ALGEBRAIC PROPERTIES OF ORDINARY DIFFERENTIAL EQUATIONS

P G L Leach

Algebraic Properties of Ordinary Differential Equations

by

PETER GAVIN LAWRENCE LEACH
BSc BA DipEd (Melb) MSc PhD (La Trobe)

Volume I: A Critical Review

This thesis
is submitted in fulfilment of the requirements for the degree of
Doctor of Science in the Faculty of Science, University of Natal

Durban
September 1995

Acknowledgments

In one's dissertation and thesis grateful thanks were made to supervisor and typiste[1] for their respectively useful contributions to getting the matter finalised, to fellow students and, maybe, colleagues for their thoughtful comments and careful reading of the manuscript and, if one were lucky, to pioneers and/or experts in the field who were kind enough to offer useful as well as encouraging advice.

In a work of what is supposed to be one's maturity it is not quite the same and yet themes persist. I must continue to thank Professor (Emeritus) C J Eliezer[2] for the inspiration which he provided me almost twenty years ago when I commenced my doctoral studies. The field has changed a little, but I am still indebted to him for the development of the personal philosophy of the appreciation of the beauty of Mathematics[3]. This is not to gainsay the importance of the practical applications of Mathematics, but to emphasise that there is a fundamentally artistic side to the subject. The sides are not fixed. So often it has happened that one generation's pleasure has been the work horse of the next.

It also happens that the scope of those to whom appreciation for assistance received must be made has broadened. It was Geoffrey Prince who started me on the road to the algebraic properties of differential equations when he came to La Trobe University as a doctoral student in the late seventies. Willy Sarlet of the Rijksuniversiteitse Gent was an excellent collaborator in the work on the linearisation of scalar nonlinear second order ordinary differential equations. It

[1]That was in the days when research students did not know how to prepare mathematical documents.

[2]Sometime Professor of Mathematics in the Department of Applied Mathematics, La Trobe University, Melbourne.

[3]The philosophy may well be a mixture of his own and that of *his* supervisor, P A M Dirac, who is reported to have remarked once that it was not the Physics which mattered but the elegance of the equations.

has been a pleasure to work with Barbara Abraham–Shrauner of Washington University in St Louis. Her insights into hidden and nonlocal symmetries have been of great value. George Flessas of the University of the Aegean has been a source of interesting problems and a fount of knowledge of Lie's writings due to the advantage of undertaking his Ph D studies in Vienna. Part of the preparation of this work was undertaken whilst I enjoyed the hospitality of his Department on the island of Samos.

I also thank Mariano Torrisi of the Università degli Studi Catania for the hospitality of his Department whilst the first draft of this thesis acted as a stand–in for Julius Caesar. The tranquility of the accommodation provided at the Villa Mater Sanctitatis[4] and the kind attentions of the dear women of the Apostollic Oblate Movement provided an ideal time for the contemplation of the correct placement of commas, split infinitives *etc.*

Those who are most deserving of my gratitude are my students in South Africa. In the areas of linear systems and Hamiltonian systems there was Vivian Gorringe. More recently Ryan Lemmer has collaborated effectively on further Hamiltonian problems and some nonlinear second order equations. However, the ones to whom I am most indebted are Fazal Mahomed, now a Senior Lecturer at the University of the Witwatersrand, Johannesburg, and Keshlan Govinder of the University of Natal, Durban. With Fazal much was done on the matters of linearisation, classification of equations by their algebras and the algebras of nth order linear ordinary differential equations. With Keshlan the algebraic properties of Ermakov systems were unravelled, the algebraic properties of the first integrals of linear higher order equations determined and the interface between the algebraic properties of differential equations and the possession of the Painlevé property explored.

Once upon a time one would thank a devoted typiste for turning hieroglyphs into clear text. It seems that those days are gone for most of us. However, I must record my appreciation of Donald Knuth for taking the time off from

[4]Via Bottego 10, 95125 Catania, Italia

his proper activities to devise TeX which has made the process of preparing mathematical, and scientific in general, text æsthetically rewarding. Keshlan Govinder's patient tutoring in the ways of TeX is a tribute to man's humanity to man.

I thank Ryan Lemmer for proofreading the whole text and Bogdan Stefanski, André Larísma, Jaynie Padayachee, Gareth Amery and Theo Pillay for subjecting sections of the final draft to close scrutiny.

La Trobe University, the University of the Witwatersrand, Johannesburg, the Rijksuniversiteitse Gent, the University of Natal, the University of the Aegean, the C S I R of South Africa and the Foundation for Research Development of South Africa are thanked for their support.

Declaration

I, Peter Gavin Lawrence Leach, affirm that the material contained in this thesis has not (to my knowledge) been published elsewhere except where due reference has been made in the text and that this thesis is not being and has not been used for the award of any other degree or diploma in any university or other institution.

P G L Leach
September 1995

Summary

In Chapter One the theoretical basis for infinitesimal transformations is presented with particular emphasis on the central theme of this thesis which is the invariance of ordinary differential equations, and their first integrals, under infinitesimal transformations. The differential operators associated with these infinitesimal transformations constitute an algebra under the operation of taking the Lie Bracket. Some of the major results of Lie's work are recalled. The way to use the generators of symmetries to reduce the order of a differential equation and/or to find its first integrals is explained. The chapter concludes with a summary of the state of the art in the mid-seventies just before the work described here was initiated.

Chapter Two describes the growing awareness of the algebraic properties of the paradigms of differential equations. This essentially *ad hoc* period demonstrated that there was value in studying the Lie method of extended groups for finding first integrals and so solutions of equations and systems of equations. This value was emphasised by the application of the method to a class of nonautonomous anharmonic equations which did not belong to the then pantheon of paradigms. The generalised Emden–Fowler equation provided a route to major development in the area of the theory of the conditions for the linearisation of second order equations. This was in addition to its own interest. The stage was now set to establish broad theoretical results and retreat from the particularism of the seventies.

Chapters Three and Four deal with the linearisation theorems for second order equations and the classification of intrinsically nonlinear equations according to their algebras. The rather meagre results for systems of second order equations are recorded.

In the fifth chapter the investigation is extended to higher order equations for which there are some major departures away from the pattern established

at the second order level and reinforced by the central rôle played by these equations in a world still dominated by Newton. The classification of third order equations by their algebras is presented, but it must be admitted that the story of higher order equations is still very much incomplete.

In the sixth chapter the relationships between first integrals and their algebras is explored for both first order integrals and those of higher orders. Again the peculiar position of second order equations is revealed.

In the seventh chapter the generalised Emden–Fowler equation is given a more modern and complete treatment.

The final chapter looks at one of the fundamental algebras associated with ordinary differential equations, the three element $s\ell(2,R)$, which is found in all higher order equations of maximal symmetry, is a fundamental feature of the Pinney equation which has played so prominent a rôle in the study of nonautonomous Hamiltonian systems in Physics and is the signature of Ermakov systems and their generalisations.

Dedication

To my students

Prologue

The diversity of subject matters which attract the intellectual interests of man is extraordinary. Many times there is a practical bent which supplies the incentive to pursue a particular interest. At others it is the sheer delight of intellectual activity. In this instance there is a blending of the two. I solve differential equations as a service to others or, even, as a self-serving occupation. I also play with differential equations to see what secrets lie within, which makes me a plaything of ordinary differential equations except when I make demands of them and they become my playthings. The blending is more than just the juxtaposition of two activities. What may one day be a little bit of pleasure with an obscure differential equation can well turn to a purposeful investigation the next when one of one's 'practically minded' colleagues has come up with the same equation in a situation which is the very opposite of 'airy fairy'. Whatever it is, it is always a pleasure to solve a differential equation and the methodical means afforded by Lie symmetries provide a gateway to that pleasure. In fact it accentuates the pleasure because of its very system. The parlor tricks of old have been replaced by a new rationalism! One is reminded of Lagrange's proud boast in the introduction to his *Méchanique Analytique* in 1788 that the avid reader would not find a single diagram in his tome [35, p 333]. Deceitful suggestiveness had been exiled to the yellow pages! In a similar fashion the recipes of the old artizens have been replaced by the systematic application of a principle which does not call on arcane trickery for its implementation.

One of the major methods used to solve differential equations is to transform them into other differential equations of a more recognisable variety. This is simply the differential equations version of transformation theory in Hamiltonian Mechanics. In the latter one uses the technique of canonical transformations which have lead to very interesting and useful results, particularly for the nonautonomous systems so beloved of plasma physicists. The Lie theory is deeply rooted in transformation theory. The algebras of differential equations are invariant under point transformations. Find an equation with a particular algebraic structure and there will be a canonical form for it which has probably been solved. The point transformation between the two representations of the algebra will lead to the solution of the original equation from that of its transformed equivalent.

This thesis is not concerned with partial differential equations. That is a separate area with many particular considerations which do not arise in the field of ordinary differential equations. Nevertheless it is a field in which the concept of the symmetries of differential equations enjoys wide usage.

Some thirty–five and more years ago a guest speaker[5] addressed the senior scholars on counting systems both primitive and sophisticated, old and new. An oft reiterated point was the primitive concept of counting as being *one, two, many*[6]. In many respects this work is concerned with the particularity of early representatives of classes of equations and the generic properties of those classes. Does the transition occur from one to two, two to three or ...? We shall see that it varies.

Mathematics is often thought of as a dry subject and yet within just the area of differential equations there is such a wonderland of enchantment. The beautiful structures imposed by the divers algebras which persist from transformation to transformation are a marvel in themselves. More recent results on the appearance and disappearance of symmetries as one transforms in a more ambitious mode makes one wonder if the alchemists of old had ideas as strange as we tend to believe.

[5]Sent out by the Mathematical Association, I presume. My recollections of the speaker are of a man of medium height and middle years with a black beard and a slight tendency to portliness. It may have been Fred Syer, but, as he died in 1993 at the age of 92, direct verification is not as easy as it once was.

[6]Clearly a hidden warning to those who were heading off to varsity in the near future!

Contents

Chapter 1

Introduction

1.1 Infinitesimal Transformations

The basis of the Lie theory is invariance under transformation. In principle the transformation may be finite, but the great advantage of the Lie theory is its infinitesimal formulation[1]. One does not have to do Lie theory infinitesimally and in 1988 Aguirre and Krause [7, 8, 9] treated the simple harmonic oscillator in terms of finite transformations. This had the advantage of demonstrating directly that the symmetry group was $SL(3, R)$. It had the disadvantage of being horrendously boring in its calculation with nonlinear partial differential equation piled upon nonlinear partial differential equation. The infinitesimal approach obtains local results and so leads only to the algebra $s\ell(3, R)$ for the simple harmonic oscillator, but it has the advantage of requiring the solution of linear partial differential equations only. This is true whether the equation being examined be linear or not. In fact the whole beauty of the Lie method is that it makes nonlinear equations linear as far as the analysis is concerned and it is for nonlinear equations that the Lie method produces the richest rewards[2].

[1]All treated in exhaustive detail by Lie in six volumes published between 1888 and 1896 [133, 134, 135, 136, 137, 138].

[2]The results may not be the richest in terms of the number of symmetries, but it is generally easier to determine the functional form of the symmetries precisely whereas in

As we are going to be considering transformations of ordinary differential equations, it makes some sense to commence with the transformations of functions of two variables. Suppose that we have an infinitesimal transformation

$$\begin{aligned}\bar{x} &= x + \varepsilon\xi \\ \bar{y} &= y + \varepsilon\eta. \end{aligned} \tag{1.1.1}$$

Under this transformation a function, $f(x, y)$, becomes

$$\begin{aligned} f(\bar{x}, \bar{y}) &= f(x + \varepsilon\xi, y + \varepsilon\eta) \\ &= f(x, y) + \varepsilon\left(\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}\right) \end{aligned} \tag{1.1.2}$$

to the leading order in ε. This can be rewritten as

$$\begin{aligned} f(\bar{x}, \bar{y}) &= f(x) + \varepsilon\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) f \\ &= (1 + \varepsilon G) f(x, y), \end{aligned} \tag{1.1.3}$$

where the differential operator, G, is defined by

$$G := \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}. \tag{1.1.4}$$

The function, $f(x, y)$, is said to possess the symmetry, G, if

$$\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right) f(x, y) = 0 \tag{1.1.5}$$

and is said to be invariant under the infinitesimal transformation generated by G. It is from this basis that we consider the invariance of functions and equations involving derivatives. The first task is to establish how derivatives transform given a transformation in the dependent and independent variables. This is easily done. Consider the transformed first derivative

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} = \frac{\mathrm{d}(y + \varepsilon\eta)}{\mathrm{d}(x + \varepsilon\xi)}$$

the case of linear equations it is necessary to be able to obtain an explicit solution for the equation before the symmetries can be displayed.

$$
\begin{aligned}
&= \frac{\mathrm{d}y + \varepsilon \mathrm{d}\eta}{\mathrm{d}x + \varepsilon \mathrm{d}\xi} \\
&= \frac{y' + \varepsilon \eta'}{1 + \varepsilon \xi'} \\
&= y' + \varepsilon(\eta' - y'\xi') \qquad (1.1.6)
\end{aligned}
$$

to the first order in ε. An infinitesimal transformation of a function containing x, y and y' is generated by the differential operator

$$
G^{[1]} = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + (\eta' - y'\xi') \frac{\partial}{\partial y'}. \qquad (1.1.7)
$$

The differential operator, $G^{[1]}$, is called the first extension of G. The presence of higher derivatives is accommodated by further extensions of G. The nth extension [153] is given by

$$
G^{[n]} = G^{[n-1]} + \left\{ \eta^{(n)} - \sum_{i=0}^{n-1} \left[\begin{pmatrix} n \\ i \end{pmatrix} y^{(n-i)} \xi^{(i+1)} \right] \right\} \frac{\partial}{\partial y^{(n)}}. \qquad (1.1.8)
$$

In the case of a differential equation invariance under the appropriate extension of a generator of symmetry must take into account the existence of the equation. Thus, for a function $f(x, y, y', \ldots, y^{(n)})$, invariance under a symmetry means

$$
G^{[n]} f\left(x, y, y', \ldots, y^{(n)}\right) = 0 \qquad (1.1.9)
$$

whereas the differential equation

$$
f\left(x, y', \ldots, y^{(n)}\right) = 0 \qquad (1.1.10)
$$

possesses a symmetry if

$$
G^{[n]} f\left(x, y, y', \ldots, y^{(n)}\right)\Big|_{f\left(x, y, y', \ldots, y^{(n)}\right) = 0} = 0. \qquad (1.1.11)
$$

Included in the class of functions containing derivatives are first integrals. As there is some variation of understanding of the expressions first integral, constant of the motion[3] and invariant[4] we adhere to the more mathematical

[3]In the context of Mechanics.

[4]Or even worse that favourite of the Physicist, *exact invariant*!

usage of the expression 'first integral' by which is meant a function of the independent variable, dependent variable and its derivatives (including one at level $(n-1)$ for a differential equation of order n) which, if differentiated with respect to the independent variable, is zero when the differential equation is taken into account[5]. One point, which should be realised, is that a first integral is not a function which contains the first derivative, but is one which contains a derivative of order one less than that of the differential equation whence it was derived. This creates an interesting situation as a first integral can be regarded as a differential equation. Indeed this is the usual way to regard them for they are used as an intermediate stage in the process of reduction to quadratures. Suffice for the moment to state that a differential equation of given order has a number of symmetries greater than or equal to the number of symmetries which the corresponding first integral has. We treat this matter further in the chapter on symmetries of first integrals. For the moment we enjoy the amusement of the one object having different algebraic properties depending upon whether it is treated as a function or an equation[6]. More seriously we relate it to the generally undeveloped field of configurational invariants [76, 195].

1.2 Lie Algebras

A Lie algebra consists of a vector space $\mathcal{G}$ over a field $\mathcal{F}$ together with a binary operation $[\,,]$ called the Lie Bracket which is defined on $\mathcal{G}$ such that the axioms

(a) Bilinearity: for any $u, v, w \in \mathcal{G}$ and $a, b \in \mathcal{F}$

$$[au + bv, w] = a[u, w] + b[v, w]$$
$$[u, av + bw] = a[u, v] + b[u, w];$$

(b) Skew–symmetry: for any $u, v \in \mathcal{G}$

$$[u, v] = -[v, u]; \tag{1.2.1}$$

[5]Or system of equations if there is more than one dependent variable.

[6]In the context of the same class of symmetries.

and

(c) Jacobi Identity: for any $u, v, w \in \mathcal{G}$

$$[[u,v],w] + [[v,w],u] + [[w,u],v] = 0 \tag{1.2.2}$$

are satisfied. It is usual to take $\mathcal{F}$ to be the field of real numbers, but it could equally well be the field of complex numbers especially if one wishes to work with differential equations in the complex plane.

The Lie Bracket $[\,,]$ on a set $\mathcal{S}$ of vector fields of the form (1.1.4) is defined as

$$[X,Y] = XY - YX \tag{1.2.3}$$

for any $X, Y \in \mathcal{S}$. The definition (1.2.3) introduces a binary operation into the space of vector fields $\mathcal{S}$ which makes it into a Lie algebra since the definitions (a), (b) and (c) above hold[7].

If a differential equation admits the vector fields X and Y in the sense of §1.1, it also admits their commutator, $[X,Y]$, as the Lie Bracket is often called. The set of all vector fields admitted by a given differential equation generates a Lie algebra. The largest admitted Lie algebra is called the full Lie algebra of the equation. The algebras one encounters in the study of ordinary differential equations are for the most part finite dimensional real algebras. We meet an exception in the infinite dimensional algebra associated with the Ermakov invariant[8].

1.3 Integrals and Reduction of Order

Suppose that a differential equation has a symmetry of the form

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{1.3.1}$$

[7] Ovsiannikov (1982) [168].

[8] See Chapter Eight.

where the functions ξ and η are restricted to be of x and y only, *ie*, we confine our consideration to point transformations. Then we can look for new variables in which the order of the equation is reduced by one and also to the determination of a first integral from the integration of the reduced form of the equation. There are two ways to introduce new variables.

The first takes the expression for the first extension

$$G^{[1]} = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + (\eta' - y'\xi')\frac{\partial}{\partial y'} \tag{1.3.2}$$

and looks for the characteristics by solving the associated Lagrange's system

$$\frac{\mathrm{d}x}{\xi} = \frac{\mathrm{d}y}{\eta} = \frac{\mathrm{d}y'}{\eta' - y'\xi'}. \tag{1.3.3}$$

The characteristics are called invariants in the language of Lie theory as each is invariant under the transformation induced by the symmetry. Since ξ and η are functions of x and y only, the characteristic obtained by the solution of the first and second of (1.3.3) above is called the zeroth order invariant. When the third of (1.3.3) is brought in, the second characteristic will contain y' and so is called the first differential invariant. From these two invariants expressions for the higher derivatives can be found. If we call the first u and the second v, in principle the expression for v can be inverted to give y' in term of u, v and x. Higher derivatives are obtained by successive differentiation and substitution of the results obtained for the lower derivatives. Eventually the differential equation is used to stop the process. What is left is a differential equation of order one lower than the original.

If the new equation has any symmetry, this can be used to reduce the order of the equation yet further. A 'good' equation will have symmetries sufficient to reduce it to the zeroth order, *ie* to an algebraic equation[9]. In this respect one needs to be careful with the symmetries which are used to reduce the order of the differential equation. The reduction of order of an equation may

[9]Usually one does not bother to reduce as far as the algebraic equation, but stops at the last quadrature.

or may not cause the loss of other symmetries depending upon the Lie Bracket properties of the symmetries. Clearly we are concerned with the case for which there is more than one symmetry. Let two of those symmetries be G_1 and G_2. Then, if

$$[G_1, G_2] = \lambda G_1, \tag{1.3.4}$$

where λ is some constant, reduction of order via G_1 will result in the reduced equation inheriting a point symmetry derived from G_2. If the constant, λ, is nonzero, this is not the case if reduction is based on G_2. If an equation has more than two symmetries, the choice of the route for the reduction of order can be made with less concern in the case of a second order equation. Clearly the higher the order of the equation the more parsimonious one must be in the unnecessary discarding of symmetries.

The equation which results after the reduction of order may be integrable, in which case a first integral is obtained for the original equation. One must note that there is a difference between reduction of order using symmetries and that obtained by using integration.

The second route is to change variables in the original differential equation. This is achieved by finding the zeroth order and first differential invariants and seeing what transformation of the independent variable will make the derivative of u a function of v and u only. The original equation is then written in terms of u as dependent variable and the new independent variable. The order is unchanged, but the variables can be made more user friendly at that order. Typically an equation with just the one symmetry would be transformed to autonomous form. This need not necessarily be the simplest form, but would, for example, be optimal for an application of the Painlevé analysis[10].

We have here confined ourselves to consideration of point symmetries and the way to make best use of them in the process of reducing the order of a

[10]Strictly speaking the Painlevé analysis is outside the scope of this work which is dedicated to symmetry, but recent developments, to which brief mention is made in the Epilogue, make it seem evident that the one cannot be considered without the other.

differential equation. The unnecessary loss of symmetry by the injudicious use of a symmetry for the reduction of order is easily explained in terms of the algebraic properties of the generators. However, there are times when the reduction of order does not result in the reduction of the number of symmetries, but can even lead to an *increase* in their number. The typical example is the reduction of a third order equation with the maximal number of point symmetries, seven, to a linear second order equation which has eight symmetries[11]. To be specific the third order equation

$$y''' = 0 \tag{1.3.5}$$

has seven symmetries, one of which is

$$G = \frac{\partial}{\partial y} \tag{1.3.6}$$

for which the zeroth order invariant is

$$u = x \tag{1.3.7}$$

and the first differential invariant is

$$v = y'. \tag{1.3.8}$$

The reduced equation is

$$v'' = 0 \tag{1.3.9}$$

and this has eight symmetries.

This example is very simple and tends not to be considered by those who look at the apparently mysterious appearance of unexpected, or hidden, symmetries. The original equation is effortlessly integrable and so does not really need to be examined closely. The equations of interest are those which have a limited number of symmetries, perhaps even an inadequate number, and yet are reducible to quadratures because of the appearance of new symmetries during

[11] In the following we are specifically concerned with the existence of point symmetries.

the process of reduction of order. The source of the new symmetries appears to be in what are called nonlocal symmetries, *ie* symmetries which contain integrals as coefficients in the generator. The story of their appearance, and disappearance, can be found in the works of Barbara Abraham–Shrauner [1, 4] and her student, Ann Guo [2, 3, 74].

1.4 The Stage Is Set

The modern knowledge and use of the symmetries of ordinary differential equations can be effectively traced to the paper of Anderson and Davison [10] in 1974. The mathematicians had looked deeply into the subject at the time of Lie and shortly thereafter. Then the mathematical content of the area seemed to be exhausted and the mathematicians moved on to newer lode. For the practitioners of the usage of mathematics in divers disciplines the area had not been greatly explored until the Russian School under the leadership of L V Ovsiannikov[12] developed in the late fifties and then the thrust was towards partial differential equations, in particular those relating to gas dynamics which for some unfortunate reason was then a subject of current interest. Anderson was the link between East and West. Anderson's paper with Davison [10], followed shortly by that of Wulfman and Wybourne [214], sparked off a cascade of papers which explored the symmetries of specific ordinary differential equations. In the process Lie and Noether symmetries were intermingled although they have separate provenances.

The ordinary differential equations treated were the standard paradigms of oscillator, Kepler problem, damped particle and the like. Then in 1981 Leach [106] wrote about the symmetry of an anharmonic problem and its use in finding a first integral for that problem which lead to the reduction of the

[12]He still contributes actively in his mature years and has been a keynote speaker at recent meetings in Russia (Ufa 1991, Sezryn 1993), Italy (Acireale 1992) and South Africa (Johannesburg 1994).

original differential equation to quadratures. This moved the goal posts in a positive fashion. Symmetries could be used for useful purposes and not just to give alternative solutions to problems already solved.

The Emden–Fowler equation had received attention from various sources and it was natural that Leach should again apply symmetry methods to this problem. In the process of this investigation a certain nonlinear second order equation occurred. The observation that this particular nonlinear second order equation possessed eight symmetries with the Lie algebra $s\ell(3, R)$ and was linearisable via a point transformation sparked off a large part of the work which is reported here. It lead to the systematic study of the conditions for a second order nonlinear differential equation to be linearisable. It was then natural to consider the forms of scalar second order equations invariant under algebras of smaller dimension. Higher order equations and systems of second order equations were considered next almost at the same time[13]. The algebraic properties of linear higher order equations were established in friendly rivalry with a Franco–Chilean team[14]. The consideration of systems of linear equations can only be described as limited and the literature has stayed that way for some years for the simple reason that it is a very complicated problem.

The properties of the first integrals of some equations proved to be of algebraic interest in their own right. Again it was found that what applied for second order equations was not the rule for higher order equations.

The paradigm of the nonlinear second order equation is the Emden–Fowler equation. Considerable attention has been given to it for two reasons. It persists in occuring in applications and it has mathematically interesting features. In its generalised form it is also very frustrating, but even in frustration new features emerge.

In the world of symmetry there are certain algebras which stand out because

[13]During the course of a week in which the author was immobilised by a large plaster cast.

[14]Jorge Krause of the Pontifical University in Santiago, Chile, and Louis Michel of the IHES at Bures–sur–Yvette, near Paris.

of the implications of their possession. In Physics the rotation algebra, $so(3)$, is doubtless notorious. In the field of ordinary differential equations one such algebra is $s\ell(2,R)$. It is common to all linear equations of order greater than the first which possess maximal symmetry. It is the algebra of the Pinney equation which plays a major rôle in the story of the time–dependent oscillator and its generalisations. It is the algebra of Ermakov systems which themselves possess an integral which has an infinite number of symmetries.

We see that there has been a great flowering of work on the symmetries of ordinary differential equations which was sparked by some very elementary beginnings. Had it not been for Geoffrey Prince's knowledge of the seventies papers on symmetries a lot of the research reported here would not have taken place. Had it not been for Ralph Lewis' constant search for more complicated time–dependent systems for which first integrals could be found a lot of the research reported here would not have taken place. Had it not been for the fact that I am fascinated by invariance under transformation none of the research reported here is likely to have ever taken place.

Chapter 2

The Ad Hoc Period

2.1 Introduction

The papers of Anderson and Davison [10] and Wulfman and Wybourne [214] created an amount of interest in the Mathematics and Physics communities which was really far beyond their fundamental merits. Anyone familiar with the works of Lie would have known that it should be so, but such persons were[1] not common in the world of applications. It is for this reason that these two papers are important. They brought back to life an area which had been left to moulder and so opened up the way to the recent explosion of interest in the symmetries of ordinary differential equations and their applications. It is true that partial differential equations have been more consistently supported due to the work of the *doyen* of the Russian School, Laurentiev Ovsiannikov, and his cotérie, but there does seem to have been an accretion of more diverse mathematical expertise since the 'extension' to ordinary as opposed to partial differential equations was made.

In hindsight the progress which was made make reinventing the wheel look technologically advanced. However, the work of the old masters was not known in detail. Some general principles were and that was about it. This chapter is

[1]And still are although the situation is improving.

dedicated to the period of the reinvention of the wheel. In a sense there was some merit in this even though we had to plough again the already harrowed field. The curious peculiar interests of those who came as neophytes to the subject affected the way the subject of ordinary differential equations and their symmetries was interpreted. In the present instance the interest was that of non–autonomous Hamiltonian systems because of their possible applications in the Plasma Physics which lay behind the theoretical work which attempted to explain and direct[2] experimental work on controlled thermonuclear fusion. Curiously, as time has passed, it has been the interest in the group theoretical properties of the equations which has persisted even as the hope of controlled thermonuclear fusion in our lifetimes has diminished and the wish for it to happen may also have decreased from fear of a radiative Armageddon.

In this chapter we consider the process of moving from the what should have been known to respectable, if paradigmic, applications to what is, as far as we know, the first instance of putting Lie's theory to work. In the process we find that there had been a certain amount of humbug in a couple of the early papers of this period. We conclude with consideration of a problem which has had considerable impact on the study of the theory of linear equations and those equations which are linearisable by a point transformation.

2.2 Linear Scalar Second Order Ordinary Differential Equations

2.2.1 The Time–Dependent Oscillator

The first problem we consider is the time–dependent oscillator which has played an important rôle in classical and quantum mechanics. It has been around a long time, in fact rather longer than some of the not quite recent literature would suggest. At the first Solvay Conference in 1911 [140, 208] Lorentz pro-

[2]One is reminded of Eddington's dictum that theory must mould experiment.

posed an adiabatic invariant for the lengthening pendulum, modelled in the standard approximation by the linearised form[3]

$$\ddot{q} + \omega^2(t)q = 0, \tag{2.2.1}$$

where as usual the overdot denotes differentiation with respect to time, to be

$$I := \frac{\dot{q}^2}{\omega} + \omega q^2 \tag{2.2.2}$$

in the case that $\omega(t)$ was slowly varying. Some fifty years later Littlewood [140, 141] provided a rigorous treatment. In particular he quantified the approximate constancy of the adiabatic invariant, I, by proving that $I(t) - I(\infty) = O(\epsilon)$ when $\omega(t)$ has the form $\phi(\epsilon t)$.

A few years later while on sabbatical in Heidelberg Ralph Lewis found[4] an exact invariant[5] for (2.2.1) in an application of Kruskal's asymptotic method [91]. The motivation for the study was from Plasma Physics in which the equation for the time–dependent oscillator arises as the linear approximation for the motion of a particle in an electromagnetic field[6]. The exact invariant of Lewis is

$$I := \frac{1}{2}\left\{(\rho\dot{q} - \dot{\rho}q)^2 + \left(\frac{q^2}{\rho^2}\right)\right\}, \tag{2.2.3}$$

where the auxiliary function, $\rho(t)$, is a solution of the equation

$$\ddot{\rho} + \omega^2\rho = \rho^{-3}. \tag{2.2.4}$$

Some years later a computationally less complicated derivation was provided by Leach [95] although the same auxiliary equation had to be solved. The

[3]The model is not appropriate for a pendulum of shortening string length as the approximation, $\sin\theta \approx \theta$, eventually breaks down. This case was analysed by Ross [188].

[4]Ralph was developing the asymptotic expansion in the parameter, ε. The first order correction was zero, then the second. Most would have stopped there, but he persisted, convinced himself that he had a first integral and then calculated $\mathrm{d}I/\mathrm{d}t$.

[5]The physicists' term. The invariant has to be labelled 'exact' lest anyone think it 'inexact'. The remark is not without point when one considers the terminology introduced by Hall [76] in his treatment of configurational invariants. Mathematicians tend to prefer 'first integral'.

[6]See also the cyclotron studies of Seymour [198].

derivation was in the context of Hamiltonian Mechanics and was simply the transformation from the time–dependent Hamiltonian

$$H = \frac{1}{2}\left(p^2 + \omega^2(t)q^2\right) \tag{2.2.5}$$

to the autonomous form

$$\bar{H} = \frac{1}{2}\left(P^2 + Q^2\right) \tag{2.2.6}$$

via the generalised canonical transformation [24, 25]

$$Q = \frac{q}{\rho} \qquad P = \rho p - \dot{\rho} q \qquad T = \int \rho^{-2}(t)\mathrm{d}t \tag{2.2.7}$$

and it is the same $\rho(t)$ as above. The only problem was to determine the function, $\rho(t)$. Fortunately the solution of (2.2.4) had been provided by Pinney in 1950 [172]. The solution of the auxiliary equation is

$$\rho^2(t) = au_1^2 + bu_2^2 + 2cu_1u_2 \qquad ab - c^2 = 1, \tag{2.2.8}$$

where $a, b, c \in \mathcal{R}$ and u_1 and u_2 are two linearly independent solutions of

$$\ddot{u} + \omega^2(t)u = 0. \tag{2.2.9}$$

Unfortunately a particular solution of the original equation of interest is required to obtain its general solution. When students first meet this happy circumstance, they tend to react negatively. However, the mere knowledge of the formal solution is often enough to keep a realistic problem on course to the next to bottom line. Furthermore the real problem at hand is usually not the lengthening pendulum, but something from a nonclassical context which reduces to the time–dependent oscillator when the final equations are obtained. Frequently this is a considerable advance, particularly in the case of quantum mechanical problems where a nonseparable partial differential equation is replaced by a separable partial differential equation and an ordinary differential equation.

There is, of course, the matter of extension to higher dimensions [73], but we are wandering somewhat from the theme of this work. We hope that enough has been said of the time–dependent oscillator to indicate that it is an important problem[7].

Our interest at the moment is the Lie algebra of (2.2.1). Leach [101] gives a full discussion of it from the point of view which existed at the time in his then work. Here we present a more current treatment. The differential equation (2.2.1) possesses a Lie point symmetry

$$G = \tau(q,t)\frac{\partial}{\partial t} + \eta(q,t)\frac{\partial}{\partial q} \tag{2.2.10}$$

if the action of the second extension of G, $G^{[2]}$, on (2.2.1) when (2.2.1) holds gives zero. The calculation, like most associated with Lie symmetries, is notable more for its tediousness than its content of elegant deductions. Suffice to say that the eight symmetries are

$$\begin{aligned}
G_1 &= \rho^2 \sin 2T\frac{\partial}{\partial t} + q(\rho\dot{\rho}\sin 2T + \cos 2T)\frac{\partial}{\partial q}\\
G_2 &= \rho^2 \cos 2T\frac{\partial}{\partial t} + q(\rho\dot{\rho}\cos 2T - \sin 2T)\frac{\partial}{\partial q}\\
G_3 &= q\rho\cos T\frac{\partial}{\partial q}\\
G_4 &= q\rho\sin T\frac{\partial}{\partial q}\\
G_5 &= \rho^2\frac{\partial}{\partial t} + q\rho\dot{\rho}\frac{\partial}{\partial q}\\
G_6 &= q\frac{\partial}{\partial q}\\
G_7 &= \rho^{-1}q\sin T\frac{\partial}{\partial T} + (\dot{\rho}\sin T + \rho^{-1}\cos T)q^2\frac{\partial}{\partial q}\\
G_8 &= \rho^{-1}q\cos T\frac{\partial}{\partial T} + (\dot{\rho}\cos T - \rho^{-1}\sin T)q^2\frac{\partial}{\partial q},
\end{aligned} \tag{2.2.11}$$

where $\rho(t)$ is a solution of (2.2.4) and the 'new time' [24, 25], T, is given by

$$T = \int^t \rho^{-2}(t')\mathrm{d}t'. \tag{2.2.12}$$

[7]No mention has been made of applications in Quantum Optics as evinced by the many papers of Abdalla and co–workers (*eg* [29, 30, 31]) as this is essentially the same as Leach's treatment [108] even if the physical parameters are somewhat different.

The Lie Brackets of these symmetries coincide with those of the time–independent problem[8] and so the algebra is $s\ell(3,R)$.

2.2.2 The Damped Free Particle

The damped free particle was considered by Prince *et al* [177] as an example to illustrate some of the ideas behind the concept of 'Lie admissibility' as espoused by Santilli [191]. It has the equation of motion

$$\ddot{x}+\gamma\dot{x}=0, \tag{2.2.13}$$

the symmetries are calculated in the usual way and are found to be

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial t}\\
G_2 &= e^{\gamma t}\frac{\partial}{\partial t}\\
G_3 &= -\frac{1}{\gamma}e^{-\gamma t}\frac{\partial}{\partial t}+xe^{-\gamma t}\frac{\partial}{\partial x}\\
G_4 &= x\frac{\partial}{\partial t}-\gamma x^2\frac{\partial}{\partial x}\\
G_5 &= xe^{\gamma t}\frac{\partial}{\partial t}\\
G_6 &= x\frac{\partial}{\partial x}\\
G_7 &= \frac{\partial}{\partial x}\\
G_8 &= e^{-\gamma t}\frac{\partial}{\partial x}.
\end{aligned} \tag{2.2.14}$$

The algebra is $s\ell(3,R)$.

2.2.3 The Forced Harmonic Oscillator

It may come as no surprise that the algebra of the forced harmonic oscillator is also $s\ell(3,R)$ when the forcing term is a function of time only. Thus the symmetries of the equation of motion

$$\ddot{q}+q+f=0, \tag{2.2.15}$$

[8]Leach (1980) [101]

where $f(t)$ is a continuous function of time over the interval of interest, are[9]

$$
\begin{aligned}
G_1 &= \sin 2t\frac{\partial}{\partial t} + \{(q-g)\cos 2t - h\sin 2t\}\frac{\partial}{\partial q} \\
G_2 &= \cos 2t\frac{\partial}{\partial t} - \{(q-g)\sin 2t + h\cos 2t\}\frac{\partial}{\partial q} \\
G_3 &= \cos t\frac{\partial}{\partial q} \\
G_4 &= \sin t\frac{\partial}{\partial q} \\
G_5 &= \frac{\partial}{\partial t} - h\frac{\partial}{\partial q} \\
G_6 &= (q-g)\frac{\partial}{\partial q} \\
G_7 &= (q-g)\sin t\frac{\partial}{\partial t} + \{(q-g)^2\cos t - (q-g)h\sin t\frac{\partial}{\partial q} \\
G_8 &= (q-g)\cos t\frac{\partial}{\partial t} - \{(q-g)^2\sin t + (q-g)h\cos t\frac{\partial}{\partial q},
\end{aligned}
\tag{2.2.16}
$$

where

$$
\begin{aligned}
g(t) &= \int_0^t \sin(\tau - t)f(\tau)\mathrm{d}\tau \\
h(t) &= \int_0^t \cos(\tau - t)f(\tau)\mathrm{d}\tau.
\end{aligned}
\tag{2.2.17}
$$

It is perhaps worth noting that in this paper[10] transformation from one equation to another by point transformations was understood to preserve the algebra! It was observed that 'the use of point transformations may make the investigation of the symmetries of other systems easier' and that 'the problem of determining whether a given dynamical system possesses this symmetry[11] is reduced to finding a point transformation relating it to a system which does.' The *naïveté* of the observation is almost beyond comprehension in the light of current understanding, but the situation was somewhat different then.

[9] Leach (1980) [103]

[10] Leach (1980) [103].

[11] The Lie algebra $s\ell(3, R)$. At that time linear systems were the chief ones under investigation although the situation was to change not much later.

2.2.4 The Repulsive Oscillator

In the course of what may, with little breach of the code of charity, be described as experimental work on the symmetries of equations of long familiarity the repulsor was one of the last of the linear scalar equations to be studied[12]. It is, perhaps, amusing to note that even still there was a reluctance to apply the method for the direct calculation of the Lie point symmetries and Noether's theorem was frequently used to find five of the eight symmetries and then the Lie method was used for the remaining three. Part of the responsibility for this can be attributed to Lutzky [143, 144] who had used the theorem to obtain the five Noetherian symmetries and so set a pattern[13]. The remaining part can be squarely identified through the interests of the more active investigators at the time. They were motivated by the search for first integrals. Apart from first integrals being on the route to the solution of a problem they were the functions of interest to the physicists as the first integrals were going to provide a source of observables in quantum mechanics or particle distribution functions in plasma physics.

The symmetries of the equation of motion[14]

$$\ddot{q} - q = 0 \tag{2.2.18}$$

were found to be

$$\begin{aligned} G_1 &= \sinh 2t\frac{\partial}{\partial t} + q\cosh 2t\frac{\partial}{\partial q} \\ G_2 &= \frac{\partial}{\partial t} \\ G_3 &= \cosh t\frac{\partial}{\partial q} \end{aligned}$$

[12] Leach (1980) [104].

[13] It did have to be shown that the Noether symmetries which were associated with the Variational Principle and not necessarily with the differential equation were in fact symmetries of the differential equation. More recent studies, *cf* Kara and Mahomed(1992) [87] have emphasised the need for care in identifying the number of symmetries of an equation with those of an Action Integral if only because of the variety of inequivalent Lagrangians.

[14] Leach (1980) [104].

$$
\begin{aligned}
G_4 &= \sinh t\frac{\partial}{\partial q}\\
G_5 &= \cosh 2t\frac{\partial}{\partial t} + q\sinh 2t\frac{\partial}{\partial q}\\
G_6 &= q\frac{\partial}{\partial q}\\
G_7 &= q\sinh t\frac{\partial}{\partial t} + q^2\cosh t\frac{\partial}{\partial q}\\
G_8 &= q\cosh t\frac{\partial}{\partial t} + q^2\sinh t\frac{\partial}{\partial q}
\end{aligned}
\tag{2.2.19}
$$

and, as should be evident by now, the Lie algebra of the symmetries is $s\ell(3,R)$.

If it is felt that there needs to be some relief from the *naïveté* of these results, Leach [104] does join the known results together and points out that all the linear scalar second order ordinary differential equations considered had $s\ell(3,R)$ symmetry. Indications from linear systems were that the symmetry was $s\ell(n+2,R)$ provided that the systems were uncoupled, undamped and unforced. In particular Prince and Eliezer [175] had demonstrated that the complete symmetry group of the $n-$dimensional time–dependent oscillator was $s\ell(n+2,R)$ and to be honest much was inferred from that. It was not until sometime later that the *somewhat more complex situation* of systems of linear equations was explored separately by Gorringe and Leach [58], González–López [56] and González–Gascón and González–López [57]. Even now it would be reasonable to state that the situation for linear systems is one of serious underdevelopment.

2.3 The Classical Systems

It comes as no surprise that after the few standard and a couple of not so standard linear scalar equations were investigated the major multi–dimensional syetems were treated. There is not an excessive number of them, just the oscillator and the Kepler problem[15].

[15]One could extend the oscillator to the repulsor, but the experience with the two in the one–dimensional case suggests that there would be little sense in it. In fact the only

At the time there was something of a debate between the use of Noether's Theorem and the Lie Theory of Extended Groups. The advantage of the former was that the integral followed without effort once the symmetry was known whereas in the latter case the remaining calculations could be highly nontrivial[16]. However, Noether's Theorem was unable to produce the Jauch–Hill–Fradkin tensor[17] for the multi–dimensional oscillator or the Laplace–Runge–Lenz vector[18] for the Kepler Problem. This was in the context of point transformations. They could be found when velocity–dependent transformations were admitted, but the price of generality was that of nonclosure. An infinite number of velocity–dependent transformations required an *ansatz* on the nature of the velocity–dependence in the symmetry. As this is related to the velocity–dependence in the integral[19], one is back at the chief obstacle to Bertrand's method [20]. It is also known as the direct method following its extension to time–dependent integrals by Lewis and Leach (1982) [130], González–Gascón *et al* (1982) [55] and Moreira (1984) [160]. The Lie method had the advantage that there was no such fixed relationship between symmetry and integral.

In fact there is much more to the story, but it should be told at the closing of this work for that is where its thoughts exist.

The use of the Lie method for the two paradigms, three–dimensional oscillator and Kepler problem, was championed by Leach [105] for a reason of

real difference between the two is to be found in the rôles played by the different conserved tensors. This was described by Leach (1980) [98], but is not directly related to the symmetry problems considered here. Naturally the relationship between symmetries and first integrals as pursued in Chapter Six does make them relevant to the study of the algebraic properties of differential equations but not in the context of the present chapter.

[16]See Gorringe and Leach (1990) [59] for an example from a later period.

[17]Jauch and Hill (1941) [83]; Fradkin (1965) [48, 49].

[18]Laplace (1798) [94]; Runge (1922) [189, p 79]; Lenz (1919) [124]. However, Goldstein [52, 53] points out that a special form was derived by Hermann in 1711 and the general form by Bernoulli in 1712. As vector notation had not been invented at that time, the vector was given in component form.

[19]See Sarlet and Cantrijn (1981) [194] for a very clear discussion.

consistency. In particular the latter problem had been a source of some disquiet. Lévy–Leblond [125] had produced the velocity-dependent symmetry which leads to the Laplace–Runge–Lenz vector with even more sleight–of–hand than the average magician would care to expose to the public gaze. Prince and Eliezer [176] were slightly subtler in that they had Lie point symmetries rather than a Noetherian velocity–dependent symmetry, but they did not demonstrate a direct derivation of the first integrals from the symmetries.

Perhaps the most significant result of Leach [105] was that the classical integrals for the two paradigms were obtained from the basic symmetry of both systems, that of invariance under time translation. It is possible that at the time the general concept was to associate an integral with a characteristic symmetry. Thus conservation of energy with invariance under time translation and conservation of angular momentum with invariance under the rotation group were typical of what was understood. Lévy–Leblond (1971) [125] moved away from the point symmetries when he associated Kepler's Third Law with invariance under what was the first extension of self–similarity in r and t. It had to be extended to give the right results, but Lévy–Leblond did not exactly explain the situation in great detail. More recently the fortuitous connection has been made painfully obvious[20]. In the case of an autonomous system possessed with a number of autonomous integrals they could be obtained from the characteristic symmetry[21]

$$G = \frac{\partial}{\partial t}. \tag{2.3.1}$$

Suppose that the system is one of n degrees of freedom. A first integral invariant under a particular generator of a symmetry transformation satisfies the equation

$$G^{[1]}I = 0 \tag{2.3.2}$$

and in the first extension of G there are $2n+1$ independent variables so that

[20]Gorringe and Leach (1993) [61].

[21]There is nothing special about the choice of this symmetry apart from its relevance to the examples under consideration. The following argument is general.

(2.3.2) has $2n$ characteristics. The further requirement of invariance under total time differentiation leads to a second linear partial differential equation. This time the $2n$ characteristics are the variables and so the first integral ends up as an arbitrary function of $2n-1$ new characteristics, each of which is a constant of the motion[22]. In the case of the autonomous three–dimensional oscillator and the Kepler problem all of the autonomous integrals were derived from the single Lie point symmetry, $\partial/\partial t$[23].

To a large extent Leach [105] was making a point rather than promoting a radical change of strategy in the approach to the search for first integrals which was the basic interest behind the investigation of symmetries at the time. In fact we are not even using hindsight as we read 'It must be emphasized that we are not promoting the use of the generator, $\partial/\partial t$, to obtain all of the time–independent first integrals. This might require considerable ingenuity, rather more, in fact, than is required when the generator directly associated with the particular first integral is used. What we do wish to demonstrate is that, in the Lie method, such first integrals are implied in the generator of time translation.'

We would like to amend that statement a little. A symmetry is a local operator. A first integral is a nonlocal expression. A symmetry will have associated with it the solution of a first order differential equation. This solution will exist under the usual conditions. However, this existence is local and does not guarantee that there does exist a 'global' function which is invariant under the symmetry. Thus a chaotic system such as the Hénon–Heiles system [78] is autonomous, but only the one integral – the energy – is known and the system exhibits chaos.

[22]The connection with Hamilton–Jacobi theory and Liouville's theorem on complete integrability is somewhat obvious.

[23]See Leach (1981) [105] for the details.

2.4 Anharmonic Systems

The symmetries of the systems described so far could be described as interesting, but it would be difficult to claim more than that as the solutions of the problems treated were already well-known from other approaches. Here we consider the application of the Lie theory to the actual solution of unsolved problems. This story must be told in two parts as this chapter is concerned with the early period. In Chapter Seven we shall take up the story with more recent work. Here we consider the problems of the time–dependent oscillator with a cubic anharmonicity and the autonomous equation which is sometimes known these days as Mahomed's equation.

