THEORETICAL STUDIES OF THE CROSSFIELD CURRENT-DRIVEN ION ACOUSTIC INSTABILITY

by

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PREFACE

The work described in this thesis was carried out in the Department of Physics, University of Natal, Durban, from January 1976 to December 1979, under the supervision of Professor Manfred A. Hellberg.

These studies represent original work by the author and, except for the part presented in Section 3.5, have not been submitted in any form to this or any other University. Where use was made of the work of others it has been duly acknowledged in the text.

The Guassian c.g.s. system of units has been used for the work undertaken in this thesis. The equations are numbered according to the section in which they appear. A list of symbols used has also been compiled.

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To My Mother

ABSTRACT

Studies on collisionless shock waves and turbulent heating problems in plasmas have aroused considerable interest in electron-ion streaming instabilities. In this thesis a theoretical investigation of the electrostatic crossfield current-driven ion acoustic instability is conducted. For the entire investigation the electrons are assumed to be hot and the ions cold, i.e., $T_e >> T_i (\sim 0)$. The lengthscales and timescales are chosen such that the electrons are magnetized and the ions unmagnetized, with the analysis usually conducted in the ion rest frame.

Using the Vlasov equation, the linear dispersion relation is solved for equilibrium particle velocity distribution functions of the general form $f_{oj}(V_\perp^2, V_\parallel)$. The results obtained are found to reduce to well known forms for the special case of Maxwellian distributions. The author's previously reported work on the effect of inhomogeneities in plasma density, plasma temperature and magnetic field on the instability, is reviewed. An explanation is offered for the reversal in the behaviour of the temperature gradient drift.

The quasilinear development of the instability is investigated. Particle diffusion equations in velocity space are set up, assuming distribution functions of the general form $f_{oj} = f_{oj}(V_\perp^2, V_\parallel, t)$. The equations are solved analytically both for particles resonating with the waves and for non-resonant particles. It is found that electron diffusion is along the external magnetic field while the ions diffuse primarily across the field. An examination of anomalous plasma resistivity indicates an enhancement of the resistivity perpendicular to the magnetic field, as compared to the field-free case. The electron heating rate is found to be greater than the ion heating rate. Under certain conditions, the ion acoustic and the electron-cyclotron drift instabilities are found to produce the same relative heating rate for the two species. Energy studies indicate an exchange

of energy between the waves and the resonant electrons. However, a comparatively small fraction of the total wave energy appears in the form of electrostatic potential energy. A similar result has been reported for electrostatic ion cyclotron waves.

The effect of a sheared magnetic field on the linear instability in a plasma with a density gradient is investigated. The analysis is restricted to the limit when wave growth due to inverse electron Landau damping is small. Magnetic shear is found to be stabilizing and the critical shear length is obtained.

Using a model corresponding to the Double Plasma device, the effect of elastic and inelastic charge-transfer ion-neutral collisions on an incident ion beam are studied. The Bhatnagar-Gross-Krook collision model is adopted and here the investigation is conducted in the electron rest frame. The elastic collisions are found to cause a slowing down of the beam, while the inelastic collisions give rise to an exponential decay in ion beam density. The effect of collisional damping on ion acoustic perturbations superimposed on the ion beam is then determined.

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LIST OF SYMBOLS

SYMBOL	DESCRIPTION
a_j	$n_{oj}/(2\pi C_j^2)^{\frac{1}{2}}$
b	$k_\perp^2 C_e^2/\Omega_e^2 = k_\perp^2 r_e^2$
$\vec{B}$	magnetic field
C	speed of light
C_e	$(T_e/m_e)^{\frac{1}{2}}$ - electron thermal speed
C_i	$(T_i/m_i)^{\frac{1}{2}}$ - ion thermal speed
C_s	$(T_e/m_i)^{\frac{1}{2}}$ - ion sound speed
$d\vec{V}$	$dV_x dV_y dV_z$ - cartesian coordinates
	$V_\perp dV_\perp dV_z d\theta$ - cylindrical coordinates
e	magnitude of electronic charge
f_j	velocity distribution function of particles of type j
f_{oj}	equilibrium distribution of particles of type j
g_{max}	maximum value of quantity g
g_{min}	minimum value of quantity g
i	$\sqrt{-1}$
I_ℓ	modified Bessel function of first kind of order ℓ
$Im(\omega)$	imaginary part of complex quantity ω
J_ℓ	Bessel function of first kind of order ℓ
$\vec{k}$	wave propagation vector
$k_\perp$	component of $\vec{k}$ perpendicular to $\vec{B}$
$k_\parallel$	component of $\vec{k}$ parallel to $\vec{B}$
	$= k_z$ when $\vec{B} = B\hat{z}$
k_D	λ_D^{-1} - inverse electron Debye length

SYMBOL	DESCRIPTION
m_e	electron mass
m_i	ion mass
n_j	density of particles of type j
$\vec{r}$	(x,y,z) - position vector
r_e	electron gyroradius
r_i	ion gyroradius
$Re(\omega)$	real part of complex quantity ω
t	time coordinate
T	temperature
T_e	Boltzmann's constant × electron temperature
T_i	Boltzmann's constant × ion temperature
$\vec{V}$	velocity vector
$\vec{V}_\phi$	wave phase velocity
$\vec{V}_B$	magnetic field gradient drift velocity
$\vec{V}_n$	density gradient drift velocity
$\vec{V}_T$	temperature gradient drift velocity
$\vec{V}_o$	$\vec{E} \times \vec{B}$ electron drift velocity
$(\hat{x},\hat{y},\hat{z})$	set of rectangular unit vectors
$Z(\lambda)$	plasma dispersion function with argument λ
z^*	complex conjugate of z
Γ_p	$e^{-b} I_p(b)$
γ_k	imaginary part of complex ω_k
ε_L	longitudinal dielectric constant
λ_D	$(T_e/4\pi n_e e^2)^{\frac{1}{2}}$ - electron Debye length
Ω_e	electron gyrofrequency
Ω_i	ion gyrofrequency

SYMBOL	DESCRIPTION
ω_k^r	complex frequency of mode with wave number k
ω_r^k	real part of ω_k
ω_{pe}	$(4\pi n_e e^2/m_e)^{\frac{1}{2}}$-electron plasma frequency
	$= k_D C_e$
ω_{pi}	ion plasma frequency
∇	$\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$
∇B	magnetic field gradient
∇n	density gradient
∇T	temperature gradient
ξ	$k_\perp V_\perp/\Omega_e$
$\Psi_{oj}(V_z,t)$	$\int f_{oj}(V_\perp^2,V_z,t)\, V_\perp\, dV_\perp$
$\Phi_{oj}(V_\perp^2,t)$	$\int f_{oj}(V_\perp^2,V_z,t)\, dV_z$

CHAPTER ONE

INTRODUCTION

1.1 PLASMA CONFINEMENT AND SHOCK WAVES

The considerable interest in, and intense study of plasma instabilities in recent years have arisen from several sources. Among these are:

(i) the structure of, and energy dissipation in collisionless shock waves in plasmas,

(ii) the stability of a plasma as regards thermonuclear fusion experiments, and

(iii) astrophysical and space phenomena such as sunspots and emission of radio waves by galaxies.

Laboratory plasmas are invariably not in thermodynamic equilibrium and this means that a certain amount of free energy is stored in the plasma. This energy may arise from, for instance, particle drifts due to gradients in plasma density, plasma temperature or magnetic fields, externally induced currents or beams of particles. The plasma can attain thermal equilibrium by allowing the growth of electromagnetic waves and a redistribution of energy among the waves and plasma particles. This process of energy conversion is called an *instability*. Plasma instabilities may be broadly classed into two types - those arising from distortions in physical space, i.e., change in shape, called macroscopic or hydromagnetic instabilities, and those involving changes in velocity space, known as microinstabilities.

At the high temperatures ($\sim 10^8$K) required for thermonuclear reactions the plasma cannot be physically contained by walls. It is therefore magnetically confined. Much of the research in plasma physics has been directed to the development of confining systems which will allow the containment of the plasma at the required density and period of time for fusion reactions to take place. The Lawson condition(1) for a power-producing reactor requires ion temperatures $T_i \geq 10$ keV, and the product of particle density (n) and energy confinement time (τ) of $n\tau > 10^{14}$ cm^{-3} s for a deuterium-tritium fusion reaction.

Plasma configuration schemes designed to confine plasmas may be broadly classed as 'open' systems or 'closed' systems. Of the former, magnetic mirrors and the linear θ- and Z-pinch devices are well known examples. In the case of magnetic mirrors, suitably arranged current-carrying coils produce a magnetic bottle, i.e., a magnetic field in which it is possible to trap plasma particles. The linear θ- and Z-pinches will be discussed later. Confinement times in the open systems are short because particles escape through the open ends. To overcome particle losses the field lines are bent to yield a toroidal magnetic field, thereby forming a closed system.

Of the closed confinement systems the Stellarator and the Tokamak have received most attention. To improve particle confinement in closed systems externally arranged conductors modify the magnetic field so that it has the property of nested magnetic flux surfaces. Such a configuration is present in the Stellarator. In the Tokamak, the conversion of the toroidal magnetic field into a toroidal-poloidal field is achieved with high internal currents coupled to the plasma by means of a transformer. This conversion has been found to improve plasma stability and confinement, and at present the Tokamak is the most promising device in which the conditions for a successful controlled thermonuclear fusion reaction can be realized.

At the high temperatures required for fusion experiments ohmic heating is not suitable since it decreases with temperature, T, as $T^{-3/2}$. There was, thus, a need for more efficient heating mechanisms. Gas dynamic studies have demonstrated that strong shock waves are highly efficient in heating a medium. This led to an interest in shock waves as a potential heating mechanism for plasmas. Shock waves arise as follows. When a wave excited in a plasma attains a large enough amplitude and propagates at speeds above the speed of sound, it steepens as it propagates to form a shock, i.e., a narrow transition layer propagating through the system, separating two regions of local thermodynamic equilibrium which have different densities, temperatures and mean velocities. Shocks with widths substantially smaller than the mean free path for binary collisions are called collisionless.

In addition to the decrease in ohmic heating, at the temperatures required for fusion experiments two-body collisions are rare and the plasma is practically collisionless[(2)]. The important implication of this is that collisional dissipation, the dissipation mechanism of conventional shocks, is no longer possible. By what process, then, is dissipation in a collisionless plasma achieved? A possible mechanism is the excitation of plasma waves by instabilities due to the relative drift between electrons and ions. Furthermore, instabilities can play a significant role in determining the behaviour of shock fronts, which are locations of gradients in magnetic field, plasma density and plasma temperature and therefore constitute a source for particle drifts.

In the open linear θ-pinch device an externally induced, rapidly rising axial magnetic field $\vec{B}$ induces an azimuthal current of density $\vec{j}$ within the plasma. The resulting $\vec{j} \times \vec{B}$ force acts to push the plasma towards the axis; this rapid radial compression produces a shock wave which may provide shock heating. In the case of the Z-pinch the roles of the current $\vec{j}$ and the magnetic field $\vec{B}$ are now reversed; $\vec{j}$ is now axial and $\vec{B}$ azimuthal with the $\vec{j} \times \vec{B}$ force again directed inwards. θ- pinch experiments are conducted in many laboratories, including, for instance, Los Alamos (U.S.A.) and Jülich (West Germany). Besides the linear θ-pinch, experiments are also performed in the toroidal θ-pinch which is designed to overcome end losses of the former. A typical Z-pinch device is Tarantula II at Culham Laboratory, U.K.

1.2 COLLISIONLESS SHOCK EXPERIMENTS

Collisionless shock waves have been produced in a number of plasma laboratories[3,4]. We shall discuss some of the experiments on the structure of shock waves propagating through a plasma perpendicular to an external magnetic field. Most of the shocks were formed in a θ- or Z-pinch.

PAUL *et al*[5] created an imploding shock wave, which propagated radially inwards, in a linear Z-pinch by producing a sharply rising axial current in a thin annular layer (∿1 cm). The measured shock width (1,4 mm) was much less than the mean free paths for ion-ion collisions (5 cm) and ion-neutral collisions (2,5 cm). Although the shock thickness does not eliminate electron-ion collisions, PAUL *et al*[6] discounted these interactions as the dominant dissipation mechanism. They found that the measured temperature of about 40 eV at shock velocity to initial Alfvén velocity ratio M ≃ 2,5 far exceeded that calculated by assuming resistive (ion-electron) dissipation. These results demonstrated that significant collisionless electron heating occurred in the shocks. The authors attributed the dissipation mechanism to plasma instabilities arising from the relative drift between electrons and ions. This process provides an anomalous resistance to the current. The electrons are heated by the subsequent damping of the waves.

Experiments in the linear Z-pinch were continued by PAUL *et al*[7] when they studied the light scattered from a 50 MW ruby laser during the passage of the shock through the laser beam. Measurements of scattered power from the shock indicated the presence of an unusually high level of fluctuation within the shock, which in turn implied the presence of instability and associated nonlinear phenomena. The source of the free energy to drive the instability was attributed to the current in the plasma or drifts due to inhomogeneities within the shock.

Using a θ-pinch device KEILHACKER *et al*[8] investigated the heating of a plasma by collisionless shock waves. From Thomson scattering of laser light the variations in magnetic field, electron density and temperature were measured. Only 20% of the observed electron heating could be explained in terms of adiabatic and collisional resistive heating. The measured electron temperature and shock width indicated an anomalously high effective collision frequency, about two orders of magnitude higher than that for classical binary collisions. The authors then suggested excitation of plasma waves, driven by electron-ion relative drifts, in the high current density of the shock front as the most probable cause of the anomalous plasma resistance.

In a later experiment KEILHACKER *et al*[9] measured the level of density fluctuations in a collisionless shock wave as a function of time for a fixed scattering angle. The maximum level of fluctuations in the shock front was found to be about 250 times the thermal level. The reversal of the frequency shift of the fluctuations with the reversal of the diamagnetic current in the shock front suggested that the electron drift provided the free energy for the instabilities causing the enhanced fluctuations. In this experiment the electron temperature T_e and the ion temperature T_i were such that $T_e \lesssim T_i$.

In a more recent experiment, ASTRAKHANTSEV *et al*[10] investigated the nature of turbulent processes and the mechanism of collisionless dissipation in the front of an electrostatic shock wave. The observed high level of turbulence was attributable to plasma instabilities arising from counter-streaming ion beams. Ion heating was explained in terms of the diffuse scattering of the ions by the oscillations which were excited.

Astrophysical observations have revealed that when the solar wind (plasma flow from the sun) encounters the earth's magnetic field, a collisionless shock is formed. This transition layer is called the bow shock. Measurements, via satellites, have indicated a high level of electrostatic fluctuations within the shock[2] and was associated with current-driven plasma instabilities.

Experiments on shock structure including those discussed above, have revealed that the shock width L and the time of passage of the shock τ satisfy the following conditions:

$$\Omega_i < \tau^{-1} < |\Omega_e|$$
$$r_e < L < r_i \qquad (1.2.1)$$

where $\Omega_i(\Omega_e)$ is the ion (electron) gyrofrequency, $\Omega_j = q_j B/m_j C$, and r_i (r_e) is the ion (electron) gyroradius.

Under such conditions the effect of the magnetic field on the ions, as compared to that on the electrons, is negligible, and the ions may be considered to be unmagnetized.

It is thus seen that plasma instabilities play an important role in collisionless shock dissipation of plasma energy.

1.3 SUMMARY OF THESIS

This thesis is concerned with a theoretical study of linear and quasilinear aspects of a particular plasma instability - the electrostatic crossfield current-driven ion acoustic instability. The principal electrostatic instabilities and associated linear investigations are summarized in Chapter Two. Experimental studies of the ion acoustic instability are discussed and the development of the quasilinear theory is reviewed.

In Chapter Three the linear dispersion relation is established and solved for any equilibrium velocity distribution function of the type $f_{oj} = f_{oj}(V_\perp^2, V_z)$, where V_z and $V_\perp$ are the components of the velocity $\vec{V}$

along and perpendicular to $\vec{B} = B\hat{z}$ respectively. For the case of Maxwellian electron and ion equilibrium velocity distributions, the general results are shown to reduce to well known forms. For the purpose of completeness, a previously reported study by the author on the effect of inhomogeneities is reviewed. The quasilinear diffusion equations for the electrons and ions are set up and solved in Chapter Four. Electron and ion heating rates and anomalous resistivity studies are presented. Total energy conservation is then discussed. Finally, a brief comparison is made with the heating rates associated with the electron-cyclotron drift instability.

The effect of a density gradient and a sheared magnetic field on the ion acoustic instability are studied in Chapter Five. In Chapter Six we examine the linear effects of elastic and inelastic charge-transfer collisions on the crossfield current-driven ion acoustic instability. Here the model is chosen to correspond to that of the Double Plasma device (DP-device) in the Plasma Physics Research Institute, University of Natal, Durban. Finally in Chapter Seven a summary is presented of the major findings in the preceding chapters. Conclusions are drawn, and possible extensions to the investigations undertaken are discussed.

CHAPTER TWO

A SURVEY OF LINEAR AND QUASILINEAR INVESTIGATIONS OF PLASMA INSTABILITIES

2.1 INTRODUCTION

In investigations of crossfield current-driven plasma instabilities more attention has been given to electrostatic instabilities, where perturbations in the magnetic field are neglected and the electric field is assumed to be derivable from a scalar potential. The reason for this is that such modes constitute the fastest growing and most destructive instabilities as far as plasma confinement is concerned.

Instabilities within a plasma may be divided into two types[11]:

(a) Dissipative instability: arising from an exchange of energy between the plasma particles and the wave. The physical mechanism for Landau damping - collisionless damping of a wave - is associated with the strong interaction between a longitudinal plasma wave and those particles in the plasma having velocities close to the wave phase velocity $\vec{V}_\phi$. Of these, particles with velocity $\vec{V}<\vec{V}_\phi$ are accelerated, while those with $\vec{V}>\vec{V}_\phi$ are decelerated. If there are more particles travelling with velocities slightly less than $\vec{V}_\phi$

than with velocities slightly larger, the net result is an extraction of energy from the wave with consequent wave damping. However, when the reverse holds, energy is transferred from the particles to the wave, which grows in amplitude. This is known as 'inverse' Landau damping and can, if sufficiently strong, lead to instability.

(b) <u>Reactive instabilities</u>: involve a coupling between two waves which carry energy of opposite sign to each other. There is an energy exchange between the two waves, but *not* between the wave and the plasma. For a linear electrostatic wave the energy density is given by:

$$\xi = \frac{1}{4} \varepsilon_o \, |\vec{E}|^2 \frac{\partial}{\partial\omega} \{\omega \; \varepsilon_L(\omega,k)\}_{\varepsilon_L} = 0$$

where ε_L is the longitudinal dielectric constant. LASHMORE-DAVIES [12] shows that for $\xi < 0$ (i.e., a negative energy wave) one requires $0 < \omega < \vec{k}.\vec{V}_D$. Negative energy waves have the unusual property that they *grow* when energy is extracted from them. Thus when a negative and a positive energy wave couple, both grow. Similarly, a negative energy wave grows when energy is extracted from it by resonant particles (dissipative case).

It should be noted that because dissipative instabilities involve relatively few particles (the resonant ones), while reactive instabilities involve the whole distribution function which supports the wave, the latter are more difficult to stabilize and in that sense are most dangerous.

2.2 THEORETICAL LINEAR INVESTIGATIONS

Of the plasma instabilities that occur, there are three which have received considerable attention. These are the ion acoustic instability, the electron-cyclotron drift instability (also known as the beam cyclotron or Bernstein instability) and the modified two-stream instability. A review of the reported linear aspects of these instabilities is now presented.

(a) Ion acoustic instability

Ion acoustic waves satisfy the dispersion relation:

$$\omega = \frac{k\, C_s}{(1 + k^2\, \lambda_D{}^2)^{\frac{1}{2}}}$$

where $C_s = (T_e/m_i)^{\frac{1}{2}}$ is the ion sound speed and $\lambda_D = (T_e/4\pi\, n_e\, e^2)^{\frac{1}{2}}$ the electron Debye length. These modes are found to grow when $V_D > C_s$, where V_D is the electron-ion relative drift. In addition, an important requirement is that $T_e \gg T_i$. The instability is of a dissipative nature, arising from the interaction of the ion sound waves with resonant electrons and ions. A necessary condition for the growth of the waves is the dominance of inverse electron Landau damping over ion Landau damping. The condition $T_e \gg T_i$ is very important, since when $T_e \sim T_i$ ion Landau damping is strong and the waves are stabilized or even damped.

In their examination of the ion acoustic instability KRALL and BOOK[13] used the gradients in electron density and magnetic field as the only sources of electron drift. Their investigation was limited to the regime $(kr_e)^2 > 1$, $(\omega/k\overline{V}_B) > (1/kr_e)^2$, where $\vec{k} = (0,k,0)$ is the wave vector and $\overline{V}_B$ the average magnetic drift. The growth rate of the instability was found to be comparable with that for the $\vec{B} = \vec{0}$ ion wave instability; the effect of the non-zero magnetic field was to reduce the growth rate.

AREFEV[14], in his treatment of the ion acoustic instability, assumed that the electron drift resulted from an externally applied electric field. The linear dispersion relation was solved for $T_e >> T_i$ by assuming $|\omega-\vec{k}\cdot\vec{V}_D| << k_z C_e$, $kC_i << \omega$ and $k_\perp{}^2 r_e{}^2 << 1$. The choice $k_z/k \sim (m_e/m_i)^{\frac{1}{2}}$ was shown to restrict the instability to drift speeds V_D such that $2 C_s > V_D > C_s$.

A detailed, numerical, linear study of electrostatic instabilities has been undertaken by LASHMORE-DAVIES and MARTIN[11]. They have assumed the $\vec{E} \times \vec{B}$ drift to be the major source of electron current and therefore neglected the effects of gradients. The analysis for the ion acoustic instability was subjected to the approximation $\vec{k} = (0, k_y, k_z)$, $k^2 \lambda_D^2 << 1$, and $(\omega - \vec{k}\cdot\vec{V}_D) << \sqrt{2}k_z C_e$ or $e^{-b} I_o(b) << 1$. Here $b = (k_y r_e)^2$ and $I_o(b)$ is the modified Bessel function of order zero. An analysis of the normalized growth rate (γ/Ω_e) as a function of $k_y r_e$ showed that for $k_y r_e \lesssim 1$ the magnetic field effects were strong, producing an enhanced growth rate compared to the magnetic field-free case. For $k_y r_e \gtrsim 4$, a surprising result was obtained; the behaviour of the instability seemed to be independent of the magnetic field.

The enhanced growth rate in the presence of the magnetic field, as discovered by LASHMORE-DAVIES and MARTIN(11) contradict the result of KRALL and BOOK(13), who found the opposite effect. Besides the choice of different parameter ranges, this may be attributed to the fact that the $\vec{E} \times \vec{B}$ drift, in the opinion of the former the dominant drift, is ignored by the latter.

PRIEST and SANDERSON(15) have differed in their approach to the ion acoustic instability problem. They have established the linear dispersion relation via the generalized Gordeyev integral(16). Using the assumption $\vec{k} = (0, k_y, k_z)$, $(k_y r_e)^2 \gg 1$ and $(k_z r_e)^2 \gtrsim 1$, they found that the inclusion of a temperature gradient produced a dramatic increase in the growth rate. This was attributed, not to a simply larger drift velocity, but to a distortion of the electron distribution function produced by ∇T; the distortion being such that it increased the slope of the electron velocity distribution function (and therefore the growth rate) in the resonant region of velocity space. The effect of a density gradient drift that was small compared to the electron thermal speed C_e, was found to be negligible, while a slight modification to the growth rate was produced by a magnetic field gradient. The authors also conjecture that a temperature gradient drift of order C_e could cause sufficient distortion of the electron distribution function to allow the ion acoustic instability even when $T_i \sim T_e$ and $V_D < C_e$, the obvious implication being that the ion acoustic instability could not be completely ruled out in experiments where the average electron and ion temperatures were of the same order[9].

The suggestion of PRIEST and SANDERSON[15] concerning ion acoustic instability at $T_e \sim T_i$ was numerically investigated, to some degree, by ALLEN and SANDERSON[17]. For a low collision frequency, $k_\perp r_e >> 1$, a significant ∇T drift but a weak ∇B drift, the authors found:

(i) maximum growth occurred for propagation perpendicular to $\vec{B}$,

(ii) a positive growth rate was obtained for $V_T = 0{,}5\ C_e$ and $T_e = T_i$, where V_T is the ∇T drift, and

(iii) for $V_T = C_e$, the wave phase velocity increased with the wave number, a behaviour exactly opposite to that of a normal ion acoustic mode.

BHARUTHRAM and HELLBERG[18] studied the ion acoustic instability, the electron-cyclotron drift instability and the intermediate transition regime for a dominant drift $\vec{V}_o$ (either external-beam type or $\vec{E} \times \vec{B}$) and weak gradients in electron density, temperature and magnetic field. For the ion acoustic mode with $k_z \neq 0$ the ∇T drift enhanced the growth rate for $k_\perp r_e > 1$, but had a stabilizing effect for $k_\perp r_e \leq 1$. The ∇n drift always had a stabilizing effect. By allowing a ∇n drift at various angles to $\vec{V}_o$, it turned out that the maximum growth was always achieved for propagation along the net drift, as found by ALLEN and

SANDERSON[17] for the particular case of a net drift across the magnetic field.

In a more recent report, SIZONENKO and STEPANOV[19] have examined the crossfield current-driven ion acoustic instability driven by gradients in plasma temperature, plasma density and magnetic field in the limit $T_e >> T_i$. Derivations of the instability growth rate are presented for both a strong and a weak ∇B drift. Wave propagation along and perpendicular to the magnetic field are separately treated.

(b) The electron-cyclotron drift instability (ECDI)

This instability is also known as the beam cyclotron or Bernstein instability. For $T_e >> T_i$, the instability is of a reactive type, arising from the resonance coupling between a Doppler-shifted negative energy electron Bernstein wave and the positive energy ion acoustic wave. As a result one has the resonance condition $\omega \simeq \vec{k}\cdot\vec{V}_D - |n|\ \Omega_e = k\ C_s$ for maximum growth rate. Electron Bernstein waves are electrostatic modes. Their propagation is independent of ion dynamics and occurs under zero drift conditions. These modes satisfy the dispersion relation $\omega \simeq \ell\ \Omega_e$ (ℓ is an integer) and propagate, with a constant amplitude, perpendicular to the magnetic field. They are severely damped for slightly off-perpendicular propagation.

In one of the first investigations on the reactive ECDI, WONG[20] worked in the rest frame of the drifting electrons, with the crossfield ion drift being produced by some external source. The linear dispersion relation was solved with the aid of the assumptions $\vec{k} = (0, k, 0)$, $k^2 r_e^2 >> 1$ and $T_e >> T_i$. Modifications to the dispersion relation introduced by $k_z \gtrsim 0$ were found to be small provided the upper limit of (k_z/k_y) was restricted. The treatment was extended to include the effect of a ∇B drift.

The results of LASHMORE-DAVIES[12] for the reactive ECDI are identical to that of WONG[20] if one takes into consideration the former's additional assumption of $k^2 \lambda_D^2 << 1$. LASHMORE-DAVIES[12] was the first to point out that the ECDI could exist as a dissipative instability when $T_e \sim T_i$. The driving mechanism was associated with the Doppler-shifted electron Bernstein mode 'seeing' a positive slope to the ion distribution function and thereby undergoing ion Landau damping, which, for a negative energy wave results in growth. In a later paper LASHMORE-DAVIES[21] included a density gradient drift. This not only reduced the net growth rate of the reactive ECDI but also restricted the instability to wave numbers lying within a specified band. A much more detailed examination of the dissipative ECDI was also undertaken.

GARY and SANDERSON[22] investigated the reactive ECDI in the presence of an $\vec{E} \times \vec{B}$ electron drift, and arrived, independently, at the result obtained by WONG[20]. They then showed that the inclusion of a ∇B drift reduced the instability growth rate. Despite this, the growth rate was still larger than that of the magnetic field-free ion acoustic instability.

FORSLUND *et al*[23] made an indepth study of the dissipative ECDI. A numerical solution of the dispersion relation revealed that the growth rate was a maximum near the point of intersection of the wave phase velocity with the maximum slope of the ion velocity distribution. The authors initially discovered the instability with a numerical simulation code.

Gary extended the work of GARY and SANDERSON[22] for $T_e \gg T_i$ (~ 0) to the regime $T_e = 10\ T_i$[24] and $T_e = T_i$[25]. The major finding in the former case was the reduction in the growth rate of the reactive ECDI, presumably due to the enhancement of ion Landau damping by a finite T_i. The latter case, demonstrated, via numerical studies, that the unstable modes were severely damped for propagation outside a few tenths of a degree off the perpendicular to $\vec{B}$.

In an extension of the theory on the dissipative ECDI ($T_e \sim T_i$), SANDERSON and PRIEST[26] included gradients in electron density and temperature. For perpendicular propagation the growth rate was increased. For oblique propagation the instability ceased to exist due to strong electron Landau damping of the Bernstein mode. In a later report SANDERSON and PRIEST[27] included a ∇B drift and analytically solved the dispersion relation for $T_e \gg T_i$ and $T_e \sim T_i$.

LASHMORE-DAVIES and MARTIN[11], besides reproducing the work of LASHMORE-DAVIES[12,21], numerically extended the results to cover the whole range of parameter space. The regime $T_e = 10\ T_i$ was also considered.

(c) Modified two-stream instability (MTSI)

This instability is associated with modes which propagate with wave numbers and frquencies satisfying[28]: $\Omega_i << \omega << \Omega_e$, $kr_e << 1$, $C_i << (\omega/k)$ and $|\omega - \vec{k}.\vec{V}_D| >> k_z C_e$, and are described by the dispersion relation:

$$1 + \frac{k_y^2 \omega_{pe}^2}{k^2 \Omega_e^2} - \frac{\omega_{pe}^2}{\omega^2} - \frac{k_z^2}{k^2} \quad \frac{\omega_{pe}^2}{(\omega - \vec{k}\cdot\vec{V}_D)^2} = 0.$$

Here, $\vec{k} = (0, k_y, k_z)$ and ω_{pe} (ω_{pi}) the electron (ion) plasma frequency.

In arriving at the above dispersion relation only the zeroth order term in the electron contribution to the dispersion relation was considered since $\omega << \Omega_e$. The MTSI differs from the well known two-stream instability for a non-diamagnetic plasma, which has the dispersion relation[29]

$$1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega - \vec{k}\cdot\vec{V}_D)^2} = 0,$$

in that the latter requires $V_D > C_e$ for instability, while it (MSTI) may exist for drifts much smaller than C_e.

In his treatment of the MTSI STEPANOV[30] took into account not only the zeroth order but also the first order term in the electron contribution to the dispersion relation. Working in the electron frame, he found

(i) for $V_D >> C_e$: growth rate $\gamma \sim (m_e/m_i)^{\frac{1}{2}}\ \omega_{pi}$;

(ii) for $V_D \lesssim C_e$, $(k_z/k) \sim (m_e/m_i)^{\frac{1}{2}}$:$\gamma \sim$ Re(ω) $\sim (\Omega_e\ \Omega_i)^{\frac{1}{2}}$ when $\omega_{pe} \gtrsim \Omega_e$, and $\gamma \sim$ Re(ω) $\sim \omega_{pi}$ when $\omega_{pe} \lesssim \Omega_e$.

ASHBY and PATON[31], in an analytical study, arrived at Re(ω) $\sim \gamma \sim \vec{k}\cdot\vec{V}_D \sim \omega_{LH} = \omega_{pi} \cdot (1 + \omega_{pe}{}^2/\Omega_e{}^2)^{-\frac{1}{2}}$ (the lower hybrid frequency) for $(k_z/k_y) \sim (m_e/m_i)^{\frac{1}{2}}$. The MTSI was experimentally observed by the authors in a low density plasma stream.

The modified two-stream instability was also studied by AREFEV[14]. Part of his results showed that instability occurred for a discrete frequency spectrum. In their investigation KRALL and LIEWER[32] used the additional assumption $k_z << k_\perp$. They found that $V_D > ((T_e + T_i)/m_i)^{\frac{1}{2}}$ was a necessary condition for a positive growth rate.

The dispersion relation for the MTSI, given above, was manipulated by LASHMORE-DAVIES and MARTIN[11] to show that the instability may be visualized as a reactive instability, arising from a resonance coupling between two modes - the lower hybrid ($\omega = \omega_{LH} = \omega_{pi} \times (1 + \omega_{pe}{}^2/\Omega_e{}^2)^{-\frac{1}{2}}$) and a Doppler-shifted electron plasma wave perpendicular to $\vec{B}$, with $\omega - k_y\ V_D \simeq -\ (k_z/k)\ \omega_{pe}$. The latter has $\omega < \vec{k}\cdot\vec{V}_D$ and is therefore a negative energy mode[12]. The authors then showed that the so called drift modes, as discussed by KRALL and LIEWER[32] for propagation across $\vec{B}$ in the presence of gradients, were just the MTSI propagating perpendicular to $\vec{B}$. Thus the effect of the gradients was to increase the 'cone of propagation' of the MTSI.

A detailed study of the MTSI in the linear and nonlinear regimes was conducted by McBRIDE *et al*[28]. The importance of the instability as a possible turbulent heating mechanism was indicated. PAPADOPOULOS *et al*[33] examined a MTSI arising from two counter-streaming ion beams through a relatively cold electron background. The quasilinear and nonlinear stages of the instability were also studied.

GLADD[34] found that the growth rate of the MTSI assumed a maximum value for a particular oblique angle of propagation. The introduction of a weak density gradient shifted the maximum growth rate to $k_z = 0$, corresponding to the lower hybrid drift instability. Electromagnetic effects were also investigated. As a follow up, DAVIDSON and GLADD[35] examined the anomalous resistivity and heating associated with the lower hybrid drift instability. However, before entering the nonlinear regime, the authors undertook an extensive parameter study in the linear domain - the instability being driven by an $\vec{E} \times \vec{B}$ drift and gradients in electron temperature, electron density and magnetic field.

The instabilities discussed above are not isolated from each other. GARY[24] has examined the reactive (electron Bernstein mode - ion wave coupling) ECDI for propagation across $\vec{B}$. However, for oblique propagation he finds that the Bernstein mode is severely damped. The coupling is no longer possible and the ion acoustic wave grows, provided inverse electron Landau damping overcomes ion Landau damping. Thus, as the propagation changes from the perpendicular to $\vec{B}$ and becomes oblique, the reactive ECDI degenerates into the dissipative ion acoustic instability. LASHMORE-DAVIES and

and MARTIN[11] and AREFEV[14] have shown from numerical studies that as (k_z/k) increases the MTSI changes to the ion acoustic instability. It has been pointed out by LASHMORE-DAVIES and MARTIN[11] and GLADD[34] that the MTSI ($k_z \neq 0$) and lower hybrid drift instability ($k_z = 0$)[32] may be considered as different aspects of the same instability.

2.3 EXPERIMENTAL INVESTIGATIONS OF THE ION ACOUSTIC INSTABILITY

Here, we briefly discuss some of the experiments involving the ion acoustic instability.

HIROSE *et al*[36] have reported an experimental observation of the crossfield ion acoustic instability in a toroidal turbulent-heating experiment. The measured values of the anomalous resistivity (about two orders of magnitude larger than the classical value) and anomalous electron thermal transport (about 25 times larger than the classical value) were explained in terms of ion acoustic waves driven by a radial temperature gradient across the toroidal magnetic field.

Observation of the crossfield ion acoustic instability in two different configurations, a streaming cesium-plasma device and a double plasma device, have been reported by BARRETT *et al*[37]. The authors have suggested that the instability could occur for $T_e \sim T_i$, a suitable k_z allowing for inverse electron Landau damping to overcome ion Landau damping. However, the experimental growth rates do not provide convincing agreement with theoretical predictions. This could possibly be due to a neglect of the finite size of the plasma, plasma inhomogeneities and inter-particle collisions. McBRIDE *et al*[28] have remarked that the results of

BARRETT *et al*[37] could be explained in terms of the modified two-stream instability.

HAMBERGER and JANCARIK[38] measured the electrostatic fluctuations in a turbulent-heating experiment performed in a small toroidal stellarator. For a hydrogen plasma in which the electron drift was relatively small, the observed fluctuation spectra appeared to be consistent with turbulence driven by ion sound waves.

WATANABE[39] observed the evolution of ion acoustic waves in a discharge tube by gradually increasing the electron drift velocity. The onset of the ion acoustic instability was seen when the bias on the grid controlling the electron drift velocity (V_g) reached 1,1 V. For $V_g \gtrsim 10$ V nonlinear effects set in, and the instability entered a turbulent state. The ion acoustic nature of the instability was confirmed by the measured dispersion relation which agreed remarkably well with the theoretical curve for ion acoustic waves.

In another turbulent-heating experiment in a magnetic mirror, WHARTON *et al*[40] found that the low frequency components of the measured turbulent spectrum followed the dispersion curve for ion acoustic waves. The heating of the plasma was attributed to the current-driven ion acoustic instability.

The crossfield current-driven ion acoustic instability was studied in an ion beam-plasma system within a double-plasma device by HAYZEN and BARRETT[41]. Allowing for the finiteness of the plasma and ion-neutral collisions the authors obtained very good agreement between the measured spatial growth rates and dispersion properties, and the equivalent theoretical estimates for the crossfield ion acoustic instability. JONES and BARRETT[42] have extended the experiments into the nonlinear domain. Quasilinear and nonlinear theories have been invoked to explain the observed nonlinear saturation of the instability.

2.4 DEVELOPMENT OF THE QUASILINEAR THEORY

When plasma instabilities are excited and the associated waves grow, after a sufficient time these waves assume such amplitudes that the nonlinear terms become important and the linearizing of the Vlasov equation (a basic equation for the study of a collisionless plasma) is no longer valid. Nonlinear behaviour is very much an inherent characteristic of laboratory plasmas[38,39,42]. The question arises as to the evolution of such instabilities; nonlinear effects may modify certain plasma parameters which in turn can cause the unstable oscillations to saturate.

A natural extension of the linear theory of plasma waves and instabilities is the weak turbulence theory - first published independently by VEDENOV *et al*[43] and DRUMMOND and PINES[44]. There are two necessary conditions on which the weak turbulence theory is based[45]. The first is that the energy density associated with the fluctuations, W, must be small compared to the plasma thermal energy, nkT, i.e. $(W/nkT) \ll 1$. The second requirement is that the spectrum of waves must be broad. If $\Delta\omega$ is the frequency spread of the wave spectrum, then it is necessary that $\Delta\omega^{-1} \ll (eEk/m)^{-\frac{1}{2}}$, where $(eEk/m)^{-\frac{1}{2}} = t_r$ is the trapping time[46] of an electron in the potential well of the electric field E having wave number k. What this means is that the correlation time of the waves $\Delta\omega^{-1}$ must be much shorter than the trapping time t_r, i.e. a particle does not 'see' a single wave long enough to be trapped by it. For a valid application of weak turbulence theory it is essential that both these conditions are independently satisfied. Regimes which do not satisfy these conditions are loosely described as strongly turbulent.

As is well known, in the linear theory terms of second order and higher are neglected. In weak turbulence theory such terms, generally up to the fourth order, are retained. In the simplest treatment of weak turbulence theory, called quasilinear theory, the coupling between the different modes is neglected. This will be discussed in Chapter Four.

Since its inception quasilinear theory has been extensively treated in several standard texts, some of which are by SAGDEEV and GALEEV[47], TSYTOVICH[48], DAVIDSON[46] and KRALL and TRIVELPIECE[1]. We shall review a few of the papers that have made significant contributions to the development of the quasilinear theory.

In the initial report of DRUMMOND and PINES[44] the fundamental equations of quasilinear theory are established. The authors then show that for electron plasma oscillations with a positive growth rate ($\gamma > 0$) one half of the energy lost by particles resonating with the waves goes into the wave electrostatic potential energy and the other into the kinetic energy of oscillations of the bulk of the non-resonant particles. In a one-dimensional application, a Maxwellian velocity distribution with a 'gentle-bump' is chosen. A numerical study of the development of the system in time shows:

(i) a flattening of the velocity distribution (also known as plateau formation) in the bump region, and

(ii) although, the initial fluctuation level does grow in time, its final level, however, is still comparatively small.

BERNSTEIN and ENGELMANN[49] formulated the quasilinear theory for growing ($\gamma > 0$) and decaying ($\gamma < 0$) modes. For the one-dimensional case of a 'bump in tail' velocity distribution they recovered the results of DRUMMOND and PINES[44]. This implied that in the one-dimensional treatment only growing modes need be considered. However, this is not true in two- or three-dimensions. The 'H-like' theorem (analogous to Boltzmann's H-theorem in thermodynamics) developed by the authors showed that the asymptotic behaviour in time was such that $\gamma < 0$ for all modes within the system, i.e. it was inadequate to consider only growing oscillations. This time asymptotic behaviour in two- and three-dimensions has been comprehensively treated by SAGDEEV and GALEEV[47].

In their treatment of the quasilinear theory VAHALA and MONTGOMERY[50] considered the fluctuations to consist of a discrete $\vec{k}$ - spectrum, which was in contrast to the case of a continuous $\vec{k}$ - spectrum as used by BERNSTEIN and ENGELMANN[49]. For such a discrete $\vec{k}$ - spectrum they found that a consistent formulation of the quasilinear theory could be undertaken without decaying modes in one-, two- or three-dimensions; a result different from that of BERNSTEIN and ENGELMANN[49] for a continuous spectrum. However, in view of the fact that one of the pre-conditions for the quasilinear theory is a sufficiently broad spectrum, the discrete spectrum of VAHALA and MONTGOMERY[50] may not fulfill this requirement. The authors then point out that the inclusion of damped waves could result in negative diffusion coefficients in velocity space and therefore lead to inconsistencies. It seems that this problem has arisen from a misinterpretation of the diffusion coefficient in the resonance region of velocity space for decaying modes and has been resolved by DAVIDSON[46], who shows that the diffusion coefficient is non-negative in the resonance region for both growing and decaying modes.

BURNS and KNORR[51] attempted to resolve some of the difficulties associated with the quasilinear theory. They investigated a one-dimensional electron plasma. In the development of the theory the authors emphasized the need for the approximate form of the perturbed velocity distribution used in the calculation of the velocity moments to be supplemented by the prescribed Landau contour of integration. If this is not done significant errors could arise. Good qualitative agreement was obtained for a discrete spectrum between quasilinear theory and,a numerical analysis of the one-dimensional Vlasov equation and Poisson's equation in the absence of mode-coupling terms. The authors also found that the requirement of VAHALA and MONTGOMERY[50] that all growth rates remain non-negative for a finite time was not necessary. Furthermore, they showed that if the Landau contour of integration is well defined then damped modes ($\gamma < 0$) need not render the velocity diffusion equation ill-posed, as argued by VAHALA and MONTGOMERY[50].

Despite the initial difficulties associated with it, quasilinear theory has been widely applied. DAVIDSON *et al*[52] investigated electron heating by two counter-streaming ion beams in a computer simulation experiment of a one-dimensional beam-plasma system. The rate of electron heating and its saturation offered good agreement with the predictions of the related quasilinear theory. A detailed study, comprising linear and nonlinear theory and computer simulation, of the electron-cyclotron drift instability was undertaken by LAMPE *et al*[53]. Quasilinear rate equations for the electron temperature, ion temperature and the average ion drift velocity were obtained. The results of the computer simulation for cold ions exhibited good agreement with the quasilinear predictions. The saturation of the instability was found to be accompanied by ion trapping effects. DAVIDSON and GLADD[35] and DAVIDSON[54] have examined the lower hybrid drift instability in considerable detail in the quasilinear regime; current relaxation, electron and ion heating rates and diffusion relaxation times have been studied.

The quasilinear behaviour of the two-stream ion cyclotron instability was theoretically studied by DRUMMOND and ROSENBLUTH[55]. Energy considerations showed that only a fraction $k^2\lambda_D{}^2 << 1$ of the energy given up by the resonant electrons appeared as wave potential energy. The bulk of the energy went into the kinetic energy of the wave motion, associated with the oscillations of the non-resonant electrons and ions in the presence of the wave. This was in contrast to the result of DRUMMOND and PINES[44] for electron plasma oscillations, where the energy was equally divided between wave potential and kinetic energies. The authors also found that the resonant interaction between waves and electrons led to an anomalous spatial diffusion across the magnetic field.

DUM *et al*[56] investigated the turbulent heating and stabilization of the crossfield current-driven ion acoustic instability via two-dimensional computer simulation. Heating and anomalous resistivity results were well in agreement with quasilinear predictions, as was the relaxation of the electron and ion energy distributions.

The nonlinear development of the ion acoustic instability in a collisionless, unmagnetized plasma was theoretically investigated by

CAPONI and DAVIDSON[57]. The analysis was conducted via the quasilinear theory, extended to include the effects of ion resonance broadening. Calculated values of the anomalous resistivity provided much better agreement with experimental values than the expression of SAGDEEV and GALEEV[47, p.94-103], which is based on the assumption that nonlinear ion Landau damping is the dominant saturation mechanism. Numerical integration provided the temporal variation of the fluctuation energy density and electron and ion heating rates.

In a more recent undertaking, APPERT and VACLAVIK[58] studied the saturation of the current-driven ion acoustic instability in a weakly ionized, uniform, unmagnetized plasma. The analysis was conducted within the context of a quasilinear model that included the effects of close two-body collisions and Coulomb collisions. The spectral distribution of the turbulence was calculated in a one-dimensional treatment. In a restricted three-dimensional treatment, where $(m_e/m_i)^{\frac{1}{2}} < k_z/k \ll 1$, an expression was derived for the total wave energy density in terms of measurable physical parameters. Numerical calculations showed that the ion velocity distribution function was virtually unaffected by the turbulent oscillations. For low degrees of ionization the effect of Coulomb collisions was negligible. The derived results provided favourable agreement with previously reported experimental observations.

We conclude this section by mentioning an experiment that has confirmed the quasilinear theory. ROBERSON *et al*[59] studied the 'gentle-bump' instability in a long plasma column into which an electron beam was injected. The observed flattening of the bump in velocity space and the saturation of the wave energy were in excellent agreement with the predictions of quasilinear theory. As a note of warning, the authors point out that the theory should be carefully applied. For, when the experimental conditions exceeded the limits of quasilinear theory, ever so slightly, a qualitatively different behaviour was observed.

CHAPTER THREE

LINEAR THEORY OF THE CROSSFIELD CURRENT-DRIVEN ION ACOUSTIC INSTABILITY

3.1 BASIC EQUATIONS

We adopt the model of GARY and SANDERSON [22] and consider a homogeneous collisionless plasma with an external electric field $\vec{E}_o$ and a magnetic field $\vec{B}_o$ as shown in Fig. 3.1 below. The analysis is conducted in the ion rest frame.

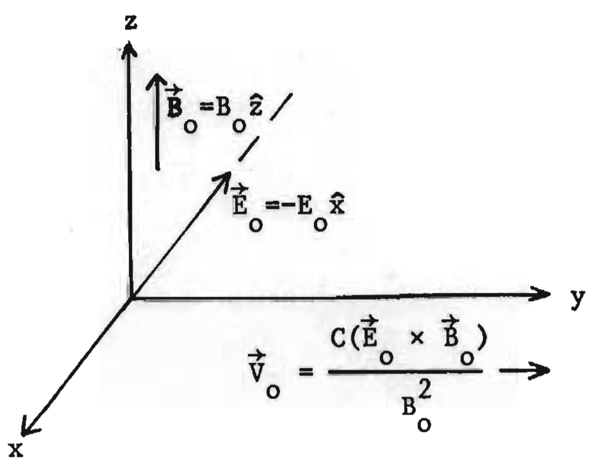


Figure 3.1

Two fundamental assumptions are made:

(a) The time scale τ and length scale ℓ of the perturbation are such that they satisfy the conditions (1.2.1), viz.,

$$\Omega_i < \tau^{-1} < |\Omega_e|$$

and

$$r_e < \ell < r_i$$

As mentioned earlier, under such conditions the ions may be considered unmagnetized, and the magnetized electrons have an $\vec{E} \times \vec{B}$ drift $\vec{V}_o$ relative to the ions.

(b) We use the electrostatic approximation, i.e., assume that the wave electric field is produced by charge separation, which in turn implies that the perturbed electric field $\vec{E}_1$ can be expressed in terms of a scalar potential ϕ_1 as $\vec{E}_1 = -\nabla\phi_1$. For a wave propagating as $\exp\{i(\vec{k} \cdot \vec{r} - \omega t)\}$, Maxwell's equation

$$\nabla \times \vec{E}_1 = -\frac{\partial \vec{B}_1}{\partial t}$$

reduces to

$$\vec{0} = \vec{k} \times \vec{E}_1 = \omega \vec{B}_1$$

Therefore, in the electrostatic approximation, the magnetic field remains unperturbed and the waves are longitudinal with $\vec{k}$ parallel to $\vec{E}_1$.

In addition, for electromagnetic effects to remain insignificant the wave phase speed V_ϕ must be such that $V_\phi = \omega/k << C$, where C is the speed of light.

In the absence of collisions, the electron and ion velocity distribution functions $f_j(j = e(\text{electron}),\ j = i(\text{ion}))$ satisfy the Vlasov equation

$$\frac{\partial f_j(\vec{r},\vec{V},t)}{\partial t} + \vec{V} \cdot \frac{\partial f_j(\vec{r},\vec{V},t)}{\partial \vec{r}} + \frac{e_j}{m_j}\left(\vec{E} + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_j(\vec{r},\vec{V},t)}{\partial \vec{V}} = 0 \tag{3.1.1}$$

We define $f_j = \langle f_j \rangle + f_{1j}$, where $f_{oj} = \langle f_j \rangle$ is the ensemble average of f_j, usually defined as an average over one or several coordinates in space or over time. f_{1j} represents the perturbation in f_j due to a set of randomly phased, rapid oscillations. Similarly $\vec{E}(\vec{r},t) = \vec{E}_o + \vec{E}_1(\vec{r},t)$.

The Vlasov equation is now averaged over an arbitrary variable; the exact variable will be specified later. We obtain

$$\frac{\partial f_{oj}}{\partial t} + \vec{V} \cdot \frac{\partial f_{oj}}{\partial \vec{r}} + \frac{e_j}{m_j}\left(\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_{oj}}{\partial \vec{V}} = -\frac{e_j}{m_j}\left\langle \vec{E}_1 \cdot \frac{\partial f_{1j}}{\partial \vec{V}} \right\rangle \tag{3.1.2}$$

Note that $\left\langle \vec{E}_o \cdot \frac{\partial f_{1j}}{\partial \vec{V}} \right\rangle = \vec{E}_o \cdot \left\langle \frac{\partial f_{1j}}{\partial \vec{V}} \right\rangle = 0$, etc.

Subtracting Eq.(3.1.2) from Eq.(3.1.1), we get

$$\frac{\partial f_{1j}}{\partial t} + \vec{V} \cdot \frac{\partial f_{1j}}{\partial \vec{r}} + \frac{e_j}{m_j}\left(\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_{1j}}{\partial \vec{V}} + \frac{e_j}{m_j}\left(\vec{E}_1 \cdot \frac{\partial f_{oj}}{\partial \vec{V}}\right)$$

$$= -\frac{e_j}{m_j}\left[\vec{E}_1 \cdot \frac{\partial f_{1j}}{\partial \vec{V}} - \left\langle \vec{E}_1 \cdot \frac{\partial f_{1j}}{\partial \vec{V}} \right\rangle\right] \tag{3.1.3}$$

If we assume periodic boundary conditions at the end of a system of length L, the perturbed quantities may be expressed in terms of their spatial Fourier transforms as

$$f_{1j}(\vec{r},\vec{V},t) = \sum_k f_{jk}(\vec{V},t)\, e^{i\vec{k}.\vec{r}}$$

$$\vec{E}_1(\vec{r},t) = \sum_k \vec{E}_k(t)\, e^{i\vec{k}.\vec{r}} \qquad (3.1.4)$$

$$\phi_1(\vec{r},t) = \sum_k \phi_k(t)\, e^{i\vec{k}.\vec{r}}$$

where

$$f_{jk}(\vec{V},t) = \frac{1}{L^3}\int f_{1j}(\vec{r},\vec{V},t)\, e^{-i\vec{k}.\vec{r}}\, d\vec{r}$$

$$\vec{E}_k(t) = \frac{1}{L^3}\int \vec{E}_1(\vec{r},t)\, e^{-i\vec{k}.\vec{r}}\, d\vec{r}$$

and $\vec{E}_1(\vec{r},t) = -\nabla\phi_1(\vec{r},t)$ in the electrostatic limit. For convenience we have written k instead of $\vec{k}$ in all summations and subscripts.

Their time dependence is taken to be of the WKB form (46, p 226), viz.,

$$\exp\left\{\int_o^t S_k(t')\, dt'\right\} \qquad (3.1.5)$$

where, in general, $S_k(t') = -i\,\omega_k(t') = -i\omega_k^r(t') + \gamma_k(t')$, where

$$\omega_k(t) = \omega_k^r(t) + i\,\gamma_k(t) \qquad (3.1.6)$$

with $\omega_k^r \equiv Re(\omega_k)$ the real part, and $\gamma_k \equiv Im(\omega_k)$ the imaginary part of ω_k. In the electrostatic limit Maxwell's equations are replaced by Poisson's equation

$$\nabla^2\phi = -4\pi \sum_j n_j\, e_j \qquad (3.1.7)$$

Defining the ensemble average as

$$<g> = \lim_{L\to\infty} \frac{1}{L^3} \int g \, d\vec{r}$$

the right hand side of Eq.(3.1.2) is manipulated with the aid of Eq.(3.1.4) to yield

$$\left\langle \vec{E}_1 \cdot \frac{\partial f_{1j}}{\partial \vec{V}} \right\rangle = \left\langle \sum_q \vec{E}_q(t)\, e^{i\vec{q}.\vec{r}} \cdot \sum_k \frac{\partial f_{jk}(\vec{V},t)}{\partial \vec{V}}\, e^{i\vec{k}.\vec{r}} \right\rangle$$

$$= \lim_{L\to\infty} \sum_q \sum_k \vec{E}_q(t) \cdot \frac{\partial f_{jk}}{\partial \vec{V}} \frac{1}{L^3} \int e^{i(\vec{k}+\vec{q}).\vec{r}} \, d\vec{r}$$

$$= \sum_k \vec{E}_{-k}(t) \cdot \frac{\partial f_{jk}(\vec{V},t)}{\partial \vec{V}}$$

since

$$\lim_{L\to\infty} \frac{1}{L^3} \int \exp\{i(\vec{k}+\vec{q}).\vec{r}\} \, d\vec{r} = \begin{cases} 1 \text{ if } \vec{q} + \vec{k} = \vec{0} \\ 0 \text{ if } \vec{q} + \vec{k} \neq \vec{0} \end{cases}$$

Thus Eq.(3.1.2) may be rewritten as

$$\frac{\partial f_{oj}}{\partial t} + \vec{V} \cdot \frac{\partial f_{oj}}{\partial \vec{r}} + \frac{e_j}{m_j}\left(\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_{oj}}{\partial \vec{V}} = -\frac{e_j}{m_j} \sum_k \vec{E}_{-k} \cdot \frac{\partial f_{jk}}{\partial \vec{V}} \tag{3.1.8}$$

In the linear approximation we neglect the products of perturbed quantities ,and hence set the right hand sides of Eqs.(3.1.3) and (3.1.8) to zero. We then write

$$f_{oj} = f_{oj}(V_\perp^2, V_z) \tag{3.1.9}$$

as the solution of Eq.(3.1.8). Note that the background distribution

f_{oj} is assumed to be spatially uniform and isotropic in the velocity plane perpendicular to $\vec{B}_o$. The reason for this particular choice of f_{oj} will be discussed in Section 3.2.

With its right hand side equated to zero, Eq.(3.1.3) reduces to

$$\frac{\partial f_{1j}}{\partial t} + \vec{V} \cdot \frac{\partial f_{1j}}{\partial \vec{r}} + \frac{e_j}{m_j}\left(\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_{1j}}{\partial \vec{V}} = \frac{e_j}{m_j} \nabla\phi_1 \cdot \frac{\partial f_{oj}}{\partial \vec{V}} \qquad (3.1.10)$$

where $\vec{E}_1(\vec{r},t)$ has been replaced by $-\nabla\phi_1(\vec{r},t)$. This equation, in turn, can be written as

$$\left[\frac{df_{1j}}{dt}\right]_u = g(\vec{r},\vec{V},t)$$

where the operator $[d/dt]_u$ is defined as the rate of change following an unperturbed orbit in phase space (16), i.e., the left hand side represents the rate of change of f_{1j} as 'seen' by a particle which at time t is at the phase point $(\vec{r},\vec{V})$ under observation, but whose motion through phase space is determined by the external fields $\vec{E}_o$ and $\vec{B}_o$.

Thus, integrating Eq.(3.1.10) along the unperturbed orbits, we find

$$f_{1j}(\vec{r},\vec{V},t) = \frac{e_j}{m_j} \int_{-\infty}^{t} \nabla\phi_1(\vec{r}\,',t'). \frac{\partial f_{oj}}{\partial \vec{V}'} dt' \qquad (3.1.11)$$

The lower limit has been set at $t' = -\infty$ under the assumption that the plasma is undisturbed in the infinite past and the perturbations grow from zero.

3.2 ELECTRON AND ION CONTRIBUTIONS TO THE DISPERSION RELATION

For the electrons Eq.(3.1.11) becomes

$$f_{1e}(\vec{r},\vec{V},t) = -\frac{e}{m_e}\int_{-\infty}^{t} \nabla\phi_1(\vec{r}',t') \cdot \frac{\partial f_{oe}}{\partial \vec{V}'}\, dt' \tag{3.2.1}$$

The equation governing electron motion in the presence of an electric field $\vec{E}_o$ and a magnetic field $\vec{B}_o$ is

$$m\ddot{\vec{r}} = -e\left[\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right] \tag{3.2.2}$$

For the configuration shown in Fig. 3.1 this yields

$$V_x(t') = V_\perp(t) \cos\{\theta(t) - \Omega_e(t'-t)\}$$
$$V_y(t') - V_o = V_\perp(t) \sin\{\theta(t) - \Omega_e(t'-t)\}$$
$$V_z = \text{constant.}$$

where $V_\perp^2(t) = V_x^2(t) + (V_y(t) - V_o)^2$ and $\vec{V}_o$ is the electron $\vec{E} \times \vec{B}$ drift. From the above equations we may construct two constants of the motion, viz.,

$$V_\perp^2 = V_x^2 + (V_y - V_o)^2 \quad \text{and} \quad V_z$$

Since any distribution that is a function of the constants of the motion is necessarily a solution of the zero order Vlasov equation [1], the equilibrium distribution function f_{oe} is chosen to depend on $V_\perp^2$ and V_z, i.e.,

$$f_{oe} = f_{oe}(V_\perp^2, V_z) \tag{3.2.3}$$

as in Eq.(3.1.9).

