integrals of the cool species densities given by (4.13).

Existence domains of ϕ_0, the soliton height over the range of cool number species density N_c, for varying Mach number, could be found through simple substitution of variables: solving $V(\phi_0) = 0$ for specific values of Mach number, cool species number density and temperature ratio $\frac{T_c}{T_h}$.

4.5 Arbitrary amplitude soliton solutions

Clearly the closed form of the Sagdeev potential, although elegant, needs to be evaluated numerically for interpretation. In this section we discuss numerical solutions of $V(\phi, \mu)$, equation (4.14), with the integrals of the cool species densities given by (4.13). We recast all relevant equations in terms of M instead of μ, remembering that $M = \frac{\mu}{\left(\frac{N_c}{N_h}+3Tc\right)^{\frac{1}{2}}}$.

Figure 4.8 and 4.9 show a typical form of the Sagdeev potential $V(\phi, M)$, and soliton profile for compressive and rarefactive solitons. It can be seen that the Sagdeev potential

$$V(\phi, M) = 2N_h(1 - \cosh\phi) + \int_0^{\phi} n_{pc}(\phi', M)d\phi' - \int_0^{\phi} n_{ec}(\phi', M)d\phi',$$

with (4.9) and (4.10), is an even function of ϕ, (Verheest, Hellberg, Gray & Mace 1996). This is obvious if we consider n_{ec}, given by equation (4.9), to be a function $F(\phi)$, then n_{pc}, given by equation (4.10), is just $F(-\phi)$, and $V(\phi, M)$ is thus even in ϕ.

The result of the even nature of $V(\phi, M)$ is an exact symmetry in the existence conditions of compressive and rarefactive solitons, evident in Figures 4.8 and 4.9. Due to the symmetrical nature of the soliton solutions we shall, for simplicity, consider only the compressive case.

Figure 4.10 and 4.11 show the Sagdeev potential and corresponding soliton profile for the specific variables $M = 1.11$, $N_c = 0.5$, for two temperature ratios $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4}. It is clear from the figures that as the temperature ratio $\frac{T_c}{T_h}$ decreases, the height of the soliton, ϕ_0, increases. Physically, an increase in the temperature ratio $\frac{T_c}{T_h}$, causes an increase in the random kinetic energies of the cool species electrons and positrons. This results in a greater interpenetration between the hot and cool species, reducing the charge separation needed to sustain soliton structures (Baboolal, Bharuthram & Hellberg 1988).

An increase in cool species temperature is equivalent to decreasing the wave dispersion (cf. Figure 4.12). This leads to a consequent decrease in nonlinearity (for balance) and subsequent breakdown of the nonlinear structure, (Baboolal *et al.* 1988).

For the case when $T_c \approx T_h$ Landau damping should play a dominant role. The phase velocity of the wave will lie within the cool species velocity distribution, so that the thermal velocity of a considerable number of the

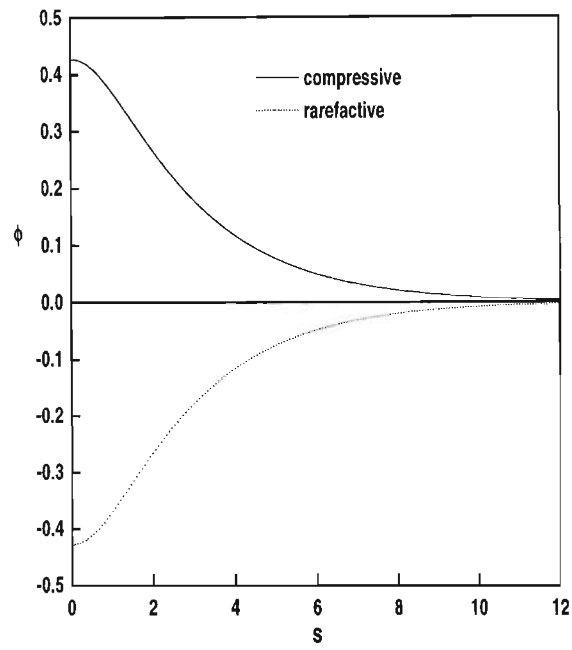


Figure 4.8: Typical half profiles for compressive and rarefactive solitons. Specific parameters are Mach number $M = 1.11$, cool species number density $N_c = 0.5$, and temperature ratio $\frac{T_c}{T_h} = 10^{-2}$.

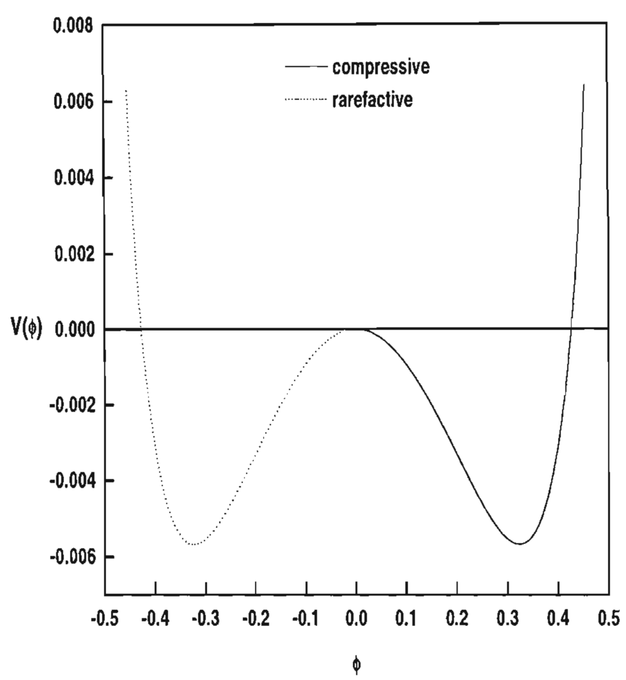


Figure 4.9: Typical form of the Sagdeev potential $V(\phi, M)$ for compressive and rarefactive solitons. Specific parameters are Mach number $M = 1.11$, cool species number density $N_c = 0.5$, and temperature ratio $\frac{T_c}{T_h} = 10^{-2}$.

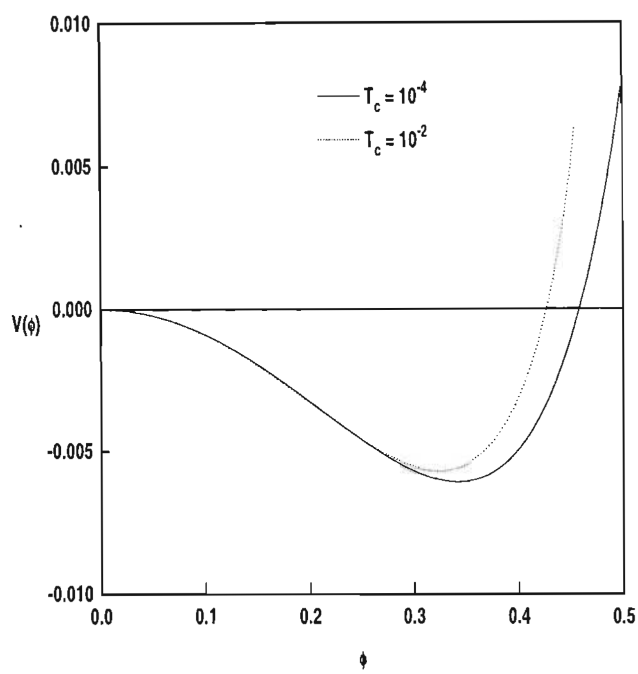


Figure 4.10: Sagdeev potentials for the specific variables $M = 1.11$, $N_c = 0.5$ for two temperature ratios $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4}.

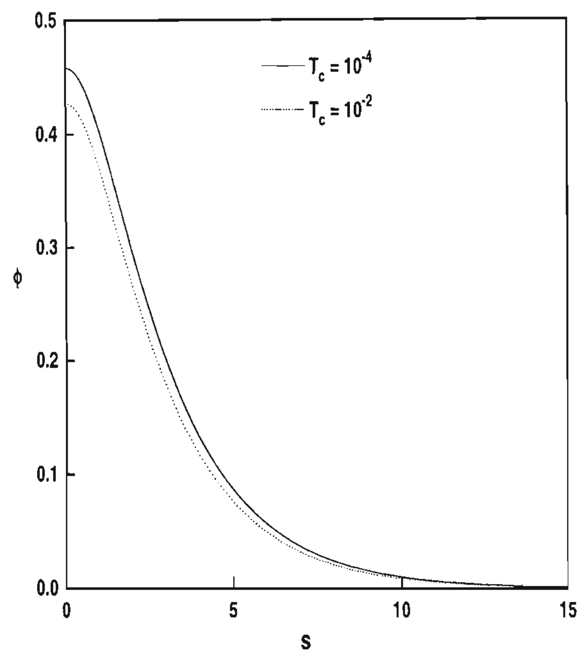


Figure 4.11: Soliton half profiles for the specific variables $M = 1.11$, $N_c = 0.5$ for two temperature ratios $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4}.

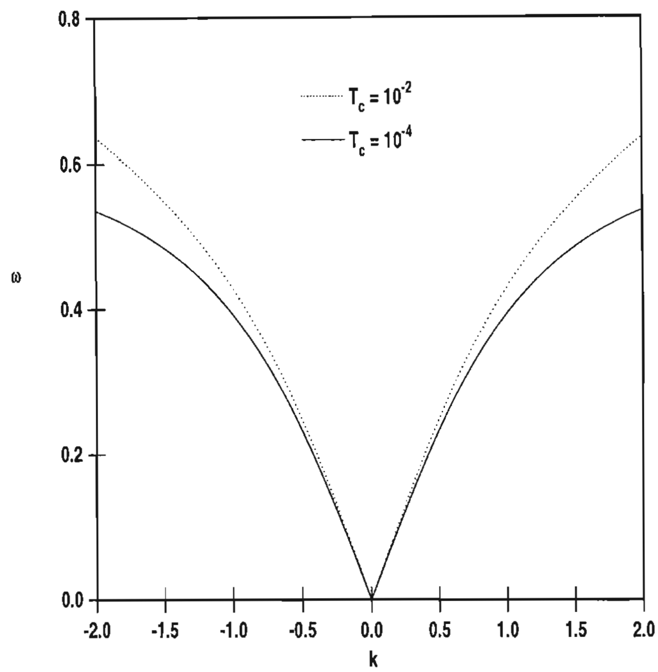


Figure 4.12: Dispersion curves for different temperature ratios $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4}, showing an decrease in dispersion as $\frac{T_c}{T_h}$ increases.

cool species particles match the phase velocity of the wave. These resonant particles travel along with the wave, and do not see a rapidly fluctuating electric field. They can therefore exchange energy with the wave effectively. The slower moving resonant particles gain energy from the wave, which loses energy and is damped (Chen 1984).

However, the model requires that

$$\left(\frac{T_c}{m}\right)^{\frac{1}{2}} \ll v_s \ll \left(\frac{T_h}{m}\right)^{\frac{1}{2}}$$

for the acoustic mode, else we cannot make the distinction between hot Boltzmann and cool adiabatic electrons and positrons. Thus allowing $T_c \approx T_h$ is prohibited. If we allowed T_c to approach T_h the cool particles could no longer be regarded as adiabatic, and massive, but would have to be considered as isothermal, Boltzmann distributions. The single species case of Boltzmann electrons and positrons has been mentioned in Section 2.3.1 where it was discovered that ω is indeterminate. Thus by virtue of the lack of inertia of the hot, isothermal particles, oscillations in this mode are unlikely.

Existence domains for solitons may be plotted in the space of soliton amplitude and cool species number density as shown in Figure 4.13, for a cool to hot temperature ratio of 10^{-2}, and in Figure 4.14, for a ratio of 10^{-4}. However, it will be seen that our model of hot, isothermal Boltzmann particles, and cool, adiabatic particles places severe constraints on the validity of the existence of arbitrary amplitude solitons.

4.5.1 Limits imposed by the model

Restrictions on ϕ

The requirement that the densities of the cool and hot species of electrons and positrons be real, imposes restrictions on the range of validity of the electrostatic potential ϕ. In order for the densities

$$n_{jc}^2 = \frac{N_c^2}{6T_c}\left[\left(\mu^2 - Z_j 2\phi + 3T_c\right) - \sqrt{(\mu^2 - Z_j 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right],$$

to be real, it is necessary that the terms under the square root sign must be greater than or equal to zero,

$$\left(\mu^2 - Z_j 2\phi + 3T_c\right)^2 - 12\mu^2 T_c \geq 0.$$

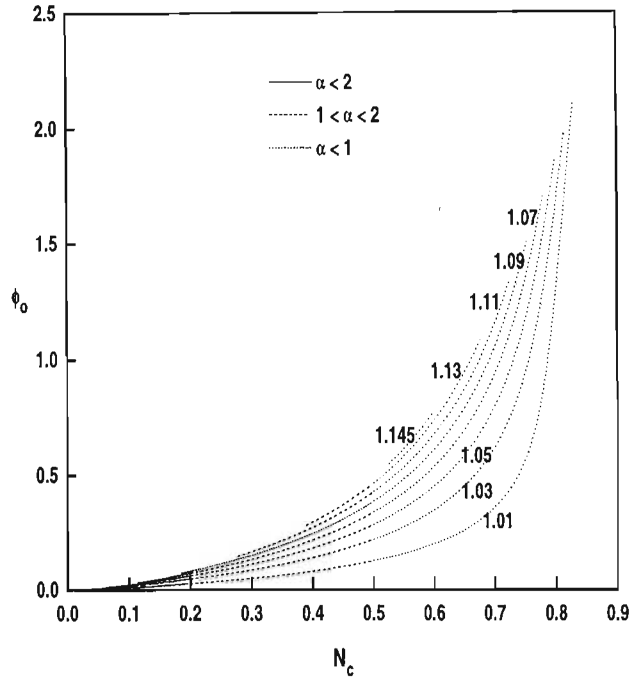


Figure 4.13: Existence domains for solitons in the space of soliton amplitude and cool species number density, for a cool to hot temperature ratio of 10^{-2}.

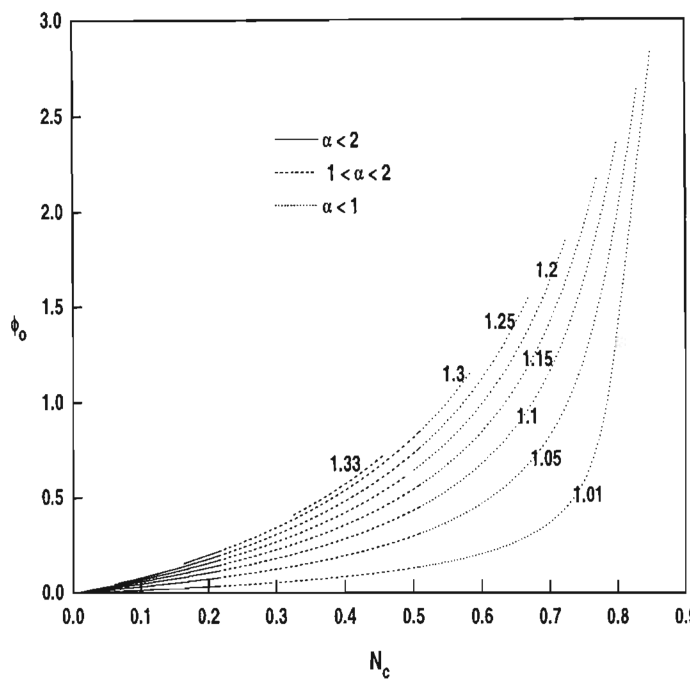


Figure 4.14: Existence domains for solitons in the space of soliton amplitude and cool species number density, for a cool to hot temperature ratio of 10^{-4}.

Rearranging in terms of ϕ yields

$$\phi_{min} = -\frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 \leq \phi \leq \frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 = \phi_{max}, \qquad (4.15)$$

which, in the limit as $T_c \to 0$ reduces to the condition

$$-\frac{\mu^2}{2} \leq \phi \leq \frac{\mu^2}{2},$$

also found for ion acoustic solitons (Chen 1984).