2.4.1 The Time–Dependent Anharmonic Oscillator

In the investigation of the behaviour of plasma one of the models which was early adopted was that of the motion of a charged particle in an axially symmetric field. Apart from the imposition of the symmetry which produces considerable simplification in the governing equations it was also easy to assume that the radial equation reduced to that of a simple harmonic oscillator when suitable approximations were made. In this context the word 'suitable' is used in the sense of making the resulting mathematics easy. Long familiarity has made the simple harmonic oscillator one of those problems which is considered to be mathematically simple. The only disadvantage of this model was that the Zeta machine and its ilk did not actually work that way. A more refined model was required. The replacement was the time–dependent oscillator which did have the attraction of paying some attention to the presence of time–varying fields. The linearity of the spatial component of the oscillator field was not really sufficient to the demands made on the model and Lewis [128] suggested that a time–dependent anharmonic oscillator with cubic anharmonicity in the Hamiltonian would be suitable as a starting point and that an invariant for such a system could be informative. It was this study of the time–dependent anharmonic problem[24] which lead to a new approach to the Lie theory in that it now became a tool in the search for solutions of new problems governed by ordinary differential equations.

A little explanation of the context will help explain the interest in the existence of an invariant for such models[25]. The motion of a charged particle in a plasma is governed by the Vlasov–Poisson equations in one dimension and the Vlasov–Maxwell equations in more than one dimension. We confine our attention to the case of one dimension and shall ignore the second named equation. The Vlasov equation describes the distribution of particles in a collisionless plasma. It can be written variously and we take the form

$$\frac{\partial f}{\partial t}+\dot{q}\frac{\partial f}{\partial q}+\ddot{q}\frac{\partial f}{\partial \dot{q}}=0 \tag{2.4.1}$$

which carefully ignores any Hamiltonian structure. Usually the system is Hamiltonian. The particle distribution function, $f(t,q,\dot{q})$, is normally a function of $(2n+1)$ variables, where n is the number of degrees of freedom of the system. We contain our attention to the case $n=1$. The distribution of the particles is known when f is known. Since (2.4.1) is a linear partial differential equation in three variables, the number of characteristics is two. It so happens that the Vlasov equation, (2.4.1), is also Liouville's equation in Classical Mechanics and describes the condition which a function must satisfy in order for it to be a first integral, or invariant as the physicists would prefer, of the system. The solution of Liouville's equation, and so Vlasov's, should provide both integrals. This is certainly the manifest desire of the physicists[26] although

[24]Leach (1981) [106].

[25]Strictly speaking this lies outside of the subject matter proper to this thesis, but research is rarely neatly packaged. It will be recalled that the starting point for the work reported here was in the theoretical study of plasmas as they occur in confinement devices. That the applicability of the results is much wider than that of fusion experiments should come as no surprise. The Physics is already more general than the Engineering context of the experiment and, naturally, the Mathematics must perforce be even more general.

[26]Leach *et al* (1993) [114].

the mathematician would be content to find one integral as then the one degree of freedom system is integrable and that is all the mathematician wants. In contrast the physicist needs both invariants explicitly so that the electrical side of the system can be solved[27].

The general form of the equation to be considered is

$$\ddot{q} + a(t)\dot{q} + b(t)q + c(t)q^2 + d(t) = 0. \tag{2.4.2}$$

This equation contains a spurious generality which can be reduced by means of suitable transformations[28] to

$$\ddot{q} + q + B(t)q^2 = 0. \tag{2.4.3}$$

It is assumed that the original equation justifies the positive sign in the transformed equation. As far as the subsequent treatment is concerned, it does not matter to any serious extent, but in terms of the initiating physical model a negative sign could lead to kinking in the plasma.

After some calculation we find that (2.4.3) possesses the symmetry

$$G = a\frac{\partial}{\partial t} + \left\{\tfrac{1}{2}\left(\dot{a} + \alpha\right)q + d\right\}\frac{\partial}{\partial q} \tag{2.4.4}$$

provided that

$$\begin{aligned} B(t) &= Ka^{-5/2}\exp\left\{-\tfrac{1}{2}\alpha\int^{t} \mathrm{d}t'/a(t')\right\} \\ d(t) &= D\sin t + E\cos t \end{aligned} \tag{2.4.5}$$

and the function, $a(t)$, satisfies the third order equation

$$\dddot{a} + 4\dot{a} + 2Bd = 0. \tag{2.4.6}$$

As an aside we note that the corresponding result for Noether's theorem demands that α be zero.

---

[27]See Lewis *et al* (1992) [131].

[28]Keeping to the spirit of the original work the model is not reduced to its elemental form. In Chapter Seven this is done in a more recent treatment of the anharmonic problem with an arbitrary power.

The original equation (2.4.3) with its symmetry (2.4.4) can be transformed to an autonomous one by requiring that the symmetry take the form

$$G = \frac{\partial}{\partial T} \tag{2.4.7}$$

in suitable coordinates. The transformation is

$$\begin{aligned} T &= \int^t a^{-1}(t')\mathrm{d}t' \\ Q &= qa^{-1/2}(t)\exp\left(-\tfrac{1}{2}\alpha\int^t a^{-1}(t')\mathrm{d}t'\right) \\ &\quad - \int^t d(t')a^{-3/2}(t')\exp\left(-\tfrac{1}{2}\alpha\int^{t'} a^{-1}(t'')\mathrm{d}t''\right)\mathrm{d}t' \end{aligned} \tag{2.4.8}$$

and the transformed equation has the autonomous form

$$Q'' + \alpha Q + KQ^2 + MQ + N = 0, \tag{2.4.9}$$

where the constants M and N are given by

$$\begin{aligned} M &= \tfrac{1}{2}\left\{a\ddot{a} - \tfrac{1}{2}(\dot{a}+\alpha)^2 + \alpha(\dot{a}+\alpha) + 2\alpha^2 - 4Kh\right\} \\ N &= h(M+Kh) + g\left\{\tfrac{1}{2}(\dot{a}-\alpha)d - a\dot{d}\right\} \end{aligned} \tag{2.4.10}$$

and $g(t)$ and $h(t)$ are the coefficients of q and the non-q term in (2.4.8) respectively. The constancy of these expressions is verified by direct differentiation and the use of the differential equation.

Since (2.4.9) is autonomous, it can be written as the Abel's equation of the second kind,

$$Q'\frac{\mathrm{d}Q'}{\mathrm{d}Q} + \alpha Q' + KQ^2 + MQ + N = 0, \tag{2.4.11}$$

and the invariant be obtained by quadrature. In the case $\alpha = 0$ the quadrature is immediate and we have the energylike integral

$$\Phi(Q, Q') = \frac{1}{2}Q'^2 + \frac{1}{3}KQ^3 + \frac{1}{2}MQ^2 + NQ. \tag{2.4.12}$$

This is a Noether invariant and the result can be put in Hamiltonian form. The form of the first integral in the original coordinates is recovered by use of the transformation (2.4.8) in (2.4.12). The case $\alpha \neq 0$ is not so simple, but some progress was made in later work which is discussed in §7.7.

2.4.2 The Lane–Emden Equation

The Lane–Emden equation[29] was introduced by Lane in an investigation of the equilibrium configuration of a spherical gas cloud under the influence of gravity and thermodynamic forces. It was re–introduced by Emden nearly thirty years later[30] in the form

$$\frac{1}{\xi^2}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = -\theta^n, \tag{2.4.13}$$

subject to the initial conditions $\theta(0) = 1, \theta'(0)$ and $\xi \geq 0$. The index n is related to the specific heats. Chandrasekhar (1957) [27] discusses the equation extensively and reports complete solutions in the cases $n = 0, 1, 5$.

The Lane–Emden equation may be considered as an equation in dynamics, *viz.*

$$\ddot{q} + \frac{2}{t}\dot{q} + q^n = 0, \tag{2.4.14}$$

which represents an anharmonic oscillator subject to velocity–dependent damping. The case $n = 5$ was treated by Logan (1977) [142, p 52] using Noether's theorem and he found the first integral

$$I = \frac{1}{6}t^3q^6 + \frac{1}{2}t^3\dot{q}^6 + \frac{1}{2}t^2q\dot{q}. \tag{2.4.15}$$

This integral was also found by Sarlet and Bahar (1980)[31] who extended their treatment to the more general equation

$$\ddot{q} + \beta(t)\dot{q} + \alpha(t)q^m = 0, \qquad m \neq -1 \tag{2.4.16}$$

and found a first integral under certain constraints on $\alpha(t)$ and $\beta(t)$.

Moriera (1984) [160], as an example of the application of the Lewis–Leach direct method [130] to Newtonian equations of motion, studied the equation

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0 \tag{2.4.17}$$

[29] Kelvin (1860–62) [203], Lane (1869–70) [93], Emden (1907) [37], Chandrasekhar (1957) [27, Chapter IV].

[30] One assumes that this was independent. After all Emden wrote in German and could not be expected to be familiar with the American literature.

[31] They introduced a time–dependent integrating factor to the equation of motion [193].

and found the first integral

$$I = \exp\left(m\int^t \alpha(t')\mathrm{d}t'\right) \times \\ \left\{\dot{q}^2 + \alpha q\dot{q}(2-m) + \frac{2q^{n+1}}{n+1} + \frac{1}{2}q^2(m-2)\left[(m-1)\alpha^2 + \dot{\alpha}\right]\right\}, \tag{2.4.18}$$

where $m = 2(n+1)/(n+3)$, provided that $\alpha(t)$ satisfied the differential equation

$$\ddot{\alpha} + (3m-2)\alpha\dot{\alpha} + \alpha^3 m(m-1) = 0. \tag{2.4.19}$$

Moreira did not provide the general solution of (2.4.19), but gave the two particular solutions

$$\alpha(t) = \frac{2}{mt}, \quad \alpha(t) = \frac{1}{(m-1)t}. \tag{2.4.20}$$

Feix and Lewis (1985) [41] examined the equation

$$\ddot{x} + \beta(t)\dot{x} + \frac{\partial\phi(x,t)}{\partial x} = 0 \tag{2.4.21}$$

in which they took $\beta(t)$ to be proportional to t^{-1} and used a specific power law expression for the x dependence in ϕ. Their approach was from the point of view of rescaling which has been a favourite method of the Orléans school [75, 26] and which is the equivalent in Newtonian mechanics of generalised canonical transformations in Hamiltonian mechanics [24, 25].

Moreira's equation (2.4.17) is a generalisation of the Lane–Emden equation. It can be thought of as arising in a circumstance under which the constant in the equation of state,

$$P = K\rho^{(n+1)/n} \tag{2.4.22}$$

is replaced by a function depending upon the radial distance r[32]. Leach (1985) [109] applied the guage–variant form of Noether's theorem[33] to the Lagrangian

[32] As is commonly used, P is the pressure and ρ is the density.

[33] The reader is referred to the excellent review of Noether's theorem by Sarlet and Cantrijn (1981) [194].

for the equation considered by Moreira, *viz.*

$$L(t,q,\dot{q}) = \frac{1}{2}A(t)\dot{q}^2 - \frac{A(t)}{n+1}q^{n+1}, \tag{2.4.23}$$

where

$$A(t) = \exp\left(\int^t \alpha(t')\mathrm{d}t'\right). \tag{2.4.24}$$

During the course of the analysis we find that the function, $\alpha(t)$, is required to satisfy (2.4.19)[34].

The first integration of this equation is straightforward[35] and yields

$$\begin{gathered} C_1\left(\dot{\alpha} + \frac{n+1}{n+3}\alpha^2\right)^{n+1} = C_2\left(\dot{\alpha} + \frac{n-1}{n+3}\alpha^2\right)^{n-1}, \\ |C_1| + |C_2| \neq 0. \end{gathered} \tag{2.4.25}$$

If one of C_1 or C_2 is zero, the integration of (2.4.25) is easy and we obtain, up to scaling and translation terms, the results given by Moreira (1984) [160] and the first integrals follow directly.

The integration of (2.4.25) is not simple in the general case. In parametric form[36] it is

$$\begin{aligned} u &= k\left[\frac{(\eta + (n+1)/(n+3))^{n+1}}{(\eta + (n-1)/(n+3))^{n-1}}\right]^{1/4} \\ t &= \frac{k}{2}\int^{\eta}\left[\frac{(\eta' + (n+1)/(n+3))^{n+1}}{(\eta' + (n-1)/(n+3))^{n-1}}\right]^{1/4}\mathrm{d}\eta', \end{aligned} \tag{2.4.26}$$

where

$$u(t) = -\alpha^{-1}(t). \tag{2.4.27}$$

The integral can be evaluated when n is an odd integer, but it is only in the case $n = 3$ that inversion is possible so that α can be expressed as an explicit function of t. This feature is considered in more detail in Chapter Seven.

[34] Strictly speaking in the case $n \neq 2$. As the case $n = 2$ comes under the exhaustive treatment of the Emden–Fowler equation of Chapter Seven, we do not consider it here.

[35] Particularly if one is a Kamke [84] *afficiando.*

[36] *cf* Kamke (1971) [84, p 30, §4.17].

The case $n = 2$ is special because $n - 1 = 1$, *ie* the nonlinearity is connected to the linear term. For the moment it suffices to note that special results were obtained. This case is very specialised and the details are best left to the consolidated treatment of the Emden–Fowler equation in Chapter Seven.

It is perhaps interesting to recall the Conclusion to Leach (1985) [109]. The differential equation

$$\ddot{q} + \alpha(t)\dot{q} + q^n = 0 \tag{2.4.28}$$

was shown to possess first integrals for more general $\alpha(t)$ than had been previously reported. Only in the case $n = 3$ was it possible to write $\alpha(t)$ as an explicit function of time. For other values of n only a parametric solution was possible. The case $n = 2$ was particularly difficult. Then we calmly reported that the equation for $\alpha(t)$ had been investigated for Lie symmetries. For $n \neq 3$ there were only two and they yielded the same information as the analysis considered here. However, for $n = 3$ there existed eight symmetries. As (2.4.19) was rather nonlinear, its possession of the maximum number of point symmetries permitted to a second order ordinary differential equation was unexpected although Jim Reid had observed a similar result some time earlier[37].

It was evident that the Lie analysis of (2.4.19) should be undertaken at some depth.

2.4.3 Mahomed's Equation

The Lie analysis of (2.4.19) was reported in detail by Mahomed and Leach (1985) [149] and such has been the impact of that study in the area of the theory of the linearisation of differential equations that the equation is sometimes

[37] J L Reid, private communication dated 20 August 1981. The equation to which he referred is

$$y'' + 4y' + 2Ky^{-5/2}\exp\left\{-\frac{k}{2}\int^x y^{-1}(x')\mathrm{d}x'\right\} \times (M\sin x + N\cos x) = 0. \tag{2.4.29}$$

It is not obvious which type of symmetries he meant as the treatment of nonpoint symmetries was not well established at the time.

referred to as Mahomed's equation. The equation does occur in applications[38], but its properties as a differential equation *per se* are more than sufficient to attract the attention of those interested in the properties of ordinary differential equations. For the moment it suffices to say that the equation

$$\ddot{\alpha} + \frac{4n}{n+3}\alpha\dot{\alpha} + \frac{2(n^2-1)}{(n+3)^2}\alpha^3 = 0 \tag{2.4.30}$$

possesses only the two symmetries

$$G_1 = \frac{\partial}{\partial t} \tag{2.4.31}$$

$$G_2 = t\frac{\partial}{\partial t} - \alpha\frac{\partial}{\partial \alpha} \tag{2.4.32}$$

for general values of n. However, when $n = 3$, which is a special case as noted above, there are eight. It is convenient to introduce the rescaling

$$\begin{aligned} y &= \frac{3(n+3)\alpha}{4n} \\ x &= t. \end{aligned} \tag{2.4.33}$$

Then, for $n = 3$, the equation becomes

$$y'' + 3yy' + y^3 = 0 \tag{2.4.34}$$

which has the eight symmetries

$$\begin{aligned}
G_1 &= \frac{x^2y}{2}\frac{\partial}{\partial x} + \left(xy^2 - \frac{x^2y^3}{2} - y\right)\frac{\partial}{\partial y} \\
G_2 &= y\frac{\partial}{\partial x} - y^3\frac{\partial}{\partial y} \\
G_3 &= xy\frac{\partial}{\partial x} + \left(y^2 - xy^3\right)\frac{\partial}{\partial y} \\
G_4 &= \left(-\frac{x^2y}{2} + x\right)\frac{\partial}{\partial x} + \left(\frac{x^2y^3}{2} - xy^2\right)\frac{\partial}{\partial y} \\
G_5 &= \left(\frac{x^3}{3} - \frac{x^4y}{4}\right)\frac{\partial}{\partial x} + \left(-x - x^3y^2 + \frac{x^4y^3}{4} + \frac{3x^2y}{4}\right)\frac{\partial}{\partial y}
\end{aligned}$$

[38] Apart from the astrophysical connection, it is also found in studies of univalued functions defined by second order differential equations [54], the Riccati equation [28] and in the modelling of the fusion of pellets [39].

$$G_6 = \left(-\frac{x^3y}{2} + x^2\right)\frac{\partial}{\partial x} + \left(xy + \frac{x^3y^3}{2} - \frac{3x^2y^2}{2}\right)\frac{\partial}{\partial y}$$
$$G_7 = \left(-\frac{x^3y}{2} + \frac{3x^2}{2}\right)\frac{\partial}{\partial x} + \left(1 + \frac{x^3y^3}{2} - \frac{3x^2y^2}{2}\right)\frac{\partial}{\partial y}$$
$$G_8 = -\frac{\partial}{\partial x}. \tag{2.4.35}$$

The generators are in the main complicated. However, when the Lie Brackets are calculated, it is seen that the algebra is that of $s\ell(3,R)$[39]. This is the same algebra as possessed by all linear second order equations[40] and, as algebras are preserved a under point transformation, it is evident that there exists a point transformation from (2.4.19) to the archtypal linear equation

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = 0. \tag{2.4.36}$$

It is observed that

$$G_4 = \rho(x,y)G_2, \tag{2.4.37}$$

where

$$\rho(x,y) = x - \frac{1}{y}, \tag{2.4.38}$$

and

$$[G_2, G_4] = G_2. \tag{2.4.39}$$

Thus these symmetries fall into Type IV of Lie's classification of all two dimensional algebras possessed by second order equations. The canonical form of the symmetries is

$$\bar{G}_2 = \frac{\partial}{\partial Y}$$
$$\bar{G}_4 = Y\frac{\partial}{\partial Y} \tag{2.4.40}$$

and the transformation between the two sets of coordinates is

$$Y = -\frac{x^2}{2} + \frac{x}{y}$$
$$X = x - \frac{1}{y}. \tag{2.4.41}$$

[39]See Table 2 of Mahomed and Leach (1985) [149].

[40]Leach (1980) [102]. This appeared in print in Leach and Mahomed (1988) [121].

In the new variables (2.4.19) takes the canonical form

$$\frac{\mathrm{d}^2Y}{\mathrm{d}X^2} = 0, \tag{2.4.42}$$

the solution of which is quite trivial. It is then straightforward to find that the solution of the original equation, (2.4.19), is

$$y = \frac{2(1+Ax)}{Ax^2 + 2x + C}, \tag{2.4.43}$$

where A and C are the arbitrary constants of integration.

The solution of (2.4.19) using the heavy machinery of the Lie analysis, complicated tables of Lie Brackets and the identification of the appropriate symmetries to obtain the correct transformation of coordinates does seem to be a little overdone when (2.4.19) can be reduced to

$$v''' = 0 \tag{2.4.44}$$

by the Riccati transformation

$$y = \frac{v'}{v}. \tag{2.4.45}$$

However, that is not the point. A nonlinear equation of unexceptional provenance turned out to have the $s\ell(3,R)$ of linear second order equations. The question which it immediately raised was what other nonlinear equations had the same property? This leads us in to the general problem of determining the criteria which establish whether or not a second order equation has $s\ell(3,R)$ symmetry and so is linearisable by means of a point transformation. It is this question which is addressed in Chapter Three.

Chapter 3

Linearisation of Second Order Scalar Ordinary Differential Equations

3.1 Introduction

Ordinary differential equations present themselves as either linear or nonlinear equations. The linearity one must accept as real. There is a battery of theorems which apply to linear equations of which the most valuable must be that, if $u_i(x)$, $i = 1, n$, are linearly independent solutions of a linear differential equation of order n, then $\Sigma_{i=1}^{n} a_i u_i(x)$, where the a_i are arbitrary constants, is the general solution. This does not actually help to find the n particular linearly independent solutions, but it does provide the consolation that the task is finished when one has. The problems associated with the solution of linear second order equations and the vast literature which is devoted to the subject do suggest that the task is passingly nontrivial, indeed so nontrivial that there are those who resort to using a computer to obtain a numerical solution. All one needs is two solutions[1] for independent initial conditions and the general

[1] For a second order equation of course. One more is needed for each increment in the order.

solution is at hand. This happy situation does not occur in the case of nonlinear equations for which the computer must be given its burden for each set of initial conditions. A serious study of a nonlinear equation is devastatingly expensive by comparison with a linear equation of the same order[2] and it makes sense to assure oneself that the equation under consideration is really nonlinear.

The nonlinearity need not necessarily be real. The equation

$$yy'' - \tfrac{1}{2}y'^2 + f(x)y^2 = 0 \tag{3.1.1}$$

is nonlinear, but it is not inherently nonlinear since the transformation

$$y = Y^2 \qquad x = X \tag{3.1.2}$$

reduces (3.1.1) to the normal form for a second order linear equation, *viz.*

$$Y'' + f(X)Y = 0. \tag{3.1.3}$$

On the other hand the equation

$$\ddot{q} + p(t)\dot{q} + r(t) = \mu\dot{q}^2q^{-1} + f(t)q^n, \tag{3.1.4}$$

although it can be rendered into the more attractive form[3]

$$Y'' + f(X)Y^2 = 0 \tag{3.1.5}$$

[2]In the research world one tends not to think of the actual monetery cost of a computer computation. The guiding cost is that of time. However, it does cost money to run computers and one must be sensitive to the real cost of computation. At one university, at least, budgets, in a nominal sense, were allocated to departments. For those who were interested it seemed to be a matter of how much more *my* department had used than cognate ones and *how* uncomputed some *funny* departments were.

[3]Lemmer and Leach (1993) [123]. In Ranganathan (1988) [178] and (1989) [179] this equation and some similar ones were presented. The discussion was more appropriate in the less exotic form obtained through transformation by Lemmer and Leach [123]. Without wishing to be more than reasonably uncharitable to quasi–mathematical savants they do tend to forget that simpler versions of their equations may exist and be accessible via elementary transformations.

by means of a Kummer–Liouville transformation [92, 139, 17, 18], is essentially nonlinear. We recognise (3.1.4) as the generalised Emden–Fowler equation of index two[4] which has been of some interest in recent relativity[5].

We have the problem of whether the nonlinearity of an equation is apparent or inherent. In some cases, such as the one above, it is easy to spot a transformation which linearises the equation. Life tends to be not that easy. In this chapter we study second order nonlinear equations for linearisability conditions. The emphasis on second order equations is motivated simply on the basis of their commonness because of the persistent influence of Newton. In a sense it is the prelude to a far more extensive study which has by no means been implemented completely. We have the criteria for scalar equations of higher order, but the results for systems of equations are essentially non–existent.

We commence with Lie's classification scheme for second order equations. Then we prove some results about the linearisability of nonlinear equations. The equivalence of all second order linear equations in an algebraic sense is proven in a very simple way. We conclude with the elements of the studies so far of systems of second order equations. We note the presently unsatisfactory state of work in systems of linear equations or linearisable systems and end our discussion by pointing out the obvious. There is still much to be done in the establishment of fundamental results in this area.

3.2 Lie's Classification Scheme

The possession of a Lie point symmetry enables the order of an equation to be reduced by one. In the case of a second order equation this leads to the reduced equation being of the first order and hence integrable. Unfortunately the integrability is formal as the determination of the integrating factor requires a knowledge of the solution of the equation for which one is seeking

[4]This equation is discussed at some length in Chapter Seven.

[5]Leach *et al* (1992) [119].

the integrating factor so that it can be solved. However, if there is a second symmetry of the second order equation and reduction is performed by the normal subgroup so that the first order equation inherits the other symmetry, the first order equation can be transformed to autonomous form and be reduced to quadratures. It was this simple consideration which led Lie to start off with the properties of second order equations invariant under a two-dimensional algebra[6].

Second order ordinary differential equations possessing two point symmetries have four canonical forms for the representations of the two-dimensional algebras. They and their associated differential equations are given in Table 3.1. There are two canonical forms for each of the two Lie algebras, $2A_1$ and A_2. In the Table they are called Types I, II, III and IV and we refer to each by its appropriate type number.

Table 3.1 is easily explained. Suppose that a second order ordinary differential equation admits a two-dimensional Lie algebra. The two symmetries have either the Abelian algebra, $2A_1$, or the solvable algebra, A_2. There is not much choice when it comes to two-dimensional algebras. However, the two symmetries can be either connected, as in the cases of Types II and IV, or unconnected, as for Types I and III. If the two symmetries have the properties, say, of being unconnected and having zero Lie Bracket, *ie* they belong to Type I, there exists a point transformation which will reduce them to the canonical form associated with Type I and the differential equation to one with the structure of the entry in the last column of the table[7].

Lie deduced that any differential equation of Type II or IV is linearisable by means of a point transformation. Hence it has the algebra, $s\ell(3, R)$, for its point symmetries and can be transformed to the *free particle* equation by means of a point transformation[8]. This means that the Type II and IV equations

[6]Lie (1891) [133].

[7]Naturally it need not be as general in appearance.

[8]These details are covered in §§3.3 and 3.4.

Table 3.1

Type	Algebra	Connectedness of G_1 and G_2	Canonical forms of G_1 and G_2	Form of equation
I	$2A_1$	$\forall\rho\ G_2 \neq \rho(x,y)G_1$ (unconnected)	$\bar{G}_1 = \dfrac{\partial}{\partial X}$ $\bar{G}_2 = \dfrac{\partial}{\partial Y}$	$Y'' = F(Y')$
II	$2A_1$	$\exists\rho : G_2 = \rho(x,y)G_1$ (connected)	$\bar{G}_1 = \dfrac{\partial}{\partial Y}$ $\bar{G}_2 = X\dfrac{\partial}{\partial Y}$	$Y'' = F(X)$
III	A_2	$\forall\rho\ G_2 \neq \rho(x,y)G_1$ (unconnected)	$\bar{G}_1 = \dfrac{\partial}{\partial Y}$ $\bar{G}_2 = X\dfrac{\partial}{\partial X} + Y\dfrac{\partial}{\partial Y}$	$XY'' = F(Y')$
IV	A_2	$\exists\rho : G_2 = \rho(x,y)G_1$ (connected)	$\bar{G}_1 = \dfrac{\partial}{\partial Y}$ $\bar{G}_2 = Y\dfrac{\partial}{\partial Y}$	$Y'' = Y'F(X)$

have an additional six symmetries. These can be found from those of the free particle equation using the inverse of the transformation which takes the original equation to the free particle equation.

This leaves the two unconnected realisations to be explored for the condition or conditions under which they will have the extra symmetries to make up the numbers for possessing $s\ell(3,R)$. The next two sections are devoted to this end.

Before we proceed to further consideration of these two cases we can give a general result for the necessary and sufficient condition for a second order equation to possess the maximal $s\ell(3,R)$ algebra. It is that the equation have the nilpotent algebra

$$[G_1, G_2] = 0 \qquad [G_2, G_3] = 0 \qquad [G_1, G_3] = G_2 \tag{3.2.1}$$

to which we refer as the algebra $\aleph$. Sufficiency proceeds by construction and necessity follows from $\aleph$ being a subalgebra of $s\ell(3,R)$[9]. To anticipate the discussion of higher order equations below the second order equations differ from those of higher order in that the dimension of the algebra sufficient for linearisability in the latter case equals the order of the equation whereas for second order equations it is three. This is because there are two inequivalent representations of the algebras $2A_1$ and A_2. One pair of representations gives linearity immediately and the other does not, but provides the cause for the next two sections.

3.3 Linearisation of Type I Equations

We assume that we have an equation with two independent commuting symmetries[10] and that they and the equation are written in canonical form. In particular the equation is

$$y'' = f(y'). \tag{3.3.1}$$

[9] Sarlet *et al* (1987) [196].

[10] The discussion follows Sarlet *et al* (1987) [196] without dwelling overly much on minor details.

A generator of the form

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{3.3.2}$$

is a symmetry of (3.3.1) if

$$\begin{aligned}&\eta_{xx} + y'(2\eta_{xy} - \xi_{xx}) + y'^2(\eta_{yy} - 2\xi_{xy}) - y'^3\xi_{yy} + (\eta_y - 2\xi_x - 3y'\xi_y)f \\ &= [\eta_x + y'(\eta_y - \xi_x) - y'^2\xi_y]f_{y'},\end{aligned} \tag{3.3.3}$$

where suffices refer to partial derivatives, which follows from the action of $G^{[2]}$ on (3.3.1).

For general f it is evident from a casual perusal of (3.3.3) that at most only two symmetries exist. These are the two we assumed. A slightly less general situation applies if we take $f(y')$ to be a polynomial in y' of degree greater than three. The lower order terms are not of much importance here. There can now be at most three symmetries. However, when $f(y')$ is of degree no higher than the third, there is no immediate loss of symmetry. Accordingly we consider the equation

$$y'' = gy'^3 + ay'^2 + by' + c, \tag{3.3.4}$$

where all coefficients are constants and g is non–zero. In this form (3.3.4) has a certain illusory degree of generality which can be removed by a rescaling of independent variable and a translation of dependent to leave the essential form

$$y'' = y'^3 + by' + c. \tag{3.3.5}$$

It is perhaps proper to reflect that we are not concerned here with the solution of (3.3.5) for it is apparent that it can be reduced to quadratures[11] with the information already at our disposal. What we really wish to determine is the circumstances under which (3.3.5) can be linearised. For then the solution is trivial rather than something which smacks like a double dose of elliptic integrals. The extraordinary thing is that the equation is linearisable no matter

[11] That *is* what those two symmetries are for!

the values of the coefficients[12]. There are always the eight symmetries of the Lie algebra, $s\ell(3,R)$. It is only the representations which vary with the properties of the roots of the cubic. We note that in the particular case for which the polynomial is a quadratic a similar analysis holds[13].

The important thing to remember is that the equation can be no more than cubic in the first derivative if it is to be linearisable. It is interesting to note that the 'free particle' equation

$$Y'' = 0, \tag{3.3.6}$$

when subjected to the point transformation

$$X = F(x,y) \qquad Y = G(x,y), \tag{3.3.7}$$

becomes

$$\begin{aligned}[F,G]_{x,y}y'' + [F,G]_{y,y^2}y'^3 + ([F,G]_{x,y^2} + 2[F,G]_{y,xy})y'^2 \\ + ([F,G]_{y,x^2} + 2[F,G]_{x,xy})y' + [F,G]_{x,x^2} = 0,\end{aligned} \tag{3.3.8}$$

where, for example, $[F,G]_{y,y^2}$ is a shorthand notation for

$$[F,G]_{y,y^2} = \frac{\partial F}{\partial y}\frac{\partial^2 G}{\partial y^2} - \frac{\partial^2 F}{\partial y^2}\frac{\partial G}{\partial y}. \tag{3.3.9}$$

The application of a further point transformation to (3.3.9) does not increase the degree of y' in the equation.

3.4 Type III Linearisation

We recall that the second order equation of Type III has the form

$$XY'' = F(Y') \tag{3.4.1}$$

[12]The details are in Sarlet *et al* (1987) [196]. The treatment depends upon the factorisation of the cubic on the right side of the equation except that the signs change just as in the case of linear second order partial differential equations.

[13]The quadratic case can be attained by a fibre–preserving transformation instead of the more general point transformation. Hsu and Kamran [80] congratulated themselves on the generality of their result compared with the linearisation of equations cubic in the first derivative achieved by Sarlet *et al* [196]!

and the two symmetries

$$\bar{G}_1 = \frac{\partial}{\partial Y} \qquad \bar{G}_2 = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}. \tag{3.4.2}$$

We can make some simplifications to the discussion by employing the results of Tresse[14]. The equation

$$y'' = H(x, y, y') \tag{3.4.3}$$

is linearisable if and only if the relative invariants defined by

$$\begin{aligned} I_1 &= H_{y'y'y'y'} \\ I_2 &= \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(H_{y'y'}\right) - 4\frac{\mathrm{d}}{\mathrm{d}x}\left(H_{y'y}\right) - 3H_yH_{y'y'} + 6H_{yy} \\ &\quad + H_{y'}\left(4H_{y'y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(H_{y'y'}\right)\right) \end{aligned} \tag{3.4.4}$$

both vanish for the equation. The vanishing of I_1 imposes the form

$$y'' = \mathcal{A}(x,y)y'^3 + \mathcal{B}(x,y)y'^2 + \mathcal{C}(x,y)y' + \mathcal{D}(x,y) \tag{3.4.5}$$

on the differential equation. The vanishing of I_2 imposes the following conditions on the four coefficient functions

$$\begin{aligned} &3\mathcal{A}_{xx} + 3\mathcal{A}_x\mathcal{C} - 3\mathcal{A}_y\mathcal{D} + 3\mathcal{A}\mathcal{C}_x + \mathcal{C}_{yy} - 6\mathcal{A}\mathcal{D}_y + \mathcal{B}\mathcal{C}_y \\ &\qquad - 2\mathcal{B}\mathcal{B}_x - 2\mathcal{B}_{xy} = 0 \end{aligned} \tag{3.4.6}$$

$$\begin{aligned} &6\mathcal{A}_x\mathcal{D} - 3\mathcal{B}_y\mathcal{D} + 3\mathcal{A}\mathcal{D}_x + \mathcal{B}_{xx} - 2\mathcal{C}_{xy} - 3\mathcal{B}\mathcal{D}_y + 3\mathcal{D}_{yy} \\ &\qquad + 2\mathcal{C}\mathcal{C}_y - \mathcal{C}\mathcal{B}_x = 0. \end{aligned} \tag{3.4.7}$$

Given these two conditions and the general form of an equation invariant under Type III symmetries we need only consider the equation

$$xy'' = ay'^3 + by'^2 + cy' + d, \tag{3.4.8}$$

where the coefficients a through d are constants. The two equations above, (3.4.6) and (3.4.7), do not permit linearisation if $a = 0$. The general cubic can be transformed to the elemental form

$$xy'' = y'^3 + cy' + d. \tag{3.4.9}$$

[14] Tresse (1896) [207]. Poincaré [173] published his paper on the existence of a conserved vector of angular momentum type for the classical magnetic monopole in the same year.

Further use of (3.4.6) and (3.4.7) reduces (3.4.9) to simply

$$xy'' = y'^3 + y' \tag{3.4.10}$$

as the only equation of this type which can be linearised by means of a point transformation.

The transformation itself is found from an analysis of the point symmetries of (3.4.10) and comparison of them with the standard set for the free particle[15]. The generators of the algebra, $\aleph$, are easily identified and the required transformation is

$$Y = \frac{1}{2}(x^2 + y^2) \qquad X = y. \tag{3.4.11}$$

3.5 The General Theorem

In the previous two sections we have treated the two cases of second order equations possessing algebras of dimension two which were not automatically linear. In this way we have treated all four cases[16] of equations admitting two–dimensional algebras of symmetries which are linearisable. If an equation of the form (3.4.3) passes the linearisation test, *ie* conditions (3.4.6) and (3.4.7) hold, one need only obtain two symmetries of the equation in order to construct a linearising point transformation for the equation. Of course it may be just as easy to calculate the full complement of symmetries as to calculate two[17], but this is to miss the point of what is possible in theory.

A simple example helps to illustrate what the procedure is. The equation

$$y'' + 3yy' + y^3 = 0 \tag{3.5.1}$$

[15]See Mahomed and Leach (1985) [149] Tables II and III for a listing of the symmetries and their Lie Brackets.

[16]Types I through IV.

[17]This is particularly the case when the equation is fed to the machine and the tender mercies of Program LIE [77]. It is not obvious how the machine can be instructed to stop when the two appropriate symmetries have been found.

arises in the study of the generalised Emden–Fowler equation[18] and in the investigation of univalued functions defined by second–order equations[19]. The equation satisfies both conditions (3.4.6) and (3.4.7). Hence it is linearisable. By observation it is invariant under time–translation and, if one looks a little more carefully, under a similarity transformation[20]. We have the two non–commuting and non–proportional symmetries

$$G_1 = \frac{\partial}{\partial x} \qquad G_2 = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \tag{3.5.2}$$

The transformation

$$X = \frac{1}{y} \qquad Y = x + \frac{1}{y} \tag{3.5.3}$$

transforms the generators to the canonical form and (3.5.1) to[21]

$$XY'' = -Y'^3 + 6Y'^2 - 11Y' + 6. \tag{3.5.4}$$

The further transformation

$$\bar{Y} = \frac{1}{2}X^2 - \frac{X}{Y} \qquad \bar{X} = X - \frac{1}{Y} \tag{3.5.5}$$

yields a linear equation.

There are times when it is easier to determine two commuting symmetries. Consider the differential equation of the conic sections, *viz.*

$$y'' = \frac{y'}{x} - \frac{y'^2}{y}. \tag{3.5.6}$$

It has the two commuting symmetries

$$G_1 = x\frac{\partial}{\partial x} \qquad G_2 = y\frac{\partial}{\partial y}. \tag{3.5.7}$$

[18]Leach (1985) [109] and Lemmer and Leach (1993) [122]. See Chapter Seven.

[19]Golubev (1950) [54].

[20]This can be obtained by assuming the structure of a similarity transformation and determining the coefficients. It can also be found by performing the first step of the Painlevé analysis.

[21]We did not claim that the canonical form was the simplest form!

The transformation which converts these to $\partial/\partial Y$ and $\partial/\partial X$ is

$$X = \log x \qquad Y = \log y. \tag{3.5.8}$$

The equation becomes

$$Y'' = 2Y' - 2Y'^2. \tag{3.5.9}$$

The free particle comes after the transformation

$$\bar{Y} = \frac{1}{2}\exp(2X) \qquad \bar{X} = -\exp(2Y - X). \tag{3.5.10}$$

3.6 The Equivalence of all Linear Second Order Ordinary Differential Equations

In Chapter Two we recalled the many *ad hoc* investigations of the symmetries of various equations, particularly the second order linear ones, which, one by one, revealed themselves to be in possession of the symmetry algebra, $s\ell(3, R)$. By the end of the seventies it was accepted folklore that all linear equations had $s\ell(3, R)$ symmetry[22] although the first proof of which we are aware is due to Mahomed[23]. Here we offer a very simple proof[24] which really does not require much understanding nor knowledge of anything[25].

The general second order linear differential equation

$$y'' + a(x)y' + b(x)y = c(x) \tag{3.6.1}$$

[22]Recall that Lie had shown that the maximal algebra was $s\ell(3, R)$. This is not the same result and, as we shall see in the case of higher order equations, does not even follow.

[23]Mahomed (1986) [147]. The present writer appreciates the contribution made by Dr Mahomed in converting folklore to proven theorem as it saved him the difficulty of coming up with a proof when confronted whilst evoking folkloristic recollections at the IHES in 1988.

[24]See Govinder and Leach (1994) [67].

[25]Nevertheless it is important to make the statement loudly and clearly as there are many out there who are unaware of the commonality of $s\ell(3, R)$ symmetry for all linear sodes and those which are linearisable.

is transformed to

$$\ddot{v}(ut'^2) + \dot{v}(2u't' + ut'' + aut') + v(u'' + au' + bu) + w'' + aw' + bw = c \quad (3.6.2)$$

under the generalised Kummer–Liouville transformation[26]

$$y = u(x)v(t) + w(x) \qquad t = t(x), \tag{3.6.3}$$

where, as usual, $'$ denotes $\mathrm{d}/\mathrm{d}x$ and $\dot{}$ denotes $\mathrm{d}/\mathrm{d}t$. The coefficient of $\dot{v}$ in (3.6.2) becomes zero if

$$2u't' + ut'' + aut' = 0, \tag{3.6.4}$$

the coefficient of v becomes zero if

$$u'' + au' + bu = 0 \tag{3.6.5}$$

and the nonhomogeneous term vanishes if

$$w'' + aw' + bw = c. \tag{3.6.6}$$

Each of (3.6.4), (3.6.5) and (3.6.6) have continuous solutions provided that the functions $a(x)$, $b(x)$ and $c(x)$ are continuous and satisfy a Lipschitz condition[27]. Hence (3.6.1) is equivalent to

$$\ddot{v} = 0 \tag{3.6.7}$$

under a point transformation. Eq (3.6.7) has the Lie algebra $s\ell(3, R)$ of its point symmetries and hence (3.6.1) does since the transformation is a point transformation. The point transformation may only have local validity, but we are here concerned with the algebra and not the group.

This proof is very elementary and it is surprising that it has not been presented before[28].

[26] Kummer (1887) [92], Liouville (1837)[139].

[27] Ince (1927) [82, p 63].

[28] It is perhaps not so surprising that no journal wants to publish it. The proof is far too easy to understand!

3.7 Systems of Linear Second Order Ordinary Differential Equations

3.7.1 Introduction

The study of the symmetry properties of systems of second order ordinary differential equations cannot be claimed to have the completeness associated with that of scalar equations. There have been studies of some particular varieties of nonlinear systems such as the Kepler problem and variations thereon[29]. When it comes to linear systems, there has been very little work[30] and it must be admitted that the present situation is not only under-developed, but it is scandalously so. The reason for the scandal is easy to see. It is the same old story. Linear equations are cheap to solve. Nonlinear ones are expensive to solve. The situation for scalar equations has been resolved[31], but the knowledge of the situation for systems is almost non-existent. The cause is simple. We do not know what the symmetry algebras of systems of linear equations are in general. Hence the knowledge of the algebra of a nonlinear system is not necessarily going to point the direction towards the determination of the

[29]For example Prince and Eliezer (1981) [176] on the Kepler problem, Moreira *et al* (1985) [161] on the magnetic monopole, Leach and Gorringe (1990) [117] on the equation $\ddot{\mathbf{r}}+f(r)\mathbf{L}+g(r))\mathbf{r}=0$, Gorringe and Leach (1991) [59] on central force problems and Gorringe and Leach (1993) [60] on Kepler's Third Law.

[30]The only studies known to us are those of González–Gascón and González–López (1983) [57] and González–López (1988) [56] apart from the one by Gorringe and Leach (1988) [58] to be summarised here. The coincidence of two groups independently working on the same problem at about the same time is not so unusual. Even within the experience of the present writer it is not unique as both González–Gascón and Leach, with different sets of co–authors, sent almost the same papers to *Journal of Mathematical Physics* in early 1982 and they appeared in the same issue [55, 130] towards the end of the year. We are satisfied that the two works were independent studies although some doubts were raised at the IUTAM meeting in Torino in June, 1982, without corroboration.

[31]The same is more or less true for higher order scalar ordinary differential equations.

possible existence of a linearising transformation[32].

Here we report what can only be described as the first, tentative, steps towards a classification of the symmetries of systems of ordinary differential equations. The systems are of the second order and are linear with a very simple structure. Future routes of investigation to pursue are more complicated linear equations and systems of higher order. The latter are not encountered commonly in applications, but results found for them could cast some light on the properties of systems of second order equations in much the same way that studies of higher order scalar equations have contributed to our understanding of scalar second order equations[33]. Another area of interest is the relationship between the symmetries of a system of equations and an equivalent higher order equation. It may be more accurate to add 'or lack of' as the two sets of equations are related via a non-point transformation. Rapidly a plethora of questions is raised which suggests that there is still much life left in the area. However, we leave the area of speculation over lines of future research and hark back to those of the not too distant past.

3.7.2 Two dimensions of a limited variety

To make the discussion as transparent as possible we commence with the two linear equations

$$\begin{aligned} \ddot{x} &= ax + by \\ \ddot{y} &= cx + dy, \end{aligned} \tag{3.7.1}$$

where a, b, c and d are constants and overdot denotes differentiation with respect to the independent variable, t. By means of elementary similarity

[32]For a system of n second order equations possessing the symmetry algebra $s\ell(n+2, R)$, linearisation is obvious. It has been argued by Marc Feix (during a seminar at PMMS, CNRS, Orléans in July, 1987) that partial linearisation would be useful, but to our knowledge the criteria for that have not been addressed.

[33]For which see Chapters Five and Six.

transformations[34], which do not affect the number of symmetries since they are point transformations, (3.7.1) may be written in the upper triangular form

$$\begin{aligned}\ddot{x} &= ax + by\\ \ddot{y} &= dy\end{aligned} \tag{3.7.2}$$

and it is this system which we analyse. We write the generator as

$$G = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}, \tag{3.7.3}$$

where τ, ξ and η are functions of x, y and t. The application of the second extension of G to the system (3.7.2) leads to fifteen partial differential equations to be satisfied by the functions τ, ξ and η. Several cases emerge.

Case (i) The coefficients of (3.7.2) are $d = a$ and $b = 0$.

There are the fifteen generators[35]

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial t}\\
G_2 &= x\frac{\partial}{\partial x}\\
G_3 &= y\frac{\partial}{\partial x}\\
G_4 &= x\frac{\partial}{\partial y}\\
G_5 &= y\frac{\partial}{\partial y}\\
G_6 &= e^{\alpha t}\frac{\partial}{\partial x}\\
G_7 &= e^{-\alpha t}\frac{\partial}{\partial x}\\
G_8 &= e^{\alpha t}\frac{\partial}{\partial y}\\
G_9 &= e^{-\alpha t}\frac{\partial}{\partial y}\\
G_{10} &= e^{2\alpha t}\left(\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y}\right)
\end{aligned}$$

[34]Wilkinson (1965) [210, p 46].

[35]$\sqrt{a}$ is written as α. Were a negative, the exponentials should be replaced by circular functions.

$$G_{11} = e^{-2\alpha t}\left(\frac{\partial}{\partial t} - \alpha x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y}\right)$$
$$G_{12} = xe^{\alpha t}\left(\frac{\partial}{\partial t} - \alpha x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y}\right)$$
$$G_{13} = xe^{-\alpha t}\left(\frac{\partial}{\partial t} - \alpha x\frac{\partial}{\partial x} - \alpha y\frac{\partial}{\partial y}\right)$$
$$G_{14} = ye^{\alpha t}\left(\frac{\partial}{\partial t} + \alpha x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y}\right)$$
$$G_{15} = ye^{-\alpha t}\left(\frac{\partial}{\partial t} - \alpha x\frac{\partial}{\partial x} + \alpha y\frac{\partial}{\partial y}\right). \qquad (3.7.4)$$

The algebra under the operation of taking the Lie Bracket is $s\ell(4, R)$[36]

Case (ii) The coefficients of (3.7.2) are $d \neq a$ and $b = 0$.

We write $\sqrt{a} = \alpha$ and $\sqrt{d} = \beta$ in the case that a and b are positive. If either or both are negative, the exponentials below are replaced by circular functions as appropriate. There are seven symmetries and they are

$$G_1 = \frac{\partial}{\partial t}$$
$$G_2 = x\frac{\partial}{\partial x}$$
$$G_3 = y\frac{\partial}{\partial y}$$
$$G_4 = e^{\alpha t}\frac{\partial}{\partial x}$$
$$G_5 = e^{-\alpha t}\frac{\partial}{\partial x}$$
$$G_6 = e^{\beta t}\frac{\partial}{\partial y}$$
$$G_7 = e^{-\beta t}\frac{\partial}{\partial y}. \qquad (3.7.5)$$

When this list is compared with that of Case (i), we see that we have lost the interchange operators, G_3 and G_4, plus G_{10} - G_{15}. The first two signify that the variables x and y are no longer equivalent. The second set is more or less the equivalent of the loss of $s\ell(2, R)$ symmetry that one finds with higher order equations[37].

---

[36] See Prince and Eliezer (1981) [176].

[37] See Chapters Five and Six.

Case (iii) The coefficients of (3.7.2) are $a = 0 = d$ and $b \neq 0$.

