ASYMPTOTIC AND BLOW-UP DYNAMICS OF KELLER-SEGEL CHEMOTAXIS EQUATIONS IN SCALE OF BANACH SPACES

By

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As the candidate's supervisor, I have/have not approved this thesis for submission.

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Abstract

In this Thesis, we study the asymptotic and blow-up dynamics of Keller-Segel (KS) chemotaxis equations in Lebesgue-Bochner spaces of underlying Banach spaces of either type $L^p(\Omega)$ or Bessel potential spaces $(I-\Delta)^{-\frac{s}{2}}L^p(\Omega) = H^{s,p}(\Omega)$. The model equations involve the attraction or minimal, and the attraction-repulsion Keller-Segel (ARKS) chemotaxis equations. The treatment yielded begins with a review of the semigroup action in Bessel potential spaces, and interpolation theory for their construction. In studying the well-posedness of the equations we establish a natural condition between the initial data spaces and spaces for the inhomogeneous terms of the equations, with which we prove the well-posedness of the dynamical system for an extended analytic semigroup in Banach spaces. The best constants of the function spaces embedding into L^ρ-spaces yield, for either Banach fixed point theorem, or global existence of solutions, no need for neither the time for a contraction mapping, nor initial data of the equations to be relatively small respectively. The global asymptotic dynamics of the system equations in time is captured in the limit set $\mathcal{M} \times \{0\}$, where $\mathcal{M} = |\Omega| L^1(\Omega)$ is of the spatial average solutions, and the approach to null states of the orthogonal to constant solutions is due to an *a priori* decay to zero of the drift chemical cues from that of the cells density. At several points in the work we have proved the existence of *a priori* uniform boundedness of the solutions to the equations in time and space, yielding via bootstrap arguments that the solutions are classical solutions. In blow-up dynamics, we obtain non-existence of solutions at the borderline space, independent of time, when the chemical coefficient or difference is above the Moser-Trudinger threshold value.

The contributions of the work, besides being at the interface of mathematical analysis and medical biology, lies in the fact that its mathematical analysis takes the most often used function spaces platform for the treatment of the equations a step further in frontier to the

customary. In this regard, some known results are obtained without strong restrictions on the initial data spaces. The generation of an extended analytic semigroup by the system equations imply other non-linear terms of bio-physical relevance can be taken into account in modelling diversified complex phenomena that might be possibly more precise to describing the pathological situations arising in nature to the system of model equations. In blow-up dynamics of the equations our analysis does not limit the scientist concerned to the case of two dimensions corresponding to the particular case of the Hilbert space setting H^1. The non-local elliptic equation reduction of the system equations still call for important other analysis to the topic in the general function space setting that insofar has been introduced, for instance the establishment of Palais-Smale condition, and Pohozaev inequality for non-existence of solutions at the borderline of the function spaces $H^{2\alpha,p}(\Omega)$. This last point we have resolved in the case of Hilbert spaces.

Declaration - Plagiarism

I, David Shituula Ila-kutse Iiyambo, declare that

1. The research reported in this thesis, except where otherwise indicated, is my original research.

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Declaration 2- Publications

DETAILS OF CONTRIBUTION TO PUBLICATIONS that form part and/or include research presented in this thesis (include publications in preparation, submitted, in press and published and give details of contributions of each author to experimental work and writing of each publication)

Publication 1.

Iiyambo D., and Willie R., Semigroup and blow-up dynamics of attraction Keller-Segel chemotaxis equations in scale of Banach spaces., (Submitted to *Analysis and Mathematical Physics* on November 2015.
(There were regular meetings between myself and my supervisors to discuss research material for publications. The outline of the research papers and discussion of the significance of the results were jointly done. The papers were mainly written by myself with some input from my supervisors.)

Student Declaration

I declare that the contents of this dissertation are original except where due reference has been made. It has not been submitted before for any degree to any other institution.

David Shituula Ila-kutse Iiyambo June 10, 2016

This Thesis is dedicated to the memory of my little brother Andreas. The pain caused by your departure is only fractionally reduced by the fond memories that we have of you.

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Introduction

0.1 Origin and Importance of the model in Biological Sciences and Mathematics

Chemotaxis is a process characterised by the directed movement or orientation of organisms or cells, in response to the concentration gradient of an external chemical signal. The chemical signals can come from external sources, or they can be secreted by the organisms themselves. The situation where the chemical is produced by the organisms themselves leads to aggregation of organisms and to the formation of patterns.

In 1970, E.F. Keller and L.A. Segel [41] proposed a mathematical model describing this chemotactic aggregation of cellular slime molds *Dictyostelium discoideum* which move (in a domain Ω) preferentially towards relatively high concentrations of a chemical substance cAMP (*cyclic adenosine monophosphate*), produced by the amoebae themselves. Their derivation of the system of equation was, briefly, as follows. Let $u(x,t)$ denote the density of amoebae, $v(x,t) := \psi_2$ denote the concentration of the chemical attractant. Then the four basic assumptions which underlie the derivations are (see [35, 41]):

1. The chemical attractant is produced per amoeba at a rate of $f(v)$.

2. There exists an extracellular enzyme that degrades the chemical attractant. The

concentration of the enzyme at time t in point x is denoted by $w(x,t) := \psi_3$. This enzyme is produced by the amoebae at a rate $g(v,w)$ per amoebae.

3. The chemical attractant and the enzyme react to form a complex of concentration $z(x,t) := \psi_4$, which dissociates into a free enzyme plus the degraded product.

4. v, w and z obey Fick's Law when diffusing.

It then follows from the balance of the cell density in open bounded domain $\Omega \subset \mathbb{R}^N$, with smooth boundary $\partial\Omega$, during aggregation that

$$\frac{d}{dt}\int_\Omega u(x,t)\,dx = \int_\Omega Q^{(u)}(x,t)\,dx - \int_{\partial\Omega} J^{(u)}(x,t)\cdot\vec{n}d\sigma, \tag{0.1}$$

where $Q^{(u)}(x,t)$ represents the mass of amoeba created/dying per unit volume per unit time, while $J^{(u)}(x,t) = u\chi_3\nabla w - u\chi_2\nabla v - \nabla u$ is the flux of amoeba mass. Note that the composition of $J^{(u)}(x,t)$ follows from using Fick's Law and Fourier's Law for the heat flow, and we have also taken a linear sensitivity function $\chi(s,t) = \chi$. If we neglect reproduction and death of the amoebae, then $Q^{(u)}(x,t) \equiv 0$. Since the chemical attractant v, the enzyme w, and the complex z diffuse, we get that

$$\frac{d}{dt}\int_\Omega \psi(t,x)\,dx = \int_\Omega Q^{(\psi)}(x,t)\,dx - \int_{\partial\Omega} J^{(\psi)}(x,t)\,\vec{n}d\sigma, \tag{0.2}$$

where $J^{(\psi)} = -\nabla\psi$ is the flux of either v, w or z, and $Q^{(\psi)}(x,t)$ is the chemical attractant or the enzyme or the complex produced per unit volume per unit time. One then uses the divergence theorem on (0.1) and (0.2) to obtain the following system of equations

$$\begin{cases} u_t = \Delta u - \sum_{i=2}^{3} \mathrm{Div}\left(u(-1)^i \chi_i \nabla \psi_i\right) & \text{in } \Omega \times (0,T), \\ v_t = \Delta v - \lambda_2 v w + b_2 z + u f(v) & \text{in } \Omega \times (0,T), \\ w_t = \Delta w - \lambda_2 v w + b_3 z + u g(v,w) & \text{in } \Omega \times (0,T), \\ z_t = \Delta z + \lambda_2 v w - (b_2 + b_3) z & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial \psi_i}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad \psi_i(x,0) = \psi_{i0}(x), & \text{in } \Omega, \end{cases} \tag{0.3}$$

where $\lambda_2,\ b_2,\ b_3$ are constants representing the reaction rates mentioned in assumption 3 above.

For the minimal Keller-Segel model [32], one may assume that the concentration of the enzyme is constant, that the complex is in a steady state with regard to the chemical reaction, and that the rate of production of the chemical attractant is constant. The model (0.3) then reduces to

$$\begin{cases} u_t = \Delta u - \mathrm{Div}\left(u \chi \nabla v\right) & \text{in } \Omega \times (0,T), \\ v_t = \Delta v - \lambda v + a u & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{in } \Omega, \end{cases} \tag{0.4}$$

in which

$$\begin{aligned} \chi &:= \text{chemotactic coefficient towards attractant,} \\ \lambda &:= \text{rate of decay of the chemical attractant,} \\ a &:= \text{rate of production of the chemical attractant.} \end{aligned}$$

For notational simplicity, we will often write

$$\begin{aligned} P(u)v &:= -\mathrm{Div}\left(u \chi \nabla v\right) \\ &= -\nabla \cdot \left(u \chi \nabla v\right). \end{aligned} \tag{0.5}$$

Note that (0.5) can be viewed in the sense of distributions as the weak form

$$P_\Omega(u)v := \langle P(u)v, \varphi \rangle_{q',q} = \int_\Omega u\chi \nabla v \nabla \varphi \, dx \tag{0.6}$$

in adequate function spaces. Similar assumptions can be made to obtain the attraction-repulsion Keller-Segel model from (0.3). For more information regarding the derivation and various variations of the Keller-Segel model, please consult [32, 35, 41, 42] among others, where particularly in [32], Hillen, T. and Painter, K., have given an encyclopaedic user's guide to infinite dimensional models of these equations.

In addition to the aggregation of cellular slime molds, chemotaxis is believed to underlie many social activities of micro-organisms. When there is an infection in the human body, white blood cells are known to move to the source of inflammation, the region where the concentration of bacteria is high [62].

The equations can assume very general formulations. For instance, multiple species competing for resources. Among these are attraction-repulsion equations, which for example, model the aggregation of cells called microglia, involved in the inflammation associated with pathology in Alzheimer's disease [50, 87].

In the system of equations (0.4), the $u-$flux moves in the direction of the concentration gradient of the chemical concentration v. Thus, as another example, chemotaxis can be regarded as a sort of negative drift, an example of which is appearing in reaction-diffusion equations of electrically charged species in semiconductors [62]. The simplified form of this process is

$$\begin{cases} a_t = \nu_1 \Delta a - \nabla \cdot (a \nabla \varphi) + f \\ b_t = \nu_2 \Delta b - \nabla \cdot (b \nabla \varphi) + g \\ -\nabla(\varepsilon \nabla \varphi) = b - a, \end{cases}$$

where a and b are the densities of electrons and holes respectively, and φ is the electrostatic potential. The ν_i are positive diffusion coefficients, and f and g are reaction terms depending of the carrier densities.

As mentioned earlier, the model (0.4) is made up of a set of coupled parabolic partial differential equations, where the first equation features a divergence-0 operator acting on a vector field $u\chi\nabla v$ of concentration of chemicals. The mathematical difficulty in handling the system (0.4) stems from the fact that the chemotactic term and the production term in the second equation carry opposite signs, and this brings in the possibility of the solution blowing-up in finite time.

The system of equations (0.4) has been studied before by many authors in this direction.[1] In their pioneering work of 1998, Gajewski, H. and Zacharias, K. [26] studied the global behaviour of the solutions of a reaction-diffusion system (0.4) (where the chemotactic coefficient was not necessarily equal to one) for two-dimensional bounded piecewise smooth domains in the plane, using Lyapunov functionals.[2] They found, for the first time, a Lyapunov functional for the system (0.4), and they proved local existence and uniqueness of solutions. They proved that the solutions of a transformed version of (0.4) asymptotically

[1]See [22, 26, 33, 37, 38, 50, 53, 54, 58, 62, 76, 83, 87, 88, 89, 90, 92] among others.

[2]Lyapunov functionals are functionals that decrease along solutions as time increases.

approximate non-trivial solutions of the problem

$$\begin{cases} -d_2\Delta v + \lambda v = \gamma(u-1) & \text{in } \Omega \\ \frac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega \\ u = \frac{|\Omega| e^v}{\int_\Omega e^v d\Omega}, \end{cases}$$

where $d_1 = 1,\ \ \gamma = a\bar{u}_0$ with $\bar{u}_0$ the spatial mean of the initial value u_0.

Then a year later, Post [62] built on the work in [26] by studying the system (0.4), where the chemotactic coefficient and a in the production term for v were functions of v. That is, she considered the system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla S(v)) & \text{in } \Omega\times(0,T), \\ v_t = d_2\Delta v - \lambda v + auS'(v) & \text{in } \Omega\times(0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{0.7}$$

The function S is referred to as the *sensitivity function*. The introduction of the sensitivity function is important because it gives a more realistic model of chemotaxis. It incorporates into the model the ability of the amoebae u to sense the $v-$chemical concentration. In this setting, she proved existence of global solutions of system (0.7) on a two-dimensional Lipschitz domain for different natural classes of sensitivity functions. This result was most significant because it enabled her to prove convergence of the trajectories of solutions to trivial and non-trivial steady state, under differing conditions on the data of the system. Uniqueness and further regularity of the solutions was shown under the assumption that $S \in C^2(\mathbb{R},\mathbb{R})$ and $|S''(v)| \le C$ for all $v \ge 0$, where $C > 0$ is a constant. She also gave results, for the first time, for the fully non-stationary chemotaxis system (with or without sensitivity functions) on higher dimensional domains.

Liu, J. and Wang, Z. A., [49] have established the existence of global classical solutions and non-trivial steady states of the one-dimensional attraction-repulsion system of equations. Extending the work of Zhang, Q., and Li, Y., in [93], one can easily obtain the two-dimensional case. Kozono, H., *et al* proved in [43] the existence and uniqueness of solutions to the system (0.4) in $\mathbb{R}^N$, $N \geq 3$, in the scaling invariant space. More recently, in [87], Willie, R., and Wacher, A., have proven in scales of Hilbert spaces the well-posedness of the system of equations for a perturbed analytic semigroup, which decays exponentially in the large time asymptotic dynamics of the problem to a subset in $\mathbb{R}^3$ of the spatial average solutions. They also provided uniform bounds in $\Omega \times (0,T)$ of the solution, and via a bootstrap argument, they argued that the solutions are in fact classical solutions.

On blow-up solutions of these equations, the ground-breaking work was done in 1973 by Nanjundiah [56], as cited in [37, 33] among others, where he suggested that

> "*the end-point (in time) of aggregation is such that the cells are distributed in form of $\delta-$function concentration.*"

After that, in their 1981 work [15], Childress and Percus came up with the following statements for space dimension $N = 2$:

- The density $u(x,t)$ cannot form a $\delta-$function singularity, if the total density on $\Omega \subset \mathbb{R}^2$ is less than some critical number d_Ω.
- The density $u(x,t)$ can form a $\delta-$function singularity, if the total density on Ω is greater than some critical number D_Ω.

It was then believed that for the above statements, $d_\Omega = D_\Omega$. It was also observed that the density $u(x,t)$ (and hence the chemical concentration $v(x,t)$) might blow-up if the total density on Ω exceeds the critical number D_Ω.

While studying the following modified Keller-Segel model, where the second equation was replaced by a stationary equation,

$$\begin{aligned} u_t &= \Delta u - \chi \nabla(u \nabla v) \\ 0 &= d_2 \Delta v - \lambda v + au, \end{aligned} \tag{0.8}$$

with homogeneous Neumann conditions and $u(x,0) = u_0$, Jäger and Luckhaus [39] proved in 1992 the existence of global radial solutions when the initial values have small mass, and they showed that the radial solutions of (0.8) blow-up at the origin in a finite time[3] T.

Following [39], in 1995, Nagai [55] studied the system (0.8), and showed that in one dimension ($N = 1$), the solution does not blow-up, but it blows-up when the dimension is greater than or equal to three ($N \geq 3$). This suggests that $N = 2$ is the borderline case. It was also shown there that if $N = 2$, the domain Ω is a ball, $u_0(x)$ is radially symmetric and

$$\frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx < \frac{8\pi}{\chi |\Omega| a},$$

then there is no blow-up for (0.8). But under some conditions, if $u_0(x)$ is radially symmetric and

$$\frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx > \frac{8\pi}{\chi |\Omega| a},$$

then blow-up does occur.

In 1996, Herrero and Velázquez [31], for the first time, studied the system (0.4) with an instationary $v-$equation. They proved existence of $\delta-$distribution blow-up in the disc center by inverting the $\Delta-$operator. In particular, they showed the existence of radially

[3]This associates blow-up with mass.

symmetric initial data such that the solution of a transformed version of (0.4) blows-up at the center of a disc in finite time when $\frac{au_0|\Omega|}{d_2} > 8\pi$.

A few years later, Gajewski and Zacharias [26] showed that if Ω is a general smooth domain, then there is no blow-up and solutions exist globally in time when

$$\frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx < \frac{4\pi}{\chi|\Omega|a},$$

while blow-up occurs when

$$\frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx > \frac{4\pi}{\chi|\Omega|a}.$$

In an earlier mentioned literature, Post [62] did not do a blow-up analysis of the system (0.7). She however put forward her belief that a realistic mathematical model for chemotaxis should be able to exclude blow-up of solutions in finite time. Hence, she does not agree with the interpretations of Nanjundiah [56], and Herrero and Velázquez [31] that a $\delta-$distribution blow-up at a point can be viewed as an approximation of the erection of fruiting body.

For the system (0.4) where no symmetry is assumed on the solution, Horstmann and Wang [37] (also see Horstmann's survey in [33]) proved the existence of blow-up solutions for a smooth domain $\Omega \subset \mathbb{R}^2$, provided that $\frac{au_0|\Omega|}{d_2} > 4\pi$ and $\frac{au_0|\Omega|}{d_2} \neq 4k\pi, \quad k \in \mathbb{N}$. Horstmann later proved in [34], where he assumed radial symmetry,[4] that there are initial data for (0.4) that lead to blow-up in finite or infinite time[5], provided that $\frac{au_0|\Omega|}{d_2} > 8\pi$, while if $\frac{au_0|\Omega|}{d_2} < 8\pi$, then the solution can only converge to a steady state as $t \to \infty$.

The system (0.4) is not an easy one to treat in the Bessel potential space setting. In the context of the semilinear evolution equations, one would prefer that the order of the

[4]He also assumed that the chemical consumption is paltry.

[5]Note that, the chemical diffusion coefficient, d_2, was not assumed to necessarily be equal to 1.

semi-linear term be strictly less than the order of the elliptic partial differential operator. However, we note that in the cell density equation (first equation) of (0.4), the semi-linear term features a divergence operator, which is of the same order as the principle elliptic partial differential operator. Furthermore, the equation variable appears in it as a diffusion coefficient (∇v), of which *a priori* boundedness in $L^\infty(\Omega)$ is not necessarily immediate. Thus, for this term to be defined, $H^1(\Omega) \cong E_2^{\frac{1}{2}}$ is only possible in $\mathbb{R}^2$.

We also note that the chemical concentrations equation (the second equation) is linear, but the reaction term (data) features the cell density variable u. The difficulty here is that the semigroup smoothness space has to be the same as the space in which the initial data to the chemical concentration equation is considered. Moreover, to control the semi-linear term in the cell density equation, we need to map it into the space in which its initial data are considered to be, while at the same time the chemical drift term is controlled appropriately so that it is well defined in adequate function spaces.

In this thesis, we work mainly in the Bessel potential space setting, in which the *a priori* compactness is lost compared to the usual Hilbert space setting. To take care of this, we employ the Concentration-Compactness Principle [48] in the Bessel potential spaces to compensate for this failure of pre-compactness. Note that if we take the reaction data to be equal to zero, then we obtain the following Liouville semi-linear elliptic equation

$$-\Delta v + \lambda v - ake^v = 0, \tag{0.9}$$

which has the non-linearity of so-called critical growth. The difficulty lies in controlling this non-linearity. We note that when $q = 2$, for a limiting case $2\alpha = \frac{N}{2}$, we have $E_2^\alpha \not\subset L^\infty(\Omega)$.

So, to prove the existence of non-trivial solutions to the above Dirichlet problem one uses the famous Trudinger-Moser inequality

$$\sup_{\|\nabla u\|_{L^2(\Omega)}\leq 1} \int_\Omega e^{\kappa u}\, dx \begin{cases} \leq c|\Omega| & \text{if} \quad \kappa \leq 4\pi \\ = \infty & \text{if} \quad \kappa > 4\pi. \end{cases} \tag{0.10}$$

One can then use Lions' [48] (see Theorem I.6 and Remark I.18) Concentration-Compactness Alternative for the Moser-Trudinger inequality, and deduce compactness of the embedding of the space into an Orlicz space (see [2] for more information on Orlicz spaces).

In the same vein, when investigating the maximal time of existence for the system (0.4), we recall a well-known nonexistence result of Pohozaev, as given in [74] among others, which asserts that if Ω is star shaped and $\lambda \leq 0$, then there is no nontrivial solution of the problem

$$\begin{cases} -\Delta u = \lambda u + |u|^{2^*-2} & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where $2^* = \frac{2N}{N-2}$. This is due to the fact that the standard variational arguments do not apply since the embedding $H_0^1 \subset L^{2^*}(\Omega)$ is not compact, and so, the corresponding functional

$$I_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right)\, dx$$

does not satisfy the Palais-Smale conditions. For this reason, when considering the system (0.4), the chemical concentration equation has to be elliptic. This allows us to decouple the system.

We mention here some alternative ideas for investigating the maximal time of existence that has been used in literature. In [34], Hortsmann investigated the existence of radially symmetric blow-up solutions for (0.4). He excluded the possibility of global boundedness of the solution for the certain class of initial data, and hence, concluded blow-up by showing that for $au_0|\Omega| > 8d_2\pi$, the Lyapunov functional corresponding to (0.4) is not bounded from below. He used results from Gajewski and Zacharias [26] and Brézis and Merle [13].

Based on the same ideas as in [34] and Wang and Wei's ([83]) generalization of Brezis and Merle's results [13], Horstmann and Wang [37] investigated the blow-up in (0.4) without symmetry assumptions. Pohozaev identity was used in that work.

In [16], Chipot illustrated the usage of the concavity method. This method is useful in proving that blow-up occurs, but it does not specify exactly what the maximal time of existence of the solutions is. There is also a treatment by K. Post in [62], in which she used the results of her existence theorem of global solutions of a chemotaxis model, where different natural classes of sensitivity functions were considered, to study the asymptotic solution behaviour. All these are alternatives to the treatment given in this work.

With regard to the limit case of large time for the system (0.4), we exploit the invariance principle, credited to J.P. LaSalle [47]. If (u, v) is a solution of the system of equations (5.3), obtained through rescaling of solutions of the system (0.4), then we can write down a Lyapunov functional, F (see (5.41)). From Theorem 3 of [47], we get that if the solutions are unique, and the Lyapunov functional is constant on the boundary of the union of all solutions in their maximal interval of definition, then these solutions are asymptotically stable. For more information on the invariance principle, also see [28].

0.2 Outline of the Thesis

In what follows, we want to give a brief outline of this thesis. The preliminary results and definitions are given in Chapter 1. Moreover, we will also describe some mathematical notations in this chapter, which we will be frequently using in the sequel. More specifically in this chapter, we write down some basics of semigroups.

In Chapter 2, we give a brief review of interpolation theory. We will be limiting ourselves to the $(L^p, W^{2,p})$ example for the real interpolation. On the complex interpolation, we give the definition, characterize some background material, and then we state some of its application to the construction of Bessel potential spaces. The significance of this chapter is in that no exact reference (that we are aware of) yields completely this construction, but in most cases they are derived as particular cases of more general spaces.

In Chapter 3, we prove the existence and uniqueness of solutions to the minimal system model (0.4) in the Bessel potential space setting, and that the system (0.4) defines a perturbed analytic semigroup to the semigroup generated by the operator $\mathcal{A}$ (see (3.3)), using abstract semigroup theory results for semi-linear evolution equations from [30, 51, 60, 68, 66]. In Section 3.3, we prove the existence of *a priori* uniform bounds in $\Omega \times (0, T)$ of solutions and gradient solutions to the problem. We conclude Chapter 3 by highlighting, in few details, the blow-up analysis of solutions to the system of equations at the borderline spaces $E_q^{\alpha},\ \alpha = \frac{N}{2q}$.

In Chapter 4, we work in a Hilbert space setting. The treatment which we will give in this chapter is that of a Keller-Segel system of equation with Attraction-Repulsion effects.

We prove, in Section 4.3, that the system model equations (4.1)-(4.4) defines a perturbed analytic semigroup to the semigroup generated by the operator $-\mathcal{A}$. We then prove the existence of *a priori* uniform bounds in $\Omega \times (0,T)$ of solutions and gradient solutions to the problem in Section 4.4. We conclude this section by using a bootstrap argument to prove that the solutions to the problem are classical solutions. In Section 4.5, we revisit the complete system of equations coupled partial differential operator, to prove that it is an infinitesimal generator of a fundamental solution operator in scales of spaces $Z_\delta, \delta \in \mathbb{R}^+$ as given by quasilinear partial differential operators. We then conclude this chapter by numerically simulating the equations using a Gradient Weighted Moving Finite Element method in Section 4.6.

In Chapter 5, we, in a Hilbert space setting, investigate the maximal time of existence for the system (0.4), by using Pohozaev's Non-existence principle, guided by [37, 83]. Conditions will be given under which blow-up occurs in finite or infinite time. We conclude this chapter by briefly doing the blow-up analysis for the system (0.4) following the Concavity method in [16].

In Chapter 6, we revisit the attraction-repulsion Keller-Segel system of equations which we studied in Chapter 4. In this case however, we study the asymptotic dynamics in Lebesgue-Bochner spaces of underlying Banach spaces either $L^p(\Omega)$, or Bessel potential spaces $H^{2\alpha,p}(\Omega)$. In Section 6.3, we prove the well-posedness of the system of equations in $L^\sigma(\dot{I}; L^p(\Omega))$, then we prove *a priori* uniform boundedness in $\Omega \times \dot{I}$ of the cells density solution in Subsection 6.3.1. In Section 6.4, we prove similar results to those of Section 6.3, but in Bessel potential spaces E_q^α, $\alpha \in \mathbb{R}, 1 < q < \infty$. Lastly, in Section 6.5, we give an overview analysis of the blow-up of solutions to the system of equations at the borderline spaces $E_p^{\alpha}, \alpha = \frac{N}{2p}$.

Chapter 1

Preliminaries

1.1 Introduction

In this chapter, we are going to state some preliminary results and definitions which will be of great use in this work. We will define some notations which will be used in this regard. We will then state the definition of semigroups and write down some fundamental results about them.

1.2 Functional Setting

We are going to let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary $\partial\Omega$. Throughout this thesis, we assume that the reader is familiar with the basic notions of Sobolev spaces (see [2, 12, 30] among others). For $1 \leq q \leq \infty$, the Sobolev space of functions on Ω will be denoted by $W^{s,q}(\Omega)$, and the standard notation of its norm is $\|\cdot\|_{W^{s,q}(\Omega)}$. In particular, we will write $H^s(\Omega) := W^{s,2}(\Omega)$.

If we choose $L^q(\Omega)$, for $1 < q < \infty$, as the base space, then the unbounded linear operator $-\mathcal{A} : D(\mathcal{A}) \subset L^q(\Omega) \to L^q(\Omega)$, with domain $D(\mathcal{A}) = H^{2,q}(\Omega)$, as defined in (3.3), generates an analytic semigroup in $L^q(\Omega)$, see [5, 30, 60, 66, 68].

The Bessel potential spaces[1] of functions on Ω will be denoted by $H^{s,q}(\Omega)$, where $s \in \mathbb{R}$ and $1 \le q \le \infty$ [30, 71, 82]. Note that $H^{s,q}(\Omega)$ coincide with $W^{s,q}(\Omega)$ for integer s if $1 < q < \infty$, or for all s if $q = 2$. The notation

$$E_q^\alpha := H^{2\alpha,q}(\Omega), \quad \alpha \in [-1, 1], \tag{1.1}$$

denote well defined scale spaces associated with the non-coupled system partial differential operator $\mathcal{A}$ in (3.3), with their norm being written as

$$\| \cdot \|_{H^{2\alpha,q}(\Omega)} = \| \cdot \|_{E_q^\alpha} = \| \cdot \|_\alpha.$$

With this in mind, we will therefore make use of the following conventions:

$$E_q^{\frac{1}{2}} = W^{1,q}(\Omega), \quad E_q^0 = L^q(\Omega), \quad E_{q'}^{-\frac{1}{2}} = W^{-1,q'}(\Omega).$$

Furthermore, if there is no danger of confusion, we will adopt the equivalent Bessel potential spaces norm notation. That is,

$$\|\cdot\|_{\frac{1}{2}} = \|\cdot\|_{1,q}, \quad \|\cdot\|_0 = \|\cdot\|_q, \quad \|\cdot\|_{-\frac{1}{2}} = \|\cdot\|_{-1,q}.$$

Sometimes, we will assume that the spaces are “nested". That is, for any α, $\beta \in \mathbb{R}$, if $\alpha \ge \beta$, we have

$$E_q^\alpha \subseteq E_q^\beta, \tag{1.2}$$

with a continuous embedding, and the norm of the embedding will be denoted by $\|i\|_{\alpha,\beta}$, where the relation i is equivalent to the identity operator $i : E_q^\alpha \to E_q^\beta$ [66]. In such a case,

[1] See Definition 2.8.

we will say that the spaces are nested, for short. This situation will be explicitly stated if needed. Note that if we consider (1.1) above, we will have $\|i\|_{\alpha,\beta} < \infty$ for all $\alpha \geq \beta$.

Occasionally, we will use the notation

$$Z_{\alpha(\beta)} := E_q^{\alpha} \times E_q^{\beta}, \quad 1 < q < \infty.$$

Lastly for this section, we recall the Banach Contraction principle [30, 68].

Definition 1.1. Let $(X, \|\cdot\|_X),\ (Y, \|\cdot\|_Y)$ be Banach spaces. A mapping $T : X \to Y$ is said to be a contraction if there exists a positive number $\theta < 1$ such that

$$\|T(x) - T(y)\|_Y \leq \theta \|x - y\|_X \quad \text{for all } x,\ y \in X.$$

We therefore have the following Theorem.

Theorem 1.1 (The Banach Contraction Mapping Theorem)**.** *Let $(X, \|\cdot\|)$ be a Banach space, and $T : X \to X$ be a contraction. Then there exists a unique fixed point of T in X: $x \in X$ such that $T(x) = x$.*

Also, for any $y \in X$, if $T^n(y) = T(T^{n-1}(y))$ is the $n-$fold composition, then $T^n(y) \to x$ as $n \to \infty$. In fact, $\|T^n(y) - x\| \leq \theta^n \|y - x\|$.

1.3 Semigroups

In this section, we will write down the definition of analytic semigroups, and state some of their abstract properties [30, 68]. To this end, let W be a Banach space and $\mathcal{L}(W)$ be a space of bounded linear operators on W. Let $\delta \in (0, \pi)$ and define an open sector $\Delta_\delta := \{z \in \mathbb{C} : |arg\ z| < \delta,\ z \neq 0\}$. If $S(t)$ is a C_0-semigroup [51, 68] on W generated by the operator $\mathcal{A}$, then $S(t)$ is called an analytic semigroup generated by $\mathcal{A}$ if there exists an extension of $S(t)$ to a mapping $S(t)$ defined for t in $\Delta_\delta \cup \{0\}$ such that

(i) $t \mapsto S(t)$ is a mapping of $\Delta_\delta \cup \{0\}$ to $\mathcal{L}(W)$,

(ii) $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1,\ t_2 \in \Delta_\delta \cup \{0\}$,

(iii) For each $w \in W$, $S(t)w \to w$ as $t \to 0$ in $\Delta_\delta \cup \{0\}$,

(iv) For each $w \in W$, $t \mapsto S(t)w$ is an analytic mapping from Δ_δ into W.

If, in addition, there exist $a \in \mathbb{R}$, $\sigma \in (0, \frac{\pi}{2})$, and $M \geq 1$, such that $\Sigma_\sigma(a) := \{z \in \mathbb{C} : |arg(z-a)| > \sigma,\ z \neq a\} \subset \rho(\mathcal{A})$, and $\|R(\lambda, \mathcal{A})\| \leq \dfrac{M}{|\lambda - a|}$ for every $\lambda \in \Sigma_\sigma(a)$, then the operator $\mathcal{A}$ is called a sectorial operator[2] on W.

We then say that, the operator $-\mathcal{A}$, as defined in (3.3) (or more precisely, a suitable realization of it) is the infinitesimal generator of an analytic semigroup,

$$\{S(t) = e^{-\mathcal{A}t} :\ t \in \mathbb{R}^+ \setminus \{0\}\} \tag{1.3}$$

in each space of the scales $H^{2\alpha,q}(\Omega)$, $\alpha \in \mathbb{R}$ [5, 30, 60, 66, 68]. This semigroup is order preserving and satisfies the smoothing estimates

$$\|S(t)u_0\|_{H^{2\alpha,q}(\Omega)} \leq \frac{M_{\alpha,\beta}e^{\mu_0 t}}{t^{\alpha-\beta}}\|u_0\|_{H^{2\beta,q}(\Omega)}, \quad t > 0, \quad u_0 \in H^{2\beta,q}(\Omega) \tag{1.4}$$

for $-1 \leq \beta \leq \alpha \leq 1$ and some $\mu_0 \in \mathbb{R}$. In addition, we have

$$\|S(t)u_0\|_{L^\tau(\Omega)} \leq \frac{M_{\tau,\rho}e^{\mu_0 t}}{t^{\frac{N}{2}(\frac{1}{\rho}-\frac{1}{\tau})}}\|u_0\|_{L^\rho(\Omega)}, \quad t > 0, \quad u_0 \in L^\rho(\Omega) \tag{1.5}$$

for $1 \leq \rho \leq \tau \leq \infty$. For any u_0 in $H^{2\beta,q}(\Omega)$ or $L^\rho(\Omega)$, the function $u(t; u_0) := S(t)u_0,\ t > 0$, is a classical solution of the problem

$$\begin{cases} u_t - \mathcal{A}u &= f(u) \\ u(0) &= u_0, \end{cases}$$

[2] $\rho(\mathcal{A})$ denotes the resolvent set of $\mathcal{A}$, while $R(\lambda, \mathcal{A}) = (\mathcal{A} - \lambda I)^{-1}$ is the resolvent of $\mathcal{A}$.

provided that $f(u)$ is locally Hölder continuous in t, and locally Lipschitzian in u. For further properties of semigroups, please see [5, 30, 60, 66, 68].

Next, we review some abstract analytic semigroup theory results proven in [5, 30, 51, 60, 68]. To this end, we note that (1.4) can be rewritten in an abstract language as

$$\|S(t)\|_{\mathcal{L}(E_q^\beta, E_q^\alpha)} \leq \frac{M_{\alpha,\beta} e^{\mu t}}{t^{\alpha-\beta}}, \quad \alpha \geq \beta, \quad t > 0.$$

We also assume that the semigroup acting on the scales satisfies, for $\alpha,\ \beta \in I$ such that $\alpha \geq \beta$,

$$\|S(t)\|_{\beta,\alpha} := \|S(t)\|_{\mathcal{L}(E_q^\beta, E_q^\alpha)} \leq \frac{M_0(\beta,\alpha)}{t^{\alpha-\beta}}, \quad \text{for all } 0 < t \leq 1, \tag{1.6}$$

for some constant $M_0(\beta,\alpha) > 0$.

From these assumptions, the following Lemma follows.

Lemma 1.2. *Assume that* (1.6) *is satisfied. Then*

(i) *For every $\alpha,\ \beta \in I$ such that $\alpha \geq \beta$, and for all $T > 0$,*

$$\|S(t)\|_{\beta,\alpha} \leq \frac{M_0(\beta,\alpha,T)}{t^{\alpha-\beta}}, \quad \textit{for all } 0 < t \leq T \tag{1.7}$$

for some constant $M_0(\beta,\alpha,T) > 0$.

(ii) *For each $\beta \in I$, there exists $\omega(\beta) \geq 0$ such that*

$$\|S(t)\|_{\beta,\beta} \leq M_0(\beta,\beta)e^{\omega t}, \quad \textit{for all } t > 0,$$

and for every $\alpha,\ \beta \in I$ such that $\alpha \geq \beta$ there exists $\omega = \omega(\beta)$ and $M(\beta,\alpha)$ such that

$$\|S(t)\|_{\beta,\alpha} \leq \frac{M(\beta,\alpha)e^{\omega t}}{t^{\alpha-\beta}}, \quad \textit{for all } 0 < t < \infty.$$

(iii) *Assume that the scales are nested, that is* (1.2) *holds. Then, if for some fixed* $\beta_0 \in I$, *we have*

$$\|S(t)\|_{\beta_0,\beta_0} \leq Me^{\omega_0 t}, \quad \textit{for all } t > 0 \tag{1.8}$$

for some $M = M(\beta_0)$ *and* $\omega_0 \in \mathbb{R}$*, then for any* $\alpha \in I$*, there exists a constant* $M(\alpha) \geq 1$ *such that*

$$\|S(t)\|_{\alpha,\alpha} \leq M(\alpha)e^{\omega_0 t}, \quad \textit{for all } t > 0. \tag{1.9}$$

Moreover, given $t_0 > 0$*, define* $\delta = \|S(t_0)\|_{\beta_0,\beta_0}$*. Then we have* (1.8) *with*

$$\omega_0 = \frac{\ln \delta}{t_0}$$

and some constant M *depending on* t_0*,* δ *and* $M_0(\beta_0, \beta_0, t_0)$ *as in* (1.7)*. In particular, if* $\delta < 1$*, then* $\omega_0 < 0$*.*

(iv) *Under the settings of (iii), for every* α*,* $\beta \in I$ *such that* $\alpha \geq \beta$ *we have*

$$\|S(t)\|_{\beta,\alpha} \leq \begin{cases} M_1(\beta,\alpha)t^{-(\alpha-\beta)} & \textit{if } 0 < t \leq 1, \\ \\ M_1(\beta,\alpha)e^{\omega_0 t} & \textit{if } t > 1. \end{cases}$$

for some positive constant $M_1(\beta, \alpha)$*.*

In particular, for all $\varepsilon > 0$ *there exists* $M_\varepsilon(\beta, \alpha) > 0$ *such that*

$$\|S(t)\|_{\beta,\alpha} \leq M_\varepsilon(\beta,\alpha)\frac{e^{(\omega_0+\varepsilon)t}}{t^{\alpha-\beta}}, \quad \textit{for all } t > 0.$$

Lemma 1.2 and its proof appeared in [66]. For completeness, we give the proof here.

Proof. **(i)** Let $T > 0$ and define n to be the smallest integer such that $T \leq n+1$. Further, for $0 < t \leq T$, define $h = \dfrac{t}{n+1} \leq 1$ and $s_j = jh$, $j = 0, \ldots, n+1$. This means that $s_{n+1} = t$ and since

$$S(t) = S(s_{n+1} - s_n) \cdots S(s_1 - s_0),$$

and $0 < s_{i+1} - s_i = h \leq 1 \ \ \forall 0 < t \leq T$ and $i = 0, \ldots, n$, we get from (1.6) that

$$\begin{aligned}
\|S(t)\|_{\beta,\alpha} &= \|S(s_{n+1} - s_n) \cdots S(s_1 - s_0)\|_{\beta,\alpha} \\
&\leq \|S(h)\|_{\beta,\alpha} \|S(h)\|^n_{\alpha,\alpha} \\
&\leq M_0(\alpha,\alpha)^n M_0(\beta,\alpha) h^{-(\alpha-\beta)} \\
&= M_0(\alpha,\alpha)^n M_0(\beta,\alpha)(n+1)^{\alpha-\beta} t^{-(\alpha-\beta)} \\
&= \frac{M_0(\beta,\alpha,T)}{t^{\alpha-\beta}} \quad \text{for all } 0 < t \leq T,
\end{aligned}$$

where $M_0(\beta,\alpha,T) = M_0(\alpha,\alpha)^n M_0(\beta,\alpha)(n+1)^{\alpha-\beta}$.

(ii) In a particular case of (i) when $\alpha = \beta$, we let $t > 0$ and define $n \in \mathbb{N}$ such that $n \leq t < n+1$ and we get as in (i), that

$$\|S(t)\|_{\beta,\beta} \leq M_0(\beta,\beta)^{n+1} \leq M_0(\beta,\beta)^{t+1} \leq M_0(\beta,\beta) e^{\ln(M_0(\beta,\beta))t}, \quad \text{for all } t > 0.$$

Note that since $M_0(\beta,\beta) \geq 1$, $\omega(\beta) := \ln(M_0(\beta,\beta)) \geq 0$.

Now, if $\alpha, \beta \in I$ such that $\alpha \geq \beta$ and $t > 1$, then we have

$$\|S(t)\|_{\beta,\alpha} \leq \|S(t-1)\|_{\alpha,\alpha} \|S(1)\|_{\beta,\alpha} \leq M_0(\alpha,\alpha) e^{\omega(\alpha)(t-1)} M_0(\beta,\alpha),$$

while for $0 < t < 1$, we have estimate (1.6). Then for any $\omega > \omega(\alpha)$, we get the result.

(iii) First we notice that from (1.6), for any $\alpha \geq \beta_0$, we have $\|S(1)\|_{\beta_0,\alpha} \leq M_0(\beta_0,\alpha)$. Now, if $t > 1$, then

$$\begin{aligned}
\|S(t)u_0\|_\alpha &\leq \|S(1)\|_{\beta_0,\alpha} \|S(t-1)u_0\|_{\beta_0} \\
&\leq M_0(\beta_0,\alpha) M e^{-\omega_0} e^{\omega_0 t} \|u_0\|_{\beta_0} \\
&\leq M_0(\beta_0,\alpha) \|i\|_{\alpha,\beta_0} M e^{-\omega_0} e^{\omega_0 t} \|u_0\|_\alpha,
\end{aligned}$$

where $\|i\|_{\alpha,\beta_0}$ denotes the norm of the inclusion $E^\alpha_q \hookrightarrow E^{\beta_0}_q$. Thus,

$$\|S(t)\|_{\alpha,\alpha} \leq K e^{\omega_0 t}, \quad \text{for all } t > 1,$$

with $K = M_0(\beta_0, \alpha)\|i\|_{\alpha,\beta_0} M e^{-\omega_0}$.

On the other hand, if $\beta_0 \geq \alpha$, then we also have from (1.6) that $\|S(1)\|_{\alpha,\beta_0} \leq M_0(\alpha, \beta_0)$, and for $t > 1$,

$$\begin{aligned}
\|S(t)u_0\|_\alpha &\leq \|i\|_{\beta_0,\alpha}\|S(t)u_0\|_{\beta_0} \\
&\leq \|i\|_{\beta_0,\alpha}\|S(t-1)\|_{\beta_0,\beta_0}\|S(1)u_0\|_{\beta_0} \\
&\leq \|i\|_{\beta_0,\alpha} M e^{-\omega_0} e^{\omega_0 t}\|S(1)\|_{\alpha,\beta_0}\|u_0\|_\alpha \\
&\leq \|i\|_{\beta_0,\alpha} M e^{-\omega_0} M_0(\alpha, \beta_0) e^{\omega_0 t}\|u_0\|_\alpha.
\end{aligned}$$

Thus, we get that

$$\|S(t)\|_{\alpha,\alpha} \leq K e^{\omega_0 t}, \qquad \text{for all } t > 1,$$

with $K = M_0(\alpha, \beta_0)\|i\|_{\beta_0,\alpha} M e^{-\omega_0}$.

Therefore, for any $\alpha \in I$, we have the estimate

$$\|S(t)\|_{\alpha,\alpha} \leq K(\alpha) e^{\omega_0 t}, \qquad \text{for all } t > 1.$$

Hence, from (1.6), if $\beta = \alpha$, then we get (1.9), with

$$M(\alpha) = \begin{cases} \max\{K(\alpha),\ M_0(\alpha, \alpha)\} & \text{if } \omega_0 \geq 0, \\ \\ \max\{K(\alpha),\ M_0(\alpha, \alpha) e^{-\omega_0}\} & \text{if } \omega_0 \leq 0. \end{cases}$$

Moreover, if for a given $t_0 > 0$ we define $\delta = \|S(t_0)\|_{\beta_0,\beta_0}$, then for $t > 0$ we write $t = nt_0 + s$, with $n \in \mathbb{N}$ and $0 \leq s < t_0$. Then

$$\|S(t)\|_{\beta_0,\beta_0} \leq \delta^n \|S(s)\|_{\beta_0,\beta_0} \leq e^{\ln(\delta)(\frac{t-s}{t_0})} M_0(\beta_0, \beta_0, t_0),$$

with $M_0(\beta_0, \beta_0, t_0)$ as in (1.7), and the result follows. In particular, if $\delta < 1$, then $\omega_0 < 0$.

(iv) We note that, if $0 < t \leq 1$, the estimate reduces to (1.6). On the other hand, if $t > 1$, then using (1.6) and part (iii), we get

$$\begin{aligned} \|S(t)\|_{\beta,\alpha} &\leq \|S(t-1)\|_{\alpha,\alpha}\|S(1)\|_{\beta,\alpha} \\ &\leq M_0(\beta,\alpha)M(\alpha)e^{-\omega_0}e^{\omega_0 t} = M_1(\beta,\alpha)e^{\omega_0 t}, \end{aligned}$$

where $M_1(\beta,\alpha) = M_0(\beta,\alpha)M(\alpha)e^{-\omega_0}$, and the result follows easily.

□

Remark 1.1. We observe that if the original constants $M_0(\beta,\alpha)$ in (1.6) do not depend on (or can be taken independent of) $\alpha,\ \beta \in I$, then the same is true for $M_0(\beta,\alpha,T)$, and $M(\alpha)$ in (1.9) depends on the scales only through the norm of the embeddings $\|i\|_{\beta_0,\alpha}$ or $\|i\|_{\alpha,\beta_0}$.

The following spaces will be used immensely henceforth.

Definition 1.2. For $T > 0,\ \gamma \in I$ and $\varepsilon \geq 0$, we denote the space of all locally essentially bounded functions, $u \in L^\infty_{loc}((0,T],E^\gamma_q)$, for which $\sup\limits_{t\in(0,T]} t^\varepsilon\|u(t)\|_\gamma < \infty$ by $\mathcal{L}^\infty_\varepsilon((0,T],E^\gamma_q)$, and define the quantity

$$|||u|||_{\gamma,\varepsilon} = \sup_{t\in(0,T]} t^\varepsilon\|u(t)\|_\gamma,$$

as its norm.

We then have the following Lemma;

Lemma 1.3. *Let $T > 0,\ \gamma \in I$ and $\varepsilon \geq 0$. Then the space $\mathcal{L}^\infty_\varepsilon((0,T],E^\gamma_q)$, equipped with the norm $|||\cdot|||_{\gamma,\varepsilon}$, is a Banach space.*

Proof. Note that $\{u^k\}_k$ is a Cauchy sequence in $\mathcal{L}^\infty_\varepsilon((0,T],E^\gamma_q)$ if and only if $v^k(t) = t^\varepsilon u^k(t)$ is a Cauchy sequence in $L^\infty([0,T],E^\gamma_q)$, and also $u^k(t)$ converges in E^γ_q to some $u(t)$ for almost all $t > 0$ and, hence, in $L^\infty([o,T],E^\gamma_q)$ topology. This implies that u^k converges to u in $\mathcal{L}^\infty_\varepsilon((0,T],E^\gamma_q)$, and hence the space $\mathcal{L}^\infty_\varepsilon((0,T],E^\gamma_q)$ is complete. □

Note that part (i) in Lemma 1.2 can be restated as

Lemma 1.4. *Assume the semigroup* $S(t) = e^{-\mathcal{A}t}$, $t > 0$ *and the scales of spaces satisfy* (1.6)*. Then, for any* α, $\beta \in I$ *such that* $\alpha \geq \beta$ *and* $T > 0$*, the mapping*

$$S(\cdot) : E_q^\beta \to \mathcal{L}^\infty_{\alpha-\beta}((0,T], E_q^\alpha), \quad u_0 \mapsto S(\cdot)u_0,$$

is linear and continuous.

Chapter 2

Interpolation theory and Scales of Banach spaces

The aim of this chapter is to give a basis of the theory of scales of Banach spaces. This theory is naturally based in the methods of interpolation theory. It has, to some extent, similarities with the theory of real number system, but in this case relating to intermediate Banach spaces. Interpolation theory in functional analysis and applications has been developed by many authors, of which we cite [4, 9, 29, 52, 79].

2.1 Interpolation theory background

In what follows, for notation simplicity, for $T \in \mathcal{L}(E, F)$, we will write $\|T\| = \|T\|_{\mathcal{L}(E,F)}$, on understanding that it is the operator norm that is being considered. Within the case of $\mathcal{L}(L^p, L^q)$, when emphasizing is necessary, we will use $\|T\|_{p,q}$.

It is worthwhile to mention that the classical results making up the basis of interpolation theory are theorems of M. Riesz with Thorin's proof, and Marcinkiewicz [9]. Thorin's proof of the Riesz-Thorin theorem contains the idea behind the complex interpolation method.

Similarly, the proof of Marcinkiewicz's theorem resembles the construction of the real interpolation method. Following these distinctions of cited pioneer works, we state them with their proofs in this initial section of the chapter.

Before we state Riesz-Thorin interpolation theorem, we first note the following:

Lemma 2.1 (Lyapunov Inequality). *Let $1 \le p_j \le \infty$, $j = 0, 1$, $\theta \in [0,1]$ and $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Then, $\bigcap L^{p_j} \subset L^{p_\theta}$ and*

$$\|f\|_{p_\theta} \le \|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}, \quad \forall f \in \bigcap L^{p_j}.$$

Proof. The proof uses Hölder's inequality., i.e. if $f \in L^p$, $g \in L^q$ such that $1 = \frac{1}{p} + \frac{1}{q}$ then $fg \in L^1$. Now let

$$x := (1-\theta)p,\ y := \theta p,\ \frac{1}{z_0} := \frac{1-\theta}{p_0}p,\ \frac{1}{z_1} := \frac{\theta}{p_1}p.$$

Then

$$x + y = p,\ \frac{1}{z_0} + \frac{1}{z_1} = 1,\ xz_0 = p_0, \quad \text{and} \quad yz_1 = p_1.$$

Now using Hölder's inequality we obtain

$$\begin{aligned} \|f\|_p^p &= \|f^p\|_1 = \|f^x f^y\|_1 \le \|f^x\|_{z_0} \|f^y\|_{z_1} \\ &= \left(\int |f|^{xz_0}\right)^{\frac{1}{z_0}} \left(\int |f|^{yz_1}\right)^{\frac{1}{z_1}} = \left(\int |f|^{p_0}\right)^{\frac{1-\theta}{p_0}p} \left(\int |f|^{p_1}\right)^{\frac{\theta}{p_1}p} = \left(\|f\|_{p_0}^{1-\theta} \|f\|_{p_1}^{\theta}\right)^p, \end{aligned}$$

and the conclusion is demonstrated to hold. □

In the proof of Riesz-Thorin theorem, we will need the following result from complex analysis (see, for instance [85]).

Proposition 2.2 (Three Lines Lemma). *Let $F : S = \{z = x + iy \in \mathbb{C} : 0 \le x \le 1\} \to \mathbb{C}$ be a bounded, continuous function, analytic on $\mathring{S} := \{z = x + iy \in \mathbb{C} : 0 < x < 1\}$. Let*

$$M_\theta := \sup_{y \in \mathbb{R}} |F(\theta + iy)| \quad \textit{for } \theta \in [0,1].$$

Then,

$$M_\theta \leq M_0^{1-\theta} M_1^\theta,$$

where M_0 *and* M_1 *are such that* $|F(z)| \leq M_0$ *when* $x = 0$ *and* $|F(z)| \leq M_1$ *when* $x = 1$.

Proof. **Case 1.** $M_0, M_1 \leq 1 \Rightarrow M_\theta \leq 1$.

Define $F_\varepsilon(z) := \frac{F(z)}{1+\varepsilon z}$ for $z = x + iy \in \mathring{S}$, $\varepsilon > 0$, and note that this function is bounded, continuous and analytic on $\mathring{S}$. Moreover,

$$\lim_{|y| \to \infty} F_\varepsilon(z) = 0, \quad \text{uniformly for} \quad x \in [0,1],$$

since $|F_\varepsilon(z)| \leq \frac{F(z)}{\varepsilon z}$ and F is bounded.

Now let $r > |y_0|$ be such that $|F(z)| \leq 1$ for $x \in [0,1]$ and $|y| \geq r$. Also let $R = [0,1] \times i[-r, r]$. This implies that $|F_\varepsilon(z)| \leq 1$ on ∂R. Thanks to Phragmén-Lindelöf maximum principle [63] we have $|F_\varepsilon(z)| \leq 1$ for all $z \in R$. In particular, if $z_0 = x_0 + iy_0$ in $\mathring{S}$ we have $|F_\varepsilon(z_0)| \leq 1$ and thus $|F(z_0)| = \lim_{\varepsilon \to 0} |F_\varepsilon(z_0)| \leq 1$.

Case 2. M_0, M_1 arbitrary. Let $G(z) = \frac{F(z)}{\alpha^{1-z}\beta^z}$ where $\alpha > M_0$, $\beta > M_1$. Then, G is continuous, bounded and analytic on $\mathring{S}$ and $|G(z)| \leq 1$ on ∂S. Thanks to Case 1, we have $|G(z)| \leq 1$ on S so $M_\theta \leq \alpha^{1-\theta}\beta^\theta$ and $M_\theta \leq M_0^{1-\theta} M_1^\theta$. □

In what follows, we let L^{p_j}, L^{q_j}, $j = 0, 1$, denote Lebesgue spaces of functions, defined on different $\sigma-$ finite measure spaces (Ω_r, μ_r), $r = p, q$.

We are now ready to state the Riesz-Thorin interpolation theorem.

