On pseudo-amenability of $C(\mathcal{X}, \mathcal{A})$ for norm irregular Banach algebra $\mathcal{A}$

by
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As the candidate's supervisor, I have approved this dissertation for submission.

Dr. O. T. Mewomo

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Dedication

To all the curious minds out there, who strive tirelessly in the pursuit of knowledge. And of course, to my darling companion, Seyi.

Abstract

In this dissertation, we discuss the amenability properties of arbitrary Banach algebras. We pay special attention to the different ways one may characterize the amenability of Banach algebras such as existence of a bounded approximate diagonal, existence of a virtual diagonal and the splitting of short exact sequence of Banach modules. Expanded proofs of some interesting results found in literature are also given. We further discuss some known notions of amenability of arbitrary Banach algebras such as weak amenability, approximate amenability and pseudo-amenability. Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a compact Hausdorff space. We give the proof of the amenability of $C(\mathcal{X})$ due to Seinberg and also discuss the construction of bounded approximate diagonals for $C(\mathcal{X})$ and $C(\mathcal{X}, \mathcal{A})$, which are results credited to Abtahi and Zhang, and Ghamarshoushtari and Zhang respectively. We show that for a Banach algebra $\mathcal{A}$ with a bounded approximate identity such that $\mathcal{A}\hat{\otimes}\mathcal{A}$ is norm irregular, if $\mathcal{A}$ has an approximate diagonal which is bounded with respect to the multiplier norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$, then $C(\mathcal{X}, \mathcal{A})$ has an approximate diagonal.

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Contents

Declaration

This dissertation, in its entirety, has not been submitted to this or any other institution in support of an application for the award of a degree.
The ideas expressed in this dissertation are those of the author and where the work of others has been used in the text, appropriate references has been made.

Ugochukwu Oliver Adiele

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Chapter 1

Introduction

The development of the theory of amenability in Banach algebras has its origins in the now classical memoirs of American Mathematical Society published by Barry Johnson in 1972 [28], in which he showed that for a locally compact group G, the Banach algbera $L^1(G)$ is amenable if and only if G is amenable as a group.
The group, G is amenable if and only if it has a left invariant mean. A left invariant mean on G is a positive linear functional $\mu \in L^\infty(G)$ such that:

(i) $\langle 1, \mu \rangle = 1$,

(ii) $\langle \delta_g \star \phi, \mu \rangle = \langle \phi, \mu \rangle$, for every $\phi \in E \subset L^\infty(G)$.

Using the above definition and some interesting cohomological properties of Banach algebras, Johnson in the afore-stated memoir further showed that an arbitrary Banach algebra, say $\mathcal{A}$, is amenable if every continuous derivation D from $\mathcal{A}$ into X^* is inner for every Banach $\mathcal{A}$-bimodule X, where X^* denotes the dual space of X. It has been realized that the above definition given by Johnson [28] is too restrictive and so does not allow for the development of a rich general theory and also too weak to include a variety of interesting examples. For this reason, by relaxing some of the constraints in the definition of amenability via restricting the class of bimodules in question or by relaxing the structure of the derivations themselves, various notions of amenability have been introduced in recent years. Some of these notions are weak amenability, approximate amenability, pseudo amenability, ideal amenability, character amenability, approximately character amenability and so on. These notions of amenability have been studied for different classes of Banach algebras (e.g. semigroup algebras, Segal algebras, Beurling algebras, group algebras, measure algebras, closed ideals of $\mathcal{B}(E)$-algebras of bounded linear operators on a Banach space E and so on). See [6], [12], [13], [14], [16], [37], [38] and [39].

By using Johnson's result above and Stone-Weierstrass theorem, M.V. Seinberg [48] showed that the algebra $C(\mathcal{X})$ of complex-valued continuous functions is amenable for any compact Hausdorff space $\mathcal{X}$. A constructive proof of this important result was later given by Abtahi and Zhang [2] in 2010. Recently in 2015, Ghamarshoushtari and Zhang [17] extended this result to the Banach algebra $C(\mathcal{X}, \mathcal{A})$ of all $\mathcal{A}$ valued continuous functions and showed that $C(\mathcal{X}, \mathcal{A})$ is amenable if and only if the range Banach algebra $\mathcal{A}$ is amenable. Zhang in [49], further showed that for a commutative Banach algebra $\mathcal{A}$, $C(\mathcal{X}, \mathcal{A})$ is weakly amenable if and only if $\mathcal{A}$ is weakly amenable. It should be noted that other known notions of amenability are yet to be studied for $C(\mathcal{X}, \mathcal{A})$. Therefore, in this work, we aim to derive the relationship between the pseudo-amenability of $\mathcal{A}$ and the Banach algebra $C(\mathcal{X}, \mathcal{A})$. Thus, providing a slight extension to the result of Ghamarshoushtari and Zhang in [17].

Through out this dissertation, we denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$, the collection of all natural, integer, real and complex numbers respectively. All Banach spaces and algebras considered in this study are defined over $\mathbb{C}$ except where it is stated otherwise.

Chapter 2 is concerned with the definition of terms and some basic results on Banach spaces, algebras and modules made use of in this work. A lot of the concepts covered can be found in [5], [8], [10], [13], [31], [36] and [47].

Let $\mathcal{A}$ be an arbitrary Banach algebra. In **Chapter 3**, we discuss some known important results on the amenability properties of $\mathcal{A}$ and also take a look at some interesting results from some generalised notions of amenability of $\mathcal{A}$, while also considering some relationships between the notions of amenability discussed.

Chapter 4 is concerned with the study of some algebraic and topological properties of $C(\mathcal{X}, \mathbb{C}) = C(\mathcal{X})$, as well as its amenability properties. We further give a motivation for studying the amenability properties of $C(\mathcal{X})$, state the proof of the amenability of $C(\mathcal{X})$ due to Seinberg and also consider the construction of a bounded approximate diagonal for $C(\mathcal{X})$, which is a result that is credited to Abathi and Zhang.

In **Chapter 5**, the result by Gharmashoustari and Zhang which showed that $C(\mathcal{X}, \mathcal{A})$ being amenable is equivalent to $\mathcal{A}$ being amenable is discussed, an expanded version of the proof is given, while we also show an alternative way of proving the assertion which follows from the fact that $C(\mathcal{X}, \mathcal{A})$ is isometrically isomorphic to $C(\mathcal{X})\check{\otimes}\mathcal{A}$.

A question of interest that naturally arises from the work of Gharmashoustari and Zhang is: Can the pseudo-amenability of $C(\mathcal{X}, \mathcal{A})$ be inferred from the pseudo-amenability of $\mathcal{A}$? In **Chapter 6**, we give a partial answer to this question for a case where $\mathcal{A}$ has a bounded approximate identity and $\mathcal{A}\hat{\otimes}\mathcal{A}$ is norm irregular. We also give a proof of an interesting property of the multiplier norm on the projective tensor product. The results in this chapter serve as our contribution to knowledge.

Chapter 2

Preliminaries

This chapter is concerned with the definition of some terms used in other areas of this work. We also state some elementary results on Banach spaces, algebras and modules that are required for other parts of this dissertation. It should be noted that the proofs of all results stated are omitted, and can be found in the references therein.

2.1 Banach spaces

2.1.1 Definitions and basic results

A *vector space* over $\mathbb{C}$ is a non empty set E equipped with the operations $E \times E \to E$ and $\mathbb{C} \times E \to E$ such that

(i) $x + y,\ \alpha x \in E$,

(ii) $x + y = y + x,\ (x + y) + z = x + (y + z)$,

(iii) $0 + x = x + 0 = x,\ x + (-x) = (-x) + x = 0$,

(iv) $(\alpha\beta)x = \alpha(\beta x),\ \alpha(x + y) = \alpha x + \alpha y,\ (\alpha + \beta)x = \alpha x + \beta x$,

(v) $1.x = x.1 = x$,

where $\alpha, \beta \in \mathbb{C}$, $x, y, z \in E$, 1 is the multiplicative identity in $\mathbb{C}$ and 0 is the additive identity in E. A *semi-norm* on E is a function $\rho : E \to \mathbb{R}$ satisfying the following properties:

(i) $\rho(x) \geq 0$,

(ii) $\rho(\alpha x) = |\alpha|\rho(x)$,

(iii) $\rho(x+y) \le \rho(x) + \rho(y)$,

where $x, y \in E$ and $\alpha \in \mathbb{C}$. If in addition to satisfying the above stated properties, $\rho(x) = 0$ if and only if $x = 0$, then ρ is referred to as a *norm* on E. In this case, we write $\rho = \|.\|$, $\rho(x) = \|x\|$ and $(E, \|.\|)$ is called a *normed vector space*, or for short a *normed space*. The natural topology on E is induced by the metric

$$d(x,y) = \|x - y\| \quad (x, y \in E).$$

If (E, d) is complete, in the sense that every Cauchy sequence in E is convergent, then $(E, \|.\|)$ is called a Banach space.

Let E and M be Banach spaces, we denote by $\mathcal{B}(E, M)$, the collection of all bounded linear operators that maps E into M, we define the operator norm on $\mathcal{B}(E, M)$ as

$$\|T\| = \sup_{\|x\| \le 1} \{\|T(x)\| : x \in E\}.$$

The linear space $\mathcal{B}(E, M)$ equipped with this norm is a Banach space. If $E = M$, then we write $\mathcal{B}(E, E) = \mathcal{B}(E)$.

Definition 2.1.1. Let M and N be closed linear subspaces of the Banach space E. Let

$$M + N = \{x + y : x \in M, y \in N\},$$

and

$$M.N = \{xy : x \in M, y \in N\}.$$

(i) The Banach space E is said to be a direct sum of M and N if $E = M + N$ and $M \cap N = 0$. Here we write $E = M \oplus N$.

(ii) The linear span of $M.N$ is

$$\operatorname{lin} M.N = \left\{ \sum_{j=1}^{n} \lambda_j x_j y_j : \lambda_j \in \mathbb{C},\ \ x_j \in M, y_j \in N,\ \ j = 1, 2, ..., n \right\}.$$

We write $\operatorname{lin} M.N = MN$. If $M = N$, we write M^2 for $\operatorname{lin} M.M$.

Definition 2.1.2. A linear subspace M of a Banach space E is said to be complemented if it is a direct summand in E.

Proposition 2.1.3. [3] *Let M and N be linear subspaces of the vector space E. If $E = M \oplus N$, then $z \in E$ has a unique representation $z = x + y$, for $x \in M,\ y \in N$.*

Definition 2.1.4. An operator $P \in \mathcal{B}(E)$ is called a projection if P is linear and $P^2 = P$. We recall that P is linear if for $x, y \in E$ and $\alpha, \beta \in \mathbb{C}$, $P(\alpha x + \beta y) = \alpha P(x) + \beta P(y)$.

Proposition 2.1.5. [3] *Let M and N be linear subspaces of a vector space E such that $E = M \oplus N$. Define $P : E \to E$ by $P(z) = x$, where $z = x + y$, $z \in E, x \in M$ and $y \in N$. Then P is an algebraic projection of E onto M along N. Moreover $P(E) = M$ and $\ker P = N$.*

Definition 2.1.6. Let E be a Banach space and N a closed linear subspace of E,

$$E/N = \{x + N : \ x \in E\},$$

equipped with the norm

$$\|x + N\| = \inf_{y \in N} \|x + y\|,$$

is a Banach space. The linear space E/N is referred to as the quotient space. The norm defined above is called the quotient norm. The *codimension* of N is the dimension of the quotient space E/N.

Definition 2.1.7. A Banach space E is said to be separable if it has a dense countable subset.

2.1.2 Dual spaces and weak topologies

Let E be a Banach space, the canonical embedding of E into its bidual E^{**} is

$$\kappa_E : E \to E^{**}, \ \ \kappa_E(x) = \hat{x}.$$

For $\lambda \in E^*$, $\langle \lambda, \hat{x} \rangle = \langle x, \lambda \rangle, x \in E$. We recall that κ_E is an injective map, hence E can be viewed as a subset of its bidual E^{**}. If κ_E is onto, then E is said to be reflexive.

Definition 2.1.8. Let M be a closed linear subspace of the Banach space E

(i) The set

$$M^{\perp} = \{\lambda \in E^* : \langle x, \lambda \rangle = 0, \ \ x \in M\}$$

is the annihilator of M.

(ii) We define the *annihilator* of the dual M^* of M as

$$M^{\perp\perp} = \{\Phi \in E^{**} : \langle \lambda, \Phi \rangle = 0, \lambda \in M^*\}.$$

Clearly $M^{\perp\perp} = (M^{\perp})^{\perp}$.

Theorem 2.1.9 (Hahn-Banach)**.** *Suppose E is a vector space over $\mathbb{C}$, and ρ is a semi-norm on E. Suppose further that M is a subspace of E and ϕ a linear functional on M such that*

$$|\langle y, \phi\rangle| \leq \rho(y) \quad (y \in M).$$

Then there exists a linear functional ψ on E such that $\psi|_M = \phi$ and

$$|\langle x, \psi\rangle| \leq \rho(x),$$

for all $x \in E$.

Corollary 2.1.10. [3] *Let E be a vector space over $\mathbb{C}$, let ρ be a semi-norm on E, and let x_0 be fixed in E. Then there exists a linear functional ϕ on E such that $\langle x_0, \phi\rangle = \rho(x_0)$ and*

$$|\langle x, \phi\rangle| \leq \rho(x),$$

for all $x \in E$.

The following result is a direct consequence of **Theorem (2.1.9)**.

Theorem 2.1.11. [44] *Let M be a closed linear subspace of the normed space E.*

(i) For each $\lambda \in M^*$, let $\phi \in E^*$ be such that $\|\lambda\| = \|\phi\|$ and $\phi|_M = \lambda$. Then the map

$$M^* \to E^*/M^{\perp}, \;\; \lambda \mapsto \phi + M^{\perp}$$

is an isometric isomorphism.

(ii) Let $q : E \to E/M$ be a quotient map. Then the map

$$q^* : (E/M)^* \to M^{\perp}$$

is an isometric isomorphism.

Definition 2.1.12. A linear subspace M of E is said to be weakly complemented if $M^{\perp}$ is a direct summand in E^*.

Remark 2.1.13. Suppose $P \in \mathcal{B}(E)$ is a projection from E onto M, then $(I_E - P)^* \in \mathcal{B}(E^*)$ is a projection from E^* onto $M^{\perp}$, where I_E is the identity operator in $\mathcal{B}(E)$. It follows that every complemented linear subspace of E is weakly complemented.

Definition 2.1.14. Let $(X, \leq)$ be a partially ordered set, X is called a directed set if for any $\alpha, \beta \in X$, there exists $\gamma \in X$ such that $\alpha, \beta < \gamma$. Let S be a non empty set, a net $(x_\alpha) \subset S$ is a function from a directed set into S.

Definition 2.1.15. Let E be a Banach space.

(i) The weak topology $\sigma(E, E^*)$ on E is the topology generated by the family of semi-norms $\{\rho_\lambda, \lambda \in E^*\}$, where

$$\rho_\lambda(x) = |\langle x, \lambda \rangle|, \quad x \in E.$$

(ii) The weak* topology $\sigma(E^*, E)$ on E^* is the topology generated by the family of semi-norms $\{\rho_{i(x)}, x \in E\}$.

The following important result shows the applicability and importance of the topologies stated above.

Theorem 2.1.16. *Let E be a Banach space.*

(i) (Banach-Alaoglu) *The unit ball $E^*_{[1]}$ is weak* compact. Every bounded net in E^* has a weak* accumulation point and a weak* convergent subnet.*

(ii) (Goldstine) *Let $\kappa_E : E \to E^{**}$ be the canonical embedding of E into E^{**}. Then for any $\Phi \in E^{**}_{[1]}$, there exists a net $(x_\alpha) \subset E_{[1]}$ such that $\kappa_E(x_\alpha) \to \Phi$ in $\sigma(E^{**}, E^*)$.*

(iii) (Mazur) *For each convex set $S \subset E$, the strong closure and the weak closure of S are the same.*

2.1.3 Tensor products of Banach spaces

Let E, M and N be vector spaces, a map $\Lambda : E \times M \to N$ is said to be bilinear if for $\alpha, \beta \in \mathbb{C}$, $x_1, x_2, x \in E$ and $y_1, y_2, y \in M$,

$$\Lambda(\alpha x_1 + \beta x_2, y) = \alpha\Lambda(x_1, y) + \beta\Lambda(x_2, y)$$

and

$$\Lambda(x, \alpha y_1 + \beta y_2) = \alpha\Lambda(x, y_1) + \beta\Lambda(x, y_2).$$

We denote by $\mathcal{B}^2(E \times M, N)$, the collection of all bounded bilinear maps from $E \times M$ into N. If $N = \mathbb{C}$, then $\mathcal{B}^2(E \times M, \mathbb{C}) = \mathcal{B}^2(E \times M)$ is complete. We may construct the tensor product $E \otimes M$ of the vector spaces E, M as a space of linear functionals on $\mathcal{B}^2(E \times M)$, in the following way. For $x \in E$, $y \in M$, we denote by $x \otimes y$ the functional given by evaluation at the point (x, y), that is, for any $\Lambda \in \mathcal{B}^2(E \times M)$, $(x \otimes y)(\Lambda) = \Lambda(x, y)$. In other words, we may consider the tensor product $E \otimes M$ as a subspace of $\mathcal{B}^2(E \times M)^*$ spanned by these elements, where $\mathcal{B}^2(E \times M)^*$ is the dual of $\mathcal{B}^2(E \times M)$. It follows that a typical tensor in $E \otimes M$ has the form $u = \sum_i \alpha_i x_i \otimes y_i$, $x_i \in E, y_i \in M$ and scalar α_i for all i. Since each $\alpha_i x_i \in E$, then without loss of generality, we may write $u = \sum_i x_i \otimes y_i$. Note that the representation is not unique. The following are some interesting properties of the tensor product $E \otimes M$;

(i) $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,

(ii) $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$,

(iii) $\alpha(x \otimes y) = (\alpha x) \otimes y = x \otimes (\alpha y)$,

(iv) $0 \otimes y = x \otimes 0 = 0$, $\quad x_1, x_2, x \in E, y_1, y_2, y \in M$ and scalar α.

The projective norm on $E \otimes M$, $\|.\|_p$ is defined as

$$\|u\|_p = \inf\left\{\sum_{i=1}^{n} \|x_i\|\|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i\right\},$$

where the infimum is taken over all finite representations of u. The linear space $E \otimes M$ equipped with this norm is denoted as $E \otimes_p M$. If E and M are Banach spaces, the completion of $E \otimes_p M$ in the projective norm is called the projective tensor product and is denoted as $E \hat{\otimes}_p M$.

