CONFORMAL SYMMETRIES: SOLUTIONS IN TWO CLASSES OF COSMOLOGICAL MODELS

by

Manikam Moodley

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Abstract

In this thesis we study the conformal symmetries in two locally rotationally symmetric spacetimes and the homothetic symmetries of a Bianchi I spacetime. The conformal Killing equation in a class AIa spacetime (MacCallum 1980), with a G_4 of motions, is integrated to obtain the general solution subject to integrability conditions. These conditions are comprehensively analysed to determine the restrictions on the metric functions. The Killing vectors are contained in the general conformal solution. The homothetic vector is obtained and the explicit functional dependence of the metric functions determined. The class AIa spacetime does not admit a nontrivial special conformal factor. We also integrate the conformal Killing equation in the anisotropic locally rotationally symmetric spacetime of class $A3$ (MacCallum 1980), with a G_4 of motions, to obtain the general conformal Killing vector and the conformal factor subject to integrability conditions. The Killing vectors are obtained as a special case from the general conformal solution. The homothetic vector is found for a nonzero constant conformal factor. The explicit functional form of the metric functions is determined for the existence of this homothetic vector. The spatially homogeneous and anisotropic $A3$ spacetime also does not admit a nontrivial special conformal vector. In the Bianchi I spacetime, with a G_3 of motions, the conformal Killing equation is integrated for a constant conformal factor to generate the homothetic symmetries. The integrability conditions are solved to determine the functional dependence of the three time-dependent metric functions.

To my parents

whose personal sacrifices in providing and acknowledging

the need for a sound education made all possible.

Preface

The study described in this thesis was carried out in the Department of Mathematics and Applied Mathematics, University of Natal, Durban, during the period December 1990 to December 1991. This thesis was completed under the careful supervision of Dr. S. D. Maharaj.

This study represents original work by the author and it has not been submitted in any form to another University nor has it been previously published. Where use was made of the work of others it has been duly acknowledged in the text.

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Contents

0 Introduction

In the theory of special relativity time is linked with 3–dimensional space to generate a single higher entity, namely 4–dimensional spacetime. When allied with quantum mechanics, special relativity provides a satisfactory description of matter with the exception of gravitational phenomena. Events in spacetime are labelled by Minkowskian coordinates (t, x, y, z) where we have set the speed of light $c = 1$. The fundamental postulates of special relativity concern inertial frames: when particle motion is formulated in terms of this reference system then Newton's first law of uniform motion is valid. The results of special relativity can be deduced from the following two fundamental postulates:

(i) The special principle of relativity states the laws of physics are the same in all inertial frames.

(ii) The speed of light is the same in all inertial frames.

Special relativity describes physics in the absence of gravitational fields. To incorporate the effects of gravitational phenomena we need to develop the theory of general relativity. Einstein found that the curvature of spacetime does not permit a global formulation of a theory of gravity in terms of coordinate systems based on inertial frames as in special relativity. Einstein postulated that special relativity has to be supplemented with the principle of general covariance in the general theory. This principle states that a physical equation of general relativity is valid in all coordinate systems provided that:

(i) The equation preserves its form under general coordinate transformations, ie. the equation is a tensor equation.

(ii) The equation must also be valid in special relativity.

As a consequence gravitation was incorporated into the spacetime structure in the formulation of the field equations of general relativity valid in all coordinate systems using the invariant description of tensors. The mathematical structure of general relativity is a 4–dimensional differentiable manifold. Note that we can always find a local coordinate neighbourhood in which the spacetime structure of general relativity is locally similar to the spacetime of special relativity. General relativity not only reduces to special relativity in the appropriate limit but also yields Newtonian gravitation as an approximation.

Gravity is no longer regarded as a force. The gravitational field of a body is contained in the curvature of spacetime that it produces. The behaviour of the gravitational field is governed by the Einstein field equations. The field equations couple the gravitational field to the matter content. The gravitational field in the Einstein field equations is contained in the Einstein tensor which is related to the curvature of spacetime via the Riemann tensor and the Ricci scalar. The matter content is represented by the anisotropic energy–momentum tensor. The matter distribution is assumed to be relativistic fluid; for a gaseous matter distribution the field equations are supplemented with the Boltzmann equation (Israel 1972, Maartens and Maharaj 1985, Maharaj and Maartens 1986, Stewart 1971). The Einstein field equations are a set of highly nonlinear partial differential equations subject to conservation laws, namely the Bianchi identities.

Exact solutions to Einstein field equations are important because of their extensive applicability in cosmology and astrophysics. A large number of solutions are known today. Many of these solutions are not physical, ie. they do not satisfy a physical equation of state (Kramer *et al* 1980). We briefly mention a few of the classical solutions:

(i) The Schwarzschild solution gives the exterior gravitational field to a static, spherically symmetric body. This solution may be smoothly matched to the interior of static stars.

(ii) The Kerr solution describes the exterior gravitational field of a rotating body. Note that a satisfactory interior solution has not yet been found (Kramer *et al* 1980, Straumann 1984).

(iii) The Robertson–Walker model is a cosmological solution which is both isotropic and homogeneous. This solution is extensively used to model the evolution of the universe as it is simple and contains most of the features of the observed universe.

Most of the approaches followed in the past to obtain solutions have been *ad hoc.* A restriction is placed on the energy–momentum tensor and/or on the behaviour of the gravitational field on mathematical or physical grounds in an attempt to simplify the field equations. Recently the approach of various authors is to impose a symmetry requirement on the manifold. In this thesis we seek solutions in a class of spacetimes by imposing a conformal symmetry on the spacetime manifold, ie. the differentiable manifold is invariant under the action of a group of the conformal symmetries. The spacetimes we consider have attracted much attention because of their physical importance and the field equations have the advantage of being mathematically tractable. They are often used in the study of anisotropic cosmological models.

In various attempts to solve the highly nonlinear field equations of general relativity it is often assumed that the spacetime admits symmetries. Recently a number of solutions have been found in various models with the assumption that the spacetime is invariant under a conformal Killing vector. The conformal Killing vectors in Minkowski spacetime have been known for some time and are listed by Choquet–Bruhat *et al* (1977). Maartens and Maharaj (1986) found the G_{15} of conformal Killing vectors in Robertson–Walker models. For a recent paper on static solutions to field equations with conformal symmetries see Maartens and Maharaj (1990). For the nonstatic case the reader is referred to Herrera and Ponce de Leon (1985a, 1985b, 1985c). Conformal symmetries in nonstatic spherically symmetric spacetimes have been considered by Dyer *et al* (1987) and Maharaj *et al* (1991). Also Maartens and Maharaj (1991) have found the conformal symmetries of *pp*–wave spacetimes which may be interpreted as plane–fronted gravitational waves with parallel rays. Maartens *et al* (1986) considered conformal Killing vectors in anisotropic fluid spacetimes and Mason and Tsamparlis (1985) analysed spacelike conformal symmetries. Spacetimes admitting inheriting conformal Killing vector fields have been studied by Coley and Tupper (1989, 1990a). Papers which consider a conformal Killing vector parallel to the fluid four–velocity include Coley and Tupper (1990b), Oliver and Davis (1977) and Duggal (1987). In addition it is worthwhile mentioning that conformal Killing vectors may be important in the study of relativistic kinetic theory (see an application for massless particles in equilibrium by Israel (1972)).

In chapter 1 we briefly consider fundamental concepts of differential geometry in general relativity. In particular we study those aspects which are necessary for the development of conformal symmetries of this thesis. We begin by introducing manifolds, coordinate transformations and tensor fields. The Lie derivative is defined and it properties are listed. Lie algebras, the Lie bracket and the Jacobi identity are also defined. The connection, the covariant derivative, the curvature tensor and the Einstein field equations are briefly introduced. The conformal Killing equation is defined on a differentiable manifold and the restrictions on the connection and associated quantities due to a conformal symmetry are given. The special cases of Killing, homothetic, special and nonspecial conformal Killing vectors are noted.

In chapter 2 we determine the conformal symmetries of a locally rotationally symmetric spacetime which is spatially homogeneous but anisotropic. This spacetime admits a G_4 of motions acting transitively on spacelike hypersurfaces S_3. The Lie algebra of Killing vectors and the positive structure constants are given. The conformal Killing equation is written as a coupled system of first order, partial differential equations. We solve this system in general to obtain the conformal Killing vector and the conformal factor subject to integrability conditions. An analysis of the integrability conditions shows that the functions of integration are either constants or constrained by differential equations. The Killing vectors are obtained as a special case from the general conformal solution. The homothetic vector is determined from the general solution as a special case. We note that the homothetic vector places restrictions on the metric functions. Also we prove that there is no nontrivial special conformal vector.

In chapter 3 we consider another example of a locally rotationally symmetric spacetime. We solve the conformal Killing equation to obtain the general Killing vector subject to integrability conditions on the metric functions and the functions of integration. The system of integrability conditions in this case is more complicated than that of chapter 2. This is a result of the positional dependence in the metric functions. We were unable to integrate the system of integrability conditions completely. The Killing vectors are contained in the general conformal solution. The homothetic vector is found and the resulting restrictions on the gravitational field are determined. As for the previous metric in chapter 2 there is no nontrivial special conformal vector.

In chapter 4 we study a Bianchi I spacetime in which the gravitational field is determined in terms of three time–dependent metric functions. Here there are only three Killing vectors so that the gravitational field has less symmetry. We solve the conformal Killing equation for a constant conformal factor and generate the homothetic vector. The integrability conditions restrict the metric functions. These conditions are integrated and the functional dependence of the metric functions is explicitly determined.

Note that the results obtained in this thesis are original. Apparently this thesis represents the first attempt at a systematic analysis of conformal symmetries in locally rotationally symmetric spacetimes. We have not found any published work in the literature on conformal Killing vectors in these models. In the conclusion we summarise the results obtained in this thesis and consider avenues for further research.

1 Manifolds, Tensor Fields and Lie Algebras

1.1 Introduction

In this chapter we briefly review only those aspects of differential geometry necessary for this thesis. We begin by considering the 4–dimensional spacetime structure of a manifold which admits a Lorentzian metric in the neighbourhood of every point. The additional structure of an affine connection is introduced on the manifold. Spacetime is a 4–dimensional pseudo–Riemannian manifold with the metric connection and the gravitational field is described by the symmetric metric tensor. Also in §1.2 we consider general coordinate transformations, tensor products and tensor fields as natural geometric objects on the manifold. For more detailed expositions the reader is referred to Bishop and Goldberg (1968), Choquet–Bruhat *et al* (1977), Hawking and Ellis (1973), Misner *et al* (1973) and Stephani (1982). The Lie derivative plays a significant role when considering symmetries in general relativity and other physical fields (Schutz 1980). We consider Lie derivatives and Lie algebras in §1.3. The additional structure of the connection is introduced on the manifold in §1.4. This enables us to define the covariant derivative and thereby introduce the curvature tensor. In this section we give the geodesic equation in a coordinate basis and briefly discuss the Einstein field equations. We seek solutions to the conformal Killing equation in a class of locally rotationally symmetric models (MacCallum 1980) which have cosmological significance. The existence of a conformal symmetry often leads to a simplification of the Einstein field equations. We require that spacetime be invariant under a conformal Killing vector. In §1.5 we consider an r–dimensional Lie group. Furthermore we define a general conformal Killing vector and list the special cases of Killing, homothetic, special and nonspecial conformal Killing vectors. Note that conformal Killing vectors generate an r–dimensional Lie algebra. The conditions that the conformal symmetry imposes on the metric connection and related quantities defined on the manifold are listed in §1.5.

1.2 Manifolds and Tensor Fields

The surface of a sphere in Euclidean space or, more generally, any m–dimensional hypersurface in an n–dimensional Euclidean space ($m \leq n$) is a manifold. Another example of a manifold is the set of all rigid rotations of Cartesian coordinates in 3–dimensional Euclidean space. We may abstractly consider a manifold as any set that can be continuously parametrised. The number of independent coordinates gives the dimension of the manifold and the parameters are the coordinates of the manifold. A manifold is essentially a topological space which locally has the structure of Euclidean space in that it may be covered by coordinate patches. Even though the local structure of a manifold and the Euclidean space are similar it is important to note that their global structures may be very different.

The fundamental features of an n–dimensional differentiable manifold are that its points may be labelled by n real coordinates x^1, x^2, x^3, $\ldots$, x^n and differentiation of functions, involving changes of coordinates valid for all points in the space, is permissible. Let

$$\Re^n = \{(x^1, x^2, x^3, \ldots, x^n) | x^a \in \Re \text{ where } 1 \leq a \leq n\}$$

denote the set of n-tuples that generates an n-dimensional Euclidean space. Also let M be any set with a subset $U \subseteq M$. Suppose that $\psi : U \longrightarrow \Re^n$ is a bijective mapping. The purpose of ψ is to attach coordinates to points in U and we refer to the open set U as a coordinate neighbourhood. The pair (U, ψ), comprising the set U and ψ, is called a chart. Now consider the collection of charts $\{(U_a, \psi_a)\}_{a \in \Lambda}$, with Λ being some index set, such that the following four properties are true:

(i) $\{U_a\}$ covers M so that every point of M is contained in at least one U_a.

(ii) $\psi_a : U_a \longrightarrow \Re^n$ with the same n for all a.

(iii) $U_a \cap U_b \neq \emptyset$ for some a and b. For this intersecting region the composite functions $\psi_a \circ \psi_b^{-1}$ and $\psi_b \circ \psi_a^{-1}$ are differentiable functions from $\Re^n$ to $\Re^n$.

(iv) $\{(U_a, \psi_a)\}$ is maximal, ie. any other chart is contained in this set.

A collection of charts $\{(U_a, \psi_a)\}_{a \in \Lambda}$ satisfying properties (i) – (iv) is called an atlas. The set M together with its atlas comprises an n-dimensional differentiable manifold.

The maps $\psi_a : U_a \longrightarrow \Re^n$ and $\psi_b : U_b \longrightarrow \Re^n$ generate two coordinate systems $x^{a'}$ and x^a, respectively. These coordinates are related by the composite functions $\psi_a \circ \psi_b^{-1} : \Re^n \longrightarrow \Re^n$ and $\psi_b \circ \psi_a^{-1} : \Re^n \longrightarrow \Re^n$ because in the overlap $U_a \cap U_b \neq \emptyset$. This can be represented by the functional relationships

$$x^{a'} = x^{a'}(x^1, x^2, x^3, \ldots, x^n)$$

and the inverse relationships

$$x^a = x^a(x^{1'}, x^{2'}, x^{3'}, \ldots, x^{n'})$$

The functions $x^{a'}$ and x^a given above are both differentiable and injective. In the overlap $U_a \cap U_b \neq \emptyset$ so that the Jacobians of the matrices

$$X^{a'}_b = \frac{\partial x^{a'}}{\partial x^b} \qquad \text{and} \qquad X^b_{a'} = \frac{\partial x^b}{\partial x^{a'}}$$

are nonzero. Conversely suppose that we have a chart (U_a, ψ_a) and the system of equations $x^{a'} = x^{a'}(x^1, x^2, x^3, \ldots, x^n)$ with the Jacobian of $X^{a'}_b$ nonzero for some point $P \in U_a$ with coordinates x^a. Then we can show, using the inverse–function theorem, that there exists a coordinate system (U_b, ψ_b) about P whose coordinates are related to those of (U_a, ψ_a) by $x^a = x^a(x^{1'}, x^{2'}, x^{3'}, \ldots, x^{n'})$. For our purposes we require that the differentiability class of the manifold M is at least C^2 to ensure that operations which depend on the continuity of partial derivatives are valid.