Theorem 2.3 (Riesz-Thorin). *Let* $1 \leq p_j, q_j \leq \infty$, $j = 0, 1$, $\theta \in [0,1]$ *be such that*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

If T *is a linear map satisfying that*

$$T : L^{p_j} \to L^{q_j} \quad \textit{with} \quad \|T\|_{p_j, q_j} = N_j$$

for each $j = 0, 1$*, then,*

$$\|Tf\|_{q_\theta} \leq C N_0^{1-\theta} N_1^{\theta} \|f\|_{p_\theta}, \quad \forall f \in \bigcap L^{p_j} \tag{2.1}$$

where $C = 1$ *if* $\mathbb{K} = \mathbb{C}$ *and* $C = 2$ *if* $\mathbb{K} = \mathbb{R}$*. In particular, the mapping* $T : L^{p_\theta} \to L^{q_\theta}$ *can be extended to a continuous linear mapping satisfying in operator norm* $\|T\|_{p_\theta, q_\theta} \leq C N_0^{1-\theta} N_1^{\theta}$*.*

Proof. Thanks to Lemma 2.1, by hypotheses, we have $\bigcap L^{p_j} \subset L^{p_\theta}$ and $\bigcap L^{q_j} \subset L^{q_\theta}$, $j = 0, 1$. Thus,

$$f \in \bigcap L^{p_j} \stackrel{T}{\longmapsto} Tf \in \bigcap L^{q_j}.$$

Case 1. $p_\theta < \infty$ and $q_\theta > 1$. First we note that since integrable step functions are dense in all L^p- spaces, they are dense in $\bigcap L^{p_j}$. Thus we show that (2.1) holds for all such functions, by showing that

$$\left| \int (Tf) g \right| \leq N_0^{1-\theta} N_1^{\theta} \tag{2.2}$$

holds for all integrable step functions f, g satisfying $\|f\|_{p_\theta} = \|g\|_{q'_\theta} = 1$, where $\frac{1}{q'_\theta} = 1 - \frac{1}{q_\theta}$ is the dual conjugate exponent of $q_\theta > 1$. Indeed (2.2) asserts that the functional $l : L^{q'} \to \mathbb{C}$ mapping $g \to \int (Tf) g$ obeys $\|l\| \leq N_0^{1-\theta} N_1^{\theta}$. Furthermore, by [11], Riesz representation $l \in (L^{q'_\theta})' \cong L^{q_\theta}$ is the isometrically isomorphic image of Tf and $\|Tf\|_{q_\theta} \leq N_0^{1-\theta} N_1^{\theta}$.

To prove the above, we define step functions

$$f = \sum_{j=1}^{J} a_j \chi_{A_j}, \quad g = \sum_{k=1}^{J'} b_k \chi_{B_k},$$
$$\|f\|_{p_\theta}^{p_\theta} = \sum_{j=1}^{J} |a_j|^{p_\theta} |A_j| = 1, \quad \|g\|_{q'}^{q'_\theta} = \sum_{k=1}^{J'} |b_k|^{q'_\theta} |B_k| = 1, \tag{2.3}$$

where $|A|$ denotes the measure of a set $A \subset \mathbb{R}^N$, and $A_j \cap A_k = B_{j'} \cap B_{k'} = \emptyset$ for all $j,\ k \in J$ and $j',\ k' \in J'$. Next for $z \in \mathbb{C}$, define

$$\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}, \ \frac{1}{q'(z)} = \frac{1-z}{q'_0} + \frac{z}{q'_1},$$

so that $p(0) = p_0$, $p(\theta) = p$ and $p(1) = p_1$ as well as $q'(0) = q_0'$, $q'(1) = q_1'$. Using the convention $\frac{0}{0} = 0$ we set

$$f_z = |f|^{\frac{p}{p(z)}} \frac{f}{|f|}, \quad \text{and} \quad g_z = |g|^{\frac{q'}{q'(z)}} \frac{g}{|g|}.$$

which are integrable step functions, in particular $f_z \in L^{p_1}$ implies that Tf_z is well-defined. Lastly, define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int (Tf_z) g_z$$

to obtain, using (2.3), that

$$F(z) = \sum_{j=1}^{J} \sum_{k=1}^{J'} |a_j|^{\frac{p}{p(z)}} \frac{a_j}{|a_j|} |b_k|^{\frac{q'}{q'(z)}} \frac{b_k}{|b_k|} \int_{B_k} T\chi_{A_j}.$$

This shows that F is a linear combination of terms of the form γ^z, $\gamma > 0$. So F is analytic and satisfies the assumption of Proposition 2.2 since every function γ^z is bounded in S (see Proposition 2.2) by

$$|\gamma^{x+iy}| = \gamma^x \leq \max\{1, \gamma\}, \quad \forall x + iy \in S.$$

Now for estimating $|F(\theta + iy)|$ for $\theta = 0, 1$, we have by Hölder's inequality, that if $\theta = 0$, then

$$|F(iy)| \leq \|Tf_{iy}\|_{q_0} \|g_{iy}\|_{q_0'} \leq N_0 \|f_{iy}\|_{p_0} \|g_{iy}\|_{q_0'},$$

and furthermore,

$$\|f_{iy}\|_{p_0}^{p_0} = \sum_{j=1}^{J} \left| |a_j|^{\frac{p}{p(iy)} p_0} \right| A_j| = \sum_{j=1}^{J} |a_j|^p |A_j| = \|f\|_p^p = 1,$$

using (2.3) and the fact that $||a_j|^{\frac{p}{p(iy)} p_0}| = |a_j|^p$. Similarly, we obtain $\|g_{iy}\|_{q_0'}^{q_0'} = 1$. Summing up, we have $\sup_{y \in \mathbb{R}} |F(iy)| \leq N_0$, and carrying out the calculation with $\theta = 1$, we get $\sup_{y \in \mathbb{R}} |F(1 + iy)| \leq N_1$. Finally, Proposition 2.2 yields that

$$\left| \int Tfg \right| = |F(\theta)| \leq \sup_{y \in \mathbb{R}} |F(\theta + iy)| \leq N_0^{1-\theta} N_1^{\theta},$$

from which the desired estimate (2.2) follows.

Case 2. $p_\theta = \infty$. This assumption immediately implies that $p_0 = p_1 = \infty$, and if $q_\theta = q_0 = q_1 = 1$, then there is nothing to prove. So suppose that $q_\theta > 1$. Now f need not be integrable and we may choose $f = f_z$ for all $z \in \mathbb{C}$. Analogously, we can handle the case $q = 1$, $p < \infty$ (now $g_z = g$).

Next we prove for $\mathbb{K} = \mathbb{R}$. Luckily, this follows from the above and the following argument. Let $U : L^r_{\mathbb{R}} \to L^r_{\mathbb{R}}$ be a continuous linear operator between real L^p spaces. Furthermore, define the canonical extension by $U_{\mathbb{C}} = Uf + iUg$. This map is $\mathbb{C}$-linear, and it holds that

$$\begin{aligned} \|U_{\mathbb{C}}\| &= \sup_{\|f+ig\|=1} \|U_{\mathbb{C}}(f+ig)\| \leq \sup_{\|f+ig\|=1} (\|U(f)\| + \|U(g)\|) \\ &\leq \sup_{\|f\|=1} \|U(f)\| + \sup_{\|g\|=1} \|U(g)\| \leq 2\|U\|. \end{aligned} \tag{2.4}$$

Applying this to the assumption of the theorem we obtain the results for T by using the extension $T_{\mathbb{C}}$ and (2.1)

$$\begin{aligned} \|Tf\|_{q_\theta} &= \|T_{\mathbb{C}}f\|_{q_\theta} \leq \|T_{\mathbb{C}}\|^{1-\theta}_{p_0,q_0} \|T_{\mathbb{C}}\|^{\theta}_{p_1,q_1} \|f\|_{p_\theta} \\ &\leq 2\|T\|^{1-\theta}_{p_0,q_0} \|T\|^{\theta}_{p_1,q_1} = 2N_0^{1-\theta} N_1^{\theta} \|f\|_{p_\theta}, \end{aligned}$$

and the proof of the theorem is complete. □

To give Marcinkiewicz interpolation theorem, we need to introduce some concepts so as to make its results accessible to the reader.

Definition 2.1. **(i)** The distribution function $\lambda(\cdot, f) : \mathbb{R}^+ \to \mathbb{R}^+$ of a measurable function f on $\mathbb{R}^N$ is defined by

$$\lambda(\sigma, f) := |\, \{x : |f(x)| > \sigma\} \,|$$

where $|\cdot|$ taken on sets represent the measure of $\mathbb{R}^N$.

(ii) The equivalent norms of L^p are defined by

$$\|f\|_p = \begin{cases} \left(\int_0^\infty \sigma^{p-1}\lambda(\sigma, f)d\sigma\right)^{\frac{1}{p}} & \text{if} \quad 1 \leq p < \infty, \\ \inf\{\sigma : \lambda(\sigma, f) = 0\} & \text{if} \quad p = \infty. \end{cases}$$

(iii) The weak L^p-spaces, denoted by $L^p_*, 1 \leq p < \infty$, consist of all f such that

$$\|f\|_{p,*} = \sup_\sigma \sigma\lambda(\sigma, f)^{\frac{1}{p}} < \infty.$$

In the case $p = \infty$, we put $L^p_* = L^\infty$. The triangle inequality of L^p in L^p_* is

$$\|f + g\|_{p,*} \leq 2^{\frac{1}{p}}(\|f\|_{p,*} + \|g\|_{p,*}).$$

Thus L^p_* is a quasi normed vector space.

(iv) For any $f \in L^p$, $1 \leq p < \infty$, we have $\|f\|_{p,*} \leq \|f\|_p$. That is, $L^p \subset L^p_*$.

(v) The decreasing rearrangement of f is the function $f^* : [0, \infty) \to [0, \infty)$ defined by

$$f^*(t) = \inf\{\sigma : \lambda(\sigma, f) \leq t\}$$

with convention that $\inf \emptyset = \infty$. f^* is a non-negative, and non-increasing function on $(0, \infty)$, continuous on the right and has property $\lambda(\rho, f^*) = \lambda(\rho, f)$ for $\rho \geq 0$. Thus f^* is equi-measurable with f.

(vi) The Lorentz space $L^{p,q}(\mathbb{R}^N)$, $1 \leq p, q \leq \infty$ is the set of all measurable functions f on $\mathbb{R}^N$ such that

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\int_0^\infty \left(t^{\frac{1}{p}} f^*(t)\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if} \quad q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t) & \text{if} \quad q = \infty \end{cases}$$

is finite. Two functions in $L^{p,q}$ are said to be equal if they are equal almost everywhere. $L^{p,\infty} = L^p_*$ and $L^{p,p} = L^p$, $L^{\infty,\infty} = L^\infty = L^\infty_*$.

(vii) For $1 \le p < \infty$, $1 \le q < r \le \infty$ there exists a constant $C_{p,q,r} = (\frac{q}{p})^{\frac{1}{q} - \frac{1}{r}}$ such that

$$\|f\|_{p,r} \le C_{p,q,r} \|f\|_{p,q}. \quad \text{That is,} \quad L^{p,q} \subset L^{p,r}.$$

(viii) The spaces $L^{p,q}, 1 \le p, q \le \infty$ are complete with respect to their quasi-norms and are therefore quasi-Banach spaces.

An operator T mapping functions from a measurable space (Ω_0, μ_0) to another measurable space (Ω_1, μ_1) is said to be quasi-linear if

(i) $T(f+g)$ is defined whenever Tf and Tg are defined,

(ii) $|T(\lambda f)(x)| \le \kappa |\lambda| |Tf(x)|$, and

(iii) $|T(f+g)(x)| \le K(|Tf(x)| + |Tg(x)|)$

for almost everywhere on x, with $\kappa, K \in \mathbb{R}^+$ being independent of f and g.

We are now ready to state the Marcinkiewicz Interpolation Theorem.

Theorem 2.4 (Marcinkiewicz Interpolation Theorem)**.** *Assume* $1 \le p_j \le q_j \le \infty$, $p_0 < p_1$, $q_0 \ne q_1$ *and* T *a quasi-linear mapping defined on* $L^{p_0} + L^{p_1}$ *which is simultaneously of weak types* (p_0, q_0) *and* (p_1, q_1)*., i.e.*

$$T : L^{p_j} \to L^{q_j}_{*} \quad \textit{for } j = 0, 1, \textit{ and} \quad \|Tf\|_{q_j,*} \le N_j \|f\|_{p_j}.$$

If $0 < \theta < 1$*, and*

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

Then, T *is of strong type* (p_θ, q_θ)*, i.e.*

$$\|Tf\|_{q_\theta} \le N \|f\|_{p_\theta}, \forall f \in L^{p_\theta},$$

where $N = C(N_j, p_j, q_j, \theta)$ *and depends neither on* T *nor* f*.*

Proof. Since this proof is quite involving and delicate we skip it and make reference to the interested reader to consult with [29]. Alternative elegant proofs are provided in [9, 52]. □

2.2 Real Interpolation

Definition 2.2. **(i)** Let $X_j = (X_0, X_1)$ be a pair of Banach spaces. A pair X_j of Banach spaces is said to be admissible if there exists a topological vector space Z such that entries satisfy $X_j \subset Z, j = 0, 1$ continuously.

(ii) The spaces

$$\left(\bigcap_{j=0,1} X_j; \max\{\|\cdot\|_j\}\right), \ (X_0 + X_1; \inf_{x_j \in X_j} \|x_0\|_0 + \|x_1\|_1)$$

are Banach spaces.

(iii) X is an intermediate space with respect to the pair X_j if

$$\bigcap X_j \subset X \subset X_0 + X_1 \tag{2.5}$$

continuously.

(iv) X, Y are interpolation spaces with respect to the pairs X_j, Y_j respectively, if they are intermediate spaces, and if for some linear mapping $T : X_0 + X_1 \to Y_0 + Y_1$ it holds, per corresponding entries, that

$$T \in \mathcal{L}(X_j, Y_j) \Rightarrow T|_X \in \mathcal{L}(X, Y). \tag{2.6}$$

(v) Interpolation spaces X, Y with respect to the pairs X_j, Y_j respectively are said to be of exponent $\theta \in [0, 1]$ if there exists $C > 0$ such that

$$\|T\|_{X\to Y} \leq C\|T\|^{1-\theta}_{X_0\to Y_0}\|T\|^{\theta}_{X_1\to Y_1}, \quad \forall T \in \mathcal{L}(X_j, Y_j) \tag{2.7}$$

If $C = 1$ then X, Y are exact interpolation spaces of exponent θ.

Firstly, we treat the $K-$ method for the real interpolation of Banach spaces.

Definition 2.3. For every $x \in X_0 + X_1$, $t > 0$ let

$$K(t, x, X_0, X_1) = \inf_{x=a+b, a\in X_0, b\in X_1} (\|a\|_0 + t\|b\|_1). \tag{2.8}$$

We define the real interpolation spaces for $0 < \theta < 1$, $1 \leq p \leq \infty$, by

$$\begin{aligned} (X_0, X_1)_{\theta,p} &= \left\{ x \in X_0 + X_1 : t \to t^{-\theta} K(t, x, X_0, X_1) \in L^p_*(0, +\infty) \right\}, \\ \|x\|_{\theta,p} &= \|t^{-\theta} K(t, x, X_0, X_1)\|_p \end{aligned}$$

where L^p_* is L^p with respect to the measure $\frac{dt}{t}$ in $(0, \infty)$, and in abbreviation, we write $K(t, x)$ instead of $K(t, x, X_0, X_1)$, and

$$(X_0, X_1)_\theta = \left\{ x \in X_0 + X_1 : \lim_{t\to 0^+} t^{-\theta} K(t, x, X_0, X_1) = \lim_{t\to\infty} t^{-\theta} K(t, x, X_0, X_1) = 0 \right\},$$

as $t \to K(t, x) \in C(0, \infty)$ for every $x \in X_0 + X_1$, we have $(X_0, X_1)_\theta \subset (X_0, X_1)_{\theta,\infty}$. So the spaces $(X_0, X_1)_\theta$ are interpolation spaces.

Important to note is that $K(t, x, X_0, X_1) = tK(t^{-1}, x, X_1, X_0)$ for all $t > 0$ and by transformation $\tau = t^{-1}$ which preserves $L^p_*(0, \infty)$, we get

$$\begin{aligned} (X_0, X_1)_{\theta,p} &= (X_1, X_0)_{1-\theta,p}, \; 0 < \theta < 1, \; 1 \leq p \leq \infty, \\ (X_0, X_1)_\theta &= (X_1, X_0)_{1-\theta}. \end{aligned}$$

So the order of the spaces is crucial. Immediate particular cases are the following;

Observation 2.1. **(i)** If $X_0 = X_1$, then $X_0 + X_1 = X_0$ and $K(t, x) \leq \min\{t, 1\}\|x\|$. Therefore,

$$X_0 = (X_0, X_1)_{\theta,p} \quad 0 < \theta < 1, \; 1 \leq p \leq \infty.$$

(ii) If $\bigcap X_j = \{0\}$, then for each $x \in X_0 + X_1$, there exist unique $a \in X_0$, $b \in X_1$ such that $x = a + b$. Hence $K(t, x) = \|a\|_0 + t\|b\|_1$ and $t \to t^{-\theta} K(t, x) \notin L^p_*(0, \infty)$ unless $x = 0$. Therefore, $(X_0, X_1)_{\theta,p} = (X_0, X_1)_\theta = \{0\}$ for any $\theta \in (0, 1)$, $1 \leq p \leq \infty$.

(iii) In the important case $X_1 \subset X_0$, we have $K(t,x) \le \|x\|_0$ for every $x \in X_0$, so $t \to t^{-\theta}K(t,x) \in L^p_*(a,\infty)$ for $a > 0$. Thus, only the behaviour near $t = 0$ is important in the definition of the interpolation spaces.

The results in the following proposition are proven in [52].

Proposition 2.5. **(i)** *If* $0 < \theta < 1$, $1 \le p_1 \le p_2 \le \infty$, *then*

$$\bigcap X_j \subset (X_0, X_1)_{\theta,p_1} \subset (X_0, X_1)_{\theta,p_2} \subset (X_0, X_1)_\theta \subset (X_0, X_1)_{\theta,\infty} \subset X_0 + X_1$$

Moreover, $(X_0, X_1)_{\theta,\infty} \subset \bigcap \bar{X}_j$, *where the closure is in* $X_0 + X_1$.

(ii) *If* $X_1 \subset X_0$, $0 < \theta_1 < \theta_2 < 1$, *then*

$$(X_0, X_1)_{\theta_2,\infty} \subset (X_0, X_1)_{\theta_1,1}.$$

Therefore, $(X_0, X_1)_{\theta_2,p} \subset (X_0, X_1)_{\theta_1,q}$ *for any* $1 \le p, q \le \infty$.

(iii) *The interpolation spaces in Definition 2.3 are Banach spaces, and condition* (2.7) *holds.*

2.3 Complex interpolation method

Let $X = X_{\mathbb{C}}$ be a complex Banach space,

$$S = \{z \in \mathbb{C} : 0 \le \mathrm{Re} z \le 1\}, \text{ and } \quad \mathring{S} = \{z \in \mathbb{C} : 0 < \mathrm{Re} z < 1\}.$$

A mapping $f : \mathring{S} \to X$ is called holomorphic if the mapping $z \to \langle f(z), x' \rangle$, $z \in \mathring{S}$, is holomorphic in the usual sense for all $x' \in X'$.

Theorem 2.6 (Maximum Principle). *Let* $f : S \to X$ *be continuous, bounded, and holomorphic in* $\mathring{S}$. *Then,*

$$\sup_{z\in S} \|f(z)\|_X \le \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_X, \sup_{t\in\mathbb{R}} \|f(1+it)\|_X\right\}.$$

As a corollary, we obtain the following three lines theorem, which is the basis for the proof of the Riesz-Thorin interpolation theorem and the complex interpolation method.

Theorem 2.7 (Three Lines Theorem). *Let $f : S \to X$ be continuous, bounded, and holomorphic in $\mathring{S}$. Then,*

$$\sup_{t\in\mathbb{R}} \|f(\theta+it)\|_X \le \left(\sup_{t\in\mathbb{R}} \|f(it)\|_X\right)^{1-\theta} \left(\sup_{t\in\mathbb{R}} \|f(1+it)\|_X\right)^{\theta}$$

for $\theta \in [0,1]$.

Definition 2.4. Let X_j be an admissible Banach space, and $\mathcal{F}(X_j)$ be the set of all mappings $f : S \to \bigcup X_j$ that are continuous, bounded, and holomorphic in $\mathring{S}$, such that

$$t \to f(it) \in C(\mathbb{R}, X_0), \quad t \to f(1+it) \in C(\mathbb{R}, X_1),$$

and equipped with norm

$$\|f\|_{\mathcal{F}(X_j)} = \max\left\{\sup_{t\in\mathbb{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbb{R}} \|f(1+it)\|_{X_1}\right\},$$

it is a Banach space. Furthermore, let $\mathcal{F}_0(X_j)$ be the set of all $f \in F(X_j)$ such that $\lim_{|t|\nearrow\infty} \|f(j+it)\|_{X_j} = 0$, so that it is a closed subspace of $F(X_j)$. The linear hull of the functions

$$\mathcal{V}(X_j) = \left\{e^{\delta z^2+\lambda z}a : a \in \bigcap X_j,\, \delta > 0,\, \lambda \in \mathbb{R}\right\}$$

is a dense subspace of $\mathcal{F}_0(X_j)$.

We then define the complex interpolation space

$$X_j^{\theta} := [X_0, X_1]_{\theta} = \{f(\theta) : f \in \mathcal{F}(X_j)\},$$

endowed with the norm

$$\|x\|_{\theta} = \inf_{f\in\mathcal{F}(X_j): f(\theta)=x} \|f\|_{\mathcal{F}(X_j)}.$$

2.4 Banach scales of Bessel potential spaces

For the construction of the Bessel potential spaces, we will use the Harmonic Analysis approach [30, 71, 82]. To this end, we start by defining the Schwartz space, $\mathcal{S}$ and its dual $\mathcal{S}^*$.

Definition 2.5. The Schwartz space of functions defined on $\Omega \subset \mathbb{R}^N$, denoted by $\mathcal{S}(\Omega)$, is defined as

$$\mathcal{S}(\Omega) := \left\{ \phi \in C^\infty(\Omega) : \sup_{x\in\Omega} \left(1+|x|^2\right)^{\frac{k}{2}} \sum_{|\alpha|\leq k} |D^\alpha \phi(x)| < \infty, \;\; k \in \mathbb{N} \right\}. \tag{2.9}$$

The dual space[1] of $\mathcal{S}(\Omega)$ is denoted by $\mathcal{S}^*(\Omega)$.

The quantities $\sup_{x\in\Omega} \left(1+|x|^2\right)^{\frac{k}{2}} \sum_{|\alpha|\leq k} |D^\alpha \phi(x)|,\; k \in \mathbb{N}$, define a countable family of semi-norms on $\mathcal{S}$. Also, $\mathcal{S}^*$ is a locally convex linear topological space, and it is said to be a tempered distribution space.

Next, we define the Fourier transform and its inverse on the Schwartz space and its dual space.

Definition 2.6. Let $\phi,\; \psi \in \mathcal{S}$. Then the Fourier transform, $\mathcal{F}$, and its inverse, $\mathcal{F}^{-1}$, are defined, respectively, as

$$\begin{aligned} \hat{\phi}(y) = (\mathcal{F}\phi)(y) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega e^{-ix\cdot y} \phi(x)\, dx, \\ \check{\psi}(x) = (\mathcal{F}^{-1}\psi)(x) &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_\Omega e^{ix\cdot y} \psi(y)\, dy, \end{aligned} \tag{2.10}$$

where $x \cdot y = x_1 y_1 + \ldots + x_n y_n$.

Recall that the convolution $\phi * \psi$ on $\mathcal{S}$ is defined as follows (see, for instance, [2]).

$$\phi * \psi(x) = \int_\Omega \phi(x-y)\psi(y)\, dy.$$

[1] $\mathcal{S}^*(\Omega)$ is the set of sequentially continuous linear functionals on $\mathcal{S}(\Omega)$.

With the definitions above in mind, we define the Bessel potential operator as follows;

Definition 2.7. Let $s \in \mathbb{R}$. Then the *Bessel potential of order* s is a sequentially continuous bijective linear operator $J_s : \mathcal{S}(\Omega) \to \mathcal{S}(\Omega)$ defined by

$$J_s u = (I - \Delta)^{-\frac{s}{2}} u = \mathcal{F}^{-1} \left(1 + |x|^2\right)^{-\frac{s}{2}} \mathcal{F}u, \tag{2.11}$$

where I is the identity operator, $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$ is the Laplace operator, and $x \in \mathbb{R}^N$.

It's is trivial to see from (2.11) that, for $s,\ t \in \mathbb{R}$, the following hold;

$$J_{s+t} = J_s J_t, \quad (J_s)^{-1} = J_{-s}, \quad \text{and } J_0 = I.$$

For $u \in \mathcal{S}(\Omega)$, we also have a natural extension of $J_s : \mathcal{S}^*(\Omega) \to \mathcal{S}^*(\Omega)$ defined by

$$\langle J_s u, \phi \rangle = \langle u, J_s \phi \rangle, \quad \forall \phi \in \mathcal{S}(\Omega).$$

We therefore have the following definition of the Bessel potential space.

Definition 2.8. Let $s \in \mathbb{R}$ and $1 < p < \infty$. The Bessel potential space on Ω, denoted by $H^{s,p}(\Omega)$, is defined by

$$\begin{aligned} H^{s,p}(\Omega) &= \{u \in \mathcal{S}^*(\Omega) : \ J_s u \in L^p(\Omega)\} \\ &= \{u \in \mathcal{S}^*(\Omega) : \ \|J_s u\|_{L^p(\Omega)} < \infty\} \\ &= (I - \Delta)^{-\frac{s}{2}} L^p(\Omega). \end{aligned}$$

The norm on $H^{s,p}(\Omega)$ is given by

$$\|u\|_{H^{s,p}(\Omega)} = \|J_s u\|_{L^p(\Omega)}.$$

We should mention at this point that the Bessel potential spaces can also be constructed using the complex interpolation-extrapolation procedure[2], where they are defined by complex interpolation between L^p spaces and Sobolev spaces $W^{m,p}$. That is, if $s > 0$ and m is

[2]See [2, 5, 78] for more details.

the smallest integer greater than s, and Ω is a domain in $\mathbb{R}^N$, then we have

$$H^{s,p}(\Omega) = [L^p(\Omega), W^{m,p}(\Omega)]_{\frac{s}{m}},$$

and if $s_0,\ s_1 \in \mathbb{N}$ such that $s_0 \neq s_1$, and $0 < \theta < 1$ such that $s = s_0(1-\theta) + s_1\theta$, then

$$H^{s,p}(\Omega) = [W^{s_0,p}(\Omega), W^{s_1,p}(\Omega)]_\theta.$$

These Bessel potential spaces will provide the basic topology in this work, especially in the well-posedness work. For this reason, we collect some of their properties below.

Proposition 2.8. *Let $s \in \mathbb{R}$ and $1 < p < \infty$. Then*

1. *$H^{s,p}$ is a Banach space;*
2. *$\mathcal{S} \subset H^{s,p} \subset \mathcal{S}^*$, and $\mathcal{S}(\Omega)$ is dense in $H^{s,p}(\Omega)$.*
3. *$H^{s+\varepsilon,p} \subset H^{s,p},\ \ \forall \varepsilon > 0$;*
4. *$H^{s,p} \subset L^\infty,\ \ \forall s > \frac{N}{p}$.*

For the proof of Proposition 2.8, see [82, 71] and the references therein.

In addition to the properties in Proposition 2.8, we have the following Theorem [2, 5].

Theorem 2.9. *Let $-\infty < s_2 \leq s_1 < \infty$ and $1 < p_1 \leq p_2 < \infty$ be such that $s_1 - \frac{N}{p_1} = s_2 - \frac{N}{p_2}$. Then*

$$H^{s_1,p_1} \subset H^{s_2,p_2}. \tag{2.12}$$

2.5 Banach scale spaces of positive operators

In this section, we work in the context of complex analysis mainly because of the definition of the resolvent and spectrum of linear operators.

Definition 2.9. A linear operator $A : D(A) \subset E \to E$ is said to be positive if the resolvent set of A contains $(-\infty, 0]$ and there exists $M > 0$ such that

$$\|R(\lambda, A)\|_{\mathcal{L}(E)} \leq \frac{M}{1+|\lambda|},\ \lambda \leq 0. \tag{2.13}$$

The power operator A^z of a bounded positive operator $A : E \to E$ is defined by

$$A^z = \frac{1}{2\pi i}\int_\gamma \lambda^z R(\lambda, A)d\lambda,$$

where γ is any piecewise smooth curve surrounding $\sigma(A)$ avoiding $(-\infty, 0]$ with index 1 with respect to every $\mu \in \sigma(A)$.

Note that for a bounded positive operator the following properties are easily verified:

(i) The mapping $z \to A^z \in \mathcal{L}(E)$ is holomorphic.

(ii) If $z = k \in \pm\mathbb{N}$, then $A^z = A^k$.

(iii) For each $z_1, z_2 \in \mathbb{C}$ we have $A^{z_1}A^{z_2} = A^{z_2}A^{z_1} = A^{z_1+z_2}$ etc.

If A is unbounded, then the theory is much more complicated. To define A^z we need to have some control over its spectral properties. In this direction, we have the following lemma [71, 78]

Lemma 2.10. *Let A be a positive operator. Then, for*

$$\Lambda = \left\{\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C} : \lambda_1 \in \mathbb{R}^-, |\lambda_2| < \frac{\lambda_1 + 1}{M}\right\} \cup \{\lambda \in \mathbb{C} : |\lambda| < \frac{1}{M}\}$$

where M is as in (2.13)*, and for every $\theta_0 \in (0, \arctan\frac{1}{M})$, $r_0 \in (0, \frac{1}{M})$ there exists $M_0 > 0$ such that*

$$\|R(\lambda, A)\|_{\mathcal{L}(E)} \leq \frac{M_0}{1+|\lambda|},\ \lambda \leq 0 \tag{2.14}$$

for all $\lambda \in \mathbb{C}$, $|\lambda| \leq r_0$, $\lambda_1 < 0$, $\frac{\lambda_1}{\lambda_2} \leq \arctan\theta_0$.

Proof. It suffices to recall that for every $\lambda_0 \in \rho(A)$, there exists an open ball $B_r(\lambda_0) \subset \rho(A)$, where $r = \frac{1}{\|R(\lambda_0,A)\|}$, and such that for $|\lambda - \lambda_0| < r$ it holds that

$$R(\lambda, A) = \sum_{n=0}^{\infty} (-1)^n (\lambda - \lambda_0)^n R(\lambda_0, A)^{n+1}.$$

The union $\bigcup B_r(\lambda_0) \supset \Lambda$ and the estimate follows easily. $\square$

Definition 2.10. Let $\theta \in (\frac{\pi}{2}, \pi)$, $r > 0$, and consider the curve $\gamma_{r,\theta} = -\gamma^1 - \gamma^2 + \gamma^3$ where γ^1, γ^3 are half lines parametrized respectively by

$$z = \xi e^{-i\theta}, \; z = \xi e^{i\theta}, \quad \xi \geq r,$$

and γ^2 is the circle arc parametrized by $z = re^{i\eta}$, $-\theta \leq \eta \leq \theta$. Then, for any

$$r \in (0, \frac{1}{M}), \; \theta \in (\pi - \arctan \frac{1}{M}, \pi), \; \alpha = \alpha_0 + i\alpha_1 \in \mathbb{C} \quad \text{with} \quad \alpha_0 < 0,$$

define

$$A^\alpha = \frac{1}{2\pi i} \int_\gamma \lambda^\alpha R(\lambda, A) d\lambda \subset \mathcal{L}(E), \tag{2.15}$$

satisfying that $\lambda \to \lambda^\alpha R(\lambda, A) \in \mathcal{L}(E)$ is holomorphic in $\Lambda \setminus (-\infty, 0]$, and the integral is independent of r, θ.

Writing down the integral in (2.15), we get that

$$\begin{aligned} A^\alpha &= \frac{1}{2\pi} \int_r^\infty \xi^\alpha \left(-e^{i\theta(\alpha+1)} R(\xi e^{i\theta}, A) + e^{-i\theta(\alpha+1)} R(\xi e^{-i\theta}, A) \right) d\xi \\ &- \frac{r^{\alpha+1}}{2\pi} \int_{-\theta}^{\theta} e^{i\eta(\alpha+1)} R(re^{i\eta}, A) d\eta \end{aligned} \tag{2.16}$$

for every $r \in (0, \frac{1}{M})$, $\theta \in (\pi - \arctan \frac{1}{M}, \pi)$, which can be worked out to get a simple expression. For instance, if $-1 < \alpha_0 < 0$, then letting $r \searrow 0$, $\theta \nearrow \pi$ leads to

$$A^\alpha x = -\frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \xi^\alpha (\xi I + A)^{-1} x d\xi. \tag{2.17}$$

The following proposition yields some basic properties of the power operators A^α [2, 71, 78].

Proposition 2.11. **(i)** *If $\alpha = -n$ for $n \in \mathbb{N}$, then $A^{\alpha} = (A^{-1})^n = A^{-n}$.*

(ii) *If $Re\, z < -k$ for $k \in \mathbb{N}$, then the range*

$$R(A^z) \subset D(A^k) \quad \text{and} \quad A^k A^z x = A^{k+z} x, \quad x \in E.$$

(iii) *If $Re\, z < 0$ and $x \in D(A^k)$ for $k \in \mathbb{N}$, then $A^z x \in D(A^k)$ and $A^z A^k x = A^k A^z x$.*

(iv) *If $Re\, z_1$, $Re\, z_2 < 0$, then $A^{z_1} A^{z_2} = A^{z_1 + z_2}$.*

Proof. **(i)** Let $\alpha = -n$. Then

$$\frac{1}{2\pi i} \int_{\gamma_{r,\theta}} \lambda^{-n} R(\lambda, A) d\lambda = \lim_{k \nearrow \infty} \frac{1}{2\pi i} \int_{\gamma_k} \lambda^{-n} R(\lambda, A) d\lambda, \tag{2.18}$$

where $\gamma_k = \{z : v|z| = \frac{1}{k}\}$.

For every $k \in \mathbb{N}$ the mapping $\lambda \to R(\lambda, A)$ is holomorphic in the bounded region surrounded by γ_k, and

$$\frac{1}{2\pi i} \int_{\gamma_k} \lambda^{-n} R(\lambda, A) d\lambda = -\frac{1}{(n-1)!} \frac{d^{n-1}}{d\lambda^{n-1}} R(\lambda, A)_{|\lambda=0} = A^{-n}.$$

Back substitution of this in (2.18) yields

$$\frac{1}{2\pi i} \int_{\gamma_{r,\theta}} \lambda^{-n} R(\lambda, A) d\lambda = A^{-n}.$$

(ii) Let $k = 1$, $\mathrm{Re}\, z < -1$. Then, since

$$\|\lambda^z A R(\lambda, A)\| = \|\lambda^z (\lambda R(\lambda, A) - I)\| \leq |\lambda|^{\mathrm{Re}\, z} (M_0 + 1),$$

the integral in (2.16) is an element of $\mathcal{L}(E, D(A))$, and

$$A \frac{1}{2\pi i} \int_{\gamma_k} \lambda^z R(\lambda, A) d\lambda = \frac{1}{2\pi i} \int_{\gamma_k} \lambda^{z+1} R(\lambda, A) d\lambda - \frac{1}{2\pi i} I \int_{\gamma_k} \lambda^z d\lambda.$$

But the last integral vanishes, so we obtain that $A \cdot A^z = A^{1+z}$ and the statement holds for $k = 1$, the rest follows by induction or recursively.

(iii) is obvious because A^k commutes with $R(\lambda, A)$ on $D(A^k)$, and this implies A^k commutes with A^z on $D(A^k)$.

(iv) Let $\theta_1 < \theta_2 < \pi$, $0 < r_2 < r_1 < \frac{1}{M}$ so that γ_{r_1,θ_1} is on the right hand side of γ_{r_2,θ_2} to find that

$$\begin{aligned}
A^{z_1}A^{z_2} &= \frac{1}{(2\pi i)^2}\int_{\gamma_{r_1,\theta_1}} \lambda^{z_1}R(\lambda, A)d\lambda \int_{\gamma_{r_2,\theta_2}} \mu^{z_2}R(\mu, A)d\mu \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_{r_1,\theta_1}\times\gamma_{r_2,\theta_2}} \lambda^{z_1}\mu^{z_2}\frac{R(\lambda, A) - R(\mu, A)}{\mu - \lambda}d\lambda d\mu \\
&= \frac{1}{(2\pi i)^2}\int_{\gamma_{r_1,\theta_1}} \lambda^{z_1}R(\lambda, A)d\lambda \int_{\gamma_{r_2,\theta_2}} \frac{\mu^{z_2}}{\mu - \lambda}d\mu - \\
&\quad -\frac{1}{(2\pi i)^2}\int_{\gamma_{r_2,\theta_2}} \mu^{z_1}R(\lambda, A)d\mu \int_{\gamma_{r_1,\theta_1}} \frac{\lambda^{z_2}}{\mu - \lambda}d\lambda \\
&= \frac{1}{2\pi i}\int_{\gamma_{r_1,\theta_1}} \lambda^{z_1+z_2}R(\lambda, A)d\lambda = A^{z_1+z_2}.
\end{aligned}$$

The proof of the proposition is complete. □

Statement (iv) of Proposition 2.11 implies that A^z is injective. Indeed, if $A^z x = 0$ and $n \in \mathbb{N}$ is such that $-n < \text{Re } z$, then $A^{-n}x = A^{-n-z}A^z x = 0$ so that $x = 0$. Therefore, it is possible to define A^α if Re $\alpha > 0$ as the inverse of $A^{-\alpha}$. But in this way the powers $A^{it}, t \in \mathbb{R}$ remain undefined. So we give a unified definition for Re $\alpha \geq 0$.

Definition 2.11. For $\alpha = \alpha_0 + i\alpha_1 \in \mathbb{C}$, $0 \leq \alpha_0 < n$, $n \in \mathbb{N}$, we define

$$D(A^\alpha) = \left\{x \in E : A^{\alpha-n} \in D(A^n)\right\}, \quad A^\alpha x = A^n A^{\alpha-n}x,$$

where the operator A^α is independent of $n \in \mathbb{N}$.

In fact, the independence of A^α is a consequence of Proposition 2.11. Since, if $n, m >$ Re α, then $A^{\alpha-m}x = A^{n-m}A^{\alpha-n}x$ for $n < m$ by part (iv), and for $n > m$ by part (ii), taking $z = \alpha - n$ and $k = n - m$ gives

$$A^{\alpha-m}x \in D(A^m) \iff A^{n-m}A^{\alpha-n}x \in D(A^m) \quad i.e. \;\; A^{\alpha-n}x \in D(A^n).$$

Now, if $\alpha = 0$ then $A^0 = I$. Moreover, if $\alpha_0 > 0$, then we get

$$D(A^\alpha) = A^{-\alpha}E;\; A^\alpha = (A^{-\alpha})^{-1}.$$

Indeed,

$$A^{\alpha-n}x \in D(A^n) \iff \exists y \in E \text{ such that } A^{\alpha-n}x = A^{-n}y.$$

The uniqueness of $y \in E$ is by definition. Furthermore,

$$A^{-n}A^{-\alpha}y = A^{-\alpha}A^{-n}y = A^{-\alpha}A^{\alpha-n}x = A^{-n}x,$$

so that $x = A^{-\alpha}y \in R(A^{-\alpha})$ and $A^\alpha = (A^{-\alpha})^{-1}$. Since A^α has a bounded inverse, it is closed and $D(A^\alpha)$ is a Banach space endowed with the graph norm equivalent to $x \to \|A^\alpha x\|$, the canonical norm of $D(A^\alpha)$.

If $\alpha_0 = 0$, then $\alpha = i\alpha_1$ and

$$A^{i\alpha_1} = (A^{-i\alpha_1})^{-1} \iff \forall x \in D(A^{i\alpha_1});\; A^{i\alpha_1}x \in D(A^{-i\alpha_1}) \quad \text{and} \quad A^{-i\alpha_1}A^{i\alpha_1}x = x$$

by definition. Therefore,

$$A^{-1-i\alpha_1}A^{i\alpha}x = A^{-1-i\alpha_1}AA^{i\alpha_1-1}x = A^{-1}x.$$

Thus

$$A^{-1-i\alpha_1}AA^{i\alpha_1-1}x = AA^{-2}x \in D(A) \Rightarrow A^{i\alpha_1}x \in D(A^{-i\alpha_1}) \quad \text{and} \quad A^{-i\alpha_1}A^{i\alpha_1}x = x.$$

Note that in general the operators $A^{i\alpha_1}$ are not bounded. Nevertheless, they are closed operators, since $A^{-1+i\alpha_1}$ is bounded and closed. Therefore, $D(A^{i\alpha})$ is a Banach space under the graph norm.

Clearly, from Proposition 2.11, we have $D(A^n) \subset D(A^{\alpha_0})$ continuously, since $x \in D(A^n)$, $A^{\alpha-n}x \in D(A^n)$ using (iii) and

$$A^{\alpha}x = A^n A^{\alpha-n}x = A^{\alpha-n}A^n x \Rightarrow \|A^{\alpha}x\| \leq \|A^{\alpha-n}\|\|A^n x\|.$$

More generally in this regard, is the following theorem [71, 78].

Theorem 2.12. *Let* $\alpha = \alpha_0 + i\alpha_1, \beta = \beta_0 + i\beta_1 \in \mathbb{C}$ *be such that* $\beta_0 < \alpha_0$. *Then,* $D(A^{\alpha}) \subset D(A^{\beta})$ *and*

$$A^{\beta}x = A^{\beta-\alpha}A^{\alpha}x, \forall x \in D(A^{\alpha}).$$

Moreover,

$$A^{\alpha-\beta}A^{\beta}x = A^{\alpha}x, \forall x \in D(A^{\alpha}),\ A^{\beta}x \in D(A^{\alpha-\beta}),$$

and the converse is true.

Proof. The spaces embedding is obvious if $\beta_0 < 0$. Thus, we prove the case $\beta_0 \geq 0$. If $n > \alpha_0$ and $x \in D(A^{\alpha})$ then $A^{-n+\alpha}x \in D(A^n)$ and

$$A^{-n+\beta}x = A^{\beta-\alpha}A^{-n+\alpha}x \in D(A^n)$$

thanks to Proposition 2.11-(iii). This implies that $x \in D(A^{\beta})$ and

$$A^{\beta}x = A^n A^{\beta-\alpha}A^{-n+\alpha}x = A^{\beta-\alpha}A^{\alpha}x.$$

Since $A^{\beta-\alpha}$ is bounded, we have that

$$\|A^{\beta}x\| \leq \|A^{\beta-\alpha}\|\|A^{\alpha}x\|$$

holds, so that the first implication of the theorem is valid.

Next we notice that if $x \in D(A^\alpha)$, and $n > \max\{\alpha_0, \alpha_0 - \beta_0\}$, then

$$A^{-n+\alpha-\beta}A^\beta x = A^{-n+\alpha-\beta}A^{\beta-\alpha}A^\alpha x = A^{-n}A^\alpha x \in D(A^n)$$

implying

$$A^\beta x \in D(A^{\alpha-\beta}) \quad \text{and} \quad A^{\alpha-\beta}A^\beta x = A^\alpha x.$$

But as well, if $x \in D(A^\beta)$ such that $A^\beta x \in D(A^{\alpha-\beta})$, then we have

$$A^{\alpha-2n}x = A^{\alpha-n-\beta}A^{-n+\beta}x = A^{\alpha-n-\beta}A^{-n}A^\beta x = A^{-n}A^{\alpha-n-\beta}A^\beta x \in D(A^{2n})$$

yielding that $x \in D(A^\alpha)$ and $A^\alpha x = A^{2n}A^{\alpha-2n}x = A^{\alpha-\beta}A^\beta x$. □

A worthwhile remark is that the condition $\beta_0 < \alpha_0$ is essential in the above theorem when $\alpha_0 > 0$. In fact, for every $\alpha_0 > 0$, $\alpha_1 \in \mathbb{R}$ we have

$$D(A^{\alpha_0}) = D(A^{\alpha_0+i\alpha_1}) \iff A^{i\alpha_1} \in \mathcal{L}(E).$$

Now we give some representation formulas for $A^\alpha x$ when $x \in D(A^\alpha)$. First we consider, the case $0 < \alpha_0 < 1$, taking $n = 1$ in the definition, and we have

$$x \in D(A^\alpha) \iff A^{\alpha-1}x \in D(A),$$

and in (2.17) we get

$$A^{\alpha-1}x = -\frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \xi^{\alpha-1}(\xi I + A)^{-1}x d\xi. \tag{2.19}$$

Therefore,

$$x \in D(A^\alpha) \iff \int_0^\infty \xi^{\alpha-1}(\xi I + A)^{-1}x d\xi \in D(A)$$

and

$$\begin{aligned} A^{\alpha}x &= -\frac{\sin(\pi\alpha)}{\pi}A\int_0^{\infty}\xi^{\alpha-1}(\xi I+A)^{-1}xd\xi \\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}A\int_0^{\infty}\xi^{\alpha-1}(\xi I+A)^{-1}xd\xi, \end{aligned} \tag{2.20}$$

which is the well-known Balakrishnan formula. Another formula holds for $-1<\alpha_0<1$ starting from (2.16) for $A^{\alpha-1}$ letting $\theta \nearrow \pi$, then integrating by parts to obtain

$$\begin{aligned} A^{\alpha-1} &= \frac{1\sin(\pi\alpha)}{\pi\alpha}\int_r^{\infty}\xi^{\alpha}(\xi I+A)^{-2}xd\xi - r^{\alpha}\frac{1\sin(\pi\alpha)}{\pi\alpha}(rI+A)^{-1}x \\ &- \frac{r^{\alpha}}{2\pi}\int_{-\pi}^{\pi}e^{i\eta\alpha}(re^{i\eta}I+A)^{-1}xd\eta \end{aligned} \tag{2.21}$$

(with $\frac{1\sin(\pi\alpha)}{\pi\alpha}=1$ if $\alpha=0$) and letting $r\searrow 0$ we get (for $-1<\alpha_0<1$) that

$$A^{\alpha-1}x=\frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_r^{\infty}\xi^{\alpha}(\xi I+A)^{-2}xd\xi. \tag{2.22}$$

Therefore, $x\in D(A^{\alpha}) \iff \int_0^{\infty}\xi^{\alpha}(\xi I+A)^{-2}xd\xi\in D(A)$. In this case

$$A^{\alpha}x=\frac{1}{\Gamma(1-\alpha)\Gamma(1+\alpha)}A\int_0^{\infty}\xi^{\alpha}(\xi I+A)^{-2}xd\xi. \tag{2.23}$$

More generally, using [79] for $n\in\mathbb{N}^+$, $m\in\mathbb{N}$, $-n<\alpha_0<m-n$ we have

$$A^{\alpha}x=\frac{\Gamma(m)}{\Gamma(\alpha+n)\Gamma(m-n-\alpha)}A^{m-n}\int_0^{\infty}\xi^{\alpha+n-1}(\xi I+A)^{-m}xd\xi, \tag{2.24}$$

for every $x\in D(A^{\alpha})$. By virtue of the above formulas, we state the following proposition from [52] Chapter 3.

Proposition 2.13. *Let $A:D(A)\subset E\to E$ be such that*

$$\rho(A)\supset(-\infty,0),\ \exists M>0 \text{ such that } \|R(\lambda,A)\|_{\mathcal{L}(E)}\le\frac{M}{1+|\lambda|},\quad \lambda\le 0. \tag{2.25}$$

Then,

$$(E, D(A))_{\theta,p} = \Big\{x \in E : \lambda \to \varphi(\lambda) = \lambda^{\theta}\|AR(\lambda, A)x\| \in L^p(0,\infty)\Big\}$$

and the norms

$$\|x\|_{\theta,p} \equiv \|x\|^*_{\theta,p}$$

are equivalent, where $\|x\|^*_{\theta,p} = \|x\| + \|\varphi\|_p$.

More precisely, we prove the following embeddings.

Proposition 2.14. *If* $\alpha = \alpha_0 + i\alpha_1 \in \mathbb{C},\ 0 < \alpha_0 < 1$, *then*

$$(E, D(A))_{\alpha_0,1} \subset D(A^{\alpha}) \subset (E, D(A))_{\alpha_0,\infty}.$$

Proof. The inclusion $(E, D(A))_{\alpha_0,1} \subset D(A^{\alpha})$ is easy, because $\xi > 0$

$$\|A\xi^{\alpha-1}(\xi I + A)^{-1}x\| = \xi^{\alpha_0 - 1}\|A(\xi I + A)^{-1}x\|$$

and for every $x \in (E, D(A))_{\alpha_0,1}$ the function

$$\xi \to \xi^{\alpha_0}\|A(\xi I + A)^{-1}x\| \in L^1_*(0,\infty),$$

using Proposition 2.13. By (2.20), we get $A^{\alpha-1}x \in D(A)$. That is, $x \in D(A^{\alpha})$ and, by (2.21),

$$\|A^{\alpha}x\| \leq \frac{1}{|\Gamma(\alpha)\Gamma(1-\alpha)|}\int_0^{\infty} \xi^{\alpha_0}\|A(A+\xi I)^{-1}\xi\|\frac{d\xi}{\xi} \leq C\|x\|_{(ED(A))_{\alpha_0,1}}.$$

Next let $x \in D(A^{\alpha})$. Then, $x = A^{-\alpha}y$ with $y = A^{\alpha}x$, and using (2.22) for $A^{-\alpha-1}y = x$, we obtain

$$x = \frac{A}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_0^{\infty} \xi^{-\alpha}(A + \xi I)^{-2}y d\xi,$$

while Proposition 2.13 imply

$$
\begin{aligned}
\|x\|_{E,D(A)_{\alpha_0,\infty}} &\leq C(\alpha)\sup_{\lambda>0}\|\lambda^{\alpha_0}A(A+\lambda I)^{-1}x\| \\
&\leq C(\alpha)\sup_{\lambda>0}\left\|\frac{\lambda^{\alpha_0}A^2(A+\lambda I)^{-1}}{\Gamma(1-\alpha)\Gamma(1+\alpha)}\int_0^\infty t^{-\alpha}(A+tI)^{-2}y\,dt\right\| \\
&\leq \lambda^{\alpha_0}\frac{M}{1+\lambda}\int_0^\lambda t^{-\alpha}(M+1)^2\|y\|dt \quad + \\
&+ \lambda^{\alpha_0}M(M+1)^2\int_\lambda^\infty \frac{1}{t^{\alpha_0}(1+t)}\|y\|dt \\
&\leq C\|y\| = C\|A^\alpha x\|. \qquad (2.26)
\end{aligned}
$$

It then follows that $(E, D(A))_{\alpha_0,\infty} \supset D(A^\alpha)$. □

The theory of positive operators can be easily extended to non-negative operators in the sense of Definition 2.9 in which $(-\infty, 0) \subset \rho(A)$ and (2.13) holds with $\frac{M}{\lambda}, \lambda > 0$ as the estimate from above. See [52] for a treatment of this situation. More delicate for our immediate consideration is the question of the comment before the Definition 2.11.

Lemma 2.15. *Let A be a positive operator such that $A^{i\alpha_1} \in \mathcal{L}(E)$ for every $\alpha_1 \in \mathbb{R}$, and $\alpha_1 \to \|A^{i\alpha_1}\|$ is locally bounded. Then, for every $x \in D(A)$ the function $\mathbb{C} \ni z = z_0 + iz_1 \to A^z x$ is continuous in the closed half-plane $z_0 \leq 0$.*

Proof. If $x \in D(A)$, then $z \to A^z x$ is holomorphic for $z_0 < 1$, so that it is trivially continuous for $z_0 \leq 0$. But in the strict sense, it holds that

$$
\begin{aligned}
\|A^z\| &\leq \frac{M}{\pi}|\sin(\pi z)|\int_0^\infty \frac{\xi^{z_0}}{\xi+1}d\xi \\
&\leq \frac{M|\sin(\pi z)|}{|\sin(\pi z_0)|},
\end{aligned}
$$

which implies that $\|A^{z_0}\| \leq M$ for $z_0 \in (-\frac{1}{2}, 0)$, and thus, $\|A^z\| \leq M\|A^{iz_0}\|$. In particular, for any $\alpha_1 \in \mathbb{R}$ and $r > 0$ sufficiently small

$$
\|A^z - A^{i\alpha_1}\| \leq C
$$

in the half circle $\{z : |z - i\alpha_1| \leq r, z_0 \leq 0\}$, where the constant is independent of z. Consequently, for every $x \in D(A)$, $\lim_{z \to i\alpha_1} A^z x = A^{i\alpha_1} x$.

$\square$

Lastly, for this section we give the construction of the Banach scale spaces.

Theorem 2.16. *Assume A is a positive operator with dense domain such that for every $\alpha_1 \in \mathbb{R}$, $A^{i\alpha_1} \in \mathcal{L}(E)$, and there exists $C, \gamma > 0$ such that*

$$\|A^{i\alpha_1}\| \leq Ce^{\gamma|\alpha_1}, \quad \alpha_1 \in \mathbb{R}.$$

Then, if $\alpha = \alpha_0 + i\alpha_1$, $\beta = \beta_0 + i\beta_1 \in \mathbb{C}$ satisfy $0 \leq \alpha_0 < \beta_0$, then it holds that

$$[D(A^\alpha), D(A^\beta)]_\theta = D(A^{(1-\theta)\alpha + \theta\beta}).$$

Proof. Without loss of generality, we assume $\alpha = 0$. Moreover, since $A^{i\beta_1} \in \mathcal{L}(E)$ for any $\beta_1 \in \mathbb{R}$ we have $D(A^\beta) = D(A^{\beta_0})$ for $\beta_0 > 0$, so that we may assume $\beta \in (0, \infty)$.