In light of the electrostatic potential cut-offs defined in (4.15), it is evident that in order to obtain a typical Sagdeev potential it is necessary that $V(\phi_{max}) > 0$ in the case of compressive solitons, or $V(\phi_{min}) > 0$ for rarefactive solitons, for the possibility of a root ϕ_0. Figure 4.15 shows typical Sagdeev potentials over the range $\phi = 0$ to $\phi = \phi_{max}$, for the existence curve $M = 1.13$, $\frac{T_c}{T_h} = 10^{-2}$. The cut-off points at both ends of the existence curve in Figure 4.13 are due to $V(\phi_{max})$ being less than zero at the corresponding value of N_c.

Figure 4.16 shows a series of Sagdeev curves over the range $\phi = 0$ to $\phi = \phi_{max}$ for $M = 1.09$, $\frac{T_c}{T_h} = 10^{-2}$ for increasing N_c values. It is thus observed that as the number of the cool species of electrons and positrons increases, a cut-off point is reached, above which solitons no longer form, since they would require electrostatic potentials higher than the ϕ_{max} limit imposed under the requirement of real electron-positron plasma densities, in order for $V(\phi)$ to cross the ϕ axis. Thus physically, the existence curves in Figures 4.13 and 4.14 terminate, due to the fact that we obtain imaginary densities above these cut-off points.

Limits on density

The physical situation of soliton formation is best explained by a plot of the natural logarithm of the positron density $n_p = n_{ph} + n_{pc}$ and electron density $n_e = n_{eh} + n_{ec}$ versus ϕ, Figure 4.17, (cf. Chen 1984). Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2} = n_e - n_p = -\frac{dV(\phi)}{d\phi}$$

gives

$$\frac{dV(\phi)}{d\phi} = n_p - n_e,$$

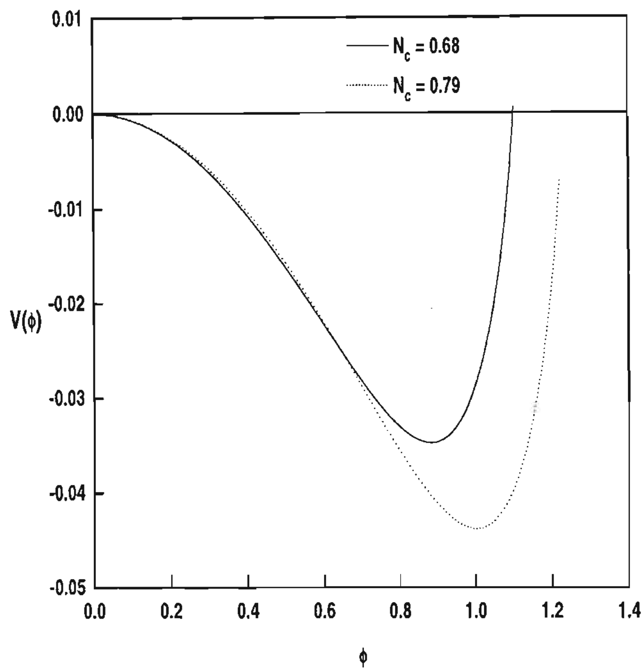


Figure 4.15: Typical Sagdeev potentials for $M = 1.13$, $\frac{T_c}{T_h} = 10^{-2}$ showing the cut-off points owing to $V(\phi_{max})$ being less than zero. The solution for $N_c = 0.68$ is valid, whereas that for $N_c = 0.79$, is not since the value of $\phi_{\max}$ is reached before $V(\phi, M) = 0$.

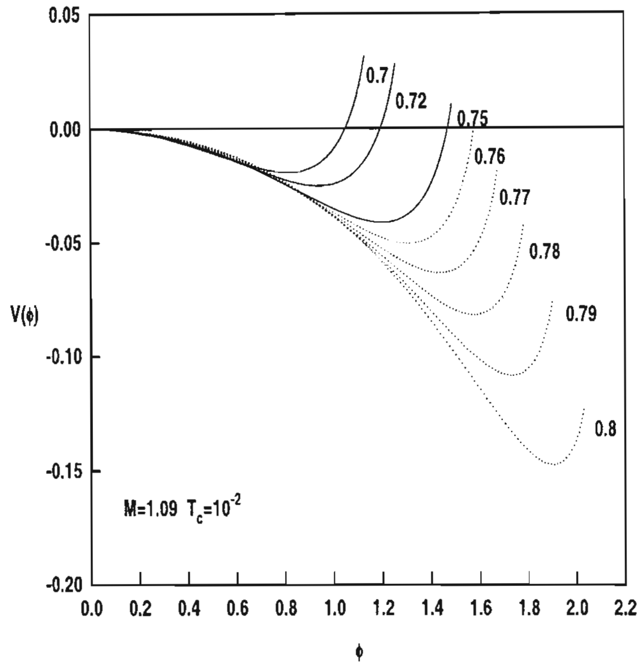


Figure 4.16: A series of Sagdeev potentials for $M = 1.09$, $\frac{T_c}{T_h} = 10^{-2}$ for increasing N_c values (numbering on curves). The existence cut-off occurs when $V(\phi_{max}) < 0$.

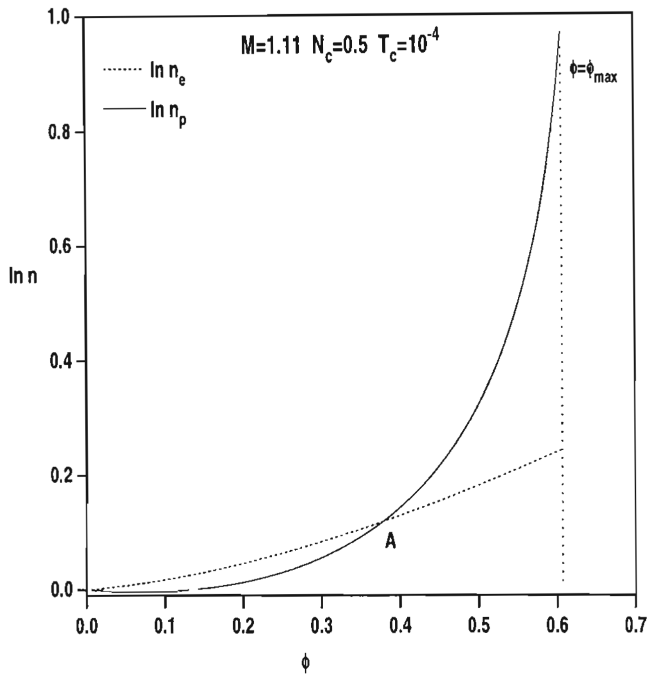


Figure 4.17: Natural logarithm curves of electron density n_e and positron density n_p versus ϕ.

which requires that n_p be less than n_e so that the Sagdeev potential $V(\phi)$ starts out with a negative curvature. Figure 4.17 shows the natural logarithm curves for n_e and n_p. The $\ln(n_p)$ curve should start out below the $\ln(n_e)$ curve for small ϕ. As ϕ tends to ϕ_0, $V(\phi_0) = 0$, implies that the areas under the two curves $\ln(n_p)$ and $\ln(n_e)$ must be equal. This suggests that at some ϕ the $\ln(n_p)$ curve must cross the $\ln(n_e)$ curve (at A). At this point where $\ln(n_e) = \ln(n_p)$, the Sagdeev potential has a minimum, corresponding to the inflection point of the $\phi(s)$ soliton profile. When ϕ is large enough so that the areas under the $\ln(n_e)$ and $\ln(n_p)$ curves are equal, the soliton reaches a peak of height ϕ_0, and the $\ln(n_e)$ and $\ln(n_p)$ curves are retraced as ϕ goes back to zero. In order that densities are real the condition

$$\phi_{min} = -\frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 \leq \phi \leq \frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 = \phi_{max}, \tag{4.16}$$

or for the compressive case only

$$\phi \leq \frac{1}{2}\left(\mu - \sqrt{3T_c}\right)^2 = \phi_{max}, \tag{4.17}$$

must hold. So in order to obtain a soliton, the positron density must exceed the electron density, so that the area under both curves $\ln(n_e)$ and $\ln(n_p)$ are equal, at some $\phi_0 < \phi_{\max}$. Thus the low Mach number cut-offs, for example (cf. Figure 4.13) the case of $T_c = 10^{-2}$ has a cut-off at $M < 1.2$, are better understood in light of the above explanation.

Figure 4.18 is a plot of $\ln(n_e)$ and $\ln(n_p)$ curves for $M = 1.2$ at $T_c = 10^{-2}$. It can be seen clearly that the area under the $\ln(n_p)$ curve is far less than that under the $\ln(n_e)$ curve when the maximum ϕ cut-off is reached. Thus no soliton solutions exist for $M > 1.15$ at $\frac{T_c}{T_h} = 10^{-2}$ since the area under the curves are never equal before ϕ_{max} is reached.

Unlike ion-acoustic solitons, in which the charge separation arises from the disparities in the densities of the ions and electrons, in the case of electron-positron soliton solutions of the four component model, the charge separation necessary for soliton formation arises because of the difference in the hot and cool electron and hot and cool positron densities.

Figure 4.19 shows density plots for the cool and hot electron species, for various Mach numbers, and Figure 4.20 show similar density plots for the cool and hot positron species, for various Mach numbers. It can be seen that the cool electron species exhibits a depletion in density, whereas the hot electron species shows a density enhancement at the soliton centre.

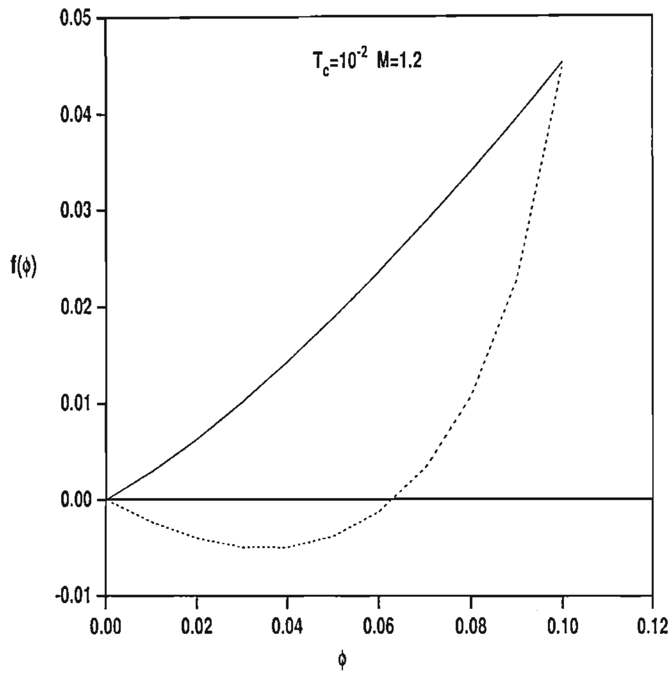


Figure 4.18: Natural logarithm curves of electron density $\ln(n_e)$ (solid curve) and positron density $\ln(n_p)$ (dashed curve) versus ϕ, for $M = 1.2$ at $T_c = 10^{-2}$.

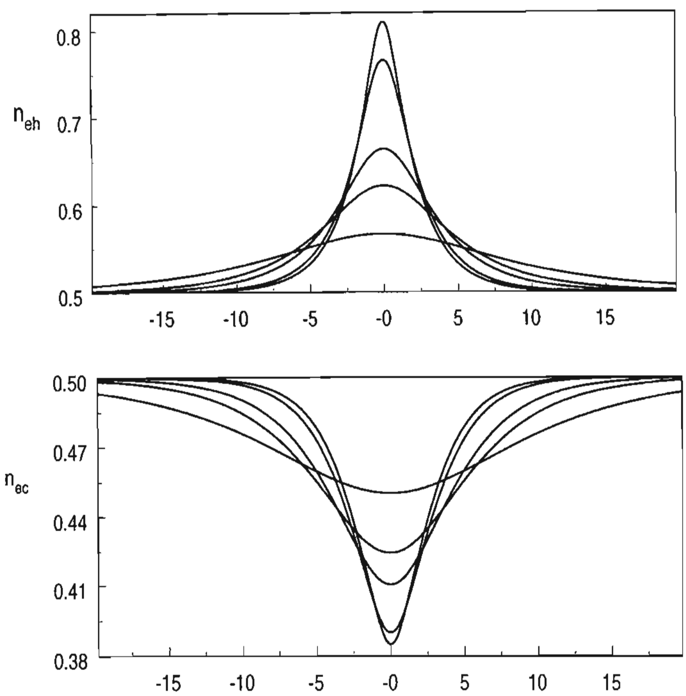


Figure 4.19: Electron hot and cool density profiles versus s, over a range of Mach numbers: $M = 1.01, 1.03, 1.05, 1.1, 1.4$. The Mach numbers increase in the order previously stated with increasing n_{eh} and decreasing n_{ec}. Other fixed parameters are $N_c = 0.5$ and $\frac{T_c}{T_h} = 10^{-2}$.

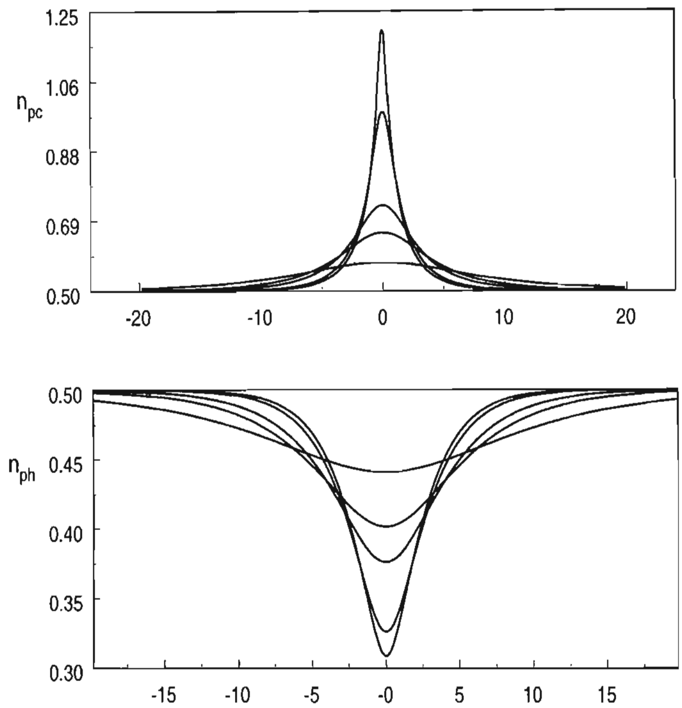


Figure 4.20: Positron hot and cool density profiles versus s, over a range of Mach numbers: $M = 1.01, 1.03, 1.05, 1.1, 1.4$. The Mach numbers increase in the order previously stated with increasing n_{pc} and decreasing n_{ph}. Other fixed parameters are $N_c = 0.5$ and $\frac{T_c}{T_h} = 10^{-2}$.

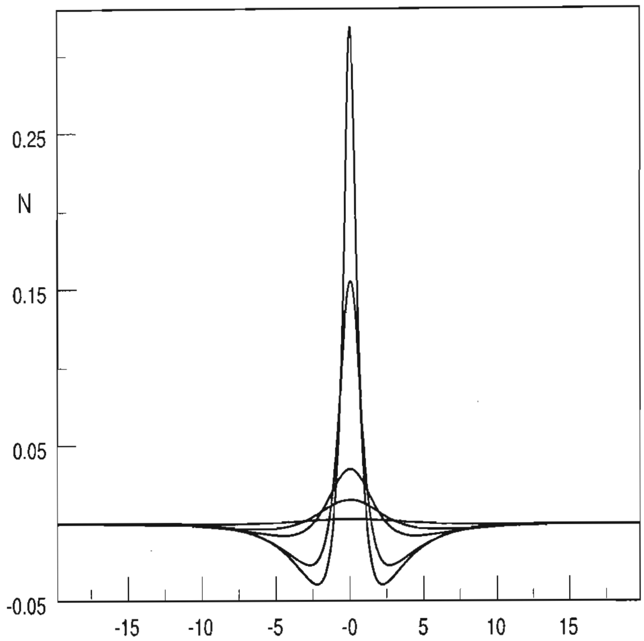


Figure 4.21: Total charge density profiles versus s, over a range of Mach numbers: $M = 1.01, 1.03, 1.05, 1.1, 1.4$. The Mach numbers increase in the order previously stated with increasing N. Other fixed parameters are $N_c = 0.5$ and $\frac{T_c}{T_h} = 10^{-2}$.