Let E and M be Banach spaces and E^*, M^* their respective duals. We denote by $E^*_{[1]}$, $M^*_{[1]}$, the respective closed unit balls of the duals . Let $\mathcal{B}^2(E^* \times M^*)$ be the space of all complex valued bounded bilinear maps on $E^* \times M^*$ equipped with the norm given by

$$\|T\| = \sup\{|T(\varphi, \psi)| : \varphi \in E^*_{[1]}, \psi \in M^*_{[1]}\}.$$

Then, $\mathcal{B}^2(E^* \times M^*)$ is complete. Let $\Lambda_{x,y}$ denote the elements of $\mathcal{B}^2(E^* \times M^*)$ defined by $\Lambda_{x,y}(\varphi, \psi) = \varphi(x)\psi(y)$, then there exists an injectve linear map from $E \otimes M$ into $\mathcal{B}^2(E^* \times M^*)$. This shows that $E \otimes M$ may be viewed as a subspace of $\mathcal{B}^2(E^* \otimes M^*)$. Hence, the injective norm on $E \otimes M$ is defined as,

$$\|u\|_\epsilon = \sup\left\{\left|\sum_{i=1}^{n} \varphi(x_i)\psi(y_i)\right| : u = \sum_{i=1}^{n} x_i \otimes y_i,\ \varphi \in E^*_{[1]},\ \psi \in M^*_{[1]}\right\},$$

where the supremum is taken over all such representations of u. The completion of $E \otimes M$ with respect to this norm is called the injective tensor product and is denoted by $E \check{\otimes}_\epsilon M$.

2.2 Banach algebras

An algebra is a vector space $\mathcal{A}$ equipped with a map $\mathcal{A}\times\mathcal{A} \to A$, $(a, b) \mapsto ab$, such that for $a, b, c \in \mathcal{A}, \alpha \in \mathbb{C}$,

(i) $a(bc) = (ab)c$

(ii) $a(b + c) = ab + ac$

(iii) $(a + b)c = ac + bc$

(iv) $(\alpha a)b = \alpha(ab) = a(\alpha b)$.

The algebra $\mathcal{A}$ is commutative if for any $a, b \in \mathcal{A}, \quad ab = ba$, and is said to be unital if it has a multiplicative identity. A normed algebra is an algebra $\mathcal{A}$ such that the vector space $\mathcal{A}$ is a normed space and

$$\|ab\| \leq \|a\|\|b\|, \text{ for all } a, b \in \mathcal{A}. \tag{2.1}$$

The inequality in **(2.1)** ensures that the multiplication in $\mathcal{A}$ is continuous. If $(\mathcal{A}, \|.\|)$ is a Banach space, then $\mathcal{A}$ is a Banach algebra. The multiplicative identity in a unital normed algebra $\mathcal{A}$, denoted by $e_{\mathcal{A}}$ satisfies, $\|e_{\mathcal{A}}\| = 1$. If $\mathcal{A}$ is not unital, then we define $\mathcal{A}^{\#} = \mathbb{C} \oplus \mathcal{A}$. Each element in $\mathcal{A}^{\#}$ is of the form (α, a), where $\alpha \in \mathbb{C}, a \in \mathcal{A}$. If we equip $\mathcal{A}^{\#}$ with the product

$$(\alpha, a)(\beta, b) = (\alpha\beta, \beta a + \alpha b + ab),$$

then $\mathcal{A}^{\#}$ becomes an algebra referred to as the unitization of $\mathcal{A}$. For any $(\alpha, a) \in \mathcal{A}$,

$$(1, 0)(\alpha, a) = (\alpha, a) \text{ and } (\alpha, a)(1, 0) = (\alpha, a).$$

This shows that the multiplicative identity in $\mathcal{A}^{\#}$ is $(1, 0)$. We define the norm

$$\|(\alpha, a)\| = |\alpha| + \|a\|_{\mathcal{A}}$$

on $\mathcal{A}^{\#}$, which turns it into a normed algebra. If $\mathcal{A}$ is a Banach algebra, then $\mathcal{A}^{\#}$ is also a Banach algebra. We denote by $\mathcal{A}^{\mathrm{op}}$, the opposite Banach algebra of $\mathcal{A}$. That is, the Banach algebra whose underlying linear space is $\mathcal{A}$, but whose multiplication is the multiplication in $\mathcal{A}$ reversed.

Definition 2.2.1. Let $\mathcal{A}$ be a normed algebra.

(i) A left (right) approximate identity for $\mathcal{A}$ is a net $(e_\alpha) \subset \mathcal{A}$ such that given any $a \in \mathcal{A}$, $\|e_\alpha a - a\| \to 0$ $(\|ae_\alpha - a\| \to 0)$, for all α.

(ii) A weak left (right) approximate identity for $\mathcal{A}$ is a net $(e_\beta) \subset \mathcal{A}$ such that given any $a \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$, $\langle ae_\beta, \phi\rangle \to \langle a, \phi\rangle$ $(\langle e_\beta a, \phi\rangle \to \langle a, \phi\rangle)$, for all β.

(iii) A left (or right) approximate identity (e_α) for $\mathcal{A}$ is said to be bounded if there exists $K > 0$, such that $\|e_\alpha\| \leq K$, for all α.

(iv) An approximate identity for $\mathcal{A}$ is a net (e_α) such that $\|ae_\alpha - a\| \to 0$ and $\|e_\alpha a - a\| \to 0$, $a \in \mathcal{A}$, for all α.

(v) A weak approximate identity for $\mathcal{A}$ is a net (e_β) such that for any $a \in \mathcal{A}$, $\phi \in \mathcal{A}^*$, $\langle ae_\beta, \phi\rangle \to \langle a, \phi\rangle$ and $\langle e_\beta a, \phi\rangle \to \langle a, \phi\rangle$, for all β.

Lemma 2.2.2. [13] *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}$ has a weak left (right) approximate identity, then $\mathcal{A}$ has a left (right) approximate identity.*

Lemma 2.2.3. [36] *Let $\mathcal{A}$ be a normed algebra with a bounded left and right approximate identity (e_α), (f_β) respectively, then $(e_\alpha + f_\beta - e_\alpha f_\beta)$ is a bounded approximate identity for $\mathcal{A}$.*

Remark 2.2.4. Clearly, if $\|e_\alpha\| \leq K_1$ and $\|f_\beta\| \leq K_2$, then $\|e_\alpha + f_\beta - e_\alpha f_\beta\| \leq K_1 + K_2 + K_1K_2$.

Definition 2.2.5. Let $\mathcal{A}$ be a normed algebra,

(i) $\mathcal{A}$ has left (right) approximate units if for each $a \in \mathcal{A}$ and $\epsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|a - ua\| \leq \epsilon(\|a - au\| \leq \epsilon)$;

(ii) $\mathcal{A}$ has approximate units if for each $a \in \mathcal{A}$ and $\epsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|a - ua\| \leq \epsilon$ and $\|a - au\| \leq \epsilon$;

(iii) $\mathcal{A}$ has (left, right) approximate unit bounded by $K > 0$ if the element u can be chosen such that $\|u\| \leq K$.

Lemma 2.2.6. [31] *Let $\mathcal{A}$ be a normed algebra, $F_\mathcal{A}$ a finite subset of $\mathcal{A}$ and $K \geq 1$. Then the following statements are equivalent*

(i) $\mathcal{A}$ has left approximate units bounded by K.

(ii) For every $a \in F$ and $\epsilon > 0$, there exists $u \in \mathcal{A}$ such that $\|u\| \leq K$ and $\|a - ua\| \leq \epsilon$.

(iii) $\mathcal{A}$ has a left approximate identity bounded by K.

Theorem 2.2.7 (Cohen-Hewitt Factorization)**.** *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. Then, $\mathcal{A} = \mathcal{A}.\mathcal{A}$.*

Definition 2.2.8. Let $\mathcal{A}$ be a Banach algebra, an *involution* on $\mathcal{A}$ is a mapping

$$* : a \mapsto a, \ \ \mathcal{A} \to \mathcal{A},$$

such that for any $a, b \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$,

(i.) $a^{**} = a$,

(ii.) $(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*$,

(iii.) $(ab)^* = b^* a^*$.

Remark 2.2.9. If the involution on $\mathcal{A}$ is isometric; that is $\|a^*\| = \|a\|$ for all $a \in \mathcal{A}$, then $\mathcal{A}$ is called a "Banach $*$-algebra".

Definition 2.2.10. A Banach algebra, say $\mathcal{A}$, with an involution * is called a *C^*-algebra* if its norm satisfy

$$\|a^* a\| = \|a\|^2, \ \ (a \in \mathcal{A}).$$

Remark 2.2.11. Let $\mathcal{A}$ be a C*-algebra. Notice that for any $a \in \mathcal{A}$,

$$\|a\|^2 = \|a^* a\| \leq \|a\| \|a^*\| \implies \|a\| \leq \|a^*\|.$$

Also,

$$\|a^*\|^2 = \|a^{**} a^*\| \leq \|a\| \|a^*\| \implies \|a^*\| \leq \|a\|.$$

Hence $\|a\| = \|a^*\|$. This shows that every C*-algebra is also a Banach $*$-algebra.

Definition 2.2.12. Let $\mathcal{A}$ be a unital Banach algebra, $a \in \mathcal{A}$ is said to be invertible if there exists $b \in \mathcal{A}$ such that

$$ab = ba = e_{\mathcal{A}}.$$

The collection of all invertible elements in $\mathcal{A}$ is denoted by $\mathrm{Inv}(\mathcal{A})$.

Theorem 2.2.13 (Gel'fand-Mazur)**.** *Let $\mathcal{A}$ be a unital Banach algebra. If* $\mathrm{Inv}\,\mathcal{A} = \mathcal{A} \setminus \{0\}$*, then $\mathcal{A}$ is isomorphic to $\mathbb{C}$.*

Definition 2.2.14. Let $\mathcal{A}$ be a unital Banach algebra

(i) The resolvent set of $a \in \mathcal{A}$ is the set

$$\rho_{\mathcal{A}}(a) = \{z \in \mathbb{C} : \ \ z e_{\mathcal{A}} - a \in \mathrm{Inv}\,\mathcal{A}\}.$$

(ii) The spectrum of $a \in \mathcal{A}$ is the set

$$\sigma_{\mathcal{A}}(a) = \{z \in \mathbb{C}, \quad ze_{\mathcal{A}} - a \notin \operatorname{Inv} \mathcal{A}\}.$$

(iii) The spectral radius of $a \in \mathcal{A}$ is

$$r_{\mathcal{A}}(a) = \sup\{|z| : z \in \sigma_{\mathcal{A}}(a)\}.$$

(iv) The resolvent function of $a \in \mathcal{A}$ is the function

$$\rho_{\mathcal{A}}(a) \to \operatorname{Inv}(\mathcal{A}), \ z \mapsto (ze_{\mathcal{A}} - a)^{-1}.$$

Theorem 2.2.15. [31] *For a unital Banach algebra $\mathcal{A}$,*

(i) $\{a \in \mathcal{A} : \|e_{\mathcal{A}} - a\| < 1\} \subset \operatorname{Inv} \mathcal{A}$.

(ii) $\operatorname{Inv} \mathcal{A}$ *is an open subset of* $\mathcal{A}$.

(iii) $\rho_{\mathcal{A}}(a)$ *is an open subset of* $\mathbb{C}$ *for each* $a \in \mathcal{A}$.

Remark 2.2.16. Clearly for each $a \in \mathcal{A}$ and $z \in \sigma_{\mathcal{A}}(a)$, $\|z^{-1}a\| \geq 1$. This further shows that

$$1 \leq \|z^{-1}a\| = |z^{-1}|\|a\| = |z|^{-1}\|a\| \implies |z| \leq \|a\|.$$

Hence

$$\sigma_{\mathcal{A}}(a) \subset \{z \in \mathbb{C} : \ |z| \leq \|a\|\}.$$

This shows that $\sigma_{\mathcal{A}}(a)$ is bounded. Further more, $\sigma_{\mathcal{A}}(a) = \mathbb{C} \setminus \rho_{\mathcal{A}}(a)$ is a closed subset of $\mathbb{C}$. Therefore $\sigma_{\mathcal{A}}(a)$ is compact for every $a \in \mathcal{A}$.

For an algebra $\mathcal{A}$:

(i) A linear subspace I of $\mathcal{A}$ is a left (right) ideal if $\mathcal{A}I \subset I$ ($I\mathcal{A} \subset I$). I is an ideal if $\mathcal{A}I \cup I\mathcal{A} \subset I$.

(ii) A left ideal M of $\mathcal{A}$ is maximal if $M \neq \mathcal{A}$ and if $M \subset I \subset \mathcal{A}$ for some left ideal I, then either $M = I$ or $I = \mathcal{A}$. The collection of all maximal ideals of $\mathcal{A}$ is denoted by $\operatorname{Max}(\mathcal{A})$.

(iii) If $\mathcal{A}$ is a Banach algebra, and I a closed ideal of $\mathcal{A}$. Then $\mathcal{A}/I$ is a Banach algebra with respect to the quotient norm.

Definition 2.2.17. An element $a \in \mathcal{A}$ is said to be quasi-nilpotent if $r_{\mathcal{A}}(a) = 0$. The collection of all quasi-nilpotent elements in $\mathcal{A}$ is $\mathcal{D}(A)$.

2.2.1 Character space of a Banach algebra

A character on a Banach algebra $\mathcal{A}$ is a multiplicative non-zero linear functional $\varphi \in \mathcal{A}^*$. The collection of all characters on $\mathcal{A}$ denoted $\Phi_{\mathcal{A}}$ is called the character space of $\mathcal{A}$. For a unital Banach algebra $\mathcal{A}$ with identity $e_{\mathcal{A}}$, notice that if $a \in \mathcal{A}, \ \varphi \in \Phi_{\mathcal{A}}$,

$$\varphi(a) = \varphi(ae_{\mathcal{A}}) = \varphi(a)\varphi(e_{\mathcal{A}}) \implies \varphi(e_{\mathcal{A}}) = 1.$$

Definition 2.2.18. Let $\mathcal{A}$ be a commutative Banach algebra. The radical of $\mathcal{A}$, $\mathrm{rad}(\mathcal{A})$, is defined by,

$$\mathrm{rad}(\mathcal{A}) = \cap\{M : M \in \mathrm{Max}(\mathcal{A})\} = \cap\{\ker\varphi : \ \ \varphi \in \Phi_{\mathcal{A}}\}.$$

If $\Phi_{\mathcal{A}} = \emptyset$, then $\mathrm{rad}(\mathcal{A}) = \mathcal{A}$.

Remark 2.2.19. Clearly, $\mathrm{rad}(\mathcal{A})$ is a closed ideal of $\mathcal{A}$ (this readily follows from the fact that the intersection of ideals of a Banach algebra is also an ideal of that Banach algebra).

Definition 2.2.20. A Banach algebra $\mathcal{A}$ is said to be semi-simple if $\mathrm{rad}(\mathcal{A}) = \{0\}$, and radical if $\mathrm{rad}(\mathcal{A}) = \mathcal{A}$.

Theorem 2.2.21. [36] *For a commutative, unital Banach algebra $\mathcal{A}$, $\Phi_{\mathcal{A}} \neq \emptyset$ and the mapping $\varphi \mapsto \ker\varphi$ is a bijection from $\Phi_{\mathcal{A}}$ onto $\mathrm{Max}(\mathcal{A})$.*

Corollary 2.2.22. [31] *Let $\mathcal{A}$ be a commutative, unital Banach algebra, and let $a \in \mathcal{A}$,*

(i) $a \in \mathrm{Inv}(\mathcal{A})$ *if and only if* $\varphi(a) \neq 0$ *for each* $\varphi \in \Phi_{\mathcal{A}}$,

(ii) $\sigma_{\mathcal{A}}(a) = \{\varphi(a) : \varphi \in \Phi_{\mathcal{A}}\}$,

(iii) $r_{\mathcal{A}} = \{|\varphi(a)| : \varphi \in \Phi_{\mathcal{A}}\}$,

(iv) $a \in \mathcal{D}(\mathcal{A})$ *if and only if* $\varphi(a) = 0$ *for each* $\varphi \in \Phi_{\mathcal{A}}$.

Proposition 2.2.23. [36] *Let φ be a character on $\mathcal{A}$. Then φ is continuous, and $\|\varphi\| \leq 1$. If $\mathcal{A}$ is unital, then $\varphi(e_{\mathcal{A}}) = 1$ and $\|\varphi\| = 1$.*

Remark 2.2.24. Proposition (2.2.23) shows that the character space of $\mathcal{A}$ is contained in the unit ball, $\mathcal{A}^*_{[1]}$ of the dual $\mathcal{A}^*$ of $\mathcal{A}$.

Let $\mathcal{A}$ be a commutative Banach algebra and $\Phi_{\mathcal{A}}$ the character space of $\mathcal{A}$, $\Phi_{\mathcal{A}}$ is endowed with the weakest topology with respect to which all functions

$$\Phi_{\mathcal{A}} \to \mathbb{C}, \ \ \varphi \mapsto \varphi(a), \ \ a \in \mathcal{A}$$

are continuous. As a result, let $F_{\mathcal{A}}$ be a finite subset of $\mathcal{A}$, a basis for a neighbourhood of $\varphi_0 \in \Phi_{\mathcal{A}}$ is given by the collection

$$V(\varphi_0, F_{\mathcal{A}}, \epsilon) = \{\varphi \in \Phi_{\mathcal{A}} : \ |\varphi(a) - \varphi_0(a)| < \epsilon, \text{ for all } a \in F_{\mathcal{A}}, \ \ \epsilon > 0\}.$$

Remark 2.2.25. The topology described above is referred to as the Gel'fand topology. The character space $\Phi_{\mathcal{A}}$ equipped with the $\mathcal{A}$ topology is called the *Gel'fand space.*

Proposition 2.2.26. [31] *Let φ_∞ be a zero linear functional. The character space $\Phi_{\mathcal{A}}$ is a locally compact Hausdorff space with one point compactification $\Phi_{\mathcal{A}}^\infty = \Phi_{\mathcal{A}} \cup \varphi_\infty$. If $\mathcal{A}$ is unital, then $\Phi_{\mathcal{A}}$ is compact.*

Let $F_{\mathcal{A}}$ be a finite subset of $\mathcal{A}$, the proof of **Proposition (2.2.26)** readily follows from the fact that

$$\left\{\phi \in \Phi_{\mathcal{A}}^\infty : |\phi(a)| \geq \frac{1}{2}|\varphi(a)|, a \in F_{\mathcal{A}}\right\}$$

is a compact neighborhood of $\varphi \in \Phi_{\mathcal{A}}$ that does not contain φ_∞. That is, $\Phi_{\mathcal{A}}$ is locally compact. Also, if $\mathcal{A}$ has identity element $e_{\mathcal{A}}$, then

$$\Phi_{\mathcal{A}} = \{\varphi \in \Phi_{\mathcal{A}}^\infty : \varphi(e_{\mathcal{A}}) = 1\}.$$

This shows that $\Phi_{\mathcal{A}}$ is closed and therefore compact.

Definition 2.2.27. Let $a \in \mathcal{A}$, we define $\hat{a} : \Phi_{\mathcal{A}} \to \mathbb{C}$ by $\hat{a}(\varphi) = \varphi(a)$. Then $\hat{a}$ is a continuous functional called the Gel'fand transform of a. The linear mapping

$$\mathcal{G} : \mathcal{A} \to C(\Phi_{\mathcal{A}}), \quad a \mapsto \hat{a}$$

is a homomorphism called the Gel'fand representation of $\mathcal{A}$. We denote $\mathcal{G}(\mathcal{A})$ by $\hat{\mathcal{A}}$.