Now $$\frac{\partial f_{oe}}{\partial \vec{V}} = \frac{\partial f_{oe}}{\partial V_x}\hat{x} + \frac{\partial f_{oe}}{\partial V_y}\hat{y} + \frac{\partial f_{oe}}{\partial V_z}\hat{z}$$

$$= \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\vec{V}_\perp + \frac{1}{V_z}\frac{\partial f_{oe}}{\partial V_z}\vec{V}_z$$

where

$$\vec{V}_\perp = V_x\,\hat{x} + (V_y - V_o)\,\hat{y}$$

Consequently

$$\nabla\phi_1 \cdot \frac{\partial f_{oe}}{\partial \vec{V}} = \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\nabla\phi_1 \cdot \vec{V}_\perp + \frac{\partial f_{oe}}{\partial V_z}\frac{\partial \phi_1}{\partial z} \qquad (3.2.4)$$

In the linear approximation, $S_k = -i\omega_k$ is a constant, i.e. independent of time. Expression (3.1.5) then reduces to

$$\exp\,(-\,i\,\omega_k\,t)$$

Thus, using Eq.(3.1.4), the perturbed quantities ϕ_1 and f_{1e} may be written as

$$f_{1e}(\vec{r},\vec{V},t) = \sum_k f_{ek}(\vec{V},t)\;e^{i\vec{k}.\vec{r}}$$

where (3.2.5a)

$$f_{ek}(\vec{V},t) = f_{ek\omega}(\vec{V})\;e^{-i\omega_k t}$$

and

$$\phi_1(\vec{r},t) = \sum_k \phi_k(t)\;e^{i\vec{k}.\vec{r}} = \sum_k \phi_{k\omega}\;e^{i(\vec{k}.\vec{r}-\omega_k t)} = \sum_k \tilde{\phi}_{k\omega}(\vec{r}(t),t)$$

where (3.2.5b)

$$\phi_k(t) = \phi_{k\omega}\;e^{-i\omega_k t}$$

$$\tilde{\phi}_{k\omega}(\vec{r}(t),t) = \phi_{k\omega}\;e^{i(\vec{k}.\vec{r}-\omega_k t)}$$

In terms of the Fourier transforms (3.1.4), Eq.(3.2.1) reduces to

$$f_{ek}(\vec{V},t) = -\frac{e}{m_e}\int_{-\infty}^{t} \phi_k(t')i\vec{k} \cdot \frac{\partial f_{oe}(\vec{V}')}{\partial \vec{V}'} \quad \exp\{i\vec{k} \cdot [\vec{r}(t')-\vec{r}(t)]\}\, dt'$$

Since $\phi_k(t') = \phi_{k\omega}\, e^{-i\omega_k t'}$, this may be rewritten as

$$f_{ek}(\vec{V},t) = -\frac{e}{m_e}\int_{-\infty}^{t} \phi_k(t)\; i\vec{k} \cdot \frac{\partial f_{oe}(\vec{V}')}{\partial \vec{V}'} \quad \exp\{i\vec{k} \cdot [\vec{r}(t')-\vec{r}(t)]-i\omega_k(t'-t)\}\, dt'$$

or (3.2.5c)

$$f_{ek\omega}(\vec{V}) = -\frac{e}{m_e}\int_{-\infty}^{t} \phi_{k\omega}\; i\vec{k} \cdot \frac{\partial f_{oe}(\vec{V}')}{\partial \vec{V}'} \quad \exp\{i\vec{k} \cdot [\vec{r}(t')-\vec{r}(t)]-i\omega_k(t'-t)\}\, dt'$$

Then

$$\frac{d\phi_1(\vec{r},t)}{dt} = \frac{\partial \phi_1}{\partial t} + \vec{V} \cdot \frac{\partial \phi_1}{\partial \vec{r}}$$

$$= \nabla\phi_1 \cdot \vec{V}_\perp + \sum_k \{-\, i(\omega_k - k_y V_o - k_z V_z)\}\, \tilde{\phi}_{k\omega}$$

from which

$$\nabla\phi_1 \cdot \vec{V}_\perp = \sum_k \left[\frac{d\tilde{\phi}_{k\omega}}{dt} + i(\omega_k - k_y V_o - k_z V_z)\tilde{\phi}_{k\omega}\right] \qquad (3.2.6)$$

With the aid of Eqs.(3.2.4), (3.2.5) and (3.2.6), Eq.(3.2.1) reduces to

$$\sum_k f_{ek\omega}(\vec{V}) \exp\{i[\vec{k}.\vec{r}(t) - \omega_k t]\}$$

$$= -\frac{e}{m_e}\sum_k \int_{-\infty}^{t} \left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\left\{\frac{d}{dt'}\{\tilde{\phi}_{k\omega}(\vec{r}(t'),t')\} + i(\omega_k - k_y V_o - k_z V_z)\tilde{\phi}_{k\omega}(\vec{r}(t'),t')\right\}\right.$$

$$\left. + ik_z \frac{\partial f_{oe}}{\partial V_z}\tilde{\phi}_{k\omega}(\vec{r}(t'),t')\right] dt'$$

which yields

$$f_{ek\omega}(\vec{V}) = -\frac{e}{m_e}\phi_{k\omega}\left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} + \left\{i(\omega_k - k_y V_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + ik_z V_z\left(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\right)\right\} f_{oe}\right.$$

$$\left. \times \int_{-\infty}^{t} \exp\{i\vec{k}\,.\,[\vec{r}(t')-\vec{r}(t)] - i\omega_k(t'-t)\}\, dt'\right]$$

(3.2.7)

The electron equation of motion (3.2.2) has as solutions [47]

$$x(t') = x(t) - (V_\perp/\Omega_e)\{\sin[\theta(t) - \Omega_e(t'-t)] - \sin\theta(t)\}$$

$$y(t') = y(t) + (V_\perp/\Omega_e)\{\cos[\theta(t) - \Omega_e(t'-t)] - \cos\theta(t)\} + V_o(t'-t)$$

$$z(t') = z(t) + V_z(t'-t)$$

(3.2.8)

with which

$$\vec{k}\,.\,[\vec{r}(t')-\vec{r}(t)] = (k_\perp V_\perp/\Omega_e)\left[-\sin\{[\theta(t) - \Omega_e(t'-t)] - \Psi\} + \sin\{\theta(t) - \Psi\}\right]$$

$$+ (k_y V_o + k_z V_z)(t'-t)$$

where

$$\vec{k} = (k_x, k_y, k_z) = (k_\perp\cos\Psi,\ k_\perp\sin\Psi,\ k_z) \qquad (3.2.9)$$

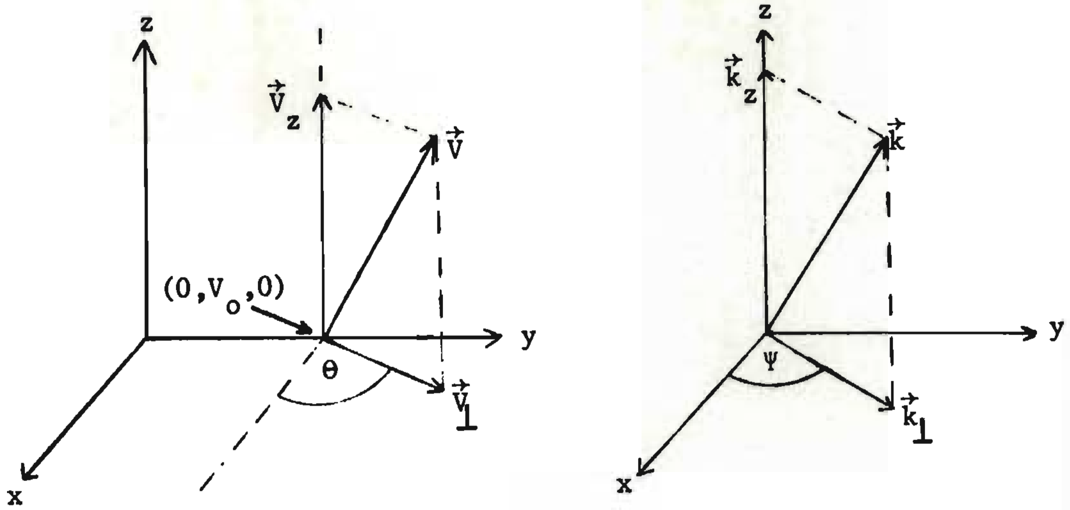


Figure 3.2

Hence, the integral in Eq.(3.2.7) may be rewritten as

$$\exp\{i\,\xi\,\sin[\theta(t)-\Psi]\}\int_{-\infty}^{t}\exp\{-\,i\,\xi\,\sin\,[\theta(t)-\Omega_e(t'-t)-\Psi]\}$$

$$\times\;\exp\{i(k_yV_o+k_zV_z-\omega_k)\,(t'-t)\}\,dt'$$

where

$$\xi = k_\perp V_\perp/\Omega_e. \tag{3.2.10}$$

This integral is manipulated with the aid of the identity [60]

$$\exp\,(i\,\alpha\,\sin\,\beta) = \sum_{n=-\infty}^{+\infty} J_n(\alpha)\,\exp\,(i\,n\,\beta) \tag{3.2.11}$$

where J_n is the Bessel function of the first kind of order n.

Recalling that the plasma is undisturbed at t = − ∞, we finally obtain

$$f_{ek\omega}(\vec{V}) = -\frac{e}{m_e}\,\phi_{k\omega}\Bigg[\frac{1}{V_\perp}\,\frac{\partial f_{oe}}{\partial V_\perp} - \exp\{i\,\xi\,\sin(\theta-\Psi)\}$$

$$\times\sum_{n=-\infty}^{+\infty}\frac{\{(\omega_k-k_yV_o)\,\frac{1}{V_\perp}\,\frac{\partial}{\partial V_\perp}+k_zV_z\,(\frac{1}{V_z}\,\frac{\partial}{\partial V_z}-\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\}f_{oe}}{(\omega_k-k_yV_o-k_zV_z-n\Omega_e)}\qquad J_n(\xi)\exp\{-i\,n(\theta-\Psi)\}\Bigg]$$

(3.2.12)

For low frequency modes, of which the ion acoustic is an example, $|\omega_k - k_y V_o - k_z V_z| << |\Omega_e|$, and therefore we keep only the n = 0 term in the summation above. Then

$$f_{ek\omega}(\vec{V}) = -\frac{e}{m_e}\,\phi_{k\omega}\left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \exp\{i\,\xi\,\sin(\theta-\Psi)\}\,J_o(\xi)\right.$$

$$\times \left.\frac{\left\{(\omega_k - k_y V_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_z V_z\left(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\right)\right\}f_{oe}}{(\omega_k - k_y V_o - k_z V_z)}\right] \qquad (3.2.12a)$$

For the ions we assume not only that because of their inertia, they are unmagnetized, but also for the same reason they do not react to the electric field. Thus the equation of motion for the ions is

$$\ddot{\vec{r}} = \vec{0}$$

with solutions

$$\dot{\vec{r}} = \vec{V} = \text{constant}$$

$$\vec{r}(t') = \vec{r}(t) + \vec{V}\,(t'-t)$$

Then for the ions, Eq.(3.2.7) modifies to

$$f_{ik\omega}(\vec{V}) = \frac{e}{m_i}\,\phi_{k\omega}\left[\frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp} + \left\{i\,\omega_k\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + i\,k_z V_z\left(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\right)\right\}f_{oi}\right.$$

$$\times \left.\int_{-\infty}^{t}\exp\{i(\vec{k}.\vec{V}-\omega_k)\,(t'-t)\}\,dt'\right]$$

It must be noted that here $V_\perp^2 = V_x^2 + V_y^2$, while for the electrons we had $V_\perp^2 = V_x^2 + (V_y - V_o)^2$.

Upon integrating with respect to t' we find

$$f_{ik\omega}(\vec{V}) = \frac{e}{m_i}\,\phi_{k\omega}\left[\frac{1}{V_\perp}\,\frac{\partial f_{oi}}{\partial V_\perp} - \frac{\left\{\omega_k \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_z V_z\left(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\right)\right\} f_{oi}}{(\omega_k - \vec{k}.\vec{V})}\right]$$

(3.2.13)

where

$$\vec{k}\,.\,\vec{V} = \vec{k}_\perp\,.\,\vec{V}_\perp + k_z V_z = k_\perp V_\perp\,\cos\phi + k_z V_z$$

3.3 THE LINEAR DISPERSION RELATION

The linear dispersion relation for the crossfield current-driven ion acoustic instability is now derived with the added assumption that the electrons are hot and the ions cold, i.e., T_e (electron temperature) >> T_i (ion temperature).

Using Eq.(3.2.5b), Poisson's equation (3.1.7) reduces to

$$\phi_{k\omega} = \frac{4\pi e}{k^2}\,(n_{ik\omega} - n_{ek\omega}) \qquad (3.3.1)$$

where $n_{jk\omega}$ represents the perturbation in density of the j^{th} species. $n_{jk\omega}$ is found with the aid of Eqs.(3.2.12a) and (3.2.13) and the relation

$$n_{jk\omega} = \int f_{jk\omega}\, d\vec{V} \qquad (3.3.1a)$$

For the electrons

$$n_{ek\omega} = -\frac{e}{m_e}\phi_{k\omega}\int\left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \exp\{i\,\xi\,\sin(\theta-\Psi)\}\,J_o(\xi)\right.$$

$$\left.\times\frac{\left\{(\omega_k-k_yV_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_zV_z\left(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\right)\right\}f_{oe}}{(\omega_k-k_yV_o-k_zV_z)}\right] V_\perp dV_\perp dV_z d\theta$$

where $d\vec{V} = V_\perp dV_\perp dV_z d\theta$ is expressed in terms of cylindrical coordinates in velocity space and $\xi = k_\perp V_\perp/\Omega_e$, as defined by Eq.(3.2.10).

We expand $\exp\{i\,\xi\,\sin(\theta-\Psi)\}$ in terms of the identity (3.2.11) and then use the result

$$\int_o^{2\pi}\exp(i\,\ell\,\theta)\,d\theta = \begin{cases}2\pi & \text{if } \ell = 0\\ 0 & \text{if } \ell \neq 0\end{cases}$$

to obtain

$$n_{ek\omega} = -\frac{2\pi e}{m_e}\phi_{k\omega}\int\left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - J_o^2(\xi)\frac{\left\{(\omega_k-k_yV_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_zV_z\left(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\right)\right\}f_{oe}}{(\omega_k-k_yV_o-k_zV_z)}\right]$$

$$\times\, V_\perp dV_\perp dV_z$$

$$= -\frac{2\pi e}{m_e}\phi_{k\omega}\int\left[\{1-J_o^2(\xi)\}\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^2(\xi)k_z(\partial f_{oe}/\partial V_z)}{(\omega_k-k_yV_o-k_zV_z)}\right] V_\perp dV_\perp dV_z \tag{3.3.2}$$

For a background distribution f_{oe} of the type (3.2.3), we define

$$\Psi_{oe}(V_z) = \int_o^\infty f_{oe}(V_\perp^2, V_z)\,V_\perp dV_\perp \tag{3.3.3}$$

and write

$$\int_0^{\infty} J_0^2(k_\perp V_\perp/\Omega_e)\, f_{oe}(V_\perp^2, V_z) V_\perp\, dV_\perp = \alpha(k_\perp/\Omega_e)\, \Psi_{oe}(V_z) \qquad (3.3.4)$$

$$\int_0^{\infty} \{1-J_0^2(k_\perp V_\perp/\Omega_e)\}\, \frac{1}{V_\perp} \frac{\partial f_{oe}(V_\perp^2, V_z)}{\partial V_\perp} V_\perp\, dV_\perp = \beta(k_\perp/\Omega_e)\, \Psi_{oe}(V_z) \qquad (3.3.5)$$

We shall later show that for Maxwellian velocity distributions $\alpha(k_\perp/\Omega_e)$ and $\beta(k_\perp/\Omega_e)$ are well known expressions.

Equation (3.3.2) then becomes

$$n_{ek\omega} = -\frac{2\pi e}{m_e} \phi_{k\omega} \int_{-\infty}^{+\infty} \left[\beta(k_\perp/\Omega_e) - \frac{\alpha(k_\perp/\Omega_e)k_z}{(\omega_k - k_y V_o - k_z V_z)} \frac{\partial}{\partial V_z}\right] \Psi_{oe}(V_z)\, dV_z \qquad (3.3.6)$$

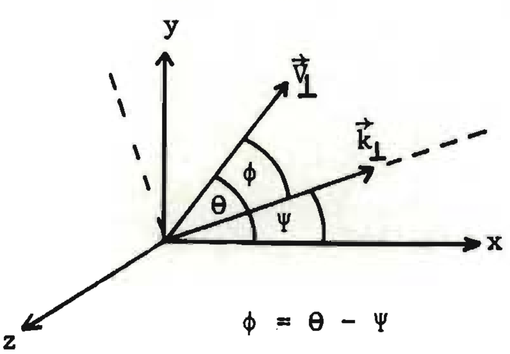
From Eq.(3.2.13) for the ions, we obtain

$$n_{ik\omega} = \int f_{ik\omega}\, d\vec{V}$$

$$= \frac{e}{m_i} \phi_{k\omega} \int \left[\frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} - \frac{\left\{\omega_k \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} + k_z V_z \left(\frac{1}{V_z} \frac{\partial}{\partial V_z} - \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp}\right)\right\} f_{oi}}{(\omega_k - k_z V_z - k_\perp V_\perp \cos\phi)}\right] V_\perp\, dV_\perp\, dV_z\, d\theta$$

$$= \frac{e\phi_{k\omega}}{m_i} \int \left[\frac{-k_\perp V_\perp \cos\phi \left(\frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp}\right) - k_z(\partial f_{oi}/\partial V_z)}{(\omega_k - k_z V_z - k_\perp V_\perp \cos\phi)}\right] V_\perp\, dV_\perp\, dV_z\, d\phi$$

upon rotating our coordinate system in the V_x-V_y plane through an angle Ψ, as shown in the accompaning figure.



$\phi = \theta - \Psi$

Integrating with respect to the phase angle ϕ, we get

$$n_{ik\omega} = \frac{2\pi e\phi_{k\omega}}{m_i}\int_{-\infty}^{+\infty}\left[\frac{k_\perp^2}{(\omega_k-k_zV_z)^2} - \frac{k_z}{(\omega_k-k_zV_z)}\frac{\partial}{\partial V_z}\right]\Psi_{oi}(V_z)\,dV_z \tag{3.3.7}$$

where

$$\Psi_{oi}(V_z) = \int_o^\infty f_{oi}(V_\perp^2,V_z)\,V_\perp\,dV_\perp \tag{3.3.8}$$

In arriving at Eq.(3.3.7) we have approximated

$$\left[1 - \frac{k_\perp^2V_\perp^2}{(\omega_k-k_zV_z)^2}\right]^{-1/2} \simeq 1 + \frac{k_\perp^2V_\perp^2}{2(\omega_k-k_zV_z)^2}$$

by assuming $\frac{k_\perp^2V_\perp^2}{(\omega_k-k_zV_z)^2} \ll 1$. This is reasonable since the average

ion speed $\langle V\rangle_i \sim C_i = (T_i/m_i)^{1/2} \ll (T_e/m_i)^{1/2} = C_s \sim (\omega_k/k)$,

for $T_e \gg T_i$, $k \sim k_\perp$ and $|k_zV_z| \ll \omega_k$.

We now substitute for $n_{ek\omega}$ and $n_{ik\omega}$ from Eqs.(3.3.6) and (3.3.7) respectively, into Eq.(3.3.1), and obtain the dispersion relation

$$\phi_{k\omega} = \frac{4\pi e}{k^2}\left[\frac{2\pi e}{m_e}\phi_{k\omega}\left\{\frac{m_e}{m_i}\int_{-\infty}^{+\infty}\left(\frac{k_\perp^2}{(\omega_k - k_z V_z)^2} - \frac{k_z}{(\omega_k - k_z V_z)}\frac{\partial}{\partial V_z}\right)\Psi_{oi}(V_z)\, dV_z\right.\right.$$

$$\left.\left. + \int_{-\infty}^{+\infty}\left[\beta(k_\perp/\Omega_e) - \frac{\alpha(k_\perp/\Omega_e)k_z}{(\omega_k - k_y V_o - k_z V_z)}\frac{\partial}{\partial V_z}\right]\Psi_{oe}(V_z)\, dV_z\right\}\right]$$

The assumption $|k_z V_z| << \omega_k$ for the ions allows us to reduce this equation to the approximate form

$$\epsilon(\omega,k) = 1 - \frac{8\pi^2 e^2}{m_e k^2}\left[\frac{m_e}{m_i}\int_{-\infty}^{\infty}\left\{\frac{k_\perp^2}{\omega_k^2}\left(1 + \frac{2k_z V_z}{\omega_k}\right) - \frac{k_z}{\omega_k}\left(1 + \frac{k_z V_z}{\omega_k}\right)\frac{\partial}{\partial V_z}\right\}\Psi_{oi}(V_z)\, dV_z\right.$$

$$\left. + \int_{-\infty}^{+\infty}\left\{\beta(k_\perp/\Omega_e) - \frac{\alpha(k_\perp/\Omega_e)k_z}{(\omega_k - k_y V_o - k_z V_z)}\frac{\partial}{\partial V_z}\right\}\Psi_{oe}(V_z)\, dV_z\right] = 0 \qquad (3.3.9)$$

In Eq.(3.1.6) we have written $\omega_k = \omega_k^r + i\,\gamma_k$. Restricting ourselves to growth rates γ_k such that $|\gamma_k| << \omega_k^r$, we may expand

$$\frac{1}{(\omega_k)^2} = \frac{1}{(\omega_k^r + i\gamma_k)^2} \simeq \frac{1}{(\omega_k^r)^2}\left(1 - \frac{2i\,\gamma_k}{\omega_k^r}\right)$$

and (3.3.10)

$$\frac{1}{\omega_k} \simeq \frac{1}{\omega_k^r}\left(1 - \frac{i\gamma_k}{\omega_k^r}\right)$$

The condition $|\gamma_k| << \omega_k^r$ also allows us to expand the integral

$$\int_{-\infty}^{+\infty}\frac{k_z}{(\omega_k - k_y V_o - k_z V_z)}\frac{\partial\Psi_{oe}}{\partial V_z}\, dV_z$$

about $\omega_k = \omega_k^r$ (i.e. $\gamma_k = 0$). Hence

$$\int_{-\infty}^{+\infty} \frac{k_z(\partial\Psi_{oe}/\partial V_z)}{(\omega_k^r+i\gamma_k-k_yV_o-k_zV_z)} dV_z$$

$$= \lim_{\epsilon\to0^+} \int_{-\infty}^{+\infty} \frac{k_z(\partial\Psi_{oe}/\partial V_z)}{(\omega_k^r-k_yV_o-k_zV_z+i\epsilon)} dV_z + i\gamma_k \frac{\partial}{\partial\omega_k^r}\left[\lim_{\epsilon\to0^+} \int_{-\infty}^{+\infty} \frac{k_z(\partial\Psi_{oe}/\partial V_z)}{(\omega_k^r-k_yV_o-k_zV_z+i\epsilon)} dV_z\right]$$

$$= \left(1 + i\gamma_k \frac{\partial}{\partial\omega_k^r}\right)\left[\lim_{\epsilon\to0^+} \int_{-\infty}^{+\infty} \frac{k_z(\partial\Psi_{oe}/\partial V_z)}{(\omega_k^r-k_yV_o-k_zV_z+i\epsilon)} dV_z\right] \qquad (3.3.11)$$

The integral term may be written as

$$-\lim_{\epsilon'\to0^+} \int_{-\infty}^{+\infty} \frac{(\partial\Psi_{oe}/\partial V_z)}{\{V_z-[\{(\omega_k^r-k_yV_o)/k_z\}+i\epsilon']\}} dV_z$$

where $\epsilon' = \epsilon/k_z$.

It is seen from the above expression that by expanding about $\omega = \omega_k^r + i\epsilon$ we have moved the pole above the real V_z axis ($\epsilon'>0$). Since the integral has to be performed according to the "Landau prescription" (1, p. 376), this enables us to integrate along the real V_z axis. The limit $\epsilon'\to0^+$ then yields the solution to the original integral.

Thus, defining

$$\overline{V}_z = (\omega_k^r - k_yV_o)/k_z \qquad (3.3.11a)$$

we have (1, p. 382)

$$-\lim_{\epsilon'\to0^+} \int_{-\infty}^{+\infty} \frac{(\partial\Psi_{oe}/\partial V_z)}{\{V_z-(\overline{V}_z+i\epsilon')\}} dV_z$$

$$= -\left[\oint\frac{(\partial\Psi_{oe}/\partial V_z)}{(V_z-\overline{V}_z)}\,dV_z + i\int_{-\infty}^{+\infty}(\partial\Psi_{oe}/\partial V_z)\,\pi\,\delta(V_z-\overline{V}_z)\,dV_z\right]$$

$$= -\oint\frac{(\partial\Psi_{oe}/\partial V_z)}{(V_z-\overline{V}_z)}\,dV_z - i\pi\left(\frac{\partial\Psi_{oe}}{\partial V_z}\right)_{V_z=\overline{V}_z} \qquad (3.3.12)$$

The second term on the right hand side, due to $\delta(V_z-\overline{V}_z)$, represents the resonant interaction between the electrons and the wave, i.e. electrons with speed along the magnetic field $\vec{B}_o$ close to the Doppler-shifted wave speed $\overline{V}_z = (\omega_k^r - k_y V_o)/k_z$. These particles are responsible for either growth or damping of the wave. The first term is the principal part of the integral, and represents the non-resonant interaction between the bulk of the electrons and the wave - the particles merely oscillating in the presence of the wave.

With the aid of the Eqs.(3.3.10) - (3.3.12) the dispersion relation (3.3.9) can be written as

$$\varepsilon(\omega,k)=1-\frac{8\pi^2e^2}{m_ek^2}\left[\frac{m_e}{m_i}\int_{-\infty}^{+\infty}\left\{\frac{k_\perp^2}{(\omega_k^r)^2}\left(1-\frac{2i\gamma_k}{\omega_k^r}\right)\left[1+\frac{2k_zV_z}{\omega_k^r}\left(1-\frac{i\gamma_k}{\omega_k^r}\right)\right]\right.\right.$$

$$\left.-\left[\frac{k_z}{\omega_k^r}\left(1-\frac{i\gamma_k}{\omega_k^r}\right)+\frac{k_z^2V_z}{(\omega_k^r)^2}\left(1-\frac{2i\gamma_k}{\omega_k^r}\right)\right]\frac{\partial}{\partial V_z}\right\}\Psi_{oi}(V_z)\,dV_z$$

$$\left.+\beta(k_\perp/\Omega_e)\int_{-\infty}^{+\infty}\Psi_{oe}(V_z)dV_z+\alpha(k_\perp/\Omega_e)\left(1+i\gamma_k\frac{\partial}{\partial\omega_k^r}\right)\left\{\oint\frac{(\partial\Psi_{oe}/\partial V_z)}{(V_z-\overline{V}_z)}dV_z+i\pi\left(\frac{\partial\Psi_{oe}}{\partial V_z}\right)_{V_z=\overline{V}_z}\right\}\right]$$

$$= 0 \qquad (3.3.13)$$

Upon resolving into real and imaginary components, we obtain for the real part

$$\varepsilon_r(\omega,k) = 1 - \frac{8\pi^2 e^2}{m_e k^2}\left[\frac{m_e}{m_i}\int_{-\infty}^{+\infty}\left\{\frac{k_\perp^2}{(\omega_k^r)^2} - \frac{k_z}{\omega_k^r}\left(1 + \frac{k_z V_z}{\omega_k^r}\right)\frac{\partial}{\partial V_z}\right\}\Psi_{oi}(V_z)\, dV_z\right.$$

$$\left. + \beta(k_\perp/\Omega_e)\int_{-\infty}^{+\infty}\Psi_{oe}(V_z)\, dV_z + \alpha(k_\perp/\Omega_e)\oint\frac{(\partial\Psi_{oe}/\partial V_z)}{(V_z-\bar{V}_z)}\, dV_z\right] = 0$$

(3.3.14)

The imaginary part can be manipulated to show that

$$\frac{\gamma_k}{\omega_k^r} = \frac{-\pi\,\alpha(k_\perp/\Omega_e)\left[\partial\Psi_{oe}(V_z)/\partial V_z\right]_{V_z=\bar{V}_z}}{\frac{m_e}{m_i}\left[\int_{-\infty}^{+\infty}\left\{\left(-\frac{2k_\perp^2}{(\omega_k^r)^2}\right)\left(1 + \frac{2k_z V_z}{\omega_k^r}\right) + \frac{k_z}{\omega_k^r}\left(1 + \frac{2k_z V_z}{\omega_k^r}\right)\frac{\partial}{\partial V_z}\right\}\Psi_{oi}(V_z)\, dV_z\right]}$$

(3.3.15)

In arriving at the above results we have neglected small terms and their products in comparison to other terms.

In addition, for the growth rate (3.3.15) we have assumed

$$|\bar{V}_z| = |(\omega_k^r - k_y V_o)/k_z| \ll |V_z| \qquad (3.3.16)$$

for the non-resonant electrons. This approximation will be discussed in the next section.

We recall that the expressions (3.3.14) and (3.3.15) hold for any equilibrium distributions of the form $f_{oj} = f_{oj}(V_\perp^2, V_z)$, with the functionals α and β given by (3.3.4) and (3.3.5) respectively.

3.4 APPLICATION TO MAXWELLIAN ELECTRON AND ION VELOCITY DISTRIBUTIONS

The theory developed above, is now applied to the case of Maxwellian electron and ion velocity distributions.

For the drifting electrons we choose

$$
\begin{aligned}
f_{oe}(V_\perp^2, V_z) &= n_o\ (2\pi C_e^2)^{-3/2}\ \exp\{-(V_x^2+(V_y-V_o)^2+V_z^2)/2C_e^2\} \\
&= a_e\ \exp\{-(V_\perp^2+V_z^2)/2C_e^2\}
\end{aligned}
$$

where

$$
a_e = n_o\ (2\pi C_e^2)^{-3/2} \tag{3.4.1}
$$

and, we recall, for the electrons $V_\perp^2 = V_x^2+(V_y-V_o)^2$.

Then, from Eq.(3.3.3),

$$
\begin{aligned}
\Psi_{oe}(V_z) &= \int_o^\infty f_{oe}\ (V_\perp^2, V_z)\ V_\perp\ dV_\perp \\
&= a_e\ \exp(-V_z^2/2C_e^2) \int_o^\infty \exp(-V_\perp^2/2C_e^2)\ V_\perp\ dV_\perp \\
&= a_e C_e^2\ \exp(-V_z^2/2C_e^2)
\end{aligned}
\tag{3.4.2}
$$

Furthermore

$$
\begin{aligned}
&\int_o^\infty J_o^2(k_\perp V_\perp/\Omega_e)\ f_{oe}\ (V_\perp^2, V_z)\ V_\perp\ dV_\perp \\
&\qquad = a_e\ \exp(-V_z^2/2C_e^2) \int_o^\infty J_o^2\ (k_\perp V_\perp/\Omega_e)\ \exp(-V_\perp^2/2C_e^2)\ V_\perp\ dV_\perp
\end{aligned}
$$

Using the relation [60]

$$\int_0^\infty \exp(-px^2)\, J_n^2(qx)\, x\, dx = \frac{1}{2p} \exp\left(-\frac{q^2}{2p}\right) I_n\left(\frac{q^2}{2p}\right) \tag{3.4.3}$$

where I_n is the modified Bessel function of the first kind of order n, it turns out that

$$\int_0^\infty J_o^2(k_\perp V_\perp/\Omega_e)\, f_{oe}(V_\perp^2, V_z)\, V_\perp\, dV_\perp = a_e C_e^2 \exp(-V_z^2/2C_e^2)\, \Gamma_o(b) \tag{3.4.4}$$

where

$$\Gamma_o(b) = e^{-b}\, I_o(b)$$

with (3.4.5)

$$b = k_\perp^2 C_e^2/\Omega_e^2$$

Thus, from Eqs.(3.3.4), (3.4.2) and (3.4.4) we see that

$$\alpha(k_\perp/\Omega_e) = \Gamma_o(b) \tag{3.4.6}$$

Similarly

$$\int_0^\infty \{1 - J_o^2(k_\perp V_\perp/\Omega_e)\} \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} V_\perp\, dV_\perp$$

$$= a_e \exp(-V_z^2/2C_e^2) \int_0^\infty \{1 - J_o^2(k_\perp V_\perp/\Omega_e)\} (-C_e^2)^{-1} \exp(-V_\perp^2/2C_e^2)\, V_\perp\, dV_\perp$$

$$= a_e C_e^2 \exp(-V_z^2/2C_e^2)\, \{\Gamma_o(b) - 1\}\, (C_e^2)^{-1} \tag{3.4.7}$$

Therefore, from Eqs.(3.3.5), (3.4.2) and (3.4.7),

$$\beta(k_\perp/\Omega_e) = (C_e^2)^{-1} \{\Gamma_o(b) - 1\} \tag{3.4.8}$$

For the stationary ions the equilibrium distribution function is chosen to be

$$f_{oi}(V_\perp^2,V_z) = n_o\ (2\pi C_i^2)^{-3/2}\ \exp\ \{-(V_x^2+V_y^2+V_z^2)/2C_i^2\}$$

$$= a_i\ \exp\ \{-(V_\perp^2+V_z^2)/2C_i^2\}$$

where

$$a_i = n_o\ (2\pi C_i^2)^{-3/2} \tag{3.4.9}$$

and $V_\perp^2 = V_x^2+V_y^2$ for the ions.

Then, using Eq.(3.3.8), we find

$$\Psi_{oi}(V_z) = a_i\ C_i^2\ \exp\ (-V_z^2/2C_i^2)$$

and (see Appendix A)

$$\int_{-\infty}^{+\infty} \Psi_{oj}(V_z)\ dV_z = a_j\ C_j^2 \int_{-\infty}^{+\infty} \exp\ (-V_z^2/2C_j^2)\ dV_z$$

$$= a_j\ C_j^2\ (2\pi C_j^2)^{1/2} \qquad (j = i,e) \tag{3.4.10a}$$

$$= (n_o/2\pi)$$

$$\int_{-\infty}^{+\infty} V_z\ \Psi_{oi}(V_z)\ dV_z = 0 \tag{3.4.10b}$$

$$\int_{-\infty}^{+\infty} \left\{1 + \frac{\ell k_z V_z}{\omega_k^r}\right\}\left(\frac{\partial \Psi_{oi}}{\partial V_z}\right) dV_z = - \frac{\ell k_z}{\omega_k^r}\ a_i\ C_i^2\ (2\pi C_i^2)^{1/2} \tag{3.4.10c}$$

$$= - \frac{n_o \ell k_z}{2\pi\omega_k^r}$$

where ℓ = integer.

Furthermore, the assumption (3.3.16) leads to the result

$$\oint \frac{(\partial\Psi_{oe}/\partial V_z)}{(V_z-\bar{V}_z)}\, dV_z = \int_{-\infty}^{+\infty} \frac{1}{V_z}\frac{\partial\Psi_{oe}}{\partial V_z}\, dV_z = - a_e(2\pi C_e^2)^{1/2}$$

$$= - \frac{n_o}{2\pi C_e^2} \qquad (3.4.10d)$$

With the results (3.4.6) — (3.4.10) the real part of the dispersion relation ((3.3.14)) reduces to

$$1 - \frac{8\pi^2 e^2}{m_e k^2}\left[\frac{m_e}{m_i}\left\{\frac{k_\perp^2}{(\omega_k^r)^2}\frac{n_o}{2\pi} + \frac{k_z^2}{(\omega_k^r)^2}\frac{n_o}{2\pi}\right\} + \frac{n_o}{2\pi C_e^2}\left\{\Gamma_o(b) - 1\right\} - \Gamma_o(b)\frac{n_o}{2\pi C_e^2}\right] = 0$$

which may be written as

$$1 - \frac{\omega_{pe}^2}{k^2}\left[\frac{m_e}{m_i}\frac{k^2}{(\omega_k^r)^2} - \frac{1}{C_e^2}\right] = 0$$

since $k^2 = k_\perp^2 + k_z^2$, and $\omega_{pe} = (4\pi n_o e^2/m_e)^{1/2}$ is the electron plasma frequency.

Solving for ω_k^r, we obtain

$$\omega_k^r = k\ C_s\ (1 + k^2\lambda_D^2)^{-1/2} \qquad (3.4.11)$$

where $\lambda_D = (T_e/4\pi n_o e^2)^{1/2}$ is the electron Debye length and $C_s = (T_e/m_i)^{1/2}$ is the ion sound speed. The expression (3.4.11) is the usual form for ω_k^r for the ion acoustic mode [1].

Similarly, expression (3.3.15) for the growth rate reduces to

$$\frac{\gamma_k}{\omega_k^r} = \frac{-\pi \quad \Gamma_o(b) \quad n_o \ (2\pi C_e^2)^{-3/2} \ C_e^2[(-\overline{V}_z/C_e^2) \exp\{-(\overline{V}_z/\sqrt{2}C_e)^2\}]}{\frac{m_e}{m_i}\left[\left(-\frac{2k_\perp^2}{(\omega_k^r)^2}\right)\frac{n_o}{2\pi} + \frac{k_z}{\omega_k^r}\left(-\frac{2k_z}{\omega_k^r}\frac{n_o}{2\pi}\right)\right]}$$

$$= \frac{-\pi \ \Gamma_o(b) \quad \overline{V}_z \quad \exp\{-(\overline{V}_z/\sqrt{2}C_e)^2\}}{\left[\frac{m_e}{m_i}(2\pi C_e^2)^{3/2} \quad \frac{2k^2}{2\pi(\omega_k^r)^2}\right]}$$

Substituting for $\overline{V}_z$ and $\Gamma_o(b)$ from Eqs.(3.3.11a) and (3.4.5) respectively, this becomes

$$\frac{\gamma_k}{\omega_k^r} = \left(\frac{\pi}{4}\right)^{1/2}\left\{\frac{k}{k_z}\left(\frac{m_e}{2m_i}\right)^{1/2}\right\}\frac{e^{-b}I_o(b)}{(1+k^2\lambda_D^2)}\left[\frac{k_yV_o-\omega_k^r}{kC_s}\right] \qquad (3.4.12)$$

In arriving at the result (3.4.12) we have assumed that for the warm electrons $\left|\frac{\omega_k^r-k_yV_o}{\sqrt{2}k_zC_e}\right| << 1$ and hence set $\exp\{-(\overline{V}_z/\sqrt{2}C_e)^2\} \simeq 1$. For $V_z \sim C_e$ this is equivalent to the approximation (3.3.16), and has been adopted by LASHMORE-DAVIES and MARTIN [11] and AREFEV [14]. Since in practice $V_o >> (\omega_k^r/k)$, it follows that $(\omega_k^r/\sqrt{2}k_zC_e) << 1$. This implies

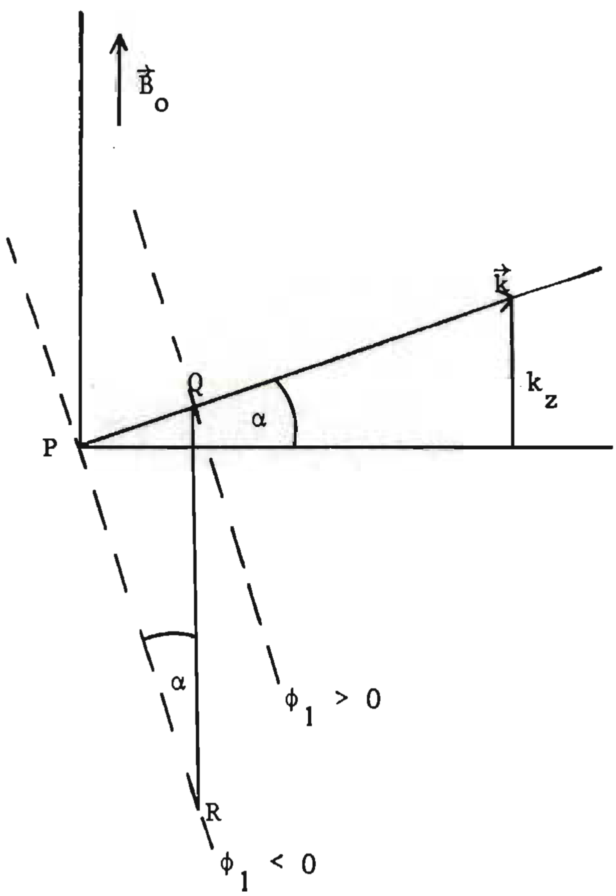
$$\text{that } \frac{\omega_k^r}{k_zC_e} \sim \frac{kC_s}{k_zC_e} << 1, \text{ or } 0 < \left(\frac{m_e}{m_i}\right)^{1/2} << \frac{k_z}{k}. \qquad (3.4.13)$$

In experimental observations of the ion acoustic instability BARRETT *et al* [37] and HAYZEN and BARRETT [41] find agreement with theory for parameters satisfying this condition. The former worked with a cesium plasma with $(m_e/m_i)^{1/2} \simeq 0{,}002$ and $(k_z/k) \simeq 0{,}03$, the latter with an argon plasma with $(m_e/m_i)^{1/2} \simeq 0{,}004$ and $(k_z/k) \simeq 0{,}03$.

The growth rate (3.4.12) is identical to that obtained from the ion acoustic dispersion relation of LASHMORE - DAVIES and MARTIN [11]. For long wavelength fluctuations, i.e. $b = k_\perp^2 r_e^2 << 1$, it agrees with the result of AREFEV [14], provided one neglects, as we have done, the contribution of the small exponential term due to ion Landau damping in his investigations.

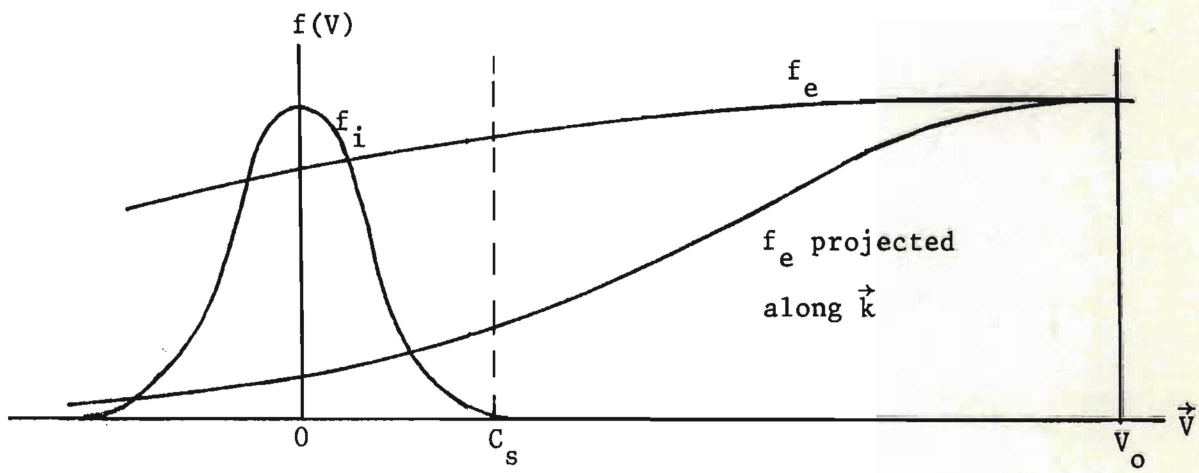
The enhancement of the growth rate by the factor (k/k_z) for wave propagation oblique to the magnetic field as compared to the field-free or field-aligned case, has been discussed by HAYZEN [61] and LEE [62]. In the absence of a magnetic field the electrons, because of their small mass, move rapidly to neutralize any potential variations produced by the ions.

However, for $\vec{B} \neq \vec{0}$ the electrons are tied to the field lines and are free to accelerate only along $\vec{B}$.



In the figure above, an electron at point P will travel a distance PQ in the field-free case to neutralize the potential perturbation. For $\vec{B} \neq \vec{0}$ the motion of the electron across $\vec{B}$ is restricted. Instead an electron at R is free to accelerate along $\vec{B}$ and reach the point Q. The longer distance travelled by the neutralizing electron (RQ > PQ) allows the perturbations to grow to a larger amplitude. From the figure $(RQ/PQ) = (k/k_z)$.

In terms of velocity distribution functions, since the electron thermal motion is along $\vec{B}$, its projection along the wave vector $\vec{k}$, gives an effective distribution with the thermal speed diminished by the ratio k_z/k, as shown in the figure below.

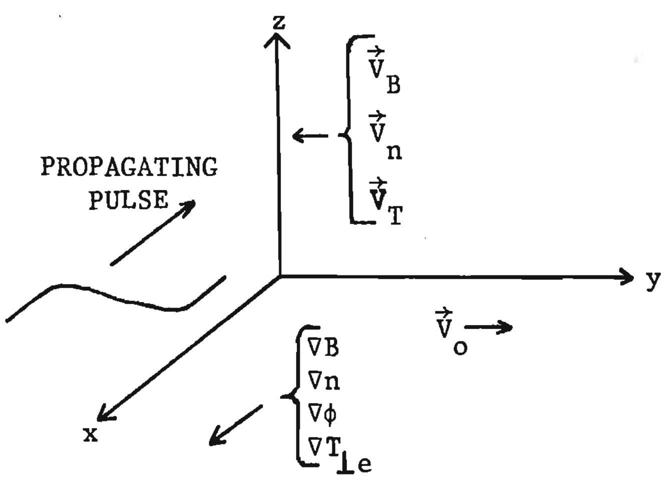


Consequently, even for small drift velocities $V_o \gtrsim C_s$, the phase velocity of the wave ($V_\phi \sim C_s$) can coincide with the location of maximum slope of the effective electron distribution function and thereby experience enhanced growth. Since the ions are unmagnetized, the slope of the ion distribution function is not affected, and ion Landau damping remains unaltered.

3.5 THE EFFECT OF INHOMOGENEITIES ON THE CROSSFIELD CURRENT-DRIVEN ION ACOUSTIC INSTABILITY

The formalism presented in arriving at the results (3.3.14) and (3.3.15) is for a general equilibrium distribution $f_{oj}(V_\perp^2, V_z)$. It has been shown to yield previously derived results for the special case of Maxwellian electron and ion velocity distributions. In the linear treatment of electrostatic plasma instabilities, where one may include inhomogeneities in plasma temperature, plasma density and magnetic field, it is a normal practice to choose a self-consistent equilibrium velocity distribution for both the electrons and the ions, and the dispersion relation is established in terms of the plasma dispersion function [63]. Since almost every reference cited in Section 2.1 adopts this approach, it is appropriate that for purpose of completeness we review the technique. The work presented here under has been previously undertaken by the author of this thesis [64] and the findings have been reported [18].

We consider a model with gradients in magnetic field $\vec{B}$, electron density n_e and electron temperature perpendicular to the magnetic field $T_{\perp e}$, as shown in the figure below.



Assuming the inhomogeneities to vary linearly, we may write

$$\vec{B} = B_o(1+\epsilon x)\hat{z} \qquad \left(\vec{V}_B = -\frac{\epsilon V_\perp^2}{2|\Omega_e|}\hat{y} = -V_B\,\hat{y}\right) \qquad \text{(a)}$$

$$n_e = n_o(1+\rho x) \qquad \left(\vec{V}_n = -\frac{\rho T_{eo}}{m_e|\Omega_e|}\hat{y} = -V_n\,\hat{y}\right) \qquad \text{(b)}$$

(3.5.1)

$$T_{\perp e} = T_{eo}(1+\delta x) \qquad \left(\vec{V}_T = -\frac{\delta T_{eo}}{m_e|\Omega_e|}\hat{y} = -V_T\,\hat{y}\right) \qquad \text{(c)}$$

$$\vec{V}_o = \frac{C(\vec{E}\times\vec{B})}{B^2} \qquad \text{(d)}$$

where the quantities in parantheses are the associated gradient drift velocities. $\vec{V}_o$ is the $\vec{E}\times\vec{B}$ drift of the magnetized electrons relative to the cold unmagnetized ions which are assumed to be at rest. From the equations of motion for the electron:

$$\ddot{x} = \frac{eE_o}{m_e} - |\Omega_e|(1+\epsilon x)\dot{y}$$

$$\ddot{y} = |\Omega_e|(1+\epsilon x)\dot{x}$$

$$\ddot{z} = 0$$

we may construct, among others, the following constants of the motion (to order ϵ)

$$V_\perp^2 = V_x^2 + (V_y - V_o)^2,\; V_z \text{ and } X = x - \frac{(V_y - V_o)}{|\Omega_e|}$$

Thus, the equilibrium distribution for the electrons is taken to be

$$f_{oe}(V_\perp^2, V_z, X) = \frac{n_o}{(2\pi C_e^2)^{3/2}}\{1 + X[\rho + \{\delta(V_\perp^2 - 2C_e^2)/2C_e^2\}]\}\exp\left\{-\frac{(V_\perp^2+V_z^2)}{2C_e^2}\right\}$$

(3.5.2)

and has been shown to be self-consistent [64].

Then

$$\frac{\partial f_{oe}}{\partial \vec{V}} = \frac{\partial f_{oe}}{\partial V_\perp^2} \nabla_V V_\perp^2 + \frac{\partial f_{oe}}{\partial X} \nabla_V X + \frac{\partial f_{oe}}{\partial V_z} \nabla_V V_z$$

$$\simeq - \frac{(\vec{V}_\perp + \vec{V}_z)}{C_e^2} f_{oe} - \frac{(V_y - V_o)}{|\Omega_e|} \frac{\delta}{C_e^2} \vec{V}_\perp f_{oe} - \frac{1}{|\Omega_e|} \{\rho + [\delta(V_\perp^2 - 2C_e^2)/2C_e^2]\}\hat{y} f_{oe}$$

where use has been made of the local approximation [1]. This involves evaluating all equilibrium features which occur in the dispersion relation ($n_e, T_{\perp e}, dn_e/dx$, etc.) at the local position $x = 0$. In addition, the inhomogeneities have been assumed to be weak, so that products of gradients are neglected.

The expression for $\partial f_{oe}/\partial \vec{V}$ is then used in Eq.(3.2.5c) to find the perturbed electron distribution $f_{ek\omega}(\vec{V})$, which in turn, via Eq.(3.3.1a), yields the perturbed electron density $n_{ek\omega}$. Using the identity (3.2.11), it turns out that

$$n_{ek\omega} = \frac{n_o e\phi_{k\omega}}{T_{eo}} \left[1 + \frac{2\pi}{(2\pi C_e^2)^{3/2}} \sum_{\ell=-\infty}^{+\infty} \int_0^{+\infty} V_\perp \, dV_\perp \int_{-\infty}^{+\infty} dV_z \; J_\ell^2\left(\frac{k_\perp V_\perp}{\Omega_e}\right) \exp\left\{ - \frac{(V_\perp^2 + V_z^2)}{2C_e^2} \right\} \right.$$

$$\left. \times \frac{\{\omega - k_y[V_o - V_n + V_T(1 - V_\perp^2/C_e^2)]\}}{(k_z V_z - [\omega - k_y(V_o - V_B)] - \ell\Omega_e)} \right] \qquad (3.5.3)$$

We may express the integration with respect to V_z in terms of the plasma dispersion function (Z-function), defined as [63]

$$Z(\lambda) = \pi^{-1/2} \int_{-\infty}^{+\infty} \frac{\exp(-x^2)}{(x - \lambda)} dx \qquad (3.5.4a)$$

for $\mathrm{Im}(\lambda) > 0$ and as the analytic continuation of this for $\mathrm{Im}(\lambda) \leq 0$.

An alternative definition is

$$Z(\lambda) = 2i \exp(-\lambda^2) \int_{-\infty}^{i\lambda} \exp(-t^2)\, dt \tag{3.5.4b}$$

Then

$$n_{ek\omega} = \frac{n_o e\phi_{k\omega}}{T_{eo}} \left[1 + \frac{1}{\sqrt{2}k_z C_e^3} \sum_{\ell=-\infty}^{+\infty} \int_0^{\infty} \{\omega - k_y[V_o - V_n + V_T(1 - V_\perp^2/C_e^2)]\}\right.$$

$$\left. \times\, Z\left[\frac{\omega - k_y(V_o + \epsilon V_\perp^2/2\Omega_e) - \ell\Omega_e}{\sqrt{2}k_z C_e}\right] J_\ell^2\left(\frac{k_\perp V_\perp}{\Omega_e}\right) \exp\left(-\frac{V_\perp^2}{2C_e^2}\right) V_\perp dV_\perp\right] \tag{3.5.5}$$

The ions, once again, are assumed to traverse straight line orbits with constant velocity, i.e., due to their inertia, they react to neither the magnetic nor the electric field . Then the ion equation of motion

$$\ddot{\vec{r}} = \vec{0}$$

with the solution

$$\vec{r}(t') - \vec{r}(t) = \vec{V}\,(t'-t)$$

and an equilibrium velocity distribution

$$f_{oi}(V_\perp^2, V_z) = \frac{n_o}{(2\pi C_i^2)^{3/2}} \exp\left\{-\frac{(V_\perp^2 + V_z^2)}{2C_i^2}\right\}$$

where $V_\perp^2 = V_x^2 + V_y^2$, are used to arrive, in a manner paralleling that for the electrons, at the result

$$n_{ik\omega} = \frac{n_o e\phi_{k\omega}}{2T_i} Z'(\omega/\sqrt{2}kC_i) \tag{3.5.6}$$

for the perturbed ion density. Here $Z'(\lambda)$, the first derivation of the plasma dispersion function, is given by (63, 16)

$$Z'(\lambda) = -2[1+\lambda Z(\lambda)] \tag{3.5.7}$$

Upon substituting for $n_{ek\omega}$ and $n_{ik\omega}$ from Eqs.(3.5.5) and (3.5.6) respectively, into Poisson's equation (3.3.1), viz.,

$$\phi_{k\omega} = \frac{4\pi e}{k^2}(n_{ik\omega}-n_{ek\omega})$$

we may write the dispersion relation in the familiar form

$$1 + K_e + K_i = 0 \tag{3.5.8a}$$

where

$$K_e = \frac{k_D^2}{k^2}\left[1 + \frac{1}{\sqrt{2}k_z C_e^3}\sum_{\ell=-\infty}^{+\infty}\int_0^{\infty}\{\omega-k_y[V_o-V_n+V_T(1-V_\perp^2/C_e^2)]\}\right.$$

$$\left.\times\, Z\left[\frac{\omega-k_y(V_o+\epsilon V_\perp^2/2\Omega_e)-\ell\Omega_e}{\sqrt{2}k_z C_e}\right] J_\ell^2(k_\perp V_\perp/\Omega_e)\exp\left(-\frac{V_\perp^2}{2C_e^2}\right)V_\perp dV_\perp\right] \tag{3.5.8b}$$

and

$$K_i = -\frac{k_D^2}{k^2}\frac{T_{eo}}{2T_i}Z'(\omega/\sqrt{2}kC_i) \tag{3.5.8c}$$

For the low frequency ion acoustic wave ($|\omega-k_y\{V_o+\epsilon V_\perp^2/2\Omega_e\}| << |\Omega_e|$) we use the result $Z_{-n}(x) = -Z_n(x)$ to retain only the $\ell = 0$ term in the summation in the expression for K_e. For a plasma with warm electrons and cold ions, i.e., $T_e >> T_i(\sim 0)$, we assume

$$|z_{oe}| = \left| \frac{\omega - k_y(V_o + \epsilon V_\perp^2/2\Omega_e)}{\sqrt{2}k_z C_e} \right| << 1$$

and use the power series expansion[63]

$$Z(\lambda) = i\,\pi^{\frac{1}{2}}\,e^{-\lambda^2} - 2\,\lambda[1 - 2\lambda^2/3 + 4\lambda^4/15 - \ldots.] \text{ for } |\lambda| << 1.$$

(3.5.9a)

The assumption of cold ions implies $|z_i| = |\omega/\sqrt{2}kC_i| >> 1$. In this limit one may employ the asympototic expansion [63]

$$Z(\lambda) = i\,\pi^{\frac{1}{2}}\,\delta\,e^{-\lambda^2} - \lambda^{-1}\,[1 + 1/2\lambda^2 + 3/4\lambda^4 + \ldots.] \tag{3.5.9b}$$

where

$$\delta = \begin{cases} 0 & \mathrm{Im}(\lambda) > 0 \\ 1 & \mathrm{Im}(\lambda) = 0 \\ 2 & \mathrm{Im}(\lambda) < 0 \end{cases}$$

For a weak magnetic field gradient, V_B is small. The assumption $|z_{oe}| << 1$ thus implies that the Doppler-shifted wave speed along the magnetic field is very much smaller than the electron thermal speed. For the ions, $|z_i| >> 1$ means that the wave phase speed is much larger than the ion thermal speed.

With the aid of the above approximations and the results [60]

$$\int_0^\infty J_o^2(k_\perp V_\perp/\Omega_e)\,\exp\left(-\frac{V_\perp^2}{2C_e^2}\right) V_\perp\,dV_\perp = C_e^2\,e^{-b}\,I_o(b)$$

$$\int_0^\infty V_\perp^2\,J_o^2(k_\perp V_\perp/\Omega_e)\,\exp\left(-\frac{V_\perp^2}{2C_e^2}\right) V_\perp\,dV_\perp = 2C_e^4\,e^{-b}\,[(1-b)\,I_o(b) + bI_1(b)]$$

where $b = k_\perp^2 C_e^2/\Omega_e^2 = k_\perp^2 r_e^2$ and $I_n(b)$ is the modified Bessel function of order n, the dispersion relation (3.5.8a) may be solved for the growth rate to yield

$$\frac{\gamma_k}{\omega_k^r} = \frac{\pi^{1/2} s[\{k_y(V_o - V_n + V_T) - \omega_k^r\}\Gamma_o - 2k_y V_T Q_o]}{2\{\alpha^{-2} kC_s - \alpha s^2 k_y \langle V_B \rangle Q_o\}} \tag{3.5.10}$$

with

$$\omega_k^r = kC_s\alpha$$

being the real part of the frequency, and where

$$\alpha = (1 + k^2\lambda_D^2)^{-\frac{1}{2}} \qquad \Gamma_n(b) = e^{-b} I_n(b)$$

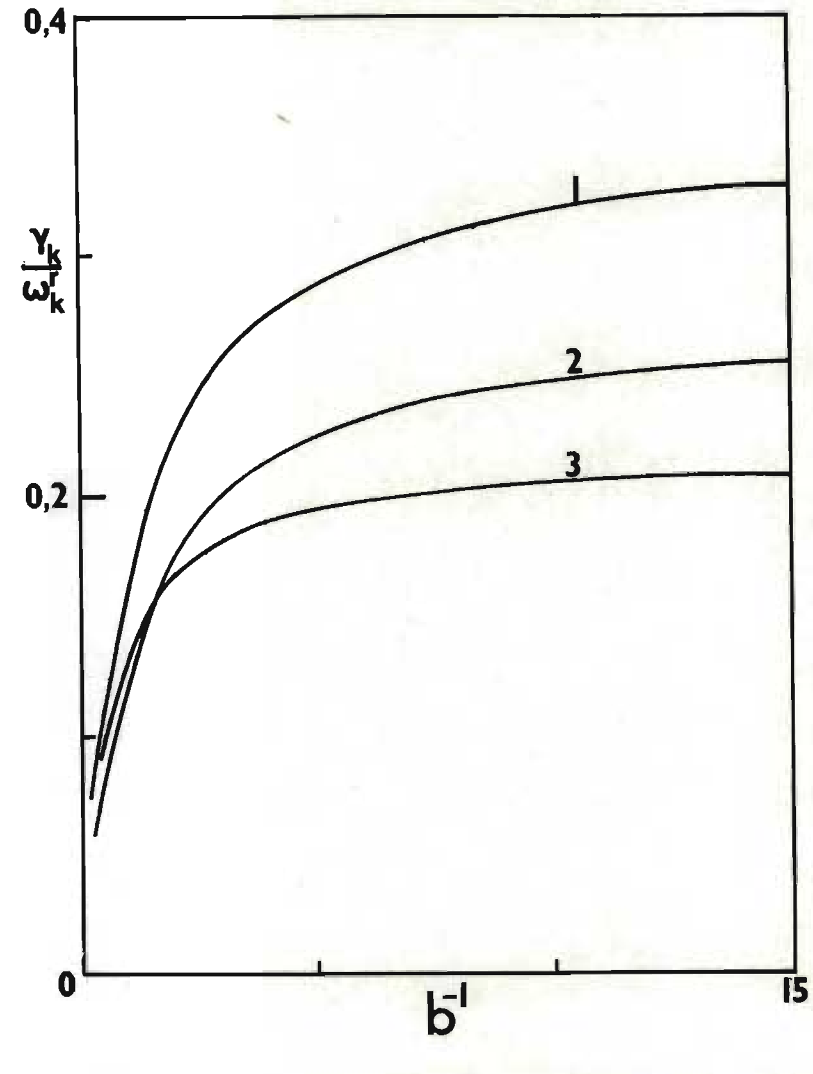
$$Q_o(b) = (1-b)\Gamma_o + b\Gamma_1 \qquad s = (k/k_z)\ (m_e/2m_i)^{\frac{1}{2}}$$

and $\langle V_B \rangle = (\varepsilon C_e^2/|\Omega_e|)$ is the average ∇B drift. In the analysis we have chosen $s \lesssim 1$. In the absence of inhomogeneities the growth rate (3.5.10) reduces to the expression (3.4.12).

The significant feature of this result is that while it confirms earlier calculations [(15)] that showed that ∇T has a destabilizing effect for large $k_\perp r_e$, for longer wavelengths ($k_\perp r_e \lesssim 1$) the temperature gradient is found to reduce the growth rate (see Fig. 3.3). One may offer the following possible explanation for the observed behaviour.

For large $k_\perp r_e$ ($\gg 1$) the distortion in the electron distribution function produced by the introduction of a temperature gradient, as explained by PRIEST and SANDERSON [(15)], increases the slope of the distribution function at the velocity corresponding to the wave phase velocity. This distortion arises from the fact that for a temperature gradient increasing with x, electrons situated at larger x values in physical space have larger gyroradii (for a fixed $\vec{B}$), since

Figure 3.3

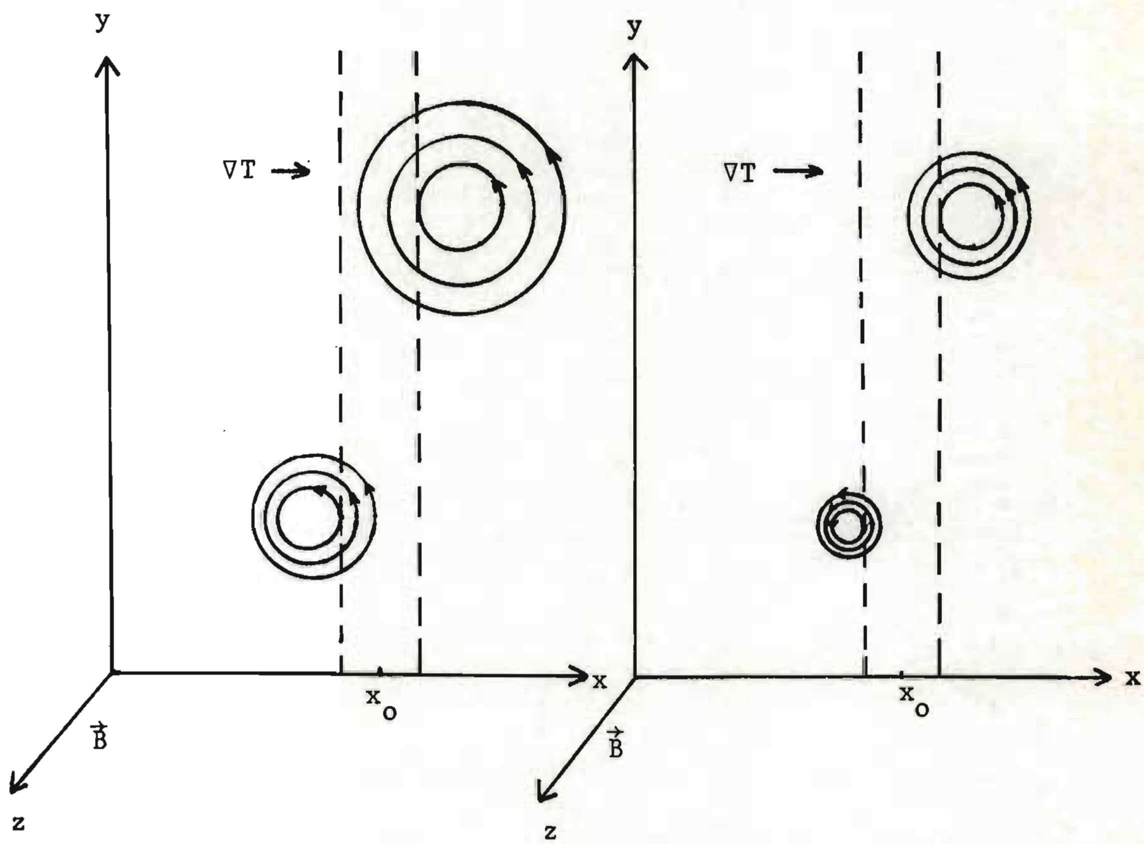


Normalized growth rate of the crossfield current-driven ion acoustic instability as a function of $b^{-1}=(k_\perp r_e)^{-2}$. Curve 1 represents $\vec{V}_o$ alone, curve 2 also includes the effects of $\vec{V}_B$ and $\vec{V}_n$, while curve 3 has $\vec{V}_o, \vec{V}_B, \vec{V}_n$ and $\vec{V}_T$. Other parameters are k_y/k = 0,9995, s = 0,045 (cesium plasma).

$$C_e^2 = T_e(x)/m_e = \Omega_e^2 r_e^2$$

This results in a non-symmetric spread in V_y, with the spread in V_y for $V_y < 0$ being larger than that for $V_y > 0$, as can be seen from Fig. 3.4(a) below. In the absence of ∇T the spread is symmetric. The consequent enhancement in growth rate is sufficient to overcome the reduction in growth rate resulting from a decrease in total particle drift speed $V_D = V_o - V_n - V_T$. A net positive change in growth results.