Conversely, the cool positrons exhibit a density enhancement, and the hot positrons a density depletion at the soliton centre. Finally, Figure 4.21 shows a plot of the total charge density, indicating that the compressive electron-positron acoustic soliton manifests itself as a slab of positive space charge "sandwiched" between adjacent wedges of negative space charge.

Limits on temperature

We may approach the Mach number cut-offs in the existence domain diagrams 4.13 and 4.14 from a different perspective, that is, as caused by a limit on the cool species temperature. Solving equation (3.82) for temperature rather than ϕ gives

$$T_c < \frac{1}{3}\left(\mu - \sqrt{2\phi}\right)^2. \tag{4.18}$$

Note that equation (3.82) is in normalized form, and thus references to T_c actually imply the ratio between T_c and T_h, $\frac{T_c}{T_h}$.

Considering (4.18) it must follow that

$$T_c < \frac{1}{3}\left(\mu - \sqrt{2 \mid \phi_0 \mid}\right)^2,$$

since this is the maximum amplitude of a particular soliton. This indicates that for a specific Mach number, a soliton with amplitude ϕ_0, will only exist for temperature ratios given by the above inequality.

4.5.2 The Boltzmann restriction

The most significant restriction placed on the solutions obtained under our four component model is due to the assumption made that the hot species are Boltzmann distributed. We assumed that the hot particles were essentially 'massless', effectively ignoring their inertia. This assumption is only justified if the thermal speed of the hot components is much larger than the sound speed of the waveform

$$v_h \gg v_s.$$

This inequality may be expressed in the form

$$v_h > \alpha v_s,$$

where α should be greater than 2, at least! Since speeds are normalized with respect to the hot species thermal speed v_h, we may write

$$\left(\frac{N_c}{N_h} + 3T_c\right) < \frac{1}{\alpha^2},$$

where $V = \frac{v_s}{v_h} = \left(\frac{N_c}{N_h} + 3T_c\right)^{\frac{1}{2}}$. As $T_c \to 0$ we obtain the condition

$$N_c < \frac{1}{\alpha^2 + 1}.$$

This imposes an upper limit on the cool species number density of $N_c \leq 0.2$, for an α of 2. As we really require $\alpha > 2$, that would mean that the maximum of N_c is actually even smaller.

Thus, in order to make the assumption of hot "massless" Boltzmann electrons and positrons, we require very small number densities of cool electrons and positrons at equilibrium (less than 20 % of the total equilibrium density!).

Considering the curves in Figures 4.13 and 4.14, only the solid portion of the curves are valid under this approximation. It is evident thus that this restriction implies that only small amplitude solitons, $\phi_0 < 0.3$, in the region of cool species number density between zero and $N_c = 0.2$, are valid. This suggests that the small amplitude mKdV perturbatory technique is sufficient to describe the existence of solitons formed in an electron-positron plasma of cool, adiabatic and hot isothermal particles.

Figure 4.22 shows a plot of Mach number M versus soliton amplitude ϕ_0 at $N_c = 0.2$ the upper limit. This plot indicates that soliton amplitude increases with increasing Mach number. It also indicates that larger amplitude solitons with larger Mach numbers are possible only at small values of the ratio $\frac{T_c}{T_h}$. We see that with a temperature ratio of $\frac{T_c}{T_h} = 0$ (an academic case), we obtain soliton solutions with Mach numbers greater than the other two cases $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4}, conforming to the upper limit of $M < 1.53$ as obtained in Section 3.4.2. This case of $\frac{T_c}{T_h} = 0$ requires either $T_c = 0$ or $T_h \to \infty$. Since we wish to have 'cool' particles, we may discard the former. A temperature ratio of $\frac{T_c}{T_h} = 0$ with $T_h \to \infty$ yielding larger Mach number cut-offs, seems understandable in the light of the Boltzmann restriction, which requires

$$v_h \gg v_s \gg v_c,$$

or in unnormalized terms

$$T_h \gg T_c.$$

Thus the larger the difference between the temperatures of the cool and hot species, the closer we are to achieving our Boltzmann requirement.

On the whole though, the soliton amplitude is rather small, $\phi_0 < 0.3$, suggesting that small amplitude modelling (mKdV) should be accurate. A comparison of existence plots for both large and small (mKdV) solitons follows.

4.5.3 Comparison of mKdV and arbitrary amplitude results

It is evident that the Boltzmann assumption for hot particles in our model leads to a very limited range of validity of soliton solutions (in $\phi - N_{ec}$ space). The amplitudes of these physical solitons are small, $\phi_0 < 0.3$. We thus speculate that the small amplitude mKdV analysis may be sufficiently accurate to describe the nature of electron-positron solitons under the conditions of our model. We present here a comparison of the small and arbitrary amplitude soliton existence domains in order to examine whether this speculation is indeed valid.

Figure 4.23 and 4.24 show existence domains for both mKdV and arbitrary amplitude solitons for temperature ratios $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4} respectively. It is immediately apparent from the figures, that the two types of curves are very similar. Closer examination shows, however, that at large values of cool species number density N_c, the mKdV curves deviate quite dramatically from their arbitrary amplitude counterparts. However, since the range of validity is such that $N_c < 0.2$, this is purely academic. On the whole, the curves for temperature ratio 10^{-2}, compare more favorably than those for $\frac{T_c}{T_h} = 10^{-4}$. This is simply because at larger temperature ratios, the soliton amplitudes are larger, and the mKdV theory breaks down. There is, in fact, very good correlation for lower Mach numbers (and hence smaller amplitudes) in the $\frac{T_c}{T_h} = 10^{-4}$ case.

Figures 4.25 and 4.26 show comparisons of the small and arbitrary amplitude solutions within the range of validity specified by the hot Boltzmann assumption. These figures make it clear that the mKdV theory is only valid for soliton amplitudes $\phi < 0.06$, above which, the correlation breaks down. The small amplitude theory is very accurate, at smaller Mach numbers of

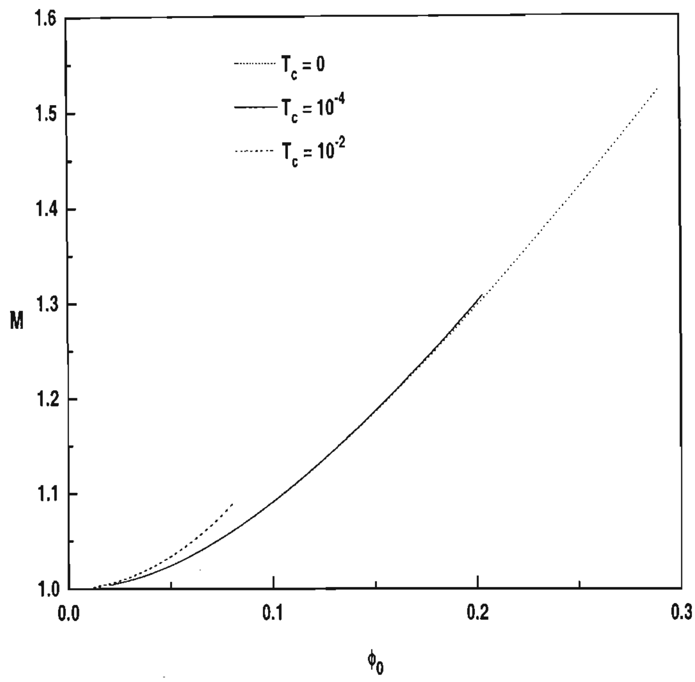


Figure 4.22: Mach number M versus soliton amplitude ϕ_0 at specific value $N_c = 0.2$ for various values of temperature ratio.

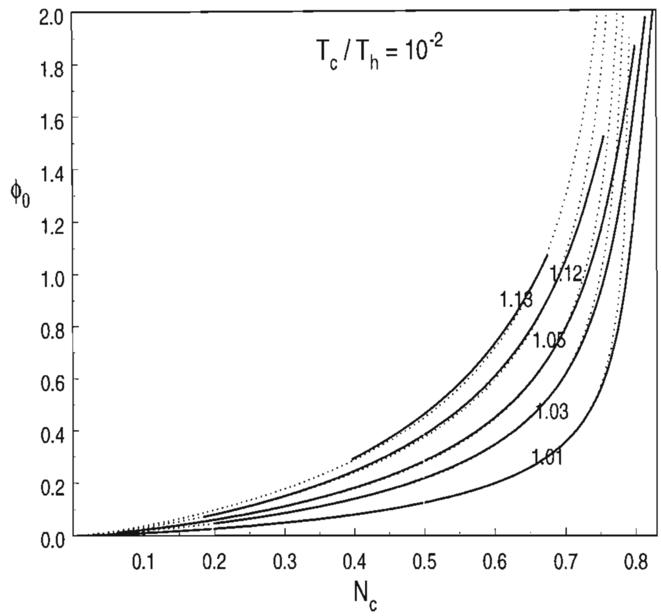


Figure 4.23: Comparison of small and arbitrary amplitude soliton existence domains for $\frac{T_c}{T_h} = 10^{-2}$. The numbers on the curves refer to Mach number.

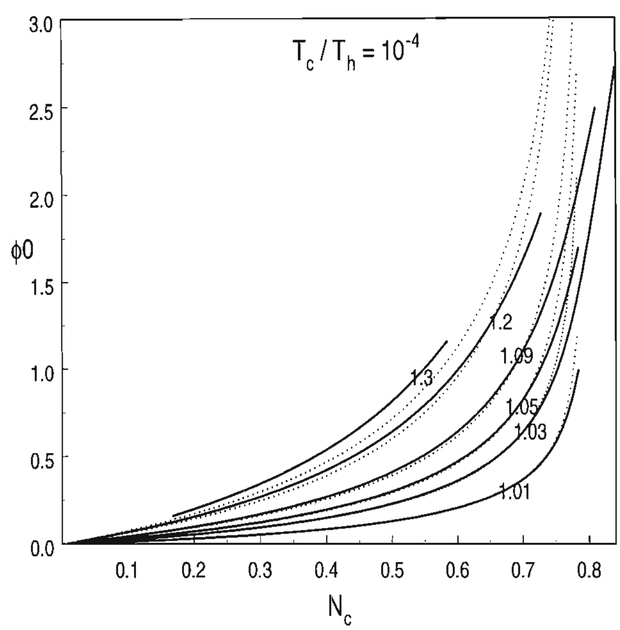


Figure 4.24: Comparison of small and arbitrary amplitude soliton existence domains for $\frac{T_c}{T_h} = 10^{-4}$. The numbers on the curves refer to Mach number.

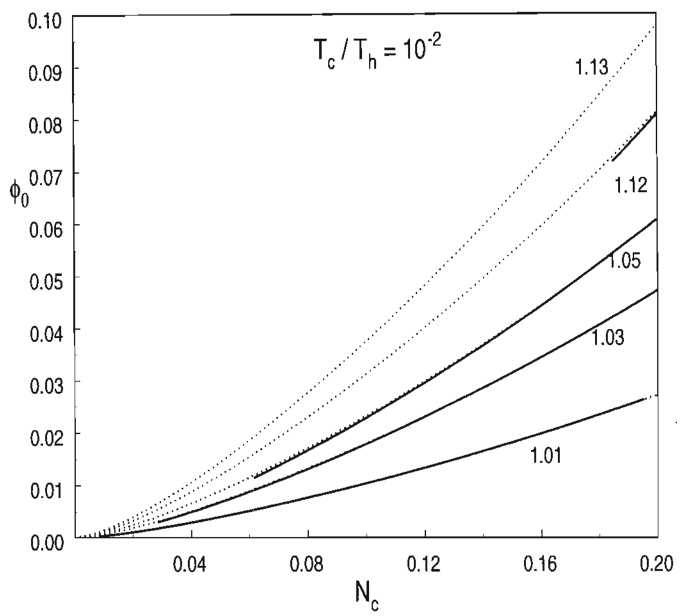


Figure 4.25: Comparison of small and arbitrary amplitude soliton existence domains for $\frac{T_c}{T_h} = 10^{-2}$ over the valid range $N_{ec} < 0.2$. The numbers on the curves refer to Mach number.

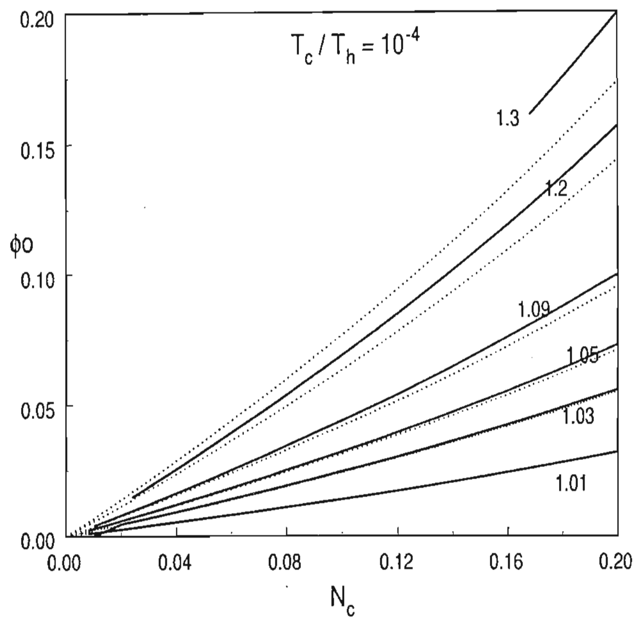


Figure 4.26: Comparison of small and arbitrary amplitude soliton existence domains for $\frac{T_c}{T_h} = 10^{-4}$ over the valid range $N_{ec} \leq 0.2$. The numbers on the curves refer to Mach number.

$M \leq 1.05$, however at higher Mach numbers it consistently predicts solitons with amplitudes smaller than the arbitrary amplitude solutions.

Most importantly, the mKdV theory does not incorporate the cut-offs predicted by the arbitrary amplitude soliton theory, as described in Sections 4.5.1 and 4.5.2 . Thus, although it shows good correlation with the arbitrary amplitude theory at low soliton amplitudes and Mach numbers, it does not give a true picture of the nature of soliton existence, and thus should not be relied on alone.

Chapter 5

Asymmetric electron-positron plasmas

As discussed in Chapter 1, a commonly used model for the magnetosphere of pulsars is an electron-positron plasma rotating in the pulsar magnetic field. The original analysis in this thesis has been concerned with symmetric electron-positron plasmas, in particular a four component fluid model. However it has been mentioned in the introductory chapter that studies of asymmetric EP plasmas have been made by a number of authors (Bharuthram 1992; Pillay & Bharuthram 1992; Srinivas *et al.* 1996). In this chapter we review the study of solitons in an asymmetric EP plasma conducted by Pillay & Bharuthram (1992), and discover that the results obtained in this work may be physically invalid in the light of restrictions imposed by the choice of model.

5.1 The four component asymmetric model

Pillay and Bharuthram (1992) invoke a fluid plasma model for the pulsar magnetosphere, in which both the electron and positron populations are subdivided into two groups of distinct temperatures, one group modelling the original (primary) EP pairs, and the other the higher energy (secondary) cascade-bred pairs. The primary particles (electrons and positrons) are treated strictly as cold fluids with $T_c = 0$, whereas secondary (more energetic) particles are described by Boltzmann distributions, which in view of their higher energies, and thus temperatures, seems a reasonable assumption.

The model used by Pillay *et al.* presents an asymmetry between the two types of charged particles, in that the number densities N_{ec} and N_{pc} of the cold electrons and positrons are not equal, similarly the hot electrons and positrons have different number densities N_{eh} and N_{ph}, and different temperatures T_e and T_p. In light of the electron-positron pair production mechanism this scenario seems unlikely, since equal numbers of electrons and positrons with equal energies are produced in the cascade process.