We define $T_P(M)$ as the set of vectors tangent to a curve at a point $P \in M$. The set of tangent vectors $T_P(M)$ generates a vector space at the point P. We create the dual tangent space $T^*_P(M)$ at the point P by defining the real-valued function $T^*_P(M) : T_P(M) \longrightarrow \Re^n$. The dual space is also a vector space. We can then build up spaces $(T^r_s)_P(M)$ of type (r,s) tensors at P by taking repeated tensor products of $T_P(M)$ and $T^*_P(M)$ (Bishop and Goldberg 1968, Hawking and Ellis 1973, Misner *et al* 1973). The space T^r_s is also a vector space at P. A type (r,s) tensor field on M is an assignment to each point $P \in M$ a member of $(T^r_s)_P(M)$. We represent the set of all type (r,s) tensor fields on M by $T^r_s(M)$. Note that we regain $T^1(M) = T(M)$ as the set of contravariant vector fields, and $T_1(M) = T^*(M)$ as the set of covariant vector fields. In this framework we define a scalar field as a real–valued function on the manifold M.

Under a change of coordinates the components $T^{a_1 a_2 \ldots a_r}{}_{b_1 b_2 \ldots b_s}$ of a type

(r, s) tensor field $\mathbf{T}$, of rank $(r+s)$, transforms as follows

$$T^{a'_1 a'_2 \ldots a'_r}{}_{b'_1 b'_2 \ldots b'_s} = X^{a'_1}_{c_1} X^{a'_2}_{c_2} \ldots X^{a'_r}_{c_r} X^{d_1}_{b'_1} X^{d_2}_{b'_2} \ldots X^{d_s}_{b'_s} T^{c_1 c_2 \ldots c_r}{}_{d_1 d_2 \ldots d_s} \tag{1.1}$$

Of particular interest in general relativity are tensor fields of rank two. In order to discuss metrical properties we need to endow M with an indefinite metric tensor field $\mathbf{g}$ of rank two. In this case the manifold M is called a pseudo–Riemannian manifold (Misner *et al* 1973, Stephani 1982). As our applications are general relativistic the dimension of M will henceforth be taken as four. Spacetime M is an oriented, smooth 4–dimensional manifold endowed with a symmetric, non–degenerate tensor field $\mathbf{g}$ of signature $(- + + +)$. The symmetric $(0, 2)$ metric tensor field g_{ab} satisfies (1.1) and is used to invariantly define the length of a curve in M. This length is defined as the integral

$$s = \int_{u_1}^{u_2} |g_{ab}\dot{x}^a \dot{x}^b|^{\frac{1}{2}} du$$

where $\dot{x}^a = dx^a/du$. The metric tensor field $\mathbf{g}$ appears in the line element or fundamental metric form

$$ds^2 = g_{ab}dx^a dx^b \tag{1.2}$$

The line element (1.2) gives a measure of the infinitesimal interval between neighbouring points x^a and $x^a + dx^a$ in the manifold. Spacetime M has the property that at any point there exists a coordinate system in which g_{ab} takes the Lorentzian form

$$[g_{ab}] = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that this is true only locally in the 4–dimensional manifold of general relativity in contrast to special relativity where there exists global coordinate systems for which g_{ab} takes the above form.

1.3 Lie Algebras and Lie Derivatives

Here we briefly summarise only those elements of Lie theory which are required for subsequent sections (Choquet–Bruhat *et al* 1977, Kramer *et al* 1980, Straumann 1984). The Lie bracket or commutator of vector fields $\mathbf{X}$ and $\mathbf{Y}$ is defined as the quantity

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX} \tag{1.3}$$

As $[\mathbf{X}, \mathbf{Y}]$ inherits the linearity properties of $\mathbf{X}$ and $\mathbf{Y}$ the Lie bracket $[\mathbf{X}, \mathbf{Y}] \in T(M)$. Further the Lie bracket defines a composition on $T(M)$ which is bilinear. The space of all vector fields on M when endowed with the composition (1.3) is a Lie algebra which is skew–symmetric by definition. This algebra is not associative and, in fact, satisfies the Jacobi identity

$$[\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [\mathbf{Z}, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]] = 0 \tag{1.4}$$

for the vector fields $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$. Also note that every Lie algebra defines a unique, simply connected Lie group G_r (see §1.5). In particular for a coordinate basis $\{\mathbf{e}_a\}$ where $\mathbf{e}_a = \partial/\partial x^a$, equation (1.3) becomes

$$\left[\frac{\partial}{\partial x^a}, \frac{\partial}{\partial x^b}\right] = 0$$

as partial derivatives commute in the smooth manifold M.

The Lie derivative of a tensor field is of particular significance because it provides a coordinate independent description of a symmetry property in the manifold M. This derivative corresponds to the change determined by an observer in going from a point P in the direction of a vector field $\mathbf{X}$ to an infinitesimally neighbouring point Q and transporting the coordinate system from P to Q. Consider an infinitesimal coordinate transformation

$$x^{a'} = x^a - \varepsilon X^a$$

and compare the components of a type $(1,0)$ tensor field T^a at a point P and at an infinitesimally neighbouring point Q. Then to first order in ε we have

$$\begin{aligned} T^{a'}(Q) &= X^{a'}_b T^b(x^c + \varepsilon X^c) \\ &= T^a(P) + \varepsilon X^b T^a{}_{,b}(P) - \varepsilon X^a{}_{,b} T^b(P) \end{aligned} \tag{1.5}$$

where the comma denotes partial differentiation. We define the Lie derivative of T^a in the direction of the vector field $\mathbf{X}$ as the limiting value

$$\mathcal{L}_{\mathbf{X}} T^a = \lim_{\varepsilon \to 0} \frac{T^{a'}(Q) - T^a(P)}{\varepsilon}. \tag{1.6}$$

On substituting (1.5) into (1.6) we obtain the equivalent expression

$$\mathcal{L}_{\mathbf{X}} T^a = T^a{}_{,b} X^b - T^b X^a{}_{,b} \tag{1.7}$$

which is the Lie derivative of the $(1,0)$ tensor field T^a. The Lie derivative of $(0,1), (1,1), (0,2)$ and $(2,0)$ types of tensor fields are given, respectively, by

$$\mathcal{L}_{\mathbf{X}} T_a = T_{a,b} X^b + T_b X^b{}_{,a} \tag{1.8}$$

$$\mathcal{L}_{\mathbf{X}} T^a{}_b = T^a{}_{b,c} X^c - T^c{}_b X^a{}_{,c} + T^a{}_c X^c{}_{,b} \tag{1.9}$$

$$\mathcal{L}_{\mathbf{X}} T_{ab} = T_{ab,c} X^c + T_{cb} X^c{}_{,a} + T_{ac} X^c{}_{,b} \tag{1.10}$$

$$\mathcal{L}_{\mathbf{X}} T^{ab} = T^{ab}{}_{,c} X^c - T^{cb} X^a{}_{,c} - T^{ac} X^b{}_{,c} \tag{1.11}$$

For an (r, s) tensor field the Lie derivative follows analogously.

The Lie derivative $\mathcal{L}_{\mathbf{X}}$ satisfies the following properties which we list here without proof:

(i) $\mathcal{L}_{\mathbf{X}}$ preserves tensor type, ie. $\mathcal{L}_{\mathbf{X}}\mathbf{T}$ is a tensor field of the same type as $\mathbf{T}$.

(ii) $\mathcal{L}_{\mathbf{X}}$ is linear and Leibniz.

(iii) $\mathcal{L}_{\mathbf{X}}$ commutes with contraction.

(iv) $\mathcal{L}_{\mathbf{X}} f = \mathbf{X} f$ where f is a real-valued smooth function on M.

(v) $\mathcal{L}_{\mathbf{X}}\mathbf{Y} = [\mathbf{X}, \mathbf{Y}]$ for all vector fields $\mathbf{Y}$.

(vi) $\mathcal{L}_{[\mathbf{X},\mathbf{Y}]} = \mathcal{L}_{\mathbf{X}}\mathcal{L}_{\mathbf{Y}} - \mathcal{L}_{\mathbf{Y}}\mathcal{L}_{\mathbf{X}}$ for all vector fields $\mathbf{X}, \mathbf{Y}$.

Properties (i) – (vi) are used extensively in later chapters where we determine the conformal symmetries in classes of anisotropic cosmological models.

The exterior derivative (which we do not consider here) and the Lie derivative may be considered as generalisations of the partial derivative. These derivatives are introduced without defining further structure on the manifold. Note that in order to define covariant derivatives we have to impose the additional structure of a connection on the manifold M (see §1.4). The Lie derivative plays an important role in describing symmetries of gravitational fields (see §1.5) and other physical fields (Schutz 1980).

1.4 Covariant Derivatives and Field equations

The exterior derivative and the Lie derivative are operations defined on a differentiable manifold without imposing additional structure on M. The exterior derivative is limited because it acts only on forms. The Lie derivative $\mathcal{L}_{\mathbf{X}}\mathbf{T}|_P$ for any tensor field $\mathbf{T}$ depends on the vector $\mathbf{X}$ not only at $P \in M$ but also at neighbouring points. To introduce derivatives which have neither of these defects we need to impose additional structure on the manifold. In particular to define covariant derivatives we need to impose the additional structure of a connection on the manifold M. A connection ∇ at a point $P \in M$ is a rule which assigns to each vector field $\mathbf{X}$ at P a differential operator $\nabla_{\mathbf{X}}$ which maps an arbitrary vector field $\mathbf{Y}$ into a vector $\nabla_{\mathbf{X}}\mathbf{Y}$. The differential operator $\nabla_{\mathbf{X}}$ satisfies the following properties:

(i) The field $\nabla_{\mathbf{X}}\mathbf{Y}$ is a tensor in the argument $\mathbf{X}$, ie.

$$\nabla_{(f\mathbf{X}_1+g\mathbf{X}_2)}\mathbf{Y} = f\nabla_{\mathbf{X}_1}\mathbf{Y} + g\nabla_{\mathbf{X}_2}\mathbf{Y}$$

for functions f, g and $\mathbf{X_1}, \mathbf{X_2} \in T(M)$.

(ii) The vector $\nabla_{\mathbf{X}}\mathbf{Y}$ is linear in $\mathbf{Y}$.

(iii) The rule

$$\nabla_{\mathbf{X}}(f\mathbf{Y}) = (\mathbf{X}f)\mathbf{Y} + f\nabla_{\mathbf{X}}\mathbf{Y}$$

holds for all real–valued smooth functions f and vector fields $\mathbf{Y}$.

The quantity $\nabla_{\mathbf{X}}\mathbf{Y}|_P$ is the covariant derivative (with respect to ∇) of $\mathbf{Y}$ in the direction of $\mathbf{X}$ at P. Since $\nabla_{\mathbf{X}}\mathbf{Y}$ is a tensor in $\mathbf{X}$ we can define $\nabla\mathbf{Y}$, the covariant derivative of $\mathbf{Y}$, as the (1,1) tensor field which when contracted with $\mathbf{X}$ yields the vector $\nabla_{\mathbf{X}}\mathbf{Y}$. A connection ∇ on the manifold M is a rule which assigns a connection ∇ to each point (Hawking and Ellis 1973).

The components of $\nabla \mathbf{Y}$, with respect to the coordinate bases $\{\mathbf{e}_a\}$ and $\{\mathbf{e}^a\}$, follow from the expression for $\nabla_{\mathbf{X}}(f\mathbf{Y})$:

$$Y^a{}_{;b} = Y^a{}_{,b} + \Gamma^a{}_{bc} Y^c \tag{1.12}$$

where the semicolon denotes covariant differentiation and the connection coefficients $\Gamma^a{}_{bc}$ are defined by

$$\nabla_{\mathbf{e}_c} \mathbf{e}_b = \Gamma^a{}_{bc} \mathbf{e}_a$$

Then we must have $\nabla_{\mathbf{X}} \mathbf{Y} = Y^a{}_{;b} X^b \mathbf{e}_a$. We are only concerned with torsion-free or symmetric connections for which

$$\Gamma^a{}_{bc} = \Gamma^a{}_{cb}$$

$$\iff \nabla_{\mathbf{X}} \mathbf{Y} - \nabla_{\mathbf{Y}} \mathbf{X} = [\mathbf{X}, \mathbf{Y}]$$

for all $\mathbf{X}, \mathbf{Y} \in T(M)$.

The definition of a covariant derivative can be extended to arbitrary tensor fields by the rules:

(i) If $\mathbf{T}$ is a tensor field of type (r, s) then the covariant derivative $\nabla \mathbf{T}$ is a tensor field of type $(r, s+1)$.

(ii) ∇ is linear and Leibniz.

(iii) ∇ commutes with contraction.

(iv) $\nabla f = df$ for all real-valued smooth functions f.

The statement that given a metric tensor field $\mathbf{g}$ there exists a unique symmetric connection ∇ such that $\nabla \mathbf{g} = \mathbf{0}$ is known as the fundamental theorem of Riemannian geometry. This connection is called the metric connection, and $\Gamma^a{}_{bc}$ takes the form

$$g_{ad}\Gamma^d{}_{bc} = \Gamma_{abc} = \mathbf{g}(\mathbf{e}_a \nabla_{\mathbf{e}_b} \mathbf{e}_c)$$

which implies

$$\Gamma^a{}_{bc} = \tfrac{1}{2} g^{ad}(g_{cd,b} + g_{db,c} - g_{bc,d})$$

The covariant derivative of a (1, 0) vector field $\mathbf{Y}$ is given by (1.12). As further examples we give the covariant derivatives of the $(0,1), (1,1), (0,2)$ and the $(2,0)$ tensor fields:

$$T_{a;b} = T_{a,b} - \Gamma^c{}_{ab} T_c \tag{1.13}$$

$$T^a{}_{b;c} = T^a{}_{b,c} + \Gamma^a{}_{dc} T^d{}_b - \Gamma^d{}_{bc} T^a{}_d \tag{1.14}$$

$$T_{ab;c} = T_{ab,c} - \Gamma^d{}_{ac} T_{db} - \Gamma^d{}_{bc} T_{ad} \tag{1.15}$$

$$T^{ab}{}_{;c} = T^{ab}{}_{,c} + \Gamma^a{}_{dc} T^{db} + \Gamma^b{}_{dc} T^{ad} \tag{1.16}$$

The formulae for arbitrary rank tensors follow the rules suggested by (1.12) – (1.16). Since the manifold has the metric connection ∇ we can replace partial derivatives by the covariant derivatives in expressions (1.7) – (1.11) for the Lie derivative of tensors.