There are now eight generators which take the forms

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t} \\
G_2 &= 2x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \\
G_3 &= 2y\frac{\partial}{\partial y} - t\frac{\partial}{\partial t} \\
G_4 &= y\frac{\partial}{\partial x} \\
G_5 &= \frac{\partial}{\partial x} \\
G_6 &= t\frac{\partial}{\partial x} \\
G_7 &= bt^2\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y} \\
G_8 &= bt^3\frac{\partial}{\partial x} + 6t\frac{\partial}{\partial y}.
\end{aligned} \tag{3.7.6}
$$

The change in the coefficients of the symmetries is rather dramatic. That this case is very easily written as a fourth order equation of the eight symmetries variety does not explain these symmetries. Indeed the connection, if any, between the symmetries of a system of equations and those of a corresponding single equation of higher order is unknown, although it is a worthy subject of investigation.

Case (iv) The coefficients are $b \neq 0$ and a, d not both zero.

This is similar to Case (ii) in that seven symmetries are found. Their precise expression depends upon the relationship between a and d. In the case that $a \neq d$ and both are non-zero they are[38]

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial t} \\
G_2 &= (x - ky)\frac{\partial}{\partial x} \\
G_3 &= y\left(ky\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)
\end{aligned}
$$

[38]The convention for both a and d as above is maintained. To keep the expressions moderately compact we write $k = b/(d - a)$.

$$
\begin{aligned}
G_4 &= e^{\alpha t}\frac{\partial}{\partial x}\\
G_5 &= e^{-\alpha t}\frac{\partial}{\partial x}\\
G_6 &= e^{\beta t}\left(k\frac{\partial}{\partial x}-\frac{\partial}{\partial y}\right)\\
G_7 &= e^{-\beta t}\left(k\frac{\partial}{\partial x}+\frac{\partial}{\partial y}\right).
\end{aligned}
\tag{3.7.7}
$$

It so happens that the commutation relations in this case are the same as those for Case (ii).

We observe that for the two-dimensional system under consideration

(i) There are three possible numbers of symmetries, *viz.* fifteen, eight and seven.

(ii) The maximal number of generators is obtained only if the coefficient matrix is a scalar multiple of the identity. This scalar may be zero. It may be time-dependent. The Lie algebra is $s\ell(4,R)$. This was already known from other studies [176].

(iii) When the coefficient matrix is diagonal with unequal entries (zero is not an excluded value), the number of generators is reduced to seven. This is also the case when the matrix is fully triangular. That the two were afforded separate consideration is a consequence of the hindsight generated by the next subsection.

(iv) When the matrix is strictly upper triangular, an additional symmetry is introduced to make eight in all.

The investigation of the simplest forms of systems of linear second order equations has indicated that there is an even more profound change in the algebraic properties of differential equations in going from one to several than there is in going from scalar linear sodes to scalar linear nodes. The variations in the algebras mean that the identification of linearisable systems will have a broader set of criteria. It will also mean that there will be a broader range of systems which can be linearised. It also suggests that there may be different degrees of difficulty in the solution of linear systems just as there are in the case of linear equations of higher order[39].

3.7.3 A system of n second order linear equations.

The n second order differential equations generalisation of (3.7.2) is

$$\begin{aligned} &\ddot{\mathbf{x}} = A\mathbf{x} \\ \Leftrightarrow\ &\ddot{x}_i = a_{ij}x_j, \qquad i = 1, n. \end{aligned} \tag{3.7.8}$$

In the sequel the summation convention is used except where specifically excluded. When the symmetry is written as

$$G = \tau\frac{\partial}{\partial t} + \eta_k\frac{\partial}{\partial x_k}, \tag{3.7.9}$$

the terms of the third and second degree in the first derivatives give that

$$\tau = b_i(y)x_i + c(t) \tag{3.7.10}$$

and

$$\eta_k = \dot{b}_i x_i x_k + d_{jk}(t)x_j + e_k(t). \tag{3.7.11}$$

The equations which come from the terms linear in the $\dot{x}_i$ and those free of derivatives lead to a number of cases. The important thing to do is to take care not to become bogged down in details. Rather the spirit of the results ought to be stressed. The general result is that there are $2n$ symmetries of the form

$$G_k = e_k\frac{\partial}{\partial x_k} \qquad k = 1, 2n, \tag{3.7.12}$$

where $e_k(t)$ is a solution of the original equation, (3.7.8)[40]. For the rest we must consider particular cases.

Case (i) $A = \alpha I$.

There are $2n$ symmetries which come from $b(t)$ of the form

$$G_k = b_i x_i\frac{\partial}{\partial t} + \dot{b}_i x_i x_k\frac{\partial}{\partial x_k} \qquad k = 1, 2n, \tag{3.7.13}$$

[39]See Chapter Six.

[40]There is no summation required in (3.7.12).

where the bs are the solutions of

$$\ddot{b}_m - \alpha b_m = 0 \qquad m = 1, n. \tag{3.7.14}$$

Three symmetries of the form

$$G_k = c_k(t)\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{c}_k x_k \frac{\partial}{\partial x_k} \qquad k = 1, 3 \tag{3.7.15}$$

come from the solutions of

$$\dddot{c} - 4\alpha\dot{c} = 0. \tag{3.7.16}$$

Finally there are n^2 symmetries of the form

$$G_{ij} = x_i \frac{\partial}{\partial x_j} \qquad i, j = 1, n \tag{3.7.17}$$

which come from the functions $d_{ij}(t)$ in the expression for η, (3.7.11). In all there are $(n+1)^2 - 1$ generators and one can identify the Lie algebra as $s\ell(n+2, R)$.

Case (ii) $A = D$.

The bs now all vanish. There is only one c and it is a constant. The $d_{ij}(t)$ can only be constants. The diagonal elements, d^0_{ii}, are arbitrary[41]. The off-diagonal terms satisfy

$$d^0_{ik}(D_{kk} - D_{ii}) = 0 \tag{3.7.18}$$

so that they are zero whenever D_{kk} and D_{ii} differ. Since at least one of the D_{ii} must differ from the rest, there are at most $[(n-1)^2 + 1]$ d^0_{ij}s. If all the elements of D differ, only the n diagonal elements of D^0 persist, where D^0 is the constant part of $[d_{ij}]$. The minimum number of generators is $3n+1$ and the maximum number is $n^2 + n + 2$. If k of the diagonal elements of D are equal, there are an additional $k(k-1)$ symmetries from the d^0_{ij}s.

Case (iii) A general matrix A.

It is more than apparent from Case (ii) that the possible plethora of cases increases dramatically with the increase in dimension of the system. In the

[41]The superscript, 0, indicates a constant.

case of a general, albeit constant, matrix A we firstly transform the system to upper triangular form. The bs are zero. There is only one c, a constant. The number of independent elements of D^0 depends upon the multiplicities of the eigenvalues of A. Since A is in upper triangular form, the multiplicities are given by the number of repeated elements on the leading diagonal. Unfortunately they are not independently associated with one eigenvalue and it does not appear to be possible to give a concise, general, statement about the number of symmetries in all cases. Furthermore there is an increase in the number of symmetries as the coefficient matrix becomes increasingly upper triangular. We saw the initial effect of this in (3.7.2) for the two-dimensional system.

3.7.4 General comments

Thus far the investigation has been confined to autonomous systems. We give a sampling of the results for nonautonomous systems by quoting the results for the two-dimensional system

$$\ddot{\mathrm{x}} = A(t)\mathrm{x}. \tag{3.7.19}$$

When $a_{11} = a_{22}$ and $a_{12} = 0$, the algebra remains as $s\ell(4, R)$ which is not unexpected[42]. When $a_{11} \neq a_{22}$ and $a_{12} = 0$, there are only six symmetries since the symmetry associated with c must disappear unless $a_{11}(t) \propto a_{22}(t) \forall t$ in which case there are seven symmetries.

If $a_{11} = 0 = a_{22}$ and $a_{12} \neq 0$, there can be ten symmetries for a suitable $a_{12}(t)$. In general there will be seven symmetries although there can be eight under special circumstances. For a general matrix A the number of symmetries can drop to five simply because of the incompatibility of the nature of the time-dependence in the elements of A.

It is curious that the introduction of a single function of time, as in the

[42] Since this is just the two-dimensional version of the case studied by Prince and Eliezer (1981) [176].

system

$$\ddot{\mathrm{x}} = \alpha(t)\mathrm{x}, \tag{3.7.20}$$

has no effect on the number of symmetries[43]. However, when there is more than one function of time present, the reduction in symmetry is not surprising unless there is a *special* relationship between the functions. Without that special relationship there will not be a point transformation to transform the time–dependent system into an autonomous one and so clearly the number of point symmetries must be different and lower seems to be the way to go.

We must conclude with the observation that the present knowledge of systems of linear equations leaves much more to be studied and understood. We shall see in the chapters on equations of higher order that the ideas to be found in scalar second order equations have to be modified. What we have seen here is that the change from a scalar second order equation to a system of second order equations with both being linear leaves us with a lot that is not easily incorporated into a general theory. Within systems there is still much to be investigated. The connection between systems and higher order equations is another matter altogether, maybe.

[43]However, it is not surprising as this time–dependent system can be transformed to an autonomous system by means of a simple Kummer–Liouville [92, 139] transformation of the type used to render the time–dependent oscillator free of explicit dependence on time.

Chapter 4

Classification of Second Order Ordinary Differential Equations by Algebras

4.1 Introduction

When one realises a real low–dimensional Lie algebra in terms of vector fields in two c oordinates, more than one canonical form may occur. We have observed this feature in the two canonical forms of generators obtained for each of the two real two–dimensional Lie algebras $2A_1$ and A_2 in §§3.2 and 3.3. Lie [133, Kap 18] showed that, if a second order ordinary differential equation admits a two–dimensional algebra of point symmetries, then the two elements can either be connected or disconnected[1]. This gives four representations of two–dimensional algebras which are the Types I – IV of Chapter Three. There we saw that the connected algebras give immediate linearisation whereas the unconnected

[1]It should be obvious that the algebras are either Abelian or solvable. There is not much scope when there are only two elements. The connectivity or otherwise depends on the existence or not of a function, $\rho(x, y)$, such that $G_1 = \rho(x, y)G_2$.

ones need further constraint on the form of the differential equation[2]. These considerations were given in the previous chapter.

In this chapter we examine scalar second order equations admitting real Lie algebras of dimension greater than or equal to two[3]. A complete treatment of the three-dimensional Lie algebras is given for the case of vector fields in the plane. The canonical realisations enable us to give the equivalence classes of all second order ordinary differential equations which admit point symmetry algebras and those specifically of dimension three. Of these there are five. A major result is that a scalar second order ordinary differential equation can admit zero, one, two, three or eight point symmetries.

4.2 Representations in two coordinates

We use the Mubarakzyanov classification of Lie algebras[4] and its notation. In the case of three- and four-dimensional algebras the list includes the decomposable algebras. An algebra is denoted by $A^a_{r,j}$ which represents the jth algebra of dimension r and the algebra depends upon a parameter[5], a. The range of the parameter(s) is restricted to avoid double counting and algebraic sums of lower algebras. The assignment of a specific value to a parameter singles out a specific algebra within a class. These may be well-known or have some special property.

There are eleven Lie algebras of dimension three (decomposable or otherwise), two of which depend upon parameters. Their algebraic properties are

[2]This is another instance of the vagaries of going one, two, many. For equations of order higher than the second the mere existence of an Abelian algebra of the same order means linearisation. See Chapter Five.

[3]Naturally with eight as the upper limit since that is the maximum number of symmetries which a second order equation can have.

[4]See Patera and Winternitz (1975) [169], Patera *et al* (1976) [170] and Patera and Winternitz (1977) [171]. The original work is found in the papers of Mubarakzyanov (1963) [163, 164, 165] and Morozov (1958) [162].

[5]Or parameters.

Table 4.1 Algebras of Dimension Three

Algebra	Nonzero commutation relations
$3A_1$	
$A_1 \oplus A_2$	$[G_1, G_2] = G_1$
$A_{3,1}$ (Weyl)	$[G_2, G_3] = G_1$
$A_{3,2}$	$[G_1, G_3] = G_1, \quad [G_2, G_3] = G_1 + G_2$
$A_{3,3}$ $(D \otimes_s T_2)$	$[G_1, G_3] = G_1, \quad [G_2, G_3] = G_2$
$A_{3,4}$ $(E(1,1))$	$[G_1, G_3] = G_1, \quad [G_2, G_3] = -G_2$
$A_{3,5}^a$ $(0 < \lvert a \rvert < 1)$	$[G_1, G_3] = G_1, \quad [G_2, G_3] = aG_2$
$A_{3,6}$ $(E(2))$	$[G_1, G_3] = -G_2, \quad [G_2, G_3] = G_1$
$A_{3,7}^b$ $(b > 0)$	$[G_1, G_3] = bG_1 - G_2, \quad [G_2, G_3] = G_1 + bG_2$
$A_{3,8}$ $(SL(2,R))$	$[G_1, G_2] = G_1, \quad [G_2, G_3] = G_3, \quad [G_3, G_1] = -2G_2$
$A_{3,9}$ $(SO(3))$	$[G_1, G_2] = G_3, \quad [G_2, G_3] = G_1, \quad [G_3, G_1] = G_2$

presented in Table 4.1. The decomposable algebras are the Abelian algebra, $3A_1$, and the non–Abelian algebra, $A_1 \oplus A_2$[6].

A number of theorems can be proved[7]. They make use of the identities

a) $[G_1, G_2] = (-G_2\rho)G_2$ if $G_1 = \rho(x,y)G_2$ for a suitable function ρ and

b) $[G_1, G_3] = (G_1\psi)G_2 + \psi[G_1, G_2]$ if $G_3 = \psi(x,y)G_2$ for a suitable function ψ,

where G_1, G_2 and G_3 are operators of the form

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}. \tag{4.2.1}$$

THEOREM 1 A second order ordinary differential equation does not admit the Abelian Lie algebra $3A_1$.

[6]The groups listed in parentheses are the Weyl group, the semidirect product of dilations and translations $D \otimes_s T_2$, the Euclidean group $E(2)$, the pseudo–Euclidean group $E(1,1)$, the special linear group $SL(2,R)$ and the special orthogonal group $SO(3)$.

[7]The details are given in Mahomed and Leach (1989) [152].

An immediate consequence is that a second order ordinary differential equation does not admit a Lie algebra which contains the the three–dimensional Abelian algebra, $3A_1$, as a subalgebra. This means that we do not have to consider all twenty–four real Lie algebras of dimension four. The nine which need to be considered are listed in Table 4.2. There are only three five–dimensional algebras which need be considered and they are listed in Table 4.3.

We discuss the Lie algebras listed in Tables 4.2 and 4.3 after considering the three–dimensional Lie algebra realisations of Table 4.1.

4.3 Equations invariant under three–dimensional algebras

The next theorem concerns the Lie algebra $s\ell(2,R)$ which features in most of the following chapters because of the central rôle it plays in the theory of linear differential equations of higher order and its association with an important class of nonlinear equations.

THEOREM 2. If a second order equation admits the Lie algebra $s\ell(2,R)$ $(A_{3,8})$, it has either three or eight symmetries.

The proof[8] reduces to the identification of two canonical forms of $s\ell(2,R)$ which can be admitted by a second order differential equation[9]. They are

$$\begin{aligned} G_1 &= \frac{\partial}{\partial y} \\ G_2 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ G_{3a} &= 2xy\frac{\partial}{\partial x} + (y^2 - x^2)\frac{\partial}{\partial y} \end{aligned}$$

[8]See Mahomed and Leach (1989) [152] for the details.

[9]There exists a third representation which is not admitted by a second order equation. It requires an equation of at least the third order and is intimately associated with the Kummer–Schwartz equation [82, p 394] [84, p 602] introduced in §5.2.1

Table 4.2 Algebras of Dimension Four

Algebra	Nonzero commutation relations
$2A_2$	$[G_1, G_2] = G_2$, $[G_3, G_4] = G_4$
$A_{3,8} \oplus A_1$	$[G_1, G_3] = 2G_2$, $[G_1, G_2] = G_1$, $[G_2, G_3] = G_3$
$A_{3,9} \oplus A_1$	$[G_1, G_2] = G_3$, $[G_2, G_3] = G_1$, $[G_3, G_1] = G_2$
$A_{4,7}$	$[G_1, G_4] = 2G_1$, $[G_2, G_4] = G_2$, $[G_3, G_4] = G_2 + G_3$ $[G_2, G_3] = G_1$
$A_{4,8}$	$[G_2, G_3] = G_1$, $[G_2, G_4] = G_2$, $[G_3, G_4] = -G_3$
$A^b_{4,9}$ $(0 < \lvert b \rvert < 1)$	$[G_2, G_3] = G_1$, $[G, G_4] = G_2$, $[G_1, G_4] = (1 + b)G_1$ $[G_3, G_4] = bG_3$
$A^1_{4,9}$	$[G_2, G_3] = G_1$, $[G_2, G_4] = G_2$, $[G_1, G_4] = 2G_1$ $[G_3, G_4] = G_3$
$A^0_{4,9}$	$[G_2, G_3] = G_1$, $[G_1, G_4] = G_1$, $[G_2, G_4] = G_2$
$A_{4,10}$	$[G_2, G_3] = G_1$, $[G_2, G_4] = -G_3$, $[G_3, G_4] = G_2$
$A^a_{4,11}$ $(a > 0)$	$[G_2, G_3] = G_1$, $[G_1, G_4] = 2aG_1$, $[G_2, G_4] = aG_2 - G_3$ $[G_3, G_4] = G_2 + aG_3$
$A_{4,12}$	$[G_1, G_3] = G_1$, $[G_2, G_3] = G_2$, $[G_1, G_4] = -G_2$ $[G_2, G_4] = G_1$

Table 4.3 Algebras of Dimension Five

Algebra	Nonzero commutation relations		
$A_{5,36}$	$[G_2, G_3] = G_1,$	$[G_1, G_4] = G_1,$	$[G_2, G_4] = G_2,$
	$[G_2, G_5] = -G_2,$	$[G_3, G_5] = G_3$	
$A_{5,37}$	$[G_2, G_3] = G_1,$	$[G_1, G_4] = 2G_1,$	$[G_2, G_4] = G_2,$
	$[G_2, G_5] = -G_3,$	$[G_3, G_4] = G_3,$	$[G_3, G_5] = G_2$
$A_{5,40}$	$[G_1, G_2] = 2G_1,$	$[G_1, G_3] = -G_2,$	$[G_1, G_4] = G_5,$
	$[G_2, G_3] = 2G_3,$	$[G_2, G_4] = G_4,$	$[G_2, G_5] = -G_5,$
	$[G_3, G_5] = G_4$		

$$G_{3b} = 2xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}. \tag{4.3.1}$$

The differential equations associated with each representation are

$$xy'' = y'^3 + y' + A(1 + y'^2)^{\frac{3}{2}} \tag{4.3.2}$$

and

$$xy'' = y'^3 - \tfrac{1}{2}y' \tag{4.3.3}$$

respectively. It is evident that (4.3.2) is linearisable whenever the constant, A, is zero. In that case there are eight symmetries and so $s\ell(3, R)$. Otherwise there are only the three symmetries of $s\ell(2, R)$.

THEOREM 3. If a second order differential equation admits the Lie algebra $A^b_{3,7}(b > 0)$ or $A_{3,6}$, it has either three or eight symmetries.

The canonical forms are

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= \frac{\partial}{\partial y} \\ G_3 &= (bx + y)\frac{\partial}{\partial x} + (by - x)\frac{\partial}{\partial y} \end{aligned} \tag{4.3.4}$$

for the differential equation

$$y'' = A(1+y'^2)^{\frac{3}{2}} \exp(b \arctan y') \tag{4.3.5}$$

and

$$\begin{aligned}
G_1 &= x\frac{\partial}{\partial y} \\
G_2 &= \frac{\partial}{\partial y} \\
G_3 &= (1+x^2)\frac{\partial}{\partial x} + (xy+by)\frac{\partial}{\partial y}
\end{aligned} \tag{4.3.6}$$

for the differential equation

$$y'' = B(1+x^2)^{-\frac{3}{2}} \exp(b \arctan x), \tag{4.3.7}$$

where A and B are constants. It is evident that the first equation is linear only when A is zero whereas the second is always linear. Furthermore the first equation is not linearisable for A nonzero.

Two remarks are in order. The canonical forms of the equations (4.3.5) and (4.3.7) were first presented by Mahomed and Leach (1989) [152]. The realisations of the Lie algebras $A_{3,7}^{b}(b > 0)$ and $A_{3,6}$ do not appear to have been considered by Lie. If the equation (4.3.5) contains a negative value of b, the basis can be changed to $V_1 = G_2$, $V_2 = G_1$ and $V_3 = -G_3$ so that the algebra is of the desired form, $A_{3,7}^{-b}$ $(-b > 0)$.

THEOREM 4. If an equation admits the Lie algebra $A_{3,2}$, it has either three or eight generators of symmetry.

The two canonical forms of the algebra are [152]

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial y} \\
G_2 &= \frac{\partial}{\partial x} \\
G_3 &= x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}
\end{aligned} \tag{4.3.8}$$

and

$$G_1 = -\frac{\partial}{\partial y}$$

$$G_2 = x\frac{\partial}{\partial y}$$

$$G_3 = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{4.3.9}$$

for the two differential equations

$$y'' = A\exp(-y') \tag{4.3.10}$$

and

$$y'' = B\exp(x) \tag{4.3.11}$$

respectively. It is obvious that (4.3.10) is linearisable only if the constant A is zero whereas there is no constraint on the constant B in (4.3.11).

THEOREM 5. If an equation admits the Lie algebra $A_{3,5}^a$ $(0 < |a| < 1)$, $A_{3,4}$ or $A_1 \oplus A_2$, it has either three or eight symmetries.

The proof proceeds as for the above[10].

We are now in a position to state the general linearisation theorem for second order equations.

THEOREM 6. For a second order ordinary differential equation, $y'' = N(x, y, y')$, to possess the Lie algebra, $s\ell(3, R)$, it is necessary and sufficient that it have the algebra (a) $A_1 \oplus A_2$ or $A_{3,5}^{\frac{1}{2}}$, (b) $A_{3,3}$, (c) $A_{3,1}$ or (d) $A_{3,9}$ $(so(3))$.

The proofs of (a), (b) and (d) are found in Mahomed and Leach (1989) [152] and of (c) in Sarlet *et al* (1987) [196].

4.4 Four– and higher–dimensional algebras

We now turn our attention to algebras of dimension higher than three. The four–dimensional algebras of interest were listed in Table 4.2. Each one of them contains a three–dimensional subalgebra which implies linearisation by Theorem 6. Hence, if a second order equation admits a four–dimensional algebra,

[10]See Mahomed and Leach (1989) [152].

it is linearisable. However, there are four-dimensional algebras which are not admitted by any second order equation and we have

THEOREM 7. A second order equation does not admit the Lie algebra (a) $A_{3,9} \oplus A_1$, (b) $A_{4,7}$, (c) $A_{4,10}$ or (d) $A_{4,11}^a$ $(a > 0)$ respectively[11].

Each of the five-dimensional algebras listed in Table 4.3 has a four-dimensional subalgebra and hence implies linearisation for a second order equation which admits it. Other five-dimensional algebras are not admitted by second order equations. The same situations apply for six- and seven-dimensional algebras. Either they are not admitted by a second order equation or the equations are linearisable. We summarise these results in

THEOREM 8. A second order ordinary differential equation does not admit exactly an $r \in \{4, 5, 6, 7\}$-dimensional point symmetry algebra.

We should remark that there is scope for further investigation here in the form of when does an algebra become a symmetry algebra of a differential equation. It is possible that some interesting equations could emerge in the case of higher order equations because then the algebra does not have to be a subalgebra of the maximal algebra as it does for second order equations[12].

4.5 Equivalence classes of equations

A second order ordinary differential equation has 0, 1, 2, 3 or 8 Lie point symmetries. We exclude the case of equations possessing no point symmetry as we cannot write representatives of equivalence classes for such equations. If an equation has one point symmetry, it can be reduced to autonomous form by means of a point transformation which brings the symmetry to the generator of translations in the independent variable. Thus an equation with a single point symmetry belongs to the equivalence class

$$y'' = f(y, y'), \tag{4.5.1}$$

[11]Mahomed and Leach (1989) [152].

[12]See Chapter Five.

where f will be a specific function, and the symmetry is

$$G = \frac{\partial}{\partial x}. \tag{4.5.2}$$

G realises the algebra A_1.

Before considering the case of equations possessing two point symmetries we look at the three symmetry case for a reason which will soon become apparent. There are five equivalence classes and their representatives are

$$\begin{aligned}
&xy'' = y'^3 + y' + A(1+y'^2)^{\frac{3}{2}} \\
&xy'' = Ay'^3 - \tfrac{1}{2}y' \\
&xy'' = (a-1)y' + Ay'^{\frac{2a-1}{a-1}} \qquad \text{or} \qquad y'' = y'^{\frac{a-2}{a-1}} \qquad a \neq 0, \tfrac{1}{2}, 1, 2 \\
&xy'' = -1 + A\exp(-y') \qquad \text{or} \qquad y'' = A\exp(-y') \\
&\;\; y'' = A(1+y'^2)^{\frac{3}{2}} \exp(a \arctan y'),
\end{aligned} \tag{4.5.3}$$

where A cannot be zero and $a \in R$. The two symmetry case is now obvious. An equation admitting two point symmetries belongs to either of the two equivalence classes

$$y'' = f(y') \tag{4.5.4}$$

or

$$xy'' = g(y'), \tag{4.5.5}$$

where f is not a polynomial which is at most cubic in y' and it is not of the form given in $(4.5.3e)$ and g is not linear in y' and is not one of the forms given in $(4.5.3a-d)$ or the eight–symmetry form given in Mahomed and Leach (1987) [150].

Equations admitting the greatest number of symmetries all belong to the equivalence class of the 'free particle' equation

$$y'' = 0. \tag{4.5.6}$$

4.6 Conclusion

In this chapter we have summarised how second order ordinary differential equations can be classified by investigating the realisations of real low–dimensional Lie algebras in terms of vector fields in two coordinates. In this way we are able to associate differential equations with those realisations which are the generators of symmetries of a second order equation. For all of the admissible dimensions of the algebra – 1, 2, 3 and 8 – we obtained the canonical representative for each possible algebra. This representative stands for all equations derivable from it by a point transformation. As the number of symmetries increases, the freedom of the form of the equation decreases. For a single symmetry there is the general function, $f(y, y')$. For two symmetries there are two possible equivalence classes, but the general function, $f(y')$, is now of one variable[13]. For three symmetries the number of equivalence classes increases to five, but the only arbitrariness is to be found in the parameters A and a. Finally the case of eight symmetries has just the single representative.

Here we have reported the general[14] structure theory for second order equations which admit Lie point symmetry algebras.

There is a number of open questions. It would be of great interest to compare the Lie classification of equations we have discussed here with the Painlevé classification which produces a catalogue of fifty equations[15]. The transformation up to which the Painlevé classification was performed is given by [32]

$$X = \alpha(x) \qquad\qquad Y = \frac{\beta(x)y + \gamma(x)}{\delta(x)y + \tau(x)}. \tag{4.6.1}$$

There is the natural question. Is there an overlap between the Lie classification and the Painlevé classification? A first step towards answering this question would be to determine the symmetries of the Painlevé equations. Alternatively

[13]Do not be confused by the use of f and g in (4.5.4) and (4.5.5) respectively.

[14]Albeit local, but we are concerned with algebras rather than groups.

[15]See Ince (1927) [82], Graham *et al* (1985) [72], Steeb and Euler (1988) [200] and references cited therein.

one could attempt to reduce some of the Lie equations to the Painlevé ones. This cannot be said to be a trivial task. For example how would one go about reducing (4.5.3e) with its transcendental function to a Painlevé equation? If the reduction cannot be performed, this suggests that the Painlevé classification is inexhaustive[16] and requires supplementation. We put it this way because the Painlevé classification was achieved under the restriction of the homographic transformation, a special case of a fibre–preserving transformation, of (4.6.1) and so one could presume incompleteness. However, the Lie classification also requires supplementation in the case of equations without symmetry as we cannot write the representative of the equivalence class of such equations. Even in the case of equations possessing one symmetry the Lie classification is too general. Clearly further investigation is required. Nevertheless we remark that a preliminary investigation reveals that the Painlevé classification does provide representatives for equations having no or one point symmetry.

Recent investigations[17] using Cartan's method of equivalence have examined the equivalence of differential equations of the form

$$y'' = F(x, y, y') \tag{4.6.2}$$

under the restricted point transformation $X = \phi(x)$, $Y = \psi(x, y)$. This approach was motivated by the Painlevé classification and as such should be viewed against the Painlevé background discussed above.

Perhaps the greatest question overhanging the matter of the comparison of the Lie classification and the Painlevé classification and also the possession of the Painlevé property is this. Should one be restricting attention to the Lie point symmetries? Contact symmetries and nonlocal symmetries may have to be taken into account to determine the correct relationships[18].

[16] In the sense of providing information about the integrability or otherwise of classes of nonlinear ordinary differential equations.

[17] Kamran *et al* (1985) [85] and Kamran and Shadwick (1986) [86].

[18] At which point a line must be drawn lest this work never be finished. Both types of symmetry are the subjects of current investigations [5, 69].

Chapter 5

Higher Order Equations

5.1 Algebras of linear nodes

5.1.1 Introduction

In the revival of interest in the study of the point symmetries of ordinary differential equations over the last two decades or so the initial investigations were motivated by physical problems such as the one–dimensional harmonic oscillator[1]. Most of the earlier works dealt with second order linear equations which all have the symmetry algebra $s\ell(3, R)$[2]. This means that any second order linear equation belongs to the equivalence class of the free particle equation

$$\ddot{q} = 0. \tag{5.1.1}$$

The classification of second order equations has been treated in Chapters Three and Four. In this chapter we investigate the Lie algebraic properties

[1] For example Wulfman and Wybourne (1976) [214], Aguirre and Krause (1988) [7, 8, 9] and Mahomed and Leach (1988) [151].

[2] In Mahomed and Leach (1990) [153] the comment made is that 'as is now well known' with reference to Mahomed (1986) [147], Aguirre and Krause (1988) [7, 8, 9] and Mahomed and Leach (1989) [152]. It may not be as well-known amongst the general mathematical fraternity as one would expect. A particularly simple proof has been given by Govinder and Leach (1994) [67] for which see §3.4.

of nth order equations, where $n \geq 3$. In fairness one must refer the reader to the papers of Krause and Michel[3]. There is a difference in substance between equations of order two and those of greater order. The former have at most eight point symmetries[4], *ie* $2 + 6$, whereas the latter have at most $n + 4$ symmetries[5]. As in all situations in which one goes one, two, many, it is a matter of determination when the generic behaviour is established. With scalar ordinary differential equations we shall see that it is at the third order[6]. A first order equation has an infinite number of point symmetries. A second order equation has at most eight point symmetries and, if it has the maximal number of symmetries, there exists a transformation which converts it to (5.1.1). For third and higher order equations we shall see that such economy of property fails to persist. Linear nth order equations can have $(n + 4)$, $(n + 2)$ or $(n + 1)$ symmetries. Not only can a linear equation not have $(n + 3)$ symmetries, but, in contrast to second order equations, an equation having $(n+3)$ symmetries has an algebra which is *not* a subalgebra of the maximal algebra of point symmetries[7].

The study of higher order equations does not find universal favour because the Newtonian world is based on second order equations. One could make

[3]Krause and Michel (1988) [89, 90]. The survey paper by Neuman (1987) [166] contains several of the recent as well as the classical references on the topic of nth order linear equations.

[4]Lie (1893) [134].

[5]Lie (1891) [133].

[6]In this respect we must emphasise that the behaviour is that of the number of point symmetries for the scalar ordinary differential equation. We do not refer to the number of contact symmetries nor make any claims about the number of symmetries associated with first integrals of those equations.

[7]An example of this type was provided by Louis Michel in a private communication in 1988 [156] with a certain degree of glee. One can easily assume that this was engendered by being able to find a chink in the virtuous armour of the present writer and his valued colleague after *they* had pointed out that not all nth order linear ordinary differential equations were members of the equivalence class of $y^{(n)} = 0$ as had been 'proven' in the 1988 preprint of Krause and Michel [89].

a plea for systems such as those found in biological modelling, which are not essentially founded in Mechanics, to be worthy of consideration and this should be a compelling plea. However, on the one hand one can be cast in the ancient ways and still be obliged to look at the higher order equations for a very simple reason. To fully utilise the properties of second order differential equations one needs to understand why they are so. For this understanding to develop it is necessary to understand the properties of differential equations in general. On the other hand there is a simpler rationale. We are so used to looking at second order equations that we tend to evaluate the merits of others in terms of them. This means that we are hidebound by a tradition which may not have been tested against the fullness of knowledge[8] or our perception of it[9].

In this section we treat the point symmetries of linear nth order ordinary differential equations. The following section deals with the classification of third order ordinary differential equations by algebras. In the next chapter we look at the algebraic properties of their integrals. We do not consider contact symmetries because they are confined to third order equations as far as is known[10]. The story for nonlinear equations is not known in general. However, in the case of third order equations it is known that the Kummer–Schwartz equation

$$2y'y''' - 3y''^2 = 0 \tag{5.1.2}$$

which possesses the Lie point symmetry algebra, $s\ell(2,R) \oplus s\ell(2,R)$, has four purely contact symmetries [138]. Thus in terms of contact symmetries it is equivalent to the linear equation of maximal symmetry although it is very different in terms of point symmetries. This is an area which is not at all understood. However, for the present we write about that which is understood[11].

[8]Whilst I am obliged to accept responsibility for the way these sentiments are expressed, I must thank K S Govinder for pointing out the writer's blinkers.

[9]Which, if we are to accept Popper's compelling argument, must always be limited and tentative. Popper (1984) [174, p7 *ff*].

[10]Certainly in the case of linear equations, *cf* Mahomed and Leach (1991) [154].

[11]There has been some recent work on this subject [5], but it falls beyond the ***terminus***

5.1.2 Symmetry conditions for linear equations

In the case of a general linear second order equation

$$y'' + a(x)y' + b(x)y = c(x) \tag{5.1.3}$$

we have seen[12] that a generalised Kummer–Liouville transformation[13] transforms it to the free particle equation. This is simply because there are three degrees of freedom in the transformation and three unwanted parts in (5.1.3). When we come to higher order equations, it is immediately evident that it is not going to be possible, generically, to reduce all nth linear order equations to the simple form

$$y^{(n)} = 0 \tag{5.1.4}$$

as Krause and Michel [89] implied. The difference between second order and higher order equations, even at the linear level, is immediately explained. What can be done for second order equations is generically impossible for higher order equations.

However, this does not mean that the Kummer–Liouville transformation should be neglected. On the contrary it enables one to transform a general linear equation into the normal form

$$y^{(n)} + \sum_{i=0}^{n-2} B_i(x)y^{(i)} = 0, \qquad n \geq 3 \tag{5.1.5}$$

which is referred to as the canonical form of a linear nth order differential equation. The point symmetry algebra of any equation related to the canonical form, (5.1.5), by a Kummer–Liouville transformation is isomorphic to that of (5.1.5).

If the generator of a point symmetry of (5.1.5) is

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{5.1.6}$$

ad quem of the present discussion.

[12] §3.4.

[13] Kummer (1887) [92], Liouville (1837) [139].

the nth extension of G, which is necessary to deal with the transformations of all derivatives of y up to and including the nth, is

$$G^{[n]} = G + \sum_{j=1}^{n} \eta^{[j]} \frac{\partial}{\partial y^{(j)}}, \tag{5.1.7}$$

where

$$\begin{aligned} \eta^{[k]} &= \frac{\mathrm{d}}{\mathrm{d}x}\eta^{[k-1]} - y^{(k)}\frac{\mathrm{d}\xi}{\mathrm{d}x}, \qquad k = 1, n \\ \eta^{[0]} &= \eta. \end{aligned} \tag{5.1.8}$$

The terms, $\eta^{[k]}$, can also be written in terms of a sum of total time derivatives which is very similar to the Leibnitz formula for the kth derivative of a product[14]. The condition for G to be a symmetry of the nth order equation

$$E(x, y, y', \ldots, y^{(n)}) = 0 \tag{5.1.9}$$

is formally written as

$$G^{[n]} E_{\big|_{E=0}} = 0. \tag{5.1.10}$$

When this is applied to the canonical form, (5.1.5), there is an awful mess because of the generality of the system. However, because the coefficient functions depend upon x and y only, it is possible, in all generality, to equate the coefficients of the terms $y''y^{(n-1)}, y'y^{(n-1)}$ and $y^{(n-1)}$ separately to zero to obtain a subset of the determining equations, *viz.*

$$\xi_y = 0, \qquad \eta_{yy} = 0, \qquad \eta_{xy} = \frac{n-1}{2}\xi_{xx}. \tag{5.1.11}$$

These three equations provide the forms of the coefficient functions ξ and η and we have

$$\begin{aligned} \xi &= a(x) \\ \eta &= \left(\frac{n-1}{2}a^{(1)} + \alpha\right) y + b(x), \end{aligned} \tag{5.1.12}$$

where a and b are as yet arbitrary functions of x and α is a constant.

---

[14] Mahomed and Leach (1990) [153].

The presence of the constant α in (5.1.12) should not come as a surprise as it is simply a reflection of the linearity and homogeneity of the canonical equation, (5.1.5). All such equations possess the symmetry

$$G_1 = y\frac{\partial}{\partial y}. \tag{5.1.13}$$

The substitution of the general expressions for ξ and η back into the determining equations does provide some modicum of simplification, but it is unwise to exaggerate its extent[15]. However, the substitution does show that the function, $b(x)$, in η separates from both the function, $a(x)$, and the constant, α. It is required to satisfy the linear equation

$$b^{(n)} + \sum_{i-0}^{n-2} B_i(x)b^{(i)} = 0. \tag{5.1.14}$$

Eq (5.1.14) has n linearly independent solutions and consequently there are n symmetries of the form

$$G_i = f_i(x)\frac{\partial}{\partial y}, \qquad i = 2, n+1 \tag{5.1.15}$$

which means that the differential equation (5.1.5) has the symmetry algebra, nA_1. In addition to G_1 above, (5.1.13), the symmetries which are obtained from the solutions of the differential equation give every nth ordinary linear differential equation the algebra $A_1 \oplus_s nA_1$.

The virtue of looking beyond the traditional second order equations is already becoming manifest as the way the symmetries are being produced is becoming very compelling in the sense of stating that there is a pattern or hierarchy of symmetries. We have the dilatation symmetry, G_1, which expresses the linearity and homogeneity of the equation. We have the 'solution' symmetries, G_2 to G_{n+1}, which simply state that, if the dependent variable, y, is replaced by one of the solutions to the equation, then the equation is satisfied.

So far we have avoided the symmetries containing the function $a(x)$ for it is in the symmetries which depend on this function that the distinctions amongst

[15] Eq (2.14) of Mahomed and Leach (1990) [153] may be consulted by those who doubt.

linear equations of given order higher than the second becomes manifest. In the sequel we shall see, perhaps in what may be perceived as a formal sense, that there are linear equations and there are other linear equations. This departure from the properties of second order equations is, in itself, sufficient to warrant the investigation of higher order equations if only to find why the second order equations are so peculiar. Needless to say it is this source of symmetries which causes the greatest trouble in terms of computation, but it does enable one to see how the differences amongst linear equations of higher order do arise.

After some rearrangements of the original equation the function $a(x)$ satisfies the set of differential equations

$$\begin{aligned}&\frac{(n+1)!(i-1)}{(n-i)!(i+1)!2}a^{(i+1)}+ia^{(1)}B_{n-i}+aB_{n-i}^{(1)}\\&+\sum_{j=2}^{i-1}B_{n-j}\frac{(n-j)![n(i-j-1)+i+j-1]}{(n-i)!(i-j+1)!2}a^{(i-j+1)}=0,\quad i=1,n.\end{aligned}\tag{5.1.16}$$

Clearly (5.1.16) vanishes identically for $i = 1$ ($B_{n-1} = 0$). This is to be expected since the result $\xi = a(x)$ follows from the equation of terms containing $y^{(n-1)}$ to zero. It should be quite obvious that (5.1.16) is not going to be simple to treat as it comprises a set of conditions on a single function and we divide the treatment into bits to make each somewhat simpler.

5.1.3 The $n+4$ dimensional symmetry algebra for linear equations

To set the flavour of the maximal symmetry case it is instructive to look at the point symmetries of the simplest equation of our general approach although it was *the* equation considered by Krause and Michel (1988) [89]. The equation is

$$y^{(n)}=0,\qquad n\geq 3.\tag{5.1.17}$$

From (5.1.16) we find that the functions $a(x)$ are simply the solutions of

$$a^{(3)}=0\tag{5.1.18}$$

which is the first non–vanishing member of the set. Thus we have three possible functions $a(x)$ and so three symmetries which happen to have the Lie algebra $s\ell(2,R)$. All in all (5.1.17) has $n+4$ point symmetries which constitute the Lie algebra $nA_1 \oplus_s (s\ell(2,R) \oplus A_1)$ which is isomorphic to the algebra $nA_1 \oplus_s g\ell(2,R)$[16].

We now revert to the general case for which the functions $B_i(x)$ are not identically zero. For $i=2$ (5.1.16) becomes

$$\frac{(n+1)!}{(n-2)!4!}a^{(3)} + a^{(1)}B_{n-2} + \frac{1}{2}B^{(1)}_{n-2}a = 0. \qquad (5.1.19)$$

This is closely related to the Lewis–Pinney equation[17] which is obtained by integrating (5.1.19) after multiplying it by the integrating factor $a(x)$ and then making the substitution $a(x) = \rho^2(x)$. Since (5.1.19) has three solutions, we still have $n+4$ symmetries provided the remaining coefficients B_i are consistent with the solutions of (5.1.16) for $i=2$. When the conditions of consistency are satisfied, the differential equation has the same algebra as (5.1.17). A whole sequence of equations of increasing order with maximal symmetry can be developed. The first three are[18]

$$\begin{aligned}
&y^{(3)} + B_1y^{(1)} + \frac{1}{2}B_1^{(1)}y = 0\\
&y^{(4)} + B_2y^{(2)} + B_2^{(1)}y^{(1)} + \left(\frac{3}{10}B_2^{(2)} + \frac{9}{100}B_2^2\right)y = 0\\
&y^{(5)} + B_3y^{(3)} + \frac{3}{2}B_3^{(1)}y^{(2)} + \left(\frac{9}{10}B_3^{(2)} + \frac{9}{100}B_3^2\right)y^{(1)} +\\
&\left(\frac{1}{5}B_3^{(3)} + \frac{16}{100}B_3B_3^{(1)}\right)y = 0.
\end{aligned} \qquad (5.1.20)$$

Even though only a single function occurs in the equations of various orders, the equations are not particularly simple in appearance. However, each one of them can be transformed to

$$y^{(n)} = 0 \qquad (5.1.21)$$

[16] $\oplus$ denotes the direct sum and $\oplus_s$ the semidirect sum.

[17] Lewis (1968) [127], Pinney (1950) [172]. The considerably earlier provenance of this equation is considered in Chapter Eight.

[18] A more complete listing is given by Mahomed and Leach (1990) [153]. Presumably the whole process could be reduced to an algorithm and fed to the computer for its consideration.

by means of a point transformation. It should not be thought that the transformation is easy to find. When the coefficients are constants, the equation with maximal symmetry can be written as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\prod_{i-1}^{(n-1)/2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{(2i)^2}{(n+1/3)}B_{n-2}\right)\right\}y=0 \tag{5.1.22}$$

when n is odd and

$$\left\{\prod_{i=1}^{n/2}\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\frac{(2i-1)^2}{(n+1)/3}B_{n-2}\right)\right\}y=0 \tag{5.1.23}$$

when n is even. Evidently both series (odd and even) are derivable from a second order equation.

5.1.4 The $n+1$ and $n+2$ dimensional symmetry algebras for linear equations

We have seen that a linear equation possesses $n+4$ symmetries under exceptional circumstances. If the coefficient functions do not have the relations prescribed by (5.1.16), the number of symmetries will be lesser. We investigate the situation more closely to determine just what numbers of symmetries can be expected. We consider the case of $n+2$ symmetries first.

The only place where there can be a reduction in the number of symmetries is from the source of the $s\ell(2,R)$ subalgebra since the other part of the algebra, $nA_1 \oplus_s A_1$, comes from the very linearity of the equation and its n linearly independent solutions. The elements of the $s\ell(2,R)$ algebra come from the three linearly independent solutions of the third order equation (5.1.19). The existence of the three solutions requires the vanishing of (5.1.16) for all $i \geq 3$. We look at the conditions under which this does not occur. Setting $i=3$ in (5.1.16) we have

$$\frac{(n+1)!}{(n-2)!4!}a^{(4)}+\frac{3}{n-2}a^{(1)}B_{n-3}+\frac{1}{n-2}aB_{n-3}^{(1)}+a^{(2)}B_{n-2}=0. \tag{5.1.24}$$

The substitution of $\mathrm{d}(5.1.19)/\mathrm{d}x$ into (5.1.24) gives

$$3a^{(1)}\Gamma_3+a\Gamma_3^{(1)}=0, \tag{5.1.25}$$

where

$$\Gamma_3 = \frac{1}{2}B_{n-2}^{(1)} - \frac{1}{n-2}B_{n-3}. \tag{5.1.26}$$

A whole sequence of Γs can be defined similarly and the condition of the vanishing of (5.1.16) becomes

$$ia^{(1)}\Gamma_i + a\Gamma_i^{(i)} = 0 \tag{5.1.27}$$

for $i = 4, \ldots$. The maximal symmetry case follows when each of the Γs is zero for then the conditions (5.1.16) do not impose any additional constraints on (5.1.19) and the three solutions persist.

If not all of the Γs are zero, (5.1.27) becomes a first order equation for a for some first value of i, say, k. The solution of this equation fixes a. If subsequent non–vanishing Γs lead to the same solution, the single a persists and there are $n+2$ symmetries of the original differential equation with the algebra $nA_1 \oplus_s (A_1 \oplus A_1)$. The $s\ell(2,R)$ subalgebra has been reduced to the one–dimensional Abelian subalgebra A_1. We conclude that the linear equation (5.1.14) admits exactly $n+2$ point symmetries when each one of its coefficients can – in principle – be expressed in terms of one arbitrary function of the independent variable.

The case of the possible existence of exactly $n+3$ point symmetries for (5.1.14) is easily treated. For (5.1.14) to possess exactly $n+3$ symmetries (5.1.24), which is a condition on a, must be implied by a second order equation in a in the sense that we have had (5.1.16) implied by the third order equation (5.1.19) for the maximal symmetry case. This cannot happen since at each stage, *ie* $i \geq 3$ in (5.1.27), one either has the first order equation (5.1.25) being nontrivial or being trivial. In the latter case the third order equation (5.1.19) applies. Hence there cannot be just two symmetries from this source and the case of exactly $n+3$ symmetries cannot arise.

Evidently the general linear equation will have $n+1$ symmetries for there will be no consistency between the various equations (5.1.27) for different i unless a is identically zero. Effectively this means that (5.1.14) will contain at least two arbitrary functions in its coefficients. In the case of constant coefficient linear equations there will always be at least $n+2$ symmetries. There will be only $n+2$ symmetries if the equation is not one of the constant coefficient forms of (5.1.22) or (5.1.23). This suggests that there may be some difference in the ease of solution of the two different types of constant coefficient equations which clearly is related to the factorisation of polynomial equations. It is also evident that equations of exactly $n+1$ symmetries are an altogether different class and of far greater degree of complexity of solution than the more symmetrical cases[19].