The symmetric model used in this thesis considers electrostatic solitons when both electrons and positrons are treated on the same footing, the hot particles by Boltzmann distributions at the same temperature T_h, and the cool particles by adiabatic fluids also at the same temperature T_c. However, the situation described in the preceding paragraphs may be applicable in the region outward of the vacuum gap where, owing to the potential in the gap, one species of high energy particles may collect, and in this way their number density may exceed that of their anti-particle. Thus it would be informative to review the findings of Pillay *et al.* in the interest of completeness.

5.2 Basic equations

Pillay & Bharuthram (1992) consider an unmagnetized electron-positron plasma consisting of a hot and cold fluid species each of electrons and positrons with equilibrium number densities N_{eh}, N_{ec}, N_{ph} and N_{pc} respectively. The condition of charge neutrality at equilibrium requires that

$$N_{eh} + N_{ec} = N_{ph} + N_{pc} = N_o, \tag{5.1}$$

where N_o is the total equilibrium number density of the undisturbed plasma.

We follow Pillay *et al.* (1992) in their derivation of the equations describing their four component asymmetrical electron-positron plasma.

Both the hot electrons and positrons are assumed to be isothermal fluids, whose densities are given by Boltzmann distributions

$$n_{eh} = N_{eh} \exp(\phi), \tag{5.2}$$

$$n_{ph} = N_{ph} \exp(-\alpha_{ph}\phi). \tag{5.3}$$

The electrostatic potential ϕ is normalized with respect to the hot electron temperature T_e so that $\tilde{\phi} = \frac{e\phi}{T_e}$ (the tilde is omitted as usual), and $\alpha_{ph} = \frac{T_e}{T_p}$ is the ratio of hot electron and positron temperatures.

The dynamics of the cold electron and positron fluids are governed by the equations of continuity and momentum

$$\frac{\partial n_{jc}}{\partial t}+\frac{\partial\left(n_{jc}u_{jc}\right)}{\partial x}=0, \tag{5.4}$$

and

$$\frac{\partial u_{jc}}{\partial t}+u_{jc}\frac{\partial u_{jc}}{\partial x}=-Z_j\frac{\partial \phi}{\partial x}, \tag{5.5}$$

where $j=e(p)$ for electrons (positrons) and $Z_j=\frac{q_j}{e}$. The system of equations is closed by Poisson's equation

$$\frac{\partial^2\phi}{\partial x^2}=N_{eh}\exp(\phi)+n_{ec}-N_{ph}\exp(-\alpha_{ph}\phi)-n_{pc}. \tag{5.6}$$

In equations (5.4)–(5.6) densities are normalized with respect to the total equilibrium value N_o, velocities are normalized by the hot electron thermal speed $v_{eh}=\left(\frac{T_e}{m}\right)^{\frac{1}{2}}$, spatial lengths by the Debye length $\lambda_{De}=\left(\frac{T_e}{4\pi N_o e^2}\right)^{\frac{1}{2}}$, and time by the inverse plasma frequency $\omega_p^{-1}=\left(\frac{4\pi N_o e^2}{m}\right)^{-\frac{1}{2}}$. Also $m_e=m_p=m$ is the mass of the electrons and positrons.

5.3 Linear acoustic waves

For a comparison with our own results, let us investigate the dispersion relation of linear acoustic waves by the process of linearization of the system of equations describing the four component asymmetric EP plasma.

Assuming that the amplitude of oscillation is small, we may use the process of linearization (Chen 1984) in which higher order amplitude factors are neglected. All variables are expressed in terms of their equilibrium value, and a perturbation, denoted by a subscript $_1$

$$\begin{aligned} n_{jk} &= N_{jk}+n_{jk1} \\ u_{jk} &= u_{jko}+u_{jk1} \\ \phi &= \phi_o+\phi_1. \end{aligned} \tag{5.7}$$

We assume an electrically neutral, uniform plasma at equilibrium before perturbation, thus

$$u_{jko}=\phi_o=0=\frac{\partial n_{jko}}{\partial x}=\frac{\partial n_{jko}}{\partial t}. \tag{5.8}$$

The exponential term in (5.2) and (5.3) may be expanded so that

$$n_{eh} = N_{eh}(1 + \phi + \cdots)$$

$$n_{ph} = N_{ph}(1 - \alpha_{ph}\phi + \cdots)$$

This becomes with (5.7) and (5.8)

$$\begin{aligned} &(N_{eh} + n_{eh1}) = N_{eh} + N_{eh}(\phi_1) \\ &\Rightarrow n_{eh1} = N_{eh}\phi_1. \end{aligned} \tag{5.9}$$

$$\begin{aligned} &(N_{ph} + n_{ph1}) = N_{ph} - \alpha_{ph}N_{ph}(\phi_1) \\ &\Rightarrow n_{eh1} = -\alpha_{ph}N_{ph}\phi_1. \end{aligned} \tag{5.10}$$

Expanding (5.5), (5.4) and (5.6) with (5.7) and neglecting quadratic and higher order terms gives

$$\frac{\partial u_{jc1}}{\partial t} = -Z_j\frac{\partial \phi_1}{\partial x}, \tag{5.11}$$

$$\frac{\partial n_{jc1}}{\partial t} + N_{jc}\frac{\partial u_{jc1}}{\partial x} = 0, \tag{5.12}$$

and

$$\frac{\partial^2 \phi_1}{\partial x^2} = (N_{eh} + n_{eh1}) + (N_{ec} + n_{ec1}) - (N_{ph} + n_{ph1}) - (N_{pc} + n_{pc1}),$$

where $N_{eh} + N_{ec} = 1 = N_{ph} + N_{pc}$. Thus

$$\frac{\partial^2 \phi_1}{\partial x^2} = (n_{eh1} + n_{ec1}) - (n_{ph1} + n_{pc1}). \tag{5.13}$$

The perturbations are assumed to be sinusoidal, that is we consider a Fourier mode (ω, k)

$$\begin{aligned} u_{jc1} &= \tilde{u_{jc1}} \exp[i(kx - \omega t)], \\ n_{jc1} &= \tilde{n_{jc1}} \exp[i(kx - \omega t)], \\ \phi_1 &= \tilde{\phi_1} \exp[i(kx - \omega t)], \end{aligned} \tag{5.14}$$

where the variables with the tilde are the amplitudes of the sinusoidal variations. We omit the tilde for simplicity, in the following derivation.

Substituting (5.14) into equations (5.11)–(5.13) yields

$$\omega u_{jc1} = Z_j k \phi_1, \tag{5.15}$$

$$u_{jc1} = \frac{\omega}{k}\frac{n_{jc1}}{N_{jc}}, \tag{5.16}$$

$$-k^2\phi_1 = n_{eh1} + n_{ec1} - n_{ph1} - n_{pc1}. \tag{5.17}$$

Further substitution of (5.16) into (5.15) gives

$$\frac{n_{jc1}}{N_{jc}} = Z_j\phi_1\frac{k^2}{\omega^2}. \tag{5.18}$$

Finally substituting (5.18), (5.9) and (5.10) into (5.17) gives

$$-k^2\phi_1 = N_{eh}\phi_1 - N_{ec}\frac{k^2}{\omega^2}\phi_1 + N_{ph}\alpha_{ph}\phi_1 - N_{pc}\frac{k^2}{\omega^2}\phi_1,$$

that is

$$-k^2 = N_{eh} - N_{ec}\frac{k^2}{\omega^2} + N_{ph}\alpha_{ph} - N_{pc}\frac{k^2}{\omega^2}. \tag{5.19}$$

Rearranging (5.19) we obtain the expression

$$\omega^2 = \frac{(N_{ec} + N_{pc})k^2}{k^2 + (N_{eh} + N_{ph}\alpha_{ph})}. \tag{5.20}$$

In the very long wavelength limit for $k \to 0$ and with V as the normalized sound speed, this equation reduces to the acoustic mode

$$\omega = Vk,$$

where

$$V = \left(\frac{N_{ec} + N_{pc}}{N_{eh} + N_{ph}\alpha_{ph}}\right)^{\frac{1}{2}}. \tag{5.21}$$

5.4 Soliton solutions - the Sagdeev potential

Seeking stationary solutions of equations (5.4)–(5.6) in the stationary frame $s = x - \mu t$, where $\mu = \frac{U}{v_{eh}}$ is defined as the Mach number, for U the speed of the soliton, Pillay *et al.* (1992) obtain the cold fluid normalized densities, from (5.4) and (5.5)

$$n_{ec}(\phi) = \frac{N_{ec}\mu}{(\mu^2 + 2\phi)^{\frac{1}{2}}}, \tag{5.22}$$

and

$$n_{pc}(\phi) = \frac{N_{pc}\mu}{(\mu^2 - 2\phi)^{\frac{1}{2}}}, \tag{5.23}$$

under the assumed boundary conditions

$$n_{jc} \to 0; \quad u_{jc} \to 0; \quad \phi(s) \to 0; \quad \frac{d\phi}{ds} \to 0;$$

as $s \to \infty$, stipulating an undisturbed plasma at infinity.

Substituting (5.22) and (5.23) into Poisson's equation (5.6) and integrating over the chosen boundary conditions yields

$$\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 + V(\phi) = 0,$$

where the Sagdeev potential (Sagdeev 1966) is given by

$$\begin{aligned} V(\phi,\mu) = \; & N_{eh}[1 - e^{(\phi)}] + \tfrac{N_{ph}}{\alpha_{ph}}[1 - e^{(-\alpha_{ph}\phi)}] \\ & + N_{ec}[\mu^2 - \mu\sqrt{\mu^2 + 2\phi}] + N_{pc}[\mu^2 - \mu\sqrt{\mu^2 - 2\phi}]. \end{aligned} \tag{5.24}$$

For the existence of solitons one requires (Sagdeev 1966) that

$$\begin{array}{ll} (i) & V(\phi) = \frac{\partial V(\phi)}{\partial \phi}|_{\phi=0} = 0 \\ (ii)(a) & V(\phi_0) = 0 \\ (ii)(b) & \frac{\partial V(\phi)}{\partial \phi}|_{\phi=\phi_0} < (>)0; \quad \phi_0 < (>)0 \\ (iii) & V(\phi) < 0; \quad 0 < |\phi| < |\phi_0| \end{array} \tag{5.25}$$

where ϕ_0 is the soliton amplitude. For condition (5.25) (ii)(b) $\phi_0 > (<)0$ refers to compressive (rarefactive) solitons.

The requirement of real densities places restrictions on the speed of the soliton profiles cf. equations (5.22) and (5.23) which imply the constraints

$$\mu^2 > 2\phi_m \tag{5.26}$$

for compressive solitons, and

$$\mu^2 > -2\phi_m \tag{5.27}$$

for rarefactive solitons, where ϕ_m is the maximum value of ϕ allowed for real densities.

5.5 Results

Pillay *et al.* (1992) numerically obtain both compressive and rarefactive soliton solutions. Figures 5.1 (compressive case) and 5.2 (rarefactive case) are reproductions of their results, showing the variation of soliton amplitude ϕ_0 with normalized cold electron density, for different values of cold positron density. For both the rarefactive and compressive case, the normalized soliton speed is chosen as $\mu = 1.2$, and the ratio of hot species temperature $\alpha_{ph} = \frac{T_e}{T_p} = 0.25$, is such that the hot positron temperature is four times that of the hot electrons.

The upper limit of soliton amplitude, $\phi = \phi_m$, represented by the dotted horizontal line in the figures, is associated with the restriction (5.26) and (5.27), so that $-\frac{\mu^2}{2} < \phi_0 < \frac{\mu^2}{2}$, preventing imaginary cold species densities.

5.5.1 The Boltzmann hot particle assumption

The assumption of Boltzmann hot particles leads to some rather stringent restrictions on soliton existence, in much the same way as in Section 4.5.2. We require that the thermal speed of the hot components be very much larger than the sound speed of the waveform

$$v_{jh} \gg v_s,$$

where the v_{jh} is the thermal speed of the hot particles $j = e(p)$. This inequality may be expressed generally, as before, in the form

$$v_{jh} > \theta v_s, \tag{5.28}$$

where the factor θ should be greater than 2, at the very least. Since speeds are normalized with respect to the hot electron thermal speed v_{eh}, (5.28) reduces to the condition

$$\left(\frac{v_{jh}}{v_{eh}}\right)^2 > \theta^2 V^2, \tag{5.29}$$

where V is given by (5.21).

For Boltzmann electrons we have from (5.29)

$$\left(\frac{N_{ec} + N_{pc}}{N_{eh} + N_{ph}\alpha_{ph}}\right) < \frac{1}{\theta^2}. \tag{5.30}$$

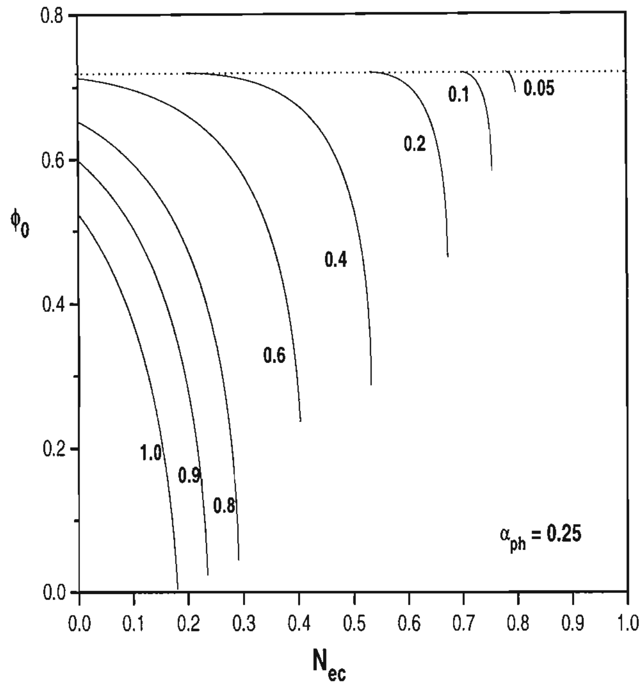


Figure 5.1: Existence domains for compressive solitons, obtained by Pillay & Bharuthram (1992). Specific parameters are $\mu = 1.2$ and $\alpha_{ph} = 0.25$. The numbering on the curves refers to the cold positron number density.

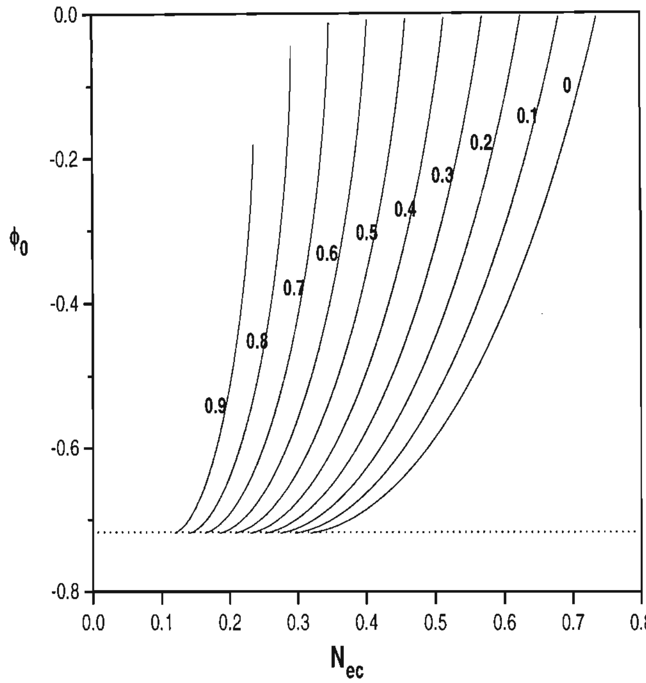


Figure 5.2: Existence domains for rarefactive solitons, obtained by Pillay & Bharuthram (1992). Specific parameters are $\mu = 1.2$ and $\alpha_{ph} = 0.25$. The numbering on the curves refers to the cold positron number density.