A tensor field $\mathbf{T}$ is said to be parallel transported along the integral curves $x^a(u)$ of a vector field $\mathbf{X}$ if

$$\nabla_{\mathbf{X}} \mathbf{T} = \mathbf{0} \tag{1.17}$$

which may be written as

$$\frac{D\mathbf{T}}{du} = \mathbf{0}$$

where D/du is the absolute derivative. Therefore in particular the metric tensor is parallel transported along all smooth curves. If $\mathbf{T}$ is a vector field then (1.17) implies

$$\frac{\partial T^a}{\partial u} + \Gamma^a{}_{bc} T^b \dot{x}^c = 0$$

A geodesic $x^a(u)$, with affine parameter u, is a curve along which the tangent vector field $\mathbf{X}$ is parallel transported:

$$\nabla_{\mathbf{X}} \mathbf{X} = 0$$

$$\Longleftrightarrow X^a{}_{;b} X^b = 0$$

$$\Longleftrightarrow \frac{d^2 x^a}{du^2} + \Gamma^a{}_{bc} \frac{dx^b}{du} \frac{dx^c}{du} = 0$$

The affine parameter u of a geodesic curve is determined up to an additive and a multiplicative constant, ie. up to transformations

$$u \longrightarrow u' = au + b$$

where $a, b \in \Re$ $(a \neq 0)$, so that $\mathbf{X}$ can be renormalised by a constant factor $\mathbf{X} \rightarrow \mathbf{X}' = (1/a)\mathbf{X}$.

The path–dependence of parallel transport provides a measure of the curvature of the manifold. This path–dependence corresponds to the non–commutativity of covariant derivatives. The Riemann curvature tensor $\mathbf{R}$ gives a measure of this non–commutativity:

$$X_{a;bc} - X_{a;cb} = R^d{}_{abc} X_d \qquad (1.18)$$

for all $\mathbf{X} \in T(M)$. The above equation is often referred to as the Ricci identity. Applying (1.18) to the basis vectors $\{\mathbf{e}_a\}$ gives

$$R^d{}_{abc} = \Gamma^d{}_{ac,b} - \Gamma^d{}_{ab,c} + \Gamma^e{}_{ac} \Gamma^d{}_{eb} - \Gamma^e{}_{ab} \Gamma^d{}_{ec}$$

for the coordinates basis components of the Riemann tensor **R**. The $R^a{}_{bcd}$ satisfy the symmetry properties

$$R_{abcd} - R_{cdab} = 0$$

$$R^a{}_{b(cd)} = 0$$

$$R^a{}_{[bcd]} = 0$$

$$R^a{}_{b[cd;e]} = 0$$

where the last equation is referred to as the Bianchi identity. The Ricci tensor R_{ab} is obtained by a contraction of the Riemann tensor:

$$R_{ab} = R^c{}_{acb}$$

The curvature of the manifold defined in terms of the metric tensor is related to matter–energy by the Einstein field equations

$$\begin{aligned} G_{ab} &\equiv R_{ab} - \tfrac{1}{2}Rg_{ab} \\ &= T_{ab} \end{aligned} \tag{1.19}$$

where T_{ab} is the energy–momentum tensor. The Einstein tensor **G** is defined in terms of the Ricci tensor R_{ab} and the Ricci scalar $R^a{}_a \equiv R$. The Einstein field equations (1.19) constitute a system of ten nonlinear partial differential equations which determines the behaviour of the gravitational field via the metric functions g_{ab}. Not all of the field equations (1.19) are independent because of the Bianchi identity

$$G^{ab}{}_{;b} = 0 \tag{1.20}$$

which is essentially a conservation law. For further information on various categories of exact solutions to the Einstein field equations (1.19) with the Bianchi identity (1.20) the reader is referred to Hawking and Ellis (1973), Kramer *et al* (1980) and Misner *et al* (1973).

1.5 Conformal Motions

When analysing the Einstein field equations we find that it is easier to solve these equations in many cases when the gravitational field possesses symmetries. Manifolds with structure may admit groups of transformations G_r preserving this structure. An r–dimensional Lie group G_r is a group which is also a smooth r–dimensional differentiable manifold whose structure is such that the group composition $G \times G \rightarrow G$ and the group inverse $G \rightarrow G$ are smooth maps (Kramer *et al* 1980). A Lie group G_r acts as a transformation group of an n–dimensional smooth manifold on the right if there exists a map $f : M \times G_r \rightarrow M$ which satisfies

$$f(P, e) = P$$

for the identity $e \in G_r$ and for all $P \in M$, and

$$f(f(P, g), g') = f(P, gg')$$

for all $g, g' \in G_r$. We assume that the action of G_r on M is effective, ie.

$$f(P, g) = P \Longrightarrow g = e$$

for all $P \in M$. Locally these transformations are given by

$$(x')^a = f^a(x^b, y^i)$$

where $a = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, r$. The quantities f^a are smooth functions and y^i are local coordinates on G_r. The coordinates in G_r are called group parameters and G_r is said to be of r (essential) parameters. The transformations are generated by r linearly independent vector fields $X_i = X_i^a \partial/\partial x^a$ on M where

$$X_i^a = \left.\frac{\partial f^a}{\partial y^i}\right|_{y^i=0} \tag{1.21}$$

The vector fields (1.21) generate a Lie algebra which is also isomorphic to the Lie algebra of left-invariant vector fields on G_r. The orbit $O(P)$ of any point $P \in M$ is the submanifold of M consisting of all points reachable by the transformations generated by G_r, ie.

$$O(P) = \{P' \in M | f(P, g) = P'\}$$

for some $g \in G_r$. The orbit of all points foliate M. The action of G_r is simply transitive on the orbits if the dimension of each orbit is r, and multiply transitive if the dimensions of the orbit are less than r.

In this thesis we are concerned with the conformal symmetries or conformal motions G_r of infinitesimal transformations. A conformal motion preserves the metric up to a factor. A conformal Killing vector is defined by

$$\mathcal{L}_{\mathbf{X}} g_{ab} = 2\phi g_{ab} \tag{1.22}$$

where $\phi = \phi(x^c)$ is the conformal factor and g_{ab} is the metric tensor. The conformal equation (1.22) preserves angles between vectors and the light cone structure on spacetime, and maps null geodesics to null geodesics. There are four categories of the conformal equation (1.22):

(i) $\mathbf{X}$ is a Killing vector when $\phi = 0$.

(ii) $\mathbf{X}$ is a homothetic Killing vector when $\phi_{,a} = 0 \neq \phi$.

(iii) $\mathbf{X}$ is a special conformal Killing vector when $\phi_{;ab} = 0$.

(iv) $\mathbf{X}$ is a nonspecial conformal Killing vector when $\phi_{;ab} \neq 0$.

The Killing vectors span a group of isometries: an isometry is a transformation which leaves the metric invariant. Killing vectors generate constants or first integrals of the motion along geodesics. The group of isometries may be utilised to systematically and invariantly characterise solutions of the Einstein field equations (Kramer *et al* 1980). A homothetic Killing vector scales all distances by the same constant factor and preserves the null geodesic affine parameters. Homothetic Killing vectors lead to self–similar spacetimes. A recent approach in seeking solutions to the Einstein field equations is to suppose that spacetime admits a group of conformal motions G_r of infinitesimal transformations. This leads to further restrictions on the metric functions and often simplifies the solution of the Einstein field equations (see for example Dyer *et al* 1987, Maharaj *et al* 1991).

The set of all conformal Killing vectors generates a Lie Algebra G_r with basis $\{\mathbf{X}_I\}$. The elements of the basis are related by

$$[\mathbf{X}_I, \mathbf{X}_J] = C^K{}_{IJ}\mathbf{X}_K \tag{1.23}$$

where the $C^K{}_{IJ}$ are the structure constants of the group. From (1.3), (1.4) and (1.23) we obtain the Lie identity

$$C^K{}_{LM}C^M{}_{IJ} + C^K{}_{IM}C^M{}_{JL} + C^K{}_{JM}C^M{}_{LI} = 0$$

The maximal order r of a group of conformal transformations G_r is given by

$$r = \tfrac{1}{2}(n+1)(n+2)$$

for an n–dimensional manifold. The maximal dimensionality of the Lie algebra in spacetime is $r = 15$. The generators of the G_{15} of conformal Killing vectors for flat

Minkowski space is given by Choquet–Bruhat *et al* (1977) and for Robertson–Walker spacetimes by Maartens and Maharaj (1986).

A vector field is said to be recurrent or parallel if its covariant derivative is parallel to itself. A recurrent vector is geodesic and nonrotating. A conformally recurrent manifold M has a real recurrent vector $\mathbf{K}$ (Kramer *et al* 1980) such that

$$\nabla_e C_{abcd} = C_{abcd} K_e$$

If $\mathbf{K} = \mathbf{0}$ then M is a conformally symmetric spacetime:

$$\nabla_e C_{abcd} = 0$$

where C_{abcd} is the conformal Weyl tensor defined by

$$C_{abcd} = R_{abcd} + g_{a[d}R_{c]b} + g_{b[c}R_{d]a} + \tfrac{1}{3} R g_{a[c}g_{d]b}$$

The manifold M is conformally flat if and only if the conformal tensor C_{abcd} vanishes.

The existence of a conformal Killing vector (1.22) places restrictions on the connection coefficients, the Riemann tensor, the Ricci tensor, the Ricci scalar, the Einstein tensor and the Weyl tensor. Yano (1957) lists the following conditions that the conformal Killing vector $\mathbf{X}$ is known to satisfy:

$$\mathcal{L}_{\mathbf{X}} \Gamma^a{}_{bc} = \delta^a{}_b \phi_{;c} + \delta^a{}_c \phi_{;c} - g_{bc} \nabla^a \phi \tag{1.24}$$

$$\mathcal{L}_{\mathbf{X}} R^a{}_{bcd} = -\delta^a{}_c \nabla_d(\phi_{;b}) + \delta^a{}_d \nabla_c(\phi_{;b}) - (\nabla_c \nabla^a \phi) g_{bd} + (\nabla_d \nabla^a \phi) g_{bc} \tag{1.25}$$

$$\mathcal{L}_{\mathbf{X}} R_{ab} = -2\nabla_a(\phi_{;b}) - (\nabla_a \nabla^a \phi) g_{ab} \tag{1.26}$$

$$\mathcal{L}_{\mathbf{X}} R = -2\phi R - 6(\nabla_a \nabla^a \phi) \tag{1.27}$$

$$\mathcal{L}_{\mathbf{X}} G_{ab} = -2\nabla_a(\phi_{;b}) + 2(\nabla_a \nabla^a \phi) g_{ab} \tag{1.28}$$

$$\mathcal{L}_{\mathbf{X}} C^a{}_{bcd} = 0 \tag{1.29}$$

The above conditions illustrate the fact that the existence of a conformal Killing vector $\mathbf{X}$ imposes severe restrictions on the spacetime manifold M.

A number of general results on the effect of conformal symmetries on the gravitational field have been obtained with the help of (1.24) – (1.29). Collinson and French (1967) established that a vacuum spacetime with a proper conformal symmetry must be of Petrov type N. Eardley *et al* (1986) showed that the only vacuum solutions with a conformal symmetry are everywhere locally flat type O spacetimes or of type N representing a class of plane waves. Also Garfinkle (1987) and Eardley *et al* (1986) proved that asymptotically flat spacetimes with reasonable energy conditions are locally flat type O spacetimes. Garfinkle and Tian (1987) have shown that vacuum spacetimes with nonzero cosmological constant and a proper conformal symmetry are constant curvature spacetimes of type O. These represent the de Sitter and anti-de Sitter models. Recently Sharma (1988) established that conformally symmetric spacetimes admitting an infinitesimal conformal symmetry are either of type O or N. Conformal motions are special cases of more general symmetry properties of spacetimes namely curvature collineations. Spacetime symmetries, including conformal motions and their relationship with curvature collineations, are discussed by Katzin *et al* (1969).

2 Conformal symmetries in models with metric

$$\mathbf{ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + dz^2]}$$

2.1 Introduction

In this chapter we consider an example of a spacetime which is spatially homogeneous but anisotropic. This is a special case of a class of locally rotationally symmetric spacetimes. These symmetric spacetimes admit a G_4 of motions acting transitively on spacelike hypersurfaces S_3. In §2.2 we discuss the spacetime geometry of locally rotationally symmetric spacetimes in general, and the above metric in particular. We give the generators of the Lie algebra of Killing vectors and the positive structure constants. In §2.3 we write the conformal equation as a coupled system of ten first order, linear partial differential equations. We explicitly solve this system of equations to obtain the conformal Killing vector $\mathbf{X}$ and the conformal factor ϕ subject to integrability conditions. These conditions place consistency requirements on the metric functions $A(t)$ and $B(t)$. The effect of the integrability conditions is comprehensively investigated in §2.4. In §2.5 we obtain the Killing vectors as a special case from our general solution. Also we find the homothetic vector from the general conformal Killing vector. The homothetic vector restricts the metric functions $A(t)$ and $B(t)$. Finally we investigate the case of a special conformal vector.