Now, for $k_\perp \simeq k$ = constant (fixed wavelength) $k_\perp r_e$ may be reduced so that $k_\perp r_e \lesssim 1$ by decreasing r_e. If we retain the same temperature gradient, it is seen from the above equation that at a given x, r_e is reduced by increasing Ω_e, i.e., increasing the magnetic field strength. This illustrated in Fig. 3.4(b) below.



(a) $k_\perp r_e \gg 1$; Weak $\vec{B}$ field (b) $k_\perp r_e \lesssim 1$; Strong $\vec{B}$ field

Figure 3.4

Thus, although the spread in V_y for $V_y < 0$, is larger than that for $V_y > 0$, the net spread in V_y is smaller for a strong $\vec{B}$ field as compared to a weak $\vec{B}$ field. The resulting distortion in the electron distribution function is smaller, which implies a weaker enhancement of wave growth. It is then possible for the reduction in growth due to a decrease in the net drift velocity to dominate over the weak enhancement and produce a smaller resultant growth rate, i.e., ∇T now has a stabilizing effect.

It is seen from Fig. 3.3 that in contrast to the temperature gradient, the introduction of a density gradient, which reduces the net drift V_D, has a stabilizing influence for all b^{-1}.

CHAPTER FOUR

THE QUASILINEAR THEORY OF THE CROSSFIELD CURRENT-DRIVEN ION ACOUSTIC INSTABILITY

4.1 ESTABLISHMENT OF THE QUASILINEAR DIFFUSION EQUATIONS

In the quasilinear limit of weak turbulence theory, the right hand side of Eq.(3.1.2), and therefore of Eq.(3.1.8), is retained, i.e., we treat $f_{oj}(\vec{V},t)$ as a slowly varying function of time. With $f_{oj}(\vec{V},t)$ still assumed to be isotropic in its velocity dependence perpendicular to $\vec{B}_o$, we may write

$$f_{oj}(\vec{V},t) = < f_j(\vec{V},t) > = f_{oj}\ (V_\perp^2,\ V_z,t) \qquad (4.1.1)$$

Expressing Eq.(3.1.8) in terms of cylindrical coordinates $(V_\perp, V_z, \theta)$ in velocity space, we have (see Appendix B)

$$\frac{\partial f_{oj}}{\partial t} + \vec{V} \cdot \frac{\partial f_{oj}}{\partial \vec{r}} - \frac{e_j}{m_j} E_o \left(\cos\theta \frac{\partial f_{oj}}{\partial V_\perp} - \frac{\sin\theta}{V_\perp} \frac{\partial f_{oj}}{\partial \theta}\right) - \Omega_j \frac{\partial f_{oj}}{\partial \theta} = - \frac{e_j}{m_j} \sum_k \vec{E}_{-k} \cdot \frac{\partial f_{jk}}{\partial \vec{V}} \qquad (4.1.2)$$

where $\Omega_j = \frac{e_j B_o}{m_j C}$ is the gyrofrequency.

In general,for a background distribution of the type (4.1.1), we average over the phase angle θ to obtain [47,65]

$$\frac{\partial f_{oj}}{\partial t} = \frac{1}{2\pi} \int_o^{2\pi} \left(- \frac{e_j}{m_j} \sum_k \ (- i\ \phi_k\ \vec{k})^* \cdot \ \frac{\partial f_{jk}}{\partial \vec{V}} \right) d\theta \qquad (4.1.3)$$

Here we have set $\vec{E}_{-k} = \vec{E}_k^* = (-i\vec{k}\,\phi_k)^*$, since $\vec{E}_1 = -\nabla\phi_1$.

The right hand side of Eq. (3.1.3), which can be shown to represent the coupling between the different plasma modes by writing $\vec{E}_1$ and f_{1j} in terms of their spatial Fourier transforms [45], is again neglected. Therefore, we may use for the perturbations $f_{jk\omega}$ the solutions found in Chapter Three.

However, the transition from linear to quasilinear is not straight forward. We recall that the f_{jk} were solutions of the basic equation (3.1.11), which for a time dependent f_{oj} may be written as

$$f_{1j}(\vec{r}, \vec{V}, t) = \frac{e_j}{m_j}\int_{-\infty}^{t} \nabla\phi_1(\vec{r}', t') \cdot \frac{\partial f_{oj}(\vec{V}', t')}{\partial \vec{V}'}\, dt'$$

In terms of the Fourier transforms (3.1.4)this becomes

$$f_{jk}(\vec{V}, t) = \frac{e_j}{m_j}\int_{-\infty}^{t} \phi_k(t')\, i\,\vec{k} \cdot \frac{\partial f_{oj}(\vec{V}', t')}{\partial \vec{V}'} \exp\left(i\,\vec{k} \cdot [\vec{r}(t')-\vec{r}(t)]\right) dt' \tag{4.1.4}$$

As mentioned in Section 2.4, a prerequisite for the validity of the quasilinear theory is that a sufficiently broad spectrum of waves must be present. If $\Delta(\omega_k - \vec{k} \cdot \vec{V})$ is the characteristic spread in $(\omega_k - \vec{k} \cdot \vec{V})$ over the range of $\vec{k}$ values in the spectrum, then

$$\tau_c = |\Delta(\omega_k - \vec{k} \cdot \vec{V})|^{-1}$$

may be considered as the correlation time of the fluctuating fields for the resonant particles.

In equation (4.1.4) we may approximate

$$f_{oj}\,(\vec{V}',\, t') = f_{oj}\,(\vec{V}',\, t - \tau)$$

$$= f_{oj}\,(\vec{V}',\, t) - \tau\,\frac{\partial f_{oj}(\vec{V}',\, t)}{\partial t}$$

$$\simeq f_{oj}\,(\vec{V}',\, t)$$

provided $\tau_{f_{oj}}$, the characteristic time of relaxation of f_{oj}, is such that

$$\tau_c << \tau_{f_{oj}} \tag{4.1.5}$$

with $$t - t' \sim \tau_c .$$

For a time dependence of the WKB type (3.1.5), we set

$$\phi_k\,(t) = \phi_{k\omega}\,\exp\,\{-\,i \int_o^t \omega_k\,(t'')\,dt''\}$$

$$f_{jk}\,(\vec{V},t) = f_{jk\omega}\,(\vec{V},t)\,\exp\,\{-\,i \int_o^t \omega_k\,(t'')\,dt''\} \tag{4.1.5a}$$

Then $$\phi_k\,(t') = \phi_{k\omega}\,\exp\,\{-\,i \int_o^{t'} \omega_k\,(t'')\,dt''\}$$

$$= \phi_{k\omega}\,\exp\,\{-\,i\,[\int_o^t - \int_{t'}^t]\,\omega_k\,(t'')\,dt''\}$$

$$= \phi_{k\omega}\,\exp\,\{-\,i \int_o^t \omega_k\,(t'')\,dt''\}\exp\{i\int_{t'}^t \omega_k\,(t'')\,dt''\}$$

$$= \phi_k\,(t)\,\exp\,\{-\,i\,\omega_k(t)\,(t' - t)\}$$

where we have also approximated

$$\omega_k\,(t'') = \omega_k\,(t - \tau)$$

$$= \omega_k(t) \;-\; \tau\,\frac{\partial \omega_k(t)}{\partial t}$$

$$\simeq \omega_k(t)$$

for $\tau_c << \tau_{f_{oj}}$. We note that this condition is satisfied if the frequency spectrum is broad, as is required.

In going over from the linear to the quasilinear behaviour we have allowed f_{oj} to change very slowly in time. Equation (4.1.4) may now be written in the familiar form (cf.Eq. (3.2.5c))

$$f_{jk}(\vec{V},t) = \frac{e_j}{m_j} \int_{-\infty}^{t} \phi_k(t) \; i\vec{k}.\frac{\partial f_{oj}(\vec{V},t)}{\partial \vec{V}} \; \exp\{i\vec{k}.[\vec{r}(t') - \vec{r}(t)]-i\omega_k(t) \; (t'-t)\} \; dt'$$

or, from Eq. (4.1.5a)

$$f_{jk\omega}(\vec{V},t)=\frac{e_j}{m_j}\int_{-\infty}^{t} \phi_{k\omega} \; i \; \vec{k} \; . \; \frac{\partial f_{oj}(\vec{V},t)}{\partial \vec{V}} \quad \exp\{i \; \vec{k} \; . \; [\vec{r}(t')-\vec{r}(t)]-i \; \omega_k(t)(t'-t)\}dt'$$

We note that here $f_{jk\omega} = f_{jk\omega}(\vec{V},t)$, where as in the linear treatment we had $f_{jk\omega} = f_{jk\omega}(\vec{V})$. The time dependence has been introduced in Eq.(4.1.5a) to allow for the slow time variation of $f_{oj}(\vec{V},t)$. This is clear from the above equation.

Hence,for the ion acoustic instability the perturbations $f_{ek\omega}$ and $f_{ik\omega}$ are given by Eqs. (3.2.12a) and (3.2.13) respectively, with $f_{oj} = f_{oj}(\vec{V},t)$ and $\omega_k = \omega_k(t)$.

Diffusion Equations

For a given f_{jk} we can manipulate the right hand side of Eq. (4.1.3) and write this equation in the form

$$\frac{\partial f_{oj}}{\partial t} = \frac{\partial}{\partial \vec{V}} \cdot \bar{D}_j \cdot \frac{\partial f_{oj}}{\partial \vec{V}}$$

Thus f_{oj} obeys a diffusion equation in velocity space. The diffusion tensor $\bar{D}_j$ is proportional to the wave energy density. We shall see that $\bar{D}_j$ can be written as the sum of two terms. The first is due to the resonant particles, i.e., those with speed V close to the wave phase speed $V_\phi = (\omega_k^r/k)$. It is the interaction between these particles and the wave that is responsible for wave growth or damping. During wave growth (damping) energy is extracted from (gained by) the resonant particles. Therefore the velocity distribution of these particles must change as they lose or gain energy - hence the diffusion in velocity space. The second term is due to the non-resonant particles which do not interact directly with the wave but merely oscillate in the presence of the wave, thereby acquiring mechanical energy. The 'diffusion' of non-resonant particles occurs because they gain (lose) energy as the wave grows (damps).

<u>Electron Diffusion Equation</u>

For the electrons we substitute for f_{ek}, which is given by Eqs.(4.1.5a) and (3.2.12a), into Eq.(4.1.3). Then

$$\frac{\partial f_{oe}}{\partial t} = -\frac{ie^2}{m_e^2}\left\langle \sum_k |\phi_k(t)|^2 \vec{k} \cdot \frac{\partial}{\partial \vec{V}}\left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \exp\{i\,\xi\sin(\theta-\Psi)\}J_o(\xi)\right.\right.$$

$$\left.\left.\times\ \frac{\{(\omega_k - k_y V_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_z V_z(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\}f_{oe}}{(\omega_k - k_y V_o - k_z V_z)}\right]\right\rangle$$

where $<--->$ denotes the average over the phase angle.

To simplify the algebra we rotate our coordinate system in the x - y plane through an angle Ψ as shown in Fig. 4.1 below.

We then have

$$\vec{k} = k_\perp\ \hat{e}_\perp + 0\hat{e}_\psi + k_z\ \hat{e}_z$$

$$(\vec{V}_\perp)_{\hat{e}_\perp} = V_\perp\ \cos\ \phi$$

$$(\vec{V}_\perp)_{\hat{e}_\psi} = V_\perp\ \sin\ \phi$$

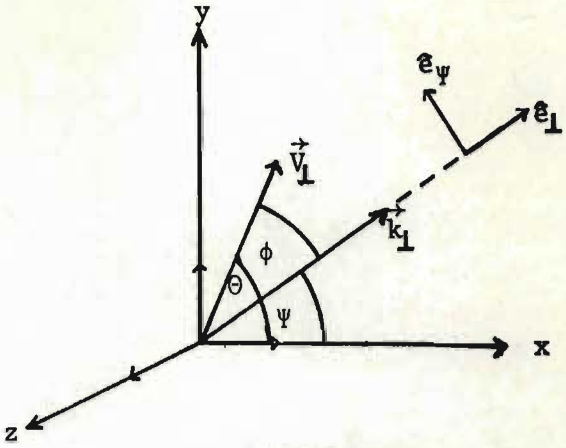


Figure 4.1

with

$$\frac{\partial}{\partial \vec{V}} = \hat{e}_\perp(\cos\ \phi\ \frac{\partial}{\partial V_\perp} - \frac{\sin\ \phi}{V_\perp}\frac{\partial}{\partial \phi}) + \hat{e}_\psi(\sin\ \phi\ \frac{\partial}{\partial V_\perp} + \frac{\cos\ \phi}{V_\perp}\frac{\partial}{\partial \phi}) + \hat{z}\ \frac{\partial}{\partial z}$$

Hence

$$\vec{k}\ .\ \frac{\partial}{\partial \vec{V}} = k_\perp\ (\cos\ \phi\ \frac{\partial}{\partial V_\perp} - \frac{\sin\ \phi}{V_\perp}\frac{\partial}{\partial \phi}) + k_z\ \frac{\partial}{\partial V_z}$$

and

$$\frac{\partial f_{oe}}{\partial t} = (-\frac{ie^2}{m_e^2})\left\langle \sum_k |\phi_k|^2 \{k_\perp(\cos\ \phi\ \frac{\partial}{\partial V_\perp} - \frac{\sin\ \phi}{V_\perp}\frac{\partial}{\partial \phi}) + k_z\ \frac{\partial}{\partial V_z}\}\left[\frac{1}{V_\perp}\ \frac{\partial f_{oe}}{\partial V_\perp}\right.\right.$$

$$\left.\left. -\ \exp(i\ \xi\ \sin\ \phi)J_o(\xi)\ .\ \frac{\{(\omega_k - k_y V_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_z V_z(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\}f_{oe}}{(\omega_k - k_y V_o - k_z V_z)}\right]\right\rangle \tag{4.1.6}$$

where <- - - -> now represents the averaging over ϕ. This may be rewritten as

$$\frac{\partial f_{oe}}{\partial t} = (-\frac{ie^2}{m_e^2})\sum_k |\phi_k|^2 \left\langle k_\perp(\cos\ \phi\ \frac{\partial g(\phi)}{\partial V_\perp} - \frac{\sin\ \phi}{V_\perp}\frac{\partial g(\phi)}{\partial \phi}) + k_z\ \frac{\partial g(\phi)}{\partial V_z} \right\rangle \tag{4.1.7}$$

where

$$g(\phi) = \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \sum_\ell J_\ell(\xi)e^{i\ell\phi}J_o(\xi)\frac{\{(\omega_k - k_yV_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_zV_z(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\}f_{oe}}{(\omega_k - k_yV_o - k_zV_z)} \tag{4.1.8}$$

In writing $g(\phi)$ in the above form we have expressed $\exp\{i\ \xi \sin\phi\}$ in terms of the identity (3.2.11).

Now

$$\left\langle -\frac{\sin\phi}{V_\perp}\frac{\partial g(\phi)}{\partial\phi}\right\rangle = \frac{1}{2\pi}\int_o^{2\pi}\frac{(-\sin\phi)}{V_\perp}\frac{\partial g(\phi)}{\partial\phi}\,d\phi$$

$$= \frac{1}{2\pi}\int_o^{2\pi}\frac{g(\phi)}{V_\perp}\cos\phi\,d\phi$$

Equation (4.1.7) then reduces to

$$\frac{\partial f_{oe}}{\partial t} = (-\frac{ie^2}{m_e^2})\sum_k |\phi_k|^2\{k_\perp(\frac{\partial}{\partial V_\perp} + \frac{1}{V_\perp})<\cos\phi\ g(\phi)> + k_z\frac{\partial}{\partial V_z}<g(\phi)>\} \tag{4.1.9}$$

Upon substituting for $g(\phi)$ from Eq. (4.1.8) it turns out that

$$<\cos\phi\ g(\phi)> = -\frac{J_o(\xi)\{(\omega_k - k_yV_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_zV_z(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\}f_{oe}}{(\omega_k - k_yV_o - k_zV_z)}\frac{\{J_1(\xi)+J_{-1}(\xi)\}}{2}$$

$$= 0$$

where we have used the result

$$\sum_{\ell=-\infty}^{+\infty} J_\ell(\xi)\int_o^{2\pi}\cos\phi\ \exp(i\ \ell\ \phi)\,d\phi = \pi\{J_1(\xi) + J_{-1}(\xi)\}$$

$$= 0 \tag{4.1.9a}$$

since [60]

$$J_{-n}(x) = (-1)^n J_n(x) \qquad (n = 1, 2, 3, \ldots.)$$

$$\langle g(\phi)\rangle = \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^{\,2}(\xi)\,\{(\omega_k - k_y V_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_z V_z(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\} f_{oe}}{(\omega_k - k_y V_o - k_z V_z)}$$

$$= \{1 - J_o^{\,2}(\xi)\}\,\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^{\,2}(\xi)\,k_z\,(\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)}$$

With these results Eq. (4.1.9) reduces to

$$\frac{\partial f_{oe}}{\partial t} = \mathrm{Re}\left[(\frac{ie^2}{m_e^2})\sum_k |\phi_k|^2\, k_z \frac{\partial}{\partial V_z}\{\frac{J_o^{\,2}(\xi)\,k_z(\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)}\}\right] \tag{4.1.10}$$

We have taken the real part on the right hand side since the left hand side represents a real time rate of change.

We note that

$$\mathrm{Re}\left[(-\frac{ie^2}{m_e^2})\sum_k |\phi_k|^2\, k_z \frac{\partial}{\partial V_z}\{(1 - J_o^{\,2}(\xi))\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\}\right] = 0 \tag{4.1.10a}$$

Recalling that $\omega_k = \omega_k^r + i\gamma_k$, we find

$$\mathrm{Re}\left[i(\omega_k^r + i\,\gamma_k - k_y V_o - k_z V_z)^{-1}\right] = \frac{\gamma_k}{(k_z V_z - \bar{\omega}_k^r)^2 + \gamma_k^2} \tag{4.1.11}$$

where

$$\bar{\omega}_k^r = \omega_k^r - k_y V_o$$

Equation (4.1.10) is then rewritten as

$$\frac{\partial f_{oe}}{\partial t} = \frac{\partial}{\partial V_z} D_e \frac{\partial f_{oe}}{\partial V_z} \tag{4.1.12}$$

with

$$D_e = \frac{e^2}{m_e^2} \sum_k \frac{k_z^2 \, \gamma_k \, |\phi_k|^2 \, J_o^2(k_\perp V_\perp/\Omega_e)}{(k_z V_z - \varpi_k^r)^2 + \gamma_k^2}$$

We observe that this implies electron diffusion in the V_z direction, i.e. along $\vec{B}_o$. This is not surprising since the magnetized electrons are tied to the field lines and are free to accelerate only along $\vec{B}_o$. For the case of a field-free plasma one would expect the electrons to diffuse in the direction of wave propagation, since motion along the wave vector provides the most energy exchange. Equation (4.1.12) thus implies electron heating along the magnetic field.

Ion Diffusion Equation

Since we assume that the ions do not react to the external fields $\vec{E}_o$ and $\vec{B}_o$ (see discussion preceeding Eq.(3.2.13)), for them Eq.(4.1.2) reduces to

$$\frac{\partial f_{oi}}{\partial t} = -\frac{e}{m_i} \sum_k \vec{E}_{-k} \cdot \frac{\partial f_{ik}}{\partial \vec{V}}$$

$$= -\frac{e}{m_i} \sum_k (-i \, \phi_k \, \vec{k})^* \cdot \frac{\partial f_{ik}}{\partial \vec{V}}$$

Upon substituting for f_{ik} which is given by Eqs.(4.1.5a) and (3.2.13), we find that

$$\frac{\partial f_{oi}}{\partial t} = \left(-\frac{ie^2}{m_i^2}\right) \sum_k |\phi_k|^2 \vec{k} \cdot \frac{\partial}{\partial \vec{V}} \left[\frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} - \frac{\left\{ \omega_k \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} + k_z V_z \left(\frac{1}{V_z} \frac{\partial}{\partial V_z} - \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} \right) \right\} f_{oi}}{(\omega_k - \vec{k} \cdot \vec{V})} \right]$$

$$= \left(-\frac{ie^2}{m_i^2}\right) \sum_k |\phi_k|^2 \vec{k} \cdot \frac{\partial}{\partial V} \left[\frac{-(\vec{k}_\perp \cdot \vec{V}_\perp) \frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} - k_z \frac{\partial f_{oi}}{\partial V_z}}{(\omega_k - \vec{k} \cdot \vec{V})} \right] \tag{4.1.13}$$

From Fig. 4.1 and the discussion preceeding Eq.(4.1.6), we see that for a background distribution f_{oi} of the type (4.1.1), viz., $f_{oi}(V_\perp^2, V_z, t)$,

$$\vec{k} \cdot \frac{\partial f_{oi}}{\partial \vec{V}} = k_\perp V_\perp \cos\phi \frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp} - \frac{k_\perp \sin\phi}{V_\perp}\frac{\partial f_{oi}}{\partial \phi} + k_z \frac{\partial f_{oi}}{\partial V_z}$$

$$= (\vec{k}_\perp \cdot \vec{V}_\perp) \frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp} + k_z \frac{\partial f_{oi}}{\partial V_z}$$

Hence Eq.(4.1.13) can be written as (taking the real part)

$$\frac{\partial f_{oi}}{\partial t} = \mathrm{Re}\left[\frac{ie^2}{m_i^2} \sum_k |\phi_k|^2 \vec{k} \cdot \frac{\partial}{\partial \vec{V}} \left\{\frac{\vec{k} \cdot (\partial f_{oi}/\partial \vec{V})}{(\omega_k - \vec{k} \cdot \vec{V})}\right\}\right] \tag{4.1.13a}$$

which, in turn, may be rewritten as

$$\frac{\partial f_{oi}}{\partial t} = \frac{\partial}{\partial \vec{V}} \cdot \bar{D}_i \cdot \frac{\partial f_{oi}}{\partial \vec{V}}$$

(4.1.14)

with

$$\bar{D}_i = \frac{e^2}{m_i^2} \sum_k \frac{\vec{k}\,\vec{k}\,\gamma_k\,|\phi_k|^2}{(\vec{k} \cdot \vec{V} - \omega_k^r)^2 + \gamma_k^2}$$

The ion diffusion equation (4.1.14) has the same form as that used by McBRIDE *et al* (28) for the modified two-stream instability. Furthermore, in the small gyroradius approximation $\xi = k_\perp V_\perp/\Omega_e << 1$, with $J_o^2(\xi) \simeq 1$. In this limit Eq.(4.1.12) differs from that of McBRIDE *et al* (28) for the electrons only by the Doppler-shifted term $k_y V_o$. For a spatially homogeneous plasma the electron diffusion equation of SAGDEEV and GALEEV(47, p. 82) reduces to Eq. (4.1.12) above.

As for the ions, the $(\partial/\partial\vec{V})$ dependence in Eq.(4.1.14) indicates a general three-dimensional diffusion. We assume, however, that $k_z \ll k$. This is reasonable since experimental measurements by HIROSE *et al* [36], BARRETT *et al* [37] and HAYZEN and BARRETT [41] display maximum growth rates when $k_z \ll k$. The approximation $\left|\frac{(\omega_k^r - k_y V_o)}{k_z C_e}\right| \ll 1$ in Section 3.4 places a lower limit on (k_z/k). Hence the discussion to follow is valid for (k_z/k) values satisfying (cf. Eq.(3.4.13))

$$\left\{\frac{m_e}{m_i}\right\}^{1/2} \ll \frac{k_z}{k} \ll 1 \qquad (4.1.15)$$

Since wave propagation is now restricted primarily perpendicular to $\vec{B}_o$, we may, as an approximation, replace $\vec{k}\,\vec{k}$ with $\vec{k}_\perp\,\vec{k}_\perp$ in Eq.(4.1.14). Thus, from this equation we conclude that ion diffusion (and therefore ion heating) is predominantly perpendicular to $\vec{B}_o$. This is a plausible implication since the ions, with their straight line trajectories, will tend to favour the direction that produces the most heating. With wave propagation restricted to a small cone about the perpendicular, ion heating will be primarily normal to $\vec{B}_o$.

4.2 SOLUTIONS OF THE ELECTRON DIFFUSION EQUATIONS

(a) Resonant electron diffusion

For the resonant electrons, which are responsible for wave growth, $\frac{\omega_k^r - k_y V_o}{k_z V_z} \simeq 1$ (see discussion following Eq.(3.3.12)) and Eq.(4.1.12) reduces to

$$\frac{\partial f_{oe}}{\partial t} = \frac{\partial}{\partial V_z}\left\{\frac{e^2}{m_e^2} \sum_k k_z^2 |\phi_k|^2 \pi \delta(k_z V_z - \varpi_k^r) J_o^2(\xi) \frac{\partial f_{oe}}{\partial V_z}\right\} \tag{4.2.1}$$

where, from Eq.(4.1.11), $\varpi_k^r = \omega_k^r - k_y V_o$, and we have, for small γ_k, made use of the identity

$$\lim_{\gamma_k \to 0^+} \frac{\gamma_k}{(k_z V_z - \varpi_k^r)^2 + \gamma_k^2} = \pi \delta(k_z V_z - \varpi_k^r) \tag{4.2.2}$$

This identity implies that Eq.(4.2.1) is valid only for unstable modes, i.e., $\gamma_k > 0$. DAVIDSON (46, p. 164), however, has discussed the extension to the stable regime. It is found that Eq.(4.2.1) is also applicable in this regime, i.e., for $\gamma_k \leq 0$.

For $k_z \ll k$ we approximate $|\phi_k|^2 \simeq |\phi_{k_\perp}|^2$ and replace $\sum_{k_z}$ by $\int dk_z$.

Then

$$\frac{\partial f_{oe}}{\partial t} \simeq \frac{\partial}{\partial V_z} \{\frac{e^2}{m_e^2} \sum_{k_\perp} |\phi_{k_\perp}|^2 J_o^2(k_\perp V_\perp/\Omega_e) \pi \int \frac{k_z^2}{V_z} \delta(k_z - \frac{\varpi_k^r}{V_z}) \frac{\partial f_{oe}}{\partial V_z} dk_z\}$$

We set $\sum_{k_\perp} \simeq \frac{L}{2\pi} \sum_k$, and using definition (3.3.3), viz.,

$$\Psi_{oe}(V_z,t) = \int_o^\infty f_{oe}(V_\perp^2, V_z, t) V_\perp dV_\perp \tag{4.2.3}$$

integrate with respect to $V_\perp dV_\perp$ to obtain

$$\frac{\partial \Psi_{oe}(V_z,t)}{\partial t} = \frac{\partial}{\partial V_z} \{\frac{L}{2\pi} \sum_k \frac{e^2 |\phi_k|^2}{m_e^2} \frac{\varpi_k^{r\,2}}{V_z^3} \pi \alpha(k_\perp/\Omega_e) \frac{\partial \Psi_{oe}(V_z,t)}{\partial V_z} \} \tag{4.2.3a}$$

In arriving at the above result definition (3.3.4) for $\alpha(k_\perp/\Omega_e)$ has been used. We recall that $|\phi_k|^2 = |\phi_k|^2(t)$ and $\bar{\omega}_k^r = \omega_k^r(t) - k_y V_o$. By replacing time with the new parameter

$$\tau = 25\,\{\frac{L}{2\pi}\int_o^t \sum_k \frac{e^2}{m_e^2}\,|\phi_k|^2(t')\,(\omega_k^r(t') - k_y V_o)^2\,\pi\,\alpha(k_\perp/\Omega_e)\,dt'\}$$

equation (4.2.3a) may be written in the form

$$\frac{\partial \Psi^R_{oe}(V_z,\tau)}{\partial \tau} = \frac{1}{25}\frac{\partial}{\partial V_z}\left\{\frac{1}{V_z^3}\frac{\partial \Psi^R_{oe}(V_z,\tau)}{\partial V_z}\right\} \tag{4.2.4}$$

where the superscript "R" indicates resonant particles. We note that τ has the dimensions of $(\text{velocity})^5$.

The "Green's function" solution of Eq.(4.2.4) is (see Appendix C)

$$\Psi^R_{oe}(V_z,\tau) = \frac{5}{\tau}\int_o^\infty V_z^2 V_z'^2\,\Psi^R_{oe}(V_z',\tau=0)\,I_{-4/5}(2V_z^{5/2}V_z'^{5/2}/\tau)\exp\{-(V_z^{\,5}+V_z'^{5})/\tau\}dV_z' \tag{4.2.5}$$

where $\Psi^R_{oe}(V_z,\tau = 0)$ is the initial electron distribution in the resonant region of velocity space and $I_{-4/5}$ is the modified Bessel function of order -4/5. For a given $\Psi^R_{oe}(V_z,\tau = 0)$, Eq.(4.2.5) may be used to determine $\Psi^R_{oe}(V_z,\tau)$ at any later time t.

On the other hand, we may in the time asymptotic limit obtain the similarity solution [66] of Eq.(4.2.4). It is a solution that gives the asymptotic distribution of velocity ($t,\tau \to \infty$) for an arbitrary initial form of the distribution function. We proceed by writing

$$\Psi^R_{oe}(V_z,\tau) = \tau^P\,\phi(\zeta) \tag{4.2.6}$$

where $$\zeta = V_z \tau^{\nu}$$

Equation (4.2.4) then becomes

$$p\,\phi(\zeta) + \nu\,\zeta\,\phi'(\zeta) = \frac{\tau^{5\nu+1}}{25}\left\{\frac{\phi''(\zeta)}{\zeta^3} - \frac{3\,\phi'(\zeta)}{\zeta^4}\right\} \tag{4.2.7}$$

where

$$\phi'(\zeta) = \frac{d\phi(\zeta)}{d\zeta}\ , \quad \phi''(\zeta) = \frac{d^2\phi(\zeta)}{d\zeta^2}$$

Since the left hand side has an explicit dependence on ζ only, it follows that $5\nu+1 = 0$, i.e., $\nu = -1/5$. Therefore

$$\zeta = V_z\,\tau^{-1/5}\ , \qquad \Rightarrow \qquad \zeta^5 = V_z^5\,\tau^{-1}.$$

Thus from Eq.(4.2.7) we have

$$p\,\phi(\zeta) = \frac{1}{25}\,\frac{\phi''(\zeta)}{\zeta^3} - \phi'(\zeta)\left\{\frac{3}{25\zeta^4} - \frac{\zeta}{5}\right\}$$

Initially, we set $p = 0$ and obtain

$$\phi(\zeta) = A'\int \zeta^3 \exp\,(-\zeta^5)\,d\zeta$$

Since the integral on the right hand side cannot be manipulated by standard methods, we try a solution of the form

$$\phi(\zeta) = A \exp\,(-\zeta^5)$$

Equation (4.2.7) then yields $p = -1/5$, and hence from Eq.(4.2.6)

$$\Psi^R_{oe}(V_z,\tau) = A\,\tau^{-1/5} \exp\,(-V_z^5/\tau) \tag{4.2.8}$$

For the ideal choice of initial distribution (so that the integral may be exactly evaluated)

$$\Psi^R_{oe}\ (V'_z, \tau = 0) = A\ C_e^{-1}\ \exp\ (-V'_z/C_e)^5 \qquad (A > 0) \tag{4.2.9}$$

the solution (4.2.5) reduces to (see Appendix C)

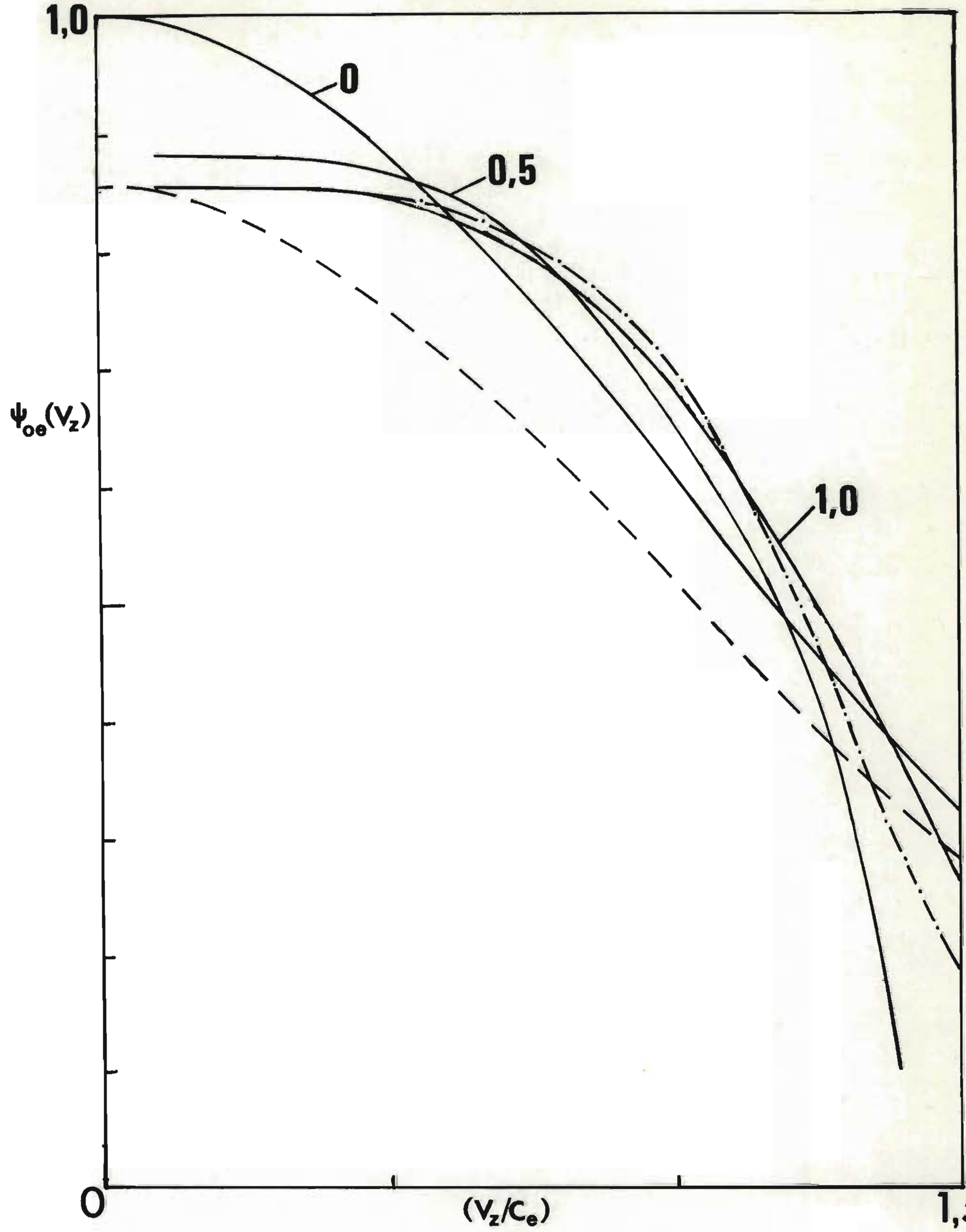
$$\Psi^R_{oe}\ (V_z, \tau) = \frac{A}{(C_e^5 + \tau)^{1/5}}\ \exp\ \{-V_z^5/(C_e^5 + \tau)\} \tag{4.2.10}$$

which for $\tau >> C_e$ (asymptotic case) reduces to the similarity solution (4.2.8). For any other initial distribution, numerical integration of Eq.(4.2.5) yields the evolution of the resonant electron distribution function in time. This is illustrated in Fig. 4.2 for an initial ($\tau = 0$) Maxwellian distribution function. The curves for $\tau = 0{,}5$ and $\tau = 1{,}0$ are seen to behave in a manner similar to the curve $f(V) \propto \exp(-aV^5)$, where $a=0{,}2$. We notice that the solutions (4.2.8) and (4.2.10) are physically acceptable for $V_z \geq 0$, but not for $V_z \to -\infty$, since they diverge in this limit. This behaviour will be discussed later. The solution (4.2.8) has the same functional form as that of the unmagnetized ion acoustic instability [47, p. 52].

(b) Non-resonant electron diffusion

By virtue of the assumption $|(\omega_k^r - k_y V_o)/k_z V_z| << 1$ for the non-resonant electrons in the linear study (cf. Eq.(3.3.16)), for them Eq.(4.1.12) reduces to

Figure 4.2



Temporal evolution of the resonant electron velocity distribution function $\Psi^R_{oe}(V_z,\tau)$. The initial distribution ($\tau = 0$) is chosen to be Maxwellian. The curve (-·-·-·-) represents $\exp(-aV_z^5)$, (a=0,2), while for comparison, the broken line (- - -) is a Maxwellian with the same peak value. The parameter labelling the curves is τ.

$$\frac{\partial f_{oe}}{\partial t} = \frac{\partial}{\partial V_z}\{\sum_k \frac{e^2}{m_e^2} \frac{\gamma_k |\phi_k|^2 J_o^2 (k_\perp V_\perp/\Omega_e)}{V_z^2} \frac{\partial f_{oe}}{\partial V_z}\} \qquad (4.2.11)$$

As in the case of the resonant electrons, we use the definitions (4.2.3) and (3.3.4) for $\Psi_{oe}(V_z,t)$ and $\alpha(k_\perp/\Omega_e)$ respectively, to rewrite this as

$$\frac{\partial\Psi_{oe}^{NR}(V_z,t)}{\partial t} = \{\sum_k \frac{e^2}{2m_e^2} \alpha(k_\perp/\Omega_e)\frac{\partial}{\partial t}|\phi_k|^2\}\frac{\partial}{\partial V_z}\left[\frac{1}{V_z^2}\frac{\partial\Psi_{oe}^{NR}(V_z,t)}{\partial V_z}\right] \qquad (4.2.12)$$

where the superscript "NR" indicates non-resonant particles, and we have, in addition, made use of the equation for wave growth

$$\frac{\partial}{\partial t} |\phi_k|^2 = 2\gamma_k |\phi_k|^2 \qquad (4.2.13)$$

Equation (4.2.13)may be derived as follows.

From Eq.(4.1.5a)

$$\phi_k(t) = \phi_{k\omega} \exp\{-i\int_o^t [\omega_k^r(t'') + i\,\gamma_k(t'')]\,dt''\}$$

since $\omega_k = \omega_k^r + i\,\gamma_k$ (Eq.(3.1.6)). Thus

$$|\phi_k(t)|^2 = \phi_k(t)\,\phi_k^*(t) = |\phi_{k\omega}|^2 \exp\{2\int_o^t \gamma_k(t'')\,dt''\}$$

with $$\frac{\partial}{\partial t}|\phi_k|^2 = 2\,\gamma_k(t)\,|\phi_k|^2$$

In terms of the parameter

$$\tau = 8\sum_k \alpha(k_\perp/\Omega_e) \frac{e^2|\phi_k|^2}{m_e^2}$$

which has dimensions of $(\text{velocity})^4$, Eq.(4.2.12) takes the form

$$\frac{\partial\Psi_{oe}^{NR}(V_z,t)}{\partial\tau} = \frac{1}{16}\frac{\partial}{\partial V_z}\left\{\frac{1}{V_z^2}\frac{\partial\Psi_{oe}^{NR}(V_z,t)}{\partial V_z}\right\} \qquad (4.2.14)$$

with a similarity solution

$$\Psi_{oe}^{NR}(V_z,\tau) = B\,\tau^{-1/4}\exp(-V_z^4/\tau) \qquad (4.2.15)$$

We have used exactly the same procedure as that used to arrive at the result (4.2.8). Paralleling the sequence from Eq.(4.2.6) to Eq.(4.2.8), we write

$$\Psi_{oe}^{NR}(V_z,\tau) = \tau^p\,\phi(\zeta) \qquad (\zeta = V_z\,\tau^\nu)$$

Equation (4.2.14) then reduces to

$$p\,\phi(\zeta) + \nu\,\zeta\,\phi'(\zeta) = \frac{\zeta^{4\nu+1}}{16}\left\{\frac{\phi''(\zeta)}{\zeta^2} - \frac{2\,\phi'(\zeta)}{\zeta^3}\right\}$$

Since the left hand side has an explicit depence on ζ only, it follows that $4\nu+1 = 0$, i.e., $\nu = -1/4$.

Upon setting $p = 0$ we arrive at the solution

$$\phi(\zeta) = B'\int\zeta^2\exp(-\zeta^4)\,d\zeta$$

in which the integral cannot be evaluated via standard methods. A trial solution of the form $\phi(\zeta) = B \exp\{-\zeta^4\}$ yields $p = -1/4$. Hence the result (4.2.15).

4.3 SOLUTIONS OF THE ION DIFFUSION EQUATIONS

(a) Non-resonant ions

We recall Eq.(4.1.14) for the ions, viz.,

$$\frac{\partial f_{oi}}{\partial t} = \frac{\partial}{\partial \vec{V}} \cdot \bar{D}_i \cdot \frac{\partial f_{oi}}{\partial \vec{V}}$$

with (4.3.1)

$$\bar{D}_i = \frac{e^2}{m_i^2} \sum_k \frac{\vec{k}\,\vec{k}\,\gamma_k\,|\phi_k|^2}{(\vec{k}\cdot\vec{V} - \omega_k^r)^2 + \gamma_k^2}$$

Since the average ion speed $\langle V\rangle_i$ is such that

$$\langle V\rangle_i \sim C_i = (T_i/m_i)^{1/2} \ll (T_e/m_i)^{1/2} = C_s \sim (\omega_k^r/k)$$

for $T_e \gg T_i$, the interaction between the ions and the ion acoustic waves is predominantly non-resonant. In the above limit and for $|\gamma_k| \ll \omega_k^r$ we may approximate

$$\frac{\gamma_k}{(\vec{k}\cdot\vec{V} - \omega_k^r)^2 + \gamma_k^2} \simeq \frac{\gamma_k}{(\omega_k^r)^2} \qquad (4.3.1a)$$

for the bulk of the (non-resonant) ions.

Rotating the coordinate system through an angle Ψ, as shown in Fig. 4.1, we have

$$\vec{k}\,\vec{k} = (k_\perp\,\hat{e}_\perp + k_z\hat{z})\,(k_\perp\,\hat{e}_\perp + k_z\hat{z})$$

$$= k_\perp^2\,\hat{e}_\perp\,\hat{e}_\perp + k_z\,k_\perp\,\hat{z}\,\hat{e}_\perp + k_\perp\,k_z\,\hat{e}_\perp\hat{z} + k_z^2\,\hat{z}\,\hat{z}$$

and

$$\frac{\partial}{\partial\vec{V}} = \hat{e}_\perp\,(\cos\phi\,\frac{\partial}{\partial V_\perp} - \frac{\sin\phi}{V_\perp}\frac{\partial}{\partial\phi}) + \hat{e}_\Psi(\sin\phi\,\frac{\partial}{\partial V_\perp} + \frac{\cos\phi}{V_\perp}\frac{\partial}{\partial\phi}) + \hat{z}\frac{\partial}{\partial V_z}$$

Equation (4.3.1) then becomes

$$\frac{\partial f_{oi}}{\partial t} = \{\hat{e}_\perp(\cos\phi\,\frac{\partial}{\partial V_\perp} - \frac{\sin\phi}{V_\perp}\frac{\partial}{\partial\phi}) + \hat{e}_\Psi(\sin\phi\,\frac{\partial}{\partial V_\perp} + \frac{\cos\phi}{V_\perp}\frac{\partial}{\partial\phi}) + \hat{z}\frac{\partial}{\partial V_z}\}$$

$$\cdot\,\frac{e^2}{m_i^2}\sum_k \frac{\gamma_k\,|\phi_k|^2}{(\omega_k^r)^2}\,(k_\perp^2\,\hat{e}_\perp\,\hat{e}_\perp + k_z\,k_\perp\,\hat{z}\,\hat{e}_\perp + k_\perp\,k_z\,\hat{e}_\perp\hat{z} + k_z^2\,\hat{z}\,\hat{z})$$

$$\cdot\,\{\hat{e}_\perp\,\cos\phi\,\frac{\partial f_{oi}}{\partial V_\perp} + \hat{e}_\Psi\,\sin\phi\,\frac{\partial f_{oi}}{\partial V_\perp} + \hat{z}\,\frac{\partial f_{oi}}{\partial V_z}\}$$

since for $f_{oi} = f_{oi}\,(V_\perp^2,\,V_z,t)$, $\frac{\partial f_{oi}}{\partial\phi} = 0$.

This equation simplifies to

$$\frac{\partial f_{oi}}{\partial t} = \frac{e^2}{m_i^2}\sum_k \frac{\gamma_k\,|\phi_k|^2}{(\omega_k^r)^2}\,\{k_\perp^2\,\cos^2\phi\,\frac{\partial^2 f_{oi}}{\partial V_\perp^2} + k_\perp^2\,\frac{\sin^2\phi}{V_\perp}\,\frac{\partial f_{oi}}{\partial V_\perp} + k_z^2\,\frac{\partial^2 f_{oi}}{\partial V_z^2}$$

$$+\,2\,k_\perp\,k_z\,\cos\phi\,\frac{\partial^2 f_{oi}}{\partial V_\perp\,\partial V_z}\}$$

Upon averaging over the angle ϕ (with $<\cos^2\phi> = <\sin^2\phi> = 1/2$, $<\cos\phi> = 0$), it turns out that

$$\frac{\partial f_{oi}}{\partial t} = \frac{e^2}{m_i^2} \sum_k \frac{\gamma_k |\phi_k|^2}{(\omega_k^r)^2} \left\{ \frac{k_\perp^2}{2} \left(\frac{\partial^2 f_{oi}}{\partial V_\perp^2} + \frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} \right) + k_z^2 \frac{\partial^2 f_{oi}}{\partial V_z^2} \right\}$$

which may be rewritten as

$$\frac{\partial f_{oi}}{\partial t} = \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} \left\{ D_{i_\perp} V_\perp \frac{\partial f_{oi}}{\partial V_\perp} \right\} + \frac{\partial}{\partial V_z} \left\{ D_{i_\parallel} \frac{\partial f_{oi}}{\partial V_z} \right\} \tag{4.3.2}$$

where

$$D_{i_\perp} = \frac{e^2}{2m_i^2} \sum_k \frac{k_\perp^2 \gamma_k |\phi_k|^2}{(\omega_k^r)^2} \text{ and } D_{i_\parallel} = \frac{e^2}{m_i^2} \sum_k \frac{k_z^2 \gamma_k |\phi_k|^2}{(\omega_k^r)^2}$$

Since we have already assumed $k_z << k_\perp$ it follows that $D_{i_\perp} >> D_{i_\parallel}$, i.e., ion diffusion across $\vec{B}_o$ is significantly stronger than along $\vec{B}_o$. Therefore, the term corresponding to ion diffusion along $\vec{B}_o$ is neglected. To obtain the non-resonant ion diffusion equation we integrate Eq.(4.3.2) with respect to V_z, and get

$$\frac{\partial \Phi_{oi}^{NR}(V_\perp^2, t)}{\partial t} = \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} \left[D_{i_\perp} V_\perp \frac{\partial \Phi_{oi}^{NR}(V_\perp^2, t)}{\partial V_\perp} \right]$$

where

$$\Phi_{oi}^{NR}(V_\perp^2, t) = \int f_{oi}^{NR}(V_\perp^2, V_z, t)\, dV_z$$

Substituting for $D_{i_\perp}$ from above, we have

$$\frac{\partial \Phi_{oi}^{NR}}{\partial t} = \sum_k \frac{e^2 k_\perp^2}{2m_i^2 (\omega_k^r)^2} \frac{1}{2} \frac{\partial}{\partial t} |\phi_k|^2 \left[\frac{1}{V_\perp} \frac{\partial \Phi_{oi}^{NR}}{\partial V_\perp} + \frac{\partial^2 \Phi_{oi}^{NR}}{\partial V_\perp^2} \right]$$

where use has also been made of the equation for wave growth (4.2.13).

This equation may in turn be expressed in terms of the new parameter τ, defined as

$$\tau = \sum_k \frac{e^2}{2m_i} \frac{k_\perp^2 \, |\phi_k|^2}{(\omega_k^r)^2}$$

and having the dimensions of temperature, as

$$\frac{\partial \Phi_{oi}^{NR}}{\partial \tau} = \frac{1}{2m_i} \left[\frac{1}{V_\perp} \frac{\partial \Phi_{oi}^{NR}}{\partial V_\perp} + \frac{\partial^2 \Phi_{oi}^{NR}}{\partial V_\perp^2} \right] \tag{4.3.3}$$

For an initial Maxwellian distribution

$$\Phi_{oi}^{NR} (V_\perp^2, \tau = 0) = \frac{n_o \, m_i}{2\pi \, T_{io}} \exp \left(- \frac{m_i \, V_\perp^2}{2 \, T_{io}}\right) \tag{4.3.4}$$

this equation has the solution (see Appendix D)

$$\Phi_{oi}^{NR} (V_\perp^2, \tau) = \frac{n_o \quad m_i}{2\pi \, (T_{io} + \tau)} \exp \left[- \frac{m_i \, V_\perp^2}{2(T_{io} + \tau)} \right] \tag{4.3.5}$$

i.e., the Maxwellian character is retained, with an effective temperature increase of τ.

Using the definition of the wave energy density, to be given later (Eq. (4.4.7a)), we find $\tau = (\sum_k W_k / n_o T_{io}) \, T_{io}$, where $W = \sum_k W_k$ is the total energy density associated with the fluctuations. Since quasi-linear theory requires $W/n_o T_{io} << 1$, it follows that $\tau << T_{io}$. Such a small modification to the ion distribution function, due to the oscillations, has also been found by APPERT and VACLAVIK[58] for an unmagnetized plasma.

(b) Resonant ions

Although the ion-wave interaction is almost entirely non-resonant, it is possible for the few high velocity ions in the tail of the distribution to resonate with the waves and produce ion Landau damping.

From the discussion following Eq.(4.1.14) and the treatment of the non-resonant ions, we neglect ion diffusion along $\vec{B}_o$ and set $\vec{k} = \vec{k}_\perp$. Since for the resonant ions $\omega_k^r \simeq \vec{k} \cdot \vec{V}$, the identity (4.2.2) allows us to write their associated diffusion equation, from Eq.(4.3.1), as

$$\frac{\partial f_{oi}}{\partial t} = \{\hat{e}_\perp(\cos\phi \frac{\partial}{\partial V_\perp} - \frac{\sin\phi}{V_\perp}\frac{\partial}{\partial\phi}) + \hat{e}_\psi(\sin\phi \frac{\partial}{\partial V_\perp} + \frac{\cos\phi}{V_\perp}\frac{\partial}{\partial\phi})\}$$

$$\cdot \sum_k \frac{e^2}{m_i^2} |\phi_k|^2 k_\perp^2 \pi \, \delta(\omega_k^r - k_\perp V_\perp \cos\phi) \, \hat{e}_\perp \hat{e}_\perp \cdot \{\hat{e}_\perp(\cos\phi \frac{\partial}{\partial V_\perp} - \frac{\sin\phi}{V_\perp}\frac{\partial}{\partial\phi}) + \hat{e}_\psi(\sin\phi \frac{\partial}{\partial V_\perp} + \frac{\cos\phi}{V_\perp}\frac{\partial}{\partial\phi})\} f_{oi}$$

with ϕ, $\hat{e}_\perp$ and $\hat{e}_\psi$ as shown in Fig. 4.1.

In analogy with the non-resonant ions, for the resonant ions we define

$$\Phi_{oi}^R (V_\perp^2,t) = \int f_{oi}^R (V_\perp^2,V_z,t) \, dV_z$$

The above equation then reduces to

$$\frac{\partial\Phi_{oi}^R}{\partial t} = (\cos\phi \frac{\partial}{\partial V_\perp} - \frac{\sin\phi}{V_\perp}\frac{\partial}{\partial\phi}) \sum_k \frac{\pi e^2}{m_i^2} k_\perp^2 |\phi_k|^2 \cos\phi \, \delta(\omega_k^r - k_\perp V_\perp \cos\phi) \frac{\partial\Phi_{oi}^R}{\partial V_\perp}$$

Upon averaging over the angle ϕ , we find

$$\frac{\partial\Phi^R_{oi}(V_\perp^2,t)}{\partial t} = \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\left[V_\perp \sum_k \frac{\pi e^2}{m_i^2} k_\perp^2|\phi_k|^2\left[\frac{1}{2\pi}\int_0^{2\pi}\cos^2\phi\ \delta(\omega_k^r - k_\perp V_\perp\cos\phi)d\phi\right]\frac{\partial\Phi^R_{oi}}{\partial V_\perp}\right]$$

$$= \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\left[\sum_k \frac{e^2}{m_i^2}\frac{\omega_k^r V_a\ |\phi_k|^2}{(V_\perp^2 - V_a^2)^{1/2}}\frac{1}{V_\perp}\frac{\partial\Phi^R_{oi}(V_\perp^2,t)}{\partial V_\perp}\right] \tag{4.3.6}$$

where $V_a = \omega_k^r/k_\perp$.

Transforming to the energy-like variable $W = V_\perp^2 - V_a^2$ and introducing the new parameter

$$\tau = 25 \sum_k \frac{e^2}{m_i^2}\omega_k^r V_a \int_0^t |\phi_k|^2(t')\ dt'$$

equation (4.3.6) may be rewritten as

$$\frac{\partial\Phi^R_{oi}(W,\tau)}{\partial\tau} = \frac{4}{25}\frac{\partial}{\partial W}\left[\frac{1}{W^{1/2}}\frac{\partial\Phi^R_{oi}(W,\tau)}{\partial W}\right] \tag{4.3.7}$$

We now determine the time-asymptotic similarity solution of this equation. As before, we write (cf. procedure from Eq.(4.2.6) to Eq.(4.2.8))

$$\Phi^R_{oi}(W,\tau) = \tau^p\ \phi(\zeta) \qquad (\zeta = W\ \tau^\nu)$$

Equation (4.3.7) then reduces to

$$p\ \phi(\zeta) + \nu\ \zeta\ \phi'(\zeta) = \frac{4\tau^{\frac{5}{2}\nu+1}}{25}\left[\frac{\phi''(\zeta)}{\zeta^{1/2}} - \frac{1}{2}\frac{\phi'(\zeta)}{\zeta^{3/2}}\right] \tag{4.3.8}$$

Since the left hand side has an explicit dependence on ζ only, it follows that $\frac{5}{2}\nu+1 = 0$, i.e., $\nu = -2/5$. Thus,

$$\zeta = W\ \tau^{-2/5} \Rightarrow \zeta^{5/2} = W^{5/2}\ \tau^{-1}$$

and, from Eq.(4.3.8)

$$p\ \phi\ (\zeta) = \frac{4}{25}\frac{\phi''(\zeta)}{\zeta^{1/2}} - 2\ \phi'(\zeta)\ \{\frac{1}{25\zeta^{3/2}} - \frac{\zeta}{5}\}$$

We set $p = 0$ and find that

$$\phi(\zeta) = A'\int \zeta^{1/2}\ \exp\ (-\ \zeta^{5/2})\ d\zeta$$

which cannot be evaluated by standard techniques.

A trial solution of the form $\phi(\zeta) = A\ \exp\ (-\zeta^{5/2})$ yields $p = -2/5$.

Thus we finally obtain

$$\begin{aligned}\phi^R_{oi}\ (W,\tau) &= A\ \tau^{-2/5}\ \exp\ (-\ W^{5/2}/\tau) \\ &= A\ \tau^{-2/5}\ \exp\ \{-(V_\perp^2 - V_a^2)^{5/2}/\tau\}\end{aligned} \tag{4.3.9}$$

having substituted back for W.

Ion heating due to Landau damping thus leads to an exp $(-V_{\perp}^5)$ type of distribution.

Discussion

For an unmagnetized plasma and a two-dimensional wave spectrum, SAGDEEV and GALEEV[47, p. 52] arrive at the similarity solution $f_e(V) = A \exp\{-(V/V_o)^5\}$ for the resonant electrons. This distribution has the same form as Eq.(4.2.8) above. Computer simulation experiments by BISKAMP *et al* [67] predict $f_e \propto \exp\{-(V/C_e)^x\}$ with $4.7 \leq x \leq 4{,}9$. The discrepancy in the exponential factor was attributed to electron-electron collisions, which the theory neglects.

We have seen in Section 4.1 that if the condition (4.1.5) is satisfied then the perturbations $f_{jk\omega}(\vec{V},t)$ are given by their linear forms, as derived in Chapter Three, with $\omega_k = \omega_k(t)$ and $f_{oj} = f_{oj}(\vec{V},t)$. It thus follows that in the quasilinear limit the plasma oscillations continue to satisfy the linear dispersion relation. Hence, from Eq.(3.3.15) the growth rate $\gamma_k(t)$ may be written as

$$\frac{\gamma_k}{\omega_k^r} = \frac{\pi\, \alpha(k_\perp/\Omega_e) \left[\dfrac{\partial \Psi_{oe}(V_z,t)}{\partial V_z}\right]_{V_z=\bar{V}_z}}{\dfrac{m_e}{m_i}\left[\displaystyle\int_{-\infty}^{+\infty}\left\{\frac{2k_\perp^2}{(\omega_k^r)^2}\left(1+\frac{2k_zV_z}{\omega_k^r}\right)-\frac{k_z}{\omega_k^r}\left(1+\frac{2k_zV_z}{\omega_k^r}\right)\frac{\partial}{\partial V_z}\right\}\Psi_{oi}(V_z,t)\,dV_z\right]} \tag{4.3.10}$$

The solution (4.2.8) is a one-dimensional projection of $f_{oe}(\vec{V},t)$. It is seen that $\Psi_{oe}^R(V_z,t)$ reaches a time asymptotic stable form with

$(\partial\Psi^R_{oe}/\partial V_z) \neq 0$ for a given non-zero V_z. It is well known, however, that in the one-dimensional quasilinear treatment $\Psi^R_{oe}(V_z,t)$ flattens in the resonance region with $(\partial\Psi^R_{oe}/\partial V_z) = 0$ (47, and references there in).

Why does this not apply in this case? The solution (4.2.8) has a very strong dependence on the existence of a broad spectrum of waves. Consider, for simplicity, the electron velocity distribution in the (V_y,V_z) plane. We assume the wave spectrum to have a sufficiently broad spread in k_z, with k_z small ($k_z \ll k$).

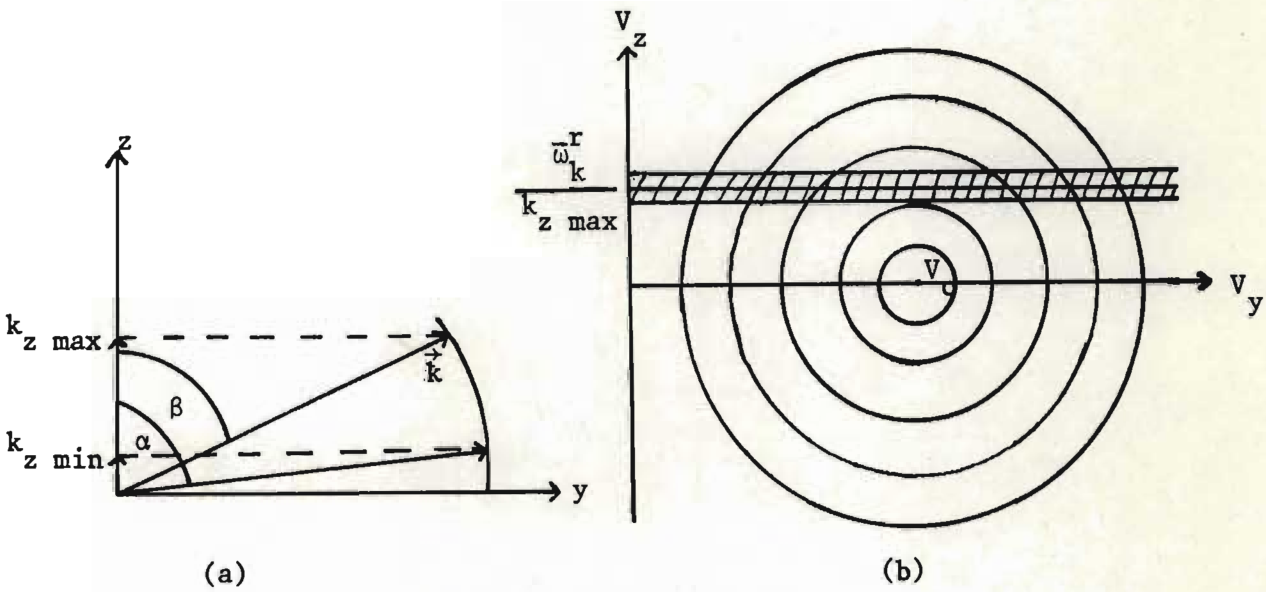


Figure 4.3

For a mode propagating at an angle β with respect to $\vec{B}_o$, $k_z = k_{z\ max}$. Using the one-dimensional analogy, all the electrons in the shaded band (Fig. 4.3(b)) will, through the term $\delta(k_zV_z - \bar{\omega}^r_k)$ in the diffusion equation (4.2.1), interact resonantly with the wave, resulting in the eventual formation of a plateau $((\partial\Psi^R_{oe}/\partial V_z) = 0)$ in the shaded region.