The following result is an important property of commutative Banach algebras.

Theorem 2.2.28. [31] *Let $\mathcal{A}$ be a commutative Banach algebra. For every $a \in \mathcal{A}$,*

$$\sigma_{\mathcal{A}}(a) \setminus \{0\} \subset \hat{a}(\Phi_{\mathcal{A}}) = \{\varphi(a) : \varphi \in \Phi_{\mathcal{A}} \subset \sigma_{\mathcal{A}}(a)\}.$$

If $\mathcal{A}$ is unital, then $\hat{a}(\Phi_{\mathcal{A}}) = \sigma_{\mathcal{A}}$.

Corollary 2.2.29. [3] *Let $a \in \mathcal{A}$. $\hat{a}$ is trivial if and only if*

$$r_{\mathcal{A}}(a) = \lim_{n \to 0} \|a^n\|^{1/n} = 0.$$

Theorem 2.2.30 (Gel'fand Representation Theorem)**.** *Let $\mathcal{A}$ be a commutative Banach algebra and $\mathcal{G}$ the Gel'fand representation of $\mathcal{A}$. Then,*

(i) $\mathcal{G}$ maps $\mathcal{A}$ into $C_0(\Phi_{\mathcal{A}})$ and is norm decreasing,

(ii) $\mathcal{G}(\mathcal{A})$ strongly seperates points of $\Phi_{\mathcal{A}}$,

(iii) $\mathcal{G}$ is isometric if and only if $\|a^2\| = \|a\|^2$.

2.2.2 Tensor products of Banach algebras

Let $\mathcal{A}$ and $\mathcal{B}$ be algebras, the vector space $\mathcal{A} \otimes \mathcal{B}$ admits a unique product with respect to which $\mathcal{A} \otimes \mathcal{B}$ is an algebra, called the algebra tensor product and which satisfies $(a \otimes b)(c \otimes d) = ac \otimes bd$ for all $a, c \in \mathcal{A}$ and $b, d \in \mathcal{B}$.

Recall that for Banach spaces $\mathcal{A}$ and $\mathcal{B}$, the projective norm on $\mathcal{A} \otimes \mathcal{B}$ is

$$\|u\|_p = \inf\left\{ \sum_{i=1}^{n} \|a_i\|\|b_i\| : u = \sum_{i=1}^{n} a_i \otimes b_i \right\}.$$

Notice that for any $u, v \in \mathcal{A} \otimes \mathcal{B}$, $u = \sum_{i=1}^{n} a_i \otimes b_i$, $v = \sum_{j=1}^{m} c_j \otimes d_j$, $uv = \sum_{i,j} a_i c_j \otimes b_i d_j$. Hence,

$$\sum_{i,j} \|a_i c_j\|\|b_i d_j\| \leq \sum_{i} \|a_i\|\|b_i\| \sum_{j} \|c_j\|\|d_j\|.$$

This shows that $\|uv\|_p \leq \|u\|_p \|v\|_p$. That is, $\|.\|_p$ is indeed an algebra norm on $\mathcal{A} \otimes \mathcal{B}$. The tensor product on $\mathcal{A} \otimes \mathcal{B}$ can be extended to $\mathcal{A} \hat{\otimes} \mathcal{B}$, so that $\mathcal{A} \hat{\otimes} \mathcal{B}$ is a Banach algebra.

The corresponding diagonal operator is defined by

$$\pi_{\mathcal{A}} : \mathcal{A} \hat{\otimes} \mathcal{A} \to \mathcal{A}, \;\; a \otimes b \mapsto ab.$$

If there is no ambiguity with regards to the Banach algebra in question, then we use π.

Due to the nature of the norm on the injective tensor product, it is not yet clear under which conditions $\mathcal{A} \check{\otimes} \mathcal{B}$ is a Banach algebra. An interesting example of an injective tensor product that is a Banach algebra will be the focus of our study in **Chapter 5.**

2.2.3 Arens products

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{A}^{**}$ be the second dual of its underlying Banach space, each $a \in \mathcal{A}$ has a canonical embedding $\hat{a} \in \mathcal{A}^{**}$ determined by

$$\langle \phi, \hat{a} \rangle = \langle a, \phi \rangle \ \ (\phi \in \mathcal{A}^*).$$

Let $\Phi \in \mathcal{A}^{**}, \ \lambda \in \mathcal{A}^*$. Then $\Phi\lambda \in \mathcal{A}^*$, determined by

$$\langle a, \Phi.\lambda \rangle = \langle \lambda.a, \Phi \rangle \ \ \ (a \in \mathcal{A}).$$

For $\Phi, \Psi \in \mathcal{A}^{**}$, $\Phi \Box \Psi$, $\Phi \diamond \Psi$, the *first Arens product* and the *second Arens product* are respectively defined by

$$\langle \lambda, \Phi \Box \Psi \rangle = \langle \Psi.\lambda, \Phi \rangle, \quad \langle \lambda, \Phi \diamond \Psi \rangle = \langle \lambda.\Psi, \Phi \rangle \ \ \ (\lambda \in \mathcal{A}^*).$$

If $\Phi \Box \Psi = \Phi \diamond \Psi$, then $\mathcal{A}$ is said to be *Arens regular*. The linear space $\mathcal{A}^{**}$ equipped with either of this products is indeed a Banach algebra. Since $a \mapsto \hat{a}$ is an isometric momonorphism, it follows that $\mathcal{A}$ can be viewed as a closed subalgebra of $\mathcal{A}^{**}$.

2.3 Examples of Banach algebras

Example 2.3.1. Let S be a non empty set, we define $\mathbb{C}^S$ as the collection of all complex valued functions on S . The product on $\mathbb{C}^S$ is defined pointwise in the sense that for any $f, g \in \mathbb{C}^S$ and $s \in S$, $fg(s) = f(s)g(s)$. Clearly $\mathbb{C}^S$ with the product defined above is an algebra. We define $I^\infty(S)$ as the set of all bounded complex valued functions on S. Notice that $I^\infty(S) \subset \mathbb{C}^S$. Consider the norm

$$\|f\|_S = \sup\{|f(s)|, \ \ s \in S, f \in I^\infty(S)\}.$$

Clearly for any $f, g \in I^\infty(S)$,

$$\|fg\|_S = \sup|f(s)g(s)| \leq \sup|f(s)| \sup|g(s)| = \|f\|_S \|g\|_S,$$

this shows that $I^\infty(S)$ is a normed algebra. The norm described above is referred to as the uniform norm. The algebra $I^\infty(S)$ with pointwise product and uniform norm is a Banach algebra.

Example 2.3.2. Let $\mathcal{X}$ be a topological space, then $C(\mathcal{X})$ is defined as the set of continuous complex valued functions on $\mathcal{X}$. That $C(\mathcal{X})$ is a linear space readily follows from the fact that for any $f, g \in C(\mathcal{X}), \ \alpha, \beta \in \mathbb{C}, \ \ \alpha f + \beta g$ is a continuous function. We denote by $C^b(\mathcal{X})$ the set of all bounded complex valued continuous functions on $\mathcal{X}$. Notice that $C^b(\mathcal{X}) \subset C(\mathcal{X}) \cap I^\infty(\mathcal{X})$. The algebra $C^b(\mathcal{X})$ equipped with the uniform norm and pointwise product is a Banach algebra. An interesting subalgebra of $C^b(\mathcal{X})$ is $C_0(\mathcal{X})$, the collection of all complex valued continuous functions that vanish at infinity.

Remark 2.3.3. We shall discuss the Banach algebra $C^b(\mathcal{X})$ and some of its subalgebras in detail in **Chapter 4.**. It should be noted that $C_0(\mathcal{X})$ is a closed subalgebra of $C^b(\mathcal{X})$ and is therefore a Banach algebra with respect to the uniform norm.

Example 2.3.4. Let S be a non empty set. For $f \in \mathbb{C}^S$, we set

$$\sum\{|f(s)| : s \in S\} = \sup_{s \in S} \sum\{|f(s)| : s \in F\}.$$

Where the supremum is taken over all finite subsets F of S. Then

$$l^1(S) = \{f \in C^S : \sum\{|f(s)| : s \in S\} < \infty\}.$$

For each $s \in S$, let δ_s be the characteristic function of $\{s\}$. Then a generic element of $l^1(S)$ can be denoted by $\sum_{s \in S} f(s)\delta_s$. We set each $f(s) = \alpha_s,\ s \in S$. Hence each $f \in l^1(S)$ can be written in the form $f = \sum_{s \in S} \alpha_s \delta_s$. We define the following norm on $l^1(S)$:

$$\|f\|_1 = \sum_{s \in S} |\alpha_s|. \tag{2.2}$$

Since each $f(s) = \alpha_s \in \mathbb{C}$ for all $s \in S$, it follows that $(l^1(S), \|.\|_1)$ is a Banach space. In the case when S is countable, then $(l^1(S), \|.\|_1)$ is seperable.

If the non empty set S is a semigroup, that is, it is equipped with the map $S \times S \to S$, $(s,t) \mapsto st$, such that $(rs)t = r(st),\ r, s, t \in S$. We define the product on $l^1(S)$ by convolution. That is for $f, g \in l^1(S)$,

$$(f \star g)(t) = \sum_{rs=t} \{f(r)g(s) : r, s \in S, rs = t \in S\}.$$

Clearly,

$$\begin{aligned} \|f \star g\|_1 = \sum \left| \sum_{rs=t} f(r)g(s) \right| \leq \sum \sum |f(r)||g(s)| = \sum |f(r)| \sum |g(s)| \\ = \|f\|_1 \|g\|_1. \end{aligned}$$

It follows that $(l^1 S, \|.\|_1, \star)$ is a Banach algebra called the semigroup algebra.

Example 2.3.5. A non empty space G is called a topological group if G satisfies the following properties:

(i) G is a group,

(ii) G is a topological Hausdorff space such that the maps $(a, b) \mapsto ab$ and $a \mapsto a^{-1}$ for all $a, b \in G$, are continuous.

The topological group G is locally compact if the topology on G is locally compact. We denote by $M(G)$ the collection of all bounded complex Borel measures on G. Let $E_i, i = 1, 2, ..., n$ be a partition of G, such that for any $\mu \in M(G)$, $\mu(E_i) < \infty, i = 1, 2, ..., n$. Then

$$\|\mu\| = \sup \left\{ \sum_{i=1}^{n} |\mu(E_i)| : \mu \in M(G),\ G = \cup_{i=1}^{n} E_i \right\}$$

is a norm on $M(G)$, where the supremum is taken over all such partitions of G. The linear space $M(G)$ equipped with this norm is a Banach space.

We define the product in $M(G)$ by convolution such that for $\mu, \nu \in M(G),\ f \in C_0(G)$,

$$\langle f, \mu \star \nu \rangle = \int_G \int_G f(ab) d\mu(a) d\nu(b),\ \ a, b \in G.$$

Notice that

$$\begin{aligned} |\langle f, \mu \star \nu \rangle| &= \left| \int_G \int_G f(ab) d\mu(a) d\nu(b) \right| \\ &\leq \int_G \int_G |f(ab)| d|\mu(a)| d|\nu(b)| \\ &\leq |f(ab)| \int_G d|\mu(a)| \int_G d|\nu(b)| \\ &\leq \|f\|_\infty |\mu|(G) |\nu|(G) \\ &\leq \|f\|_\infty \|\mu\| \|\nu\|. \end{aligned}$$

This shows that $M(G)$ equipped with the given norm is indeed a Banach algebra referred to as the measure algebra.

Example 2.3.6. Given a locally compact group G, a regular Borel measure μ on G is called a left Haar measure if it is left translation invariant. That is for every $B \subset G$, $\mu(aB) = \mu(B)$, for all $a \in G$. We define $L^1(G)$ as the set of all μ integrable functions f on G such that

$$\int_G \|f(s)\| d\mu(s) < \infty,\ \ s \in G.$$

$(L^1(G), \|.\|_1)$ is a Banach space where,

$$\|f\|_1 = \int_G \|f(s)\| d\mu(t), \;\; f \in L^1(G), s \in G.$$

We define the convolution product on $L^1(G)$ by

$$(f \star g)(t) = \int_G f(s) g(s^{-1}t) d\mu(s), f, g \in L^1(G), \;\; s, t \in G.$$

Clearly,

$$\begin{aligned}\|f \star g\|_1 &= \int_G \left\| \int_G f(s) g(s^{-1}t) d\mu(s) \right\| d\mu(t) \leq \int_G \int_G \|f(s) g(s^{-1}t)\| d\mu(s) d\mu(t) \\ &\leq \int_G \|f(s)\| d\mu(s) \int_G \|g(s^{-1}t)\| d\mu(t) = \|f\|_1 \|g\|_1.\end{aligned}$$

It therefore follows that $(L^1(G), \|.\|_1)$ is indeed a Banach algebra where the product is defined by convolution. This Banach algebra is called the group algebra.

Example 2.3.7. Let E be a normed linear space. A linear map $T : E \to E$ is called a linear operator on E. The collection of all linear operators of this form is denoted $\mathcal{L}(E)$. We denote the collection of all bounded linear operators on E by $\mathcal{B}(E) \subset \mathcal{L}(E)$. Recall that if E is a Banach space, then $\mathcal{B}(E)$ equipped with the operator norm is a Banach space. We define the product on $\mathcal{B}(E)$ by composition. That is, for any $S, T \in \mathcal{B}(E)$, $x \in E$, $(S \circ T)x = S(Tx)$. Notice that

$$\|S(Tx)\| \leq \|S\| \|Tx\| \leq \|S\| \|T\| \|x\|_E.$$

It therefore folows that

$$\|ST\| = \sup_{\|x\|_E \leq 1} \{\|S(Tx)\| : x \in E\} \leq \|S\| \|T\|.$$

Hence, $\mathcal{B}(E)$ equipped with the operator norm and the product defined by composition is a Banach algebra. Clearly, $\mathcal{B}(E)$ is a unital Banach algebra (the unit element here is the identity operator).

Example 2.3.8. Let $\{\mathcal{A}_i : i \in I\}$ be a collection of Banach algebras. We denote by $\Pi_{i \in I} \mathcal{A}_i$, the product space of the collection. This space consists of all mappings $a : I \to \bigcup\limits_{i \in I} \mathcal{A}_i$ such that $a(i) \in \mathcal{A}_i$ for all $i \in I$, the linear operator given coordinate-wise.

The l_p direct sum of the collection is

$$\bigoplus_{i\in I}^{p}\mathcal{A}_i = \left\{ a \in \Pi_{i\in I}\mathcal{A}_i : \sum_i \|a(i)\|^p < \infty,\ \ p \geq 1 \right\}.$$

We equip $\bigoplus_{i\in I}^{p}\mathcal{A}_i$ with the norm

$$\|a\|_p = \left(\sum_i \|a(i)\|^p \right)^{\frac{1}{p}}.$$

It follows $\bigoplus_{i\in I}^{p}\mathcal{A}_i$ is a Banach space. We define a coordinate-wise product on $\bigoplus_{i\in I}^{p}\mathcal{A}_i$. Notice that for any $a, b \in \bigoplus_{i\in I}^{p}\mathcal{A}_i$,

$$\begin{aligned}\|ab\|_p^p = \sum_i \|a(i)b(i)\|^p &\leq \sum_i \|a(i)\|^p \|b(i)\|^p \\ &\leq \sum_i \|a(i)\|^p \sum_i \|b(i)\|^p.\end{aligned}$$

Hence,

$$\|ab\|_p \leq \left(\sum_i \|a(i)\|^p \right)^{\frac{1}{p}} \left(\sum_i \|b(i)\|^p \right)^{\frac{1}{p}} = \|a\|_p \|b\|_p.$$

This shows that $\bigoplus_{i\in I}^{p}\mathcal{A}_i$ is indeed a Banach algebra.

The c_0 direct sum of the collection is

$$\bigoplus_{i\in I}^{0}\mathcal{A}_i = \left\{ a \in \Pi_{i\in I}\mathcal{A}_i : \max_i \|a(i)\| < \infty,\ \lim_i a(i) = 0 \right\}.$$

$\bigoplus_{i\in I}^{0}\mathcal{A}_i$ equipped with the norm;

$$\|a\|_\infty = \max_i \|a(i)\|$$

is a Banach space. That $\bigoplus_{i\in I}^{0}\mathcal{A}_i$ equipped with this norm is a Banach algebra readily follows from the fact that each $\mathcal{A}_i,\ i \in I$ is a Banach algebra.

2.4 Modules

Definition 2.4.1. Let $\mathcal{A}$ be a Banach algebra and X an additive group, X is said to be a left $\mathcal{A}$-module if it is also equipped with an operation, $\mathcal{A} \times X \to X$, defined by

(i) $(a+b)x = ax + bx$,

(ii) $a(x+y) = ax + ay$,

(iii) $a(bx) = (ab)x$,

(iv) $\alpha(ax) = (\alpha a)x$,

$a, b \in \mathcal{A}, x \in X$ and $\alpha \in \mathbb{C}$.
The additive group X is said to be a right $\mathcal{A}$-module if it is also equipped with an operation, $X \times \mathcal{A} \to X$ such that,

(i) $x(a+b) = xa + xb$,

(ii) $(x+y)a = xa + ya$,

(iii) $(xa)b = x(ab)$,

(iv) $\alpha(xa) = x(\alpha a)$.

X is an $\mathcal{A}$-bimodule if it is both a left and right $\mathcal{A}$-module.

Definition 2.4.2. Let X be a left $\mathcal{A}$-module, X is called a normed left $\mathcal{A}$-module if it is a normed vector space and for $a \in \mathcal{A}$, $x \in X$,

$$\|a.x\| \leq K\|a\|\|x\|, \;\; K > 0.$$

A normed left $\mathcal{A}$-module X is called a Banach left $\mathcal{A}$-module if $(X, \|.\|)$ is Banach space.
Let X be a right $\mathcal{A}$-module. Then X is a normed right $\mathcal{A}$-module if it is a normed vector space and for $a \in \mathcal{A}$, $x \in X$,

$$\|x.a\| \leq K\|a\|\|x\|, \; K > 0.$$

A normed right $\mathcal{A}$-module is a Banach right $\mathcal{A}$-module if $(X, \|.\|)$ is a Banach space. If X is both a left Banach $\mathcal{A}$-module and a right Banach $\mathcal{A}$-module, then X is a Banach $\mathcal{A}$-bimodule.

Definition 2.4.3. Let X and Y be Banach modules. A linear map $f : X \to Y$ is a left $\mathcal{A}$-module homomorphism if for any $a \in \mathcal{A}$,

$$f(a.x) = a.f(x), \;\; (x \in X),$$

and a right $\mathcal{A}$-module homomorphism if

$$f(x.a) = f(x).a.$$

In the case where X and Y are Banach $\mathcal{A}$-bimodules, then f is an $\mathcal{A}$-bimodule homomorphism if it is a left and right $\mathcal{A}$-module homomorphism.