2.2 Spacetime Geometry

In this chapter we consider the spatially homogeneous and anisotropic spacetime described by the metric

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)[dy^2 + dz^2] \tag{2.1}$$

with coordinates $x^a = (t, x, y, z)$. This spacetime is called class AIa in the MacCallum (1980) classification, and is referred to as type VI_4 by Petrov (1969). Since groups G_r of motions transitive on orbits are admitted each point has an isotropy group. The term locally isotropic has been used by Cahen and Defrise (1968) where every point has a nontrivial isotropy group. As this group consists of spatial rotations (2.1) is an example of a locally rotationally symmetric spacetime. The Lie algebras of the isometry groups are obtained by using tetrads defined up to isotropy by properties of the curvature tensor, and the invariance under the isotropy is then imposed. For a detailed analysis of the symmetries of locally rotationally symmetric spacetimes see Maartens and Maharaj (1985) and MacCallum (1980). Ellis (1967) and Stewart and Ellis (1968) give all locally rotationally symmetric metrics with perfect fluid and electromagnetic field. A G_4 of motions is admitted which acts transitively on 3–dimensional spacelike hypersurfaces S_3 (the local isotropy is a spatial rotation about a spacelike direction). There are simply transitive subgroups G_3 acting on hypersurfaces S_3 of Bianchi type I or VII_0. Also there is a multiply transitive G_3 subgroup of Bianchi type VII_0 acting on flat surfaces S_2. The Lie algebra of Killing vectors $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ is spanned by

$$\mathbf{X}_1 = \frac{\partial}{\partial x}$$

$$\mathbf{X}_2 = \frac{\partial}{\partial y}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial z}$$

$$\mathbf{X}_4 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

For this Lie algebra the following positive structure constants are obtained from equation (1.23):

$$C^2{}_{43} = C^3{}_{24} = 1$$

2.3 Conformal equation

For the metric (2.1) the conformal equation (1.22) reduces to the following system of ten equations:

$$X^0_t = \phi \tag{2.2}$$

$$A^2X^1_t - X^0_x = 0 \tag{2.3}$$

$$B^2X^2_t - X^0_y = 0 \tag{2.4}$$

$$B^2X^3_t - X^0_z = 0 \tag{2.5}$$

$$\dot{A}X^0 + AX^1_x = A\phi \tag{2.6}$$

$$B^2 X^2_x + A^2 X^1_y = 0 \tag{2.7}$$

$$B^2 X^3_x + A^2 X^1_z = 0 \tag{2.8}$$

$$\dot{B} X^0 + B X^2_y = B\phi \tag{2.9}$$

$$X^3_y + X^2_z = 0 \tag{2.10}$$

$$\dot{B} X^0 + B X^3_z = B\phi \tag{2.11}$$

where the subscripts denote differentiation with respect to t, x, y and z. The equations (2.2) – (2.11) are a coupled system of ten first order, linear partial differential equations. We have to integrate this system to obtain $\mathbf{X} = (X^0, X^1, X^2, X^3)$ and ϕ in terms of the metric functions $A(t)$ and $B(t)$. This is achieved by taking combinations of derivatives of (2.2) – (2.11) to generate identities that simplify the integration process.

On subtracting the derivative of (2.5) with respect to y from the derivative of (2.4) with respect to z we obtain $X^2_{zt} - X^3_{yt} = 0$. This result together with the derivative of (2.10) with respect to t yields the identities

$$X^2_{zt} = 0 \tag{2.12}$$

$$X^3_{yt} = 0 \tag{2.13}$$

The difference of the derivative of (2.7) with respect to z and the derivative of (2.8) with respect to y gives $X^2_{zx} - X^3_{yx} = 0$. This equation and the derivative of (2.10) with respect to x gives

$$X^2_{zx} = 0 \tag{2.14}$$

$$X^3_{yx} = 0 \tag{2.15}$$

The equations (2.12) and (2.14) imply that X^2_z is a function only of y and z. Similarly (2.13) and (2.15) imply that X^3_y is also a function only of y and z. Differentiating (2.4) with respect to z and using (2.12) gives

$$X^0_{yz} = 0. \tag{2.16}$$

Then equation (2.2) differentiated with respect to y and z implies that

$$\phi_{yz} = 0 \tag{2.17}$$

Now differentiate equations (2.9) and (2.11) by both y and z to obtain the following partial differential equations

$$\left(X^2_z\right)_{yy} = 0 \tag{2.18}$$

$$\left(X^3_y\right)_{zz} = 0 \tag{2.19}$$

where we have used (2.16) and (2.17)

Integrating (2.18) and (2.19) gives

$$X^2_z = \mathcal{A}(z)y + \mathcal{B}(z) \tag{2.20}$$

$$X^3_y = \mathcal{C}(y)z + \mathcal{D}(y) \tag{2.21}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are functions of integration. On substituting (2.20) and (2.21) into (2.10) we obtain

$$\mathcal{A}(z)y + \mathcal{B}(z) + \mathcal{C}(y)z + \mathcal{D}(y) = 0$$

which necessarily implies

$$\mathcal{A}(z) = \alpha z + \beta$$

$$\mathcal{B}(z) = -\gamma z + \delta$$

$$\mathcal{C}(y) = -\alpha y + \gamma$$

$$\mathcal{D}(y) = -\beta y - \delta$$

where α, β, γ and δ are constants. On substituting these results into (2.20) and (2.21) and integrating we obtain the components

$$X^2 = \tfrac{1}{2}\alpha y z^2 + \beta y z - \tfrac{1}{2}\gamma z^2 + \delta z + \mathcal{E}(t, x, y) \tag{2.22}$$

$$X^3 = -\tfrac{1}{2}\alpha z y^2 + \gamma y z - \tfrac{1}{2}\beta y^2 - \delta y + \mathcal{F}(t, x, z) \tag{2.23}$$

where $\mathcal{E}$ and $\mathcal{F}$ are functions of integration.

Differentiating (2.9) and (2.11) by y, differentiating (2.10) by z, and forming a sum gives an equation in X^2:

$$X^2_{yy} + X^2_{zz} = 0 \tag{2.24}$$

Similarly, differentiating (2.9) and (2.11) by z, differentiating (2.10) by y, and forming a sum gives an equation in X^3:

$$X^3_{yy} + X^3_{zz} = 0 \tag{2.25}$$

To find the function $\mathcal{E}(t,x,y)$ substitute (2.22) into (2.24) to obtain

$$\mathcal{E}_{yy} + \alpha y - \gamma = 0$$

which has the solution

$$\mathcal{E}(t,x,y) = -\tfrac{1}{6}\alpha y^3 + \tfrac{1}{2}\gamma y^2 + y\mathcal{G}(t,x) + \mathcal{H}(t,x) \tag{2.26}$$

where $\mathcal{G}$ and $\mathcal{H}$ result from integration. Similarly to find $\mathcal{F}(t,x,z)$ substitute (2.23) into (2.25) to obtain

$$\mathcal{F}_{zz} - \alpha z - \beta = 0$$

which has the solution

$$\mathcal{F}(t,x,z) = \tfrac{1}{6}\alpha z^3 + \tfrac{1}{2}\beta z^2 + z\mathcal{I}(t,x) + \mathcal{J}(t,x) \tag{2.27}$$

where $\mathcal{I}$ and $\mathcal{J}$ are functions of the integration process. Now the difference of (2.9) and (2.11) yields

$$X^2_y - X^3_z = 0 \tag{2.28}$$

We substitute (2.22) and (2.23), taking into account (2.26) and (2.27), into (2.28) to obtain the result

$$\mathcal{G} = \mathcal{I}$$

Thus (2.22) and (2.23) assume the following forms respectively

$$X^2 = \tfrac{1}{2}\alpha y z^2 + z(\beta y + \delta) + \tfrac{1}{2}\gamma\left(y^2 - z^2\right) - \tfrac{1}{6}\alpha y^3 + y\mathcal{G} + \mathcal{H} \tag{2.29}$$

$$X^3 = -\tfrac{1}{2}\alpha z y^2 + y(\gamma z - \delta) - \tfrac{1}{2}\beta\left(y^2 - z^2\right) + \tfrac{1}{6}\alpha z^3 + z\mathcal{G} + \mathcal{J} \tag{2.30}$$

Thus far we have obtained the components X^2 and X^3 in terms of the integration functions $\mathcal{G}(t,x)$, $\mathcal{H}(t,x)$ and $\mathcal{J}(t,x)$. It remains to find X^0, X^1 and ϕ.

Now from (2.29) and (2.4) we find the component

$$X^0 = B^2(\tfrac{1}{2}y^2\mathcal{G}_t + y\mathcal{H}_t) + \mathcal{K}(t,x,z) \tag{2.31}$$

where $\mathcal{K}$ is a function of integration. By substituting (2.30) and (2.31) into (2.5) we obtain $\mathcal{K}$:

$$\mathcal{K}_z = B^2(z\mathcal{G}_t + \mathcal{J}_t)$$

$$\mathcal{K}(t,x,z) = B^2(\tfrac{1}{2}z^2\mathcal{G}_t + z\mathcal{J}_t) + \mathcal{L}(t,x)$$

with $\mathcal{L}$ being a function of integration. Therefore (2.31) becomes

$$X^0 = \tfrac{1}{2}B^2\mathcal{G}_t(y^2+z^2) + B^2(y\mathcal{H}_t + z\mathcal{J}_t) + \mathcal{L} \tag{2.32}$$

Now from the equations (2.29) and (2.7) we have

$$X^1 = -\left(\frac{B}{A}\right)^2 (\tfrac{1}{2}y^2\mathcal{G}_x + y\mathcal{H}_x) + \mathcal{M}(t,x,z) \tag{2.33}$$

where the function $\mathcal{M}$ results from integration. Equations (2.30), (2.33) and (2.8) give a restriction on $\mathcal{M}$:

$$\mathcal{M}_z = -\left(\frac{B}{A}\right)^2 (z\mathcal{G}_x + \mathcal{J}_x)$$

$$\mathcal{M}(t,x,z) = -\left(\frac{B}{A}\right)^2 (\tfrac{1}{2}z^2\mathcal{G}_x + z\mathcal{J}_x) + \mathcal{N}(t,x)$$

where $\mathcal{N}$ is a function of integration. Then (2.33) can be written as

$$X^1 = -\tfrac{1}{2}\left(\frac{B}{A}\right)^2 \mathcal{G}_x(y^2+z^2) - \left(\frac{B}{A}\right)^2 (y\mathcal{H}_x + z\mathcal{J}_x) + \mathcal{N} \tag{2.34}$$

Hence (2.32), (2.34), (2.29) and (2.30) give X^0, X^1, X^2 and X^3 respectively. Note that the y and z dependence of the components of $\mathbf{X}$ have been completely determined at this stage. With these forms of X^0, X^1, X^2 and X^3 equations (2.4), (2.5), (2.7), (2.8) and (2.10) are satisfied. The remaining equations (2.2), (2.3), (2.6), (2.9) and (2.11) will restrict the functions of integration and the metric functions. Also the conformal factor ϕ has to be determined.

Taking into account (2.32) and (2.34), equation (2.3) will be satisfied provided the following integrability conditions hold:

$$2B^2\mathcal{G}_{xt} + A^2\left[\left(\frac{B}{A}\right)^2\right]_t \mathcal{G}_x = \left[\frac{B}{A}\mathcal{G}_x\right]_t = 0 \tag{2.35}$$

$$2B^2\mathcal{H}_{xt} + A^2\left[\left(\frac{B}{A}\right)^2\right]_t \mathcal{H}_x = \left[\frac{B}{A}\mathcal{H}_x\right]_t = 0 \tag{2.36}$$

$$2B^2\mathcal{J}_{xt} + A^2\left[\left(\frac{B}{A}\right)^2\right]_t \mathcal{J}_x = \left[\frac{B}{A}\mathcal{J}_x\right]_t = 0 \tag{2.37}$$

$$\mathcal{L}_x - A^2\mathcal{N}_t = 0 \tag{2.38}$$

The conformal factor

$$\phi = \tfrac{1}{2}(y^2+z^2)(B^2\mathcal{G}_t)_t + y(B^2\mathcal{H}_t)_t + z(B^2\mathcal{J}_t)_t + \mathcal{L}_t \tag{2.39}$$

follows from (2.2) and (2.32). Now taking into account (2.32), (2.34) and (2.39), equation (2.6) will be satisfied provided the following set of integrability conditions holds:

$$(B^2\mathcal{G}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{G}_t + \left(\frac{B}{A}\right)^2\mathcal{G}_{xx} = 0 \tag{2.40}$$

$$(B^2\mathcal{H}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{H}_t + \left(\frac{B}{A}\right)^2 \mathcal{H}_{xx} = 0 \qquad (2.41)$$

$$(B^2\mathcal{J}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{J}_t + \left(\frac{B}{A}\right)^2 \mathcal{J}_{xx} = 0 \qquad (2.42)$$

$$\frac{\dot{A}}{A}\mathcal{L} - \mathcal{L}_t + \mathcal{N}_x = 0 \qquad (2.43)$$

Also substituting (2.29), (2.32) and (2.39) into (2.9) we obtain the following set of integrability conditions:

$$(B^2\mathcal{G}_t)_t - \dot{B}B\mathcal{G}_t + \alpha = 0 \qquad (2.44)$$

$$(B^2\mathcal{G}_t)_t - \dot{B}B\mathcal{G}_t - \alpha = 0 \qquad (2.45)$$

$$(B^2\mathcal{H}_t)_t - \dot{B}B\mathcal{H}_t - \gamma = 0 \qquad (2.46)$$

$$(B^2\mathcal{J}_t)_t - \dot{B}B\mathcal{J}_t - \beta = 0 \qquad (2.47)$$

$$\frac{\dot{B}}{B}\mathcal{L} - \mathcal{L}_t + \mathcal{G} = 0 \qquad (2.48)$$

At this point we note that equation (2.11) gives the same integrability conditions as equation (2.9). It immediately follows from equations (2.44) and (2.45) that

$$\alpha = 0 \qquad (2.49)$$

Thus the coupled system (2.2) – (2.11) has the general solution (2.29), (2.30), (2.32), (2.34) and (2.39) subject to the integrability conditions (2.35) – (2.38) and (2.40) – (2.49). Collecting these results we have the following solution with the integration functions $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{L}$ and $\mathcal{N}$ dependent only on the timelike coordinate t and the spacelike coordinate x:

$$X^0 = \tfrac{1}{2}B^2\mathcal{G}_t(y^2+z^2) + B^2(y\mathcal{H}_t + z\mathcal{J}_t) + \mathcal{L} \tag{2.50}$$

$$X^1 = -\tfrac{1}{2}\left(\frac{B}{A}\right)^2 \mathcal{G}_x(y^2+z^2) - \left(\frac{B}{A}\right)^2 (y\mathcal{H}_x + z\mathcal{J}_x) + \mathcal{N} \tag{2.51}$$

$$X^2 = z(\beta y + \delta) + \tfrac{1}{2}\gamma\left(y^2 - z^2\right) + y\mathcal{G} + \mathcal{H} \tag{2.52}$$

$$X^3 = y(\gamma z - \delta) - \tfrac{1}{2}\beta\left(y^2 - z^2\right) + z\mathcal{G} + \mathcal{J} \tag{2.53}$$

$$\phi = \tfrac{1}{2}(y^2+z^2)(B^2\mathcal{G}_t)_t + y(B^2\mathcal{H}_t)_t + z(B^2\mathcal{J}_t)_t + \mathcal{L}_t \tag{2.54}$$

subject to the following integrability conditions

$$\left[\frac{B}{A}\mathcal{G}_x\right]_t = 0 \tag{2.55}$$

$$\left[\frac{B}{A}\mathcal{H}_x\right]_t = 0 \tag{2.56}$$

$$\left[\frac{B}{A}\mathcal{J}_x\right]_t = 0 \tag{2.57}$$

$$(B^2\mathcal{G}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{G}_t + \left(\frac{B}{A}\right)^2 \mathcal{G}_{xx} = 0 \tag{2.58}$$

$$(B^2\mathcal{H}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{H}_t + \left(\frac{B}{A}\right)^2 \mathcal{H}_{xx} = 0 \tag{2.59}$$

$$(B^2\mathcal{J}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{J}_t + \left(\frac{B}{A}\right)^2 \mathcal{J}_{xx} = 0 \tag{2.60}$$

$$(B^2\mathcal{G}_t)_t - \dot{B}B\mathcal{G}_t = 0 \tag{2.61}$$

$$(B^2\mathcal{H}_t)_t - \dot{B}B\mathcal{H}_t - \gamma = 0 \tag{2.62}$$

$$(B^2\mathcal{J}_t)_t - \dot{B}B\mathcal{J}_t - \beta = 0 \tag{2.63}$$

$$\frac{\dot{A}}{A}\mathcal{L} - \mathcal{L}_t + \mathcal{N}_x = 0 \tag{2.64}$$

$$\frac{\dot{B}}{B}\mathcal{L} - \mathcal{L}_t + \mathcal{G} = 0 \tag{2.65}$$

$$\mathcal{L}_x - A^2\mathcal{N}_t = 0 \tag{2.66}$$

Thus the existence of the conformal Killing vector $\mathbf{X}$ with components (2.50) – (2.53) and the conformal factor (2.54) restricts the metric functions $A(t)$ and $B(t)$ by the integrability conditions (2.55) – (2.66)

2.4 Integrability conditions

The integrability conditions (2.55) – (2.66) comprise a coupled system of partial differential equations for the integration functions $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{L}$ and $\mathcal{N}$. These equations severely restrict the metric functions $A(t)$ and $B(t)$. In the analysis of a conformally symmetric gravitational field we should consider these restrictions together with the Einstein field equations (1.19). This is the subject of ongoing research.