As $\vec{k}$ moves through the angle $(\alpha - \beta)$ to the angle α in Fig. 4.3(a), k_z decreases. Therefore $(\bar{\omega}^r_k/k_z)$ increases and reaches the maximum value of $(\varpi_k^r/k_{z\,min})$. For a sufficiently small $k_{z\,min}$ this value could extend to very high velocities resulting in possible flattening over the region shown below.

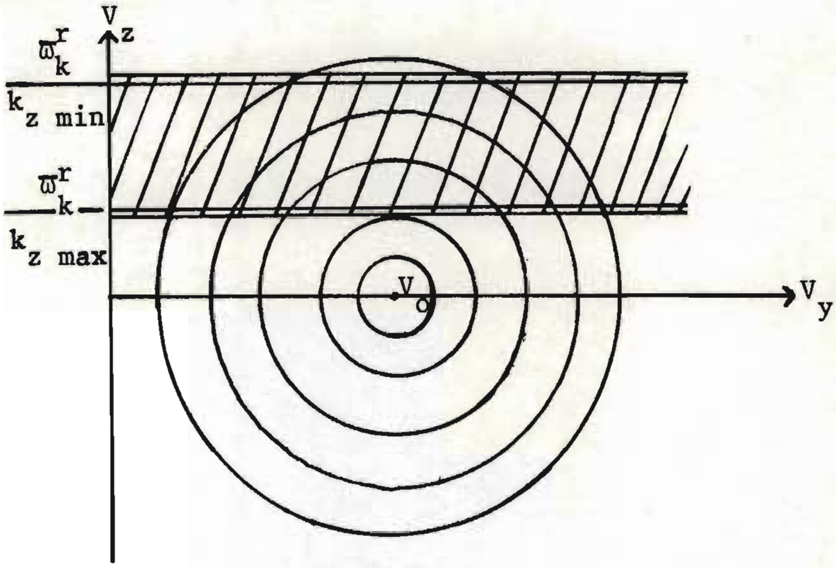


Figure 4.4

The above discussion refers to the $V_y - V_z$ plane only. The inclusion of a V_x component will extend the resonance region to a much larger volume in velocity space, the flattening of which will require a significant amount of energy and will therefore be physically unattainable. Hence for a quasi-stationary equilibrium we require, from the diffusion equation (4.1.12), $D_e^R = 0$. We see from Eq.(4.2.3a) that

$$D_e^R = \frac{L}{2\pi} \sum_k \frac{e^2 |\phi_k|^2}{m_e^2} \frac{\varpi_k^{r\,2}}{V_z^3} \pi\, \alpha(k_\perp/\Omega_e)$$

Therefore $D_e^R = 0$ holds only if $|\phi_k|^2 \to 0$ for all k. Thus the wave spectrum must damp to zero with $\gamma_k < 0$. We have seen in Section 3.4 that for a Maxwellian ion velocity distribution, which from Eq.(4.3.5) holds true for the bulk of the (non-resonant) ions, the denominator on the right hand side of Eq.(4.3.10) above is positive. Hence $\gamma_k < 0$

requires $(\partial\Psi_{oe}/\partial V_z) < 0$ for all resonant V_z, i.e.,the distribution function $\Psi^R_{oe}(V_z,t)$ must tend, in the time-asymptotic limit, to a monotonically decreasing form, as,for example, given by Eq.(4.2.8).

Even if the spread in k_z is not broad enough, we can have a sufficiently large spread in the frequency ω for resonance over the region shaded in Fig. 4.4. Experimental studies of turbulent heating by VIRKO and KIRICHENKO [68] indicated the presence of the ion sound instability. The observed frequency spectrum was found to be three-dimensional with a wide spread in the phase velocity (ω/k_z) along the magnetic field. This enabled the authors to make a quasilinear estimate of the electron heating rate.

The diffusion in velocity space of the non-resonant electrons is associated with the growth and damping of the electrostatic fluctuations, since the kinetic energy of the non-resonant electrons is associated with their 'sloshing' motion in the presence of the waves. As these electrons do not interact directly with the waves, one would expect their velocity distribution to display a Maxwellian - type of behaviour, i.e., $\propto \exp(-V_z^2)$. The $\exp(-V_z^4)$ dependence in Eq.(4.2.15) is somewhat surprising. However, SAGDEEV and GALEEV [47, p. 70] have pointed out that although the interaction is adiabatic, i.e.,non-resonant, considerable modification of the velocity distribution could occur.

As in the case of the non-resonant electrons, the diffusion of the non-resonant ions is attributed to their oscillations in the presence of the waves. Eq.(4.3.5) shows that this results in an effective heating with a temperature increment of $\tau = \sum_k \frac{e^2}{2m_i} \frac{k_\perp^2 |\phi_k|^2}{(\omega_k^r)^2}$.

The asymptotic distribution for the resonant high energy ions can be explained in a manner analogous to that for the resonant electrons, strongly depending on the existence of a broad spread in $k_\perp$ or ω. The form of the distribution will be discussed later when we investigate particle heating rates.

4.4 ANOMALOUS PLASMA RESISTIVITY

Experimental investigations of plasma instabilities have inferred the presence of an anomalous resistivity of a few orders of magnitude larger than the classical collisional value [6,8]. The $\vec{E} \times \vec{B}$ drift ($\vec{V}_o$) of the electrons relative to the ions provides the necessary free energy to drive the instability. When an instability occurs, the wave momentum and energy grow at the expense of electron momentum and kinetic energy respectively, in particular of the resonant electrons (see the discussion following Eq.(3.3.12)). We therefore associate the loss rate of electron momentum (due to the radiation of ion sound waves) with the effective wave-particle collision frequency, i.e.,

$$- m_e n_o \vec{V}_o(t) \nu_{ef} = m_e n_o \frac{\partial \langle \vec{V} \rangle}{\partial t} \tag{4.4.1}$$

From Eqs.(4.1.1) and (4.1.2),

$$m_e n_o \frac{\partial \langle \vec{V} \rangle}{\partial t} = m_e n_o \{\frac{1}{n_o} \int \vec{V} \frac{\partial f_{oe}}{\partial t} d\vec{V}\}$$

$$= m_e \text{ Re } \{\frac{ie}{m_e} \sum_k \phi_k^* \int \vec{V}\, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} d\vec{V}\} \qquad (4.4.1a)$$

as the other terms in Eq.(4.1.2) vanish due to the choice of f_{oe} and since $\vec{V}_o = V_o \hat{y}$.

Now $\int \vec{V}\, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} d\vec{V}$

$$= \hat{x} \int V_x \{k_x \frac{\partial}{\partial V_x} + k_y \frac{\partial}{\partial V_y} + k_z \frac{\partial}{\partial V_z}\} f_{ek}\, dV_x\, dV_y\, dV_z$$

$$+ \hat{y} \int V_y \{k_x \frac{\partial}{\partial V_x} + \text{---} + \text{---}\} f_{ek} dV_x\, dV_y\, dV_z + \hat{z} \int V_z \{k_x \frac{\partial}{\partial V_x} + \text{---} + \text{---}\} f_{ek} dV_x\, dV_y\, dV_z$$

$$= - k_x\, \hat{x} \int f_{ek}\, d\vec{V} - k_y\, \hat{y} \int f_{ek}\, d\vec{V} - k_z\, \hat{z} \int f_{ek}\, d\vec{V}$$

$$= - \vec{k} \int f_{ek}(\vec{V})\, d\vec{V}$$

where we have used the fact that $f_{ek}\ (\vec{V}) = 0$ at $V = \pm \infty$.

Therefore

$$m_e\, n_o \frac{\partial \langle\vec{V}\rangle}{\partial t} = \text{Re } \{\sum_k (- i\, e\, \vec{k}\, \phi_k^*) \int f_{ek}\, d\vec{V}\}$$

Substituting for $f_{ek}(\vec{V})$ from Eqs.(4.1.5a) and (3.2.12a), we see that

$$\int f_{ek}\ (\vec{V})\ d\vec{V} = \frac{-e}{m_e} \phi_k \int\Bigg[\frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} - \exp \{i\, \xi \sin (\theta - \Psi)\}\, J_o\ (\xi)$$

$$\times \frac{\{(\omega_k - k_y V_o) \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} + k_z V_z (\frac{1}{V_z} \frac{\partial}{\partial V_z} - \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp})\} f_{oe}}{(\omega_k - k_y\, V_o - k_z\, V_z)}\Bigg] V_\perp\, dV_\perp\, dV_z\, d\theta$$

$$= -\frac{2\pi e}{m_e}\phi_k \iint\left[\beta(k_\perp/\Omega_e) - \frac{\alpha(k_\perp/\Omega_e)\,k_z}{(\omega_k - k_y V_o - k_z V_z)}\frac{\partial}{\partial V_z}\right]\Psi_{oe}(V_z,t)\,dV_z$$

as shown in the manipulation leading from Eq.(3.3.1a) to Eq.(3.3.6), with $\Psi_{oe}(V_z,t)$ defined in Eq.(4.2.3).

Then

$$m_e n_o \frac{\partial\langle\vec{V}\rangle}{\partial t} = -\frac{2\pi e^2}{m_e}\,\mathrm{Re}\left\{\sum_k i\,\vec{k}\,|\phi_k|^2\int\frac{\alpha(k_\perp/\Omega_e)\,k_z\,(\partial\Psi_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)}\,dV_z\right\}$$

where the result (4.1.10a) has been used.

With the aid of the result (4.1.11), the right hand side becomes

$$-\frac{2\pi e^2}{m_e}\sum_k \vec{k}\,|\phi_k|^2\int\frac{\alpha(k_\perp/\Omega_e)\,\gamma_k k_z(\partial\Psi_{oe}/\partial V_z)}{(k_z V_z - \varpi_k^r)^2 + \gamma_k^2}\,dV_z \tag{4.4.2}$$

For the resonant electrons, we use the identity (4.2.2) to arrive at

$$m_e n_o \frac{\partial\langle\vec{V}\rangle}{\partial t} = \frac{-2\pi^2 e^2}{m_e}\sum_k \vec{k}\,\alpha(k_\perp/\Omega_e)\,|\phi_k|^2\int_{-\infty}^{+\infty}\delta(k_z V_z - \varpi_k^r)\,k_z\frac{\partial\Psi_{oe}}{\partial V_z}\,dV_z$$

$$= \frac{-2\pi^2 e^2}{m_e}\sum_k \vec{k}\,\alpha(k_\perp/\Omega_e)\,|\phi_k|^2\left[\frac{\partial\Psi_{oe}(V_z,t)}{\partial V_z}\right]_{V_z = \overline{V}_z} \tag{4.4.3}$$

where, as in Eq.(3.3.11a), $\overline{V}_z = \varpi_k^r/k_z = (\omega_k^r - k_y V_o)/k_z$.

If it is assumed that the ion velocity distribution is Maxwellian, which in the light of the discussion in Section 4.3 pertaining to the bulk of the ions, viz., the non-resonant ones, is reasonable, then we can use the results in Section 3.4 to manipulate the denominator of growth rate (4.3.10).

The results (3.4.10a) - (3.4.10c) then allow us to write the growth rate as

$$\frac{\gamma_k(t)}{\omega_k^r} = \frac{\pi\,\alpha(k_\perp/\Omega_e)\ \left(\partial\Psi_{oe}(V_z,t)/\partial V_z\right)_{V_z=\overline{V}_z}}{\frac{m_e}{m_i}\left[\frac{2k_\perp^2}{(\omega_k^r)^2}\frac{n_o}{2\pi} + \frac{k_z}{\omega_k^r}\frac{2k_z}{\omega_k^r}\frac{n_o}{2\pi}\right]}$$

from which

$$\alpha(k_\perp/\Omega_e)\left[\partial\Psi_{oe}(V_z,t)/\partial V_z\right]_{V_z=\overline{V}_z} = 2\,\frac{\gamma_k(t)}{\omega_k^r}\,\frac{n_o}{2\pi^2}\,\frac{k^2}{(\omega_k^r)^2}\,\frac{m_e}{m_i} \qquad (4.4.4)$$

Equation (4.4.3) then becomes

$$m_e\, n_o \frac{\partial\langle\vec{V}\rangle}{\partial t} = -\frac{2\, n_o\, e^2}{m_i}\sum_k \frac{\gamma_k\, k^2\, |\phi_k|^2}{(\omega_k^r)^3}\,\vec{k}$$

$$= -\frac{1}{2\pi}\sum_k \frac{\omega_{pi}^2\,\gamma_k\, k^2\, |\phi_k|^2}{(\omega_k^r)^3}\,\vec{k} \qquad (4.4.5)$$

where $\omega_{pi} = (4\pi n_o e^2/m_i)^{\frac{1}{2}}$ is the ion plasma frequency.

From Eq.(3.3.14), the results (3.3.10a) - (3.4.10c) for a Maxwellian ion distribution, and the approximation (3.3.16) for the non-resonant electrons, we see that

$$\varepsilon_r(\omega,k) = 1 - \frac{\omega_{pi}^2}{(\omega_k^r)^2} + \rho$$

where ρ is independent of ω_k^r. Therefore

$$\frac{\partial\,\varepsilon_r(\omega,k)}{\partial\omega_k^r} = \frac{2\,\omega_{pi}^2}{(\omega_k^r)^3} \qquad (4.4.6)$$

Thus, from Eqs.(4.4.1) and (4.4.5) we find that

$$\nu_{ef} = \frac{1}{4\pi} \sum_k \left(\frac{\partial \varepsilon_r}{\partial \omega_k^r}\right) \frac{\gamma_k k^2 |\phi_k|^2 (\vec{k} \cdot \vec{V}_o)}{m_e n_o V_o^2}$$

$$= \sum_k 2 \gamma_k \left\{\omega_k^r \frac{\partial \varepsilon_r}{\partial \omega_k^r} \frac{k^2 |\phi_k|^2}{8\pi}\right\} \left(\frac{\vec{k} \cdot \vec{V}_o}{\omega_k^r}\right) \frac{1}{m_e n_o V_o^2}$$

i.e.,

$$\nu_{ef} = \sum_k \gamma_k \left(\frac{\vec{k} \cdot \vec{V}_o}{\omega_k^r}\right) \frac{W_k}{\frac{1}{2} m_e n_o V_o^2} \qquad (4.4.7)$$

where

$$W_k = \omega_k^r \frac{\partial \varepsilon_r}{\partial \omega_k^r} \frac{k^2 |\phi_k|^2}{8\pi} \qquad (4.4.7a)$$

is the total wave energy density[47].

This expression compares favourably with that of SAGDEEV and GALEEV[47], and for $\vec{k}$ parallel to $\vec{V}_o$ is identical to that of HASEGAWA[65].

Since Eq.(4.4.7) represents the momentum transfer parallel to $\vec{V}_o$, i.e., across $\vec{B}_o$, it (Eq.(4.4.7)) describes the effective collision frequency and associated anomalous resistivity perpendicular to the magnetic field. To emphasize this point we write the frequency as $\nu_{ef\perp}$.

If, as an approximation, we replace γ_k by its linear expression as given by Eq.(3.4.12), then

$$\frac{\nu_{ef\perp}}{\omega_{pe}} = \left(\frac{\pi}{4}\right)^{1/2} \sum_k \frac{k}{k_z} \left\{\frac{m_e}{2m_i}\right\}^{\frac{1}{2}} \frac{\Gamma_o(b)}{(1 + k^2\lambda_D^2)} \frac{(k_y V_o - \omega_k^r)}{k C_s} \frac{\omega_k^r}{\omega_{pe}} \frac{k_y V_o}{\omega_k^r} \frac{W_k}{\frac{1}{2} m_e n_o V_o^2}$$

For $V_o \gg C_s$, $k_y = k_\perp \simeq k$, $b = k_\perp^2 r_e^2 \ll 1$, and $k\lambda_D \sim 1$, which are typical experimental parameters, this reduces to

$$\frac{\nu_{ef\perp}}{\omega_{pe}} \simeq \left(\frac{\pi}{4}\right)^{1/2} \sum_k \frac{k}{k_z} \left\{\frac{m_e}{2m_i}\right\}^{\frac{1}{2}} \frac{k}{k_D} \frac{W_k}{n_o m_e C_s C_e}$$

$$= \left(\frac{\pi}{8}\right)^{1/2} \sum_k \frac{k}{k_z} \frac{W_k}{n_o T_e}$$

Here, the relation $\omega_{pe} = k_D C_e$ and the result $\Gamma_o(b) \simeq 1$ for $b \ll 1$ have been used. Thus

$$\nu_{ef\perp} \sim \omega_{pe} \sum_k \frac{k}{k_z} \frac{W_k}{n_o T_e} \tag{4.4.8}$$

For the case of wave propagation and particle drift along the magnetic field, HASEGAWA (65, p. 67) uses the linear expression for the growth rate and the approximations $V_o \simeq C_s$ and $k\lambda_D \sim 1$ to arrive at the result

$$\nu_{ef\parallel} \sim \omega_{pe} \sum_k \frac{W_k}{n_o T_e} \tag{4.4.9}$$

He then discusses the fact that this effective collision frequency is larger than the classical electron-ion collision frequency given by

$$\nu_{ei} = \omega_{pe} \frac{W_T}{n_o T_e}$$

where W_T is the energy density of the fluctuating field in thermal equilibrium, since this is the minimum value of the field energy density $W = \sum_k W_k$ due to the instability.

From Eqs.(4.4.8) and (4.4.9) we see that in the case of propagation oblique to the magnetic field and particle drift across the magnetic field the effective collision frequency $\nu_{ef\perp}$ is enhanced by the factor (k/k_z) when compared to $\nu_{ef\parallel}$. This is probably due to the increase in the growth rate by an amount $\frac{k}{k_z}$ in the presence of the magnetic field, as discussed in Section 3.4. Since the resistivity is proportional to ν_{ef} it is also enhanced. This is not surprising, as in a magnetized plasma the electrons, drifting across $\vec{B}$, are bound to the field lines. Thermal runaway across $\vec{B}$ is lowered with a consequent decrease in conductivity.

4.5 ELECTRON AND ION HEATING RATES

The concept of kinetic temperature is introduced in terms of the mean random kinetic energy of the plasma components.

From gas kinetic theory we have

$$\frac{3}{2} n_j T_j = \frac{1}{2} n_j m_j \langle w_j^2 \rangle \qquad (j = i,e) \tag{4.5.1}$$

$$= \frac{1}{2} n_j m_j \int w_j^2 f_{oj} \, d\vec{V}$$

where $\vec{w}_j$, the random velocity, equals $(\vec{V} - \vec{V}_o)$ for the drifting electrons and $\vec{V}$ for the stationary ions.

(a) Electron Heating Rate

For the electrons, the above equation reduces to

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = n_o m_e \frac{\partial}{\partial t} \langle w_e^2 \rangle$$

$$= \frac{n_o m_e}{2} \{ \frac{1}{n_o} \int (\vec{V} - \vec{V}_o)^2 \frac{\partial f_{oe}}{\partial t} d\vec{V} \}$$

For a background distribution of the type (4.1.1), we substitute for $\frac{\partial f_{oe}}{\partial t}$ from the quasilinear diffusion equation (4.1.2). It then turns out that

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = \frac{m_e}{2} \text{Re} \left\{ \frac{ie}{m_e} \sum_k \phi_k^* \int (\vec{V} - \vec{V}_o)^2 \, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} \, d\vec{V} \right\} \tag{4.5.2}$$

Now

$$\int (\vec{V} - \vec{V}_o)^2 \, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} \, d\vec{V}$$

$$= \int \{V_x^2 + (V_y - V_o)^2 + V_z^2\} \, \{k_x \frac{\partial f_{ek}}{\partial V_x} + k_y \frac{\partial f_{ek}}{\partial V_y} + k_z \frac{\partial f_{ek}}{\partial V_z}\} \, dV_x \, dV_y \, dV_z$$

Considering the first term,

$$k_x \int V_x^2 \frac{\partial f_{ek}}{\partial V_x} \, dV_x \, dV_y \, dV_z = -2k_x \int V_x \, f_{ek} \, dV_x \, dV_y \, dV_z$$

and

$$k_x \int (V_y - V_o)^2 \frac{\partial f_{ek}}{\partial V_x} \, d\vec{V} = k_x \int V_z^2 \frac{\partial f_{ek}}{\partial V_x} \, d\vec{V} = 0$$

since $f_{ek} = 0$ at $V_x = \pm \infty$.

In a similar manner we may manipulate the other integrals to find that

$$\begin{aligned} \int (\vec{V} - \vec{V}_o)^2 \, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} \, d\vec{V} &= -2 \int \{k_x V_x + k_y (V_y - V_o) + k_z V_z\} \, f_{ek} \, d\vec{V} \\ &= -2 \int (\vec{k}_\perp \cdot \vec{V}_\perp + k_z V_z) \, f_{ek} \, d\vec{V} \end{aligned} \tag{4.5.3}$$

where, as previously defined, $\vec{k}_\perp = (k_x, k_y)$ and $\vec{V}_\perp = V_x \hat{x} + (V_y - V_o) \hat{y}$

for the electrons.

Furthermore

$$\int \vec{k}_\perp \cdot \vec{V}_\perp \, f_{ek} \, d\vec{V}$$

$$= - \frac{e}{m_e} \phi_k \int k_\perp V_\perp \cos\phi \left[\frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} - \exp(i\,\xi \sin\phi)\, J_o(\xi) \right.$$

$$\left. \times \frac{\{(\omega_k - k_y V_o) \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} + k_z V_z (\frac{1}{V_z} \frac{\partial}{\partial V_z} - \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp})\} f_{oe}}{(\omega_k - k_y V_o - k_z V_z)} \right] V_\perp \, dV_\perp \, dV_z \, d\phi$$

(4.5.4)

where a substitution for f_{ek} has been made from Eqs.(4.1.5a) and (3.2.12a), and the coordinate system has been rotated through the angle Ψ as shown in Fig. 4.1.

With the aid of the identity (3.2.11) and then the result (4.1.9a), we find that the integral (4.5.4) vanishes.

Equation (4.5.3) then reduces to

$$\int (\vec{V} - \vec{V}_o)^2 \, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} \, d\vec{V} = - 2 \int k_z V_z \, f_{ek} \, d\vec{V} \qquad (4.5.5)$$

As previously, we substitute for f_{ek} and then integrate with respect to ϕ to find

$$- 2 \int k_z V_z \, f_{ek} \, d\vec{V} = \frac{2e\phi_k}{m_e} \int k_z V_z \left[\frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} - \sum_\ell J_o(\xi)\, J_\ell(\xi) \exp(i\,\ell\,\phi) \right.$$

$$\left. \times \frac{\{(\omega_k - k_y V_o) \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp} + k_z V_z (\frac{1}{V_z} \frac{\partial}{\partial V_z} - \frac{1}{V_\perp} \frac{\partial}{\partial V_\perp})\} f_{oe}}{(\omega_k - k_y V_o - k_z V_z)} \right] V_\perp \, dV_\perp \, dV_z \, d\phi$$

$$= \frac{4\pi e\, \phi_k}{m_e} \int k_z V_z \left[\{1 - J_o^2(\xi)\} \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^2(\xi)\, k_z\, (\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)} \right] V_\perp\, dV_\perp\, dV_z$$

Thus from Eqs.(4.5.2) and (4.5.5) we see that

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = \frac{2\pi e^2}{m_e} \mathrm{Re}\, \{ i \sum_k |\phi_k|^2 \int k_z V_z \left[\{1 - J_o^2(\xi)\} \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} \right.$$

$$\left. - \frac{J_o^2(\xi)\, k_z\, (\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)} \right] V_\perp\, dV_\perp\, dV_z$$

$$= \frac{2\pi e^2}{m_e} \mathrm{Re} \left[- i \sum_k |\phi_k|^2 \int \frac{k_z^2\, V_z\, J_o^2(\xi)\, (\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)} V_\perp\, dV_\perp\, dV_z \right]$$

since

$$\mathrm{Re} \left[i \int \{1 - J_o^2(\xi)\} \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} V_\perp\, dV_\perp\, dV_z \right] = 0$$

Using Eq.(4.1.11) this becomes

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = \frac{-2\pi\, e^2}{m_e} \sum_k |\phi_k|^2 \int \frac{k_z^2 V_z\, \gamma_k\, J_o^2(\xi)\, (\partial f_{oe}/\partial V_z)}{(\varpi_k^r - k_z V_z)^2 + \gamma_k^2} V_\perp\, dV_\perp\, dV_z$$

With the definitions (3.3.4) and (4.2.3) for $\alpha(k_\perp/\Omega_e)$ and $\Psi_{oe}(V_z,t)$ respectively, the above equation reduces to

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = - \frac{2\pi e^2}{m_e} \sum_k |\phi_k|^2 \alpha(k_\perp/\Omega_e) \int \frac{k_z^2 V_z \gamma_k (\partial \Psi_{oe}/\partial V_z)}{(k_z V_z - \varpi_k^r)^2 + \gamma_k^2} dV_z \tag{4.5.6}$$

Both the resonant and the non-resonant electrons contribute to the right hand side of Eq.(4.5.6). For the resonant electrons we use the identity (4.2.2) to write their contribution as

$$- \frac{2\pi e^2}{m_e} \sum_k |\phi_k|^2 \alpha(k_\perp/\Omega_e) \int_{\frac{\varpi_k^r}{k_z} - \Delta V}^{\frac{\varpi_k^r}{k_z} + \Delta V} k_z^2 V_z (\partial \Psi_{oe}/\partial V_z) \pi \delta(k_z V_z - \varpi_k^r) dV_z$$

$$= - \frac{2\pi^2 e^2}{m_e} \sum_k |\phi_k|^2 \alpha(k_\perp/\Omega_e) \varpi_k^r \left[\frac{\partial \Psi_{oe}}{\partial V_z}\right]_{V_z = \frac{\varpi_k^r}{k_z} = \overline{V}_z}$$

where $2\Delta V$ is the width of the resonance region.

For a Maxwellian ion distribution, the result (4.4.4) allows us to write this as

$$- \frac{2n_o e^2}{m_e} \sum_k \frac{\varpi_k^r}{(\omega_k^r)^3} \frac{m_e}{m_i} \gamma_k k^2 |\phi_k|^2$$

$$= - \sum_k \frac{\omega_{pi}^2}{2\pi (\omega_k^r)^2} \left\{\frac{\partial \varepsilon_r}{\partial \omega_k^r} \frac{(\omega_k^r)^3}{2 \omega_{pi}^2}\right\} \frac{\varpi_k^r}{\omega_k^r} \gamma_k k^2 |\phi_k|^2$$

having used Eq.(4.4.6). Finally, definition (4.4.7a) for the wave energy density is used to write the resonant-electron heating rate as

$$\sum_k 2 \gamma_k W_k \left[\frac{\vec{k} \cdot \vec{V}_o}{\omega_k^r} - 1\right] \tag{4.5.7}$$

since $-\dfrac{\varpi_k^r}{\omega_k^r} = -\dfrac{(\omega_k^r - k_y V_o)}{\omega_k^r} = \dfrac{\vec{k} \cdot \vec{V}_o}{\omega_k^r} - 1$.

As for the non-resonant electrons, since they do not interact directly with the waves it is not expected that the heating experienced by them would be as significant as that of the resonant electrons. The so called 'fake diffusion' of the non-resonant electrons, as discussed earlier, describes their adjustment to changes in wave amplitude. During the growth phase of the wave the kinetic energy associated with the 'sloshing motion' of the non-resonant particles in the presence of the wave increases and the non-resonant electrons appear to be heated. However, since an important requirement for weak turbulence theory, and therefore quasilinear theory, is that the electron field fluctuations be sufficiently small (refer to Section 2.4), it follows that the heating of the non-resonant electrons is very weak. This will be shown later from energy considerations. Hence, to a fairly good approximation, from Eqs.(4.5.6) and (4.5.7) the total electron heating rate may be written as

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = \sum_k 2 \gamma_k \left[\frac{\vec{k} \cdot \vec{V}_o}{\omega_k^r} - 1\right] W_k \tag{4.5.8}$$

(b) <u>Ion Heating Rate</u>

It has been pointed out at the beginning of Section 4.3 that the interaction between the ions and the waves is primarily non-resonant. Thus, the discussion above implies a low level of ion heating. This is analytically shown to be so.

For the ions, Eq.(4.5.1) gives

$$\frac{3}{2} n_o \frac{\partial T_i}{\partial t} = \frac{n_o m_i}{2} \{\frac{1}{n_o} \int \vec{V}^2 \frac{\partial f_{oi}}{\partial t} d\vec{V}\} \tag{4.5.9}$$

Upon substituting for $\frac{\partial f_{oi}}{\partial t}$ from Eq.(4.1.13a), the integral on the right hand side reduces to

$$\text{Re}\left[\frac{ie^2}{m_i^2} \sum_k |\phi_k|^2 \int \vec{V}^2 \vec{k} \cdot \frac{\partial}{\partial \vec{V}} \left\{\frac{\vec{k} \cdot (\partial f_{oi}/\partial \vec{V})}{(\omega_k - \vec{k} \cdot \vec{V})}\right\} d\vec{V}\right]$$

$$= - 2 \text{ Re}\left[\frac{ie^2}{m_i^2} \sum_k |\phi_k|^2 \int \frac{(\vec{k} \cdot \vec{V}) \{\vec{k} \cdot (\partial f_{oi}/\partial \vec{V})\}}{(\omega_k - \vec{k} \cdot \vec{V})} d\vec{V}\right]$$

by virtue of the result (4.5.3),as there are no particles at $V = \pm \infty$. We recall that for the ions $\vec{V}_\perp = V_x \hat{x} + V_y \hat{y}$.

With the result (4.1.11) and the approximation (4.3.1a) applicable to the non-resonant ions, it can be shown that

$$\text{Re}\left[i \int \frac{(\vec{k} \cdot \vec{V}) \{\vec{k} \cdot (\partial f_{oi}/\partial \vec{V})\}}{(\omega_k - \vec{k} \cdot \vec{V})} d\vec{V}\right] = \int (\vec{k} \cdot \vec{V}) \{\vec{k} \cdot \frac{\partial f_{oi}}{\partial \vec{V}}\} \frac{\gamma_k}{(\omega_k^r)^2} d\vec{V}$$

Therefore

$$\int \vec{V}^2 \frac{\partial f_{oi}}{\partial t} d\vec{V} = - \frac{2e^2}{m_i^2} \sum_k \frac{\gamma_k}{(\omega_k^r)^2} |\phi_k|^2 \int (\vec{k} \cdot \vec{V}) \{\vec{k} \cdot \frac{\partial f_{oi}}{\partial \vec{V}}\} d\vec{V} \tag{4.5.10}$$

Upon rotating our coordinate axes in the x-y plane through the angle Ψ as shown in Fig. 4.1, we find that

$$\int (\vec{k} \cdot \vec{V}) \{\vec{k} \cdot \frac{\partial f_{oi}}{\partial \vec{V}}\} \, d\vec{V}$$

$$= \int (k_\perp V_\perp \cos\phi + k_z V_z) \{k_\perp \left[\cos\phi \frac{\partial f_{oi}}{\partial V_\perp} - \frac{\sin\phi}{V_\perp} \frac{\partial f_{oi}}{\partial \phi}\right] + k_z \frac{\partial f_{oi}}{\partial V_z}\} V_\perp \, dV_\perp \, dV_z \, d\phi$$

$$= \pi \, k_\perp^2 \int V_\perp \frac{\partial f_{oi}}{\partial V_\perp} V_\perp \, dV_\perp \, dV_z + 2\pi \, k_z^2 \int V_z \frac{\partial f_{oi}}{\partial V_z} V_\perp \, dV_\perp \, dV_z$$

for an f_{oi} of the form (4.1.1).

Using the fact that there are no particles at $V_\perp = + \infty$, it turns out that

$$\pi \, k_\perp^2 \int V_\perp^2 \frac{\partial f_{oi}}{\partial V_\perp} dV_\perp \, dV_z = - 2\pi \, k_\perp^2 \int f_{oi} V_\perp \, dV_\perp \, dV_z$$

$$= - n_o \, k_\perp^2$$

since the equilibrium ion density n_o is given by

$$n_o = \int f_{oi} V_\perp \, dV_\perp \, dV_z \, d\phi = 2\pi \int f_{oi} V_\perp \, dV_\perp \, dV_z$$

Similarly

$$2\pi \, k_z^2 \int V_z \frac{\partial f_{oi}}{\partial V_z} dV_z \, V_\perp \, dV_\perp = - n_o \, k_z^2$$

Thus,

$$\int (\vec{k} \cdot \vec{V}) \{\vec{k} \cdot \frac{\partial f_{oi}}{\partial \vec{V}}\} \, d\vec{V} = - n_o \, (k_\perp^2 + k_z^2) = - n_o k^2$$

and from Eqs.(4.5.9) and (4.5.10),

$$\frac{3}{2} n_o \frac{\partial T_i}{\partial t} = - \frac{e^2}{m_i} \sum \frac{\gamma_k}{k(\omega_k^r)^2} |\phi_k|^2 \; (- n_o k^2) \tag{4.5.11a}$$

$$= \sum_k \gamma_k \frac{\omega_{pi}^2}{4\pi} \frac{k^2 |\phi_k|^2}{(\omega_k^r)^2}$$

Recalling that $\dfrac{\partial \varepsilon_r}{\partial \omega_k^r} = \dfrac{2\omega_{pi}^2}{(\omega_k^r)^3}$ and the wave energy density

$W_k = \omega_k^r \dfrac{\partial \varepsilon_r}{\partial \omega_k^r} \dfrac{k^2 |\phi_k|^2}{8\pi}$, we may write the above equation as

$$\frac{3}{2} n_o \frac{\partial T_i}{\partial t} = \sum_k \gamma_k W_k \tag{4.5.11}$$

From Eqs.(4.5.8) and (4.5.11) we see that

$$\frac{\partial T_e/\partial t}{\partial T_i/\partial t} = \frac{\sum_k 2\gamma_k W_k \left[\dfrac{\vec{k} \cdot \vec{V}_o}{\omega_k^r} - 1\right]}{\sum_k \gamma_k W_k}$$

$$\simeq 2 \left[\frac{\vec{k} \cdot \vec{V}_o}{\omega_k^r} - 1\right] \tag{4.5.12}$$

Measurements of electron and ion heating rates in the Double Plasma device by JONES [86] agree reasonably well with this expression. Since for appreciable growth rates the drift speed V_o is several times larger than the wave phase speed $(\omega_k^r/k) \simeq C_s$, it follows that for small k_z

$$\frac{\vec{k}\cdot\vec{V}_o}{\omega_k^r} \gg 1$$

Therefore Eq.(4.5.12) shows that the increase in electron temperature (electron thermal energy) is significantly larger than that in ion temperature (ion thermal energy). This is in keeping with the experimentally observed phenomenon that ion acoustic waves have maximum growth for [36, 41]

$$\frac{k_z}{k} > \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}}$$

Then the effective electron mass $k^2 m_e/k_z^2 < m_i$ and therefore the electrons experience a larger increase in thermal energy.

To estimate the characteristic heating time of the electrons (t_{he}) we adopt the approximate formula

$$\frac{1}{t_{he}} \simeq \frac{1}{T_e}\,\frac{dT_e}{dt}$$

Then from Eq.(4.5.8) for the heating rate and Eq.(3.4.12) for the linear growth rate,

$$\frac{1}{t_{he}} \simeq \frac{2}{3}\left(\frac{\pi}{4}\right)^{\frac{1}{2}} \frac{k}{k_z}\left(\frac{m_e}{2m_i}\right)^{\frac{1}{2}}\left(\frac{V_o}{C_s}-1\right)^2 \omega_k^r \left\{\frac{\sum\limits_k W_k}{n_o T_e}\right\}$$

For the typical values of $(V_o/C_s) = 8$, $\omega_k^r = 1{,}0$ MHz, $kV_o/k_z C_e \sim 1$, as measured by HAYZEN and BARRETT [41] and $\sum\limits_k W_k/n_o T_e \simeq 10^{-2}$, it terms out that $t_{he} \simeq 40\mu s$. This compares favourably with the value

$t_{he} \sim 10^{-4}$s, as quasilinearly estimated from the experimental results of VIRKO and KIRICHENKO [68].

As for the characteristic ion heating time, it is seen from the above ratio of the heating rates that

$$\frac{t_{hi}}{t_{he}} \simeq \frac{(1/T_i)\ (\partial T_i/\partial t)}{(1/T_e)\ (\partial T_e/\partial t)} \simeq 2 \left(\frac{V_o}{C_s} - 1\right) \frac{T_i}{T_e}$$

For $T_e = 10\ T_i$ and $(V_o/C_s) = 8$ [41], we find $t_{hi} \simeq 1{,}4\ t_{he} \simeq 64\mu s$.

The heating of electrons is via scattering of electrons by ion sound waves, being related to the linear wave-particle resonances. In the assessment of ion heating we have neglected the high-energy ions in the tail of the distribution which resonate with the waves. For the ion acoustic mode with $T_e \gg T_i$ there are very few such ions. Thus within the confines of the quasilinear theory the only possible ion heating is associated with the sloshing motion of the non-resonant ions in the presence of the waves. This motion, however, comprises ordered energy. Therefore in speaking of thermal heating which involves random energy, we are assuming the presence of some other nonlinear processes, such as particle trapping, which convert ordered energy into random energy.

The small increase in ion temperature due to the non-resonant nature of the ion - wave interactions implies that the change in the ion distribution should be small. This has been confirmed in Section

4.3, where we saw that for an initial Maxwellian distribution the only change in the non-resonant ion distribution with time was a small increase in ion temperature. However, it has been observed in experiments [40] and in computer simulations [56] that the ions have a two-temperature distribution, showing the marked presence of a high-energy tail. At the outset one may attribute this effect to linear Landau damping of the waves on the few ions in the tail of the distribution. We recall that this resonance process was not considered in arriving at Eq.(4.5.11). However, in Section 4.3 we have noticed that a quasilinear treatment of resonant ion - wave interactions leads to an asymptotic distribution with an exp $(- V^5)$ dependence on velocity. Such a dependence was not observed in either experiments or simulations [40, 56].

Thus, we conclude that linear ion Landau damping may be a contributory process to ion heating, but is certainly not the dominant one. To explain the observed ion heating we have to invoke other non-linear processes such as particle trapping and nonlinear Landau damping. In the latter process two or more modes of the wave spectrum may couple to produce beats which resonate with the ions and cause damping. For two modes $(\omega_k, \vec{k})$ and $(\omega_{k'}, \vec{k}')$ the resonance condition is

$$(\omega_k - \omega_{k'}) - (\vec{k} - \vec{k}') \cdot \vec{V} = 0$$

where $\vec{V}$ is the ion velocity.

4.6 ENERGY STUDIES

The rate of change of the average kinetic energy of the electrons is given by

$$\frac{\partial}{\partial t}\left(\frac{1}{2} n_o m_e \langle V^2 \rangle\right) = \frac{n_o m_e}{2} \left\{\frac{1}{n_o} \int V^2 \frac{\partial f_{oe}}{\partial t} d\vec{V}\right\} \qquad (4.6.1)$$

In a manner paralleling the development from Eq.(4.5.1) to Eq.(4.5.3) we see that

$$\frac{\partial}{\partial t}\left(\frac{1}{2} n_o m_e \langle V^2 \rangle\right) = \frac{m_e}{2} \mathrm{Re} \left\{\frac{ie}{m_e} \sum_k \phi_k^* \int V^2 \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} d\vec{V}\right\} \qquad (4.6.2)$$

with

$$\int V^2 \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} d\vec{V} = -2 \int (k_x V_x + k_y V_y + k_z V_z) f_{ek} \, d\vec{V}$$

$$= -2 \int (\vec{k}_\perp \cdot \vec{V}_\perp + k_y V_o + k_z V_z) f_{ek} \, d\vec{V} \qquad (4.6.3)$$

where we recall that $\vec{V}_\perp = V_x \hat{x} + (V_y - V_o) \hat{y}$.

From the discussion following Eq.(4.5.4) we have that

$$\int \vec{k}_\perp \cdot \vec{V}_\perp f_{ek} \, d\vec{V} = 0$$

Upon substituting for f_{ek} from Eqs.(4.1.5a) and (3.2.12a), we find that

$$k_y V_o \int f_{ek}\, d\vec{V} = k_y V_o \left(-\frac{e\phi_k}{m_e}\right) \int\left[\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \sum_\ell J_\ell\,(\xi)\, J_o\,(\xi)\, \exp\,(i\,\ell\,\phi)\right.$$

$$\times \left.\frac{\{(\omega_k - k_y V_o)\frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} + k_z V_z\,(\frac{1}{V_z}\frac{\partial}{\partial V_z} - \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp})\} f_{oe}}{(\omega_k - k_y V_o - k_z V_z)}\right] V_\perp\, dV_\perp\, dV_z\, d\phi$$

$$= -\,k_y V_o \frac{2\pi e\phi_k}{m_e} \int\left[\{1 - J_o^2\,(\xi)\}\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^2\,(\xi)\, k_z\, (\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)}\right] V_\perp\, dV_\perp\, dV_z \tag{4.6.4}$$

Similarly

$$\int k_z V_z\, f_{ek}\, d\vec{V} = -\frac{2\pi e\phi_k}{m_e}\int k_z V_z \left[\{1 - J_o^2\,(\xi)\}\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^2(\xi) k_z (\partial f_{oe}/\partial V_z)}{(\omega_k - k_y V_o - k_z V_z)}\right] V_\perp\, dV_\perp\, dV_z \tag{4.6.5}$$

The results (4.6.4) and (4.6.5) are used to rewrite the right hand side of Eq.(4.6.3), which is then used to express Eq.(4.6.2) as

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\, n_o\, m_e \langle V^2\rangle\right) = \frac{2\pi e^2}{m_e} \mathrm{Re}\left\{i \sum_k |\phi_k|^2 \int (k_y V_o + k_z V_z)\left[\frac{J_o^2(\xi) k_z (\partial f_{oe}/\partial V_z)}{\{k_z V_z - (\omega_k - k_y V_o)\}}\right] V_\perp\, dV_\perp\, dV_z\right\}$$

Now

$$\frac{k_y V_o + k_z V_z}{(k_z V_z + k_y V_o - \omega_k)} = 1 + \frac{\omega_k}{(k_z V_z + k_y V_o - \omega_k)}$$

and the integral above becomes

$$\int_0^\infty \int_{-\infty}^{+\infty} J_o^2\left(\frac{k_\perp V_\perp}{\Omega_e}\right) k_z \frac{\partial f_{oe}}{\partial V_z}\, dV_z\, V_\perp\, dV_\perp + \omega_k \int_{-\infty}^{+\infty}\int_0^\infty \frac{J_o^2(k_\perp V_\perp/\Omega_e) k_z}{\{k_z V_z - (\omega_k - k_y V_o)\}} \frac{\partial f_{oe}}{\partial V_z}\, V_\perp\, dV_\perp\, dV_z$$

Since

$$\int_{-\infty}^{+\infty} \frac{\partial f_{oe}}{\partial V_z} dV_z = \left\{ f_{oe} \right\}_{V_z = \pm\infty} = 0$$

as there are no particles at $V_z = \pm \infty$, the first term vanishes. Thus

$$\frac{\partial}{\partial t}\left(\frac{1}{2} n_o m_e \langle V^2 \rangle\right) = \frac{2\pi e^2}{m_e} \mathrm{Re}\left\{ i\sum_k |\phi_k|^2 \int_{-\infty}^{\infty} \omega_k k_z \frac{\frac{\partial}{\partial V_z}\left[\int_o^\infty J_o^2\left(\frac{k_\perp V_\perp}{\Omega_e}\right) f_{oe}(V_\perp^2, V_z, t) V_\perp \, dV_\perp\right]}{\{k_z V_z - (\omega_k - k_y V_o)\}} dV_z \right\}$$

With the definitions (3.1.6), (3.3.4) and (4.2.3) for ω_k, $\alpha(k_\perp/\Omega_e)$ and $\Psi_{oe}(V_z,t)$ respectively, this may be further reduced to

$$\frac{\partial}{\partial t}\left(\frac{1}{2} n_o m_e \langle V^2 \rangle\right) = -\frac{2\pi e^2}{m_e} \sum_k |\phi_k|^2 k_z \alpha(k_\perp/\Omega_e) \int \left[\frac{\omega_k^r \gamma_k}{(k_z V_z - \varpi_k^r)^2 + \gamma_k^2} + \frac{\gamma_k (k_z V_z - \varpi_k^r)}{(k_z V_z - \varpi_k^r)^2 + \gamma_k^2} \right] \left(\frac{\partial \Psi_{oe}}{\partial V_z}\right) dV_z$$

The term on the right hand side above is made up of contributions from both the resonant and the non-resonant electrons. If K_e^R represents the average kinetic energy of the resonant electrons, then we may use the identity (4.2.2) to arrive at

$$\frac{\partial K_e^R}{\partial t} = -\frac{2\pi e^2}{m_e} \sum_k |\phi_k|^2 \alpha(k_\perp/\Omega_e) \omega_k^r \pi \left[\frac{\partial \Psi_{oe}}{\partial V_z}\right]_{V_z = \frac{\varpi_k^r}{k_z} = \overline{V}_z}$$

With the aid of Eq.(4.4.4) it turns out that

$$\frac{\partial K_e^R}{\partial t} = -\sum_k 2 \gamma_k \left(\frac{2\omega_{pi}^2}{(\omega_k^r)^3} \omega_k^r \frac{k^2 |\phi_k|^2}{8\pi} \right)$$

i.e.,
$$\frac{\partial K_e^R}{\partial t} = - \sum_k 2\,\gamma_k\, W_k$$

In arriving at this result we have used Eqs.(4.4.6) and (4.4.7a). The equation for wave growth (4.2.13) finally allows us to write it in the form

$$\frac{\partial}{\partial t}\{K_e^R + \sum_k W_k\} = 0 \tag{4.6.6}$$

This tells us that the total energy in our wave-particle system is conserved, i.e., the kinetic energy lost by the resonant electrons is converted into total wave energy. The latter is made up of the wave electrostatic energy plus the kinetic energy associated with the oscillations of the non-resonant electrons and ions in the presence of the wave.

From Eqs.(4.4.6) and (4.4.7a) we see that the rate of change of total wave energy density may be written as

$$\frac{\partial W}{\partial t} = \frac{\partial\left(\sum_k W_k\right)}{\partial t} = \sum_k 2\,\gamma_k\, W_k$$

$$= 2\sum_k \gamma_k \frac{2\,\omega_{pi}^2}{(\omega_k^r)^2}\frac{k^2|\phi_k|^2}{8\pi}$$

Upon substituting $\omega_k^r = kC_s\,(1 + k^2\lambda_D^2)^{-1/2}$ (the usual form for ω_k^r, cf. Eq.(3.4.11)) this reduces to

$$\frac{\partial W}{\partial t} = 4\sum_k \left\{1 + \frac{k_D^2}{k^2}\right\}\gamma_k \frac{|E_k|^2}{8\pi} \tag{4.6.7}$$

The contributions to the right hand side of Eq.(4.6.7) are as follows:

(a) A fraction

$$2 \sum_k \gamma_k \frac{|E_k|^2}{8\pi} \tag{4.6.8}$$

which represents the rate of change of the electrostatic energy density of the waves.

(b) From Eq.(4.5.11) we see that a fraction

$$2 \sum_k \left(1 + \frac{k_D^2}{k^2}\right) \gamma_k \frac{|E_k|^2}{8\pi} \tag{4.6.9}$$

is the rate at which energy is fed into the oscillations of the non-resonant ions, i.e., the ion kinetic energy of the waves.

(c) It thus follows from Eqs.(4.6.7) - (4.6.9) that the fraction

$$2 \sum_k \frac{k_D^2}{k^2} \gamma_k \frac{|E_k|^2}{8\pi} \tag{4.6.10}$$

represents the rate at which energy is fed into the oscillations of the non-resonant electrons, i.e., the electron kinetic energy of the waves.

Thus for $k^2\lambda_D^2 \ll 1$ (i.e $(k_D^2/k^2) \gg 1$), as is common in practice, it is seen from Eqs.(4.6.7) and (4.6.8) that the waves have only a fraction

$$\frac{k^2\lambda_D^2}{2} \ll 1$$

of their energy as potential energy. A similar result has been found for electrostatic ion cyclotron waves by DRUMMOND and ROSENBLUTH [55]. As pointed out by the authors, this behaviour is in contrast to the case of electron plasma oscillations for which the energy lost by the resonant electrons was equally divided into wave potential and kinetic energies [44].

Furthermore, for $k^2\lambda_D^2 << 1$, $k_z << k \simeq k_\perp = k_y$ and $V_o >> C_s \sim \omega/k$, we see from Eqs.(4.5.7), (4.6.7) and (4.6.10) that the heating rate of the resonant electrons far exceeds that of the non-resonant electrons, thereby justifying our assertion in the discussion preceeding Eq.(4.5.8) that the heating of the latter is very weak.

4.7 HEATING RATES ASSOCIATED WITH THE ELECTRON-CYCLOTRON DRIFT INSTABILITY (ECDI)

A quasilinear derivation of electron and ion heating rates associated with the reactive ECDI has been presented by LAMPE *et al* [53], for wave propagation across the magnetic field, i.e., $k_z = 0$. We recall that the reactive ECDI arises in the limit $T_e >> T_i$ from a resonance coupling between a negative energy Doppler-shifted electron Bernstein mode and the positive energy ion acoustic wave. The resonance condition satisfied is

$$\omega_k^r = k_y V_o + n\,\Omega_e \simeq k\,C_s\,(1 + k^2\lambda_D^2)^{-1/2} \tag{4.7.1}$$

We shall briefly describe the procedure followed. The geometry of our model corresponds to that shown in Fig. 3.1, i.e., a reference frame in which the magnetized electrons drift with an $\vec{E} \times \vec{B}$ drift $\vec{V}_o$ relative to the unmagnetized ions. For the electrons, the heating rate is given by Eq.(4.5.2), viz.,

$$\frac{3}{2} n_o \frac{\partial T_e}{\partial t} = \frac{m_e}{2} \mathrm{Re} \left\{ \frac{ie}{m_e} \sum_k \phi_k^* \int (\vec{V} - \vec{V}_o)^2 \, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} d\vec{V} \right\}$$

where from Eq.(4.5.3),

$$\int (\vec{V} - \vec{V}_o)^2 \, \vec{k} \cdot \frac{\partial f_{ek}}{\partial \vec{V}} d\vec{V} = -2 \int \vec{k}_\perp \cdot \vec{V}_\perp \, f_{ek} \, d\vec{V}$$

for modes propagating across the magnetic field ($\vec{k} = \vec{k}_\perp$).

We substitute for f_{ek} from Eqs.(4.1.5a) and (3.2.12). The expression on the right hand side in the above equation then becomes

$$\frac{2e}{m_e} \phi_k \int \vec{k} \cdot \vec{V}_\perp \left[\frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} - \exp\{ i\xi \sin(\theta - \Psi)\} \left\{ \frac{(\omega_k - k_y V_o)}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} \right\} (\omega_k - k_y V_o - n\Omega_e)^{-1} \right.$$
$$\left. \times \, J_n(\xi) \exp\{- i \, n(\theta - \Psi)\} \right] d\vec{V}$$

where in the summation over n we retain only the term satisfying the resonance condition (4.7.1), since this is the most dominant term.

Then for a Maxwellian electron velocity distribution (as assumed by LAMPE *et al* [53]) the identity (3.2.11) and the recurrence relation[60]

$$J_{n-1}(x) + J_{n+1}(x) = (2n/x) \, J_n(x)$$

are used to arrive at the result

$$\frac{\partial T_e}{\partial t} = \left(\frac{2e^2 m_i^{1/2}}{3} \right) \frac{1}{T_e^{3/2}} \sum_k \left[k_y V_o - \frac{k C_s}{(1 + k^2 \lambda_D^2)^{1/2}} \right] \frac{(1 + k^2 \lambda_D^2)^{3/2}}{k} \frac{\partial}{\partial t} |\phi_k|^2$$

For $k_y = k_\perp = k$ and $V_o >> V_\phi$, this reduces to

$$\frac{\partial T_e}{\partial t} = \left(\frac{2e^2 m_i^{1/2}}{3}\right)\frac{V_o}{T_e^{3/2}} \sum_k (1 + k^2\lambda_D^2)^{3/2} \frac{\partial}{\partial t} |\phi_k|^2 \tag{4.7.2}$$

which, apart from the factor (2/3) is that found by LAMPE *et al* [53].

In a similar manner, by following the analysis from Eq.(4.5.9) to Eq.(4.5.11a), it can be shown that the ion heating rate is given by

$$\frac{3}{2}\frac{\partial T_i}{\partial t} = \left(\frac{e^2}{2m_i}\right) \sum_k \frac{k^2}{(\omega_k^r)^2}\frac{\partial}{\partial t} |\phi_k|^2$$

i.e.,

$$\frac{\partial T_i}{\partial t} = \left(\frac{e^2}{3m_i}\right) \sum_k \frac{(1 + k^2\lambda_D^2)}{C_s^2} \frac{\partial}{\partial t} |\phi_k|^2 \tag{4.7.3}$$

This differs by the factor (1/3) from the result of LAMPE *et al* [53].

For $k^2\lambda_D^2 << 1$ and $V_o >> C_s$, as is common in practice, from Eqs.(4.7.2) and (4.7.3) we have

$$\frac{\dfrac{\partial T_e}{\partial t}}{\dfrac{\partial T_i}{\partial t}} = \frac{\left(\dfrac{2e^2 m_i^{1/2}}{3}\right)\dfrac{V_o}{T_e^{3/2}}}{\left(\dfrac{e^2}{3m_i}\right)\dfrac{1}{C_s^2}} = \frac{2V_o}{C_s}$$

In the same limits Eq.(4.5.2) for the ion acoustic instability reduces to

$$\frac{\partial T_e/\partial t}{\partial T_i/\partial t} \simeq \frac{2\,\vec{k}\cdot\vec{V}_o}{\omega_k^r}$$

$$\simeq \frac{2V_o}{C_s}$$

for $k_z \ll k$ and $\vec{k}_\perp \parallel \vec{V}_o$.

Thus we conclude that the reactive ECDI propagating across the magnetic field ($k_z = 0$) and the ion acoustic instability propagating slightly off the perpendicular ($k_z \ll k$) provide the same relative electron and ion heating rates. This is not surprising, since for wave propagation off the perpendicular to $\vec{B}$ the electron Berstein modes are severely damped and the ECDI transforms into just the positive energy ion acoustic mode [24, 53].

CHAPTER FIVE

THE CROSSFIELD CURRENT-DRIVEN ION ACOUSTIC INSTABILITY IN A PLASMA WITH A DENSITY GRADIENT IN A SHEARED MAGNETIC FIELD

5.1 INTRODUCTION

If, instead of a constant field in the z direction, the equilibrium magnetic field $\vec{B}_o$ is chosen to be

$$\vec{B}_o = B_{oz}(x)\,\hat{z} + B_{oy}(x)\,\hat{y} \tag{5.1.1}$$

then the lines of force of the field remain straight. However, because of the x dependence of B_{oz} and B_{oy} the lines are not parallel to each other; the direction of the line of force is, in general, a function of the coordinate x. Thus, the field (5.1.1) is an example of a field with straight but non-parallel lines of force. It will be seen in the next section that such a field leads to a coordinate dependent $k_{\parallel}$(component of the wave vector $\vec{k}$ along the magnetic field $\vec{B}_o$). Magnetic fields giving rise to such a behaviour of $k_{\parallel}$ are called sheared magnetic fields.

Most of the study on the effect of magnetic shear on plasma instabilities has been concentrated on drift waves. This is probably due to the broad range of plasma conditions under which such modes are unstable. Since they are driven by inhomogeneities within the plasma, e.g., a density gradient, drift waves are easily excited in most high temperature plasma devices. As far as is known, very little evidence exists of theoretical studies of the ion acoustic instability in a sheared magnetic field. A study of the crossfield current-driven ion acoustic instability in a plasma with an electron density gradient in a magnetic field of this nature is the subject of the present chapter.

5.2 ELECTRON CONTRIBUTION TO THE DISPERSION RELATION

We consider a model in which the magnetic field exhibits shear in the x direction. Thus, we write[69, 70]

$$\vec{B} = B_{oz}\,\hat{z} + B_{oy}\,\hat{y}$$
$$= B_o\{\,\hat{z} + (x/L_s)\,\hat{y}\,\} \tag{5.2.1}$$

with $B_{oz} >> B_{oy}$, where $L_s = \left[\frac{1}{B}\ \frac{d\,B_{oy}}{dx}\right]^{-1}$, called the shear length, is the characteristic length over which the magnetic field changes direction. The electron density is also assumed to vary in the x direction, with

$$n_e = n_o(1 + \varepsilon\, x) \tag{5.2.2}$$

where $\varepsilon^{-1} = \left[\frac{1}{n_e}\ \frac{dn_e}{dx}\right]^{-1} = L_n$ is the density gradient scale length.

As in Chapter Three, we assume that the length and time scales are such that the electrons are magnetized and the ions are not. Retaining the additional approximation that the ions do not react to the electric field $\vec{E}_o$, it turns out, once again, that the electrons have an $\vec{E} \times \vec{B}$ drift $\vec{V}_o = C(\vec{E}_o \times \vec{B})/B^2$ relative to the ions. In examining the motion of the electrons we shall closely follow the mathematical formalism presented by DAVIDSON and KAMMASH[71].

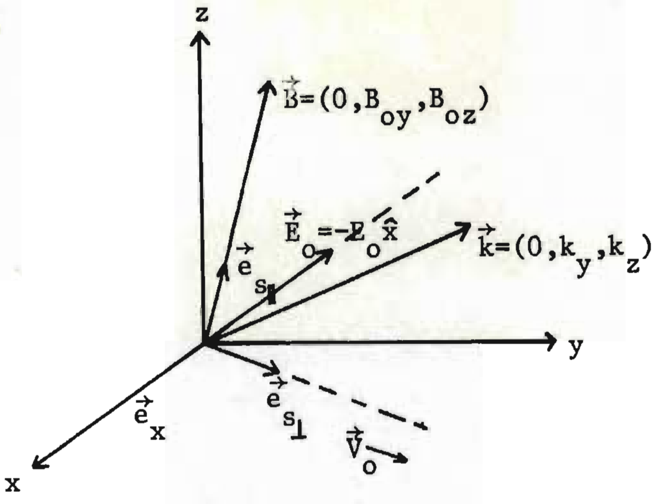


Figure 5.1

The geometry considered is shown in Fig. 5.1 above, where, in addition to the sheared magnetic field, there is a constant external electric field in the negative x direction. The investigation is conducted in the ion rest frame. In the absence of shear one normally works in the (x, y, z) cartesian coordinate system as shown above. However, to simplify the calculations of the unperturbed particle orbits in a sheared field we introduce a rotating coordinate system ($\vec{e}_x$, $\vec{e}_{s_\parallel}(x)$, $\vec{e}_{s_\perp}(x)$) defined by

$$\vec{e}_x = \hat{x}$$

$$\vec{e}_{s_\parallel}(x) = \frac{\vec{B}(x)}{|\vec{B}|} \qquad (5.2.3)$$

$$\vec{e}_{s_\perp}(x) = \vec{e}_{s_\parallel} \times \vec{e}_x$$

From Fig. 5.1 we see that

$$\vec{e}_{s_\parallel} = h_y\,\hat{y} + h_z\,\hat{z}$$

$$\vec{e}_{s_\perp} = h_z\,\hat{y} - h_y\,\hat{z}$$

where in general, (5.2.4)

$$h_y(x) = \frac{B_{oy}(x)}{B}\ , \quad h_z(x) = \frac{B_{oz}(x)}{B}$$

Since the plasma density and magnetic shear are chosen to have a spatial dependence on x only, the perturbation potential is taken to be of the form

$$\phi_1(\vec{r},t) = \phi_1(x) \exp\{ i[\vec{k}.\vec{r} - \omega_k t]\} \quad (5.2.5a)$$

where

$$\begin{aligned}\vec{k} &= k_z\hat{z} + k_y\hat{y} \\ &= k_{\parallel}\vec{e}_{s_{\parallel}} + k_{\perp}\vec{e}_{s_{\perp}}\end{aligned} \quad (5.2.5b)$$

In contrast to the shearless case $\vec{B}_o \parallel \hat{z}$, the wave numbers k_z and k_y no longer represent the components of $\vec{k}$ along and perpendicular to the magnetic field respectively. These roles are now played by the quantities $k_{\parallel}$ and $k_{\perp}$, From Eqs. (5.2.4) and (5.2.5b) these are given by

$$\begin{aligned}k_{\parallel}(x) &= \vec{k}.\vec{e}_{s_{\parallel}} = k_y h_y(x) + k_z h_z(x) \\ k_{\perp}(x) &= \vec{k}.\vec{e}_{s_{\perp}} = k_y h_z(x) - k_z h_y(x)\end{aligned} \quad (5.2.5c)$$

In the stationary coordinate system ($\hat{x}$, $\hat{y}$, $\hat{z}$) the position vector of a particle is given by $\vec{r}(t) = x(t)\,\hat{x} + y(t)\,\hat{y} + z(t)\,\hat{z}$, while in the rotating frame, $\vec{r}(t) = x(t)\,\vec{e}_x + s_{\perp}(t)\,\vec{e}_{s_{\perp}} + s_{\parallel}(t)\,\vec{e}_{s_{\parallel}}$.

In solving the electron equation of motion

$$m_e\ddot{\vec{r}} = -e\{\vec{E}_o + (\vec{V} \times \vec{B})/C\} \quad (5.2.6)$$

it is simpler to calculate x(t), $s_{\perp}(t)$ and $s_{\parallel}(t)$, instead of the more complicated solutions x(t), y(t) and z(t). In doing so, we shall assume that a single electron is not affected by the shear, but that its motion is determined by a constant local field. Thus, in terms of typical lengths, we assume

$$r_e \ll L_s \quad (5.2.7a)$$

where r_e is the electron gyroradius.