For Boltzmann positrons we have

$$\frac{T_p}{T_e} > \theta^2 V^2,$$

that is,

$$\left(\frac{N_{ec} + N_{pc}}{N_{eh} + N_{ph}\alpha_{ph}}\right) < \frac{1}{\alpha_{ph}\theta^2}. \tag{5.31}$$

We may rearrange (5.30) and (5.31) with the normalized charge neutrality condition

$$N_{ec} + N_{eh} = N_{pc} + N_{ph} = 1;$$

to obtain the following inequalities linking the cool particle number density and the constant θ. For Boltzmann electrons

$$(\theta^2 + 1)N_{ec} + (\theta^2 + \alpha_{ph})N_{pc} < 1 + \alpha_{ph}, \tag{5.32}$$

and for Boltzmann positrons

$$(\alpha_{ph}\theta^2 + 1)N_{ec} + \alpha_{ph}(\theta^2 + 1)N_{pc} < 1 + \alpha_{ph}. \tag{5.33}$$

It is evident from (5.32) and (5.33) that N_{ec}, α_{ph} and N_{pc} are inexorably linked.

If we assume $\theta = 2$ so that $v_{jh} > 2v_s$, then (5.32) becomes

$$5N_{ec} + (4 + \alpha_{ph})N_{pc} < 1 + \alpha_{ph}; \tag{5.34}$$

and (5.33) becomes

$$(4\alpha_{ph} + 1)N_{ec} + 5\alpha_{ph}N_{pc} < 1 + \alpha_{ph}. \tag{5.35}$$

In order to have physical values of cool species number density it is necessary that $N_{ec} \geq 0$ and $N_{pc} \geq 0$. For Boltzmann electrons, rearranging (5.34) so that

$$5N_{ec} < (1 + \alpha_{ph}) - (4 + \alpha_{ph})N_{pc},$$

it is immediately obvious that

$$N_{pc} \leq \frac{1 + \alpha_{ph}}{4 + \alpha_{ph}} = N_{pc(MAX)}, \tag{5.36}$$

for $N_{ec} \geq 0$. Similarly, rearranging (5.34) thus

$$(4+\alpha_{ph})N_{pc} < (1+\alpha_{ph}) - 5N_{ec},$$

we must have

$$N_{ec} \leq \frac{1+\alpha_{ph}}{5} = N_{ec(MAX)}, \tag{5.37}$$

for $N_{pc} \geq 0$.

On the other hand, for Boltzmann positrons, rearranging (5.35) so that

$$(4\alpha_{ph}+1)N_{ec} < (1+\alpha_{ph}) - 5\alpha_{ph}N_{pc},$$

it is obvious that

$$N_{pc} \leq \frac{1+\alpha_{ph}}{5\alpha_{ph}} = N_{pc(MAX)}, \tag{5.38}$$

for $N_{ec} \geq 0$. Similarly, rearranging (5.35) thus

$$5\alpha_{ph}N_{pc} < (1+\alpha_{ph}) - (4\alpha_{ph}+1)N_{ec},$$

$$N_{ec} \leq \frac{1+\alpha_{ph}}{4\alpha_{ph}+1} = N_{ec(MAX)}, \tag{5.39}$$

for $N_{pc} \geq 0$.

Figures 5.3 and 5.4 show plots of these maximum values of N_{pc} and N_{ec} for varying temperature ratios, α_{ph}, for both electron and positron Boltzmann restrictions. Figure 5.3 shows that for values of $\alpha_{ph} > 1$, the N_{pc} cut-off imposed by the assumption of Boltzmann positrons, tends to a value of $N_{pc(MAX)} = 0.2$ as $\alpha_{ph} \to \infty$, whereas the cut-off value imposed by the assumption of Boltzmann electrons, tends to unity. Since both conditions must be satisfied, we have in general, an imposed cold positron density cut-off of $N_{pc} \leq 0.2$. Similarly, Figure 5.4 shows that the N_{ec} cut-off, imposed by Boltzmann electrons, tends to infinity as $\alpha_{ph} \to \infty$, and $N_{ec(MAX)} \to 0.25$, for Boltzmann positrons. Again, since both the electron and positron Boltzmann conditions must hold true, we have, in general, $N_{ec} \leq 0.25$.

We see thus that the assumption of Boltzmann hot particles leads to severe restrictions on the allowed numbers of cool particles. The Boltzmann assumption will only apply then, in the case of very hot electrons ($v_{eh} \to \infty$), if there exists only very small equilibrium numbers of cold electrons and positrons (less than or equal to 20%–25% of the equilibrium plasma density). This corresponds to the requirement of very high percentages (75% – 80%) of

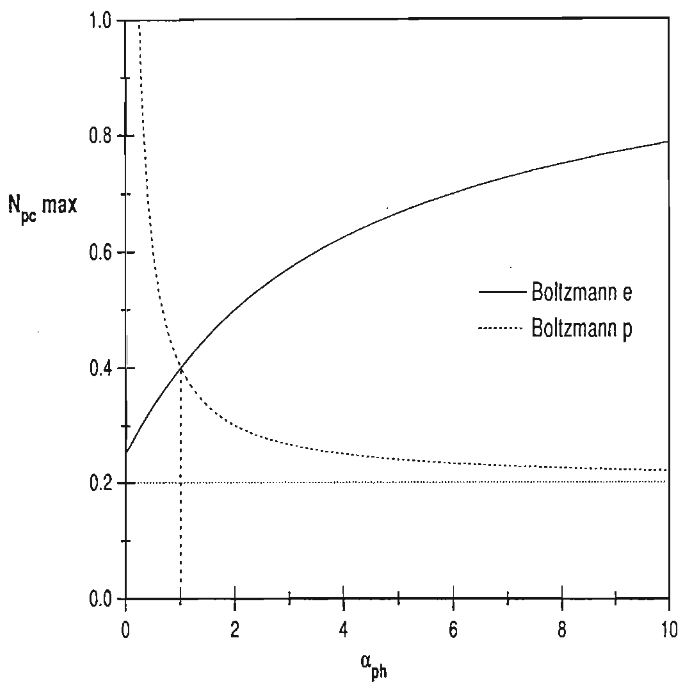


Figure 5.3: Maximum values for N_{ec} for varying α_{ph}, the solid curve refers to the electron Boltzmann restriction and the dashed curve to the positron Boltzmann restriction.

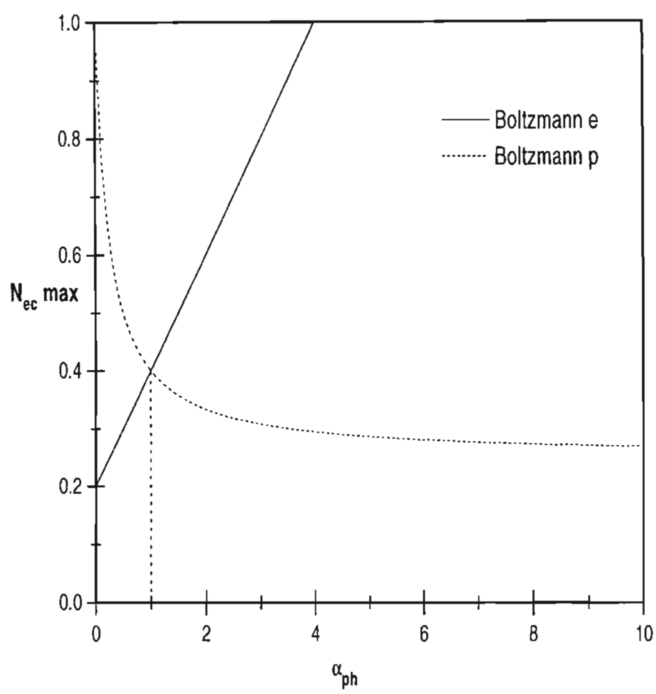


Figure 5.4: Maximum values for N_{ec} for varying α_{ph}, the solid curve refers to the electron Boltzmann restriction and the dashed curve to the positron Boltzmann restriction.

hot particles at equilibrium, if their Boltzmann nature is to be valid as $\frac{T_e}{T_p} \to \infty$. This seems to suggest that at high temperature ratios the hot particles tend to play the dominant part in soliton formation, with the cool particles having very little effect. Perhaps we should question this assumption of $\alpha_{ph} \to \infty$, however. Remembering that $\alpha_{ph} = \frac{T_e}{T_p}$, represents the temperature ratio of the hot particles, if we then consider $\alpha_{ph} \to \infty$, this suggests that either $T_e \to \infty$, or $T_p \to 0$. In either case, with such large temperature differences, it would be more probable then to describe the positron species as adiabatic, in relation to the much hotter electrons. Such high α_{ph}'s are thus invalid under the twin Boltzmann model. This does not cast doubt on the cold species density limits discussed above, however, since these hold even at relatively small temperature ratios ($\alpha_{ph} = 2 \to 5$).

It is also interesting to note, that in the range $\alpha_{ph} > 1$, it is the cold positron Boltzmann restriction which provides the cold species limits, in both cases.

As $\alpha_{ph} \to 0$, $N_{pc(MAX)} \to 0.25$, and $N_{ec(MAX)} \to 0.2$. Again, this is a limiting case and is questionable in terms of the twin Boltzmann model. Worthy of note is the case for $\alpha_{ph} = 1$ ($T_e = T_p$) where both $N_{pc(MAX)}$ and $N_{ec(MAX)}$ reach a maximum value of 0.4. It might then be valuable to investigate the case of equal temperature hot electrons and positrons to optimise the soliton solution existence space. It is also interesting to note that for $\alpha_{ph} = 1$ both electron and positron Boltzmann conditions reduce to

$$N_{ec} < \frac{2}{5} - N_{pc}, \tag{5.40}$$

so that the curves in Figures 5.3 and 5.4 intersect.

Let us first see what effect the restrictions (5.34) and (5.35) have on those results obtained by Pillay & Bharuthram (1992). Figure 5.1 shows a range of N_{pc} existence curves for compressive solitons, in $\phi_0 - N_{ec}$ space, using a Mach number $\mu = 1.2$ and a temperature ratio of $\alpha_{ph} = \frac{T_e}{T_p} = 0.25$, that is, the hot positrons are four times hotter than the hot electrons. Using this value of α_{ph} in equations (5.36), (5.37), (5.38) and (5.39) we find for

$$N_{ec} > 0 \Rightarrow N_{pc} \leq 0.29 \tag{5.41}$$

and for

$$N_{pc} > 0 \Rightarrow N_{ec} \leq 0.25. \tag{5.42}$$

Note that both of the above restrictions result from the electron Boltzmann condition. The positron Boltzmann condition gives higher cut-off values, but since both conditions must hold true, we take the minimum of both sets of cut-offs. We can see that these two restrictions together imply that the entire figure is invalid, since at low values of N_{ec}, only solutions with high values of N_{pc} are found. We then attempt to find valid curves for lower μ values.

5.5.2 Some valid results

Trying solutions with lower μ values, we find that solutions exist for $\mu < 1$. We follow the method of Baboolal *et al.* (1990) to obtain a condition for a stationary rest point at $\phi = 0$, which should lead to an lower μ limit. For the existence of a stationary rest point at $\phi = 0$

$$\frac{\partial^2 V(\phi, \mu)}{\partial \phi^2} < 0. \tag{5.43}$$

Since

$$\frac{\partial^2 V(\phi, \mu)}{\partial \phi^2} = -\frac{\partial N(\phi, \mu)}{\partial \phi}, \tag{5.44}$$

equation (5.43) becomes with (5.24)

$$N_{eh} + \alpha_{ph} N_{ph} > \frac{N_{ec} + N_{pc}}{\mu^2},$$

yielding a lower limit for μ as

$$\mu > \left(\frac{N_{ec} + N_{pc}}{N_{eh} + \alpha_{ph} N_{ph}} \right)^{\frac{1}{2}}. \tag{5.45}$$

Here the RHS of the equation is simply the normalized sound speed of the wave, V, cf. equation (5.21), calculated from the linear dispersion relation in the long wavelength limit as $k \to 0$. In order to better correlate Pillay *et al.*'s results with those obtained in the symmetric model, let us introduce a Mach number so that

$$M = \frac{\mu}{V},$$

as we did in Section 3.4.2, so that soliton speeds are measured with respect to the normalized sound speed, and thus will always be greater than unity.

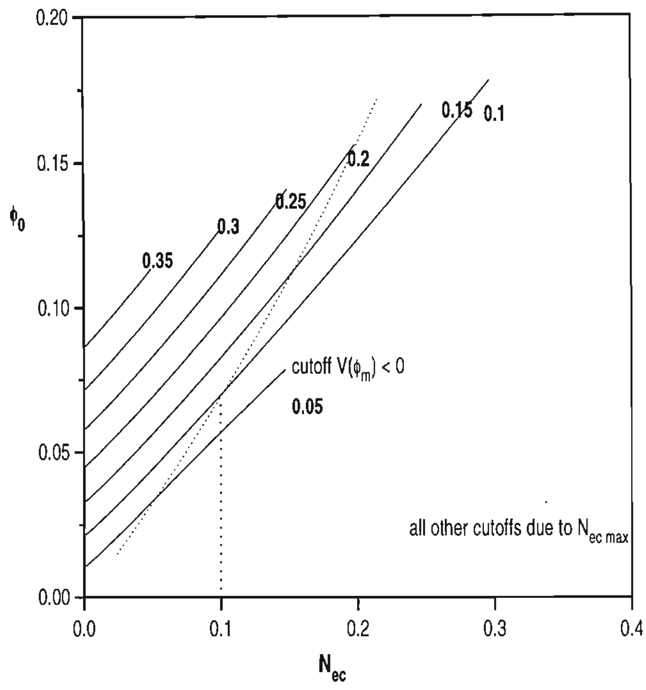


Figure 5.5: Existence domains for compressive solitons, in cold electron number density - soliton amplitude space. Parameters for the curves are $M = 1.2$ and $\alpha_{ph} = 1$. The numbering on the curves refers to cold positron number density, N_{pc}.

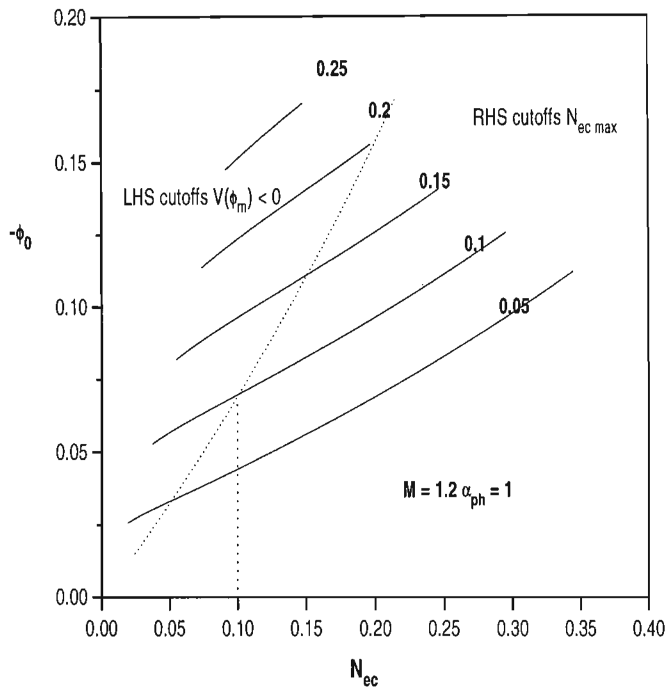


Figure 5.6: Existence domains for rarefactive solitons, in soliton amplitude - cold electron number density space. Parameters for the curves are $M = 1.2$ and $\alpha_{ph} = 1$. The numbering on the curves refers to cold positron number density, N_{pc}.

Let us now consider the case of $\alpha_{ph} = 1$, in order to maximise the soliton existence domain. Using a Mach number $M = 1.2$, we find solutions shown in Figures 5.5 and 5.6 for compressive and rarefactive solitons, respectively.

At first sight it may appear as if these solutions are in complete contradiction to the Pillay *et al.* results. However, it must be borne in mind that our definition of soliton Mach number M is different to that of Pillay *et al.* We have defined M, so that

$$\mu = M \left(\frac{N_{ec} + N_{pc}}{N_{eh} + \alpha_{ph} N_{ph}} \right)^{\frac{1}{2}},$$

and thus for a constant M and N_{pc}, as N_{ec} changes so does the value μ. This value μ was previously held constant by Pillay *et al.* Thus the normalization of μ with respect to the normalized sound speed V, changes the shape of the curves to a large degree.