We first consider the effect of equations (2.55), (2.58) and (2.61) on the function $\mathcal{G}(t,x)$. Equation (2.55) is immediately integrated to yield

$$\mathcal{G} = \frac{A}{B}\mathcal{O}(x) + \mathcal{P}(t) \tag{2.67}$$

where $\mathcal{O}$ and $\mathcal{P}$ are functions of integration. On substituting (2.67) into (2.61) and simplifying we obtain

$$\mathcal{O}\left[B\left(\frac{A}{B}\right)^{\cdot}\right]^{\cdot} = -(B\dot{\mathcal{P}})^{\cdot} \tag{2.68}$$

Two cases arise, namely $[B(A/B)^{\cdot}]^{\cdot} \neq 0$ and $[B(A/B)^{\cdot}]^{\cdot} = 0$.

Case I: $[B(A/B)^{\cdot}]^{\cdot} \neq 0$

In this case (2.68) is a variable separable equation which implies

$$\mathcal{O} = \epsilon \tag{2.69}$$

$$\epsilon B\left(\frac{A}{B}\right)^{\cdot} + B\dot{\mathcal{P}} = \varepsilon \tag{2.70}$$

where ϵ and ε are constants. Then with the help of (2.67), (2.69) and (2.70), equation (2.58) becomes

$$(B^2\mathcal{G}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{G}_t = 0$$

$$\varepsilon\left(\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right) = 0$$

which implies that $\varepsilon = 0$ otherwise the spacetime degenerates to Robertson–Walker. Then equation (2.70) is integrated to give

$$\mathcal{P} = -\frac{A}{B}\epsilon + \zeta$$

where ζ is a constant. Thus from (2.67) the function of integration $\mathcal{G}$ has the form

$$\mathcal{G} = \zeta \tag{2.71}$$

Case II: $\quad [B(A/B)\dot{}\,]\dot{} = 0$

In this case, equation (2.68) gives the results:

$$B\left(\frac{A}{B}\right)^{\cdot} = \eta$$

$$B\dot{\mathcal{P}} = \theta$$

where η and θ are constants. Then with the help of these two results and (2.67), equation (2.58) reduces to the ordinary differential equation

$$\mathcal{O}_{xx} - \eta^2\mathcal{O} - \eta\theta = 0 \tag{2.72}$$

We have integrated (2.55), (2.58) and (2.61): either $\mathcal{G}$ is a constant as in (2.71), or $\mathcal{O}(x)$ of (2.67) has to satisfy the differential equation (2.72).

Now we consider the effect of equations (2.56), (2.59) and (2.62) on the function $\mathcal{H}(t, x)$. We integrate (2.56) to obtain

$$\mathcal{H} = \frac{A}{B}\mathcal{Q}(x) + \mathcal{R}(t) \tag{2.73}$$

where $\mathcal{Q}$ and $\mathcal{R}$ are functions of integration. On substituting (2.73) into (2.62) and simplifying we obtain

$$\mathcal{Q}\left[B\left(\frac{A}{B}\right)^{\cdot}\right]^{\cdot} = \frac{\gamma}{B} - (B\dot{\mathcal{R}})^{\cdot} \tag{2.74}$$

Again two cases arise, namely $[B(A/B)^{\cdot}]^{\cdot} \neq 0$ and $[B(A/B)^{\cdot}]^{\cdot} = 0$

Case I: $[B(A/B)^{\cdot}]^{\cdot} \neq 0$

We immediately obtain from (2.74) that

$$\mathcal{Q} = \vartheta \tag{2.75}$$

$$\vartheta\left[B\left(\frac{A}{B}\right)^{\cdot}\right]^{\cdot} = \frac{\gamma}{B} - (B\dot{\mathcal{R}})^{\cdot} \tag{2.76}$$

where ϑ is a constant. Then with the help of (2.73), (2.75) and (2.76) the integrability condition (2.59) becomes

$$(B^2\mathcal{H}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{H}_t = 0$$

$$\mathcal{H}_{tt} = \mathcal{H}_t\left[\frac{\dot{A}}{A} - 2\frac{\dot{B}}{B}\right]$$

There are two possibilities: $\mathcal{H}_t = 0$ or $\mathcal{H}_t \neq 0$. If $\mathcal{H}_t \neq 0$ then we have

$$\mathcal{H}_t = \iota\frac{A}{B^2} \tag{2.77}$$

where ι is a constant of integration. Equations (2.73) and (2.77) give the result

$$\dot{\mathcal{R}} = \iota\frac{A}{B^2} - \vartheta\left(\frac{A}{B}\right)^{\cdot}$$

On substituting this equation in (2.76) we find that $[B(A/B)^{\cdot}]^{\cdot} = 0$ which is a contradiction to our assumption. Hence we must conclude that $\mathcal{H}_t = 0$ so that

$$\mathcal{H} = \kappa \tag{2.78}$$

is a constant (this forces $\mathcal{R} = -(A/B)\iota + \kappa)$. The constant $\mathcal{H}$ of (2.78) implies that γ vanishes from (2.62).

<u>Case II</u>: $\quad [B(A/B)^{\cdot}]^{\cdot} = 0$

Here we find that (2.74) yields

$$B\left(\frac{A}{B}\right)^{\cdot} = \lambda$$

$$(B\dot{\mathcal{R}})^{\cdot} = \frac{\gamma}{B}$$

where λ is a constant. Now taking into account these results and (2.73), equation (2.59) becomes

$$\mathcal{Q}_{xx} - \lambda^2\mathcal{Q} = \lambda B\dot{\mathcal{R}} - \gamma\frac{A}{B}$$

where the variables t and x have separated. Thus in this case we have

$$\mathcal{Q}_{xx} - \lambda^2\mathcal{Q} - \mu = 0 \tag{2.79}$$

$$\lambda B\dot{\mathcal{R}} - \gamma\frac{A}{B} - \mu = 0 \tag{2.80}$$

where μ is a separation constant. Thus we have integrated equations (2.56), (2.59) and (2.62): either $\mathcal{H}$ is a constant, or $\mathcal{Q}$ and $\mathcal{R}$ have to satisfy the ordinary differential equations (2.79) and (2.80).

Next we consider the effect of equations (2.57), (2.60) and (2.63) on the function $\mathcal{J}(t,x)$. The results in this case have the same form obtained for the function $\mathcal{H}(t,x)$. Equation (2.57) gives

$$\mathcal{J} = \frac{A}{B}\mathcal{S}(x) + \mathcal{T}(t) \tag{2.81}$$

where $\mathcal{S}$ and $\mathcal{T}$ are functions resulting from integration. The analogue of (2.74) is

$$\mathcal{S}\left[B\left(\frac{A}{B}\right)^{\cdot}\right]^{\cdot} = \frac{\beta}{B} - (B\dot{\mathcal{T}})^{\cdot}$$

<u>Case I</u>: $\quad [B(A/B)^{\cdot}]^{\cdot} \neq 0$

The function $\mathcal{J}$ in this case is given by

$$\mathcal{J} = \nu \tag{2.82}$$

where ν is a constant. The constant $\mathcal{J}$ of (2.82) implies that β vanishes from equation (2.63).

<u>Case II</u>: $\quad [B(A/B)^{\cdot}]^{\cdot} = 0$

On setting

$$B(A/B)^{\cdot} = \xi$$

$$(B\dot{\mathcal{R}})^{\cdot} = \beta/B$$

$(B\dot{T})^{\cdot}$

where ξ is a constant we find that $\mathcal{S}$ and $\mathcal{T}$ satisfy the ordinary differential equations

$$\mathcal{S}_{xx} - \xi^2 \mathcal{S} - o = 0 \tag{2.83}$$

$$\xi B \dot{\mathcal{T}} - \beta \frac{A}{B} - o = 0 \tag{2.84}$$

where o is a separation constant. Therefore we have integrated (2.57), (2.60) and (2.63): either $\mathcal{J}$ is a constant, or $\mathcal{S}$ and $\mathcal{T}$ satisfy the ordinary differential equations (2.83) and (2.84).

By the above argument we have integrated nine of the twelve integrability conditions (2.55) – (2.66). If $[B(A/B)\dot{}]\dot{} \neq 0$ then the quantities $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{J}$ are constants, or $[B(A/B)\dot{}]\dot{} = 0$ implies a set of five ordinary differential equations given by

$$\mathcal{O}_{xx} - \eta^2 \mathcal{O} - \eta\theta = 0$$

$$\mathcal{Q}_{xx} - \lambda^2 \mathcal{Q} - \mu = 0$$

$$\mathcal{S}_{xx} - \xi^2 \mathcal{S} - o = 0$$

$$\lambda B \dot{\mathcal{R}} - \gamma \frac{A}{B} - \mu = 0$$

$$\xi B \dot{\mathcal{T}} - \beta \frac{A}{B} - o = 0$$

To completely integrate the entire set of integrabliity conditions we also have to solve the three equations (2.64) – (2.66) to obtain $\mathcal{L}$ and $\mathcal{N}$. Equation (2.65) is a first order differential equation in $\mathcal{L}$ with general solution

$$\mathcal{L} = B\int \frac{\mathcal{G}}{\mathcal{B}}dt + B\mathcal{U}(x)$$

where $\mathcal{U}$ is a function of integration. Substituting this form of $\mathcal{L}$ in (2.64) and integrating gives

$$\mathcal{N} = x\left(B\int \frac{\mathcal{G}}{\mathcal{B}}dt\right)^{\cdot} + \dot{B}\int \mathcal{U}dx - x\frac{\dot{A}B}{A}\int \frac{\mathcal{G}}{\mathcal{B}}dt - \frac{\dot{A}}{A}\int \mathcal{U}dx + \mathcal{V}(t)$$

where $\mathcal{V}$ results from the integration process. Therefore equations (2.64) and (2.65) are satisfied with the above functional representations for $\mathcal{L}$ and $\mathcal{N}$. It remains to consider equation (2.66). Now on substituting the above forms of $\mathcal{L}$ and $\mathcal{N}$ into (2.66) we obtain the equation

$$\dot{\mathcal{V}} = \frac{B}{A^2}\mathcal{U}_x + x\left[\left(\frac{\dot{A}B}{A}\int \frac{\mathcal{G}}{\mathcal{B}}dt\right)^{\cdot} - \left(B\int \frac{\mathcal{G}}{\mathcal{B}}dt\right)^{\cdot\cdot}\right] + \left[\left(\frac{\dot{A}}{A}\right)^{\cdot} - \ddot{B}\right]\left(\int \mathcal{U}dx\right)$$

which is a consistency requirement on the integrability functions $\mathcal{U}$ and $\mathcal{V}$. Thus we have fully integrated the integrability conditions (2.55) – (2.66).

2.5 Special cases

To obtain the Killing vectors of the spacetime (2.1) from the general conformal Killing vector (2.50) – (2.53) we need to set $\phi = 0$ in (2.54). This gives the conditions

$$(B^2\mathcal{G}_t)_t = 0$$

$$(B^2\mathcal{H}_t)_t = 0$$

$$(B^2\mathcal{J}_t)_t = 0$$

$$\mathcal{L}_t = 0$$

which together with the integrability conditions (2.55) – (2.66) imply

$$\mathcal{G} = 0$$

$$\mathcal{H} = \text{constant}$$

$$\mathcal{J} = \text{constant}$$

$$\mathcal{L} = 0$$

$$\mathcal{N} = \text{constant}$$

$$\beta = 0$$

$$\gamma = 0$$

Then the components of the general Killing vector **X** are given by

$$X^0 = 0$$

$$X^1 = \mathcal{N}$$

$$X^2 = \delta z + \mathcal{H}$$

$$X^3 = -\delta y + \mathcal{J}$$

so that

$$\mathbf{X} = \mathcal{N}\frac{\partial}{\partial x} + (\delta z + \mathcal{H})\frac{\partial}{\partial y} + (-\delta y + \mathcal{J})\frac{\partial}{\partial z}$$

The Killing vectors given in §2.2 are obtained from $\mathbf{X}$ above by appropriate choices of $\mathcal{H}$, $\mathcal{J}$, $\mathcal{N}$ and δ.