In the rotating coordinate system

$$\vec{V}_o = \frac{cE_o}{B} \vec{e}_{s_\perp} \qquad (5.2.7b)$$

and the density gradient (5.2.2) produces an electron diamagnetic drift[1, p.422]

$$\vec{V}_n = - \frac{\varepsilon\, T_e}{e\, B} \vec{e}_{s_\perp}$$

$$= \frac{\varepsilon\, T_e}{m_e \Omega_e} \vec{e}_{s_\perp} = \frac{\varepsilon\, C_e^2}{\Omega_e} \vec{e}_{s_\perp} \qquad (5.2.7c)$$

where, as previously defined, $\Omega_e = (-eB/m_e C)$ is the electron gyrofrequency. A set of solutions to Eq. (5.2.6) is (cf. Eq. (3.2.8))

$$\begin{aligned} x(t') &= x(t) - (V_\perp/\Omega_e) \{ \mathrm{Sin} [\theta(t) - \Omega_e(t'-t)] - \mathrm{Sin}\, \theta(t)\} \\ s_\perp(t') &= s_\perp(t) + (V_\perp/\Omega_e) \{\mathrm{Cos}[\theta(t) - \Omega_e(t'-t)] - \mathrm{Cos}\theta(t)\} + V_o(t'-t) \qquad (5.2.8) \\ s_\parallel(t') &= s_\parallel(t) + V_\parallel\, (t'-t) \end{aligned}$$

where $V_\perp^{\;2} = V_x^{\;2} + (V_{s_\perp} - V_o)^2$, $\theta(t) = \tan^{-1} \{ (V_{s_\perp}(t) - V_o)/V_x(t) \}$ and $V_\parallel = \vec{V}.\vec{e}_{s_\parallel}$.

From the Eqs. (5.2.8) we may construct, among others, the following constants of motion (as in Section 3.5)

$$V_\perp^{\;2},\ V_\parallel \quad \text{and } X = x + \frac{(V_{s_\perp} - V_o)}{\Omega_e}$$

Then from Eqs. (5.2.5) and (5.2.8)

$$\phi_1(\vec{r}',t') = \phi_1(x') \exp \{i\vec{k}(x').\vec{r}(t') - i\omega_k t'\}$$

$$= \phi_1(x') \exp \{i\vec{k}(x).\vec{r}(t) - i\omega_k t\} \exp\left\{\frac{ik_\perp(x)V_\perp}{\Omega_e} [\mathrm{Cos}\{\theta - \Omega_e(t'-t)\} - \mathrm{Cos}\theta]\right\}$$

$$\times \exp \{- i\, [\omega_k - k_\perp(x)V_o - k_\parallel(x)V_\parallel\,]\, (t'-t)\} \qquad (5.2.9)$$

where it has been assumed that $k_\perp$ and $k_\parallel$ depend on x rather than x'. This is reasonable since the difference is of the order of (r_e/L_s), which by assumption (5.2.7a) is negligibly small.

The equilibrium velocity distribution for the electrons is chosen to be a function of the above constants of motion, viz.,

$$f_{oe} = f_{oe}\,(V_\perp^2,\ V_\parallel\ ,\ X)$$
$$= \frac{n_o}{(2\pi C_e^2)^{3/2}}\left[1 + \varepsilon\left\{x + \frac{(V_{s\perp} - V_o)}{\Omega_e}\right\}\right]\exp\left[-\frac{[V_x^2+(V_{s\perp}-V_o)^2+V_\parallel^2]}{2C_e^2}\right] \tag{5.2.10}$$

We can readily show that the above distribution is self-consistent. It is, in fact, the form to which the distribution (3.5.2) reduces in the absence of a temperature gradient.

Following the technique adopted in Chapter Three, we may express the electron and ion distribution functions and the electric field as sums of their respective equilibrium values and a perturbation term due to the presence of oscillations. Eq. (3.1.11) then gives us the perturbed distribution $f_{ij}\ (\vec{r},\ \vec{V},\ t)$ in terms of the perturbation potential ϕ_1 and the equilibrium distribution f_{oj}, viz.,

$$f_{ij}(\vec{r},\vec{V},t) = \frac{e_j}{m_j}\int_{-\infty}^{t} \nabla\phi_1(\vec{r}\,',t').\ \frac{\partial f_{oj}}{\partial \vec{V}'}\ dt'$$

with Poisson's equation

$$\nabla^2\phi_1 = -4\,\pi\ \sum_j n_j\ e_j$$

$$=-4\pi\ \sum_j\ \frac{e_j^2}{m_j}\int d\vec{V}\int_{-\infty}^{t}\nabla\phi_1(\vec{r}\,',t').\ \frac{\partial f_{oj}}{\partial \vec{V}'}\,dt' \qquad (j = i,e)$$

For perturbations of the form (5.2.5a), this reduces to

$$\left(\frac{\partial^2}{\partial x^2} - k^2\right) \phi_1(\vec{r},t) = -4\pi \sum_j \frac{e_j^2}{m_j} \int d\vec{V} \int_{-\infty}^{t} \nabla\phi_1(\vec{r}',t') \cdot \frac{\partial f_{oj}}{\partial \vec{V}'} dt' \qquad (5.2.11)$$

where $k^2 = k_\parallel^{\,2} + k_\perp^{\,2} = k_y^{\,2} + k_z^{\,2}$.

For $f_{oe} = f_{oe}(V_\perp^{\,2}, V_\parallel, X)$ and $\vec{V}_\perp = V_x \vec{e}_x + (V_{s_\perp} - V_o)\vec{e}_{s_\perp}$, it can be shown that

$$\frac{\partial f_{oe}}{\partial \vec{V}} = \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\vec{V}_\perp + \frac{1}{\Omega_e}\frac{\partial f_{oe}}{\partial X}\vec{e}_{s_\perp} + \frac{\partial f_{oe}}{\partial V_\parallel}\vec{e}_{s_\parallel}$$

and

$$\nabla\phi_1 \cdot \frac{\partial f_{oe}}{\partial \vec{V}} = \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\nabla\phi_1 \cdot \vec{V}_\perp + i\frac{k_\perp}{\Omega_e}\frac{\partial f_{oe}}{\partial X}\phi_1 + i k_\parallel \frac{\partial f_{oe}}{\partial V_\parallel}\phi_1$$

Furthermore,

$$\begin{aligned}\frac{d\phi_1(\vec{r},t)}{dt} &= \frac{\partial \phi_1}{\partial t} + \vec{V} \cdot \nabla\phi_1 \\ &= \nabla\phi_1 \cdot \vec{V}_\perp - i(\omega_k - k_\perp V_o - k_\parallel V_\parallel)\phi_1\end{aligned}$$

Combining these results, we have

$$\nabla\phi_1 \cdot \frac{\partial f_{oe}}{\partial \vec{V}} = \frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\frac{d\phi_1}{dt} + i\left\{(\omega_k - k_\perp V_o - k_\parallel V_\parallel)\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\right.$$

$$\left. + \frac{k_\perp}{\Omega_e}\frac{\partial f_{oe}}{\partial X} + k_\parallel \frac{\partial f_{oe}}{\partial V_\parallel}\right\}\phi_1(r,t)$$

Thus, in the electron contribution to the right hand side of Eq. (5.2.11), viz.,

$$-\frac{4\pi e^2}{m_e}\int d\vec{V}\int_{-\infty}^{t} \nabla\phi_1(\vec{r}',t') \cdot \frac{\partial f_{oe}}{\partial \vec{V}'} dt' \qquad (5.2.12a)$$

the integral with respect to time reduces to

$$\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\phi_1(\vec{r},t) + i\left\{(\omega_k - k_\perp V_o - k_\parallel V_\parallel)\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} + \frac{k_\perp}{\Omega_e}\frac{\partial f_{oe}}{\partial X} + k_\parallel\frac{\partial f_{oe}}{\partial V_\parallel}\right\}$$

$$\times \int_{-\infty}^{t} \phi_1(\vec{r}\,',t')\,dt' \qquad (5.2.12b)$$

with $\phi_1(\vec{r}\,',t')$ given by Eq. (5.2.9). The unknown function $\phi_1(x')$ in the said equation is now expanded about $x' = x$ as follows:

$$\phi_1(x') = \phi_1(x) + (x'-x)\left.\frac{d\phi_1(x')}{dx'}\right|_{x'=x} + \frac{(x'-x)^2}{2}\left.\frac{d^2\phi_1(x')}{dx'^2}\right|_{x'=x} + \ldots$$

$$(5.2.13)$$

With the aid of Eqs. (5.2.9) and (5.2.13)we see that

$$\int_{-\infty}^{t}\phi_1(\vec{r}\,',t')\,dt' = \int_{-\infty}^{t}\left[\phi_1(\vec{r},t) + (x'-x)\frac{\partial\phi_1(\vec{r},t)}{\partial x} + \frac{(x'-x)^2}{2}\frac{\partial^2\phi_1(\vec{r},t)}{\partial x^2}\right]$$

$$\times \exp\left\{\frac{ik_\perp V_\perp}{\Omega_e}[\mathrm{Cos}\{\theta-\Omega_e(t'-t)\}-\mathrm{Cos}\theta]\right\}\exp\{-i(\omega_k - k_\perp V_o - k_\parallel V_\parallel)(t'-t)\}\,dt'$$

$$(5.2.14)$$

With the aid of the identities[60]

$$\exp\left\{i\frac{k_\perp V_\perp}{\Omega_e}\mathrm{Cos}\{\phi - \Omega_e(t'-t)\}\right\} = \sum_{n=-\infty}^{+\infty} i^n J_n(\xi)\exp\{-in[\phi - \Omega_e(t'-t)]\}$$

and

$$\exp\left\{-i\frac{k_\perp V_\perp}{\Omega_e}\mathrm{Cos}\,\phi\right\} = \sum_{n=-\infty}^{+\infty} i^{-n} J_n(\xi)\exp(i\,n\,\phi)$$

where, as previously defined, $\xi = k_\perp V_\perp/\Omega_e$, Eq. (5.2.14) permits us to write the electron term (5.2.12a) as

$$-\frac{4\pi e^2}{m_e}\left[\int\!\!\int V_\perp\,dV_\perp\,dV_\parallel\,d\theta\,\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp}\phi_1(\vec{r},t)\right.$$

$$+ \; i \int V_\perp \, dV_\perp \, dV_\parallel d\theta \left\{ (\bar{\omega}_k - k_\parallel V_\parallel) \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} + \frac{k_\perp}{\Omega_e} \frac{\partial f_{oe}}{\partial X} + k_\parallel \frac{\partial f_{oe}}{\partial V_\parallel} \right\}$$

$$\times \sum_n \sum_m J_n(\xi) \, J_m(\xi) \left\{ \int_{-\infty}^{t} \left(\phi_1(\vec{r},t) + (x'-x) \frac{\partial \phi_1(\vec{r},t)}{\partial x} + \frac{(x'-x)^2}{2} \frac{\partial^2 \phi_1(\vec{r},t)}{\partial x^2} \right) \right.$$

$$\left. \times \; i^{(n-m)} \, e^{i(m-n)\theta} \exp\{-i(\bar{\omega}_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)\} \, dt' \right\} \Bigg] \quad (5.2.15a)$$

where

$$\bar{\omega}_k = \omega_k - k_\perp V_o \qquad (5.2.15b)$$

For $f_{oe}(V_\perp^{\;2}, V_\parallel, X)$ as defined in Eq. (5.2.10),

$$\frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} = - \frac{n_o(1+\varepsilon x)}{C_e^{\;2}(2\pi C_e^{\;2})^{3/2}} \exp\left\{\frac{-V^2}{2C_e^{\;2}}\right\} + \frac{\mathrm{Sin}\theta}{(2\pi C_e^{\;2})^{3/2}} \left[\frac{n_o \varepsilon}{\Omega_e V_\perp} \left(1 - \frac{V_\perp^{\;2}}{C_e^{\;2}}\right) \exp\left\{\frac{-V^2}{2C_e^{\;2}}\right\} \right]$$

(5.2.16a)

$$\frac{\partial f_{oe}}{\partial X} = \frac{n_o \varepsilon}{(2\pi C_e^{\;2})^{3/2}} \exp\left\{\frac{-V^2}{2C_e^{\;2}}\right\} \qquad (5.2.16b)$$

$$\frac{\partial f_{oe}}{\partial V_\parallel} = \frac{-n_o(1+\varepsilon x) V_\parallel}{C_e^{\;2}(2\pi C_e^{\;2})^{3/2}} \exp\left[\frac{-V^2}{2C_e^{\;2}}\right] - \frac{n_o \varepsilon \, V_\parallel \, V_\perp \, \mathrm{Sin}\phi}{C_e^{\;2} \, \Omega_e \, (2\pi C_e^{\;2})^{3/2}} \exp\left\{\frac{-V^2}{2C_e^{\;2}}\right\}$$

(5.2.16c)

Then

$$(\bar{\omega}_k - k_\parallel V_\parallel) \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} + \frac{k_\perp}{\Omega_e} \frac{\partial f_{oe}}{\partial X} + k_\parallel \frac{\partial f_{oe}}{\partial V_\parallel}$$

$$= - \frac{\bar{\omega}_k n_o (1+\varepsilon x)}{C_e^{\;2}(2\pi C_e^{\;2})^{3/2}} \exp\left\{\frac{-V^2}{2C_e^{\;2}}\right\} + \frac{\varepsilon k_\perp n_o}{\Omega_e (2\pi C_e^{\;2})^{3/2}} \exp\left\{\frac{-V^2}{2C_e^{\;2}}\right\}$$

$$+ \frac{(\bar{\omega}_k - k_\parallel V_\parallel)\varepsilon n_o}{(2\pi C_e^2)^{3/2}\Omega_e V_\perp} \operatorname{Sin}\theta \exp\left\{\frac{-v^2}{2C_e^2}\right\} - \frac{\bar{\omega}_k}{(2\pi C_e^2)^{3/2}} \frac{\varepsilon n_o}{\Omega_e} \frac{V_\perp \operatorname{Sin}\theta}{C_e^2} \exp\left\{\frac{-v^2}{2C_e^2}\right\} \tag{5.2.16d}$$

With the aid of Eq. (5.2.16a) it turns out that

$$\int V_\perp dV_\perp dV_\parallel d\theta \frac{1}{V_\perp} \frac{\partial f_{oe}}{\partial V_\perp} \phi_1(\vec{r},t)$$

$$= \frac{-n_o(1+\varepsilon x)\ \phi_1(\vec{r},t)}{C_e^2} \tag{5.2.17}$$

We notice that if we substitute for (x'-x) from Eq. (5.2.8) and use Eq. (5.2.16d), then the second term within [- -] in Eq. (5.2.15a) involves integrals of the type

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \int_0^{2\pi} d\phi \int_{-\infty}^{t} \exp\{-i(\bar{\omega}_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)\} \exp\{i\ h(n,m)\phi\} dt'$$

provided we express the sines and cosines in terms of exponentials. All such integrals which are necessary to manipulate the said term completely are explicity shown in Appendix E.

With the result (5.2.17) and the integrals in Appendix E, Eq. (5.2.15a) reduces to

$$\frac{-4\pi_e^2}{m_e}\left[-\frac{n_o(1+\varepsilon x)}{C_e^2}\phi_1(\vec{r},t) + i\int \frac{V_\perp dV_\perp dV_\parallel}{(2\pi C_e^2)^{3/2}}\left\{\phi_1(\vec{r},t)\left[\left(\frac{-\bar{\omega}_k n_o(1+\varepsilon x)}{C_e^2} + \frac{\varepsilon k_\perp n_o}{\Omega_e}\right)\right.\right.\right.$$

$$\times \sum_p \frac{2\pi J_p^2(\xi) \exp(-v^2/2C_e^2)}{\{-i(\bar{\omega}_k - k_\parallel V_\parallel - p\,\Omega_e)\}} + \left(\frac{(\bar{\omega}_k - k_\parallel V_\parallel)\ \varepsilon n_o}{\Omega_e V_\perp} - \frac{\bar{\omega}_k \varepsilon n_o}{C_e^2 \Omega_e} V_\perp\right)$$

$$\times \sum_p \frac{2\pi p\, J_p^{\,2}(\xi)\exp(-V^2/2C_e^{\,2})}{\xi\{-i(\bar{\omega}_k - k_\parallel V_\parallel - p\Omega_e\}}\Bigg] + \frac{\partial\phi_1(\vec{r},t)}{\partial x}\left[\left(\frac{(\bar{\omega}_k - k_\parallel V_\parallel)\varepsilon n_o}{\Omega_e V_\perp} - \frac{\bar{\omega}_k}{C_e^{\,2}}\,\frac{\varepsilon n_o}{\Omega_e}V_\perp\right)\right.$$

$$\left.\times \frac{1}{2}\sum_p \frac{2\pi\Omega_e V_\perp}{k_\perp^{\,2}}\,\frac{\exp(-V^2/2C_e^{\,2})}{\{i(\bar{\omega}_k - k_\parallel V_\parallel - p\Omega_e)\}}\,\frac{1}{V_\perp}\,\frac{\partial}{\partial V_\perp}\{J_p^{\,2}(\xi)\}\right]$$

$$+\frac{1}{2}\,\frac{\partial^2\phi_1(\vec{r},t)}{\partial x^2}\left[\left(\frac{-\bar{\omega}_k\, n_o(1+\varepsilon x)}{C_e^{\,2}} + \frac{\varepsilon k_\perp n_o}{\Omega_e}\right)\sum_p \frac{2\pi}{k_\perp^{\,2}}\,\frac{\exp(-V^2/2C_e^{\,2})}{\{i(\bar{\omega}_k - k_\parallel V_\parallel - p\Omega_e)\}}\,\frac{1}{V_\perp}\,\frac{\partial}{\partial V_\perp}\{J_p^2(\xi)\}\right.$$

$$+\frac{1}{2}\left(\frac{\bar{\omega}_k - k_\parallel V_\parallel)\,\varepsilon n_o}{\Omega_e V_\perp} - \frac{\bar{\omega}_k}{C_e^{\,2}}\,\frac{\varepsilon n_o}{\Omega_e}V_\perp\right)\sum_p \frac{2\pi}{k_\perp^{\,2}}\,\frac{\exp(-V^2/2C_e^{\,2})}{\{-i(\bar{\omega}_k - k_\parallel V_\parallel - p\Omega_e)\}}$$

$$\left.\times\left\{-\frac{p\Omega_e}{k_\perp}\,\frac{\partial}{\partial V_\perp}\{J_p^2(\xi)\} + \frac{2p\Omega_e}{k_\perp V_\perp}J_p^2(\xi)\right\}\right]\Bigg\}\Bigg]$$

As in Section 3.2, we restrict ourselves to low frequency modes for which $|\bar{\omega}_k - k_\parallel V_\parallel| \ll \Omega_e$. Therefore, only the p = o term is retained in the summations above.

We then use the results

$$\int_o^\infty J_o^{\,2}(k_\perp V_\perp/\Omega_e)\exp(-V_\perp^{\,2}/2C_e^{\,2})\,V_\perp\,dV_\perp \int_{-\infty}^{+\infty}\frac{\exp(-V_\parallel^{\,2}/2C_e^{\,2})}{(\bar{\omega}_k - k_\parallel V_\parallel)}\,dV_\parallel$$

$$= -\,C_e^{\,2}\,e^{-b}\,I_o(b)(\pi^{\frac{1}{2}}/k_\parallel)\,Z(\lambda_e)$$

where $b = k_\perp^{\,2}C_e^{\,2}/\Omega_e^{\,2}$, $\lambda_e = \bar{\omega}_k/\sqrt{2}\,k_\parallel C_e$, $I_o(b)$ has already been defined as the modified Bessel function of the first kind of order zero, and $Z(\lambda)$ is the plasma dispersion function as defined by the equations (3.5.4),

$$\int_{-\infty}^{+\infty} \exp\left(-V_\perp{}^2/2C_e{}^2\right) V_\perp \frac{\partial}{\partial V_\perp} \{J_o{}^2(k_\perp V_\perp/\Omega_e)\} V_\perp \, dV_\perp \int_{-\infty}^{+\infty} \frac{\exp(-V_\parallel{}^2/2C_e{}^2)}{(\bar{\omega}_k - k_\parallel V_\parallel)} dV_\parallel$$

$$= 2\, C_e{}^2\, b\, e^{-b} \left[I_o(b) - I_o'(b)\right] \; (\pi^{\frac{1}{2}}/k_\parallel)\, Z(\lambda_e)$$

and

$$\int_o^{\infty} \exp(-V_\perp{}^2/2C_e{}^2) \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp} \{J_o{}^2(k_\perp V_\perp/\Omega_e)\} V_\perp \, dV_\perp \int_{-\infty}^{+\infty} \exp(-V_\parallel{}^2/2C_e{}^2)\, dV_\parallel$$

$$= (2\pi C_e{}^2)^{\frac{1}{2}} \left[e^{-b} I_o(b) - 1\right]$$

to write the above expression as

$$\frac{-4\pi e^2}{m_e}\left[\frac{-n_o(1+\epsilon x)}{C_e{}^2}\,\phi_1(\vec{r},t) + \frac{2\pi\, n_o}{C_e{}^2(2\pi C_e{}^2)^{3/2}}\left\{\phi_1(\vec{r},t)\left[\left(-\bar{\omega}_k(1+\epsilon x) + \frac{k_\perp \epsilon C_e{}^2}{\Omega_e}\right)\right.\right.\right.$$

$$\left.\times \{C_e{}^2 \Gamma_o(b)(\pi^{\frac{1}{2}}/k_\parallel)\, Z(\lambda_e)\}\right] + \frac{\partial \phi_1(\vec{r},t)}{\partial x}\left[\epsilon\, C_e{}^2 \frac{1}{2k_\perp{}^2}\{(2\pi C_e{}^2)^{\frac{1}{2}}(\Gamma_o(b)-1)\}\right.$$

$$\left. - \bar{\omega}_k\, \epsilon \frac{1}{2k_\perp{}^2}\; \{2C_e{}^2\, b\left[\Gamma_o(b) - \Gamma_1(b)\right] \; (\pi^{\frac{1}{2}}/k_\parallel)\, Z(\lambda_e)\}\right]$$

$$\left.\left. + \frac{1}{2}\frac{\partial^2 \phi_1(\vec{r},t)}{\partial x^2}\left[\left(-\bar{\omega}_k(1+\epsilon x) + \frac{k_\perp \epsilon C_e{}^2}{\Omega_e}\right)\frac{1}{k_\perp{}^2}\{2C_e{}^2 b\left[\Gamma_o(b) - \Gamma_1(b)\right] \; (\pi^{\frac{1}{2}}/k_\parallel)\, Z(\lambda_e)\}\right]\right\}\right]$$

where we have used the result [(60)] $I_o'(b) = (dI_o(b)/db) = I_1(b)$ and have written $\Gamma_n(b) = e^{-b} I_n(b)$. This may be further reduced to

$$\frac{-4\pi\, n_o\, e^2}{m_e C_e^{\,2}} \Bigg[\phi_1(\vec{r},t) \Big\{ -(1+\varepsilon x) - \Big[\frac{\bar{\omega}_k}{\sqrt{2}k_{\parallel}C_e} (1+\varepsilon x) - \frac{k_{\perp}}{\sqrt{2}k_{\parallel}C_e} \Big(\frac{\varepsilon C_e}{\Omega_e} \Big)^2 \Big] \Gamma_o(b)\, Z(\lambda_e) \Big\}$$

$$+ \frac{\varepsilon}{2k_{\perp}^{\,2}} \frac{\partial \phi_1(\vec{r},t)}{\partial x} \Big\{ \big[\Gamma_o(b)-1\big] - \frac{\bar{\omega}_k}{\sqrt{2}k_{\parallel}C_e} \{ 2b \big[\Gamma_o(b) - \Gamma_1(b)\big]\; Z(\lambda_e) \} \Big\}$$

$$- \frac{1}{k_{\perp}^{\,2}} \frac{\partial^2 \phi_1(\vec{r},t)}{\partial x^2} \Big\{ \Big[\frac{\bar{\omega}_k(1+\varepsilon x)}{\sqrt{2}k_{\parallel}C_e} - \frac{k_{\perp}}{\sqrt{2}k_{\parallel}C_e} \Big(\frac{\varepsilon C_e}{\Omega_e} \Big)^2 \Big] b\, [\Gamma_o(b) - \Gamma_1(b)]\; Z(\lambda_e) \Big\} \Bigg]$$

(5.2.18)

5.3 THE ION TERM

It has already been assumed that, due to their inertia, the ions react to neither the magnetic field $\vec{B}$ nor the electric field $\vec{E}_o$. They assume straight line orbits with $\dot{\vec{r}} = \vec{V} = $ constant.

Since the ions are unmagnetized, it is not expected that their motion will be affected by the magnetic shear. In other words, if an ion does not perceive the original field $\vec{B}_o$ at x = o, then one does not expect it to sense the rotation of the $\vec{B}$ field in the y-z plane as it moves on its straight line trajectory.

An alternative view point arises from the fact that we are interested in the ion acoustic instability with frequency $\omega \sim kC_s$ in the limit $T_e >> T_i$. Then

$$(T_i/m_i)^{\frac{1}{2}} = C_i << C_s = (T_e/m_i)^{\frac{1}{2}} << (T_e/m_e)^{\frac{1}{2}} = C_e$$

i.e.,

$$C_i << \omega/k << C_e$$

or

$$t'C_i \ll \lambda \ll t'C_e$$

where t' is the period and λ the wavelength of the wave. Therefore, during one wave period the distance travelled by a typical ion is very small compared to the wavelength, which in turn, is much less than the distance traversed by a typical electron. We thus conclude that over the time scales involved, for the ions the distance $(x'-x)$ in expansion (5.2.13) is negligibly small and $\phi(x') \simeq \phi(x)$.

The equilibrium ion distribution is chosen to a stationary Maxwellian, viz.,

$$f_{oi}(V_\perp^{\,2}, V_\parallel) = n_o(2\pi C_i^{\,2})^{-3/2} \exp\{-(V_\perp^{\,2}+V_\parallel^{\,2})/2C_i^{\,2}\} \qquad (5.3.1)$$

with $V_\perp^{\,2} = V_x^{\,2} + V_{s_\perp}^{\,2}$. It is seen from the equations (5.2.12) that the ion contribution to the right hand side of Eq. (5.2.11) is

$$\frac{-4\pi e^2}{m_i}\int d\vec{V}\left[\frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp}\phi_1(\vec{r},t) + i\left\{\omega_k \frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp} + k_\parallel V_\parallel\left(\frac{1}{V_\parallel}\frac{\partial f_{oi}}{\partial V_\parallel} - \frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp}\right)\right\} \times \int_{-\infty}^{t}\phi_1(\vec{r}',t')dt'\right]$$

In the light of the above discussion, and by definition (5.2.5a) for $\phi_1(\vec{r},t)$, this reduces to

$$\frac{-4\pi e^2}{m_i}\phi_1(\vec{r},t)\int d\vec{V}\left[\frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp} + i\left\{\omega_k \frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp} + k_\parallel V_\parallel\left(\frac{1}{V_\parallel}\frac{\partial f_{oi}}{\partial V_\parallel} - \frac{1}{V_\perp}\frac{\partial f_{oi}}{\partial V_\perp}\right)\right\} \times \int_{-\infty}^{t}\exp\{i\left[\vec{k}.(\vec{r}(t')-\vec{r}(t))-\omega_k(t'-t)\right]\}\,dt'\right]$$

With the aid of the solution

$$\vec{r}(t') = \vec{r}(t) + \vec{V}(t'-t)$$

to the ion equation of motion $\ddot{\vec{r}} = \vec{0}$, the time integral above is manipulated to yield

$$\frac{-4\pi e^2}{m_i} \phi_1(\vec{r},t) \int d\vec{V} \left[\frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} - (\omega_k - \vec{k}.\vec{V})^{-1} \left\{ \omega_k \frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} \right.\right.$$

$$\left.\left. + k_\parallel V_\parallel \left(\frac{1}{V_\parallel} \frac{\partial f_{oi}}{\partial V_\parallel} - \frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp}\right)\right\}\right] \tag{5.3.2}$$

For the velocity distribution (5.3.1),

$$\int \frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} d\vec{V} = \frac{-n_o}{(2\pi C_i^{\,2})^{3/2}} \frac{1}{C_i^{\,2}} \int \exp\{-(V_\perp^{\,2} + V_\parallel^{\,2})/2C_i^{\,2}\} \; V_\perp \, dV_\perp \, dV_\parallel \, d\theta$$

$$= \frac{-n_o}{C_i^{\,2}}$$

$$\frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} - \frac{1}{V_\parallel} \frac{\partial f_{oi}}{\partial V_\parallel} = 0$$

$$\int \frac{\omega_k}{(\omega_k - \vec{k}.\vec{V})} \frac{1}{V_\perp} \frac{\partial f_{oi}}{\partial V_\perp} d\vec{V} = \frac{n_o \lambda_i}{C_i^{\,2}} \left[Z(\lambda_i) + \frac{k_\perp^{\,2}}{4k_\parallel^{\,2}} \frac{d^2 Z(\lambda_i)}{d\lambda_i^{\,2}} \right]$$

where $\lambda_i = \omega_k / \sqrt{2} k_{\parallel} C_i$.

The techniques employed in arriving at these results are the same as those used in Section 3.3. For the last result we have approximated

$$\left[1 - \frac{k_{\perp}^2 V_{\perp}^2}{(\omega_k - k_{\parallel} V_{\parallel})^2} \right]^{-\frac{1}{2}} \simeq 1 + \frac{k_{\perp}^2 V_{\perp}^2}{2(\omega_k - k_{\parallel} V_{\parallel})^2}$$

which for the ion acoustic wave is physically plausible (see discussion following Eq. (3.3.8)).

The ion term (5.3.2) then reduces to

$$\frac{4\pi n_o e^2}{m_i C_i^2} \phi_1(\vec{r},t) \left[1 + \lambda_i \left\{ Z(\lambda_i) + \frac{k_{\perp}^2}{4k_{\parallel}^2} \frac{d^2 Z(\lambda_i)}{d\lambda_i^2} \right\} \right] \quad (5.3.3)$$

5.4 THE EIGENVALUE EQUATION

From Eq. (5.2.18) for the electrons and Eq. (5.3.3) for the ions, Eq. (5.2.11) may be written as

$$\frac{1}{k_{\perp}^2} \left[\frac{k_{\perp}^2}{k^2} - \frac{k_D^2}{k^2} \left(\frac{\bar{\omega}_k (1+\varepsilon x)}{\sqrt{2}\, k_{\parallel} C_e} + \frac{k_{\perp} V_n}{\sqrt{2} k_{\parallel} C_e} \right) b\{\Gamma_o(b) - \Gamma_1(b)\} Z(\lambda_e) \right] \frac{\partial^2 \phi_1}{\partial x^2}$$

$$+ \frac{1}{k_{\perp}} \left[\frac{\varepsilon}{2k_{\perp}} \frac{k_D^2}{k^2} \{ [\Gamma_o(b) - 1] - 2b [\Gamma_o(b) - \Gamma_1(b)] \lambda_e Z(\lambda_e) \} \right] \frac{\partial \phi_1}{\partial x}$$

$$+ \left[-1 - \frac{k_D^2}{k^2} \left\{ (1+\varepsilon x) + \left(\frac{\bar{\omega}_k (1+\varepsilon x)}{\sqrt{2} k_{\parallel} C_e} + \frac{k_{\perp} V_n}{\sqrt{2} k_{\parallel} C_e} \right) \Gamma_o(b) Z(\lambda_e) \right\} \right.$$

$$- \frac{T_e}{T_i} \frac{k_D^2}{k^2} \left\{ 1 + \lambda_i \left(Z(\lambda_i) + \frac{k_\perp^2}{4k_\parallel^2} \frac{d^2 Z(\lambda_i)}{d\lambda_i^2} \right) \right\} \Big] \phi_1 = 0$$

where we recall

$$\lambda_e = \frac{\bar{\omega}_k}{\sqrt{2} k_\parallel C_e} = \frac{\omega_k - k_\perp V_o}{\sqrt{2} k_\parallel C_e}$$

$$\lambda_i = \omega_k / \sqrt{2} k_\parallel C_i \quad , \qquad b = k_\perp^2 C_e^2 / \Omega_e^2$$

$$k_D = \lambda_D^{-1} = (4\pi n_o e^2 / T_e)^{+\frac{1}{2}}$$

and

$$V_n = \frac{\varepsilon T_e}{m_e |\Omega_e|}$$ is the magnitude of the diamagnetic

drift $\vec{V}_n$ ((5.2.7c)).

The above equation may be rewritten as

$$\frac{1}{k_\perp^2} A(x) \frac{d^2\phi(x)}{dx^2} + \frac{1}{k_\perp} B(x) \frac{d\phi(x)}{dx} + C(x)\,\phi(x) = 0 \tag{5.4.1}$$

where

$$A(x) = \frac{k_\perp^2}{k^2} - \frac{k_D^2}{k^2} \left(\frac{\bar{\omega}_k (1+\varepsilon x)}{\sqrt{2} k_\parallel C_e} + \frac{k_\perp V_n}{\sqrt{2} k_\parallel C_e} \right) b \left[\Gamma_o(b) - \Gamma_1(b) \right] Z(\lambda_e)$$

$$B(x) = \frac{\varepsilon}{k_\perp} \frac{k_D^2}{2k^2} \{ [\Gamma_o(b) - 1] - 2b \left[\Gamma_o(b) - \Gamma_1(b) \right] \lambda_e Z(\lambda_e) \}$$

$$C(x) = -1 - \frac{k_D^2}{k^2} \left[(1+\varepsilon x) - \left(\frac{\bar{\omega}_k (1+\varepsilon x)}{\sqrt{2} k_\parallel C_e} + \frac{k_\perp V_n}{\sqrt{2} k_\parallel C_e} \right) \Gamma_o(b) Z(\lambda_e) \right]$$

$$- \frac{T_e}{T_i} \frac{k_D^2}{k^2} \left[1 + \lambda_i \left(Z(\lambda_i) + \frac{k_\perp^2}{4k_\parallel^2} \frac{d^2 Z(\lambda_i)}{d\lambda_i^2} \right) \right]$$

To obtain analytical results we resort to further approximations. It follows from Eq. (5.2.2) that

$$\varepsilon x = x/L_n$$

Thus, for the purpose of simplification, we assume that Eq. (5.4.1) is only applicable over a range of x which is much less than L_n. This imposes the restriction that the potential $\phi(x)$ is very localized, vanishing for x values not very far from x = 0. Then

$$(x/L_n) << 1 \tag{5.4.2a}$$

and the coefficients A(x), B(x) and C(x) reduce to

$$A(x) = \frac{k_\perp^2}{k^2} - \frac{k_D^2}{k^2} \frac{[\omega_k - k_\perp(V_o - V_n)]}{\sqrt{2}k_\parallel C_e} b \{\Gamma_o(b) - \Gamma_1(b)\} Z(\lambda_e) \tag{5.4.2b}$$

$$B(x) = \frac{\varepsilon}{k_\perp} \frac{k_D^2}{2k^2} \{[\Gamma_o(b) - 1] - 2b [\Gamma_o(b) - \Gamma_1(b)] \lambda_e Z(\lambda_e)\} \tag{5.4.2c}$$

$$C(x) = -1 - \frac{k_D^2}{k^2} \left[1 + \frac{\{\omega_k - k_\perp(V_o - V_n)\}}{\sqrt{2}k_\parallel C_e} \Gamma_o(b) Z(\lambda_e) \right]$$

$$- \frac{T_e}{T_i} \frac{k_D^2}{k^2} \left\{ 1 + \lambda_i \left[Z(\lambda_i) + \frac{k_\perp^2}{4k_\parallel^2} \frac{d^2 Z(\lambda_i)}{d\lambda_i^2} \right] \right\} \tag{5.4.2d}$$

For warm electrons and cold ions, i.e., $T_e >> T_i$, we assume as in Section 3.5,

$$|\lambda_e| = |(\omega_k - k_\perp V_o)/\sqrt{2}k_\parallel C_e| << 1 \tag{5.4.3a}$$

and

$$|\lambda_i| = |\omega_k|/\sqrt{2}k_{\parallel}C_i >> 1 \tag{5.4.3b}$$

A term common to A(x) and C(x) is

$$\frac{[\omega_k - k_{\perp}(V_o - V_n)]}{\sqrt{2}k_{\parallel}C_e} \; Z\,(\lambda_e)$$

$$= \frac{[\omega_k - k_{\perp}(V_o - V_n)]}{\sqrt{2}k_{\parallel}C_e} \; Z\left(\frac{\omega_k - k_{\perp}V_o}{\sqrt{2}k_{\parallel}C_e}\right) \tag{5.4.4}$$

For $B_{oy} << B_{oz}$ and $k_z << k_y$, we see from Eqs. (5.2.5c), (5.2.4) and (5.2.1) that in the presence of magnetic shear

$$\begin{aligned} k_{\parallel}(x) &\simeq k_z + (x/L_s)k_y \\ k_{\perp} &\simeq k_y = \text{constant} \end{aligned} \tag{5.4.5}$$

Now

$$\lambda_e = \frac{\omega_k + i\gamma_k - k_{\perp}V_o}{\sqrt{2}k_{\parallel}C_e}$$

$$= \frac{\omega_k - k_{\perp}V_o}{\sqrt{2}k_{\parallel}C_e} + \frac{i\gamma_k}{\sqrt{2}k_{\parallel}C_e} \tag{5.4.6}$$

If we assume $\gamma_k > 0$ (growing wave) then the sign of $\mathrm{Im}\lambda_e = \gamma_k/\sqrt{2}k_{\parallel}C_e$ can be changed by reversing the sign of $k_{\parallel}$. It is seen from Eq. (5.4.5) that this implies x assuming an appropriate negative value. However, such a change leads to difficulties as far as the Z-function is concerned, since it is a sectionally regular function of two sheets with a discontinuity along the real axis. For real λ it behaves as follows[71]:

$$\lim_{\delta\to o^+} Z(\lambda+i\delta) = \pi^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{(x-\lambda)} dx + i\pi^{\frac{1}{2}} e^{-\lambda^2} \tag{5.4.7a}$$

$$\lim_{\delta\to o^+} Z(\lambda+i\delta) = \pi^{-\frac{1}{2}} \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{(x-\lambda)} dx - i\pi^{\frac{1}{2}} e^{-\lambda^2} \tag{5.4.7b}$$

Thus, a derivation of the dispersion relation for different initial signs of $Im\lambda_e$ will lead to two relations involving different sheets of the Z-function.

We notice from Eq.(5.4.6) that changing the sign of $Im(\lambda_e)$ through k also changes the sign of $Re\lambda_e$. A similar behaviour holds for $[\omega_k-k_\perp(V_o-V_n)]/\sqrt{2}k_\parallel C_e$. Therefore, we may compensate for the discontinuity of $Z(\lambda_e)$ by taking $k_\parallel>0$ (i.e., $x>0$ for k_z small) in the first term of the product in Eq. (5.4.4), and for $Z(\lambda_e)$ use the power series expansion in the limit $|\lambda_e|<<1$ corresponding to Eq.(5.4.7a) above. This yields

$$\frac{[\omega_k - k_\perp(V_o - V_n)]}{\sqrt{2}k_\parallel C_e} Z(\lambda_e)$$

$$\simeq \frac{[\omega_k - k_\perp(V_o - V_n)]}{\sqrt{2}|k_\parallel| C_e} \{i\pi^{\frac{1}{2}} - 2\lambda_e^2\} \tag{5.4.8a}$$

where, from Eq. (5.4.5), $|k_\parallel|$ implies $k_\parallel > 0$, and therefore $x \geq 0$ provided k_z is very small.

The problem associated with the discontinuity of the Z-function did not arise earlier in the work undertaken in Section 3.5 , because there we have assumed $k_z > 0$, without actually stating so. Therefore, in the consequent manipulations the power series expansion corresponding to Eq. (5.4.7a) was used (cf. Eq. (3.5.9a)).

For $|\lambda_i| >> 1$, we have the asymptotic expansion (3.5.9b), viz.,

$$Z(\lambda_i) \simeq - \left(\frac{1}{\lambda_i} + \frac{1}{2\lambda_i^3} + \ldots\right) \tag{5.4.8b}$$

with

$$\frac{\lambda_i}{2} \frac{d^2 Z(\lambda_i)}{d\lambda_i^2} \simeq - \frac{1}{\lambda_i^2} \tag{5.4.8c}$$

With these results the coefficients A(x) ((5.4.2b)) and C(x) ((5.4.2d)) reduce to

$$A(x) = \frac{k_\perp^2}{k^2} - i\pi^{\frac{1}{2}} \frac{k_D^2}{k^2} \frac{\{\omega_k - k_\perp(V_o - V_n)\}}{\sqrt{2}|k_{||}|C_e} b[\Gamma_o(b) - \Gamma_1(b)] \tag{5.4.9a}$$

$$C(x) = -1 - \frac{k_D^2}{k^2}\left\{1 + i\pi^{\frac{1}{2}} \frac{[\omega_k - k_\perp(V_o - V_n)]}{\sqrt{2}|k_{||}|C_e} \Gamma_o(b)\right\} - \frac{T_e}{T_i}\frac{k_D^2}{k^2}\left\{- \frac{1}{2\lambda_i^2} - \frac{k_\perp^2}{2k_{||}^2}\frac{1}{\lambda_i^2}\right\}$$

$$= -1 - \frac{k_D^2}{k^2}\left\{1 + i\pi^{\frac{1}{2}} \frac{[\omega_k - k_\perp(V_o - V_n)]}{\sqrt{2}|k_{||}|C_e} \Gamma_o(b)\right\} + \frac{k_D^2 C_s^2}{\omega_k^2} \tag{5.4.9b}$$

For physically meaningful solutions we require $\lambda < x$ where x is the range over which $\phi(x)$ is non-zero. By virtue of assumption (5.4.2a) we have

$$\frac{\varepsilon}{k_\perp} < \frac{x}{L_n} << 1$$

Since $|\Gamma_o(b)| \leq 1, |\Gamma_o(b) - \Gamma_1(b)| \leq 1$ (72) and by assumption (5.4.3a) $|\lambda_e| << 1$, it thus follows that the coefficient B(x) is small compared to A(x) and C(x) and is therefore neglected.

Equation (5.4.1) may then be written as

$$\frac{1}{k_\perp^2} A(x) \frac{d^2\phi(x)}{dx^2} + C(x)\,\phi(x) = 0 \qquad (5.4.10)$$

with A(x) and C(x) given by Eqs. (5.4.9a) and (5.4.9b) respectively. This equation is called the <u>eigenvalue equation</u>.

We notice that since in the absence of shear ($L_s \to \infty$) ϕ is a constant, Eq. (5.4.10) then implies C = 0, C no longer having an x dependence since $k_\parallel$ is also constant. This gives us the usual non-shear linear dispersion relation

$$C = -1 - \frac{k_D^2}{k^2}\left\{1 + i\pi^{\frac{1}{2}} \frac{[\omega_k - k_\perp(V_o - V_n)]}{\sqrt{2}k_\parallel C_e} \Gamma_o(b)\right\} + \frac{k_D^2 C_s^2}{\omega_k^2} = 0 \qquad (5.4.11a)$$

For $\omega_k = \omega_k^r + i\gamma_k$ with $|\gamma_k| << \omega_k^r$, the real part is given by (1, p. 389)

$$\mathrm{Re}C(\omega_k^r) = 0$$

This yields

$$\omega_k^r = kC_s(1 + k^2\lambda_D^2)^{-\frac{1}{2}} \qquad (5.4.11b)$$

For the growth rate, the equation

$$\gamma_k = -\frac{\mathrm{Im}C(\omega_k^r)}{\frac{\partial}{\partial \omega_k^r}\mathrm{Re}C(\omega_k^r)}$$

gives

$$\frac{\gamma_k}{\omega_k^r} = \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \left\{\frac{k}{k_\parallel}\left(\frac{m_e}{2m_i}\right)^{\frac{1}{2}}\right\} \frac{\Gamma_o}{(1+k^2\lambda_D^2)} \left[\frac{k_\perp(V_o-V_n) - \omega_k^r}{kC_s}\right] \tag{5.4.11c}$$

which is a special case of the result (3.5.10) when one ignores gradients in plasma temperature and magnetic field.

For $V_n < V_o$, we have by assumption (5.4.3a) that

$$\left|\frac{\omega_k - k_\perp(V_o-V_n)}{\sqrt{2}\, k_\parallel C_e}\right| << 1$$

Since $|\Gamma_o(b) - \Gamma_1(b)| \leq 1$ (72), it is seen from Eq. (5.4.9a) that

$$A(x) \simeq \frac{k_\perp^{\,2}}{k^2} \tag{5.4.12}$$

for

$$\frac{bk_D^2}{k^2} \simeq \frac{\omega_{pe}^{\,2}}{\Omega_e^2} \lesssim 1 \tag{5.4.13}$$

The effect of magnetic shear is now introduced. We substitute for $k_\perp$ and $k_\parallel$ from the equations (5.4.5). In the limit of approximation (5.4.12), Eq. (5.4.10) may be written as

$$\frac{d^2\phi(x)}{dx^2} - \frac{1}{\lambda_D^2}\left[1 + k^2\lambda_D^2 + \frac{i\pi^{\frac{1}{2}}\{\omega_k - k_y(V_o - V_n)\}\Gamma_o(b)}{\sqrt{2}|k_z + xk_yL_s^{-1}|C_e} - \frac{k^2C_s^2}{\omega_k^2}\right]\phi(x) = 0$$

The necessary boundary condition is that $\phi(x) \to 0$, as $x \to \pm\,\infty$. Since for k_z small (~ 0) this equation is symmetric in x we may restrict it to $x \geq 0$, with the added condition that $\phi(x)$ be regular at $x = 0$.

We transform to the new variable $z = k_y(x + (k_z/k_y)L_s)$, to obtain

$$\frac{d^2\phi(z)}{dz^2} - \frac{1}{k_y^2\lambda_D^2}\left[1 + k_y^2\lambda_D^2\left(1 - \frac{k_D^2C_s^2}{\omega_k^2}\right) + \frac{i\pi^{\frac{1}{2}}\{\omega_k - k_yV_D\}\Gamma_o(b)}{\sqrt{2}|z|L_z^{-1}C_e}\right.$$

$$\left. + z^2\frac{\lambda_D^2}{L_s^2}\left(1 - \frac{k_D^2C_s^2}{\omega_k^2}\right)\right]\phi(z) = 0 \tag{5.4.14}$$

where $V_D = V_o - V_n$. This equation, in turn, may be written as

$$\frac{d^2\phi(z)}{dz^2} - \frac{1}{k_y^2\lambda_D^2}Q(z)\,\phi(z) = 0 \tag{5.4.15a}$$

where

$$Q(z) = Q_R(z) + i\,Q_I(z) \tag{5.4.15b}$$

with

$$Q_R(z) = 1 + k_y^2\lambda_D^2\left(1 - \frac{k_D^2C_s^2}{\omega_k^2}\right)\left\{1 + \frac{z^2}{k_y^2L_s^2}\right\} \tag{5.4.15c}$$

$$Q_I(z) = \left(\frac{\pi}{2}\right)^{\frac{1}{2}}\{\omega_k - k_yV_D\}\,\Gamma_o(b)/(|z|\,L_s^{-1}C_e) \tag{5.4.15d}$$

To analyse this equation we temporarily ignore $Q_I(z)$, i.e.,we treat the inverse-electron Landau damping term, which, from Eq. (5.4.11c), gives rise to a positive growth rate when $V_D > \omega_k^r/k_\perp \sim C_s$ ($k^2\lambda_D^2 \ll 1$), as a perturbation, i.e.,we assume the magnetic shear effect to dominate. Such an approach has been adopted by PEARLSTEIN and BERK(70) and by GLADD and HORTON(73) in the study of drift waves. Equation (5.4.15a) then reduces to

$$\frac{d^2\phi}{dz^2} + \{\beta - \alpha^2 z^2\}\, \phi(z) = 0 \qquad (5.4.16a)$$

where

$$\beta = -(k_y^2\lambda_D^2)^{-2}\, \{1 + k_y^2\lambda_D^2\, (1 - k_D^2 C_s^2/\omega_k^2)\} \qquad (5.4.16b)$$

and

$$\alpha = i(k_y L_s)^{-1}\, \{(k_D^2 C_s^2/\omega_k^2) - 1\}^{\frac{1}{2}} = i\delta \qquad (5.4.16c)$$

with

$$\delta = (k_y L_s)^{-1}\, \{(k_D^2 C_s^2/\omega_k^2) - 1\}^{\frac{1}{2}}$$

The above differential equation is similar to that encountered in a study of the harmonic oscillator in wave mechanics. A solution that satisfies the required boundary condition of waves with outgoing energy flux at large z, and therefore large x, is given by[70]

$$\phi(z) = H_n\, \{(i\delta)^{\frac{1}{2}}\, z\}\, \exp(-\tfrac{1}{2}\, i\delta z^2) \qquad (5.4.17a)$$

where H_n $(n = 0,1,2, \ldots)$ is the Hermite polynomial of order n, and

$$\beta = (2n+1)\, \alpha \qquad (5.4.17b)$$

Substituting for β and α in the above equation, we have

$$\left[\left(\frac{k_D^2 C_s^2}{\omega_k^2} - 1\right) - \frac{1}{k_y^2\lambda_D^2}\right] = \frac{i(2n+1)}{k_y L_s} \left\{\frac{k_D^2 C_s^2}{\omega_k^2} - 1\right\}^{\frac{1}{2}}$$

Setting $\omega_k = \omega_k^r + i\gamma_k^s$ with $|\gamma_k^s| << \omega_k^r$, this becomes

$$1 + k_y^2\lambda_D^2 - \frac{k_y^2 C_s^2}{(\omega_k^r)^2}\left(1 - \frac{2i\gamma_k}{\omega_k^r}\right) = - \frac{i\,(2n+1)}{k_y L_s} \frac{k_D C_s}{\omega_k^r} k_y^2\lambda_D^2 \tag{5.4.18a}$$

where it has been assumed, in addition, that

$$|\omega_k^2| \ll k_D^2 C_s^2 \tag{5.4.18b}$$

This approximation will be discussed later.

Resolving Eq. (5.4.18a) into real and imaginary components, we find

$$(\omega_k^r)^2 = \frac{k_y^2 C_s^2}{1 + k_y^2 \lambda_D^2} \tag{5.4.19a}$$

and

$$\gamma_k^s = - \frac{(n+\frac{1}{2})}{k_y L_s} \frac{(\omega_k^r)^2}{k_D C_s} \tag{5.4.19b}$$

We see that $\gamma_k^s < 0$. Therefore shear has a stabilizing effect.

Further, for $k_y^2 \lambda_D^2 \ll 1$, as is usually the case in practice, $(\omega_k^r)^2 \simeq k_y^2 C_s^2$, and in this limit the assumption $|\omega_k^2| \ll k_D^2 C_s^2$ holds true.

We now estimate the wave growth produced by inverse electron Landau damping by evaluating $Q_I(z)$ at the WKB turning point[69], which, in the case of a shear dominated situation, is given by $Q_R(z_T) = 0$, i.e.,

$$1 + k_y^2\lambda_D^2 \{1 - (k_D^2 C_s^2/\omega_k^2)\} = - k_y^2\lambda_D^2 \{1 - (k_D^2 C_s^2/\omega_k^2)\} (z_T^2/k_y^2 L_s^2)$$

With the aid of the results (5.4.19), it turns out that

$$\frac{z_T^2}{k_y^2 L_s^2} = \frac{2i\gamma_k^s}{\omega_k^r} = - \frac{i(2n+1)}{k_y L_s} \frac{\omega_k^r}{k_D C_s} \tag{5.4.20}$$

Thus, upon evaluating the electron term $Q_I(z)$ at $z = z_T$, we add it to the coefficient β in Eq.(5.4.16b) and obtain the consequent dispersion relation

relation

$$\left[\left(\frac{k_D^2 C_s^2}{\omega_k^2} - 1\right) - \frac{1}{k_y^2 \lambda_D^2}\right] - \frac{i}{k_y^2 \lambda_D^2}\left(\frac{\pi}{2}\right)^{\frac{1}{2}}\left[\frac{(\omega_k - k_y V_D)\Gamma_o(b) L_s}{|z_T| C_e}\right] = \frac{i(2n+1)}{k_y L_s}\left(\frac{k_D^2 C_s^2}{\omega_k^2} - 1\right)^{\frac{1}{2}} \tag{5.4.21}$$

An approximate stability criterion may be obtained by setting $\omega_k \simeq \omega_k^r$, as given by Eq.(5.4.19a).

Then, using the assumption (5.4.18b), we have

$$\frac{\pi(k_y V_D - \omega_k^r)^2 \Gamma_o^2}{2k_y^4 \lambda_D^4 \; k_y^2 C_e^2} \; \frac{k_y^2 L_s^2}{|z_T|^2} = \frac{(2n+1)^2}{k_y^2 L_s^2} \; \frac{k_D^2 C_s^2}{\omega_k^2}$$

With $(|z_T| / k_y L_s)^2$ given by Eq. (5.4.20), it turns out that

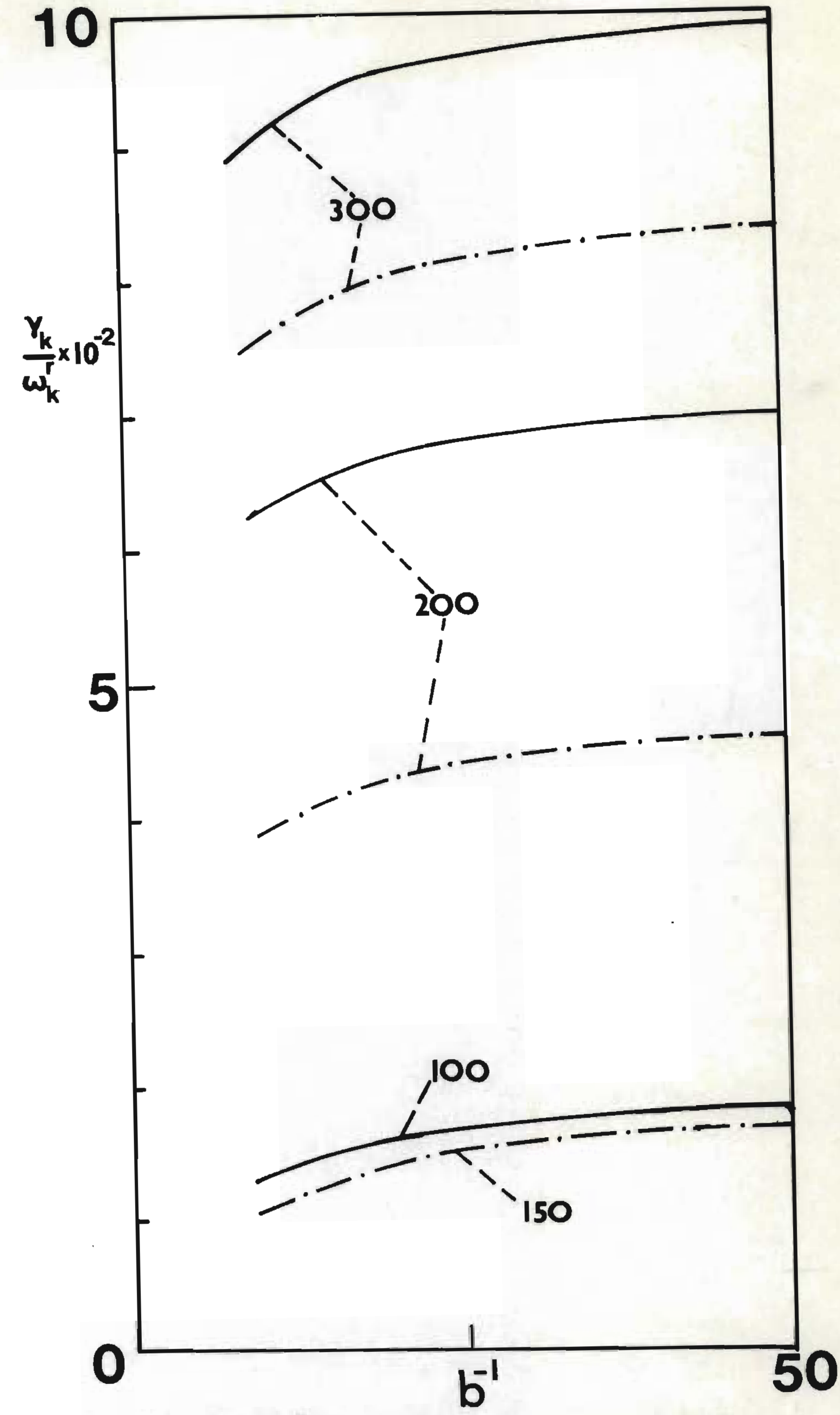
$$\frac{L_s}{\lambda_D} = \left(\frac{2}{\pi}\right)^{1/3} (2n+1)\left(\frac{m_i}{m_e}\right)^{1/3}\left(\frac{V_D}{C_s} - 1\right)^{-2/3} \left(\Gamma_o(b)\right)^{-2/3} \equiv \left(\frac{L_s}{\lambda_D}\right)_c$$

Therefore for a fixed λ_D, $(L_s/\lambda_D) < (L_s/\lambda_D)_c$ will result in a completely damped wave.

Figure 5.2 represents a plot of the normalized growth rate γ_k/ω_k^r against $b^{-1} = (k_\perp r_e)^{-2}$, as obtained from Eq. (5.4.21). Since shear damping increases with decreasing shear length, the associated net growth rate is seen to decrease. For $k\lambda_D = 0{,}1$, it turns out that the critical shear length $(L_s/\lambda_D)_c = 130$, and for $k\lambda_D = 0{,}2$, $(L_s/\lambda_D)_c = 80$.

An attempt to find an analytical solution to the eigenvalue equation (5.4.10) in the opposite limit, viz.,

Figure 5.2



Growth rate γ_k/ω_k^r for the ion acoustic instability in the presence of magnetic shear as a function of $b^{-1}=(k_\perp r_e)^{-2}$. $m_i/m_e=73440$, $V_o/C_s=5$, $V_n/C_s=1$. The solid lines (——) are for $k\lambda_D=0,2$, and the discontinuous lines (-·-·-) correspond to $k\lambda_D=0,1$. The parameter labelling the curves is L_s/λ_D.

$$\left| \pi^{\frac{1}{2}} \frac{k_D^2 b}{k^2} \frac{\{\omega_k - k_\perp V_D\}}{\sqrt{2}\,|k_\parallel| C_e} [\Gamma_o(b) - \Gamma_1(b)] \right| \quad >> \quad 1 \qquad (5.4.22)$$

which requires

$$\frac{bk_D^{\ 2}}{k^2} \simeq \frac{\omega_{pe}^{\ 2}}{\Omega_e^{\ 2}} >> 1$$

leads to difficulties. For, the usual WKB technique[74] used in solving an equation of the form

$$\frac{d^2\phi(x)}{dx^2} + Q(x)\ \phi(x) = 0$$

can no longer be employed. This technique involves setting up solutions at the WKB turning points x_1 and x_2, where $Q(x_1) = Q(x_2) = 0$, which in the asymptotic limit reduce to exponentially decreasing solutions at $x \to \pm \infty$, where Q assumes a constant complex value. The eigenvalue ω is determined by the requirement that the solutions at x_1 and x_2 match in the region between x_1 and x_2.

The equation (5.4.10) can be readily written in the above form, with

$$Q(x) = \frac{k_\perp^{\ 2} C(x)}{A(x)}$$

As $x \to \pm \infty$, C(x) is given by Eq. (5.4.2d), since from the expression (5.4.5) for $k_\parallel$, it is seen that the expansion (5.4.3b) used in arriving at the result (5.4.9b) is not valid at these limits. It can be shown that C(x) reduces to a constant as $x \to \pm \infty$. However, A(x), which is still given by Eq.(5.4.9a), excluding the $(k_\perp^2/k^2)$

contribution (by virtue of assumption (5.4.22)), vanishes at these extremeties. This causes $Q(x)$ to diverge at $x \to \pm \infty$.

A close inspection of the mathematical formalism shows that the ions do not contribute a term to the coefficient $A(x)$. This is so since the ions have been assumed to be unmagnetized. It thus seems feasible that the presence of magnetized ions could produce a term which, in the limit (5.4.22), would make $A(x)$ non-zero at $x \to \pm \infty$, and thereby allow an approximate WKB solution.

CHAPTER SIX

THE EFFECT OF ELASTIC AND INELASTIC CHARGE-TRANSFER COLLISIONS ON THE CROSSFIELD CURRENT-DRIVEN ION ACOUSTIC INSTABILITY

Here, we initially concern ourselves with the effect of elastic and inelastic collisions on an incident ion beam. We then study the effect of such collisions on ion acoustic wave perturbations superimposed on the ion beam.

6.1 SPATIAL EVOLUTION OF AN INCIDENT ION BEAM IN THE PRESENCE OF ELASTIC AND INELASTIC COLLISIONS

We adopt the model of LEE [62] which describes the experimetal arrangement in the Double Plasma (DP) device in the Plasma Physics Research Institute, University of Natal, Durban. In the model, a beam of thermally isotropic ions, density N_o, enters the region $y \geq 0$ of an (x,y,z) Cartesian coordinate system with an initial drift $\vec{U}(y = 0) = U_o\hat{y}$ in the laboratory frame. This region is called the target plasma, while the ion beam originates from the source or driver plasma. To maintain quasi-neutrality we have a thermally isotropic electron distribution of equal density. These electrons are at rest in the laboratory frame. This is in contrast to the investigations conducted far, where we have worked in the ion rest frame.
The half-space defined by $-\infty \leq x \leq +\infty$, $y \geq 0$,

$-\infty \leq z \leq +\infty$, is filled with neutral atoms of the same atomic species as the ions in the beam, with n_N (neutral atom density) >> n_B (ion beam density). The system is subjected to a constant magnetic field $\vec{B}_o$ in the z direction. No external electric field is present. As before, the length and time scales are such that the electrons are magnetized and the ions are not. The latter therefore assume straight line orbits with constant velocity.