For a Banach algebra $\mathcal{A}$:

(i) $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with the product of $\mathcal{A}$ giving the two module multiplication.

(ii) Each left (right) ideal of $\mathcal{A}$ is a normed left (right) $\mathcal{A}$-bimodule, the product of $\mathcal{A}$ giving the module multiplication.

(iii) Let X be a normed left $\mathcal{A}$- module with dual space X^*. Then X^* with the module multiplication given by

$$\langle x, \phi.a \rangle = \langle a.x, \phi \rangle, \quad a \in \mathcal{A}, \;\; x \in X, \phi \in X^*,$$

is a Banach right $\mathcal{A}$-module called the dual Banach left $\mathcal{A}$-module X^*.

(iv) Let X be a normed right $\mathcal{A}$-module with dual space X^*. Then X^* with the module multiplication given by

$$\langle x, a.\phi \rangle = \langle x.a, \phi \rangle, \quad a \in \mathcal{A}, \;\; x \in X, \phi \in X^*,$$

is a Banach left $\mathcal{A}$-module called the dual Banach right $\mathcal{A}$-module X^*.

(v) Let X be a normed $\mathcal{A}$-bimodule, with dual X^*. The operations defined in *(iii)* and *(iv)* turns X^* into a Banach $\mathcal{A}$-bimodule referred to as the dual Banach $\mathcal{A}$-bimodule. It should be noted that each Banach $\mathcal{A}$-bimodule X has a corresponding dual Banach $\mathcal{A}$-bimodule.

(vi) Let L be a closed left ideal of $\mathcal{A}$, $X = \mathcal{A} \setminus L$, and $a \mapsto a^*$ denote the canonical mapping of $\mathcal{A}$ onto X. Then the normed vector space X becomes a normed left $\mathcal{A}$-module.

(vii) The tensor product $\mathcal{A}\hat{\otimes}\mathcal{A}$ equipped with the left and right action

$$a.(b \otimes c) = ab \otimes c, \;\; (b \otimes c).a, \;\; a \in \mathcal{A}, \;\; b \otimes c \in \mathcal{A}\hat{\otimes}\mathcal{A}$$

is a Banach $\mathcal{A}$-bimodule.

(viii) Let $\varphi, \psi \in \Phi_{\mathcal{A}} \cup \varphi_\infty$. Then $\mathbb{C}$ is an $\mathcal{A}$-bimodule when equipped with the operation

$$a.z = \varphi(a)z, \;\; z.a = \psi(a)z \;\; (a \in \mathcal{A}, z \in \mathbb{C}).$$

We denote this module by $\mathbb{C}_{\varphi,\psi}$.

(ix) The product map $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{A}$ is an $\mathcal{A}$-bimodule homomorphism with respect to the module structure on $\mathcal{A}\hat{\otimes}\mathcal{A}$.

(x) Let X and Y be left (right) $\mathcal{A}$-module. A map $T \in \mathcal{L}(X, Y)$ is a left (right) $\mathcal{A}$-module homomorphism if

$$T(a.x) = a.Tx \ \ (T(x.a) = Tx.a), \quad a \in \mathcal{A}, \ x \in X.$$

(xi) Let X and Y be $\mathcal{A}$-bimodules. A map $T \in \mathcal{L}(X, Y)$ is an $\mathcal{A}$-bimodule homomorphism if it is a left and right $\mathcal{A}$-module homomorphism.

We denote by ${}_{\mathcal{A}}\mathcal{L}(X, Y)$, $\mathcal{L}_{\mathcal{A}}(X, Y)$, and ${}_{\mathcal{A}}\mathcal{L}_{\mathcal{A}}(X, Y)$, the linear spaces of left and right $\mathcal{A}$-modules and $\mathcal{A}$-bimodule homomorphisms respectively. For a commutative Banach algebra $\mathcal{A}$, the following result holds.

Proposition 2.4.4. [8] *Let $\mathcal{A}$ be a commutative algebra, and let X and Y be $\mathcal{A}$-modules. Then ${}_{\mathcal{A}}\mathcal{L}(X, Y)$ is an $\mathcal{A}$-module for the map*

$$(a, T) \to a.T.$$

Proposition 2.4.5. [8] *Let $\mathcal{A}$ be a Banach algebra and let X and Y be left and right Banach $\mathcal{A}$-modules respectively. Then $(X \otimes Y)^* \simeq \mathcal{L}(X, Y^*)$ as $\mathcal{A}$-bimodules.*

Definition 2.4.6. Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule.

(i) If $\mathcal{A}$ is unital with unit e, X is said to be unital if for any $x \in X$, $ex = xe = x$.

(ii) If the Banach algebra $\mathcal{A}$ is not unital, X is said to be neo-unital if $X = \mathcal{A}.X.\mathcal{A}$.

(iii) X is said to be essential if $X = \overline{\mathcal{A}X\mathcal{A}}$.

Remark 2.4.7. Clearly, all neo-unital Banach algebra are essential. The converse is not necessarily true, except for the case where the Banach algebra $\mathcal{A}$ has a bounded approximate identity.

Chapter 3

Amenability properties of Banach algebras

Barry Johnson's characterisation of the amenability of $L^1(G)$ was to show that for amenable locally compact G, the first cohomology group with coefficient in X^* is identically zero for every Banach $L^1(G)$-bimodule X. Much of what is referred to as amenability of Banach algebra is simply a generalisation of this characterisation to any arbitrary Banach algebra. Due to the difficulty in applying this characterisation, several authors [6], [25], [27] etc, came up with other less tedious ways of showing that a Banach algebra is amenable. In the following section, we discuss some of these characterisations and also point out some interesting results therein.

3.1 Amenable Banach algebras

Definition 3.1.1. Let $\mathcal{A}$ be a Banach algebra, and X a Banach $\mathcal{A}$-bimodule. A map $D : \mathcal{A} \to X$ is called a derivation if:

$$D(ab) = a.D(b) + D(a).b \ \ (a, b \in \mathcal{A}).$$

Consider the map $\delta_x : \mathcal{A} \to X$ determined by:

$$\delta_x(a) = ax - xa.$$

For any $a, b \in \mathcal{A}$, notice that

$$\begin{aligned} \delta_x(ab) &= abx - xab = abx - axb + axb - xab = a(bx - xb) + (ax - xa)b \\ &= a\delta_x(b) + \delta_x(a)b. \end{aligned}$$

This shows that δ_x is indeed a derivation. Derivations of this kind are referred to as inner derivations.

We denote the collection of all continuous derivations from $\mathcal{A}$ to X as $\mathcal{Z}^1(\mathcal{A}, X)$ and the collection of all continuous inner derivations from $\mathcal{A}$ to X by $\mathcal{B}^1(\mathcal{A}, X)$. Let $\mathcal{H}^1(\mathcal{A}, X) = \mathcal{Z}^1(\mathcal{A}, X)/\mathcal{B}^1(\mathcal{A}, X)$, $\mathcal{H}^1(\mathcal{A}, X)$ is called the first cohomology group with coefficients in X. Clearly, $\mathcal{H}^1(\mathcal{A}, X) = \{0\}$ if and only if $\mathcal{Z}^1(\mathcal{A}, X) = \mathcal{B}^1(\mathcal{A}, X)$. That is, every continuous derivation from $\mathcal{A}$ to X is inner.

Definition 3.1.2. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{A}$ is said to be amenable if $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X.

Proposition 3.1.3. [47] *Let $\mathcal{A}$ be a Banach algebra with a bounded right approximate identity. Let X be a Banach $\mathcal{A}$-bimodule such that $\mathcal{A}.X = \{0\}$, then $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$.*

Proof. It suffices to show that every derivation from $\mathcal{A}$ into X^* is inner. Consider the following. Notice that for any $\phi \in X^*$,

$$\langle x, \phi.a \rangle = \langle a.x, \phi \rangle = \langle 0, \phi \rangle = 0,$$

for all $a \in \mathcal{A}, \ x \in X$. Therefore $X^*.\mathcal{A} = \{0\}$. Let $D \in \mathcal{Z}^1(\mathcal{A}, X^*)$, then clearly,

$$D(ab) = a.D(b).$$

Let (e_α) be a bounded right approximate identity for $\mathcal{A}$. Let $\phi \in X^*$ be chosen such that $\phi = w^* - \lim_\alpha D(ae_\alpha)$. Hence:

$$D(a) = \lim_\alpha D(ae_\alpha) = \lim_\alpha a.D(e_\alpha) = a.\phi.$$

Therefore $D = \delta_\phi$. □

Remark 3.1.4. The proposition above shows that if $\mathcal{A}$ has a bounded approximate identity and trivial left action on every Banach $\mathcal{A}$-bimodule, then $\mathcal{A}$ is amenable.

The following result shows the connection between the amenability of a Banach algebra and its possession of a bounded approximate identity.

Proposition 3.1.5. [47] *Every amenable Banach algebra has a bounded approximate identity.*

Proof. Let $\mathcal{A}$ be an amenable Banach algebra and B the Banach $\mathcal{A}$-bimodule whoose underlying space is $\mathcal{A}$ such that B has trivial left action, and right action determined by,

$$a.x = ax,$$

for all $a \in \mathcal{A}$, $x \in B$. Let $D : \mathcal{A} \to B^{**}$ be the canonical embedding of $\mathcal{A}$ into its bidual. Clearly, $D \in \mathcal{Z}^1(\mathcal{A}, B^{**})$. Since $\mathcal{A}$ is amenable, then there exists $E \in B^{**}$ such that $a = a.E$ for all $a \in \mathcal{A}$. Let (e_α) be a bounded net in $\mathcal{A}$ such that E is the w^* limit of (e_α), it then follows that

$$a = w - \lim_{\alpha} ae_\alpha,$$

for all $a \in \mathcal{A}$. Passing to convex combination, it follows that $a = \lim_\alpha ae_\alpha$ by applying **Theorem (2.1.16 (iii))**. That is, (e_α) is a bounded right approximate identity for $\mathcal{A}$. In a similar manner, we obtain a bounded left approximate identity, say (f_β) for $\mathcal{A}$. It follows that $(g_{\alpha,\beta}) = (e_\alpha + f_\beta - e_\alpha f_\beta)$ is a bounded approximate identity for $\mathcal{A}$. □

Remark 3.1.6. The result above shows that the possession of a bounded approximate identity is a necessary condition for a Banach algebra to be amenable.

The following result is stated without proof.

Proposition 3.1.7. [47] *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and suppose that $\mathcal{A}$ is a closed ideal of a Banach algebra $\mathcal{B}$. Let X be a neo-unital Banach $\mathcal{A}$-bimodule and D a continuous derivation on $\mathcal{A}$ with $D(\mathcal{A}) \subset X^*$. Then X is a Banach $\mathcal{B}$-bimodule in the canonical sense and there exists a unique derivation $\tilde{D}$ on $\mathcal{B}$ with $\tilde{D}(\mathcal{B}) \subset X^*$ such that*

(i) $\tilde{D}|_{\mathcal{A}} = D$,

(ii) $\tilde{D}$ *is continuous with respect to the strong topology on $\mathcal{B}$ and the weak topology on X^*.*

For a unital Banach algebra, the following holds.

Proposition 3.1.8. [28] *Let $\mathcal{A}$ be a unital Banach algebra. If $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every unital Banach $\mathcal{A}$-bimodule X, then $\mathcal{A}$ is amenable.*

Proof. Let $e \in \mathcal{A}$ be the identity element in $\mathcal{A}$. Let X be a Banach $\mathcal{A}$-bimodule and $D \in \mathcal{Z}^1(\mathcal{A}, X^*)$. Let $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$, where:

$$Y_1 = eX^*e,\ Y_2 = (1-e)X^*e,\ Y_3 = eX^*(1-e),\ Y_4 = (1-e)X^*(1-e).$$

Let $\Delta_i : X^* \to Y_i$ be the associated projection map. Set $D_i = \Delta_i \circ D$. Clearly:

$$D_i \in \mathcal{Z}^1(\mathcal{A}, Y_i),\ \ i = 1, 2, ..., 4.$$

Also, Y_1 is unital and is isometrically isomorphic to $(e.X.e)^*$. Hence, $D_1 = \delta_{\phi_1}$, $\phi_1 \in Y_1$. Notice further that $Da = D(ea) = eDa + D(e)a$ and $aD_2e =$

$a(1-e)(De)e = aDe - aDe = 0$. Hence for $a \in \mathcal{A}$, $-D_2e = \phi_2,\ \phi_2 \in Y_2$. It follows that :

$$D_2a = (1-e)(De)ea = -(D_2e)a = \delta_{\phi_2}(a).$$

Also, $(D_3e)a = eD(1-e)ea = eDa - eDa = 0$. We set $D_3e = \delta_{\phi_3}, \phi_3 \in Y_3$. Hence:

$$D_3e = eD(a-e) = \delta_{\phi_3}.$$

In a similar manner, we obtain

$$D_4e = \delta_{\phi_4},\ \phi_4 \in Y_4.$$

It follows that

$$D = \delta_\phi = \delta_{\phi_1} + \delta_{\phi_2} + \delta_{\phi_3} + \delta_{\phi_4} = \delta_{\phi_1+\phi_2+\phi_3+\phi_4}.$$

□

Proposition 3.1.9. [47] *Let $\mathcal{A}$ be a Banach algebra with a bounded aproximate identity, then the following statements are equivalent.*

(i) $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X.

(ii) $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every neo-unital Banach $\mathcal{A}$-bimodule X.

Proof. $(i) \implies (ii)$ Since X every neo-unital Banach $\mathcal{A}$-bimodule is also an $\mathcal{A}$-bimodule. It follows that if (i) holds, so does (ii)
$(ii) \implies (i)$ Suppose X is a Banach $\mathcal{A}$-bimodule and $D \in \mathcal{Z}^1(\mathcal{A}, X^*)$. Let

$$X_0 = \{axb : a, b \in \mathcal{A}, x \in X\}.$$

Let $\Delta : X^* \to X_0^*$ be the associated restriction map. Clearly, Δ is a module epimorphism so that $\Delta \circ D \in \mathcal{Z}^1(\mathcal{A}, X_0^*)$. But $\mathcal{H}^1(\mathcal{A}, X_0^*) = \{0\}$, so that there exists $\phi_0 \in X_0^*$ such tha $\Delta \circ D = \delta_{\phi_0}$. Choose $\phi \in X^*$ such that $\phi|_{X_0^*} = \phi_0$. Set $\tilde{D} = D - \delta_\phi$. Notice that for $a, b \in \mathcal{A}, x \in X_0$,

$$\begin{aligned}\langle x, \tilde{D}a\rangle &= \langle x, Da - \delta_{\phi_0}a\rangle = \langle x, D\rangle - \langle x, a.\phi_0 - \phi_0.a\rangle \\ &= \langle x, D|_{X_0^*}a\rangle - \langle x, a.\phi_0 - \phi_0.a\rangle = \langle x, a.\phi_0 - \phi_0.a\rangle = 0.\end{aligned}$$

This shows that $\tilde{D} = D - \delta_\phi \in \mathcal{Z}^1(\mathcal{A}, X_0^\perp)$. By applying **Theorem (2.1.11)**, $X_0^\perp \simeq (X/X_0)^*$. Notice that for any $y \in \mathcal{A}.(X/X_0)$, there exists $a \in \mathcal{A}, x \in X$, such that $y = ax + X_0 \subset X_0$, so that $\mathcal{A}.(X/X_0) = \{0\}$. Hence by **Proposition (3.1.3)**,

$$\mathcal{H}^1(\mathcal{A}, X^\perp) = \mathcal{H}^1(\mathcal{A}, (X/X_0)^*) = \{0\}.$$

It follows that there exists $\psi \in X^\perp$, such that $\tilde{D} = \delta_\psi$ and $D = \delta_{\phi+\psi}$. □

For a non unital Banach algebra $\mathcal{A}$, the following result is analogous to **Proposition (3.1.8)**.

Theorem 3.1.10. [28] *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity. $\mathcal{A}$ is amenable if and only if $\mathcal{H}^1(\mathcal{A}, X_0^*) = \{0\}$ for every neo-unital Banach $\mathcal{A}$-bimodule X_0.*

Proof. Suppose $\mathcal{A}$ is amenable. Then $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X. This further implies that $\mathcal{H}^1(\mathcal{A}, X_0^*) = \{0\}$ for every neo-unital Banach $\mathcal{A}$-bimodule X_0. Conversely, suppose $\mathcal{H}^1(\mathcal{A}, X_0^*) = \{0\}$ for every neo-unital Banach $\mathcal{A}$-bimodule X. Then by **Proposition (3.1.9)**, $\mathcal{A}$ is amenable. □

It is often difficult to show that a Banach algebra is amenable by applying the afore-stated definition, hence several authors, for example see [28], [25], have developed equivalent ways of showing the amenability of a given Banach algebra. Some of these characterisations are discussed in the next two subsections.

3.1.1 Existence of bounded approximate diagonal and virtual diagonal

Definition 3.1.11. Let $\mathcal{A}$ be a Banach algebra.

(i) A bounded approximate diagonal for $\mathcal{A}$ is a norm bounded net $(m_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ such that,

$$a.m_\alpha - m_\alpha.a \to 0, \;\; \pi(m_\alpha)a \to a,$$

for every $a \in \mathcal{A}$.

(ii) A virtual diagonal for $\mathcal{A}$ is an element $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that,

$$a.M - M.a = 0, \; \pi^{**}(M)a = a, a \in \mathcal{A}.$$

One of the ways to establish the amenability of a given Banach algebra, say $\mathcal{A}$ is to show that $\mathcal{A}$ has a bounded approximate diagonal or a virtual diagonal. The following result shows that the existence of a bounded approximate diagonal guarantees the existence of a virtual diagonal and vice versa.

Lemma 3.1.12. [27] *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ has a bounded approximate diagonal if and only if it has a virtual diagonal.*

Proof. Suppose $\mathcal{A}$ has a bounded approximate diagonal, say (m_α). Let $(\hat{m_\alpha})$ be the canonical embedding of (m_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ and M the w^*-accumulation point of $(\hat{m_\alpha})$. Hence, for any $a \in \mathcal{A}$,

$$0 = \lim_\alpha \{a.m_\alpha - m_\alpha.a\} = w^* - \lim_\alpha \{a.\hat{m}_\alpha - \hat{m}_\alpha.a\} = a.M - M.a.$$

Also,

$$0 = \lim_\alpha \{\pi(m_\alpha)a - a\} = w^* - \lim_\alpha \{\pi^{**}(\hat{m}_\alpha)a - a\} = \pi^{**}(M)a - a.$$

Hence M is a virtual diagonal.
Conversely, Suppose M is virtual diagonal for $\mathcal{A}$. Then by applying **Theorem (2.1.16 (ii))**, we obtain a bounded net $(x_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $w^*{-}\lim_\alpha \hat{x}_\alpha = M$, where $(\hat{x}_\alpha)$ is the canonical embedding of (x_α) in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$. Hence, for any $a \in \mathcal{A}$,

$$0 = a.M - M.a = w^* - \lim_\alpha \{a.\hat{x}_\alpha - \hat{x}_\alpha.a\} = w - \lim_\alpha \{a.x_\alpha - x_\alpha.a\}.$$

Also, by applying convex combination and **Theorem (2.1.16(iii))**, we have that

$$\begin{aligned} 0 &= \pi^{**}(M)a - a = w^* - \lim_\alpha \{\pi^{**}(\hat{x}_\alpha)a - a\} = w - \lim_\alpha \{\pi(x_\alpha)a - a\} \\ &= \lim_\alpha \{\pi(x_\alpha)a - a\}. \end{aligned}$$

□

Theorem 3.1.13. [27] *The following statements are equivalent for a Banach algebra $\mathcal{A}$.*

(i) $\mathcal{A}$ is amenable.