Let $x \in D(A^{\theta\beta})$ and,

$$f(z) = e^{(z-\theta)^2} A^{-(z-\theta)} x, \quad 0 \leq z_0 \leq 1.$$

To prove that $f \in \mathcal{F}(E, D(A^\beta))$, we observe that f is holomorphic in the strip $z_0 \in (0,1)$ and continuous up to $z_0 = 1$ taking values in E. Since $D(A)$ is dense in E, f is a continuous function up to $z_0 = 0$ with values in E. Indeed, $A^{-(z-\theta)\beta} x = A^{-z\beta} A^{\theta\beta} x$ and the mapping $\nu \to A^\nu y \in E$ is continuous for $\nu \leq 0$ for any $y \in \overline{D(A)} = E$. Similarly, $t \to f(1 + it) \in D(A^\beta)$ is continuous. On the other hand,

$$\|A^{-(z-\theta)\beta} x\| = \|A^{-\beta z_1} A^{-\beta z_0} A^{\theta\beta} x\| \leq \|A^{-\beta z_0}\| Ce^{\gamma\beta z_1} \|A^{\theta\beta} x\|,$$

implying that f is bounded. Therefore, $f \in \mathcal{F}(E, D(A^\beta))$, while $f(\theta) = x$ imply $x \in [E, D(A^\beta)]_\theta$ and

$$\begin{aligned} \|x\|_\theta &\leq \max\left\{\sup_{t\in\mathbb{R}} \|e^{-t^2+\theta^2} A^{-(it-\theta)\beta} x\|, \sup_{t\in\mathbb{R}} \|e^{-t^2+(1-\theta)^2} A^{-(1+it-\theta)\beta} x\|_\beta\right\} && (2.27) \\ &\leq C' \|A^{\theta\beta} x\| \Rightarrow D(A^{\theta\beta}) \subset [E, D(A^\beta)]_\theta. && (2.28) \end{aligned}$$

Conversely, let $x \in D(A^{\theta\beta}), f \in \mathcal{F}(E, D(A^{\beta}))$ be such that $f(\theta) = x$. Then the function,

$$F(z) = e^{(z-\theta)^2} A^{z\beta} f(z) \in E$$

is continuous for $z_0 = 0, 1$, and we have

$$\begin{aligned} \sup_{t\in\mathbb{R}} \|F(it)\| &\leq \sup_{t\in\mathbb{R}} e^{-t^2+\theta^2} C e^{\gamma\beta|t|} \sup_{t\in\mathbb{R}} \|f(it)\| \leq C'\|f\|_{\mathcal{F}(E,D(A^{\beta}))}, \quad \text{and} \\ \sup_{t\in\mathbb{R}} \|F(1+it)\| &\leq \sup_{t\in\mathbb{R}} e^{-t^2+(1-\theta^2)} C e^{\gamma\beta|t|} \sup_{t\in\mathbb{R}} \|f(1+it)\| \leq C'\|f\|_{\mathcal{F}(E,D(A^{\beta}))}. \end{aligned}$$

Thus F is bounded with values in E, for $z_0 = 0, 1$. If F was holomorphic in the interior of S and continuous in S we could apply the maximum principle so as to get $\|A^{\theta}x\| \leq C'\|f\|_{\mathcal{F}(X,D(A^{\beta}))}$. But in general, F is not even defined in S interior, as it takes values in E and not in the domain of the power operator of A. So we have to modify the approach.

By definition

$$\|x\|_{\theta} = \inf\{\|f\|_{\mathcal{F}(E,D(A^{\beta}))} : f \in \mathcal{V}(E, D(A^{\beta})), f(\theta) = x\}.$$

Next, define

$$F(z) = e^{(z-\theta)^2} A^{z\beta} f(z), \quad 0 \leq z_0 \leq 1,$$

which is properly defined, holomorphic if $z_0 \in (0, 1)$ and continuous taking values in E, up to $z = 0, 1$. As it is further bounded, the maximum principle implies that

$$\|A^{\theta\beta}x\| = \|f(\theta)\| \leq \max\{\sup_{t\in\mathbb{R}} \|F(it)\|, \sup_{t\in\mathbb{R}} \|F(1+it)\} \leq C'\|f\|_{\mathcal{F}(E,D(A^{\beta})},$$

where the last estimates follow from ones in above paragraph. The conclusion of the theorem thus, is obtained through the fact that $D(A^{\beta})$ is dense in $[E, D(A^{\beta})]_{\theta}$. $\square$

Chapter 3

Minimal KS Equation in Bessel Potential Spaces

3.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary $\partial\Omega = \Gamma$. In this chapter, we are going to prove the local existence and uniqueness of solutions for the following minimal prototype of the Keller-Segel models, describing the aggregation of amoebae by chemotaxis;

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\chi\nabla v) & \text{in } \Omega \times (0,T), \\ v_t = \Delta v - \lambda v + au & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x). \end{cases} \tag{3.1}$$

where

$$\begin{aligned} u &:= \text{cell density of the amoebae}, \\ v &:= \text{chemical attractant concentration}, \\ \lambda &:= \text{rate of decay of chemical}, \\ a &:= \text{rate of production of chemical}, \end{aligned}$$

$$\begin{aligned} \chi &:= \text{chemotactic sensitivity coefficient}, \\ \nabla\cdot &:= \text{div}, \\ \vec{n} &:= \text{unit normal vector pointing outwards of } \Gamma. \end{aligned}$$

For notational simplicity, we let $I = [0, T)$, $\dot{I} = (0, T)$.

We will be working in the Bessel potential spaces in $L^q(\Omega)$ $(1 < q < \infty)$, hereafter denoted by E_q^α (see Definition 2.8 and (1.1)). The system of equations (3.1) can therefore be written in matrix form as follows:

$$\begin{cases} U_t + \mathcal{A}U &= P(u)U \\ U(0) &= U_0 \in E_q^\beta \times E_q^\gamma, \quad \beta \leq \gamma < \beta + 1, \end{cases} \tag{3.2}$$

where $U = (u, v)^\top$, $U_0 = (u_0, v_0)^\top$, and

$$\begin{cases} \mathcal{A} &= \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta + \lambda \end{pmatrix}, \\ \\ P(u)U &= \begin{pmatrix} -\nabla \cdot (u\chi\nabla v) \\ au \end{pmatrix}. \end{cases} \tag{3.3}$$

The domain of the operator $\mathcal{A}$ in (3.3), denoted by $\mathcal{D}(\mathcal{A})$, is taken to be

$$\mathcal{D}(\mathcal{A}) := \left\{ U \in H^{2,q}(\Omega, \mathbb{R}^2) : \partial_{\vec{n}} U = \vec{0} \text{ on } \Gamma, \ \ 1 \leq q < \infty \right\}. \tag{3.4}$$

Furthermore, recall from Chapter 1 that the operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^q(\Omega) \to L^q(\Omega)$ in (3.3) is sectorial (C^+ operator for short). Thus, by the complex interpolation-extrapolation theory [2, 5, 30, 67, 79], the scale spaces

$$E_q^\alpha := H^{2\alpha,q}(\Omega) = (I - \Delta)^{-\alpha} L^q(\Omega)$$

are well defined, subject to the boundary conditions, for $-1 \leq \alpha \leq 1$, $1 < q < \infty$.

Next, we recall the embedding relations for the Bessel potential spaces as in [2, 5, 30, 67, 71]. Suppose that the boundary of Ω, $\partial\Omega$, is C^1-smooth. Then we have that

$$H^{s,q}(\Omega) \subset \begin{cases} L^p(\Omega),\ s - \frac{N}{q} \geq -\frac{N}{p},\ 1 \leq p < \infty, & \text{if } s - \frac{N}{q} < 0, \\ L^p(\Omega),\ 1 \leq p < \infty, & \text{if } s - \frac{N}{q} = 0, \\ C^\theta(\Omega) & \text{if } s - \frac{N}{q} > \theta > 0, \end{cases} \tag{3.5}$$

with continuous inclusions. These embeddings are known to be optimal. The best space embeddings constant which we will use relatively often is [27, 91]

$$C_\alpha = \begin{cases} (2\sqrt{\pi})^{-\frac{2\alpha}{2}} \frac{\Gamma\left(\frac{N-2\alpha}{2}\right)}{\Gamma\left(\frac{N+2\alpha}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{\frac{2\alpha}{N}} \simeq \left(2(Ne\pi)^{-1}\right)^\alpha & \text{if } 1 < q < \infty,\ 0 < 2\alpha < \frac{N}{q}, \\ \\ \dfrac{2\Gamma\left(\frac{N}{2q'}\right)|\Omega|^{\frac{1}{p}}}{2^{\frac{1}{q}+\frac{N}{q}} \pi^{\frac{N}{2q}} N^{\frac{1}{q'}} \left(1 - \frac{1}{q'}\left(\frac{1}{q'}+\frac{1}{p}\right)\right)^{\frac{1}{q'}+\frac{1}{p}} \left[\Gamma\left(\frac{N}{2}\right)\right]^{\frac{1}{q'}} \Gamma\left(\frac{N}{2q}\right)} & \text{if } 2\alpha = \frac{N}{q},\ q \leq p < \infty, \end{cases} \tag{3.6}$$

obtained by using Stirling's formula for large N, with $1 < p < \infty$, and $0 < 2\alpha < \frac{N}{p}$.

With regards to the dual spaces for these Bessel potential spaces, we first recall from Proposition 2.8 that if $s \in \mathbb{R}$ and $1 < p < \infty$, then $\mathcal{S}(\Omega)$ is dense in $H^{s,p}(\Omega)$. Thus, a continuous linear functional on $H^{s,p}(\Omega)$ can be interpreted in the usual way as an element of $\mathcal{S}^*(\Omega)$. Moreover, it is known that the scales of Bessel potential spaces with negative exponents satisfy $H^{-s,p}(\Omega) = \left(H^{s,p'}(\Omega)\right)^*$. See [5, 6] for more details.

We can then easily obtain that for $s > 0$ we have that

$$H^{-s,q}(\Omega) \supset \begin{cases} L^p(\Omega),\ -s - \frac{N}{q} \leq -\frac{N}{p},\ 1 < p \leq \infty, & \text{if } -s - \frac{N}{q} > -N, \\ L^p(\Omega),\ 1 < p \leq \infty, & \text{if } -s - \frac{N}{q} = -N, \\ \mathcal{M}(\Omega) & \text{if } -s - \frac{N}{q} < -N. \end{cases}$$

In (3.3), $P(u)U$ is a linearly coupled vector function, with the first entry featuring a divergence-0 operator acting on a vector field $u\chi\nabla v$ of concentration of chemicals, while in the second component we have the productive effects of the amoebae. Note that, from the system of equations (3.1)-(3.3), neither proliferation nor death of the amoebae has been considered[1].

It is known that (see [30, 68, 66]), corresponding to the equations in (3.1) are the variation of coefficient formulae (also known as Integral formulae) given as

$$\begin{aligned} u(t) &= e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}\nabla(u(s)\chi\nabla v(s))\,ds, \\ v(t) &= e^{(\Delta-\lambda)t}v_0 + a\int_0^t e^{(\Delta-\lambda)(t-s)}u(s)\,ds, \end{aligned} \tag{3.7}$$

assuming that they are well defined for given functions u and v, respectively, defined on $(0,T]$.

The well-posedness conditions for the system (3.1)-(3.3) depend on the three embedding cases in (3.5). We are going to find conditions for the system (3.1) to be well-posed in the super-critical case ($0 < 2\alpha < \frac{N}{q}$).

Furthermore, we are going to prove the existence of *a priori* uniform bounds in $\Omega \times (0,T)$ of solutions and their gradients (3.1). We will then use a bootstrap argument to prove that the solutions to the system (3.1) are in fact classical.

Lastly for this chapter, we are going to give some highlights on the blow-up dynamics of the system of equations (3.1) at the borderline spaces E_q^α, $\alpha = \frac{N}{2q}$.

3.2 Well-posedness of the System

In this section, we will prove the well-posedness for the system (3.1)-(3.3) with the bounded domain $\Omega \subset \mathbb{R}^N$ in the Bessel potential spaces E_q^α, where $0 < 2\alpha < \dfrac{N}{q}$. To this end, we

[1] Refer to the paragraph after (0.1).

have the following lemma.

Lemma 3.1. *Consider the system* (3.1) *and assume that* (1.2) *holds. Let* $\alpha,\ \gamma \in \mathbb{R},\ \ \alpha \geq \frac{1}{2},\ \ \gamma \leq \alpha < \gamma + 1$ *such that*

$$\alpha + \gamma \geq \frac{1}{2} + \frac{N}{2q}, \quad \textit{and} \ \ 2\alpha + \gamma \geq 1 + \frac{N}{2q}. \tag{3.8}$$

If $u \in E_q^{\gamma},\ \ v \in E_q^{\alpha}$*, then* $P(U) \in E_q^{-\beta}$ *as a weak form in the sense given by*

$$\langle P(U), \varphi\rangle_{q,q'} = \langle u\chi\nabla v, \nabla\varphi\rangle_{q,q'} = \chi\int_{\Omega} u\nabla v\nabla\varphi \in \mathbb{R}, \quad \forall\varphi \in E_{q'}^{\alpha}, \tag{3.9}$$

is well defined. Moreover,

$$\|P\|_{\mathcal{L}_{lip}(E_q^{\alpha}, E_{q'}^{-\gamma})} := \sup_{\|\varphi\|_{\alpha,q'}\leq 1} \frac{\left|\langle P(U), \varphi\rangle_{E_q^{\alpha}, E_{q'}^{\gamma}}\right|}{\chi\|u\nabla v\|_{\alpha,q}} \leq \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}. \tag{3.10}$$

In particular, $P \in \mathcal{L}_{lip}(E_q^{\alpha}, E_{q'}^{\gamma})$ *is true.*

Proof. Let $\varphi \in E_{q'}^{\alpha}$ be a test function to the operator $-\mathrm{Div}(u\nabla v)$ in the duality passing of $L^q(\Omega)$. Using the Sobolev type embeddings (3.5) and Hölder's inequality, we have that the mapping

$$E_q^{\gamma} \times E_q^{\alpha} \times E_{q'}^{\alpha} \ni (u, v, \varphi) \mapsto \langle -\mathrm{Div}(u\chi\nabla v), \varphi\rangle_{q,q'} = \chi\int_{\Omega} u\nabla v\nabla\varphi \in \mathbb{R} \tag{3.11}$$

is well defined and continuous, since, if $\gamma - \dfrac{1}{2} \geq 0$, then $\nabla v \in E_q^{\gamma-\frac{1}{2}} \subset E_q^0$. Given that $q \geq \dfrac{N}{2\alpha}$, we have from (3.5) that $u\nabla v \in E_q^0$. We further see from (3.5), regarding embeddings into $E_q^{\alpha-\frac{1}{2}}$, that

$$\frac{1}{q} \geq \frac{N - 2\gamma q}{qN} + \frac{N - 2\alpha q + q}{qN} \Leftrightarrow N \geq 2N - 2q(\alpha + \gamma) + q, \tag{3.12}$$

from which we obtain that $2q(\alpha+\gamma) \geq N + q \Leftrightarrow \alpha + \gamma \geq \dfrac{1}{2} + \dfrac{N}{2q}$, which is the first hypothesis in (3.8).

Furthermore, applying again (3.5) and Hölder's inequality, we obtain that

$$\frac{N - 2\gamma q}{qN} + \frac{N - 2q(\alpha - \frac{1}{2})}{qN} + \frac{N - 2q'(\alpha - \frac{1}{2})}{q'N} \leq 1,$$

which yields the second hypothesis in (3.8). As a result, we get, by using Hölder's inequality, (3.5) and (3.6), that

$$
\begin{aligned}
|\int_\Omega \chi u \nabla v \nabla \varphi \, dx| &\leq \|u\chi\nabla v\|_q \|\nabla\varphi\|_{q'} \\
&\leq \chi \left(\frac{2}{Ne\pi}\right)^{\alpha-\frac{1}{4}} \|u\nabla v\|_{\alpha,q}\|\nabla\varphi\|_{\alpha-\frac{1}{2},q'} \\
&\leq \chi \left(\frac{2}{Ne\pi}\right)^{\alpha-\frac{1}{2}} \|u\|_{p_0}\|\nabla v\|_{\alpha-\frac{1}{2},q}\|\nabla\varphi\|_{\alpha-\frac{1}{2},q'} \\
&\leq \chi \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} \|u\|_{\gamma,q}\|v\|_{\alpha,q}\|\varphi\|_{\alpha,q'},
\end{aligned}
\tag{3.13}
$$

in which we supposed that $\frac{1}{p} = \frac{1}{p_0} + \frac{1}{p_1}$. Moreover, we get by linearity of the mapping that it is Lipschitz continuous, and the Lemma is established. □

A worthwhile observation is that the bounding estimate in (3.10) changes with the space embedding $E_q^{\frac{N}{2q}} \subset L^p(\Omega)$ in the critical case $2\alpha = \frac{N}{q}$ (see (3.6)). Furthermore, the yielding condition in (3.8) is a special case of the condition

$$
\frac{1}{2} + \frac{N}{2}\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{\rho}\right) \leq \beta + \gamma \quad \text{and} \quad 1 + \frac{N}{2}\left(\frac{1}{q} + \frac{1}{p} - \frac{1}{\rho}\right) \leq 2\beta + \gamma, \tag{3.14}
$$

pertinent to initial data spaces in all different exponents, implying $P_\Omega = P : E_\rho^\gamma \mapsto E_{\rho'}^{-\beta}$, $P_\Omega \in \mathcal{L}_{lip}(E_\rho^\gamma, E_{\rho'}^\beta)$, and (3.8) is obtained when $\rho = q$. Since $\gamma \geq \frac{1}{2}$, $\beta \leq 1$, the condition (3.14) yields Young's inequality for convolutions, and $\rho \geq q$, p. Furthermore, the sharp optimal Bessel potential space inclusions (3.5) are verified, with $\rho = p$.

Following from Lemma 3.1, we have the main result of this section as the following Theorem.

Theorem 3.2. *Suppose that for the second equation in* (3.1), $u \in L^\sigma(\dot{I}, E_q^\alpha)$ *for* $1 \leq \sigma \leq \infty$ *and* $0 \leq \gamma - \alpha < \frac{1}{\sigma'}$. *Then*

(i) *we have that*

$$v \in C(I; E_q^{\alpha}) \cap C(\dot{I}; E_q^{\gamma}) \cap C(\dot{I}; E_q^{\alpha+1}) \cap C^1(\dot{I}; E_q^{\gamma'}), \tag{3.15}$$

for any $\gamma' < \alpha + \frac{1}{\sigma'}$, $\alpha \leq \gamma < \alpha + 1$. *Furthermore, since the operator in* (3.3) *is a* C^+ *operator, we have that* $u \in L^{\sigma}(0, \infty; E_q^{\alpha})$, $v \in L^{\sigma}(0, \infty; E_q^{\gamma})$, $\alpha \leq \gamma < \alpha + 1$, *for any* $\alpha \in \mathbb{R}$, *and*

$$\limsup_{t \to \infty} \|u\|_{\gamma} = 0 \quad \textit{and} \quad \limsup_{t \to \infty} \|\nabla v\|_{\gamma - \frac{1}{2}} = 0 \tag{3.16}$$

hold.

(ii) *If Lemma 3.1 holds, then*

$$u \in C(I; E_q^{\beta}) \cap C(\dot{I}; E_q^{\gamma}) \cap C(\dot{I}; E_q^{\beta+1}) \cap C^1(\dot{I}; E_q^{\gamma'}), \tag{3.17}$$

for any $\gamma' < \beta + 1$.

(iii) *The evolution equation* (3.2) *admits a unique global strong solution given by* (3.7), *and the converse is true. Furthermore, if*

$$\frac{2\chi + a}{q} \left(\frac{2}{Ne\pi} \right)^{\alpha + \frac{\beta}{2} - \frac{1}{2}} < 1, \tag{3.18}$$

then the complete system differential operator

$$\mathcal{A} - \tilde{P}(\phi) = \begin{pmatrix} -\Delta & \nabla(\phi\chi\nabla\cdot) \\ -a & -\Delta + \lambda \end{pmatrix} : Z_q^{\beta+\alpha} \to Z_q^{-\beta-\alpha}, \tag{3.19}$$

with $\beta \leq \alpha \leq \gamma < \alpha + 1$, *defines a perturbed analytic semigroup in* $Z_q^{\beta+\alpha}$, *and the global asymptotic dynamics satisfy*

$$\limsup_{t \to \infty} \|(u(t), v(t))^{\top}\|_{\alpha+\gamma} = A^* \in \mathcal{M} \cup \{0\}, \tag{3.20}$$

where $\mathcal{M}$ *is the limit set*

$$\mathcal{M} = \{A \in \mathbb{R}^2 : A = \left(\frac{a|\Omega|\bar{u}_0}{\lambda}, |\Omega|\bar{v}_0 \right)^{\top} \}, \tag{3.21}$$

which corresponds to $|\Omega|L^1-$*spatial average solutions to the system of equations* (3.1) *in the distributions sense.*

(iv) *The global solution in (iii) is a classical solution.*

We remark here that (3.18) has been shown to hold even using numerical experimental data (see [50, 87]). Before we give a proof for Theorem 3.2, we state that with the integral formulae (3.7) in mind, we have the following definition:

Definition 3.1.

(i) If $u \in L^\sigma(\dot{I}, E_q^\alpha)$ for $1 \leq \sigma \leq \infty$, then a function v is a strong solution if it satisfies (3.15), (3.7) and (3.1) in distribution sense as an identity in E_q^α.

(ii) A function $u \in E_q^\alpha$ is a strong solution if (3.11) is verified and well defined in E_q^γ, with $0 \leq \alpha - \gamma < 1$, and (3.17) holds so as the equation in distribution sense as an identity in E_q^γ.

(iii) If (i) and (ii) above are satisfied, then $U = (u, v)^\top$ is a strong solution to (3.2), and the equation is verified in distribution sense as an identity in $Z_q^{\alpha+\gamma} = E_q^\alpha \times E_q^\gamma$.

We are now ready to give the proof for our main result.

3.2.1 Proof of Theorem 3.2

The proof of Theorem 3.2 will be given in steps. Firstly, we observe that in (3.15)-(3.17), the initial smoothness of solutions are due to the fact that the analytic semigroup (1.3) is also a C^0-semigroup, and hence [67, 68] yield the assertions using (3.7).

We also have, using the second integral formula in (3.7), and the estimate in (1.6) for

$\gamma = \alpha$ if $\sigma = 1$, or for $0 \le \gamma - \alpha < \frac{1}{\sigma'}$ if $1 < \sigma \le \infty$, that

$$\begin{aligned}
\|v(t)\|_\gamma &\le \|e^{(\Delta-\lambda)t}v_0\|_\gamma + a\int_0^t \|e^{(\Delta-\lambda)(t-s)}u(s)\|_{\gamma\, ds} \\
&\le Mt^{-(\gamma-\alpha)}\|v_0\|_\alpha + Ma\int_0^t (t-s)^{-(\gamma-\alpha)}\|u(s)\|_\alpha\, ds \\
&\le M\left[t^{-(\gamma-\alpha)}\|v_0\|_\alpha + a\left(\int_0^t (t-s)^{-\sigma'(\gamma-\alpha)}\, ds\right)^{\frac{1}{\sigma'}}\left(\int_0^t \|u(s)\|_\alpha^\sigma\, ds\right)^{\frac{1}{\sigma}}\right] \\
&\le M\left[t^{-(\gamma-\alpha)}\|v_0\|_\alpha + a\left(\frac{1}{1-\sigma'(\gamma-\alpha)}\right)^{\frac{1}{\sigma'}} t^{\frac{1}{\sigma'}-(\gamma-\alpha)}\|u(t)\|_{L^\sigma(0,T,E_q^\alpha)}\right],
\end{aligned}$$

where the second inequality above is obtained from (1.6) and the third one from Hölder's inequality. This means that $v(t)$ is bounded on finite intervals away from $t = 0$, and $v(t) \in E_q^\gamma$ for $t > 0$.

In particular,

$$\begin{aligned}
|||v|||_{\gamma,\gamma-\alpha} &= \sup_{t\in(0,T]} t^{\gamma-\alpha}\|v(t)\|_\gamma \\
&\le \sup_{t\in(0,T]} t^{\gamma-\alpha}M\left[t^{-(\gamma-\alpha)}\|v_0\|_\alpha + t^{\frac{1}{\sigma'}-(\gamma-\alpha)}\|u(t)\|_{L^\sigma(0,T,E_q^\alpha)}\right] \\
&= \sup_{t\in(0,T]} M\left[\|v_0\|_\alpha + t^{\frac{1}{\sigma'}}\|u(t)\|_{L^\sigma(0,T,E_q^\alpha)}\right] \\
&\le C\left(\|v_0\|_\alpha + \|u\|_{L^\sigma((0,T),E_q^\alpha)}\right),
\end{aligned}$$

for some $C > 0$, which proves that $v \in \mathcal{L}^\infty_{\gamma-\alpha}(\dot{I}, E_q^\gamma)$.

To prove continuity, fix $t > 0$ (or even $t = 0$ if $v_0 \in E_q^\gamma$), $h > 0$. Then, from the second formula in (3.7) we compute

$$v(t+h) - v(t) = e^{(\Delta-\lambda)h}v(t) - v(t) + a\int_t^{t+h} e^{(\Delta-\lambda)(t+h-s)}u(s)\, ds,$$

so that, when we take the norm, (1.6) gives

$$\begin{aligned} \|v(t+h)-v(t)\|_\gamma &\le \|e^{(\Delta-\lambda)h}v(t)-v(t)\|_\gamma \\ &\quad + aM\int_t^{t+h}(t+h-s)^{-(\gamma-\alpha)}\|u(s)\|_\alpha\,ds. \end{aligned} \tag{3.22}$$

Now, as $h\to 0$, $e^{(\Delta-\lambda)h}v(t)\to v(t)$, and thus the first term goes to zero. By the Hölder's inequality, the second term is bounded by

$$aM\left(\int_t^{t+h}(t+h-s)^{-\sigma'(\gamma-\alpha)}\,ds\right)^{\frac{1}{\sigma'}}\left(\int_t^{t+h}\|u(s)\|_\alpha^\sigma\,ds\right)^{\frac{1}{\sigma}} \le M_{1-\sigma'(\gamma-\alpha)}\|u\|_{\sigma,\alpha}h^{\frac{1}{\sigma'}-(\gamma-\alpha)},$$

which also goes to zero as $h\to 0$. Hence, continuity of the $v-$solution component follows.

Next, for any $\alpha,\ \gamma\in\mathbb{R}$ such that $\alpha\le\gamma\le\alpha+\frac{1}{\sigma'}$, if we let

$$c_{\alpha,\gamma}(t)=e^{(\Delta-\lambda)t}\in L^1(0,\infty),$$

then it is not bounded at $t=0$, unless $\alpha=\gamma$. Also, if $\sigma=1$, we let

$$\phi(t)=\int_0^t e^{(\Delta-\lambda)(t-s)}u(s)\,ds,$$

then, since

$$e^{(\Delta-\lambda)(t-s)}v_0\in L^1(0,\infty;\,E_q^\gamma),$$

provided that $v_0\in E_q^\gamma$, we only need to prove that $\phi(t)\in L^1(0,\infty;\,E_q^\gamma)$. Thus, if $s=t\rho$ for $\rho\in[0,1]$ fixed, then we get

$$\begin{aligned} \|\phi(t)\|_{1,\gamma} &\le \int_0^t \|\phi(t)\|_{1,\gamma}\,d\rho = \int_0^t\int_0^\infty \left\|e^{(\Delta-\lambda)t(1-\rho)}u(t\rho)\right\|_\gamma\,dt \\ &\le \int_0^t\int_0^\infty \frac{r}{\rho^2}c_{\alpha,\gamma}(r(1-\rho)\rho^{-1})\,\|u(r)\|_\alpha\,drd\rho \\ &\le \left(\int_0^\infty c_{\alpha,\gamma}(s)\,ds\right)\left(\int_0^\infty \|u(r)\|_\alpha\,dr\right), \end{aligned}$$

following other changes to the time variables $r = t\rho,\ s = r\left(\frac{1-\rho}{\rho}\right)$, then integrating with respect to ρ. It then follows that

$$\|v(t)\|_{1,\gamma} \leq \|c_{\gamma,\gamma}(t)\|_1 \|v_0\|_\gamma + \|c_{\alpha,\gamma}(s)\|_1 \|u(r)\|_{1,\alpha}.$$

Note that for $\sigma = \infty$ the second term in (3.22) is bounded by

$$aM\left(\|u(t)\|_{L^\infty([0,T],E_q^\alpha)}\right)\int_t^{t+h}(t+h-s)^{-(\gamma-\alpha)}\,ds \leq M_{1-\sigma'(\gamma-\alpha)}\|u\|_{\infty,\alpha}h^{1-(\gamma-\alpha)}.$$

If $0 \leq \gamma - \alpha \leq 1$, then we have continuity. The rest follows by using interpolation, thus, the second from last result in (3.16) is valid.

Furthermore, we note that if we apply ∇ to the second equation in (3.7) and take the norm in $\gamma - \frac{1}{2}$, we get

$$\begin{aligned}
\|\nabla v\|_{\gamma-\frac{1}{2}} &\leq \|\nabla\left(e^{(\Delta-\lambda)t}v_0\right)\|_{\gamma-\frac{1}{2}} + a\int_0^t \|\nabla\left(e^{(\Delta-\lambda)(t-s)}u(s)\right)\|_{\gamma-\frac{1}{2}}\,ds \\
&\leq Mt^{-(\gamma-\alpha)}\|v_0\|_\alpha + Ma\int_0^t (t-s)^{-(\gamma-\alpha)}\|u(s)\|_\alpha\,ds \\
&\leq M\left[t^{-(\gamma-\alpha)}\|v_0\|_\alpha + a\left(\int_0^t (t-s)^{-\sigma'(\gamma-\alpha)}\,ds\right)^{\frac{1}{\sigma'}}\left(\int_0^t \|u(s)\|_\alpha^\sigma\,ds\right)^{\frac{1}{\sigma}}\right] \\
&\leq M\left[t^{-(\gamma-\alpha)}\|v_0\|_\alpha + a\left(\frac{1}{1-\sigma'(\gamma-\alpha)}\right)^{\frac{1}{\sigma'}} t^{\frac{1}{\sigma'}-(\gamma-\alpha)}\|u(t)\|_{L^\sigma(0,T,E_q^\alpha)}\right],
\end{aligned} \tag{3.23}$$

since (3.8) is assumed so that if $\alpha \geq \frac{N}{2q}$, then $\gamma - \frac{1}{2} \geq \alpha$. This implies that $\nabla v \in \mathcal{L}^\infty_{\gamma-\alpha}(0,\infty; E_q^{\gamma-\frac{1}{2}})$.

If we then consider the first formula in (3.7) and set $h(t) = t^{\gamma-\alpha}\|\nabla v\|_{\gamma-\frac{1}{2}}$, then we have

$$\begin{aligned}
\|u(t)\|_\gamma &\leq Mt^{-(\gamma-\beta)}\|u_0\|_\beta + \int_0^t \|\nabla e^{\Delta(t-s)}(u\chi\nabla v)(s)\|_\gamma\,ds \\
&\leq Mt^{-(\gamma-\beta)}\|u_0\|_\beta + \chi M\int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\alpha)}\|(u\nabla v)(s)\|_\alpha\,ds
\end{aligned}$$

$$
\begin{aligned}
&\leq Mt^{-(\gamma-\beta)}\|u_0\|_\beta + \chi M \left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} \times \\
&\quad \times \int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\alpha)} \|u(s)\|_\beta \|\nabla v(s)\|_{\gamma-\frac{1}{2}} \, ds \\
&\leq Mt^{-(\gamma-\beta)}\|u_0\|_\beta + \chi M \left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} \times \\
&\quad \times \int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\alpha)} h(s) s^{-(\gamma-\alpha)} \|u(s)\|_\beta \, ds \\
&= Mt^{-(\gamma-\beta)}\|u_0\|_\beta + \chi M \left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} \Phi,
\end{aligned}
$$

where $\Phi = \int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\alpha)} h(s) s^{-(\gamma-\alpha)} \|u(s)\|_\beta \, ds$. If we then make the change $s = \rho t$ in the time variable, then we see that

$$
\begin{aligned}
\Phi &\leq \sup_{t>0} h(t) \left(\int_0^1 t^{-\sigma'(\frac{1}{2}+\gamma-\alpha)} t^{-\sigma'(\gamma-\alpha)} \left(\frac{1}{(1-\rho)^{\sigma'(\frac{1}{2}+\gamma-\alpha)} \rho^{\sigma'(\gamma-\alpha)}} \right) d\rho \right)^{\frac{1}{\sigma'}} \|u\|_{\sigma,\alpha} \\
&\leq t^{-(\frac{1}{2}+2(\gamma-\alpha))} \sup_{t>0} h(t) \left(\int_0^1 \frac{1}{(1-\rho)^{\sigma'(\frac{1}{2}+\gamma-\alpha)} \rho^{\sigma'(\gamma-\alpha)}} d\rho \right) \|u\|_{\sigma,\alpha} \qquad (3.24) \\
&\leq t^{-(\frac{1}{2}+(\gamma-\alpha))} \sup_{t>0} \|\nabla v\|_{\gamma-\frac{1}{2}} \left(\int_0^1 \frac{1}{(1-\rho)^{\sigma'(\frac{1}{2}+\gamma-\alpha)} \rho^{\sigma'(\gamma-\alpha)}} d\rho \right) \|u\|_{\sigma,\alpha} .
\end{aligned}
$$

It then follows by a backward substitution into (3.24) that $\limsup_{t\to\infty} \|u\|_\gamma = 0$. If we then allow the exponential decay effect of the semigroup (1.4) in the norm estimates of (3.23) and let $\sigma = \infty$, we conclude that the last statement in (3.16) is valid. In fact:

$$
\limsup_{t\to\infty} \|\nabla v\|_{\gamma-\frac{1}{2}} \leq aM \left(\int_0^\infty \frac{e^{-\omega t}}{t^{\gamma-\alpha}} dt \right) \limsup_{t\to\infty} \|u\|_\alpha = 0,
$$

from which the result follows.

To complete the proof of (i), we need some extra results on (ii).

Lemma 3.3. *Let $u \in E_q^\beta$ be as given in (3.7). Then $u \in C_{loc}^\xi(0,T;E_q^\alpha)$ is of exponent $\xi = \gamma - \alpha \in (0,1)$. That is, it is Hölder continuous in time.*

Proof. Let $0 < t < t+h < T$. Then we have that

$$u(t+h) - u(t) = (e^{\Delta h} - I)e^{\Delta t}u_0 + \int_0^t (e^{\Delta h} - I)e^{\Delta(t-s)}\nabla(u(s)\chi\nabla v(s))\, ds + \\ + \int_t^{t+h} e^{\Delta(t+h-s)}\nabla(u(s)\chi\nabla v(s))\, ds,$$

Now, by virtue of Lemma 3.1, taking $\beta = \alpha$, and taking the norm of E_q^α on both sides, and estimating, we get that

$$\begin{aligned}
\|u(t+h) - u(t)\|_\alpha &\leq \|(e^{\Delta h} - I)e^{\Delta t}u_0\|_\alpha + \int_0^t \|(e^{\Delta h} - I)e^{\Delta(t-s)}\nabla(u(s)\chi\nabla v(s))\|_\alpha ds + \\
&+ \int_t^{t+h} \|e^{\Delta(t+h-s)}\nabla(u(s)\chi\nabla v(s))\|_\alpha ds \\
&\leq M_{\gamma-\alpha}h^{\gamma-\alpha}\left\|e^{\Delta t}u_0\right\|_\gamma + M_{\gamma-\alpha}h^{\gamma-\alpha} + \int_0^t \|\nabla(u(s)\chi\nabla v(s))\|_\gamma ds + \\
&+ \int_t^{t+h} \|e^{\Delta(t+h-s)}\nabla(u(s)\chi\nabla v(s))\|_\gamma ds \\
&\leq M_{\gamma-\alpha}Mh^{\gamma-\alpha}t^{-(\gamma-\alpha)}\left\|u_0\right\|_\alpha + \chi M_{\gamma-\alpha}Mh^{\gamma-\alpha}\int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\alpha)} \times \\
&\times \left\|u(s)\nabla v(s)\right\|_\alpha ds + \chi M \int_t^{t+h} (t+h-s)^{-\frac{1}{2}-(\gamma-\alpha)}\left\|u(s)\nabla v(s)\right\|_\alpha ds \\
&\leq M_{\gamma-\alpha}Mh^{\gamma-\alpha}t^{-(\gamma-\alpha)}\left\|u_0\right\|_\alpha + \chi M\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\alpha}{2}-\frac{1}{2}} M_{\gamma-\alpha}h^{\gamma-\alpha} \times \\
&\times \int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\alpha)} \times \left\|u(s)\right\|_\alpha \left\|\nabla v(s)\right\|_{\gamma-\frac{1}{2}} ds + \\
&+ \chi M\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\alpha}{2}-\frac{1}{2}} \int_t^{t+h} (t+h-s)^{-\frac{1}{2}-(\gamma-\alpha)}\left\|u(s)\right\|_\alpha \left\|\nabla v(s)\right\|_{\gamma-\frac{1}{2}} ds \\
&\leq \Bigg(M_{\gamma-\alpha}Mt^{-(\gamma-\alpha)}\left\|u_0\right\|_\alpha + \chi\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\alpha}{2}-\frac{1}{2}} M_{\gamma-\alpha}M_{1-(\gamma-\alpha)}t^{1-(\gamma-\alpha)} + \\
&+ \chi\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\alpha}{2}-\frac{1}{2}} M_{1-(\gamma-\alpha)} \sup_{t\in(0,T)}\left\{\left\|u(s)\right\|_\alpha \left\|\nabla v(s)\right\|_{\gamma-\frac{1}{2}}\right\}\Bigg) h^{\gamma-\alpha},
\end{aligned}$$

which thus furnishes the desired Hölder continuity of the $u-$integral solution form in (3.7), and thus the proof of the lemma is complete. $\square$

Lemma 3.4. *Consider the set*

$$W := \left\{\psi \in C(I; E_q^\gamma) : \sup_{t\in(0,T)} \left\|\psi(t)\right\|_\gamma \leq C\left\|\psi_0\right\|_\beta\right\}, \tag{3.25}$$

and set the left-hand side of the first equation in (3.7) *to* $\mathcal{F}(u)(t)$. *Then*

(i) $\mathcal{F}(W) \subset W$. *That is,* $\mathcal{F}$ *maps* W *onto itself.*

(ii) *The mapping* $\mathcal{F} : E_q^\alpha \to E_q^\gamma$ *is a contraction.*

(iii) *There exists a unique* $u \in W$ *such that* $\mathcal{F}(u)(t) = u(t)$ *is a solution to* (3.1) *up to maximal time* $T^*(\|u_0\|_\beta)$ *of existence of solutions of* (3.2).

Proof. We first note that we can read the right-hand side of (3.7) in taking the norm of $E_q^\gamma = E_q^\beta \times E_q^{\gamma-\beta}$ as in the scale spaces product, whereas, by virtue of Lemma3.1, we have that $u\chi\nabla v$ is well defined in $E_q^0 \cong L^q(\Omega)$. Therefore, if $u \in W$, then we find that

$$\begin{aligned}
\|\mathcal{F}(u)(t)\|_\gamma &\le M\|u_0\|_\beta + M\int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\beta)}\|u\chi\nabla v\|_0\, ds\\
&\le M\|u_0\|_\beta + \chi\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} M\int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\beta)}\|u\|_\beta\|\nabla v\|_{\gamma-\frac{1}{2}}\, ds\\
&\le M\|u_0\|_\beta + \chi MC\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} \sup_{t\in(0,T)}\|\nabla v\|_{\gamma-\frac{1}{2}}\|u_0\|_\beta \times\\
&\times\int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\beta)} ds\\
&\le M\|u_0\|_\beta + \chi MC\left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} \sup_{t\in(0,T)}\|\nabla v\|_{\gamma-\frac{1}{2}}\|u_0\|_\beta T^{-\frac{1}{2}-(\gamma-\beta)}.
\end{aligned}$$

Thus, for

$$T = \left(\left(\frac{1}{M}-\frac{1}{C}\right)\frac{1}{\chi\sup_{t\in(0,T)}\|\nabla v\|_{\gamma-\frac{1}{2}}}\left(\frac{2}{Ne\pi}\right)^{\frac{1-2\gamma-\beta}{2}}\right)^{\frac{2}{1-2(\gamma-\beta)}},$$

we obtain that (i) is satisfied.

To prove (ii), we let $u_1,\ u_2 \in W$. Then, for $0 \le t \le T$ and using the same initial data, we have that

$$\begin{aligned}
\|\mathcal{F}(u_1)(t) - \mathcal{F}(u_2)(t)\|_\gamma &\le \int_{t_0}^t \|e^{\Delta(t-s)}\nabla(u_1\chi\nabla v - u_2\chi\nabla v)\|_\gamma\, ds\\
&\le M\int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\beta)}\|(u_1-u_2)\chi\nabla v\|_\beta\, ds
\end{aligned}$$

$$\leq \chi M \left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} \int_0^t (t-s)^{-\frac{1}{2}-(\gamma-\beta)} \|u_1-u_2\|_\beta \|\nabla v\|_{\gamma-\frac{1}{2}} \, ds$$
$$\leq \chi M \left(\frac{2}{Ne\pi}\right)^{\gamma+\frac{\beta}{2}-\frac{1}{2}} T^{\frac{1}{2}-(\gamma-\beta)} \sup_{t\in(0,T)} \|\nabla v\|_{\gamma-\frac{1}{2}} \sup_{t\in(0,T)} \|u_1-u_2\|_\beta ,$$

so that for

$$T < \left(\frac{1}{\chi \sup_{t\in(0,T)} \|\nabla v\|_{\gamma-\frac{1}{2}}} \left(\frac{2}{Ne\pi}\right)^{\frac{1-2\gamma-\beta}{2}}\right)^{\frac{2}{1-2(\gamma-\beta)}},$$

we have that $\mathcal{F}$ is a contraction on W.

Thus, viewed together with (i) of this lemma, by the Banach Contraction Mapping Theorem (Theorem 1.1) and Picard's method, or classical continuation allows the extension of the finite existence time to maximal time $T^* = T(\|u_0\|_\beta)$, yielding the last assertion of the lemma. □

It now remains to prove (iii) of the theorem. For this, we observe on the smoothness of solutions as given in the theorem, that (3.15) follows by [30, 67, 68]. Since (3.3) is a C^+ operator, and Lemma 3.3 holds, $u(t) \in L^\sigma(\dot{I}; E_q^\alpha)$ is Hölder continuous. Consequently, linear non-homogeneous evolution equation results imply the time regularity of the solution component with even T at ∞. In a similar manner, writing the weak form (3.11) as

$$f(t) = \langle (u\chi\nabla v)(t), \nabla\varphi\rangle_{q,q'} \quad \text{for any } \varphi \in E_{q'}^{\gamma}, \tag{3.26}$$

we conclude that (3.17) also holds, because $\nabla v \in E_q^{\gamma-\frac{1}{2}}$ is bounded, and by Lemma 3.3, $u \in C^\xi(\dot{I}; E_q^\alpha)$ for $0 \leq \xi = \gamma - \alpha < 1$, as a result, imply that (3.26) is Hölder continuous in time. We therefore get by Lemma 3.2.1 and Theorem 3.2.2 in [30] the existence and uniqueness of the solution to (3.1)-(3.2). The converse to the fact that the solution is given by (3.7) is given by Definition 3.1.

To prove the generation of a perturbed analytic semigroup, we have the following lemma.

Lemma 3.5. *The operator in* (3.19) *is an infinitesimal generator of a perturbed analytic semigroup in scale spaces* $Z_q^{\beta+\alpha}$*, and the strong solution of the theorem coincides with that generated by* (3.19).

Proof. We firstly observe from what has been proven up to now that $v \in \mathcal{L}^{\infty}_{\gamma-\alpha}(0,T;E_q^{\gamma})$ and $\limsup_{t\to\infty} t^{\gamma-\alpha}\|v(t)\|_{\gamma} \leq M\|v_0\|_{\alpha}$ using (3.24) with $\sigma = \infty$, while still with (3.24), we obtain $\limsup_{t\to\infty} t^{\alpha-\beta}\|u(t)\|_{\alpha} \leq M\|u_0\|_{\beta}$, and the assertion should follow. More precisely, to complete ideas, we prove that (3.19) is well defined, continuously, coercive, strictly monotone and is a sectorial operator in $E_q^0 \cong L^q(\Omega)$.

To this end, define $b : Z_q^{\beta+\alpha} \times Z_q^{\beta+\alpha} \mapsto \mathbb{R}$ by

$$b(U,\Upsilon) = \int_{\Omega} \nabla v \nabla \varphi + \lambda \int_{\Omega} v\varphi + \int_{\Omega} \nabla u \nabla \psi - \chi \int_{\Omega} u \nabla v \nabla \psi - a \int_{\Omega} u\varphi, \tag{3.27}$$

where $\Upsilon = (\varphi,\psi)^{\top}$, and note that since Lemma 3.1-(3.8) is assumed, continuity of the mapping (3.27) is clear. We therefore need to prove only the coercivity, (since to apply Browder-Minty theorem strictly monotonicity can be easily deduced).

Thus, taking $\Upsilon = U$, we find that

$$\begin{aligned}
b(U,U) &\geq \|\nabla v\|^2_{\alpha-\frac{1}{2}} + \|\nabla u\|^2_{\beta-\frac{1}{2}} - \chi\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \|u\|_{\beta}\|\nabla v\|_{\alpha-\frac{1}{2}}\|\nabla u\|_{\beta-\frac{1}{2}} + \\
&\quad + \lambda\|v\|^2_{\alpha} - \frac{a}{2q}\|u\|^2_{\beta} - \frac{a}{2q}\|v\|^2_{\alpha} \\
&\geq \left(1 - \frac{\chi}{2q}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}}\right)\|\nabla v\|^2_{\alpha-\frac{1}{2}} + \left(1 - \frac{\chi}{q}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} - \frac{a}{2q}\right) \times \\
&\quad \times \|\nabla u\|^2_{\beta-\frac{1}{2}} + \left(\lambda - \frac{a}{2q}\right)\|v\|^2_{\alpha} \\
&\geq \left(1 - \frac{\chi}{2q}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} + \left(\lambda - \frac{a}{2q}\right)\right)\|\nabla v\|^2_{\alpha-\frac{1}{2}} + \\
&\quad + \left(1 - \frac{\chi}{q}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} - \frac{a}{2q}\right)\|\nabla u\|_{\beta-\frac{1}{2}} \\
&\geq \left(1 - \frac{\chi}{q}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} - \frac{a}{2q}\right)\|U\|^2_{\beta+\alpha},
\end{aligned} \tag{3.28}$$

implying the coercivity of (3.27), using (3.18). Thus, (3.19) is uniquely invertible using Browder-Minty's theorem, and is a sectorial operator in $E_q^0 \cong L^q(\Omega)$, since

$$\begin{aligned}
\left\|(\mathcal{A}+\mu)^{-\gamma}\tilde{P}\right\|_0 &= \sup_{\|U\|\le 1}\left\{\frac{\left\|(\mathcal{A}+\mu)^{-\gamma}\tilde{P}(U)\right\|_0}{\|U\|_0}\right\} \\
&\le \frac{C}{\mu^\gamma}(a+\|P\|_{\gamma,0}) \\
&\le \frac{C}{\mu^\gamma}\left(a+\left(\frac{2}{Nee\pi}\right)^{\gamma-\frac{1}{2}}\right),
\end{aligned}$$

for any $0 \le \gamma < 1$ satisfying Lemma 3.1-(3.8), for some $C \in \mathbb{R}^+\backslash\{0\}$, $|\pi - \arg\mu| \ge \vartheta$, $\vartheta < \frac{\pi}{2}$, and the conclusion of the lemma is obtained using Corollary 1.4.5 in [30]. Clearly, (3.21) and (3.16) imply (3.20). So, the proof of the lemma is complete. □

To complete the proof of part (iv) of the theorem, if suffices to note that since $\gamma - \frac{1}{2} \ge \frac{N}{2q}$, we have $E_q^{\gamma-\frac{1}{2}} \subset L^\infty(\Omega)$ by virtue of (3.5), and Theorem 3.2-(3.16) imply that $\nabla v \in L^\infty(\Omega)$ is bounded for all $t > 0$. As $u \in E_q^0 \cong L^q(\Omega)$, $q > \frac{N}{2}$ and $1 \ge \gamma - \frac{1}{2} > \frac{N}{2q}$, viewing the weak form (3.26) in L^q as well as the equation in elliptic form by passing u_t to the right hand side, using [67], we get $u \in L^\infty(\Omega)$ is bounded for all $t > 0$. The rest is trivial or immediate. Thus, Theorem 3.2 has been established.

□

3.3 Uniform Bounds of Solutions

In this section, we study the existence of *a priori* uniform bounds in $\Omega \times (0, T)$ for solutions to the system of equations (3.1)-(3.3), and hence, proving an alternative to part (iv) of Theorem 3.2 without using the space embeddings. We will be using the Moser-Nash-De Giorgi technique [3, 77, 44, 67], as illustrated in [87]. Thus, we have the following theorem.

Theorem 3.6. *Suppose that the minimal condition* (3.8) *is attained strictly. If* $\beta = 0$, *then* $U = (u, v)^{\top} \in L^{\infty}((0, \infty; L^{\infty}(\Omega) \times W^{1,\infty}(\Omega)),$

$$\sup_{t>0} \|U\|_{\frac{1}{2},\infty;\infty} \leq M\left(t^{-(\gamma-\alpha)} \|v_0\|_{\alpha} + t^{-\alpha} \|u_0\|_0\right) + C, \tag{3.29}$$

and the solution semigroup to (3.1) *is a classical solution semigroup.*

Proof. Let's assume that $u_0 = 0$ and let $\frac{N}{2} < q \leq N$, $|u|^{q-2}u \in E_2^{\frac{1}{2}} \cong H^1(\Omega)$. Then we get from the second line from above of (3.13) and [30] Gagliardo-Nirenberg's inequality that,

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_{\Omega} |u|^q + \frac{4(q-1)}{q^2}\int_{\Omega} |\nabla |u|^{\frac{q}{2}}|^2 &= \chi \int_{\Omega} u \nabla v \nabla(|u|^{q-2}u) \\
&\leq (q-1)\chi \int_{\Omega} |\nabla v |u|^{q-1} \nabla u| = \frac{2\chi(q-1)}{q}\int_{\Omega} |u|^{\frac{q}{2}} \nabla |u|^{\frac{q}{2}} |\nabla v| \\
&\leq \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}} \|\nabla v\|_{\infty,\infty}\left(\left\|\nabla |u|^{\frac{q}{2}}\right\|_2 \left\||u|^{\frac{q}{2}}\right\|_2\right) \\
&\leq \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}} \|\nabla v\|_{\infty,\infty}\left\|\nabla |u|^{\frac{q}{2}}\right\|_2 \left(\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}} \left\|\nabla |u|^{\frac{q}{2}}\right\|_2^{\frac{N}{N+2}} \times\right. \\
&\quad \left.\times \left\|\nabla |u|^{\frac{q}{2}}\right\|_1^{1-\frac{N}{N+2}} + \int_{\Omega} |u|^{\frac{q}{2}}\right) \\
&\leq \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}} \|\nabla v\|_{\infty,\infty}\left\|\nabla |u|^{\frac{q}{2}}\right\|_2^{1+\frac{N}{N+2}} \left\|\nabla |u|^{\frac{q}{2}}\right\|_1^{1-\frac{N}{N+2}} + \\
&\quad + \frac{2\chi(q-1)}{q}\left(\frac{2}{Ne\pi}\right)^{\frac{1}{4}} \|\nabla v\|_{\infty,\infty}\left\|\nabla |u|^{\frac{q}{2}}\right\|_2 \int_{\Omega} |u|^{\frac{q}{2}}.
\end{aligned}$$

After multiplying throughout by q and using Young's inequality, we get that

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega} |u|^q + \frac{4}{q'}\int_{\Omega} |\nabla |u|^{\frac{q}{2}}|^2 &\leq \left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\left(1 + \frac{N}{N+2}\right)\int_{\Omega} |\nabla |u|^{\frac{q}{2}}|^2 + \\
&\quad + \left((2q\chi \|\nabla v\|_{\infty,\infty})^{N+2} + (2q\chi \|\nabla v\|_{\infty,\infty})^2\right)\left(\int_{\Omega} |u|^{\frac{q}{2}}\right)^2 \\
&\leq \left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\left(1 + \frac{N}{N+2}\right)\int_{\Omega} |\nabla |u|^{\frac{q}{2}}|^2 + (2\chi N \|\nabla v\|_{\infty,\infty})^2 \times \\
&\quad \times (1 + q^N)\left(\int_{\Omega} |u|^{\frac{q}{2}}\right)^2,
\end{aligned}$$

where we have used the fact that for $T \gg 1$ sufficiently large, $\|\nabla v\|_{\infty,\infty} \ll 1$ is absolutely small.

Thus, on setting

$$\omega = \frac{4}{q'} - \left(\frac{2}{Ne\pi}\right)^{\frac{1}{2}}\left(1 + \frac{N}{N+2}\right) > 0, \quad \text{and } C_\Omega = (2\chi N)^2,$$

we get

$$\begin{aligned} \frac{d}{dt}\int_\Omega |u|^q + \omega \int_\Omega |u|^q &\le C_\Omega (1+q)^N \left(\int_\Omega |u|^{\frac{q}{2}}\right)^2 \\ \Rightarrow \int_\Omega |u|^q &\le C_\Omega (1+q)^N \sup_{t>0}\left(\int_\Omega |u|^{\frac{q}{2}}\right)^2. \end{aligned}$$

Consequently,

$$\Lambda(p) \le \left[C_\Omega (1+q)^N\right]^{\frac{1}{q}} \Lambda\left(\frac{q}{2}\right), \quad \forall\, q \ge 2.$$

If we let $q_i = 2^i,\ i \in \mathbb{N}^*$, we conclude the following;

$$\begin{aligned} \Lambda(2^i) &\le C_\Omega^{2^{-i}}(1+2^i)^{\frac{N}{2^i}}\Lambda(2^{i-1}) \le \ldots \le C_\Omega^{\sum_{k=1}^{i} 2^{-k}}(1+2^i)^{2^{-i}N}\ldots(1+2)^{2^{-1}N}\Lambda(1) \\ &\le C_\Omega \left[2^{i2^{-i}N}(2^{-i})^{2^{-i}N}\right]\cdots\left[2^{2^{-1}N}(2^{-1})^{2^{-1}N}\right]\Lambda(1) \\ &\le C_\Omega\, 2^{N\sum_{k=1}^{i} k2^{-k}} \times 2^{N\sum_{k=1}^{i} 2^{-k}}\Lambda(1) \\ &\le C_\Omega 2^{3N}\Lambda(1). \end{aligned}$$

Thus, taking the limit as $i \to \infty$ gives

$$\|u(t)\|_\infty \le C_\Omega 2^{3N}\Lambda(1) \le C_\Omega 2^{3N}\|u_0\|_1, < \infty. \tag{3.30}$$

In what's remaining of the proof, we write $u(t) = \psi_1(t) + \psi_2(t)$, where $\psi_1(t)$ verifies the homogeneous equation in (3.1) with $u(0) = u_0$ and $\psi_2(t)$ verifies the non-homogeneous equation with $u_0 = 0$. It then follows, by (3.5) and (1.4), that $\|\psi_1(t)\|_\infty \le Mt^{-\frac{N}{2q}}\|u_0\|_0$ for all $t > 0$, while (3.30) implies that $\|\psi_2(t)\|_\infty \le C$. Thus, we obtain $\|u(t)\|_\infty \le Mt^{-\alpha}\|u_0\|_0 + C$, with $\alpha = \frac{N}{2q}$, and combining with the $v-$solution gives (3.29). □

3.4 Blow-up Dynamics

In this section, we give some highlights on the blow-up dynamics of the system of equations (3.1) at the borderline spaces E_q^{α}, $\alpha = \frac{N}{2q}$. To this end, we first notice the stationary equations to the system can be derived as the limit process at time ∞, to the Lyapunov function

$$J(u,v) = \int_\Omega u \ln u - \chi \int_\Omega uv + \frac{\chi}{aq}\int_\Omega (|\nabla v|^q + \lambda |v|^q), \tag{3.31}$$

using [30] La-Salle-Hale-Henry invariance principle. For the following Theorem, let $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ denote the measure of a unit sphere in $\mathbb{R}^N$.