Figure 5.5 shows existence curves in $N_{ec} - \phi_0$ space for the compressive case, with parameters $M = 1.2$, and $\alpha_{ph} = 1$. The cut-offs (excluding $N_{pc} = 0.05$) in the figure are due to the Boltzmann restrictions, so that for a specific value of N_{pc} the curve terminates for a value of N_{ec} satisfying

$$N_{ec} < \frac{2}{5} - N_{pc}.$$

The curve for $N_{pc} = 0.05$ terminates due to the fact that $V(\phi_m, M) < 0$, preventing complete the return of the pseudo-particle via reflection in the potential well of $V(\phi, M)$, and thus not allowing for fully formed soliton solutions at higher N_{ec} values.

The dotted curve shows the corresponding existence curve for $M = 1.2$ for our symmetric case. It is clear that this curve and the Pillay *et al.* curve for $N_{pc} = 0.1$ intersect when $N_{ec} = 0.1$ (the same occurs for $N_{ec} = N_{pc} = 0.2$). At these points $N_{ec} = N_{pc}$ and the Pillay *et al.* (1992) model becomes our symmetric model in the limit as $T_c \to 0$. This would be the case for all $N_{ec} = N_{pc}$, however, as the value of N_{pc} increases, so the N_{ec} cut-off decreases. Figure 5.6 shows the rarefactive case where it can be seen that our curve and the Pillay $N_{pc} = 0.1$ curve coincide at exactly the same soliton amplitude ϕ_0 for $N_{ec} = 0.1$ also. Thus the symmetry is restored. For the rarefactive case the left hand cut-offs are due to $V(\phi_m, M) < 0$, and the right hand cut-offs due to the Boltzmann cool density condition (5.40).

In light of these results it is once again evident, as in the case for symmetric EP plasmas, that the assumption of Boltzmann hot particles severely restricts the range of validity of soliton solutions. We find for the asymmetric case, that valid solutions are found only for small cold species densities. Only small amplitude soliton solutions are found ($\phi_0 < 0.2$). We also find that the cold positron and electron number densities are directly related to the existence of solitons, in that, as the positron number density increases, so the domain of validity, in terms of cold electron number density, decreases.

Chapter 6

Conclusion

6.1 Summary

This thesis has been largely concerned with nonlinear electrostatic electron-positron acoustic waves, in particular solitons. We attempted to model the magnetosphere of a pulsar, which is considered (Goldreich & Julian 1969; Beskin 1993) to consist of an electron-positron plasma of primary particles produced near the surface of the pulsar, and secondary particles, cascade-bred in the vacuum gap. The cascade process relies on two distinct pair production mechanisms: curvature radiation, producing first generation pairs, and synchroradiation, producing second generation electrons and positrons. The energy ratio of the particles created by these two processes is given by (Beskin 1993)

$$\frac{\varepsilon_{min}}{\varepsilon_{max}} \sim \frac{10^1}{10^5} \sim 10^{-4}.$$

We applied a four component symmetric fluid model of a nonrelativistic unmagnetized plasma, consisting of species of hot Boltzmann and cool adiabatic electrons and positrons, to describe the pulsar magnetosphere. The hot Boltzmann electrons and positrons, at temperature T_h, and number density N_h, model the secondary (first-generation) particles born of curvature radiation, and the cool adiabatic electrons and positrons, at temperature T_c, and number density N_c, model the secondary (second-generation) particles born of synchroradiation. We assumed that the plasma was unmagnetized, which, in light of the large pulsar magnetic field, required justification. The motion of the particles along the magnetic field lines is mainly longitudinal

(Kaplan, Tsytovich & Ter Haar 1972), thus in their reference frame, they do not "see" a magnetic field.

We investigated linear electron-positron waves, considering five plasma models for completeness. The simplest model was the single species pair plasma of electrons and positrons of the same type. The assumption of isothermal, Boltzmann distributed particles, implies that we may ignore their mass, and in retrospect, the resulting dispersion relation of indeterminate ω, seems obvious, since particle oscillations require a restoring force, and thus inertia. Since we found no wave with low phase velocity, $v_\phi \ll v_{th}$, we thus investigated high velocity linear waves ($v_\phi \gg v_{th}$) in a simple pair plasma of cool adiabatic electrons and positrons, obtaining a dispersion relation of the form as that of an electron plasma wave. The final simple plasma consisted of isothermal electrons, and adiabatic positrons. This model was purely academic, since we are of the belief that the pulsar pair production mechanisms result in pairs of the same energy, and thus temperature. We obtained a wave of the form $\omega = v_s k$, with sound speed $v_h \gg v_s \gg v_c$ (Watanabe *et al.* 1977), which is clearly analogous to the usual electron-acoustic wave. We reviewed a more complex, asymmetric three component model (Srinivas *et al.* 1996) of cold inertial electron and positron fluids, with a component of energetic isothermal positrons, resulting in a linear acoustic mode. Finally we considered linear waves with respect to our symmetric four component model. We obtained a dispersion relation of the form of an acoustic mode, with sound speed $v_s = v_h(\frac{N_c}{N_h} + 3T_c)^{\frac{1}{2}}$, in the long wavelength limit. As the number density of the cool particles N_c tends to zero, this dispersion relation becomes purely acoustic, whereas as $N_h \to 0$, the plasma begins to support a plasma-like wave. Our assumption of $v_h \gg v_s \gg v_c$ (Watanabe *et al.* 1977) prohibits the latter limiting case, however, prescribing an upper limit on the cool species density given by $\left(\frac{N_c}{N_h}\right)^{\frac{1}{2}} \ll 1$.

We then considered nonlinear electron-positron waves, of both small and arbitrary amplitude. We derived a Korteweg-de Vries equation to investigate small, finite amplitude waves, but found that the reductive perturbation technique (Washimi & Taniuti 1966; Baboolal, Bharuthram & Hellberg 1989) resulted in a purely dispersive equation, with no nonlinear term. We thus followed Das *et al.* (1975) and Verheest (1988) and considered higher orders of nonlinearity, obtaining a modified KdV equation (Watanabe 1984). Stationary solutions to this mKdV equation were of the form of a soliton profile

$$\phi = \pm \left(\frac{2(M-1)}{B} \right)^{\frac{1}{2}} \operatorname{sech} \left[\left(\frac{M-1}{A} \right)^{\frac{1}{2}} s \right],$$

in terms of a defined a Mach number $M = \frac{\mu}{V}$, where μ is the normalized soliton speed, and V the normalized sound speed of the linear acoustic wave above. A and B were defined as the coefficients of the mKdV equation.

Arbitrary amplitude soliton solutions could then be obtained via the generalized Sagdeev potential $V(\phi, M)$ (Sagdeev 1966)

$$V(\phi, M) = 2N_h (1 - \cosh \phi) - \int_0^{\phi} n_{ec}(\phi', M) d\phi' + \int n_{pc}(\phi', M) d\phi',$$

with

$$n_{jc} = \frac{N_c}{\sqrt{6T_c}} \left[(\mu^2 - 2Z_j\phi + 3T_c) \pm \sqrt{(\mu^2 - 2Z_j\phi + 3T_c)^2 - 12\mu^2 T_c} \right]^{\frac{1}{2}},$$

and $M = \frac{\mu}{V}$. In order for soliton solutions, $V(\phi_{\max}, M) > 0$, that is, the function $V(\phi, M)$ must cross the ϕ axis for some $\phi > 0$ (Chen 1984). This condition places restrictions on the soliton Mach number, so that for the existence of solitons in our four component electron-positron plasma, we require $M < 1.53$.

It was thought initially that the Sagdeev potential could not be solved analytically, and thus it would be necessary to attempt numerical analysis using algorithms to approximate the integrals. However it was discovered (Chatterjee *et al.* 1994), that the exact pseudopotential $V(\phi, M)$ could be obtained in the case of non-zero temperature, in analytic form. Numerical analysis of this closed potential yielded both rarefactive and compressive solitons. The even nature of $V(\phi, M)$ with respect to ϕ, resulted in an exact symmetry in the profiles and potentials of the compressive and rarefactive soliton solutions of the same Mach number, number density and temperature. We considered the existence domains of solitons at temperature ratios $\frac{T_c}{T_h} = 10^{-2}$ and 10^{-4}, in accordance with the ratio of particle energies described above. Cut-offs for these Mach number curves in amplitude–cool species number density space are imposed due to the condition that the particle densities must be real. We found that at lower temperature ratios, with the same values of Mach number and number density, the amplitude of the soliton increases. Physically, the smaller amplitudes at higher temperature ratios may be understood in terms of increased kinetic energy of the cool species electrons and positrons, leading to greater interpenetration between the hot and cool species. This reduces the charge separation required the sustain soliton structures (Baboolal, Bharuthram & Hellberg 1988).

Perhaps the most important result obtained in this thesis is the limiting effect on the existence of soliton solutions owing to the assumption of isothermal Boltzmann hot particles. In order to make this assumption, the thermal speed of the hot particles must be very much larger than the sound speed of a waveform supported by the plasma, $v_h \gg v_s$. We expressed this inequality in terms of a constant α, such that $v_h > \alpha v_s$, where α should be large, but at least greater than 2. This imposed an upper limit on the cool species number density, so that $N_c < \frac{1}{\alpha^2+1}$, which for an $\alpha = 2$ gives $N_c < 0.2$, a very small range of validity in density space. Thus all soliton existence domains are subject to this cut-off. The soliton solutions that are valid under this restriction are of small amplitude, $\phi_0 < 0.25$. We conducted a comparison between the small and arbitrary amplitude results, in an attempt to discover if the mKdV theory was sufficient to describe the existence of solitons in our electron-positron plasma. We found that correlation was very good at a temperature ratio $\frac{T_c}{T_h} = 10^{-2}$, but less so at 10^{-4}. The small amplitude theory is very accurate, at smaller Mach numbers of $M \leq 1.05$, but at higher Mach numbers (and thus amplitudes) it consistently predicts solitons with amplitudes smaller than the arbitrary amplitude solutions. Importantly, it does not incorporate the cut-offs predicted by the arbitrary amplitude soliton theory, mentioned above, and thus it does not give a true picture of the nature of soliton existence.

Finally we considered asymmetric electron-positron plasmas, reviewing a four component fluid plasma model (Pillay *et al.* 1992) in which both electrons and positrons are subdivided into two groups: one at a temperature $T_c = 0$, and another, Boltzmann distributed, at T_j $(j = e, p)$. The model is asymmetric, in that the number densities N_{ec} and N_{pc}, of the cold electrons and positrons are not equal; similarly the hot electrons and positrons have different number densities N_{eh} and N_{ph}, at temperatures T_e and T_p. Although unlikely, in light of pair production mechanisms producing equal numbers of particles at equal temperatures, this scenario may be applicable in the region of the pulsar magnetosphere, above the vacuum gap. Reproducing the results obtained by Pillay *et al.* (1992), we then applied the Boltzmann restriction $v_h > \theta v_s$, for $\theta = 2$. Once again this condition leads to severe upper limits on the cool species density, given by the equations for Boltzmann electrons and positrons respectively,

$$5N_{ec} + (4 + \alpha_{ph})N_{pc} < 1 + \alpha_{ph};$$

$$(4\alpha_{ph} + 1)N_{ec} + 5\alpha_{ph}N_{pc} < 1 + \alpha_{ph},$$

where $\alpha_{ph} = \frac{T_e}{T_p}$.

In conclusion, thus, it is of great import to be aware of the consequences of assumptions made in the choice of model, as they have a significant effect on the domain of solutions obtained.

6.2 Limitations of the present and suggestions for further research

6.2.1 Relativistic plasmas

We have remarked earlier that the model used in this thesis, namely the non-relativistic four component model, is a simplistic representation of the physics behind pulsar magnetospheres. According to Ruderman & Sutherland (1975) (and the majority of authors) the pulsar magnetosphere is dominated by a relativistic electron-positron plasma. Thus it would be of interest to consider relativistic effects when considering a model of a pulsar magnetosphere. In fact, both electromagnetic (Mikhailovskii 1980; Sakai & Kawata 1980; Yu, Shukla & Rao 1984; Yu & Rao 1985; Mofiz, de Angelis & Forlani 1985; Mikhailovskii, Onishchenko & Tatarinov 1985; Mofiz & Mamun 1993; Tsintsadze & Berezhiani 1993; Shukla 1993; Verheest 1996a; Verheest & Lakhina 1996) and longitudinal (Lominadze, Mikhailovskii & Sagdeev 1979; Mamradze, Machabeli & Melikidze 1980; Sakai & Kawata 1980; Tsintsadze 1992) waves in relativistic electron-positron plasmas have been investigated extensively in the literature, to this end.

Specifically, electron-acoustic solitons in weakly relativistic plasmas have been investigated (Mace, Hellberg, Bharuthram & Baboolal 1992). Small (KdV) and arbitrary amplitude analyses, considering a plasma consisting of relativistically streaming fluid components and a hot Boltzmann component, yield only rarefactive acoustic soliton solutions. Relativistic beam effects are shown to increase the soliton amplitude beyond its nonrelativistic value, and a finite cool-electron temperature destroys the balance between nonlinearity and dispersion. Our study of electron-positron acoustic waves could similarly be expanded to include weakly relativistic effects, and thus consider similarities with or disparities from the electron-acoustic model.

6.2.2 Relativistic kinetic theory

Further relativistic study could involve kinetic theory, specifically with regard to particles with Lorentzian velocity distributions. Space plasmas frequently contain superthermal particles, accelerated to velocities greater than the thermal velocity, by mechanisms such as Fermi acceleration by shock waves (Drury 1983), and nonlinear acceleration by EM waves (pertinent to pulsar plasmas). These acceleration mechanisms remove particles from the thermal 'hump' of the distribution increasing their velocity into the region $v > v_{th}$, so that they reside in the tail. The particle distributions which result from such acceleration mechanisms usually have the form of a power law

$$4\pi v^2 f(v) \propto v^{-\alpha},$$

for $v > v_{th}$ where v is the magnitude of the particle velocity and α, some real constant. The velocity distribution then takes the form (Summers & Thorne 1991)

$$f(v) = (\pi\kappa\theta)^{-\frac{3}{2}} \frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\frac{1}{2})} \left(1 + \frac{v^2}{\kappa\theta^2}\right)^{-(\kappa+1)}$$

where $\theta^2 = \frac{(2\kappa-3)}{\kappa}\left(\frac{T}{m}\right)$ is a modified thermal speed and $\Gamma(z)$ is the gamma function. This kappa distribution is employed to define a dispersion function (Summers & Thorne 1991) dependent on κ, analogous to the Fried & Conte Z function. The parameter κ, which must be integer valued, and exceed $\frac{3}{2}$ for $\theta > 0$ (Summers & Thorne 1991) is a measure of the inverse of the efficiency of the acceleration mechanism, the smaller the value of κ the larger the tail of the distribution. In the limit as $\kappa \to \infty$ the distribution function reduces to the three dimensional Maxwellian, and thus represents the cut-off for the acceleration mechanism.

Such enhanced superthermal particle distributions give rise to some interesting wave physics, specifically the significant damping of Langmuir oscillations (Summers & Thorne 1991; Mace & Hellberg 1995). The superthermal electrons couple strongly to the Langmuir waves with large phase velocities (or small wavenumbers) resulting in strong Landau damping in this k region.

Mace & Hellberg (1995) have also introduced a more general Z_κ dispersion function removing the constraint that κ be an integer. The Z_κ function plays an important role in the dispersion relation of waves and instabilities in plasmas, and it would be worthwhile to extend our model of the electron-positron pulsar plasma to include the effects of superthermal particles.