5.1.5 Linearity and Abelian structure

From the above we see that all linear equations have at least $n+1$ point symmetries with the Lie algebra $nA_1 \oplus_s A_1$. Contained within this is the result that all linear equations[20] have a generic Abelian structure in the form of the Abelian n-dimensional algebra nA_1. It is a small step to assert that, if an nth order equation

$$y^{(n)} = H(x, y, \ldots, y^{(n-1)}) \tag{5.1.28}$$

is linearisable via a point transformation, it admits the Abelian algebra nA_1. The proof follows from the invariance of the Lie Bracket under a point transformation. It is probably not surprising that the converse also applies[21].

[19]As far as we know, there has been no exploration of this point to any extent. Some of the examples given in Chapter Six do provide an illustration of the increasing degree of complexity in the solution of these three classes of linear equations, but that is all. Whether there is anything of interest in these differences is an open question.

[20]Naturally we mean equations written in the canonical form (5.1.14) to keep matters precise.

[21]See Mahomed and Leach (1990) [153] for the details of this and other related theorems.

5.1.6 Nonexistence of an $n+3$ dimensional subalgebra of $nA_1 \oplus_s g\ell(2,R)$

In the case of second order equations we have seen that the number of symmetries is 0, 1, 2, 3 or 8. So far the investigation of linear nodes[22] has revealed that higher order equations have different classes near the top end of the number of symmetries range. The missing algebra is of dimension $n+3$, *ie* one below that of the maximal number of point symmetries. This algebra is not found because of a theorem the statement of which is that there does not exist any nth ($n \geq 3$) order ordinary differential equation having exactly an $n+3$ dimensional point symmetry algebra which is a subalgebra of $nA_1 \oplus_s g\ell(2,R)$. The basis of the proof is that the only algebra which has to be considered is $(n-1)A_1 \oplus_s (s\ell(2,R) \oplus A_1)$[23].

This has no bearing on the existence of algebras of dimension $n+3$ for nonlinear equations of order n. Indeed, at the third order there exist some interesting equations with algebras of dimension six. For example the equation

$$2y'y''' - 3y''^2 = 0 \tag{5.1.29}$$

has the algebra $s\ell(2,R) \oplus_s s\ell(2,R)$[24].

---

[22]A fairly transparent abbreviation of nth order ordinary differential equations. One has *fode(s), sode(s)* and *tode(s)* in common usage.

[23]It is dealt with at length in Mahomed and Leach (1990) [153].

[24]This was originally brought to our attention by Professor Louis Michel of the IHES at Bures-sur-Yvette (1988) [156]. Subsequently the equation has been found to have a more ancient lineage and is known as the Kummer–Schwartz equation, for which see Govinder and Leach (1995) [71]. Somewhat similar equations have the algebras $s\ell(2,R) \oplus_s so(2,1)$ and $so(2,1) \oplus_s so(2,1)$ although they were reported by Mahomed (1989) [148] as examples of equations with four-dimensional algebras. It is true that they do have four-dimensional subalgebras. An interesting feature of the three equations is that they all have ten contact symmetries which is the maximal number for a third order equation. This remains one of the many details of the properties of non-point symmetries of ordinary differential equations which await elucidation. See, however, Abraham–Shrauner *et al* [5].

5.2 Classification of third order ordinary differential equations by algebras

5.2.1 Introduction

The study of the algebraic properties of ordinary differential equations initiated by Lie towards the end of the nineteenth century [133] was directed towards the integrability of equations which admit a one– or multi–parameter group of invariance point transformations, be the equations linear or nonlinear. The algebraic properties of the infinitesimal generators of these transformations under the operation of taking the Lie Bracket, or commutator as this operation is commonly termed, have an intrinsic interest of their own. It is almost comforting to come across a representation of a familiar group which is the set of symmetries of a differential equation[25]. In the last section those two amazing algebras, $s\ell(2,R)$ and $so(2,1)$(the noncompact version of $so(3)$), showed that their capacity has not yet been exceeded.

In the previous chapter we considered the classification of all second order ordinary differential equations by means of their algebras. One of the impetuses for this study was to identify those equations which were really linear equations in disguise. For those which did not fall into that class there was the value of knowing the forms of the representative equations of each admissible algebra. In the case of second order equations Lie[26] established the canonical forms of the two–dimensional algebras[27]. In this section[28] we consider the anal-

[25]In all of this we take the symmetries to be point symmetries and so the algebraic properties are invariant under point transformations. We admit that this is a restriction, but the more general area has not been explored to the extent required to provide a cohærent theory. The more than somewhat scatter-brained ideas of Bluman and Kumei (1989) [21, 379 *ff*] on potential symmetries are typical of the case in point.

[26]See Mahomed and Leach (1988) [151].

[27]Recall that an algebra of dimension equal to that of the order can lead to reduction of the equation to quadratures as far as the solution of the equation is concerned.

[28]Which is based on Mahomed and Leach (1988) [151].

ogous problem for third order ordinary differential equations. We commence with linear equations and their Abelian structure thereby supplementing the material of the previous section. We then move to the consideration of equations invariant under the admissible algebras of dimension three. Naturally this includes the Abelian case which we have already discussed.

5.2.2 Linearity and Abelian structure

Consider the general third order linear equation

$$y''' + B_2y'' + B_1y' + B_0y = g(x), \tag{5.2.1}$$

where the coefficients may depend upon the independent variable, x. Under a generalised Kummer–Liouville transformation[29] this equation is reduced to the canonical form[30]

$$y''' + B_1y' + B_0y = 0. \tag{5.2.2}$$

The Lie point symmetries of (5.2.2) have the form

$$G = a(x)\frac{\partial}{\partial x} + (b(x)y + c(x))\frac{\partial}{\partial y}, \tag{5.2.3}$$

where the functions $a(x)$, $b(x)$ and $c(x)$ satisfy

$$\begin{aligned} b' &= a'' \\ 3b'' - a''' + 2B_1a' + aB_1' &= 0 \\ b''' + 3a'B_0 + aB_0' + B_1b' &= 0 \\ c''' + B_1c' + B_0c &= 0. \end{aligned} \tag{5.2.4}$$

As we saw in the previous section[31], there will be the four symmetries which correspond to the three solutions of the linear equation and the symmetry due

[29]Vide §3.6. Kummer (1887) [92] and Liouville (1837) [139].

[30]The coefficients B_1, B_0 of (5.2.1) are not to be identified with those of (5.2.2).

[31]When the restriction to third order equations is made.

to its linearity and homogeneity. If we substitute (5.2.4a) into (5.2.4b) and (5.2.4c), we obtain

$$
\begin{aligned}
2a''' + 2B_1 a' + aB_1' &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}x} a''' + 3a'B_0 + aB_0' + B_1 a'' &= 0.
\end{aligned}
\tag{5.2.5}
$$

There are two possible outcomes. If $B_1' \neq 2B_0$,

$$a = \mathcal{A}(B_1' - 2B_0)^{-1/3}, \tag{5.2.6}$$

where $\mathcal{A}$ is a constant. On the other hand, if $B_0 = B_1'/2$, (5.2.5b) is simply the derivative of (5.2.5a) and so there is the single condition, (5.2.5a). In this case there are seven symmetries whereas in the other case there are five. The rôle of (5.2.5a) in its integrated form as the Lewis–Pinney equation has already been noted in §2.2.1. When there are seven symmetries, the algebra is $3A_1 \oplus_s (s\ell(2,R) \oplus A_1)$ and, when there are five, it is $3A_1 \oplus_s (A_1 \oplus A_1)$. The three–dimensional Abelian algebra persists, but the three–dimensional $s\ell(2,R)$ is reduced to the one–dimensional Abelian algebra, A_1.

The important result is that a third order linear equation is reducible to

$$y''' = 0 \tag{5.2.7}$$

only when the coefficients are related by[32]

$$B_0 = \frac{1}{2} B_1'. \tag{5.2.8}$$

As a physical example of an equation of the third order for which the analysis above applies consider the Langevin equation[33]

$$m\ddot{x} = eE(t) + F(t, x, \dot{x}) + m\tau \dddot{x}\,, \tag{5.2.9}$$

where $\tau = 2e^2/3mc^3$, m is the mass and e the charge of the particle, F is the given external field and E is the electric field. For a force defined by[34]

$$F(t, x, \dot{x}) = F_1(t)x + F_2(t)\dot{x} + F_3(t) \tag{5.2.10}$$

[32] Krause and Michel (1988) [90], Mahomed and Leach (1990) [153].

[33] de la Peña–Auerbach and Cetto (1977) [33].

[34] *cf* Soares Neto and Vianna (1988) [199].

(5.2.9) takes the form of (5.2.2) and can be analysed in terms of the general linear equation[35] and so its invariance properties are just those delineated above.

5.2.3 Equations with three symmetries

Here we give the canonical forms of all third order equations possessing three point symmetries. All equations of the third order which have a three–dimensional point symmetry algebra can be transformed to one of these forms by means of a point transformation. The realisations of real three–dimensional Lie algebras in terms of vector fields in two coordinates are given by Mahomed and Leach (1988) [151] and all we need do is to associate with each realisation its canonical third order differential equation. We assume that the equation is in the standard form

$$y''' = H(x, y, y', y'')^{36}. \tag{5.2.11}$$

To determine the structure of an equation invariant under a particular algebra we simply solve the partial differential equation

$$G^{[3]}(y''' - H(x, y, y', y''))\Big|_{(y''' - H(x,y,y',y''))=0} = 0 \tag{5.2.12}$$

for each G belonging to the algebra. There is no necessity for a given realisation of an algebra to leave invariant any third order equation[37]. Such happens in the case of the realisations of $so(3)$ ($A_{3,9}$) found in Table 5.2.

[35]An area which has not been explored is the relative ease of the solution of linear equations with, in the case of third order equations, four, five and seven symmetries. The question of ease is probably very much in the eye of the beholder. However, the examples of §6.2 are suggestive.

[36]We could take the more general expression $E(x, y, y', y'', y''') = 0$, but we are then always caught up with the requirements of the implicit function theorem. These are not onerous in equations of practical origin for they tend to have the standard form anyway. There does not seem to be much sense in making matters more difficult for what is really a nominal gain in generality.

[37]The same was noted in the case of second order equations in §4.2.

There are eleven real Lie algebras of dimension three[38]. They are listed in Table 5.1. The realisations of these algebras in terms of vector fields in two coordinates are given in Table 5.2. In the table the notations p and r represent the operators $\partial/\partial x$ and $\partial/\partial y$ respectively. We note that most of the algebras have more than one realisation. To distinguish amongst the different realisations we adopt the notation $\mathcal{A}^{I}$, $\mathcal{A}^{II}$ *etc* whenever there is more than one.

We associate third order ordinary differential equations with each of the realisations except that of $A_{3,9}$ which is not a subalgebra of $3A_1 \oplus_s (s\ell(2,R) \oplus A_1)$. This involves the solution of a system of three linear first order partial differential equations which arise from the imposition of the symmetry requirement for each of the three operators of a given realisation. The results are summarised in Table 5.3. In Table 5.3 γ is an arbitrary function of its argument. There are fifteen classes of equations. Each is reducible to a second order equation and some are reducible to a first order equation. Four of them depend upon a parameter. The representation of the generators in canonical form does not necessarily yield the simplest form of the differential equation invariant under a given algebra. By way of example the equation associated with $A_{3,8}^{II}$ in Table 5.3 takes the simpler and more elegant form

$$Q^2Q''' + \Gamma\left(\frac{1}{2}Q'^2 - QQ''\right) = 0, \tag{5.2.13}$$

in which $'$ denotes $\mathrm{d}/\mathrm{d}T$, under the transformation

$$Q = t \qquad T = q. \tag{5.2.14}$$

We note that two of the classes of equation in Table 5.3 are linear and have the result that, if a third order ordinary differential equation possesses a three-dimensional algebra with the properties that $G_1 = \rho(x,y)G_2$ and $G_3 = \psi(x,y)G_2$ for some functions ρ and ψ with $\rho\psi \neq 1$, it is linearisable by a point transformation. There are only three algebras with proportional generators,

[38]Patera and Winternitz (1977) [171].

TABLE 5.1

Algebra	Nonzero commutation relations		
$3A_1$			
$A_1 \oplus A_2$	$[G_1, G_3] = G_1$		
$A_{3,1}$ (Weyl)	$[G_2, G_3] = G_1$		
$A_{3,2}$	$[G_1, G_3] = G_1,$	$[G_2, G_3] = G_1 + G_2$	
$A_{3,3}$ $(D \otimes_s T_2)$	$[G_1, G_3] = G_1,$	$[G_2, G_3] = G_2$	
$A_{3,4}$ $(E(1,1))$	$[G_1, G_3] = G_1,$	$[G_2, G_3] = -G_2$	
$A^a_{3,5}$ $(0 < \|a\| < 1)$	$[G_1, G_3] = G_1,$	$[G_2, G_3] = aG_2$	
$A_{3,6}$ $(E(2))$	$[G_1, G_3] = -G_2,$	$[G_2, G_3] = G_1$	
$A^b_{3,7}$ $(b > 0)$	$[G_1, G_3] = bG_1 - G_2,$	$[G_2, G_3] = G_1 + bG_2$	
$A_{3,8}$ $(s\ell(2, R))$	$[G_1, G_2] = G_1,$	$[G_2, G_3] = G_3,$	$[G_3, G_1] = -2G_2$
$A_{3,9}$ $(so(3))$	$[G_1, G_2] = G_3,$	$[G_2, G_3] = G_1,$	$[G_3, G_1] = G_2$

TABLE 5.2

Algebra	Elements of the realisation			
$3A_1$	r	tr	$h(x)r$	
$A_{3,1}$	r	p	xr	
$A_{3,2}$	r	p	$xp + (x + y)r$	
	r	$-(\log x)r$	$xp + yr$	
$A_1 \oplus A_2$ $(a = 0)$	p	r	$xp + ayr$	
$A_{3,3}$ $(a = 1)$				
$A_{3,4}$ $(a = -1)$	r	xr	$(1 - a)xp + yr$	
$A_{3,5}^a$ $(0 < \|a\| < 1)$				
$A_{3,6}$ $(b = 0)$	p	r	$(bx + y)p + (by - x)r$	
$A_{3,7}^b$ $(b > 0)$	xr	r	$(1 + x^2)p + (xy + by)r$	
$A_{3.8}$	r	yr	y^2r	
	r	$xp + yr$	$2xyp + y^2r$	
	r	$xp + yr$	$2xyp + (y^2 - x^2)r$	
$A_{3,9}$	$i(\sin y)r$	r	$-i(\cos y)r$	$i = \sqrt{-1}$
	$(1 + x^2)p + xyr$	$xyp + (1 + y^2)r$	$yp - xr$	

TABLE 5.3

Realisation	**Canonical Equation**
$3A_1$	$y''' + \alpha(x)y'' + \gamma(x) = 0$
$A_{3,1}$	$y''' = \gamma(y'')$
$A^I_{3,2}$	$y''' = y''^2\gamma(y'' \exp y')$
$A^{II}_{3,2}$	$x^2y''' = \gamma(xy'' + y') + 2y'$
$A^I_{3,3}$	$y''' = y''^2\gamma(y')$
$A^{II}_{3,3}$	$y''' = y''\gamma(x)$
$A_1 \oplus A^I_2$, $A^I_{3,4}$, $A^{aI}_{3,5}$	$y''' = y''^{\frac{a-3}{a-2}}\gamma(y''y'^{\frac{2-a}{a-1}}) \qquad a \neq 1, 2$
$A^{\frac{1}{2}I}_{3,5}$	$y'y''' = \gamma(y'') \qquad a = 2$
$A_1 \oplus A^{II}_2$, $A^{II}_{3,4}$, $A^{aII}_{3,5}$	$y''' = y''^{\frac{2-3a}{1-2a}}\gamma\left(x^{\frac{2a-1}{a-1}}y''\right) \qquad a \neq 1, \frac{1}{2}$
$A^{\frac{1}{2}II}_{3,5}$	$xy''' = \gamma(y'') \qquad a = \frac{1}{2}$
$A^I_{3,6}$, $A^{bI}_{3,7}$	$y''' = 3y'y''^2(1 + y'^2)^{-1} + (1 + y'^2)^2 \exp(2b \tan^{-1} y') \times$
	$\gamma(y''(1 + y'^2)^{-3/2} \exp(-b \tan^{-1} y'))$
$A^{II}_{3,6}$, $A^{II}_{3,7}$	$y''' = 3xy''(1 + x^2)^{-1} + (1 + x^2)^{-5/2} \exp(b \tan^{-1} x) \times$
	$\gamma(y''(1 + x^2)^{3/2} \exp(-b \tan^{-1} x))$
$A^I_{3,8}$	$y'y''' = \frac{3}{2}y''^2 + y'^2\gamma(x)$
$A^{II}_{3,8}$	$x^2y'y''' = y'^5\gamma((xy'' + \frac{1}{2}y')/y'^3) + 3x^2y''^2$
$A^{III}_{3,8}$	$x^2(1 + y'^2)y''' = (1 + y'^2)^3\gamma((xy'' - y' - y'^3)/(1 + y'^2)^{3/2})$
	$+3x^2y''^2y'$

viz. $3A_1$, $A_{3,3}^{II}$ and $A_{3,8}^{I}$ and the last is excluded by violation of the condition $\rho\psi \neq 1$.

5.2.4 Conclusion

The classification of differential equations by means of their algebras is useful for identifying the equivalence class to which a particular equation belongs by virtue of the algebra of its symmetries. Although it need not be an elementary task to solve the equations for the transformation, the procedure is simple enough in principle. One looks at the Lie Bracket relations of the canonical form and those of the symmetries of the equation in question and identifies which symmetries are which and uses these to determine the transformation. If the algebra is small, it is easy to identify the relationships amongst the two sets of symmetries. If the number of symmetries is large, the identification is not necessarily so simple[39]. On the other hand a thin algebra does not bode well for reduction of order to quadratures whereas a generous one has an amplitude of routes to its solution. The ideal would seem to be an algebra of dimension equal to the order of the equation provided that the Lie Bracket relations do not cause ruin to the descending symmetries. It is for this reason that the Abelian algebras of the correct dimension are so useful. There is no loss of descendants. It may be that there is no accident in the association of the nA_1 algebra with linear equations. They are solvable by a linear superposition of linearly independent solutions. The sad part is that one has to be able to solve the equation to find those Abelian symmetries[40].

[39] Do bear in mind that one is dealing with the symmetries as they come out of the calculation and some imaginative reconstruction is usually required.

[40] As always we except nonlinear equations from this difficulty. The determining equations for the symmetries of a nonlinear equation are linear.

Chapter 6

The Algebras of first integrals

6.1 Introduction

We saw in §1.1 that with a symmetry of a differential equation there can be associated a first integral of the differential equation. We recall that, if the symmetry is[1]

$$G = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{6.1.1}$$

and the differential equation

$$y^{(n)} = f(x,y,\ldots,y^{(n-1)}), \tag{6.1.2}$$

the first integral, $I(x,y,\ldots,y^{(n-1)})$, in which the dependence on the $(n-1)$th derivative is nontrivial, satisfies the two conditions

$$G^{[n-1]}I = 0 \tag{6.1.3}$$

$$\frac{\mathrm{d}I}{\mathrm{d}x} = 0. \tag{6.1.4}$$

The former represents the invariance under the infinitesimal transformation induced by the symmetry and the latter the fact that I is a first integral.

In this chapter we look at the symmetries associated with first integrals. In a way this is not related to matters of integrability of the differential equation

[1]As usual we confine our attention to point symmetries from practical considerations.

or the like, but is an investigation of the properties of first integrals and, in a very real sense, was motivated more from considerations of æsthetics rather than utilitarianism[2]. In particular we look at the symmetries of a certain class of first integrals, *viz.* those which belong to linear or linearisable differential equations. Not surprisingly the first class of integrals and associated symmetries considered are those of second order equations with $s\ell(3, R)$ symmetry, which were first discussed by Leach and Mahomed[3]. The extension to higher order equations was only made recently by Govinder and Leach [66] and Flessas *et al* [43]. The results show the typical atypicality of second order equations which themselves already demonstrate the atypicality of first order equations. There is always some point of separation in going from one to many whether it be in the order of a differential equation, from a scalar equation to a system of equations, from one to many physical dimensions or, indeed, in the very process of counting. In the case of scalar differential equations it is not until one has reached third order equations that the pattern for the higher order equations is established. In this the integrals follow the equations themselves.

The results reported here are necessarily sketchy once we move to higher order equations although in the case of second order equations they are complete. There is a simple explanation. All second order equations possess an algebra of symmetries which is a subset of the maximal symmetry algebra, $s\ell(3, R)$. This is not the case for equations of higher order. Consequently what is said of the integrals associated with linear equations of higher order cannot be applied to integrals associated with nonlinear equations[4]. They remain an open subject for investigation as there has not even been speculation about them.

For the first part we consider the first integrals of second order equations of maximal symmetry. We follow with a detailed discussion of linear third order

[2]For which no apology is made. The whole concept of symmetry reeks of æsthetics.

[3]Leach and Mahomed (1988) [121]. The genesis of the paper goes back several years before the year of publication [102].

[4]As always this use of nonlinear means nonlinearisable by a point transformation.

equations and conclude with a brief discussion of the general nth order linear equation. As we have remarked above, there is a change in behaviour when one moves from second order to third. It is, we believe, interesting. We also believe that an explanation would be even more interesting.

6.2 Second order ordinary differential equations

As all linearisable second order differential equations are equivalent, it really does not matter which linear or linearisable equation we consider. The free particle equation says it all. However, we give a variation to avoid the necessity to see how the free particle results need to be transformed. The example considered[5] is that work horse of Mechanics, the simple harmonic oscillator with equation of motion

$$\ddot{q} + q = 0. \tag{6.2.1}$$

We summarise the symmetries of this equation and their associated first integrals.

$$\begin{aligned}
G_1 &= \sin 2t\frac{\partial}{\partial t} + q\cos 2t\frac{\partial}{\partial q} & J_1 &= \tfrac{1}{2}(\dot{q}^2 - q^2)\sin 2t - q\dot{q}\cos 2t\\
G_2 &= \cos 2t\frac{\partial}{\partial t} - q\sin 2t\frac{\partial}{\partial q} & J_2 &= \tfrac{1}{2}(\dot{q}^2 - q^2)\cos 2t - q\dot{q}\sin 2t\\
G_3 &= \cos t\frac{\partial}{\partial q} & J_3 &= q\sin t + \dot{q}\cos t\\
G_4 &= \sin t\frac{\partial}{\partial q} & J_4 &= q\cos t - \dot{q}\sin t\\
G_5 &= \frac{\partial}{\partial t} & J_5 &= \tfrac{1}{2}(\dot{q}^2 + q^2)\\
G_6 &= q\frac{\partial}{\partial q} & J_6 &= t + \arctan\frac{\dot{q}}{q}\\
G_7 &= q\sin t\frac{\partial}{\partial t} + q^2\cos t\frac{\partial}{\partial q} & J_7 &= (q\cos t - \dot{q}\sin t)/(q\sin t + \dot{q}\cos t)\\
G_8 &= q\cos t\frac{\partial}{\partial t} - q^2\sin t\frac{\partial}{\partial q} & J_8 &= (q\cos t - \dot{q}\sin t)/(q\sin t + \dot{q}\cos t).
\end{aligned} \tag{6.2.2}$$

[5]The reader is referred to Leach and Mahomed (1988)[121].

Now we reverse the inquiry and ask what symmetries are associated with the various first integrals listed above. In each case we solve the equations given in §6.1. For each of J_1, J_2 and J_5 only one generator is found. For each of J_3, J_4 and, not surprisingly, J_6, J_7 and J_8 a three parameter solution is found. Our interest is in those integrals of maximum symmetry and we list the integrals and symmetries in (6.2.3).

$I_1 = \dfrac{q\cos t - \dot{q}\sin t}{q\sin t + \dot{q}\cos t}$	$X_{11} = -q\dfrac{\partial}{\partial q}$
	$X_{12} = -q\sin t\dfrac{\partial}{\partial t} - q^2\cos t\dfrac{\partial}{\partial q}$
	$X_{13} = q\cos t\dfrac{\partial}{\partial t} - q^2\sin t\dfrac{\partial}{\partial q}$
$I_2 = q\sin t + \dot{q}\cos t$	$X_{21} = \cos t\dfrac{\partial}{\partial q}$
	$X_{22} = \sin t\cos t\dfrac{\partial}{\partial t} + q\cos^2 t\dfrac{\partial}{\partial q}$
	$X_{23} = -\cos^2 t\dfrac{\partial}{\partial t} + q\sin t\cos t\dfrac{\partial}{\partial q}$
$I_3 = q\cos t - \dot{q}\sin t$	$X_{31} = \sin t\dfrac{\partial}{\partial q}$
	$X_{32} = \sin^2 t\dfrac{\partial}{\partial t} + q\sin t\cos t\dfrac{\partial}{\partial q}$
	$X_{33} = -\sin t\cos t\dfrac{\partial}{\partial t} + q\sin^2 t\dfrac{\partial}{\partial q}$

(6.2.3)

We compare these symmetries with those of the differential equation, (6.2.1), given above in (6.2.2) and see that

$$
\begin{aligned}
G_1 &= X_{22} - X_{33}\\
G_3 &= X_{21}\\
G_5 &= -X_{23} + X_{32}\\
G_7 &= -X_{12}\\
G_2 &= -X_{23} - X_{32}\\
G_4 &= X_{31}\\
G_6 &= -X_{11} = X_{22} + X_{33}\\
G_8 &= X_{13}.
\end{aligned}
\tag{6.2.4}
$$

The linear dependence of the Xs is seen in the expressions for G_6 which are equivalent to the relation

$$X_{11} + X_{22} + X_{33} = 0. \tag{6.2.5}$$

This relationship is necessary as there are only eight linearly independent generators of symmetry for a linear second order equation whereas the integrals throw up nine symmetries.

The symmetries of the integrals have their own relations amongst each other and we see that

$$\begin{aligned} q\, X_{i1} &= \cos t X_{i2} + \sin t X_{i3}, \qquad & i = 1,3 \\ X_{1i} + q \cos t X_{2i} + q \sin t X_{3i} &= 0, \qquad & i = 1,3. \end{aligned} \tag{6.2.6}$$

By inspection the first integrals listed in (6.2.2) and (6.2.3) are related according to

$$\begin{aligned} J_1 &= -\frac{1}{2} I_2 I_3 \\ J_2 &= \frac{1}{2}(I_3^2 - I_2^2) \\ J_3 &= I_2 \\ J_4 &= I_3 \\ J_5 &= \frac{1}{2}(I_2^2 + I_3^2) \\ J_6 &= I_1 (= J_7 = J_8). \end{aligned} \tag{6.2.7}$$

The members of each of the three classes of first integrals – linear, quadratic and quotient – constitute a complete set in each case. In the case of the quotient set, I_1, it will be appreciated that the theory of the determination of the integral does not distinguish between I_1 and its reciprocal.

Leach and Mahomed (1988) [121] go on to discuss the damped free particle with equation of motion

$$\ddot{q} + k\dot{q} = 0, \tag{6.2.8}$$

the differential equation

$$\ddot{q} + 3q\dot{q} + q^3 = 0, \tag{6.2.9}$$

which occurs in the study of the modified Emden–Fowler equation[6] of astrophysical relevance, as well as other problems of a more mundane nature, and the equation[7]

$$t\ddot{q} = \dot{q}^3 + \dot{q}. \tag{6.2.10}$$

In all cases results of like nature to those given here for the oscillator are obtained. This is not surprising as each of these systems is related to the free particle equation by means of a point transformation.

Each of the triplets $\{X_{1i}\}$, $\{X_{2i}\}$ and $\{X_{3i}\}$ constitutes a Lie subalgebra. The commutation relations are

$$\begin{array}{lll} [X_{11}, X_{12}] = -X_{12} & [X_{11}, X_{13}] = -X_{13} & [X_{12}, X_{13}] = 0 \\ [X_{21}, X_{22}] = \pm X_{21} & [X_{21}, X_{23}] = 0 & [X_{22}, X_{23}] = \mp X_{23} \\ [X_{31}, X_{32}] = 0 & [X_{31}, X_{33}] = \pm X_{31} & [X_{32}, X_{33}] = \pm X_{32} \end{array} \tag{6.2.11}$$

Clearly each of the three sets of commutation relations given above can be written in the form

$$[Z_1, Z_2] = 0, \quad [Z_1, Z_3] = Z_1, \quad [Z_2, Z_3] = Z_2 \tag{6.2.12}$$

and so the algebraic properties of the triplets are identical.

The Lie algebra represented by the commutation relations (6.2.12) is denoted by $A_{3,3}$[8]. The question now arises as to whether the three triplets of generators, having isomorphic algebras (6.2.11a), (6.2.11b) and (6.2.11c), associated with the integrals considered above can be transformed into a canonical triplet of generators by a point transformation. Not surprisingly the answer is yes since Mahomed and Leach (1989) [152] prove that for a second order ordinary differential equation to possess $s\ell(3, R)$ symmetry it is necessary and sufficient that it have the algebra $A_{3,3}$.

[6]Leach (1985) [109]; Mahomed and Leach (1985) [149]; Duarte, Duarte and Moreira (1987) [34].

[7]Mahomed and Leach (1989) [152].

[8]Patera *et al* (1976) [170] and Patera and Winternitz (1977) [171]

This last result has its own importance as far as the algebras of the symmetries of the first integrals is concerned. For a second order ordinary differential equation the only possibility for the existence of symmetries is as a subset of those which have the algebra $s\ell(3,R)$. We have seen that the maximal number of symmetries is three for those particular integrals, the set of which is composed of the initial conditions and their ratios. We have also seen that the possession of these three symmetries is necessary and sufficient for a second order equation to have the maximal symmetry given by the algebra, $s\ell(3,R)$. Consequently we can confidently claim that only the first integrals of linear or linearisable second order equations will have three symmetries and they will have to be the particular integrals referred to above[9].

6.3 Third order ordinary differential equations

6.3.1 Introduction

Third order equations are the first in which the pattern for higher order equations becomes manifest. To summarise these[10] an nth order ordinary differential equation possesses at most $(n+4)$ point symmetries. This is achieved in

$$y^{(n)} = 0 \tag{6.3.1}$$

and any equation related to it by a point transformation. When $n = 3$, the algebra is $3A_1 \oplus_s (s\ell(2,R) \oplus A_1)$. Unlike the case of second order equations third order equations of lower symmetry need not have an algebra which is a subset of the maximal algebra. The best-known example is the Kummer–Schwartz

[9]Subject, as always, to a point transformation. Mahomed and Leach (1989) [152] consider the matter in further detail.

[10]A very detailed treatment is given in Mahomed and Leach (1990) [153]; see also Chapter Five.

equation[11]

$$\frac{1}{2}\frac{y'''}{y'} - \frac{3}{4}\frac{y''^2}{y'^2} = 0 \tag{6.3.2}$$

which has the beautifully symmetric algebra $s\ell(2,R)\oplus_s s\ell(2,R)$ with generators

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= x\frac{\partial}{\partial x} \\ G_3 &= x^2\frac{\partial}{\partial x} \\ G_4 &= \frac{\partial}{\partial y} \\ G_5 &= y\frac{\partial}{\partial y} \\ G_6 &= y^2\frac{\partial}{\partial y}. \end{aligned} \tag{6.3.3}$$

As Mahomed and Leach (1990) [153] prove, the number of symmetries for even linear systems is not uniform. Thus linear equations of the third order can have four or five symmetries instead of the maximal seven. They cannot have six which further emphasises the difference in nature of the Kummer–Schwartz equation[12].

6.3.2 The general case

The treatment of the symmetries of the first integrals of linear third order equations[13] has been done in a more systematic fashion than that of the second order equations[14]. The greater complexity of the subject demanded it. We assume that the original differential equation has been cast into normal form before the analysis is commenced. We therefore confine our analysis to the first integral

$$I = ay'' - a'y' + cy \tag{6.3.4}$$

[11]Berkovič and Nechaevsky (1985) [19].

[12]This is in the context of point symmetries. For contact symmetries see Lie (1896) [138, p 148] and Abraham–Shrauner *et al* (1994) [5].

[13]Govinder and Leach (1994) [66].

[14]Leach and Mahomed (1988) [121].

and its differential equation

$$y''' + \frac{c - a''}{a} y' + \frac{c'}{a} y = 0. \tag{6.3.5}$$

For a general third order linear equation in normal form

$$y''' + f(x)y' + g(x)y = 0 \tag{6.3.6}$$

it follows that $a(x)$ and $c(x)$ are determined from the third order system

$$a'' + fa = c \tag{6.3.7a}$$

$$c' = ag, \tag{6.3.7b}$$

ie the solution set contains three arbitrary functions.

On application of the procedure explained in §6.1 we find that the first integral (6.3.4) has a symmetry of the form (6.1.1) *iff*

$$\begin{aligned}\xi a'y'' - \xi a''y' + \xi c'y + a\Bigg\{&\frac{\partial^2\eta}{\partial x\partial y} + 2y'\frac{\partial^2\eta}{\partial x\partial y} + y'^2\frac{\partial^2\eta}{\partial y^2} + y''\frac{\partial\eta}{\partial y} - 2y''\left(\frac{\partial\xi}{\partial x} + y'\frac{\partial\xi}{\partial y}\right)\\ &- y'\left(\frac{\partial^2\xi}{\partial x^2} + 2y'\frac{\partial^2\xi}{\partial x\partial y} + y'^2\frac{\partial^2\xi}{\partial y^2} + y''\frac{\partial\xi}{\partial y}\right)\Bigg\}\\ &- a'\left(\frac{\partial\eta}{\partial x} + y'\frac{\partial\eta}{\partial y} - y'\frac{\partial\xi}{\partial x} - y'^2\frac{\partial\xi}{\partial y}\right) + c\eta = 0.\end{aligned} \tag{6.3.8}$$

The coefficient of y'' in (6.3.8) gives the functional forms of ξ and η as

$$\xi = p(x) \tag{6.3.9}$$

$$\eta = \left(2p' - \frac{pa'}{a}\right) y + q(x) \tag{6.3.10}$$

which imply that the coefficient of y'^2 in (6.3.8) is identically zero. The terms in (6.3.8) that do not involve y or any of its derivatives now give

$$aq'' - a'q' + cq = 0, \tag{6.3.11}$$

which always has two solutions. Thus (6.3.4) will always have at least two symmetries of the form

$$\begin{aligned}G_1 &= q_1(x)\frac{\partial}{\partial y}\\ G_2 &= q_2(x)\frac{\partial}{\partial y} = q_1(x)\int \frac{a(x)}{q_1(x)^2},\end{aligned} \tag{6.3.12}$$

where $q_1(x)$ and $q_2(x)$ are the solutions of (6.3.11) and $a(x)$ is a solution of (6.3.7). Note that (6.3.11) can be differentiated to give an equation of the form (6.3.5). Thus $q_1(x)$ and $q_2(x)$ are also solutions of (6.3.5). In fact all three solutions of (6.3.5) are applicable. We choose them pairwise for a given integral. Hence the three independent first integrals of (6.3.5) will each have a pair of solutions of (6.3.5) in their symmetries.

The coefficient of y' in (6.3.8) gives a second order equation for p with solution

$$p = Aa + Ba\int\frac{1}{a}, \tag{6.3.13}$$

where A and B are constants of integration. We will in general, then, obtain two symmetries from p of the form

$$G_3 = a\frac{\partial}{\partial x} + a'y\frac{\partial}{\partial y} \tag{6.3.14a}$$

$$G_4 = a\int\frac{1}{a}\frac{\partial}{\partial x} + \left(a'\int\frac{1}{a} + 2\right)y\frac{\partial}{\partial y}. \tag{6.3.14b}$$

However, the solutions of (6.3.13) must be consistent with

$$pc' + a\left(2p' - \frac{pa'}{a}\right)'' - a'\left(2p' - \frac{pa'}{a}\right)' - c\left(2p' - \frac{pa'}{a}\right) = 0, \tag{6.3.15}$$

the coefficient of y in (6.3.8). Substitution of (6.3.13) into (6.3.15) gives

$$\begin{aligned} &A\left[a'c + ac' + aa''' - a'a''\right] \\ &+B\left\{\left[a'c + ac' + aa''' - a'a''\right]\int\frac{1}{a} + 2\left[a'' - \frac{a'^2}{a} + c\right]\right\} = 0. \end{aligned} \tag{6.3.16}$$

For both solutions of p to persist (and hence lead to two symmetries) the coefficients of both A and B must vanish, *ie*,

$$a'c + ac' + aa''' - a'a'' = 0 \tag{6.3.17}$$

$$a'' - \frac{a'^2}{a} + c = 0. \tag{6.3.18}$$

From (6.3.18)

$$c = -a\left(\frac{a'}{a}\right)' \tag{6.3.19}$$

and (6.3.17) is satisfied identically. Thus, if (6.3.19) holds, p gives rise to two symmetries. Note that (6.3.18) is a second order equation and so only two of the three solutions for $a(x)$ in (6.3.7) apply. This implies that we obtain two first integrals with four symmetries. Note further, that (6.3.19) implies that

$$\left(\frac{c-a''}{a}\right)' = 2\left(\frac{c'}{a}\right) \tag{6.3.20}$$

in (6.3.5) which is now

$$y''' - \left[\left(\frac{a'}{a}\right)' + \frac{a''}{a}\right]y' - \frac{1}{a}\left[a\left(\frac{a'}{a}\right)'\right]' y = 0. \tag{6.3.21}$$

The relationship (6.3.20) implies that (6.3.21) has maximal (seven) symmetry [153].

Suppose, now, that (6.3.17) is satisfied giving

$$c = -a\left(\frac{a'}{a}\right)' + \frac{k}{a}, \tag{6.3.22}$$

where k is a constant. Clearly, for $k \neq 0$, (6.3.18) is not satisfied and p has only one solution. Note that (6.3.22) still gives (6.3.20). Hence a third order equation with maximal symmetry has two first integrals which have four symmetries and one with three. The first integral with three symmetries will involve the remaining solution for $a(x)$ (in (6.3.7)) that does not satisfy (6.3.18). The differential equation now becomes

$$y''' - \left[\left(\frac{a'}{a}\right)' + \frac{a''}{a}\right]y' - \frac{1}{a}\left[a\left(\frac{a'}{a}\right)'\right]' y + k\left[\frac{y'}{a} - \frac{a'y}{a^2}\right] = 0. \tag{6.3.23}$$

The three independent first integrals of (6.3.23) have the symmetries (for $k = 0$)

$$G_1' = q_1(x)\frac{\partial}{\partial y} \tag{6.3.24a}$$

$$G_2' = a_1(x)\frac{\partial}{\partial y} \tag{6.3.24b}$$

$$G_3' = a_1(x)\frac{\partial}{\partial x} + a_1' y\frac{\partial}{\partial y} \tag{6.3.24c}$$

$$G_4' = a_1\int\frac{1}{a_1}\frac{\partial}{\partial x} + \left(a_1'\int\frac{1}{a_1} + 2\right) y\frac{\partial}{\partial y}, \tag{6.3.24d}$$

$$G_1'' = q_2(x)\frac{\partial}{\partial y} \tag{6.3.25a}$$

$$G_2'' = a_2(x)\frac{\partial}{\partial y} \tag{6.3.25b}$$

$$G_3'' = a_2(x)\frac{\partial}{\partial x} + a_2' y\frac{\partial}{\partial y} \tag{6.3.25c}$$

$$G_4'' = a_2\int\frac{1}{a_2}\frac{\partial}{\partial x} + \left(a_2'\int\frac{1}{a_2} + 2\right)y\frac{\partial}{\partial y} \tag{6.3.25d}$$

and (for $k \neq 0$)

$$\tilde{G}_1 = q_1(x)\frac{\partial}{\partial y} \tag{6.3.26a}$$

$$\tilde{G}_2 = q_2(x)\frac{\partial}{\partial y} \tag{6.3.26b}$$

$$\tilde{G}_3 = a_3(x)\frac{\partial}{\partial x} + a_3' y\frac{\partial}{\partial y}, \tag{6.3.26c}$$

where $q_1(x)$ and $q_2(x)$ are solutions of (6.3.11) and hence (6.3.23) and the a_is ($i = 1, 2, 3$) are solutions of (6.3.7).

In a constructive approach one firstly solves (6.3.7) for $a(x)$ and $c(x)$ and then substitutes into (6.3.18) to find the relationship which the constants of integration must satisfy for $p(x)$ and hence yield four symmetries. There is a doubly infinite family of integrals with four symmetries, but only two linearly independent ones. The third linearly independent integral, obtained by a selection of constants of integration which does not satisfy (6.3.18), has only three symmetries.

Suppose, now, that the coefficient of B in (6.3.16) is zero. This gives

$$c = -a\left(\frac{a'}{a}\right)' + \frac{k}{a(\int a^{-1})^2}. \tag{6.3.27}$$

When $k \neq 0$, the coefficient of A in (6.3.16) is not zero. This implies that p produces only one symmetry of the form

$$G_3 = a\int\frac{1}{a}\frac{\partial}{\partial x} + \left(a'\int\frac{1}{a} + 2\right)y\frac{\partial}{\partial y}. \tag{6.3.28}$$

Note that, while c and a are related via (6.3.27), (6.3.20) does not hold in (6.3.5). This is the case for which (6.3.5) has five symmetries [153]. The three independent first integrals of a third order linear equation with five symmetries have the symmetries

$$G_1' = q_1(x)\frac{\partial}{\partial y} \tag{6.3.29a}$$

$$G_2' = q_2(x)\frac{\partial}{\partial y} \tag{6.3.29b}$$

$$G_3' = a_1\int\frac{1}{a_1}\frac{\partial}{\partial x} + \left(a_1'\int\frac{1}{a_1} + 2\right)y\frac{\partial}{\partial y}, \tag{6.3.29c}$$

$$G_1'' = q_1(x)\frac{\partial}{\partial y} \tag{6.3.30a}$$

$$G_2'' = q_3(x)\frac{\partial}{\partial y} \tag{6.3.30b}$$

$$G_3'' = a_2\int\frac{1}{a_2}\frac{\partial}{\partial x} + \left(a_2'\int\frac{1}{a_2} + 2\right)y\frac{\partial}{\partial y} \tag{6.3.30c}$$

and

$$\tilde{G}_1 = q_2(x)\frac{\partial}{\partial y} \tag{6.3.31a}$$

$$\tilde{G}_2 = q_3(x)\frac{\partial}{\partial y} \tag{6.3.31b}$$

$$\tilde{G}_3 = a_3\int\frac{1}{a_3}\frac{\partial}{\partial x} + \left(a_3'\int\frac{1}{a_3} + 2\right)y\frac{\partial}{\partial y}, \tag{6.3.31c}$$

where the q_is and a_is ($i = 1, 2, 3$) are solutions of (6.3.11) and (6.3.7) respectively.

Finally, when both the coefficients of A and B in (6.3.16) are nonzero, $p = 0$ and contributes no symmetries. This implies that the coefficients of y' and y in (6.3.5) are unrelated. This is the case for which (6.3.5) has four symmetries [153]. The three independent first integrals of (6.3.5) then have the pairs of symmetries (6.3.29a–6.3.29b), (6.3.30a–6.3.30b) and (6.3.31a–6.3.31b) respectively.

6.3.3 Example I: $n = 7$

We firstly consider the generic third order equation with maximal (seven) symmetry, *viz.*

$$y''' = 0 \tag{6.3.32}$$

with symmetries [77]

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial y}\\
G_2 &= x\frac{\partial}{\partial y}\\
G_3 &= x^2\frac{\partial}{\partial y}\\
G_4 &= y\frac{\partial}{\partial y}\\
G_5 &= \frac{\partial}{\partial x}\\
G_6 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\\
G_7 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}
\end{aligned} \tag{6.3.33}$$

which has the Lie algebra $3A_1 \oplus_s (s\ell(2,R) \oplus A_1)$ [153]. It is easily verified that (6.3.32) has the three independent first integrals

$$\begin{aligned}
I_1 &= y''\\
I_2 &= xy'' - y'\\
I_3 &= \tfrac{1}{2}x^2y'' - xy' + y.
\end{aligned} \tag{6.3.34}$$

We call the first integrals (6.3.34) initial condition first integrals as, for $x = 0$, we have

$$I_1 = y_0'' \tag{6.3.35}$$

$$I_2 = -y_0' \tag{6.3.36}$$

$$I_3 = y_0. \tag{6.3.37}$$

In (6.3.34) I_1 has the symmetries

$$G_1 = \frac{\partial}{\partial y}\cdot$$

$$
\begin{aligned}
G_2 &= x\frac{\partial}{\partial y} \\
G_3 &= \frac{\partial}{\partial x} \\
G_4 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y},
\end{aligned}
\tag{6.3.38}
$$

I_2,

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial y} \\
X_2 &= x^2\frac{\partial}{\partial y} \\
X_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}
\end{aligned}
\tag{6.3.39}
$$

and I_3,

$$
\begin{aligned}
Y_1 &= x\frac{\partial}{\partial y} \\
Y_2 &= x^2\frac{\partial}{\partial y} \\
Y_3 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y} \\
Y_4 &= x\frac{\partial}{\partial x}.
\end{aligned}
\tag{6.3.40}
$$

The Z_1 and the Z_2 symmetries (where Z refers to G, X and Y in turn) contain the solutions of the equations (6.3.32) and correspond to the two symmetries in (6.3.12). Note that the linear solution implies that $k = 0$ in (6.3.22) and hence does not appear in the three symmetry case (6.3.39). Note further that the Z_3 and Z_4 symmetries also contain solutions of the original equation. This is because the functions $a_i(x)$ are solutions of the adjoint equation of (6.3.5) and in the case of maximal symmetry this equation is self-adjoint.

The Lie Brackets of the symmetries of I_1 are

$$
\begin{array}{lll}
[G_1, G_2] = 0 & [G_2, G_3] = -G_1 & [G_3, G_4] = G_3 \\
[G_1, G_3] = 0 & [G_2, G_4] = G_2 & \\
[G_1, G_4] = 2G_1 & &
\end{array}
\tag{6.3.41}
$$

(which is the Lie algebra $A_{4,9}^1$ [147]) and of I_3 are

$$\begin{aligned} &[Y_1, Y_2] = 0 && [Y_2, Y_3] = 0 && [Y_3, Y_4] = Y_4 \\ &[Y_1, Y_3] = Y_2 && [Y_2, Y_4] = -2Y_2 && \\ &[Y_1, Y_4] = -Y_1. && && \end{aligned} \tag{6.3.42}$$

Clearly the correspondence is

$$\begin{aligned} Y_1 &\longrightarrow G_2 \\ Y_2 &\longrightarrow -G_1 \\ Y_3 &\longrightarrow G_3 \\ Y_4 &\longrightarrow -G_4 \end{aligned}$$

and (6.3.42) is also the Lie algebra $A_{4,9}^1$. If we let the coordinates in I_3 be (x, y) and in I_1, $(\mathcal{X}, \mathcal{Y})$, the transformation which converts I_3 to I_1 (and *vice versa*) is

$$\mathcal{X} = -\frac{1}{x} \qquad \mathcal{Y} = -\frac{y}{x^2}. \tag{6.3.43}$$

This transformation just maps I_2 to itself and

$$\begin{aligned} X_1 &\longrightarrow -X_1 \\ X_2 &\longrightarrow -X_2 \\ X_3 &\longrightarrow -X_3. \end{aligned}$$

Note that (6.3.39) forms the Lie algebra $A_{3,4}$ which is better known as the algebra of the pseudo–Euclidean group $E(1,1)$. Equation (6.3.32) is obviously invariant under (6.3.43) and, in the new variables, is

$$\mathcal{Y}''' = 0. \tag{6.3.44}$$

It is instructive to consider a slightly less trivial example of an equation with seven symmetries to see how the constructive approach works in practice. Consider

$$y''' + y' = 0 \tag{6.3.45}$$

for which (cf (6.3.6)) $f = 1$ and $g = 0$. Then

$$\begin{aligned} a &= A_1 \sin x + A_2 \cos x + C_0 \\ c &= C_0. \end{aligned} \tag{6.3.46}$$

Four symmetries exist when (6.3.18) is satisfied, *ie*

$$A_1^2 + A_2^2 = C_0^2. \tag{6.3.47}$$

Two choices[15] are $C_0 = 1$ and $A_2 = \pm 1$ which give the integrals with four symmetries

$$I_1 = (1 + \cos x)y'' + \sin xy' + y \tag{6.3.48}$$

$$I_2 = (1 - \cos x)y'' - \sin xy' + y. \tag{6.3.49}$$

A integral with three symmetries can be obtained by the choice, say, of $A_1 = 1$ and $C_0 = 0$ is

$$I_3 = \sin xy'' - \cos xy'. \tag{6.3.50}$$

The symmetries follow from (6.3.24), (6.3.25) and (6.3.26) respectively.