Theorem 3.7. *The dynamical system defined by the equations* (3.1) *accepts the Lyapunov function* (3.31)*, and* (5.22) *is verified at* $T = \infty$*, with, if the initial data is in spaces* E_q^{α}, $\alpha = \frac{N}{2q}$ *such that* $\chi > \chi_{N,\alpha} = \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{q'}} \left[\frac{\pi^{\frac{N}{2}} 2^{2\beta}\Gamma(\beta)}{\Gamma(\frac{N-2\beta}{2})}\right]$*, then*

$$\|(u,v)^\top\|_{\beta+\alpha} = \infty \quad \textit{for any } t \in (0,\infty).$$

That is, the system solution semigroup blow-up independent of time.

Proof. To show that (3.31) is a Lyapunov function, we take the dual spaces product in (3.1) with $\ln u - \chi v \in E_{q'}^{\beta}$ as a test function, in the $u-$equation, and let $v_t \in E_{q'}^{\alpha}$, to find

$$\begin{aligned}
\frac{dJ(U)}{dt} &= \int_\Omega u_t \ln u + \int_\Omega u_t - \chi \int_\Omega u_t v - \kappa \int_\Omega v_t u + \\
&\quad + \frac{\chi}{a}\left(\int_\Omega |\nabla v|^{q-2}\nabla v \nabla v_t + \lambda \int_\Omega |v|^{q-2} v v_t\right) \\
&= \int_\Omega u_t(\ln u - \chi v) - \frac{\chi}{a}\int_\Omega |v_t|^q \\
&= \int_\Omega \nabla(\nabla u - \chi u \nabla v)(\ln u - \chi v) - \frac{\chi}{a}\int_\Omega |v_t|^q \\
&= -\int_\Omega (\nabla u - \chi u \nabla v)\nabla(\ln u - \chi v) - \frac{\chi}{a}\int_\Omega |v_t|^q \\
&= -\int_\Omega u|\nabla(\ln u - \chi v)|^q - \frac{\chi}{a}\int_\Omega |v_t|^q \le 0,
\end{aligned}$$

having used the dual space function characterization for functions in $L^{q'}$, and the fact that

$$\int_\Omega u_t = 0, \quad \nabla(\ln u - \chi v) = u\left(\frac{\nabla u}{u} - \chi\nabla v\right),$$

to yield that (3.31) is a Lyapunov function for the system of equation (3.1). The proof asserts that it decreases along trajectories of the orthogonal to constant solutions of the equations as time increases to infinity.

To prove the blow-up of solutions, we note that (3.28) holds using the best constant of the inclusion E_q^α, $\alpha = \frac{N}{2q}$ in (3.6), while, associated to (5.22) is the energy functional

$$E(v) = \frac{1}{q}\|\nabla v\|_{\alpha-\frac{1}{2}}^q + \frac{\lambda}{q}\|v\|_\alpha^q - \mu \ln\left(\int_\Omega e^{\chi v}\right) \geq 0. \tag{3.32}$$

Consequently, (3.28) yields

$$b(U,U) \geq \omega\|\nabla u\|_{\beta-\frac{1}{2}}^q + \frac{\mu}{\omega}\ln\left(\int_\Omega e^{\chi v}\right)$$

using the second embedding condition in (3.6), implying the conclusion on taking $U \in E_q^\alpha \times E_q^\beta$ as a test function in the complete system of equations (3.2) then integrating in time $t \in (0,T)$ using a reduction to absurd argument.

In fact, supposing that the conclusion was false, it would follow from

$$\begin{aligned}
0 &= \frac{d}{dt}\|U\|_{\beta+\alpha}^\rho + b(U,U) \geq \frac{d}{dt}\|U\|_{\beta+\alpha}^\rho + \omega\|\nabla u\|_{\beta-\frac{1}{2}}^q + \frac{\mu}{\omega}\ln\left(\int_\Omega e^{\chi v}\right) \\
&\Leftrightarrow \|U_0\|_{\beta+\alpha}^\rho \geq \|U\|_{\beta+\alpha}^\rho + \frac{\mu}{\omega}\int_0^t \ln\left(\int_\Omega e^{\chi v}\right)\, ds \\
&\geq \frac{\mu}{\omega}\int_0^t \ln\left(\int_\Omega e^{\chi v}\right)\, ds = \infty,
\end{aligned}$$

using [46, 91], that we have the contrary to the premises holds since the norm $\|U_0\|_{\beta+\alpha}^\rho = \|v_0\|_\alpha^q + \|u_0\|_\beta^q$ is finite. This imply that the assertion in the theorem is valid.

For an alternative, much fine approach, see [26, 36], which can be adapted to our situation from their results in the case of $Z_{\alpha+\beta}$, $\alpha = \beta = \frac{1}{2}$, $q = 2$, using the Lyapunov function (3.31), embedding into Orlicz spaces [23, 61] and properties.

The proof of the theorem is complete. $\square$

Chapter 4

Attraction-Repulsion KS Equations in Scale of Hilbert Spaces

4.1 Introduction

In this chapter, we study the well-posedness and asymptotic global dynamics in scales of Hilbert spaces $E^\alpha, \alpha \in \mathbb{R}$ defined by the non-coupled system partial differential operator of the following chemotaxis system of equations modelling the aggregation of microglia in Alzheimer's disease. The treatment which we will give in this chapter is that of the Attraction-Repulsion equations of the following form.

$$\begin{cases} U_t + \mathcal{A}U &= P(u)U, \\ U(0) &= U_0 \in E^\beta \times E^\gamma \times E^\gamma, \beta \leq \gamma < \beta + 1, \end{cases} \tag{4.1}$$

where $U = (u, v, w)^\top$ with components holding the following meaning

$$\begin{aligned} u &:= \text{cell density of activated microglia}, \\ v &:= \text{chemical concentrations of attractant}, \\ w &:= \text{chemical concentrations of repellent}, \end{aligned} \tag{4.2}$$

$$\tag{4.3}$$

$$\left\{\begin{array}{rcl} \mathcal{A} & = & \left(\begin{array}{ccc} -d_1\Delta & 0 & 0 \\ 0 & -d_2\Delta+\lambda_2 & 0 \\ 0 & 0 & -d_3\Delta+\lambda_3 \end{array}\right), \\ \\ P(u)U & = & \left(\begin{array}{c} -\mathrm{Div}(u\vec{d}(\nabla v,\nabla w)) \\ a_2 u \\ a_3 u \end{array}\right), \quad \vec{d}(\nabla v,\nabla w)=\chi_2\nabla v-\chi_3\nabla w \end{array}\right. \tag{4.4}$$

Here, $d_i, \lambda_j, a_j, \chi_j \in \mathbb{R}^+ \setminus \{0\}, i = 1, 2, 3 = j \neq 1$ are all different constants of biophysical importance, with the following meanings,

$$\begin{aligned} d_1 &:= \text{motility coefficient}, \\ d_j &:= \text{diffusion coefficients}, \\ \chi_2 &:= \text{chemotactic coefficient towards attractant}, \\ \chi_3 &:= \text{chemotactic coefficient away from repellent}, \\ \lambda_j &:= \text{rates of decay of chemicals, and} \\ a_j &:= \text{rates of production of chemicals.} \end{aligned} \tag{4.5}$$

Let Ω be a smooth open and bounded subset of $\mathbb{R}^N$ with boundary $\partial\Omega = \Gamma$. We consider as domain for the operator $\mathcal{A}$ in (4.4) the following:

$$D(\mathcal{A}) = \left\{ \left(\begin{array}{c} z_1 \\ z_2 \\ z_3 \end{array}\right) \in H^2(\Omega) : \left(\begin{array}{c} d_1\partial_{\vec{n}} z_1 \\ d_2\partial_{\vec{n}} z_2 \\ d_3\partial_{\vec{n}} z_3 \end{array}\right) = \vec{0} \quad \text{on} \quad \Gamma \right\}, \tag{4.6}$$

where $\vec{n}$ is a unit normal vector pointing outwards of Γ. Still in (4.4), $P(u)U$ is a linearly coupled vector function, with, in the first component, featuring a divergence-0 operator acting on a vector field $\vec{d}$ of concentrations of chemicals, and in the second and third components, the productive effects on activated microglia cells.

In this chapter, we prove, in twofold, that the model system of equations (4.1)-(4.4) partial differential operator is an infinitesimal generator of an analytic semigroup acting on $U_0 \in Z_{\delta=\beta+2\gamma} = E^{\beta} \times E^{\gamma} \times E^{\gamma}$, where $E^{\alpha}, \alpha \in \mathbb{R}$ are scales of Banach spaces in $L^2(\Omega)$ defined by the operator in (4.4). In this context Section 4.2 gives some preliminaries, in addition to the ones given in Chapter 1. In Section 4.3, we prove that the system model equations (4.1)-(4.4) defines a perturbed analytic semigroup to the semigroup generated by the operator $-\mathcal{A}$, using abstract semigroup theory results for evolution equations from [30, 60, 66]. Section 4.4, is devoted to proving the existence of *a priori* uniform bounds in $\Omega \times (0, T)$ of solutions and gradient solutions to the problem. It concludes using a bootstrap argument in proving that the solutions to the problem are classical solutions. In Section 4.5, we revisit the complete system of equations coupled partial differential operator (i.e. in (4.1) we consider the contribution of the term $P(u)U$ of (4.4) appearing in the left hand side of the equations), to prove that it is an infinitesimal generator of a fundamental solution operator in scales of spaces $Z_{\delta}, \delta \in \mathbb{R}^{+}$, in as given by quasilinear partial differential operators. Since we are considering positive time, the results agree with and are much finer to those of Section 4.3. An immediate consequence of our results is that the large time asymptotic dynamics of the system of equations (4.1)-(4.4) are well-defined and captured by a subset K in $\mathbb{R}^3$ of spatial average solutions. This conclusion coincide with other well known results [53, 88, 59, 64, 76] related to the minimal chemotaxis model or Keller-Segel chemotactic problem.

In appreciation, the results of this chapter imply nonlinear diffusion, proliferation and death of cells can be incorporated into the system of equations. A proposition which agrees with the study given in [57], we suppose also that this citation is among others. In Section 4.6, to visualize the aggregation of microglia as in the model equations, we numerically simulate the equations using a Gradient Weighted Moving Finite Element method. For the simulations shown in this chapter we use the code developed in [81] using a set of model parameters found in [50], where the parameters used there are calculated from dimensional values found in Biology, Immunology and Neuroscience publications referenced therein. In Section 4.6 we discuss the results of the numerical simulation.

Lastly, we point out that throughout the chapter we work in a slightly general set-up., i.e. without loss of particularity, we do not immediately assume positivity of the initial data to the system of equations, which naturally imply positivity of the solutions. If positivity of solutions is assumed note that most of the calculations in Section 4.4 are very much simplified and are relatively easier.

4.2 Preliminaries

Now for a brief review of the functional setting. To this end, clearly by Lax-Milgram's Theorem [12, 70], $\mathcal{A}$ in (4.4) is a maximal monotone, self adjoint, sectorial operator in $L^2(\Omega)$ with spectrum

$$\sigma(\mathcal{A}) = \bigcup_{i=1}^{3} \sigma(-d_i\Delta + \lambda_i) = \{\mu_n; n \in \mathbb{N}\} \subset \mathbb{R}^+, \lambda_1 = 0, \tag{4.7}$$

such that

$$0 < \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \nearrow \infty \quad \text{as} \quad n \nearrow \infty, \text{ and } 0 \in \sigma(\mathcal{A}). \tag{4.8}$$

As $\mu \in \sigma(\mathcal{A})$, if for some $i = 1, 2, 3 = j \neq 1$, $\mu \in \sigma(-d_i\Delta + \lambda_j)$, we can choose associated eigenfunctions

$$\varphi_n = \varphi_n \cdot \vec{e}_i, \quad \text{where} \quad \{\vec{e}_i; i = 1, 2, 3\} \subset \mathbb{R}^3 \tag{4.9}$$

is a canonic basis of $\mathbb{R}^3$, orthonormal in $L^2(\Omega)$ and a Hilbert basis of this function space. Thus, by [5, 30, 60, 68] the scales of Banach spaces $E^\alpha, \alpha \in \mathbb{R}$ are well defined. Note that the spaces $E^\alpha, \alpha \in \mathbb{R}^-$ define the dual spaces of the scales of spaces $E^\alpha, \alpha \in \mathbb{R}^+$, and in equivalent of norms, we can identify the spaces

$$E^1 \equiv D(\mathcal{A}), \quad E^{1/2} \equiv H^1(\Omega) \quad \text{and} \quad E^0 \equiv L^2(\Omega), \quad E^{-1/2} \cong H^{-1}(\Omega).$$

In general, $E^\alpha \cong H^{2\alpha}(\Omega)$ and Sobolev type space embeddings [2, 10, 12, 25, 30, 68, 60],

$$E^\alpha \subset L^r(\Omega) \iff r \begin{cases} \leq & \infty \quad \text{if } N = 1 \\ < & \infty \quad \text{if } N = 2 \\ \leq & \frac{2N}{N-4\alpha} \quad \text{if } N \geq 3 \end{cases} \tag{4.10}$$

are satisfied. Also,

$$E^\alpha \subset C^\theta(\Omega), \quad \theta \in (0, 1) \iff 2\alpha - \frac{N}{2} > \theta. \tag{4.11}$$

In addition, it holds that for any $\alpha, \beta \in \mathbb{R}$,

$$\text{if } \alpha \geq \beta, \text{ then } E^\alpha \subset E^\beta \tag{4.12}$$

continuously, densely, compactly if $\alpha > \beta$, and constant of the inclusions is as given by (3.6). Furthermore, if $\alpha, \beta \in \mathbb{R}$ and $\theta \in [0, 1]$, then for every $u \in E^\gamma$, $\gamma = \max\{\alpha, \beta\}$ we have

$$\|u\|_{\theta\alpha + (1-\theta)\beta} \leq \|u\|_\alpha^\theta \|u\|_\beta^{1-\theta}. \tag{4.13}$$

Next, for every $\alpha, \varepsilon \in \mathbb{R}$, $\mathcal{A}^\varepsilon : E^{\alpha+\varepsilon} \to E^\alpha$ is a surjective isometry with $(\mathcal{A}^\varepsilon)^{-1} = \mathcal{A}^{-\varepsilon}$. Moreover, for every $\alpha, \beta, \gamma \in \mathbb{R}$, $\mathcal{A}^\alpha \mathcal{A}^\beta = \mathcal{A}^{\alpha+\beta}$ as operators between the spaces $E^{\alpha+\beta+\gamma}$

and E^{γ}. In particular, for every $\delta \in \mathbb{R}$ we can define the $\delta-$ product

$$\prec\prec u, v \succ\succ_{\delta} := \sum_{n=1}^{\infty} \mu_n^{\delta} u_n v_n \tag{4.14}$$

for every $\varepsilon \in \mathbb{R}$, $u \in E^{\delta-\varepsilon}, v \in E^{\varepsilon}$. Clearly, if $\alpha + \beta + 2\gamma = \delta$, then for every $u \in E^{\alpha+\gamma}$ and $v \in E^{\beta+\gamma}$,

$$\prec\prec u, v \succ\succ_{\delta} = \langle \mathcal{A}^{\alpha} u, \mathcal{A}^{\beta} v \rangle_{\gamma}$$

and the $0-$ product describes all the dualities between the E^{α} spaces, while the $\delta-$ describes among others the scalar product in $E^{\frac{\delta}{2}}$. Occasionally, we will use the notation

$$Z_{\delta} := E^{\alpha+\beta+2\gamma} = E^{\alpha+\gamma} \times E^{\beta+\gamma} = E^{\alpha} \times E^{\beta} \times E^{2\gamma}.$$

If there is no confusion caused we will simply write $\varphi \in E^{\alpha}$ with understanding that $\nabla\varphi \in E^{\alpha-\frac{1}{2}}$ whenever its derivatives are involved.

Now, recall from Chapter 1 that the operator $-\mathcal{A}$ generates an analytic semigroup

$$\left\{S(t) = e^{-\mathcal{A}t}; t \in \mathbb{R}^{+} \setminus \{0\}\right\} \tag{4.15}$$

in spaces $E^{\alpha}, \alpha \in \mathbb{R}$. We refer the reader to Section 1.3 for more on analytic semigoups.

Getting, back to (4.7)-(4.8) since $0 \in \sigma(\mathcal{A})$, if we take $V = (1,1,1)^{\top}$ in (4.1)-(4.4) as a test function, then integrating over Ω followed by over $(0,t)$, we get as $t \nearrow \infty$ that

$$\begin{aligned} U \;\; = \;\; (u,v,w)^{\top} \in K := \Bigg\{ & (\phi,\varphi,\psi) \in [L^1(\Omega)]^3 : \int_{\Omega} \phi dx = \int_{\Omega} \phi_0 = |\Omega|\overline{\phi}_0, \\ & \|(\varphi,\psi)\|_{L^1(\Omega)\times L^1(\Omega)} \leq \left(\frac{a_2}{\lambda_2} + \frac{a_3}{\lambda_3}\right)|\Omega|\overline{\phi}_0 \Bigg\}, \end{aligned} \tag{4.16}$$

which turns out [67, 86] to be a closely approximate limit set for the long time asymptotic dynamics of the system of equations in large diffusion. Throughout this chapter, all generic constants will be denoted by $C \geq 0$, unless a distinction is necessary.

4.3 Well-posedness of the system of equations

In this section, we first recall some abstract analytic semigroup theory results proved in [30, 51, 68, 60], in addition to the ones stated in Section 1.3. We will then prove the well-posedness of the problem (4.1)-(4.4) in the product scales of Banach spaces $Z_\delta, \delta \in \mathbb{R}^+$. To this end, consider the Cauchy problem

$$\begin{cases} \varphi_t + A\varphi &= f(t), \\ \varphi(t_0) &= \varphi_0 \in E^\beta, \end{cases} \tag{4.17}$$

where $f : [t_0, t_1) \to E^\beta$, $\beta \in \mathbb{R}$, and A is a maximal monotone, self adjoint and sectorial operator with compact resolvent in $L^2(\Omega)$. Then we have the following definition.

Definition 4.1. If $\varphi_0 \in E^\beta$, $\varphi(\cdot)$ is a strong solution of (4.17) on $[t_0, t_1)$ if and only if $\varphi : [t_0, t_1) \to E^\beta$ is a continuous function satisfying that $\varphi_t \in E^\beta$, $\varphi(t) \in E^{\beta+1}$ on (t_0, t_1), $\varphi(t_0) = \varphi_0$ and the differential equation in (4.17) is verified on the open interval (t_0, t_1) as an equality in $E^\beta, \beta \in \mathbb{R}$.

The well posedness of the evolution problem (4.17) is given in the following theorem.

Theorem 4.1. *Consider the Cauchy problem* (4.17)*, and assume that* $f \in L^p(t_0, t_1, E^\beta), 1 \leq p \leq \infty$. *Then, the solution to the problem* (4.17) *given by*

$$\varphi(t) = e^{-A(t-t_0)}\varphi_0 + \int_{t_0}^{t} e^{-A(t-s)} f(s) ds \tag{4.18}$$

satisfies that

(i) $\varphi \in C(t_0, t_1, E^\gamma)$ *with* $\gamma < \beta + \frac{1}{p'}$ *where* $\frac{1}{p} + \frac{1}{p'} = 1$. *If* $\varphi_0 \in E^\gamma$, *then* $\varphi \in C([t_0, t_1), E^\gamma)$, *and the mapping*

$$E^\gamma \times L^p(t_0, t_1, E^\beta) \ni (\varphi_0, f) \to \varphi \in C([t_0, t_1]; E^\gamma)$$

is Lipschitz continuous.

(ii) *For any $\beta \in \mathbb{R}$ and $\gamma \in [\beta, \beta+1)$, the mapping*

$$E^\gamma \times L^p(t_0, \infty, E^\beta) \ni (\varphi_0, f) \to \varphi \in L^p(t_0, \infty; E^\gamma)$$

is Lipschitz continuous. In particular, if $p = 2$, and $\gamma = \beta + \frac{1}{2}$, then, the mapping,

$$E^{\beta+\frac{1}{2}} \times L^2(t_0, t_1, E^\beta) \ni (\varphi_0, f)$$
$$\longrightarrow (\varphi, \varphi_t) \in \left(C([t_0, t_1], E^{\beta+\frac{1}{2}}) \cap L^2(t_0, t_1, E^{\beta+1})\right) \times L^2(t_0, t_1, E^\beta),$$

is continuous and the problem (4.17) is verified almost everywhere on (t_0, t_1).

(iii) *If $f : (t_0, t_1) \to E^\beta$ is locally Hölder continuous of exponent $0 < \theta \le 1$ and if*

$$\int_{t_0}^{t_0+\rho} \|f(s)\|_\beta ds < \infty, \quad \textit{for some} \quad \rho > 0$$

then φ in (4.18) is a unique solution of (4.17) such that

$$\varphi \in C([t_0, t_1), E^\beta) \cap C(t_0, t_1, E^{\beta+1}) \cap C^1(t_0, t_1, E^\gamma) \quad \textit{for any} \quad \gamma < \beta + \theta.$$

Proof. The proof of the theorem is classical [30, 51, 60, 68], with of most recently in [51] where Bessel potential function spaces have been used.

Thus in the case of (i) if we consider the formula (4.18) and let $\gamma \ge \beta$, then in estimating from above we get that

$$\|\varphi(t)\|_\gamma \le \|e^{-A(t-t_0)}\varphi_0\|_\gamma + \int_{t_0}^t \|e^{-A(t-s)}\|_{\beta,\gamma} \|f(s)\|_\beta ds$$

where $\|e^{-A(t-s)}\|_{\beta,\gamma}$ denotes the norm of $\mathcal{L}(E^\beta, E^\gamma)$. Since

$$\|e^{-A(t-s)}\|_{\beta,\gamma} \le \frac{M}{(t-s)^{\gamma-\beta}}$$

on finite time intervals, it follows, with $\gamma = \beta$ if $p = 1$ or with $\beta \le \gamma < \beta + \frac{1}{p'}$ if $1 < p < \infty$, that

$$\|\varphi(t)\|_\gamma \le \|e^{-A(t-t_0}\varphi_0\|_\gamma + b(t) \left(\int_{t_0}^t \|f\|_\beta^p\right)^{\frac{1}{p}}$$

where

$$b(t) = M\left(\int_{t_0}^{t} (t-s)^{-p'(\gamma-\beta)} ds\right)^{\frac{1}{p'}} \approx t^{\frac{1}{p'}-(\gamma-\beta)}.$$

So (4.18) is bounded on finite intervals. Consequently, $\varphi(t) \in E^{\gamma}$ for any $t > 0$. To prove the continuity, fix $t > t_0$ (or even $t = t_0$ if $\varphi_0 \in E^{\gamma}$). Then

$$\begin{aligned} &\|\varphi(t+h) - \varphi(t)\|_{\gamma} \\ &\qquad \leq \quad \|\left(e^{-Ah} - I\right)\varphi(t)\|_{\gamma} + \int_{t}^{t+h} \|e^{-A(t+h-s)}\|_{\beta,\gamma}\|f(s)\|_{\beta} ds. \end{aligned}$$

Since the linear semigroup is continuous, we have that

$$\|\left(e^{-Ah} - I\right)\varphi(t)\|_{\gamma} \to 0 \quad \text{as} \quad h \to 0,$$

while also

$$\begin{aligned} &\int_{t}^{t+h} \|e^{-A(t+h-s)}\|_{\beta,\gamma}\|f(s)\|_{\beta} ds \\ &\qquad \leq \quad M\left(\int_{t}^{t+h} (t+h-s)^{-p'(\gamma-\beta)} ds\right)^{\frac{1}{p'}} \left(\int_{t}^{t+h} \|f\|_{\beta}^{p}\right)^{\frac{1}{p}} = 0(h^{\frac{1}{p'}-(\gamma-\beta)}), \end{aligned}$$

and we obtain the continuity of (4.18). Furthermore, if $\varphi_0 \in E^{\gamma}$, then we have

$$\|\varphi(t)\|_{C[t_0,t_1],E^{\gamma})} \leq b(t_1)\left(\|\varphi_0\|_{\gamma} + \|f\|_{L^p(t_0,t_1,E^{\beta})}\right),$$

which proves the Lipschitz continuity of the mapping $(\varphi_0, f) \to \varphi$. The proof if $p = \infty$ follows the same lines with obvious modifications and therefore we shall skip it.

To prove (ii) of the theorem, note that for every $\beta \in \mathbb{R}$ and γ such that $\beta \leq \gamma < \beta + 1$, we have that,

$$c_{\beta,\gamma}(t) := \|e^{-At}\|_{\beta,\gamma} \leq \frac{Me^{-\omega t}}{t^{\gamma-\beta}}$$

and $c_{\beta,\gamma}(t) \in L^1(0,\infty)$ but unbounded at zero, unless $\gamma = \beta$. Let $p = 1$, $\varphi_0 \in E^{\gamma}$ and $f \in L^1(t_0,\infty,E^{\beta})$. Since

$$e^{-A(t-t_0)}\varphi_0 \in L^1(t_0,\infty,E^{\gamma}),$$

we just need to prove that

$$\psi(t) := \int_{t_0}^{t} e^{-A(t-s)} f(s)ds \in L^1(t_0, \infty, E^\gamma) = Z.$$

To this end, we set $s = (t - t_0)\sigma + t_0$, to get that

$$\psi(t) = \int_0^1 e^{-A(t-t_0)(1-\sigma)} f((t - t_0)\sigma + t_0)(t - t_0)d\sigma.$$

Therefore,

$$\|\psi\|_Z \leq \int_0^1 \|e^{-A(t-t_0)(1-\sigma)} f((t - t_0)\sigma + t_0)(t - t_0)\|_Z d\sigma.$$

But for any fixed $\sigma \in [0, 1]$,

$$\begin{aligned} &\|e^{-A(t-t_0)(1-\sigma)} f((t - t_0)\sigma + t_0)(t - t_0)\|_Z \\ &\qquad = \int_{t_0}^{\infty} \|e^{-A(t-t_0)(1-\sigma)} f((t - t_0)\sigma + t_0)(t - t_0)\|_\gamma dt. \end{aligned}$$

Therefore, setting $r = (t - t_0)\sigma + t_0$, we find that

$$\|\psi(t)\|_Z \leq \int_0^1 \int_{t_0}^{\infty} \frac{r - t_0}{\sigma^2} c_{\beta,\gamma} \left((r - t_0) \left(\frac{1-\sigma}{\sigma} \right) \right) \|f(r)\|_\beta dr d\sigma.$$

Again, letting $s = (r - t_0)\frac{(1-\sigma)}{\sigma}$, and integrating over σ we get that

$$\|\psi(t)\|_Z \leq \left(\int_0^{\infty} c_{\beta,\gamma}(s)ds \right) \left(\int_{t_0}^{\infty} \|f(r)\|_\beta dr \right),$$

yielding that

$$\|\varphi\|_Z \leq \|c_{\gamma,\gamma}\|_1 \|\varphi_0\|_\gamma + \|c_{\beta,\gamma}\|_1 \|f\|_{L^1(t_0,\infty,E^\beta)},$$

where $\|c_{\beta,\gamma}\|_1 = \|c_{\beta,\gamma}\|_{L^1(0,\infty)}$, and the result is proven.

The case of $p = \infty$, follows exactly as in (i), thus we have that

$$\|\varphi\|_{L^\infty(t_0,\infty,E^\gamma)} \leq \|c_{\gamma,\gamma}\|_\infty \|\varphi_0\|_\gamma + \|c_{\beta,\gamma}\|_1 \|f\|_{L^\infty(t_0,\infty,E^\beta)}.$$

Note that, in fact it holds that $\varphi \in C_b([t_0, \infty), E^\gamma)$. Now from what is proven above, we get by interpolation that the results are valid for any $1 < p < \infty$. We skip the proof of (iii) as it is exactly as in [30, pp. 50-52], with which the proof of the theorem is complete. □

Next, we consider the case of perturbations of analytic semigroups. For this, assume that $P \in \mathcal{L}_{lip}(E^\alpha, E^\beta), 0 \leq \alpha - \beta < 1$ and consider the evolution problem

$$\left\{ \begin{array}{rcl} u_t + Au &=& Pu, \\ u(t_0) &=& u_0 \in E^\beta, t_0 > 0. \end{array} \right. \tag{4.19}$$

Then, following [30, 51, 60, 66] abstract semigroup theory results for semilinear equations, let $Y \subset E^\alpha$ and $P : Y \to E^\beta$ be locally Lipschitz continuous. We define a solution to (4.19) as follows:

Definition 4.2. A continuous function $u : [t_0, t_1) \to E^\alpha$ satisfying that $u(t) \in E^\alpha$, $u(t_0) = u_0$, $u(t) \in E^{\beta+1}$, $u_t \in E^\beta$ on (t_0, t_1) and the evolution problem (4.19) holds on (t_0, t_1) as an identity in E^β, is called a strong solution to the problem (4.19).

On existence of solutions to (4.19) we have the following proposition.

Proposition 4.2. *Consider the problem* (4.19) *with* $P \in \mathcal{L}_{lip}(E^\alpha, E^\beta), 0 \leq \alpha - \beta < 1$, *and let* $u \in C([t_0, t_1), E^\alpha)$ *verify*

$$u(t, u_0) = e^{-A(t-t_0)}u_0 + \int_{t_0}^{t} e^{-A(t-s)}Pu(s)ds. \tag{4.20}$$

Then,

(i) $u \in C^\theta_{loc}(t_0, t_1, E^\alpha)$ *for some* $\theta \in (0, 1)$.

(ii) $u \in C([t_0, t_1), E^\alpha)$ *is a solution of the problem* (4.19) *if and only if* (4.20) *is verified.*

(iii) $u(t, u_0)$ *given by* (4.20) *is a* C^1 *strong solution of* (4.19) *in* E^β, *and*

(iv) $-A + P$ *is an infinitesimal generator of an analytic semigroup* $\{S_p(t); t > 0\}$ *in the spaces* E^β, *with* $\beta \in (\alpha - 1, \alpha]$.

Proof. See [66] Proposition 3.12 and Theorem 3.20. $\square$

A *priori* yielding the main theorem of this section, note that following Theorem 4.1 - Proposition 4.2 we look for a solution to the problem (4.1)-(4.4) of the form

$$U(t;U_0) = e^{-\mathcal{A}(t-t_0)}U_0 + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)}P(u(s))U(s)ds, \tag{4.21}$$

where

$$P(u) = \begin{pmatrix} 0 & -\mathrm{Div}(u\chi_2\nabla\cdot) & \mathrm{Div}(u\chi_3\nabla\cdot)) \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{pmatrix}, \tag{4.22}$$

so that

$$P(u)U := \begin{pmatrix} \Pi(u)(v,w) \\ a_2 u \\ a_3 u \end{pmatrix},$$

in which we have set $\Pi(u)(v,w) := -\mathrm{Div}(u\vec{d}(\nabla v, \nabla w))$ as in (4.4) and $U = (u,v,w)^\top$. It is also interesting to note that the system of equations (4.1)-(4.4) have nice regularity features debited to their nature of coupledness, see Remark 4.1. As for the system well-posedness we have the following theorem.

Theorem 4.3. *Consider the system of equations* (4.1)-(4.4) *for any* $\beta, \gamma \in \mathbb{R}$ *such that* $\beta \leq \gamma < \beta + 1$. *Assume that* $v_0, w_0 \in E^\gamma$ *and* $u \in C(t_0, t_1, E^\beta)$. *Then,* $v, w \in C(t_0, t_1, E^\gamma)$.

Conversely, for any $\alpha' \in \mathbb{R}$ *such that* $\alpha' \geq \alpha, 0 \leq \alpha' - \beta < 1$ *and* $2\alpha + \gamma \geq 1 + \frac{N}{4}$, *let* $u \in C(t_0, t_1, E^\alpha)$, $v, w \in C([t_0, t_1), E^\gamma)$. *Then,*

$$\Pi := Div : E^{\alpha'} \to E^\beta \quad \textit{is well defined, } \Pi(u) := Div(u\vec{d}(\nabla\cdot, \nabla\cdot)) \in \mathcal{L}_{lip}(E^{\alpha'}, E^\beta) \tag{4.23}$$

and the solution of (4.1)-(4.4), $u \in C(t_0, t_1, E^\beta)$. *If* $u_0 \in E^{\alpha'}$, *then*

$$U \in C([t_0, t_1), Z_{\alpha'(\gamma)}) \cap C(t_0, t_1, Z_{\beta+1(\gamma+1)}) \cap C^1(t_0, t_1, Z_\delta), \tag{4.24}$$

where $Z_{\delta(\nu)} := E^\delta \times E^\nu \times E^\nu$, *and* $Z_\delta := E^{\delta_0} \times E^{\delta_1} \times E^{\delta_1}$ *for any* $\delta_0 < \beta + 1, \delta_1 < \beta + \theta, \theta \in$

$(0,1)$. *Moreover,* (4.1)-(4.4) *defines a globally well-posed strong solution, which, if*

$$\Lambda = \max\left\{\{\chi_2,\chi_3\}\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}, \{a_2,a_3\}\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\right\} \le 1 \tag{4.25}$$

holds, is a perturbed analytic semigroup in the spaces $Z_{\delta(\nu)}, \delta(\nu) \in \mathbb{R}$ *satisfying* $\alpha'(\gamma) > \alpha'(\beta)$.

It is worthwhile pointing out that unlike in Proposition 4.2 the converse statements of this theorem require an additional condition to be verified i.e. $2\alpha + \gamma \ge 1 + \frac{N}{4}$ for the proper-posedness of the Div operator, which if the test function space, say E^ν, is chosen different from E^α, then this condition reads as $\nu + \alpha + \gamma \ge 1 + \frac{N}{4}$. Based on this condition α' in the theorem can always be associated adequately in such a way that $0 < \alpha - \beta < 1$. Most important is that in view of the experimental data given in the numerical section of the chapter the assumption (4.25) is not restrictive but consistent with that data.

Proof. The first part of the theorem follows by Theorem 4.1-(i). To prove the converse, let $\varphi \in E^\alpha$ be a test function to, for example, the operator $-\mathrm{Div}(u\chi_2\nabla v)$ in the scalar product of $L^2(\Omega)$. Then we conclude, using the Sobolev type embeddings (4.10) and Hölder's inequality, that the mapping

$$E^\alpha \times E^\gamma \times E^\alpha \ni (u,v,\varphi) \longmapsto \langle -\mathrm{Div}(u\chi_2\nabla v), \varphi\rangle = \chi_2 \int_\Omega u\nabla v\nabla\varphi \in \mathbb{R} \tag{4.26}$$

is well defined and continuous, provided $\alpha + \gamma \ge \frac{1}{2} + \frac{N}{4}$, $u\nabla v \in L^r(\Omega)$ with $r = 2$ if $\alpha \ge \frac{1}{2}$, and $r > 2$ if $\frac{1}{2} > \alpha$. Also that $-\mathrm{Div}(u\chi_2\nabla v) \in L^p(\Omega)$, with $p \ge \frac{2N}{N+4\alpha} \ge 2$ if $\alpha \le 0$. More concretely, as $u \in E^\alpha \subset L^{r_0}(\Omega)$, $\nabla v \in L^{r_1}(\Omega)$ then $u\nabla v \in L^2(\Omega)$ if and only if

$$\frac{1}{2} = \frac{1}{r_0} + \frac{1}{r_1} \ge \frac{N-4\alpha}{2N} + \frac{N-4\gamma+2}{2N} \Rightarrow N \ge 2N - 4(\alpha+\gamma) + 2 \tag{4.27}$$

of which we obtain $4(\alpha+\gamma) \ge 2 + N$, i.e. $\alpha + \gamma \ge \frac{1}{2} + \frac{N}{4}$. But also $\nabla\varphi \in E^{\alpha-\frac{1}{2}} \subset L^{r_2}(\Omega)$, $r_2 \ge 2$, which implies that

$$\frac{1}{2} \ge \frac{1}{r_2} \ge \frac{N-4\alpha+2}{2N} \Rightarrow N \ge N - 4\alpha + 2.$$

Consequently $\alpha \geq 1/2$ and from $\alpha + \gamma \geq \frac{1}{2} + \frac{N}{4}$ it is implied that $\gamma \geq \frac{N}{4}$. In the strict case, as by Hölder inequality we need $\frac{1}{r_2} + \frac{1}{r} = 1$, we get, using (4.10) embeddings, that $r \geq \frac{2N}{N+4\alpha-2}$ and $r > 2$ yields $2N > 2N + 8\alpha - 4$ of which as a result implies $1/2 > \alpha$. On the other hand, replacing $1/2$ by $1/r$ in (4.27) gives $2 < r \leq \frac{2N}{2N-4(\alpha+\gamma)+2}$ with yielding condition $\alpha + \gamma > \frac{1}{2} + \frac{N}{4}$. Thus taking into account either of the conditions on α leads to $\frac{2N}{N+4\alpha-2} \leq r \leq \frac{2N}{2N-4(\alpha+\gamma)+2}$, the yielding condition in the theorem $2\alpha + \gamma \geq 1 + \frac{N}{4}$ is obtained. The also part follows using (4.10) and Hölder's inequality directly from the inner product expression in (4.27) without passing the partial derivative to the test function.

Now considering (4.22) we get that $P(u)U \in L^p(\Omega) \times E^\alpha \times E^\alpha \subset E^{-\alpha'} \times E^\alpha \times E^\alpha = Z_{-\alpha'(\alpha)}$, for $p \geq 2$, $\alpha \geq 0$, and $U = (u, v, w)^\top \in Z_{\alpha(\gamma)} \subset Z_{\beta(\gamma)}$ since $0 \leq \alpha - \beta < 1$. Next if we let $V = (\phi, \varphi, \psi) \in Z_{\alpha(\gamma)}$ in the scalar product of $L^2(\Omega)$, thanks to the space embeddings (4.12), we get by Hölder's inequality that the mapping

$$V = (\phi, \varphi, \psi) \in Z_{\alpha(\gamma)} \longmapsto \langle P(u)U, V\rangle \in [L^1(\Omega)]^3 \tag{4.28}$$

is well defined and continuous. Therefore, linearity implies, for any $U_1, U_2 \in Z_{\alpha(\gamma)}$ of finite norm, that $P(u)U \in Z_{-\alpha'(\alpha)}$ is Lipschitz continuous. Thus Proposition 4.2 or abstract semilinear evolution equations results [30, 51, 60, 68] yield the conclusion of the theorem including (4.24). Moreover, see Theorem 4.8-(4.50)- (4.51) in the next sections.

$$\|P\|_{\mathcal{L}(Z_{\alpha(\gamma)}, Z_{-\alpha'(\alpha)})} := \left\{ |\langle P(u)U, U\rangle|; \|U\|_{\alpha(\gamma)} \leq 1 \right\} \leq \Lambda \tag{4.29}$$

and the solution to the problem (4.1)-(4.4) using (4.25) defines an analytic perturbed semigroup in the scales of spaces $Z_{\delta(\nu)}, \delta(\nu)) \in \mathbb{R}$ satisfying $\alpha'(\gamma) > \alpha'(\beta)$.

As the proofs are non-trivial due to the coupled nature of system of equations (4.1)-(4.4) we produce them for completeness in what follows. Assume $U \in C([t_0, t_1), Z_{\alpha(\gamma)})$ verifies (4.21). We show that $U : (t_0, t_1) \to Z_{\alpha(\gamma)}$ is locally Hölder continuous. To this end, let

$t_0 < t < t+h < t_1$. Then

$$U(t+h) - U(t) = \left(e^{-\mathcal{A}h} - I\right) e^{-\mathcal{A}(t-t_0)} U_0 \quad + \\ + \int_{t_0}^{t} \left(e^{-\mathcal{A}h} - I\right) e^{-\mathcal{A}(t-s)} P(u(s))U(s)ds + \int_{t}^{t+h} e^{-\mathcal{A}(t+h-s)} P(u(s))U(s)ds.$$

Since (4.29) holds using the semigroup estimates (1.4) we get that,

$$\begin{aligned} &\|e^{-\mathcal{A}(t+h-s)} P(u(s))U(s)\|_{\alpha(\gamma)} = \\ &\quad = \|e^{-\mathcal{A}(t+h-s)} \Pi(u)(v,w)(s))\|_{\alpha} + (a_2+a_3)\|e^{-\mathcal{A}(t+h-s)} u(s)\|_{\gamma} \\ &\quad \leq M(t-s)^{-(\alpha'-\alpha)} \|\Pi\|_{-\alpha',\alpha} \|(v,w)\|_{\gamma} + (a_2+a_3) M(t-s)^{-(\gamma-\alpha)} \|u(s)\|_{\alpha}, \end{aligned}$$

and, in addition, we also have

$$\begin{aligned} &\| \left(e^{-\mathcal{A}h} - I\right) e^{-\mathcal{A}(t-t_0)} U_0 \|_{\alpha(\gamma)} \\ &\quad \leq C_\varepsilon h^\varepsilon \left(\|e^{-\mathcal{A}(t-t_0)} u_0\|_{\alpha+\varepsilon} + \|e^{-\mathcal{A}(t-t_0)} (v_0,w_0)\|_{\gamma+\varepsilon} \right) \\ &\quad \leq M C_\varepsilon h^\varepsilon (t-t_0)^{-\varepsilon} \|U_0\|_{\alpha}, \end{aligned}$$

for some $\varepsilon \leq 1$.

Similarly, we get that

$$\begin{aligned} &\| \left(e^{-\mathcal{A}h} - I\right) e^{-\mathcal{A}(t-s)} P(u(s))U(s)\|_{\alpha(\gamma)} = \\ &= \| \left(e^{-\mathcal{A}h} - I\right) e^{-\mathcal{A}(t-s)} \Pi(u)(v,w)(s)\|_{\alpha} + \|(a_2+a_3) \left(e^{-\mathcal{A}h} - I\right) e^{-\mathcal{A}(t-s)} u(s)\|_{\gamma} \\ &\leq M C_\varepsilon h^\varepsilon \left((t-s)^{-(\alpha'+\varepsilon-\alpha)} \|\Pi\|_{-\alpha,\alpha} \|(v,w)\|_{\gamma} + (a_2+a_3)(t-s)^{-(\gamma+\varepsilon-\alpha)} \|u(s)\|_{\alpha} \right) \\ &\leq M C_\varepsilon h^\varepsilon \max\{(t-s)^{-(\alpha'+\varepsilon-\alpha)} \|\Pi\|_{-\alpha',\alpha}, (a_2+a_3)(t-s)^{-(\gamma-\alpha)}\} \left(\|(v,w)\|_{\gamma} + \|u(s)\|_{\alpha}\right) \\ &= M C_\varepsilon h^\varepsilon \max\{(t-s)^{-(\alpha'+\varepsilon-\alpha)} \|\Pi\|_{-\alpha',\alpha}, (a_2+a_3)(t-s)^{-(\alpha-\beta)}\} \|U\|_{\alpha(\gamma)}. \end{aligned}$$

As $\|U\|_{\beta(\gamma)}$ is bounded on proper subintervals of $[t_0, t_1]$ we get that $\|U(t+h) - U(t)\|_{\gamma(\beta)} = O(h^\theta)$, for some $\theta \in (0,1)$, on proper subintervals of $[t_0, t_1]$, and U given by (4.21) is locally Hölder continuous. In continuation, if U is a solution to the problem (4.1)-(4.4), as noted from Theorem 4.1 and Proposition 4.2, it verifies (4.21) and is a continuous function in Z_α.

Conversely from what was just proved in the above statements $U : (t_0, t_1) \to Z_\alpha$ is locally Hölder continuous, thus $f(t) = P(u(t))U(t) : (t_0, t_1) \to Z_{\beta(\gamma)}$ is locally Hölder continuous and integrable on the time interval. Consequently, Proposition 4.2 concludes and Theorem 4.1 yields the regularity results. As for the globally well-posedness of the semigroup, we note that from (4.21) it follows that

$$\begin{aligned}
\|U(t)\|_{\beta(\gamma)} &\leq \|e^{-\mathcal{A}t}U_0\|_{\beta(\gamma)} + \int_0^t \|e^{-\mathcal{A}(t-s)}P(u(s))U(s)\|_{\beta(\gamma)}ds \\
&\leq \frac{Me^{-\omega t}}{t^{\alpha-\beta}}\left[\|u_0\|_\alpha + \|(v_0, w_0)\|_{\alpha+\frac{1}{2}}\right] + M\max\{a_2 + a_3, \|\Pi(u)\|_{L^\infty(\mathbb{R}^+;\mathcal{L}(Z_\alpha, Z_{\beta(\gamma)}))}\}\times \\
&\quad \times \int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{\alpha-\beta}}\left(\|(v,w)\|_{\alpha+\frac{1}{2}} + \|u\|_\alpha\right) \\
&\leq \frac{Me^{-\omega t}}{t^{\alpha-\beta}}\|U_0\|_\alpha + M\max\{a_2 + a_3, \|\Pi(u)\|_{L^\infty(\mathbb{R}^+;\mathcal{L}(Z_\alpha, Z_{\beta(\gamma)}))}\}\times \\
&\quad \times \int_0^t \frac{e^{-\omega(t-s)}}{(t-s)^{\alpha-\beta}}\|U(s)\|_\alpha,
\end{aligned}$$

as long as $(t-s)^{-(\alpha+\frac{1}{2}-\gamma)} \leq (t-s)^{-(\alpha-\beta)}$ (with results valid in the inverse case), and the singular Gronwall-Henry inequality [30] concludes that $e^{\omega t}\|U(t)\|_{\beta(\gamma)} \leq ME_{1-\alpha+\beta}(\pi t)\|U_0\|_\alpha$ for $t > 0$, where $\pi = M\max\{a_2 + a_3, \|\Pi(u)\|_{L^\infty(\mathbb{R}^+;\mathcal{L}(Z_\alpha, Z_{\beta(\gamma)}))}\}\Gamma(1-\alpha+\beta)$.

The proof of the theorem is complete. $\square$

Now, for some remarks on the main condition yielding Theorem 4.3 we have the following.

Remark 4.1. First we note that if $\beta = 0$, we require that $\nabla v, \nabla w \in L^\infty(\Omega)$ i.e. $\gamma > \frac{1}{2} + \frac{N}{4}$ and since $\gamma < 1$ this implies solvability of the problem (4.1)-(4.4) in space dimensions of $\Omega \subset \mathbb{R}^N, N = 1$. Next, if in (4.23) we take $\alpha = \beta$, the necessary condition reads $2\beta+\gamma \geq 1+\frac{N}{4}$ but $\beta \leq \gamma < \beta+1$. If we assume $\gamma = \beta > 0$ we get that $3\beta \geq 1+\frac{N}{4}$ and if $\beta = \frac{1}{2}$ then $N \leq 2$. If $\gamma = \frac{3}{4} > \beta = \frac{1}{2}$ then $N \leq 3$. If $\gamma = \beta = \frac{3}{4}$ then $N \leq 5$. If $\gamma = \frac{5}{4} > \beta = \frac{3}{4}$ then $N \leq 7$. Thus the higher the regularity assumed on that data, the higher the space dimensions in which it is possible to solve the problem (4.1)-(4.4). Lastly, we note that if $2\beta + \gamma > \frac{3N}{4}$ then $Z_{\delta=\beta+\gamma} := E^\beta \times E^\gamma \subset C(\Omega) \times C(\Omega)$ using (4.11) and also if $2\beta + \gamma > \frac{1}{2} + \frac{3N}{4}$, then

$Z_{\delta=\beta+\gamma-\frac{1}{2}} = E^{\beta} \times E^{\gamma-\frac{1}{2}} \subset C(\Omega) \times C(\Omega)$. In both of these cases we can solve the problem in any space dimension.

We conclude this section with the following corollary.

Corollary 4.4. *Consider the system of equations* (4.1)-(4.4). *Assume the hypotheses of Theorem 4.3 holds within* (4.23)

$$\alpha = \theta\beta + (1-\theta)(\gamma - \frac{1}{2}),\ \theta \in [0,1]. \tag{4.30}$$

Then,

(i) (4.24) *holds in* $Z_{\beta} = E^{\beta} \times E^{\beta+\frac{1}{2}} \times E^{\beta+\frac{1}{2}}$.

(ii) *If* $2\beta + \gamma > \frac{3N}{4}$, *then the solution to the problem* (4.1)-(4.4), *satisfies*

$$U \in C(0,\infty, C^{\theta}(\Omega)) \cap C(0,\infty, C^{2+\theta}(\Omega)) \cap C^{1}(0,\infty, C^{\theta}(\Omega)),$$

for some $\theta \in (0,1)$ *and is a classical solution.*

Proof. To prove (i), it suffices to note that if α is as given in (4.30), then $\beta > \alpha$ if and only if $\beta > \gamma - \frac{1}{2}$, and also $\alpha > \beta$ if and only if $\gamma - \frac{1}{2} > \beta$. Combining the two we find that $\gamma = \beta + \frac{1}{2}$. We prove (ii) of this corollary in the next section of the paper. As an alternative, using a classical approach, since by (4.11), $U_0 \in L^{\infty}(\Omega)$, the conclusion follows by [5, 38, 51, 76, 88, 92] and Theorem 4.3. The proof of the corollary is complete. □

4.4 Uniform bounds of solutions

In this section, we study the existence of *a priori* uniform bounds in $\Omega \times (0,T)$ of solutions to the system of equations (4.1)-(4.4). As an approach to this end, we use the Moser-Nash-De Giorgi [3, 77, 44, 67] technique, and our first lemma is the following:

Lemma 4.5. *Consider the evolution problem* (4.1)-(4.4) *in context of the Theorem 4.3. Assume that the initial data of the system of equations* $U_0 = (u_0, v_0, w_0)^\top \in Z_{\beta(\gamma)} \cap [L^\infty(\Omega)]^3$, *and that* $u \in L^\infty(0, T, L^r(\Omega))$, *for some* $r > \frac{N}{2}$ *are finitely bounded in norms. Then* $v, w \in L^\infty(\Omega \times (0,T))$ *are also finitely bounded in norm of the given function space.*

Proof. It suffices only to consider one of either of the last two equations of (4.1)-(4.4) in v or w. We adopt here for simplification to use the function space $H^1(\Omega)$. Thus, considering the equation in v and taking the inner product of $L^2(\Omega)$ with $|v|^{r-1}v, r > 1$ we get that

$$\begin{aligned}\frac{1}{r+1}\frac{d}{dt}\int_\Omega |v|^{r+1} + \left(\frac{2\sqrt{d_2 r}}{r+1}\right)^2 \int_\Omega |\nabla |v|^{\frac{r+1}{2}}|^2 + \lambda_2 \int_\Omega |v|^{r+1} &\leq a_2 |\int_\Omega u|v|^{r-1}v| \\ \leq \quad a_2 \left(\int_\Omega |v|^{\frac{N(r+1)}{N-2}}\right)^{\Theta_1} \left(\int_\Omega |u|^r\right)^{\Theta_2} \left(\int_\Omega |v|^{r+1}\right)^{\Theta_3} \\ \leq a_2 C \left(\int_\Omega (|\nabla |v|^{\frac{r+1}{2}}|^2 + (|v|^{\frac{r+1}{2}})^2\right)^{\frac{N\Theta_1}{N-2}} \left(\int_\Omega |u|^r\right)^{\frac{2\Theta_2}{N-2}} \left(\int_\Omega |v|^{r+1}\right)^{\Theta_3},\end{aligned} \tag{4.31}$$

where in the second inequality above we have used the Nakao-Hölder -Sobolev inequality [3, 77], since there exists $\vartheta > 0$ such that $r = \frac{N}{2} + \vartheta$ and

$$\Theta_1 = \frac{N-2}{N+\vartheta}, \quad \Theta_2 = \frac{2}{N+\vartheta}, \quad \Theta_3 = \frac{\vartheta}{N+\vartheta},$$

the third inequality is due to Sobolev space embeddings [2, 12, 25], i.e. (4.10), in $\alpha = 1/2$.

In what follows we first note that $2r > r+1 > 2$, hence after multiplying throughout in (4.31) by $r+1$, and using in the right hand side Young's inequality [12, 77] i.e.

$$ab \leq \eta a^s + C_\eta b^{s'}, a, b \geq 0, \eta \in (0,1)$$

since

$$\frac{N\Theta_1}{N-2} = \frac{N}{N+\vartheta}, \frac{N\Theta_1}{N-2} + \Theta_3 = 1$$

we obtain if we let

$$\beta_0 = \inf\{\mu_1, 2d_2 - \eta, 2\lambda_2 - \eta\} > 0, \quad \mu_1 \in \sigma(-\Delta + 1), \quad C_{a_2} = a_2 C$$

that

$$\frac{d}{dt}\int_\Omega |v|^{r+1}+\beta_0\left(\int_\Omega |\nabla|v|^{\frac{r+1}{2}}|^2+\int_\Omega |v|^{r+1}\right)$$
$$\leq (2rC_{a_2})^{\frac{1}{\Theta_3}}\left(\sup_{(0,T)}\int_\Omega |u|^r\right)^{\frac{4}{\vartheta(N-2)}}\int_\Omega |v|^{r+1}\leq (2rC_{a_2})^{\frac{1}{\Theta_3}}\int_\Omega |v|^{r+1}, \tag{4.32}$$

since the term in brackets from the last inequality right to left is finitely bounded from above, and $\frac{1}{\Theta_3}>\frac{4}{\vartheta(N-2)}$ we have incorporated the bounding from above constant with and/or the given $C_{a_2}\geq 0$.