(ii) $\mathcal{A}$ has a bounded approximate diagonal.

(iii) $\mathcal{A}$ has a virtual diagonal.

Proof. $(iii) \implies (i)$ Suppose $\mathcal{A}$ has a virtual diagonal, say M. Let (m_α) be the associated bounded approximate diagonal, such that M is a w^*-accumulation point of (m_α). Without loss of generality, we may further assume that $\mathcal{A}$ is unital, and that X is a unital Banach $\mathcal{A}$-bimodule. Let $D \in \mathcal{Z}^1(\mathcal{A}, X^*)$. Given $x \in X$, let μ_x be a continuous linear functional on $\mathcal{A}\hat{\otimes}\mathcal{A}$ determined by:

$$\langle a \otimes b, \mu_x \rangle = \langle x, aDb \rangle \;\; (a, b \in \mathcal{A}).$$

Let $\langle x, f\rangle = \langle \mu_x, M\rangle$, then $f \in X^*$. We want to show that $D = \delta_f$. Given $a \in \mathcal{A}, x \in X$, we have for $b, c \in \mathcal{A}$,

$$\begin{aligned}
\langle b\otimes c, \mu_{xa-ax}\rangle &= \langle xa - ax, bDc\rangle = \langle xa, bDc\rangle - \langle ax, bDc\rangle \\
&= \langle x, abDc\rangle - \langle x, (bDc)a\rangle \\
&= \langle x, abDc\rangle - \langle x, (bDc)a\rangle - \langle x, bcDa\rangle + \langle x, bcDa\rangle \\
&= \langle x, abDc\rangle - \langle x, (bDc)a + bcDa\rangle + \langle x, bcDa\rangle \\
&= \langle x, abDc\rangle - \langle x, bDca\rangle + \langle x, bcDa\rangle \\
&= \langle ab\otimes c, \mu_x\rangle - \langle b\otimes c, \mu_x\rangle + \langle x, bcDa\rangle \\
&= \langle b\otimes c, \mu_x a\rangle - \langle b\otimes c, a\mu_x\rangle + \langle x, bcDa\rangle \\
&= \langle b\otimes c, \mu_x a - a\mu_x\rangle + \langle x, bcDa\rangle.
\end{aligned}$$

This implies that for any $m \in \mathcal{A}\hat{\otimes}\mathcal{A}$,

$$\langle m, \mu_{xa-ax}\rangle = \langle m, \mu_x a - a\mu_x\rangle + \langle x, \pi(m)Da\rangle.$$

Therefore,

$$\begin{aligned}
\langle x, af - fa\rangle = \langle xa - ax, f\rangle &= \langle \mu_{xa-ax}, M\rangle \\
&= \langle \mu_x a - a\mu_x, M\rangle + \lim_\alpha \langle x, \pi(m_\alpha)Da\rangle \\
&= 0 + \langle x, eDa\rangle = \langle x, Da\rangle.
\end{aligned}$$

This shows that $\mathcal{A}$ is amenable.
$(i) \implies (ii)$ Since $\mathcal{A}$ is amenable, then by **Proposition (3.1.5)**, $\mathcal{A}$ has a bounded approximate identity, say(e_α). Suppose $(\hat{e}_\alpha \otimes \hat{e}_\alpha)$ is the canonical embedding of $(e_\alpha \otimes e_\alpha)$ in $(\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$, and suppose further that $E \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ is a w^*-accumulation point for $(\hat{e}_\alpha \otimes \hat{e}_\alpha)$. Then for any $a \in \mathcal{A}$,

$$\begin{aligned}
\pi^{**}(a.E - E.a) &= w^* - \lim_\alpha \pi^{**}(a.(\hat{e}_\alpha \otimes \hat{e}_\alpha) - (\hat{e}_\alpha \otimes \hat{e}_\alpha).a) \\
&= w - \lim_\alpha \pi(ae_\alpha \otimes e_\alpha - e_\alpha \otimes e_\alpha) = \lim_\alpha (ae_\alpha^2 - e_\alpha^2 a) = 0.
\end{aligned}$$

Hence, $\delta_E(\mathcal{A}) \subset \ker \pi^{**}$. That π^{**} is a bimodule homomorphism readily follows from the fact that π is a bimodule homomorphism. Hence $\ker \pi^{**}$ is a Banach $\mathcal{A}$-bimodule. Since $\mathcal{A}$ has a bounded approximate identity, by applying **Theorem (2.2.7)**, π is surjective and therefore an open map, so that $\ker \pi^{**} \simeq (\ker \pi)^{**}$. It readily follows that $\ker \pi^{**}$ is indeed a dual Banach

$\mathcal{A}$-bimodule. By the amenability of $\mathcal{A}$, there exists $N \in \ker \pi^{**}$ such that $\delta_E = \delta_N$. We set $M = E - N$, so that for any $a \in \mathcal{A}$,

$$\pi^{**}(M)a = \pi^{**}(E - N) = \pi^{**}(E) - 0 = \lim_{\alpha} e_\alpha a = a.$$

Also,

$$a.M - M.a = \delta_M = \delta_{E-N}(a) = \delta_E(a) - \delta_N(a) = 0.$$

Thus M is a virtual diagonal for $\mathcal{A}$.
$(ii) \implies (iii)$ and $(iii) \implies (ii)$ This is **Lemma (3.1.12)**.
$(ii) \implies (i)$ Let (m_α) be a bounded approximate diagonal for $\mathcal{A}$, without loss of generality, we may suppose $(\pi(m_\alpha))$ is a bounded approximate identity in $\mathcal{A}$. We want to show that $\mathcal{H}^1(\mathcal{A}, X^*) = \{0\}$ for every Banach $\mathcal{A}$-bimodule X. Without loss of generality, we may assume that X is neo-unital. Let $D \in \mathcal{Z}^1(\mathcal{A}, X^*)$ and suppose $m_\alpha = \sum_{n=1}^{\infty} a_n^{(\alpha)}.Db_n^{(\alpha)}$ such that $\sum_{n=1}^{\infty} \|D\|\|a_n^{(\alpha)}\|\|b_n^{(\alpha)}\| < \infty$. Notice that,

$$\left\| \sum_{n=1}^{\infty} a_n^{(\alpha)}.Db_n^{(\alpha)} \right\| \leq \sum_{n=1}^{\infty} \|D\|\|a_n^{(\alpha)}\|\|b_n^{(\alpha)}\| < \infty,$$

which shows that $\left(\sum_{n=1}^{\infty} a_n^{(\alpha)}.Db_n^{(\alpha)} \right)$ is a bounded net in X^*. Let $\phi \in X^*$ be a w^*-accumulation point of $\sum_{n=1}^{\infty} a_n^{(\alpha)}.Db_n^{(\alpha)}$ such that $\phi = w^* - \lim_\alpha \left(\sum_{n=1}^{\infty} a_n^{(\alpha)}.Db_n^{(\alpha)} \right)$. Then for any $a \in \mathcal{A}, x \in X$,

$$\begin{aligned}
\langle x, a.\phi \rangle &= \lim_{\alpha} \left\langle x, \sum_{n=1}^{\infty} aa_n^{(\alpha)}.Db_n^{(\alpha)} \right\rangle \\
&= \lim_{\alpha} \left\langle x, \sum_{n=1}^{\infty} a_n^{(\alpha)}.D(b_n^{(\alpha)}a) \right\rangle \\
&= \lim_{\alpha} \left\langle x, \sum_{n=1}^{\infty} (a_n^{(\alpha)}b_n^{(\alpha)}.Da + a_n^{(\alpha)}(Db_n^{(\alpha)}).a) \right\rangle \\
&= \lim_{\alpha} \left\langle x, \sum_{n=1}^{\infty} a_n^{(\alpha)}.(Db_n^{(\alpha)}).a \right\rangle + \lim_{\alpha} \left\langle x, \sum_{n=1}^{\infty} a_n^{(\alpha)}b_n^{(\alpha)}.Da \right\rangle \\
&= \langle x, \phi.a \rangle + \lim_{\alpha} \left\langle x, \sum_{n=1}^{\infty} a_n^{(\alpha)}b_n^{(\alpha)}.Da \right\rangle \\
&= \langle x, \phi.a \rangle + \langle x, Da \rangle.
\end{aligned}$$

This shows that $D = \delta_\phi$, so that $\mathcal{A}$ is amenable. □

The following important result due to Barry Johnson is stated without proof.

Theorem 3.1.14. [28] *Let $\mathcal{A}$ and $\mathcal{B}$ be amenable Banach algebra. Then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is an amenable Banach algebra.*

The following is very useful.

Theorem 3.1.15. [37] *Let $\mathcal{A}$ be a Banach algebra. Then there exists a continuous linear mapping $\Psi : \mathcal{A}^{**}\hat{\otimes}\mathcal{A}^{**} \to (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that for $a, b, c \in \mathcal{A}$ and $m \in \mathcal{A}^{**}\hat{\otimes}\mathcal{A}^{**}$, the following holds.*

(i) $\Psi(a \otimes b) = a \otimes b$,

(ii) $\Psi(m).c = \Psi(m.c)$,

(iii) $c.\Psi(m) = \Psi(c.m)$,

(iv) $\Psi^{**}_{\mathcal{A}}(m) = \Psi_{\mathcal{A}^{**}}(m)$.

The following is a direct application of **Lemma (3.1.15)**.

Theorem 3.1.16. [6] *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{**}$ is amenable. Then $\mathcal{A}$ is also amenable.*

Proof. Let $(M_\alpha) \subset \mathcal{A}^{**}\hat{\otimes}\mathcal{A}^{**}$ be an approximate diagonal for $\mathcal{A}^{**}$. By **Lemma (3.1.15)**, there exists a $\Psi : \mathcal{A}^{**}\hat{\otimes}\mathcal{A}^{**} \to (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that for any $a \in \mathcal{A}$,

$$\begin{aligned} a\Psi(M_\alpha) - \Psi(M_\alpha)a &= \Psi(aM_\alpha) - \Psi(M_\alpha a) \\ &= \Psi(aM_\alpha - M_\alpha a) \\ &\to \Psi(0) = 0, \end{aligned}$$

and,

$$\pi^{**}_{\mathcal{A}}(\Psi(M_\alpha)a) = \pi_{\mathcal{A}^{**}}(M_\alpha)a \to a.$$

Choose $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ such that $M = w^* - \lim_\alpha \Psi(M_\alpha)$. Then for every $a \in \mathcal{A}$,

$$aM - Ma = w^* - \lim_\alpha \Psi(aM_\alpha - M_\alpha a) \to \Psi(0) = 0,$$

and

$$\pi^{**}(M_\alpha)a = w^* - \lim_\alpha \pi^{**}_{\mathcal{A}}(\Psi(M_\alpha)a) = w - \lim_\alpha \pi_{\mathcal{A}^{**}}(M_\alpha)a = a.$$

It follows that M is a virtual diagonal for $\mathcal{A}$. □

3.1.2 Splitting of exact sequence of $\mathcal{A}$-modules

The introduction of homological algebra as a tool for deriving interesting outcomes on the amenability properties of Banach algebras is credited to Helemskii [26]. Much of the results he obtained can be found in [25]. Curtis and Loy in [6], without delving too deeply into the homological algebra "machinery" used in the derivations by Helemskii, came up with interesting proofs to the results stated in this section. It should be noted that the results are mostly due to Helemskii and Seinberg.

Definition 3.1.17. Suppose X, Y, Z are Banach $\mathcal{A}$-bimodules, such that $f : X \to Y$, $g : Y \to Z$ are Banach $\mathcal{A}$-bimodule homomorphisms. Let

$$\sum : 0 \to X \xrightarrow{f} Y \xrightarrow{g} X \to 0$$

be a short sequence of Banach $\mathcal{A}$-bimodules.

(i) $\sum$ is said to be exact if f is injective, g is surjective and $\operatorname{Im} f = \ker g$.

(ii) The exact sequence $\sum$ is admissible if there exists a bounded linear map $F : Y \to X$, such that $Ff = I_X$.

(iii) The exact sequence $\sum$ splits if there exists a Banach $\mathcal{A}$-bimodule homomorphism $F : Y \to X$, such that $Ff = I_X$.

The following important result about a homological property of Banach spaces/algebras is stated without proof.

Proposition 3.1.18. [6] *Let $\sum : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ be a short admissible sequence of Banach spaces. If there exists a bounded linear operator $F : Y \to X$ which is a left inverse on f, then there exists a unique bounded linear operator $G : Z \to Y$ which is a right inverse on g. The converse also holds, and $fF + Gg$ is an identity map on Y. If $\mathcal{A}$ is a Banach algebra, and X, Y, Z are Banach $\mathcal{A}$-modules, and f, g are Banach $\mathcal{A}$-module homomorphisms, then F is a Banach $\mathcal{A}$-module homomorphism if and only if G is.*

In the characterisation of the amenability of a Banach algebra $\mathcal{A}$ in terms of short exact sequences of $\mathcal{A}$-modules, the following sequence plays an important role.

$$\Pi : 0 \to K \xrightarrow{i} \mathcal{A}\hat{\otimes}\mathcal{A} \xrightarrow{\pi} \mathcal{A} \to 0, \tag{3.1}$$

$$\Pi^* : 0 \to \mathcal{A}^* \xrightarrow{\pi^*} (\mathcal{A}\hat{\otimes}\mathcal{A})^* \xrightarrow{i^*} K^* \to 0. \tag{3.2}$$

where Π^* is the dual of Π and $K = \ker \pi$ and i is the injection of $\ker \pi$ into $\mathcal{A}\hat{\otimes}\mathcal{A}$.

Lemma 3.1.19. [6] *Let $\mathcal{A}$ be a Banach algebra.*

(i) If $\mathcal{A}$ is unital, then Π is admissible.

(ii) If $\mathcal{A}$ has a bounded approximate identity, then Π^* is admissible.

Proof. (i) Let e be the identity element in $\mathcal{A}$, we define the map

$$\theta : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}, \;\; a \mapsto a \otimes e.$$

Clearly, for any $a, b \in \mathcal{A}$,

$$\theta(a+b) = (a+b) \otimes e = a \otimes e + b \otimes e = \theta(a) + \theta(b),$$

and

$$\|\theta\| = \sup\{\|\theta(a)\|_p\} \leq K, \; K > 0.$$

It follows that θ is a bounded linear map. Also, notice that

$$\pi\theta(a) = \pi(\theta(a)) = \pi(a \otimes e) = a.$$

This further implies that $\theta^*\pi^* = I_{\mathcal{A}^*}$. By **Proposition (3.1.18)**, Π is admissible.
(ii) Suppose $\mathcal{A}$ has a bounded approximate identity, say (e_α). Let $M \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ be a w^*-accumulation point of $(e_\alpha \otimes e_\alpha)$. Suppose further that $\lim_\alpha \langle e_\alpha \otimes e_\alpha, M\rangle = \langle \lambda, M\rangle$, for all $\lambda \in (\mathcal{A}\hat{\otimes}\mathcal{A})^*$. Define the map:

$$\sigma : (\mathcal{A}\hat{\otimes}\mathcal{A})^* \to \mathcal{A}^*$$

by:

$$\langle a, \sigma(\lambda)\rangle = \langle a.\lambda, M\rangle, \;\; a \in \mathcal{A}, \; \lambda \in (\mathcal{A}\hat{\otimes}\mathcal{A})^*.$$

Let $\lambda = \pi^*\phi, \;\; \phi \in \mathcal{A}^*$, then:

$$\begin{aligned} \langle a, \sigma\pi^*\phi\rangle &= \langle a.(\pi^*\phi), M\rangle = \langle \pi^*(a.\phi), M\rangle \\ &= \lim_\alpha \langle e_\alpha \otimes e_\alpha, \pi^*(a.\phi)\rangle = \lim_\alpha \langle \pi(e_\alpha \otimes e_\alpha), a.\phi\rangle = \lim_\alpha \langle e_\alpha^2, a.\phi\rangle \\ &= \lim_\alpha \langle e_\alpha^2.a, \phi\rangle = \langle a, \phi\rangle. \end{aligned}$$

Thus $\sigma\pi^* = I_{\mathcal{A}^*}$ as required. □

Theorem 3.1.20. [6] *Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is amenable if and only if*

(i) $\mathcal{A}$ has a bounded approximate identity,

(ii) the admissible sequence Π^* *splits.*

Proof. Suppose $\mathcal{A}$ is amenable, then by **Proposition (3.1.5)**, $\mathcal{A}$ has a bounded approximate identity, say (e_α), and also the amenability of $\mathcal{A}$ implies $\mathcal{A}$ has a virtual diagonal, say M. Hence by **Lemma (3.1.18)**, π^* has a left inverse, say θ, where

$$\theta : (\mathcal{A}\hat{\otimes}\mathcal{A})^* \to \mathcal{A}^*,$$

is determined by

$$\langle a, \lambda\theta\rangle = \langle \lambda.a, M\rangle, \;\; a \in \mathcal{A}, \lambda \in (\mathcal{A}\hat{\otimes}\mathcal{A})^*$$

is linear. We claim that θ is a Banach $\mathcal{A}$-bimodule homomorphism. Let $b \in \mathcal{A}$, then:

$$\begin{aligned}\langle a, \theta(b.\lambda)\rangle &= \langle (b.\lambda).a, M\rangle = \langle \lambda.a, M.b\rangle \\ &= \langle \lambda.a, b.M\rangle = \langle \lambda.(ab), M\rangle \\ &= \langle ab, \theta\lambda\rangle = \langle a, b.\theta\lambda\rangle.\end{aligned}$$

Therefore, $\theta.(b.\lambda) = b\theta(\lambda)$. Similarly,

$$\begin{aligned}\langle a, \theta(\lambda).b\rangle &= \langle (\lambda.b).a, M\rangle = \langle \lambda.(ba), M\rangle \\ &= \langle ba, \theta\lambda\rangle = \langle a, \theta\lambda.b\rangle.\end{aligned}$$

Hence, $\theta(\lambda.a) = \theta(\lambda).a$. We thus conclude that Π^* splits.
Conversely, suppose $\mathcal{A}$ has a bounded approximate identity, (e_α) and θ is an $\mathcal{A}$-bimodule homomorphism with $\theta\pi^* = I_{\mathcal{A}^*}$. Suppose further that $u \in (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$ is such that $u = w^* - \lim(e_\alpha \otimes e_\alpha)$. Set $M = \theta^*\pi^{**}u$. We claim M is a virtual diagonal for $\mathcal{A}$. Consider the following.
Let $a \in \mathcal{A}, \lambda \in (\mathcal{A}\hat{\otimes}\mathcal{A})^*$, then

$$\begin{aligned}\langle \lambda, a.M\rangle &= \langle \lambda, a\theta^*\pi^{**}u\rangle \\ &= \langle \pi^*\theta(\lambda.a), u\rangle = \lim_\alpha \langle e_\alpha \otimes e_\alpha, \pi^*\theta(\lambda.a)\rangle \\ &= \lim_\alpha \langle ae_\alpha^2, \theta\lambda\rangle = \langle a, \theta\lambda\rangle \\ &= \lim_\alpha \langle e_\alpha^2 a, \theta\lambda\rangle = \lim_\alpha \langle e_\alpha \otimes e_\alpha, \pi^*\theta(a.\lambda)\rangle \\ &= \langle \pi^*\theta(a.\lambda), u\rangle = \langle a.\lambda, \theta^*\pi^{**}u\rangle = \langle \lambda, M.a\rangle.\end{aligned}$$

Also,

$$\pi^{**}Ma = \pi^{**}\theta^*\pi^{**}ua = \pi^{**}ua = \lim_\alpha e_\alpha^2 a = a.$$

□

The amenability property of a Banach algebra $\mathcal{A}$ can also be characterised by the splitting of short exact sequence of arbitrary Banach $\mathcal{A}$-modules. The following results are quite helpful in that regard.