6.3.4 Example II: $n = 5$

Consider now the linear equation

$$y''' - y = 0 \tag{6.3.51}$$

with the five symmetries

$$\begin{aligned} G_1 &= \mathrm{e}^{x}\frac{\partial}{\partial y} \\ G_2 &= \mathrm{e}^{\omega x}\frac{\partial}{\partial y} \\ G_3 &= \mathrm{e}^{\omega^2 x}\frac{\partial}{\partial y} \\ G_4 &= y\frac{\partial}{\partial y} \\ G_5 &= \frac{\partial}{\partial x}, \end{aligned} \tag{6.3.52}$$

[15]Provided the constraint (6.3.47) is satisfied, the choice of values is one of taste.

where $1+\omega+\omega^2=0$. The Lie brackets of (6.3.52) constitute the Lie algebra $3A_1 \oplus_s (2A_1)$. We choose the three independent first integrals of (6.3.51) to be

$$\begin{aligned} I_1 &= \mathrm{e}^{-x}\left(y+y'+y''\right) \\ I_2 &= \mathrm{e}^{-\omega x}\left(\omega y+y'+\omega^2 y''\right) \\ I_3 &= \mathrm{e}^{-\omega^2 x}\left(\omega y+\omega^2 y'+y''\right). \end{aligned} \tag{6.3.53}$$

I_1 has the symmetries

$$\begin{aligned} G_1 &= \mathrm{e}^{\omega x}\frac{\partial}{\partial y} \\ G_2 &= \mathrm{e}^{\omega^2 x}\frac{\partial}{\partial y} \\ G_3 &= \frac{1}{\omega}\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right), \end{aligned} \tag{6.3.54}$$

I_2,

$$\begin{aligned} X_1 &= \mathrm{e}^{x}\frac{\partial}{\partial y} \\ X_2 &= \mathrm{e}^{\omega^2 x}\frac{\partial}{\partial y} \\ X_3 &= \frac{1}{\omega^2}\left(\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right) \end{aligned} \tag{6.3.55}$$

and I_3,

$$\begin{aligned} Y_1 &= \mathrm{e}^{\omega x}\frac{\partial}{\partial y} \\ Y_2 &= \mathrm{e}^{x}\frac{\partial}{\partial y} \\ Y_3 &= \frac{\partial}{\partial x}+y\frac{\partial}{\partial y}. \end{aligned} \tag{6.3.56}$$

The Lie brackets of (6.3.54), (6.3.55) and (6.3.56) each form the Lie algebra $A_{3,3}$ which is the algebra of the group comprising the semi–direct product of dilations and translations $D \otimes_s T_2$. We can transform the first integrals (and hence their symmetries) in a cyclic manner by setting

$$x \longrightarrow \omega X.$$

This transformation leaves (6.3.51) invariant.

6.3.5 Example III: $n = 4$

As the final linear example we consider[16]

$$y''' + f(x)y'' + y' + f(x)y = 0, \tag{6.3.57}$$

where $f(x)$ is an arbitrary function of x, with symmetries

$$\begin{aligned} G_1 &= \sin x \frac{\partial}{\partial y} \\ G_2 &= \cos x \frac{\partial}{\partial y} \\ G_3 &= z(x) \frac{\partial}{\partial y} \\ G_4 &= y \frac{\partial}{\partial y}, \end{aligned} \tag{6.3.58}$$

where

$$z(x) = \int_0^x \exp\left(-\int f(u)\mathrm{d}u\right) \sin(x-u)\mathrm{d}u, \tag{6.3.59}$$

which possess the Lie algebra $3A_1 \oplus_s A_1$. Three independent first integrals of (6.3.57) are

$$\begin{aligned} I_1 &= y' \sin x - y \cos x - z(x)(y'' + y) \exp\left(\int f(u)\mathrm{d}u\right) \\ I_2 &= y' \cos x + y \sin x - z(x)(y'' + y) \exp\left(\int f(u)\mathrm{d}u\right) \\ I_3 &= (y'' + y) \exp\left(\int f(u)\mathrm{d}u\right) \end{aligned} \tag{6.3.60}$$

with the symmetries

$$\begin{aligned} G_1 &= \sin \frac{\partial}{\partial y} \\ G_2 &= z(x) \frac{\partial}{\partial y}, \end{aligned} \tag{6.3.61}$$

$$\begin{aligned} X_1 &= \cos x \frac{\partial}{\partial y} \\ X_2 &= z(x) \frac{\partial}{\partial y} \end{aligned} \tag{6.3.62}$$

---

[16]Kamke [84, p 512, 3.23].

and

$$
\begin{aligned}
Y_1 &= \sin x \frac{\partial}{\partial y} \\
Y_2 &= \cos x \frac{\partial}{\partial y}
\end{aligned}
\tag{6.3.63}
$$

respectively. Each pair forms the Lie algebra $2A_1$ and contains the solutions of the original equation. The search for the transformation to cycle through the first integrals (6.3.60) requires an Abel's formula for third order equations which, to our knowledge, has yet to be discovered.

6.3.6 Nonlinear examples

The situation in respect of nonlinear equations of the third order is not at all clear. We return to the Kummer–Schwarz equation mentioned above, *viz.*

$$
2y'y''' - 3y''^2 = 0 \tag{6.3.64}
$$

with the symmetries

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial x} \\
G_2 &= x \frac{\partial}{\partial x} \\
G_3 &= x^2 \frac{\partial}{\partial x} \\
G_4 &= \frac{\partial}{\partial y} \\
G_5 &= y \frac{\partial}{\partial y} \\
G_6 &= y^2 \frac{\partial}{\partial y}
\end{aligned}
\tag{6.3.65}
$$

which form the Lie algebra $s\ell(2, R) \oplus s\ell(2, R)$. Three independent first integrals are

$$
\begin{aligned}
I_1 &= \frac{y''^2}{y'^3} \\
I_2 &= \frac{yy'' - 2y'^2}{y''} \\
I_3 &= x + \frac{2y'(yy'' - y'^2)}{y''(yy'' - 2y'^2)}
\end{aligned}
\tag{6.3.66}
$$

with the symmetries

$$\begin{aligned} G_1 &= \frac{\partial}{\partial y} \\ G_2 &= \frac{\partial}{\partial x} \\ G_3 &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \end{aligned} \tag{6.3.67}$$

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} \\ X_2 &= x\frac{\partial}{\partial x} \end{aligned} \tag{6.3.68}$$

and

$$Y_1 = y\frac{\partial}{\partial y} \tag{6.3.69}$$

respectively.

Consider also a nonlinear third order equation with four symmetries[17], *viz.*

$$(1 + y'^2)y''' = (3y' + 1)y''^2. \tag{6.3.70}$$

The symmetries of (6.3.70) are

$$\begin{aligned} G_1 &= \frac{\partial}{\partial x} \\ G_2 &= \frac{\partial}{\partial y} \\ G_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ G_4 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \end{aligned} \tag{6.3.71}$$

Three independent first integrals are

$$\begin{aligned} I_1 &= \frac{y''}{(1 + y'^2)^{3/2}} \exp\left(-\arctan y'\right) \\ I_2 &= 2y + \frac{(1 + y')(1 + y'^2)}{y''} \\ I_3 &= 2x + \frac{(1 + y'^2)(1 - y')}{y''} \end{aligned} \tag{6.3.72}$$

[17]Kamke [84, p 603, 7.12].

with the symmetries

$$\begin{aligned}
G_1 &= \frac{\partial}{\partial x} \\
G_2 &= \frac{\partial}{\partial y} \\
G_3 &= (x+y)\frac{\partial}{\partial x} + (y-x)\frac{\partial}{\partial y},
\end{aligned} \tag{6.3.73}$$

$$X_1 = \frac{\partial}{\partial x} \tag{6.3.74}$$

and

$$Y_1 = \frac{\partial}{\partial y} \tag{6.3.75}$$

respectively. It is apparent from the above that the relationship between the first integrals (6.3.66) and (6.3.72) and their respective symmetries (6.3.67–6.3.69) and (6.3.73–6.3.75) is not at all obvious. Indeed there is some question as to where the beginning of the resolution of this problem lies. Similar to the linear case, each nonlinear equation will have a different normal form that depends on its Lie algebra. However, while there is just one Lie algebra for a particular dimension (any one of four, five or seven) that relates to third order linear equations, in the case of nonlinear equations the number is as yet undetermined. This needs to be resolved before a search for a relationship similar to the linear case can be commenced. It would seem that Gat [50] has provided some classification for third order nonlinear equations that could be used as a starting point.

As a further illustration of the importance of first integrals of third order equations we mention that (6.3.70) can be solved using the first integrals (6.3.72). Setting

$$\begin{aligned}
X &= I_3 - 2x \\
Y &= I_2 - 2y
\end{aligned} \tag{6.3.76}$$

we have (using the ratio of I_2 to I_3 and integrating)

$$KX = \frac{1}{(1+V^2)^{1/2}}\exp\left(\arctan V\right), \tag{6.3.77}$$

where K is a constant of integration and $Y = VX$. The solution (6.3.77) is implicit, but is still an improvement over the parametric solution provided in Kamke[18].

6.3.7 Conclusion

We have shown that the symmetries of first integrals of third order linear equations are related to the number of symmetries of the equation. For the maximal case two first integrals have four symmetries and one has three. When the equation has five symmetries, all three first integrals have three symmetries and for four symmetries the first integrals have two symmetries each. The relationship is rather intriguing and bears further investigation in a generalization of the result to higher order linear equations. Unfortunately the relationship for nonlinear equations is not as yet obvious.

In the case of second order ordinary differential equations with the maximal symmetry, $s\ell(3, R)$, there are three first integrals which each have three symmetries[19]. To take the example of the free particle with equation

$$y'' = 0 \tag{6.3.78}$$

those integrals are

$$\begin{aligned} I_1 &= y' \\ I_2 &= y - xy' \\ I_3 &= \frac{y}{y'} - x. \end{aligned} \tag{6.3.79}$$

In each case the algebra of the symmetries is $A_{3,3}$ (or $D \oplus_s T_2$). We note that the first integrals with this property are $y(0), y'(0)$ and their ratio.

Third order linear equations (and others transformable to one by a point transformation) differ in two respects. In the first instance all such equations

[18]Kamke [84, p 603, 7.12].

[19]See §6.2.

do not have the same number of symmetries, but have four, five or seven depending on the internal structure of the equation. In the case of four the maximum number of symmetries for the first integrals is two; five, three and seven, four. The last of these is the closest to the second order case in that it is the case of maximal symmetry. It could be anticipated that the maximum number of symmetries would be four. What is unexpected is that this occurs only for two first integrals because the equation, (6.3.18), acts as a constraint. The third integral has only three symmetries. It is also of interest to note that the maximum number of symmetries does not necessarily occur for the initial condition integrals (*cf* the second of the maximal symmetry examples). The property of the ratio of the two first integrals also having the same number of symmetries is also lost as can easily be seen from an analysis of I_1/I_2 of the second of the maximal symmetry examples. In fact the ratio has only the two symmetries

$$\begin{aligned} G_1 &= \sin x \frac{\partial}{\partial y} \\ G_2 &= y \frac{\partial}{\partial y}. \end{aligned} \tag{6.3.80}$$

The significance and theoretical basis behind this are not obvious, but become more transparent under an investigation of the first integrals of higher order linear equations which is reported below.

6.4 nth Order Ordinary Differential Equations

6.4.1 Introduction

A scalar ordinary differential equation

$$E(x, y, y', \ldots, y^{(n)}) = 0, \tag{6.4.1}$$

where $'$ denotes differentiation of the dependent variable, y, with respect to the independent variable, x, and $y^{(n)}$ the nth derivative, possesses a Lie point symmetry

$$G = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} \tag{6.4.2}$$

if

$$G^{[n]}E_{\big|_{E=0}} = 0, \tag{6.4.3}$$

where $G^{[n]}$ is the nth extension of G given by [153]

$$G^{[n]} = G + \sum_{i=1}^{n}\left\{\eta^{(i)} - \sum_{j=0}^{i-1}\begin{pmatrix} i \\ j \end{pmatrix} y^{(j+1)}\xi^{(i-j)}\right\}\frac{\partial}{\partial y^{(i)}}. \tag{6.4.4}$$

(The extension is needed to give the infinitesimal transformations in the derivatives up to $y^{(n)}$ induced by the infinitesimal transformations which G produces in x and y.) The symmetries of (6.4.1) constitute a Lie algebra under the operation of taking the Lie Bracket

$$[G_1, G_2] = G_1G_2 - G_2G_1. \tag{6.4.5}$$

A first integral of (6.4.1) associated with the symmetry (6.4.2) is a function, $f(x, y, y', \ldots, y^{(n-1)})$, in which the dependence on $y^{(n-1)}$ is nontrivial, satisfying the two conditions

$$G^{[n-1]}f = 0 \tag{6.4.6}$$

and

$$\frac{\mathrm{d}f}{\mathrm{d}x}_{\big|_{E=0}} = 0. \tag{6.4.7}$$

The association of f with G, as stated in (6.4.6), and (6.4.7) means that f is a first integral of (6.4.1). Equally a first integral, $f(x, y, y', \ldots, y^{(n-1)})$, of (6.4.1) has a symmetry of the form of (6.4.2) if

$$G^{[n-1]}f = 0. \tag{6.4.8}$$

In recent years a number of papers has been devoted to the algebraic properties of first integrals of scalar ordinary differential equations associated with its symmetries. The number of symmetries associated with a differential equation depends upon its internal structure up to an upper limit which is fixed by the order of the equation and the type of symmetry under consideration. Here we are concerned with point and contact symmetries only. The symmetry (6.4.2) is a Lie point symmetry if the coordinate functions ξ and η are functions of x and y only. It is a contact symmetry if ξ and η depend upon x, y and y' subject to the constraint that [138, p 94]

$$\frac{\partial f}{\partial \eta}y' = y'\frac{\partial f}{\partial \xi}y' \tag{6.4.9}$$

which means that the first extension of G also depends upon x, y and y' only. A point symmetry is always a contact symmetry. Note that we do not restrict ξ to be a function of x only which is the case for the so–called Cartan symmetries [80].

Lie showed that the maximum number of point symmetries of a scalar ordinary differential equation was infinite for equations of the first order [133, p 114], eight for equations of the second order [133, p 405] and $n+4$ for equations of the nth order [134, p 298]. He also [137] classified all the invariance algebras of dimension one, two and three for second order equations. Mahomed and Leach [153] showed that higher order linear equations could have $n+1$ or $n+2$ point symmetries instead of the $n+4$ for the maximal symmetry case. Note that, whenever reference is made to linear equations, we include nonlinear equations which are linearisable by a point (resp contact) transformation when point (resp contact) symmetries are being considered. Lie also showed that second order equations possessed an infinite number of contact symmetries [138, p 84] and third order at most ten [138, p 241]. For higher order (in standard form) it has been shown [148] that equations of maximal symmetry only admit $n+4$ contact symmetries. Abraham–Shrauner *et al* [5] showed that the maximal number of contact symmetries of third order equations could be found in equations which did not have seven (the maximum) point symmetries in addition to the Kummer–Schwarz equation given by Lie [138, p 148].

Leach and Mahomed [121] discussed the algebraic properties of the first integrals of equations of maximal point symmetry which are represented by the single equivalence class

$$y'' = 0. \tag{6.4.10}$$

It was found that the two functionally independent first integrals

$$I_1 = xy' - y \tag{6.4.11}$$

$$I_2 = y' \tag{6.4.12}$$

each possessed three symmetries with the three–dimensional algebra $A_{3,3}$ in the Mubarakzyanov classification [163, 164, 165, 162, 169, 170, 171]. Furthermore the related integral

$$I_3 = \frac{I_1}{I_2} \tag{6.4.13}$$

also possessed three symmetries with the same algebra. In the case of (6.4.13) the three symmetries are

$$G_1 = y\frac{\partial}{\partial y} \tag{6.4.14}$$

$$G_2 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{6.4.15}$$

$$G_3 = y\frac{\partial}{\partial x}. \tag{6.4.16}$$

G_1 is the homogeneity symmetry and for any integral to possess it the integral must be homogeneous of degree zero in y. G_2 and G_3 are the nonCartan symmetries of (6.4.10). Note that, although the integrals have nine symmetries in all, there are only eight linearly independent symmetries. The interesting algebraic structure of the symmetries associated with the linear integrals of (6.4.10) led to the study of the corresponding algebras of third order equations with four, five and seven symmetries [66] and nth order equations with $n+1$, $n+2$ and $n+4$ symmetries [70]. The pattern of the second order equations was not maintained. The demonstration by Abraham–Shrauner *et al* [5] that contact symmetries are the appropriate ones to be used in treatments of third order equations led to a study [42] of the algebraic properties of the contact symmetries associated with the integrals of third order equations with the symmetry algebra $sp(4)$ which was more extensive than the comparable study of second order equations by Leach and Mahomed [121].

Second and third order ordinary differential equations have properties which are peculiar to each and which mark them off from the general nth order equation. This is the case even for the representative equations of maximal symmetry. For

$$y'' = 0 \tag{6.4.17}$$

there are eight point symmetries, six of which are of Cartan form and two of which are not. In the case of

$$y''' = 0 \tag{6.4.18}$$

the seven point symmetries are all of Cartan form, but there are three additional purely contact symmetries and the consideration of third order equations without contact symmetries is as incomplete as the consideration of second order equations without the nonCartan point symmetries. However,

$$y^{(n)} = 0 \tag{6.4.19}$$

has only $n + 4$ point symmetries all of which are of Cartan type [134, p 298]. It is this variation in the type, rather than in the number (apart from the fact that it is maximal), of symmetries which suggests that the generic behaviour for ordinary differential equations of maximal symmetry is not to be found at the second or third order, but at the fourth order. It is the intention of this section to demonstrate this feature and to describe the generic properties of the algebras of the symmetries of first integrals of scalar ordinary differential equations.

6.4.2 Methodology

There is a certain ambiguity in the treatment of the symmetries of the first integrals. In the case of the second order equation, (6.4.10), we recalled that the three first integrals, (6.4.11), (6.4.12) and (6.4.13) each had three point symmetries and that the algebras were isomorphic. However, this is not the case for any combination of (6.4.11) and (6.4.12) [121]. We do seek those first integrals which have, in some sense, a maximal number of symmetries. To make comparison from one order to another feasible it is necessary to use as much of the common structure for the algebras of the differential equations as is possible. To this end we use the form given by Mahomed and Leach [153], $nA_1 \oplus_s (s\ell(2,R) \oplus A_1)$, to which the two nonCartan symmetries are to be added when $n = 2$ and the three purely contact symmetries are to be added when $n = 3$. To each of these symmetries there corresponds a first integral and it is the symmetries of these integrals which we consider. For an nth order equation there are $n-1$ integrals associated with each symmetry of the nth order equation and the first integral mentioned above is an arbitrary function of these. We select $n-1$ of these arbitrary independent functions in such a way as to have the maximum number of symmetries possible.

In the case of (6.4.18) there are three functionally independent first integrals which we take to be [42]

$$I_1 = \tfrac{1}{2}x^2y'' - xy' + y \tag{6.4.20}$$

$$I_2 = xy'' - y' \tag{6.4.21}$$

$$I_3 = y'' \tag{6.4.22}$$

which have been shown [66] to possess maximal symmetry. In terms of I_1, I_2 and I_3 the integrals associated with the symmetries of (6.4.18) are given [42] in Table 6.1. Note that each symmetry has two integrals (denoted by p and q) since the equation is of the third order. In Tables 6.2 and 6.3 we list the symmetries [42] associated with each integral given in Table 6.1 in terms of the symmetries of (6.4.18) (and combinations thereof) and the corresponding algebra according to the Mubarakzyanov classification. We observe that the algebras are either three–dimensional or four–dimensional. The latter is always $A_{4,9}^1$, but the former are either $A_{3,4}$ (also known as $E(1,1)$, the algebra of the pseudo–Euclidean group in the plane) or $A_{3,8}$ (much better known as $s\ell(2,R)$).

Table 6.1: The first integrals associated with the ten contact symmetries of $y''' = 0$

Symmetry	p	q
G_1	I_2	I_3
G_2	I_1	I_3
G_3	I_1	I_2
G_4	I_3	$I_1I_3 - \frac{1}{2}I_2^2$
G_5	I_2	$I_1I_3 - \frac{1}{2}I_2^2$
G_6	I_1	$I_1I_3 - \frac{1}{2}I_2^2$
G_7	I_2/I_3	I_3/I_1
G_8	I_2/I_3	$(I_1I_3 - \frac{1}{2}I_2^2)/I_3$
G_9	I_3/I_1	$(I_1I_3 - \frac{1}{2}I_2^2)/I_1$
G_{10}	I_1/I_2	$(I_1I_3 - \frac{1}{2}I_2^2)/I_2$

This already indicates two departures from the results for (6.4.17) in that the dimensions of the algebras are not the same and also the three–dimensional algebras differ not only from that of the second order case, $A_{3,3}$, but within themselves. Perhaps even more surprising is that $A_{3,3}$ is not a subalgebra of $A_{4,9}^1$ [147].

With the example of the third order equation before us we shall present the results for the fourth order equation of maximal symmetry, *viz.*

$$y^{iv} = 0, \tag{6.4.23}$$

in §§6.4.3 and 6.4.4. In §6.4.5 our concluding remarks address, amongst a number of observations, the matter of the general equation

$$y^{(n)} = 0. \tag{6.4.24}$$

Table 6.2: The contact symmetries and algebras of the integrals I_1 through I_5

Integral	Symmetries	Nonzero Lie Brackets	Algebra
I_1	$X_{11} = G_2$	$[X_{11}, X_{13}] = X_{12}$	$A^1_{4,9}$
	$X_{12} = G_3$	$[X_{11}, X_{14}] = -X_{11}$	
	$X_{13} = G_7$	$[X_{12}, X_{14}] = -2X_{12}$	
	$X_{14} = G_6 - G_4$	$[X_{13}, X_{14}] = -X_{13}$	
I_2	$X_{21} = G_1$	$[X_{21}, X_{23}] = X_{21}$	$A_{3,4}$ $(E(1,1))$
	$X_{22} = G_3$	$[X_{22}, X_{23}] = -X_{22}$	
	$X_{23} = G_6$		
I_3	$X_{31} = G_1$	$[X_{31}, X_{34}] = 2X_{31}$	$A^1_{4,9}$
	$X_{32} = G_2$	$[X_{32}, X_{33}] = -X_{31}$	
	$X_{33} = G_5$	$[X_{32}, X_{34}] = X_{32}$	
	$X_{34} = G_6 + G_4$	$[X_{33}, X_{34}] = X_{33}$	
$I_4 = I_3/I_2$	$X_{41} = G_1$	$[X_{41}, X_{42}] = X_{41}$	$A_{3,4}$ $(E(1,1))$
	$X_{42} = G_4$	$[X_{42}, X_{43}] = X_{43}$	
	$X_{13} = G_8$		
$I_5 = I_3/I_1$	$X_{51} = G_4$	$[X_{51}, X_{52}] = -X_{52}$	$A_{3,8}$ $(s\ell(2,R))$
	$X_{52} = G_2$	$[X_{51}, X_{53}] = X_{53}$	
	$X_{53} = G_9$	$[X_{52}, X_{53}] = -2X_{51}$.	

Table 6.3: The contact symmetries and algebras of the integrals I_6 through I_{10}

Integral	Symmetries	Nonzero Lie Brackets	Algebra
$I_6 = I_2/I_1$	$X_{61} = G_4$	$[X_{61}, X_{62}] = -X_{62}$	$A_{3,4}$ $(E(1,1))$
	$X_{62} = G_3$	$[X_{61}, X_{63}] = X_{63}$	
	$X_{63} = G_{10}$		
$I_7 = I_1I_3 - \frac{1}{2}I_2^2$	$X_{71} = G_5$	$[X_{71}, X_{72}] = X_{71}$	$A_{3,8}$ $(s\ell(2,R))$
	$X_{72} = G_6$	$[X_{71}, X_{73}] = 2X_{72}$	
	$X_{73} = G_7$	$[X_{72}, X_{73}] = X_{73}$	
$I_8 = I_7/I_3$	$X_{81} = G_5$	$[X_{81}, X_{82}] = X_{81}$	$A^1_{4,9}$
	$X_{82} = G_6 - G_4$	$[X_{81}, X_{84}] = 2X_{83}$	
	$X_{83} = G_8$	$[X_{82}, X_{83}] = -X_{83}$	
	$X_{84} = G_9$	$[X_{82}, X_{84}] = -X_{84}$	
$I_9 = I_7/I_2$	$X_{91} = G_8$	$[X_{91}, X_{92}] = X_{91}$	$A_{3,4}$ $(E(1,1))$
	$X_{92} = G_6$	$[X_{92}, X_{93}] = X_{93}$	
	$X_{93} = G_{10}$		
$I_{10} = I_7/I_1$	$X_{101} = G_6 + G_4$	$[X_{101}, X_{102}] = X_{102}$	$A^1_{4,9}$
	$X_{102} = G_7$	$[X_{101}, X_{103}] = X_{103}$	
	$X_{103} = G_9$	$[X_{101}, X_{104}] = 2X_{104}$	
	$X_{104} = G_{10}$	$[X_{102}, X_{103}] = -2X_{104}$	

6.4.3 Symmetries and their integrals

Eq (6.4.23) is the representative equation of the fourth order with the maximum number of symmetries which is eight with the algebraic structure $(A_1 \oplus s\ell(2,R)) \oplus_s 4A_1$. The symmetries are

$$
\begin{aligned}
G_1 &= \frac{\partial}{\partial y} \\
G_2 &= x\frac{\partial}{\partial y} \\
G_3 &= \tfrac{1}{2}x^2\frac{\partial}{\partial y} \\
G_4 &= \tfrac{1}{6}x^3\frac{\partial}{\partial y} \\
G_5 &= \frac{\partial}{\partial x} \\
G_6 &= x\frac{\partial}{\partial x} + \tfrac{3}{2}y\frac{\partial}{\partial y} \\
G_7 &= x^2\frac{\partial}{\partial x} + 3xy\frac{\partial}{\partial y} \\
G_8 &= y\frac{\partial}{\partial y}.
\end{aligned} \tag{6.4.25}
$$

The first four symmetries are called the solution symmetries since the coefficient of $\partial/\partial y$ is a solution of the original differential equation. (This is one of the banes of linear differential equations. It is necessary to be able to solve the equation before most of the symmetries can be determined. In this respect nonlinear equations are more amenable to treatment provided they are treatable.) G_5 through G_7 are the elements of $s\ell(2,R)$ appropriate to (6.4.23). For an nth order equation of maximal symmetry they have the form [153]

$$G = a(x)\frac{\partial}{\partial x} + \frac{n-1}{2}a'(x)y\frac{\partial}{\partial y}, \tag{6.4.26}$$

where $a(x)$ is one of the three solutions of the self–adjoint equation

$$\frac{(n+1)!}{(n-2)!4!}a''' + B_{n-2}a' + \tfrac{1}{2}B'_{n-2}a = 0, \tag{6.4.27}$$

where B_{n-2} is the coefficient of $y^{(n-2)}$ when the equation is cast into normal form. The final symmetry, G_8, follows from the homogeneity of the differential equation which happens to coincide with linearity in this case.

Table 6.4: The first integrals associated with the eight point symmetries of $y^{iv} = 0$

Symmetry	First integrals		
G_1	I_2	I_3	I_4
G_2	I_3	I_4	I_1
G_3	I_4	I_1	I_2
G_4	I_1	I_2	I_3
G_5	I_4	$I_2I_4 - \frac{1}{2}I_3^2$	$I_2I_3I_4 - \frac{1}{3}I_3^3 - I_1I_4^2$
G_6	I_1I_4	$I_2I_4^{\frac{1}{3}}$	$I_3I_4^{-\frac{1}{3}}$
G_7	I_1	$I_1I_3 - \frac{2}{3}I_2^2$	$I_1I_2I_3 - \frac{4}{3}I_2^3 - I_1^2I_4$
G_8	I_1/I_2	I_2/I_3	I_3/I_4

In calculating the first integrals associated with each of G_1 through G_8 according to eqq (6.4.6) and (6.4.7) we find that four functionally independent linear first integrals occur. To make the reportage of our results more compact we express all other integrals in terms of them. The four integrals are

$$\begin{aligned} I_1 &= \tfrac{1}{6}x^3y''' - \tfrac{1}{2}x^2y'' + xy' - y \\ I_2 &= \tfrac{1}{2}x^2y''' - xy'' + y' \\ I_3 &= xy''' - y'' \\ I_4 &= y'''. \end{aligned} \tag{6.4.28}$$

With each symmetry there will be associated three functionally independent first integrals. This follows from the solutions to the two first order partial differential equations (6.4.6) and (6.4.7) for the first integral associated with a particular symmetry. In (6.4.6) there are five variables, x, y, y', y'' and y''', and so four characteristics. This means that (6.4.7) has four variables and hence three characteristics each of which is a first integral. The integrals belonging to the symmetries are listed in Table 6.4.

In comparison with the integrals listed in Table 6.1 (less those associated with the purely contact symmetries, $G_8 - G_{10}$) for $y''' = 0$ we note a number of similarities and differences. The solution symmetries, $G_1 - G_4$, simply form a permutation with three of the linear integrals. (The labeling was chosen to highlight this feature.) The homogeneity symmetry, G_8, has the three independent ratios of the linear integrals. Any other independent set of three could be equally be chosen. The ratios are the only integrals possible for G_8 since any integral associated with it must be of zero degree in y.

These are the anticipated generalisations of the corresponding results for $y''' = 0$ and, indeed, for $y'' = 0$. For (6.4.24) we may infer that each of the n solution symmetries will have $n-1$ of the n functionally independent linear integrals associated with it and that, by a suitable choice of labels as was made for (6.4.18) and (6.4.23), the one symmetry label and the $n-1$ integral labels will be a permutation of the integers 1 through n. Equally confidently we infer that the homogeneity symmetry, G_{n+4}, will have $n-1$ independent ratios of the linear integrals associated with it.

In the case of the representation of $s\ell(2,R)$, which is common to all linear equations of maximal symmetry, the situation is not so clear. To give more scope for observation we list the corresponding relationships for $y'' = 0$. They are

$$\begin{aligned} G_3 &= \frac{\partial}{\partial x} && I_2 \\ G_4 &= x\frac{\partial}{\partial x} + \tfrac{1}{2}y\frac{\partial}{\partial y} && I_1 I_2 \\ G_5 &= x^2\frac{\partial}{\partial x} + \tfrac{1}{2}xy\frac{\partial}{\partial y} && I_1, \end{aligned} \tag{6.4.29}$$

where, in the spirit adopted for (6.4.18) and (6.4.23), the $s\ell(2,R)$ symmetries have been listed after the solution symmetries and

$$\begin{aligned} I_1 &= xy' - y \\ I_2 &= y'. \end{aligned} \tag{6.4.30}$$

One point should be made before we continue. Since the first and third of the two symmetries in $s\ell(2,R)$ are related by the transformation

$$x \longrightarrow -\frac{1}{x} \qquad y \longrightarrow \frac{y}{x^{n-1}} \tag{6.4.31}$$

and the second is invariant under (6.4.31), we need only consider the first two symmetries, invariance under translation in the independent variable and a self–similar transformation. (The integrals of the so–called conformal symmetry follow from those of invariance under translations in x by the application of (6.4.31).)

There are two problems. The first is the form of the expressions for the more complicated integrals associated with $\partial/\partial x$. Under the labeling scheme adopted I_n can always be taken as the first representative. The first representative for the self–similar symmetry differs from the even order equations to the odd order equation. This provides the necessary hint and we have

Proposition 1: *One of the integrals associated with the self–similar symmetry*

$$G = x\frac{\partial}{\partial x} + \frac{n-1}{2}y\frac{\partial}{\partial y} \tag{6.4.32}$$

of

$$y^{(n)} = 0 \tag{6.4.33}$$

is

$$J_1 = I_1 I_n \tag{6.4.34}$$

when n is even and

$$J_1 = I_{(n+1)/2} \tag{6.4.35}$$

when n is odd, where the numbering of the functionally independent linear first integrals of (6.4.33) is according to the scheme

$$I_i = \sum_{k=0}^{n-i} \frac{(-1)^k}{(n-i-k)!} x^{n-i-k} y^{(n-1-k)}, \qquad i = 1, n. \tag{6.4.36}$$

The result follows trivially from **Proposition 3**.

Associated with the self–similar symmetry there is another problem. We choose this symmetry so that the subalgebra $s\ell(2,R)$ occurs naturally within the list of symmetries of the equation. However, the homogeneity symmetry, G_{n+4}, can be added without changing the *nature* of the self–similar symmetry.

6.4.4 Integrals and their symmetries

The Lie point symmetries associated with each of the first integrals listed in Table 6.4 are calculated following the prescription of (6.4.6) using the symbolic code Program LIE [77]. The integrals, symmetries, nonzero Lie Brackets and algebras are given in Tables 6.5 and 6.6. The symmetries are written as X_{ij} in which the label i refers to the integral number and the label j to the number of the symmetry within the integral's algebra. The relationship of these symmetries to those of the differential equation (6.4.23) is also given.

6.4.5 Discussion

The linear integrals, as expected [70], have either five or four symmetries. Three of these are solution symmetries and, in the way the labeling has been arranged, the solution symmetry not included is the one of the number of the integral. (The action of the omitted symmetry on the integral is a constant, $+1$ for I_2 and I_4 and -1 for I_1 and I_3 due to the way the integrals have been defined. The integrals with five symmetries have either G_5 or G_7 which are equivalent under the transformation

$$x \longrightarrow -\frac{1}{x} \qquad y \longrightarrow \frac{y}{x^3}. \tag{6.4.37}$$

The remaining symmetry is a combination of the self-similar G_6 and the homogeneity G_8 of the form

$$X_i = G_6 + \frac{2i-5}{2} G_8, \qquad i = 1, 4. \tag{6.4.38}$$

Similar combinations occur for some of the other integrals. This does suggest that the choice of the form of the self-similar symmetry is at our disposal and that we should not be constrained by the form of G_6 as it occurs in the list of symmetries for (6.4.23). This has some impact on the expressions for the integrals associated with the self-similar symmetry. If we take it to be of the form

$$G = x\frac{\partial}{\partial x} - ay\frac{\partial}{\partial y}, \tag{6.4.39}$$

Table 6.5: The point symmetries and algebras of the integrals I_1 through I_6

Integral	Symmetries	Nonzero Lie Brackets	Algebra
I_1	$X_{11} = G_2$	$[X_{11}, X_{14}] = -X_{11}$	$A^1_{4,9}$
	$X_{12} = G_3$	$[X_{11}, X_{15}] = 4X_{12}$	
	$X_{13} = G_4$	$[X_{12}, X_{14}] = -2X_{12}$	
	$X_{14} = G_6 - \frac{3}{2}G_8$	$[X_{12}, X_{15}] = \frac{3}{2}X_{13}$	
	$X_{15} = G_7$	$[X_{14}, X_{15}] = X_{15}$	
I_2	$X_{21} = G_1$	$[X_{21}, X_{24}] = X_{21}$	$A_{3,4}$ $(E(1,1))$
	$X_{22} = G_3$	$[X_{22}, X_{24}] = -X_{22}$	
	$X_{23} = G_4$	$[X_{23}, X_{24}] = -2X_{23}$	
	$X_{24} = G_6 - \frac{1}{2}G_8$		
I_3	$X_{31} = G_1$	$[X_{31}, X_{34}] = 2X_{31}$	$A^1_{4,9}$
	$X_{32} = G_2$	$[X_{32}, X_{34}] = X_{32}$	
	$X_{33} = G_4$	$[X_{33}, X_{34}] = -X_{33}$	
	$X_{34} = G_6 + \frac{1}{2}G_8$		
I_4	$X_{41} = G_1$	$[X_{41}, X_{45}] = 3X_{41}$	$A^1_{4,9}$
	$X_{42} = G_2$	$[X_{42}, X_{44}] = -X_{41}$	
	$X_{43} = G_3$	$[X_{42}, X_{45}] = 2X_{42}$	
	$X_{44} = G_5$	$[X_{43}, X_{44}] = -2X_{42}$	
	$X_{45} = G_6 + \frac{3}{2}G_8$	$[X_{43}, X_{45}] = X_{43}$	
		$[X_{44}, X_{45}] = X_{44}$	
$I_5 = I_1/I_2$	$X_{51} = G_3$	$[X_{51}, X_{52}] = X_{51}$	$A_{3,4}$ $(E(1,1))$
	$X_{52} = G_4$	$[X_{52}, X_{53}] = X_{52}$	
	$X_{33} = G_8$		
$I_6 = I_2/I_3$	$X_{61} = G_1$	$[X_{61}, X_{63}] = X_{61}$	$A_{3,3}$ $(D \otimes_s T_2)$
	$X_{62} = G_4$	$[X_{62}, X_{63}] = X_{62}$	

Table 6.6: The point symmetries and algebras of the integrals I_7 through I_{14}

Integral	Symmetries	Nonzero Lie Brackets	Algebra
	$X_{63} = G_8$		
$I_7 = I_3/I_4$	$X_{71} = G_1$	$[X_{71}, X_{73}] = X_{71}$	$A_{3,4}$ $(E(1,1))$
	$X_{72} = G_2$	$[X_{72}, X_{73}] = X_{73}$	
	$X_{73} = G_8$		
$I_8 = I_2I_4 - \frac{1}{2}I_3^2$	$X_{81} = G_1$	$[X_{81}, X_{83}] = X_{81}$	$A_{3,5}^{\frac{1}{2}}$
	$X_{82} = G_5$	$[X_{82}, X_{83}] = \frac{1}{2}X_{82}$	
	$X_{83} = \frac{1}{2}(G_6 + \frac{1}{2}G_8)$		
$I_9 = I_2I_3I_4 - \frac{1}{3}I_3^3$	$X_{91} = G_5$	$[X_{91}, X_{92}] = X_{91}$	$2A_1$
$-I_1I_4^2$	$X_{92} = G_6 + \frac{1}{2}G_8$		
$I_{10} = I_1I_4$	$X_{101} = G_2$	$[X_{101}, X_{103}] = X_{101}$	$A_{3,4}$
	$X_{102} = G_3$	$[X_{102}, X_{103}] = -X_{102}$	
	$X_{103} = 2G_6$	$(E(1,1))$	
$I_{11} = I_1^{\frac{1}{3}}I_3$	$X_{111} = G_1$	$[X_{111}, X_{113}] = X_{111}$	$A_{3,5}^{-\frac{1}{3}}$
	$X_{112} = G_3$	$[X_{112}, X_{113}] = \frac{1}{3}X_{112}$	
	$X_{113} = \frac{2}{3}G_6$		
$I_{12} = I_1^{-\frac{1}{3}}I_2$	$X_{121} = G_1$	$[X_{121}, X_{123}] = X_{121}$	$A_{3,5}^{\frac{1}{3}}$
	$X_{122} = G_2$	$[X_{122}, X_{123}] = \frac{1}{3}X_{122}$	
	$X_{123} = \frac{2}{3}G_6$		
$I_{13} = I_1I_3 - \frac{2}{3}I_2^2$	$X_{131} = G_4$	$[X_{131}, X_{133}] = X_{131}$	$A_{3,5}^{\frac{1}{2}}$
	$X_{132} = G_7$	$[X_{132}, X_{133}] = \frac{1}{2}X_{133}$	
	$X_{133} = -\frac{1}{2}(G_6 - \frac{1}{2}G_8)$		
$I_{14} = I_1I_2I_3 - \frac{4}{3}I_2^3$	$X_{141} = G_6 - \frac{1}{2}G_8$	$[X_{141}, X_{142}] = X_{142}$	$2A_1$
$-I_1^2I_4$	$X_{142} = G_7$		

where a is a parameter, the three integrals are

$$J_1 = I_4^{-\frac{a}{a+3}} I_1 \tag{6.4.40}$$

$$J_2 = I_4^{-\frac{a+1}{a+3}} I_2 \tag{6.4.41}$$

$$J_3 = I_4^{-\frac{a+2}{a+3}} I_3. \tag{6.4.42}$$

Equivalently other combinations may be taken. For example, if I_1 were taken as the integral to have the fractional power associated with it, we would have

$$K_1 = I_1^{-\frac{a+3}{a}} I_4 \tag{6.4.43}$$

$$K_2 = I_1^{-\frac{a+2}{a}} I_3 \tag{6.4.44}$$

$$K_3 = I_1^{-\frac{a+1}{a}} I_2. \tag{6.4.45}$$

In the cases of the four forms of the self–similarity symmetry found in the algebras we find the following sets of integrals

$$\begin{aligned}
X_{14}: \quad & J_1 = I_1 \\
& J_2 = I_4^{-\frac{1}{3}} I_2 \\
& J_3 = I_4^{-\frac{2}{3}} I_3 \\
\\
X_{24}: \quad & J_1 = I_4^{\frac{1}{2}} I_1 \\
& J_2 = I_2 \\
& J_3 = I_4^{-\frac{1}{2}} I_3 \\
\\
X_{34}: \quad & J_1 = I_4^2 I_1 \\
& J_2 = I_4 I_2 \\
& J_3 = I_4 I_3 \\
\\
X_{44}: \quad & J_1 = I_4 \\
& J_2 = I_1^{-\frac{2}{3}} I_2 \\
& J_3 = I_1^{-\frac{1}{3}} I_3.
\end{aligned} \tag{6.4.46}$$

The ratio integrals all have the same algebras of their symmetries. There are three symmetries with the algebra $A_{3,3}$ which represents dilatations and translations in the plane. Two of the symmetries are the solution symmetries of subscript not that of the two integrals in the ratio and the third is the expected homogeneity symmetry, G_8. We note that this algebra is the same as for the ratio integral of (6.4.17) although that algebra used nonCartan symmetries. The three ratio integrals of (6.4.18) are different. Although they are three-dimensional, the algebras are $A_{3,4}$ (the algebra of the pseudo-Euclidean group $E(1,1)$) for the ratios I_1/I_2 and I_3/I_2. For I_1/I_3 it is $A_{3,8}$ (or $s\ell(2,R)$). These algebras involved one truly contact symmetry. Given the conflicting information from the three equations as far as the algebras of their ratio integrals are concerned it is not immediately evident what the general situation is. However, if we consider just the Cartan symmetries, we find the following pattern for the algebras of the ratio integrals. If we denote a ratio integral by R^k_{ij}, where i and j are the labels of the two linear integrals comprising the ratio integral and k is the order of the equation, we have

$$\begin{aligned}
R^2_{12}: &\quad \{G\} = \{y\frac{\partial}{\partial y}\}\\
R^3_{ij}: &\quad \{G\} = \{y\frac{\partial}{\partial y}\} + \{s_m\frac{\partial}{\partial y}; m \neq i,j\}\\
R^4_{ij}: &\quad \{G\} = \{y\frac{\partial}{\partial y}\} + \{s_m\frac{\partial}{\partial y}, s_n\frac{\partial}{\partial y}; m,n \neq i,j\},
\end{aligned}$$

where s_m and s_n are solutions of (6.4.18) or (6.4.23). Thus we are led to

Proposition 2: *The number of symmetries associated with the ratio of any two linear first integrals of an nth order scalar linear ordinary differential equation is $n-1$, where $n \geq 4$. The symmetries consist of the homogeneity symmetry $y\frac{\partial}{\partial y}$ and $n-2$ solution symmetries. The Lie algebra of the symmetries is $(n-2)A_1 \oplus_s A_1$.*

Proof: Since the linear integrals are each homogeneous of degree one in y, the ratio integral is of degree zero in y and so possesses $y\partial/\partial y$ as a symmetry.

Let the linear integrals be labeled $I_1 - I_n$ and the solution symmetries $G_1 - G_n$ in such a way that

$$G_i^{[n-1]}I_j = 0 \qquad i \neq j$$

(as noted above $G_i^{[n-1]}I_i = \pm 1$ depending on the value of i). Then

$$\begin{aligned} G_i^{[n-1]}\left(\frac{I_j}{I_k}\right) &= \frac{\left(G_i^{[n-1]}I_j\right)I_k - I_j\left(G_i^{[n-1]}I_k\right)}{I_k^2} \\ &= 0 \end{aligned}$$

provided

$$\left(G_i^{[n-1]}I_j\right)I_k - I_j\left(G_i^{[n-1]}I_k\right) = 0. \tag{6.4.47}$$

If $i \neq j, k$, this is true. Hence there are at least $n-2$ solution symmetries for I_j/I_k. There are also at most $n-2$ solution symmetries since $j \neq k$ and, if $i = k$ (say), (6.4.47) would require I_j to be zero. The symmetries of the ratio integral comprise solution symmetries of the form $s_m(x)\partial/\partial y$, where $s_m(x)$ is one of the fundamental solutions of $y^{(n)} = 0$, and the homogeneity symmetry, $y\partial/\partial y$. The algebra of the solution symmetries admitted by the ratio integral is $(n-2)A_1$ and, since

$$[s_m(x)\frac{\partial}{\partial y}, y\frac{\partial}{\partial y}] = s_m(x)\frac{\partial}{\partial y},$$

the algebra is $(n-2)A_1 \oplus_s A_1$. QED

We have now been able to explain the algebras of the linear and ratio integrals in general. It now remains to deal with those associated with the three symmetries of $s\ell(2,R)$. It is well to recall them. For $y'' = 0$ we have

$$\begin{aligned} G_3 &= \frac{\partial}{\partial x} & J_1 &= y' \\ G_4 &= x\frac{\partial}{\partial x} + \tfrac{1}{2}y\frac{\partial}{\partial y} & J_2 &= y'(xy'-y) = J_1J_3 \\ G_5 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} & J_3 &= xy' - y. \end{aligned}$$

J_1 and J_3 are the two linear integrals and have three Cartan symmetries whereas J_2 has only one Cartan symmetry. They are

$$\begin{aligned} J_1: \quad X_{11} &= \frac{\partial}{\partial y} \\ X_{12} &= \frac{\partial}{\partial x} \\ X_{13} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \end{aligned}$$

$$J_2: \quad X_{21} = x\frac{\partial}{\partial x} + \tfrac{1}{2}y\frac{\partial}{\partial y}$$

$$\begin{aligned} J_3: \quad X_{31} &= x\frac{\partial}{\partial y} \\ X_{32} &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y} \\ X_{33} &= x\frac{\partial}{\partial x}. \end{aligned}$$

The algebras are $A_{3,3}$ for the symmetries of J_1 and J_3 and A_1 for that of J_2.