6.2.3 Dusty plasmas

In recent years there has been growing interest in the study of dusty plasmas (Verheest 1993, 1996b; Mace & Hellberg 1993b, 1993c). Dusty plasmas contain charged dust grains besides the usual constituents of normal plasmas. These dust grains are the natural extension of clusters of molecules. Most of the solid matter in the universe may be represented by grains of dust. These dust particles having charge, will thus be subject to the influence of electromagnetic phenomena. When dust is introduced into a standard plasma it becomes negatively charged (in the absence of charging effects other than the plasma), since the electron flux to the dust particle surface is larger than the ion flux, because of the electron's higher thermal speed. In order for electromagnetic forces to become significant, the dust particles must be no larger than micrometre size. If the dust number density is such that the average intergrain distance is comparable with the Debye length, then collective effects among the dust grains (Whipple *et al.* 1985), may lead to new low-frequency wave modes (Rao *et al.* 1990). Linear and weakly nonlinear properties of the dust-acoustic wave have been investigated (Rao *et al.* 1990) as well as dust-acoustic double layers and solitons (Mace & Hellberg 1993b, 1993c; Verheest 1993). Perhaps the inclusion of dust in the electron-positron plasma can remove the inherent symmetry which prevents the existence of acoustic double layer solutions (cf. Bharuthram 1992).

6.2.4 The early Universe

An alternative path of study could involve the investigation of the behaviour of electron-positron plasma in the early Universe. As is well known, conditions at $t = 10^{-2}$s to the time of recombination, at $t = 10^{14}$s, are such that the matter content in the Universe will be in the form of plasma. Before $t = 1$s the plasma is dominated by electrons and positrons, whereafter it would consist of electrons and protons, with an admixture of ions of other light elements. The plasma is believed to be in thermal equilibrium with photons (Weinberg 1972), thus prompting the term "radiation epoch".

Plasma theory and Cosmology have not met well in the past. This is perhaps due to the importance of general relativity and the role of gravity in Cosmology. The early Universe is subject to high rates of expansion, with the result that general relativity will have a significant effect on the physics. However, existing plasma theories are based on Newtonian physics, with the inclusion of some special relativity. Herein lies the rub. It would be necessary to incorporate general-relativistic effects into the analytical methods of plasma processes. The solution to this problem is provided by Thorne and Macdonald (1982) who have written the general-relativistic electromagnetic equations in a form similar to the Newtonian equivalent. This allows for an understanding of general relativistic plasmas without the need to discard the prior Newtonian methods.

A summary of the analysis of linear collective phenomena pertaining to general relativistic plasmas in the early Universe (Holcomb & Tajima 1989) follows. Holcomb *et al.* assume that the plasma exists in a space-time described by the standard solution for a radiation dominated Friedmann-Robertson-Walker cosmology. The analysis is based on the "3+1" formulation of general relativity (Arnowitt, Desner and Misner 1962) whereby the Maxwell equations, and particle equations of motion may be written in a form which resembles closely their Newtonian counterparts.

Holcomb *et al.* (1989) reproduce the standard linear results of small amplitude waves for a warm plasma, using the general relativistic equations. The solution is far from trivial, however. The space-time is expanding, and one cannot make the usual assumption of plane wave dependence for the amplitudes, since the cosmological wave must decay in time and its frequency must redshift. Holcomb *et al.* (1989) find that acoustic oscillations obey a dispersion relation with a form identical to the Newtonian case. Interesting is the fact that the sound speed is constant in time even for an expanding background. The plasma wave dispersion relation is modified by a relativistic enthalpy term, and it is found that the frequencies of the fundamental oscillations decay, due to the expansion of the background.

Further applications of this Cosmological Plasma theory would be particularly relevant to the study of matter fluctuations in the early Universe, as well as the quark-gluon plasma believed to exist in the very early Universe. Pertaining to the older Universe, the inclusion of dust particles could simulate the formation of galaxies under the influence of self gravitation (Jeans

Instability).

Appendix A

The modified Korteweg-de Vries approximation

The system is described by the equations

$$n_{eh} = N_h \exp(\phi), \qquad n_{ph} = N_h \exp(-\phi), \tag{A.1}$$

$$\frac{\partial n_{jc}}{\partial t} + \frac{\partial}{\partial x}(n_{jc} u_{jc}) = 0, \tag{A.2}$$

$$\frac{\partial u_{jc}}{\partial t} + u_{jc}\frac{\partial u_{jc}}{\partial x} + 3T_c \frac{n_{jc}}{N_c^2}\frac{\partial n_{jc}}{\partial x} = -Z_j \frac{\partial \phi}{\partial x}, \tag{A.3}$$

$$\frac{\partial^2 \phi}{\partial x^2} = n_{eh} - n_{ph} + \sum_j -Z_j n_{jc}, \tag{A.4}$$

where $Z_j = \frac{q_j}{e}$.

The expanded form of the densities, velocities and electrostatic potential is given by

$$\begin{aligned} n_{jc} &= N_c + \epsilon n_{jc1} + \epsilon^2 n_{jc2} + \epsilon^3 n_{jc3} + \cdots \\ u_{jc} &= \epsilon u_{jc1} + \epsilon^2 u_{jc2} + \epsilon^3 u_{jc3} + \cdots \\ \phi &= \epsilon \phi_1 + \epsilon^2 \phi_2 + \epsilon^3 \phi_3 + \cdots \end{aligned} \tag{A.5}$$

Equations (A.1) may be written in terms of an exponential power series

$$\begin{aligned} n_{eh} &= N_h \left(1 + \phi + \tfrac{1}{2}\phi^2 + \tfrac{1}{3!}\phi^3 + \cdots\right), \\ n_{ph} &= N_h \left(1 - \phi + \tfrac{1}{2}\phi^2 - \tfrac{1}{3!}\phi^3 + \cdots\right), \end{aligned} \tag{A.6}$$

so that (A.4) becomes

$$\frac{\partial^2\phi}{\partial x^2}=2N_h\left(\phi+\frac{1}{3!}\phi^3+\frac{1}{5!}\phi^5+\cdots\right)-\sum_j Z_j n_{jc}. \tag{A.7}$$

Employing the stretched variables (Verheest 1988)

$$\xi=\epsilon\left(x-Vt\right),\quad \tau=\epsilon^3 Vt,$$

implies the transformation

$$\frac{\partial}{\partial x}=\epsilon\frac{\partial}{\partial\xi},\qquad \frac{\partial}{\partial t}=-\epsilon V\frac{\partial}{\partial\xi}+\epsilon^3 V\frac{\partial}{\partial\tau},$$

so that equations (A.2), (A.3) and (A.7) become

$$-\epsilon V\frac{\partial n_{jc}}{\partial\xi}+\epsilon^3 V\frac{\partial n_{jc}}{\partial\tau}+\epsilon\frac{\partial}{\partial\xi}\left(n_{jc}u_{jc}\right)=0, \tag{A.8}$$

$$-\epsilon V\frac{\partial u_{jc}}{\partial\xi}+\epsilon^3 V\frac{\partial u_{jc}}{\partial\tau}+u_{jc}\epsilon\frac{\partial u_{jc}}{\partial\xi}+\frac{3T_c}{N_c^2}\epsilon n_{jc}\frac{\partial n_{jc}}{\partial\xi}=-Z_j\epsilon\frac{\partial\phi}{\partial\xi}, \tag{A.9}$$

$$\epsilon^2\frac{\partial^2\phi}{\partial\xi^2}=2N_h\left(\phi+\frac{1}{3!}\phi^3+\frac{1}{5!}\phi^5+\cdots\right)-\sum_j Z_j n_{jc}. \tag{A.10}$$

With expanded quantities (A.5), equations (A.8), (A.9) and (A.10) become

$$\begin{aligned}
&-\epsilon V\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\xi}+\epsilon^3\tfrac{\partial n_{jc3}}{\partial\xi}+..\right]+\epsilon^3 V\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\tau}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\tau}+\epsilon^3\tfrac{\partial n_{jc3}}{\partial\tau}+..\right]\\
&+\epsilon\left[N_c+\epsilon n_{jc1}+\epsilon^2 n_{jc2}+\epsilon^3 n_{jc3}+..\right]\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\xi}+\epsilon^3\tfrac{\partial u_{jc3}}{\partial\xi}+..\right]\\
&+\epsilon\left[\epsilon u_{jc1}+\epsilon^2 u_{jc2}+\epsilon^3 u_{jc3}+..\right]\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\xi}+\epsilon^3\tfrac{\partial n_{jc3}}{\partial\xi}+..\right]=0;
\end{aligned} \tag{A.11}$$

$$\begin{aligned}
&-\epsilon V\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\xi}+\epsilon^3\tfrac{\partial u_{jc3}}{\partial\xi}+..\right]+\epsilon^3 V\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\tau}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\tau}+\epsilon^3\tfrac{\partial u_{jc3}}{\partial\tau}+..\right]\\
&+\epsilon\left[\epsilon u_{jc1}+\epsilon^2 u_{jc2}+\epsilon^3 u_{jc3}+..\right]\left[\epsilon\tfrac{\partial u_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial u_{jc2}}{\partial\xi}+\epsilon^3\tfrac{\partial u_{jc3}}{\partial\xi}+..\right]\\
&+\tfrac{3T_c}{N_c^2}\epsilon\left[N_c+\epsilon n_{jc1}+\epsilon^2 n_{jc2}+\epsilon^3 n_{jc3}+..\right]\left[\epsilon\tfrac{\partial n_{jc1}}{\partial\xi}+\epsilon^2\tfrac{\partial n_{jc2}}{\partial\xi}+\epsilon^3\tfrac{\partial n_{jc3}}{\partial\xi}+..\right]\\
&=-Z_j\epsilon\left[\epsilon\tfrac{\partial\phi_1}{\partial\xi}+\epsilon^2\tfrac{\partial\phi_2}{\partial\xi}+\epsilon^3\tfrac{\partial\phi_3}{\partial\xi}+..\right];
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
&\epsilon^2\left[\epsilon\tfrac{\partial^2\phi_1}{\partial\xi^2}+\epsilon^2\tfrac{\partial^2\phi_2}{\partial\xi^2}+\epsilon^3\tfrac{\partial^2\phi_3}{\partial\xi^2}+..\right]=2N_h\left(\epsilon\phi_1+\epsilon^2\phi_2+\epsilon^3\phi_3+..\right)\\
&+\tfrac{2N_h}{3!}\left(\epsilon^3\phi_1^3+3\epsilon^4\phi_1^2\phi_2+3\epsilon^5\phi_1^2\phi_3+3\epsilon^5\phi_1\phi_2^2+6\epsilon^6\phi_1\phi_2\phi_3+\epsilon^6\phi_2^3+..\right)\\
&-\textstyle\sum_j Z_j\left(N_c+\epsilon n_{jc1}+\epsilon^2 n_{jc2}+\epsilon^3 n_{jc3}+..\right).
\end{aligned} \tag{A.13}$$

Solving order by order in ϵ, we have from (A.11)

$$O\left(\epsilon^2\right) \quad -V\frac{\partial n_{jc1}}{\partial \xi} + N_c\frac{\partial u_{jc1}}{\partial \xi} = 0,$$

which is, through integration,

$$Vn_{jc1} = N_c u_{jc1}; \tag{A.14}$$

$$O(\epsilon^3) \quad -V\frac{\partial n_{jc2}}{\partial \xi} + N_c\frac{\partial u_{jc2}}{\partial \xi} + n_{jc1}\frac{\partial u_{jc1}}{\partial \xi} + u_{jc1}\frac{\partial n_{jc1}}{\partial \xi} = 0,$$

that is,

$$-Vn_{jc2} + N_c u_{jc2} + n_{jc1}u_{jc1} = 0; \tag{A.15}$$

$$O\left(\epsilon^4\right) \quad V\frac{\partial n_{jc1}}{\partial \tau} - V\frac{\partial n_{jc3}}{\partial \xi} + N_c\frac{\partial u_{jc3}}{\partial \xi} + \frac{\partial}{\partial \xi}\left(n_{jc1}u_{jc2}\right) + \frac{\partial}{\partial \xi}\left(n_{jc2}u_{jc1}\right) = 0. \tag{A.16}$$

From (A.12) we have

$$O\left(\epsilon^2\right) \quad -V\frac{\partial u_{jc1}}{\partial \xi} + \frac{3T_c}{N_c}\frac{\partial n_{jc1}}{\partial \xi} = -Z_j\frac{\partial \phi_1}{\partial \xi},$$

which through integration becomes

$$-Vu_{jc1} + \frac{3T_c}{N_c}n_{jc1} = -Z_j\phi_1; \tag{A.17}$$

$$O\left(\epsilon^3\right) \quad -V\frac{\partial u_{jc2}}{\partial \xi} + u_{jc1}\frac{u_{jc1}}{\partial \xi} + \frac{3T_c}{N_c^2}\left[N_c\frac{\partial n_{jc2}}{\partial \xi} + n_{jc1}\frac{\partial n_{jc1}}{\partial \xi}\right] = -Z_j\frac{\partial \phi_2}{\partial \xi},$$

that is

$$-Vu_{jc2} + \frac{1}{2}u_{jc1}^2 + \frac{3T_c}{N_c}n_{jc2} + \frac{3T_c}{2N_c^2}n_{jc1}^2 = -Z_j\phi_2; \tag{A.18}$$

$$\begin{aligned} O\left(\epsilon^4\right) \quad & V\tfrac{\partial u_{jc1}}{\partial \tau} - V\tfrac{\partial u_{jc3}}{\partial \xi} + \tfrac{\partial}{\partial \xi}\left(u_{jc1}u_{jc2}\right) + \tfrac{3T_c}{N_c}\tfrac{\partial n_{jc3}}{\partial \xi} + \tfrac{3T_c}{N_c^2}\tfrac{\partial}{\partial \xi}\left(n_{jc1}n_{jc2}\right) \\ & = -Z_j\tfrac{\partial \phi_3}{\partial \xi}; \end{aligned} \tag{A.19}$$

(A.13) gives

$$O\left(\epsilon^0\right) \quad \textstyle\sum_j Z_j N_c = 0; \tag{A.20}$$

$$O(\epsilon) \quad 2N_h\phi_1 - \textstyle\sum_j Z_j n_{jc1} = 0; \tag{A.21}$$

$$O\left(\epsilon^2\right) \quad 2N_h\phi_2 - \textstyle\sum Z_j n_{jc2} = 0; \tag{A.22}$$

$$O\left(\epsilon^3\right) \quad \frac{\partial^2\phi_1}{\partial\xi^2} = 2N_h\phi_3 + \frac{N_h}{3}\phi_1^3 - \sum_j Z_j n_{jc3}. \tag{A.23}$$

We obtain the relationship between n_{jc1} and u_{jc1} from (A.14) as

$$n_{jc1} = \frac{N_c}{V}u_{jc1}, \qquad u_{jc1} = \frac{V}{N_c}n_{jc1}. \tag{A.24}$$

Substituting (A.24) into (A.17) gives expressions for n_{jc1} and u_{jc1} respectively, in terms of ϕ_1

$$n_{jc1} = \frac{Z_j N_c}{V^2 - 3T_c}\phi_1, \tag{A.25}$$

$$u_{jc1} = \frac{Z_j V}{V^2 - 3T_c}\phi_1. \tag{A.26}$$

(A.25) into (A.21) gives

$$\left[2N_h - \sum_j \frac{Z_j^2 N_c}{V^2 - 3T_c}\right]\phi_1 = 0, \tag{A.27}$$

which, for $\phi_1 \neq 0$, gives the small k linear dispersion relation

$$V^2 = \frac{N_c}{N_h} + 3T_c,$$

where $V = \frac{\omega}{k}$.

Eliminating u_{jc2} from (A.18) by substituting (A.15)

$$(A.15): \quad N_c u_{jc2} = V n_{jc2} - n_{jc1}u_{jc1},$$

$$N_c \cdot (A.18): \quad -VN_c u_{jc2} + \tfrac{1}{2}N_c u_{jc1}^2 + 3T_c n_{jc2} + \tfrac{3T_c}{2N_c}n_{jc1}^2 = -Z_j N_c \phi_2,$$

gives

$$\left(V^2 - 3T_c\right)n_{jc2} = Vn_{jc1}u_{jc1} + \frac{1}{2}N_c u_{jc1}^2 + \frac{3T_c}{2N_c}n_{jc1}^2 + Z_j N_c \phi_2.$$

Substituting (A.25) and (A.26) into the above equation gives n_{jc2} in terms of ϕ only

$$n_{jc2} = \frac{Z_j N_c}{V^2 - 3T_c}\phi_2 + \frac{\frac{3}{2}Z_j^2 N_c}{\left(V^2 - 3T_c\right)^3}\left(V^2 + T_c\right)\phi_1^2. \tag{A.28}$$

Using this expression for n_{jc2} in equation (A.22) gives

$$\left[2N_h - \frac{Z_j^2 N_c}{V^2 - 3T_c}\right]\phi_2 - \sum_j \frac{3}{2}\frac{Z_j^3 N_c\left(V^2+T_c\right)}{\left(V^2-3T_c\right)^3}\phi_1^2 = 0,$$

which reduces to

$$\sum_j \frac{3}{2} Z_j^3 \frac{N_c\left(V^2+T_c\right)}{\left(V^2-3T_c\right)^3}\phi_1^2 = 0, \tag{A.29}$$

with (A.27).