Proposition 3.1.21. [6] *Let X, Z be left Banach $\mathcal{A}$-modules. Then $\mathcal{B}(Z, X)$ is a Banach $\mathcal{A}$-bimodule, where the module operation is determined by,*

$$(a.T)z = a.T(z), \ \ (T.a)z = T(a.z), \ \ z \in Z, T \in \mathcal{B}(Z, X).$$

Proof. That the relevant right and left operations are indeed module operations is trivial. Notice that:

$$\|a.T(z)\| \leq \|a\|\|T(z)\| \leq \|a\|\|T\|\|z\| \leq \|a\|\|T\|, \ \ \|z\| \leq 1.$$

It follows that,

$$\|a.T\| \leq \|a\|\|T\|.$$

In a similar manner,

$$\|T.a\| \leq \|a\|\|T\|.$$

□

Proposition 3.1.22. [6] *Let Z be a left Banach $\mathcal{A}$-module and X a right Banach $\mathcal{A}$-module, then $W = Z\hat{\otimes}X$ is a Banach $\mathcal{A}$-bimodule with module action determined by*

$$a.(z \otimes x) = (a.z) \otimes x, \ \ (z \otimes x).a = z \otimes (x.a), \ \ a \in \mathcal{A}.$$

Further more, the map $T : W^* \to \mathcal{B}(Z, X^*), \ \ \phi \mapsto T_\phi$ given by

$$\langle x, T_\phi \rangle = \langle z \otimes x, \phi \rangle$$

is an isometric Banach $\mathcal{A}$-bimodule homomorphism.

Theorem 3.1.23. [6] *Let $\mathcal{A}$ be an amenable Banach algebra, and*

$$\sum : 0 \to X^* \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

an admissible short exact sequence of left or right Banach $\mathcal{A}$-modules with X^* a dual Banach $\mathcal{A}$-module. Then $\sum$ splits.

Proof. Suppose that $\sum$ is a sequence of left Banach $\mathcal{A}$-modules. Since $\sum$ is admissible, there exists $\tilde{G} \in \mathcal{B}(Z, Y)$ such that $g\tilde{G} = I_Z$. Define $D(a) =$

$a.\tilde{G} - \tilde{G}.a$. Then D is a derivation from $\mathcal{A}$ into the bimodule $\mathcal{B}(Z, Y)$. For any $z \in Z$,

$$\begin{aligned} g(Da)(z) &= g(a.\tilde{G} - \tilde{G}.a)(z) \\ &\quad = a.I_Z(z) - I_Z(z) = I_Z(az) - I_Z(az) = az - az = 0. \end{aligned}$$

That is, $D(\mathcal{A}) \subset B(Z, \ker g) = \mathcal{B}(Z, \operatorname{Im} f)$, which shows that:

$$f^{-1}D : \mathcal{A} \to \mathcal{B}(Z, X^*) \simeq (Z \otimes X)^*$$

is a derivation into a dual Banach $\mathcal{A}$-bimodule. Since $\mathcal{A}$ is amenable, $f^{-1}D$ is inner, so that there exists $Q \in \mathcal{B}(Z, X^*)$ such that

$$Da = a.\tilde{G} - \tilde{G}.a = a.fQ - fQ.a.$$

Set $G = \tilde{G} - fQ$. Clearly $a.G = G.a$ and

$$\begin{aligned} gG(z) &= g\tilde{G}(z) - gfQ(z) = g\tilde{G}(z) - g(fQ(z)) \\ &= g\tilde{G}(z) - 0 = g\tilde{G}(z) = z. \end{aligned}$$

That is G is a right inverse for g. Hence $\sum$ splits. $\square$

3.1.3 Some hereditary properties

Below we discuss some of the hereditary properties of amenable Banach algebras.

Theorem 3.1.24. [28] *Let $\mathcal{A}$ be an amenable Banach algebra, and $\mathcal{B}$ a Banach algebra. Let $\theta : \mathcal{A} \to \mathcal{B}$ be a continuous homomorphism with dense range, then $\mathcal{B}$ is also amenable.*

Proof. Let X be a Banach $\mathcal{A}$-bimodule, let $D \in \mathcal{Z}^1(\mathcal{B}, X^*)$. Then X is a Banach $\mathcal{B}$-bimodule, where the module operation is given by

$$ax = \theta(a)x, \;\; xa = x\theta(a), \;\; x \in X.$$

Notice that

$$\begin{aligned} D \circ \theta(ab) &= D(\theta(a)\theta(b)) = \theta(a)D(\theta(b)) + D(\theta(a))\theta(b) \\ &\quad = \theta(a)D \circ \theta(b) + (D \circ \theta(a))\theta(b) = a(D \circ \theta)(b) + (D \circ \theta(a))b, \\ & a, b \in \mathcal{A}. \end{aligned}$$

Hence $D \circ \theta \in \mathcal{Z}^1(\mathcal{A}, X^*)$. Since $\mathcal{A}$ is amenable, there exists $\phi \in X^*$ such that $D \circ \theta = \delta_\phi$. Therefore $Db = bf - fb$, $b \in \theta(\mathcal{A})$. Since $\theta(\mathcal{A})$ is dense in $\mathcal{B}$ and by applying the continuity of θ, it follows that this is true for any $b \in \mathcal{B}$, so that $D = \delta_f$. $\square$

Corollary 3.1.25. [47] *Suppose $\mathcal{A}$ is an amenable Banach algebra and I is a closed ideal of $\mathcal{A}$, then $\mathcal{A}/I$ is amenable as well.*

Proof. Notice that $\theta : \mathcal{A} \to \mathcal{A}/I$ is a continuous epimorphism. Hence the rest of the proof follows from **Theorem (3.1.24)**. $\square$

The following theorem is quite useful.

Proposition 3.1.26. [47] *For an amenable Banach algebra $\mathcal{A}$ with closed ideal I, the following statements are equivalent.*

(i) I is amenable.

(ii) I has a bounded approximate identity.

Proof. Clearly $(i) \implies (ii)$ holds.
$(ii) \implies (i)$ Let X be a Banach I-bimodule. Without loss of generality we may assume X is neo-unital. By **Proposition (3.1.7)**, the module action of I on X extends to $\mathcal{A}$ in a canonical sense. Let $D \in \mathcal{Z}^1(I, X^*)$. Then again by **Proposition (3.1.7)**, D has an extension $\tilde{D} \in \mathcal{Z}^1(\mathcal{A}, X^*)$. Since $\mathcal{A}$ is amenable, we have that $\tilde{D} \in \mathcal{H}^1(\mathcal{A}, X^*)$. It follows that $D = \tilde{D}|_I$. $\square$

Lemma 3.1.27. [47] *Let $\mathcal{A}$ be an amenable Banach algebra and J a closed left ideal of $\mathcal{A}$. Then the following statements are equivalent.*

(i) J has a bounded right approximate identity.

(ii) J is weakly complemented.

Proof. $(i) \implies (ii)$ Let $(e_\alpha)_{\alpha\in\Gamma}$ be a bounded right approximate identity for J, and $\mathcal{U}$ an ultra filter on Γ. We define

$$P : \mathcal{A}^* \to \mathcal{A}^*, \;\; \phi \mapsto w^* - \lim_{\mathcal{U}}(\phi - e_\alpha.\phi).$$

Notice that for any $a \in J$, $\phi \in \mathcal{A}^*$,

$$\begin{aligned}
\langle a, P\phi\rangle &= \lim_{\mathcal{U}}\langle a, \phi - e_\alpha.\phi\rangle \\
&= \langle a, \phi\rangle - \lim_{\mathcal{U}}\langle a, e_\alpha.\phi\rangle \\
&= \langle a, \phi\rangle - \lim_{\mathcal{U}}\langle ae_\alpha, \phi\rangle \\
&= \langle a, \phi\rangle - \langle a, \phi\rangle \\
&= 0.
\end{aligned}$$

This shows that $P(\mathcal{A}^*) \subset J^\perp$. Recall that since J is a left ideal of $\mathcal{A}$, then for any $a \in \mathcal{A}$, $e_\alpha a \in J$ for all $\alpha \in \Gamma$. Hence for any $a \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$,

$$\begin{aligned}\langle a, P\phi\rangle &= \lim_{\mathcal{U}}(\langle a, \phi\rangle - \langle a, e_\alpha \phi\rangle)\\ &= \langle a, \phi\rangle - \lim_{\mathcal{U}}\langle a e_\alpha, \phi\rangle\\ &= \langle a, \phi\rangle.\end{aligned}$$

That is, $P^2 = P$. We therefore conclude that $J^\perp$ is a direct summand in $\mathcal{A}^*$, which further implies that J is weakly complemented.
$(ii) \implies (i)$ Since $\mathcal{A}$ is amenable, then it has a bounded approximate diagonal, say (m_β), were $m_\beta = \sum_{n=1}^\infty a_n^{(\beta)} \otimes b_n^{(\beta)}$, with $\sum_{n=1}^\infty \|a_n^{(\beta)}\|\|b_n^{(\beta)}\| < \infty$. Let $P : \mathcal{A}^* \to \mathcal{A}^*$ be a projection onto $J^\perp$, let $Q = \delta_{\mathcal{A}^{**}} - P^{**}$. Then clearly $Q : \mathcal{A}^{**} \to \mathcal{A}^{**}$ is a projection onto $J^{\perp\perp} \simeq J^{**}$. Define

$$E_\beta = \sum_{n=1}^\infty a_n^{(\beta)}.Qb_n^{(\beta)}.$$

Then for $a \in J$, we have

$$\begin{aligned}\lim_\beta aE_\beta &= \lim_\beta \sum_{n=1}^\infty a a_n^{(\beta)}.Qb_n^{(\beta)}\\ &= \lim_\beta \sum_{n=1}^\infty a_n^{(\beta)}.Q(b_n^{(\beta)}a) \quad (b_n^{(\beta)}a \in J)\\ &= \lim_\beta \sum_{n=1}^\infty a_n^{(\beta)} b_n^{(\beta)} a\\ &= \lim_\beta \pi(m_\beta)a = a.\end{aligned}$$

Let $E \in J^{**}$ be a w^*-accumulation point of (E_β). Then $a.E = a$. By applying **Theorem (2.1.16 (ii))**, there exists a bounded net in $(e_\alpha) \subset \mathcal{A}$ such that $e_\alpha \to E$. Therefore (e_α) is a bounded weak right approximate identity for J, which further implies (i) holds. □

The folowing result is a direct application of **Lemma (3.1.27)**.

Theorem 3.1.28. [8] *For an amenable Banach algebra with a closed ideal I, the following are equivalent.*

(i) I is amenable.

(ii) I has a bounded approximate identity.

(iii) I is weakly complemented.

Proof. $(i) \implies (ii)$ Follows from **Proposition (3.1.26)**.
$(ii) \implies (iii)$ Follows from **Lemma (3.1.27)**.
$(iii) \implies (i)$ Suppose I is weakly complemented. By applying **Lemma (3.1.27)**, I has a bounded right approximate identity. By passing to $\mathcal{A}^{\mathrm{op}}$ and applying **Lemma (3.1.9)** on $\mathcal{A}^{\mathrm{op}}$, we obtain a bounded left approximate identity for I. That is, I has a bounded approximate identity and is therefore amenable. □

Corollary 3.1.29. [47] *Let $\mathcal{A}$ be an amenable Banach algebra, and I a closed ideal of finite codimension. Then I is amenable.*

Theorem 3.1.30. [47] *Let $\mathcal{A}$ be a Banach algebra and I a closed two sided ideal of $\mathcal{A}$ such that both I and $\mathcal{A}/I$ are amenable. Then $\mathcal{A}$ is amenable.*

Proof. Let X be a Banach $\mathcal{A}$-bimodule, and $D \in \mathcal{Z}^1(\mathcal{A}, X^*)$. Then $D|_I \in \mathcal{Z}^1(I, X^*)$. Since I is amenable, then there exists $\psi_1 \in X^*$ such that

$$Da = \delta_{\psi_1}(a) \quad (a \in I).$$

Let $\tilde{D} = D - \delta_{\psi_1}$. Then clearly $\tilde{D}|_I = 0$ and thus induces a map from $\mathcal{A}/I$ into X^*, which we also denote by $\tilde{D}$. Let

$$F = \{\phi \in X^* : a.\phi = \phi.a = 0 \text{ for all } a \in I\},$$

and

$$X_0 = \overline{IX + XI}.$$

Then $F \simeq (X/X_0)^*$ is a dual Banach $\mathcal{A}/I$-bimodule. Since I is a two sided ideal of $\mathcal{A}$, then for any $a \in I$, $b \in \mathcal{A}$, $ab \in I$. Also since $\tilde{D}$ vanishes on I, it follows that

$$a.\tilde{D}b = \tilde{D}(ab) - \tilde{D}(a).b = 0.$$

In a similar manner, we show that $\tilde{D}b.a = 0$. This further shows that $\tilde{D}(\mathcal{A}/I) \subset F$. Since $\mathcal{A}/I$ is amenable, there exists $\psi_2 \in F$ such that $\tilde{D} = \delta_{\psi_2}$. This implies that $D = \delta_{\psi_1 + \psi_2}$. □

Corollary 3.1.31. [8] *A Banach algebra $\mathcal{A}$ is amenable if and only if $\mathcal{A}^{\#}$ is amenable.*

Let $\mathcal{A}$ be a Banach algebra. The following section deals with some interesting results involving the notion of amenability derived by restricting the class of Banach $\mathcal{A}$-bimodules to $\mathcal{A}$ itself.

3.2 Weak amenability in Banach algebras

The introduction of the notion of weak amenability for commutative Banach algebras is credited to W. G. Bade, P. C. Curtis and H. G. Dales [7]. It was further expanded to include some interesting examples by Grønbæk [20] and B.E. Johnson [29]. It is motivated by certain behaviours of some known Banach algebras. For instance, given a locally compact group G, $\mathcal{H}^1(L^1(G), L^1(G)^*) = \{0\}$, but the same cannot be said for any arbitrary Banach $L^1(G)$-bimodule except for the case where G is amenable as a group. In this section, we state some interesting results on weakly amenable Banach algebras and also discuss some interesting hereditary properties therein.

Definition 3.2.1. Let $\mathcal{A}$ be a Banach algebra. $\mathcal{A}$ is said to be weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$.

Definition 3.2.2. Let $\mathcal{A}$ be a Banach algebra, and $\Phi_{\mathcal{A}}$ the character space of $\mathcal{A}$. $d \in \mathcal{A}^*$ is called a point derivation if for $a, b \in \mathcal{A}$,

$$d(ab) = \varphi(a)d(b) + d(a)\varphi(b), \;\; \varphi \in \Phi_{\mathcal{A}}.$$

Below is a variant of weak amenability.

Definition 3.2.3. Let $\mathcal{A}$ be a Banach algebra, and $n \in \mathbb{N}$. Then $\mathcal{A}$ is n-weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}^{(n)}) = \{0\}$, where $\mathcal{A}^{(n)}$ is the nth dual of $\mathcal{A}$. The Banach algebra $\mathcal{A}$ is said to be *permanently weakly amenable* if $\mathcal{A}$ is n-weakly amenable for each $n \in \mathbb{N}$.

Proposition 3.2.4. [37] *Let $\mathcal{A}$ be a weakly amenable Banach algebra. Then*

(i) $\mathcal{A}^2$ is dense in $\mathcal{A}$.

(ii) $\mathcal{A}$ does not admit a zero point derivation.

(iii) If $\mathcal{A}$ is commutative, then every derivation from $\mathcal{A}$ into each Banach $\mathcal{A}$-bimodule is trivial.

Proof. (i) Suppose $\mathcal{A}^2$ is not dense in $\mathcal{A}$, that is, $\overline{\mathcal{A}^2} \neq \mathcal{A}$, choose $\phi_0 \in \mathcal{A}^*$ such that $\phi_0|_{\mathcal{A}^2} = 0$, and $\langle a_0, \phi_0 \rangle = 1$ for $a_0 \in \mathcal{A} \setminus \overline{\mathcal{A}^2}$. Define $D = \phi_0 \otimes \phi_0 : a \mapsto \langle a, \phi_0 \rangle, \;\; \mathcal{A} \to \mathcal{A}^*$. Clearly, D is a continuous linear map. Notice that since $\phi_0|_{\mathcal{A}^2} = 0$, then $D(ab) = 0$, for all $a, b \in \mathcal{A}$ and,

$$\begin{aligned}\langle c, a.Db + Da.b \rangle &= \langle c, a.Db \rangle + \langle c, Da.b \rangle \\ &= \langle ca, Db \rangle + \langle bc, Da \rangle \\ &= \langle ca, \phi_0 \rangle \langle b, \phi_0 \rangle + \langle bc, \phi_0 \rangle \langle a, \phi_0 \rangle \\ &= 0, \;\; (c \in \mathcal{A}).\end{aligned}$$

This shows that D is indeed a derivaion from $\mathcal{A}$ into $\mathcal{A}^*$. Clearly $\langle a_0, Da_0\rangle = 1$, but $\langle a_0, \delta_\phi(a_0)\rangle = 0,\ \ \phi \in \mathcal{A}^*$. It therefore implies that D is not inner, which contradicts our assumption that $\mathcal{A}$ is weakly amenable. Thus $\mathcal{A}^2$ is dense in $\mathcal{A}$.