Therefore, if $r_i=2^i, i\in\mathbb{N}$, and

$$\Theta_i=\frac{2(r_i+1)}{N(r_i+1)-(N-2)(r_{i-1}+1)}, \Theta_i'=1-\Theta_i,$$

then by the Hölder's inequality as well as from the Sobolev type inclusions [2, 12, 25] and Young's inequality [12, 77], one obtains that

$$\begin{aligned}\int_\Omega |v|^{r_i+1} &\leq \left(\int_\Omega |v|^{\frac{N(r_i+1)}{N-2}}\right)^{\Theta_i'}\left(\int_\Omega |v|^{r_i+1}\right)^{\Theta_i}\\ &\leq C\left(\int_\Omega (|v|^{\frac{r_i+1}{2}})^2+\int_\Omega |\nabla|v|^{\frac{r_i+1}{2}}|^2\right)^{\frac{N\Theta_i'}{N-2}}\left(\int_\Omega |v|^{r_i+1}\right)^{\Theta_i}.\end{aligned}$$

Thus, from (4.32) while still setting $C_{a_2}=C_{a_2}C$, it follows that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |v|^{r_i+1}+\beta_0(\eta)\left(\int_\Omega |\nabla|v|^{\frac{r_i+1}{2}}|^2+\int_\Omega |v|^{r_i+1}\right)\\ &\quad\leq (2rC_{a_2})^{\frac{1}{\Theta_3}}\left(\int_\Omega (|v|^{\frac{r_i+1}{2}})^2+\int_\Omega |\nabla|v|^{\frac{r_i+1}{2}}|^2\right)^{\frac{N\Theta_i'}{N-2}}\left(\int_\Omega |v|^{r_i+1}\right)^{\Theta_i}\\ &\quad\leq \eta\left(\int_\Omega (|v|^{\frac{r_i+1}{2}})^2+\int_\Omega |\nabla|v|^{\frac{r_i+1}{2}}|^2\right)+(2r_iC_{a_2})^{\frac{N+2}{2\Theta_3}}\left(\int_\Omega |v|^{r_{i-1}+1}\right)^{s_i},\end{aligned}$$

and because $\frac{N\Theta_i'}{N-2}<1$, we have used Young's inequality [12, 77].

Now set $s_i=\frac{r_i+1}{r_{i-1}+1}$ and since $\frac{N+2}{2}\geq\frac{Nr_{i-1}+r_i+2}{r_i+2}$, we get

$$\frac{d}{dt}\int_\Omega |v|^{r_i+1}+\beta\left(\int_\Omega |\nabla|v|^{\frac{r_i+1}{2}}|^2+\int_\Omega |v|^{r_i+1}\right)\leq (2r_iC_{a_2})^{\sigma}\left(\int_\Omega |v|^{r_{i-1}+1}\right)^{s_i},$$

where $\sigma = \frac{N+2}{2\Theta_3}, \beta = \beta(\mu_1, d_2, \lambda_2, 2\eta) > 0$. Applying Poincaré inequality and defining $y_i(t) = \int_\Omega |v|^{r_i+1}$, we obtain

$$\frac{dy_i}{dt} + \beta y_i \leq (r_i C)^\sigma \left(y_{i-1}\right)^{s_i}. \tag{4.33}$$

If $M = M(\|v_0\|_\infty) > 0$ is such that $y_i(0) \leq M^{(r_{i-1}+1)s_i)}$. Then, solving (4.33), we get

$$y_i(t) \leq (r_i C)^\sigma \left(y_i(0) + \left(\sup_{t\in(0,T)} y_{i-1}(t) \right)^{s_i} \right).$$

We obtain from $(a+b)^p \leq 2^p(a^p + b^p), a, b, p \geq 0$ with $i = k \geq 1$ that

$$\begin{aligned}
y_k(t) &\leq (2C)^{1+2s_k+2s_{k-1}s_k+\ldots+2s_2s_3\ldots s_k}(2C)^{k\sigma+(k-1)\sigma s_k+\ldots+\sigma s_2 s_3\ldots s_k} M^{2s_1s_2\ldots s_k} + \\
&+ (2C)^{1+2s_k+2s_{k-1}s_k+\ldots+2s_2s_3\ldots s_k}(2C)^{k\sigma+(k-1)\sigma s_k+\ldots+\sigma s_2 s_3\ldots s_k} \left(\sup_{t\in(0,T)} \int_\Omega |v|^2 \right)^{s_1s_2\ldots s_k} \\
&\leq (2C)^{2A_k}(2C)^{\sigma B_k} M^{2\chi_k} + (2C)^{2A_k}(2C)^{\sigma B_k} \left(\sup_{t\in(0,T)} \int_\Omega |v|^2 \right)^{\chi_k},
\end{aligned}$$

where $\chi_k = s_1 \ldots s_k \leq \frac{r_k+1}{2}$,

$$\begin{aligned}
A_k &= 1 + s_k + s_k s_{k-1} + \ldots + s_k s_{k-1} \ldots s_1 \leq (r_k+1) \sum_{i=1}^\infty \frac{1}{r_i+1}, \\
B_k &= k + (k-1)s_k + (k-2)s_k s_{k-1} + \ldots + s_k s_{k-1} \ldots s_1 \leq (r_k+1) \sum_{i=1}^\infty \frac{i}{r_i+1},
\end{aligned}$$

and the series in the right hand sides converge since $r_i = 2^i$. We then let

$$\omega_1 = \sum_{i=1}^\infty \frac{1}{r_i+1}, \quad \omega_2 = \sum_{i=1}^\infty \frac{i}{r_i+1},$$

to conclude that

$$\begin{aligned}
y_k(t) &\leq \left((2C)^{2\omega_1}(2C)^{\sigma\omega_2} M + (2C)^{2\omega_1}(2C)^{\sigma\omega_2} \left(\sup_{t\in(0,T)} \int_\Omega |v|^2 \right)^{\frac{1}{2}} \right)^{r_k+1} \\
&\leq \left((2C)^{2\omega_1}(2C)^{\sigma\omega_2} M \left(\sup_{t\in(0,T)} \left(\int_\Omega |v|^2 \right)^{\frac{1}{2}} + 1 \right) \right)^{r_k+1}.
\end{aligned}$$

This implies

$$\begin{aligned}
\sup_\Omega |v(t, v_0)| &\leq \lim_{k\to 0} \left(\int_\Omega |v|^{r_k+1} \right)^{\frac{1}{r_k+1}} \\
&\leq (2C)^{2\omega_1}(2C)^{\sigma\omega_2} M \left(\sup_{t\in(0,T)} \left(\int_\Omega |v|^2 \right)^{\frac{1}{2}} + 1 \right),
\end{aligned}$$

and the proof of the lemma is complete. Note that this proof in general scales of spaces $E^\alpha, \alpha \in \mathbb{R}$ implies the results since $\gamma \geq \beta \geq 1/2$, but this will be more complicated. $\square$

Next we observe that due to the linearity of the system of equations in v, w and Lemma 4.5 we have, as a corollary, the following:

Corollary 4.6. *Consider the evolution problem* (4.1)-(4.4) *in the context of Theorem 4.3 and Lemma 4.5. Assume the initial data* $u_0 \in L^\infty(\Omega)$*,* $\nabla v_0, \nabla w_0 \in [L^\infty(\Omega)]^N$*, and that* $\nabla u \in L^\infty(0,T,L^r(\Omega)), r > \frac{N}{2}$ *are finitely bounded in norms of the given spaces. Then, the gradient solutions* $\nabla v, \nabla w \in L^\infty(\Omega \times (0,T))$*, and* $u \in L^\infty(\Omega \times (0,T)$ *are also finitely bounded in norms.*

Proof. It suffices to note that from one of v, w system equations of (4.1)-(4.4), if one differentiate these equations with respect to the x variable, and takes as test function say $|\nabla v|^{r-1}\nabla v \in H^1(\Omega), r > 1$. Then, Lemma 4.5 holds due to the linearity of these system equations and weak coupledness.

So we only need to prove that $u \in L^\infty(\Omega \times (0,T))$ is finitely bounded in norm. To this end, consider the u equation of the system (4.1)-(4.4) and take the inner product of $L^2(\Omega)$ with test function $|u|^{r-1}u \in H^1(\Omega), r > 1$ to find that

$$\begin{aligned}
&\frac{1}{(r+1)}\frac{d}{dt}\int_\Omega |u|^{r+1} + \left(\frac{2\sqrt{d_1 r}}{r+1}\right)^2 \int_\Omega |\nabla |u|^{\frac{r+1}{2}}|^2 \\
&\qquad = \chi_2 r \int_\Omega |u|^{r-1} u \nabla u \nabla v - \chi_3 r \int_\Omega |u|^{r-1} u \nabla u \nabla w \\
&\qquad \leq (C_1\chi_2 + C_2\chi_3) r \int_\Omega ||u|^r \nabla u| = \frac{2(C_1\chi_2 + C_2\chi_3)r}{r+1} \int_\Omega ||u|^{\frac{r+1}{2}} \nabla |u|^{\frac{r+1}{2}}|.
\end{aligned}$$

This yields that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega |u|^{r+1} + 2d_1 \int_\Omega |\nabla |u|^{\frac{r+1}{2}}|^2 &\leq 2(C_1\chi_2 + C_2\chi_3) r \int_\Omega ||u|^{\frac{r+1}{2}} \nabla |u|^{\frac{r+1}{2}}| \\
&\leq \eta \int_\Omega |\nabla |u|^{\frac{r+1}{2}}|^2 + \frac{(C_1\chi_2 + C_2\chi_3) r}{2\eta} \int_\Omega |u|^{r+1}, \qquad (4.34)
\end{aligned}$$

by Young's inequality

$$ab \leq \eta a^s + (\eta s)^{-\frac{s'}{s}} s'^{-1} b^{s'}, a, b \geq 0, \frac{1}{s} + \frac{1}{s'} = 1,$$

and if $\beta_0 := 2d_1 - \eta > 0$ or $0 < \eta \ll 1$ is adequately chosen, then Poincaré inequality implies that

$$\frac{d}{dt} \int_\Omega |u|^{r+1} + \beta \int_\Omega |u|^{r+1} \leq \frac{(C_1\chi_2 + C_2\chi_3)r}{2\eta} \int_\Omega |u|^{r+1}, \tag{4.35}$$

where for $\mu_1 \in \sigma(-\Delta)$, we defined $\beta := \mu_1\beta_0 > 0$.

From this point, one can proceed as in the proof of Lemma 4.5 to conclude that $u \in L^\infty(\Omega)$ is finitely bounded in norm.

Alternatively, we notice that by interpolation, for $\varphi \in H^1(\Omega)$, it holds that

$$\|\varphi - \overline{\varphi}\|^2_{L^2(\Omega)} \leq C\|\nabla\varphi\|^{2\theta}_{L^2(\Omega)}\|\varphi\|^{2(1-\theta)}_{L^1(\Omega)}, \tag{4.36}$$

where $\theta = \frac{N}{N+2}$, and Young's inequality yields

$$\|\varphi\|^2_{L^2(\Omega)} \leq \eta_0\|\nabla\varphi\|^2_{L^2(\Omega)} + C(1 + \eta_0^{-\frac{N}{2}})\|\varphi\|^2_{L^1(\Omega)}.$$

Consequently, setting $\varphi = |u|^{\frac{r+1}{2}}$, $\eta_0 = \frac{r}{(r+1)^2 C_\eta}$ with $C_\eta = \frac{(C_1\chi_2 + C_2\chi_3)}{2\eta}$, we obtain that

$$rC_\eta \int_\Omega |u|^{r+1} \leq \frac{r}{(r+1)^2} \int_\Omega |\nabla |u|^{\frac{r+1}{2}}|^2 + (2^2 C_\eta)^{\frac{N}{2}} C(1 + r^N) \left(\int_\Omega |u|^{\frac{r+1}{2}} \right)^2,$$

since $r > 1, 2r > r + 1$, and $0 < \eta \ll 1$ sufficiently small implies $C_\eta \gg 1$.

Therefore, $C_\eta \geq C$ and, because $1 + r^N \leq (1 + r)^N$, we obtain from (4.34) the following iterative inequality type of (4.35), with $\beta_0 - \frac{1}{4} > 0$,

$$\frac{d}{dt} \int_\Omega |u|^{r+1} + \beta \int_\Omega |u|^{r+1} \leq (2C_\eta)^N (1 + r)^N \left(\int_\Omega |u|^{\frac{r+1}{2}} \right)^2$$
$$\overset{implying}{\Longrightarrow} \quad \int_\Omega |u|^{r+1} \leq \int_\Omega u_0^{r+1} + (2C_\eta)^N (1 + r)^N \sup_{(0,T)} \left(\int_\Omega |u|^{\frac{r+1}{2}} \right)^2.$$

Next, defining

$$K(p) := \max \left\{ \|u_0\|_{L^\infty(\Omega)}, \sup_{(0,T)} \left(\int_\Omega |u|^p \right)^{\frac{1}{p}} \right\} \tag{4.37}$$

leads to that

$$K(r+1) \leq [(2C_\eta)^N (1+r)^N]^{\frac{1}{r+1}} K(\frac{r+1}{2}), \ \forall r \geq 1,$$

so that, if we let $r_i + 1 = 2^i, i \in \mathbb{N}^*$, then we conclude that

$$\begin{aligned} K(2^i) &\leq (2C_\eta)^{N2^{-i}} (2^i)^{\frac{N}{2^i}} K(2^{i-1}) \leq \ldots \leq (2C_\eta)^{N\sum_{k=1}^{i} 2^{-k}} (2^i)^{2^{-i}N} \ldots (1+2)^{2^{-1}N} K(1) \\ &\leq (2C_\eta)^N \left[2^{i2^{-i}N} (2^{-i})^{2^{-i}N}\right] \ldots\ldots \left[2^{2^{-1}N} (2^{-1})^{2^{-1}N}\right] K(1) \\ &\leq (2C_\eta)^N 2^{N\sum_{k=1}^{i} k2^{-k}} \times 2^{N\sum_{k=1}^{i} 2^{-k}} K(1) \leq C2^{3N} K(1). \end{aligned}$$

Consequently, taking the limit as $i \to \infty$ yields

$$\|u\|_{L^\infty(\Omega)} \leq C2^{3N} K(1) \leq C2^{3N} \max\left\{\|u_0\|_{L^\infty(\Omega)}, \|u_0\|_{L^1(\Omega)}\right\} < \infty,$$

and the proof of the corollary is complete. □

The following is a particular converse lemma to Corollary 4.6, since its conclusion holds for all $r \in [2, \infty]$.

Lemma 4.7. *Consider the evolution problem* (4.1)-(4.4). *Assume that the hypotheses of Corollary 4.6 hold, and that* $\nabla u_0 \in L^\infty(\Omega)$ *is finitely bounded in norm. If* $\nabla v, \nabla w \in L^\infty(\Omega \times (0,T))$ *are finitely bounded in norm, then* $\nabla u \in L^\infty(0, T, L^r(\Omega)), \forall r > \frac{N}{2}$ *is also finitely bounded in norm.*

Proof. Differentiate the system equation in u with respect to x. Then, take the inner product of $L^2(\Omega)$ with the test function $|\nabla u|^r \nabla u \in H^1(\Omega), r \geq 0$ to find that

$$\begin{aligned}
&\frac{1}{r+2}\frac{d}{dt}\int_\Omega |\nabla u|^{r+2} + \frac{4d_1(r+1)}{(r+2)^2}\int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}}|^2 \\
&\quad = \frac{r+2}{2}\chi_2 \int_\Omega \left(|\nabla u|^{\frac{r+2}{2}} \nabla v \Delta u + u|\nabla u|^{\frac{r+2}{2}-2}\nabla u \Delta u \Delta v\right) - \\
&\quad - \frac{r+2}{2}\chi_3 \int_\Omega \left(|\nabla u|^{\frac{r+2}{2}} \nabla w \Delta u + u|\nabla u|^{\frac{r+2}{2}-2}\nabla u \Delta u \Delta w\right) \\
&\quad \leq \frac{r+2}{2}\chi_2 \left(\int_\Omega ||\nabla u|^{\frac{r+2}{2}} \nabla v \Delta u| + \int_\Omega |u|\nabla u|^{\frac{r+2}{2}-2}\nabla u \Delta u \Delta v|\right) + \\
&\quad + \frac{r+2}{2}\chi_3 \left(\int_\Omega ||\nabla u|^{\frac{r+2}{2}} \nabla w \Delta u| + \int_\Omega |u|\nabla u|^{\frac{r+2}{2}-2}\nabla u \Delta u \Delta w|\right) \\
&\quad \leq \frac{r+2}{2}\left(C_\chi \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})\nabla u| + C_u \int_\Omega ||\nabla u|^{\frac{r}{2}} \Delta u(\Delta v| + \Delta w|)\right),
\end{aligned} \tag{4.38}$$

where $C_\chi = (\chi_2 C_{\nabla v} + \chi_3 C_{\nabla w}) \geq 0$, with $C_{\nabla v}, C_{\nabla w}, C_u$ constants for the upper bounds of the variables in $L^\infty(\Omega)$. Multiplying throughout by $r+2$, and since $\frac{1}{2} + \frac{1}{r+2} \leq 1$, for any $r \geq 0$, we get, by Holder's inequality, that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla u|^{r+2} + 2d_1 \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}}|^2 \\
&\leq \quad \frac{(r+2)^2}{2}\left(C_\chi \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})\nabla u| + C_u \int_\Omega ||\nabla u|^{\frac{r}{2}} \Delta u(\Delta v| + \Delta w|)\right) \\
&\leq \frac{(r+2)^2}{2}\left(C_\chi \left(\int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^{r+2}\right)^{\frac{1}{r+2}} \quad + \right. \\
&+ \quad \left. \frac{2C_u}{r+2}\left(\int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2\right)^{\frac{1}{2}} \left[\left(\int_\Omega |\Delta v|^2\right)^{\frac{1}{2}} + \left(\int_\Omega |\Delta w|^2\right)^{\frac{1}{2}}\right]\right).
\end{aligned} \tag{4.39}$$

We recall at this point the Young's inequality

$$ab \leq \eta a^s + \eta^{-\frac{s'}{s}} s'^{-1} b^{s'}, a, b \geq 0, \eta \in (0,1), \frac{1}{s} + \frac{1}{s'} = 1$$

with which, if we let $\eta_1 := \frac{2\eta_2}{(r+2)^2 C_\chi}$, where by Nirenberg-Gagliardo's inequality [30, 25] $0 < \eta_2 \leq 1$ and

$$\int_\Omega |\nabla u|^{r+2} \leq \eta_2 \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2 + C_\Omega \eta_2^{-m}\left(\int_\Omega |\nabla u|^{\frac{r+2}{2}}\right)^2,$$

for $m > \frac{N}{2}$, $C_\Omega = C(\Omega, m)$, $\eta_3 := \frac{16\eta_2^2}{(r+2)^4 C_\chi^2 |\Omega|^{1-\frac{2}{r+2}}}$, then we get that

$$
\begin{aligned}
\frac{(r+2)^2}{2} C_\chi \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})\nabla u| &\leq \frac{(r+2)^2}{2} C_\chi \left(\int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} \\
&\leq \frac{(r+2)^2}{2} C_\chi \left(\eta_1 \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2 + (4\eta_1)^{-1} \int_\Omega |\nabla u|^2\right) \\
&\leq \eta_2 \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2 + \frac{(r+2)^2}{2} C_\chi |\Omega|^{1-\frac{2}{r+2}} \left(\frac{8\eta_2}{(r+2)^2 C_\chi}\right)^{-1} \int_\Omega |\nabla u|^{r+2} \\
&\leq 2\eta_2 \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2 + \frac{(r+2)^2}{2} C_\chi |\Omega|^{1-\frac{2}{r+2}} \left(\frac{8\eta_2}{(r+2)^2 C_\chi}\right)^{-1} C_\Omega (\eta_3)^{-m} \times \\
&\quad \times \left(\int_\Omega |\nabla u|^{\frac{r+2}{2}}\right)^2 \\
&= 2\eta_2 \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2 + (r+2)^{4m} \Gamma_1 C_\Omega \left(\int_\Omega |\nabla u|^{\frac{r+2}{2}}\right)^2,
\end{aligned}
$$

where $\Gamma_1 = \left(\frac{C_\chi^2 |\Omega|^{1-\frac{2}{r+2}}}{16\eta_2^2}\right)^{m+1}$.

As for the last expression in (4.39), we need a control from above of the integrals involving $-\Delta$ of v, w. To this end, multiplying either stationary equations in v or w by $-\Delta$ of the variable, to obtain that

$$
\begin{aligned}
d_2 \int_\Omega |\Delta v|^2 + \lambda_2 \int_\Omega |\nabla v|^2 &= a_2 \int_\Omega \nabla u \nabla v \leq a_2 \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}} \left(\int_\Omega |\nabla v|^2\right)^{\frac{1}{2}} \\
&\leq \eta_4 \int_\Omega |\nabla u|^2 + a_2 (4\eta_4)^{-1} \int_\Omega |\nabla v|^2 \leq \eta_4 |\Omega|^{1-\frac{2}{r+2}} \int_\Omega |\nabla u|^{r+2} + a_2 (4\eta_4) C_{\nabla v}^2 |\Omega| \\
&\leq \eta_4 \frac{|\Omega|^{1-\frac{2}{r+2}}}{\mu_1} \int_\Omega |\nabla(|\nabla u|^{\frac{r+2}{2}})|^2 + a_2 (4\eta_4)^{-1} C_{\nabla v}^2 |\Omega|.
\end{aligned}
$$

Note that this remains true even if one had considered the entire equation involving the time derivative, since by Theorem 4.3, the solutions are continuous in time. Let $\eta_4 \leq \eta_5$ and set

$$
\eta_6 := \frac{1}{(r+2)C_u} \left(2\eta_5 + \frac{\eta_4}{2\mu_1 \eta_5} \left(\frac{1}{d_2} + \frac{1}{d_3}\right) |\Omega|^{1-\frac{2}{r+2}}\right).
$$

Then we find that

$$
\begin{aligned}
&(r+2)C_u\left(\int_\Omega|\nabla(|\nabla u|^{\frac{r+2}{2}})|^2\right)^{\frac{1}{2}}\left[\left(\int_\Omega|\Delta v|^2\right)^{\frac{1}{2}}+\left(\int_\Omega|\Delta w|^2\right)^{\frac{1}{2}}\right]\\
&\quad\le(r+2)C_u\left(2\eta_5\int_\Omega|\nabla(|\nabla u|^{\frac{r+2}{2}})|^2+\frac{1}{4\eta_5}\left(\int_\Omega|\Delta v|^2+\int_\Omega|\Delta w|^2\right)\right)\\
&\quad\le(r+2)C_u\left(\left(2\eta_5+\frac{\eta_4}{2\mu_1\eta_5}\left(\frac{1}{d_2}+\frac{1}{d_3}\right)|\Omega|^{1-\frac{2}{r+2}}\right)\int_\Omega|\nabla(|\nabla u|^{\frac{r+2}{2}})|^2\quad+\right.\\
&\qquad\quad+\quad\left.\left(\frac{a_2(C_{\nabla v}+C_{\nabla w})|\Omega|}{16\eta_5\eta_4}\right)\right)\\
&\quad\le\quad\eta_6\int_\Omega|\nabla(|\nabla u|^{\frac{r+2}{2}})|^2+\left(\frac{a_2(r+2)(C_{\nabla v}+C_{\nabla w})|\Omega|C_u}{16\eta_4^2}\right).
\end{aligned}
$$

Thus, from (4.39), if we let $\eta_7=2\eta_2+\eta_6$, $\Gamma_2=\frac{a_2(C_{\nabla v}+C_{\nabla w})|\Omega|C_u}{16\eta_4^2}$ and $\Gamma=\max\{\Gamma_1,\Gamma_2\}$, then we are led to conclude that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega|\nabla u|^{r+2}+2d_1\int_\Omega|\nabla(|\nabla u|^{\frac{r+2}{2}}|^2\\
&\qquad\le\eta_7\int_\Omega|\nabla(|\nabla u|^{\frac{r+2}{2}})|^2+(r+2)^{4m}\Gamma\left(\left(\int_\Omega|\nabla u|^{\frac{r+2}{2}}\right)^2+1\right)
\end{aligned}
$$

$$
\overset{implying}{\Longrightarrow}\ \frac{d}{dt}\int_\Omega|\nabla u|^{r+2}+\beta\int_\Omega|\nabla u|^{r+2}\le(r+2)^{4m}\Gamma\left(\left(\int_\Omega|\nabla u|^{\frac{r+2}{2}}\right)^2+1\right)\tag{4.40}
$$

$$
\overset{implying}{\Longrightarrow}\ \int_\Omega|\nabla u|^{r+2}\le\int_\Omega|\nabla u_0|^{r+2}+(r+2)^{4m}\Gamma\left(\left(\sup_{(0,T)}\int_\Omega|\nabla u|^{\frac{r+2}{2}}\right)^2+1\right),
$$

following a use of Poincaré inequality and that $\beta:=2d_1-\eta_7>0$.

Next, proceeding as either in the proof of Lemma 4.5 or as in the proof of Corollary 4.6, yields that $\nabla u\in L^\infty(\Omega\times(0,T))$ is finitely bounded in norm.

To complete the ideas, consider (4.37) in gradient functions and take $p=r+2$, to get that

$$
K(r+2)\le[\Gamma(r+2)^{4m}]^{\frac{1}{r+2}}K(\frac{r+2}{2}),\forall r\ge0.
$$

Now, if we let $r_i + 2 = 2^i, i \in \mathbb{N}$, then we obtain that

$$\begin{aligned} K(2^i) &\leq \Gamma^{2^{-i}}(2^i)^{4m2^{-i}} K(2^{i-1}) \leq \ldots \leq \Gamma^{\sum_{k=1}^{i} 2^{-k}} (2^i)^{2^{-i}4m} \ldots (2)^{2^{-1}4m} K(1) \\ &\leq \Gamma \left[2^{i2^{-i}4m} (2^{-i})^{2^{-i}4m} \right] \ldots\ldots \left[2^{2^{-1}4m} (2^{-1})^{2^{-1}4m} \right] K(1) \\ &\leq \Gamma 2^{4m \sum_{k=1}^{i} k2^{-k}} \times 2^{4m \sum_{k=1}^{i} 2^{-k}} K(1) \leq C 2^{12m} K(1). \end{aligned}$$

Thus taking the limit as $i \to \infty$ yields

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C 2^{12m} K(1) \leq C 2^{12m} \max \left\{ \|\nabla u_0\|_{L^\infty(\Omega)}, \|\nabla u_0\|_{L^1(\Omega)} \right\} < \infty,$$

and the proof of the lemma is complete. □

Now we prove (ii) of Corollary 4.4.

Proof. We use a bootstrap argument. By Theorem 4.3 taking into account that the space inclusions (4.11) imply $E^\beta, E^\gamma \subset C(\Omega)$, we get that $u \in C(\Omega \times (0, T))$. Thus, viewing either equations in v or w variables, we get for example that $v_t \in L^\infty(\Omega)$, consequently $g(t) := a_2 u - v_t \in L^p(\Omega)$ for all $p \geq 1$. Thus, $v \in W^{2,p}(\Omega)$ for all $p \geq 1$ and $\nabla v \in W^{1,p}(\Omega)$ for all $p \geq 1$ In particular, $\nabla v \in W^{1,p}(\Omega)$ for some $p > N$, yielding $\nabla v \in C^\theta(\Omega)$, for some $\theta > 0$.

In fact, if $p < N$, as $\nabla v \in W^{1,p}(\Omega) \subset L^{q_1}(\Omega)$, $q_1 = \frac{pN}{N-p}$ if $p > \frac{N}{2}$, then $q_1 > N$ and the above statements hold. If not we repeat the process, with $W^{1,p}(\Omega) \subset L^{q_2}(\Omega)$, $q_2 = \frac{q_1 N}{N - q_1} = \frac{pN}{N-2p}$ and if $p > \frac{N}{3}$ we are done as $q_2 > N$. In the otherwise case we repeat the iterative process to find $q_m = \frac{q_{m-1}N}{N - q_{m-1}} = \frac{pN}{N-mp}$ and if $p > \frac{N}{m+1}$ then we are done. Thus in a finite number of steps it is always possible to get $q_m > N$ and the above Hölder smoothness of gradient solutions are obtained.

The next immediate result from Corollary 4.6 is also that $u \in L^\infty(\Omega)$. Now, if $\nabla u_0 \in L^\infty(\Omega)$, then Lamma 4.7 implies that $\nabla u \in L^\infty(\Omega)$. Furthermore,

$$f(t) = -\operatorname{div}(u\vec{d}(\nabla v, \nabla w)) \in W^{2,p}(\Omega), \quad \forall\, p > 1.$$

Thus viewing the equation in u as an elliptic problem, we get $\nabla u \in W^{1,p}(\Omega) \subset C^{\theta}(\Omega)$ for some $\theta > 0$, since, in particular, using a bootstrap iteration argument as in above lines $W^{1,p}(\Omega) \subset L^{q_m}(\Omega)$, it is possible to get $q_m > N$, provided that $p > \frac{N}{m+1}$. Consequently, $u \in C^{2+\theta}(\Omega)$. Getting back to the v equation, we obtain $g(t) \in C^{\theta}(\Omega)$ and so $v \in C^{2+\theta}(\Omega)$ for some $\theta > 0$. By similarity, of the equations we also have $w \in C^{2+\theta}(\Omega)$ for some $\theta > 0$.

Combining all of the above, we conclude that the solution to the problem (4.1)-(4.4) verifies regularity properties given in Corollary 4.4, and that it is a classical solution. $\square$

4.5 Equations in system coupled elliptic differential operator

In this section, we view the problem (4.1)-(4.4) in the form

$$\begin{cases} U_t + \mathcal{A}(t)U &= \vec{0} \\ U(0) &= U_0 \in E^{\beta} \times E^{\gamma} \times E^{\gamma}, \\ 1/2 \leq \beta &\leq \gamma < \beta + 1 \end{cases} \tag{4.41}$$

where $\mathcal{A}(t) = \mathcal{A}(u)$ is the coupled elliptic partial differential operator associated with the problem by passing all terms in the right hand side to the left hand side of the system of equations i.e.

$$\begin{aligned} \mathcal{A}(u) &= \begin{pmatrix} -d_1\Delta & \mathrm{Div}(u\chi_2\nabla\cdot) & -\mathrm{Div}(u\chi_3\nabla\cdot)) \\ -a_1 & -d_2\Delta + \lambda_2 & 0 \\ -a_2 & 0 & -d_3\Delta + \lambda_3 \end{pmatrix} \\ &= \begin{pmatrix} -d_1\Delta & 0 & 0 \\ 0 & -d_2\Delta + \lambda_2 & 0 \\ 0 & 0 & -d_3\Delta + \lambda_3 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{Div}(u\chi_2\nabla\cdot) & -\mathrm{Div}(u\chi_3\nabla\cdot)) \\ -a_2 & 0 & 0 \\ -a_3 & 0 & 0 \end{pmatrix} \\ &= \mathcal{A} - P(u). \end{aligned} \tag{4.42}$$

Now, if in (4.42) we let the left hand side of the operator be a function of $\Theta \in E^\beta$, and set $U = (u, v, w)^\top$ then

$$\mathcal{A}(\Theta)U = \begin{pmatrix} -d_1\Delta u + \mathrm{Div}(\Theta\chi_2\nabla v) - \mathrm{Div}(\Theta\chi_3\nabla w) \\ -d_2\Delta v + \lambda_2 v - a_2 u \\ -d_3\Delta w + \lambda_3 w - a_3 u \end{pmatrix}.$$

Consequently, if we define

$$B : Z_{\gamma(\beta)} \times Z_{\gamma(\beta)} \to \mathbb{R},\ Z_{\gamma(\beta)} := E^\beta \times E^\gamma \times E^\gamma,\ 2\beta + \gamma \geq 1 + \frac{N}{4} \tag{4.43}$$

by

$$\begin{aligned} B(\Theta; U, V) &:= \langle A(\Theta)U, V\rangle = \langle \mathcal{A}U, V\rangle + \langle P(\Theta)U, V\rangle \\ &= d_1\int_\Omega \nabla u\nabla\phi + d_2\int_\Omega \nabla v\nabla\varphi + d_3\int_\Omega \nabla w\nabla\psi - \chi_2\int_\Omega \Theta\nabla v\nabla\phi + \\ &\quad + \chi_3\int_\Omega \Theta\nabla w\nabla\phi + \lambda_2\int_\Omega v\varphi + \lambda_3\int_\Omega w\psi - a_2\int_\Omega u\varphi - a_3\int_\Omega u\psi \end{aligned} \tag{4.44}$$

where $V = (\phi, \varphi, \psi)^\top \in Z_{\gamma(\beta)}$, then w have the following theorem:

Theorem 4.8. *Let $\Theta \in E^\beta$ be fixed. Then, there exist constants*

$$\begin{cases} M(\|\Theta\|_\beta) &= \max\Big\{d_1\left(\frac{2}{Ne\pi}\right)^{2\beta-1}, d_2\left(\frac{2}{Ne\pi}\right)^{2\gamma-1} + \lambda_2\left(\frac{2}{Ne\pi}\right)^{2\gamma}, d_3\left(\frac{2}{Ne\pi}\right)^{2\gamma-1} + \\ &\quad \lambda_3\left(\frac{2}{Ne\pi}\right)^{2\gamma}, \{\chi_2, \chi_3\}\,\|\Theta\|_\beta\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}, \{a_2, a_3\}\,\|\Theta\|_\beta\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\Big\} > 0, \\ \omega(\|\Theta\|_\beta) &= \min\Big\{\big\{d_1, d_i + \lambda_i\frac{2}{Ne\pi} : i = 2, 3\big\} - 2\Lambda_1\|\Theta\|_\beta\Big\} > 0, \end{cases} \tag{4.45}$$

where $\Lambda_1 = \max\Big\{\{\chi_2, \chi_3\}\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}, \{a_2, a_3\}\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\Big\} > 0$, *such that*

(i) $|B(\Theta; U, V)| \leq M(\|\Theta\|_\beta)\|U\|_{\gamma(\beta)}\|V\|_{\gamma(\beta)}$,

(ii) $B(\Theta; U, U) \geq \omega(\|\Theta)\|_\beta)\|U\|^2_{\gamma(\beta)}$,

(iii) $\langle \mathcal{A}(\Theta)U - \mathcal{A}(\Theta)V, U - V\rangle > 0, \forall U, V \in Z_{\gamma(\beta)}$.

Moreover, for fixed $(U, F) \in Z_{\gamma(\beta)} \times [Z_{\gamma(\beta)}]^*$ arbitrary, and if we consider (4.44) for any $V \in Z_{\gamma(\beta)}$, then

(iv) *$\mathcal{A}(\Theta)U = F \in [Z_{\gamma(\beta)}]^*$ has one and only one solution $U = T_F(\Theta) \in Z_{\gamma(\beta)}$,*

(v) *$\mathcal{A}(\Theta)U$ depends continuously on Θ for each $U \in Z_{\gamma(\beta)}$ fixed.*

(vi) *$T_F(\cdot) \in \mathcal{L}(Z_{\gamma(\beta)})$ is well-posed and $U = T_F(u)$ is a unique solution of $\mathcal{A}(u)U = F \in [Z_{\gamma(\beta)}]^*$.*

(vii) *$T_F(\cdot) \in \mathcal{K}([Z_{\gamma(\beta)}]^*, Z_{\gamma(\beta)})$ is a compact operator.*

Proof. First we notice that for any $\beta, \gamma \geq 1/2$, by Sobolev type space embeddings [5, 30, 60, 66], i.e. (4.10), the mapping

$$(U, V) \ni Z_{\gamma(\beta)} \times Z_{\gamma(\beta)} \to \langle \mathcal{A}U, V\rangle \in L^1(\Omega) \tag{4.46}$$

is well defined and continuous. In fact, it holds that

$$\begin{aligned}
|\langle \mathcal{A}U, V\rangle| \leq\ & d_1\|\nabla u\|_{L^{r_1}(\Omega)}\|\nabla \phi\|_{L^{r_1}(\Omega)} + d_2\|\nabla v\|_{L^{r_2}(\Omega)}\|\nabla \varphi\|_{L^{r_2}(\Omega)} + \\
& +d_3\|\nabla w\|_{L^{r_2}(\Omega)}\|\psi\|_{L^{r_2}(\Omega)} + \lambda_2\|v\|_{L^{r_3}(\Omega)}\|\varphi\|_{L^{r_3}(\Omega)} + \\
& +\lambda_3\|w\|_{L^{r_3}(\Omega)}\|\psi\|_{L^{r_3}(\Omega)} \\
\leq\ & d_1\left(\frac{2}{Ne\pi}\right)^{2\beta-1}\|u\|_{\beta-\frac{1}{2}}\|\phi\|_{\beta-\frac{1}{2}} + d_2\left(\frac{2}{Ne\pi}\right)^{2\gamma-1}\|v\|_{\gamma-\frac{1}{2}}\|\varphi\|_{\gamma-\frac{1}{2}} \\
& +d_3\left(\frac{2}{Ne\pi}\right)^{2\gamma-1}\|w\|_{\gamma-\frac{1}{2}}\|\psi\|_{\gamma-\frac{1}{2}} + \lambda_2\left(\frac{2}{Ne\pi}\right)^{2\gamma}\|v\|_{\gamma}\|\varphi\|_{\gamma} + \\
& +\lambda_3\left(\frac{2}{Ne\pi}\right)^{2\gamma}\|w\|_{\gamma}\|\psi\|_{\gamma} \\
\leq\ & d_1\left(\frac{2}{Ne\pi}\right)^{2\beta-1}\|u\|_{\beta}\|\phi\|_{\beta} + d_2\left(\frac{2}{Ne\pi}\right)^{2\gamma-1}\|v\|_{\gamma}\|\varphi\|_{\gamma} + \qquad (4.47)\\
& +d_3\left(\frac{2}{Ne\pi}\right)^{2\gamma-1}\|w\|_{\gamma}\|\psi\|_{\gamma} + \lambda_2\left(\frac{2}{Ne\pi}\right)^{2\gamma}\|v\|_{\gamma}\|\varphi\|_{\gamma} + \\
& +\lambda_3\left(\frac{2}{Ne\pi}\right)^{2\gamma}\|w\|_{\gamma}\|\psi\|_{\gamma}
\end{aligned}$$

$$\begin{aligned}
&\leq \max\left\{d_1\left(\frac{2}{Ne\pi}\right)^{2\beta-1}, d_2\left(\frac{2}{Ne\pi}\right)^{2\gamma-1} + \lambda_2\left(\frac{2}{Ne\pi}\right)^{2\gamma},\right. \\
&\qquad\qquad \left. d_3\left(\frac{2}{Ne\pi}\right)^{2\gamma-1} + \lambda_3\left(\frac{2}{Ne\pi}\right)^{2\gamma}\right\} \times \qquad (4.48)\\
&\times \quad (\|u\|_\beta\|\phi\|_\beta + \|v\|_\gamma\|\varphi\|_\gamma + \|w\|_\gamma\|\psi\|_\gamma) \leq \Lambda_0\|U\|_{\gamma(\beta)}\|V\|_{\gamma(\beta)},
\end{aligned}$$

where $\Lambda_0 \in \mathbb{R}^+ \setminus \{0\}$ is the value expressed in the max argument, and since the norm of $\|W\|_{\gamma(\beta)}$ is greater than or equal to the partial summed norms of elements constituting the product space sum of norms.

Also we have, for any $\Theta \in E^\beta$, that the mapping

$$(U, V) \in Z_{\gamma(\beta)} \times Z_{\gamma(\beta)} \to \langle P(\Theta)U, V\rangle \in L^1(\Omega) \qquad (4.49)$$

is well defined and continuous, provided that $2\beta + \gamma \geq 1 + \frac{N}{4}$ again by Sobolev type space embeddings (4.10). Note that this implies, from (4.46), that if $\beta = \gamma = 1/2$, then (4.49) holds only for $N \leq 2$. Now proceeding as above, we have that

$$\begin{aligned}
|\langle P(\Theta)U, V\rangle| &\leq \chi_2|\langle\Theta\nabla v, \nabla\phi\rangle| + |\chi_3|\langle\Theta\nabla w, \nabla\phi\rangle + a_2|\langle u, \varphi\rangle| + a_3|\langle u, \psi\rangle| \\
&\leq \chi_2\|\Theta\|_{L^{r_4}(\Omega)}\|\nabla v\|_{L^{r_2}(\Omega)}\|\nabla\phi\|_{L^{r_1}(\Omega)} + |\chi_3|\|\Theta\|_{L^{r_4}(\Omega)}\|\nabla w\|_{L^{r_2}(\Omega)} \times \\
&\qquad \times\|\nabla\phi\|_{L^{r_1}(\Omega)} + a_2\|u\|_{L^{r_5}(\Omega)}\|\varphi\|_{L^{r_3}(\Omega)} + a_3\|u\|_{L^{r_5}(\Omega)}\|\psi\|_{L^{r_3}(\Omega)} \\
&\leq \chi_2\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}\|\Theta\|_\beta\|v\|_{\gamma-\frac{1}{2}}\|\phi\|_{\beta-\frac{1}{2}} + \chi_3\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1} \times \qquad (4.50)\\
&\qquad \times\|\Theta\|_\beta\|w\|_{\gamma-\frac{1}{2}}\|\phi\|_{\beta-\frac{1}{2}} \quad + \quad a_2\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\|u\|_\beta\|\|\varphi\|_\gamma + \\
&\qquad +a_3\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\|u\|_\beta\|\|\varphi\|_\gamma.
\end{aligned}$$

Consequently,

$$\begin{aligned}
|\langle P(\Theta)U, V\rangle| \quad &\leq \quad \chi_2\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}\|\Theta\|_\beta\|v\|_\gamma\|\phi\|_\beta + |\chi_3|\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}\|\Theta\|_\beta\|w\|_\gamma\|\phi\|_\beta + \\
&+ \quad a_2\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\|u\|_\beta\|\|\varphi\|_\gamma + a_3\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\|u\|_\beta\|\|\psi\|_\gamma.
\end{aligned}$$

Next if we set

$$\Lambda_1 = \max\left\{\{\chi_2,\chi_3\}\left(\frac{2}{Ne\pi}\right)^{2\beta+\gamma-1}, \{a_2,a_3\}\left(\frac{2}{Ne\pi}\right)^{\beta+\gamma}\right\}$$

then

$$\begin{aligned}
|\langle P(\Theta)U,V\rangle| &\leq \Lambda_1\|\Theta\|_\beta\left(\|v\|_\gamma+\|w\|_\gamma\right)\|\phi\|_\beta+\left(\|\varphi\|_\gamma+\|\psi\|_\gamma\right)\|u\|_\beta\\
&\leq \Lambda_1\|\Theta\|_\beta\left(\|u\|_\beta+\|v\|_\gamma+\|w\|_\gamma\right)\|V\|_{\gamma(\beta)}+\|V\|_{\gamma(\beta)}\|u\|_\beta\\
&\leq \Lambda_1\|\Theta\|_\beta\Big(\left(\|u\|_\beta+\|v\|_\gamma+\|w\|_\gamma\right)\|V\|_{\gamma(\beta)}+\|V\|_{Z_{\gamma(\beta)}}\|U\|_{\gamma(\beta)}\Big) \quad (4.51)\\
&\leq 2\Lambda_1\|\Theta\|_\beta\|U\|_{\gamma(\beta)}\|V\|_{\gamma(\beta)}.
\end{aligned}$$

Combining this with the estimate from above in (4.48), taking $M := \max\{\Lambda_0, 2\Lambda_1\} \in \mathbb{R}^+\setminus\{0\}$ we conclude that (i) holds.

Next, observing that $Z_{\gamma(\beta)}$ is endowed with the norm $\langle (u,v,w)^\top,(u,v,w)^\top\rangle_{\gamma(\beta)} = \|u\|_\beta^2 + \|v\|_\gamma^2+\|w\|_\gamma^2$, if we take in (4.43) the scalar product $V=U\in Z_{\gamma(\beta)}$, then we get from (4.51) that

$$\begin{aligned}
B(\Theta;U,U) &\geq d_1\|u\|^2_{\beta-\frac{1}{2}}+d_2d_3\left(d_3^{-1}\|v\|^2_{\gamma-\frac{1}{2}}+d_2^{-1}\|w\|^2_{\gamma-\frac{1}{2}}\right)+\lambda_2\frac{2}{Ne\pi}\|v\|^2_{\gamma-\frac{1}{2}}\\
&+ \lambda_3\frac{2}{Ne\pi}\|w\|^2_{\gamma-\frac{1}{2}}-2\Lambda_1\|\Theta\|_\beta\|U\|^2_{\gamma(\beta)-\frac{1}{2}}\\
&\geq \min\left\{\left\{d_1, d_i+\lambda_i\frac{2}{Ne\pi}: i=2,3\right\}-2\Lambda_1\|\Theta\|_\beta\right\}\|U\|^2_{\gamma(\beta)-\frac{1}{2}}\\
&= \omega(\|\Theta\|_\beta)\|U\|^2_{\gamma(\beta)-\frac{1}{2}},
\end{aligned}$$

since $u\in E^\beta\subset E^{\beta-\frac{1}{2}}$, $v,w\in E^\gamma\subset E^{\gamma-\frac{1}{2}}$, using the inclusions (4.10), and hence (ii) is verified, taking $V=U\in Z_{\gamma(\beta)+1/2}$. From, (ii) if $U\neq V$ we also get the conclusion (iii) of the theorem. To obtain (iv), it suffices to notice from (i)-(ii) that for each $\Theta\in E^\beta$ fixed, (4.43) defines an isomorphism

$$\mathcal{A}(\Theta)U := B(\Theta;U,\cdot)\in Z^*_{\gamma(\beta)} \text{ for any } U\in Z_{\gamma(\beta)} \text{ by } \langle\mathcal{A}(\Theta)U,V\rangle=\langle F,V\rangle, \quad \forall V\in Z_{\gamma(\beta)},$$

and $F\in Z^*_{\gamma(\beta)}$. This proves (iv) with uniqueness of the solutions being given by (ii).

Also we get, using (4.51), that (v) is proved for any two $\Theta_1, \neq \Theta_2 \in Z_{\gamma(\beta)}$ and $U \in Z_{\gamma(\beta)}$ fixed. To prove (vi) we observe that the mapping $F \in Z^*_{\gamma(\beta)} \to U \in Z_{\gamma(\beta)}$, by (ii) is continuous. It is also compact, since the space inclusions $Z_{\gamma(\beta)} \subset Z^*_{\gamma(\beta)}$ are compact, and this proves (vii). Thus, the mapping $F \in Z_{\gamma(\beta)} \to T_F(\cdot) = \mathcal{A}^{-1}(\cdot)F \in \mathcal{L}(Z_{\gamma(\beta)})$ and the problem $U = T_F(u) = \mathcal{A}^{-1}(u)U$, by Schauder -Tychonoff theorem (see [12, 10, pp.179, pp.120] respectively) has a unique fixed point $U \in Z_{\gamma(\beta)}$. The remainder of the proof is trivial, and the theorem is proven. □

If in what follows, we let $D(\mathcal{A}(t)) = Z_{\gamma(\beta)+1/2} := E^{\beta+\frac{1}{2}} \times E^{\gamma+\frac{1}{2}} \times E^{\gamma+\frac{1}{2}}$, then, the operator $\mathcal{A}(t) : Z_{\gamma(\beta)+1/2} \subset Z_{\gamma(\beta)} \to Z^*_{\gamma(\beta)} \subset E^{-1/2} \times E^{-1/2} \times E^{-1/2}$ is closed and densely defined. Also by Theorem 4.8 for each $t \in \mathbb{R}^+$, the resolvent operator

$$R(\mathcal{A}(t), \kappa) = (\mathcal{A}(t) - \kappa I)^{-1} : Z^*_{\gamma(\beta)} \to Z_{\gamma(\beta)}$$

exists for any $\kappa \in \mathbb{C}$, with $\mathcal{R}e(\kappa) \leq 0$ such that

$$\|R(\mathcal{A}(t), \kappa)\|_{\mathcal{L}(Z^*_{\gamma(\beta)}, Z_{\gamma(\beta)})} \leq \frac{C}{|\kappa| + 1}.$$

Furthermore, by Theorem 4.3, Hölder continuity of the solution for any $0 \leq s \leq \tau \leq t < \infty$, we have that

$$\|[\mathcal{A}(t) - \mathcal{A}(s)]\mathcal{A}^{-1}(\tau)\|_{\mathcal{L}(Z_{\gamma(\beta)}, Z^*_{\gamma(\beta)})} \leq C(t-s)^{\theta} \tag{4.52}$$

for $\theta \in (0, 1)$. Consequently, by [5, 18, 21, 25, 30, 60], we obtain that (4.42) is an infinitesimal generator of an analytic semigroup or fundamental solution operator $\{G(t, s) : t > s\}$: $Z_{\alpha_0} \to Z_{\alpha_1}$ satisfying the following:

Lemma 4.9. *Let $J_s := (s, T)$, $s \geq 0$. Then, $G(t, s) \in \mathcal{L}(Z_{\alpha_0}, Z_{\alpha_1})$ uniformly for any $t \in J_s$ verifies that*

$$\|G(t, s)\|_{\alpha_0, \alpha_1} \leq c(\alpha_0, \alpha_1)e^{-\omega(t-s)}(t-s)^{\alpha_0-\alpha_1} \quad \textit{and} \quad G(\cdot, s) \in C^{\beta-\alpha}(J_s, \mathcal{L}(Z_{\alpha_1}, Z_{\alpha_0}))$$

whenever $-1 \leq \alpha_0 \leq \alpha_1 \leq 1$, *where* $\omega \in \mathbb{R}^+ \setminus \{0\}$, *and if*

$$U_t + \mathcal{A}(t)U = \vec{0}, \quad in\ J_s, \quad U(s) = U_s \in Z_{\alpha_0}, \tag{4.53}$$

then

$$U(t,s,U_s) = G(t,s)U_s \in C^1(J_s, Z_{\alpha_0}) \cap C^{\alpha_1-\alpha_0}(J_s, Z_{\alpha_1})$$

is a unique solution of (4.53).

Moreover, if $U_s \in Y$, *where either* $Y = Z^*_{\alpha_0}$, *or* $[L^r(\Omega)]^3$, *then it holds that*

$$\|G(t,s)U_s\|_Y \begin{cases} \leq \ C\frac{e^{-\omega(t-s)}}{(t-s)}\|U_s\|_{Z^*_{\alpha_0}}, \quad t > s \\ \\ \leq \ C\frac{e^{-\omega(t-s)}}{(t-s)^{\frac{N}{2}(\frac{1}{q}-\frac{1}{r})}}\|U_s\|_{L^q(\Omega)}, \quad t > s, \end{cases} \tag{4.54}$$

within last estimates, $Y = L^r(\Omega)$ *following a bootstrap argument for any* $1 \leq q \leq r \leq \infty$ *and the evolution operator is in* $L^q(\Omega)$ *for any* $1 < q < \infty$.

4.6 Numerical simulation

To visualize the aggregation of microglia as in the model equations, we numerically simulate the equations using a Gradient Weighted Moving Finite Element method.

Gradient Weighted Moving Finite Element methods (GWMFE) are numerical moving mesh methods which are designed for tracking moving shocks and complex structures with a fixed number of mesh nodes. These methods are well suited to modelling aggregation of microglial cells, where the cells aggregate into sharp peaks which need to be resolved. Also see [80] for a comparison of SGWMFE and a Parabolic Moving Mesh Partial Differential Equation method, for solutions of Partial Differential Equations.

In [81] the authors extend the String Gradient Weighted Moving Finite Element (SG-WMFE) method in order to include the non-linear diffusion of different variables, necessary for the chemo-attraction-repulsion model equations. For the simulations shown in this chapter we use the code developed in [81] using a set of model parameters found in [50].

4.6.1 Parameter values

The parameter values used in the numerical simulations are calculated in [50], where the parameters used there are calculated from dimensional values found in Biology, Immunology and Neuroscience publications referenced therein. From the set of data found in [50], the corresponding parameter values chosen for the simulations in this chapter are summarized in Tables 4.6.1 and 4.6.1.

The equations are defined on a real and bounded domain Ω, where the boundary is denoted by Γ. Our numerical domain is a two dimensional square of length and width 10. The boundary conditions which hold are zero flux through the boundary Γ. No proliferation or death of microglial cells is considered in this model.

The contour plots for the three unknown variables are shown in Fig 4.1. The corresponding evolving meshes are shown in Fig 4.2 where we also show a slice of the solutions, where the slice is taken along $y = 7$ of the computational domain.

Table 4.1: Biological parameters from [50], found in literature or calculated therein.

Parameter	Description	Value
μ	Microglia random motility	$33\frac{\mu\ \mathrm{m}^2}{\mathrm{min}}$
$\tilde{\chi_1}$	Chemoattraction	$6 - 780\frac{\mu\ \mathrm{m}^2}{nM \cdot \mathrm{min}}$
$\tilde{\chi_2}$	Chemorepulsion	Not available
D_1	IL-1β diffusion	$900\ \frac{\mu\ \mathrm{m}^2}{\mathrm{min}}$
D_2	TNF-α diffusion	$900\ \frac{\mu\ \mathrm{m}^2}{\mathrm{min}}$
$\tilde{a_1}$	IL-1β production rate per microglia cell	$6.25 \times 10^{-6}\frac{\mathrm{pg}}{\mathrm{min}}$
$\tilde{a_2}$	TNF-α production rate per microglia cell	$8.33 \times 10^{-6}\frac{\mathrm{pg}}{\mathrm{min}}$
b_1	IL-1β decay rate	$0.003 - 0.03\mathrm{min}^{-1}$
b_2	TNF-α decay rate	$0.002 - 0.03\mathrm{min}^{-1}$
L_1	Spatial range for chemoattraction	$\sqrt{D_1/b_1}$
L_2	Spatial range for chemorepulsion	$\sqrt{D_2/b_2}$
$\bar{m}$	Average microglial cell density	$10^{-6} - 10^{-4}\frac{\mathrm{cells}}{\mu\ \mathrm{m}^3}$

Table 4.2: Model parameter values in relation to biological parameters from Table 4.1.

Dimensionless variable	Expression in terms of variables in Table 4.6.1	Variable values from data Set 3, Table 10 in [50]
χ_2	$\frac{\tilde{\chi}_1 \tilde{a}_1 \bar{m}}{\mu b_1}$	37.14
χ_3	$\frac{\tilde{\chi}_2 \tilde{a}_2 \bar{m}}{\mu b_2}$	27
ϵ_1	$\frac{\mu}{D_1}$	0.0367
ϵ_2	$\frac{\mu}{D_2}$	0.0367
a	$\frac{L_2}{L_1}$	1.1
a_2	$\frac{a^2}{\epsilon_1}$	32.970027248
a_3	$\frac{1}{\epsilon_2}$	27.2479564033
d_2	$\frac{1}{\epsilon_1}$	27.2479564033
d_3	$\frac{1}{\epsilon_2}$	27.2479564033
λ_2	$\frac{a^2}{\epsilon_1}$	32.970027248
λ_3	$\frac{1}{\epsilon_2}$	27.2479564033

Summarizing the relation between the non-dimensional variables used in the model equations in this chapter and the dimensional variables (as derived from [50]): the characteristic cell density used is the average cell density $\bar{m}$. One can calculate the dimensional variable for density, from the non-dimensional density u as $u_{dim} = \bar{m}u$. The average chemical concentrations at which production and decay balance, form the characteristic scales for chemical concentrations $\hat{v} = a_1\bar{m}/b_1$ and $\hat{w} = a_2\bar{m}/b_2$. In order to obtain the dimensional chemical concentrations one can then calculate $v_{dim} = \hat{v}v$ and $w_{dim} = \hat{w}w$.