There are several possible collision processes within the half-space defined above. These are electron-electron, electron-ion, electron-neutral, ion-neutral and ion-ion. The starting point of a statistical description of a plasma with collisions is an equation of the form [1,16]

$$\frac{\partial f}{\partial t} + \vec{V} \cdot \frac{\partial f}{\partial \vec{r}} + \vec{a} \cdot \frac{\partial f}{\partial \vec{V}} = \left(\frac{\partial f}{\partial t}\right)_c \qquad (6.1.1)$$

where $(\partial f/\partial t)_c$ is the time rate of change of the distribution function as a result of collisions. In the absence of collisions the right hand side of Eq.(6.1.1) vanishes and it reduces to the Vlasov equation (cf. Eq.(3.1.1)). The actual construction of the collision term $(\partial f/\partial t)_c$ presents considerable difficulty. Furthermore, it differs in form for the various types of collisions. There are two well known models of the collision term. The first, the Boltzmann collision integral [1], is based on the assumption that the collisions are short-range and binary. It is appropriate for the study of a weakly ionized plasma. The second, the Fokker-Planck model, is suited to the examination of a fully ionized plasma where the deflection of a particle is more likely due to the cumulative effect of a large number of small-angle scatterings than a single close collision. Here, the long range Coulomb-interactions are of importance.

In our investigations we shall adopt the 'Boltzmann model'. Thus, Coulomb-interactions between charged particles are neglected. Electron-neutral collisions are also ignored. LEE [62] has shown that the effect of such collisions are negligible when compared to ion-neutral collisions. He found that for typical experimental parameters the mean free path for electron-neutral collisions was twice as long as the width of the DP device measured along the magnetic field. Our analysis is thus restricted to ion-neutral collisions only. Of these, the elastic collisions between the incident beam ions and the stationary neutrals are of the 'billiard-ball' type. The total kinetic energy of the interaction remains constant and the internal states of the colliding particles are unchanged. The inelastic collision is assumed to be of a charge-transfer type, whereby an incident beam ion, travelling with a velocity $\vec{U}$, 'absorbs' an electron from a neutral atom (of the same atomic species) at rest. The end products of the interaction are therefore a neutral atom with approximate velocity $\vec{U}$ and an ion with zero drift velocity in the laboratory frame, which, henceforth, will be referred to as a 'rest' ion.

The Boltzmann collision integral is complicated by nature and is mathematically intractable. In practice it is necessary to simplify the mathematical procedures in order to interpret experimental measurements. Therefore, we adopt the model of BHATNAGAR *et al* [75] and write the Boltzmann equation for the beam ions as

$$\frac{\partial f_B}{\partial t} + \vec{V} \cdot \frac{\partial f_B}{\partial \vec{r}} + \frac{e}{m_i}\left[\vec{E}_o + \frac{\vec{V} \times \vec{B}}{C}\right] \cdot \frac{\partial f_B}{\partial \vec{V}} = -\frac{V}{\lambda_{ce}}\left(f_B - f_{oB'}\right) - \frac{V}{\lambda_{ci}} f_B$$

(6.1.2)

Here, the first team on the right hand side is the elastic collision term, while the second represents inelastic charge-transfer collisions. The effect of the elastic collisions is to drive the velocity distribution $f_B(\vec{r},\vec{V},t)$ towards a stationary Maxwellian $f_{oB'}(\vec{r},\vec{V},t)$ given by

$$f_{oB'}(\vec{r},\vec{V},t) = \frac{n_B(y)}{(2\pi C_{io}^2)^{3/2}} \exp\left[-\frac{(V_x^2 + V_y^2 + V_z^2)}{2\, C_{io}^2}\right] \qquad (6.1.3)$$

where $C_{io} = (T_{io}/m_i)^{1/2}$ is the thermal speed of an ion with zero drift. This is plausible since the condition $n_N \gg n_B$ implies that the background neutrals may be considered to act as an infinite sink for momentum and energy. In Eq.(6.1.2) we have written the collision frequencies as

$$\nu_{i(e)} = \frac{V}{\lambda_{ci(e)}} \qquad (6.1.4)$$

where ν_e (ν_i) is the elastic (inelastic) collision frequency and λ_{ce} (λ_{ci}) the mean free path for elastic (inelastic) collisions. In general the mean free path λ is related to the collision cross section σ and the background neutral density n_N according to

$$\lambda = (n_N\, \sigma)^{-1}$$

Although the density n_N changes (due to charge-transfer collisions) as the ion beam traverses the region $y \geq 0$, the condition $n_N \gg n_B$

implies that this change, as seen by the beam ions, is negligibly small. The cross section σ is usually a function of velocity. However, if we restrict ourselves by the assumption that σ is independent of velocity, then we see that λ assumes a constant value. The experimental measurements of BROWN [76] and OHNUMA and FUJITA [77] exhibit a wide velocity range over which the change in σ is small. Hence, Eq.(6.1.4) tells us that although the mean free paths λ_{ci} and λ_{ce} are constants, the collision frequencies ν_i and ν_e have an explicit dependence on speed.

In writing down Eq.(6.1.3) we have assumed spatial homogeneity in the x and z directions, and neglected any spatial variation of the temperature. It has already been assumed that the ion beam enters the region $y \geq 0$ with density N_o and drift $\vec{U}(y = 0) = U_o\hat{y}$. We assume, in addition, that they enter with a Maxwellian distribution. Thus we may write

$$f_{oB}(y = 0,\vec{V}) = \frac{N_o}{(2\pi C_{iB}^2)^{3/2}} \exp\left[- \frac{\{V_x^2 + (V_y - U_o)^2 + V_z^2\}}{2\, C_{iB}^2}\right] \qquad (6.1.5)$$

with $n_B(y = 0) = N_o$.

We note that here $C_{iB} = (T_{iB}/m_i)^{1/2}$ is the thermal speed of a drifting beam ion and is usually different from that of a stationary ion, since in general $T_{iB} \neq T_{io}$. The effect of the charge-transfer collisions is to modify the density of the ion beam as it progresses into the region $y \geq 0$. This change is represented by the term $n_B(y)$ in Eq.(6.1.3). As for the elastic collisions, they cause a slowing down of the beam. Hence, from the point of view of the elastic collisions, allowing for density modification by the inelastic collisions, we may expand the distribution $f_B(y,\vec{V})$ about an equilibrium $f_{oB}(y,\vec{V})$ defined by

$$f_{oB}(y,\vec{V}) = n_B(y) \left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left[-\frac{m_i \{V_x^2 + (V_y - U_o)^2 + V_z^2\}}{2\, T_{iB}}\right] \tag{6.1.6}$$

Since there is no external electric field and the ions are unmagnetized, we have

$$\frac{e}{m_i}\left(\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_B}{\partial \vec{V}} = 0 \tag{6.1.7a}$$

Furthermore, we seek a stationary solution of Eq.(6.1.2). Thus

$$\frac{\partial}{\partial t} \equiv 0 \tag{6.1.7b}$$

Using the expansion

$$f_B(y,\vec{V}) = f_{oB}(y,\vec{V}) + \epsilon f_1(y,\vec{V}) + \epsilon^2 f_2(y,\vec{V}) + \ldots. \tag{6.1.8}$$

and the results (6.1.7a) and (6.1.7b), Eq.(6.1.2) becomes

$$V_y \frac{\partial}{\partial y}\{f_{oB} + \epsilon f_1 + \epsilon^2 f_2 + \ldots.\} = -\frac{\epsilon V}{\lambda_{ce}}\{(f_{oB} + \epsilon f_1 + \epsilon^2 f_2 + \ldots) - f_{oB}'\} - \frac{V}{\lambda_{ci}}(f_{oB} + \epsilon f_1 + \epsilon^2 f_2 + \ldots) \tag{6.1.9}$$

In writing the above equation we have assumed that the inelastic collisions occur much more frequently than the elastic ones. Therefore the collision term of the latter is of an order (in ϵ) higher than that of the former. An equivalent statement is that the collision mean free paths are such that

$$\lambda_{ci} << \lambda_{ce} \tag{6.1.10}$$

To lowest order in ϵ, Eq.(6.1.9) yields

$$V_y \frac{\partial f_{oB}}{\partial y} = - \frac{V}{\lambda_{ci}} f_{oB} \tag{6.1.11}$$

We notice that the choice (6.1.6) for $f_{oB}(\vec{y},\vec{V})$ and the expansion (6.1.8) for $f_B(y,\vec{V})$ have restricted possible spatial variations in U and T_{iB} to higher orders of magnitude than that in the density n(y). This means that we consider the spatial variation of n(y) to be much more rapid than that of U(y) and $T_{iB}(y)$. In fact the variation of T_{iB} is ignored. This behaviour has been experimentally observed by JONES and BARRETT [42].

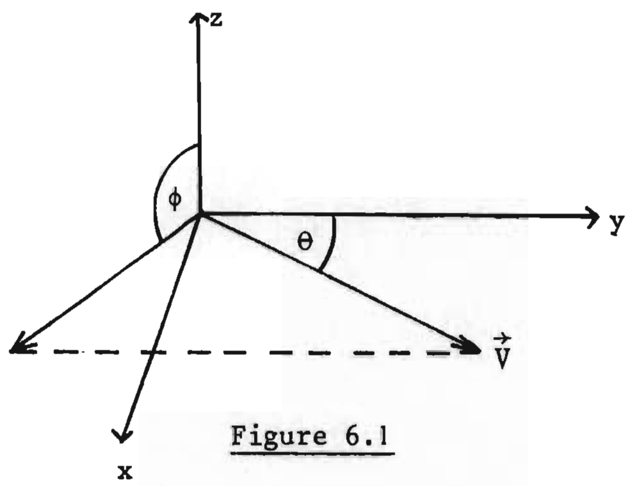


Figure 6.1

Choosing spherical coordinates in velocity space as shown above, Eq.(6.1.11) reduces to

$$V \cos \theta \frac{\partial f_{oB}}{\partial y} = - \frac{V}{\lambda_{ci}} f_{oB}$$

i.e.,

$$\cos\theta \frac{\partial f_{oB}}{\partial y} = \frac{-1}{\lambda_{ci}} f_{oB}$$

Upon substituting for f_{oB} from Eq.(6.1.6), with T_{iB} treated as a constant, we have

$$\cos\theta \frac{\partial n_B(y)}{\partial y} \left\{\frac{m_i}{2\pi\ T_{iB}}\right\}^{3/2} \exp\left[-\frac{m_i\{V_x^2 + (V_y - U_o)^2 + V_z^2\}}{2\ T_{iB}}\right]$$

$$= -\frac{1}{\lambda_{ci}} n_B(y) \left\{\frac{m_i}{2\pi\ T_{iB}}\right\}^{3/2} \exp\left[-\frac{m_i\{V_x^2 + (V_y - U_o)^2 + V_z^2\}}{2\ T_{iB}}\right] \qquad (6.1.12)$$

Next, we integrate over velocity space with $d\vec{V} = V^2 \sin\theta\ dV\ d\theta\ d\phi$. The left hand side of Eq.(6.1.12) then becomes

$$\frac{\partial n_B(y)}{\partial y}\left\{\frac{m_i}{2\pi T_{iB}}\right\}^{3/2}\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \int_{U_o}^{\infty} dV \cos\theta \exp\left(-\frac{m_i V^2}{2T_{iB}}\right)\exp\left(\frac{m_i U_o}{T_{iB}} V\cos\theta\right)\exp\left(-\frac{m_i U_o^2}{2T_{iB}}\right)V^2 \sin\theta$$

$$(6.1.13)$$

In general the integration with respect to θ ranges from 0 to π. However, the beam ions entering the region $y>0$ and proceeding in the positive y direction have $V_y > 0$. Since these are the ions detected, we see from Fig. 6.1 that the upper limit of θ is restricted to $\pi/2$. The lower limit of V is chosen as U_o and not zero since in the absence of any thermal motion the ions have the externally imposed drift velocity $\vec{U}_o$.

Using the result

$$\int_0^{\pi/2} \cos\theta \sin\theta \quad \exp(x\cos\theta)\ d\theta = \frac{1}{x}\left(e^x - \frac{1}{x} e^x \right) + \frac{1}{x^2}$$

we see that

$$\int_0^{\pi/2} \cos\theta \sin\theta \exp\left\{\left(\frac{m_i U_o V}{T_{iB}}\right)\cos\theta\right\} d\theta$$

$$= \frac{T_{iB}}{m_i U_o V}\left\{\exp\left(\frac{m_i U_o V}{T_{iB}}\right) - \frac{T_{iB}}{m_i U_o V}\exp\left(\frac{m_i U_o V}{T_{iB}}\right)\right\} + \frac{T_{iB}^2}{m_i^2 U_o^2 V^2}$$

We substitute this result into Eq.(6.1.13) and perform the trivial integration with respect ϕ to obtain

$$2\pi\left[\int_{U_o}^{\infty}\exp\left\{-\frac{m_i(V-U_o)^2}{2T_{iB}}\right\}\left(\frac{T_{iB}}{m_i U_o V}\right)V^2 dV - \int_{U_o}^{\infty}\exp\left\{-\frac{m_i(V-U_o)^2}{2T_{iB}}\right\}\left(\frac{T_{iB}}{m_i U_o V}\right)^2 V^2\, dV\right.$$

$$\left. + \int_{U_o}^{\infty}\exp\left(-\frac{m_i V^2}{2T_{iB}}\right)\exp\left(-\frac{m_i U_o^2}{2T_{iB}}\right)\frac{T_{iB}^2}{m_i^2 U_o^2 V^2}\, V^2 dV\right]$$

Defining $x = \left(\frac{m_i}{2T_{iB}}\right)^{1/2}(V - U_o)$, this reduces to

$$2\pi\left[\frac{T_{iB}}{m_i U_o}\int_0^{\infty}\exp(-x^2)\left\{\left(\frac{2T_{iB}}{m_i}\right)^{1/2} x + U_o\right\}\left(\frac{2T_{iB}}{m_i}\right)^{1/2} dx - \left(\frac{T_{iB}}{m_i U_o}\right)^2\int_0^{\infty}\exp(-x^2)\left(\frac{2T_{iB}}{m_i}\right)^{1/2} dx\right.$$

$$\left. + \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{T_{iB}}{m_i}\right)^{1/2}\frac{T_{iB}^2}{m_i^2 U_o^2}\exp\left(-\frac{m_i U_o^2}{2T_{iB}}\right)\left[1 - \operatorname{erf}\left(\frac{U_o}{\sqrt{2}C_{iB}}\right)\right]\right]$$

$$= 2\pi\left[\left(\frac{T_{iB}}{m_i U_o}\right)\left\{\frac{1}{2}\left(\frac{2T_{iB}}{m_i}\right) + \frac{\pi}{2}^{1/2} U_o\left(\frac{2T_{iB}}{m_i}\right)^{1/2}\right\} - \frac{\pi}{2}^{1/2}\left(\frac{T_{iB}}{m_i U_o}\right)^2\left(\frac{2T_{iB}}{m_i}\right)^{1/2}\right.$$

$$+ \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{T_{iB}}{m_i}\right)^{1/2} \frac{T_{iB}^2}{m_i^2 U_o^2} \exp\left(-\frac{m_i U_o^2}{2T_{iB}}\right) \left[1 - \mathrm{erf}\left(\frac{U_o}{\sqrt{2} C_{iB}}\right)\right]\Bigg]$$

$$= 2\pi \left[\frac{T_{iB}^2}{m_i^2 U_o} + \left(\frac{\pi}{2}\right)^{1/2} \left(\frac{T_{iB}}{m_i}\right)^{3/2} \left\{1 - \frac{T_{iB}}{m_i U_o^2} + \frac{T_{iB}}{m_i U_o^2} \exp\left(-\frac{U_o^2}{2C_{iB}^2}\right)\left[1 - \mathrm{erf}\left(\frac{U_o}{\sqrt{2} C_{iB}}\right)\right]\right\}\right]$$

$$= 2\pi\, C_{iB}^3 \left[\frac{C_{iB}}{U_o} + \left(\frac{\pi}{2}\right)^{1/2} \left\{1 - \frac{C_{iB}^2}{U_o^2}\left(1 - e^{-U_o^2/2C_{iB}^2}\left[1 - \mathrm{erf}\left(\frac{U_o}{\sqrt{2} C_{iB}}\right)\right]\right)\right\}\right]$$

(6.1.14)

where erf(x) is the well known error function, defined as

$$\mathrm{erf}(x) = \frac{2}{\pi^{1/2}} \int_o^x e^{-t^2}\, dt$$

From Eqs.(6.1.13) and (6.1.14) we see that integration of the left hand side of Eq.(6.1.12) over velocity space gives

$$\frac{\partial n_B(y)}{\partial y} \left\{\frac{m_i}{2\pi T_{iB}}\right\}^{3/2} 2\pi \left(\frac{T_{iB}}{m_i}\right)^{3/2} \left[\frac{C_{iB}}{U_o} + \left(\frac{\pi}{2}\right)^{1/2} \left\{1 - \frac{C_{iB}^2}{U_o^2}\left(1 - e^{-U_o^2/2C_{iB}^2}\left[1 - \mathrm{erf}\left(\frac{U_o}{\sqrt{2} C_{iB}}\right)\right]\right)\right\}\right]$$

$$= \left[\frac{C_{iB}}{(2\pi)^{1/2} U_o} + \frac{1}{2}\left\{1 - \frac{C_{iB}^2}{U_o^2}\left(1 - e^{-U_o^2/2C_{iB}^2}\left[1 - \mathrm{erf}\left(\frac{U_o}{\sqrt{2} C_{iB}}\right)\right]\right)\right\}\right] \frac{\partial n_B(y)}{\partial y}$$

(6.1.15)

Next, we integrate the right hand side of Eq.(6.1.12) over velocity space. In terms of the chosen spherical coordinates, this becomes

$$- \frac{n_B(y)}{\lambda_{ci}} \left\{\frac{m_i}{2\pi T_{iB}}\right\}^{3/2} \int_o^{2\pi} d\phi \int_o^{\pi/2} d\theta \int_{U_o}^{\infty} dV \exp\left(-\frac{m_i V^2}{2T_{iB}}\right) \exp\left(\frac{m_i U_o V \cos\theta}{T_{iB}}\right) \exp\left(-\frac{m_i U_o^2}{2T_{iB}}\right) V^2 \sin\theta$$

(6.1.16)

Now

$$\int_0^{\pi/2} \exp\left\{\left(\frac{m_i U_o V}{T_{iB}}\right)\cos\theta\right\}\sin\theta\, d\theta = \left(\frac{T_{iB}}{m_i U_o V}\right)\left[\exp\left(\frac{m_i U_o V}{T_{iB}}\right) - 1\right]$$

Together with this result and the trivial integration with respect to ϕ, Eq.(6.1.16) reduces to

$$-\frac{n_B(y)}{\lambda_{ci}}\left\{\frac{m_i}{2\pi T_{iB}}\right\}^{3/2}\left(\frac{2\pi T_{iB}}{m_i U_o}\right)\left[\int_{U_o}^{\infty}\exp\left\{-\frac{m_i}{2T_{iB}}(V-U_o)^2\right\}V\,dV - \int_{U_o}^{\infty}\exp\left(-\frac{m_i U_o^2}{2T_{iB}}\right)\exp\left(-\frac{m_i V^2}{2T_{iB}}\right)V\,dV\right]$$

$$= -\frac{n_B(y)}{\lambda_{ci}}\left\{\frac{m_i}{2\pi T_{iB}}\right\}^{3/2}\left(\frac{2\pi T_{iB}}{m_i U_o}\right)\left[\int_0^{\infty}\exp(-x^2)\left\{\left(\frac{2T_{iB}}{m_i}\right)^{1/2} x + U_o\right\}\left(\frac{2T_{iB}}{m_i}\right)^{1/2} dx - C_{iB}^2\exp\left(-\frac{U_o^2}{C_{iB}^2}\right)\right]$$

where $x = \left(\frac{m_i}{2T_{iB}}\right)^{1/2}(V - U_o)$.

This gives

$$-\frac{n_B(y)}{\lambda_{ci}}\left\{\frac{m_i}{2\pi T_{iB}}\right\}^{3/2}\left(\frac{2\pi T_{iB}}{m_i U_o}\right)\left\{\frac{1}{2}\left(\frac{2T_{iB}}{m_i}\right) + \frac{\pi}{2}^{1/2} U_o\left(\frac{2T_{iB}}{m_i}\right)^{1/2} - \left(\frac{T_{iB}}{m_i}\right)\exp\left(-\frac{U_o^2}{C_{iB}^2}\right)\right\}$$

$$= -\frac{1}{\lambda_{ci}}\left[\frac{C_{iB}}{(2\pi)^{1/2}U_o}\left\{1 + \left(\frac{\pi}{2}\right)^{1/2}\frac{U_o}{C_{iB}} - \exp\left(-\frac{U_o^2}{C_{iB}^2}\right)\right\}\right]n_B(y) \qquad (6.1.17)$$

Hence from Eqs.(6.1.15) and (6.1.17) we see that

$$\left[\frac{C_{iB}}{(2\pi)^{1/2}U_o} + \frac{1}{2}\left\{1 - \frac{C_{iB}^2}{U_o^2}\left(1 - e^{-U_o^2/2C_{iB}^2}\left[1 - \mathrm{erf}\left(\frac{U_o}{\sqrt{2}C_{iB}}\right)\right]\right\}\right]\frac{\partial n_B(y)}{\partial y}$$

$$= -\frac{1}{\lambda_{ci}}\left[\frac{C_{iB}}{(2\pi)^{1/2}U_o}\left\{1 + \left(\frac{\pi}{2}\right)^{1/2}\frac{U_o}{C_{iB}} - \exp\left(-\frac{U_o^2}{C_{iB}^2}\right)\right\}\right]n_B(y)$$

The solution to this differential equation is

$$n_B(y) = A \exp\left(-\alpha \frac{y}{\lambda_{ci}}\right)$$

where

$$\alpha = \frac{\dfrac{C_{iB}}{(2\pi)^{1/2}U_o}\left[1 - \exp\left(-\dfrac{U_o^2}{C_{iB}^2}\right)\right] + \dfrac{1}{2}}{\dfrac{C_{iB}}{(2\pi)^{1/2}U_o} + \dfrac{1}{2} - \dfrac{C_{iB}^2}{2U_o^2}\left[1 - \exp\left(-\dfrac{U_o^2}{2C_{iB}^2}\right)\left\{1 - \text{erf}\left(\dfrac{U_o}{\sqrt{2}C_{iB}}\right)\right\}\right]} \tag{6.1.18a}$$

Since $n_B(y = 0) = N_o$, it follows that

$$n_B(y) = N_o \exp\left(-\frac{\alpha y}{\lambda_{ci}}\right) \tag{6.1.18b}$$

It is seen that for $C_{iB} << U_o$, as is usually the case in practice,

$$\alpha \simeq 1$$

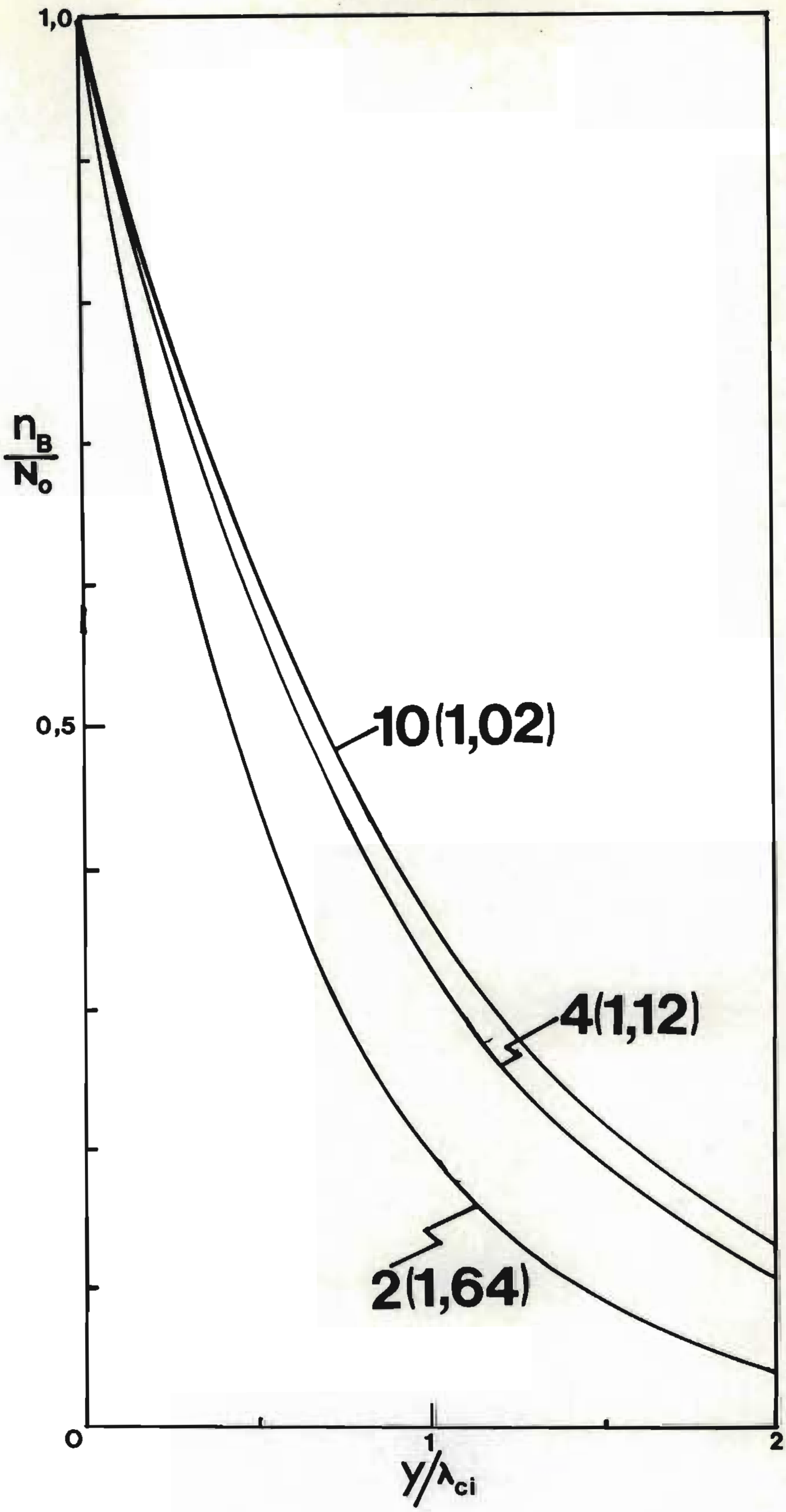
and

$$n_B(y) = N_o \exp(-y/\lambda_{ci}) \tag{6.1.19}$$

It is important to note that the result (6.1.19) is equivalent to replacing V by V_y on the right hand side of Eq.(6.1.11) above. Since $C_{iB} \sim C_{io}$, this holds true for $U_o >> C_i$ in general.

Figure 6.2 illustrates $n_B(y)/N_o$ as a function of y/λ_{ci}. The parameter labelling the curves is $(U_o/\sqrt{2}C_{iB})$. The corresponding α values are given in parentheses. The n_B/N_o values for $\alpha = 1$ are less than 0,5% larger than those for $\alpha = 1{,}02((U_o/\sqrt{2}C_{iB}) = 10)$ and cannot be separately indicated. For the DP device, a typical value of $(U_o/\sqrt{2}C_{iB})$ is ~ 17 [41]. The expression (6.1.19) is therefore a good indication of the spatial variation of ion beam density in the device.

Figure 6.2



Variation of normalized ion beam density $n_B(y)/N_o$ with distance into the plasma. The parameter labelling the curves is $(U_o/\sqrt{2}C_{iB})$. The corresponding α values, as defined in Eq.(6.1.18a), are given in parantheses.

6.2 THE BACKGROUND REST IONS

It has been shown that if $U_o >> C_i$ then the stationary distribution of the beam ions may be written as

$$f_{oB'}(y,\vec{V}) = N_o \exp(-y/\lambda_{ci}) \left\{\frac{m_i}{2\pi T_{io}}\right\}^{3/2} \exp\left[-\frac{m_i(V_x^2 + V_y^2 + V_z^2)}{2T_{io}}\right] \quad (6.2.1)$$

Since the rest ions are created by charge-transfer inelastic collisions, the Boltzmann equation for them may be written as

$$\frac{\partial f_R}{\partial t} + \vec{V} \cdot \frac{\partial f_R}{\partial \vec{r}} + \frac{e}{m_i}\left[\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right] \cdot \frac{\partial f_R}{\partial \vec{V}} = \frac{V}{\lambda_{ci}} f_{oB'} \quad (6.2.2)$$

The choice of the collision term in Eq.(6.2.2) not only allows us to maintain constant local ion density, but is also consistent with our assumption, and an experimentally observed phenomenon, that the ions created during the collision have zero drift velocity in the laboratory frame.

In view of the results (6.1.7a) and (6.1.7b), and spatial homogeneity in the x and z directions, Eq.(6.2.2) modifies to

$$V_y \frac{\partial f_R}{\partial y} = \frac{V}{\lambda_{ci}} f_{oB'}$$

As mentioned at the end of Section 6.1, the assumption $C_i << U_o$ allows us to write $n_B(y) = N_o \exp(-y/\lambda_{ci})$, as in Eq.(6.2.1), and is equivalent to setting $V = V_y$ on the right hand side above.

Thus, we have

$$V_y \frac{\partial f_R}{\partial y} = \frac{V_y}{\lambda_{ci}} f_{oB'}$$

i.e.,

$$\frac{\partial f_R(y,\vec{V})}{\partial y} = \frac{1}{\lambda_{ci}} N_o \exp\,(-\,y/\lambda_{ci}) \left\{\frac{m_i}{2\pi T_{io}}\right\}^{3/2} \exp\left(-\frac{m_i V^2}{2T_{io}}\right)$$

We integrate with respect to y and use the fact that there are no rest ions at y = 0, to find

$$f_R(y,\vec{V}) = N_o \left\{1 - \exp\left(-\frac{y}{\lambda_{ci}}\right)\right\} \left\{\frac{m_i}{2\pi T_{io}}\right\}^{3/2} \exp\left(-\frac{m_i V^2}{2T_{io}}\right) \tag{6.2.3}$$

with rest ion density

$$n_R(y) = \int f_R(y,\vec{V})\, d\vec{V}$$

$$= N_o \,\{1 - \exp\,(-\,y/\lambda_{ci})\} \tag{6.2.4}$$

It is clear from Eqs.(6.1.19) and (6.2.4) that

$$n_B(y) + n_R(y) = N_o \tag{6.2.5}$$

which is consistent with the mechanism of the charge-transfer process, whereby for each beam ion lost, a rest ion is 'born'. Therefore, a constant total ion density is maintained.

6.3 VARIATION OF MEAN BEAM SPEED WITH DISTANCE INTO THE PLASMA

To first order in ϵ, Eq.(6.1.9) yields

$$V_y \frac{\partial f_1}{\partial y} = - \frac{V}{\lambda_{ce}} (f_{oB} - f_{oB'}) - \frac{V}{\lambda_{ci}} f_1 \qquad (6.3.1)$$

where f_{oB} and $f_{oB'}$ are given by Eqs.(6.1.6) and (6.1.3) respectively.

As in the previous undertaking, in the limit $C_i << U_o$ we have

$$n_B(y) = N_o \exp(- y/\lambda_{ci})$$

and set $V = V_y$ on the right hand side of Eq.(6.3.1). Thus, upon substituting for f_{oB} and $f_{oB'}$ we obtain

$$\frac{\partial f_1}{\partial y} + \frac{1}{\lambda_{ci}} f_1 = -\frac{N_o}{\lambda_{ce}} \exp(- y/\lambda_{ci}) \left[\left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left\{- \frac{m_i(V_x^2+[V_y-U_o]^2+V_z^2)}{2T_{iB}}\right\} - \left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{- \frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\}\right]$$

i.e.,

$$\frac{\partial}{\partial y} \left\{f_1 \exp(y/\lambda_{ci})\right\} = - \frac{N_o}{\lambda_{ce}} \left[\left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left\{- \frac{m_i(V_x^2+[V_y-U_o]^2+V_z^2)}{2T_{iB}}\right\} - \left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{- \frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\}\right]$$

We integrate with respect to y, and since $f_1(y = 0,\vec{V}) = 0$, it turns out that

$$f_1(y,\vec{V}) = - \frac{y}{\lambda_{ce}} N_o \exp(- y/\lambda_{ci}) \left[\left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left\{- \frac{m_i(V_x^2+[V_y-U_o]^2+V_z^2)}{2T_{iB}}\right\} - \left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{- \frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\}\right] \qquad (6.3.2)$$

From Eqs.(6.1.8), (6.1.6) and (6.3.2), we have (to first order in ϵ)

$$f_B(y,\vec{V}) = f_{oB}(y,\vec{V}) + f_1(y,\vec{V})$$

$$= n_B(y)\{1-(y/\lambda_{ce})\} \left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left\{-\frac{m_i(V_x^2+[V_y-U_o]^2+V_z^2)}{2T_{iB}}\right\}$$

$$+ \frac{y}{\lambda_{ce}} n_B(y) \left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{-\frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\} \tag{6.3.3}$$

Equation (6.3.3) tells us that after a distance y, a fraction y/λ_{ce} of the ions in the charge-transfer modulated ion beam have undergone elastic collisions. These ions are lost to the drifting beam and, as shown, become part of the eventually stationary distribution f_{oB}, given by Eq.(6.1.3).

The mean drift velocity $\vec{U}(y)$ of the ion beam is calculated with the aid of Eq.(6.3.3). It is found that

$$\vec{U}(y) = \frac{1}{n_B(y)} \int \vec{V} f_B(y,\vec{V})\, d\vec{V} = U(y)\, \hat{y}$$

where

$$\begin{aligned} U(y) &= U_o \{1 - (y/\lambda_{ce})\} \\ &= U_o \{1 - (\lambda_{ci}/\lambda_{ce})\,(y/\lambda_{ci})\} \end{aligned} \tag{6.3.4}$$

Since $\lambda_{ci} << \lambda_{ce}$ by assumption (6.1.10), we see from Eqs.(6.1.19) and (6.3.4) that the spatial variation of the ion beam density is indeed much more rapid than that of the mean flow velocity.

The loss of beam ions by charge-transfer interactions and the spreading of the beam with distance, in the absence of any instability, have been observed in the DP device[41,42]. At an argon pressure of 8×10^{-4} Torr, the following measurements were recorded at a distance of 16 cm into the plasma[78]:

Ion beam density : $\dfrac{n_B(y)}{N_o} = 0{,}467$

Ion beam speed : $0{,}10 < \dfrac{\Delta U}{U_o} < 0{,}14$

where $\Delta U = U_o - U(y)$.

In Fig. 6.3 plots of $n_B(y)/N_o$ and $U(y)/U_o$ are displayed, satisfying Eqs.(6.1.19) and (6.3.4) respectively. We recall that these curves are for $C_i \ll U_o$, a condition well satisfied by the experiment. The parameter labelling the different curves of $U(y)/U_o$ is $(\lambda_{ci}/\lambda_{ce})$. Measurements show that $\lambda_{ci} \approx 20$ cm[78]. Thus, at $(y/\lambda_{ci}) = (16/20) = 0{,}80$, we notice from the figure that when

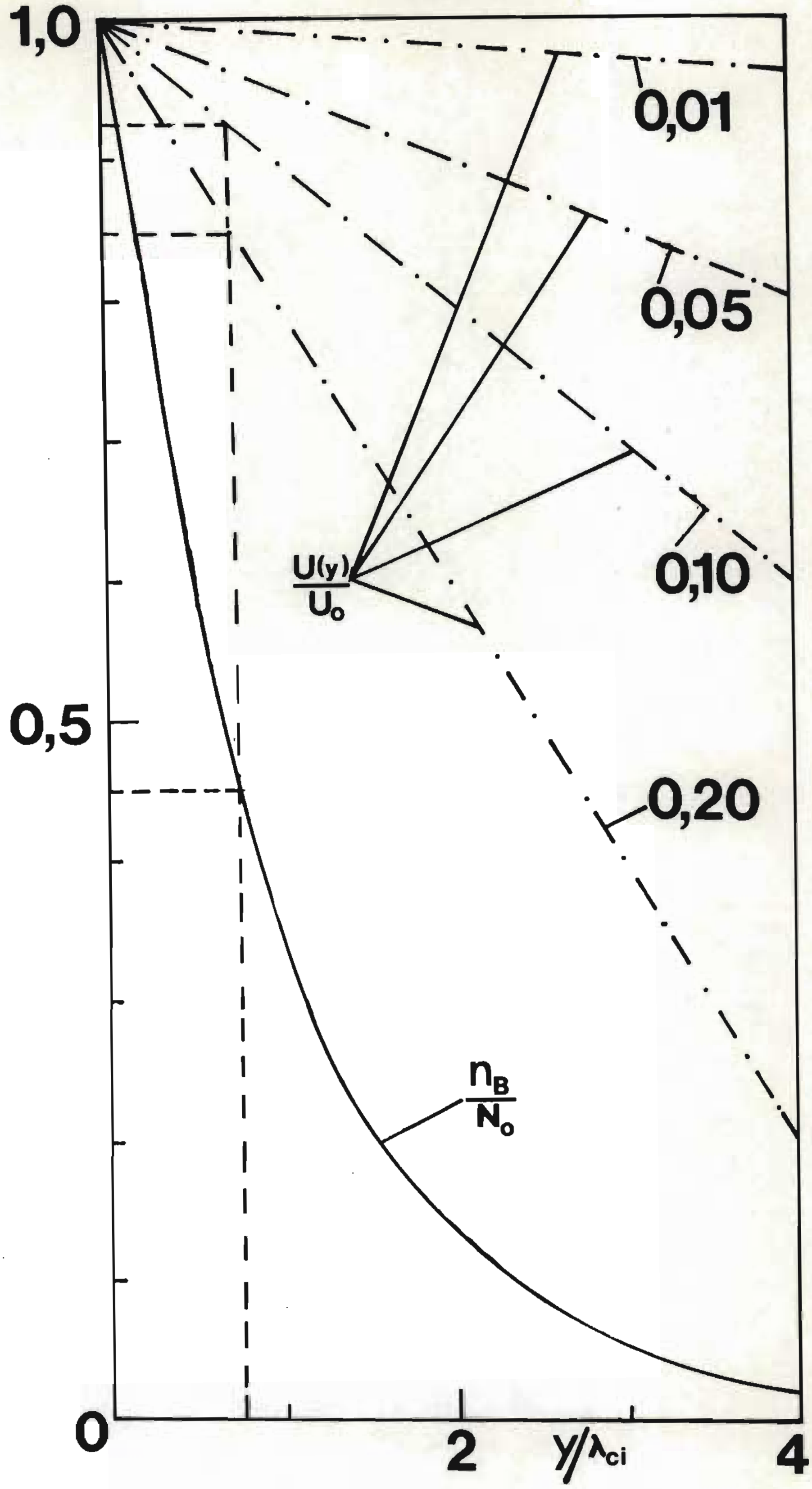
(a) $\underline{(\lambda_{ci}/\lambda_{ce}) = 0{,}10}$

$\dfrac{n_B(y)}{N_o} = 0{,}45$ $\qquad\qquad \dfrac{\Delta U}{U_o} = 0{,}08$

(b) $\underline{(\lambda_{ci}/\lambda_{ce}) = 0{,}20}$

$\dfrac{n_B(y)}{N_o} = 0{,}45$ $\qquad\qquad \dfrac{\Delta U}{U_o} = 0{,}16$

Figure 6.3



Variation of normalized ion beam density $n_B(y)/N_o$ (———) and average beam speed $U(y)/U_o$ (-·-·-·-), as defined by Eqs.(6.1.19) and (6.3.4) respectively, with distance into the plasma. The parameter labelling the different $U(y)/U_o$ curves is $\lambda_{ci}/\lambda_{ce}$.

Comparing these values with the experimental results we see that a reasonable estimate of $(\lambda_{ci}/\lambda_{ce})$ for the DP device under the given operating conditions is $0{,}10 < (\lambda_{ci}/\lambda_{ce}) < 0{,}20$. Thus, it appears that our assumption that the inelastic charge-transfer collisions are more dominant than the elastic collisions is justified by experiment.

6.4 WAVE PERTURBATIONS SUPERIMPOSED ON THE INCIDENT ION BEAM

Before proceeding with our investigations on the effect of ion-neutral collisions on wave perturbations superimposed on the ion beam, we review some of theoretical studies already undertaken using the collision model of BHATNAGAR, GROSS and KROOK [(75)], henceforth, referred to as the BGK model. A few experimental observations are also discussed.

(a) Theoretical Studies

KAW [(79)] investigated the propagation of ion waves in a weakly ionized collisional plasma with crossfield electron drift. With the assumption $\nu_e \ll \Omega_e$ and $\nu_i \gg \Omega_i$, where $\nu_e(\nu_i)$ is the electron (ion)-neutral collision frequency, he used the fluid equations to describe electron motion along $\vec{B}$ and a kinetic equation with a BGK collision model for the ions. Upon using the quasi-neutrality approximation it turned out that the perturbation growth rate was a factor (k^2/k_z^2) larger than that in an unmagnetized plasma. Electron-neutral collisions were found to promote wave growth, while ion-neutral collisions caused damping.

CUSSENOT and FABRY [80] studied the effect of ion-neutral collisions on ion waves. Numerical solutions of the dispersion relation showed that collisions were important in the low frequency range, i.e.,$(\omega/\nu) < 1$, where ω is the wave frequency and ν the collision frequency. In this domain collisional damping was much more significant than ion Landau damping. The transition to ion Landau damping occurred over the range $\nu < \omega < \omega_{pi}$.

The investigations of KAW [79] were carried a step further by SHARMA and BHATNAGAR [81] who included the effects of perturbations in the average drift velocity and temperature of the ions. Using the complete BGK collision model for the ions they found that ion-neutral collisions did not contribute to wave damping, in contrast to the findings of KAW. The inclusion of collisions was found to reduce the critical drift velocity required for the onset of the instability.

(b) Experimental Observations

To compare with their measurements made in the positive column of a helium discharge, FENNEMAN *et al* [82] developed a one dimensional linear theory of the ion acoustic instability. The approach was via the Boltzmann equation with a BGK type elastic collision term for both ion-neutral and electron-neutral collisions. The authors showed that in the region of parameter space where the ion acoustic waves were well defined electron-neutral collisions could be neglected. Their experimental results provided quite good agreement with theory. The waves were found to grow spatially and saturate. The saturation mechanism was not well understood.

Working in a DP device, KIWAMOTO [83] injected a high density ion beam into a homogeneous unmagnetized plasma. The charge-transfer mean free path for the beam ions was measured to be ~ 15 cm. Propagation of test waves showed that for beam speeds $V_b < 2\,C_s$ the fast-beam ion acoustic mode damped exponentially, while the initially growing slow-beam mode saturated further down the target chamber.

SATO *et al* [84] conducted their experiments in a Q-machine operated as a DP device. They found that as the positive bias on the grid separating the driver plasma from the target plasma was increased the damping distance δ and the wavelength λ of the perturbations also increased, as did the ratio δ/λ. Both the fast and the slow-beam ion acoustic modes were detected and the measured phase speeds agreed well with theoretical estimates.

Dispersion measurements by HAYZEN and BARRETT [41] in the Double Plasma device, on which the model in Section 6.1 is based, indicate the propagation of the slow-beam ion acoustic mode. Good agreement between theoretical and experimental growth rates is obtained for a fitted ion-neutral collision frequency of $1{,}7 \times 10^{-5}\ s^{-1}$. The overall effect of the collisions was to reduce the growth rate. The authors found that the calculated growth rate corresponding to the above collision frequency was less than the collisionless value by a factor of two.

We have seen that in the absence of wave perturbations, and for $C_i << U_o$, the effect of elastic and inelastic collisions was to modify the background distribution

$$f_{oB}(y = 0,\vec{V}) = N_o \left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left\{-\frac{m_i(V_x^2+(V_y-U_o)^2+V_z^2)}{2T_{iB}}\right\} \qquad (6.4.1)$$

with which the ions enter the region $y \geq 0$, into the form ((6.3.3))

$$f_{oB}(y,\vec{V})=N_o \exp(-y/\lambda_{ci})\{1-y/\lambda_{ce}\}\left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2} \exp\left\{-\frac{m_i(V_x^2+(V_y-U_o)^2+V_z^2)}{2T_{iB}}\right\}$$

$$+ \frac{y}{\lambda_{ce}} N_o \exp\left(-\frac{y}{\lambda_{ci}}\right)\left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{-\frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\}$$

(6.4.2)

The distribution (6.4.2) is taken as a pseudo-equilibrium for the wave perturbations, i.e., from the point of view of the waves, it is selected to be the zero-order solution of the Boltzmann equation

$$\frac{\partial f_B}{\partial t} + \vec{V} \cdot \frac{\partial f_B}{\partial \vec{r}} + \frac{e}{m_i}\left[\vec{E} + \frac{\vec{V} \times \vec{B}_o}{C}\right] \cdot \frac{\partial f_B}{\partial \vec{V}} = -\frac{V_y}{\lambda_{ce}}(f_B - f_{oB'}) - \frac{V_y}{\lambda_{ci}} f_B$$

(6.4.3)

and is therefore written as $f_{oB}(y,\vec{V})$.

We recall that

$$f_{oB'} = N_o \exp(-y/\lambda_{ci}) \left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{-\frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\} \qquad (6.4.4)$$

is the stationary Maxwellian towards which the system is driven by the collisions.

Thus, we write

$$f_B(\vec{r},\vec{V},t) = f_{oB}(y,\vec{V}) + f_{1B}(\vec{r},\vec{V},t)$$
$$\vec{E}(\vec{r},t) = \vec{E}_1(\vec{r},t) \qquad (6.4.5)$$

where $f_{1B}(\vec{r},\vec{V},t)$ and $\vec{E}_1(\vec{r},t)$ represent the perturbations due to the waves. Once again, the electrostatic approximation has been used, with $\vec{B}_1(\vec{r},t) = \vec{0}$.

Similarly, for the 'rest' ions the distribution

$$f_{oR}(\vec{V},y) = N_o\,\{1 - \exp\,(-\,y/\lambda_{ci})\}\left(\frac{m_i}{2\pi T_{io}}\right)^{3/2} \exp\left\{-\,\frac{m_i(V_x^2+V_y^2+V_z^2)}{2T_{io}}\right\} \qquad (6.4.6)$$

is the assumed pseudo-equilibrium for wave perturbations, since it is the zero-order solution of the associated Boltzmann equation

$$\frac{\partial f_R}{\partial t} + \vec{V}\cdot\frac{\partial f_R}{\partial \vec{r}} + \frac{e}{m_i}\left[\vec{E} + \frac{\vec{V}\times\vec{B}_o}{C}\right]\cdot\frac{\partial f_R}{\partial \vec{V}} = \frac{V_y}{\lambda_{ci}}\, f_{oB'}(\vec{r},\vec{V},t) \qquad (6.4.7)$$

We have seen that in the absence of any instability the distributions (6.4.2) and (6.4.6) maintain a constant total ion density (cf. Eq.(6.2.5)). However, to maintain a constant local total ion density in the presence of oscillations, we modify Eqs.(6.4.3) and (6.4.7) as follows.

For the beam ions we write

$$\frac{\partial f_B}{\partial t} + \vec{V}\cdot\frac{\partial f_B}{\partial \vec{r}} + \frac{e}{m_i}\left[\vec{E} + \frac{\vec{V}\times\vec{B}_o}{C}\right]\cdot\frac{\partial f_B}{\partial \vec{V}} = -\,\nu_e\left(f_B - \frac{n_B(\vec{r},t)}{n_{oB}(y)}\, f_{oB'}\right) - \nu_i\, f_B \qquad (6.4.8)$$

where, in general, $\nu_e(\nu_i)$ is the elastic (inelastic) collision frequency.

Here

$$n_{oB}(y) = N_o \exp(-y/\lambda_{ci}) \tag{6.4.9}$$

and, from Eq.(6.4.5),

$$n_B(\vec{r},t) = \int f_B(\vec{r},\vec{V},t)\, d\vec{V}$$

$$= n_{oB}(y) + n_{1B}(\vec{r},t)$$

where

$$n_{1B}(\vec{r},t) = \int f_{1B}(\vec{r},\vec{V},t)\, d\vec{V}$$

is the perturbation in the ion beam density due to the waves.

The Boltzmann equation for the rest ions is written as

$$\frac{\partial f_R}{\partial t} + \vec{V} \cdot \frac{\partial f_R}{\partial \vec{r}} + \frac{e}{m_i}\left(\vec{E} + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_R}{\partial \vec{V}} = \nu_i \frac{n_B(\vec{r},t)}{n_{oB}(y)} f_{oB}, \tag{6.4.10}$$

The reason for the choice of the above forms can be easily seen for the case where the collision frequencies are independent of velocity. Upon adding Eqs.(6.4.8) and (6.4.10), and integrating over velocity space, we find that the localized total ion density is, indeed, conserved.

We substitute for f_B from Eq.(6.4.5) into Eq.(6.4.8), express $\nu_e(\nu_i)$ explicity in terms of V_y and $\lambda_{ce}(\lambda_{ci})$, and linearize about $f_{oB}(\vec{V},y)$, to obtain

$$\frac{\partial f_{1B}}{\partial t} + \vec{V} \cdot \frac{\partial f_{1B}}{\partial \vec{r}} - \frac{e}{m_i} \nabla\phi_1 \cdot \frac{\partial f_{oB}}{\partial \vec{V}} = - \frac{V_y}{\lambda_{ce}} \left(f_{1B} - \frac{n_{1B}}{n_{oB}(y)} f_{oB'} \right) - \frac{V_y}{\lambda_{ci}} f_{1B} \qquad (6.4.11)$$

where $\vec{E}_1 = -\nabla\phi_1$ in the electrostatic limit. The result (6.1.7a) has also been used. Assuming the perturbed quantities to be harmonic in space and time, we may write

$$\phi_1(\vec{r},t) = \sum_k \phi_{k\omega} \exp\{i(\vec{k} \cdot \vec{r} - \omega_k t)\}$$

$$f_{1B}(\vec{r},\vec{V},t) = \sum_k f_{Bk\omega}(\vec{V}) \exp\{i(\vec{k} \cdot \vec{r} - \omega_k t)\} \qquad (6.4.12)$$

and

$$n_{1B}(\vec{r},t) = \sum_k n_{Bk\omega} \exp\{i(\vec{k} \cdot \vec{r} - \omega_k t)\}$$

Then, from Eq.(6.4.11),

$$- i \omega_k f_{Bk\omega} + i \vec{k} \cdot \vec{V} f_{Bk\omega} - \frac{e}{m_i} \phi_{k\omega} i \vec{k} \cdot \frac{\partial f_{oB}}{\partial \vec{V}} = - \frac{V_y}{\lambda_{ce}} f_{Bk\omega} + \frac{V_y}{\lambda_{ce}} \frac{n_{Bk\omega}}{n_{oB}(y)} f_{oB'} - \frac{V_y}{\lambda_{ci}} f_{Bk\omega}$$

Solving for $f_{Bk\omega}$,

$$f_{Bk\omega} = \frac{- \frac{e}{m_i} \phi_{k\omega} \vec{k}\cdot(\partial f_{oB}/\partial \vec{V})}{\{\omega_k - \vec{k}\cdot\vec{V} + i(k_{ce}+k_{ci})V_y\}} + \frac{i V_y n_{Bk\omega} f_{oB'}}{\lambda_{ce} n_{oB}(y)\{\omega_k - \vec{k}\cdot\vec{V} + i(k_{ce}+k_{ci})V_y\}}$$

where $k_{cj} = \lambda_{cj}^{-1}$ $(j = i,e)$ and $n_{Bk\omega} = \int f_{Bk\omega} \, d\vec{V}$.

We integrate the above equation over velocity space and obtain

$$n_{Bk\omega} = - \frac{e}{m_i} \phi_{k\omega} \int \frac{\vec{k}\cdot(\partial f_{oB}/\partial \vec{V}) d\vec{V}}{\{\omega - \vec{k}\cdot\vec{V} + i(k_{ce}+k_{ci})V_y\}} + \frac{i \, n_{Bk\omega}}{\lambda_{ce} \, n_{oB}(y)} \int \frac{V_y f_{oB'} \, d\vec{V}}{\{\omega_k - \vec{k}\cdot\vec{V} + i(k_{ce}+k_{ci})V_y\}} \qquad (6.4.13)$$

Since the unmagnetized ions assume straight line trajectories we may simplify the algebra by an appropriate rotation of the coordinate axes so that $\vec{V}_y$ is parallel to $\vec{k}$. Upon substituting for f_{oB} from Eq.(6.4.2), the first term on the right hand side in the above equation becomes

$$\frac{e}{m_i}\,\phi_{k\omega}\,n_{oB}(y)\left[\{1-y/\lambda_{ce}\}\left(\frac{m_i}{2\pi T_{iB}}\right)^{3/2}\frac{k}{C_{iB}^2}\int\frac{(V_y-U_o)\exp\{-(V_x^2+(V_y-U_o)^2+V_z^2)/2C_{iB}^2\}}{(\omega_k-[k-i(k_{ce}+k_{ci})]V_y)}d\vec{V}\right.$$

$$\left.+\frac{y}{\lambda_{ce}}\left(\frac{m_i}{2\pi T_{io}}\right)^{3/2}\frac{k}{C_{io}^2}\int\frac{V_y\exp\{-(V_x^2+V_y^2+V_z^2)/2C_{io}^2\}}{(\omega_k-[k-i(k_{ce}+k_{ci})]V_y)}d\vec{V}\right]$$

$$=\frac{e}{m_i}\phi_{k\omega}n_{oB}(y)\left[\{1-y/\lambda_{ce}\}\frac{1}{(2\pi C_{iB}^2)^{1/2}}\frac{k}{C_{iB}^2}\int\frac{(V_y-U_o)\exp\{-(V_y-U_o)^2/2C_{iB}^2\}}{(\omega_k-k'V_y)}dV_y\right.$$

$$\left.+\left(\frac{y}{\lambda_{ce}}\right)\frac{1}{(2\pi C_{io}^2)^{1/2}}\frac{k}{C_{io}^2}\int\frac{V_y\exp\{-V_y^2/2C_{io}^2\}}{(\omega_k-k'V_y)}dV_y\right] \tag{6.4.14}$$

where

$$k' = k - i\,(k_{ce}+k_{ci})$$

Defining $x = \dfrac{(V_y - U_o)}{\sqrt{2}\,C_{iB}}$, the first integral becomes

$$-\frac{\sqrt{2}C_{iB}}{k'}\int_{-\infty}^{+\infty}\frac{x\exp(-x^2)}{\left[x-\left(\dfrac{\omega_k-k'U_o}{\sqrt{2}k'C_{iB}}\right)\right]}dx$$

$$=\frac{(2\pi C_{iB}^2)^{1/2}}{k'}\;\frac{Z'(\theta)}{2}$$

where (6.4.15)

$$\theta = \frac{\omega_k - k'U_o}{\sqrt{2}\, k'C_{iB}}$$

and $Z'(\theta)$, the derivative of the Z-function, is defined by Eq.(3.5.7).

Similarly, the second integral may be manipulated to yield

$$\frac{(2\pi\, C_{io}^2)^{1/2}}{k'} \, \frac{Z'(\theta')}{2}$$

where (6.4.16)

$$\theta' = \frac{\omega_k}{\sqrt{2}\, k'C_{io}}$$

With the results (6.4.15) and (6.4.16), Eq.(6.4.14) reduces to

$$\frac{e}{m_i} \phi_{k\omega}\, n_{oB}(y) \frac{k}{k'} \left[\frac{\{1-(y/\lambda_{ce})\}}{2\, C_{iB}^2} Z'(\theta) + \frac{(y/\lambda_{ce})}{2\, C_{io}^2} Z'(\theta') \right] \qquad (6.4.17)$$

In a manner paralleling the one above, the integral in the second term in Eq.(6.4.13) reduces to

$$\frac{n_{oB}(y)}{2k'} Z'(\theta') \qquad (6.4.18)$$

Thus, from Eqs.(6.4.13), (6.4.17) and (6.4.18), we have

$$n_{Bk\omega} = \frac{e}{m_i} \phi_{k\omega}\, n_{oB}(y) \frac{k}{k'} \left[\frac{\{1-(y/\lambda_{ce})\}}{2\, C_{iB}^2} Z'(\theta) + \frac{(y/\lambda_{ce})}{2\, C_{io}^2} Z'(\theta') \right]$$

$$+ \frac{i\, n_{Bk\omega}}{2k'\lambda_{ce}} Z'(\theta')$$

Solving for $n_{Bk\omega}$,

$$n_{Bk\omega} = \frac{\frac{e\phi_{k\omega}}{m_i} n_{oB}(y) \frac{k}{k'} \left[\frac{\{1-(y/\lambda_{ce})\}}{2\, C_{iB}^2} Z'(\theta) + \frac{(y/\lambda_{ce})}{2\, C_{io}^2} Z'(\theta') \right]}{\left[1 - \frac{i}{2k'\lambda_{ce}} Z'(\theta') \right]} \tag{6.4.19}$$

For the rest ions we write $f_R = f_{oR} + f_{1R}$, and linearize Eq.(6.4.10) about f_{oR}. Upon substituting $\nu_i = (V_y/\lambda_{ci})$, we obtain

$$\frac{\partial f_{1R}}{\partial t} + \vec{V} \cdot \frac{\partial f_{1R}}{\partial \vec{r}} - \frac{e}{m_i} \nabla\phi_1 \cdot \frac{\partial f_{oR}}{\partial \vec{V}} = \frac{V_y}{\lambda_{ci}} \frac{n_{1B}}{n_{oB}(y)} f_{oB'}$$

For perturbations of the form (6.4.12), this reduces to

$$- i\, \omega_k f_{Rk\omega} + i\, \vec{k} \cdot \vec{V} f_{Rk\omega} - \frac{e}{m_i} \phi_{k\omega}\, i\vec{k} \cdot \frac{\partial f_{oR}}{\partial \vec{V}} = \frac{V_y}{\lambda_{ci}} \frac{n_{Bk\omega}}{n_{oB}(y)} f_{oB'}$$

from which

$$f_{Rk\omega}(\vec{V}) = \frac{-\frac{e}{m_i} \phi_{k\omega} \vec{k} \cdot (\partial f_{oR}/\partial\vec{V})}{(\omega_k - \vec{k} \cdot \vec{V})} + \frac{i\, n_{Bk\omega} V_y f_{oB'}}{\lambda_{ci}\, n_{oB}(y) \{\omega_k - \vec{k} \cdot \vec{V}\}}$$

where, in analogy with the equations (6.4.12),

$$f_{1R}(\vec{r},\vec{V},t) = \sum_k f_{Rk\omega}(\vec{V}) \exp\{i(\vec{k} \cdot \vec{r} - \omega_k t)\}$$

The density perturbation associated with $f_{Rk\omega}$ is given by

$$n_{Rk\omega} = \int f_{Rk\omega}\, d\vec{V}$$

$$= - \frac{e}{m_i} \phi_{k\omega} \int \frac{\vec{k}.(\partial f_{oR}/\partial\vec{V})}{(\omega_k - \vec{k}.\vec{V})} d\vec{V} + \frac{i\, n_{Bk\omega}}{\lambda_{ci}\, n_{oB}(y)} \int \frac{V_y f_{oB'}}{(\omega_k - \vec{k}.\vec{V})} d\vec{V} \tag{6.4.20}$$

Once again taking $\vec{k} \parallel \vec{V}_y$ and substituting for f_{oR} from Eq.(6.4.6), we find that the first integral on the right hand side reduces to

$$- \frac{n_{oR}(y)}{(2\pi C_{io}^2)^{1/2}} \frac{k}{C_{io}^2} \int_{-\infty}^{+\infty} \frac{V_y \exp(-V_y^2/2C_{io}^2)}{(\omega_k - k V_y)} dV_y$$

$$= - \frac{n_{oR}(y)}{2C_{io}^2} Z'(\omega_k/\sqrt{2}kC_{io})$$

where (6.4.21)

$$n_{oR}(y) = N_o\{1 - \exp(-y/\lambda_{ci})\}$$

With the aid of Eq.(6.4.4), the second integral in Eq.(6.4.20) modifies to

$$\frac{n_{oB}(y)}{(2\pi C_{io}^2)^{1/2}} \int_{-\infty}^{+\infty} \frac{V_y \exp(-V_y^2/2C_{io}^2)}{(\omega_k - k V_y)} dV_y$$

$$= \frac{n_{oB}(y)}{2\ k} Z'(\omega_k/\sqrt{2}kC_{io}) \qquad (6.4.22)$$

From Eqs.(6.4.19) - (6.4.22), we have

$$n_{Rk\omega} = \frac{e\phi_{k\omega}}{m_i} \frac{n_{oR}(y)}{2C_{io}^2} Z'(\theta'') + \frac{i\ n_{Bk\omega}}{2k\lambda_{ci}} Z'(\theta'')$$

$$= \frac{e\phi_{k\omega}}{m_i} Z'(\theta'') \left[\frac{n_{oR}(y)}{2C_{io}^2} + \frac{i}{2k\lambda_{ci}} n_{oB}(y) \frac{k}{k'} \left\{1 - \frac{i}{2k'\lambda_{ce}} Z'(\theta')\right\}^{-1}\right.$$

$$\left. \times \left\{\frac{[1 - (y/\lambda_{ce})]\ Z'(\theta)}{2C_{iB}^2} + \frac{(y/\lambda_{ce})}{2C_{io}^2} Z'(\theta')\right\}\right]$$

(6.4.23)

with

$$\theta'' = \frac{\omega_k}{\sqrt{2}kC_{io}} \tag{6.4.24}$$

6.5 THE ELECTRON TERM

We assume that the stationary neutralizing background electrons have a Maxwellian distribution, given by

$$f_{oe}(\vec{V}) = N_o(2\pi C_e^2)^{-3/2} \exp\{-(V_x^2+V_y^2+V_z^2)/2C_e^2\} \tag{6.5.1}$$

The perturbed electron density is given by Eq.(3.3.2) of Chapter Three, viz.,

$$n_{ek\omega} = -\frac{2\pi e}{m_e}\phi_{k\omega}\iint\left[\{1 - J_o^2(\xi)\}\frac{1}{V_\perp}\frac{\partial f_{oe}}{\partial V_\perp} - \frac{J_o^2(\xi)\, k_z\,(\partial f_{oe}/\partial V_z)}{(\omega_k - k_z V_z)}\right] V_\perp\, dV_\perp\, dV_z \tag{6.5.2}$$

where $V_\perp^2 = V_x^2+V_y^2$ and $\xi = k_\perp V_\perp/\Omega_e$. We have allowed for the fact that here the electrons, in contrast to the study undertaken in Chapter Three, no longer have a drift $\vec{V}_o$.