(ii) Suppose that $\mathcal{A}$ admits a non zero point derivation, say d at $\varphi \in \Phi_{\mathcal{A}}$. Then clearly, $D : a \mapsto d(a)\varphi,\ \ \mathcal{A} \to \mathcal{A}^*$ is a continuous linear operator, and also D is a derivation. Since $\mathcal{A}$ is weakly amenable, there exists a $\psi \in \mathcal{A}^*$ such that $D(a) = a.\psi - \psi.a$. Let $x_1 \in \mathcal{A}$ with $\psi(x_1) = 1$, $x_2 \in \ker\varphi$ with $d(x_2) = 1$. We set $x_0 = x_1 + (1 - d(x_1))x_2$. Then

$$\begin{aligned}
\varphi(x_0) &= \varphi(x_1 + (1 - d(x_1))x_2) \\
&= \varphi(x_1) + (1 - d(x_1))\varphi(x_2) \\
&= \varphi(x_1) = 1.
\end{aligned}$$

It follows that,

$$\begin{aligned}
1 &= \langle x_0, Dx_0\rangle \\
&= \langle x_0, x_0\phi - \phi x_0\rangle \\
&= \langle x_0, x_0\phi\rangle - \langle x_0, \phi x_0\rangle \\
&= \langle x_0^2, \phi\rangle - \langle x_0^2, \phi\rangle \\
&= 0,
\end{aligned}$$

which is a contradiction, therefore $\mathcal{A}$ does not admit a point derivation.

(iii) Suppose there exists $D \in \mathcal{Z}^1(\mathcal{A}, X)$ with $D \neq 0$. By (i), $\overline{\mathcal{A}^2} = \mathcal{A}$, and so that there exists $a_o \in \mathcal{A}$ with $D(a_0^2) \neq 0$. We then have that $a_0.D(a_0) \neq 0$, so that there exists $\phi \in X^*$ such that $\langle a_0.Da_0, \phi\rangle = 1$. Let $R_\phi \in_{\mathcal{A}} B(\mathcal{A}, X^*)$. Then $R_\phi \circ D \in \mathcal{Z}^1(\mathcal{A}, X^*)$ and $\langle a_0, (R_\phi \circ D)(a_0)\rangle = \langle a_0.Da_0, \phi\rangle = 1$, so that $R_\phi \circ D \neq 0$, a contradiction of the fact that $\mathcal{A}$ is weakly amenable. $\square$

Theorem 3.2.5. [37] *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{**}$ is weakly amenable and $\hat{\mathcal{A}}$ the image of $\mathcal{A}$ under the canonical embedding, such that $\hat{\mathcal{A}}$ is a left ideal in $\mathcal{A}^{**}$. Then $\mathcal{A}$ is weakly amenable.*

Proof. Let $D : \mathcal{A} \to \mathcal{A}^*$ be a continuous derivation, and D^{**} its second adjoint. Let $E, F \in \mathcal{A}^{**}$ and $(a_i), (b_j)$ be bounded nets in $\mathcal{A}$ such that $E = w^*{-}\lim_i a_i$, $F = w^*{-}\lim_j b_j$. Then clearly for any $x \in \mathcal{A}$, $Fx = w^*{-}\lim_j b_j x$. If $\hat{a}_i, \hat{b}_j$ are the canonical images of a_i, b_j respectively for all i, j, then

$$\begin{aligned}
D^{**}(EF) &= w^* - \lim_i \lim_j D^{**}(\hat{a}_i \hat{b}_j) \\
&= w^* - \lim_i \lim_j (a_i D(b_j) + D(a_i)b_j\hat{)} \\
&= w^* - \lim_i \hat{a}_i D^{**}(G) + D^{**}(E)F.
\end{aligned}$$

Let $R : \mathcal{A}^{**} \to \mathcal{A}^{*}$ be the restriction determined by

$$\langle R(a, \Phi)\rangle = \langle \hat{a}, \Phi\rangle, \ \ \Phi \in \mathcal{A}^{***},$$

and Γ be the subsequent extension

$$\langle \Phi, \Gamma\rangle = (R(\Phi))\hat{}.$$

From the above, we have

$$\Gamma \circ D^{**}(EF) = \Gamma(w^{*} - \lim_{i} a_i D^{**}(F)) + \Gamma D^{**}(E)F.$$

Hence for any $x \in \mathcal{A}$,

$$\langle \Gamma(D^{**}(E)F), \hat{x}\rangle = \langle D^{**}(E)F, \hat{x}\rangle = \langle D^{**}(E), F\hat{x}\rangle.$$

Since $\hat{\mathcal{A}}$ is assumed to be a left ideal in $\mathcal{A}^{**}$, $F\hat{x} \in \hat{\mathcal{A}}$. Then,

$$\langle D^{**}(E), F\hat{x}\rangle = \langle \Gamma(D^{**}(E)), F\hat{x}\rangle = \langle \Gamma \circ D^{**}(E)F, \hat{x}\rangle.$$

That is,

$$\Gamma(D^{**}(E)F) = (\Gamma \circ D^{**})(E)F.$$

It follows that for any $x \in \mathcal{A}$,

$$\begin{aligned}
\langle \Gamma(w^{*} - \lim_{i} \hat{a}_i D^{**}(F)), \hat{x}\rangle &= \langle w^{*} - \lim_{i} \hat{a}_i D^{**}(F), \hat{x}\rangle \\
&= \lim_{i} \langle \hat{a}_i D^{**}(F), \hat{x}\rangle \\
&= \lim_{i} \langle D^{**}(F), \hat{x}\hat{a}_i\rangle \\
&= \lim_{i} \langle \hat{x}\hat{a}_i, R(D^{**}(F))\rangle \\
&= \langle \hat{x}E, R(D^{**}(F))\rangle.
\end{aligned}$$

This shows that $\Gamma \circ D^{**}$ is a derivation from $\mathcal{A}^{**}$ to $\mathcal{A}^{***}$. Since $\mathcal{A}^{**}$ is weakly amenable, there exists $\Lambda \in \mathcal{A}^{***}$ such that

$$\Gamma \circ D^{**}(\Psi) = \Psi.\Lambda - \Lambda.\Psi,$$

for all $\Psi \in \mathcal{A}^{**}$. Set $\phi = R(\Psi)$, then for any $a \in \mathcal{A}$,

$$D(a) = R(a.\Psi - \Psi.a) = a.R(\Psi) - R(\Psi).a = a.\phi - \phi.a.$$

This shows that D is inner. $\square$

The following are some interesting hereditary properties of weakly amenable Banach algebras.

Proposition 3.2.6. [37] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras and $\theta : \mathcal{A} \to \mathcal{B}$ a continuous homomorphism such that $\overline{\theta(\mathcal{A})} = \mathcal{B}$. If $\mathcal{A}$ is commutative and weakly amenable, then $\mathcal{B}$ is weakly amenable.*

Proof. Let $D \in \mathcal{Z}^1(\mathcal{B}, \mathcal{B}^*)$. Clearly for any $a, b \in \mathcal{A}$,

$$D \circ \theta(a) = D(\theta(a)(b)) = \theta(a)D(\theta(b)) + D(\theta(a))\theta(b) = aD \circ \theta(b) + (D \circ \theta(a))b.$$

It follows that $D \circ \theta \in \mathcal{Z}^1(\mathcal{A}, \mathcal{B}^*)$. Since $\mathcal{A}$ is weakly amenable and commutative, $D \circ \theta = 0$. Since $\overline{\theta(\mathcal{A})} = \mathcal{B}$, it follows that $D = 0$. □

Proposition 3.2.7. [8] *Let $\mathcal{A}$ be a Banach algebra and I a closed ideal of $\mathcal{A}$. If I and $\mathcal{A}/I$ are weakly amenable. Then $\mathcal{A}$ is weakly amenable.*

Proof. Let $D \in \mathcal{Z}^1(\mathcal{A}, \mathcal{A}^*)$ and $i : I \to \mathcal{A}$ the natural embedding with dual $i^* : I^* \to \mathcal{A}^*$. Clearly $i^* \circ D \circ i \in \mathcal{Z}^1(I, I^*)$. Since I is weakly amenable, there exists $\phi \in I^*$ such that $i^* \circ D = \delta_\phi$. We replace D with $D - \delta_\phi$, and suppose that $(i^* \circ D)|_I = 0$. For $a, b \in I$ and $c \in \mathcal{A}$,

$$\begin{aligned}\langle c, D(ab)\rangle &= \langle c, aDb + Da.b\rangle = \langle ca, Db\rangle + \langle bc, Da\rangle \\ &= \langle ca, (i^* \circ D)b\rangle + \langle bc, (i^* \circ D)a\rangle = 0.\end{aligned}$$

This shows that $D|_{I^2} = 0$. But I is weakly amenable implies $\overline{I^2} = I$, which shows that $D|_I = 0$. The rest of the proof readily follows from the proof of **Theorem (3.1.30)**, where we replace X with $\mathcal{A}$. □

The following lemma is quite useful.

Lemma 3.2.8. [8] *Let $\mathcal{A}$ be a weakly amenable commutative Banach algebra, let I be a closed ideal of $\mathcal{A}$, and X a Banach I-module. Then $D|_{I^4} = 0$ for each $D \in \mathcal{Z}^1(I, X)$.*

Theorem 3.2.9. [8] *Let $\mathcal{A}$ be a weakly amenable Banach algebra and I a closed ideal of $\mathcal{A}$. Then:*

(i) I is weakly amenable if and only if I^2 is dense in I.

(ii) If I has a finite codimesion in $\mathcal{A}$, then I is weakly amenable.

Proof. (i) Suppose I is weakly amenable, then clearly I^2 is dense in I. Conversely, suppose I^2 is dense in I. Then I^4 is also dense in I. Let $D \in \mathcal{Z}^1(I, I^*)$. By **Lemma (3.2.8)**, $D|_{I^4} = 0$, so that $D = 0$. Since $\mathcal{A}$ is commutative and it follows that I is weakly amenable.

(ii) Suppose that I has codimension 1 in $\mathcal{A}$. Since $\mathcal{A}$ is weakly amenable, then $\mathcal{A}^2$ is dense in $\mathcal{A}$, it follows that $\mathcal{A} \not\subset I$, so that $I = M_\varphi,\ \varphi \in \Phi_{\mathcal{A}}$. It follows that I^2 is dense in I. Hence by (i), I is weakly amenable. □

Corollary 3.2.10. [8] *Let $\mathcal{A}$ be a commutative Banach algebra. Then $\mathcal{A}$ is weakly amenable if and only if $\mathcal{A}^\#$ is weakly amenable.*

3.3 Some generalised notions of amenability in Banach algebras

Some interesting amenability like properties of Banach algebras can be obtained by relaxing some of the conditions required for a Banach algebra to be amenable. This has become quite necessary due to the fact that amenability of Banach algebra as a concept, is a bit restrictive and does not allow for a rich collection of examples. In this section, we shall discuss some of these generalised notions of amenability in Banach algebras. Cases involving specific Banach algebras are also considered.

3.3.1 Approximate amenability of Banach algebras

The notion of approximate amenability was introduced in 2004 by Ghahramani and Loy [13], and was further expanded by Ghahramani, Loy, and Zhang [38] in 2008. The concept is based entirely on a certain behaviour of continuous derivations on a Banach algebra. In this section we state some interesting results on the approximate amenability of Banach algebras.

Definition 3.3.1. Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-bimodule. A derivation

$$D : \mathcal{A} \to X$$

is said to be approximately inner if there exists a net $(\eta_v) \subset X$ such that

$$D(a) = \lim_v (a.\eta_v - \eta_v.a) \ \ (a \in \mathcal{A}).$$

That is, $D = \lim_v \delta_{\eta_v}$ in the strong topology on $\mathcal{B}(\mathcal{A}, X)$.

Definition 3.3.2. Let $\mathcal{A}$ be a Banach algebra.

(i) $\mathcal{A}$ is said to be approximately amenable if every continuous derivation $D : \mathcal{A} \to X^*$ is approximately inner for every Banach $\mathcal{A}$-bimodule X.

(ii) $\mathcal{A}$ is approximately contractible if for every continuous derivation $D : \mathcal{A} \to X$ is approximately inner for every Banach $\mathcal{A}$-bimodule X

The limits in the definitions above are taken in the norm. In (i), it should be noted that if it is only required that the net be taken from $\mathcal{A}^*$, then we say $\mathcal{A}$ is weakly approximately amenable.

It has long been established that a necessary condition for a Banach algebra to be amenable is that it must possess a bounded approximate identity, see **Lemma (3.1.5)**. In the case of an approximately amenable Banach algebra, it is still unknown if this condition holds. The following result provides a partial answer.

Lemma 3.3.3. [13] *Let $\mathcal{A}$ be an approximately amenable Banach algebra. Then $\mathcal{A}$ has left and right approximate identities. In particular $\mathcal{A}^2$ is dense in $\mathcal{A}$.*

Proof. Let $a \mapsto \hat{a}$ be the canonical injection of $\mathcal{A}$ into $\mathcal{A}^{**}$. Notice that for $b \in \mathcal{A}$, $\phi \in \mathcal{A}^*$,

$$\langle \phi, \hat{ab} \rangle = \langle ab, \phi \rangle = \langle b, \phi.a \rangle = \langle \phi.a, \hat{b} \rangle = \langle \phi, a\hat{b} \rangle.$$

This shows that $a \mapsto \hat{a}$ is a derivation with usual left action and trivial right action. Since $\mathcal{A}$ is approximately amenable, there exists a net $(E_v) \subset \mathcal{A}^{**}$ with $a.E_v \to \hat{a}$ for each $a \in \mathcal{A}$.

We take finite sets $F \subset \mathcal{A}, \Phi \subset \mathcal{A}^*$, and $\epsilon > 0$. Let $H = \{\phi.a : a \in F, \phi \in \Phi\}$, $K = \max\{\|\psi\|, \|\phi\| : \psi \in H, \phi \in \Phi\}$. Then there is $v = v(F, \Phi, \epsilon)$ such that

$$\|\hat{a} - a.E_v\| < \frac{\epsilon}{2K},$$

for $a \in F$. By **Theorem (2.1.16 (ii))**, there exists $(b_v) \subset \mathcal{A}$ such that

$$|\langle \psi, b_v \rangle - \langle E_v, \psi \rangle| < \frac{\epsilon}{2}, \ (\psi \in H).$$

Hence for any $a \in F, \phi \in \Phi$,

$$\begin{aligned}
|\langle ab_v, \phi \rangle - \langle a, \phi \rangle| &= |\langle ab_v, \phi \rangle - \langle a, \phi \rangle - \langle \phi, a.E_v \rangle + \langle \phi, a.E_v \rangle| \\
&= |\langle ab_v, \phi \rangle - \langle \phi, \hat{a} \rangle - \langle \phi, a.E_v \rangle + \langle \phi, a.E_v \rangle| \\
&= |\langle ab_v, \phi \rangle - \langle \phi, a.E_v \rangle + \langle \phi, a.E_v - \hat{a} \rangle| \\
&\leq |\langle ab_v, \phi \rangle - \langle \phi, a.E_v \rangle| + |\langle \phi, a.E_v - \hat{a} \rangle| \\
&\leq |\langle b_v, \phi.a \rangle - \langle \phi.a, E_v \rangle| + \|\phi\| \|a.E_v - \hat{a}\| \\
&< \frac{\epsilon}{2} + K\frac{\epsilon}{2K} = \epsilon.
\end{aligned}$$

This shows that (b_v) is a weak right appproximate identity for $\mathcal{A}$, so that $\mathcal{A}$ has a right approximate identity. In a similar manner, we can show that $\mathcal{A}$ has a left approximate identity. $\square$

Consider the following lemma.

Lemma 3.3.4. [13] *Let $\mathcal{A}$ be a unital Banach algebra with identity e, X a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to X$ a continuous derivation. Then there exists a derivation $D_1 : \mathcal{A} \to X$, and $\phi \in X^*$, such that*

(i) $\|\phi\| \leq 2C_X\|D\|$,

(ii) $D = D_1 + \delta_\phi$.

Proof. Set $Y_1 = e.X^*.e, Y_2 = (1-e).X^*.e, Y_3 = e.X^*.(1-e), Y_4 = (1-e).X^*.(1-e)$, let $\Delta_j : X^* \to Y_j$ be the associated projections, $j = 1, 2, 3, 4$. Clearly $X^* = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4$. Set $D_j = \Delta_j \circ D$, then clearly, D_j is a derivation for $j = 1, ..., 4$, and $D = D_1 + D_2 + D_3 + D_4$. As is the case in **Proposition (3.1.8)**, Y_2 has trivial right action, Y_3 has trivial left action and Y_4 has left and right trivial actions. That is, D_j is inner for $j = 2, 3, 4$. Hence, there exists $\phi_j \in Y_j$,$j = 2, 3, 4$, such that $D_j = \phi_j$. Hence $D_2 = \delta_{-D_2e}, D_3 = \delta_{D_3e}, D_4 = 0$. Set $\phi = D_3e - D_2e$. Also notice that

$$\phi = D_3e - D_2e = eDe(1-e) - (1-e)De - De.e.$$

It follows that

$$\begin{aligned}\|\phi\| &= \|D_3e - D_2e\| = \|eDe(1-e) - (1-e)De - De.e\| = \|eDe - De.e\| \\ &\leq \|eDe\| + \|De.e\| \leq \|e\|\|De\| + \|De\|\|e\| \\ &= 2\|e\|\|De\| \leq 2C_X\|D\|.\end{aligned}$$

$\square$

The following is an interesting application of **Lemma (3.3.4)**.

Lemma 3.3.5. [13] *Let $\mathcal{A}$ be a unital approximately amenable Banach algebra, X a Banach $\mathcal{A}$-bimodule, $D : \mathcal{A} \to X^*$ a continuous derivation. Then there exists a net $(\eta_v) \subset e.X^*.e$ and $\phi \in X^*$ such that*

(i) $\|\phi\| \leq 2C_X\|D\|$,

(ii) $D = \delta_\phi + st - \lim \delta_{\eta_v}$.

Proof. By applying **Lemma (3.3.4)**, it suffices to show that D_1 is approximately inner. Notice that $e.X^*.e \simeq (e.X.e)^*$ isometrically. Since $\mathcal{A}$ is approximately amenable, there exists a net of inner derivations $(\eta_v) \subset (e.X.e)^*$ such that $D_1 = st - \lim_v \eta_v$. $\square$

We apply **Lemma (3.3.5)** in proving the following important result.

Proposition 3.3.6. [13] *The Banach algebra $\mathcal{A}$ is approximately amenable if and only if $\mathcal{A}^{\#}$ is approximately amenable.*

Proof. Suppose $\mathcal{A}$ is aproximately amenable. Let $D : \mathcal{A}^{\#} \to X^*$ be a derivation. By **Lemma (3.3.4)**, $D = D_1 + \delta_\phi$, where $D_1 : \mathcal{A}^{\#} \to e.X^*.e$. Notice that

$$D_1(e) = e.De.e = e(De - eDe) = eDe - eDe = 0,$$

and also since $\mathcal{A}$ is approximately amenable, $D|_{\mathcal{A}}$ is approximately inner. Therefore D is approximately inner. This shows that $\mathcal{A}^{\#}$ is approximately amenable.