L_2 is close in value to L_1, and so L_2 is taken as the characteristic length scale of the problem, with $L_2 = \sqrt{900/0.01} = 300\mu$ m. This value corresponds to the distance over which chemicals spread during the characteristic time of decay. The 10 by 10 non-dimensional domain used for the simulations corresponds to a physical domain of length and width equal to $3,000\mu$ m.

The characteristic time scale for the problem is $\hat{t} = L_2^2/\mu$, which is the time needed for a cell to move over one unit of the characteristic length scale L_2 [50]. Then in order to calculate the dimensional time t_{dim}, from the non-dimensional time t found in the equations of the model, we calculate $t_{dim} = \hat{t}t$. In the simulations shown in this chapter, we compute up to a non-dimensional time $t = 0.8$, which corresponds to a dimensional time of

$$t_{dim} = ((300\mu \text{ m})^2 \text{min}/33\mu \text{ m}^2) \times 0.8 = (2.727 \times 10^3 \text{min}) \times 0.8 \approx 1.5 \text{days},$$

i.e. one and a half days.

4.7 Discussion of results

Fig 4.1 shows the contour plots of the microglia, attractant and repellent solutions to the equations in system (4.1), at five different times, $t = 0$, $t = 0.2$, $t = 0.4$, $t = 0.6$ and $t = 0.8$. As in the one dimensional results found in [50], we see similar behaviour in that, small initial perturbations increase in amplitude and decrease in spatial frequency, so that a few peaks evolve in each of the solutions. We observe that the microglial cells merge locally due to the attractant and form sharp peaks. This feature can also be observed in the slice plot in Figure 4.2.

The numerical solutions, resulting from the application of SGWMFE, shown in Fig 4.1, mimic the behavior of microglia observed in both in *vitro* and in *vivo* experiments, specifically the migration in response to chemoattraction.

We show numerical simulations up to a time t=0.8, corresponding to a dimensional time computation of 36hrs. This time frame is of interest because studying the early changes in the Alzheimer's disease affected brain is critical, especially given the prospect of new disease-modifying drugs. It should be noted that this time frame is believed to be sufficient to induce early Alzheimer's disease pathology in experimental models, as is recently shown in the development of AD-like pathology at 24hrs in a novel model for sporadic Alzheimer's disease [40].

The equations were solved with SGWMFE using a mesh of 21 by 21 nodes. At time t = 0 the cells and concentrations of attractant and repellent are initialized randomly in the interval (0.998, 1.002).

Mesh plots (a) to (e) and slice plots (f) to (j) corresponding to the numerical solutions in Figure 4.2. The slice is taken along the line $y = 7$ of the computational domain. The microglia, attractant and repellent are represented by the starred line, the solid line and the dash and dot line respectively.

Remark 4.2. To conclude this chapter, we point out that the chemical attraction and the chemical repulsion equations of system (4.1) are identical, and they are both similar to the second equation of the minimal model studied in Chapter 3. Moreover, the cell density equations for the two systems are also similar, with the only difference being the extra coupled term. It therefore follows that the well-posedness result (Theorem 4.3) for system (4.1) implies the well-posedness result for the minimal system (3.1) in the Hilbert space setting.

Figure 4.1: Contour plots of the numerical solutions of the aggregation of microglia model equations.

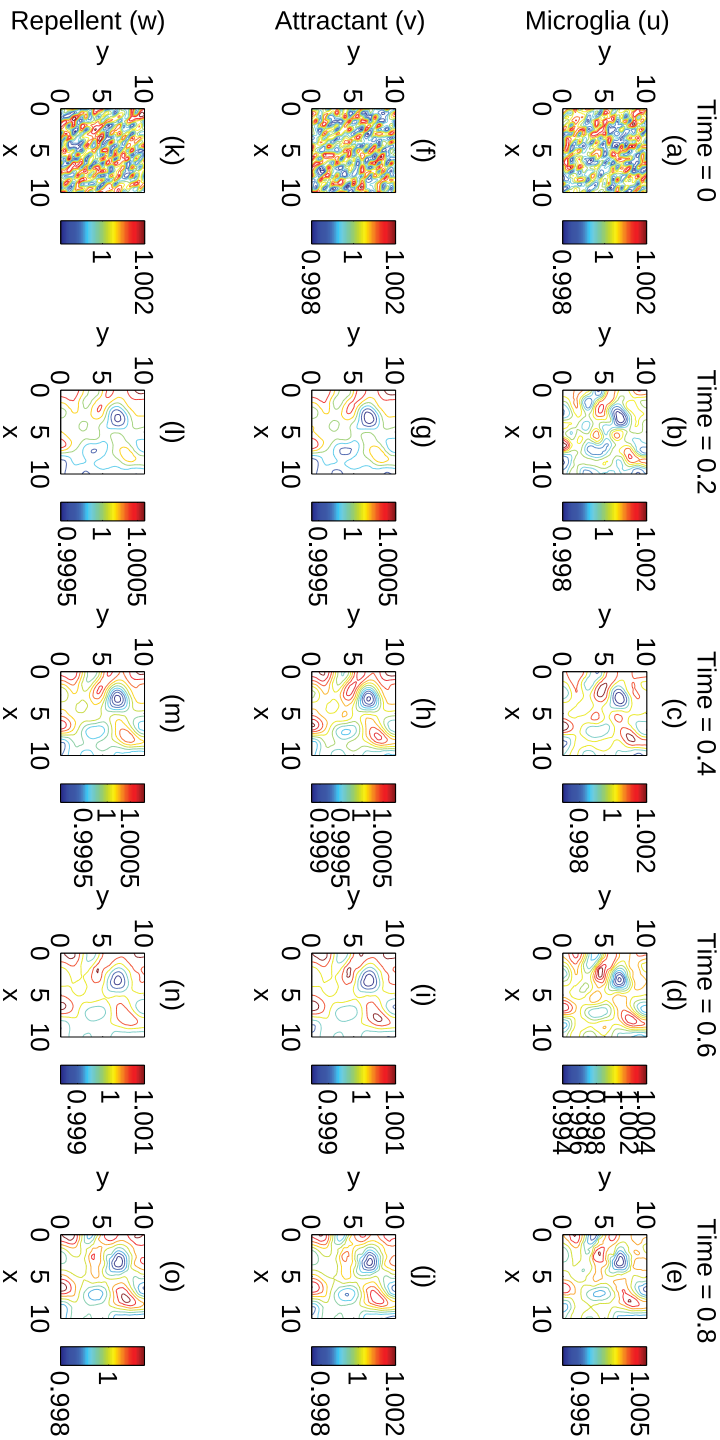
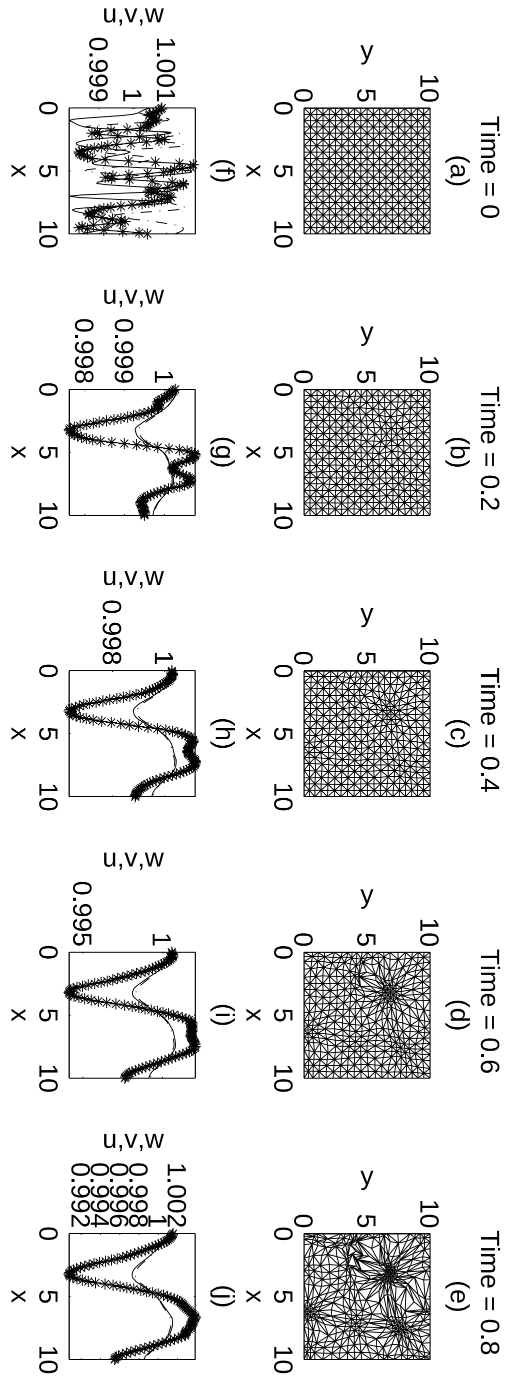


Figure 4.2: Mesh and slice plots



Chapter 5

Minimal KS Equations Blow-up Analysis in Hilbert Spaces

5.1 Introduction

Parabolic equations may experience solutions that may not exist globally. In such cases, solutions are said to blow up either in finite or in infinite time. There has been some work done on the Blow-up analysis for the Keller-Segel system, among which we cite [33, 37, 39, 69, 83]. In this Chapter, we will be carrying out the blow-up analysis in the Hilbert space setting.

5.2 Rescaling Solutions

Let's consider the Keller-Segel system again, where the motility coefficient of the amoebae, d_1, and the diffusion coefficient of the chemical attractant, d_2, are not necessarily equal to

1.

$$\begin{cases} u_t = d_1 \Delta u - \nabla \cdot (u \chi \nabla v) & \text{in } \Omega \times (0,T), \\ v_t = d_2 \Delta v - \lambda v + au & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \vec{n}} = \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \end{cases} \tag{5.1}$$

In this section, we rescale the blow-up solutions for the system (5.1). To this end, let's denote the spatial mean for a function $\omega = \omega(x,t)$ by

$$\bar{\omega}(t) = \frac{1}{|\Omega|} \int_\Omega \omega(x,t)\, dx,$$

where $|\Omega|$, as usual, denotes the measure (volume) of the set Ω.

The following transformation of the system can be found in [26, 39]. Also see [83]. We integrate in (5.1) to obtain that

$$\bar{u}(t) = \bar{u}_0, \quad \frac{d\bar{v}}{dt} + \lambda \bar{v} = a\bar{u}_0, \quad \bar{v}(0) = \bar{v}_0.$$

Through rescaling, we introduce new unknown functions u^* and v^* defined as

$$u^*(x,t) = \frac{u(x,t)}{\bar{u}_0}, \quad v^*(x,t) = v(x,t) - \bar{v}(t),$$

and let γ be a new constant given by

$$\gamma = a\bar{u}_0. \tag{5.2}$$

We thus get the transformed system as

$$\begin{cases} u^*_t = d_1 \Delta u^* - \nabla \cdot (u^* \chi \nabla v^*) & \text{in } \Omega \times (0,T), \\ v^*_t = d_2 \Delta v^* - \lambda v^* + \gamma(u^* - 1) & \text{in } \Omega \times (0,T), \\ \dfrac{\partial u^*}{\partial \vec{n}} = \dfrac{\partial v^*}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma \\ u^*(x,0) = u^*_0(x) = \frac{u_0(x)}{\bar{u}_0}, \ v^*(x,0) = v^*_0(x) = v_0(x) - \bar{v}_0, \end{cases} \tag{5.3}$$

with

$$\bar{u}_0^* = \frac{1}{|\Omega|}\int_\Omega u_0^*(x)\,dx = 1, \quad \bar{v}_0^* = \frac{1}{|\Omega|}\int_\Omega v_0^*(x)\,dx = 0.$$

It's trivial to see that $\bar{u}^*(t) = 1,\ \bar{v}^*(t) = 0\ \ \forall t \geq 0$. Also, by the maximum principle, we see that for any solution (u^*, v^*), $u^* \geq 0$, while v^* may change signs. For simplicity, we will drop the $*$ on u and v in what follows.

For the sake of clarity, we will state here the definition of a blow-up solution for the problem (5.3) [37].

Definition 5.1. We say that a solution for (5.3) blows up or is a blow-up solution for (5.3) if there exists a time $T_{max} \leq \infty$ such that

$$\limsup_{t\to T_{max}} \|u(x,t)\|_{L^\infty(\Omega)} = \infty \quad \text{or} \quad \limsup_{t\to T_{max}} \|v^+(x,t)\|_{L^\infty(\Omega)} = \infty,$$

in which $v^+(x,t)$ denotes the positive part of the function $v(x,t)$. If $T_{max} < \infty$, then we say that the solution for (5.3) blows up in finite time. Otherwise, if $T_{max} = \infty$, then we say that the solution blows up in infinite time.

Next, let's consider the stationary system of (5.3). From the first equation, we have that

$$\begin{aligned} d_1\Delta u - \nabla\cdot(u\chi\nabla v) = 0 \quad &\Rightarrow \quad d_1\frac{\nabla u}{u} - \chi\nabla v = 0 \\ &\Rightarrow \quad \ln u = \frac{\chi v}{d_1} + C \\ &\Rightarrow \quad u = Ke^{\frac{\chi v}{d_1}}. \end{aligned}$$

for some constant K. If we let $\kappa = \frac{\chi}{d_1}$, $\alpha = \frac{\lambda}{d_2}$, $\beta = \frac{|\Omega| a\bar{u}_0}{d_2} = \frac{|\Omega|\gamma}{d_2}$ and $u = \frac{e^{\kappa v}}{\int_\Omega e^{\kappa v}dx}$, then we reduce the second equation of the stationary system of (5.3) to

$$\begin{cases} \Delta v - \alpha v + \beta\left(\frac{e^{\kappa v}}{\int_\Omega e^{\kappa v}dx} - \frac{1}{|\Omega|}\right) = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma, \end{cases} \tag{5.4}$$

If we then multiply through (5.4) by $v \in A := \{v \in H^1(\Omega) | \int_\Omega v\, dx = 0\}$ and integrate, then we obtain the following energy functional,

$$E_\beta(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\, dx + \frac{\alpha}{2}\int_\Omega v^2\, dx - \beta\left(\frac{1}{|\Omega|}\int_\Omega e^{\kappa v} dx\right). \tag{5.5}$$

In the next section, we will try to determine whether there exist non-trivial solutions of the nonlinear elliptic equation in (5.4) for large values of β.

5.3 Global Existence of Blow-up solutions

In [26], Gajewski and Zacharias have shown, using Cherrier's extension of Moser-Trudinger's inequality, that the minimizer of E_β over $\{v \in H^1(\Omega) | \int_\Omega v\, dx = 0\}$ exists for $\beta \in (0, 4\pi)$.

In this Section, we will try to determine the existence of nontrivial solutions for (5.4) for large β. To do this, we will follow [37, 83]. To this end, we set

$$\beta_0 := \inf_{\substack{v \in H_0^1 \\ \int_\Omega v\, dx = 0}} \left\{ \int_\Omega |\nabla v|^2\, dx \ : \ \int_\Omega v^2\, dx = 1 \right\}. \tag{5.6}$$

If we consider $E_\beta(v)$ as defined in (5.5), then it is trivial to see that $E_\beta(0) = 0$, and $v = 0$ is always a solution of (5.4). Furthermore, the mapping $\beta \to E_\beta(v)$ is monotone decreasing for any $v \in A$, and $\int_\Omega e^{\kappa v} dx \geq |\Omega| \ \ \forall v \in A$ by Jensen's inequality. So, we prove first the following Lemma.

Lemma 5.1. *Let $\frac{\beta}{|\Omega|} - \beta_0 < \alpha$. Then $v = 0$ is a strict local minimum for E_β.*

Proof. It suffices to show that $E''_\beta(0)(v, v) > 0$ in any nonzero direction $v \in A$. Indeed, using

(5.6), we have

$$\begin{aligned} E''_\beta(0)(v,v) &= \int_\Omega (|\nabla v|^2 + \alpha v^2)\, dx - \frac{\beta}{|\Omega|}\int_\Omega v^2\, dx \\ &\geq (\alpha + \beta_0 - \frac{\beta}{|\Omega|})\int_\Omega v^2\, dx \\ &> 0, \qquad \text{since } \alpha + \beta_0 - \frac{\beta}{|\Omega|} > 0, \end{aligned}$$

and the Lemma is established. □

Next, we fix $P = 0 \in \partial\Omega$, and for $x \in \Omega$, set

$$v_\varepsilon = \log \frac{32\varepsilon^2}{(\varepsilon^2 + |x|^2)^2},$$

and

$$u_\varepsilon(x) = v_\varepsilon - \frac{1}{|\Omega|}\int_\Omega \log \frac{32\varepsilon^2}{(\varepsilon^2 + |x|^2)^2}\, dx. \tag{5.7}$$

We see that $\{u_\varepsilon(x)\}_{\varepsilon>0}$ is a sequence in the set $\{u \in H^1(\Omega) | \int_\Omega u\, dx = 0\}$. We thus have the following Lemma.

Lemma 5.2.

$$E_\beta(u_\varepsilon) = 2(4\pi - \beta)\log\frac{1}{\varepsilon} + O(1),$$

where $|O(1)| \leq K$ *as* $\varepsilon \to 0$.

Proof. Firstly, it is routine to calculate that

$$\int_\Omega v_\varepsilon\, dx = 2\log\varepsilon|\Omega| + O(1), \quad \int_\Omega |\nabla u_\varepsilon|^2\, dx = \int_\Omega \frac{16|x|^2}{(\varepsilon^2 + |x|^2)^2}\, dx.$$

If we carry out the transformation $y = \frac{x}{\varepsilon}$, then we get

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx = 16\int_{\Omega_\varepsilon} \frac{|y|^2}{(1 + |y|^2)^2}\, dy = 16\pi\log\frac{1}{\varepsilon} + O(1),$$

and

$$\int_\Omega |u_\varepsilon|^2\, dx = \int_\Omega \left|\log\left(\varepsilon^2 + |x|^2\right)^2\right|^2 dx = O(1).$$

Furthermore, we have

$$\log \int_\Omega e^{\kappa u_\varepsilon} dx = \log \int_\Omega e^{\kappa v_\varepsilon} dx - \int_\Omega \kappa v_\varepsilon \, dx = O(1) - 2|\Omega| \log \frac{1}{\varepsilon}.$$

If we put all these together, then we get our result that

$$E_\beta(u_\varepsilon) = 2(4\pi - \beta) \log \frac{1}{\varepsilon} + O(1),$$

□

In what follows, let's assume that $\beta > 4\pi$. Using Lemma 5.2, we see that $\exists \varepsilon_0 = \varepsilon_0(\beta) > 0$ sufficiently small, such that for $v_0 = v_{\varepsilon_0}$,

$$E_\beta(v_0) < 0 \quad \text{and } \|v_0\| \geq 1.$$

Thus, $\forall \psi \geq \beta$, we have that $E_\psi(v_0) \leq E_\beta(v_0) < 0$.

Also in what follows, we will be using a set Θ_β, defined as follows.

$$\Theta_\beta := \{\omega : [0,1] \to A | \ \omega \text{ is continuous, } \ \omega(0) = 0, \ \omega(1) = v_0^\beta\}. \tag{5.8}$$

If we let

$$\sigma_\beta = \inf_{\omega \in \Theta_\beta} \max_{t \in [0,1]} E_\beta(\omega(t)),$$

then, by Lemma 5.1, there exists a constant $c_0 > 0$, not dependent on β, such that

$$0 < \left(\alpha + \beta_0 - \frac{\beta}{|\Omega|} \right) c_0 \leq \sigma_\beta.$$

The number σ_β defines a Mountain-Pass value (see [74] among others). We want to show that σ_β is achieved by a solution of (5.3). We will use a technique introduced by Struwe [72, 73], also used in [83], to show that there exists a dense subset I of $(4\pi, +\infty)$ such that for any $\beta \in I$, σ_β is achieved by a solution u_β of (5.3). We start by looking at the Palais-Smale (P. S.) sequence for σ_β. To this end, we have the following two Lemmas[1].

Lemma 5.3. *Let u_i be a Palais-Smale sequence for E_β. That is, u_i satisfies*

$$|E_\beta(u_i)| \leq c < \infty, \tag{5.9}$$

and

$$dE_\beta(u_i) \to 0 \quad \textit{strongly in } H^{-1,2}(\Omega). \tag{5.10}$$

Also, suppose that

$$\int_\Omega (|\nabla u_i|^2 + |u_i|^2)\, dx \leq c_0, \quad \textit{for } i = 1, 2, \ldots \tag{5.11}$$

for c_0 a constant not dependent of i. Then there exists a subsequence of $\{u_i\}$, still denoted by $\{u_i\}$ for simplicity, which converges strongly to a critical point of E_β, u_0, in $H^1(\Omega)$.

Lemma 5.4. *The mapping $\beta \mapsto \frac{\sigma_\beta}{\beta}$ is non-increasing in I.*

For this reason, we have that the mapping $\beta \mapsto \frac{\sigma_\beta}{\beta}$ is differentiable almost everywhere. So, let

$$T = \{\beta \in I | \ \frac{\sigma_\beta}{\beta} \text{ is differentiable at } \beta\}.$$

Then it's known, [73], that T is a dense subset of $(0, +\infty)$. So, let $\beta \in T$ and choose $\beta_k \nearrow \beta$ such that

$$0 \leq \lim_{k\to\infty} -\frac{1}{(\beta - \beta_k)} \left(\frac{\sigma_\beta}{\beta} - \frac{\sigma_{\beta_k}}{\beta_k} \right) \leq c_1, \tag{5.12}$$

for some constant c_1 not dependent of k. Then we have the following Lemma, which was proven in [75] (Lemma 3.3).

Lemma 5.5. *The Mountain-Pass value σ_β is achieved by u_β, for any $\beta \in T$.*

---

[1] See [72, 73, 74] for the proofs

Proof. The proof is by contradiction. Thus, suppose that the Lemma is false. That is, suppose that there exists $\beta \in I$ such that σ_β is not achieved by u_β. Then by Lemma 5.3, there exist $\delta > 0$ such that

$$\|dE_\beta(u)\|_{H^{-1,2}(\Omega)} \geq 2\delta \tag{5.13}$$

in the set

$$N_\delta := \{u \in A| \int_\Omega |\nabla u|^2 \, dx \leq c_2, \ |E_\beta(u) - \sigma_\beta| < \delta\},$$

where c_2 is any fixed constant such that $N_\delta \neq \emptyset$. Now, let $X_\beta : N_\delta \to A$ be a pseudo-gradient vector field for E_β in N_δ [74]. That is, a locally Lipschitz vector field of norm $\|X_\beta\|_{H_0^{1,2}(\Omega)} \leq 1$, with

$$\langle dE_\beta(u), X_\beta(u)\rangle < -\delta. \tag{5.14}$$

Since

$$\begin{aligned}\|dE_\beta(u) - dE_{\beta_k}(u)\| &= \left\|dE_\beta(u) - \frac{\beta}{\beta_k} dE_{\beta_k}(u)\right\| + \left\|(1 - \frac{\beta}{\beta_k}) dE_{\beta_k}(u)\right\| \\ &\leq \frac{1}{2}\left(1 - \frac{\beta}{\beta_k}\right)\int_\Omega |\nabla u|^2 \, dx + c\left(1 - \frac{\beta}{\beta_k}\right)\int_\Omega |\nabla u|^2 \, dx \\ &\to 0\end{aligned} \tag{5.15}$$

uniformly in $\{u \in A| \int_\Omega |\nabla u|^2 \, dx \leq c_2\}$, X_β is also a pseudo-gradient vector field for E_{β_k} in N_δ, with

$$\langle dE_{\beta_k}(u), X_\beta(u)\rangle < -\frac{\delta}{2}, \tag{5.16}$$

for $u \in N_\delta$, provided that k is sufficiently large.

For any sequence $\{\omega_k\}$, $\omega_k \in \Theta_{\beta_k} \subset \Theta_\beta$ such that

$$\sup_{u \in \omega_k(\Theta_\beta)} E_{\beta_k}(u) \leq \sigma_{\beta_k} + \beta - \beta_k \tag{5.17}$$

and all $u \in \omega_k(\Theta_{\beta_k})$ such that

$$E_\beta(u) \geq \sigma_\beta - (\beta - \beta_k), \tag{5.18}$$

by (5.11), (5.17) and (5.18), we have the following estimate

$$
\begin{aligned}
\frac{1}{2}\int_\Omega |\nabla u|^2\,dx &= \beta\cdot\beta_k\frac{\frac{E_{\beta_k}(u)}{\beta_k}-\frac{E_\beta(u)}{\beta}}{\beta-\beta_k} \\
&\le \beta\cdot\beta_k\frac{\frac{\sigma_{\beta_k}}{\beta_k}-\frac{\sigma_\beta}{\beta}}{\beta-\beta_k}+(\beta+\beta_k) \\
&\le C,
\end{aligned}
\tag{5.19}
$$

where $C=(16\pi)^2c_1+32\pi$.

Now we consider the following pseudo-gradient flow for E_β in N_δ. Choose a Lipschitz continuous cut-off function η, such that $0\le\eta\le 1,\ \eta=0$ outside $N_\delta,\ \eta=1$ in $N_{\delta/2}$. Then consider the following flow in A generated by $\eta = X_\beta$

$$
\begin{aligned}
\frac{\partial\phi}{\partial t}(u,t) &= \eta(\phi(u,t))X_\beta(\phi(u,t)) \\
\phi(u,0) &= u.
\end{aligned}
$$

By (5.13) and (5.15), we have

$$\frac{d}{dt}E_\beta(\phi(u,t))_{|t=0}\le -\delta, \tag{5.20}$$

for $u\in N_{\delta/2}$, and

$$\frac{d}{dt}E_{\beta_k}(\phi(u,t))_{|t=0}\le -\frac{\delta}{2}, \tag{5.21}$$

for large k.

It is now clear that for any $\omega\in\Theta_{\beta_k},\ \omega(r,\theta)\notin N_\delta$ for r close to 1. Thus, $\phi(\omega,t)\in\Theta_{\beta_k}$ for any $t>0$. In particular, $\phi(\cdot,t)$ preserves the class of $\omega_k\in\Theta_{\beta_k}$ with condition (5.16). On the other hand, for any $\omega\in\Theta_\beta$,

$$\sup_{u\in\omega(\Theta_\beta)} E_\beta(u)\ge\sigma_\beta$$

by definition. Thus, for any $\omega_k\in\Theta_{\beta_k}$, (5.16) guarantees that $\sup\limits_{u\in\phi(\omega(\Theta_\beta),t)} E_\beta(u)$ is achieved in $N_{\delta/2}$, provided that k is large enough. As a result, we have by (5.19) that

$$\frac{d}{dt}\sup\{E_\beta(u)|\ u\in\phi(\omega(D),t)\}\le -\delta,$$

for all $t \geq 0$, and this is a contradiction. Thus, our initial assumption is false, and we have established the Lemma. $\square$

Given the above-stated Lemmas, we can therefore prove the following Lemma.

Lemma 5.6. *For almost all $\beta > 4\pi$, σ_β is a critical value for E_β. Moreover, problem* (5.4) *admits a nontrivial solution for almost all $\beta > 4\pi$.*

Proof. This Lemma is established directly by Lemma 5.3. $\square$

5.4 Blow-up

Let $\beta_{00} > 4\pi$ be a point where σ_β is not differentiable, and suppose that $\beta_{00} \notin \{4m\pi | \, m = 1,\ 2, \ldots\}$. Then by Lemma 5.6, there exists a sequence $\beta_n \to \beta_{00}$ and solutions v_n to (5.4) associated with β_n. Furthermore,

$$h_1 \geq E_{\beta_n}(v_n) \geq (\alpha + \beta_0 - \beta_n)c_0 \geq h_0 > 0,$$

where h_0 and h_1 do not depend on n.

Now, if $\max\limits_{x\in\bar{\Omega}} v_n(x) \leq C$, then $v_n(x) \to v_0$ by the standard elliptic regularity arguments, and v_0 is a solution of (5.4) associated with β_{00}. Since $E_{\beta_n}(v_n) \to E_{\beta_{00}}(v_0) \geq h_0$, we have that $v_0 \neq 0$. Hence, v_0 is a non-trivial solution of (5.4).

It remains to exclude the case when $\max\limits_{x\in\bar{\Omega}} v_n(x) \to +\infty$. To this end, we note that v_n satisfies

$$\begin{cases} \Delta v_n - \alpha v_n + \beta \left(\frac{e^{\kappa v_n}}{\int_\Omega e^{\kappa v_n} dx} - \frac{1}{|\Omega|} \right) = 0 & \text{in } \Omega, \\ \int_\Omega v_n \, dx = 0, \quad \dfrac{\partial v_n}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma. \end{cases} \tag{5.22}$$

Thus we have $\int_\Omega e^{v_n} \, dx \to +\infty$ (otherwise, since $E_{\beta_n}(v_n) \leq h_1$, we would deduce that $\|v_n\|_{H^1(\Omega)}$ and v_n would converge by Lemma 5.3). Now, let $v_n^* = v_n + \frac{\beta_n}{\alpha|\Omega|}$ and $\mu_n = \frac{\beta_n}{\int_\Omega e^{\kappa v_n^*} dx}$.

Then, using (5.22), we have

$$\begin{cases} \Delta v_n^* - \alpha v_n^* + \mu_n e^{\kappa v_n^*} = 0 & \text{in } \Omega, \\ \int_\Omega v_n^* \, dx = \dfrac{\beta_n}{\alpha}, \quad \dfrac{\partial v_n^*}{\partial \vec{n}} = 0 & \text{on } \partial\Omega = \Gamma, \end{cases} \tag{5.23}$$

where $\mu_n \to 0$. As in the previous section, for simplicity, we will drop the $*$ on v_n. By the Maximum Principle for elliptic operator, $v_n > 0$ in Ω. We want to show, following [37, 83], that as $n \to \infty$,

$$\mu_n \int_\Omega e^{\kappa v_n} \, dx \to 4m\pi, \tag{5.24}$$

for some integer m. This means that $\beta_n \to 4m\pi$ and $\beta_{00} = 4m\pi$, which is a contradiction since we assumed earlier that $\beta_{00} \neq 4m\pi$. It remains to verify (5.24). We will be following an approach in [37, 83, 84]. We start by stating the following Lemma which was proven by Chanillo and Yangan Li in [14].

Lemma 5.7. *Let $L = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ be a uniformly elliptic operator, namely*

$$\nu_0 I \leq (a_{ij})_{1 \leq i,j \leq 2} \leq \nu_1 I.$$

Then there exists a constant $\phi = \phi(\nu_0, \nu_1)$ such that for any solution v of the problem

$$Lv = f(x) \quad \text{in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega,$$

we have

$$\int_\Omega \exp\left(\frac{\phi |v(x)|}{\|f\|_{L^1(\Omega)}} \right) dx \leq C.$$

Next, we recall that by elliptic estimates, we have

$$\int_\Omega |\nabla v_n|^q \, dx + \int_\Omega |v_n|^q \, dx \leq C, \tag{5.25}$$

for any $1 < q < 2$.

To analyse v_n, we first introduce the following set. Let

$$S := \left\{ x \in \bar{\Omega} \;\middle|\; \begin{array}{l} \text{there exists } \mu_n \to 0, \text{ solutions } v_n \text{ of (5.23)}, \\ x_n \in \bar{\Omega} \text{ such that } v_n(x_n) \to \infty,\ x_n \to x \end{array} \right\}. \tag{5.26}$$

Let $\Lambda_n = \int_\Omega \mu_n e^{\kappa v_n}\, dx$.[2] Now, since v_n satisfies

$$\int_\Omega \frac{\mu_n e^{\kappa v_n}}{\Lambda_n}\, dx = 1,$$

we can extract a subsequence of v_n, still denoted by v_n (for simplicity) such that there exists a positive finite measure μ in the set of all real bounded Borel measures on $\bar{\Omega}$, $\mathcal{M}(\bar{\Omega})$, such that as $n \to \infty$,

$$\int_\Omega \frac{\mu_n e^{\kappa v_n}}{\Lambda_n} \varphi\, dx \to \int_\Omega \varphi\, d\mu, \tag{5.27}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$. Let

$$v_n^* = \frac{v_n}{\Lambda_n}, \quad g_n = \frac{\mu_n e^{\kappa v_n}}{\Lambda_n}.$$

For each $x_0 \in \partial\Omega$, we can find a smooth function Φ_{x_0} and a small positive constant $r_{x_0} > 0$ such that

$$\Phi_{x_0}(x) : B_{r_{x_0}}(x_0) \cap \bar{\Omega} \to B_{r_{x_0}}(0) \cap \mathbb{R}^2_+, \tag{5.28}$$

in which $\mathbb{R}^2_+ = \{(x_1, x_2) | x_2 > 0\}$. Then the Laplace operator Δ becomes $L_{x_0} + \sum_{l=1}^{2} b_l \frac{\partial}{\partial x_l}$, where $L_{x_0} = \sum_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ is a uniformly elliptic operator and $|b_l| \le C = const.$ By the compactness of the boundary, we can choose a uniform $\phi = \phi_0$ in Lemma 5.7 for all L_{x_0}, $x_0 \in \partial\Omega$.

For the sake of clarity, we now state the definition of $\delta-$regular points of $\bar{\Omega}$ [37].

---
[2] Note that $\Lambda_n = \beta_n$.

Definition 5.2. For any $\delta > 0$, we say that x_0 is a $\delta-$regular point if there is a function $\varphi \in C_0^\infty(\mathbb{R}^2)$, $0 \leq \varphi \leq 1$, with $\varphi = 1$ in a neighbourhood of x_0, such that

$$\int_\Omega \varphi \, d\mu < \frac{\phi_0}{1 + 3\delta}. \tag{5.29}$$

Let's denote the set of all points in $\bar{\Omega}$ which are not $\delta-$regular by

$$\Lambda(\delta) = \{x_0 | \, x_0 \text{ is not a } \delta-\text{regular point}\}. \tag{5.30}$$

If no confusion is foreseen in the following, we will use the reference 'regular', 'irregular' and 'Λ' without mentioning δ.

Also, in the following, we notice that, by assumption, the set S defined in (5.26), is not empty. We therefore have the following Lemma which we will be using later in the proof for Lemma 5.11. See [37] among others.

Lemma 5.8. *Let $1 < q < 2$. Then there exists a constant C_q, not dependent of n, such that $\|\nabla v_n\|_{L^q(\Omega)} \leq C_q$.*

Proof. Let $q' = \frac{q}{q-1} > 2$. Then we know that

$$\|\nabla v_n\|_{L^q(\Omega)} \leq \sup \left\{ | \int_\Omega \nabla v_n \cdot \nabla \varphi \, dx| \ : \ \varphi \in L_1^{q'}(\Omega), \ \int_\Omega \varphi \, dx = 0, \ \|\varphi\|_{L_1^{q'}(\Omega)} = 1 \right\}.$$

By using the Sobolev embedding theorem, we get that

$$\|\varphi\|_{L^\infty(\Omega)} \leq C_1.$$

Using the fact that $v_n > 0$, we see that

$$\begin{aligned} | \int_\Omega \nabla v_n \cdot \nabla \varphi \, dx| &= | \int_\Omega \Delta v_n \varphi \, dx| \\ &= | \int_\Omega (\alpha v_n - \mu_n e^{\kappa v_n}) \, \varphi \, dx| \\ &\leq C_1 \int_\Omega (v_n + \mu_n e^{\kappa v_n}) \, dx \\ &\leq C_2. \end{aligned}$$

□

We thus have the following Lemmas from [83].

Lemma 5.9. *If x_0 is a $\delta-$regular point, then $\{v_n\}$ is bounded in $L^\infty(B_{R_0}(x_0))$ for some $R_0 > 0$.*

Proof. Let x_0 be a regular point. We will give the proof for the case when $x_0 \in \partial\Omega$. The case for when x_0 is an interior point is simpler and can be proven in a similar way.

It follows from the definition of a regular point that there exists $R_1 > 0$ such that

$$\int_{B_{R_1}(x_0)\cap\bar{\Omega}} g_n \, dx < \frac{\phi_0}{1+3\delta}.$$

Let's pick $r < R_1$, a small number. At x_0, since $\frac{\partial v_n}{\partial \vec{n}} = 0$, we can strengthen the boundary near $B_{R_1}(x_0) \cap \bar{\Omega}$ by Φ_{x_0} defined in (5.28), and then extend v_n by even extension (still denoted by v_n) to

$$L_{x_0} v_n^* + \sum_{l=1}^{2} b_l \frac{\partial v_n^*}{\partial x_l} - \alpha v_n^* + \frac{\mu_n e^{\kappa v_n}}{\Lambda_n} = 0 \quad \text{in } B_\rho(0),$$

where $L_{x_0} = \sum\limits_{i,j=1}^{2} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ is a uniformly elliptic operator, $|b_l| \le C$ and $\rho = \rho(r) \to 0$ as $r \to 0$. We can pick r so small that

$$\int_{B_\rho(0)} g_n \, dx < \frac{\phi_0}{1+3\delta}$$

and, by (5.25)

$$\begin{aligned} \int_{B_\rho(0)} |\sum_{l=1}^{2} b_l \frac{\partial v_n^*}{\partial x_l}| + \alpha v_n^* &\le C\alpha \|v_n^*\|_{W^{1,q}(\Omega)} \rho^{\frac{q-1}{q}} \\ &< C\rho^{\frac{q-1}{q}} \\ &< \frac{\phi_0 \delta}{(1+2\delta)(1+\delta)}. \end{aligned}$$

Thus

$$\int_{B_\rho(0)} \left(|\sum_{l=1}^{2} b_l \frac{\partial v_n^*}{\partial x_l}| + \alpha v_n^* + g_n \right) dx < \frac{\phi_0}{1+\delta}. \tag{5.31}$$

Next, we split v_n^* into two parts such that

$$v_n^* = v_{1n}^* + v_{2n}^*,$$

in which v_{1n}^* is the solution for

$$\begin{cases} L_{x_0} v_{1n}^* = -\sum_{l=1}^{2} b_l \frac{\partial v_{1n}^*}{\partial x_l} + \alpha v_{1n}^* - g_n & \text{in } B_\rho(0) \\ v_{1n}^* = 0 & \text{on } \partial B_\rho(0), \end{cases} \tag{5.32}$$

while v_{2n}^* is the solution for

$$\begin{cases} L_{x_0} v_{2n}^* = 0 & \text{in } B_\rho(0) \\ v_{2n}^* = v_n^* & \text{on } \partial B_\rho(0), \end{cases} \tag{5.33}$$

Note that by the Maximum principle, $v_{2n}^* > 0$, and by (5.30) and (5.31), $\|v_{1n}^*\|_{L^1(B_\rho(0))} \leq C$. Thus, we get that $\int_{B_\rho(0)} v_{2n}^* \, dx < \int_{B_\rho(0)} (v_n^* + |v_{1n}^*|) \, dx \leq C$. Using Harnack inequality [19], we obtain that

$$\|v_{2n}^*\|_{L^\infty(B_{\rho/2}(0))} \leq C \|v_{2n}^*\|_{L^1(B_\rho(0))} \leq C \|v_n^*\|_{L^1(\Omega)} \leq C.$$

Therefore, we only need to consider v_{1n}^*.

Now, using (5.31) and Lemma 5.7, we get that

$$\int_{B_\rho(0)} \exp\left[\left(1 + \frac{\delta}{2}\right) |v_{1n}^*| \right] \, dx \leq C. \tag{5.34}$$

Since v_{2n}^* is uniformly bounded, we thus have

$$\int_{B_{\rho/2}(0)} \left(|\sum_{l=1}^{2} b_l \frac{\partial v_n^*}{\partial x_l}| + \alpha v_n^* + g_n \right)^{1+\delta} \, dx \leq C \tag{5.35}$$

on $B_{\rho/2}(0)$. From the elliptic estimates, we get $\|v_{1n}^*\|_{L^\infty(B_{\rho/4}(0))} \leq C$, and the proof is complete. $\square$

From above, the following result is immediate.

Lemma 5.10. $S = \Lambda(\delta)\ \ \forall \delta > 0$.

Proof. Let $\delta > 0$ and suppose that $x_0 \notin \Lambda(\delta)$. Then x_0 is a regular point, so that, by Lemma 5.9, $\{v_n^*\}$ is bounded in $L^\infty(B_R(x_0)) \cap \bar{\Omega}$ for some $R > 0$. That is, $x_0 \notin S$ and thus $S \subset \Lambda(\delta)$.

Conversely, suppose that $x_0 \in \Lambda(\delta)$. Then for every $R > 0$, we have that

$$\lim_{x \to \infty} \|v_n\|_{L^\infty(B_R(x_0))} = \infty. \tag{5.36}$$

Otherwise, there would be some $R_0 > 0$ and a subsequence, still denoted by v_n, such that

$$\|v_n\|_{L^\infty(B_R(x_0))} < C,$$

for some constant C, not dependent on n. This would imply that

$$\mu_n e^{v_n} \leq C\mu_n$$

uniformly as $n \to \infty$ on $B_{R_0}(x_0)) \cap \bar{\Omega}$, so that

$$\int_{B_R(x_0) \cap \bar{\Omega}} \mu_n e^{v_n}\, dx \leq C\mu_n \leq \varepsilon_0 < \frac{\phi_0}{1 + 3\delta}.$$

This implies that x_0 is a regular point, and so $x_0 \notin \Lambda(\delta)$, which is a contradiction. Equation (5.36), by definition of S, (5.26), implies then that $x_0 \in S$.

Hence, $S = \Lambda(\delta)$, and the proof is complete. $\square$

The statements in Lemmas 5.9 and 5.10 give that $1 \leq n(S) < \infty$, where $n(S)$ denotes the cardinality of set S. Let's now decompose S into $S_1 = S \cap \partial\Omega$ and $S_2 = S \cap \Omega$. Let $S = \{p_1, \ldots, p_N\}$, r be a small constant and $\theta_j^n(r) = \int_{B_r(p_j)} \mu_n e^{v_n}\, dx$. Then $\lim\limits_{n \to +\infty} \int_\Omega \mu_n e^{v_n}\, dx = \sum\limits_{j=1}^{N} \lim\limits_{n \to +\infty} \theta_j^n(r)$, for all small r, which implies that

$$\lim_{n \to +\infty} \int_\Omega \mu_n e^{v_n}\, dx = \sum_{j=1}^{N} \lim_{r \to 0} \lim_{n \to +\infty} \theta_j^n(r).$$

By Lemma 5.9, $\theta_j^n(r) \geq \dfrac{\phi_0}{1 + 3\delta}$. In fact, it can be proven that

Lemma 5.11. *If $p_j \in S_1$, then $\lim_{r\to 0}\lim_{n\to+\infty}\theta_j^n(r) = 4\pi$, and if $p_j \in S_2$, then $\lim_{r\to 0}\lim_{n\to+\infty}\theta_j^n(r) = 8\pi$. In particular, $\beta_0 = 4m\pi$ for some integer $m > 0$.*

Proof. We first consider the case for when $p \in S_1$. We will make use of the following Pohozaev's identity. Recall that for v satisfying

$$\Delta v - \alpha v + f(v) = 0, \quad \text{in } U \subset \mathbb{R}^2,$$

we have the Pohozaev's identity [74]

$$\begin{aligned}&\int_U (-\alpha v^2 + 2F(v))\, dx \\ &\quad = \int_{\partial U} \left[(x\cdot\nabla v)\frac{\partial v}{\partial \vec{n}} - (x\cdot\vec{n})\frac{|\nabla v|^2}{2} + (x\cdot\vec{n})(-\alpha\frac{v^2}{2} + F(v))\right] dS,\end{aligned} \tag{5.37}$$

where $F(v) = \int_0^v f(s)\, ds$.

Let $f(v) = \mu e^{\kappa v}$, where $\mu = \frac{\beta}{\int_\Omega e^{\kappa v} dx}$. Without loss of generality, we can assume that $p = 0$. Let $U_r = B^r(0) \cap \bar{\Omega}$ and consider the function w_n which is a solution for the problem

$$\begin{cases} \Delta w - \alpha w = 0 & \text{in } U_r \\ \frac{\partial w}{\partial \vec{n}} = \frac{\partial v_n}{\partial \vec{n}} & \text{on } \partial U_r, \end{cases} \tag{5.38}$$

Then it is trivial to see that $w_n = O(1)$ in $C^2(U_r)$, since $|\frac{\partial v_n}{\partial \vec{n}}| \le C$ on ∂U_r. Now, fix $\omega_n = \frac{(v_n - w_n)}{\theta_j^n(r)}$. Then by regularity theory [20], we have that $\omega_n \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0)\cap\bar{\Omega}\setminus\{0\})$, where $G(\cdot,0)$ satisfies

$$\begin{cases} -\Delta G + \alpha G = \delta_0 & \text{in } U_r \\ \frac{\partial G}{\partial \vec{n}} = 0 & \text{on } \partial U_r. \end{cases}$$

By potential theory [65], we see that for $|x|$ small

$$G(\cdot, 0) = \frac{1}{\pi}\log|x| + O(1).$$

Thus, we have

$$v_n = \frac{\theta_j^n(r)}{\pi}\log|x| + O(1)$$

in $C^1(\partial U_r)$. Note that here $O(1)$ may depend on r, but is uniform in n.

Now, by Pohozaev's identity, we have

$$\begin{aligned}
&\int_{U_r} (-\alpha v_n^2 + 2\mu_n(e^{\kappa v_n} - 1))\, dx \\
&= \int_{\partial U_r} \left[(x \cdot \nabla v_n)\frac{\partial v_n}{\partial \vec{n}} - (x \cdot \vec{n})\frac{|\nabla v_n|^2}{2} + (x \cdot \vec{n})(-\alpha \frac{v_n^2}{2} + \mu_n(e^{\kappa v_n} - 1))\right] dS.
\end{aligned} \tag{5.39}$$

We now use Lemma 5.8 to estimate each term on both sides of (5.39) as follows; For the first term on the left-hand side, we have

$$\int_{U_r} v_n^2\, dx = O(r^{1/2}\|v_n\|_{L^4(U_r)}) = O(r^{1/2}\|v_n\|_{W^{1,3/2}(\Omega)}) = O(r^{1/2}).$$

For the second term on the left-hand side,

$$\begin{aligned}
\int_{U_r} 2\mu_n(e^{\kappa v_n} - 1)\, dx &= 2\mu_n \int_{U_r} e^{\kappa v_n}\, dx + O(\mu_n) \\
&= 2\theta_j^n(r) + O(\mu_n).
\end{aligned}$$

Looking at the first term on the right-hand side, we have

$$\begin{aligned}
\int_{\partial U_r} (x \cdot \nabla v_n)\frac{\partial v_n}{\partial \vec{n}}\, dS &= \left(\frac{\theta_j^n(r)}{\pi}\right)^2 \int_{\partial U_r} \left(\frac{(x\cdot\vec{n})}{|x|^2} + O(1)\right) \\
&= \left(\frac{\theta_j^n(r)}{\pi}\right)^2 (\pi + O(r)).
\end{aligned}$$

For the second term on the right-hand side,

$$\int_{\partial U_r} (x \cdot \vec{n})\frac{|\nabla v_n|^2}{2}\, dS = \left(\frac{\theta_j^n(r)}{\pi}\right)^2 \left(\frac{\pi}{2} + O(r)\right).$$

From the third term on the right-hand side,

$$\int_{\partial U_r} v_n^2\, dS = O(r).$$

Lastly, for the last term on the right-hand side, we have

$$\int_{\partial U_r} (x \cdot \vec{n})\mu_n(e^{\kappa v_n} - 1)\, dS = O\left(\mu_n \max_{x \in \partial U_r} e^{\kappa v_n}\right) = O(\mu_n).$$

Now, if we first let $n \to +\infty$, and then we let $r \to 0$, we get that

$$2 \lim_{r\to 0} \lim_{n\to+\infty} \theta_j^n(r) = \frac{1}{\pi^2}\frac{\pi}{2}\left(\lim_{r\to 0}\lim_{n\to+\infty}\theta_j^n(r)\right)^2,$$

which implies that

$$\lim_{r\to 0}\lim_{n\to+\infty}\theta_j^n(r) = 4\pi.$$

The interior blow up case ($p \in S_2$) is proven in a similar way, with the following modifications. Instead of (5.38), we will consider w_n satisfying the problem

$$\begin{cases} \Delta w - \alpha w = 0 & \text{in } U_r \\ w = v_n & \text{on } \partial U_r. \end{cases} \tag{5.40}$$

We fix $\omega_n = \frac{(v_n - w_n)}{\theta_j^n(r)}$ and assume that $p = 0 \in \Omega$. Then, similarly, $\omega_n \to G(\cdot, 0)$ in $C^2_{loc}(B_r(0))/\{0\}$, where G is now a Green function with Dirichlet boundary data

$$\begin{cases} -\Delta G + \alpha G = \delta_0 & \text{in } B_r \\ G = 0 & \text{on } \partial U_r. \end{cases}$$

In this case, the Green function has the following expansion near 0;

$$G(\cdot, 0) = -\frac{1}{2\pi}\log|x| + O(1).$$

We then obtain the same estimates as in the case when $p \in S_2$, except with the following two. Looking at the first term on the right-hand side, we have

$$\begin{aligned} \int_{\partial U_r} (x\cdot\nabla v_n)\frac{\partial v_n}{\partial \vec{n}}\, dS &= \left(\frac{\theta_j^n(r)}{2\pi}\right)^2 \int_{\partial U_r}\left(\frac{(x\cdot\vec{n})}{|x|^2} + O(1)\right) \\ &= \left(\frac{\theta_j^n(r)}{2\pi}\right)^2 (2\pi + O(r)), \end{aligned}$$

and for the second term on the right-hand side,

$$\int_{\partial U_r} (x\cdot\vec{n})\frac{|\nabla v_n|^2}{2}\, dS = \left(\frac{\theta_j^n(r)}{2\pi}\right)^2 (\pi + O(r)).$$

Now, applying Pohozaev's identity again, we obtain in this case that

$$2 \lim_{r\to 0} \lim_{n\to+\infty} \theta_j^n(r) = \frac{1}{4\pi^2}\pi \left(\lim_{r\to 0} \lim_{n\to+\infty} \theta_j^n(r) \right)^2,$$

which implies that

$$\lim_{r\to 0} \lim_{n\to+\infty} \theta_j^n(r) = 8\pi.$$

The proof of the Lemma is complete. $\square$

We now note that, by Lemma 5.11, the statement in (5.24) holds true. We have therefore proven the following Theorem.

Theorem 5.12. *Suppose that* $\frac{\beta}{|\Omega|} - \beta_0 < \alpha,\ \beta > 4\pi,$ *and* $\beta \in \mathbb{R} \setminus \{4m\pi|\ m = 1,\ 2, \ldots\}$*. Then* (5.4) *has a non-constant solution.*

In the following discussions, we will use a Lyapunov functional. Let (u, v) be a solution for (5.3), with $u \geq 0$. We introduce the following Lyapunov functional [26, 37].

$$F(u, v) = \int_\Omega \left[\frac{1}{2\gamma} \left(d_2|\nabla v|^2 + \lambda v^2 \right) + u(\log u - 1) + 1 - (u-1)v \right] \, dx. \tag{5.41}$$

By Lemma 4.7 in [26], we know that if we let

$$f(v) = \frac{1}{2\gamma} \int_\Omega \left(d_2|\nabla v|^2 + \lambda v^2 \right) \, dx - |\Omega| \log \frac{\int_\Omega e^{\kappa v} dx}{|\Omega|}, \tag{5.42}$$

then for $t \geq 0$,

$$f(v(t)) \leq F(u(t), v(t)). \tag{5.43}$$

From Lemma 5.11 and Theorem 5.12, we obtain the following Lemma.

Lemma 5.13. *Assume that* $\beta > 4\pi$ *and* $\beta \in \mathbb{R} \setminus \{4m\pi|\ m = 1,\ 2, \ldots\}$*. Then there exists a constant* $\hat{K} \leq 0$ *such that*

$$f(v) \geq \hat{K} > -\infty$$

holds for all solutions v *of* (5.23)*.*

We therefore have the following Theorem, which summarizes some known facts about blow-up solutions.

Theorem 5.14. *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain, and $\hat{K}$ be the constant from Lemma 5.13. Suppose further that $\beta > 4\pi$ and $\beta \in \mathbb{R} \setminus \{4m\pi |\ m = 1,\ 2, \ldots\}$. Then there exist initial data (u_0, v_0) such that*

$$F(u_0, v_0) < \hat{K},$$

and the corresponding solution of (5.3) *blows up in finite or infinite time. For these blow-up solutions, the following statements hold;*

1. $\lim\limits_{t \to T_{max}} \|u(x,t)\|_{L^2(\Omega)} = \infty$
2. $\lim\limits_{t \to T_{max}} \int_\Omega u(x,t)v(x,t)\ dx = \infty$
3. $\lim\limits_{t \to T_{max}} \|\nabla v(x,t)\|_{L^2(\Omega)} = \infty$
4. $\lim\limits_{t \to T_{max}} \int_\Omega e^{v(x,t)} dx = \infty$
5. $\lim\limits_{t \to T_{max}} \|u(x,t)\|_{L^\infty(\Omega)} = \lim\limits_{t \to T_{max}} \|v(x,t)\|_{L^\infty(\Omega)} = \infty$
6. *If $4\pi < \beta < 8\pi$ and Ω is a simply connected domain, then*

$$\lim_{t \to T_{max}} \int_{\partial\Omega} e^{\frac{v(x,t)}{2}}\ dS = \infty$$

Proof. We first note that for v_ε as defined in (5.7), it is clear that as $\varepsilon \to 0$,

$$f(v_\varepsilon) \to -\infty \tag{5.44}$$

and

$$\|v_\varepsilon\|_{L^2(\Omega)} \to \infty. \tag{5.45}$$

Thus, by Lemma 5.13, (5.44) and (5.45), the existence of a blow-up solution is established.