The homothetic Killing vector of the spacetime (2.1) is obtained as a special case of (2.50) – (2.53) by taking ϕ to be a nonzero constant in (2.54). We obtain from (2.54) the following conditions

$$(B^2\mathcal{G}_t)_t = 0$$

$$(B^2\mathcal{H}_t)_t = 0$$

$$(B^2\mathcal{J}_t)_t = 0$$

$$\mathcal{L}_t = \phi$$

which together with the integrability conditions (2.55) – (2.66) imply

$$\mathcal{G} = \text{constant}$$

$$\mathcal{H} = \text{constant}$$

$$\mathcal{J} = \text{constant}$$

$$\mathcal{L} = \phi t + \pi$$

$$\mathcal{N} = \varpi x + \rho$$

$$\beta = 0$$

$$\gamma = 0$$

$$\frac{\dot{A}}{A} = \frac{\varpi - \phi}{\phi t + \pi}$$

$$\frac{\dot{B}}{B} = \frac{\mathcal{G} - \phi}{\phi t + \pi}$$

where π, ϖ and ρ are constants. Therefore the existence of a homothetic vector places the following restrictions on the gravitational field:

$$A = \varrho(\phi t + \pi)^{(\varpi - \phi)/\phi}$$

$$B = \sigma(\phi t + \pi)^{(\mathcal{G} - \phi)/\phi}$$

where ϱ and σ are constants. The components of the homothetic Killing vector $\mathbf{X}$ are given by

$$X^0 = \phi t + \pi$$

$$
\begin{aligned}
X^1 &= \varpi x + \rho \\
X^2 &= \delta z + \mathcal{G} y + \mathcal{H} \\
X^3 &= -\delta y + \mathcal{G} z + \mathcal{J}
\end{aligned}
$$

so that

$$\mathbf{X} = (\phi t + \pi)\frac{\partial}{\partial t} + (\varpi x + \rho)\frac{\partial}{\partial x} + (\delta z + \mathcal{G} y + \mathcal{H})\frac{\partial}{\partial y} + (-\delta y + \mathcal{G} z + \mathcal{J})\frac{\partial}{\partial z}$$

is the general homothetic Killing vector.

The special conformal Killing vector is obtained from (2.54) with the restrictions

$$
\begin{aligned}
\phi_{;ab} &= 0 \\
\phi_{,ab} - \Gamma^c{}_{ab}\phi_{,c} &= 0
\end{aligned}
$$

on the conformal factor ϕ. These equations provide a further constraint on ϕ in terms of the ten partial differential equations:

$$
\begin{aligned}
\phi_{xx} &= A\dot{A}\phi_t \\
\phi_{yy} &= B\dot{B}\phi_t \\
\phi_{zz} &= B\dot{B}\phi_t
\end{aligned}
$$

$$
\begin{aligned}
\phi_{xt} &= \frac{\dot{A}}{A}\phi_x \\
\phi_{yt} &= \frac{\dot{B}}{B}\phi_y \\
\phi_{zt} &= \frac{\dot{B}}{B}\phi_z \\
\phi_{tt} &= 0 \\
\phi_{yx} &= 0 \\
\phi_{zx} &= 0 \\
\phi_{yz} &= 0
\end{aligned}
$$

The above system of equations has to be solved together with the integrability conditions (2.55) – (2.66) to obtain the special conformal vector **X**. However on analysing this system of ten equations we find that we must have

$$\phi = \text{constant}$$

so that the conformal vector **X** reduces to a homothetic Killing vector. Hence we conclude that the spacetime (2.1) does not admit a nontrivial special conformal vector. This is consistent with the result of Coley and Tupper (1990a): orthogonal synchronous perfect fluid spacetimes, other than Robertson–Walker, admit no proper inheriting conformal Killing vector.

3 Conformal symmetries in models with metric

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{2x}B^2(t)[dy^2 + dz^2]$$

3.1 Introduction

In this chapter we study another locally rotationally symmetric spacetime which is spatially homogeneous but anisotropic. As in chapter 2 there is a G_4 of motions acting transitively on spacelike hypersurfaces S_3 but here the gravitational field is dependent on the coordinates t and x. A consequence of the positional dependence in the metric functions is that the integration process is much more complicated. In §3.2 we consider the geometry of the spacetime of the above metric. We give the generators of the Lie algebra of Killing vectors and the positive structure constants. The conformal Killing equation is given in the form of a coupled system of ten first order, linear partial differential equations in §3.3. We obtain the general conformal Killing vector $\mathbf{X}$ and the conformal factor ϕ by solving this system of equations subject to integrability conditions. These conditions place consistency requirements on the metric functions $A(t)$ and $B(t)$. It is interesting to observe that even though the gravitational field depends also on position it is still possible to integrate the coupled system to obtain the conformal Killing vector $\mathbf{X}$. However the x dependence of the metric functions complicates the integrability conditions as stated in §3.4. We are unable to fully integrate these conditions as in chapter 2. In §3.5 we obtain the Killing vectors as a special case from our general solution. Also we find the homothetic vector from the general conformal Killing vector with ϕ as a nonzero constant. The existence of a homothetic vector severely restricts the metric functions $A(t)$ and $B(t)$. We investigate the case of the existence of a special conformal vector.

3.2 Spacetime Geometry

In this chapter we consider another spatially homogeneous and anisotropic spacetime which is described by the metric

$$ds^2 = -dt^2 + A^2(t)dx^2 + e^{2x}B^2(t)[dy^2 + dz^2] \tag{3.1}$$

with coordinates $x^a = (t, x, y, z)$. The spacetime (3.1) is called class $A3$ in the MacCallum (1980) classification scheme and is referred to as type V by Petrov (1969). The metric (3.1) is another example of a locally rotationally symmetric spacetime (see §2.2). The difference from (2.1) is that the gravitational field here depends on the spacelike coordinate x, via the exponential factor e^{2x}, in addition to the timelike coordinate t. As in chapter 2 a G_4 of motions is admitted which acts transitively on spacelike hypersurfaces S_3. There are simply transitive subgroups G_3 acting on hypersurfaces S_3 of Bianchi type V and VII_h. Also there is a multiply transitive G_3 subgroup of Bianchi type VII_0 acting on S_2. The Lie algebra of Killing vectors $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_4\}$ is spanned by

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z} \\
\mathbf{X}_2 &= \frac{\partial}{\partial y}
\end{aligned}$$

$$\mathbf{X}_3 = \frac{\partial}{\partial z}$$

$$\mathbf{X}_4 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}$$

For the above Lie algebra we obtain the following positive structure constants from equation (1.23):

$$C^2{}_{12} = C^2{}_{43} = 1$$

$$C^3{}_{13} = C^3{}_{24} = 1$$

3.3 Conformal equation

For the metric (3.1) the conformal equation (1.22) reduces to the following system of ten equations:

$$X^0_t = \phi \tag{3.2}$$

$$A^2 X^1_t - X^0_x = 0 \tag{3.3}$$

$$e^{2x} B^2 X^2_t - X^0_y = 0 \tag{3.4}$$

$$e^{2x} B^2 X^3_t - X^0_z = 0 \tag{3.5}$$

$$\dot{A}X^0 + AX^1_x = A\phi \tag{3.6}$$

$$e^{2x}B^2X^2_x + A^2X^1_y = 0 \tag{3.7}$$

$$e^{2x}B^2X^3_x + A^2X^1_z = 0 \tag{3.8}$$

$$\dot{B}X^0 + BX^1 + BX^2_y = B\phi \tag{3.9}$$

$$X^3_y + X^2_z = 0 \tag{3.10}$$

$$\dot{B}X^0 + BX^1 + BX^3_z = B\phi \tag{3.11}$$

where we follow the notation of chapter 2. The equations (3.2) – (3.11) are a coupled system of ten first order, linear partial differential equations which we have to integrate to obtain $\mathbf{X} = (X^0, X^1, X^2, X^3)$ and ϕ in terms of the metric functions $A(t)$ and $B(t)$. A difference between (3.2) – (3.11) and (2.2) – (2.11) is the presence of the exponential factor e^{2x}. It seems as though this factor would greatly complicate the integration procedure. However it turns out that the integration process closely parallels that of §2.3. We present all steps in the integration process for completeness and consistency. As in chapter 2 we take combinations of derivatives of (3.2) – (3.11) to yield identities that will simplify the integration process.

The derivative of (3.5) with respect to y is subtracted from the derivative of (3.4) with respect to z to obtain $X^2_{zt} - X^3_{yt} = 0$. This result together with the derivative of (3.10) with respect to t yields the identities

$$X^2_{zt} = 0 \tag{3.12}$$

$$X^3_{yt} = 0 \tag{3.13}$$

A combination of the derivative of (3.7) with respect to z and the derivative of (3.8) with respect to y gives $X^2_{zx} - X^3_{yx} = 0$. This result and the derivative of (3.10) with respect to x yields

$$X^2_{zx} = 0 \tag{3.14}$$

$$X^3_{yx} = 0 \tag{3.15}$$

The equations (3.12) and (3.14) imply that X^2_z is a function only of y and z. Similarly (3.13) and (3.15) imply that X^3_y is also a function only of y and z. Differentiating (3.4) with respect to z and using (3.12) gives

$$X^0_{yz} = 0 \tag{3.16}$$

Then equation (3.2) differentiated with respect to y and z implies that

$$\phi_{yz} = 0 \tag{3.17}$$

Now equation (3.7) gives $X^1_{yz} = 0$ where we have utilised (3.14). Differentiate equations (3.9) and (3.11) by both y and z to obtain the equations

$$\left(X^2_z\right)_{yy} = 0 \tag{3.18}$$

$$\left(X^3_y\right)_{zz} = 0 \tag{3.19}$$

where we have used (3.16), (3.17) and $X^1_{yz} = 0$.

Integrating (3.18) and (3.19) gives

$$X^2_z = \mathcal{A}(z)y + \mathcal{B}(z) \tag{3.20}$$

$$X^3_y = \mathcal{C}(y)z + \mathcal{D}(y) \tag{3.21}$$

where $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are functions of integration. By an abuse of notation we use the same symbols for the functions of integration as in section §2.3 – this should not lead to any ambiguity. On substituting (3.20) and (3.21) into (3.10) we obtain

$$\mathcal{A}(z)y + \mathcal{B}(z) + \mathcal{C}(y)z + \mathcal{D}(y) = 0$$

which necessarily implies

$$\begin{aligned} \mathcal{A}(z) &= \alpha z + \beta \\ \mathcal{B}(z) &= -\gamma z + \delta \\ \mathcal{C}(y) &= -\alpha y + \gamma \\ \mathcal{D}(y) &= -\beta y - \delta \end{aligned}$$

where α, β, γ and δ are constants. We substitute these results into (3.20) and (3.21) and integrate to obtain the components

$$X^2 = \tfrac{1}{2}\alpha y z^2 + \beta y z - \tfrac{1}{2}\gamma z^2 + \delta z + \mathcal{E}(t, x, y) \tag{3.22}$$

$$X^3 = -\tfrac{1}{2}\alpha z y^2 + \gamma y z - \tfrac{1}{2}\beta y^2 - \delta y + \mathcal{F}(t, x, z) \tag{3.23}$$

where $\mathcal{E}$ and $\mathcal{F}$ are functions of integration. Note that (3.22) and (3.23) have the same form as (2.22) and (2.23).

Differentiating (3.9) and (3.11) by y, differentiating (3.10) by z, and forming a sum gives an equation with X^2 as the dependent variable:

$$X^2_{yy} + X^2_{zz} = 0 \tag{3.24}$$

Similarly, differentiating (3.9) and (3.11) by z, differentiating (3.10) by y, and forming a sum gives an equation in X^3:

$$X^3_{yy} + X^3_{zz} = 0 \tag{3.25}$$

The function $\mathcal{E}(t,x,y)$ is found by substituting (3.22) into (3.24) to obtain the differential equation

$$\mathcal{E}_{yy} + \alpha y - \gamma = 0$$

which has the solution

$$\mathcal{E}(t,x,y) = -\tfrac{1}{6}\alpha y^3 + \tfrac{1}{2}\gamma y^2 + y\mathcal{G}(t,x) + \mathcal{H}(t.x) \tag{3.26}$$

where $\mathcal{G}$ and $\mathcal{H}$ result from integration. Similarly we find $\mathcal{F}(t,x,z)$ by substituting (3.23) into (3.25) to obtain

$$\mathcal{F}_{zz} - \alpha z - \beta = 0$$

which has the solution

$$\mathcal{F}(t,x,z) = \tfrac{1}{6}\alpha z^3 + \tfrac{1}{2}\beta z^2 + z\mathcal{I}(t,x) + \mathcal{J}(t,x) \tag{3.27}$$

where $\mathcal{I}$ and $\mathcal{J}$ are functions of the integration process. Now the difference of (3.9) and (3.11) yields

$$X^2_y - X^3_z = 0 \tag{3.28}$$

We obtain the result

$$\mathcal{G} = \mathcal{I}$$

by substituting (3.22) and (3.23) in (3.28), and using (3.26) and (3.27). Thus equations (3.22) and (3.23) take the forms

$$X^2 = \tfrac{1}{2}\alpha y z^2 + z(\beta y + \delta) + \tfrac{1}{2}\gamma\left(y^2 - z^2\right) - \tfrac{1}{6}\alpha y^3 + y\mathcal{G} + \mathcal{H} \tag{3.29}$$

$$X^3 = -\tfrac{1}{2}\alpha z y^2 + y(\gamma z - \delta) - \tfrac{1}{2}\beta\left(y^2 - z^2\right) + \tfrac{1}{6}\alpha z^3 + z\mathcal{G} + \mathcal{J} \tag{3.30}$$

where the components X^2 and X^3 are expressed in terms of the integration functions $\mathcal{G}(t,x), \mathcal{H}(t,x)$ and $\mathcal{J}(t,x)$. We note that up to this stage the results are similar to the case of the metric (2.1) in chapter 2. The exponential factor e^{2x} of the metric (3.1) has an influence in the remainder of the calculation. Also note that the y and z dependence of the components X^2 and X^3 of $\mathbf{X}$ have been completely determined. We have to find the remaining components X^0, X^1 of $\mathbf{X}$, and ϕ.