For a distribution of the type (6.5.1), it is seen from Eqs.(3.4.4) and (3.4.7) that we may write

$$n_{ek\omega} = -\frac{e\phi_{k\omega}}{m_e}\frac{N_o}{(2\pi C_e^2)^{1/2}}\left\{C_e^{-2}\,[\Gamma_o(b) - 1]\int_{-\infty}^{+\infty}\exp(-V_z^2/2C_e^2)\, dV_z\right.$$

$$\left. - \Gamma_o(b)\int_{-\infty}^{+\infty}\frac{k_z(\partial/\partial V_z)\{\exp(-V_z^2/2C_e^2)\}}{(\omega_k - k_z V_z)}\, dV_z\right\} \tag{6.5.3}$$

where we recall, $\Gamma_o(b) = \exp(-b)\, I_o(b)$ with $b = k_\perp^2 C_e^2/\Omega_e^2$.

With the aid of the integrals in Appendix A, this may be expressed in terms of the plasma dispersion function as

$$n_{ek\omega} = - \frac{e\phi_{k\omega}}{m_e} \frac{N_o}{C_e^2} \left[\Gamma_o(b) \left\{ 1 + \frac{Z'}{2}(\omega_k/\sqrt{2}k_z C_e) \right\} - 1 \right]$$

or using the relation (3.5.7),

$$n_{ek\omega} = \frac{e\phi_{k\omega}}{m_e} \frac{N_o}{C_e^2} \left[1 + \Gamma_o(b) \left(\frac{\omega_k}{\sqrt{2}k_z C_e} \right) Z(\omega_k/\sqrt{2}k_z C_e) \right] \tag{6.5.4}$$

6.6 SOLUTION OF THE DISPERSION RELATION

With the aid of Eqs.(6.4.19), (6.4.23) and (6.5.4), Poisson's equation

$$\phi_{k\omega} = \frac{4\pi e}{k^2} (n_{Bk\omega} + n_{Rk\omega} - n_{ek\omega})$$

becomes

$$1 - \frac{4\pi e^2}{k^2} \left[\frac{n_{oB}(y)}{m_i} \frac{k}{k'} \left\{ \frac{[1-(y/\lambda_{ce})]Z'(\theta)}{2C_{iB}^2} + \frac{(y/\lambda_{ce})}{2C_{io}^2} Z'(\theta') \right\} \left\{ 1 - \frac{i}{2k'\lambda_{ce}} Z'(\theta') \right\}^{-1} \right.$$

$$+ \frac{Z'(\theta'')}{m_i} \left\{ \frac{n_{oR}(y)}{2C_{io}^2} + \frac{i}{2k\lambda_{ci}} n_{oB}(y) \frac{k}{k'} \left[1 - \frac{i}{2k'\lambda_{ce}} Z'(\theta') \right]^{-1} \left[\frac{[1-(y/\lambda_{ce})]\, Z'(\theta)}{2C_{iB}^2} \right. \right.$$

$$\left. \left. \left. + \frac{(y/\lambda_{ce})}{2C_{io}^2} Z'(\theta') \right] \right\} - \frac{N_o}{m_e C_e^2} \left\{ 1 + \Gamma_o(b) \left(\frac{\omega_k}{\sqrt{2}k_z C_e} \right) Z(\omega_k/\sqrt{2}k_z C_e) \right\} \right] = 0$$

(6.6.1)

where, from Eqs.(6.4.14), (6.4.15), (6.4.16) and (6.4.24) respectively,

$$k' = k - i\,(k_{ce} + k_{ci}) \quad ; \quad \theta = \frac{\omega_k - k'U_o}{\sqrt{2}k'C_{iB}}$$

$$\theta' = \frac{\omega_k}{\sqrt{2}k'C_{io}} \quad ; \quad \theta'' = \frac{\omega_k}{\sqrt{2}kC_{io}}$$

In solving the dispersion relation we retain the assumption of warm electrons and cold ions, i.e., $T_e >> T_{iB}, T_{io}$.

For the warm electrons we use the approximation

$$\left|\frac{\omega_k}{\sqrt{2}k_z C_e}\right| << 1 \tag{6.6.2}$$

i.e., the wave phase speed along $\vec{B}$ is much smaller than the electron thermal speed, and hence the power series expansion [63]

$$Z(\lambda) = i\,\pi^{1/2} \exp\,(-\lambda^2) - 2\lambda\left[1 - \frac{2\lambda^2}{3} + \ldots\right] \tag{6.6.3}$$

for $|\lambda| << 1$.

For the cold ions, we assume

$$|\theta| = \left|\frac{\omega_k - k'U}{\sqrt{2}k'C_{iB}}\right| >> 1$$

$$|\theta'| = \left|\frac{\omega_k}{\sqrt{2}k'C_{io}}\right| >> 1$$

and

$$|\theta''| = \left|\frac{\omega_k}{\sqrt{2}kC_{io}}\right| >> 1$$

Then the asymptotic expansion [63]

$$Z(\lambda) \simeq -\lambda^{-1}\left(1 + \frac{1}{2\lambda^2}\right)$$

for $|\lambda| >> 1$, permits us to approximate

$$Z'(\lambda) = -2[1 + \lambda Z(\lambda)]$$

$$\simeq \lambda^{-2} \qquad (6.6.4)$$

With the approximations (6.6.3) and (6.6.4) the dispersion relation (6.6.1) simplifies to

$$1 - \frac{4\pi e^2}{k^2}\left[\frac{n_{oB}(y)}{m_i}\frac{k}{k'}\left\{\frac{[1-(y/\lambda_{ce})]}{2C_{iB}^2}\left(\frac{\sqrt{2}k'C_{iB}}{(\omega_k - k'U_o)}\right)^2 + \frac{(y/\lambda_{ce})}{2C_{io}^2}\left(\frac{\sqrt{2}k'C_{io}}{\omega_k}\right)^2\right\}\right.$$

$$\times\left\{1 - \frac{i}{2k'\lambda_{ce}}\left(\frac{\sqrt{2}k'C_{io}}{\omega_k}\right)^2\right\}^{-1}$$

$$+ \frac{1}{m_i}\left(\frac{\sqrt{2}kC_{io}}{\omega_k}\right)^2\left\{\frac{n_{oR}(y)}{2C_{io}^2} + \frac{i}{2k\lambda_{ci}}n_{oB}(y)\frac{k}{k'}\left[1 - \frac{i}{2k'\lambda_{ce}}\left(\frac{\sqrt{2}k'C_{io}}{\omega_k}\right)^2\right]^{-1}\right.$$

$$\left.\times\left[\frac{[1-(y/\lambda_{ce})]}{2C_{iB}^2}\left(\frac{\sqrt{2}k'C_{iB}}{\omega_k - k'U_o}\right)^2 + \frac{(y/\lambda_{ce})}{2C_{io}^2}\left(\frac{\sqrt{2}k'C_{io}}{\omega_k}\right)^2\right]\right\}$$

$$\left. - \frac{N_o}{m_eC_e^2}\left\{1 + i\,\pi^{1/2}\,\Gamma_o(b)\frac{\omega_k}{\sqrt{2}k_zC_e}\right\}\right] = 0 \qquad (6.6.5)$$

To proceed further we assume, in addition to $\lambda_{ce} >> \lambda_{ci}$, that

$$k\,\lambda_{ce} >> k\,\lambda_{ci} >> 1 \tag{6.6.6}$$

In terms of the definition $k_{cj} = \lambda_{cj}^{-1}$ $(j = i,e)$ we have

$$k >> k_{ci} >> k_{ce}.$$

This means that the wavelength of the fluctuations is assumed to be much smaller than the inelastic charge-transfer mean free path, which, in turn, is smaller than the elastic mean free path. Typical measured values in the DP device [(78)] are $\lambda \sim 2$ cm and $\lambda_{ci} \sim 20$ cm, and the investigations in Section 6.3 have inferred $0{,}1 < (\lambda_{ci}/\lambda_{ce}) < 0{,}2$. Thus, (6.6.6) is a reasonable approximation.

Then

$$\frac{1}{k'} = \frac{1}{k-i(k_{ce}+k_{ci})} \simeq \frac{1}{k}\left\{1 + \frac{i(k_{ce}+k_{ci})}{k}\right\}$$

For $\omega_k = \omega_k^r + i\,\gamma_k$, with $|\gamma_k| << \omega_k^r$, we use the approximation (3.3.10), viz.,

$$\frac{1}{\omega_k^2} \simeq \frac{1}{(\omega_k^r)^2}\left(1 - \frac{2i\gamma_k}{\omega_k^r}\right)$$

The dispersion relation (6.6.5) then becomes

$$1 - \frac{4\pi e^2}{k^2}\left[\frac{n_{oB}(y)}{m_i}\left\{1 + \frac{i(k_{ce}+k_{ci})}{k}\right\}\left\{\frac{[1-(y/\lambda_{ce})]}{2C_{iB}^2}\,\frac{2[k^2-2ik(k_{ce}+k_{ci})]C_{iB}^2}{[(\omega_k^r-kU_o)+i(\gamma_k+\{k_{ce}+k_{ci}\}U_o)]^2}\right.\right.$$

$$+ \frac{(y/\lambda_{ce})}{2C_{io}^2} \frac{2C_{io}^2[k^2-2ik(k_{ce}+k_{ci})]}{(\omega_k^r)^2} \left(1- \frac{2i\gamma_k}{\omega_k^r}\right)\Bigg\}$$

$$+ \frac{2k^2C_{io}^2}{m_i(\omega_k^r)^2}\left(1- \frac{2i\gamma_k}{\omega_k^r}\right) \left\{\frac{n_{oR}(y)}{2C_{io}^2} + \frac{i\ n_{oB}(y)}{2k\lambda_{ci}} \left[\frac{\{1-(y/\lambda_{ce})\}}{2C_{iB}^2}\right.\right.$$

$$\left.\left.\times \frac{2C_{iB}^2[k^2-2ik(k_{ce}+k_{ci})]}{[(\omega_k^r-kU_o)+i(\gamma_k+\{k_{ce}+k_{ci}\}U_o)]^2} + \frac{(y/\lambda_{ce})}{2C_{io}^2} \frac{[k^2-2ik(k_{ce}+k_{ci})]}{(\omega_k^r)^2} 2C_{io}^2\left(1- \frac{2i\gamma_k}{\omega_k^r}\right)\right]\right\}$$

$$\left.- \frac{N_o}{m_eC_e^2}\left\{1 + \frac{i\ \pi^{1/2}\ \Gamma_o(b)\ (\omega_k^r + i\gamma_k)}{\sqrt{2}k_zC_e}\right\}\right] = 0 \qquad (6.6.7)$$

where terms of second order and higher have been neglected.

If we assume, in addition, that

$$\left|\frac{\gamma_k+ \{k_{ce}+k_{ci}\}U_o}{(\omega_k^r-kU_o)}\right| << 1 \qquad (6.6.8)$$

then

$$\frac{1}{[(\omega_k^r-kU_o)+i(\gamma_k+\{k_{ce}+k_{ci}\}U_o)]^2} \simeq \frac{1}{(\omega_k^r-kU_o)^2}\left\{1- \frac{2i(\gamma_k+\{k_{ce}+k_{ci}\}U_o)}{(\omega_k^r-kU_o)}\right\}$$

and Eq.(6.6.7) modifies to

$$1 - \frac{4\pi e^2}{k^2}\left[\frac{n_{oB}(y)}{m_i}\left\{1 + \frac{i(k_{ce}+k_{ci})}{k}\right\}\left\{\frac{[1-(y/\lambda_{ce})]k^2}{(\omega_k^r-kU_o)^2}\left(1 - \frac{2i(k_{ce}+k_{ci})}{k}\right)\right.\right.$$

$$\left.\times\left(1 - \frac{2i(\gamma_k+\{k_{ce}+k_{ci}\}U_o)}{(\omega_k^r-kU_o)}\right) + \frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\left(1 - \frac{2i(k_{ce}+k_{ci})}{k}\right)\left(1 - \frac{2i\gamma_k}{\omega_k^r}\right)\right\}$$

$$+\frac{2k^2C_{io}^2}{m_i(\omega_k^r)^2}\left(1-\frac{2i\gamma_k}{\omega_k^r}\right)\left\{\frac{n_{oR}(y)}{2C_{io}^2}+\frac{i\,n_{oB}(y)}{2k\lambda_{ci}}\left[\frac{\{1-(y/\lambda_{ce})\}k^2}{(\omega_k^r-kU_o)^2}\left(1-\frac{2i(k_{ce}+k_{ci})}{k}\right)\right.\right.$$

$$\left.\left.\times\left(1-\frac{2i(\gamma_k+\{k_{ce}+k_{ci}\}U_o)}{(\omega_k^r-kU_o)}\right)+\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\left(1-\frac{2i(k_{ce}+k_{ci})}{k}\right)\left(1-\frac{2i\gamma_k}{\omega_k^r}\right)\right]\right\}$$

$$-\frac{N_o}{m_eC_e^2}\left\{1+\frac{i\,\pi^{1/2}\,\Gamma_o(b)\,(\omega_k^r+i\,\gamma_k)}{\sqrt{2}k_zC_e}\right\}\Bigg]=0 \qquad (6.6.9)$$

Upon neglecting the product of small quantities, the real part turns out to be

$$1-\frac{4\pi e^2}{k^2}\left[\frac{n_{oB}(y)}{m_i}\left\{\frac{\{1-(y/\lambda_{ce})\}k^2}{(\omega_k^r-kU_o)^2}+\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\right\}+\frac{n_{oR}(y)k^2}{m_i(\omega_k^r)^2}-\frac{N_o}{m_eC_e^2}\right]=0$$

i.e.,

$$\frac{m_i}{4\pi e^2}+\frac{N_om_i}{m_ek^2C_e^2}=n_{oB}(y)\left\{\frac{1}{(\omega_k^r-kU_o)^2}+(y/\lambda_{ce})\left[\frac{1}{(\omega_k^r)^2}-\frac{1}{(\omega_k^r-kU_o)^2}\right]\right.$$

$$\left.+\frac{\{N_o-n_{oB}(y)\}}{(\omega_k^r)^2}\right\}$$

where we have used Eqs.(6.4.9) and (6.4.21) to replace $n_{oR}(y)$. This may be rewritten as

$$n_{oB}(y)\left[\left\{\frac{1}{(\omega_k^r-kU_o)^2}-\frac{1}{(\omega_k^r)^2}\right\}\{1-(y/\lambda_{ce})\}\right]=N_o\left[\frac{1}{k^2C_s^2}-\frac{1}{(\omega_k^r)^2}\right]+\frac{N_o}{\omega_{pi}^2}$$

where ω_{pi} is the ion plasma frequency and C_s the ion sound speed. We now make the assumptions

$$(\omega_k^r)^2 \gg (\omega_k^r-kU_o)^2 \qquad (6.6.10a)$$

and

$$(\omega_k^r)^2 >> k^2C_s^2 \tag{6.6.10b}$$

The physical implications of these approximations will be discussed *a posteriori*. The above equation then reduces to

$$\frac{N_o \exp(-y/\lambda_{ci}) \{1 - (y/\lambda_{ce})\}}{(\omega_k^r - kU_o)^2} = N_o \left\{\frac{1}{k^2C_s^2} + \frac{1}{\omega_{pi}^2}\right\}$$

from which

$$(\omega_k^r - kU_o)^2 = k^2C_s^2 \left\{1 + \frac{k^2C_s^2}{\omega_{pi}^2}\right\}^{-1} \exp(-y/\lambda_{ci}) \{1 - y/\lambda_{ce}\}$$

Solving for ω_k^r, we have

$$\omega_k^r = k\left[U_o \pm \frac{C_s}{(1+k^2\lambda_D^2)^{1/2}} \exp(-y/2\lambda_{ci}) \{1 - (y/\lambda_{ce})\}^{1/2}\right] \tag{6.6.11}$$

Thus, the assumption (6.6.10b) implies

$$k^2\left[U_o \pm \frac{C_s}{(1+k^2\lambda_D^2)^{1/2}} \exp(-y/2\lambda_{ci}) \{1 - (y/\lambda_{ce})\}^{1/2}\right]^2 >> k^2C_s^2$$

Since $(1+k^2\lambda_D^2)^{-1/2} < 1$, $0 < \exp(-y/2\lambda_{ci}) \leq 1$ for $y \geq 0$, and for $0 \leq y \leq \lambda_{ce}$, $\{1 - (y/\lambda_{ce})\}^{1/2} \leq 1$, this restriction is satisfied if

$$U_o >> C_s \tag{6.6.12}$$

i.e., the ion beam speed is much larger than the ion sound speed - as is usually the case in experimental studies of the ion sound instability. It is easy to show that this condition also justifies the assumption (6.6.10a).

The positive sign in Eq.(6.6.11) corresponds to the so called fast-beam ion acoustic mode, and the negative sign to the slow-beam ion acoustic mode. Since electron density $n_e = N_o$, the said equation may be written as

$$\frac{\omega_k^r}{k} = U_o \pm \frac{\{N_o \exp(-y/\lambda_{ci})\}^{1/2}}{N_o^{1/2}} \frac{C_s}{(1+k^2\lambda_D^2)^{1/2}} \{1 - (y/\lambda_{ce})\}^{1/2}$$

i.e.,

$$\frac{\omega_k^r}{k} = U_o \pm \left(\frac{n_{oB}}{n_e}\right)^{1/2} \frac{C_s}{(1+k^2\lambda_D^2)^{1/2}} \{1 - (y/\lambda_{ce})\}^{1/2} \qquad (6.6.11a)$$

where from Eq(6.4.9) n_{oB} is the ion beam density. In the absence of elastic collisions ($\lambda_{ce} \to \infty$), this reduces to

$$\frac{\omega_k^r}{k} = U_o \pm \left(\frac{n_{oB}}{n_e}\right)^{1/2} \frac{C_s}{(1+k^2\lambda_D^2)^{1/2}}$$

For $k^2\lambda_D^2 << 1$, this corresponds to the expression derived by SATO *et al* [84]. Moreover, in the complete absence of collisions ($\lambda_{ce}, \lambda_{ci} \to \infty$), Eq.(6.6.11) modifies to

$$\omega_k^r = k\{U_o \pm C_s (1+k^2\lambda_D^2)^{-1/2}\} \qquad (6.6.13)$$

which is the usual collisionless ion acoustic wave frequency as determined in the electron rest-frame [37].

The imaginary part of the dispersion relation (6.6.9) gives

$$\frac{n_{oB}(y)}{m_i}\left\{\frac{[1-(y/\lambda_{ce})]k^2}{(\omega_k^r-kU_o)^2}\left(-\frac{2(k_{ce}+k_{ci})}{k}-\frac{2(\gamma_k+\{k_{ce}+k_{ci}\}U_o)}{(\omega_k^r-kU_o)}\right)\right.$$

$$\left.+\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\left(-\frac{2\{k_{ce}+k_{ci}\}}{k}-\frac{2\gamma_k}{\omega_k^r}\right)+\left(\frac{k_{ce}+k_{ci}}{k}\right)\left[\frac{[1-(y/\lambda_{ce})]k^2}{(\omega_k^r-kU_o)^2}+\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\right]\right\}$$

$$-\frac{2\gamma_k}{\omega_k^r}\frac{k^2\, n_{oR}(y)}{m_i(\omega_k^r)^2}+\frac{2k^2C_{io}^2}{m_i(\omega_k^r)^2}\frac{n_{oB}(y)}{2k\lambda_{ci}}\left\{\frac{[1-(y/\lambda_{ce})]k^2}{(\omega_k^r-kU_o)^2}+\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\right\}$$

$$-\frac{N_o}{m_eC_e^2}\pi^{1/2}\,\Gamma_o(b)\,\frac{\omega_k^r}{\sqrt{2}k_zC_e}=0$$

i.e.,

$$\frac{n_{oB}(y)}{m_i}\left\{-\frac{(k_{ce}+k_{ci})}{k}\left[\frac{k^2}{(\omega_k^r-kU_o)^2}-(y/\lambda_{ce})\left\{\frac{1}{(\omega_k^r-kU_o)^2}-\frac{1}{(\omega_k^r)^2}\right\}k^2\right]\right.$$

$$\left.+\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^2}\left[-\frac{2\gamma_k}{\omega_k^r}\right]\right\}+\frac{n_{oB}(y)}{m_i}\frac{\{1-(y/\lambda_{ce})\}k^2}{(\omega_k^r-kU_o)^2}\left[-\frac{2\gamma_k}{(\omega_k^r-kU_o)}-\frac{2(k_{ce}+k_{ci})U_o}{(\omega_k^r-kU_o)}\right]$$

$$-\frac{2\gamma_k}{\omega_k^r}\frac{k^2n_{oR}(y)}{m_i(\omega_k^r)^2}+\frac{2k^2C_{io}^2}{m_i(\omega_k^r)^2}\frac{n_{oB}(y)}{2k\lambda_{ci}}\left[\frac{k^2}{(\omega_k^r-kU_o)^2}-(y/\lambda_{ce})\left\{\frac{1}{(\omega_k^r-kU_o)^2}-\frac{1}{(\omega_k^r)^2}\right\}k^2\right]$$

$$-\frac{N_o\pi^{1/2}}{m_eC_e^2}\Gamma_o\frac{\omega_k^r}{\sqrt{2}k_zC_e}=0$$

Using approximation (6.6.10a) this reduces to

$$2\gamma_k\left[-\frac{n_{oB}(y)}{m_i}\frac{(y/\lambda_{ce})k^2}{(\omega_k^r)^3}-\frac{k^2n_{oR}(y)}{m_i(\omega_k^r)^3}-\frac{n_{oB}(y)\{1-(y/\lambda_{ce})\}k^2}{m_i\,(\omega_k^r-kU_o)^3}\right]$$

$$=\frac{n_{oB}(y)}{m_i}\left\{\frac{(k_{ce}+k_{ci})}{k}\left[\frac{k^2\{1-(y/\lambda_{ce})\}}{(\omega_k^r-kU_o)^2}\right]\right\}$$

$$+\frac{2n_{oB}(y)}{m_i}\frac{k^2\{1-(y/\lambda_{ce})\}(k_{ce}+k_{ci})U_o}{(\omega_k^r-kU_o)^3}-\frac{2k^2C_{io}^2}{m_i(\omega_k^r)^2}\frac{n_{oB}(y)}{2k\lambda_{ci}}\frac{k^2\{1-(y/\lambda_{ce})\}}{(\omega_k^r-kU_o)^2}$$

$$+\frac{N_o}{m_eC_e^2}\pi^{1/2}\Gamma_o\frac{\omega_k^r}{\sqrt{2}k_zC_e}$$

Thus

$$\gamma_k=\frac{-\dfrac{n_{oB}(y)}{m_i}\dfrac{\{1-(y/\lambda_{ce})\}}{(\omega_k^r-kU_o)^2}\left\{\left(\dfrac{k_{ce}+k_{ci}}{k}\right)\left[1+\dfrac{2kU_o}{(\omega_k^r-kU_o)}\right]-\dfrac{k^2C_{io}^2}{(\omega_k^r)^2k\lambda_{ci}}\right\}-\dfrac{N_o}{m_eC_e^2}\dfrac{\pi^{1/2}\Gamma_o}{k^2}\dfrac{\omega_k^r}{\sqrt{2}k_zC_e}}{2\left[\dfrac{n_{oB}(y)}{m_i}\left\{\dfrac{1}{(\omega_k^r-kU_o)^3}+(y/\lambda_{ce})\left(\dfrac{1}{(\omega_k^r)^3}-\dfrac{1}{(\omega_k^r-kU_o)^3}\right)\right\}+\dfrac{n_{oR}(y)}{m_i(\omega_k^r)^3}\right]}$$

(6.6.14)

In the absence of collisions ($\lambda_{ce}, \lambda_{ci} \to \infty$),

$k_{ce}, k_{ci} \to 0$

$n_{oB} = N_o \exp(-y/\lambda_{ci}) = N_o$

$n_{oR} = N_o\{1-\exp(-y/\lambda_{ci})\} = 0$

Then for the slow-beam ion acoustic mode (corresponding to the negative sign in Eq.(6.6.13)), the growth rate (6.6.14) reduces to

$$\frac{\gamma_k}{\omega_k^r}=\left(\frac{\pi m_e}{8m_i}\right)^{1/2}\frac{k}{k_z}\frac{\Gamma_o(b)}{(1+k^2\lambda_D^2)^{3/2}} \qquad (6.6.15)$$

A.J. HAYZEN [61] has shown from theoretical considerations that for $U_o >> C_s$,

$$\frac{\gamma_k}{\omega_k^r}\simeq\frac{k_i}{k_r}$$

where $k = k_r + i\,k_i$ is complex when ω is real (as in experiments). Computational studies relaxed this result into the region $U_o > C_s$. Hence from Eq.(6.6.15) we may write

$$\frac{\gamma_k}{\omega_k^r} = \frac{k_i}{k_r} = \left(\frac{\pi m_e}{8m_i}\right)^{1/2} \frac{k}{k_z} \frac{\Gamma_o(b)}{(1+k^2\lambda_D^2)^{3/2}}$$

This is identical to the result of BARRETT *et al* [37].

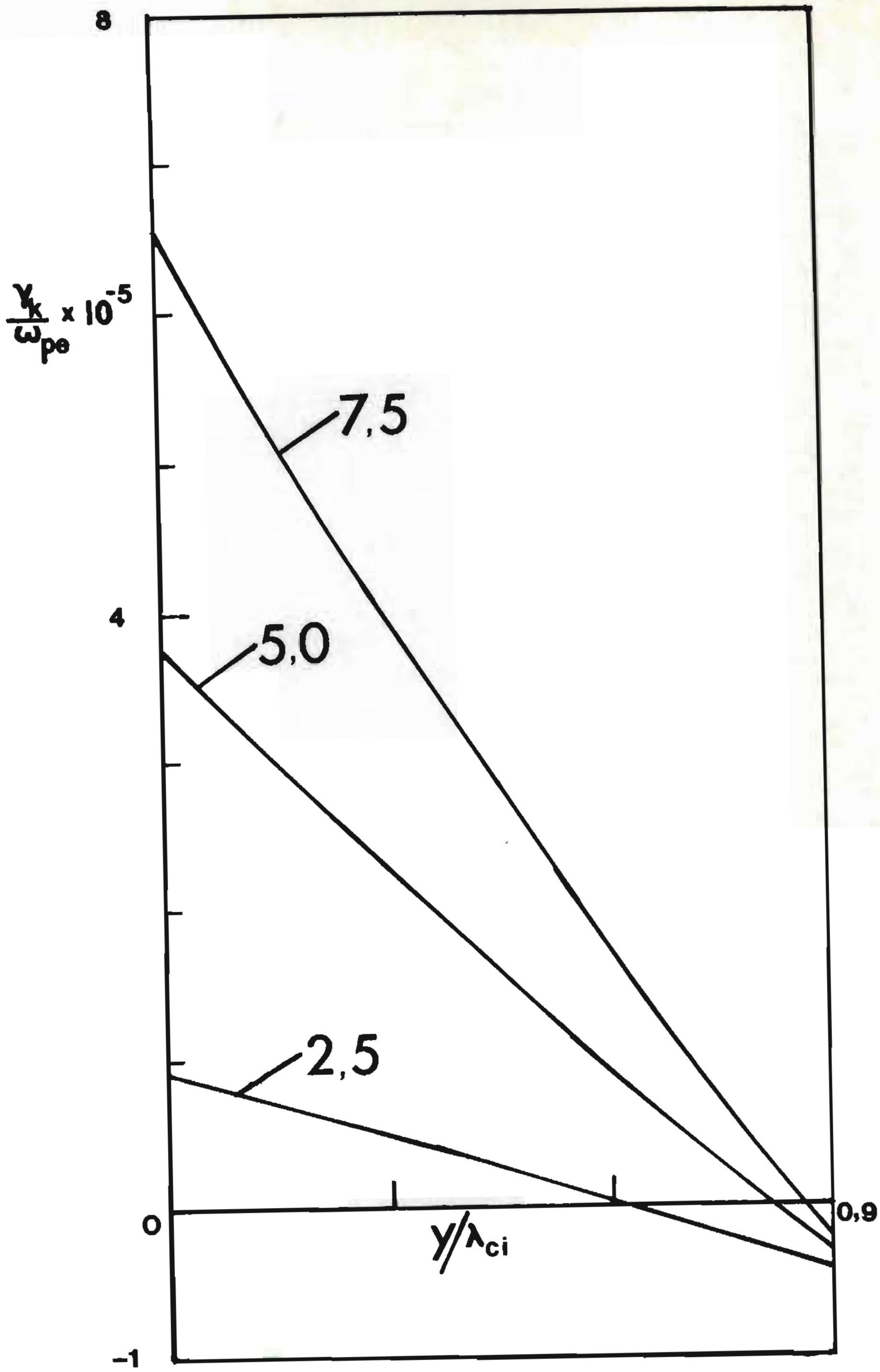
The growth rate (6.6.14) which describes the linear behaviour of the crossfield ion acoustic instability in the presence of elastic and inelastic charge-transfer collisions will be numerically considered in the next section.

6.7 NUMERICAL STUDIES

A graphical study of the variation of the normalized growth rate (γ_k/ω_{pe}) with distance into the plasma (y/λ_{ci}) has been conducted as a function of the different variable parameters. The examination is restricted to the slow-beam ion acoustic wave since measurements in the DP device have revealed this to be the mode travelling down the target plasma [41, 62]. Wherever possible, parameter values typical of the argon plasma (with $(m_i/m_e) = 73440$) in the DP device have been chosen.

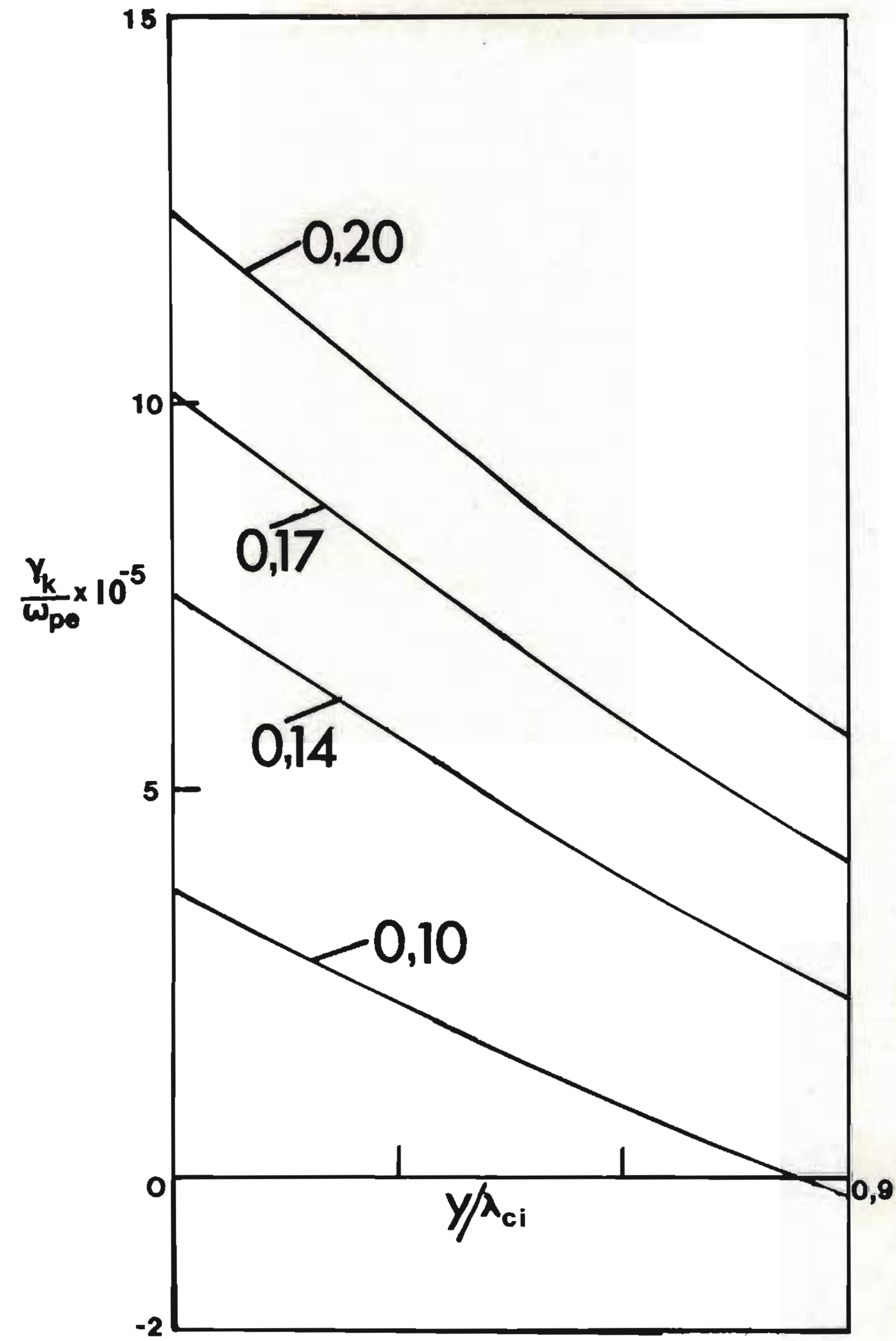
The curves in Fig. 6.4 correspond to different values of (U_o/C_s). For a given (y/λ_{ci}), the growth rate is found to increase with (U_o/C_s). This may be explained as follows. An increase in ion

Figure 6.4



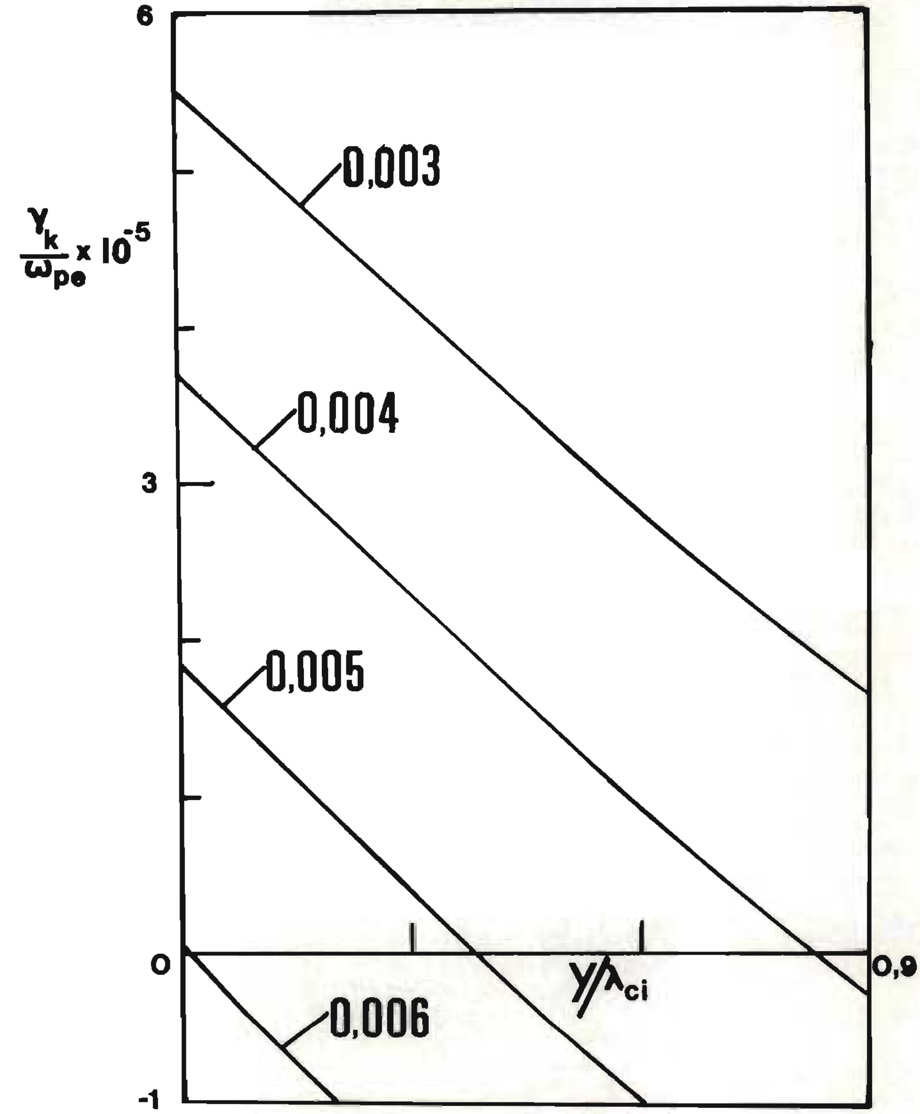
Normalized growth rate γ_k/ω_{pe} for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. $m_i/m_e = 73440$, $T_{io}/T_e = 0,1$, $\lambda_{ci}/\lambda_{ce} = 0,1$, $k\lambda_D = 0,1$, $\lambda_D/\lambda_{ci} = 0,004$, $k_z/k = 0,03$, $\omega_{pe}/\Omega_e = \sqrt{2}$. The parameter labelling the curves is U_o/C_s.

<u>Figure 6.5</u>



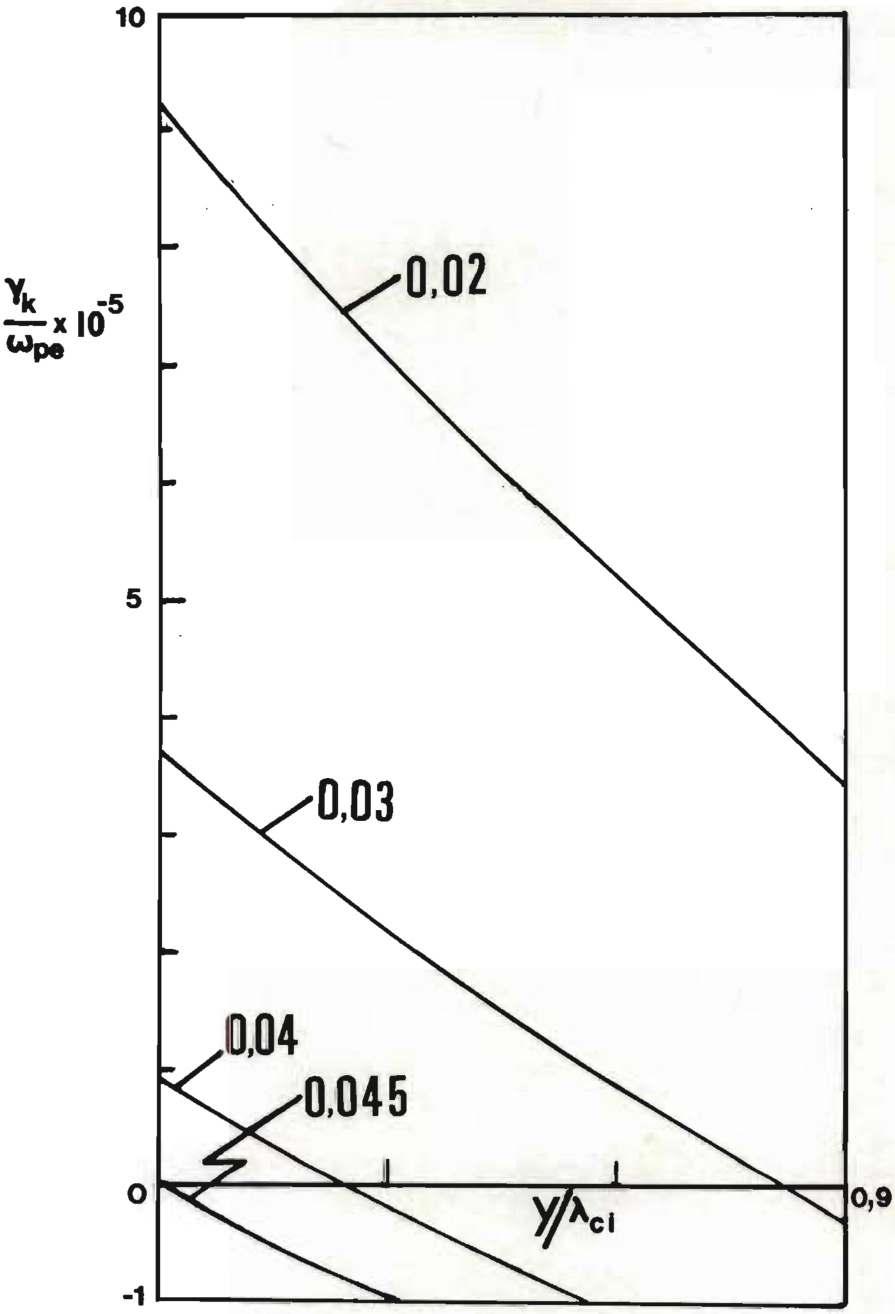
Normalized growth rate γ_k/ω_{pe} for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. $m_i/m_e = 73440$, $T_{io}/T_e = 0,1$, $\lambda_{ci}/\lambda_{ce} = 0,1$, $\lambda_D/\lambda_{ci} = 0,004$, $k_z/k = 0,03$, $U_o/C_s = 5$, $\omega_{pe}/\Omega_e = \sqrt{2}$. The parameter labelling the curves is $k\lambda_D$.

Figure 6.6



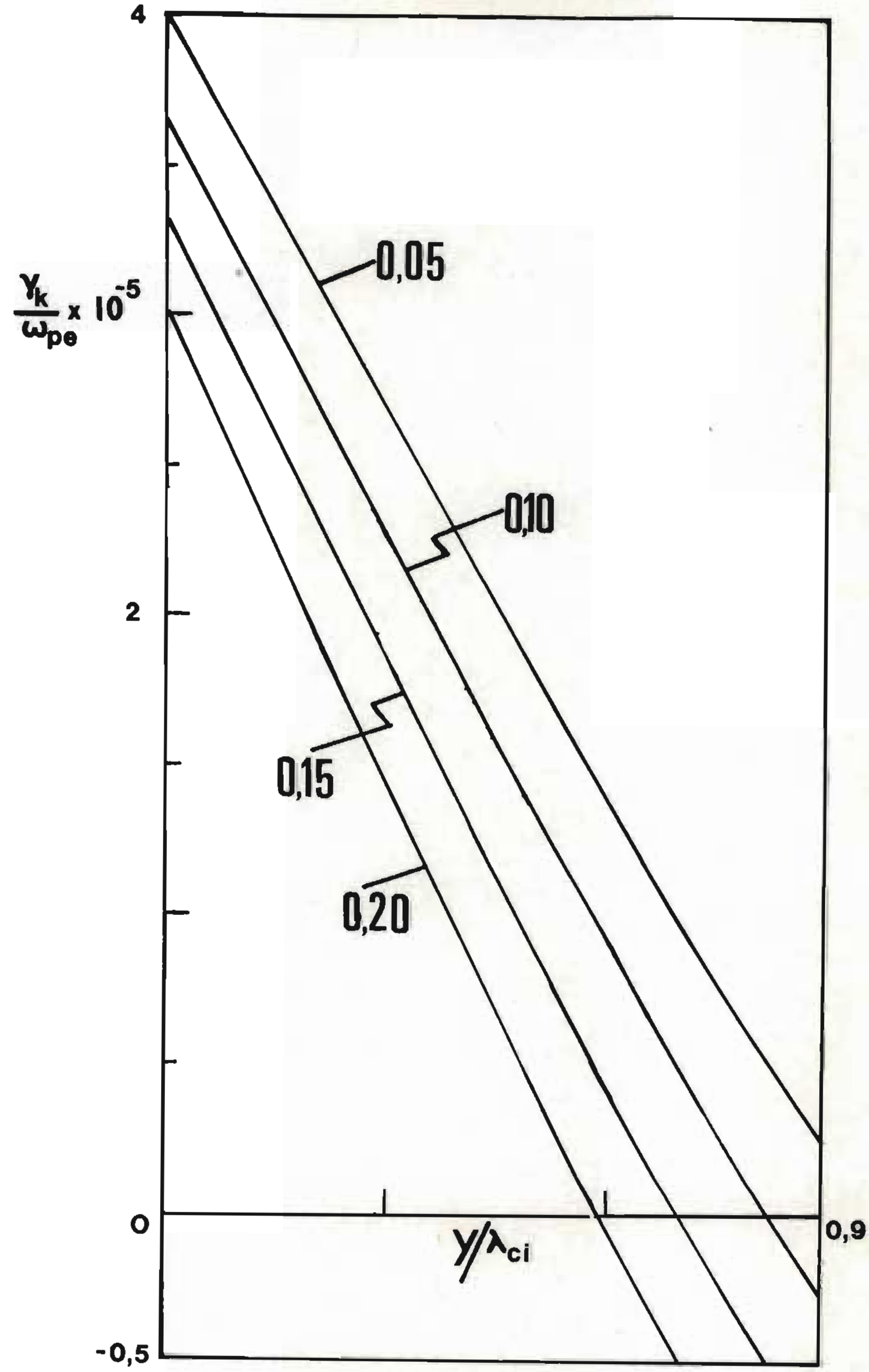
Normalized growth rate γ_k/ω_{pe} for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. m_i/m_e = 73440, T_{io}/T_e = 0,1, U_o/C_s = 5, $k\lambda_D$ = 0,1, k_z/k = 0,03, ω_{pe}/Ω_e = $\sqrt{2}$. The parameter labelling the curves is λ_D/λ_{ci}.

Figure 6.7



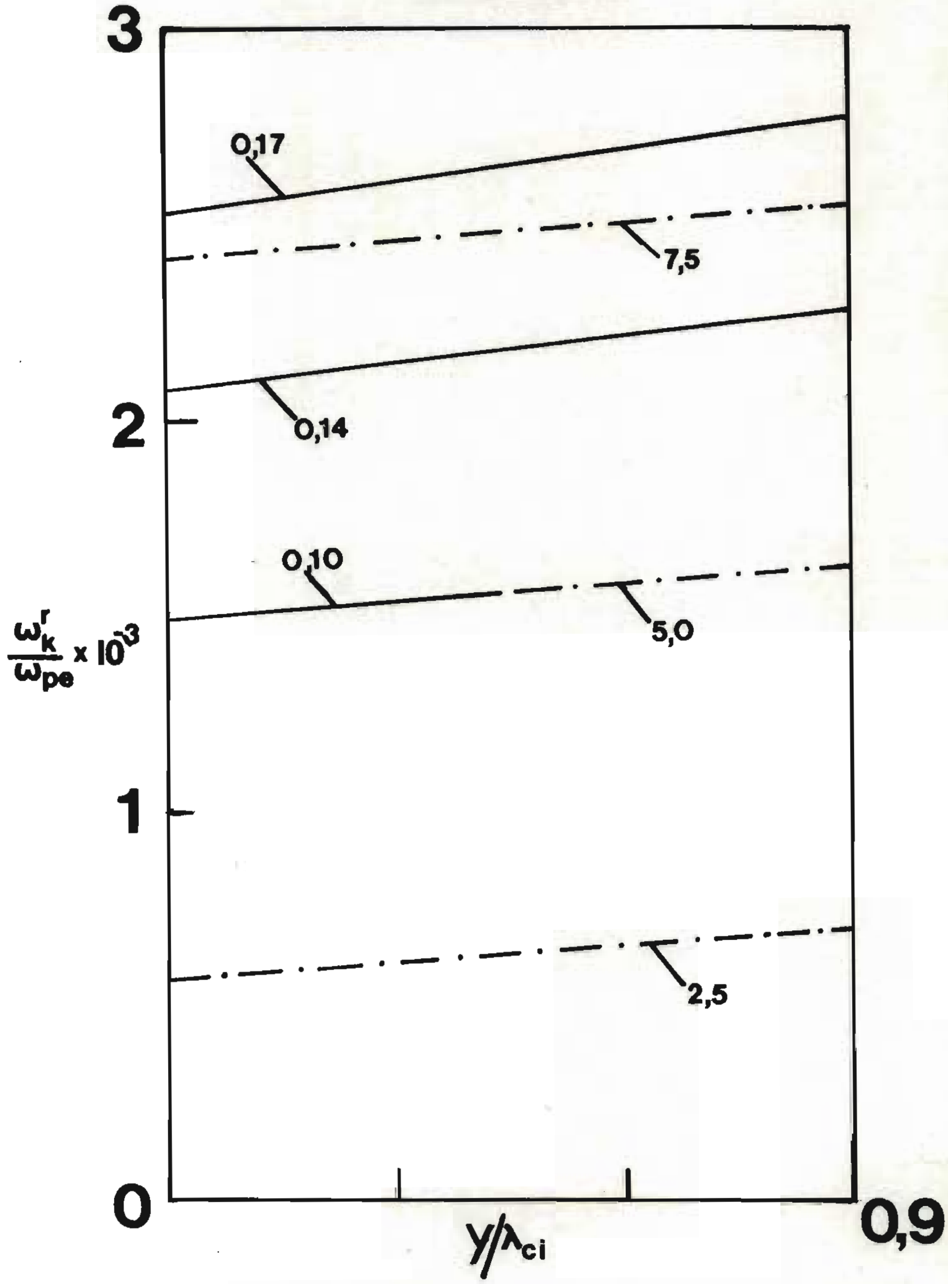
Normalized growth rate γ_k/ω_{pe} for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. $m_i/m_e = 73440$, $T_{io}/T_e = 0,1$, $\lambda_{ci}/\lambda_{ce} = 0,1$, $U_o/C_s = 5$, $k\lambda_D = 0,1$, $\lambda_D/\lambda_{ci} = 0,004$, $\omega_{pe}/\Omega_e = \sqrt{2}$. The parameter labelling the curves is k_z/k.

Figure 6.8



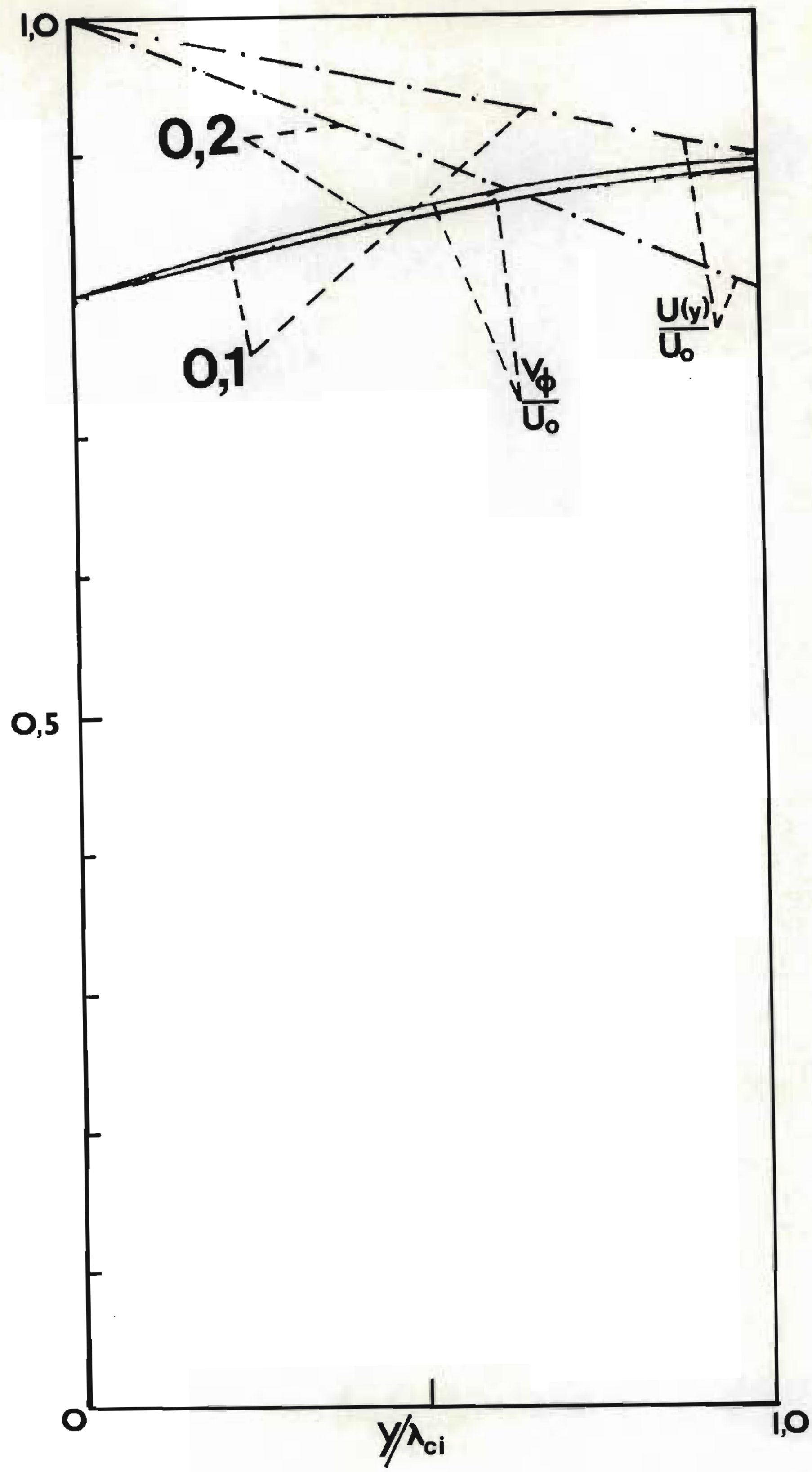
Normalized growth rate γ_k/ω_{pe} for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. $m_i/m_e = 73440$, $T_{io}/T_e = 0,1$, $U_o/C_s = 5$, $k\lambda_D = 0,1$, $\lambda_D/\lambda_{ci} = 0,004$, $k_z/k = 0,03$, $\omega_{pe}/\Omega_e = \sqrt{2}$. The parameter labelling the curves is $\lambda_{ci}/\lambda_{ce}$.

Figure 6.9(a)



Normalized frequency ω_k^r/ω_{pe} for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. $m_i/m_e = 73440$, $\lambda_{ci}/\lambda_{ce} = 0,1$. For the solid lines (——) $U_o/C_s = 5$ and the parameter labelling the curves is $k\lambda_D$; the broken lines (—·—·—) are for $k\lambda_D = 0,1$, with U_o/C_s as the variable parameter.

Figure 6.9(b)



Normalized wave phase speed V_ϕ/U_o(———) and normalized average beam speed $U(y)/U_o$(-··-·-··-) for the ion acoustic instability as a function of the normalized distance y/λ_{ci}. $m_i/m_e = 73440$, $U_o/C_s = 5$, $k\lambda_D = 0{,}1$. The parameter labelling the curves is $\lambda_{ci}/\lambda_{ce}$.

beam speed results in more free energy being available to drive the instability. Thus wave growth is enhanced. The observed reduction in growth rate with (y/λ_{ci}), for a given (U_o/C_s), may be attributed to the decrease in the ion beam speed $U(y)$ $(= U_o\{1-(\lambda_{ci}/\lambda_{ce})\ (y/\lambda_{ci})\})$ and the number of beam ions supporting the waves, viz., $n_B(y) = N_o \exp(-y/\lambda_{ci})$, as we move down the plasma into the region $y \geq 0$.

The parameter labelling the curves in Fig. 6.5 is $k\lambda_D$. An explanation of the increase in growth rate with $k\lambda_D$, for a given (y/λ_{ci}), may be offered in terms of the wave phase velocity. We notice from Eq.(6.6.11) that in the limit $k_z \ll k$ the slow mode satisfies the condition $0 < \omega_k < \vec{k}.\vec{U}_o$, and is therefore a negative energy mode (12). Such a mode grows when energy is extracted from it, i.e.,when the wave 'sees' a negative gradient to the particle distribution function. It damps when it 'sees' a positive gradient (energy gained by the wave). From Eq.(6.3.3) we see that the ion beam distribution function in the V_y direction, at a distance y into the plasma, may be qualitatively represented as follows.

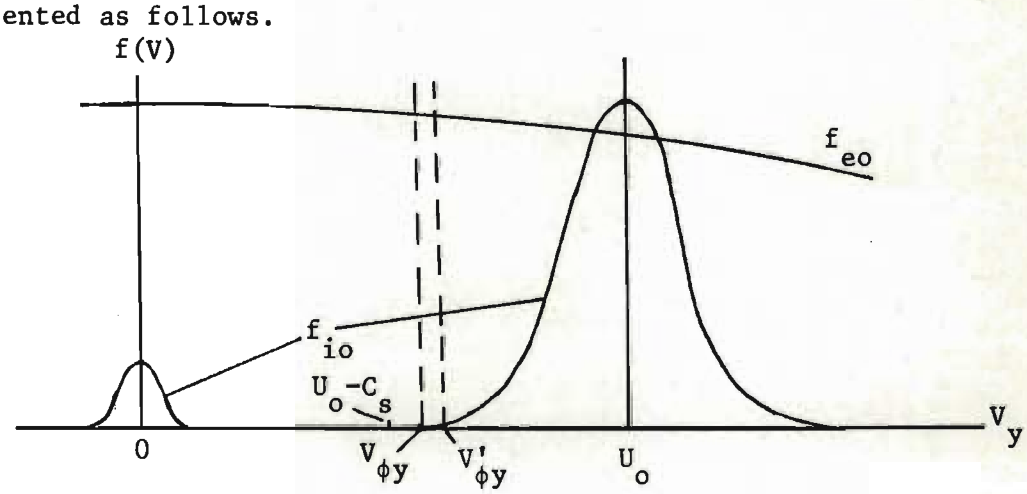


Figure 6.10

The projection of the slow wave phase velocity (6.6.11) in the y direction is

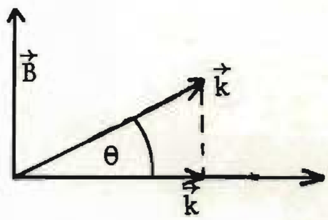
$$V_{\phi y} = \frac{\omega_k^r}{k_y} = \frac{k}{k_y}\left[U_o - \frac{C_s}{(1+k^2\lambda_D^2)^{1/2}} \exp\,(-y/2\lambda_{ci})\,\{1-(y/\lambda_{ce})\}^{1/2}\right]$$

For fixed (U_o/C_s), (k/k_z) and (y/λ_{ci}), $V_{\phi y}$ increases with $k\lambda_D$. $V_{\phi y}$ shifts to $V'_{\phi y}$ in the figure above. We see that the electrons, with an associated negative velocity distribution slope, promote wave growth, while the ions, with a positive slope, cause damping. However, for $T_e >> T_{iB}$ ion Landau damping is very small, and the shift from $V_{\phi y}$ to $V'_{\phi y}$ does very little to alter this. On the other hand, the change in $\partial f_e/\partial V$ is much more significant, and since it $(\partial f_e/\partial V)$ becomes more negative as we move from $V_{\phi y}$ to $V'_{\phi y}$, electron Landau damping increases and wave growth (for a negative energy mode) is enhanced. Thus an increase in $V_{\phi y}$, corresponding to an increase in $k\lambda_D$, results in a positive change in growth rate.

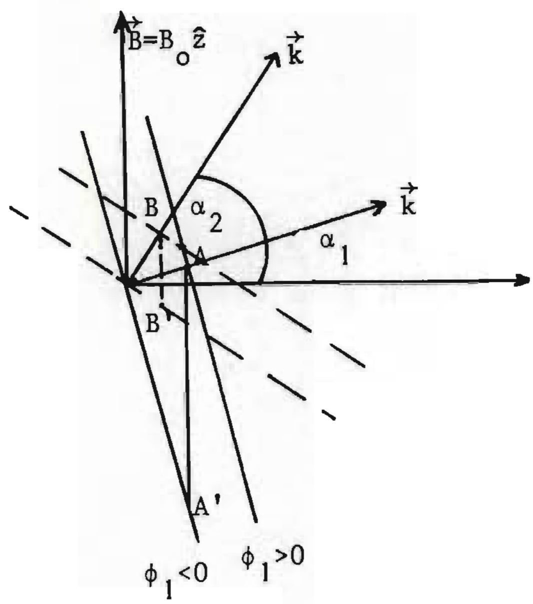
In Fig. 6.6 the variable parameter is (λ_D/λ_{ci}). Our results show that for $(\lambda_D/\lambda_{ci}) > 0{,}006$, the wave is completely damped. This may be so because for a fixed λ_D (i.e., constant electron density and temperature), (λ_D/λ_{ci}) increases as λ_{ci} gets smaller. Hence collisions get stronger and it is possible to reach the state where collisional damping dominates over wavegrowth due to Landau damping and a net damping of the wave results. Experimentally, λ_{ci} may be reduced by increasing the neutral atom density in the target plasma.

The behaviour of the growth rate curves in Fig.6.7 is similar to that in Fig.6.6. Here we notice that for $(k_z/k) > 0{,}045$ the mode is completely damped.

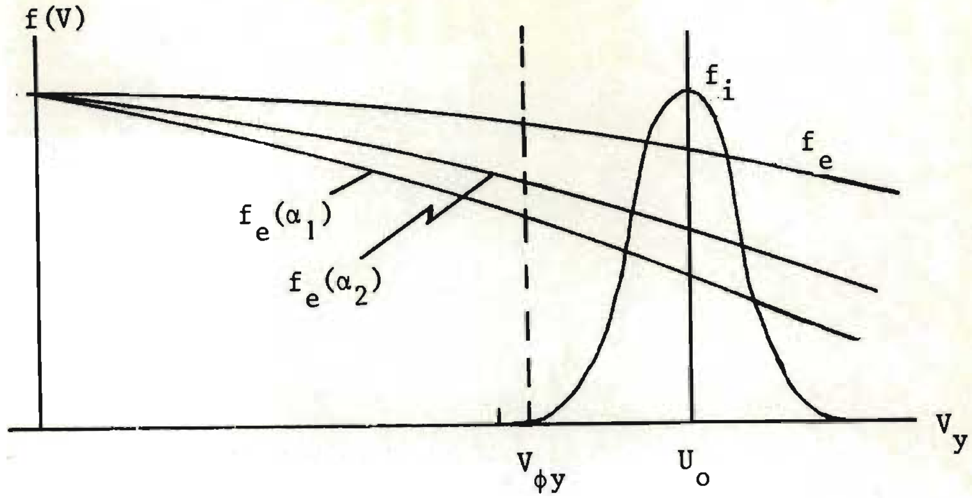
This corresponds to angles of propagation $\theta > 2{,}6^{\circ}$ off the perpendicular to $\vec{B}$.