The situation for the third order equation presents an anomaly due to the behaviour of the symmetries of $s\ell(2,R)$ and the solution integrals under the transformation (6.4.35) for which

$$\begin{array}{lll} G_5 \longrightarrow G_7 & G_6 \longrightarrow G_6 & G_7 \longrightarrow G_5 \\ I_1 \longrightarrow I_3 & I_2 \longrightarrow I_2 & I_3 \longrightarrow I_1. \end{array}$$

This does not have much effect on the single integrals, I_1, I_2 and I_3, but it does on the quadratic integral, $I_1I_3 - \frac{1}{2}I_2^2$, which is invariant under the transformation (6.4.37). (The reader will appreciate the resemblance to the generalised Kummer–Schwarz equation [71].) This is why this integral possesses the $s\ell(2,R)$ algebra of its symmetries in contrast to the $A_{3,3}$, $A_{3,4}$ and $A_{3,5}^{\frac{1}{2}}$ found for the integrals of the second and fourth order equations. In fact this labeling is a little misleading. The nonzero Lie Brackets of the three algebras, $A_{3,3}$, $A_{3,4}$

and $A_{3,5}^{\frac{1}{2}}$, are respectively

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = G_2$$

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = -G_2$$

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = aG_2 \quad 0 < |a| < 1.$$

By expanding the range of definition of a to unit modulus these three algebras are variations of the one form which we denote by $A_{3,5}^{\alpha}$ with Lie Brackets

$$[G_1, G_3] = G_1 \quad [G_2, G_3] = \alpha G_2 \quad 0 < |\alpha| \leq 1.$$

to avoid confusion with the more standard notation. In this way we obviate a plethora of labels for what is essentially the one algebra. The numerical coefficient other than 1 depends upon the order of the equation and the number of the integral being considered.

We have noted that for $n \geq 4$ the treatment of the two symmetries,$\partial/\partial x$ and $x^2\partial/\partial x + (n-1)xy\partial/\partial y$, and their associated integrals is equivalent under the transformation (6.4.37). Consequently the only two symmetries with which we must deal are $\partial/\partial x$ and $x\partial/\partial x + \frac{1}{2}(n-1)y\partial/\partial y$. It is not surprising that these should be the most complex of all the symmetries. In fact there is insufficient information in the cases considered and we add the results for $y^{(v)} = 0$ and $y^{(vi)} = 0$. We firstly note some of the properties of the linear integrals as they are defined in (6.4.36). For the linear integrals of $y^{(n)} = 0$ we have

$$G_i^{[n-1]} I_j = (-1)^{n+j}\delta_{ij} \qquad i, j = 1, n \tag{6.4.48}$$

for G_i a solution symmetry,

$$G_{n+1}^{[n-1]} I_j = \begin{cases} I_{j+1} & 1 \leq j \leq n-1 \\ 0 & j = n, \end{cases} \tag{6.4.49}$$

where $G_{n+1} = \partial/\partial x$, and

$$G_{n+2}^{[n-1]} I_j = \frac{n+1-2j}{2} I_j, \tag{6.4.50}$$

where $G_{n+2} = x\partial/\partial x + \frac{1}{2}(n-1)y\partial/\partial y$. This accounts for the two members of $s\ell(2,R)$ with which we must treat. In addition we have

$$G_{s1}^{[n-1]}I_j = [n-j]I_j, \tag{6.4.51}$$

where $G_{s1} = x\partial/\partial x + (n-1)y\partial/\partial y$, and

$$G_{s2}^{[n-1]}I_j = [n-j-1]I_j, \tag{6.4.52}$$

where $G_{s2} = x\partial/\partial x + (n-2)y\partial/\partial y$. The necessity for considering G_{s1} and G_{s2} becomes evident shortly.

As we do not have labels for all of the algebras to be considered, we simply label them by the two subalgebras they contain. The first subalgebra consists of solution symmetries and the second of elements of $s\ell(2,R)$. In Table 6.7 we list the integrals associated with G_{n+1} for $n = 4$, 5 and 6 and the algebras associated with the integrals. The A_2 algebra consists of G_{n+1} and G_{s1} for the G_{n+1} first integral and G_{n+1} and G_{s2} for the subsequent integrals. We note that all of the integrals have both G_{n+1} and G_{n+2} as symmetries. The number of solution symmetries decreases as the number of linear integrals included in each integral increases. To take the case of $y^{(v)} = 0$, for example, I_5 is invariant under four solution symmetries, the second integral contains three of the linear integrals and these can only have two solution symmetries in common. Hence the number of these drops from four to two. The third integral introduces the linear integral I_2 and the number of solution symmetries is reduced to one. This single symmetry is lost in the fourth integral when I_1 is introduced.

The pattern for the algebraic structures of the integrals associated with G_{n+1} is clear from Table 6.7. It is also evident that, as the order of the equation increases by one, an additional integral is added to the list for the previous equation and that the subscripts for the existing ones are increased by one. Hence the problem of determining the integrals is reduced to finding a homogeneous polynomial of degree $n-1$ in I_1 - I_n which is invariant under the action of $G_{n+1}^{[n-1]}$.

Table 6.7: Integrals and associated algebras of G_{n+1} for $y^{iv} = 0$, $y^{v} = 0$ and $y^{vi} = 0$

Equation	Integrals	Algebras
$y^{iv} = 0$	I_4	$3A_1 \oplus_s A_2$
	$I_2 I_4 - \frac{1}{2} I_3^2$	$A_1 \oplus_s A_2$
	$I_2 I_3 I_4 - \frac{1}{3} I_3^3 - I_1 I_4^2$	A_2
$y^{v} = 0$	I_5	$4A_1 \oplus_s A_2$
	$I_3 I_5 - \frac{1}{2} I_4^2$	$2A_1 \oplus_s A_2$
	$I_3 I_4 I_5 - \frac{1}{3} I_4^3 - I_2 I_5^2$	$A_1 \oplus_s A_2$
	$\frac{1}{2} I_3 I_4^2 I_5 - \frac{1}{8} I_4^4 - I_2 I_4 I_5^2 + I_1 I_5^3$	A_2
$y^{vi} = 0$	I_6	$5A_1 \oplus_s A_2$
	$I_4 I_6 - \frac{1}{2} I_5^2$	$3A_1 \oplus_s A_2$
	$I_4 I_4 I_6 - \frac{1}{3} I_5^3 - I_3 I_6^2$	$2A_1 \oplus_s A_2$
	$\frac{1}{2} I_4 I_5^2 I_6 - \frac{1}{8} I_5^4 - I_3 I_5 I_6^2 + I_2 I_6^3$	$A_1 \oplus_s A_2$
	$\frac{1}{6} I_4 I_5^3 I_6 - \frac{1}{30} I_5^5 - I_3 I_5^2 I_6^2 + I_2 I_5 I_6^2 - I_1 I_6^4$	A_2

Table 6.8: Integrals and associated algebras of G_{n+2} for $y^{iv} = 0$, $y^{v} = 0$ and $y^{vi} = 0$

Equation	Integrals	Algebras
$y^{iv} = 0$	$I_1 I_4$	$2A_1 \oplus_s G_6(A_{3,4})$
	$I_2 I_4^{\frac{1}{3}}$	$2A_1 \oplus_s G_6(A_{3,5}^{\frac{1}{3}})$
	$I_3 I_4^{-\frac{1}{3}}$	$2A_1 \oplus_s G_6(A_{3,5}^{\frac{1}{3}})$
$y^{v} = 0$	$I_1 I_5$	$3A_1 \oplus_s G_7$
	$I_2 I_5^{\frac{1}{2}}$	$3A_1 \oplus_s G_7$
	I_3	$4A_1 \oplus_s G_7$
	$I_4 I_5^{-\frac{1}{2}}$	$3A_1 \oplus_s G_7$
$y^{vi} = 0$	$I_1 I_6$	$4A_1 \oplus_s G_8$
	$I_2 I_6^{\frac{3}{5}}$	$4A_1 \oplus_s G_8$
	$I_3 I_6^{\frac{1}{5}}$	$4A_1 \oplus_s G_8$
	$I_4 I_6^{-\frac{1}{5}}$	$4A_1 \oplus_s G_8$
	$I_5 I_6^{-\frac{3}{5}}$	$4A_1 \oplus_s G_8$

We turn now to the self-similar symmetry, G_{n+2}. The integrals and the algebras of their symmetries are listed in Table 6.8, also for the three equations $y^{iv} = 0$, $y^{v} = 0$ and $y^{vi} = 0$. Recalling that in the case of $y'' = 0$ the integral was $I_1 I_2$ with the single Cartan symmetry $G_4 = x\partial/\partial x + \frac{1}{2}y\partial/\partial y$ and for $y''' = 0$ the integral was I_2 with the anomalous $A_{3,4}$ algebra the pattern for the self-similar symmetry is clear to see. We have

Proposition 3: *For*

$$y^{(n)} = 0 \qquad n \geq 4 \tag{6.4.53}$$

there are $n-1$ integrals associated with the symmetry

$$G = x\frac{\partial}{\partial x} + \frac{n-1}{2}y\frac{\partial}{\partial y} \tag{6.4.54}$$

of the form

$$J_i = I_i I_n^{(n+1-2i)/(n-1)}, \qquad i = 1, \ldots, n-1. \tag{6.4.55}$$

Proof: We first confirm that (6.4.54) is a symmetry of (6.4.53). The nth extension of (6.4.54) is (using (6.4.4)),

$$G^{[n]} = x\frac{\partial}{\partial x} + \sum_{i=0}^{n}\left(\frac{n-1}{2} - i\right) y^{(i)}\frac{\partial}{\partial y^{(i)}}. \tag{6.4.56}$$

Operating on (6.4.53) with (6.4.56) gives

$$\left(\frac{n-1}{2} - n\right) y^{(n)} = 0, \tag{6.4.57}$$

which, given (6.4.53) is identically satisfied.

For the determination of the first integrals associated with (6.4.54) we require the $(n-1)$th extension of G, *viz.*

$$G^{[n-1]} = x\frac{\partial}{\partial x} + \sum_{i=0}^{n-1}\left(\frac{n-1}{2} - i\right) y^{(i)}\frac{\partial}{\partial y^{(i)}}. \tag{6.4.58}$$

If we assume the form

$$I = f(x, y, y', \ldots, y^{(n-1)}) \tag{6.4.59}$$

for the first integral, the associated Lagrange's system of

$$G^{[n-1]} I = 0 \tag{6.4.60}$$

is

$$\frac{\mathrm{d}x}{x} = \frac{\mathrm{d}y}{(n-1)y/2} = \cdots = \frac{\mathrm{d}y^{(i)}}{((n-1)/2 - i)y^{(i)}} = \cdots = \frac{\mathrm{d}y^{(n-1)}}{-(n-1)y^{(n-1)}/2}. \tag{6.4.61}$$

The first set of (n) characteristics are (taking combinations of the first and ith terms in (6.4.61))

$$\begin{aligned} u_1 &= x^{(1-n)/2} y \\ u_2 &= x^{(3-n)/2} y' \\ &\vdots \\ u_i &= x^{(2i-n-1)/2} y^{(i-1)} \end{aligned}$$

$$
\begin{aligned}
u_{i+1} &= x^{(2i-n+1)/2}y^{(i)} \\
&\vdots \\
u_{n-1} &= x^{(n-3)/2}y^{(n-2)} \\
u_n &= x^{(n-1)/2}y^{(n-1)}.
\end{aligned}
\tag{6.4.62}
$$

The first integral (6.4.59) now has the form

$$
I = g(u_i), \qquad i = 1, \ldots, n \tag{6.4.63}
$$

where the u_i are given in (6.4.62).

The final requirement

$$
\frac{\mathrm{d}I}{\mathrm{d}x}\bigg|_{y^{(n)}=0} = 0 \tag{6.4.64}
$$

gives the linear first order partial differential equation

$$
\begin{aligned}
&\left[\left(\frac{1-n}{2}\right)x^{(1-n)/2-1}y + x^{(1-n)/2}y'\right]\frac{\partial f}{\partial f}u_1 \\
&+\left[\left(\frac{3-n}{2}\right)x^{(3-n)/2-1}y' + x^{(3-n)/2}y''\right]\frac{\partial f}{\partial f}u_2 + \cdots \\
&+\left[\left(\frac{2i-n-1}{2}\right)x^{(2i-n-1)/2-1}y^{(i-1)} + x^{(2i-n-1)/2}y^{(i)}\right]\frac{\partial f}{\partial f}u_i \\
&+\left[\left(\frac{2i-n+1}{2}\right)x^{(2i-n+1)/2-1}y^{(i)} + x^{(2i-n+1)/2}y^{(i+1)}\right]\frac{\partial f}{\partial f}u_{i+1} + \cdots \\
&+\left[\left(\frac{n-3}{2}\right)x^{(n-3)/2-1}y^{(n-2)} + x^{(n-3)/2}y^{(n-1)}\right]\frac{\partial f}{\partial f}u_{n-1} \\
&+\left[\left(\frac{n-1}{2}\right)x^{(n-1)/2-i}y^{(n-1)}\right]\frac{\partial f}{\partial f}u_n = 0,
\end{aligned}
\tag{6.4.65}
$$

where we have substituted for $y^{(n)}$ from (6.4.53). Multiplying (6.4.65) by x and using (6.4.62) we obtain

$$
\begin{aligned}
&\left[\left(\frac{1-n}{2}\right)u_1 + u_2\right]\frac{\partial f}{\partial f}u_1 + \left[\left(\frac{3-n}{2}\right)u_2 + u_3\right]\frac{\partial f}{\partial f}u_2 + \cdots \\
&+\left[\left(\frac{2i-n-1}{2}\right)u_i + u_{i+1}\right]\frac{\partial f}{\partial f}u_i + \left[\left(\frac{2i-n+1}{2}\right)u_{i+1} + u_{i+2}\right]\frac{\partial f}{\partial f}u_{i+1} + \cdots \\
&+\left[\left(\frac{n-3}{2}\right)u_{n-2} + u_n\right]\frac{\partial f}{\partial f}u_{n-1} + \left[\left(\frac{n-1}{2}\right)u_n\right]\frac{\partial f}{\partial f}u_n = 0
\end{aligned}
\tag{6.4.66}
$$

with the associated Lagrange's system

$$
\frac{\mathrm{d}u_1}{(1-n)u_1/2 + u_2} = \frac{\mathrm{d}u_2}{(3-n)u_2/2 + u_3} = \cdots = \frac{\mathrm{d}u_i}{(2i-n-1)u_i/2 + u_{i+1}}
$$

$$= \frac{du_{i+1}}{(2i-n+1)u_{i+1}/2+u_{i+2}} = \cdots = \frac{du_{n-1}}{(n-3)u_{n-1}/2+u_n} = \frac{du_n}{(n-1)/2u_n}. \tag{6.4.67}$$

The second set of $(n-1)$ characteristics are obtained by taking combinations of the ith and final terms in (6.4.67). Starting with $i = n-1$ we have to solve

$$\frac{du_{n-1}}{du_n} = \frac{n-3}{n-1}\frac{u_{n-1}}{u_n} + \frac{2}{n-1}. \tag{6.4.68}$$

This linear equation is easily integrated to

$$u_{n-1}u_n^{(3-n)/(n-1)} = u_n^{2/(n-1)} + v_{n-1}, \tag{6.4.69}$$

where v_{n-1} is the constant of integration which we take to be the first characteristic

$$v_{n-1} = u_{n-1}u_n^{(3-n)/(n-1)} - u_n^{2/(n-1)}. \tag{6.4.70}$$

Substituting for u_i from (6.4.62) we have

$$\begin{aligned} v_{n-1} &= x^{(n-3)/2}y^{(n-2)}\left(x^{(n-1)/2}y^{(n-1)}\right)^{(3-n)/(n-1)} - \left(x^{(n-1)/2}y^{(n-1)}\right)^{2/(n-1)} \\ &= \left(y^{n-1}\right)^{(3-n)/(n-1)}\left(y^{(n-2)} - xy^{(n-1)}\right) \\ &= -I_{n-1}I_n^{(3-n)/(n-1)}, \end{aligned} \tag{6.4.71}$$

where we have used (6.4.36) to determine the I_i, which is just (6.4.55) with $i = n-1$.

In general, the equation to be solved for the ith characteristic is

$$\frac{du_i}{du_n} = \frac{2i-n-1}{n-1}\frac{u_i}{u_n} + \frac{2}{n-1}\frac{u_{i+1}}{u_n}. \tag{6.4.72}$$

We can write u_{i+1} in terms of v_i $(i = i+1, \ldots, n-1)$ and u_n. Thus (6.4.72) is always a linear equation in u_i. However, it is not at all obvious as to how the general formula for u_{i+1} can be determined as a repeated substitution needs to be effected. It can be verified, from (6.4.67), that (6.4.55) holds by using a symbolic manipulation package, for example, `Mathematica` [211].

For our purposes we will simply demonstrate that (6.4.54) is a symmetry of (6.4.55). To this end we operate on (6.4.55) with the $(n-1)$th extension of

(6.4.54), *viz.*

$$
\begin{aligned}
G^{[n-1]}J_i &= G^{[n-1]}\left(I_i I_n^{(n+1-2i)/(n-1)}\right)\\
&= \left(G^{[n-1]}I_i\right) I_n^{(n+1-2i)/(n-1)} + \frac{n+1-2i}{n-1} I_i I_n^{(2-2i)/(n-1)}\left(G^{[n-1]}I_n\right)\\
&= I_n^{(2-2i)/(n-1)}\left[\left(G^{[n-1]}I_i\right) I_n + \frac{n+1-2i}{n-1} I_i\left(G^{[n-1]}I_n\right)\right]\\
&= \left(y^{(n-1)}\right)^{(2-2i)/(n-1)}\left\{\sum_{k=0}^{n-i}\frac{(-1)^k}{(n-i-k)!}\left[(n-i-k)x^{(n-i-k)}y^{(n-1-k)}\right.\right.\\
&\quad \left. + \left(\frac{2k-n+1}{2}\right) x^{(n-i-k)}y^{(n-1-k)}\right]\\
&\quad \left. + \left(\frac{n+1-2i}{n-1}\right)\sum_{k=0}^{n-1}\frac{(-1)^k}{(n-i-k)!}x^{(n-i-k)}\left(\frac{1-n}{2}\right)y^{(n-1-k)}\right\}\\
&= 0. \qquad (6.4.73)
\end{aligned}
$$

QED

Each of the integrals, J_i has $n-1$ symmetries with the 'algebra' $(n-2)A_1 \oplus_s G_{n+2}$ unless n is an odd integer in which case $J_{(n+1)/2}$ has the 'algebra' $(n-1)A_1 \oplus_s G_{n+2}$. The algebraic properties follow directly from the symmetries of the constituent linear integrals. The solution symmetries are those apart from the two (one if n is odd) associated with the two (one) integrals in the expression for J_i. We note that the structure given for the integrals is not unique. For example we could equally use $I_4 I_1$, $I_3 I_1^{1/3}$ and $I_2 I_1^{-1/3}$ for $y^{(iv)} = 0$.

6.4.6 Conclusion

In this section we have treated at length the integrals and the algebras of the symmetries associated with them in the case of scalar ordinary differential equations of maximal symmetry. The pattern for the general equation $y^{(n)} = 0$ has been established with the exception of a formula for the integral associated with the symmetry $\partial/\partial x$ of homogeneous degree $n-1$. To elaborate this formula does not appear to be a feasible proposition. We have seen that the cases $n = 2$ and $n = 3$ are anomalous and that the pattern is established at $n = 4$ when the only symmetries are of Cartan form. However, there is still a distinction between equations of even and odd degree of a number theoretic origin. In the context of linear equations this problem has, in a sense, been the easier one since equations of maximal symmetry are equivalent to $y^{(n)} = 0$ and so the solution symmetries are trivial to determine. This is not the case with linear equations of lower symmetry, particularly in the case of the linear equation of least symmetry. That one had to contend with the additional complications of the $s\ell(2, R)$ subalgebra did compensate for the ease of solution of the equations.

Chapter 7

The Emden–Fowler Equation

7.1 Introduction

In Chapter Two[1] the Emden–Fowler equation was given a brief treatment. Here we consider it in greater detail and with the benefit of the considerable insight which has developed as a result of several studies of the equation from the viewpoint of applications in Cosmology [119, 120]. It is an equation of great interest in its own right and has prompted investigations of extensions to more complex equations [146, 120].

The generalised Emden-Fowler equation

$$y'' + p(x)y' + r(x)y = f(x)y^n \tag{7.1.1}$$

is the simplest second order ordinary differential equation which contains a single nonlinear term. It arises frequently in the modelling of problems in one dimension and as the radial equation in spherically symmetric problems[2]. The

[1] §2.3.3.

[2] See Wong (1975) [213] for a review which, even then, was very selective in its list of references. Leach *et al* (1992) [119] note that Wong listed 144 references. One of them was in Vol **91** of the *Monthly Notices of the Royal Astronomical Society* in which Fowler (1930) [46] provides an astrophysically motivated discussion. Wong failed to mention that in the same volume there were also papers by Milne (1930, 1931) [157, 158], Fairclough (1930) [40], Hopf (1931) [79] and Russell (1931) [190] which doubtless reflect the strong interest in stellar

origin of the name is found in the works on stellar structure by Lane[3] and Emden (1907) [37] and the more mathematical analyses initiated by Fowler[4]. The more generalised form of (7.1.1) can be found in the papers of Feix and Lewis (1985) [41], Leach (1981) [106] and Basu and Ray (1990) [22]. Variations on (7.1.1) have been considered by Ranganathan[5] and further discussed by Kara and Mahomed (1992) [87], but they do not constitute any real generalisation as was clearly demonstrated by Lemmer and Leach (1994) [123]. We follow the exhaustive treatment given by Mellin *et al* (1994) [155].

Indeed the degree of generalisation is a real question. It is a well-known fact that the second and third terms of (7.1.1) can be removed by a Kummer–Liouville transformation [92, 139]

$$y(x) = u(x)v(t) \qquad\qquad t = t(x) \tag{7.1.2}$$

and this approach is found in, for example, the papers of Leach[6] and Leach *et al* (1992) [119]. In this treatment we do not remove these terms by means of a preliminary transformation for a very specific reason. By keeping these terms we find that the analysis gives rise to a particular type of third order linear differential equation which provides significant insight into the properties of the mathematical problem under consideration. This is by no means critical to the analysis, but it is a nice point to be appreciated by those who enjoy the study of the structure of differential equations.

The purpose of our treatment is to solve (7.1.1) in the sense of reduction to quadratures. To this end we examine the equation for Lie point symmetries. We are well-familiar with the concept that the possession of a point symmetry

structure at the time and the value of Emden's model.

[3]Lane (1870) [93]. His name did not stick to the equation, possibly because the American literature was not well-known in Europe at the time. It is a not unusual occurrence although these days it tends to be due to an oppositely directed lack of knowledge of the literature.

[4]Fowler (1914) [45] and (1931) [47]

[5]Ranganathan (1988) [178] and (1989) [179].

[6]Leach (1981) [106] for the particular case $n = 2$.

enables one to reduce the order of an ordinary differential equation by one. Equally, if unfortunately, well–familiar to us is that this does not guarantee integrability and, indeed, in the case of second order equations the reduction usually leads to an Abel's equation of the second kind from which little joy is to be expected. However, the exceptional case occurs, as we well know, when a second order equation possesses two symmetries, G_1 and G_2, with the property that $[G_1, G_2] = (cst)G_2$, for then reduction of order using G_2 leads to a first order equation which inherits a symmetry from G_1 and so is reducible to zeroth order, *ie* the solution is given by a quadrature[7]. Consequently we look for the subset of equations of type (7.1.1) which have two symmetries.

The requirement that (7.1.1) possess one symmetry imposes a relationship amongst the functions $p(x)$, $r(x)$ and $f(x)$ which we shall regard as a constraint on the last, *ie*, we require the nonlinearity to fit in with the linear structure[8]. Under this constraint (7.1.1) can be transformed to autonomous form by requiring that the symmetry take the form $\partial/\partial X$ in the new coordinates. This is the approach found in several papers on the subject[9]. In this chapter we investigate the conditions under which the autonomous form of (7.1.1) possesses a second point symmetry. When this does occur, it is this second symmetry which plays the rôle of G_2 referred to above. The reduction to quadratures becomes very simple.

The price of integrability is the imposition of a further constraint on the freedom of choice of the function, $f(x)$[10].

We find that the analysis naturally separates into two cases, $n \neq 2$ and $n = 2$[11]. The latter gives a richer result compared with a general value of n.

[7]In general two symmetries with this algebra reduce an nth order equation to one of $(n-2)$th order.

[8]This is simply a matter of arbitrary choice. One could equally demand that the damping term be consistent with the 'potential' terms.

[9]*cf* Leach (1981) [106], Feix and Lewis (1985) [41] and Leach *et al* (1992) [119].

[10]In the sense noted below at the end of §7.7.

[11]We exclude $n = 0, 1$ for obvious reasons. Linear equations were given more than adequate

However, when $n = -3$, the results are distinct from all other values of n. Two symmetries simply cannot occur. Either there is one and the problem of dealing with an intractable Abel's equation of the second kind remains or there are three. In the latter case the equation is of Ermakov–Pinney type[12] and the quadrature is reduced to an explicit formula.

We should note that recently Berkovich[13] has also mentioned the existence of a second symmetry, but as an aside and in a context of intent different to that of the present discussion.

7.2 The basic equations

We recall that a second order differential equation

$$N(x, y, y', y'') = 0 \tag{7.2.1}$$

possesses a Lie point symmetry

$$G = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{7.2.2}$$

if

$$G^{[2]}N\Big|_{N=0} = 0, \tag{7.2.3}$$

where

$$G^{[2]} := G + (\eta' - y'\xi')\frac{\partial}{\partial y'} + (\eta'' - 2y''\xi' - y'\xi'')\frac{\partial}{\partial y''} \tag{7.2.4}$$

is the second extension of G which is necessary to deal with the infinitesimal transformations in y' and y'' induced by the action of G. It is a standard procedure to show that the application of (7.2.4) to (7.1.1) requires that G necessarily take the form[14]

$$G = a(x)\frac{\partial}{\partial x} + (c(x)y + d(x))\frac{\partial}{\partial y}. \tag{7.2.5}$$

---

attention in the earlier chapters.

[12] Ermakov (1880) [38], Pinney (1950) [172].

[13] Berkovich (1992) [17, 18].

[14] *cf* Leach (1981) [105].

Two possible cases emerge immediately. They are

(i) $n \neq 2$, for which

$$\begin{aligned} 2c' - a'' + ap' + a'p &= 0 \\ c'' + c'p + ar' + 2a'r &= 0 \\ af' + [2a' + (n-1)c]\, f &= 0 \\ d'' + d'p + dr &= 0 \\ dnf &= 0 \end{aligned} \tag{7.2.6}$$

and

(ii) $n = 2$, for which

$$\begin{aligned} 2c' - a'' + ap' + a'p &= 0 \\ c'' + c'p + ar' + 2a'r &= 2df \\ af' + (2a' + c)\, f &= 0 \\ d'' + d'p + dr &= 0. \end{aligned} \tag{7.2.7}$$

We consider each in turn.

7.3 Case $n \neq 2$

From the last of (7.2.6) it is immediately evident that $d = 0$. The functions a and c satisfy the system of equations

$$2c' - a'' + ap' + a'p = 0 \tag{7.3.1}$$

$$c'' + c'p + ar' + 2a'r = 0 \tag{7.3.2}$$

and f is given by

$$\frac{f'}{f} = -\frac{2a' + (n-1)c}{a}. \tag{7.3.3}$$

From (7.3.1)

$$c' = \tfrac{1}{2}a'' - \tfrac{1}{2}(ap' + a'p) \tag{7.3.4}$$

so that (7.3.2) is

$$\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)' a = 0. \qquad (7.3.5)$$

Eq (7.3.5) is a linear equation of the form

$$y''' + By' + \tfrac{1}{2}B'y = 0 \qquad (7.3.6)$$

which is self-adjoint and has the maximal symmetry for a third order ordinary differential equation, *viz.* $3A_1 \oplus_s (s\ell(2, R) \oplus A_1)$[15]. Eq (7.3.6) has an integrating factor, y, and the integrated equation reduces to the Ermakov–Pinney equation[16] on the substitution $y = \rho^2$.

Eq (7.3.4) is readily integrated to give

$$c = C_0 + \tfrac{1}{2}a' - \tfrac{1}{2}ap \qquad (7.3.7)$$

and (7.3.3) becomes

$$\frac{f'}{f} = -\left\{\frac{n+3}{2}\frac{a'}{a} + (n-1)\frac{C_0}{a} - \frac{n-1}{2}p\right\}. \qquad (7.3.8)$$

7.4 The special case $n = -3$, $C_0 = 0$

From (7.3.8) it is evident that the case $n = -3$ and $C_0 = 0$ is special since then

$$\frac{f'}{f} = -2p, \qquad (7.4.1)$$

ie, f is independent of a and there are the three symmetries

$$G_i = a_i\frac{\partial}{\partial x} + \tfrac{1}{2}\left(a_i' - a_ip\right)\frac{\partial}{\partial y}, \qquad (7.4.2)$$

where the three functions, $a_i(x)$, are the linearly independent solutions of the third order equation (7.3.5). The algebra of the symmetries is $s\ell(2, R)$.

---

[15]Mahomed and Leach (1990) [153].

[16]Ermakov (1880) [38], Pinney (1950) [172].

Under the standard representation of $sl(2, R)$ the three symmetries have the form

$$\begin{aligned} G_1 &= \frac{\partial}{\partial X} \\ G_2 &= 2X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} \\ G_3 &= X^2\frac{\partial}{\partial X} + XY\frac{\partial}{\partial Y} \end{aligned}$$

and the equation is

$$Y'' = KY^{-3}. \tag{7.4.3}$$

The solution of (7.4.3) is

$$Y = \left[A + 2BX + CX^2\right]^{\frac{1}{2}}, \tag{7.4.4}$$

where $AC - B^2 = K$. This follows from the well-known result[17] that, if $u(x)$ and $v(x)$ are linearly independent solutions of

$$y'' + \omega^2(x)y = 0, \tag{7.4.5}$$

the solution of

$$y'' + \omega^2(x)y = \frac{K}{y^3} \tag{7.4.6}$$

is

$$y = \left[Au^2 + 2Buv + Cv^2\right]^{\frac{1}{2}} \tag{7.4.7}$$

with $AC - B^2 = K/W^2$, where W is the value of the Wronskian of the solutions $u(x)$ and $v(x)$.

The solution of (7.4.1) is

$$f(x) = K\exp\left[-2\int p\mathrm{d}x\right] \tag{7.4.8}$$

and (7.1.1) becomes

$$y'' + py' + ry = K\exp\left[-2\int p\mathrm{d}x\right]y^{-3}. \tag{7.4.9}$$

[17] Pinney (1950) [172].

The transformation from (7.4.9) to the autonomous form, (7.4.3), is

$$X = \frac{1}{2\sqrt{-M}}\exp\left[2\sqrt{-M}\int\frac{\mathrm{d}x}{a}\right] \tag{7.4.10}$$

$$Y = \exp\left[\sqrt{-M}X\right] ya^{-\frac{1}{2}}\exp\left[\tfrac{1}{2}\int p\mathrm{d}x\right], \tag{7.4.11}$$

where M is a parameter, and the solution follows[18].

The value of M is found from the integration of (7.3.5), *viz.*

$$\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)'a = 0.$$

When (7.3.5) is multiplied by the integrating factor a, it is trivially integrated to give

$$\tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2 = M. \tag{7.4.12}$$

7.5 Case $n = -3$ and $C_0 \neq 0$.

In this case (7.3.8) yields

$$f = K\exp\left[-2\int\left(p - \frac{2C_0}{a}\right)\mathrm{d}x\right] \tag{7.5.1}$$

and there is the single symmetry

$$G = a\frac{\partial}{\partial x} + \left(C_0 + \tfrac{1}{2}(a' - ap)\right)y\frac{\partial}{\partial y}. \tag{7.5.2}$$

However, recall that there are three functions, $a(x)$, from the solution of (7.3.5) and so three independent functions, $f(x)$, for a given C_0.

With f as in (7.5.1) the autonomous form of (7.1.1) is

$$Y'' + 2C_0Y' + \left(M + C_0^2\right)Y = KY^{-3} \tag{7.5.3}$$

and the transformation is

$$X = \int\frac{\mathrm{d}x}{a}$$

$$Y = ya^{-\frac{1}{2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\mathrm{d}x\right]. \tag{7.5.4}$$

[18]For $M > 0$ the relevant exponentials are replaced by trigonometric functions. The usual adjustment is made for the particular case $M = 0$.

Eq (7.5.3) has only the one symmetry given by $\partial/\partial X$ and the standard reduction of order via $\eta = Y$ and $\zeta = Y'$ gives

$$\zeta\zeta' + 2C_0\zeta + \left(M + C_0^2\right)\eta = K\eta^{-3} \tag{7.5.5}$$

which is an Abel's equation of the second kind and no closed form solution is apparent.

7.6 Case $n \neq -3 \text{or} 2$

We return to the consideration of the case for general index $n(\neq 0, 1, 2, -3)$. We recall that we have the equation

$$y'' + py' + ry = fy^n \tag{7.6.1}$$

with the symmetry

$$G = a\frac{\partial}{\partial x} + cy\frac{\partial}{\partial y} \tag{7.6.2}$$

and that

$$c = C_0 + \tfrac{1}{2}\left(a' - ap\right) \tag{7.6.3}$$

$$f = Ka^{-(n+3)/2}\exp\left[\frac{n-1}{2}\int\left(p - \frac{2C_0}{a}\right)\right], \tag{7.6.4}$$

where $a(x)$ is a solution of

$$\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)'a = 0 \tag{7.6.5}$$

or, equivalently, the integrated form

$$\tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2 = M. \tag{7.6.6}$$

The autonomous form of (7.6.1), with f as given in (7.6.4), is

$$Y'' + 2C_0Y' + \left(M + C_0^2\right)Y = KY^n, \tag{7.6.7}$$

where the transformation from (7.6.1) to (7.6.7) is

$$X = \int\frac{\mathrm{d}x}{a}$$

$$Y = ya^{-\frac{1}{2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\mathrm{d}x\right]. \tag{7.6.8}$$

The symmetry (7.6.2) now has the form

$$G_1 = \frac{\partial}{\partial X}. \tag{7.6.9}$$

The standard reduction of (7.6.7) under (7.6.9) leads to an Abel's equation of the second kind from which very little joy can be expected. However, it is valid to pose the question *"Are there any circumstances under which (7.6.7) has two symmetries?"*. Any symmetry of (7.6.7) apart from (7.6.9) must have the form

$$G_2 = a\frac{\partial}{\partial X} + cY\frac{\partial}{\partial Y}, \tag{7.6.10}$$

where $a(X)$ and $c(X)$ have to be determined. Such a symmetry exists if there is a nontrivial solution to the system

$$2c' - a'' + 2C_0 a' = 0 \tag{7.6.11}$$

$$(n-1)c = -2a' \tag{7.6.12}$$

$$c'' + 2C_0 c' + 2\left(M + C_0^2\right) a' = 0. \tag{7.6.13}$$

Eqq (7.6.11) and (7.6.12) combine to give

$$\frac{n+3}{n-1}a'' - 2C_0 a' = 0 \tag{7.6.14}$$

which explains the peculiarity of the $n = -3$ case. Either C_0 is zero which leads to the three symmetries given in (7.4.2) or a' is zero which leads to only one symmetry.

In the general case ($n \neq -3$) we solve (7.6.14) for a and c follows from (7.6.12). We find

$$a = A_0 + A_1 \exp\left[2C_0\left(\frac{n-1}{n+3}\right)X\right] \tag{7.6.15}$$

$$c = -\frac{4C_0 A_1}{n+3}\exp\left[2C_0\left(\frac{n-1}{n+3}\right)X\right]. \tag{7.6.16}$$

However, we also require consistency with (7.6.13) and this imposes the constraint

$$M = -C_0^2\left(\frac{n-1}{n+3}\right)^2. \tag{7.6.17}$$

Hence

$$Y'' + 2C_0Y' + \frac{8(n+1)}{(n+3)^2}C_0^2Y = KY^n \tag{7.6.18}$$

has the two symmetries

$$G_1 = \frac{\partial}{\partial X} \tag{7.6.19}$$

$$G_2 = \exp\left[2C_0\left(\frac{n-1}{n+3}\right)X\right]\left(\frac{\partial}{\partial X} - \frac{4C_0}{n+3}Y\frac{\partial}{\partial Y}\right). \tag{7.6.20}$$

Since $[G_1, G_2] = (\mathrm{cst})G_2$, reduction should be done using G_2[19]. This is made easier by the change of variables

$$\mathcal{X} = \frac{1}{2C_0}\left(\frac{n+3}{n-1}\right)\exp\left[-2C_0\left(\frac{n+1}{n-3}\right)X\right] \tag{7.6.21}$$

$$\mathcal{Y} = Y\exp\left[\frac{2C_0X}{n+3}\right] \tag{7.6.22}$$

which gives

$$G_2 = \frac{\partial}{\partial \mathcal{X}} \tag{7.6.23}$$

and

$$\mathcal{Y}'' = K\mathcal{Y}^n. \tag{7.6.24}$$

The reduction to quadratures is obvious.

For the existence of one symmetry $a(x)$ is a solution of

$$\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)'a = 0 \tag{7.6.25}$$

or its integrated version

$$\tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2 = M \tag{7.6.26}$$

and there are three such functions, $a(x)$. For the existence of two symmetries $a(x)$ is a solution of

$$\tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2 = -C_0^2\left(\frac{n-1}{n+3}\right)^2 \tag{7.6.27}$$

[19] Olver (1993)[167, p 148].

which, with $a = \rho^2$, has the Ermakov–Pinney form

$$\rho'' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)\rho = -\frac{C_0^2}{\rho^3}\left(\frac{n-1}{n+3}\right)^2, \tag{7.6.28}$$

ie, $a(x)$ is now a two parameter function (excluding C_0). Note that the two parameter function, $a(x)$, which is the solution of (7.6.27) satisfies (7.6.25), but it is not a general solution.

7.7 The case $n = 2$

Eq (7.1.1) is now

$$y'' + py' + ry = fy^2 \tag{7.7.1}$$

and it has a symmetry of the form

$$G = a(x)\frac{\partial}{\partial x} + [c(x)y + d(x)]\frac{\partial}{\partial y} \tag{7.7.2}$$

provided

$$2c' - a'' + ap' + a'p = 0 \tag{7.7.3}$$

$$c'' + c'p + ar' + 2a'r = 2df \tag{7.7.4}$$

$$af' + (2a' + c)f = 0 \tag{7.7.5}$$

$$d'' + d'p + dr = 0. \tag{7.7.6}$$

Note that the fifth equation for the $n \neq 2$ case is now not separate, but coalesces with the second. The solution of (7.7.6) gives d. From (7.7.3) we have

$$c = C_0 + \tfrac{1}{2}(a' - ap) \tag{7.7.7}$$

and from (7.7.5)

$$f = Ka^{-\frac{5}{2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]. \tag{7.7.8}$$

The remaining equation, (7.7.4), is now

$$\begin{aligned}\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)'a \\ = 2Kda^{-\frac{5}{2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right].\end{aligned} \tag{7.7.9}$$

Multiplication of (7.7.9) by a and integration gives

$$M = \tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2$$
$$-2K\int \frac{d}{a^{3/2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]. \tag{7.7.10}$$

Multiplication of (7.7.9) by $a\int d/a^{3/2}$ and integration gives

$$N = K\left\{\int \frac{d}{a^{3/2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]\right\}^2$$
$$-\int\left\{\left[\tfrac{1}{2}aa''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)aa' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2\right]\right.$$
$$\left.\times\left[\int \frac{d}{a^{3/2}}\exp\left[\tfrac{1}{2}\int\left(p - \frac{2C_0}{a}\right)\right]\right]\right\}. \tag{7.7.11}$$

The autonomous form of (7.7.1) with f as given in (7.7.8), *viz.*

$$Y'' + 2C_0Y' + \left(M + C_0^2\right)Y + N = KY^2, \tag{7.7.12}$$

is obtained by the transformation

$$X = \int \frac{\mathrm{d}x}{a}$$
$$Y = y\exp\left(-\int\frac{c}{a}\right) - \int\left[\frac{d}{a}\exp\left(-\int\frac{c}{a}\right)\right], \tag{7.7.13}$$

where, as usual,

$$c = C_0 + \tfrac{1}{2}\left(a' - ap\right). \tag{7.7.14}$$

The standard analysis of (7.7.12) shows that it has two symmetries if the constraint

$$\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) + 4KN = 0 \tag{7.7.15}$$

applies. The symmetries of

$$Y'' + 2C_0Y' + \left(M + C_0^2\right)Y - \frac{1}{4K}\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) = KY^2 \tag{7.7.16}$$

are

$$G_1 = \frac{\partial}{\partial X} \tag{7.7.17}$$
$$G_2 = \exp\left[\frac{2C_0X}{5}\right]\left\{\frac{\partial}{\partial X} - \left[\frac{4C_0}{5}Y - \frac{2C_0}{5K}\left(M + \frac{C_0^2}{25}\right)\right]\frac{\partial}{\partial Y}\right\}. \tag{7.7.18}$$

The transformation (X, Y) to $(\mathcal{X}, \mathcal{Y})$ which gives

$$G_2 = \frac{\partial}{\partial \mathcal{X}} \tag{7.7.19}$$

transforms (7.7.16) to the standard form

$$\mathcal{Y}'' = K\mathcal{Y}^2 \tag{7.7.20}$$

the solution of which is straightforward.

7.8 Conclusion

In this Chapter we have examined the Emden–Fowler equation in some detail. During the course of our investigations we have seen that for $n \neq -3$or2 the nonlinear equation

$$y'' + py' + ry = fy^n \tag{7.8.1}$$

has a Lie point symmetry provided that

$$f = Ka^{-(n+3)/2} \exp\left[\frac{n-1}{2} \int \left(p - \frac{2C_0}{a}\right)\right], \tag{7.8.2}$$

where

$$\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)' a = 0 \tag{7.8.3}$$

or

$$\tfrac{1}{2}aa'' - \tfrac{1}{4}a'^2 - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)a^2 = M. \tag{7.8.4}$$

Eq (7.8.1) has two point symmetries provided the constraint

$$M = -C_0^2\left(\frac{n-1}{n+3}\right)^2 \tag{7.8.5}$$

be satisfied. The solution of eq (7.8.1) is then trivially reduced to a quadrature. In general there are five parameters, *viz.* C_0, K, A_1, A_2 and A_3. Two symmetries exist on the family of hypersurfaces in the five dimensional parameter space defined by (7.8.5).

For $n = 2$ the equation

$$y'' + py' + ry = fy^2 \tag{7.8.6}$$

has

$$f = Ka^{-5/2} \exp\left[\tfrac{1}{2}\int \left(p - \frac{2C_0}{a}\right)\right], \tag{7.8.7}$$

where

$$\begin{aligned} &\tfrac{1}{2}a''' - \left(p' + \tfrac{1}{2}p^2 - 2r\right)a' - \tfrac{1}{2}\left(p' + \tfrac{1}{2}p^2 - 2r\right)' a \\ &= 2Kda^{-5/2} \exp\left[\tfrac{1}{2}\int \left(p - \frac{2C_0}{a}\right)\right] \end{aligned} \tag{7.8.8}$$

and

$$d'' + pd' + rd = 0. \tag{7.8.9}$$

Eq (7.8.6) has two symmetries provided the constraint

$$\left(M + \frac{C_0^2}{25}\right)\left(M + \frac{49C_0^2}{25}\right) + 4KN = 0, \tag{7.8.10}$$

where M and N are the values of two of the integrals of (7.8.8), be satisfied. Since (7.8.8) can be rewritten as a fourth order ordinary differential equation, the two symmetries exist on the family of hypersurfaces defined by (7.8.10) in an *eight* dimensional parameter space.

The case $n = -3$ has either one symmetry ($C_0 \neq 0$) or three symmetries ($C_0 = 0$). In the former case a solution in closed form via the reduction of order to an Abel's equation of the second kind is not at all obvious. In the latter case the solution is trivial as the equation is now an Ermakov–Pinney equation.

One can consider the question of the integrability of (7.1.1) from the viewpoint of Lie point symmetries of the differential equation as a matter of levels of constraints. If there is no constraint imposed on the relationship amongst $p(x), r(x)$ and $f(x)$, there is no inference provided by considerations of symmetry. When $f(x)$ is constrained by (7.6.4) and (7.6.5) [20], a single symmetry exists. When $n \neq 2$, the parameter space is five dimensional and, when $n = 2$, it is eight dimensional[21]. In both cases integrability is guaranteed on a hypersurface in each of the parameter spaces. In this respect the additional

[20]For $n \neq 2$; for $n = 2$ the relevant equations are (7.7.8) and (7.7.9).

[21]There are two parameters in $d(x)$.

constraint plays a rôle similar to that of a first integral of prescribed value in the case of configurational invariants[22].

As a final remark we note that $n = 2$ marks a transitional case from the integrable linear equation to general values of n. This is indicated by integrability[23] on a hypersurface in an eight dimensional parameter space whereas for general values of n the space is only five dimensional. The exception to this is when $n = -3$ and $C_0 = 0$. The equation is then the Ermakov–Pinney equation which can be interpreted as arising from the integration of a third order linear equation of maximal symmetry or the radial equation of a higher order system with rotational symmetry[24]

[22]Sarlet *et al* (1985) [195].

[23]In the sense of reduction to quadratures.

[24]As was proposed by Eliezer and Gray (1976) [36].

Chapter 8

The Algebra $s\ell(2, R)$

8.1 Introduction

In this chapter we consider differential equations which are invariant under the algebra $s\ell(2, R)$. This algebra has already featured several times[1] and we are quite prepared to make a case for it to be one of the more significant of the elementary algebras[2]. The first instance of the global nature of the algebra is that all linear equations with maximal symmetry of order greater than one possess it[3]. The second is that it is the algebra of the Pinney equation which plays such a significant rôle in the study of first integrals for time–dependent Hamiltonian systems[4], but which also has an important position in the algebraic theory of differential equations of maximal symmetry[5]. Finally it is the characteristic feature of Ermakov systems[6]. This chapter is mainly

[1]See §§5.1.3; 6.3; 7.4.

[2]See Znojil and Leach (1992) [217] and the references cited therein for its rôle in the context of quasi exact solutions of the Schrödinger equation.

[3]Mahomed and Leach (1990) [153].

[4]Lewis (1967, 1968) [126, 127]; Lewis and Riesenfeld (1969) [132]; Lewis and Leach (1982) [129, 130]; Eliezer and Gray (1976) [36].

[5]The third order version provides the three generators with the $s\ell(2, R)$ algebra. See §5.1.3.

[6]Certainly that is the opinion expressed in Leach (1991) [112]. Other writers on this fascinating topic have not been known to express an opinion on this rather obvious feature.

devoted to Ermakov systems, but begins with the first two features noted above.