Substituting (3.29) into (3.4) yields the component

$$X^0 = e^{2x}B^2(\tfrac{1}{2}y^2\mathcal{G}_t + y\mathcal{H}_t) + \mathcal{K}(t,x,z) \tag{3.31}$$

where $\mathcal{K}$ is a function of integration. By substituting (3.30) and (3.31) into (3.5) we obtain the function $\mathcal{K}$:

$$\mathcal{K}_z = e^{2x}B^2(z\mathcal{G}_t + \mathcal{J}_t)$$

$$\mathcal{K}(t,x,z) = e^{2x}B^2(\tfrac{1}{2}z^2\mathcal{G}_t + z\mathcal{J}_t) + \mathcal{L}(t,x)$$

with $\mathcal{L}$ being a function of integration. Therefore (3.31) can be written as

$$X^0 = \tfrac{1}{2}e^{2x}B^2\mathcal{G}_t(y^2 + z^2) + e^{2x}B^2(y\mathcal{H}_t + z\mathcal{J}_t) + \mathcal{L} \tag{3.32}$$

We now substitute (3.29) into (3.7) to obtain the component

$$X^1 = -e^{2x}\left(\frac{B}{A}\right)^2(\tfrac{1}{2}y^2\mathcal{G}_x + y\mathcal{H}_x) + \mathcal{M}(t,x,z) \tag{3.33}$$

where the function $\mathcal{M}$ results from integration. The equations (3.30), (3.33) and (3.8) give the following restriction of $\mathcal{M}$:

$$
\begin{aligned}
\mathcal{M}_z &= -e^{2x}\left(\frac{B}{A}\right)^2 (z\mathcal{G}_x + \mathcal{J}_x) \\
\mathcal{M}(t,x,z) &= -e^{2x}\left(\frac{B}{A}\right)^2 (\tfrac{1}{2}z^2\mathcal{G}_x + z\mathcal{J}_x) + \mathcal{N}(t,x)
\end{aligned}
$$

with $\mathcal{N}$ being a function of integration. Thus (3.33) becomes

$$
X^1 = -\tfrac{1}{2}e^{2x}\left(\frac{B}{A}\right)^2 \mathcal{G}_x(y^2+z^2) - e^{2x}\left(\frac{B}{A}\right)^2 (y\mathcal{H}_x + z\mathcal{J}_x) + \mathcal{N} \qquad (3.34)
$$

Hence (3.32), (3.34), (3.29) and (3.30) give X^0, X^1, X^2 and X^3 respectively. With these forms of X^0, X^1, X^2 and X^3 equations (3.4), (3.5), (3.7), (3.8) and (3.10) are satisfied. It remains to consider equations (3.2), (3.3), (3.6), (3.9) and (3.11). These equations will restrict the functions of integration, the metric functions and the conformal factor ϕ.

By substituting (3.32) and (3.34) into (3.3) we obtain the following integrability conditions:

$$
\left(\frac{B}{A}\mathcal{G}_x\right)_t + \frac{B}{A}\mathcal{G}_t = 0 \qquad (3.35)
$$

$$
\left(\frac{B}{A}\mathcal{H}_x\right)_t + \frac{B}{A}\mathcal{H}_t = 0 \qquad (3.36)
$$

$$
\left(\frac{B}{A}\mathcal{J}_x\right)_t + \frac{B}{A}\mathcal{J}_t = 0 \qquad (3.37)
$$

$$
\mathcal{L}_x - A^2\mathcal{N}_t = 0 \qquad (3.38)
$$

We obtain the conformal factor

$$\phi = \tfrac{1}{2}(y^2 + z^2)e^{2x}(B^2\mathcal{G}_t)_t + ye^{2x}(B^2\mathcal{H}_t)_t + ze^{2x}(B^2\mathcal{J}_t)_t + \mathcal{L}_t \tag{3.39}$$

from (3.2) and (3.32). Equation (3.6) gives the following set of integrability conditions

$$(B^2\mathcal{G}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{G}_t + \left(\frac{B}{A}\right)^2(2\mathcal{G}_x + \mathcal{G}_{xx}) = 0 \tag{3.40}$$

$$(B^2\mathcal{H}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{H}_t + \left(\frac{B}{A}\right)^2(2\mathcal{H}_x + \mathcal{H}_{xx}) = 0 \tag{3.41}$$

$$(B^2\mathcal{J}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{J}_t + \left(\frac{B}{A}\right)^2(2\mathcal{J}_x + \mathcal{J}_{xx}) = 0 \tag{3.42}$$

$$\frac{\dot{A}}{A}\mathcal{L} - \mathcal{L}_t + \mathcal{N}_x = 0 \tag{3.43}$$

where we have used the equations (3.32), (3.34) and (3.39). Equation (3.9) together with (3.29), (3.32), (3.34) and (3.39) gives the integrability conditions

$$(B^2\mathcal{G}_t)_t - \dot{B}B\mathcal{G}_t + \left(\frac{B}{A}\right)^2\mathcal{G}_x - \alpha = 0 \tag{3.44}$$

$$(B^2\mathcal{G}_t)_t - \dot{B}B\mathcal{G}_t + \left(\frac{B}{A}\right)^2\mathcal{G}_x + \alpha = 0 \tag{3.45}$$

$$(B^2\mathcal{H}_t)_t - \dot{B}B\mathcal{H}_t + \left(\frac{B}{A}\right)^2\mathcal{H}_x - e^{-2x}\gamma = 0 \tag{3.46}$$

$$(B^2\mathcal{J}_t)_t - \dot{B}B\mathcal{J}_t + \left(\frac{B}{A}\right)^2\mathcal{J}_x - e^{-2x}\beta = 0 \tag{3.47}$$

$$\frac{\dot{B}}{B}\mathcal{L} - \mathcal{L}_t + \mathcal{G} + \mathcal{N} = 0 \tag{3.48}$$

Note that (3.11) also gives the integrability conditions (3.44) – (3.48). Equations (3.44) and (3.45) imply that

$$\alpha = 0 \tag{3.49}$$

Thus (3.44) and (3.45) are equivalent.

Thus the system (3.2) – (3.11) of partial differential equations has the general solution (3.29), (3.30), (3.32), (3.34) and (3.39) subject to the integrability conditions (3.35) – (3.38) and (3.40) – (3.49). Collecting these results we have the following solution set with the integration functions $\mathcal{G}, \mathcal{H}, \mathcal{L}, \mathcal{J}$ and $\mathcal{N}$ being functions of t and x only:

$$X^0 = \tfrac{1}{2}e^{2x}B^2\mathcal{G}_t(y^2+z^2) + e^{2x}B^2(y\mathcal{H}_t + z\mathcal{J}_t) + \mathcal{L} \tag{3.50}$$

$$X^1 = -\tfrac{1}{2}e^{2x}\left(\frac{B}{A}\right)^2\mathcal{G}_x(y^2+z^2) - e^{2x}\left(\frac{B}{A}\right)^2(y\mathcal{H}_x + z\mathcal{J}_x) + \mathcal{N} \tag{3.51}$$

$$X^2 = z(\beta y + \delta) + \tfrac{1}{2}\gamma\left(y^2 - z^2\right) + y\mathcal{G} + \mathcal{H} \tag{3.52}$$

$$X^3 = y(\gamma z - \delta) - \tfrac{1}{2}\beta\left(y^2 - z^2\right) + z\mathcal{G} + \mathcal{J} \tag{3.53}$$

$$\phi = \tfrac{1}{2}(y^2+z^2)e^{2x}(B^2\mathcal{G}_t)_t + ye^{2x}(B^2\mathcal{H}_t)_t + ze^{2x}(B^2\mathcal{J}_t)_t + \mathcal{L}_t \tag{3.54}$$

subject to the following integrability conditions:

$$\left(\frac{B}{A}\mathcal{G}_x\right)_t + \frac{B}{A}\mathcal{G}_t = 0 \tag{3.55}$$

$$\left(\frac{B}{A}\mathcal{H}_x\right)_t + \frac{B}{A}\mathcal{H}_t = 0 \tag{3.56}$$

$$\left(\frac{B}{A}\mathcal{J}_x\right)_t + \frac{B}{A}\mathcal{J}_t = 0 \tag{3.57}$$

$$(B^2\mathcal{G}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{G}_t + \left(\frac{B}{A}\right)^2 (2\mathcal{G}_x + \mathcal{G}_{xx}) = 0 \tag{3.58}$$

$$(B^2\mathcal{H}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{H}_t + \left(\frac{B}{A}\right)^2 (2\mathcal{H}_x + \mathcal{H}_{xx}) = 0 \tag{3.59}$$

$$(B^2\mathcal{J}_t)_t - \frac{\dot{A}B^2}{A}\mathcal{J}_t + \left(\frac{B}{A}\right)^2 (2\mathcal{J}_x + \mathcal{J}_{xx}) = 0 \tag{3.60}$$

$$(B^2\mathcal{G}_t)_t - \dot{B}B\mathcal{G}_t + \left(\frac{B}{A}\right)^2 \mathcal{G}_x = 0 \tag{3.61}$$

$$(B^2\mathcal{H}_t)_t - \dot{B}B\mathcal{H}_t + \left(\frac{B}{A}\right)^2 \mathcal{H}_x - e^{-2x}\gamma = 0 \tag{3.62}$$

$$(B^2\mathcal{J}_t)_t - \dot{B}B\mathcal{J}_t + \left(\frac{B}{A}\right)^2 \mathcal{J}_x - e^{-2x}\beta = 0 \tag{3.63}$$

$$\frac{\dot{A}}{A}\mathcal{L} - \mathcal{L}_t + \mathcal{N}_x = 0 \tag{3.64}$$

$$\frac{\dot{B}}{B}\mathcal{L} - \mathcal{L}_t + \mathcal{G} + \mathcal{N} = 0 \tag{3.65}$$

$$\mathcal{L}_x - A^2\mathcal{N}_t = 0 \tag{3.66}$$

Hence the conformal Killing vector $\mathbf{X}$ given by (3.50) – (3.53) with the conformal factor (3.54) restricts the metric functions $A(t)$ and $B(t)$ by the integrability conditions (3.55) – (3.66).

3.4 Integrability Conditions

The integrability conditions (3.55) – (3.66) comprise a coupled system of partial differential equations for the functions of integration $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{L}$ and $\mathcal{N}$ which severely restrict the metric functions $A(t)$ and $B(t)$. We have not been able to analyse the system of integrability conditions in a comprehensive manner as for the metric (2.1) in §2.4. The system (3.55) – (3.66) is more complicated than the corresponding system (2.55) – (2.66) of chapter 2, and we have not succeeded in particular to integrate (3.55), (3.56) and (3.57) to obtain $\mathcal{G}, \mathcal{H}$ and $\mathcal{J}$. There are particular solutions to this system which we do not pursue here. These need to be taken into account when considering a conformally symmetric gravitational field satisfying Einstein's field equations (1.19).

3.5 Special cases

We obtain the Killing vectors of the spacetime (3.1) from the general conformal Killing vector (3.50) – (3.53) by setting $\phi = 0$ in (3.54). This gives the following set of conditions

$$
\begin{aligned}
(B^2\mathcal{G}_t)_t &= 0 \\
(B^2\mathcal{H}_t)_t &= 0 \\
(B^2\mathcal{J}_t)_t &= 0
\end{aligned}
$$

$$\mathcal{L}_t = 0$$

These together with the integrability conditions (3.55) – (3.66) imply

$$\mathcal{G} = \text{constant}$$

$$\mathcal{H} = \text{constant}$$

$$\mathcal{J} = \text{constant}$$

$$\mathcal{N} = -\mathcal{G}$$

$$\mathcal{L} = 0$$

$$\beta = 0$$

$$\gamma = 0$$

Hence the components of the general Killing vector $\mathbf{X}$ are given by

$$X^0 = 0$$

$$X^1 = \mathcal{N}$$

$$X^2 = \delta z + \mathcal{G}y + \mathcal{H}$$

$$X^3 = -\delta y + \mathcal{G}z + \mathcal{J}$$

so that we have

$$\mathbf{X} = \mathcal{N}\frac{\partial}{\partial x} + (\delta z + \mathcal{G}y + \mathcal{H})\frac{\partial}{\partial y} + (-\delta y + \mathcal{G}z + \mathcal{J})\frac{\partial}{\partial z}$$

Now we can obtain the Killing vectors given in §3.2 from $\mathbf{X}$ by appropriate choices of $\mathcal{G}, \mathcal{H}, \mathcal{J}, \mathcal{N}$ and δ.

The homothetic Killing vector of the metric (3.1) is obtained as a special case of (3.50) – (3.53) by setting ϕ to be a nonzero constant in (3.54). Thus from (3.54) we obtain the following conditions

$$(B^2\mathcal{G}_t)_t = 0$$

$$(B^2\mathcal{H}_t)_t = 0$$

$$(B^2\mathcal{J}_t)_t = 0$$

$$\mathcal{L}_t = \phi$$

which together with the integrability conditions (3.55) – (3.66) imply

$$\mathcal{G} = \text{constant}$$

$$\mathcal{H} = \text{constant}$$

$$\mathcal{J} = \text{constant}$$

$$\mathcal{N} = \text{constant}$$

$$\mathcal{L} = \phi t + \epsilon$$

$$\beta = 0$$

$$\gamma = 0$$

$$\frac{\dot{A}}{A} = \frac{\phi}{\phi t + \epsilon}$$

$$\frac{\dot{B}}{B} = \frac{\phi - \mathcal{G} - \mathcal{N}}{\phi t + \epsilon}$$

with ϵ being a constant. Therefore the existence of a homothetic Killing vector imposes the following restrictions on the gravitational field:

$$A = \varepsilon(\phi t + \epsilon)$$

$$B = \zeta(\phi t + \epsilon)^{(\phi - \mathcal{G} - \mathcal{N})/\phi}$$

where ε and ζ are constants. The components of the homothetic Killing vector $\mathbf{X}$ are in this case given by

$$X^0 = \phi t + \epsilon$$

$$X^1 = \mathcal{N}$$

$$X^2 = \delta z + \mathcal{G} y + \mathcal{H}$$

$$X^3 = -\delta y + \mathcal{G}z + \mathcal{J}$$

so that

$$\mathbf{X} = (\phi t + \epsilon)\frac{\partial}{\partial t} + \mathcal{N}\frac{\partial}{\partial x} + (\delta z + \mathcal{G}y + \mathcal{H})\frac{\partial}{\partial y} + (-\delta y + \mathcal{G}z + \mathcal{J})\frac{\partial}{\partial z}$$

is the general homothetic Killing vector.

The special conformal Killing vector is obtained from (3.54) with the restrictions

$$\phi_{;ab} = 0$$

$$\phi_{,ab} - \Gamma^c{}_{ab}\phi_{,c} = 0$$

on the conformal factor ϕ. These two equations provide a further constraint on ϕ in terms of the following ten partial differential equations:

$$\phi_{xx} = A\dot{A}\phi_t$$

$$\phi_{yy} = e^{2x}B\dot{B}\phi_t$$

$$\phi_{zz} = e^{2x}B\dot{B}\phi_t$$

$$\phi_{xt} = \frac{\dot{A}}{A}\phi_x$$

$$\phi_{yt} = \frac{\dot{B}}{B}\phi_y$$

$$\phi_{zt} = \frac{\dot{B}}{B}\phi_z$$

$$\phi_{tt} = 0$$

$$\phi_{yx} = \phi_y$$

$$\phi_{zx} = \phi_z$$

$$\phi_{yz} = 0$$

As in chapter 2 the above system of equations has to be solved together with the integrability conditions (3.55) – (3.66) to obtain the special conformal vector **X**. On analysing this system of ten equations we however find that we must have

$$\phi = \text{constant}$$

so that the conformal vector **X** again reduces to a homothetic Killing vector (see §2.5). Hence we conclude that the spacetime (3.1) also does not admit a nontrivial special conformal vector. This is another example of a spacetime satisfying the theorem of Coley and Tupper (1990a): orthogonal synchronous perfect fluid spacetimes, other than Robertson–Walker, admit no proper inheriting conformal Killing vector.