The numerical results of Lee (62 - Fig.4.8A) exhibit a similar behaviour, with $\theta = 0{,}37^{\circ}$ for the fastest growing mode. The drop off in growth rate as the angle θ increases is discussed below.



It is seen that as the wave vector $\vec{k}$ rotates from the angle α, through the angle $(\alpha_2-\alpha_1)$ the distance an electron has to move along the magnetic field to 'short out' the perturbation decreases from AA' to BB'; consequently, wave growth is hindered. The discussion at the end of Section 3.4 allows us to interpret this in terms of velocity distribution functions. As we rotate from angle α_1 to α_2, the effective electron distribution along $\vec{k}$ has a thermal speed that increases from $(k_{z\alpha_1}/k)C_e$ to $(k_{z\alpha_2}/k)C_e$, and the effective distribution function changes as shown below.



Therefore, as we move from α_1, through the angle $(\alpha_2-\alpha_1)$, to α_2, the gradient of the electron distribution as 'seen' by the wave becomes less negative. Being a negative energy mode, this implies a reduction in electron Landau damping with a consequent decrease in growth rate. For $T_e >> T_{iB}$, ion Landau is very small and effectively remains unaltered.

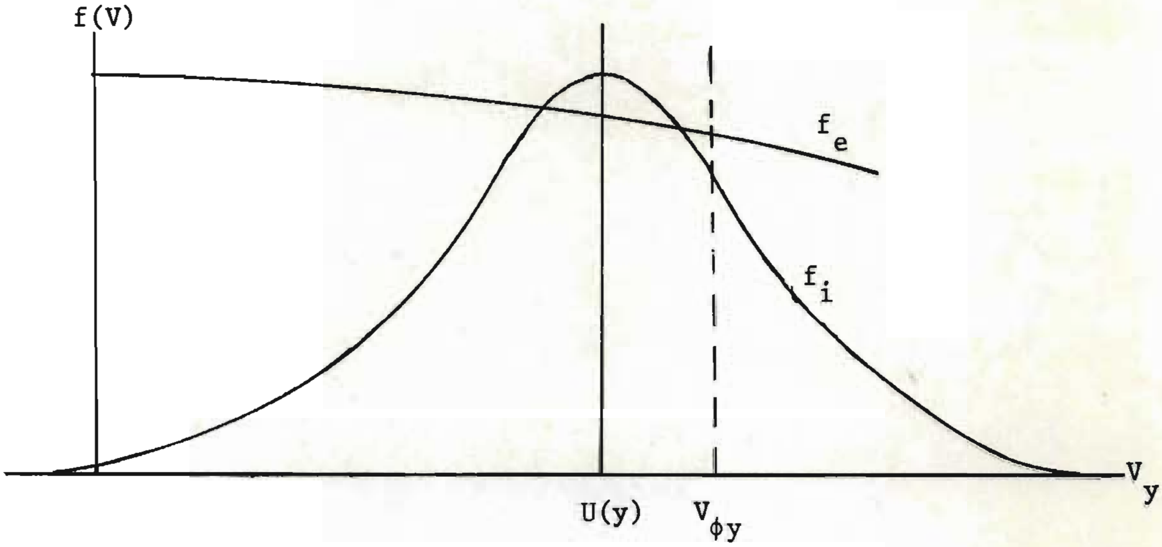
The parameter labelling the curves in Fig.6.8 is $(\lambda_{ci}/\lambda_{ce})$. For a given (y/λ_{ci}) we observe that the growth rate decreases with increasing $(\lambda_{ci}/\lambda_{ce})$. At the same observation point, i.e. fixed y and λ_{ci}, λ_{ce} has to decrease in order that $(\lambda_{ci}/\lambda_{ce})$ increases. As λ_{ce} gets smaller, a sharper drop in the ion beam drift speed $U(y)(= U_o\{1 - (\lambda_{ci}/\lambda_{ce})(y/\lambda_{ci})\})$results. Thus, the free energy available to drive the instability diminishes faster, with a corresponding drop off in growth rate.

On the other hand, it is seen from the expression (6.6.11) for the phase velocity that V_ϕ increases with $(\lambda_{ci}/\lambda_{ce})$. Thus, $V_{\phi y}$ shifts to $V'_{\phi y}$ in figure 6.10 and growth is enhanced. This contradicts the above discussion. The paradox may be somewhat resolved by an examination of the computed results.

For $(U_o/C_s) = 5$ and $k\lambda_D = 0{,}1$, (V_ϕ/U_o) changes from 0,8112 for $(\lambda_{ci}/\lambda_{ce}) = 0{,}05$ to 0,8126 for $(\lambda_{ci}/\lambda_{ce}) = 0{,}20$ at $(y/\lambda_{ci}) = 0{,}1$, i.e., an increase of 0,17%. At the same location, $(U(y)/U_o)$ changes from 0,995 for $(\lambda_{ci}/\lambda_{ce}) = 0{,}05$ to 0,98 for $(\lambda_{ci}/\lambda_{ce}) = 0{,}20$, i.e., a drop of 1,51%. Therefore, the negative effect of $(U(y)/U_o)$ is much more significant than the positive effect of (V_ϕ/U_o), for increasing $(\lambda_{ci}/\lambda_{ce})$. Overall wave damping results. An inspection of Fig.6.9(b) shows this to be the case in general.

We see in Fig. 6.8, as well as in the others, that the growth rate decreases with distance and becomes negative, i.e.,total wave damping sets in. For $(\lambda_{ci}/\lambda_{ce}) = 0{,}2$, the transition from positive to negative occurs at $(y/\lambda_{ci}) \simeq 0{,}595$, while for $(\lambda_{ci}/\lambda_{ce}) = 0{,}1$ it takes place at $(y/\lambda_{ci}) \simeq 0{,}825$.

For the chosen parameters, we find, form calculation, that the ion beam speed U(y) equals the wave phase speed in the direction of the drift, $V_{\phi y}$, at $(y/\lambda_{ci}) = 0{,}664$ for $(\lambda_{ci}/\lambda_{ce}) = 0{,}2$, and $(y/\lambda_{ci}) = 1{,}09$ for $(\lambda_{ci}/\lambda_{ce}) = 0{,}1$. For values of (y/λ_{ci}) larger than these critical values $V_{\phi y}$ exceeds U(y). Thus, the condition $0 < \omega_k < \vec{k}.\vec{U}$ is not satisfied any more and the wave is no longer a negative energy mode. Instead, it becomes a positive energy wave. If at such values of (y/λ_{ci}) the ion distribution function can be represented by an averaged Maxwellian, the situation may be visualized as follows:



Thus, the wave 'sees' a negative slope to both the electron and the ion distributions. Since it is now a positive energy mode, this results in electron and ion Landau damping. Under the influence of the joint damping effect the initially positive growth rate decreases and becomes negative.

The fact that the graphical transition of the growth rate from positive to negative values occurs at a value lower than that calculated, may be attributed to a possible breakdown of the assumption (6.6.8), viz.,

$$\varepsilon = \left| \frac{\gamma_k + \{k_{ce} + k_{ci}\}U_o}{(\omega_k^r - kU_o)} \right| << 1$$

It is found that for $(U_o/C_s) = 5$, $(\lambda_D/\lambda_{ci}) = 0{,}004$, $k\lambda_D = 0{,}1$, and $(\lambda_{ci}/\lambda_{ce}) = 0{,}2$, $\varepsilon \simeq 0{,}35$ at $(y/\lambda_{ci}) = 0{,}6$. Similarly, for $(\lambda_{ci}/\lambda_{ce}) = 0{,}1$, all other parameters as above, $\varepsilon \simeq 0{,}34$ at $(y/\lambda_{ci}) = 0{,}8$. Therefore, the assumption $\varepsilon << 1$ is not strictly satisfied, which means that the plotted curves have to be accepted with some reservation at larger values of (y/λ_{ci}).

Since growth rate measurements can only be recorded after the wave has travelled a few wavelengths, the condition $\lambda<<\lambda_{ci}$ implies that in reality we begin observations close to y = 0, but not at y = 0. Thus, the growth rate values at y = 0 in the above figures are of no physical consequence.

The curves in Fig.6.9(a) represent the normalized frequency (ω_k^r/ω_{pe}) against (y/λ_{ci}). The parameters labelling the curves are $k\lambda_D$ and (U_o/C_s). The behaviour of the curves follows from the expression for the normalized frequency, which, from Eq.(6.6.11a), may be written as

$$\frac{\omega_k^r}{\omega_{pe}} = (k\lambda_D) \left(\frac{U_o}{C_s}\right)\left(\frac{m_e}{m_i}\right)^{1/2} \left[1 - \left(\frac{n_{oB}}{n_e}\right)^{1/2} \frac{(1+k^2\lambda_D^2)^{-1/2}}{(U_o/C_s)} \quad \{1 - (\lambda_{ci}/\lambda_{ce})\ (y/\lambda_{ci})\}^{1/2}\right]$$

The exponential drop in the ion beam density n_{oB} with (y/λ_{ci}) is the primary cause of the gradual increase in (ω_k^r/ω_{pe}) with distance.

LEE (62, Fig. 4.7B) has measured the spatial growth rate k_i as a function of distance into the target plasma ($y\geq 0$). It is found that starting from a positive value, k_i initially rises, then decreases and eventually assumes negative values. The initial increase in k_i was explained by LEE in terms of beating between the slow-beam mode and the rapidly decaying fast-beam mode.

As discussed earlier for the ion acoustic wave HAYZEN [61] has shown that

$$\frac{k_i}{k_r} \simeq \frac{\gamma_k}{\omega_k^r}$$

i.e.,

$$\gamma_k \simeq V_\phi \, k_i$$

Since V_ϕ increases gradually with (y/λ_{ci}) (Fig.6.9), this tells us that γ_k and k_i should exhibit approximately the same behaviour as we move into the region $y \geq 0$. This has been somewhat confirmed by the growth rate curves displayed above, as they provide qualitative agreement with the k_i measurements of LEE[62].

As our investigations have led to the prediction $0,1 < \lambda_{ci}/\lambda_{ce} < 0,2$, calculations show that for $\lambda_{ci}/\lambda_{ce} = 0,2$ the transition from positive to negative growth rate occurs at $y/\lambda_{ci} = 0,66$, while for $\lambda_{ci}/\lambda_{ce} = 0,1$ it occurs at $y/\lambda_{ci} = 1,09$. Lee's measurements indicate the transition occuring at $y \simeq 13,6$ cm. If one uses $\lambda_{ci} = 20$cm , as measured by Jones [78] , for argon pressures of the order of 10^{-4} Torr, then $y/\lambda_{ci} \simeq 0,68$. This value lies within the expected range $0,66 < y/\lambda_{ci} < 1,09$. Thus, the theory also provides quantitative agreement with the experimental observations of LEE[62].

As a remark, we note that for most of the calculations the wave number k is fixed ($k\lambda_D$ = constant). In the majority of plasma experiments this is not the case. For a given excited frequency the wave selects the k value that yields the largest growth rate, i.e., k is not an externally determined parameter. However, studies with a fixed k may prove useful in comparisons with experiments where standing wave-like perturbations are generated, e.g., the oscillating instability [85].

CHAPTER SEVEN

SUMMARY AND CONCLUSION

In this thesis we have examined several linear aspects and a quasilinear development of the electrostatic crossfield current-driven ion acoustic instability.

The linear dispersion relation has been established and formally solved for any general equilibrium particle velocity distribution function of the form $f_{oj}(V_\perp^2, V_\parallel)$. For the particular case of Maxwellian ion and electron distributions, the results are shown to reduce to well known forms. For the purpose of completeness, studies on the effect of plasma inhomogeneities on the instability, which were earlier undertaken and reported by the author, have been reviewed. In addition, an explanation has been offered for the reversal in the behaviour of the temperature gradient drift.

In a quasilinear investigation the electron and ion velocity diffusion equations have been established and analytically solved. For the electrons resonating with the waves, the projection of the distribution along the magnetic field is of the form $\Psi_e^R(V_z) = A \exp(-a\, V_z^5)$, a = constant. The difference between this and the usual one-dimensional quasilinear behaviour $(\partial\Psi_e^R(V_z)/\partial V_z) = 0)$ has been explained in terms of our assumption of a three-dimensional wave spectrum. The non-resonant electrons, rather surprisingly, have the form $\Psi_e^{NR}(V_z) = B \exp(-b\, V_z^4)$. For the ions, which diffuse primarily across $\vec{B}$, the non-resonant portion retains its Maxwellian character. The few high energy resonant ions, which produce linear Landau damping assume a distribution with an exp $(-V^5)$ velocity dependence. Since such a behaviour was not observed in either experiments [40] or simulations [56], we conclude that linear ion Landau damping is not the principal ion heating mechanism.

Investigations into electron and ion heating rates have led to the results

$$\frac{\partial T_e/\partial t}{\partial T_i/\partial t} \simeq 2\left[\frac{\vec{k}\cdot\vec{V}_o}{\omega_k^r} - 1\right]$$

which compares, reasonably well with the measurements of JONES [86]. In the limit $V_o >> C_s$, the crossfield ion acoustic and the reactive electron-cyclotron drift instabilities are found to produce the same relative electron/ion heating rates. An examination of anomalous plasma resistivity yields the result

$$\nu_{ef} = \sum_k \frac{\gamma_k W_k}{\frac{1}{2}m_e n_o V_o^2}\left(\frac{\vec{k}\cdot\vec{V}_o}{\omega_k^r}\right)$$

for the effective electron/wave collision frequency perpendicular to $\vec{B}$. Under a suitable set of approximations this expression is found to reduce to $(k/k_z)\nu_{ef_{\parallel}}$ where $\nu_{ef_{\parallel}}$ is the collision frequency for the field-free case. The (k/k_z) enhancement has been associated with the restriction in electron motion across $\vec{B}$, since they are bound to the field lines.

Energy studies have revealed that energy exchange occurs between the waves and the resonant electrons. However, only a fraction $k^2\lambda_D^2/2$, which is usually small in practice, of the total wave energy appears as electrostatic potential energy.

The effect of a sheared magnetic field on the instability has been examined in the limit in which wave growth due to inverse Landau damping is small. Shear is found to have a stabilizing effect on the wave perturbations.

In a model corresponding to the Double Plasma device, inelastic charge-transfer collisions are found to cause an exponential drop in the density of the incident ion beam, while the elastic collisions give rise to a linear decrease in average beam speed. The results provide good agreement with experimental observations. In the presence of these collisions the real part of the frequency of ion acoustic perturbations superimposed on the ion beam turns out to be (Eq.(6.6.11a))

$$\frac{\omega_k^r}{k} = U_o \pm \frac{\{N_o \exp(-y/\lambda_{ci})\}^{\frac{1}{2}}}{N_o^{\frac{1}{2}}} \frac{C_s}{(1+k^2\lambda_D^2)^{\frac{1}{2}}} \{1 - (y/\lambda_{ce})\}^{\frac{1}{2}}$$

If one neglects elastic collisions and assume $k^2\lambda_D^2 \ll 1$, then this reduces to the expression derived by SATO *et al* [84]. Numerical studies of the instability growth rate for the slow-beam mode provide resonable agreement with the measurements of LEE [62].

Measurements in the DP device have confirmed the presence of elastic and inelastic collisions [41, 42]. On the other hand, JONES and BARRETT [42] have invoked the quasilinear theory to explain the observed electron and ion heating. The question arises as to which is the dominant effect in the DP device. An investigation of electron and ion diffusion in Section 4.1 has revealed that the particle diffusion rate in velocity space as a result of interaction with the waves is proportional to the electrostatic wave energy density ξ. The analogous diffusion of the particles due to collisions with neutrals is proportional to N_D^{-1}, where $N_D = n\lambda_D^3$ is the number of particles in a Debye sphere. Parameters within the DP device are such that [78]

$$1 >> (\xi/n_o T_e) >> N_D^{-1}$$

Thus diffusion due to wave-particle interactions is greater than that due to collisions. In fact for pressures of the order of 10^{-4} Torr, the perturbation wavelength $\lambda \sim 1 - 2$ cm, $\lambda_{ci} \sim 20$ cm and $\lambda_{ce} > \lambda_{ci}$. Therefore, for the first few wavelengths the plasma is practically collisionless. Hence we conclude that quasilinear diffusion effects are more significant than collisional effects.

The investigations undertaken in this thesis allow only a partial understanding of the growth, saturation and nonlinear behaviour of the ion acoustic instability. For example in the DP device, JONES and BARRETT [42] find that wave-wave coupling between the two launched waves produces an increasing number of harmonics, with a consequent broadening of the wave spectrum. Moreover, in the experiment of VIRKO and KIRICHENKO [68] the slowing down of the incident ion beam was much more rapid than that found by JONES and BARRETT[42]. This behaviour was attributed to particle trapping effects, i.e., the capture of ions by the ion sound waves. Both the above mentioned effects were not considered in our studies. Therefore, our investigations may be extended to include, among others, wave-wave coupling effects, particle trapping in the potential troughs of large amplitude waves and resonance broadening, whereby the perturbation of the resonant particle orbits, due to the growing waves, causes the 'sharp' wave-particle resonance to be broadened, i.e.,a particular mode may exchange energy with particles within a finite velocity interval, rather than with particles with a particular velocity.

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APPENDICES

Appendix A.

We have the recurrence relationship

$$\int_{-\infty}^{+\infty} x^{n+2}\, e^{-x^2}\, dx \;=\; \frac{n+1}{2} \int_{-\infty}^{+\infty} x^{n} e^{-x^2}\, dx$$

with

$$\int_{-\infty}^{+\infty} x\, e^{-x^2}\, dx \;=\; 0, \quad \int_{-\infty}^{+\infty} e^{-x^2}\, dx \;=\; \pi^{\frac{1}{2}}$$

Thus

$$\int_{-\infty}^{+\infty} V \exp(-V^2/2C_e^2)\, dV \;=\; 2C_e^2 \int_{-\infty}^{+\infty} p e^{-p^2}\, dp \;=\; 0 \qquad (p = V/\sqrt{2}C_e)$$

$$\int_{-\infty}^{+\infty} \exp(-V^2/2C_e^2)\, dV$$

$$= \sqrt{2}\, C_e \int_{-\infty}^{+\infty} e^{-p^2}\, dp$$

$$= (2\pi C_e^2)^{\frac{1}{2}}$$

$$\int_{-\infty}^{+\infty} \ell\, V \frac{\partial}{\partial V}\{ \exp\, (-V^2/2C_e^2) \}\, dV \qquad (\ell \;=\; \text{interger})$$

$$= -\, \frac{\ell}{C_e^2} \int_{-\infty}^{+\infty} V^2 \exp\, (-V^2/2C_e^2)\, dV$$

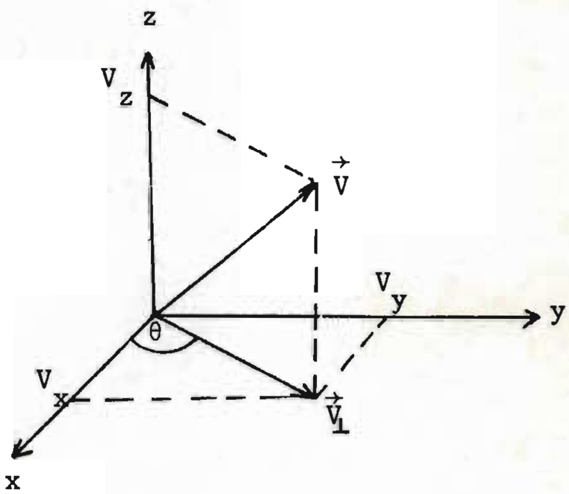
$$= -\, \ell\, (2\pi\, C_e^2)^{\frac{1}{2}}$$

Appendix B

Here we express

$$\frac{e_j}{m_j}\left(\vec{E}_o + \frac{\vec{V} \times \vec{B}_o}{C}\right) \cdot \frac{\partial f_{oj}}{\partial \vec{V}}$$

in terms of cylindrical coordinates in velocity space.



$$\frac{\partial}{\partial \vec{V}} = \hat{x}\frac{\partial}{\partial V_x} + \hat{y}\frac{\partial}{\partial V_y} + \hat{z}\frac{\partial}{\partial V_z}$$

$$\frac{\partial}{\partial V_x} = \frac{\partial V_\perp}{\partial V_x}\frac{\partial}{\partial V_\perp} + \frac{\partial \theta}{\partial V_x}\frac{\partial}{\partial \theta}$$

$$= \frac{V_x}{V_\perp}\frac{\partial}{\partial V_\perp} - \frac{V_y}{V_\perp^2}\frac{\partial}{\partial \theta}$$

$$= \cos\theta \frac{\partial}{\partial V_\perp} - \frac{\sin\theta}{V_\perp}\frac{\partial}{\partial \theta}$$

$$V_x = V_\perp \cos\theta$$

$$V_y = V_\perp \sin\theta$$

Similarly

$$\frac{\partial}{\partial V_y} = \sin\theta \frac{\partial}{\partial V_\perp} + \frac{\cos\theta}{V_\perp}\frac{\partial}{\partial \theta}$$

with

$$V_\perp = (V_x^2+V_y^2)^{1/2}$$

$$\theta = \tan^{-1}(V_y/V_x).$$

From Fig. 3.1, $\vec{E}_o = -E_o\,\hat{x}$. Therefore

$$\frac{e_j}{m_j}\vec{E}_o \cdot \frac{\partial f_{oj}}{\partial \vec{V}} = -\frac{e_j}{m_j}E_o\,\hat{x} \cdot \left\{\left(\cos\theta\frac{\partial f_{oj}}{\partial V_\perp} - \frac{\sin\theta}{V_\perp}\frac{\partial f_{oj}}{\partial \theta}\right)\hat{x} + \left(\sin\theta\frac{\partial f_{oj}}{\partial V_\perp} + \frac{\cos\theta}{V_\perp}\frac{\partial f_{oj}}{\partial \theta}\right)\hat{y} + \frac{\partial f_{oj}}{\partial V_z}\hat{z}\right\}$$

$$= -\frac{e_j}{m_j}E_o\left(\mathrm{Cos}\theta\frac{\partial f_{oj}}{\partial V_\perp} - \frac{\mathrm{Sin}\theta}{V_\perp}\frac{\partial f_{oj}}{\partial \theta}\right)$$

For $\vec{B}_o = (0,0,B_o)$, $\vec{V} \times \vec{B}_o = (B_oV_y, -B_oV_x, 0)$ and

$$\frac{e_j}{m_j}\frac{\vec{V} \times \vec{B}_o}{C} \cdot \frac{\partial f_{oj}}{\partial \vec{V}} = \frac{e_jB_o}{m_jC}\left\{ V_\perp \operatorname{Sin}\theta \left(\operatorname{Cos}\theta \frac{\partial f_{oj}}{\partial V_\perp} - \frac{\operatorname{Sin}\theta}{V_\perp}\frac{\partial f_{oj}}{\partial \theta}\right) - V_\perp \operatorname{Cos}\theta \left(\operatorname{Sin}\theta \frac{\partial f_{oj}}{\partial V_\perp} + \frac{\operatorname{Cos}\theta}{V_\perp}\frac{\partial f_{oj}}{\partial \theta}\right)\right\}$$

$$= -\frac{e_jB_o}{m_jC}\frac{\partial f_{oj}}{\partial \theta}$$

$$= -\Omega_j \frac{\partial f_{oj}}{\partial \theta}$$

where $\Omega_j = \dfrac{e_jB_o}{m_jC}$ is the gyrofrequency of the j^{th} species.

Appendix C. Green's Function Solution of the Resonant Electron Diffusion Equation

We seek a solution of Eq. (4.2.4.), viz.,

$$\frac{\partial \Psi(V,\tau)}{\partial \tau} = \frac{1}{25} \frac{\partial}{\partial V}\left[\frac{1}{V^3} \frac{\partial \Psi(V,\tau)}{\partial V}\right] \tag{C.1}$$

where, for convenience, all super- and subscripts have been omitted. Equation (C.1) is rewritten as

$$\frac{\partial^2 \Psi}{\partial V^2} - \frac{3}{V}\frac{\partial \Psi}{\partial V} = 25\, V^3 \frac{\partial \Psi}{\partial \tau}$$

We Laplace transform with respect to τ, and obtain

$$\frac{d^2\tilde{\Psi}(V,s)}{dV^2} - \frac{3}{V}\frac{d\tilde{\Psi}(V,s)}{dV} = 25\, V^3\{s\tilde{\Psi}(V,s) - \Psi(V,\tau = 0)\}$$

i.e.,

$$\frac{d^2\tilde{\Psi}}{dV^2} - \frac{3}{V}\frac{d\tilde{\Psi}}{dV} - 25\, V^3 s\tilde{\Psi} = -\,25\, V^3\Psi(V,\tau=0) \tag{C.2}$$

The definition of the Laplace transform [1]

$$\mathcal{L}\{f(\tau)\} = \int_0^\infty f(\tau)\, e^{-s\tau}\, d\tau \equiv \tilde{f}(s)$$

and the result

$$\mathcal{L}\{\partial f(\tau)/\partial\tau\} = \int_0^\infty \frac{\partial f(\tau)}{\partial \tau}\, e^{-s\tau}\, d\tau = s\tilde{f}(s) - f(\tau=0)$$

have been used to arrive at (C.2). Consider now the homogeneous equation (from (C.2))

$$\frac{d^2\tilde{\Psi}}{dV^2} - \frac{3}{V}\frac{d\tilde{\Psi}}{dV} - 25\, s\, V^3\, \tilde{\Psi} = 0 \tag{C.3}$$

The general solution of the differential equation

$$\frac{d^2 f}{dx^2} - \frac{\alpha}{x}\frac{df}{dx} - k_1^2\, x^\gamma\, f = 0 \tag{C.4}$$

is given by [2]

$$f(x) = x^{\nu/q}\{A_1' I_\nu(kx^{1/q}) + B_1' K_\nu(kx^{1/q})\} \tag{C.4a}$$

where
$$\nu = \frac{\alpha + 1}{\gamma + 2},\ q = \frac{2}{\gamma + 2},\ k = \frac{2k_1}{\gamma + 2},$$ and I_ν and K_ν

are modified Bessel functions of the first and second kind respectively, of order ν.

Comparing Eqs.(C.3) and (C.4) we see that

$$\alpha = 3\ ;\ \gamma = 3\ ;\ k_1 = 5s^{\frac{1}{2}}.$$

and therefore

$$\nu = \frac{4}{5}\quad ;\quad q = \frac{2}{5}\quad ;\ k = 2s^{\frac{1}{2}}.$$

Thus from Eq. (C.4a), the solution to Eq. (C.3) may be written as

$$\bar{\Psi}(V,s) = V^2\{A_1' I_{4/5}(2s^{\frac{1}{2}}V^{5/2}) + B_1' K_{4/5}(2s^{\frac{1}{2}}V^{5/2})\}$$

It can also be shown that

$$\bar{\Psi}(V,s) = V^2\{A_1 I_{-4/5}(2s^{\frac{1}{2}}V^{5/2}) + B_1 K_{-4/5}(2s^{\frac{1}{2}}V^{5/2})\}$$

is a solution of Eq. (C.3).

We adopt the latter solution and write

$$\bar{\Psi}(V,s) = A_1 g_1(V) + B_1 g_2(V)$$

with (C.5)

$$g_1(V) = V^2 I_{-4/5}(2s^{\frac{1}{2}}V^{5/2})$$

$$g_2(V) = V^2 K_{-4/5}(2s^{\frac{1}{2}}V^{5/2})$$

Prior to finding the solution of Eq.(C.2) we require the solution of the "Green's Function" differential equation

$$\frac{d^2\tilde{\Psi}}{dV^2} - \frac{3}{V}\frac{d\tilde{\Psi}}{dV} - 25\ V^3\ s\tilde{\Psi} = -\ \delta(V-V') \qquad (C.6)$$

Now, the general solution of the Green's function differential equation

$$\frac{d^2g}{dx^2} + p(x)\ \frac{dg}{dx} + q(x)\ g = -\ \delta(x-x')$$

is[1]

$$g(x/x') \quad = \quad C_1 g_1(x) \quad + C_2 g_2(x) \quad + \quad G(x/x') \qquad (C.7)$$

where g_1 and g_2 are solutions of the homogeneous equation

$$\frac{d^2g}{dx^2} + p(x)\ \frac{dg}{dx} + q(x)\ g = 0$$

and

$$G(x/x') \quad = \quad -\ \frac{g_1(x_<)\ g_2(x_>)}{W\ \{g_1,\ g_2,\ x'\}}$$

where $x_>$ = greater of (x, x') , $x_<$ = lesser of (x, x'),

and

$$W = g_1(x').\frac{dg_2(x')}{dx'} \quad - \quad \frac{dg_1(x')}{dx'}.g_2(x') \qquad (C.7a)$$

is the Wronskian of the two functions $g_1(x')$ and $g_2(x')$.

We now determine the solution of (C.6) subject to the boundary conditions

$$\lim_{V \to \infty} \quad \tilde{\Psi}(V,s) \quad = 0 \qquad (C.8a)$$

$$\left[\frac{d\tilde{\Psi}(V,s)}{dV}\right]_{V=0} = 0 \qquad (C.8b)$$

The first condition is true for all velocity distributions since there are no particles at $V = +\infty$. The second is true, for example, for distributions of the form exp $(-aV^n)$, $a > 0$, $n \geq 2$.

Using equation (C.5) and (C.7), the solution of Eq.(C.6) may be written as

$$\tilde{\Psi}(V/V') = C_1 V^2 I_{-4/5}(2s^{\frac{1}{2}}V^{5/2}) + C_2 V^2 K_{-4/5}(2s^{\frac{1}{2}}V^{5/2}) + G(V/V') \tag{C.9}$$

The Wronskian reduces to

$$\begin{aligned} W\{g_1, g_2, x'\} &= V'^2 I_{-4/5}(kV'^{5/2}) . \frac{d}{dV'}\{V'^2 K_{-4/5}(kV'^{5/2})\} \\ &\quad - \frac{d}{dV'}\{V'^2 I_{-4/5}(kV'^{5/2})\} . \ V'^2 K_{-4/5}(kV'^{5/2}) \\ &= V'^4 \{I_{-4/5} \frac{d}{dV'} K_{-4/5}(kV'^{5/2}) - K_{-4/5} \frac{d}{dV'} I_{-4/5}(kV'^{5/2})\} \\ &= -(5/2)V'^3 \end{aligned}$$

where $k = 2s^{\frac{1}{2}}$ and we have used the result[2] $W\{K_\nu(z), I_\nu(z)\} = (1/z)$.

Therefore

$$G(V/V') = 2\,(5V'^3)^{-1}\, g_1(V_<)\, g_2(V_>)$$

and Eq.(C.9) reduces to

$$\tilde{\Psi}(V/V') = C_1 V^2 I_{-4/5}(2s^{\frac{1}{2}}V^{5/2}) + C_2 V^2 K_{-4/5}(2s^{\frac{1}{2}}V^{5/2}) + 2(5V'^3)^{-1} g_1(V_<) g_2(V_>) \tag{C.10}$$

Since[2]

$$\lim_{V \to \infty} \left[V^2\, I_{-4/5}(2s^{\frac{1}{2}}V^{5/2})\right] \to \infty$$

the boundary condition (C.8a) requires that $C_1 = 0$.

It can also be shown that in the limit as $V \to 0$,

$\frac{d}{dV}\{V^2 K_{-4/5}(2s^{\frac{1}{2}}V^{5/2})\} \to \infty$. Thus, the boundary condition (C.8b)

requires that $C_2 = 0$.

With $C_1 = C_2 = 0$, Eq.(C.10) reduces to

$$\bar{\Psi}(V/V') = 2(5V'^3)^{-1} g_1(V_<) g_2(V_>) \tag{C.11}$$

The Green's function solution g(x/x'), given by Eq.(C.7), may now be used to find the solution of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = -f(x) \tag{C.12}$$

The solution at any point x', a<x'<b, where a and b are fixed limits, is[1]

$$y(x') = \int_a^b g(x/x')\, f(x) \exp\{-[L(x') - L(x)]\}\, dx$$

$$+ \Big[W\{g,y\} \exp\{-[L(x') - L(x)]\} \Big]_{x=a}^{x=b} \tag{C.13}$$

where

$$L(x) = \int_{x_o}^{x} p(z)dz \;,\; a \le x_o \le x < b.$$

The equivalent equation for $\bar{\Psi}(V',s)$ is

$$\bar{\Psi}(V',s) = \int_0^\infty \bar{\Psi}(V/V')\, 25V^3\, \Psi(V,\tau = 0) \exp\{-[L(V') - L(V)]\}\, dV$$

$$+ \Big[W\{\bar{\Psi}(V/V'), \bar{\Psi}(V,s\} \quad \exp\{-[L(V') - L(V)]\} \Big]_{V=o}^{V=\infty} \tag{C.14}$$

where, the manipulation from Eqs. (C.7) to (C.11) yields g(x/x'), and comparisons of Eqs.(C.2) and (C.12) yields f(x) .

We have set our fixed limits at $a = 0$ and $b = \infty$. This will be discussed later.

From Eqs.(C.2) and (C.12),

$$p(V) = -3/V$$

Therefore

$$L(V) = -\int_{V_o}^{V} \frac{3dV''}{V''} = -3\ \ln(V/V_o)$$

Similarly $L(V') = -3\ \ln(V'/V_o)$, and hence

$$\exp\{-[L(V') - L(V)]\} = (V'/V)^3 \tag{C.15}$$

The first term in Eq.(C.14) then becomes

$$\int_0^{\infty} 2(5V'^3)^{-1} g_1(V_<) g_2(V_>)\ 25V^3 \psi(V,\tau=0)\ (V'/V)^3\ dV \tag{C.16}$$

The second term is Eq.(C.14) may be written as

$$\Big[W\{\tilde{\Psi}(V/V'),\tilde{\Psi}(V,s)\}\exp\{-[L(V') - L(V)]\} \Big]_{V=0}^{V=\infty}$$

$$= \Big[\{\tilde{\Psi}(V/V') \frac{d\tilde{\Psi}(V,s)}{dV} - \frac{d\tilde{\Psi}(V/V')}{dV} \tilde{\Psi}(V,s)\}(V'/V)^3 \Big]_{V=0}^{V=\infty}$$

where the result (C.15), and the definition (C.7a) of the Wronskian have been used. For $0<V'<\infty$, when $V=\infty$, $V'<V$, and when $V=0$, $V'>V$. Therefore, upon substituting for $\tilde{\Psi}(V/V')$ from Eqs.(C.11) and (C.5), we obtain

$$\Big[(2/5V'^3)V'^2 I_{-4/5}(kV'^{5/2}) \Big\{ V^2 K_{-4/5}(kV^{5/2}) \frac{d\tilde{\Psi}(V,s)}{dV}$$

$$- \frac{d}{dV}[V^2 K_{-4/5}(kV^{5/2})]\ \tilde{\Psi}(V,s)\Big\}(V'/V)^3 \Big]_{V=\infty}$$

$$- \Big[(2/5V'^3) V'^2 K_{-4/5}(kV'^{5/2}) \Big\{ V^2 I_{-4/5}(kV^{5/2}) \frac{d\tilde{\Psi}(V,s)}{dV}$$

$$- \frac{d}{dV} [V^2 I_{-4/5}(kV^{5/2})] \tilde{\Psi}(V,s) \Big\} (V'/V)^3 \Big]_{V=0}$$

$$= (2V'^2/5) I_{-4/5}(kV'^{5/2}) \Big[\lim_{V \to \infty} \frac{K_{-4/5}(kV^{5/2})}{V} \cdot \frac{d\tilde{\Psi}(V,s)}{dV}$$

$$- 0 \cdot \Big\{ \lim_{V \to \infty} \frac{2}{V^2} K_{-4/5}(kV^{5/2}) + \lim_{V \to \infty} \frac{1}{V} \frac{d}{dV} \{ K_{-4/5}(kV^{5/2}) \} \Big\} \Big]$$

$$- (2V'^2/5) K_{-4/5}(kV'^{5/2}) \Big[0 . \Big\{ \lim_{V \to 0} \frac{I_{-4/5}(kV^{5/2})}{V} . \tilde{\Psi}(V,s) \Big\}$$

$$- \Big\{ \lim_{V \to 0} \{ \frac{2}{V^2} I_{-4/5}(kV^{5/2}) . \tilde{\Psi}(V,s) \}$$

$$+ \lim_{V \to 0} \frac{1}{V} \frac{d}{dV} \{ I_{-4/5}(kV^{5/2}) \} . \tilde{\Psi}(V,s) \Big\} \Big]$$

where $k = 2s^{\frac{1}{2}}$ and we have made use of the boundary conditions (C.8a) and (C.8b).

From the definitions of $I_\nu(x)$ and $K_\nu(x)$ and the asymptotic expansion of $K_\nu(x)$ [(2, pp. 201 - 204)], all the above limits can be shown to vanish. Thus, the second term in Eq.(C.14) also vanishes, and from Eq.(C.16) we have

$$\tilde{\Psi}(V',s) = 10 \int_0^\infty g_1(V_<) g_2(V_>) \; \Psi(V,\tau = 0) \; dV$$

To manipulate this further, we write it as

$$\tilde{\Psi}(V',s) = 10 \Big\{ \int_0^{V'} g_1(V_<) g_2(V_>) \; \Psi(V,\tau = 0) \; dV$$

$$+ \int_{V'}^\infty g_1(V_<) g_2(V_>) \Psi(V,\tau = 0) \; dV \Big\}$$

Recalling that $V_<$ = lesser of (V,V') and $V_>$ = greater of (V,V'), with the aid of the definitions (C.5) we have

$$\bar{\Psi}(V,s) = 10\int_0^{V'} V^2\, I_{-4/5}(2s^{\frac{1}{2}}V^{5/2})V'^2 K_{-4/5}(2s^{\frac{1}{2}}V'^{5/2})\ \Psi(V,\tau=0)\ dV$$

$$+\ 10\int_{V'}^{\infty} V'^2 I_{-4/5}(2s^{\frac{1}{2}}V'^{5/2})V^2 K_{-4/5}(2s^{\frac{1}{2}}V^{5/2})\Psi(V,\tau=0)\ dV$$

(C.17)

In determining the inverse-Laplace transform, we use the result(3)

$$\pounds^{-1}\{2K_\nu([a^{\frac{1}{2}}+b^{\frac{1}{2}}]\,s^{\frac{1}{2}})\ \ I_\nu([a^{\frac{1}{2}}-b^{\frac{1}{2}}]s^{\frac{1}{2}})\} = \frac{1}{\tau}I_\nu\{(a-b)/2\tau\}\exp\{-(a+b)/2\tau\}$$

Therefore

$$\pounds^{-1}\{2K_{-4/5}(2s^{\frac{1}{2}}V'^{5/2})I_{-4/5}(2s^{\frac{1}{2}}V^{5/2})\} = \frac{1}{\tau}I_{-4/5}\{(a-b)/2\tau\}\exp\{-(a+b)/2\tau\}$$

where

$$a^{\frac{1}{2}} + b^{\frac{1}{2}} = 2V'^{5/2}$$

$$a^{\frac{1}{2}} - b^{\frac{1}{2}} = 2V^{5/2}$$

We then have

$$a - b = 4V'^{5/2}V^{5/2}$$

and

$$a + b = 2(V'^5 + V^5)$$

Hence

$$\pounds^{-1}\{2K_{-4/5}(2s^{\frac{1}{2}}V'^{5/2})I_{-4/5}(2s^{\frac{1}{2}}V^{5/2})\} = \frac{1}{\tau}\,I_{-4/5}(2V'^{5/2}\ V^{5/2}/\tau)$$

$$\times\ \exp\{-(V'^5 + V^5)/\tau\}$$

Similarly, the inverse-Laplace transform of $2K_{-4/5}(2s^{\frac{1}{2}}V^{5/2})I_{-4/5}(2s^{\frac{1}{2}}V'^{5/2})$ can be shown to be just equal to that of $2K_{-4/5}(2s^{\frac{1}{2}}V'^{5/2})I_{-4/5}(2s^{\frac{1}{2}}V^{5/2})$.

Therefore, upon taking the inverse-Laplace transform on both sides of Eq.(C.17), we obtain

$$\Psi(V',\tau) = 5\int_0^{\infty}\Psi(V,\tau=0)V^2\, V'^2\tau^{-1}I_{-4/5}(2V^{5/2}V'^{5/2}/\tau)\,\exp\{-(V^5+V'^5)/\tau\}dV \tag{C.18}$$

This solution is similar to that quoted by SAGDEEV and GALEEV (4, p. 65) for the loss-cone instability.

We choose an idealized initial distribution

$$\Psi(V,\tau=0) = \frac{A}{C_e}\exp(-V/C_e)^5 \tag{C.19}$$

in order to manipulate the integral in Eq.(C.18) exactly.

Then

$$\Psi(V',\tau) = \frac{5AV'^2}{C_e}\exp(-\alpha V'^5)\alpha\int_0^{\infty}V^2 I_{-4/5}(2\alpha V'^{5/2}V^{5/2})\exp\{-(\alpha+\rho)V^5\}dV \tag{C.20}$$

where $\alpha = \tau^{-1}$ and $\rho = C_e^{-5}$.

Defining $x = V^5$, the integral reduces to

$$(1/5)\int_0^{\infty}x^{-2/5}\, I_{-4/5}(2\alpha V'^{5/2}\sqrt{x})\,\exp\{-(\alpha+\rho)x\}\,dx$$

$$= (1/5)\int_0^{\infty}x^{\mu-\frac{1}{2}}\, I_{2\nu}(2\beta\sqrt{x})\,\exp(-\gamma x)\,dx \tag{C.21a}$$

where

$$\mu - \tfrac{1}{2} = -2/5 \quad , \Rightarrow \quad \mu = 1/10$$

$$2\nu = -4/5 \quad , \Rightarrow \quad \nu = -\,2/5$$

$$2\beta = 2\alpha V'^{5/2} , \Rightarrow \quad \beta = \alpha V'^{5/2}$$

$$\gamma = \alpha + \rho$$

We use the identity(5)

$$\int_0^\infty x^{\mu-\frac{1}{2}}\, I_{2\nu}(\,2\beta\sqrt{x})\ \exp\,(-\gamma x)\ dx$$

$$= \frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)}\ \beta^{-1}\ \gamma^{-\mu}\ \exp\,(\beta^2/2\gamma)\ M_{-\mu,\nu}(\beta^2/\gamma) \qquad \text{(C.21b)}$$

where Γ is the Gamma function. $M_{\mu,\nu}$ (x) is the Whittaker function, and is given, in terms of the Kummer function ${}_1F_1$, by(5)

$$M_{\mu,\nu}(y) = y^{\frac{1}{2}+\nu}\ \exp\,(-y/2)\ {}_1F_1\{\,\tfrac{1}{2}+\nu-\mu,\ 2\nu+1, y\}$$

(C.21c)

For $\mu = 1/10$ and $\nu = -\,2/5$ (as found above), it turns out that

$$M_{-\mu,\nu}(y) = M_{-1/10,-2/5}(y) = y^{1/10} e^{-y/2}\ {}_1F_1\{1/5, 1/5, y\}$$

$$= y^{1/10}\ e^{y/2}$$

since(5)

$${}_1F_1\{a,a,y\} = e^y \qquad \text{(C.21d)}$$

With the aid of the results (C.21b) - (C.21d), the integral (C.21a) reduces to

$$(1/5)\ (\alpha V'^{5/2})^{-1}\ \ (\alpha+\rho)^{-1/10}\ \{\alpha^2 V'^5/(\alpha+\rho)\}^{1/10}(\exp\{\alpha^2 V'^5/2(\alpha+\rho)\})^2$$

and Eq.(C.20) becomes

$$\psi(V',\tau) = A\ \rho^{1/5}\ \frac{\alpha V'^2}{\alpha V'^{5/2}}\ \frac{\alpha^{1/5} V'^{1/2}}{(\alpha+\rho)^{1/5}} \exp\{-\alpha V'^5\ [1-\alpha/(\alpha+\rho)]\}$$

$$= \frac{A}{(1/\alpha + 1/\rho)^{1/5}} \exp\{-V'^5/(1/\alpha+1/\rho)\}$$

$$= \frac{A}{(\tau + C_e^5)^{1/5}} \exp\{-V'^5/(\tau+C_e^5)\} \qquad (C.22)$$

The solution to the differential equation (C.1) is clearly restricted to the resonance region ($\omega_k^r/k_z - \Delta V$, $\omega_k^r/k_z + \Delta V$) of width $2\Delta V$. By assuming this interval to lie in the region $V_z \geq 0$, we have extended the limits of integration in Eq.(C.14) to 0 and $+\infty$. This is in order, since there are no resonant particles in the intervals (0, $\omega_k^r/k_z - \Delta V$) and ($\omega_k^r/k_z + \Delta V, \infty$) and $\psi(V_z,\tau) = 0$ there. The resonance region is restricted to the domain $V_z \geq 0$ for modes with $k_z > 0$. The latter condition is usually assumed in theoretical studies.

Appendix D. Green's Function Solution of the Non-Resonant Ion Diffusion Equation.

We require the solution of Eq.(4.3.3), viz.,

$$\frac{\partial \Phi}{\partial \tau} = \frac{1}{2m_i}\left[\frac{1}{V}\frac{\partial \Phi}{\partial V} + \frac{\partial^2 \Phi}{\partial V^2}\right] \tag{D.1}$$

where, for convenience, all super- and subscripts have been omitted. The technique in solving Eq.(D.1) is analogous to that adopted in solving the resonant electron diffusion equation (Appendix C). We solve Eq.(D.1)subject to the initial condition

$$\Phi(V,\tau = 0) = N_o(m_i/2\pi T_{io})\exp(-m_iV^2/2T_{io}) \tag{D.2}$$

Taking the Laplace-transform of Eq.(D.1) with respect to τ, we obtain

$$s\tilde{\Phi}(V,s) - \Phi(V,\tau = 0) = \frac{1}{2m_i}\left[\frac{d^2\,\tilde{\Phi}(V,s)}{dV^2} + \frac{1}{V}\frac{d\tilde{\Phi}(V,s)}{dV}\right]$$

Rearranging

$$\frac{d^2\tilde{\Phi}}{dV^2} + \frac{1}{V}\frac{d\tilde{\Phi}}{dV} - 2m_is\tilde{\Phi} = -\,2m_i\;\Phi(V,\tau = 0) \tag{D.3}$$

The homogeneous equation associated with Eq.(D.3) is

$$\frac{d^2\tilde{\Phi}}{dV^2} + \frac{1}{V}\frac{d\tilde{\Phi}}{dV} - 2m_is\tilde{\Phi} = 0 \tag{D.4}$$

Comparing Eq.(D.4) with Eq.(C.4), we see that

$$\alpha = -1 \ ; \ \gamma = 0 \ ; \ k_1 = (2m_i s)^{\frac{1}{2}}$$

Therefore

$$\nu = \frac{\alpha + 1}{\gamma + 2} = 0 \ , \quad q = \frac{2}{\gamma + 2} = 1 \ , \quad \text{and} \quad k = \frac{2k_1}{\gamma + 2} = k_1 = (2m_i s)^{\frac{1}{2}},$$

and hence, from Eq.(C.4a), the solution of Eq.(D.4) is

$$\tilde{\Phi}(V,s) = A_1' \, I_o\{(2m_i s)^{\frac{1}{2}}V\} + B_1' \, K_o\{(2m_i s)^{\frac{1}{2}}V\}$$

$$= A_1' g_1(V) + B_1' g_2(V) \tag{D.5}$$

where

$$g_1(V) = I_o\{(2m_i s)^{\frac{1}{2}}V\}$$

$$g_2(V) = K_o\{(2m_i s)^{\frac{1}{2}}V\} \tag{D.6}$$

Using the functions $g_1(V)$ and $g_2(V)$ we may write down the solutions of the Green's function differential equation as (cf.manipulation from Eq.(C.6) to Eq.(C.9))

$$\tilde{\Phi}(V,V') = C_1 I_o(kV) + C_2 K_o(kV) + G(V/V') \tag{D.7}$$

where

$$G(V/V') = \frac{-g_1(V_<)g_2(V_>)}{W\{g_1,g_2,V'\}} \quad \text{and} \quad k = (2m_i s)^{\frac{1}{2}}.$$

with

$$W\{g_1, g_2, V'\} = I_o(kV') \frac{d}{dV'} K_o(kV') - \frac{dI_o}{dV'}(kV')\, K_o(kV')$$

$$= - k\, W\{K_o(z), I_o(z)\} \qquad (z = kV')$$

$$= - (k/z)$$

$$= -(1/V') \tag{D.8}$$

since[2] $\quad W\{K_\nu(z), I_\nu(z)\} = (1/z)$.

As in Appendix C, the boundary conditions

$$\lim_{V \to \infty} \tilde{\Phi}(V,s) = 0 \tag{D.9a}$$

$$\left[\frac{d\, \tilde{\Phi}(V,s)}{dV}\right]_{V=0} = 0 \tag{D.9b}$$

require that $C_1 = C_2 = 0$ in Eq.(D.7) (cf. ref. 1, p.271 for limiting forms of $I_o(x)$, $K_o(x)$ and $K_o'(x) = - K_1(x)$). Therefore

$$\tilde{\Phi}(V/V') = G(V/V')$$
$$= g_1(V_<)\, g_2(V_>)\, V' \tag{D.10}$$

This, in turn, yields the solution of our original equation (D.3), which, following the procedure leading from Eq.(C.12) to Eq.(C.14), may be written as

$$\tilde{\Phi}(V',s) = \int_0^\infty g_1(V_<) g_2(V_>)\, V' \,\{2m_i \Phi(V,\tau=0)\} \exp\{-[L(V') - L(V)]\}\, dV$$
$$+ \Big[W\{\tilde{\Phi}(V/V'), \tilde{\Phi}(V,s)\} \exp\{-[L(V') - L(V)]\} \Big]_{V=0}^{V=\infty} \tag{D.11}$$

Since $V = V_\perp$ (≥ 0), we have extended the limits of integration to cover the entire $V_\perp$ - space, as outside the non-resonant domain $\Phi^{NR} = 0$.

Comparing Eq.(D.3) with Eq.(C.12), we see that

$$p(V) = (1/V)$$

and therefore

$$L(V) = \int_{V_o}^{V} p(y)dy = \ln(V/V_o)$$

Similarly $L(V') = \ln(V'/V_o)$, and hence

$$\exp\{-[L(V') - L(V)]\} = (V/V') \tag{D.12}$$

Then, from Eqs.(D.6), (D.10) and (D.11),

$$\left[W\{\tilde{\Phi}(V/V'), \tilde{\Phi}(V,s)\} \exp\{-[L(V') - L(V)]\}\right]_{V=0}^{V=\infty}$$

$$= \left[V\left\{I_o(kV')K_o(kV)\frac{d}{dV}\tilde{\Phi}(V,s) - I_o(kV').\frac{d}{dV}K_o(kV).\tilde{\Phi}(V,s)\right\}\right]_{V=\infty}$$

$$- \left[V\left\{K_o(kV')I_o(kV)\frac{d}{dV}\tilde{\Phi}(V,s) - K_o(kV')\frac{d}{dV}I_o(kV).\tilde{\Phi}(V,s)\right\}\right]_{V=0}$$

since for $0 < V' < \infty$, when $V = \infty$, $V' < V$ and when $V = 0$, $V' > V$.

As in Appendix C, with the aid of the boundary conditions (D.9a) and (D.9b), and the limiting values of $I_o(x)$ and $K_o(x)$, and of their derivatives, the right hand side of the above equation can be shown to vanish.

Hence, from Eqs.(D.11) and (D.12), we have

$$\tilde{\Phi}(V,s) = \int_0^\infty g_1(V_<)g_2(V_>) \quad 2m_i \; \Phi(V,\tau = 0) \; V \; dV$$

$$= m_i \int_0^{V'} 2I_o\{(2m_i)^{\frac{1}{2}}Vs^{\frac{1}{2}}\} \; K_o\{(2m_i)^{\frac{1}{2}}V's^{\frac{1}{2}}\} \quad \Phi(V,\tau = 0) \; V \; dV$$

$$+ m_i \int_{V'}^\infty 2I_o\{(2m_i)^{\frac{1}{2}}V's^{\frac{1}{2}}\} \; K_o\{(2m_i)^{\frac{1}{2}}Vs^{\frac{1}{2}}\} \; \Phi(V,\tau = 0) \; V \; dV$$

With the inverse-Laplace transform (3)

$$\pounds^{-1}\{2K_\nu([a^{\frac{1}{2}}+b^{\frac{1}{2}}]s^{\frac{1}{2}})I_\nu([a^{\frac{1}{2}}-b^{\frac{1}{2}}]s^{\frac{1}{2}})\} = \frac{1}{\tau} I_\nu\{(a-b)/2\tau\} \; \exp\{-(a+b)/2\tau\}$$

we obtain

$$\pounds^{-1}\{2K_o([(2m_i)^{\frac{1}{2}}V']s^{\frac{1}{2}})I_o([(2m_i)^{\frac{1}{2}}V]s^{\frac{1}{2}})\} = \frac{1}{\tau} \; I_o(m_iVV'/\tau)\exp\{-m_i(V'^2+V^2)/2\tau\}$$

where

$$\left.\begin{aligned} a^{\frac{1}{2}} + b^{\frac{1}{2}} &= (2m_i)^{\frac{1}{2}}V' \\ a^{\frac{1}{2}} - b^{\frac{1}{2}} &= (2m_i)^{\frac{1}{2}}V \end{aligned}\right\} \Rightarrow \quad \begin{aligned} a + b &= m_i(V'^2+V^2) \\ a - b &= 2m_iV'V \end{aligned}$$

Similarly

$$\pounds^{-1}\{2K_o([(2m_i)^{\frac{1}{2}}V]s^{\frac{1}{2}})I_o([(2m_i)^{\frac{1}{2}}V']s^{\frac{1}{2}})\} = \frac{1}{\tau} \; I_o(m_iVV'/\tau)\exp\{-m_i(V'^2+V^2)/2\tau\}$$

Using these results, and upon substituting for $\Phi(V,\tau=0)$ from Eq.(D.2), it turns out that

$$\Phi(V',\tau) = N_o(m_i/2\pi T_{io})(m_i/\tau) \; \exp(-\, m_iV'^2/2\tau)$$
$$\times \int_0^\infty I_o(m_iV'V/\tau)\exp\{-[(m_i/2\tau)+(m_i/2T_{io})]V^2\} \; VdV \tag{D.13}$$

In order to evaluate the integral, we set $w = V^2/2$.

The integral then modifies to

$$\int_0^\infty \exp(-\gamma w) I_o(2\beta w^{\frac{1}{2}}) \quad dw \tag{D.14}$$

where

$$\gamma = \frac{m_i(T_{io} + \tau)}{T_{io}\ \tau} \qquad \beta = \frac{m_i V'}{\sqrt{2}\tau}$$

Using the identity (C.21b), viz.,

$$\int_0^\infty x^{\mu-\frac{1}{2}} I_{2\nu}(2\beta\sqrt{x}) \exp(-\gamma x)\, dx$$

$$= \frac{\Gamma(\mu+\nu+\frac{1}{2})}{\Gamma(2\nu+1)} \beta^{-1}\ \gamma^{-\mu} \exp(\beta^2/2\gamma)\ M_{-\mu,\nu}(\beta^2/\gamma)$$

we find that

$$\int_0^\infty \exp(-\gamma w) I_o(2\beta w^{\frac{1}{2}}) dw = \beta^{-1}\ \gamma^{-\frac{1}{2}} \exp(\beta^2/2\gamma)\ M_{-\frac{1}{2},o}(\beta^2/\gamma) \tag{D.15}$$

From Eqs.(C.21c) and (C.21d),

$$M_{-\frac{1}{2},o}(\beta^2/\gamma) = (\beta^2/\gamma)^{\frac{1}{2}} \exp(-\beta^2/2\gamma)\ {}_1F_1\{1,1,\beta^2/\gamma\}$$

$$= (\beta^2/\gamma)^{\frac{1}{2}} \exp(\beta^2/2\gamma)$$

The integral in Eq.(D.13) then reduces to

$$\beta^{-1}\ \gamma^{-\frac{1}{2}} \exp(\beta^2/2\gamma)\ (\beta^2/\gamma)^{\frac{1}{2}} \exp(\beta^2/2\gamma) = \gamma^{-1} \exp(\beta^2/\gamma)$$

We replace γ and β with the expressions given in Eq.(D.14). Then Eq.(D.13) finally yields

$$\Phi(V',\tau) = N_o(m_i/2\pi T_{io})(m_i/\tau)\ \exp(-\ m_i V'^2/2\tau)$$
$$\times\ \{\tau T_{io}/[m_i(T_{io}+\tau)]\}\exp\{m_i^2 V'^2 T_{io}\tau/[2\tau^2 m_i(T_{io}+\tau)]\}$$

i.e.,

$$\Phi(V',\tau) = N_o\ \{m_i/[2\pi(T_{io}+\tau)]\}\exp\{-\ m_i V'^2/[2(T_{io}+\tau)]\} \qquad (D.16)$$

A similar result has been found by SAGDEEV and GALEEV(4,p.68) for the non-resonant electrons in the case of Langmuir waves.

Appendix E.

a) $$\sum_{n,m=-\infty}^{+\infty} J_n(\xi) J_m(\xi) \int_0^{2\pi} d\theta \int_{-\infty}^{t} i^{(n-m)} \, e^{i(m-n)\theta} \, e^{-i(\omega_k - k_\| V_\| - n\Omega_e)(t'-t)} \, dt'$$

$$= \sum_{n,m} J_n J_m i^{(n-m)} \{-i(\omega_k - k_\| V_\| - n\Omega_e)\}^{-1} \int_0^{2\pi} e^{i(m-n)\theta} \, d\theta$$

$$= \sum_n 2\pi \, J_n^2 \{i(\omega_k - k_\| V_\| - n\Omega_e)\}^{-1} \tag{E.1}$$

since

$$\int_0^{2\pi} e^{i(m-n)\theta} \, d\theta = \begin{cases} 2\pi & \text{if} \quad m=n \\ 0 & \text{otherwise} \end{cases}$$

b) $$\sum_{n,m} J_n J_m \int_0^{2\pi} \text{Sin}\theta \; i^{(n-m)} \, e^{i(m-n)\theta} \{-i(\omega_k - k_\| V_\| - n\Omega_e)\}^{-1} \, d\theta$$

$$= \sum_{n,m} J_n J_m (-1/2) \left[\int_0^{2\pi} \frac{i^{(n-m)} \, e^{i(m-n+1)\theta}}{\{-(\omega_k - k_\| V_\| - n\Omega_e)\}} \, d\theta \right.$$

$$\left. - \int_0^{2\pi} \frac{i^{(n-m)} \, e^{i(m-n-1)\theta}}{\{-(\omega_k - k_\| V_\| - n\Omega_e)\}} \, d\theta \right]$$

$$= (1/2) \sum_n 2\pi \{-i(\omega_k - k_\| V_\| - n\Omega_e)\}^{-1} J_n \, (J_{n-1} + J_{n+1})$$

$$= (2\pi/\xi) \sum_n \{-i(\omega_k - k_\| V_\| - n\Omega_e)\}^{-1} \, n J_n^2(\xi) \tag{E.2}$$

since[6]

$$J_{n-1}(x) + J_{n+1}(x) = (2n/x) \, J_n(x)$$

Similarly

c) $$\sum_{n,m} J_n J_m \int_0^{2\pi} d\theta \int_{-\infty}^{t} (x' - x)\, i^{(n-m)}\, e^{i(m-n)\theta}\, e^{-i(\omega_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)}\, dt'$$

$$= (-V_\perp/\Omega_e) \sum_{n,m} J_n J_m \int_0^{2\pi} d\theta \int_{-\infty}^{t} [\{\mathrm{Sin}(\theta - \Omega_e[t'-t]) - \mathrm{Sin}\theta\}$$

$$\times\ i^{(n-m)}\, e^{i(m-n)\theta}\, e^{-i(\omega_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)}]dt'$$

$$= 0 \tag{E.3}$$

d) $$\sum_{n,m} J_n J_m \int_0^{2\pi} d\theta \int_{-\infty}^{t} (x'-x)\, \mathrm{Sin}\theta\, i^{(n-m)} e^{i(m-n)\theta}\, e^{-i(\omega_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)}\, dt'$$

$$= (\pi\Omega_e V_\perp/k_\perp^2) \sum_n \{i(\omega_k - k_\parallel V_\parallel - n\Omega_e)\}^{-1}\ \frac{1}{V_\perp}\frac{\partial}{\partial V_\perp}\{J_n^2(k_\perp V_\perp/\Omega_e)\} \tag{E.4}$$

e) $$\sum_{n,m} J_n J_m \int_0^{2\pi} d\theta \int_{-\infty}^{t} (x'-x)^2\, i^{(n-m)}\, e^{i(m-n)\theta}\, e^{-i(\omega_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)}\, dt'$$

$$= (2\pi/k_\perp^2) \sum_n \{i(\omega_k - k_\parallel V_\parallel - n\Omega_e)\}^{-1}\ V_\perp \frac{\partial}{\partial V_\perp}\{J_n^2(k_\perp V_\perp/\Omega_e)\} \tag{E.5}$$

f) $$\sum_{n,m} J_n J_m \int_0^{2\pi} d\theta \int_{-\infty}^{t} (x'-x)^2\, \mathrm{Sin}\theta\, i^{(n-m)} e^{i(m-n)\theta}\, e^{-i(\omega_k - k_\parallel V_\parallel - n\Omega_e)(t'-t)}\, dt'$$

$$= (-\pi/k_\perp^2) \sum_n \{i(\omega_k - k_\parallel V_\parallel - n\Omega_e)\}^{-1}$$

$$\times \left\{-\frac{n\Omega_e}{k_\perp}\frac{\partial}{\partial V_\perp}\{J_n^2(\xi)\} + \frac{2n\Omega_e}{k_\perp V_\perp} J_n^2(\xi)\right\} \tag{E.6}$$

where $\xi = k_\perp V_\perp/\Omega_e$.

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