Next, suppose that $\hat{K}$ is a constant as in Lemma 5.13, and choose ε_0 arbitrary but fixed[3], and a fixed $x_0 \in \partial\Omega$ such that if we let

$$v_{\varepsilon_0}(x) = \log \frac{32\varepsilon_0^2}{(\varepsilon_0^2 + |x - x_0|^2)^2} - \frac{1}{|\Omega|} \int_\Omega \log \frac{32\varepsilon_0^2}{(\varepsilon_0^2 + |x - x_0|^2)^2}\, dx,$$

then

$$f(v_{\varepsilon_0}(x)) < \hat{K}.$$

We note that

$$v_{\varepsilon_0}(x) \in W^{1,\infty}(\Omega).$$

Now, set

$$u_\varepsilon(x) = \frac{|\Omega| e^{v_{\varepsilon_0}(x)}}{\int_\Omega e^{v_{\varepsilon_0}(x)} dx}.$$

Then $u_\varepsilon(x)$ belongs to $L^\infty_+(\Omega)$, and

$$F(u_\varepsilon(x), v_{\varepsilon_0}(x)) = f(v_{\varepsilon_0}(x)) < \hat{K}.$$

If we then choose our initial data such that $u_0(x) = u_{\varepsilon_0}(x)$ and $v_0(x) = v_{\varepsilon_0}(x)$, then we see that the corresponding solution for the Keller-Segel model has to blow up in finite or infinite time.

For the remaining statements of the Theorem, we recall the Lyapunov function (5.41). Let $\mathcal{S}_u$ and $\mathcal{S}_{v^+}$ denote the blow-up sets for $u(x,t)$ and the positive part of v respectively. Then it is known, [33], that if there are initial data (u_0, v_0) such that the solution of (5.3) blows up, then

$$\mathcal{S}_u \cap \mathcal{S}_{v^+} \neq \emptyset,$$

and

$$\lim_{t \to T_{max}} \int_\Omega |\nabla v|^2 \, dx = \infty \quad \text{and} \quad \lim_{t \to T_{max}} \int_\Omega e^v dx = \infty.$$

This establishes 3. and 4. above for a blow-up solution of (5.3).

[3] The existence of this ε_0 is guaranteed by (5.44)

Moreover, by the properties of (5.41),

$$F(u(x,t), v(x,t)) \leq F(u_0(x), v_0(x)) \leq \hat{K},$$

so that

$$\int_\Omega \frac{1}{2\gamma}\left(d_2|\nabla v|^2 + \lambda v^2\right)\, dx \leq \int_\Omega (u-1)v\, dx + \hat{K} \tag{5.46}$$

is true. From this inequality, statement 2. is established, while statement 1. follows by employing Cauchy's inequality. Combining statements 1. and 4. establishes statement 5.

It remains to prove statement 6. To this end, we remark that, Horstmann [33] showed in Lemma 3 that if $\beta < 8\pi,\ v \in H^1(\Omega)$, and $p \in (1, \frac{8\pi}{\beta})$ is arbitrary but fixed, then

$$\begin{aligned} \log\left(\frac{1}{|\Omega|}\int_\Omega e^v dx\right) \leq \frac{p}{16\pi}\int_\Omega |\nabla v|^2\, dx + \frac{2}{p}\log\left(\int_{\partial\Omega} e^{\frac{p'v}{2}}\, dS\right) \\ + K(p, p', \beta), \end{aligned} \tag{5.47}$$

where p' is the conjugate exponent of p, and $K(p,p',\beta)$ is a constant dependent of $p,\ p'$ and β. If we then use this inequality, we can estimate (5.41) from below, for $p \in (1, \frac{8\pi}{\beta})$ arbitrary but fixed, by

$$\begin{aligned} F(u,v) &\geq \int_\Omega \frac{1}{2\gamma}\left(d_2|\nabla v|^2 + \lambda v^2\right)\, dx - |\Omega|\log\left(\frac{1}{|\Omega|}\int_\Omega e^v dx\right) \\ &\geq \int_\Omega \left[\left(\frac{d_2}{2\gamma} - \frac{p|\Omega|}{16\pi}\right)|\nabla v|^2 + \frac{\lambda}{2\gamma}v^2\right] dx \\ &\quad - \frac{2|\Omega|}{p'}\log\left(\int_{\partial\Omega} e^{\frac{p'v}{2}}\, dS\right) + K_0(p,p',\beta), \end{aligned}$$

where $K_0(p,p',\beta)$ is a constant dependent of $p,\ p'$ and β. With statement 3. in mind, we see that

$$\lim_{t\to T_{max}} \int_{\partial\Omega} e^{\frac{p'v}{2}}\, dS = \infty,$$

for every $p' \in (\frac{8\pi d_2}{8\pi d_2 - \gamma|\Omega|},\ \infty)$.

Furthermore, it has been proven in [69] that if $\Omega \subset \mathbb{R}^2$ is a simply connected, smooth domain, and $v \in H^1(\Omega)$, then

$$\begin{aligned}\log\left(\frac{1}{|\Omega|}\int_\Omega e^v dx\right) \le \frac{1}{16\pi}\int_\Omega |\nabla v|^2\, dx + \frac{1}{2|\partial\Omega|}\int_{\partial\Omega} v\, dS \\ + \log\left(\frac{1}{|\partial\Omega|}\int_{\partial\Omega} e^{\frac{v}{2}}\, dS\right) + K_1,\end{aligned} \tag{5.48}$$

where K_1 is an absolute constant. If we use this inequality instead of (5.47), we get that

$$\begin{aligned}F(u,v) &\ge \int_\Omega \frac{1}{2\gamma}\left(d_2|\nabla v|^2 + \lambda v^2\right)\, dx - |\Omega|\log\left(\frac{1}{|\Omega|}\int_\Omega e^v dx\right) \\ &\ge \int_\Omega \left[\left(\frac{d_2}{2\gamma} - \frac{|\Omega|}{16\pi}\right)|\nabla v|^2 + \frac{\lambda}{2\gamma}v^2\right] dx \\ &\quad - \frac{|\Omega|}{2|\partial\Omega|}\int_{\partial\Omega} v\, dS - |\Omega|\log\left(\int_{\partial\Omega} e^{\frac{v}{2}}\, dS\right) - K_1|\Omega|,\end{aligned}$$

and this gives us statement 6., and the Theorem is established.

□

The following Lemma gives us another result for a blow-up solution.

Lemma 5.15. *If the solution* $(u(t),v(t))$ *of system* (5.3) *blows up, then we have that*

$$\lim_{t\to T_{max}} \|u(t)\log u(t)\|_{L^1(\Omega)} = \infty. \tag{5.49}$$

Proof. Let $(u(t),v(t))$ be the blow-up solution for (5.3). Since

$$\int_\Omega u(t)\log u(t)\, dx \ge -\frac{|\Omega|}{e}$$

we see that

$$F(u(t),v(t)) \ge -\frac{|\Omega|}{e} - \int_\Omega (u(t)-1)v(t)\, dx + \frac{d_2}{2\gamma}\|\nabla v(t)\|^2_{L^2(\Omega)}.$$

Moreover, it is known, [17], that

$$\|v(t)\|^2_{L^\Phi(\Omega)} \le \tilde{K}\|\nabla v(t)\|^2_{L^2(\Omega)},$$

where $\Phi(s) = e^s - s - 1$, keeping in mind that $\int_\Omega v(t)\ dx = 0$. Here and in what follows, $L^\Phi(\Omega)$ denotes the Orlicz space which corresponds to the Young function $\Phi(s)$, and $\|\cdot\|_{L^\Phi(\Omega)}$ as its norm. Let's denote the Young function complementary to Φ by Ψ, so that $L^\Psi(\Omega)$ denotes the Orlicz space, with $\|\cdot\|_{L^\Psi(\Omega)}$ is its norm. It is also known that $\Psi(s) = (s+1)\log(s+1) - s$.

Using Hölder's inequality for Orlicz spaces [2, 78], we see that

$$\begin{aligned} F(u(t), v(t)) &\geq -\frac{|\Omega|}{e} - \int_\Omega (u(t) - 1)v(t)\ dx + \frac{d_2}{2\gamma}\|\nabla v(t)\|^2_{L^2(\Omega)} \\ &\geq -\frac{|\Omega|}{e} - \|v(t)\|_{L^\Phi(\Omega)}\|u(t) - 1\|_{L^\Psi(\Omega)} + \frac{d_2}{2\gamma}\|\nabla v(t)\|^2_{L^2(\Omega)} \\ &\geq -\frac{|\Omega|}{e} - \frac{\tilde{K}}{4\varepsilon}\|u(t) - 1\|_{L^\Psi(\Omega)} + \left(\frac{d_2}{2\gamma} - \varepsilon\right)\|\nabla v(t)\|^2_{L^2(\Omega)}, \end{aligned}$$

where $\varepsilon < \frac{d_2}{2\gamma}$. Combining this with Lemma 6.3 of [26] and the fact that

$$\int_\Omega u(t)\ dx = |\Omega| \quad \forall t \geq 0,$$

we get the result. □

Remark 5.1. We remark that, except statement 6., all the statements of Theorem 5.14 and Lemma 5.15 are also true for blow-up solutions of (5.3) when $\Omega \subset \mathbb{R}^2$ has a piecewise C^2 boundary.

5.5 Blow-up by the Concavity Method

With regard to the blow up time for the solutions to our problem, we will follow the concavity method in [16]. To this end, we have the following Lemma.

Lemma 5.16. *Suppose that $\nu > 0$, and consider a positive-valued function $F = F(t)$ such that*

$$(F^{-\nu})'' \leq 0, \quad and \quad (F^{-\nu})'(0) > 0. \tag{5.50}$$

Then there exists a time t^* after which F blows up, and the inequality

$$FF'' - (\nu + 1)(F')^2 \geq 0$$

holds.

Proof. Firstly, we observe that (5.50) means that $F^{-\nu}$ is concave with an initial positive derivative, and thus, the inequality

$$F^{-\nu}(t) \leq F^{-\nu}(0) + (F^{-\nu})'(0)t \;\Leftrightarrow F^{-\nu}(t) \leq F^{-\nu}(0) - \nu F^{-\nu-1}(0)F'(0)t,$$

is true. From this, we get that

$$t \leq t^* = \frac{F^{-\nu}(0)}{\nu F^{-\nu-1}(0)F'(0)} = \frac{F(0)}{\nu F'(0)}. \tag{5.51}$$

So, t^* is an upper bound for the blow-up time of the function F.

Secondly, we note that

$$\begin{aligned}
(F^{-\nu})' &= -\nu F^{-\nu-1}F' \\
(F^{-\nu})'' &= -\nu(-\nu - 1)F^{-\nu-2}(F')^2 - \nu F^{-\nu-1}F'' \\
&= -\nu F^{-\nu-2}[FF'' - (\nu + 1)(F')^2].
\end{aligned}$$

From the conditions in (5.50), we obtain the inequality

$$FF'' - (\nu + 1)(F')^2 \geq 0,$$

and the proof of the Lemma is complete.

□

We therefore have the following theorem regarding the system (3.1).

Theorem 5.17. *Consider the problem* (3.1). *Assume that* $v \in L^\infty(\Omega)$, $v_0 \in W^{1,\infty}(\Omega)$, $u_0 \in L^2(\Omega)$, *and set*

$$F(t) = F = \int_0^t \int_\Omega u^2(x, s)dx\, ds + \omega, \tag{5.52}$$

for some constant $\omega > 0$. Then Lemma 5.16 holds true.

Proof. We start by differentiating F with respect to t to get

$$\begin{aligned} F' = \int_\Omega u^2\,dx &= \int_0^t \frac{d}{dt}\left(\int_\Omega u^2\,dx\right)\,ds + \int_\Omega u_0^2\,dx \\ &= 2\int_0^t \int_\Omega u u_t\,dx\,ds + \int_\Omega u_0^2\,dx \\ &= 2\int_0^t \int_\Omega u u_t\,dx\,ds + C_0, \end{aligned} \tag{5.53}$$

where $C_0 = \int_\Omega u_0^2\,dx$. Differentiating in (5.53) with respect to t gives

$$\begin{aligned} F'' &= 2\int_\Omega u u_t\,dx \\ &= 2\int_0^t \frac{d}{dt}\left(\int_\Omega u u_t\,dx\right)\,ds + 2\int_\Omega u u_t\,dx\bigg|_0 \\ &= 2\int_0^t \frac{d}{dt}\left(\int_\Omega u u_t\,dx\right)\,ds + C_1, \end{aligned} \tag{5.54}$$

where

$$\begin{aligned} C_1 = 2\int_\Omega u u_t\,dx\bigg|_0 &= 2\int_\Omega u(d_1\Delta u - \nabla\cdot(u\chi\nabla v))\,dx\bigg|_0 \\ &= 2\left[\chi\int_\Omega u_0\nabla u_0\nabla v_0\,dx - d_1\int_\Omega |\nabla u_0|^2\,dx\right] \\ &= 2\left[\chi\int_\Omega u_0\nabla u_0\nabla v_0\,dx - d_1\int_\Omega |\nabla u_0|^2\,dx\right]. \end{aligned} \tag{5.55}$$

Note that (5.54) can be rewritten as

$$F'' = C\int_0^t\int_\Omega u_t^2\,dx\,ds + \left[2\int_0^t \frac{d}{dt}\left(\int_\Omega u u_t\,dx\right)\,ds - C\int_0^t\int_\Omega u_t^2\,dx\,ds\right] + C_1.$$

Squaring both sides of (5.53) and using Young's inequality, we write that

$$\begin{aligned} (F')^2 &= \left(2\int_0^t\int_\Omega u u_t\,dx\,ds + C_0\right)^2 \\ &= 4\left(\int_0^t\int_\Omega u u_t\,dx\,ds\right)^2 + 4C_0\int_0^t\int_\Omega u u_t\,dx\,ds + C_0^2 \\ &\le (4+\varepsilon)\left(\int_0^t\int_\Omega u u_t\,dx\,ds\right)^2 + C_\varepsilon C_0^2, \end{aligned}$$

for some constant C_ε. It then follows, by applying the Cauchy-Schwarz inequality, that

$$\begin{aligned} (F')^2 &\le (4+\varepsilon)\int_0^t\int_\Omega u_t^2\,dx\,ds\cdot\int_0^t\int_\Omega u^2\,dx\,ds + C_\varepsilon C_0^2 \\ &\le (4+\varepsilon)F\int_0^t\int_\Omega u_t^2\,dx\,ds + C_\varepsilon C_0^2. \end{aligned}$$

Thence,

$$FF'' - \frac{C}{4+\varepsilon}(F')^2 \geq F\left[2\int_0^t \frac{d}{dt}\left(\int_\Omega uu_t\, dx\right)\, ds - C\int_0^t\int_\Omega u_t^2\, dx\, ds + C_1\right] - \frac{C}{4+\varepsilon}C_\varepsilon C_0^2.$$

The expression between the square brackets above can be rewritten as follows;

$$\begin{aligned}
&2\int_0^t \frac{d}{dt}\left(\int_\Omega uu_t\, dx\right)\, ds - C\int_0^t\int_\Omega u_t u_t\, dx\, ds\\
&\quad = 2\int_0^t \frac{d}{dt}\left(\int_\Omega u(d_1\Delta u - \nabla\cdot(u\chi\nabla v))\, dx\right)\, ds - C\int_0^t\int_\Omega u_t(d_1\Delta u - \nabla\cdot(u\chi\nabla v))\, dx\, ds\\
&\quad = 2d_1\int_0^t (-\|\nabla u\|_2^2)' ds + 2\int_0^t \frac{d}{dt}\left(\chi\int_\Omega u\nabla u\nabla v\, dx\right)\, ds\\
&\quad - C\chi\int_0^t\int_\Omega u\nabla u_t\nabla v\, dx\, ds + \frac{Cd_1}{2}\int_0^t (\|\nabla u\|_2^2)' ds\\
&\quad = 2d_1\int_0^t (-\|\nabla u\|_2^2)' ds + 2\chi\int_0^t\int_\Omega (u_t\nabla u\nabla v + u\nabla u_t\nabla v + u\nabla u\nabla v_t)\, dx\, ds\\
&\quad - C\chi\int_0^t\int_\Omega u\nabla u_t\nabla v\, dx\, ds + \frac{Cd_1}{2}\int_0^t (\|\nabla u\|_2^2)' ds.
\end{aligned}$$

If we choose $C = 2$, then we get

$$\begin{aligned}
&2\int_0^t \frac{d}{dt}\left(\int_\Omega uu_t\, dx\right)\, ds - C\int_0^t\int_\Omega u_t^2\, dx\, ds\\
&\quad = d_1\left[\|\nabla u_0\|_{L^2(\Omega)}^2 - \|\nabla u\|_{L^2(\Omega)}^2\right] + 2\chi\int_0^t\int_\Omega (u_t\nabla u\nabla v + u\nabla u\nabla v_t)\, dx\, ds.
\end{aligned}$$

Since $F \geq \omega$, we get

$$\begin{aligned}
FF'' - \frac{2}{4+\varepsilon}(F')^2 \geq F\Bigg\{ & d_1\left[\|\nabla u_0\|_{L^2(\Omega)}^2 - \|\nabla u\|_{L^2(\Omega)}^2\right] + 2\chi\int_0^t\int_\Omega (u_t\nabla u\nabla v + u\nabla u\nabla v_t)\, dx\, ds\\
& + 2\|\nabla v_0\|_{L^\infty(\Omega)}\|u_0\|_{L^2(\Omega)}\|\nabla u_0\|_{L^2(\Omega)} - 2d_1\|\nabla u_0\|_{L^2(\Omega)}^2 - \frac{2C_\varepsilon\|u_0\|_{L^2(\Omega)}^4}{(4+\varepsilon)\omega}\Bigg\}\\
\geq\ & Fd_1\left\{\|\nabla u_0\|_{L^2(\Omega)}^2 - \|\nabla u\|_{L^2(\Omega)}^2\right\} + 2\|\nabla v_0\|_{L^\infty(\Omega)}\|u_0\|_{L^2(\Omega)}\|\nabla u_0\|_{L^2(\Omega)}\\
& - 2d_1\|\nabla u_0\|_{L^2(\Omega)}^2 - \frac{2C_\varepsilon\|u_0\|_{L^2(\Omega)}^4}{(4+\varepsilon)\omega}\\
\geq\ & 0,
\end{aligned}$$

and the proof of the Theorem is complete. □

Chapter 6

Attraction-Repulsion KS Equations in Scale of Banach Spaces

6.1 Introduction

In this chapter, we study the well-posedness and asymptotic global dynamics of the attraction-repulsion Keller-Segel system of equations admitting the following abstract formulation:

$$\begin{cases} U_t + \mathcal{A}_p U = P(u)U, \\ U(0) = U_0 \in E_q^\alpha \times E_r^\beta \times E_r^\beta, 0 \leq \alpha - \beta < 1, q, r \geq 1, \end{cases} \tag{6.1}$$

modelling aggregation of microglia in Alzheimer's disease, where $U = (u, v, w)^\top$ have entries holding meanings as in (4.2). For notational convenience, we will set $v = \psi_2,\ w = \psi_3$. In (6.1), we have that

$$\begin{aligned} \mathcal{M}_{3\times 3}(L^p(\Omega, \mathbb{R}^3)) \ni \mathcal{A}_p &= \operatorname{diag}[-\Delta, -\Delta + \lambda_2, -\Delta + \lambda_3] \\ &: D(\mathcal{A}_p) \subset L^p(\Omega, \mathbb{R}^3) \to L^p(\Omega, \mathbb{R}^3) \end{aligned} \tag{6.2}$$

with domain

$$D(\mathcal{A}_p) := \left\{\varphi \in H^{2,p}(\Omega, \mathbb{R}^3) : \partial_{\vec{n}}\varphi = 0 \quad \text{on} \quad \Gamma\right\}, \tag{6.3}$$

considered for real valued vector functions defined on an open bounded subset $\Omega \subset \mathbb{R}^N$ possessing a smooth boundary $\Gamma = \partial\Omega$, $\vec{n}$ denotes the unit outward pointing normal vector to Γ,

$$P(u)U = \left(-\sum_{j=2}^{3} \mathrm{Div}(u(-1)^j \chi_j \nabla \psi_j), a_2 u, a_3 u \right)^{\top}, \tag{6.4}$$

and the biophysical constants are as defined in (4.5), with $d_i = 1,\ i = 1,\, 2,\, 3.$

More precisely, the evolutionary equation (6.1) reads the following chemotaxis system of equations

$$\begin{cases} u_t &= \Delta u - \sum_{j=2}^{3} \mathrm{Div}(u(-1)^j \chi_j \nabla \psi_j), \\ v_t &= \Delta v - \lambda_2 v_2 + a_2 u, \\ w_t &= \Delta w - \lambda_3 w + a_3 u, \qquad\qquad \text{in } \Omega \times \dot{I}, \\ 0 &= \partial_{\vec{n}} u = \partial_{\vec{n}} \psi_j \quad \text{on } \Gamma \times \dot{I}, \\ u(0) &= u_0,\ \psi_j(0) = \psi_0 \qquad \text{in } \Omega, \end{cases} \tag{6.5}$$

where $\dot{I} = (0, T)$, $I = [0, T)$, and in simplification we have written

$$-\sum_{j=2}^{3} \mathrm{Div}(u(-1)^j \chi_j \nabla \psi_j) = -\nabla \cdot (u\,(\chi_2 \nabla v - \chi_3 \nabla w)) =: P(u)\psi. \tag{6.6}$$

Recall that (6.6) can be viewed in the sense of distributions as the weak form

$$P_{\Omega}(u)\psi := \langle P(u)\psi, \varphi \rangle_{p',p} = \sum_{j=2}^{3} \int_{\Omega} u(-1)^j \chi_j \nabla \psi_j \nabla \varphi \tag{6.7}$$

in adequate function spaces.

It is clear to see that the system of equations (6.5) has L^1- spatial integrable solutions in taking the L^p- dual product with as test function $\varphi = (1,1,1)^{\top}$ in distributions sense. More concretely,

$$\begin{aligned} &\frac{d}{dt}\int_{\Omega} u = 0 \Rightarrow u_{\Omega}(t) = \int_{\Omega} u_0(x), \quad \forall t \in \dot{I}, \\ &\frac{d}{dt}\int_{\Omega} \psi = -\lambda \int_{\Omega} \psi + a \int_{\Omega} u \Rightarrow \psi_{\Omega}(t) = e^{-\lambda t}\psi_0 + \frac{a|\Omega|\bar{u}_0}{\lambda}(1 - e^{-\lambda t}), \quad \forall t \in \dot{I}, \end{aligned}$$

where $u_\Omega = \int_\Omega u = |\Omega|\overline{u}$. Thus, if $T = \infty$ we obtain

$$\mathcal{M} = \left\{ A \in \mathbb{R}^3 : A = \left(|\Omega|\bar{u}_0, \frac{a_2|\Omega|\bar{u_0}}{\lambda_2}, \frac{a_3|\Omega|\bar{u_0}}{\lambda_3} \right)^\top \right\}, \tag{6.8}$$

as the time limit set of L^1- spatially integrable solutions.

On the other hand, the stationary equations to (6.5), using La-Salle- Hale-Henry [30] invariance principle, can be deduced associated to the system of equations (6.5) in following the works of [36] by means of the Lyapunov function

$$J(u,\psi) = \int_\Omega u \ln u - \kappa \int_\Omega u\psi + \frac{\kappa}{ar} \int_\Omega \left(|\nabla\psi|^r + \lambda|\psi|^r \right), \tag{6.9}$$

where $\psi = (\psi_2, \psi_3)$, $\lambda = (\lambda_2, \lambda_3)$, $a = (a_2, a_3)$, $\kappa = \operatorname{sgn} \sum_{j=2}^{3} (-1)^j \chi_j > 0$ to which holding onto, if the system of equations is globally well-posedness in time, then it implies studying of the non-local elliptic problem

$$\begin{cases} \Delta\psi - \lambda\psi + \mu e^{\kappa\psi} = 0 \quad \text{in } \Omega, \\ \frac{\partial\psi}{\partial\vec{n}} = 0 \quad \text{on } \Gamma = \partial\Omega, \end{cases} \tag{6.10}$$

where

$$\mu = a \frac{\int_\Omega u}{\int_\Omega e^{\sum_{j=2}^{3}(-1)^j \chi_j \psi}} = a \frac{\int_\Omega u_0}{\int_\Omega e^{\sum_{j=2}^{3}(-1)^j \chi_j \psi}},$$

using the implied conclusion in (6.8). Henceforth, in alternatives we can distinguish the following possible situations:

$$e^{\sum_{j=2}^{3}(-1)^j \chi_j} \begin{cases} \ll \; 1 \quad \text{if } \chi_2 \ll \chi_3, \\ \geq \; 1 \quad \text{if } \chi_2 \geq \chi_3, \\ \gg \; 1 \quad \text{if } \chi_2 \gg \chi_3 \end{cases} \tag{6.11}$$

corresponding respectively to that:

$$\begin{gathered}\text{Repulsion coefficient dominating strongly the attraction coefficient.}\\ \text{Attraction coefficient dominating mildly the repulsion coefficient.}\\ \text{Attraction coefficient dominating significantly the repulsion coefficient.}\end{gathered}\tag{6.12}$$

We point-out here that, as technical basis for our analysis, we use abstract dynamical systems theory for evolutionary equations [30, 68, 66], where the approach dictates that, to solve the equations (6.5) and to understand their qualitative properties one has to seeks for solutions satisfying the integral equations

$$\begin{gathered}\mathcal{F}(u,u_0)(t) := e^{\Delta t}u_0 + \int_0^t e^{\Delta(t-s)}P(u)\psi(s)ds,\\ \psi(t,\psi_0) := e^{(\Delta-\lambda)t}\psi_0 + a\int_0^t e^{(\Delta-\lambda)(t-s)}u(s)ds, \quad \psi = \psi_j, j = 2, 3,\end{gathered}\tag{6.13}$$

and vice-versa. Note that if (6.13) is to be properly defined, then the non homogeneous terms of the equations (6.5) need to be such that they are mapped into the spaces of the initial data.

This chapter is organized as follows. In Section 6.2, we give some preliminaries on the function spaces, and $E_q^\alpha - E_p^\beta$ heat kernel estimates of the semigroup associated to the operator (6.2), which might not have been covered in Chapter 1.

Section 6.3, is devoted to the well-posedness of the system of equations (6.5) in $L^\sigma(\dot{I}; L^p(\Omega))$ taking $\alpha, \beta = 0$, i.e. $u_0 \in L^q(\Omega), v_0, w_0 \in L^r(\Omega)$. In order, to gain control over the coupled term in (6.6) of the cell density equation in (6.5) we introduce the Banach space

$$Z_{q'} := \Big\{ \nabla z \in L^p(\Omega); -\mathrm{Div}(a(x)\nabla z) \in L^{q'}(\Omega),\ a(x) \in L^{\Theta}(\Omega) \supset H^{1,q}(\Omega) \quad \text{fixed}, \\ \frac{1}{q} = \frac{1}{\Theta} + \frac{1}{p} \Big\} \subset H^{-1,q'}(\Omega), \tag{6.14}$$

endowed with the norm

$$\|a(x)\nabla z\|_{Z_{q'}} = \|a(x)\nabla z\|_q + \|\mathrm{Div}(a(x)\nabla z)\|_{q'} \cong \|\mathrm{Div}(a(x)\nabla z)\|_{q'} < \infty.$$

Then we prove the following lemma.

Lemma 6.1. *Assume in* (6.6) *that* $u\chi\nabla\psi \in Z_{q'}, \quad q' \geq \frac{2N}{N-2}$. *Then,* $P_\Omega(u)\psi \in H^{-1,q'}(\Omega)$ *is well defined, and*

$$\|P_\Omega\|_{\mathcal{L}(H^{1,q}(\Omega),H^{-1,q'}(\Omega))} = \sup_{\|\nabla\varphi\|_{q'}\leq 1} \frac{|\langle P(u)\psi, \varphi\rangle_{q,q'}|}{\|u\chi\nabla\psi\|_q} \leq 2(Ne\pi)^{-1} < 1. \tag{6.15}$$

Moreover,

$$p \geq q \geq \frac{p}{2} \iff N \geq q \geq \frac{N}{2}, \tag{6.16}$$

is a valid Sobolev spaces embedding relation, with $q' \geq p$ *if* $N < 4$ *and* $p > q' \geq q$ *if* $N \geq 4$.

Important to take note of is that, (6.16) imply studying the cells density equation up to the critical space $H^{1,N}(\Omega)$, and the reduction of the system of equations in the large time asymptotic dynamics to the non-local elliptic problem (6.10), to which the Moser-Trudinger inequality imply well-posedness only if

$$\kappa \leq N\omega_{N-1}^{\frac{1}{N-1}}, \quad \text{where} \quad \omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(N/2)}, \tag{6.17}$$

denote the measure of the unit sphere in $\mathbb{R}^N$, $N \geq 2$. In the context of Lemma 6.1, we obtain that the system of equations (6.5) admits a unique solution of at least class

$$\begin{aligned} X^p_{q,r}(I) &:= C(I;L^q(\Omega)) \cap \mathcal{L}^{\infty}_{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}(I;L^p(\Omega)) \cap L^{\sigma}(I;L^p(\Omega)) \times \\ &\quad \times [C(I;L^r(\Omega)) \cap C(I;H^{1,p}(\Omega)) \cap C^1(I;L^p(\Omega))]^2 \\ &= V \times Z \times Z. \end{aligned} \tag{6.18}$$

More precisely, we have the following theorem.

Theorem 6.2. *Consider the system of equations* (6.5) *with* $u_0 \in L^q(\Omega)$,$\psi_0 \in L^r(\Omega)$, *and assume that Lemma 6.1 holds. Let* $u \in L^{\sigma}(\dot{I};L^{\Theta}(\Omega))$, *for* $r,p \geq \Theta$, *be such that*

$$\frac{1}{\sigma} + \frac{N}{2\Theta} \leq \min\left\{1 + \frac{N}{2r}, \frac{1}{2} + \frac{N}{2p}\right\}. \tag{6.19}$$

Then,

(i) $\psi \in C(I;L^r(\Omega)) \cap C(\dot{I};H^{1,p}(\Omega))$.

(ii) *If (i) holds, then* $\mathcal{F}(u,u_0) \in \mathcal{L}^{\infty}_{\frac{N}{2\Theta}}(\dot{I};L^p(\Omega))$ *satisfies that the mapping*

$$L^q(\Omega) \times L^{\sigma}(\dot{I};L^{\Theta}(\Omega)) \ni (u_0,u) \to \mathcal{F}(u,u_0) \in \mathcal{L}^{\infty}_{\frac{N}{2\Theta}}(\dot{I};L^p(\Omega)) \tag{6.20}$$

is linear and continuous. Furthermore, $\mathcal{F}(u,u_0)$ *is locally Hölder continuous with values in* $L^p(\Omega)$.

(iii) *The system of equations* (6.5) *admits a unique solution in the class* (6.18). *That is* $U \in X^p_{q,r}(I)$.

A priori uniform boundedness in $\Omega \times \dot{I}$ of the cells density solution is proven in Subsection 6.3.1, yielding, as a result, that the complete system solution is a global classical solution. Independent to this conclusion, we obtain the following proposition.

Proposition 6.3. *Consider the system of equations* (6.5) *in the context of Theorem 6.2. Then,*

$$\limsup_{t\nearrow+\infty} \|u(t)\|_p = 0, \quad \limsup_{t\nearrow+\infty} \|\nabla\psi(t)\|_p = 0, \tag{6.21}$$

and the system of equations define an extended or perturbed analytic semigroup in L^p- spaces. Moreover, in the global asymptotic dynamics ,it holds that

$$\limsup_{t\nearrow\infty} \|(u(t), v(t), w(t))^\top\|_p = A^* \in \mathcal{M} \cup \{0\}, \tag{6.22}$$

where the limit set $\mathcal{M}$, as defined in (6.8)*, corresponds to the L^1- spatial integrable solutions of the system equations in distributions sense.*

In Section 6.4, we prove similar results to those of Section 6.3, but in a much more general function space setting, which includes one used in Chapter 4 of the scale of Hilbert spaces. More precisely, we give a treatment of the equations in Bessel potential spaces E_q^α, $\alpha \in \mathbb{R}, 1 < q < \infty$. To this end, we first establish the following nesting relation between the spaces;

$$E_p^\alpha \longmapsto E_q^\alpha \longmapsto E_p^\beta \longmapsto E_q^\beta \longmapsto E_r^\beta, \tag{6.23}$$

and

$$E_{q'}^\alpha \rightleftarrows E_p^\alpha \longmapsto E_q^\alpha \longmapsto E_{p'}^\beta \longmapsto E_{q'}^\beta \rightleftarrows E_p^\beta. \tag{6.24}$$

Then, prove the following counterpart to Lemma 6.1.

Lemma 6.4. *Assume* (6.23)-(6.24) *hold, and let $u \in E_q^\alpha, \psi \in E_p^\beta$, $\beta \geq \frac{1}{2}$, $0 \leq \alpha - \beta < 1, p \geq q$. Then, if*

$$\frac{1}{2} + \frac{N}{2p} \leq \alpha + \beta, \quad \textit{and} \quad 1 + \frac{N}{2p} \leq 2\alpha + \beta, \tag{6.25}$$

hold, then the product $u\chi\nabla\psi \in E_q^\alpha$*, and the weak form* $P_\Omega(u) \in E_{q'}^0 \subseteq E_{p'}^{-\beta}$*, for* $q' \geq \frac{2N}{N-4\alpha}$*,* $p \geq \frac{N}{2\alpha}$*, are well defined, and*

$$\|P_\Omega\|_{\mathcal{L}(E_q^\alpha, E_{q'}^{-\beta})} := \sup_{\|\varphi\|_{\alpha,q'} \leq 1} \frac{|\langle P(u)\psi, \varphi\rangle_{E_q^\alpha, E_{q'}^{-\beta}}|}{\chi\|u\nabla\psi\|_{\alpha,q}} \leq \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}}. \tag{6.26}$$

In particular, $P_\Omega \in \mathcal{L}_{lip}(E_q^\alpha, E_{q'}^\beta)$ *is satisfied.*

To conclude, we give similar results to those in Theorem 6.2-Proposition 6.3 in the following theorem.

Theorem 6.5. *Assume in the system of equations* (6.1)-(6.4) *that Lemma 6.4 holds. Then,*

(i) *The system of equations admits a unique* C^1- *strong solution. Furthermore, there exists a constant*

$$\begin{aligned}\omega &= \min\Bigg\{1 - \left(\frac{2}{Ne\pi}\right)^{\alpha+\beta-\frac{1}{2}} \sum_{j=2}^{3}\left(\chi_j + \frac{a_j}{q}\right), \\ &\qquad 1 - \left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \sum_{j=2}^{3}\left(\chi_j + \frac{a_j}{r}\right)\Bigg\} > 0 \end{aligned} \tag{6.27}$$

such that the coupled system differential operator in (6.5) *defines a perturbed analytic semigroup in* $Z^{\alpha+\beta} = E_q^\alpha \times E_r^\beta \times E_r^\beta$ *spaces, and*

$$\limsup_{t\nearrow\infty} \|(u(t), v(t), w(t))^\top\|_{\alpha+\beta} = A^* \in \mathcal{M} \cup \{0\} \tag{6.28}$$

where the limit set $\mathcal{M}$ *is defined as in* (6.8) *corresponds to* L^1- *spatial integrable solutions of the system (6.5) of equations in distributions sense.*

(ii) *Assume that the first condition in* (6.25) *is verified strictly. Then, the solution semigroup is a classical solution.*

To make the proof of the results in above theorems accessible to the reader we give, in the next section, some preliminaries.

6.2 Preliminaries

In the sequel, we use, as function spaces setting, the inhomogeneous Sobolev spaces [5, 30, 66, 78] in terms of Bessel potential spaces $H^{s,p}(\Omega) = (I-\Delta)^{-\frac{s}{2}}L^p(\Omega)$, $s \in \mathbb{R}, 1 < p < \infty$, see Section 2.4. In this regard, the scale spaces $E_p^\alpha := H^{2\alpha,p}(\Omega)$, $\alpha \in [-1,1]$ associated with the operator (6.2) are well defined, using the complex interpolation-extrapolation method, with dual spaces $[E_p^\alpha]^* := E_{p'}^{-\alpha}, \frac{1}{p}+\frac{1}{p'} = 1$ and norm notation either $\|\cdot\|_{\alpha,p}$, or simply $\|\cdot\|_\alpha$ if there is no confusion caused, while $\|\cdot\|_p$ if $\alpha = 0$ for L^p spaces equipped with dual spaces product

$$\langle \cdot,\cdot\rangle_{p,p'} := \int_\Omega \cdot, \text{ of functions } \varphi \in L^p \text{ and } |\varphi|^{p-2}\varphi \in L^{p'}. \tag{6.29}$$

Similarly, we will adopt, for the spaces $L^\sigma(\dot{I}; E_p^\alpha)$, the norm notation $\|\cdot\|_{\sigma,\alpha,p}$, and $\|\cdot\|_{\sigma,p}$ if $\alpha = 0$.

Next we observe that for any $f(\cdot) \in H^{s,p}(\Omega)$, it holds that

$$f(\lambda\cdot) \in H^{s,p}(\Omega) \Rightarrow \|f(\lambda\cdot)\|_{s,p} = \lambda^{s-\frac{N}{p}}\|f(\cdot)\|_{s,p}, \tag{6.30}$$

and by comparing the behaviour of the norms at $\lambda = \infty$ for $f(\lambda\cdot) \in H^{s_j,p_j}$, with together that at $\lambda = 0$ for $f(\lambda\cdot) \in L^{p_j}$, $j = 1,2$, we establish that the spaces embedding conditions;

$$s_1 \geq s_2 \geq 0, \quad 1 < p_1 \leq p_2 < \infty, \quad s_1 - \frac{N}{p_1} \geq s_2 - \frac{N}{p_2}, \quad -\frac{N}{p_1} \leq -\frac{N}{p_2}, \tag{6.31}$$

are verified, whenever

$$H^{s_1,p_1}(\Omega) \subset H^{s_2,p_2}(\Omega) \quad \text{with } s_j = 0,\, L^{p_1}(\Omega) \subset L^{p_2}(\Omega) \iff p_1 = p_2, \tag{6.32}$$

continuously, and if $s_1 > s_2$, $p_1 \leq p_2$ the inclusions are compact. Important, to take consideration of is that (6.31) implying (6.32) does not obey the standard Sobolev spaces embedding pattern, except if $p_1 = p_2$ in $H^{s,p}$.

As usual the conditions (6.31) are related with the degrees

$$\deg(H^{s,p}) = s - \frac{N}{p}, \quad \text{and} \quad \deg(L^p) = -\frac{N}{p},$$

of smoothness of the spaces in ascertaining the embeddings (6.32) validity. In addition, the interpolation inequality

$$\|\varphi\|_{s,p} \leq C \|\varphi\|^{\theta}_{s_1,p_1} \|\varphi\|^{1-\theta}_{s_2,p_2}, \tag{6.33}$$

where $\theta \in [0,1]$, $\quad \frac{1}{p} \leq \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$, $\quad 1 < p, p_1, p_2 < \infty$ and

$$s - \frac{N}{p} \leq \theta \left(s_1 - \frac{N}{p_1} \right) + (1-\theta) \left(s_2 - \frac{N}{p_2} \right),$$

is attained for the Bessel potential spaces. In particular, we have the Sobolev type embeddings (3.5) hold, with best constants (3.6).

In view to solving the homogeneous equations of (6.5), it is well known, [5, 30, 68], that the operator $\mathcal{A}_p$ in (6.2)-(6.3) is an infinitesimal generator of an analytic semigroup

$$\{S(t) := \exp(-\mathcal{A}_p t)\,; t > 0\}$$

on the spaces $H^{s,p}(\Omega)$, satisfying, if $p_2 > p_1 > 1$, $s_2 > s_1$, that the mapping

$$S(t) : H^{s_1,p_1}(\Omega) \longmapsto H^{s_2,p_2}(\Omega)$$

verify, for any $\varphi_0 \in H^{s_1,p_1}(\Omega)$, the estimate

$$\|S(t)\varphi_0\|_{s_2,p_2} \leq \frac{Me^{-\omega t}}{t^{\frac{s_2-s_1}{2} + \frac{N}{2}\left(\frac{1}{p_1} - \frac{1}{p_2}\right)}} \|\varphi_0\|_{s_1,p_1}, \forall t > 0, \tag{6.34}$$

where $0 < \omega < \inf\{\mu \in \mathrm{Re}\ \sigma(\mathcal{A}_p)\}$, with immediate special cases when $s_1, s_2 = 0$, $s_2 = 2\alpha$, $s_1 = 2\beta$, $p_1 = p_2 = p$. Moreover, if $\frac{1}{\sigma} = \frac{N}{2}\left(\frac{1}{r} - \frac{1}{p}\right) \geq 0$, $1 \leq r \leq p \leq \infty$, $\varphi \in L^r(\Omega)$, then the mapping

$$L^r(\Omega) \ni \varphi \to S(t)\varphi \in L^\sigma([0,\infty); L^p(\Omega)) \cap C_b([0,\infty); L^r(\Omega))$$

is well- defined and

$$\|S(t)\varphi\|_{\sigma,p} \leq M\|\varphi\|_r, \quad I \subseteq [0,\infty) \tag{6.35}$$

holds, with $M \in \mathbb{R}^+ \setminus \{0\}$ independent of φ. Essential in proving the Hölder continuity of solutions is the following lemma.

Lemma 6.6 ([30, 68]). *If $p \geq q > 1$, $0 < \alpha - \beta < 1$, $0 < \gamma = \alpha - \beta + \frac{N}{2}(\frac{1}{q} - \frac{1}{p}) < 1$, then the mapping*

$$E_p^\beta \ni \varphi \to (S(t) - I)\,\varphi \in E_q^\alpha$$

satisfy that

$$\|\,(S(t) - I)\,\varphi\|_{\alpha,q} \leq C_\gamma t^\gamma \|\varphi\|_{\beta,p} \quad \forall t > 0, \tag{6.36}$$

where $C_\gamma = \frac{C_{1-\gamma}}{\gamma} > 0$ is a constant.

Let E be a Banach space, as in Chapter 1 (also see [66]), then we define by

$$\mathcal{L}_\vartheta^\infty(0,T;E) := \left\{\Phi \in E, \vartheta \in [0,1);\ \sup_{t\in(0,T)} t^\vartheta \|\Phi\|_E < \infty\right\} \tag{6.37}$$

the Lebesgue-Bochner space $t^\vartheta L^\infty(0,T;E)$, endowed with the norm

$$|||\Phi|||_{E,\vartheta} := \sup_{t\in(0,T)} t^\vartheta \|\Phi\|_E < \infty.$$

Throughout, this chapter, generic constants will be denoted by $C \geq 0$. An extension of D.R. Adams [1] 1998, results on Trudinger-Moser inequality to be useful in connection with the blow-up dynamics at the borderline Bessel spaces is the following lemma.

Lemma 6.7 ([46, 91])**.** *Let $\gamma \in (0, N)$ be positive real number, $1 < q = \frac{N}{\gamma} < \infty$ and $\tau > 0$. Then,*

$$\sup_{f \in E_q^{\gamma}, \|(I-\Delta)^{\frac{\gamma}{2}} f\|_q \leq 1} \int_\Omega e^{\kappa |f|} dx \begin{cases} \leq & C_{q,N}|\Omega| \quad \text{if } \kappa < \kappa_{N,\gamma}, \\ \\ = & +\infty \qquad \text{if } \kappa \geq \kappa_{N,\gamma}, \end{cases} \tag{6.38}$$

where, with $\omega_{N-1} = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}$ representing the measure of a unit sphere in $\mathbb{R}^N$,

$$\kappa_{N,\gamma} = \left(\frac{N}{\omega_{N-1}}\right)^{\frac{1}{q'}} \left[\frac{\pi^{\frac{N}{2}} 2^{\gamma} \Gamma(\frac{\gamma}{2})}{\Gamma(\frac{N-\gamma}{2})}\right]. \tag{6.39}$$

Lastly, we recall some notions of Orlicz spaces.

Definition 6.1 ([2, 23]; R. A. Adams, D. Edmunds)**.**

(i) A function $\Phi : \mathbb{R}^+ \longmapsto \mathbb{R}^+$ satisfying that it is increasing, convex, $\Phi(0) = 0$ and $\lim_{t \nearrow \infty} \frac{\Phi(t)}{t} = \infty$ is called a Young function.

(ii) The space L^Φ denotes the Orlicz space due to Φ, with norm

$$\|f\|_{L^\Phi(\Omega)} = \inf \left\{ \mu > 0 : \int_\Omega \Phi\left(\frac{|f(x)|}{\mu}\right) dx \leq \Phi(1) \right\}.$$

(iii) $L^\Psi(\Omega)$ is the dual space of $L^\Phi(\Omega)$ if $\Phi(1) + \Psi(1) = 1$, and

$$\int_\Omega |f(y)g(y)| dy \leq \|f\|_{L^\Phi(\Omega)} \|g\|_{L^\Psi(\Omega)}$$

holds.

(iv) The space embeddings $E_q^\alpha \subset L^\Phi(\Omega) \subset L^q(\Omega) \subset L_*^q(\Omega) \subset L^1(\Omega)$ are satisfied.

6.3 Well-posedness in $L^\sigma(\dot{I}; L^p(\Omega))$ spaces

In this section, we assume $\alpha, \beta = 0$ in the system of equations (6.1)-(6.4) and have, as our first task, to prove of the well-posedness of the weak form (6.7) in Lemma 6.1.

Proof. of Lemma 6.1. Clearly, by Hölder's inequality, the mapping

$$\begin{aligned}&(u,\psi,\varphi)\in L^{\Theta}(\Omega)\times L^{p}(\Omega)\times L^{q'}(\Omega)\\&\quad := L^{q}(\Omega)\times L^{q'}(\Omega)\ni(u\chi\nabla\psi,\varphi)\longmapsto b(u;\psi,\varphi):=\chi\int_{\Omega}u\nabla\psi\nabla\varphi\in\mathbb{R},\end{aligned}$$

for any $\varphi\in H^{1,q'}(\Omega)$, is well defined and continuous, with (6.15) holding due to (3.5).

Next, if we consider the space condition given in (6.14), using the space embeddings (3.5), since $\frac{1}{\Theta}\geq 0$, then we have $p\geq q$, and thus,

$$H^{1,p}(\Omega)\overset{(2)}{\subset}H^{1,q}(\Omega)\overset{(1)}{\subset}L^{\Theta}(\Omega). \tag{6.40}$$

Taking the embedding (1) in the space condition given in (6.14) yields $p\geq N$, while simultaneously the embedding condition $\frac{1}{p}-\frac{1}{N}\geq 0$ implies $p\leq N$. Consequently, $p=N$. But (6.40) gives $\Theta\geq p\geq q$, so that holding on the space condition in (6.14), one gets $q\geq\frac{N}{2}$. Thus we have established (6.16) must hold. The rest follows by interpolation, see (6.32), and distinguishing cases. The proof of the lemma is complete. □

Proof. of Theorem 6.2. To prove (i), we consider the second integral formula in (6.13). Thus, by the semigroup estimates (6.34) the mapping $L^{r}(\Omega)\ni\psi_0\longmapsto S(t)\psi_0=e^{(\Delta-\lambda)t}\psi_0\in L^{r}(\Omega)$ is well defined, linear and continuous. Similarly, setting

$$\mathcal{G}(u)(t)=a\int_0^t e^{(\Delta-\lambda)(t-s)}u(s)ds,$$

we get that the mapping $L^{\sigma}(\dot{I};L^{\Theta}(\Omega))\ni u\longmapsto\mathcal{G}(u)(t)\in C(\dot{I};L^{r}(\Omega))$ is linear and continuous. In fact,

$$\|\mathcal{G}(u)(t)\|_r\leq aM\int_0^t(t-s)^{-\frac{N}{2}(\frac{1}{\Theta}-\frac{1}{r})}\|u(s)\|_{\Theta}\leq aM_{1-\sigma'}t^{\frac{1}{\sigma'}-\frac{N}{2}(\frac{1}{\Theta}-\frac{1}{r})}\|u\|_{\sigma,\Theta},$$

where $M_{1-\sigma'}=aM\left(1-\frac{\sigma' N}{2}(\frac{1}{\Theta}-\frac{1}{r})\right)^{-\frac{1}{\sigma'}}$, together with, by hypothesis, that $\frac{1}{\sigma'}-\frac{N}{2}\left(\frac{1}{\Theta}-\frac{1}{r}\right)\geq 0$. Consequently,

$$\|\psi(t)\|_r\leq M\|\psi_0\|_r+M_{1-\sigma'}t^{\frac{1}{\sigma'}-\frac{N}{2}(\frac{1}{\Theta}-\frac{1}{r})}\|u\|_{\sigma,\Theta}. \tag{6.41}$$

But

$$\nabla S(t) = \nabla e^{(\Delta-\lambda)t} \in \mathcal{L}(L^r(\Omega), L^p(\Omega)), \quad \mathcal{G}(\nabla u) \in \mathcal{L}(L^\sigma(I; L^\Theta(\Omega))), L^p(\Omega))$$

by semigroup estimates (6.34), from which we get

$$\|\nabla\psi(t)\|_p \leq Mt^{-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\psi_0\|_r + M_{1-\sigma'}t^{\frac{1}{\sigma'}-\frac{1}{2}-\frac{N}{2}\left(\frac{1}{\Theta}-\frac{1}{p}\right)}\|u\|_{\sigma,\Theta}, \tag{6.42}$$

of which, by hypothesis, it holds that

$$\frac{1}{\sigma'} - \frac{1}{2} - \frac{N}{2}\left(\frac{1}{\Theta} - \frac{1}{p}\right) \geq 0, \text{ and } \quad M_{1-\sigma'} = aM\left(1 - \frac{\sigma' N}{2}\left(\frac{1}{N} + \frac{1}{\Theta} - \frac{1}{p}\right)\right)^{-\frac{1}{\sigma'}}.$$

In either of the cases, the continuity follows easily. Since, for example, with t, $h > 0$ fixed one concludes, using (6.41), that

$$\|\psi(t+h) - \psi(t)\|_r \leq \|\left(e^{(\Delta-\lambda)h} - I\right)\psi(t)\|_r + M_{1-\sigma'}\|u\|_{\sigma,\Theta}h^{\frac{1}{\sigma'}-\frac{N}{2}(\frac{1}{\Theta}-\frac{1}{r})} \searrow 0$$

as $h \searrow 0$, and the continuity of the solution is proven. This further implies, in taking $t = 0$, and $h = t$, that $\psi(t) \to \psi_0$ in $L^r(\Omega)$ as $t \searrow 0^+$.

Now to prove (ii), since (i) holds, we have from the first integral formula in (6.13) and by Lemma 6.1 that

$$\begin{aligned}
\|\mathcal{F}(u,u_0)(t)\|_p &\leq Mt^{-\frac{N}{2\Theta}}\|u_0\|_q + \sum_{j=2}^{3}\int_0^t \|e^{\Delta(t-s)}\nabla\left(u(-1)^j\chi_j\nabla\psi_j\right)\|_p ds \\
&\leq Mt^{-\frac{N}{2\Theta}}\|u_0\|_q + M\sum_{j=2}^{3}\chi_j\int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{N}{2\Theta}\right)}\|u\nabla\psi_j\|_q ds \\
&\leq Mt^{-\frac{N}{2\Theta}}\|u_0\|_q + K_{\chi_j}\int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{N}{2\Theta}\right)}\|u(s)\|_\Theta ds \\
&\leq Mt^{-\frac{N}{2\Theta}}\|u_0\|_q + K_{\chi_j}\left(\int_0^t (t-s)^{-\sigma'\left(\frac{1}{2}+\frac{N}{2\Theta}\right)} ds\right)^{\frac{1}{\sigma'}}\|u\|_{\sigma,\Theta} \\
&\leq Mt^{-\frac{N}{2\Theta}}\|u_0\|_q + K_{\chi_j}t^{\frac{1}{\sigma'}-\left(\frac{1}{2}+\frac{N}{2\Theta}\right)}\|u\|_{\sigma,\Theta},
\end{aligned} \tag{6.43}$$

where

$$K_{\chi_j} = \left(1 - \sigma'\left(\frac{1}{2} + \frac{N}{2\Theta}\right)\right)^{-\frac{1}{\sigma'}} M\sum_{j=2}^{3}\chi_j \sup_{t\in(0,T)}\|\nabla\psi_j\|_p,$$

while the hypothesis imply $\frac{1}{\sigma'} - \left(\frac{1}{2} + \frac{N}{2\Theta}\right) > 0$. Consequently, from (6.43) we get that

$$t^{\frac{N}{2\Theta}} \|\mathcal{F}(u, u_0)\|_p \leq M\|u_0\|_q + K_{\chi_j} t^{\frac{1}{\sigma'} - \frac{1}{2}} \|u\|_{\sigma,\Theta}, \tag{6.44}$$

which proves the first part of (ii) in the theorem.

It only remains to prove the Hölder continuity of the mapping (6.13). To this end, fix $h > 0$ small such that $0 < t < t + h \leq T$. Thanks to (6.15) of Lemma 6.1 we have from

$$\begin{aligned} \mathcal{F}(u)(t+h) - \mathcal{F}(u)(t) &= \left(e^{\Delta h} - I\right) e^{\Delta t} u_0 + \int_0^t (e^{\Delta h} - I) e^{\Delta(t-s)} P(u)\psi(s) ds \\ &+ \int_t^{t+h} e^{\Delta(t+h-s)} P(u)\psi(s) ds \end{aligned}$$

using (6.36), if we let $\delta = \frac{N}{2}\left(\frac{1}{q} - \frac{1}{p}\right)$, $M_\delta = \frac{M}{\delta}$, that

$$\begin{aligned} &\|\mathcal{F}(u)(t+h) - \mathcal{F}(u)(t)\|_p \leq \| \left(e^{\Delta h} - I\right) e^{\Delta t} u_0\|_p \quad + \\ &+ \int_0^t \|(e^{\Delta h} - I) e^{\Delta(t-s)} P(u)\psi(s)\|_p ds + \int_t^{t+h} \|e^{\Delta(t+h-s)} P(u)\psi(s)\|_p ds \\ &\leq C_{1-\delta}(M_\delta) h^\delta \left(\|e^{\Delta t} u_0\|_q + \int_0^t \|e^{\Delta(t-s)} P(u)\psi(s)\|_p ds \right) \quad + \\ &\qquad + \int_t^{t+h} (t+h-s)^{-\delta} \|P(u)\psi(s)\|_q ds \\ &\leq C_{1-\delta}(M_\delta) h^\delta \left(\|u_0\|_q + M_\pi \int_0^t (t-s)^{-\delta} \left(\sum_{j=2}^3 \chi_j \|\nabla\psi_j\|_p \right) \|u(s\|_\Theta ds \right) \quad + \\ &\qquad + M_\pi \sum_{j=2}^3 \chi_j \sup_{t \in (0,T)} \|\nabla\psi_j\|_p \int_t^{t+h} (t+h-s)^{-\delta} \|u(s)\|_\Theta ds \\ &\leq C_{1-\delta}(M_\delta) h^\delta \left(\|u_0\|_q + M_\pi \sum_{j=2}^3 \chi_j \sup_{t \in (0,T)} \|\nabla\psi_j\|_p t^{\frac{1}{\sigma'} - \delta} \|u\|_{\sigma,\Theta} \right) + \\ &\qquad + M_\pi \sum_{j=2}^3 \chi_j \sup_{t \in (0,T)} \|\nabla\psi_j\|_p h^{1-\delta} \|u\|_{\sigma,\Theta} \\ &\leq K_\chi C_{1-\delta}(M_\delta) \left(\|u_0\|_q + (t^{\frac{1}{\sigma'} - \delta} + 1) \|u\|_{\sigma,\Theta} \right) h^{\aleph}, \end{aligned} \tag{6.45}$$

where $\aleph = \min\{\delta, 1-\delta\}$, $M_\pi = \frac{2M}{Ne\pi}$, $K_\chi = \max\{1, M_\pi \sum_{j=2}^3 \chi_j \sup_{t \in (0,T)} \|\nabla\psi_j\|_p\}$, and we have used the fact that $h > 0$ is small. The proof of (iii) will be in continuation. $\square$

Remark 6.1. Note that, if we take $\nabla u \in L^{\sigma}(\dot{I}; L^{q}(\Omega))$ then

$$\|\mathcal{G}(\nabla u)(t)\|_p \leq aM_{1-\sigma'} t^{\frac{1}{\sigma'}-\frac{N}{2\Theta}} \|\nabla u\|_{\sigma,q}$$

and $\frac{1}{\sigma}+\frac{N}{2\Theta} \leq 1$ must hold. In addition, if this condition is strict, then $\nabla\psi \in L^{\infty}(\Omega\times(0,T))$, using [45, 67]. Furthermore, in (6.43) using (6.15), the yielding condition is less restrictive, i.e. $1 \geq \frac{1}{\sigma}+\frac{N}{2\Theta}$.