To obtain an expression for u_{jc2} in terms of ϕ we follow a similar procedure. Rearranging (A.18), and substituting (A.25), (A.26) and (A.28) gives

$$V u_{jc2} = \frac{1}{2} Z_j^2 \left[\frac{V^2+3T_c}{\left(V^2-3T_c\right)^2} + \frac{9T_c\left(V^2+T_c\right)}{\left(V^2-3T_c\right)^3}\right]\phi_1^2 + Z_j\left[1 + \frac{3T_c}{\left(V^2-3T_c\right)}\right]\phi_2, \tag{A.30}$$

which gives

$$u_{jc2} = Z_j^2 \frac{V\left(V^2+9T_c\right)}{2\left(V^2-3T_c\right)^3}\phi_1^2 + Z_j \frac{V}{\left(V^2-3T_c\right)}\phi_2.$$

Now rearranging (A.16) such that

$$N_c \frac{\partial u_{jc3}}{\partial \xi} = -V\frac{\partial n_{jc1}}{\partial \tau} + V\frac{\partial n_{jc3}}{\partial \xi} - \frac{\partial}{\partial \xi}\left(n_{jc1}u_{jc2}\right) - \frac{\partial}{\partial \xi}\left(n_{jc2}u_{jc1}\right),$$

and substituting the subsequent expression for $\frac{\partial u_{jc3}}{\partial \xi}$ into $N_c \cdot$ (A.19):

$$N_c V\frac{\partial u_{jc1}}{\partial \tau} - N_c V\frac{\partial u_{jc3}}{\partial \xi} + N_c\frac{\partial}{\partial \xi}\left(u_{jc1}u_{jc2}\right) + 3T_c\frac{\partial n_{jc3}}{\partial \xi} + \frac{3T_c}{N_c}\frac{\partial}{\partial \xi}\left(n_{jc1}n_{jc2}\right) = -Z_j N_c\frac{\partial \phi_3}{\partial \xi},$$

gives

$$\begin{aligned}\left(V^2-3T_c\right)\tfrac{\partial n_{jc3}}{\partial \xi} &= N_c V\tfrac{\partial u_{jc1}}{\partial \tau} + V^2\tfrac{\partial n_{jc1}}{\partial \tau} + V\tfrac{\partial}{\partial \xi}\left(n_{jc1}u_{jc2} + n_{jc2}u_{jc1}\right)\\ &+ N_c\tfrac{\partial}{\partial \xi}\left(u_{jc1}u_{jc2}\right) + \tfrac{3T_c}{N_c}\tfrac{\partial}{\partial \xi}\left(n_{jc1}n_{jc2}\right) + Z_j N_c\tfrac{\partial \phi_3}{\partial \xi}.\end{aligned} \tag{A.31}$$

Substituting (A.25) and (A.26) into (A.31) gives the expression for n_{jc3} in terms of ϕ, u_{jc2} and n_{jc2}

$$\begin{aligned}\tfrac{\partial n_{jc3}}{\partial \xi} = \;& 2Z_j\tfrac{N_c V^2}{\left(V^2-3T_c\right)^2}\tfrac{\partial \phi_1}{\partial \tau} + 2Z_j\tfrac{N_c V}{\left(V^2-3T_c\right)^2}u_{jc2}\tfrac{\partial \phi_1}{\partial \xi} + 2Z_j\tfrac{N_c V}{\left(V^2-3T_c\right)^2}\phi_1\tfrac{\partial u_{jc2}}{\partial \xi}\\ &+ Z_j\tfrac{\left(V^2+3T_c\right)}{\left(V^2-3T_c\right)^2}n_{jc2}\tfrac{\partial \phi_1}{\partial \xi} + Z_j\tfrac{\left(V^2+3T_c\right)}{\left(V^2-3T_c\right)^2}\phi_1\tfrac{\partial n_{jc2}}{\partial \xi} + Z_j\tfrac{N_c}{\left(V^2-3T_c\right)}\tfrac{\partial \phi_3}{\partial \xi}.\end{aligned} \tag{A.32}$$

We may now substitute (A.28) and (A.30) into $Z_j\cdot$ (A.32), summing over j, to obtain an expression for n_{jc3} in terms of ϕ only

$$\begin{aligned}&\tfrac{\partial}{\partial\xi}\textstyle\sum_j Z_j n_{jc3} = 2\sum_j Z_j^2\tfrac{N_c V^2}{(V^2-3T_c)^2}\tfrac{\partial\phi_1}{\partial\tau}\\&+3\textstyle\sum_j Z_j^4\tfrac{N_c}{(V^2-3T_c)^2}\left[\tfrac{(V^2+3T_c)}{(V^2-3T_c)^2}+\tfrac{9T_c(V^2+T_c)}{(V^2-3T_c)^3}+\tfrac{3}{2}\tfrac{(V^2+3T_c)(V^2+T_c)}{(V^2-3T_c)^3}\right]\phi_1^2\tfrac{\partial\phi_1}{\partial\xi}\\&+\textstyle\sum_j Z_j^3\tfrac{N_c}{(V^2-3T_c)^2}\left[3\tfrac{(V^2+T_c)}{(V^2-3T_c)}\right]\tfrac{\partial}{\partial\xi}(\phi_1\phi_2)+\sum_j Z_j^2\tfrac{N_c}{(V^2-3T_c)}\tfrac{\partial\phi_3}{\partial\xi}.\end{aligned}\quad\text{(A.33)}$$

We obtain another expression for n_{jc3} by taking the derivative with respect to ξ of (A.23)

$$\tfrac{\partial}{\partial\xi}\textstyle\sum_j Z_j n_{jc3} = -\tfrac{\partial^3\phi_1}{\partial\xi^3}+2N_h\tfrac{\partial\phi_3}{\partial\xi}+\tfrac{N_h}{3}\tfrac{\partial}{\partial\xi}\phi_1^3. \quad\text{(A.34)}$$

Equating (A.33) and (A.34) we may obtain an expression in terms of $\frac{\partial\phi_3}{\partial\xi}$, $\frac{\partial}{\partial\xi}(\phi_1\phi_2)$, $\frac{\partial\phi_1^3}{\partial\xi}$, $\frac{\partial\phi_1}{\partial\tau}$ and $\frac{\partial^3\phi_1}{\partial\xi}$. The coefficient of $\frac{\partial\phi_3}{\partial\xi}$ is

$$\sum_j Z_j^2\frac{N_c}{(V^2-3T_c)}-2N_h=0,$$

since this is just the dispersion relation (A.27). The coefficient of $\frac{\partial}{\partial\xi}(\phi_1\phi_2)$ is

$$3\sum_j Z_j^3\frac{N_c(V^2+T_c)}{(V^2-3T_c)^3}=0,$$

for $Z_j=\frac{q_j}{e}$ for $j=e,p$. The coefficient of $\frac{\partial\phi_1^3}{\partial\xi}$ is

$$\textstyle\sum_j Z_j^4 N_c\left[\frac{(V^2+3T_c)}{(V^2-3T_c)^4}+\frac{9T_c(V^2+T_c)}{(V^2-3T_c)^5}+\frac{3(V^2+3T_c)(V^2+T_c)}{2(V^2-3T_c)^5}\right]-\frac{N_h}{3}.$$

The coefficient of $\frac{\partial\phi_1}{\partial\tau}$ is

$$2\sum Z_j^2\frac{N_c V^2}{(V^2-3T_c)^2}.$$

Thus we are left with an equation of the form

$$\frac{\partial\phi_1}{\partial\tau}+A\frac{\partial^3\phi_1}{\partial\xi^3}+B\frac{\partial\phi_1^3}{\partial\xi},$$

in ϕ_1 only. This is the modified KdV equation (Watanabe 1984; Verheest 1988) where $A = \frac{1}{a}$, and

$$a = 2\sum_j Z_j^2 \frac{N_c V^2}{\left(V^2 - 3T_c\right)^2}$$

and $B = \frac{b}{a}$, where

$$b = \sum_j Z_j^4 \frac{N_c \left(5V^4 + 30T_c V^2 + 9T_c^2\right)}{2\left(V^2 - 3T_c\right)^5} - \frac{N_h}{3}.$$

Expanding the summations for a and b we obtain

$$B = \frac{3N_c(5V^4 + 30T_c V^2 + 9T_c^2) - N_h(V^2 - 3T_c)^5}{3(V^2 - 3T_c)^5} \cdot \frac{(V^2 - 3T_c)^2}{4N_c V^2}.$$

Substituting $V^2 = \frac{N_c}{N_h} + 3T_c$ from (A.27) gives

$$A = \frac{\left(\frac{N_c}{N_h}\right)^2}{4N_c\left(\frac{N_c}{N_h} + 3T_c\right)},$$

$$B = -\frac{\left(\frac{N_c}{N_h}\right)^4 - 15\left(\frac{N_c}{N_h}\right)^2 - 180\frac{N_c}{N_h}T_c - 432T_c^2}{12\left(\frac{N_c}{N_h} + 3T_c\right)\left(\frac{N_c}{N_h}\right)^3}.$$

Appendix B

Physical root of cool species density

Regarding equation (3.68)

$$\mu^2 \left(\frac{N_c^2}{n_{jc}^2} - 1 \right) + 3T_c \left(\frac{n_{jc}^2}{N_c^2} - 1 \right) = -2Z_j\phi,$$

in the limit as $T_c \to 0$ we obtain an expression for n_{jc}^2

$$n_{jc}^2 = \frac{N_c^2}{\left(1 - \frac{2Z_j\phi}{\mu^2}\right)}. \tag{B.1}$$

The generalized form of (3.69) and (3.70) is as follows

$$n_{jc}^2 = \frac{N_c^2}{6T_c} \left[\left(\mu^2 - 2Z_j\phi + 3T_c\right) \pm \sqrt{\left(\mu^2 - 2Z_j\phi + 3T_c\right)^2 - 12\mu^2 T_c} \right] \tag{B.2}$$

It must follow then that taking the limit as $T_c \to 0$ in equation (B.2) it should reduce to (B.1). Applying L'Hospital's rule (Spiegel 1974), differentiation of numerator and denominator with respect to T_c yields

$$n_{jc}^2 = \frac{N_c^2}{2} \left(1 \pm \frac{(\mu^2 - 2Z_j\phi + 3T_c) - 2\mu^2}{\sqrt{(\mu^2 - 2Z_j\phi + 3T_c)^2 - 12\mu^2 T_c}} \right).$$

As $T_c \to 0$ this becomes

$$n_{jc}^2 = \frac{N_c^2}{2} \left(\frac{(\mu^2 - 2Z_j\phi) \pm (-2Z_j\phi - \mu^2)}{(\mu^2 - 2Z_j\phi)} \right),$$

which reduces to (B.1) only if the negative root is applied. Thus we may discard the positive root as spurious.

Appendix C

Condition for a stationary rest point at $\phi = 0$

We follow the method of (Baboolal, Bharuthram & Hellberg 1990) to yield a lower limit for μ. For solitons we require

$$\frac{\partial^2 V(\phi,\mu)}{\partial \phi^2} < 0 \quad \text{at} \quad \phi = 0, \tag{C.1}$$

implying that there exists a stationary rest point at $\phi = 0$. Since

$$\frac{\partial V(\phi,\mu)}{\partial \phi} = -N(\phi,\mu) = N_h \exp(-\phi) + n_{pc}(\phi,\mu) - N_h \exp(\phi) - n_{ec}(\phi,\mu), \tag{C.2}$$

we have

$$-\frac{\partial N(\phi,\mu)}{\partial \phi} = -N_h \exp(-\phi) + \frac{\partial n_{pc}(\phi)}{\partial \phi} - N_h \exp(\phi) - \frac{\partial n_{ec}(\phi)}{\partial \phi}. \tag{C.3}$$

The generalized formula for cool species density is

$$n_{jc} = \frac{N_c}{\sqrt{6T_c}} \left[(\mu^2 - Z_j 2\phi + 3T_c) - \sqrt{(\mu^2 - Z_j 2\phi + 3T_c)^2 - 12\mu^2 T_c} \right]^{\frac{1}{2}}.$$

Thus

$$\begin{aligned} \frac{\partial n_{jc}(\phi,\mu)}{\partial \phi} &= \frac{N_c}{\sqrt{6T_c}} \frac{1}{\left[(\mu^2 - Z_j 2\phi + 3T_c) - \sqrt{(\mu^2 - Z_j 2\phi + 3T_c)^2 - 12\mu^2 T_c}\right]^{\frac{1}{2}}} \\ &\cdot \left[\frac{Z_j(\mu^2 - Z_j 2\phi + 3T_c)}{\sqrt{(\mu^2 - Z_j 2\phi + 3T_c)^2 - 12\mu^2 T_c}} - Z_j \right]. \end{aligned} \tag{C.4}$$

At $\phi = 0$ equation (C.4) becomes

$$\frac{\partial n_{jc}(0)}{\partial \phi} = \frac{Z_j N_c}{\mu^2 - 3T_c},$$

noting that

$$(\mu^2 + 3T_c)^2 - 12\mu^2 T_c = (\mu^2 - 3T_c)^2.$$

Equation (C.3) at $\phi = 0$ then becomes

$$\frac{\partial^2 V(0)}{\partial \phi^2} = -2N_h + \frac{2N_c}{\mu^2 - 3T_c}.$$

Equation (C.1) thus reduces to

$$-2N_h + \frac{2N_c}{\mu^2 - 3T_c} < 0,$$

or

$$\frac{N_c}{\mu^2 - 3T_c} < N_h.$$

Solving for μ this yields the condition

$$\begin{aligned} \mu^2 &> \tfrac{N_c}{N_h} + 3T_c \\ \mu &> V, \end{aligned}$$

where $V = \frac{v_s}{v_h}$ is the normalized sound speed.

Appendix D

Reflection of the pseudo-particle at $\phi = \phi_0$

We follow the method of Chen (1984), and note that for reflection of the pseudo-particle we require

$$V(\phi, \mu) > 0 \quad \text{for} \quad \text{some} \quad \phi > \phi_0.$$

Since $\phi_{max} = \frac{\mu^2}{2}$ (for $T_c \to 0$) is the largest valid value for ϕ, $V(\phi_{max}) > 0$ must be true for reflection.

Now

$$V(\phi, \mu) = 2N_h(1 - \cosh\ \phi) - \int_0^{\phi} n_{ec}(\phi', \mu) d\phi' + \int_0^{\phi} n_{pc}(\phi', \mu) d\phi'$$

and the cool species' density reduces to

$$n_{jc} = \frac{N_c}{\sqrt{1 - Z_j \frac{2\phi}{\mu^2}}},$$

for $T_c \to 0$ (cf. equation (B.1)). Thus we have

$$\int_0^{\phi} n_{ec}(\phi', \mu) d\phi' = N_c \mu^2 \left(1 + \frac{2\phi}{\mu^2}\right)^{\frac{1}{2}} - N_c \mu^2$$

and

$$\int_0^{\phi} n_{pc}(\phi', \mu) d\phi' = -N_c \mu^2 \left(1 - \frac{2\phi}{\mu^2}\right)^{\frac{1}{2}} + N_c \mu^2.$$

Thus $V(\phi_{max})$ becomes

$$V(\frac{\mu^2}{2}) = 2N_h(1 - \cosh\frac{\mu^2}{2}) - \sqrt{2}N_c\mu^2 + 2N_c\mu^2 > 0$$

or

$$2N_h(\cosh\frac{\mu^2}{2} - 1) < 2N_c\mu^2 - \sqrt{2}N_c\mu^2.$$

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