4 Homothetic symmetries in models with metric

$$\mathbf{ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2}$$

4.1 Introduction

In the previous two chapters we have studied the conformal symmetries of spacetimes with four Killing vectors. In this chapter we consider a gravitational field with less symmetry, namely a spacetime with three Killing vectors. We briefly analyse the homothetic symmetries of the Bianchi I spacetime which is a generalisation of the locally rotationally symmetric metric studied in chapter 2. The metric and the Lie algebra of Killing vectors of the Bianchi I spacetime are given in §4.2. The ten components of the conformal Killing equation (1.22) are listed in §4.3. This system is integrated for a constant conformal factor ϕ subject to integrability conditions on the three time–dependent metric functions. In §4.4 the general homothetic Killing vector admitted by a Bianchi I spacetime is obtained and the integrability conditions are solved to generate the explicit functional dependence of the metric functions. The existence of a homothetic symmetry severely restricts the functional form of the metric functions when the conformal factor is a nonzero constant. We have not succeeded in fully integrating the conformal Killing equation in the general case when the conformal factor is dependent on the spacetime coordinates t, x, y and z.

4.2 Spacetime Geometry

The spatially homogeneous and anisotropic spacetime described by the metric

$$ds^2 = -dt^2 + A^2(t)dx^2 + B^2(t)dy^2 + C^2(t)dz^2 \tag{4.1}$$

is a generalisation of the locally rotationally symmetric spacetime (2.1). This spacetime is often used in the study of anisotropic cosmological models. The given metric is a Bianchi I spacetime (Kramer *et al* 1980) with a G_3 of motions. The Lie algebra of Killing vectors $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ is spanned by

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial x} \\
\mathbf{X}_2 &= \frac{\partial}{\partial y} \\
\mathbf{X}_3 &= \frac{\partial}{\partial z}
\end{aligned}$$

As this Lie algebra is Abelian all the structure constants vanish. For a detailed analysis of the group structure and classification of Bianchi cosmologies see Ellis and MacCallum (1969) and Ryan and Shepley (1975).

4.3 Conformal equation

For the metric (4.1) the conformal equation (1.22) reduces to the following system of ten equations:

$$X^0_{\ t} = \phi \tag{4.2}$$

$$A^2X^1_t - X^0_x = 0 \tag{4.3}$$

$$B^2X^2_t - X^0_y = 0 \tag{4.4}$$

$$C^2X^3_t - X^0_z = 0 \tag{4.5}$$

$$\dot{A}X^0 + AX^1_x = A\phi \tag{4.6}$$

$$B^2X^2_x + A^2X^1_y = 0 \tag{4.7}$$

$$C^2X^3_x + A^2X^1_z = 0 \tag{4.8}$$

$$\dot{B}X^0 + BX^2_y = B\phi \tag{4.9}$$

$$C^2X^3_y + B^2X^2_z = 0 \tag{4.10}$$

$$\dot{C}X^0 + CX^3_z = C\phi \tag{4.11}$$

The equations (4.2) – (4.11) are a coupled system of ten first order, linear partial differential equations. This system is more complicated than those of chapters 2 and 3 because the gravitational field has less symmetry. We have only managed to fully integrate the system (4.2) – (4.11) in the case where ϕ is a nonzero constant and thereby obtained the general homothetic Killing vector admitted by the metric (4.1).

In the remainder of this section we provide the explicit argument that leads to the homothetic vector $\mathbf{X}$.

On subtracting the derivative of (4.7) with respect to z from the derivative of (4.8) with respect to y the equation $B^2X^2_{zx} - C^2X^3_{yx} = 0$ is obtained. This result together with the derivative of (4.10) with respect to x yields the identities

$$X^2_{zx} = 0 \tag{4.12}$$

$$X^3_{yx} = 0 \tag{4.13}$$

Differentiating (4.11) by x, y and (4.9) by x, z we obtain

$$X^0_{xy} = 0 \tag{4.14}$$

$$X^0_{xz} = 0 \tag{4.15}$$

where we have used (4.13) and (4.12). Also (4.2) implies that

$$X^0_{xt} = 0 \tag{4.16}$$

Now (4.14), (4.15) and (4.16) imply that X^0_x is a function of x only. Using this fact and integrating (4.2) gives

$$X^0 = \phi t + \mathcal{A}(x) + \mathcal{B}(y, z) \tag{4.17}$$

where $\mathcal{A}$ and $\mathcal{B}$ are functions of integration. Now differentiating (4.3) by y, (4.4) by x and using (4.14) we obtain

$$X^1_{yt} = 0$$

$$X^2_{xt} = 0$$

Thus X^1_y and X^2_x are independent of t. Hence (4.7) will be satisfied only if the following is valid

$$X^1_y = 0 \tag{4.18}$$

$$X^2_x = 0 \tag{4.19}$$

Similarly by differentiating (4.3) with z, (4.5) with x and using (4.15) we obtain the equations

$$X^1_{zt} = 0$$

$$X^3_{xt} = 0$$

Thus X^1_z and X^3_x are independent of the variable t. Hence (4.8) will be satisfied only if we have

$$X^3_x = 0 \tag{4.20}$$

$$X^1_z = 0 \tag{4.21}$$

Then equations (4.17), (4.6) and (4.11) give the results

$$\mathcal{B}_y = 0$$

$$\mathcal{B}_z = 0$$

where we have used (4.18) and (4.20). Therefore equation (4.17) is reduced to the form

$$X^0 = \phi t + \mathcal{A}(x) + \alpha$$

where α is a constant. The constant α is redundant and this equation may be written in the form

$$X^0 = \phi t + \mathcal{C}(x) \tag{4.22}$$

where we have set $\mathcal{C}(x) = \mathcal{A}(x) + \alpha$.

We obtain from equations (4.4), (4.5) and (4.22) the following two restrictions on X^2 and X^3:

$$X^2_t = 0 \tag{4.23}$$

$$X^3_t = 0 \tag{4.24}$$

We observe from (4.23) and (4.24) that X^2 and X^3 are independent of t which means that (4.10) is valid only if

$$X^2_z = 0 \tag{4.25}$$

$$X^3_y = 0 \tag{4.26}$$

Thus we have from (4.19), (4.23) and (4.25) that X^2 is a function only of y. From (4.20), (4.24) and (4.25) we note that X^3 is dependent only on z. Of course it is clear from the above that at this stage X^0 and X^1 are functions both of t and x. Thus far we have integrated the six equations (4.2), (4.4), (4.5), (4.7), (4.8) and (4.10). It remains to integrate (4.3), (4.6), (4.9) and (4.11) of the original coupled system (4.2) - (4.11).

Taking into account (4.22) we rewrite (4.3), (4.6), (4.9) and (4.11) as

$$A^2 X^1_t - \mathcal{C}_x = 0 \tag{4.27}$$

$$\frac{\dot{A}}{A}(\phi t + \mathcal{C}) + X^1_x = \phi \tag{4.28}$$

$$\frac{\dot{B}}{B}(\phi t + \mathcal{C}) + X^2_y = \phi \tag{4.29}$$

$$\frac{\dot{C}}{C}(\phi t + \mathcal{C}) + X^3_z = \phi \tag{4.30}$$

From the system (4.27) – (4.30) we observe that

$$\mathcal{C} = \text{constant}$$

$$X^1_t = 0$$

$$X^1_{xx} = 0$$

$$X^2_{yy} = 0$$

$$X^3_{zz} = 0$$

Thus equation (4.27) is identically satisfied and the components of the general Killing vector $\mathbf{X}$ are given by

$$X^0 = \phi t + \mathcal{C} \tag{4.31}$$

$$X^1 = \beta x + \gamma \tag{4.32}$$

$$X^2 = \delta y + \epsilon \tag{4.33}$$

$$X^3 = \varepsilon z + \zeta \tag{4.34}$$

where $\beta, \gamma, \delta, \epsilon, \varepsilon$ and ζ are constants of integration.

We note at this stage that from the above we may obtain the components of the Killing vector $\mathbf{X}$:

$$X^0 = 0$$

$$X^1 = \gamma$$

$$X^2 = \epsilon$$

$$X^3 = \zeta$$

if we set $\phi = 0$ and do not place any restrictions on the metric functions. With these components we observe that the coupled system (4.2) – (4.11) is identically satisfied. The general Killing vector $\mathbf{X}$ of the spacetime (4.1) is represented by

$$\mathbf{X} = \gamma\frac{\partial}{\partial x} + \epsilon\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}$$

which contains the Lie algebra of Killing vectors given in §4.2.

If $\phi \neq 0$ and on taking into account the components (4.31) – (4.34) we find that the equations (4.28) – (4.30) may be expressed as

$$\frac{\dot{A}}{A} = \frac{\phi - \beta}{\phi t + C} \tag{4.35}$$

$$\frac{\dot{B}}{B} = \frac{\phi - \delta}{\phi t + \mathcal{C}} \tag{4.36}$$

$$\frac{\dot{C}}{C} = \frac{\phi - \varepsilon}{\phi t + \mathcal{C}} \tag{4.37}$$

which are consistency conditions for the integrability of the coupled system of partial differential equations (4.2) – (4.11).

4.4 Homothetic Vector

In the above we have generated the general solution of (4.2) – (4.11) which is given by the system of equations (4.31) – (4.34) for the components X^0, X^1, X^2 and X^3 of the vector $\mathbf{X}$, with nonzero constant homothetic factor ϕ, subject to the integrability conditions (4.35) – (4.37). The general homothetic vector admitted by the metric (4.1) is of the form

$$\mathbf{X} = (\phi t + \mathcal{C})\frac{\partial}{\partial t} + (\beta x + \gamma)\frac{\partial}{\partial x} + (\delta y + \epsilon)\frac{\partial}{\partial y} + (\varepsilon z + \zeta)\frac{\partial}{\partial z}$$

The equations (4.35) – (4.37) govern the functions $A(t), B(t)$ and $C(t)$. These equations are easily integrated to obtain the following restrictions on the metric functions generating the gravitational field:

$$A = \eta(\phi t + \mathcal{C})^{(\phi-\beta)/\phi}$$

$$B = \theta(\phi t + \mathcal{C})^{(\phi-\delta)/\phi}$$

$$C = \vartheta(\phi t + \mathcal{C})^{(\phi - \varepsilon)/\phi}$$

where η, θ and ϑ are constants. Therefore the existence of a homothetic symmetry severely restricts the functional form of the metric functions $A(t), B(t)$ and $C(t)$. The Einstein field equations (1.19) will further restrict the behaviour of the gravitational field.

To obtain the general conformal vector **X** requires the general solution of the coupled system (4.2) – (4.11) where ϕ is no longer a constant but dependent on the spacetime coordinates t, x, y and z. This is much more complicated than the corresponding situations in chapters 2 and 3 because of the appearance of a new metric function $C(t)$. This is related to the fact that the metric (4.1) has less symmetry than the metrics (2.1) and (3.1), ie. only three Killing vectors in contrast to the four Killing vectors of chapters 2 and 3.

5 Conclusion

In this thesis we have analysed the conformal symmetries in a class of anisotropic spacetimes that are spatially homogeneous and locally rotationally symmetric. We have also studied the homothetic symmetries of a Bianchi I spacetime. These spacetimes are often utilised as anisotropic cosmological models. The results obtained in this thesis are original and we have not found any reference to conformal symmetries on locally rotationally symmetric spacetimes in the literature. It should be clear from the results obtained that the study of conformal symmetries in general relativity is interesting, a fertile area of research and much work remains to be done in this field.

In chapter 1 we provided those aspects of differential geometry and general relativity necessary for later chapters. We impose the condition of a conformal symmetry on the spacetime manifold. The form of the metric connection and associated quantities is restricted by a conformal symmetry. These properties are explicitly given in chapter 1.

The conformal symmetries of a class AIa (MacCallum 1980) locally rotationally symmetric spacetime were comprehensively analysed in chapter 2. The conformal Killing equation was fully integrated to obtain the general conformal Killing vector $\mathbf{X}$ in class AIa models subject to integrability conditions on the metric functions. These conditions were analysed in detail and we found that the functions of integration are constants or restrict the metric functions by differential equations.

The Killing vectors are contained in the general conformal solution. We obtain the general homothetic vector as a particular case from the general conformal vector. The functional dependence of the metric functions was determined explicitly from the integrability conditions for the homothetic vector. We also showed in a class AIa spacetime that there is no nontrivial special conformal vector (see also Coley and Tupper (1990a)).

Another anisotropic conformally symmetric spacetime was analysed in chapter 3. This spatially homogeneous spacetime is of class $A3$ (MacCallum 1980) with a G_4 of motions acting on spacelike hypersurfaces S_3. We were able to successfully solve the conformal Killing equation to obtain the general conformal Killing vector $\mathbf{X}$ and the conformal factor which were subject to integrability conditions. It is interesting to note that the integration of the coupled system was not hampered by the fact that the conformally symmetric gravitational field is also dependent on position. However this positional dependence in the metric functions complicates the form of the integrability conditions. We could not comprehensively analyse these conditions as in §2.4. Nevertheless we obtained the Killing vectors as a special case from the general solution. We found the homothetic vector from the general conformal Killing vector for a nonzero constant conformal factor. The explicit functional form of the metric functions is determined for the existence of a homothetic vector. Also we showed that there is no nontrivial special conformal vector in the class $A3$ spacetime (see §2.5 and Coley and Tupper (1990a)).

In chapter 4 we studied a Bianchi I spacetime which possesses less symmetry than those spacetimes in chapters 2 and 3. We could not solve the general conformal Killing equation (1.22) in this spacetime. However we have integrated this coupled system of equations for a constant conformal factor and have generated the homothetic symmetries of the Bianchi I spacetime. We solved the integrability conditions to explicitly determine the functional dependence of the three time-dependent metric functions. A homothetic symmetry severely restricts the metric functions in this spacetime.

We have analysed the conformal symmetries of a class of locally rotationally symmetric spacetimes and the homothetic symmetries of a Bianchi I spacetime. This work may be extended by analysing the conformal symmetries of other locally rotationally symmetric spacetimes, other Bianchi spacetimes and more models of astrophysical and cosmological significance. Also general results on conformal symmetries in general relativity need to be established using the properties (1.24) – (1.29) on the metric connection and associated quantities (see for example Garfinkle (1987) and Sharma (1988)). We may also consider more general symmetries than conformal symmetries on spacetime manifold. Katzin *et al* (1969) consider curvature collineations in general relativity. We have not analysed the Einstein field equations with a conformal symmetry in locally rotationally symmetric spacetimes. This is an area for further research. For solutions to spherically symmetric gravitational fields with a conformal motion the reader is referred to Dyer *et al* (1987) and Maharaj *et al* (1991).

We hope that we have demonstrated that the study of conformal symmetries is a fertile area of research and warrants further investigation.

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