Now to complete the proof of the above theorem, we have the following.

Proof. of Theorem 6.2-(iii). First we note that from (ii), we only need to prove that $\mathcal{F}(u,u_0) \in L^{\sigma}(\dot{I}; L^{p}(\Omega))$ since , that it is in $C(I; L^{q}(\Omega))$ can be deduced from (6.20). To this end, we notice that (6.35) and (6.43) imply that

$$\|\mathcal{F}(u,u_0)\|_{\sigma,p} \leq M\|u_0\|_q + M_{1-\sigma}\sum_{j=2}^{3}\chi_j \sup_{t\in\dot{I}}\|\nabla\psi_j\|_p t^{\frac{1}{2}-\frac{N}{2\Theta}}\|u\|_{\sigma,\Theta} \tag{6.46}$$

holds.

Next, we define the complete metric space

$$W := \{\Psi \in V; \|\Psi\|_V \leq C = 6M\|u_0\|_q\},$$

and prove that (6.13) is a contraction mapping on W. In view to this task, we initially observe from

$$\|\mathcal{F}(u,u_0)\|_q \leq M\|u_0\|_q + M\sum_{j=2}^{3}\chi_j \sup_{t\in I}\|\nabla\psi_j\|_p T^{\frac{1}{2}-\frac{1}{\sigma}}\|u\|_{\sigma,\Theta}$$

that we have $\sup_{t\in I}\|\mathcal{F}(u,u_0)\|_q \leq 6^{-1}C\left(1+K_{\chi_j}T^{\frac{1}{2}-\frac{1}{\sigma}}\right)$, while given (6.44), we find that

$$\sup_{t\in I} t^{\frac{N}{2\Theta}}\|\mathcal{F}(u,u_0)\|_p \leq 6^{-1}C\left(1+K_{\chi_j}T^{\frac{1}{2}-\frac{1}{\sigma}}\right),$$

with, as lastly from (6.46), that

$$\|\mathcal{F}(u,u_0)\|_{\sigma,p} \leq 6^{-1}C\left(1+K_{\chi_j}T^{\frac{1}{2}-\frac{N}{2\Theta}}\right).$$

Consequently,

$$\|\mathcal{F}(u;u_0)\|_V \leq 2^{-1}C + 3^{-1}CK_{\chi_j}T^{\frac{1}{2}-\frac{1}{\sigma}} + 6^{-1}CK_{\chi_j}T^{\frac{1}{2}-\frac{N}{2\Theta}},$$

with which, if $T^{\max\{\frac{1}{2}-\frac{1}{\sigma},\frac{1}{2}-\frac{N}{2\Theta}\}} \leq \frac{1}{K_{\chi_j}}$, then we have that (6.13) maps W to itself. Furthermore,

$$\|\mathcal{F}(u_1;u_0) - \mathcal{F}(u_2;u_0)\|_V \leq 2^{-1}CK_{\chi_j}T^{\max\{\frac{1}{2}-\frac{1}{\sigma},\frac{1}{2}-\frac{N}{2\Theta}\}}\|u_1-u_2\|_V,$$

with, if $T^{\max\{\frac{1}{2}-\frac{1}{\sigma},\frac{1}{2}-\frac{N}{2\Theta}\}} < \frac{2}{CK_{\chi_j}}$, then (6.13) is a contraction mapping. Therefore, Banach Contraction Principle (Theorem 1.1, also see [8]), yields that there exists a unique fixed point $u = \mathcal{F}(u, u_0) \in W$ which solves the activated cells density equation, within a maximum time $T^* = T(\|u_0\|_q)$ implied by Picard's method.

Moreover, as proved in Theorem 6.2-(ii)., this solution is Hölder continuous, yielding as a result [30, 68] the $Z-$ regularity of the solution components in chemical attraction-repulsion concentration equations of the system. The proof of the Theorem is complete. □

6.3.1 Boundedness in $\Omega \times \dot{I}$ and asymptotic global dynamics

On global existence we have the following lemma:

Lemma 6.8. *Consider the activated cells equation of the system* (6.5)*, with* $u_0 \in L^q(\Omega)$*,* $q \in (\frac{N}{2}, N]$*. Let the best constant in* (3.6) *be* $C_{s=1}$*, and*

$$\begin{aligned} 0 < K_\chi \quad &= \quad \sum_{j=2}^{3}(-1)^j\chi_j \sup_{t\in(0,T)}\|\nabla\psi_j\|_N, \quad \tau^{\pm} = 1 \pm \tau, \\ &\quad\quad \tau = N\left(\frac{1}{2} - \frac{N}{\Theta}\right), \quad 0 < \eta \leq \frac{4}{pC_{s=1}K_\chi\tau^-} - 1, \end{aligned}$$

be such that

$$\left(\frac{\tau^+ + \tau^-\eta}{2}\right)C_{s=1}K_\chi \leq \frac{2}{p}. \tag{6.47}$$

Then, $u \in L^\infty(\Omega \times (0,\infty))$ *verifies the estimate*

$$\sup_{t>0}\|u\|_\infty \leq Mt^{-\frac{N}{2q}}\|u_0\|_q + C. \tag{6.48}$$

Moreover, the system of equations (6.5) *admits a globally defined classical solution.*

Proof. Consider, initial data $u_0 = 0$ for the cells equation in (6.5). Then, take $|u|^{p-2}u \in H^1(\Omega)$ as a test function, to find that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u|^p + \frac{4}{p'}\int_\Omega |\nabla |u|^{\frac{p}{2}}|^2 &= 2(p-1)\left(\sum_{j=2}^{3}\int_\Omega |u|^{\frac{p}{2}}\nabla |u|^{\frac{p}{2}}(-1)^j\chi_j\nabla\psi_j\right)\\ &\le \quad 2(p-1)K_\chi\|\nabla|u|^{\frac{p}{2}}\|_2\||u|^{\frac{p}{2}}\|_\Theta \le 2(p-1)C_{s=1}K_\chi\|\nabla|u|^{\frac{p}{2}}\|_2^{\tau^+}\||u|^{\frac{p}{2}}\|_2^{\tau^-}\\ &\le \quad 2(p-1)C_{s=1}K_\chi\left(\frac{\tau^+ + \tau^-\eta}{2}\|\nabla|u|^{\frac{p}{2}}\|_2^2 + |\Omega|(1+\eta^{-\frac{N}{2}})\||u|^{\frac{p}{2}}\|_1^2\right),\end{aligned}$$

following from a use of Hölder, Young, and Nirenberg-Gagliardo inequalities [2, 30, 68]. Also note that the first second line inequality follows by adding a zero term of $2\chi_3(p-1)\|\nabla|u|^{\frac{p}{2}}|_2^2\|\nabla\psi_3\|_N$, taking the negative to the right hand side, then estimating from above only the first expression of the identity and next removing the zero term to establish the estimate.

Since $p \ge 2$, taking the upper limit of $\eta = \frac{4}{p^2K_\chi C_{s=1}\tau^-} - 1$ and letting

$$C_\Omega = \left(\sqrt{\frac{(|\Omega|)^{\frac{2}{N}}C_{s=1}K_\chi\tau^-}{4(1-C_{s=1}K_\chi\tau^-)}}\right),$$

thanks to the condition (6.47), there exists an $\omega := c(p) > 0$ such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u|^p + \omega\int_\Omega |u|^p &\le C_\Omega^N(1+p^N)\left(\int_\Omega |u|^{\frac{p}{2}}\right)^2\\ \overset{implying}{\Longrightarrow} \quad \int_\Omega |u|^p &\le C_\Omega^N(1+p)^N\sup_{(0,T)}\left(\int_\Omega |u|^{\frac{p}{2}}\right)^2.\end{aligned}$$

Next define $\Lambda(r) := \sup_{(0,T)}\left(\int_\Omega |u|^r\right)^{\frac{1}{r}}$, to find that

$$\Lambda(p) \le [C_\Omega)^N(1+p)^N]^{\frac{1}{p}}\Lambda\left(\frac{p}{2}\right), \quad \forall p \ge 2,$$

from which, if we let $p_i = 2^i, i \in \mathbb{N}^*$, we obtain

$$\begin{aligned}
\Lambda(2^i) &\leq C_\Omega^{N2^{-i}}(1+2^i)^{\frac{N}{2^i}}\Lambda(2^{i-1}) \leq \ldots \leq C_\Omega^{N\sum_{k=1}^{i}2^{-k}}(1+2^i)^{2^{-i}N}\ldots \\
&\qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1+2)^{2^{-1}N}\Lambda(1) \\
&\leq C_\Omega^N\left[2^{i2^{-i}N}(2^{-i})^{2^{-i}N}\right]\ldots\ldots\left[2^{2^{-1}N}(2^{-1})^{2^{-1}N}\right]\Lambda(1) \\
&\leq C_\Omega^N 2^{N\sum_{k=1}^{i}k2^{-k}} \times 2^{N\sum_{k=1}^{i}2^{-k}}\Lambda(1) \leq C_\Omega^N 2^{3N}\Lambda(1).
\end{aligned}$$

Consequently, taking the limit as $i \to \infty$ yields

$$\|u\|_\infty \leq C_\Omega^N 2^{3N}\Lambda(1) \leq C_\Omega^N 2^{3N}\|u_0\|_1 < \infty. \tag{6.49}$$

Now holding on this, if we decompose the solution into $u = \varphi^1 + \varphi^2$, where φ^1 verifies the equation in u with $P(u)\psi = 0$, $u(0) = u_0$ and φ^2 the same equation but with $P(u)\psi \neq 0, u_0 = 0$, then it follows, using the semi-group estimates (6.34) in an iteration, that $\|\varphi^1\|_\infty \leq Mt^{-\frac{N}{2q}}\|u_0\|_q$ for all $t > 0$ while (6.49) implies $\|\varphi^2\|_\infty \leq C$.

Thus, combining these yields that (6.48) holds. The moreover conclusion of the lemma follows as in Chapter 4, using a bootstrap argument, or alternatively [24] Proposition 1. The proof of the lemma is complete. □

Independent of Lemma 6.8, the large time behaviour of the system of equations in (6.5) suggest that the system solution defines a perturbed analytic semigroup.

Proof. of Proposition 6.3. Consider the estimates (6.42), and let $h(t) = t^{\frac{1}{2}+\frac{N}{2}\left(\frac{1}{r}-\frac{1}{p}\right)}\|\nabla\psi(t)\|_p$, Then we obtain that

$$\begin{aligned}
&\sum_{j=2}^{3}\int_0^t \|e^{\Delta(t-s)}\nabla\left(u(-1)^j\chi_j\nabla\psi_j\right)\|_p ds \\
&\qquad \leq \quad M\sum_{j=2}^{3}\chi_j\int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{N}{2\Theta}\right)}\|u(s)\|_\Theta\|\nabla\psi_j\|_p ds
\end{aligned}$$

$$\begin{aligned}
&\leq \quad M\sum_{j=2}^{3}\chi_j\int_0^t (t-s)^{-\left(\frac{1}{2}+\frac{N}{2\Theta}\right)}h(s)s^{-\left(\frac{1}{2}+\frac{N}{2}\left(\frac{1}{r}-\frac{1}{p}\right)\right)}\|u(s)\|_\Theta ds &(6.50)\\
&\leq \quad t^{-\frac{N}{2}\left(\frac{1}{\Theta}+\frac{1}{r}-\frac{1}{p}\right)}M\sum_{j=2}^{3}\chi_j\sup_{s\in(0,t]}h(s)\left(\int_0^1\frac{1}{(1-\rho)^{\frac{1}{2}\left(1+\frac{N}{\Theta}\right)}\rho^{\frac{1}{r}-\frac{1}{p}}}\rho\right)\|u\|_{\sigma,\Theta},
\end{aligned}$$

after a change of variable $s=\rho t$. As this imply

$$\begin{aligned}
\|u(t)\|_p \quad &\leq \quad Mt^{-\frac{N}{2\Theta}}\|u_0\|_q+\\
&+ \quad t^{-\frac{N}{2}\left(\frac{1}{\Theta}+\frac{1}{r}-\frac{1}{p}\right)}M\sum_{j=2}^{3}\chi_j\sup_{s\in(0,t]}h(s)\left(\int_0^1\frac{1}{(1-\rho)^{\frac{1}{2}\left(1+\frac{N}{\Theta}\right)}\rho^{\frac{1}{r}-\frac{1}{p}}}d\rho\right)\|u\|_{\sigma,\Theta},
\end{aligned}$$

we get $\limsup_{t\nearrow+\infty}\|u(t)\|_p=0$, yielding, as well from (6.42), the second conclusion. Next, from (6.43), since

$$t^{\frac{N}{\Theta}}\|u\|_p\leq M\|u_0\|_q+K_{\chi_j}t^{\frac{1}{\sigma'}-\frac{1}{2}}t^{\frac{N}{\Theta}}\|u\|_p,$$

we get that, if $0<t\leq K_{\chi_j}^{-\frac{1}{\sigma'}+\frac{1}{2}}=\tau_0$, then the semigroup estimates (6.34) hold, while for $t\geq\tau_0$ we decompose $t=n\tau_0+s$ for some $0\leq s<\tau_0$ iterating n times (6.34) with $q=p$, we get

$$\begin{aligned}
\|u(t)\|_p \quad &\leq \quad M_0e^{\omega s}\|u(n\tau_0)\|_p\\
&\leq \quad M_0e^{\omega s}(M_0e^{\omega\tau_0})^{n-1}\|u(\tau_0)\|_p\leq(M_0e^{\omega\tau_0})^n\|u(\tau_0)\|_p,
\end{aligned}$$

for some $M_0=M+1$, $\omega\in\mathbb{R}$. Therefore,

$$\|u(t)\|_p\leq(Me^{\omega\tau_0})^{n+1}\tau_0^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_q\leq M_1e^{\bar{\omega}t}t^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_q,\quad 0<t\leq T^*.$$

Consequently, the result holds for all $t\in(0,\infty)$ and the system of equations solution generates a perturbed analytic semigroup, see [7, 66] for the transfer of analyticity. Lastly, since $u,\psi\perp 1$ in $L^p(\Omega)$ dual spaces product, (6.21) implies that (6.28) holds. The proof of the proposition is complete. $\square$

We now proceed to the next section of the chapter.

6.4 Well -posedness in E_p^α spaces.

In this section, we study the existence and uniqueness of solutions to the system of equations (6.1)-(6.4) as given with initial data in scales of Bessel potential spaces. Thanks, to Lemma 6.1 and Theorem 6.2, we establish using (6.32) that the space nesting relation (6.23) holds. In particular, (6.16) first condition must hold by interpolation, with together that

$$\begin{array}{ccccc} & E_p^\alpha & \longmapsto & E_{q'}^\beta & \longrightarrow (6.32) \\ (p \geq q) \leftarrow & \downarrow & \overset{(6.32)}{\nearrow} & \downarrow & \rightarrow (p \leq q') \\ & E_q^\alpha & \longmapsto & E_p^\beta & \longrightarrow (6.32) \end{array}, \qquad \text{and}$$

$$\begin{array}{ccccc} & E_q^\alpha & \longmapsto & E_{q'}^\beta & \longrightarrow (6.32) \\ (q' \geq q) \leftarrow & \uparrow & \overset{(6.32)}{\searrow} & \uparrow & \rightarrow (q' \leq p) \\ & E_{q'}^\alpha & \longmapsto & E_p^\beta & \longrightarrow (6.32) \end{array},$$

$$E_{q'}^\alpha \rightleftarrows E_p^\alpha \longmapsto E_q^\alpha \longmapsto E_{p'}^\beta \longmapsto E_{q'}^\beta \rightleftarrows E_p^\beta$$

are valid space nesting relations between the scale spaces.

Proof. of Lemma 6.4. The proof follows by space embeddings (3.5) and Hölder's inequality. In fact, the mapping

$$\begin{aligned} (u, \psi, \varphi) \quad &\in \quad E_q^\alpha \times E_p^\beta \times E_{q'}^\alpha \\ ;= \quad & E_q^\alpha \times E_{q'}^\alpha \ni (u\chi\nabla\psi, \varphi) \longmapsto b(u, \psi, \varphi) := \chi \int_\Omega u\nabla\psi\nabla\varphi \in \mathbb{R} \end{aligned} \tag{6.51}$$

is well defined and continuous, since $\nabla\psi \in E_p^{\beta-\frac{1}{2}} \subset E_p^0$ if $\beta \geq \frac{1}{2}$. Thus, $u\nabla\psi \in E_q^0$, using (3.5) of E_q^α provided that $p \geq \frac{N}{2\alpha}$, as one needs that $\frac{1}{q} - \frac{2\alpha}{N} + \frac{1}{p} \leq \frac{1}{q}$.

Furthermore, relaxing the embedding into space for $E_p^{\beta-\frac{1}{2}}$, yields $\frac{1}{q} - \frac{2\alpha}{N} + \frac{1}{p} - \frac{2\beta}{N} + \frac{1}{N} \leq \frac{1}{q}$, concluding as long as the first condition in (6.25) is verified. Thanks again to (3.5) and

Hölder's inequality we have that

$$1 \geq \frac{N-2\alpha q}{qN} + \frac{N-2(\beta-\frac{1}{2})p}{pN} + \frac{N-2(\alpha-\frac{1}{2})q'}{q'N},$$

resulting in the second hypothesis of (6.25), satisfied. Consequently, Hölder's inequality imply that

$$\begin{aligned} |\int_\Omega \chi u \nabla\psi\nabla\varphi| &\leq \|u\chi\nabla\psi\|_\Theta \|\nabla\varphi\|_{\Theta'} \\ &\leq \chi\left(\frac{2}{Ne\pi}\right)^{\alpha-\frac{1}{4}} \|u\nabla\psi\|_{\alpha,q}\|\nabla\varphi\|_{\alpha-\frac{1}{2},q'} \\ &\leq \chi\left(\frac{2}{Ne\pi}\right)^{\frac{\alpha}{2}+\frac{\beta}{2}-\frac{1}{2}} \|u\|_{\Theta_0}\|\nabla\psi\|_{\beta-\frac{1}{2},p}\|\nabla\varphi\|_{\alpha-\frac{1}{2},q'} \\ &\leq \chi\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \|u\|_{\alpha,q}\|\psi\|_{\beta,p}\|\varphi\|_{\alpha,q'}, \end{aligned} \tag{6.52}$$

using (3.5) and (3.6), taking $\frac{1}{\Theta} = \frac{1}{\Theta_0} + \frac{1}{\Theta_1}$, giving together by linearity that the mapping is Lipschitz continuous. The proof of the lemma is complete. □

Remark 6.2. A worthwhile comment is to note in the yielding condition in (6.25) that:

- The spaces embedding $E_p^{\beta-\frac{1}{2}} \subset L^\infty(\Omega)$ holds. This is because $\alpha > 0$ imply $\beta - \frac{1}{2} > \frac{N}{2p}$ and (3.5) yields the conclusion. But as $\alpha \geq \beta$ similarly $\beta > 0$ imply $E^{\alpha-\frac{1}{2}} \subset L^\infty(\Omega)$.

- The yielding condition in (6.25) is a particular case of the following

$$\frac{1}{2} + \frac{N}{2}\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\rho}\right) \leq \alpha+\beta \quad \text{and} \quad 1 + \frac{N}{2}\left(\frac{1}{q}+\frac{1}{p}-\frac{1}{\rho}\right) \leq 2\alpha+\beta, \tag{6.53}$$

  implying $P_\Omega : E_\rho^\alpha \longmapsto E_{\rho'}^{-\beta}$, $P_\Omega \in \mathcal{L}_{lip}(E_\rho^\alpha, E_{\rho'}^\beta)$, and (6.25) is obtained when $\rho = q$.

- Since $\frac{1}{2} \leq \alpha, \beta \leq 1$, the condition (6.53) yields Young's equality for convolutions, and $\rho \geq q, p$. Moreover, the inclusions (3.5) are verified with $\rho = \Theta$.

The objective of this section is to prove Theorem 6.5.

Proof. of Theorem 6.5. We will prove the theorem in stages. The first part will consist in proving that the solution to the system of equations (6.1)-(6.4) is of class

$$\begin{aligned} U &\in C(I;E_q^\alpha)\cap C(\dot I;E_q^{\alpha+1})\cap C^1(\dot I;E_{p'}^{-\beta})\times \\ &\times C(I;E_r^\beta)\cap C(\dot I;E_p^\beta)\cap C(\dot I;E_r^{\beta+1})\cap C^1(\dot I;E_r^\beta)\times \\ &\times C(I;E_r^\beta)\cap C(\dot I;E_p^\beta)\cap C(\dot I;E_r^{\beta+1})\cap C^1(\dot I;E_r^\beta). \end{aligned} \tag{6.54}$$

In this direction, we have that the next proposition is valid.

Proposition 6.9. *Assume in* (6.1)-(6.4)*,that Lemma 6.4 holds, and let* $u\in L^\sigma(\cdot I;E_q^\alpha)$ *for* σ,α,q *such that*

$$\alpha+\frac{N}{2q}\le\beta+\frac{N}{2r}+\frac{1}{\sigma'}, \tag{6.55}$$

is true. Then,

(i) $\psi\in C(I;E_r^\beta)\cap\mathcal{L}^\infty_{\frac{N}{2}\left(\frac1p-\frac1r\right)}(\dot I;E_p^\beta)\cap L^\sigma(0,\infty;E_r^\gamma)$ *for any* $\gamma\in[\beta,\beta+1)$.

(ii) *The mapping*

$$L^\sigma(\dot I;E_q^\alpha)\times E_q^\alpha\ni(u,u_0)\longmapsto\mathcal{F}(u,u_0)\in\mathcal{L}^\infty_{\frac{N}{2}\left(\frac1q-\frac1p\right)}(\dot I;E_p^\alpha)) \tag{6.56}$$

is linear Lipschitz and Hölder continuous. Moreover, $\mathcal{F}(u,u_0)(t)\searrow u_0$ *in* E_p^α *as* $t\searrow 0$.

Proof. of Proposition 6.9-(i). Consider the second integral formula given in (6.13) and let

$$S(t)\psi_0=e^{(\Delta-\lambda)t}\psi_0,\quad \mathcal{G}(u)(t)=a\int_0^t e^{(\Delta-\lambda)(t-s)}u(s)ds.$$

Thanks to the semigroup estimates (6.34), $S(t)\in\mathcal{L}(\dot I;\mathcal{L}(E_r^\beta,E_r^\beta))$ is well defined, while concurrently

$$G(u)(t)\in\mathcal{L}(\dot I;\mathcal{L}(L^\sigma(\dot I;E_q^\alpha);E_r^\beta))$$

is of continuity boundedness from above $M_{1-\sigma'}t^\vartheta$ where

$$\begin{aligned} M_{1-\sigma'} &= aM\left(1-\sigma'\left(\alpha-\beta+\frac{N}{2}(\frac1q-\frac1r)\right)\right)^{-\frac{1}{\sigma'}} \\ \vartheta &= \frac{1}{\sigma'}-\left(\alpha-\beta+\frac{N}{2}\left(\frac1q-\frac1r\right)\right)\ge 0 \quad\text{by hypothesis (6.55).} \end{aligned}$$

On the other hand, we have that

$$
\begin{aligned}
\|\psi(t+h)-\psi(t)\|_{\beta,r} &\leq \|\left(e^{(\Delta-\lambda)h}-I\right)\psi(t)\|_{\beta,r}+aM\|u\|_{\sigma,\alpha,q}\times \\
&\times \left(\int_t^{t+h}(t+h-s)^{-\sigma'\left(\alpha-\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{r})\right)}ds\right)^{\frac{1}{\sigma'}} \\
&\leq \|\left(e^{(\Delta-\lambda)h}-I\right)\psi(t)\|_{\beta,r}+aM\|u\|_{\sigma,\alpha,q}h^{\vartheta}\to 0,
\end{aligned}
$$

as $h\to 0$. In other words, the mapping

$$E_r^{\beta}\times L^{\sigma}(\dot{I};E_q^{\alpha})\ni(\psi_0,u)\to\psi\in C(\dot{I};E_r^{\beta})$$

is proper linearly defined and continuous.

Similarly, $S(t)\in\mathcal{L}(\dot{I};\mathcal{L}(E_r^{\beta},E_p^{\beta}))$ is of continuity boundedness from above $Mt^{-\frac{N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}$, whereas $\mathcal{G}(u)(t)\in\mathcal{L}(I;\mathcal{L}(L^{\sigma}(\dot{I};E_q^{\alpha}),E_p^{\beta}))$ is of a bound from above $M_{1-\sigma'}t^{\frac{1}{\sigma'}-\left(\alpha-\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})\right)}$, where

$$
\begin{aligned}
M_{1-\sigma'} &= aM\left(1-\sigma'\left(\alpha-\beta+\frac{N}{2}(\frac{1}{q}-\frac{1}{p})\right)\right)^{-\frac{1}{\sigma'}}, \\
&\frac{1}{\sigma'}-\left(\alpha-\beta+\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)\right)>0 \quad \text{by hypothesis (6.55).}
\end{aligned}
$$

Thus,

$$t^{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|\psi(t)\|_{\beta,p}\leq M\|\psi_0\|_{\beta,r}+M_{1-\sigma'}t^{\frac{1}{\sigma'}-\left(\alpha-\beta-\frac{N}{2}(\frac{1}{r}-\frac{1}{q})\right)}\|u\|_{\sigma,\alpha,q}$$

holds, i.e. the mapping $E_r^{\beta}\times L^{\sigma}(\dot{I};E_q^{\alpha})\ni(\psi_0,u)\to\psi(t)\in\mathcal{L}^{\infty}_{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}(\dot{I};E_p^{\beta})$ is linear and continuous.

Since (6.34) holds for any $\gamma,\beta\in\mathbb{R}$ satisfying $\beta\leq\gamma<\beta+1$, from (6.34) we have,

$$c_{\beta,\gamma}(t):=\|S(t)\|_{\mathcal{L}(E_q^{\beta},E_r^{\gamma})}\leq\frac{M}{t^{\gamma-\beta+\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}}\in L^1(0,\infty),$$

but unbounded at zero, unless $\gamma=\beta, q=r$. Thus we need to prove that if $u\in L^1(0,\infty;E_q^{\beta})$, then $\mathcal{G}u(t) \in L^1(0,\infty; E_r^\gamma)$. To this end, set $s = t\rho, \rho \in [0,1]$, to get that

$$\begin{aligned}
\|\mathcal{G}u(t)\|_{1,\gamma,r} &\leq a\int_0^1 \|e^{(\Delta-\lambda)(1-\rho)t}u(t\rho)t\|_{1,\gamma,r}d\rho. \\
&\leq a\int_0^1\int_0^\infty \frac{\tau}{\rho^2}c_{\beta,\gamma}\left(\tau\left(\frac{1-\rho}{\rho}\right)\right)\|u(\tau)\|_{\beta,q}d\tau d\rho \qquad \text{(on setting } \tau = t\rho) \\
&\leq a\left(\int_0^\infty c_{\beta,\gamma}(s)ds\right)\left(\int_0^\infty \|u(\tau)\|_{\beta,q}d\tau\right) \qquad \text{(on seting } s = \tau\tfrac{(1-\rho)}{\rho}).
\end{aligned}$$

As a result we obtain that

$$\|\psi(t)\|_{1,\gamma,r} \leq \|c_{\gamma,\gamma}(t)\|_1\|\psi_0\|_\gamma + \|c_{\beta,\gamma}(s)\|_1\|u\|_{1,\beta,q},$$

where $\|c_{\beta,\gamma}(s)\|_1 = \|c_{\beta,\gamma}(s)\|_{L^1(0,\infty)}$. The case $\sigma = \infty$, follows exactly as in the first lines of this proof. Thus, it holds that

$$\|\psi(t)\|_{\infty,\gamma,r} \leq \|c_{\gamma,\gamma}(t)\|_\infty\|\psi_0\|_{\gamma,r} + \|c_{\beta,\gamma}(s)\|_1\|u\|_{\infty,\beta,q},$$

and by interpolation, we conclude the result for any $1 < \sigma < \infty$. This completes the proof of (i) in the proposition. □

Proof. of Proposition 6.9-(ii). Taking the second integral formula of the solution in (6.13) we find, using (6.34) and (6.26), that

$$\begin{aligned}
\|\mathcal{F}(u,u_0)(t)\|_{\alpha,p} &\leq Mt^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{\alpha,q} + M\|P_\Omega\|_{\mathcal{L}(E_q^\alpha, E_{p'}^{-\beta})} \quad \times \\
&\quad \times\sum_{j=2}^3 \chi_j \sup_{t\in(0,T)}\|\nabla\psi_j\|_{\beta-\frac{1}{2},p}\int_0^t (t-s)^{-\left(\alpha+\beta+\frac{N}{2}\left(1-\frac{2}{p}\right)\right)}\|u(s)\|_{\alpha,q}ds \\
&\leq Mt^{-\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|u_0\|_{\alpha,q} + M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}}\sum_{j=2}^3\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\beta-\frac{1}{2},p} \\
&\qquad \times \quad t^{\frac{1}{\sigma'}-\left(\alpha+\beta+\frac{N}{2}\left(1-\frac{2}{p}\right)\right)}\|u\|_{\sigma,\alpha,q}, \qquad (6.57)
\end{aligned}$$

where (6.25) imply $\frac{1}{\sigma'} - \left(\alpha+\beta+\frac{N}{2}\left(1-\frac{2}{p}\right)\right) > 0$. Note that the result remains true using the semigroup estimates (6.34) directly to control the contribution of ∇ in $P(u)\psi \in E_{p'}^\beta$

with obvious modifications. Consequently, the desired conclusion follows from the fact that

$$\begin{aligned} t^{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}\|\mathcal{F}(u,u_0)\|_{\alpha,p} &\leq M\|u_0\|_{\alpha,q} \quad + M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \times \\ &\times \sum_{j=2}^{3}\chi_j \sup_{t\in(0,T)}\|\nabla\psi_j\|_{\beta-\frac{1}{2},p}t^{\gamma(p,q)}\|u\|_{\sigma,\alpha,q}, \qquad (6.58) \end{aligned}$$

where $\gamma(p,q) = \frac{1}{\sigma'} + \frac{N}{2}\left(\frac{1}{p}+\frac{1}{q}\right) \geq \alpha+\beta+\frac{N}{2}$.

To prove the Hölder continuity, let us fix $h>0$ small such that $0<t<t+h\leq T$, then compute

$$\begin{aligned} \mathcal{F}(u)(t+h)-\mathcal{F}(u)(t) &= \left(e^{\Delta h}-I\right)e^{\Delta t}u_0 + \int_0^t (e^{\Delta h}-I)e^{\Delta(t-s)}P(u)\psi(s)ds \\ &+ \int_t^{t+h} e^{\Delta(t+h-s)}P(u)\psi(s)ds. \qquad (6.59) \end{aligned}$$

Thus, using (6.36), if we let

$$\delta = \frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right), \quad \gamma(p) = \left(\alpha+\beta+\frac{N}{2}\left(1-\frac{2}{p}\right)\right),$$

then we obtain that

$$\begin{aligned} &\|\mathcal{F}(u)(t+h)-\mathcal{F}(u)(t)\|_{\alpha,p} \leq \|\left(e^{\Delta h}-I\right)e^{\Delta t}u_0\|_{\alpha,p} \quad + \\ &\quad + \int_0^t \|(e^{\Delta h}-I)e^{\Delta(t-s)}P(u)\psi(s)\|_{\alpha,p}ds + \int_t^{t+h}\|e^{\Delta(t+h-s)}P(u)\psi(s)\|_{\alpha,p}ds \\ &\leq C_{1-\delta}(M_\delta)h^\delta\left(\|e^{\Delta t}u_0\|_{\alpha,q} + \int_0^t\|e^{\Delta(t-s)}P(u)\psi(s)\|_{\alpha,p}ds\right) \quad + \\ &\quad + M\int_t^{t+h}(t+h-s)^{-\gamma(p)}\|P(u)\psi(s)|_{-\beta,p'}ds \\ &\leq C_{1-\delta}(M_\delta)h^\delta\Bigg(\|u_0\|_{\alpha,q} + M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}}\sum_{j=2}^{3}\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\beta-\frac{1}{2},p} \times \\ &\quad \times \int_0^t (t-s)^{-\gamma(p)}\|u(s)\|_{\alpha,q}ds\Bigg) + M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}}\sum_{j=2}^{3}\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\beta-\frac{1}{2},p} \times \\ &\quad \times \int_t^{t+h}(t+h-s)^{-\gamma(p)}\|u(s)\|_{\alpha,q}ds \end{aligned}$$

$$
\begin{aligned}
&\leq C_{1-\delta}(M_\delta)h^\delta \Bigg(\|u_0\|_{\alpha,q} + M_{1-\sigma'}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \sum_{j=2}^{3}\chi_j \sup_{t\in(0,T)} \|\nabla\psi_j\|_{\beta-\frac{1}{2},p} \times \\
&\quad \times \quad t^{\frac{1}{\sigma'}-\gamma(p)}\|u\|_{\sigma,\alpha,q}\Bigg) \qquad + \\
&\qquad + \quad M_{1-\sigma'}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \sum_{j=2}^{3}\chi_j \sup_{t\in(0,T)} \|\nabla\psi_j\|_{\beta-\frac{1}{2},p} h^{\frac{1}{\sigma'}-\gamma(p)}\|u\|_{\sigma,\alpha,q} \\
&\leq CC_{1-\delta}(M_\delta)\left(\|u_0\|_{\alpha,q} + (t^{\frac{1}{\sigma'}-\gamma(p)}+1)\|u\|_{\sigma,\alpha,q}\right) h^{\aleph},
\end{aligned} \tag{6.60}
$$

where

$$
\begin{aligned}
\aleph &= \min\left\{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right), \frac{1}{\sigma'} - \left(\alpha+\beta+\frac{N}{2}\left(1-\frac{2}{p}\right)\right)\right\}, \\
C &= \max\left\{1, M_{1-\sigma'}\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} \sum_{j=2}^{3}\chi_j \sup_{t\in(0,T)} \|\nabla\psi_j\|_{\beta-\frac{1}{2},p}\right\}
\end{aligned}
$$

and we have used, as assumed in the first lines, that $h > 0$ is small.

Lastly, in (6.59) taking $t = 0$ then $h = t$ and proceeding as in arguments above leading to (6.60), we obtain the convergence at $t \searrow 0^+$ in (ii) of the proposition. □

To complete the proof of the Theorem 6.5-(ii), we need the following lemma.

Lemma 6.10. *Consider the subset*

$$
W := \left\{\xi \in C(I; E_q^\alpha);\ \sup_{t\in(0,T)} \|\xi(t)\|_{\alpha,q} \leq C\|\xi_0\|_{\gamma,q}\right\},
$$

for any $\gamma \in [\beta, \alpha)$ and the $u-$ integral equation in (6.13). Then,

(i) *$\mathcal{F}W \subset W$, i.e. it maps W to itself.*

(ii) *The mapping $\mathcal{F} : E_q^\gamma \to E_q^\alpha$ is a contraction mapping.*

(iii) *There exists a unique $u \in W$ such that $u = \mathcal{F}(u)$ is a solution to (6.5) up to a maximal time $T^*(\|u_0\|_\gamma)$ of existence of solutions of (6.1)-(6.4).*

Proof. Thanks to (6.23) - (6.24), we can read the right hand side sum of (6.13) in taking the norm of $E_q^\alpha = E_q^\gamma \times E_q^{\alpha-\gamma}$ as in the scale spaces product, whereas by virtue of the Lemma 6.4, we get that $u\chi\nabla\psi$ is well defined in $E_q^0 \cong L^q(\Omega)$. Therefore, if $u \in W$ one obtains the following

$$\begin{aligned}
\|\mathcal{F}(u)(t)\|_\alpha &\le M\|u_0\|_\gamma + M\sum_{j=2}^{3}\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|u\chi_j\nabla\psi_j\|_q ds \\
&\le M\|u_0\|_\gamma + M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}\sum_{j=2}^{3}\chi_j\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|u\|_\alpha\|\nabla\psi_j\|_{\gamma-\frac{1}{2}} ds \\
&\le M\|u_0\|_\gamma + MC\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}\sum_{j=2}^{3}\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\gamma-\frac{1}{2}}\|u_0\|_\gamma \quad \times \\
&\quad \times \int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)} ds \\
&\le M\|u_0\|_\gamma + MC\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}\sum_{j=2}^{3}\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\gamma-\frac{1}{2}}\|u_0\|_\gamma T^{\frac{1}{2}-(\alpha-\gamma)}.
\end{aligned}$$

Thus, for

$$T = \left(\left(\frac{1}{M}-\frac{1}{C}\right)\frac{1}{\sum_{j=2}^{3}\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\gamma-\frac{1}{2}}}\left(\frac{2}{Ne\pi}\right)^{\frac{1-2\alpha-\gamma}{2}}\right)^{\frac{2}{1-2(\alpha-\gamma)}},$$

we get (i) of the lemma is satisfied.

To prove (ii) of the lemma, we evaluate $\mathcal{F}$ at $u_1, u_2 \in W$ using the same initial data to conclude that

$$\begin{aligned}
\|\mathcal{F}(u_1)(t)-\mathcal{F}(u_2)(t)\|_\alpha &\le \sum_{j=2}^{3}\int_0^t \|\nabla e^{\Delta(t-s)}((u_1-u_2)\chi_j\nabla\psi_j)(s)\|_\alpha ds \\
&\le M\sum_{j=2}^{3}\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|(u_1-u_2)\chi_j\nabla\psi_j\|_\gamma ds \\
&\le M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}}\sum_{j=2}^{3}\chi_j\int_0^t (t-s)^{-\frac{1}{2}-(\alpha-\gamma)}\|u_1-u_2\|_\alpha\|\nabla\psi_j\|_{\gamma-\frac{1}{2}} \\
&\le M\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\gamma}{2}-\frac{1}{2}} T^{\frac{1}{2}-(\alpha-\gamma)}\sum_{j=2}^{3}\chi_j\sup_{t\in(0,T)}\|\nabla\psi_j\|_{\gamma-\frac{1}{2}}\sup_{t\in(0,T)}\|u_1-u_2\|_\alpha.
\end{aligned}$$

This proves that (ii) of the lemma is valid on taking

$$T < \left(\frac{1}{M \sum_{j=2}^{3} \chi_j \sup_{t\in(0,T)} \|\nabla \psi_j\|_{\gamma-\frac{1}{2}}} \left(\frac{2}{Ne\pi} \right)^{\frac{1-2\alpha-\gamma}{2}} \right)^{\frac{2}{1-2(\alpha-\gamma)}}.$$

Hence viewed concurrently with (i) of the same lemma, we obtain (iii) using Banach contraction mapping theorem and Picard's method, or classical continuation method to allow for the extension of the existence time to maximal time $T^* = T(\|u_0\|_\gamma)$ of existence of the equations. $\square$

To complete the proof of the first part of (i) of the theorem, we note that the Hölder continuity of u in view of the complete system of equations, since the restriction $f(t) = P(u)\psi(t) \in E^{\beta}_{p'}$ is locally Hölder continuous, we get using Lemma 3.2.1 in [30] that solutions to the system of equations are C^1 strong solutions and the regularity properties in (6.54) are verified.

As for the second part (i) of the Theorem 6.5 we proceed to prove that the coupled system elliptic equations

$$\mathcal{A}_p(\phi)\eta = \begin{pmatrix} -\Delta u + \nabla \cdot (\phi\chi_2 \nabla v) - \nabla \cdot (\phi \chi_3 \nabla w) \\ -\Delta v + \lambda_2 v - a_2 u \\ -\Delta w + \lambda_3 w - a_3 u \end{pmatrix}, \tag{6.61}$$

where $\nabla\cdot = \mathrm{Div}$, with $\phi \in E^{\alpha}_q$ possibly fixed, define a perturbed analytic semigroup. To this end, let $\eta = (u, v, w)^\top, z = (\varphi_1, \varphi_2, \varphi_3)^\top$, then define

$$b : Z_{\alpha+\beta} \times Z_{\alpha+\beta} \to \mathbb{R}, Z_{\alpha+\beta} := E^{\alpha}_q \times E^{\beta}_r \times E^{\beta}_r,\ \alpha + \beta \geq \frac{1}{2} + \frac{N}{2r} \tag{6.62}$$

by

$$
\begin{aligned}
b(\phi;\eta,z) &:= \langle \mathcal{A}(\phi)\eta, z\rangle \\
&= \int_\Omega \nabla u \nabla \varphi_1 + \int_\Omega \nabla v \nabla \varphi_2 + \int_\Omega \nabla w \nabla \varphi_3 \\
&\qquad - \chi_2 \int_\Omega \phi \nabla v \nabla \varphi_1 + \chi_3 \int_\Omega \phi \nabla w \nabla \varphi_1 \quad + \\
&+ \lambda_2 \int_\Omega v\varphi_2 - a_2 \int_\Omega u\varphi_2 + \lambda_3 \int_\Omega w\varphi_3 - a_3 \int_\Omega u\varphi_3. \qquad (6.63)
\end{aligned}
$$

Note that continuity of the bilinear form (6.63) follows easily using Hölder's inequality, space embeddings (3.5) with (3.6) as the best constant, and the fact that $\|z\|_{\alpha+\beta} > \|\varphi_j\|_\alpha,\ \|\varphi_j\|_\beta$, $\alpha \geq \beta$. Thus, to apply Browder-Minty Theorem, it only remains to prove that (6.63) is coercive, since from this the strictly monotonicity of the operator

$$
\begin{aligned}
\mathcal{A}_p(\phi) &= \mathcal{A}_p - P(\phi) \\
&= \begin{pmatrix} -\Delta & \nabla\cdot(\phi\chi_2\nabla\quad) & -\nabla\cdot(\phi\chi_3\nabla\quad) \\ -a_2 & -\Delta+\lambda_2 & 0 \\ -a_3 & 0 & -\Delta+\lambda_3 \end{pmatrix} \\
&\qquad : Z^{\alpha+\beta-\frac{1}{2}}_{q,r} = E^\alpha_q \times E^{\beta-\frac{1}{2}}_r \times E^{\beta-\frac{1}{2}}_r \to Z^{-\alpha-\beta+\frac{1}{2}}_{q',r;} \qquad (6.64)
\end{aligned}
$$

follows easily, and thus is invertible. To the cited task, we note that thanks to (6.24)-(6.26) and Young's inequality, we have that the integrals in (6.63) involving chemo-attraction-repulsion coefficients are well-controlled from below. In addition,

$$
\begin{aligned}
\int_\Omega uv &\leq \|u\|_{\Theta_1}\|v\|_{\Theta_2} \leq \left(\frac{2}{Ne\pi}\right)^{\alpha+\beta-\frac{1}{2}} \|\nabla u\|_{\alpha-\frac{1}{2}}\|\nabla w\|_{\beta-\frac{1}{2}} \\
&\leq \left(\frac{2}{Ne\pi}\right)^{\alpha+\beta-\frac{1}{2}} \left(\frac{1}{q}\|\nabla u\|^q_{\alpha-\frac{1}{2}} + \frac{1}{r}\|\nabla v\|^r_{\beta-\frac{1}{2}}\right),
\end{aligned}
$$

using Hölder and Young's inequalities, together with the first hypothesis in (6.27) yields that

$$
\begin{aligned}
b(\phi;\eta,\eta) &\geq \|\nabla u\|^q_{\alpha-\frac{1}{2}} + \|\nabla v\|^r_{\beta-\frac{1}{2}} + \|\nabla w\|^r_{\beta-\frac{1}{2}} \\
&- \left(\frac{2}{Ne\pi}\right)^{\alpha+\beta-\frac{1}{2}} \sum_{j=2}^{3}\left(\chi_j + \frac{a_j}{q}\right)\|\nabla u\|^q_{\alpha-\frac{1}{2}} \\
&- \left(\frac{2}{Ne\pi}\right)^{\alpha+\beta-\frac{1}{2}} \sum_{j=2}^{3}\left(\chi_j + \frac{a_j}{r}\right)\|\nabla \psi_j\|^r_{\beta-\frac{1}{2}} \qquad (6.65) \\
&\geq \omega\|\nabla\eta\|^{2\rho}_{\gamma-\frac{1}{2}},
\end{aligned}
$$

where $\gamma = \alpha, \beta$ and $\rho = q, r$ depending on the variable. Consequently, Browder -Minty theorem [8, 10] asserts that the operator (6.64) is invertible. Moreover, it is a sectorial operator in $E^0_q \times E^0_r \times E^0_r \cong L^q(\Omega) \times L^r(\Omega) \times L^r(\Omega)$, since it holds that

$$
\begin{aligned}
&\|(\mathcal{A}_p + \mu)^{-\alpha} P(\phi)\|_{q\times r\times r} \\
&\quad = \|(-\Delta+\mu)^{-\alpha} P(\phi)\|_q + \sum_{j=2}^{3}\|(-\Delta+\lambda_j+\mu)^{-\beta} a_j\|_r \\
&\quad \leq C\max\left\{\frac{1}{\mu^\alpha}, \frac{1}{\mu^\beta}\right\}\sum_{j=2}^{3}\left(\chi_j\left(\frac{2}{Ne\pi}\right)^{\alpha+\frac{\beta}{2}-\frac{1}{2}} + a_j\right),
\end{aligned}
$$

for any $0 \leq \alpha < 1$ satisfying the Lemma 6.4-(6.25), for some $C \in \mathbb{R}^+ \setminus \{0\}$, $|\pi - \arg\mu| \geq \vartheta$, $\vartheta < \frac{\pi}{2}$, using Corollary 1.4.5 in [30] or Theorem 7.1.3. This imply that (6.28) is valid, since (6.64) is an infinitesimal generator of analytic semigroup.

Alternatively, thanks to Proposition 6.9, $\psi \in \mathcal{L}^\infty_{\frac{N}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}(\dot{I}; E^\beta_p)$, $u \in \mathcal{L}^\infty_{\frac{N}{2}\left(\frac{1}{q}-\frac{1}{p}\right)}(\dot{I}; E^\alpha_p)$, and using (6.8), we obtain that (6.28) is verified, in the large time asymptotic dynamics of the system equations (6.5).

To round off, we observe that the proof of (ii) of the theorem is straight-forward from the first assertion in Remark 6.2. In fact, since $\alpha - \frac{1}{2} > \frac{N}{2q}$ we have $E^{\alpha-\frac{1}{2}}_q \subset L^\infty(\Omega)$ holds by virtue of (3.5), and (6.28) imply that $\nabla\psi_j \in L^\infty(\Omega)$ is bounded for all $t > 0$. As $u \in E^0_q \cong L^q(\Omega)$, $q > \frac{N}{2}$ and $1 \geq \alpha - \frac{1}{2} > \frac{N}{2q}$, viewing the weak form (6.51) in L^q as well as the equation in elliptic form by passing u_t to the right hand side, using [67], we get

$u \in L^{\infty}(\Omega)$ is bounded for all $t > 0$. The rest is trivial or immediate. $\square$

6.5 Blow-up dynamics

In this section, we give an overview analysis of the blow-up of solutions to the system equations (6.1) at the borderline spaces E_p^{α}, $\alpha = \frac{N}{2p}$.

Theorem 6.11. *The system of equations* (6.5) *admits* (6.9) *as a Lyapunov function. Moreover, if in* (6.11) *we assume that* (6.12) *is satisfied such that*

$$\kappa = \sum_{j=2}^{3} (-1)^j \chi_j \geq \kappa_{N,\beta} \iff (6.39), \quad \text{then} \quad \|(u,v,w)^{\top}\|_{\alpha+\beta} = \infty \tag{6.66}$$

for any $t \in (0,\infty)$, *i.e. the system solution blows-up independent of time,*

Proof. To show that (6.9) is a Lyapunov function, we write the $v, w-$ equations as a single equation in ψ. Then, taking the dual spaces product (6.29) with, as test function, $\eta = \ln u - \kappa\psi \in E_{q'}^{\alpha}$ in the $u-$ equation and setting $\psi_t \in E_{r'}^{\beta}$, we obtain that,

$$\begin{aligned}
\frac{dJ(t)}{dt} &= \int_{\Omega} u_t \ln u + \int_{\Omega} u_t - \kappa \int_{\Omega} u_t\psi - \kappa \int_{\Omega} \psi_t u \quad + \\
&\quad + \frac{\kappa}{a}\left(\int_{\Omega} |\nabla\psi|^{r-2}\nabla\psi\nabla\psi_t + \lambda\int_{\Omega} |\psi|^{r-2}\psi\psi_t\right) \\
&= \int_{\Omega} u_t(\ln u - \kappa\psi) - \frac{\kappa}{a}\int_{\Omega} |\psi_t|^r \\
&= \int_{\Omega} \nabla\left(\nabla u - \kappa u\nabla\psi\right)\left(\ln u - \kappa\psi\right) - \frac{\kappa}{a}\int_{\Omega} |\psi_t|^r \\
&= -\int_{\Omega} \left(\nabla u - \kappa u\nabla\psi\right)\nabla\left(\ln u - \kappa\psi\right) - \frac{\kappa}{a}\int_{\Omega} |\psi_t|^r \\
&= -\int_{\Omega} u|\nabla(\ln u - \kappa\psi)|^r - \frac{\kappa}{a}\int_{\Omega} |\psi_t|^r \leq 0,
\end{aligned} \tag{6.67}$$

having used the dual space function representation in (6.29), and the fact that

$$\int_{\Omega} u_t = 0, \quad \nabla(\ln u - \kappa\psi) = u\left(\frac{\nabla u}{u} - \kappa\nabla\psi\right),$$

to yield the first assertion of the theorem. The results conclude that the Lyapunov function (6.9) is decreasing along the trajectories of the orthogonal to constant solutions of the equations in (6.5) as time increases to infinity.

Now, to prove (6.66) of theorem, we note that (6.65) holds using the best constant of the inclusion $E_p^\alpha, \alpha = \frac{N}{2p}$ in (3.6), while associated to (6.10), is the energy functional

$$E(\psi) = \frac{1}{r}\|\nabla\psi\|^r_{\beta-\frac{1}{2}} + \frac{\lambda}{r}\|\psi\|^r_\beta - \mu\ln\left(\int_\Omega e^{\kappa\psi}\right) \geq 0. \tag{6.68}$$

Consequently, (6.65) yields

$$b(\phi;\eta,\eta) \geq \omega\|\nabla u\|^q_{\alpha-\frac{1}{2}} + \mu\ln\left(\int_\Omega e^{\kappa\psi}\right),$$

using the second embedding condition in (3.6), and (6.38) implies the conclusion, on taking $\eta \in Z_{\alpha+\beta}$ as a test function in the complete system equations (6.5), then integrating in time $t \in (0,T)$ using a reduction to absurd argument.

In fact supposing that the conclusion was false, it follows from

$$\begin{aligned} 0 \quad &= \quad \frac{d}{dt}\|U\|^\rho_{\alpha+\beta,\rho} + b(u;\eta,\eta) \\ &\geq \quad \frac{d}{dt}\|U\|^\rho_{\alpha+\beta,\rho} + \omega\|\nabla u\|^q_{\alpha-\frac{1}{2}} + \mu\ln\left(\int_\Omega e^{\kappa\psi}\right) \\ \iff &\quad \|U_0\|^\rho_{\alpha+\beta,\rho} \geq \|U\|^\rho_{\alpha+\beta,\rho} + \mu\int_0^t \ln\left(\int_\Omega e^{\kappa\psi(s)}\right)ds \\ &\geq \quad \mu\int_0^t \ln\left(\int_\Omega e^{\kappa\psi(s)}\right)ds = \infty, \end{aligned}$$

that the contrary to the premises is true, since the norm $\|U_0\|^\rho_{\alpha+\beta,\rho} = \|u_0\|^q_{\alpha,q} + \|v_0\|^r_{\beta,r} + \|w_0\|^r_{\beta,r}$ is finite. This therefore, imply that the last assertion of the theorem is valid. For an alternative, much finer approach, see [26, 36], which can easily be adapted to our situation from their results in the case of $Z_{\alpha+\beta}, \alpha = \beta = \frac{1}{2}, q = r = 2$, the Lyapunov function (6.9) and using the Definition 6.1. $\square$

Conclusion

In conclusion of this thesis, we remark that the importance of the results of this thesis is in the role played by the best constant of the scale spaces into the $L^{\Theta}-$spaces. This has yielded in the the studies of the well-posedness of the system of equation neither the need of the initial data to the system of equation, nor time for a contraction mapping in application of Banach fixed point theorem to be small respectively. We, however, point out that, much more still need to be done in relation to the complete analysis of the semilinear eigenvalue problem (6.10) at the borderline space E_r^{β} i.e. $2\beta = \frac{N}{r}$, in context of establishing the Palais-Smale condition in view of the Trudinger-Moser inequality, and Pohozaev's identity for nonexistence of solutions. It is the hope of the author that this task should complete elegantly the treatment of the blow-up analysis concerned with the system equations (6.5) in the general function spaces setting insofar provided for studying of the ARKS equations in (6.1).

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