8.2 Linear equations of maximal symmetry

The study of the symmetries of differential equations of order greater than one has been detailed in Chapters Three, Four and Five at least to the extent of all third order equations and all nth order *linear* equations. It would be useful to have a complete classification of all algebras and the differential equations associated with them, but the calculations are tedious and the algebras many. It is always dangerous to claim that equations of only limited order need be considered. It is almost a dare to the physicist, engineer or whomever to come up with an equation of higher order. As a simple example the Emden–Fowler equation of index two[7],

$$y'' + f(x)y^2 = 0, \tag{8.2.1}$$

has a symmetry in which the main function is $a(x)$. The $f(x)$ of (8.2.1) is defined in terms of $a(x)$ and a constant α. The equation satisfied by $a(x)$ is of the fourth order and is

$$2aa'''' + 5a'a''' + \alpha a''' = 0. \tag{8.2.2}$$

A knowledge of the algebraic properties of fourth order equations plays an important rôle in the discussion of the solution of a second order equation. Even though the models of natural phenomena tend to produce differential equations of the second order, the standard for all equations is not necessarily to be found in second order equations. The case of the algebras of the first integrals[8] is an obvious instance of two not being many.

Given the above considerations there is no doubt that it would be of mathematical interest to know just what are the algebraic properties of all differential

[7]The Emden–Fowler equation is studied in detail in Chapters Two and Seven. The discussion here is based on the approach found in Govinder and Leach (1994) [68].

[8]See Chapter Six.

equations. It would also be of potentially practical use. We walk before we run. The study of linear equations is easier by comparison with nonlinear equations[9] and yet the higher order equations have not been kind. However, Mahomed and Leach (1990) [153] proved that the detailed algebraic structure of an nth order equation of maximal symmetry was $nA_1 \oplus_s (s\ell(2, R) \oplus A_1)$. This required the linear differential equation to have a certain structure as far as the coefficients of its lower derivatives were concerned[10]. A curious feature is that in the case of linear equations with fewer than the maximal number of symmetries it is the $s\ell(2, R)$ symmetry which goes first.

8.3 The Pinney Equation

Pinney's half page paper[11] in which he states that the solution of

$$y'' + \omega^2(x)y = \frac{c}{y^3} \tag{8.3.1}$$

is

$$y = \left(Au^2 + 2Buv + Cv^2\right)^{1/2}, \tag{8.3.2}$$

where $u(x)$ and $v(x)$ are two linearly independent solutions of

$$z'' + \omega(x)^2 z = 0, \tag{8.3.3}$$

the constants A, B and C are related by

$$AC - B^2 = \frac{c}{W^2} \tag{8.3.4}$$

and W is the Wronskian of u and v, and omits the proof as trivial is an oft-quoted masterpiece intended to send the neophyte into paroxysms of disbelief at the blatant mendacity of mathematicians and their fellow travelers[12]. Part of

[9] As always we mean fundamentally nonlinear equations and not equations which are linearisable by means of a point transformation.

[10] In a sense this point was missed by Krause and Michel (1988) [89, 90].

[11] Pinney (1950) [172].

[12] The sentiments if not the actual words are common to all students when they first come upon this delectable result.

the problem is that the paradigmatic example presented is that of the classical time–dependent oscillator which can be solved provided it can be solved. To be honest it is the tendency of the instructor to take pity on the instructed which is the major cause of the problem. The real applications of the Pinney equation are not in the time–dependent oscillator of classical mechanics. Rather it is in time–dependent and generally nonlinear problems of virtually every other mathematically quantifiable subject in which the transformation

$$Q = \frac{q}{\rho} \qquad\qquad T = \int \rho^{-2}\mathrm{d}t \tag{8.3.5}$$

or its Hamiltonian equivalent[13] plays a useful rôle[14].

The beauty of these problems, in which the essence that remains is the Pinney equation, is that almost all conclusions needed can be obtained without any knowledge of the solution of the equation[15] subject to a numerical computation at the last but one line. Apart from the æsthetic appeal of obtaining an analytical solution to a problem[16] there can be a tremendous saving in computer time especially if the type of transformation contemplated here occurs somewhere in the central parts of the calculations relevant to the solution of the problem. The lack of necessity to resort to numbers early in the calculation means that many different cases can be subsumed into one.

The value of the Pinney equation was first realised when Lewis[17] applied Kruskal's asymptotic method (1962) [91] to the time–dependent oscillator which he was using as a model for the radial part of an axially symmetric magnetic field in the study of the motion of a charged particle in a plasma[18]. An in-

---

[13]The generalised canonical transformation (GCT) of Burgan [24, 25].

[14]See, for example, Leach (1977) [95, 96, 97] (1978) [99, 100] and (1983) [118]; Lewis and Leach (1982) [129, 130].

[15]The treatment of Berry's Phase as it affects time–dependent systems is a case in point. See Leach (1990) [111].

[16]The solution is regarded as analytical if only a 'trivial' numerical implementation to obtain a solution is required.

[17]Lewis (1967) [126], (1969) [127]; Lewis and Riesenfeld (1969) [132].

[18]The calculations, which Ralph Lewis recalls as horrendously long and complicated, were

terpretation of the equation and a number of solutions in addition to those given by Lewis were given by Eliezer and Grey (1976) [36] in a paper which has often been cited and has provided inspiration for further investigations[19]. The interpretation was that the Pinney equation is the radial equation of a two dimensional time–dependent oscillator for which the value of the conserved angular momentum is 1. It was a very pretty interpretation, but it did not stand the generalisation to three dimensions[20] in which the conserved quantity was not the square of the angular momentum in four space, but a combination of components which is best left unmentioned, such is its inelegance.

With one of those curious twists the idea of the Pinney equation as being part of a system of higher dimension was not in the least bit original even if the identification of the system was. In the coming sections we deal with the rôle of the Pinney equation and its generalisations in systems. However, to conclude this discussion we should point out that the symmetries of the Pinney equation are

$$G_1 = a_1\frac{\partial}{\partial x} + \tfrac{1}{2}a_1'y\frac{\partial}{\partial y} \tag{8.3.6}$$

$$G_2 = a_2\frac{\partial}{\partial x} + \tfrac{1}{2}a_2'y\frac{\partial}{\partial y} \tag{8.3.7}$$

$$G_3 = a_3\frac{\partial}{\partial x} + \tfrac{1}{2}a_3'y\frac{\partial}{\partial y}, \tag{8.3.8}$$

performed in Heidelberg in 1966 while he was there on sabbatical leave. The well-known Lewis invariant for the time–dependent oscillator came out as the first part of the expansion in terms of the function $\rho(t)$ in (8.3.5) above. For the first order in the expansion parameter ϵ the correction was zero. For the second order in the expansion parameter ϵ the correction was zero. For the third order in the expansion parameter ϵ the correction was zero. Ralph was becoming excited. The calculations were increasingly complex, but he persisted for a few more terms until he was convinced that the zeroth order result was *the* result. This insipid prose fails miserably to do justice to the sense of excitement of one man's discovery, particularly one with a sense of history. Unfortunately this work is not the place in which to trace the influence of Lewis' result.

[19]One need only mention the present author's works on time–dependent systems, *cf* Leach (1977) [95, 96, 97]; (1978) [99, 100] and (1983) [108].

[20]Günther and Leach (1977) [73].

where a_1, a_2 and a_3 are the linearly independent solutions of

$$a''' + 2\omega^2 a' + 2\omega\omega' a = 0^{21} \tag{8.3.9}$$

and that the algebra is $s\ell(2, R)$.

8.4 Ermakov Systems

Some eighty-six years before Ralph Lewis laboured long and hard over his exact invariant for the time-dependent oscillator high above the swiftly flowing Rhine V P Ermakov, in the style typical of a mathematician, solved the problem. In 1880 Ermakov[38] obtained a first integral for the time-dependent oscillator with equation of motion

$$\ddot{q} + \omega^2(t)q = 0 \tag{8.4.1}$$

by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3}, \tag{8.4.2}$$

eliminating the ω^2 term between (8.4.1) and (8.4.2), multiplying the resulting equation by the integrating factor $\rho\dot{q} - \dot{\rho}q$ and performing a trivial integration. The integral is

$$I = \frac{1}{2}\left[(\rho\dot{q} - \dot{\rho}q)^2 + \left(\frac{q}{\rho}\right)^2\right]. \tag{8.4.3}$$

In the West this integral is usually called the Lewis invariant after Ralph Lewis whose derivation of the integral at a later stage has been mentioned above.

Ermakov's prior claim was hindered by the lack of wide dissemination of the Reports of the University of Kiev, a general inability amongst non-Russians to read Russian, the Cold War and a time gap. Lewis' work in the sixties was in a new field and, as often happens, it was almost necessary that the result be rediscovered. Ermakov was not forgotten by those at the University of

[21] This is the same equation as that which provides the $s\ell(2, R)$part of the algebra of an nth order linear equation of maximal symmetry. See eq (5.1.19).

Kiev who had a love for differential equations. As far as can be gleaned from contemporary sources[22], L M Berkovich can be held responsible for keeping alive the knowledge of Ermakov's work and passing it on to James Reid who was a mature–age student of John Ray[23].

Ray and Reid spotted a veritably untouched lode of new results and proceeded both to popularise Ermakov systems in the West and to extend and generalise them. The first generalisation was to introduce the two equations

$$\begin{aligned} \ddot{x}+\omega^2(t)x &= \frac{1}{x^3}f(y/x) \\ \ddot{y}+\omega^2(t)y &= \frac{1}{y^3}g(y/x), \end{aligned} \tag{8.4.4}$$

where f and g are arbitrary functions of the argument y/x. The ω^2 terms are eliminated *à la Ermakov* as related above. Two points can be made on this context. In the first instance the ω^2 can be replaced by *anything*[24] and the anything can still be eliminated in the same way between the two equations. The second is that the presence of the ω^2 term is suggestive of a generality which is more apparent than real since the transformation

$$\begin{aligned} &T = \cot\left(\int \rho^{-2}\mathrm{d}t\right) \\ &X = \rho^{-1}x\csc T, \qquad Y = \rho^{-1}y\csc T, \end{aligned} \tag{8.4.5}$$

where $\rho(t)$ is a solution of the Pinney equation (8.4.2), transforms (8.4.4) to

$$\begin{aligned} \ddot{X} &= \frac{1}{X^3}f(Y/X) \\ \ddot{Y} &= \frac{1}{Y^3}g(Y/X). \end{aligned} \tag{8.4.6}$$

[22]Basically conversations with John Ray and James Reid of Clemson University, South Carolina, in 1990 and Lev Berkovich, formerly of the University of Kiev and now at Samara State University, in 1991, 1993 and 1994. The second set presented some technical difficulties since Professor Berkovich has no English and the present writer essentially has only English.

[23]Somewhere along the line the Library of Congress comes into the story, but precisely where has been forgotten by the present writer.

[24]An observation of no great originality in Leach (1991) [112] as it had already been made by Ray (1980) [180]; Ray and Reid (1980) [183, 184] and Goedert (1989) [51].

The elimination procedure yields the second order equation in two dependent variables

$$x\ddot{y} - \ddot{x}y = \frac{x}{y^3}g(y/x) - \frac{y}{x^3}f(y/x) \tag{8.4.7}$$

for which the first integral is

$$I = \frac{1}{2}\left(x\dot{y} - \dot{x}y\right)^2 + \int^{y/x}\left[uf(u) - u^{-3}g(u)\right]\mathrm{d}u. \tag{8.4.8}$$

Ermakov systems and their generalisations attracted much interest particularly in the late seventies and early eighties with significant contributions by Wollenberg, Ray, Reid and others[25] although interest continues with solidly mathematical contributions by Athorne and his collaborators[26]. As far as is known the first study of the Lie symmetries of Ermakov systems was made by Leach[27] although Korsch (1979) [88] had considered the rôle of the dynamical group $SO(2,1)$ in a study of the Hamiltonian form of (8.4.4).

If we confine our attention to the fundamental equations (8.4.6) in which the distraction of a time-dependent oscillator has been removed, there are three Lie symmetries, *viz.*

$$G_1 = \frac{\partial}{\partial t},$$

[25]Wollenberg (1980) [212]; Ray (1980) [181]; Ray and Reid (1981) [185]; Ray, Reid and Lutzky (1981) [186]; Korsch (1979) [88]; Lutzky (1980) [145]; Sarlet (1981) [192] and Sarlet and Ray (1981) [197].

[26]Athorne (1990) [11] and Athorne *et al* (1990) [14].

[27]Leach (1991) [112]. In fact the present author kept well clear of Ermakov systems for the whole period of intense attention by the various authors referenced above. It appeared that the serious work had been done and that Ermakov systems were the province of Ray and Reid. The work which completes the remainder of this chapter had its origins in a few quiet weeks between the end of lectures and the the script-marking season in late 1990. It is more than a little unfortunate that there was not some more time available as the calculations were set aside for a time. When they were resumed, some not so minor details were lost and there is an understatement of results in Leach (1991) [112]. Fortunately the lost details were recovered by K S Govinder in his M Sc dissertation of 1993 [63] and the parallel publications, Govinder *et al* (1993) [64] and Govinder and Leach (1993) [65]. See also Govinder (1992) [62].

$$G_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$
$$G_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} \tag{8.4.9}$$

which have the Lie Brackets

$$[G_1, G_2] = 2G_1, \quad [G_1, G_3] = G_2 \text{ and } [G_2, G_3] = 2G_3. \tag{8.4.10}$$

The Lie algebra is $A_{3,8}$, better known as $s\ell(2,R)$.

One observes that the equation which is integrated to give the Ermakov invariant and the invariant itself are very suggestive of a modification of the equation for angular momentum and the angular momentum itself. Recasting of the equations in terms of plane polar coordinates is very informative. The differential equations are

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3} \tag{8.4.11}$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{G(\theta)}{r^3} \tag{8.4.12}$$

and the Lie point symmetries are

$$G_1 = \frac{\partial}{\partial t}$$
$$G_2 = 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}$$
$$G_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r}. \tag{8.4.13}$$

The symmetries represent, respectively, invariance under the time–translation transformation, under the self–similar transformation and under the so–called conformal transformation[28]. The proper representation for the Ermakov system is in plane polar coordinates[29] and the Ermakov invariant takes the form

$$I = \frac{1}{2}\left(r^2\dot{\theta}\right)^2 - \int G(\theta)\mathrm{d}\theta. \tag{8.4.14}$$

[28]The three symmetries may be divided into the two classes $\{G_1, G_3\}$ and $\{G_2\}$. G_1 and G_3 are essentially the same. It is G_2 which is different. This is seen most easily in the reduction of order of the Kummer–Schwartz equation which has a double lot of $s\ell(2,R)$ symmetry. Reduction by G_1 or G_3 leads to a linear sode whereas reduction by G_2 leads to the Pinney equation.

[29]In the case of two dimensions.

The correct interpretation of the Ermakov invariant is that of a generalisation of the conservation of the magnitude of angular momentum[30].

There is a certain irony in that the angular momentum interpretation of the Lewis invariant proposed by Eliezer and Grey (1976) [36] has been revived for the situation in which the two variables can be identified with space coordinates[31].

8.5 Ermakov Systems: the algebraic approach

The algebraic approach to the study of differential equations takes a representation of an algebra and finds the general form of the differential equation which is invariant under the action of the appropriate extensions of the elements of the algebra. There is no necessity for a particular representation to have an associated differential equation of given order. A notable example is the set of operators

$$G_1 = \frac{\partial}{\partial q} \tag{8.5.1}$$

$$G_2 = 2q\frac{\partial}{\partial q} \tag{8.5.2}$$

$$G_3 = q^2\frac{\partial}{\partial q} \tag{8.5.3}$$

which does not have a scalar second order equation associated with it even though it is a representation of the algebra $s\ell(2, R)$. It does have a third order equation which is invariant under its action, *viz.*

$$2\dot{q}\dddot{q} - 3\ddot{q}^2 = f(t), \tag{8.5.4}$$

[30]In this respect the study of Ermakov systems can be placed in the context of the wider problem of the study of systems in which some feature of the conservation of angular momentum is relaxed. Several of these systems have been treated by Leach (1987) [110]; Gorringe and Leach (1987) [60] and Leach and Gorringe (1987) [116].

[31]However, this should not be interpreted as a departure from the identification of the invariant with the Hamiltonian in a suitable space–time frame when one of the variables is time. See Leach (1977) [96].

where $f(t)$ is an arbitrary function of its argument. The close relationship with the Kummer–Schwartz equation is evident. That equation has a second $s\ell(2,R)$ algebra of the form above but with q replaced by t.

We have already seen that Ermakov systems are best expressed in terms of (plane) polar coordinates. The form that $s\ell(2,R)$ takes is

$$\begin{aligned} G_1 &= \frac{\partial}{\partial t} \\ G_2 &= 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \\ G_3 &= t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r} \end{aligned} \tag{8.5.5}$$

when we take the form of the algebra of the generalised Ermakov system in which the $\omega^2 r$ term has been transformed away. The natural question of the most general form of second order equation invariant under (8.5.5) is easy to answer. One assumes an equation of the form

$$f(t, r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0 \tag{8.5.6}$$

and applies the second extension of each of (8.5.5) in turn. We find[32] that the general structure of a second order differential equation invariant under this representation of $s\ell(2,R)$ is

$$f(\theta, r^2\dot{\theta}, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta}, r^4\ddot{r}) = 0. \tag{8.5.7}$$

With an eye to applications in Classical Mechanics[33] we need two equations in two dimensions and each should have the appropriate acceleration term. Thus we consider the system of equations

$$\begin{aligned} \ddot{r} - r\dot{\theta}^2 &= \frac{1}{r^3} f(\theta, r^2\dot{\theta}) \\ r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{1}{r^3} g(\theta, r^2\dot{\theta}). \end{aligned} \tag{8.5.8}$$

[32]Govinder and Leach (1993) [65].

[33]That is where Ermakov systems started and is the primary area of our interest, if not necessarily the most valuable.

The usual Ermakov systems have f and g free of $r^2\dot{\theta}$. Note that the system (8.5.8) is not strictly equivalent to (8.5.7) and another equation of like form since inversion of the latter two to give the system (8.5.8) assumes that the implicit function theorem applies more than locally if one is to have (8.5.8) represent a meaningful classical system.

In order to make the structure of these equations more transparent we introduce new time τ defined by

$$\tau = \int r^{-2}\mathrm{d}t \tag{8.5.9}$$

and the inverse radial distance $\chi = 1/r$. If derivatives with respect to τ are denoted by $'$, $''$ *etc*, the system (8.5.8) becomes

$$\chi'' + [\theta'^2 + f(\theta,\theta')]\chi = 0 \tag{8.5.10}$$

$$\theta'' = g(\theta,\theta'). \tag{8.5.11}$$

Eq (8.5.11) is effectively a first order equation for θ' with θ as dependent variable. In terms of the Lie theory for the integration of a first order ordinary differential equation an integrating factor can be found such that

$$g(\theta,\theta') = -\theta'\frac{\partial M(\theta,\theta')}{\partial\theta} \bigg/ \frac{\partial M(\theta,\theta')}{\partial\theta'} \tag{8.5.12}$$

so that (8.5.11) integrates to

$$M(\theta,\theta') = h, \tag{8.5.13}$$

where h is a constant[34]. Given the structure assumed for g the implicit function theorem[35] guarantees inversion[36] of (8.5.13) to

$$\theta' = N(\theta,h)$$

[34]Note: We are here concerned with principle. In practice there may be technical difficulties!

[35]Brand (1955) [23, p 165].

[36]At least locally.

so that

$$\tau - \tau_0 = \int \frac{\mathrm{d}\theta}{N(\theta, h)}. \tag{8.5.14}$$

This can also be inverted[37] to give

$$\theta = J(\tau, h, \tau_0). \tag{8.5.15}$$

Now that θ is known, (8.5.10) becomes the differential equation in (χ, τ) space of the classical time-dependent linear oscillator if the coefficient of χ is positive, the free particle if zero and the linear repulsor if negative. We remark that the new time defined in (8.5.9) is almost familiar except that $r(t)$ is used instead of $\rho(t)$[38]. Another way to look at the definition of new time is as

$$\tau = \int r^{-2}\mathrm{d}t = \int r^{-2}\frac{\mathrm{d}t}{\mathrm{d}\theta}\mathrm{d}\theta = \int (r^2\dot{\theta})^{-1}\mathrm{d}\theta \tag{8.5.16}$$

so that τ is the measure of time in which the time rate of change of angle is the angular momentum. This angular momentum interpretation and resulting oscillator equation (8.5.10) remind one of the interpretation of the Pinney equation [172] by Eliezer and Gray [36].

Eq (8.5.8) reduces to that for a Newton-Cotes spiral in the case f = constant and $g = 0$[39]. The qualitative features of a spiral are maintained for the generalized Ermakov system in the cases that (i) $\theta'^2 + f(\theta, \theta') < 0$ since $\chi(\tau)$ is unbounded and so $r \to 0$ and (ii) $\theta'^2 + f(\theta, \theta') > 0$ since $\chi(\tau)$ passes through zero and so $r \to \infty$. However, it is possible to obtain closed orbits[40].

Some general comments are in order. The reduction of the nonlinear equation (8.5.8) to that of the linear time–dependent oscillator combines the method of Whittaker [209, p 83] and the introduction of the 'new time' τ. In the case that the angular momentum L ($:= r^2\dot{\theta}$) is conserved, the new time is just $L\theta$ which Whittaker uses. In the general case the procedure adopted here is very similar to that found in Athorne *et al* [14].

---

[37]Again locally.

[38]Leach *et al* (1988) [115] and Lewis and Leach (1982) [130].

[39]Whittaker (1944) [209].

[40]See Govinder and Leach (1993) [65].

8.6 Ermakov Systems in Three Dimensions

8.6.1 Introduction

In this section we extend the consideration of systems of differential equations invariant under $sl(2, R)$ to three dimensions[41]. The generalisation of the considerations of the previous section to higher dimensions is trivial. However, we find that the imposition of rotational invariance by making the invariance algebra $sl(2, R) \oplus so(3)$ yields an interesting class of differential equations which includes the classical equation for the magnetic monopole. The invariance of this equation under the elements of the algebra $sl(2, R) \oplus so(3)$ has already been reported[42]. The monopole is also known to possess a conserved vector[43]. We shall see that the general system to be discussed here possesses three such vectors and that the solution of the system of equations reduces to the determination of the three Poincaré vectors and the solution of the radial equation corresponding to (8.5.10). We should point out that in the case of the monopole the vector usually referred to as the Poincaré vector is obtained by elementary vectorial manipulation of the equation of motion. The derivation of the two other vectors which, because of their nature, we also term Poincaré vectors is by no means transparent even in this simple case. We also consider weak generalized Ermakov systems in three dimensions.

8.6.2 Equations invariant under $sl(2, R) \oplus so(3)$.

In spherical polar coordinates the equation invariant under the representation (8.5.5) of $s\ell(2, R)$ is

$$F(\theta,\, \phi,\, r^2\dot{\theta},\, r^2\dot{\phi},\, r^3\ddot{r},\, r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta},\, r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi}) = 0. \qquad (8.6.1)$$

[41] Govinder *et al* (1993) [64].

[42] Moreira *et al* (1985) [161]. They preferred to use the isomorphic algebra $so(2, 1) \oplus so(3)$.

[43] Poincaré (1896) [173].

To make reasonable sense as a system of second order differential equations in three dependent variables we need a system of three equations of the form[44]

$$r^3\ddot{r} = f(\theta,\, \phi,\, r^2\dot{\theta},\, r^2\dot{\phi}) \tag{8.6.3}$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = g(\theta,\, \phi,\, r^2\dot{\theta},\, r^2\dot{\phi}) \tag{8.6.4}$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = h(\theta,\, \phi,\, r^2\dot{\theta},\, r^2\dot{\phi})\,. \tag{8.6.5}$$

In terms of the new time T and inverse radial distance χ eqq (8.6.3 – 8.6.5) are

$$\chi'' = -f(\theta,\, \phi,\, \theta',\, \phi')\chi \tag{8.6.6}$$

$$\theta'' = g(\theta,\, \phi,\, \theta',\, \phi') \tag{8.6.7}$$

$$\psi'' = h(\theta,\, \phi,\, \theta',\, \phi'). \tag{8.6.8}$$

In contrast to the pair of equations (8.5.10) and (8.5.11) for which (8.5.11) was 'in principle' integrable and so (8.5.10) reduced to the time–dependent oscillator, the situation with the system (8.6.6 – 8.6.8) is much more complex. Given θ and ϕ as functions of T, (8.6.6) is straightforward enough as it is linear in χ.

We confine our attention to systems for which, in addition to invariance under $sl(2,R)$, there is also rotational invariance, *ie*, the system of equations is also invariant under the action of the generators of $so(3)$, *viz.*

$$G_4 = \frac{\partial}{\partial\phi} \tag{8.6.9}$$

[44]One could conceive of variations on this. By way of example – not definitive nor intended to be exclusive – (8.6.5) could be replaced by

$$H(\theta,\, \phi,\, r^2\dot{\theta},\, r^2\dot{\phi},\, I) = 0, \tag{8.6.2}$$

where I is a parameter which may be taken to be the value of a first integral. If I has a particular value, I_0, in which case it could just as well be omitted from (8.6.2), we are in the realm of configurational invariants. To keep the discussion concise we do not digress into this specialized area. The reader is referred to Sarlet *et al* (1985) [195] for a discussion of the relationship between systems of second order equations, first integrals and configurational invariants.

$$G_5 = \sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi} \tag{8.6.10}$$

$$G_6 = \cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}. \tag{8.6.11}$$

We add this constraint from considerations of possible physical applications. We find that the most general system of the form (8.6.3 – 8.6.5) invariant under $sl(2,R) \oplus so(3)$ is

$$r^3\ddot{r} = A_1(L) \tag{8.6.12}$$

$$r^4\ddot{\theta} + 2r^3\dot{r}\dot{\theta} = r^4\dot{\phi}^2\sin\theta\cos\theta + B(L)r^2\dot{\theta} - C(L)r^2\dot{\phi}\sin\theta \tag{8.6.13}$$

$$r^4\ddot{\phi} + 2r^3\dot{r}\dot{\phi} = -2r^4\dot{\theta}\dot{\phi}\cot\theta + \frac{1}{\sin\theta}\left[B(L)r^2\dot{\phi}\sin\theta + C(L)r^2\dot{\theta}\right], \tag{8.6.14}$$

where A_1, B and C are arbitrary functions of their argument, L, and

$$L^2 := r^4(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) \tag{8.6.15}$$

is the square of the magnitude of the angular momentum. The three equations (8.6.12 – 8.6.14) may be written in the compact vectorial form

$$\ddot{\mathbf{r}} = \frac{1}{r^3}\left\{A(L)\hat{\mathbf{r}} + B(L)\hat{\omega} + C(L)\hat{\mathbf{L}}\right\}, \tag{8.6.16}$$

where we have replaced $A_1(L)$ by $A(L) + L^2$. In an obvious notation $\hat{\mathbf{r}}$ and $\hat{\mathbf{L}}$ are the unit vectors in the direction of the radius vector and the angular momentum vector $\mathbf{L}\,(:= \mathbf{r} \times \dot{\mathbf{r}})$. The unit vector $\hat{\omega} := \hat{\mathbf{L}} \times \hat{\mathbf{r}}$ is in the direction of the rate of change of $\hat{\mathbf{r}}$ and is the natural generalization of $\hat{\theta}$ in plane polar coordinates.

In terms of the definition of generalized and weak generalized Ermakov systems (8.6.3 – 8.6.5) represents the three–dimensional form of the generalized Ermakov system. The addition of some extra term to (8.6.3) would be in the spirit of the meaning of weak generalized Ermakov system as given by Leach[112]. However, two points should be made. The first is that under suitable (for example analyticity) conditions (8.6.3 – 8.6.5) have integrals, *ie* constants of integration, defined over some local neighbourhood. The existence of one or more global first integrals for (8.6.4), (8.6.5) or a combination of (8.6.4) and (8.6.5) would require some constraints on the functions g and h. The second is that we have chosen the radial equation to be the one which leads to the symmetry breaking. It made sense in two dimensions as we were guaranteed the 'in principle' existence of an Ermakov–Lewis invariant provided that the system maintained $sl(2, R)$ symmetry in the angular equation. This of course is lost in the general three–dimensional case and further thought needs to be given to a correct terminology.

To conclude this subsection we make some observations about (8.6.16). For B and C zero and $A(L)$ a constant (L is conserved) we have the equation for a Newton–Cotes spiral[45] which, in essence, is the free particle in the plane with an excess or deficit of angular momentum. For A and B zero and $C(L)$ proportional to L ($= \lambda L$) a constant[46] we have the classic equation of a particle moving in the field of a magnetic monopole. In this case it is well–known that there exists the first integral

$$\mathbf{P} = \mathbf{L} + \lambda\hat{\mathbf{r}} \tag{8.6.17}$$

and the motion is on the surface of a cone of semi–vertex angle given by $\arccos(C/PL)$[47]. It is only more recently that Moreira *et al* [161] demonstrated that the algebra was $so(2,1) \oplus so(3)$ [48]. We note that the classical monopole is a Hamiltonian system and the components of the Poincaré vector possess the algebra $so(3)$ under the operation of taking the Poisson Bracket[49].

[45]Whittaker (1944) [209, p 83].

[46]L is again conserved.

[47]Poincaré (1896) [173].

[48]Which is isomorphic to $s\ell(2, R) \oplus so(3)$. We prefer this version of the algebra since the $s\ell(2, R)$ part has no real semblance to physical rotation.

[49]Mladenov (1988) [159].

8.6.3 Poincaré vector for (8.6.16)

The combination of the existence of the Poincaré vector, (8.6.17), and the symmetry algebra $s\ell(2,R) \oplus so(3)$ for the classical monopole equation

$$\ddot{\mathbf{r}} = \frac{C(L)\hat{\mathbf{L}}}{r^3} \tag{8.6.18}$$

suggests that it may be fruitful to look for a similar vector for the general equation (8.6.16). We assume the existence of a vector of Poincaré type given by

$$\mathbf{P} := I\hat{\mathbf{r}} + J\hat{\boldsymbol{\omega}} + K\hat{\mathbf{L}}, \tag{8.6.19}$$

where I, J and K are functions to be determined. Requiring that $\dot{\mathbf{P}}$ be zero when (8.6.16) is satisfied leads to the system of equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} I \\ J \\ K \end{pmatrix} = r^{-2}\begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix}\begin{pmatrix} I \\ J \\ K \end{pmatrix} \tag{8.6.20}$$

which, in terms of new time T, is

$$\begin{pmatrix} I \\ J \\ K \end{pmatrix}' = \begin{pmatrix} 0 & L & 0 \\ -L & 0 & C/L \\ 0 & -C/L & 0 \end{pmatrix}\begin{pmatrix} I \\ J \\ K \end{pmatrix}. \tag{8.6.21}$$

Equations (8.6.21) have a geometrical interpretation. They are the Serret–Frenet formulæ[50] associated with a curve of curvature L and torsion $C(L)/L$, parametrized by T. An orthonormal triad of solution vectors represents the principle triad of the curve, consisting of tangent, normal and binormal vectors.

As an aside we note that this approach is not feasible for the two dimensional system of equations since then $\dot{\hat{\mathbf{r}}}$ and $\dot{\hat{\theta}}$ are multiples of $\dot{\theta}$ and each multiple is a property of the geometry of the plane and is independent of the mechanics. The only way to make progress would be to specify the $\dot{r}$ and $\dot{\theta}$ dependence in

[50] Struik (1961) [201, p 18].

P. This has not been necessary in the present case because the dynamics is introduced via $\hat{\omega}$.

The scalar product of (8.6.16) with $\hat{\mathbf{r}}$ is

$$\ddot{r} - r(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) = r^{-3}A(L) \tag{8.6.22}$$

which in terms of χ and T is

$$\chi'' + \left\{L^2 + A(L)\right\}\chi = 0. \tag{8.6.23}$$

The vector product of $\mathbf{r}$ with (8.6.16) gives

$$\dot{\mathbf{L}} = r^{-2}\left\{B\hat{\mathbf{L}} - C\hat{\omega}\right\} \tag{8.6.24}$$

so that

$$L\dot{L} = r^{-2}BL \tag{8.6.25}$$

$$\text{or} \qquad L' = B(L) \tag{8.6.26}$$

which gives the first integral

$$M = T - \int \frac{\mathrm{d}L}{B(L)}. \tag{8.6.27}$$

This can be interpreted as an equation defining T in terms of L or L in terms of T. Naturally, if B is zero, the magnitude of the angular momentum is constant.

By virtue of (8.6.27), (8.6.23) becomes the by now familiar time–dependent oscillator which characterizes the radial equation for generalized Ermakov systems expressed in the appropriate coordinates.

In like fashion (8.6.21) is now a three–dimensional nonautonomous first order system of differential equations. Its structure is suggestive of a time–dependent oscillator written as a system of first order equations. However, the analogy only helps for a constant L. Without going into the details of the method of solution of (8.6.21) one comment is appropriate. As a three–dimensional first order linear system it has three linearly independent solutions. This means that there are in fact three 'Poincaré' vectors[51]. We, of course, would expect to find three conserved vectors as the form posited for $\mathbf{P}$ spans the space[52].

By construction $\mathbf{P}$ is a constant vector and I, J and K are not independent when the magnitude of $\mathbf{P}$ is specified[53]. Only two dependent variables are needed and we introduce the transformation[54]

$$\xi = \frac{I+iJ}{1-K} \qquad \eta = -\frac{I+iJ}{1+K}. \tag{8.6.28}$$

Together with the normalisation of $\mathbf{P}$, (8.6.28) leads to a common differential equation for ξ and η which is of Riccati form, *viz.*

$$w' + iLw + \frac{iL}{2C}(1-w^2) = 0, \tag{8.6.29}$$

where w stands for ξ and η in turn. The transformation

$$w = \frac{2iCy'}{Ly} \tag{8.6.30}$$

yields the linear second order equation

$$y'' + \left(\frac{C'}{C} - \frac{L'}{L} + iL\right)y' + \frac{L^2}{4C^2}y = 0 \tag{8.6.31}$$

which is trivially related via

$$y = \left(\frac{L}{C}\right)^{1/2} ue^{-i/2\int L} \tag{8.6.32}$$

to the standard time-dependent harmonic oscillator (TDHO)

$$u'' + \left\{\frac{1}{4}\left(\frac{C'}{C} - \frac{L'}{L} + iL\right)^2 - \tfrac{1}{2}\left(\frac{C'}{C} - \frac{L'}{L} + iL\right)' + \frac{L^2}{4C^2}\right\}u = 0. \tag{8.6.33}$$

---

[51] Some examples are given in Govinder *et al* (1993) [64].

[52] Apart from exceptional points where degeneracy occurs. One is reminded of the work of Fradkin (1965) [48] and (1967) [49] and Yoshida (1987) [215] and (1989) [216] on the existence of Laplace–Runge–Lenz vectors for central force and other three-dimensional problems. There does seem to be more of an element of reality in the present context compared with their attempts at complete generality. The main problem is that their results are very much local and useful results need to be global, *cf* Bacry (1991) [15].

[53] In the case of the single vector there is not much point to it, but, when there are three vectors spanning the space, there is no small appeal in specifying unit vectors.

[54] The so-called Weierstrass transformation of Forsyth (1904) [44, Part IV, p 280]; see also Kamke (1971) [84, p 618, 8.50].

Given the solution for u, ξ and η follow through (8.6.30) and (8.6.32). The components of $\mathbf{P}$ are given by

$$I = \frac{1-\xi\eta}{\xi-\eta}, \qquad J = \frac{i(1+\xi\eta)}{\xi-\eta}, \qquad K = \frac{\xi+\eta}{\xi-\eta}. \tag{8.6.34}$$

Needless to remark the tricky business is always the solution of the TDHO equation (8.6.33).

8.6.4 Some 'weak' considerations

Leach [112] proposed that systems with Ermakov invariants which did not possess $sl(2,R)$ symmetry should be termed 'weak'. Athorne [12], although not disagreeing with the distinction, noted that other classifications – such as Hamiltonian and non–Hamiltonian – were also important. Indeed, the point of that letter was that those (non–Hamiltonian) systems described, which had only *one* global invariant, could be understood as 'linear extensions' of an underlying Hamiltonian system with appropriate choice of time–variable. Here we wish to consider a few examples of systems for which only the angular equations possess $s\ell(2,R)$ symmetry. We maintain $so(3)$ symmetry overall so that the radial equation has the form

$$\ddot{r} - \frac{L^2}{r^3} = \frac{1}{r^3}A(L) + f(r\,,\,L)\,, \tag{8.6.35}$$

where $f(r\,,\,L)$ is the symmetry-breaking term. The analysis of the angular equations is the same which means that 'in principle' we have $L = L(T)$ and the three Poincaré vectors. In terms of the inverse radial variable χ and new time (8.6.35) is

$$\chi'' + \left[A(L) + L^2\right]\chi + \frac{1}{\chi^2}f\left(\frac{1}{\chi},\,L\right) = 0\,. \tag{8.6.36}$$

When f is zero, (8.6.36), as the equation for the TDHO, is transformed to autonomous form by the transformation

$$J = \frac{\chi}{\rho} \qquad \tau = \int \rho(T)^{-2}\mathrm{d}T\,, \tag{8.6.37}$$

where ρ is a solution of the Pinney equation [172]

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}T^2} + \left[A(L) + L^2\right]\rho = \rho^{-3} \tag{8.6.38}$$

and $L = L(T)$ through (8.6.27). One could hope that for some functions f that the transformation (8.6.37) would render it autonomous. For this to happen it is necessary for $\rho = g(L)$ and the argument of f to be $\chi^{-1}g(L)$, where g is a solution to a Pinney–type equation with L as independent variable containing $A(L)$ and $B(L)$.

8.6.5 Some open questions

In the case that (8.6.16) has a Hamiltonian representation the Poincaré vectors will have the Lie algebra $so(3)$ under the operation of taking the Poisson Bracket. The question is under what circumstances does it have a Hamiltonian? One would not expect the usual Poisson Bracket relation $[z_\mu\,,\,z_\nu]_{PB} = J_{\mu\nu}$[55], but more the monopole type of relation, *ie*, one should seek an $H : \dot{q} = [q\,,\,H]_{PB}$ and $\dot{p} = [p,\,H]_{PB}$ lead to the equations of motion). There are two cases of (8.6.16) to consider: (i) when (8.6.16) is itself Hamiltonian, and (ii) when (8.6.16) possesses a global invariant which is not, however, a Hamiltonian function for the system. In the latter case the possibility arises that this invariant is a Hamiltonian function for a subsystem on an appropriate phase space[56].

[55] $z_i = q_i, z_{n+i} = p_i$ and J is the $2n \times 2n$ symplectic matrix.

[56] Athorne (1991) [13].

8.7 Integrals with an Infinite Number of Symmetries

8.7.1 Introduction

We have seen that the study of the equation of motion for the harmonic oscillator with variable frequency, *viz.*

$$\ddot{q} + \omega^2(t)q = 0, \tag{8.7.1}$$

is part and parcel of the study of Ermakov systems and their generalisations which have the form

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^3}f(y/x) \tag{8.7.2a}$$

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3}g(y/x), \tag{8.7.2b}$$

where f and g are arbitrary functions of their argument. One can obtain a first integral for the system (8.7.2) in a manner similar to that used by Ermakov by eliminating the ω^2 term and multiplying by an integrating factor, *viz.*

$$x\dot{y} - \dot{x}y.$$

The first integral obtained is

$$I = \tfrac{1}{2}\left(x\dot{y} - \dot{x}y\right)^2 + \int^{y/x}\left[uf(u) - u^{-3}g(u)\right]\mathrm{d}u. \tag{8.7.3}$$

Leach [112] showed that the presence of the ω^2 terms in (8.7.2) is misleading. Firstly the transformation

$$T = \cot\left(\int \rho^{-2}\mathrm{d}t\right)$$

$$X = \rho^{-1}x\csc T, \qquad Y = \rho^{-1}y\csc T,$$

together with (8.4.2), transforms (8.7.2) to

$$\ddot{X} = \frac{1}{X^3}f\left(\frac{Y}{X}\right) \tag{8.7.4a}$$

$$\ddot{Y} = \frac{1}{Y^3}g\left(\frac{Y}{X}\right). \tag{8.7.4b}$$

Secondly, the ω^2 can be replaced by *anything*[57] and the terms can still be eliminated giving

$$x\ddot{y} - \ddot{x}y = \frac{x}{y^3} g\left(\frac{y}{x}\right) - \frac{y}{x} f\left(\frac{y}{x}\right). \tag{8.7.5}$$

It is only recently that the Lie algebra of the Ermakov system (8.7.2) was calculated as $s\ell(2,R)$[58]. However, in that paper, Leach also stated that the Lie algebra of the Ermakov–Lewis invariant was also $s\ell(2,R)$. That was misleading and in this section we set the record straight. We also find the general second order ordinary differential equation invariant under the Lie algebra of the Ermakov–Lewis invariant. Finally we provide the remaining three first integrals for Ermakov systems.

8.7.2 Lie algebra of Ermakov systems

We use the standard Lie method to analyse what we call the compact form of the Ermakov system (8.7.5) by requiring

$$G^{[2]} F\big|_{F=0} = 0, \tag{8.7.6}$$

where

$$F = x\ddot{y} - \ddot{x}y - \frac{x}{y^3} g\left(\frac{y}{x}\right) + \frac{y}{x} f\left(\frac{y}{x}\right) = 0. \tag{8.7.7}$$

After a lengthy calculation we find that the equation (8.7.7) has the symmetry

$$G = a\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{a}x\frac{\partial}{\partial x} + \tfrac{1}{2}\dot{a}y\frac{\partial}{\partial y}. \tag{8.7.8}$$

As a is an arbitrary function of time, (8.7.7) admits an infinite dimensional Lie algebra[59].

Recalling that (8.7.7) was obtained by eliminating the $\omega^2(t)$ terms from the equations

$$\ddot{x} + \omega^2(t)x = \frac{1}{x^3} f\left(\frac{y}{x}\right) \tag{8.7.9a}$$

[57] Ray (1980) [181]; Ray and Reid (1980) [184]; Leach (1991) [112] and Goedert (1989) [51].

[58] Leach (1991) [112].

[59] Thereby making it a somewhat larger class than that reported by Leach (1991) [112].

$$\ddot{y} + \omega^2(t)y = \frac{1}{y^3}g\left(\frac{y}{x}\right), \tag{8.7.9b}$$

we examine these equations for invariance under the symmetry (8.7.8). We do this by applying the second extension of G (8.7.8) to (8.7.9) and obtain that $a(t)$ must be a solution of the equation

$$\dddot{a} + 4\omega^2\dot{a} + 4\omega\dot{\omega}a = 0, \tag{8.7.10}$$

the third order form of the Ermakov–Pinney equation[60]. As the ω^2 can be transformed away, we can take it to be zero without any loss of generality to obtain

$$\dddot{a} = 0.$$

Hence the system (8.7.9) (with $\omega^2 = 0$) has the symmetries

$$G_1 = \frac{\partial}{\partial t} \tag{8.7.11a}$$

$$G_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{8.7.11b}$$

$$G_3 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}, \tag{8.7.11c}$$

which is just the Lie algebra $s\ell(2,R)$ as reported by Leach [112]

The above development would have been greatly simplified had we used the plane polar form of (8.7.9), *viz.*

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3} \tag{8.7.12a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{G(\theta)}{r^3}. \tag{8.7.12b}$$

The change to plane polar coordinates is suggested by the fact that the Ermakov system has an integral which is of angular momentum type[61]. The Ermakov–Lewis invariant is obtained from the angular component of (8.7.12)

---

[60]Leach (1993) [113].

[61]Leach (1991) [112].

and is

$$I = \tfrac{1}{2}\left(r^2\dot{\theta}\right)^2 - \int G(\theta)\mathrm{d}\theta$$
$$= \tfrac{1}{2}\left[p_\theta^2 + F(\theta)\right] \tag{8.7.13}$$

if the system is Hamiltonian.

We proceed with our analysis in this coordinate system and rewrite our symmetries (8.7.11) as

$$G_1 = \frac{\partial}{\partial t} \tag{8.7.14a}$$
$$G_2 = 2t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} \tag{8.7.14b}$$
$$G_3 = t^2\frac{\partial}{\partial t} + tr\frac{\partial}{\partial r}. \tag{8.7.14c}$$

8.7.3 Equations invariant under a generalised similarity symmetry

We now find the class of general second order ordinary differential equations invariant under the symmetry

$$G = a\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{a}r\frac{\partial}{\partial r} \tag{8.7.15}$$

of the Ermakov–Lewis invariant (the so-called 'inverse problem'). The procedure is the exact opposite of the standard Lie method. We require that the second extension of (8.7.15), *viz.*

$$G^{[2]} = a\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{a}r\frac{\partial}{\partial r} + \left(\tfrac{1}{2}\ddot{a}r - \tfrac{1}{2}\dot{a}\dot{r}\right)\frac{\partial}{\partial \dot{r}} - \dot{a}\dot{\theta}\frac{\partial}{\partial \dot{\theta}}$$
$$+ \left(\tfrac{1}{2}\dddot{a}\,r - \tfrac{3}{2}\dot{a}\ddot{r}\right)\frac{\partial}{\partial \ddot{r}} + \left(-2\dot{a}\ddot{\theta} - \ddot{a}\dot{\theta}\right)\frac{\partial}{\partial \ddot{\theta}} \tag{8.7.16}$$

act on some arbitrary function

$$f(t, r, \theta, \dot{r}, \dot{\theta}, \ddot{r}, \ddot{\theta}) = 0. \tag{8.7.17}$$

This results in a partial differential equation with associated Lagrange's system

$$\frac{\mathrm{d}t}{a} = \frac{\mathrm{d}r}{\tfrac{1}{2}\dot{a}r} = \frac{\mathrm{d}\theta}{0} = \frac{\dot{r}}{\tfrac{1}{2}\ddot{a}r - \tfrac{1}{2}\dot{a}\dot{r}} = \frac{\dot{\theta}}{-\dot{a}\dot{\theta}} = \frac{\ddot{r}}{\tfrac{1}{2}\dddot{a}\,r - \tfrac{3}{2}\dot{a}\ddot{r}} = \frac{\ddot{\theta}}{-2\dot{a}\ddot{\theta} - \ddot{a}\dot{\theta}}. \tag{8.7.18}$$

(We keep the $d\theta/0$ to remind us that θ is a characteristic.) The system (8.7.18) requires that (8.7.17) have the form

$$f\left(\frac{r}{a^{1/2}},\theta,\dot{r}a^{1/2}-\tfrac{1}{2}\frac{r\dot{a}}{a^{1/2}},\dot{\theta}a,\ddot{r}a^{3/2}-\tfrac{1}{2}(a\ddot{a}-\tfrac{1}{2}\dot{a}^2)r\dot{a}^{1/2},a^2\ddot{\theta}+a\dot{a}\dot{\theta}\right)=0. \tag{8.7.19}$$

If we set

$$a=\rho^2,$$

we have

$$f(u,v,w,x,y,z)=0, \tag{8.7.20}$$

where

$$\begin{aligned} u&=r/\rho, & v&=\theta\\ w&=\rho\dot{r}-\dot{\rho}r, & x&=\rho^2\dot{\theta}\\ y&=\rho^3\ddot{r}-\rho^2\ddot{\rho}r \quad \text{and} & z&=\rho^4\ddot{\theta}+2\rho^3\dot{\rho}\dot{\theta}. \end{aligned}$$

However, the original equation (8.7.5) that gave rise to the symmetry (8.7.15) is autonomous. If we impose this condition on (8.7.20), becomes

$$f\left(\theta,r^2\dot{\theta},2r^3\ddot{r},r^4\ddot{\theta}+2r^3\dot{r}\dot{\theta}\right)=0. \tag{8.7.21}$$

This is just the functional form of the general autonomous second order ordinary differential equation invariant under the Lie algebra $s\ell(2,R)$[62].

8.7.4 First integrals for the Ermakov system

We have obtained a restriction on $a(t)$ in (8.7.8) by requiring that (8.7.7) and (8.7.9) be invariant under the same symmetries. We now investigate the possibilities of restrictions imposed on $a(t)$ by the the alternative method of finding the first integrals (and in so doing attempt to find all (four) first integrals for (8.7.12)).

[62]Govinder and Leach (1993) [65].

We calculate the first integrals for an nth order system having the symmetry G by taking the $(n-1)th$ extension of G and applying it to some arbitrary function. In this way we obtain the functional form of the first integrals and by requiring that its total derivative be zero we obtain the first integrals. For our problem we require the first extension of our symmetry (8.7.8), *viz.*

$$G^{[1]} = a\frac{\partial}{\partial t} + \tfrac{1}{2}\dot{a}r\frac{\partial}{\partial r} + 0\frac{\partial}{\partial \theta} + \left(\tfrac{1}{2}\ddot{a}r - \tfrac{1}{2}\dot{a}\dot{r}\right)\frac{\partial}{\partial \dot{r}} - \dot{a}\dot{\theta}\frac{\partial}{\partial \dot{\theta}}$$

to operate on the function

$$f(t, r, \theta, \dot{r}, \dot{\theta}).$$

Proceeding in a manner similar to that in §8.5 we find that the first integrals have the functional form

$$I = f(u, v, w, x),$$

where u, v, w and x are the characteristics

$$u = ra^{-\frac{1}{2}}, \quad v = \theta, \quad w = \dot{r}a^{\frac{1}{2}} - \tfrac{1}{2}\dot{a}a^{-\frac{1}{2}}r, \quad x = r^2\dot{\theta}.$$

The requirement that

$$\dot{I} = 0$$

results in a partial differential equation with associated Lagrange's system

$$\frac{\mathrm{d}u}{w/a} = \frac{\mathrm{d}v}{x/r^2} = \frac{\mathrm{d}w}{a^{1/2}\ddot{r} - \frac{1}{2}a^{-1/2}\ddot{a}r + \frac{1}{4}a^{-3/2}\dot{a}^2 r} = \frac{\mathrm{d}x}{G(v)/r^2}, \tag{8.7.22}$$

where we have substituted for $\ddot{\theta}$ from (8.7.12). Using the second and fourth terms we obtain one first integral as

$$I_1 = \tfrac{1}{2}\left(r^2\dot{\theta}\right)^2 - \int G(\theta)\mathrm{d}\theta \tag{8.7.23}$$

which is just our Ermakov–Lewis invariant (8.7.13). This implies that (8.7.23) has an infinite–dimensional Lie algebra (as expected) as no restriction on a is imposed. It is the calculation of another first integral (using the first and third terms of (8.7.22))

$$J = a\left(\tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\int G(\theta)}{r^2}\right) - \tfrac{1}{2}\dot{a}r\dot{r} + \tfrac{1}{4}r^2\ddot{a} \tag{8.7.24}$$

that imposes a restriction on a, *viz.*

$$\begin{aligned} \left(a^{1/2}\right)^{\cdot\cdot} &= \frac{\alpha}{a^{3/2}} \\ \Longrightarrow \qquad \dddot{a} &= 0. \end{aligned}$$

Hence

$$a = \left\{1, t, t^2\right\}.$$

(Note that this development is only possible if the system is Hamiltonian (*ie* $F = -2\int G$). If we set

$$\ddot{r} - r\dot{\theta}^2 = \frac{F(\theta)}{r^3} = \frac{1}{r^3}\left(-2\int G(\theta)\right) = -\frac{\partial V}{\partial r} \tag{8.7.25a}$$

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r^3}G(\theta) = -\frac{1}{r}\frac{\partial V}{\partial \theta}, \tag{8.7.25b}$$

we find the Hamiltonian to be

$$\begin{aligned} H &= \tfrac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\int G}{r^2} \\ &= \tfrac{1}{2}\dot{r}^2 + \frac{I}{r^2}. \end{aligned} \tag{8.7.26}$$

We pause for a moment to consider the physical interpretations of both the Ermakov–Lewis invariant (8.7.23) and the Hamiltonian (8.7.26). The former has always been considered as an expression for the energy of a particle moving in a time–dependent linear field[63]. However, if we consider it to be (more correctly) the equation for the angular momentum of the particle, we realise that we need to rewrite it as

$$I' = 2I = \left(r^2\dot{\theta}\right)^2 - 2\int G(\theta)\mathrm{d}\theta. \tag{8.7.27}$$

[63]Loosely speaking. Another way to look at it is as follows. What one means by an energy integral when the particulate energy is not conserved needs careful consideration. The most satisfactory explanation, in the understanding of the present writer, is that it does represent the conservation of energy in a particular time–space in which the time variation of the frequency is zero. One could ask, 'Will the argument ever cease?'

(Later calculations also bear this out.) Our Hamiltonian can be rewritten as

$$\begin{aligned} H' = 2H &= \dot{r}^2 + \frac{I'}{r^2} \\ &= \left(\dot{r}^2 - \frac{2\int G}{r^2}\right) + \frac{L^2}{r^2} \end{aligned} \tag{8.7.28}$$

(However, we do not call H' the 'new' Hamiltonian.) which is just the Hamiltonian for the Newton–Cotes spiral[64]. Replacing J by $\frac{1}{2}J'$ gives

$$\begin{aligned} J' &= a\left[\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{2\int G}{r^2}\right] - \dot{a}r\dot{r} + \tfrac{1}{2}\ddot{a}r^2 \\ &= 2aH - \dot{a}r\dot{r} + \tfrac{1}{2}\ddot{a}r^2. \end{aligned} \tag{8.7.29}$$

The expression for our third first integral (obtained from the first and second terms of (8.7.22)) is

$$K = \frac{1}{2\sqrt{2I}}\text{arcsin}\frac{Jr^2/a - 2I}{(r^2/a)\sqrt{J^2 - 2I\alpha}} - \int \frac{1}{r^2}\mathrm{d}t, \tag{8.7.30}$$

which we can rewrite as

$$K' = 2K = \frac{1}{\sqrt{I'}}\text{arcsin}\frac{J'r^2 - 2aI'}{r^2\sqrt{J'^2 - 4\alpha I'}} - 2\int \frac{1}{r^2}\mathrm{d}t. \tag{8.7.31}$$

Note that both J' and K' depend on $a(t)$. We therefore have seven first integrals, *viz.*

$$I' = \left(r^2\dot{\theta}\right)^2 - 2\int G(\theta)\mathrm{d}\theta \tag{8.7.32}$$

$$J_1' = \dot{r}^2 + \frac{I'}{r^2} = H' \tag{8.7.33}$$

$$J_2' = tH' - r\dot{r} \tag{8.7.34}$$

$$J_3' = t^2H' - 2tr\dot{r} + r^2 \tag{8.7.35}$$

$$K_1' = \frac{1}{\sqrt{I'}}\text{arcsin}\left(1 - \frac{2I'}{r^2J_1'}\right) - 2\int \frac{1}{r^2}\mathrm{d}t \tag{8.7.36}$$

[64]Whittaker (1944) [209, p 83].

$$K_2' = \frac{1}{\sqrt{I'}}\arcsin\left[\frac{1}{\sqrt{J_1'J_3'}}\left(J_2' - \frac{2I't}{r^2}\right)\right] - 2\int\frac{1}{r^2}\mathrm{d}t \tag{8.7.37}$$

$$K_3' = \frac{1}{\sqrt{I'}}\arcsin\left(1 - \frac{2I't^2}{r^2J_3'}\right) - 2\int\frac{1}{r^2}\mathrm{d}t. \tag{8.7.38}$$

(Note that J_1' is twice the Hamiltonian.) However, we can relate three of these first integrals to the others in the following manner:

$$J_3' = \frac{J_2'^2 + I'}{J_1'}, \tag{8.7.39}$$

$$K_2' = K_1' - \sqrt{I'}\arcsin\sqrt{\frac{I'}{J_1'J_3'}} \tag{8.7.40}$$

$$K_3' = K_1' - \sqrt{I'}\arcsin\frac{2\sqrt{I'}J_2'}{J_1'J_3'}. \tag{8.7.41}$$

Thus the system (8.7.12) possesses the four independent first integrals I', J_1', J_2' and K_1' where I' has an infinite-dimensional Lie algebra given by (8.7.8), J_1', and K_1' have the symmetry $\partial/\partial t$ and J_2' has the symmetry $2t\partial/\partial t + r\partial/\partial r$. We remark that while there have been other attempts to find first integrals, other than the Ermakov–Lewis invariant, for (8.7.12) (or its cartesian equivalent)[65], these efforts impose artificial constraints on the system which the above approach does not.

8.7.5 Conclusion

The fact that the Ermakov–Lewis invariant has an infinite-dimensional Lie algebra is quite peculiar. The reason for this is not obvious, but it may be due to the presence of the Wronskian in the expression for the integral which is quite obvious when the cartesian form is used.. This is just conjecture and requires further investigation that includes other first integrals of angular momentum type.

The search for first integrals was confined to Hamiltonian Ermakov systems. This was suggested by the fact that the original Ermakov system (eqq (8.4.1)

[65]Ray and Reid (1979) [182]; Ray (1980) [180]; Goedert (1989) [51].

and (8.4.2)) was Hamiltonian. To persist beyond Hamiltonian systems when the algebra is the same would be to depart from the algebraic, in contrast to integral, theme of this work.

Epilogue

In a subject in which one is actively engaged in research and which itself is evolving through the efforts of dedicated savants it is impossible to close off a discussion with any air of finality. Even as these pages were being written new developments occurred.

There are some exciting current developments. The use of contact symmetries as the norm for third order equations has been promoted in a recent study by Abraham–Shrauner *et al* (1994) [5] in which the maximal algebra of contact symmetries has been shown to be $sp(4)$. This can be possessed by equations which are *generically* nonlinear when regarded in the light of point symmetries and point transformations. Contact symmetries can be used for linearisation purposes. Hidden symmetries and their nonlocal or contact manifestations have been used for the reduction of order by Abraham–Shrauner *et al* (1995) [6] and a systematic approach to their determination for sodes with one point symmetry has been developed by Govinder and Leach (1995) [69]. One looks for further interesting developments in this area.

A theme which has lurked on the sidelines of this work has been the connection between the Painlevé Property and Lie symmetries. It has been observed by Lemmer and Leach (1993) [122] that a class of sodes with two point symmetries (and thereby integrable) dependent upon a parameter possesses the Painlevé Property only for certain values of the parameter. In another problem Richard and Leach (1994) [187] observed that there occurred an increase in the number of point symmetries to two and the possession of the Painlevé Property when a parameter took a precise value. Govinder and Leach (1994)

[69] have reported that an equation with only one point symmetry possessed the Painlevé Property. It was found to possess a 'useful' nonlocal symmetry as well. These are simply suggestive, but a connection between the number of Lie symmetries of an equation and the possession of the Painlevé Property may exist. It is evident that the possession of a suitable number of point symmetries is not the answer. The actual criteria – assuming the correctness of the statement – remain an open question. That it may not be a valid statement is suggested by recent results of Hua *et al* (1994) [81] in which possession of the Painlevé Property was not necessary for the existence of an invariant in certain Lotka–Volterra and Quadratic Systems or, indeed, integrability in particular cases. The Lie symmetries of these systems have still to be identified.

An area which is in a poor state of development and lack of current attention is that of systems of equations. The meagre results of §3.7 call out for a systematic and concerted approach to the classification of linear systems by means of their algebras and particularly the fundamental criteria for a system to be linear. We have referred to Marc Feix' comment that even partial linearisability would be a boon to the world of large scale computation. In a sense a related subject is that of the connection between the symmetries of systems of equations and those of higher order equations from which the systems are derived or to which the systems are reduced. Here we have referred only to point symmetries. The world of nonlocal symmetries of systems of ordinary differential equations has not begun to be explored.

In terms of expanding approaches to the determination of symmetries the papers of Torrisi *et al* (1994) on equivalence symmetries [204, 205] and on the elucidation [206] of the mysteries of the potential symmetries as expounded by Bluman and Reid (1988) [22] display a delicacy which is a joy to behold.

Karlovassi

February

1995

Bibliography

[1] Abraham–Shrauner B 1993 Hidden symmetries and linearization of the modified Painlevé–Ince equation *Journal of Mathematical Physics* **34** 4809–4816

[2] Abraham–Shrauner B and Guo A 1992 Hidden symmetries associated with the Projective group of nonlinear first–order ordinary differential equations *Journal of Physics A: Mathematical and General* **25** 5597–5608

[3] Abraham–Shrauner B and Guo A 1993 Hidden and nonlocal symmetries of nonlinear differential equations in *Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics* N H Ibragimov, M Torrissi and A Valenti edd (Kluwer, Dordrecht) 1–5

[4] Abraham–Shrauner B and Leach P G L 1993 Hidden symmetries of nonlinear ordinary differential equations in *Exploiting Symmetry in Applied and Numerical Analysis* E Allgower, K George and R Miranda edd (Lecture Notes in Applied Mathematics **29** American Mathematical Society, Providence) 1–10

[5] Abraham–Shrauner B, Ratcliffe G, Govinder K S and Leach P G L 1995 Linearisation of third order ordinary differential equations by contact transformations (preprint: Department of Electrical Engineering, Washington University, St Louis)

[6] Abraham–Shrauner B, Leach P G L, Govinder K S and Ratcliffe 1995 Hidden and contact symmetries of ordinary differential equations

(preprint: Department of Electrical Engineering, Washington University, St Louis)

[7] Aguirre M and Krause J 1988 Finite point transformations and linearisation of $\ddot{x} = f(x)$ *Journal of Physics A: Mathematical and General* **21** 2841–2845

[8] Aguirre M and Krause J 1988 SL(3,R) as the group of symmetry transformations for all one–dimensional linear systems *Journal of Mathematical Physics* **29** 9–15

[9] Aguirre M and Krause J 1988 SL(3,R) as the group of symmetry transformations for all one–dimensional linear systems. II Realizations of the Lie algebra *Journal of Mathematical Physics* **29** 1746–1752

[10] Anderson R L and Davison S M 1974 A generalization of Lie's 'counting' theorem for second–order ordinary differential equations *Journal of Mathematical Analysis and Applications* **48** 301–315

[11] Athorne C 1990 Polyhedra Ermakov systems *Physics Letters A* **151** 407–411

[12] Athorne C 1991 On generalised Ermakov systems *Physics Letters A* **159** 375–378

[13] Athorne C 1992 Stability and periodicity in coupled Pinney equations *Journal of Differential Equations* **100** 82–101

[14] Athorne C, Rogers C, Ramgulam U and Osbaldestin A 1990 On linearization of the Ermakov system *Physics Letters A* **143** 207–212

[15] Bacry H 1991 Comment on the relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{\mathbf{r}}+f(r)\mathbf{L}+g(r)\mathbf{r} = \mathbf{0}$ *Journal of Physics A: Mathematical and General* **24** 1157–1159

[16] Basu A and Ray D 1990 Solution of an equation that leads to solutions of a modified Emden equation and a coupled Korteweg–de Vries equation *Journal of Mathematical Physics* **31** 1152–1154

[17] Berkovich L M 1992 Transformation of variables as a method of finding exact invariant solutions of the Kolmogorov–Petrovskiĭ–Piskunov equation and related nonlinear heat equations *Soviet Mathematics Doklady* **45** 146–151

[18] Berkovich L M 1992 Factorization as a method of finding exact invariant solutions of the Kolmogorov–Petrovskiĭ–Piskunov equation and the related Semenov and Zel'dovich equations *Soviet Mathematics Doklady* **45** 162–167

[19] Berkovič L M and Nechaevsky M L 1985 On the group properties and integrability of the Fowler–Emden equations in *Group Theoretical Methods in Physics: Vol I* M A Markov, V I Man'ko and A E Shabat edd (Harwood, Chur–London–Paris–New York) 477–488

[20] Bertrand J 1873 Théorème relatif au mouvement d'un point attiré vers un centre fixe *Comptes Rendus du Académie des Sciences de Paris* **LXXVII** 849–853

[21] Bluman G W and Kumei S 1989 *Symmetries and Differential Equations* (Springer–Verlag, New York)

[22] Bluman G W and Reid G 1988 New symmetries for ordinary differential equations *IMA Journal of Applied Mathematics* 40 87–94

[23] Brand L 1955 *Advanced Calculus* (John Wiley, New York)

[24] Burgan J–R 1978 *Sur des groupes de transformation en physique mathématique. Application aux fluides de l'éspace des phases, et à la mécanique quantique* (Ph D thesis, University of Orléans, Orléans)

[25] Burgan J–R, Feix M R, Fijalkow E, Gutierrez J and Munier A 1978 Utilisation des groupes de transformation pour la resolution des équations aux derivées partielles *Applied inverse problems* ed P C Sabatier (Springer, Berlin) 248–273

[26] Burgan J–R, Munier A, Fijalkow E and Feix M R 1982 Groupes de redimensionnement et transformations canoniques généralisees en Physique non–lineaire *Problemes Inverses* P C Sabatier ed (Cahiers Mathématiques Montpellier) 67–76

[27] Chandrasekhar S 1957 *An Introduction to the Study of Stellar Structure* (Dover, New York)

[28] Chislom J S R and Common A K 1987 A class of second–order differential equations and related first–order systems *Journal of Physics A: Mathematical and General* **20** 5459–5472

[29] Colegrave R K and Abdalla M S 1981 Field fluctuations in a Fabry–Pérot cavity in resonance with a reservoir of two–level atoms *Optica Acta* **28** 495–501

[30] Colegrave R K and Abdallah M S 1981 Harmonic oscillator with exponentially decaying mass *Journal of Physics A: Mathematical and General* **14** 2269–2280

[31] Colegrave R K and Abdallah M S 1982 *Journal of Physics A: Mathematical and General* **15** 1549–1559

[32] Conte R 1993 *Singularities of Differential Equations and Integrability* (Éditions Frontières, Gif–sur–Yvette)

[33] de la Peña–Auerbach L and Cetto A M 1976 The harmonic oscillator in a random electromagnetic field: Schrödinger's equation and radiative corrections *Revista Mexicana de Física* **25** 1–21

[34] Duarte L G S, Duarte S E S and Moreira I de C 1987 *Journal of Physics A: Mathematical and General* **20** L701–L704

[35] Dugas René 1988 *A History of Mechanics* Maddox R J trans (Dover, New York)

[36] Eliezer C J and Gray A 1976 A note on the time–dependent harmonic oscillator *SIAM Journal of Applied Mathematics* **30** 463–468

[37] Emden R 1907 *Gaskugeln, Anwedungen der mechanishen Warmentheorie auf Kosmologie und meteorologische Probleme* (B G Teubner, Leipzig)

[38] Ermakov V 1880 Second order differential equations. Conditions of complete integrability *Univ Izvestia Kiev Ser III* **9** 1–25, trans A O Harin

[39] Ervin V J, Ames W F and Adams E 1984 Nonlinear waves in pellet fusion in *Wave Phenomena: Modern Theory and Applications* C Rodgers and T B Moodie edd (North Holland, Amsterdam)

[40] Fairclough N 1930 Numerical integration of Emden's polytropic equation of index three *Monthly Notices of the Royal Astronomical Society* **91** 55–63

[41] Feix M R and Lewis H R 1985 Invariants for dissipative nonlinear systems by using rescaling *Journal of Mathematical Physics* **26** 68–73

[42] Flessas G P, Govinder K S and Leach P G L 1994 Remarks on the symmetry Lie algebras of first integrals of scalar third order equations with maximal symmetry (preprint: Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece)

[43] Flessas GP, Govinder K S and Leach P G L 1995 Characterisation of the algebraic properties of first integrals of scalar ordinary differential equations (preprint: Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece)

[44] Forsyth A R 1959 *Theory of Differential Equations* (Dover, New York)

[45] Fowler R H 1914 The form near infinity of real, continuous solutions of a certain differential equation of the second order *Quarterly Journal of Mathematics* **45** 289–350

[46] Fowler R H 1930 The solutions of Emden's and similar differential equations *Monthly Notices of the Royal Astronomical Society* **91** 63–91

[47] Fowler R H 1931 Further studies of Emden's and similar differential equations *Quarterly Journal of Mathematics (Oxford)* **2** 259–288

[48] Fradkin D M 1965 Three–dimensional isotropic oscillator and SU_3 *American Journal of Physics* **33** 207–211

[49] Fradkin D M 1967 Existence of the dynamic symmetries O_4 and SU_3 for all classical potential problems *Progress in Theoretical Physics* **37** 798–812

[50] Gat Omri 1992 Symmetry algebras of third–order ordinary differential equations *Journal of Mathematical Physics* **33** 2966–2971

[51] Goedert J 1989 Second constant of motion for generalised Ermakov systems *Physics Letters A* **136** 391–394

[52] Goldstein H 1975 Prehistory of the 'Runge–Lenz' vector *American Journal of Physics* **43** 737–738

[53] Goldstein H 1976 More on the prehistory of the Laplace or Runge–Lenz vector *American Journal of Physics* **44** 1123–1124

[54] Golubev V V 1950 *Lectures on Analytical Theory of Differential Equations* (Gosteekhizdat, Moscow–Leningrad)

[55] González–Gascón F, Ramos F B and Aguirre–Daban E 1982 On the polinomial first integrals of certain second order differential equations *Journal of Mathematical Physics* **23** 2281–22

[56] González–López A 1988 Symmetries of linear systems of second order ordinary differential equations *Journal of Mathematical Physics* **29** 1097–1105

[57] González–Gascón F and González–López A 1983 Symmetries of differential equations IV *Journal of Mathematical Physics* **24** 2006–2021

[58] Gorringe V M and Leach P G L 1988 Lie point symmetries for systems of second order linear ordinary differential equations *Quæstiones Mathematicæ* **11** 95–117

[59] Gorringe V M and Leach P G L 1991 First integrals associated with the additional symmetry of central force problems with power law potentials *Quæstiones Mathematicæ* **14** 277–289

[60] Gorringe V M and Leach P G L 1987 Conserved vectors for the autonomous system $\ddot{\mathbf{r}} + g(r,\theta)\hat{\mathbf{r}} + h(r,\theta)\hat{\theta} = 0$ *Physica D* **27** 243–248

[61] Gorringe V M and Leach P G L 1993 Kepler's third law and the oscillator's isochronism *American Journal of Physics* **61** 991–995

[62] Govinder K S 1992 *Generalized Ermakov systems and* $s\ell(2,R)$ (Honours Project, University of Natal, Durban)

[63] Govinder K S 1993 *Ermakov Systems: a group theoretic approach* (M Sc dissertation, University of Natal, Durban)

[64] Govinder K S, Athorne C and Leach P G L 1993 The algebraic structure of generalised Ermakov systems in three dimensions *Journal of Physics A: Mathematical and General* **26** 4035–4046

[65] Govinder K S and Leach P G L 1993 Generalized Ermakov systems in terms of $s\ell(2,R)$ invariants *Quæstiones Mathematicæ* **16** 405–412

[66] Govinder K S and Leach P G L 1995 The algebraic structure of the first integrals of third order linear equations *Journal of Mathematical Analysis and Applications* **193** 114–133

[67] Govinder K S and Leach P G L 1994 An elementary demonstration of the existence of $s\ell(3, R)$ symmetry for all second order linear ordinary differential equations (preprint: Department of Mathematics and Applied Mathematics, University of Natal, Durban)

[68] Govinder K S and Leach P G L 1994 Integrability analysis of the Emden–Fowler equation (preprint: Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece)

[69] Govinder K S and Leach P G L 1995 On the determination of nonlocal symmetries *Journal of Physics A: Mathematical and General* (to appear)

[70] Govinder K S and Leach P G L 1995 Group properties of first integrals of linear scalar ordinary differential equations (preprint: Department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece)

[71] Govinder K S and Leach P G L 1995 Linearisation by nonlocal contact transformation and the generalised Kummer–Schwarz equation (preprint: department of Mathematics, University of the Aegean, Karlovassi 83 200, Greece)

[72] Graham R, Roekaerts D and Tél T 1985 Integrability of Hamiltonians associated with Fokker–Plank equations *Physical Review A* **31** 3364–3375

[73] Günther N J and Leach P G L 1977 Generalised invariants for the time–dependent harmonic oscillator *Journal of Mathematical Physics* **18** 572–576

[74] Guo A and Abraham–Shrauner B 1993 Hidden symmetries of energy conserving differential equations *IMA Journal of Applied Mathematics* **51** 147–153

[75] Gutierrez J, Munier A, Burgan J–R, Feix M R and Fijalkow 1979 Non-linear transport equations: properties deduced through transformation groups *Nonlinear Problems in Theoretical Physics* Rañada A F ed (Lecture Notes in Physics **98**, Springer–Verlag, Berlin) 151–204

[76] Hall L S 1983 A theory of exact and approximate configurational invariants *Physica* **8D** 90–105

[77] Head A K 1993 LIE, a PC program for Lie analysis of differential equations *Computer Physics Communications* **77** 241–248

[78] Hénon M and Heiles C 1964 The applicability of the third integral of motion: some numerical experiments *Astronomical Journal* **69** 73–79

[79] Hopf Eberhard 1931 On Emden's differential equation *Monthly Notices of the Royal Astronomical Society* **91** 653–663

[80] Hsu L and Kamran N 1989 Classification of second–order ordinary differential equations admitting Lie groups of fibre–preserving point symmetries *Proceedings of the London Mathematical Society* **58** 387–416

[81] Hua D D, Cairó L, Feix M R, Govinder K S and Leach P G L 1994 Connection between the existence of first integrals and the Painlevé property in two–dimensional Lotka–Volterra and Quadratic Systems *Proceedings of the Royal Society, London* (to appear)

[82] Ince E L 1927 *Ordinary differential equations* (Longmans, Green and Co, London)

[83] Jauch J M and Hill E L 1940 On the problem of degeneracy in quantum mechanics *Physical Review* **57** 641–645

[84] Kamke E 1983 *Differentialgleichungen Lösungsmethoden und Lösungen* (B G Teubner, Stuttgart)

[85] Kamran N, Lamb K G and Shadwick W F 1985 The local equivalence problem for $\mathrm{d}^2y/\mathrm{d}x^2 = F(x, y, \mathrm{d}x/\mathrm{d}y)$ and the Painlevé transcendants *Journal of Differential Geometry* **22** 139–150

[86] Kamran N and Shadwick W F 1986 The solution of the Cartan Equivalence Problem for $y'' = f(x, y, y')$ under the pseudo-group $\bar{x} = \phi(x), \bar{y} = \psi(x, y)$ in *Quantum Gravity, Field Theory and Strings* Lecture Notes in Physics **246** (Springer–Verlag, New York)

[87] Kara A and Mahomed F M 1992 Equivalent Lagrangians and the solution of some classes of nonlinear equations $\ddot{q} + p(t)\dot{q} + r(t)q = \mu\dot{q}^2q^{-1} + f(t)q^n$ *International Journal for Nonlinear Mechanics* **27** 919–927

[88] Korsch H J 1979 Dynamical invariants and time–dependent harmonic systems *Physics Letters* **74A** 294–296

[89] Krause J and Michel L 1988 Symmetry algebra of ordinary differential equations of order n (Preprint: IHÉS, Bures–sur–Yvette, France)

[90] Krause J and Michel L 1988 Équations différentielles linéaires d'ordre $n \geq 2$ ayant une Algèbre de Lie de symétrie de dimension $n + 4$ *Comptes Rendus du Académie des Sciences de Paris Série I* **307** 905–910

[91] Kruskal M 1962 Asymptotic theory of Hamiltonian and other systems with all solutions nearly periodic *Journal of Mathematical Physics* **3** (1962) 806–828

[92] Kummer E E 1887 De generali quadam æquatione differentiali tertii ordinis *Journal für Mathematik* **100** 1–9 (reprinted from the Programm des evangelischen Königl. und Stadtgymnasiums in Liegnitz for the year 1834)

[93] Lane I J Homer 1869–70 On the theoretical temperature of the Sun under the hypothesis of a gaseous mass maintaining its volume by its internal

heat and depending on the laws of gases known to terrestrial experiment *American Journal of Science & Arts* **4** 57–74

[94] Laplace P S 1798 *Traité de Méchanique Céleste* (Villars, Paris) Vol 1

[95] Leach P G L 1977 On a direct method for the determination of an exact invariant for the time–dependent harmonic oscillator *Journal of the Australian Mathematical Society (Series B)* **20** 97–105

[96] Leach P G L 1977 On the theory of time–dependent linear canonical transformations as applied to Hamiltonians of harmonic oscillator type *Journal of Mathematical Physics* **18** 1608–1611

[97] Leach P G L 1977 Invariants and wave–functions for some time–dependent harmonic oscillator–type Hamiltonians *Journal of Mathematical Physics* **18** 1902–1907

[98] Leach P G L 1978 Quadratic Hamiltonians, quadratic invariants and the symmetry group $SU(n)$ *Journal of Mathematical Physics* **19** 446–451

[99] Leach P G L 1978 On a generalization of the Lewis invariant for the time–dependent harmonic oscillator *SIAM Journal of Applied Mathematics* **34** 496–503

[100] Leach P G L 1978 Note on the time–dependent damped and forced harmonic oscillator *American Journal of Physics* **46** 1247–1249

[101] Leach P G L 1980 The complete symmetry group of the one–dimensional time–dependent harmonic oscillator *Journal of Mathematical Physics* **21** 300–304

[102] Leach P G L 1980 Reflections on the symmetries and invariants of a one–dimensional linear system (Research Report, Department of Applied Mathematics, La Trobe University)

[103] Leach P G L 1980 The complete symmetry group of a forced harmonic oscillator *Journal of the Australian Mathematical Society* **22** (Series B) 12–21

[104] Leach P G L 1980 $s\ell(2, R)$ and the repulsive oscillator *Journal of Physics A: Mathematical and General* **13** 1991–2000

[105] Leach P G L 1981 Application of the Lie theory of extended groups in Hamiltonian mechanics: the oscillator and the Kepler problem *Journal of the Australian Mathematical Society (Series B)* **23** 173–186

[106] Leach P G L 1981 An exact invariant for a class of time–dependent anharmonic oscillators with cubic anharmonicity *Journal of Mathematical Physics* **22** 465–470

[107] Leach P G L 1981 A further note on the Hénon–Heiles problem *Journal of Mathematical Physics* **22** 679–682

[108] Leach P G L 1983 Harmonic oscillator with variable mass *Journal of Physics A: Mathematical and General* **16** 3261–3269

[109] Leach P G L 1985 First integrals for the modified Emden equation $\ddot{q} + \alpha(t)\dot{q} + q^n = 0$ *Journal of Mathematical Physics* **26** 2510–2514

[110] Leach P G L 1987 The first integrals and orbit equation for the Kepler problem with drag *Journal of Physics A: Mathematical and General* **20** 1997–2002

[111] Leach P G L 1990 Berry's phase and wave functions for time dependent Hamiltonian systems *Journal of Physics A: Mathematical and General* **23** 2695–2669

[112] Leach P G L 1991 Generalized Ermakov systems *Physics Letters A* **158** 102–106

[113] Leach P G L 1993 $s\ell(2, R)$, Ermakov systems and the magnetic monopole *Modern Group Analysis: Advanced analytical and computational methods in mathematical physics* N H Ibragimov, M Torrisi and A Valenti edd (Kluwer, Dordrecht) 255–264

[114] Leach P G L, Bouquet S and Dewisme A 1993 Symmetries of Hamiltonian one–dimensional systems *International Journal for Nonlinear Mechanics* **28** 705-712

[115] Leach P G L, Feix M R and Bouquet S 1988 Analysis and solution of a nonlinear second order differential equation through rescaling and through a dynamical point of view *Journal of Mathematical Physics* **29** 2563–2569

[116] Leach P G L and Gorringe V M 1987 Variations on Newton's Keplerian theme *South African Journal of Science* **83** 550–555

[117] Leach P G L and Gorringe V M 1990 The relationship between the symmetries of and the existence of conserved vectors for the equation $\ddot{\mathbf{r}} + f(r)\mathbf{L} + g(r)\hat{\mathbf{r}} = 0$ *Journal of Physics A: Mathematical and General* **23** 2765–2774

[118] Leach P G L, Govinder K S, Brazier T I and Schei C S 1993 The ubiquitous time–dependent simple harmonic oscillator *South African Journal of Science* **89** 126–130

[119] Leach P G L, Maartens R and Maharaj S D 1992 Self–similar solutions of the generalized Emden–Fowler equation *International Journal for Nonlinear Mechanics* **27** 575–582

[120] Leach P G L and Maharaj S D 1992 A first integral for a class of time–dependent anharmonic oscillators with multiple anharmonicities *Journal of Mathematical Physics* **33** 2023–2030

[121] Leach P G L and Mahomed F M 1988 Maximal subalgebra associated with a first integral of a system possessing $s\ell(3, R)$ algebra *Journal of Mathematical Physics* **29** 1807–1813

[122] Lemmer R L and Leach P G L 1993 The Painlevé test, hidden symmetries and the equation $y'' + yy' + ky^3 = 0$ *Journal of Physics A: Mathematical and General* **26** 5017–5024

[123] Lemmer R L and Leach P G L 1994 The Lie analysis and solutions for a class of second order nonlinear ordinary differential equations *International Journal of Nonlinear Mechanics* **29** 177–185

[124] Lenz W 1924 Über den Bewegungsverlauf und die Quantenzustände der gestörten Keplerbewung *Zeitscrift für Physik* **24** 107–204

[125] Lévy–Leblond J M 1971 Conservation laws for gauge–variant Lagrangians in classical mechanics *American Journal of Physics* **39** 502–506

[126] Lewis H R Jr 1967 Classical and quantum systems with time–dependent harmonic oscillator–type Hamiltonians *Physical Review Letters* **18** 510–512

[127] Lewis H R Jr 1968 Class of exact invariants for classical and quantum time–dependent harmonic oscillators *Journal of Mathematical Physics* **9** 1976–1986

[128] Lewis H R 1977 Private communication dated Oct 18 1977

[129] Lewis H R and Leach P G L 1982 Exact invariants for a class of time–dependent nonlinear Hamiltonian systems *Journal of Mathematical Physics* **23** 165–175

[130] Lewis H R and Leach P G L 1982 A direct approach to finding exact invariants for one–dimensional time–dependent classical Hamiltonians *Journal of Mathematical Physics* **23** 2371–2374

[131] Lewis H R Jr, Leach P G L, Bouquet S and Feix M R 1992 Representations of one–dimensional Hamiltonians in terms of their invariants *Journal of Mathematical Physics* **33** 591–598

[132] Lewis H R Jr and Reisenfeld W B 1969 An exact quantum theory of the time–dependent harmonic oscillator and of a charged particle in an electromagnetic field *Journal of Mathematical Physics* **10** 1458–1473

[133] Lie Sophus 1891 *Differentialgleichungen* (Teubner, Leipzig) (reprinted: Chelsea, New York 1967)

[134] Lie Sophus 1893 *Vorlesungen über Continuerliche Gruppen mit geometrischen und anderen Anwendungen* Bearbeitet und herausgegeben von Dr G Scheffers (Teubner, Liepsig) (reprinted: Chelsea, New York, 1971)

[135] Lie Sophus 1888 *Theorie der Transformationsgruppen: Vol I* Unter Mitwirkung von Pr Dr F Engel (Teubner, Leipzig) (reprinted: Chelsea, New York, 1970)

[136] Lie Sophus 1890 *Theorie der Transformationsgruppen: Vol II* Unter Mitwirkung von Pr Dr F Engel (Teubner, Leipzig) (reprinted: Chelsea, New York, 1970)

[137] Lie Sophus 1893 *Theorie der Transformationsgruppen: Vol III* Unter Mitwirkung von Pr Dr F Engel (Teubner, Leipzig) (reprinted: Chelsea, New York, 1970)

[138] Lie Sophus 1896 *Geometrie der Berührungstransformationen* (Teubner, Leipzig) (reprinted: Chelsea, New York, 1977)

[139] Liouville J 1837 Sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équatione differentielle du second ordre contenant un paramètre variable *Journal de Mathématiques pures et appliquées* **II** 16–35

[140] Littlewood J E 1963 Lorentz's pendulum problem *Annals of Physics* **21** 233–242

[141] Littlewood J E 1964 Adiabatic invariance IV: Note on a new method for Lorentz's pendulum problem *Annals of Physics* **29** 13–18

[142] Logan J D 1977 *Invariant Variational Principles* (Academic, New York)

[143] Lutzky M 1978 Noether's theorem and the time–dependent harmonic oscillator *Physics Letters* **68A** 3–4

[144] Lutzky M 1978 Symmetry groups and conserved quantities for the harmonic oscillator *Journal of Physics A: Mathematical and General* **11** 249–258

[145] Lutzky M 1980 Generalized Ray–Reid systems *Physics Letters* **78A** 301–303

[146] Maharaj S D, Leach P G L and Maartens R 1991 Shear–free spherically symmetric solutions with conformal symmetry *General Relativity and Gravitation* **23** 261–267

[147] Mahomed F M 1986 *Classification of nonlinear ordinary differential equations according to their symmetries* (M Sc dissertation, University of Witwatersrand, Johannesburg)

[148] Mahomed F M 1989 *Symmetry Lie algebras of nth order ordinary differential equations* (Ph D thesis, University of Witwatersrand, Johannesburg)

[149] Mahomed F M and Leach P G L 1985 The linear symmetries of a nonlinear differential equation *Quæstiones Mathematicæ* **8** 241–274

[150] Mahomed F M and Leach P G L 1987 Equivalence classes of differential equations in *Nonlinear Evolutions* J P Lions and F Calogero edd (World Scientific, Singapore) 791–790

[151] Mahomed F M and Leach P G L 1988 Normal forms for third order equations in *Proceedings of the Workshop on Finite Dimensional Integrable Nonlinear Dynamical Systems* P G L Leach and W–H Steeb edd (World Scientific, Singapore) 178–189

[152] Mahomed F M and Leach P G L 1989 Lie algebras associated with scalar second–order ordinary differential equations *Journal of Mathematical Physics* **30** 2770–2775

[153] Mahomed F M and Leach P G L 1990 Symmetry Lie algebras of nth order ordinary differential equations *Journal of Mathematical Analysis and Applications* **151** 80–107

[154] Mahomed F M and Leach P G L 1991 Contact symmetry algebras of scalar second–order ordinary differential equations *Journal of Mathematical Physics* **32** 2051–2055

[155] Mellin C M, Mahomed F M and Leach P G L 1994 Solution of generalized Emden–Fowler equations with two symmetries *International Journal of Nonlinear Mechanics* **29** 529–538

[156] Michel L 1988 Private communication (Dated 22:10:1988)

[157] Milne E A 1930 The analysis of stellar structure *Monthly Notices of the Royal Astronomical Society* **91** 4–55

[158] Milne E A 1931 Note on steady–state distributions which are given by solutions of Emden's differential equation *Monthly Notices of the Royal Astronomical Society* **91** 751–756

[159] Mladenov J M 1988 Scattering of charged particles off dyons *Journal of Physics A: Mathematical and General* **21** L1–L4

[160] Moreira I de C 1983 Comments on 'A direct approach to finding exact invariants for one–dimensional time–dependent classical Hamiltonians' Research Report IF/UFRJ/83/25, Universidade Federal do Rio de Janeiro,

Instituto de Física, Cidade Universitaria - Ilha do Fundaõ, Rio de Janeiro - CEP. : 21,944–RJ, Brasil

[161] Moreira I de C, Ritter O M and Santos F C 1985 Lie symmetries for the charge monopole problem *Journal of Physics A: Mathematical and General* **18** L427–L430

[162] Morozov V V 1958 Classification of six–dimensional nilpotent Lie algebras *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **5** 161–171

[163] Mubarakzyanov G M 1963 On solovable Lie algebras *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **32** 114–123

[164] Mubarakzyanov G M 1963 Classification of real structures of five–dimensional Lie algebras *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **34** 99–106

[165] Mubarakzyanov G M 1963 Classification of solvable six–dimensional Lie algebras with one nilpotent base element *Izvestia Vysshikh Uchebn Zavendeniĭ Matematika* **35** 104–116

[166] Neuman F 1987 Global theory of ordinary linear homogeneous differential equations in the real domain I *Aequationes Mathematicæ* **33** 123–149

[167] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (Springer-Verlag, New York)

[168] Ovsiannikov L V 1982 *Group Analysis of Differential Equations* (Academic Press, New York) (trans W F Ames)

[169] J Patera and P Winternitz 1975 Continuous subgroups of the fundamental groups of Physics. I. General method and the Poincaré group *Journal of Mathematical Physics* **16** 1597–1614

[170] J Patera, R T Sharp, P Winternitz and H Zassenhaus 1976 Invariants of real low dimension Lie algebras *Journal of Mathematical Physics* **17** 986–994

[171] J Patera and P Winternitz 1977 Subalgebras of real three– and four–dimensional Lie algebras *Journal of Mathematical Physics* **18** 1449–1455

[172] Pinney E 1950 The nonlinear differential equation $y''(x)+p(x)y+cy^{-3}=0$ *Proceedings of the Americam Mathematical Society* **1** 681

[173] Poincaré H 1896 Remarques sur une expérience de M Birkeland *Comptes Rendus du Académie des Sciences de Paris* **123** 530–533

[174] Popper K 1984 *The Open Society and its Enemies, Book I* (Routledge & Kegan Paul, London)

[175] Prince G E and Eliezer C J 1981 Symmetries of the time–dependent N–dimensional oscillator *Journal of Physics A: Mathematical and General* **13** 815–823

[176] Prince G E and Eliezer C J 1981 On the Lie symmetries of the classical Kepler problem *Journal of Physics A: Mathematical and General* **14** 587–596

[177] Prince G E, Leach P G L, Kalotas T M, Eliezer C J and Santilli R M 1979 The Lie and Lie–admissible symmetries of dynamical systems *Hadronic Journal* **3** 390–439

[178] Ranganathan P V 1988 Solutions of some classes of second order non–linear ordinary differential equations *International Journal for Nonlinear Mechanics* **23** 421–429

[179] Ranganathan P V 1989 A direct procedure for deriving the invariants of certain non–linear dynamical systems *International Journal for Nonlinear Mechanics* **24** 19–27

[180] Ray J R 1980 Nonlinear superposition law for generalized Ermakov systems *Physics Letters A* **78** 4–6

[181] Ray J R 1980 Time-dependent invariants with applications in Physics *Lettere al Nuovo Cimento* **27** 424–428

[182] Ray J R and Reid J L 1979 More exact invariants for the time-dependent harmonic oscillator *Physics Letters A* **71** 317–318

[183] Ray J R and Reid J L 1980 Noether's theorem and Ermakov systems for nonlinear equations of motion *Il Nuovo Cimento* **59A** 134–140

[184] Ray J R and Reid J L 1980 Ermakov systems, nonlinear superposition and solutions of nonlinear equations of motion *Journal of Mathematical Physics* **21** 1583–1585

[185] Ray J R and Reid J L 1981 Ermakov systems, velocity dependent potentials and nonlinear superposition *Journal of Mathematical Physics* **22** 91–95

[186] Ray J R, Reid J L and Lutzky M 1981 New nonlinear dynamical systems possessing invariants *Physics Letters A* **84** 42–44

[187] Richard A C and Leach P G L 1994 The integrability of $y''' + y'' + yy' = 0$ (preprint: Department of Mathematics and Applied Mathematics, University of Natal, Durban)

[188] Ross D K 1979 The behaviour of a simple pendulum with uniformly shortening string *International Journal for Nonlinear Mechanics* **14** 175–182

[189] Runge C 1923 *Vector Analysis* (Metheun, London)

[190] Russell H N 1931 Composite polytropic gas spheres *Monthly Notices of the Royal Astronomical Society* **91** 739–751

[191] Santilli R M 1978 *Foundations of Theoretical Mechanics I* (Springer-Verlag, New York)

[192] Sarlet W 1981 Further generalizations of Ray–Reid systems *Physics Letters* **82A** 161–164

[193] Sarlet W and Bahar L 1980 A direct construction of first integrals for certain nonlinear dynamical systems *International Journal for Nonlinear Mechanics* **15** 133–146

[194] Sarlet W and Cantrijn F 1981 Generalizations of Noether's theorem in classical mechanics *SIAM Review* **23** 467–494

[195] Sarlet W, Leach P G L and Cantrijn F 1985 Exact versus configurational invariants and a weak form of complete integrability *Physica* **17D** 87–98

[196] Sarlet W, Mahomed F M and Leach P G L 1987 Symmetries of nonlinear differential equations and linearisation *Journal of Physics A: Mathematical and General* **20** 277-292

[197] Sarlet W and Ray J R 1981 Classification scheme for two-dimensional Ermakov systems and generalizations *Journal of Mathematical Physics* **22** 2504–2511

[198] Seymour P W 1963 Motion of charged particles in plasmas *International Journal of Engineering Science* **1** 423–451

[199] Soares Neto J J and Vianna J D M 1988 Infinitesimal symmetry transformations of the Langevin equation *Journal of Physics A: Mathematical and General* **21** 2487–2490

[200] Steeb W-H and Euler N 1988 *Nonlinear Field Equations and Painléve Test* (World Scientific, Singapore)

[201] Struik D J 1961 *Lectures on Classical Differential Geometry* (Dover, New York)

[202] Srivastava D C 1987 Exact solutions for shear–free motion of spherically symmetric perfect fluid distributions in general relativity *Classical and Quantum Gravity* **4** 1093–1117

[203] Thompson W (Lord Kelvin) 1860–1862 On the convective equilibrium of temperature in the atmosphere *Manchester Philosophical Society Proceedings* **2** 170–176

[204] Torrisi M, Tracinà R and Valenti A 1994 A semidirect approach to the search for symmetries of a nonlinear second order system of partial differential equations *Lie Groups and their Applications* **1** 226–231

[205] Torrisi M and Tracinà 1994 Equivalence transformations for systems of first order quasilinear partial differential equations *Lie Groups and their Applications* (to appear)

[206] Torrisi M, Valenti A and Leach P G L 1995 On the symmetries of the equation $(2K(u)u')' + xu' = 0$ (preprint: Dipartimento di Matematica, Universitá di Catania)

[207] Tressé A 1896 *Determination des invariants punctuels de l'equation differentielle ordinaire de second ordre* (Hirzel, Leipsig)

[208] Werner A and Eliezer C J 1969 The lengthening pendulum *Journal of the Australian Mathematical Society* **9** 331–336

[209] Whittaker E T 1944 *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies* (Fourth edition; reprinted Dover, New York)

[210] Wilkinson G F 1965 *The Algebraic Eigenvalue Problem* (OUP, Oxford)

[211] Wolfram Research Inc 1993 *Mathematica: Version 2.2* (Wolfram Research, Urbana–Champaign, Illinois)

[212] Wollenberg L 1980 Characterisation of homogeneous quadratic invariants for one–dimensional homogeneous quadratic Hamiltonians *Physics Letters A* **79** 269–272

[213] Wong J S W 1975 On the generalized Emden–Fowler equation *SIAM Review* **17** 339–360

[214] Wulfman C E and Wybourne B G 1976 The Lie group of Newton's and Lagrange's equations for the harmonic oscillator *Journal of Physics A: Mathematical and General* **9** 507–518

[215] Yoshida T 1987 Two methods of generalization of the Laplace–Runge–Lenz vector *European Journal of Physics* **8** 258–259

[216] Yoshida T 1989 Determination of the generalized Laplace–Runge–Lenz vector by an inverse matrix method *American Journal of Physics* **57** 376–377

[217] Znojil M and Leach P G L 1992 On the elementary Schrödinger bound states and their multiplets *Journal of Mathematical Physics* **33** 2785–2794