Conversely, suppose $\mathcal{A}^{\#}$ is approximately amenable. Let $D : \mathcal{A} \to X^*$ be a derivation. Since $\mathcal{A}$ is a closed ideal of $\mathcal{A}^{\#}$, D has an extension $\tilde{D}$ on $\mathcal{A}^{\#}$. By setting $D(e) = 0$, and identity module action by e, we have that $\mathcal{A}^{\#}$ is approximately amenable, $\tilde{D}$ is approximately inner. It then follows that D is approximately inner. □

Theorem 3.3.7. [13] *A Banach algebra $\mathcal{A}$ is approximately amenable if and only if either of these conditions holds.*

(i) There exists a net $(M_v) \subset (\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^{**}$ such that for each $a \in \mathcal{A}^{\#}$, $a.N_v - M_v \to 0$ and $\pi^{**}(M_v) \to e$.

(ii) There exists a net $(M_v^{'}) \subset (\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^{**}$ such that for each $a \in \mathcal{A}^{\#}$, $a.M_v^{'} - M_v^{'}.a \to 0$ and $\pi^{**}(M_v^{'}) = e$. for every v.

Proof. Suppose $\mathcal{A}$ is approximately amenable, then by **Proposition (3.3.5)**, $\mathcal{A}^{\#}$ is approximately amenable. By a similar argument to the proof of **Theorem (3.1.13)**, let $u = e \otimes e$. Since $\mathcal{A}$ is approximately amenable, there exists a net $(e_v) \subset \ker \pi^{**}$ such that for any $a \in \mathcal{A}$, $\delta_u(a) = \lim_v \delta_{e_v}(a)$. Set $M_v^{'} = u - e_v$. Then for any $a \in \mathcal{A}$,

$$\begin{aligned} a.M_v^{'} - M_v^{'}.a &= a(u - e_v) - (u - e_v)a \\ &= au - ua + (ae_v - e_v a) \\ &\to 0. \end{aligned}$$

Also,

$$\begin{aligned} \pi^{**}(M_v^{'}) &= \pi^{**}(u - e_v) = \pi^{**}(e_v) \\ &= \pi(u) - 0 = \pi(u) = e. \end{aligned}$$

That is (ii) holds.

Suppose (i) holds. Let $D : \mathcal{A}^{\#} \to X^*$ be a derivation. Without loss of generality, we may suppose X^* is neo-unital. Let (f_v) be a net in X^*. By applying the argument in **Lemma (3.1.8)**, we set

$$\langle x, f_v \rangle = \langle \mu_x, M_v \rangle,$$

where $\mu_x \in (\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^*$ is determined by $\langle a \otimes b, \mu_x \rangle = \langle x, aDb \rangle$. Let $(m_v^\alpha) \subset \mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$ be such that $m_v^\alpha = w^* - \lim_\alpha M_v$, for each v. Also recall that for $m \in \mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$,

$$\langle m, \mu_{x.a-a.x} \rangle = \langle m, \mu_x.a - a.\mu_x \rangle + \langle x, \pi(m)Da \rangle.$$

Hence,

$$\begin{aligned}
\langle x, a.f_v - f_v.a \rangle &= \langle x.a - a.x, f_v \rangle \\
&= \langle \mu_{x.a-a.x}, M_v \rangle \\
&= \lim_\alpha \langle m_v^\alpha, \mu_{x.a-a.x} \rangle \\
&= \langle \mu_x.a - a.\mu_x, M_v \rangle + \lim_\alpha \langle x, \pi(m_v^\alpha)Da \rangle \\
&= \langle \mu_x, a.M_v - M_v.a \rangle + \langle x, \pi^{**}(M_v).Da \rangle.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\langle x, a.f_v - f_v.a \rangle - \langle x, Da \rangle| &= |\langle \mu_x, a.M_v - M_v.a \rangle + \langle x, \pi^{**}(M_v)Da \rangle - \langle x, Da \rangle| \\
&= |\langle \mu_x, a.M_v - M_v.a \rangle + \langle x, (\pi^{**}(M_v) - e)Da \rangle| \\
&\le |\langle \mu_x, a.M_v - M_v.a \rangle| + |\langle x, (\pi^{**}(M_v) - e)Da \rangle| \\
&\le \|\mu_x\| \|a.M_v - M_v.a\| + \|x\| \|\pi^{**}(M_v) - e\| \|D\| \|a\| \\
&\to 0.
\end{aligned}$$

This shows that $Da = st - \lim \delta_{f_v}$. Therefore $\mathcal{A}^{\#}$ is approximately amenable and so is $\mathcal{A}$. That $(ii) \implies (i)$ is obvious. Hence the equivalence holds. $\square$

Remark 3.3.8. Notice that by applying **Theorem (2.1.16(ii))**, we may choose a net $(m_v^\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $M_v = w^* - \lim_\alpha m_v^\alpha$ for all v, so that

$$a.m_v^\alpha - m_v^\alpha.a \to 0, \pi(m_v^\alpha)a \to a, \ (a \in \mathcal{A}).$$

The implication of this outcome will be discussed in the next section.

Corollary 3.3.9. [13] *A Banach algebra $\mathcal{A}$ is approximately amenable if and only if there are nets $(M'')\subset (\mathcal{A}\hat{\otimes}\mathcal{A})^{**}$, $(F_v), (G_v) \subset \mathcal{A}^{**}$, such that for each $a \in \mathcal{A}$,* *(i)* $a.M_v^{''} - M_v^{''}.a + F_v \otimes a - a \otimes G_v \to a$,

(ii) $a.F_v \to a, G_v.a \to a$, *and*

(iii) $\pi^{**}(M_v^{''}).a - F_v.a - G_v.a \to 0$.

We state the following important result on approximately contractible Banach algebras without proof.

Theorem 3.3.10. [13] *The Banach algebra $\mathcal{A}$ is approximately contractible if and only if any of the following equivalent conditions holds:*

(i) *there exists a net* $(M_v) \subset \mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$ *such that for each* $a \in \mathcal{A}^{\#}$, $a.M_v - M_v.a \to 0$ *and* $\pi(M_v) \to e$;

(ii) *there exists a net* $(M_v^{'}) \subset \mathcal{A}^{\#}\hat{\otimes}\mathcal{A}$ *such that for each* $a \in \mathcal{A}$, $a.M_v^{'} - M_v^{'}.a \to 0$ *and* $\pi(M_v^{'}) = e$;

(iii) *there exist nets* $(M_v^{''}) \subset \mathcal{A}\hat{\otimes}\mathcal{A}^{\#}$, $(F_v), (G_v) \subset \mathcal{A}$, *such that for each* $a \in \mathcal{A}$,

  (a) $a.M_v^{''} - M_v^{''}.a + F_v \otimes a - a \otimes G_v \to 0$

  (b) $a.F_v \to a, G_v.a \to a$; *and*

  (c) $\pi(M_v^{''}).a - F_v.a - G_v.a \to 0$.

Definition 3.3.11. Let

$$\sum : 0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an admissible short exact sequence of left Banach $\mathcal{A}$-modules. Then $\sum$ approximately splits if there exists a net $G_v : Z \to Y$ of right inverse maps to g such that $\lim_v(a.G_v - G_v.a) = 0$ for $a \in \mathcal{A}$ and a net $F_v : Y \to X$ of left inverse maps to f such that $\lim_v(a.F_v - F_v.a) = 0$ for $a \in \mathcal{A}$.

The approximate amenability of a Banach algebra can also be characterised in terms of short exact sequences of Banach modules. Below is an interesting and insightful result of such a characterisation.

Theorem 3.3.12. [13] *Let $\mathcal{A}$ be an approximately amenable Banach algebra and*

$$\sum : 0 \to X^* \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an admissible short exact sequence of left Banach $\mathcal{A}$-modules. Then $\sum$ approximately splits.

Proof. The proof of the theorem follows from the argument in the proof of **Theorem (3.1.22)** for a right inverse $\tilde{G}$ for g. Since $\mathcal{A}$ is approximately amenable, then there exists a net $(Q_v) \subset B(Z, X^*)$ such that

$$a.\tilde{G} - \tilde{G}.a = \lim_v (a.fQ_v - fQ_v.a), \ \ a \in \mathcal{A}.$$

Set $G_v = \tilde{G} - fQ_v$. Clearly (G_v) is a net of right inverse maps for G and

$$\lim_v (a.G - G.a) = 0.$$

By applying **Proposition (3.1.18)**, we have a net (F_v) of left inverse for f such that

$$\lim_v (a.F_v - F_v.a) = 0.$$

Therefore, $\sum$ approximately splits. $\square$

Corollary 3.3.13. [13] *Let $\mathcal{A}$ be an approximately amenable Banach algebra and let J a weakly complemented left ideal of $\mathcal{A}$. Then J has a right approximate identity. In particular, J^2 is dense in J.*

Theorem 3.3.14. [13] *Let $\mathcal{A}$ be a Banach algebra. If $\mathcal{A}^{**}$ is approximately amenable, so is $\mathcal{A}$.*

Proof. In order to simplify notation, we set $\mathcal{B} = \mathcal{A}^{\#}$. Let Ψ be the continuous linear mapping in **Lemma (3.1.15)**. Since $B^{**} = (\mathcal{A}^{**})^{\#}$, and by the approximate amenability of $\mathcal{A}^{\#}$, there exists a net $(M_v) \subset (\mathcal{B}^{**}\hat{\otimes}\mathcal{B}^{**})^{**}$ such that for all $m \in \mathcal{A}^{**}$,

$$m.N_v - N_v.m \to 0, \ \ \pi^{**}_{\mathcal{B}^{**}}(N_v)m = m.$$

Since $\mathcal{A}$ can be viewed as a subset of $\mathcal{A}^{**}$ under the canonical embedding. Then for any $a \in \mathcal{A}$,

$$a.N_v - N_v.a \to 0, \ \ \pi^{**}_{\mathcal{B}^{**}}(N_v)a = a.$$

Let $\theta : (\mathcal{B}\hat{\otimes}\mathcal{B})^* \to (\mathcal{B}\hat{\otimes}\mathcal{B})^{***}$ be the canonical embedding of $\mathcal{B}^*$ into its bidual. Since θ is an θ is an $\mathcal{A}$-bimodule homomorphism, so is θ^*. It follows that, for each $a \in \mathcal{A}$,

$$\begin{aligned} a.\theta^*\Psi^{**}(N_v) - \theta^*\Psi^{**}(N_v).a &= \theta^*\Psi^{**}(aN_v) - \theta^*\Psi^{**}(N_va) \\ &= \theta^*\Psi^{**}(aNv - N_va) \\ &\to \theta^*\Psi^{**}(0) = 0. \end{aligned}$$

For fixed v, we choose net $N^\mu \subset \mathcal{B}^{**}\hat{\otimes}\mathcal{B}^{**}$ such that $N_v = w^*\!-\lim \hat{N}^\mu$. Hence

$$\begin{aligned}
(\pi_{\mathcal{B}})^{**}(\theta^*\Psi(N_v))a &= a.w^* - \lim_\mu (\pi_{\mathcal{B}})^{**}(\theta^{**}\Psi^{**}(\hat{N}^\mu)).a \\
&= w^* - \lim_\mu (\pi_{\mathcal{B}})^{**}(\theta^*(\Psi(\hat{N}^\mu))).a \\
&= w^* - \lim_\mu (\pi_{\mathcal{B}})^{**}(\Psi(N^\mu)).a \\
&= w^* - \lim_\mu (\pi_{\mathcal{B}^{**}}(N^\mu)).a \\
&= w^* - \lim_\mu (\pi^{**}_{\mathcal{B}^{**}}(\hat{N}^\mu)).a \\
&= (\pi^{**}_{\mathcal{B}^{**}}(N_v)).a = a.
\end{aligned}$$

$\square$

The following are some results on approximate amenability of some known Banach algebras defined on a locally compact group G.

Theorem 3.3.15. [13] *Let G be a locally compact group. Then,*

(i) $M(G)$ is approximately amenable if and only if G is discrete and amenable.

(ii) $L^1(G)$ is approximately amenable if and only if G is amenable.

(iii) $L^1(G)^{**}$ is approximately amenable if and only if G is finite.

By slightly modifying some of the conditions for a Banach algebra to be the approximately amenable, the following notions of amenability can be derived.

Definition 3.3.16. Let $\mathcal{A}$ be a Banach algebra, X a Banach $\mathcal{A}$-bimodule and $D : \mathcal{A} \to X^*$ a continuous derivation,

(i) [13] $\mathcal{A}$ is sequentially approximately amenable if there exists a sequence $(\zeta_n) \subset X^*$ such that
$$D(a) = \lim_n \delta_{\zeta_n},$$

(ii) [38] $\mathcal{A}$ is uniformly approximately amenable if there exists a net $(\eta_v) \subset X^*$ such that
$$D(a) = \lim_v \delta_{\eta_v},$$
for every Banach $\mathcal{A}$-bimodule X, where the limit is taken in the unit ball $\mathcal{A}_{[1]}$ of $\mathcal{A}$,

(iii) [38] $\mathcal{A}$ is boundedly approximately amenable if there exists a net $(\eta_v) \subset X^*$ such that

$$D(a) = \lim_v \delta_{\eta_v}, \quad \|\delta_{\eta_v}\| \leq L\|a\|, \;\; a \in \mathcal{A}, L > 0,$$

for every Banach $\mathcal{A}$-bimodule X.

The following, stated without proof, are some insightful results on the afore-stated notions of amenability.

Theorem 3.3.17. *Let $\mathcal{A}$ be a Banach algebra.*

(i) [38] *$\mathcal{A}$ is uniformly approximately amenable if and only if it is amenable.*

(ii) [38] *$\mathcal{A}$ is boundedly approximately amenable if and only if there exists a constant $L_b > 0$ such that for any $\mathcal{A}$-bimodule X, and any continuous derivation $D : \mathcal{A} \to X^*$, $\sup_i \|a\delta_{\eta_i}\| \leq L_b\|D\|$, and $D(a) = \lim_i \delta_{\eta_i}(a)$, $(a \in \mathcal{A})$.*

(iii) [38] *If $\mathcal{A}$ is boundedly approximately amenable, then there exists a net $(M_v) \subset (\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^{**}$ and a constant $L > 0$ such that for each $a \in \mathcal{A}^{\#}$, $a.M_v - M_v.a \to 0, \pi^{**}(M_v) \to e$ and $\|a.M_v - M_v.a\| \leq L\|a\|$. Conversely, if the later property holds and $(\pi^{**}(M_v))$ is bounded, then $\mathcal{A}$ is boundedly approximately amenable.*

(iv) [38] *If $\mathcal{A}$ is a boundedly approximately amenable such that $\mathcal{A}$ is seperable as a Banach space, then it is sequentially approximately amenable.*

(v) [38] *$\mathcal{A}$ is boundedly approximately amenable if and only if $\mathcal{A}^{\#}$ is boundedly approximately amenable.*

(vi) [38] *$\mathcal{A}$ is boundedly approximately amenable if and only if there exists a net $(\alpha_i) \subset (\ker \pi)^{**}$ and $M > 0$ such that*

(i) *$k.\alpha_i \to k$ for each $k \in \ker \pi$,*

(ii) *$\|k.\alpha_i\| \leq M\|k\|$ for all $k \in \ker \pi$, for all i.*

(vii) [12] *If $\mathcal{A}$ is boundedly approximately amenable with a bounded approximate identity, and suppose $\mathcal{B}$ is an amenable Banach algebra. Then $\mathcal{A}\hat{\otimes}\mathcal{B}$ is boundedly approximately amenable.*

(viii) [12] *The tensor product of two boundedly approximately amenable Banach algebras need not be approximately amenable.*

3.3.2 Pseudo-amenability of Banach algebras

Definition 3.3.18. A Banach algebra $\mathcal{A}$ is said to be pseudo-amenable if it has an approximate diagonal.

Recall that an approximate diagonal for $\mathcal{A}$ is a net $(m_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ such that

$$a.m_\alpha - m_\alpha a \to 0 \text{ and } \pi(m_\alpha)a \to a \text{ for all } \alpha \text{ and } a \in \mathcal{A}.$$

It should be noted that for pseudo-amenable Banach algebras, the approximate diagonal need not be bounded, as a result the class of pseudo-amenable Banach algebras is larger than the class of amenable Banach algberas. Below, we state some results on pseudo-amenable Banach algebras. These results are due to Ghahramani and Zhang [14], who are credited with introducing this notion of amenability and Choi, Ghahramani and Zhang [16].

Proposition 3.3.19. [14] *Suppose each $\mathcal{A}_i$, $i \in I$ is pseudo-amenable. Then $\bigoplus_{i\in I}^p \mathcal{A}_i,\ p \geq 0$ is also pseudo-amenable.*

Proof. Let $\mathcal{A} = \bigoplus_{i\in I}^p \mathcal{A}_i, p \geq 0$ and P_J be the projection from $\bigoplus_{i\in I}^p \mathcal{A}_i, p \geq 0$ onto $\bigoplus_{i\in J}^p \mathcal{A}_i,\ p \geq 0$, where J is a finite subset of I. Let $\epsilon > 0$ be given. We choose finite set $F \subset \mathcal{A}$ such that:

$$\|P_J(a) - a\| < \frac{\epsilon}{2},\ \ a \in F.$$

Since each $\mathcal{A}_i$ is pseudo-amenable, there exists $u_i \in \mathcal{A}_i\hat{\otimes}\mathcal{A}_i,\ i \in J$ such that

$$\sum_{i\in J} \|P_i(a)u_i - u_iP_i(a)\| < \epsilon$$

and

$$\sum_{i\in J} \|\pi(u_i)P_i(a) - P_i(a)\| < \frac{\epsilon}{2},$$

where each P_i is the projection $P_{\{i\}}$. Since each $\mathcal{A}_i$ is complemented in $\mathcal{A}$, then $\mathcal{A}_i\hat{\otimes}\mathcal{A}_i$ can be viewed as an element of $\mathcal{A}\hat{\otimes}\mathcal{A}$. Clearly,

$$au = P_J(a)u = \sum_{i\in I} P_i(a)u_i \text{ and } ua = uP_J(a) = \sum_{i\in J} u_iP_i(a).$$

It follows that

$$\|au - ua\| = \left\| \sum_{i\in J} P_i(a)u_i - \sum_{i\in J} u_iP_i(a) \right\| \leq \sum_{i\in J} \|P_i(a)u_i - u_iP_i(a)\| < \epsilon.$$

Also since π is an $\mathcal{A}$-bimodule homomorphism,

$$\begin{aligned}
\|\pi(u)a - a\| &= \|\pi(ua) - a\| \\
&= \left\|\pi\Big(\sum_{i\in J} u_i P_i(a)\Big) - a\right\| = \left\|\sum_{i\in J} \pi(u_i P_i(a)) - a\right\| \\
&= \left\|\sum_{i\in J}(\pi(u_i P_i(a)) - P_i(a) + P_i(a)) - a\right\| \\
&= \left\|\sum_{i\in J}(\pi(u_i P_i(a) - P_i(a))) + \sum_{i\in J} P_i(a) - a\right\| \\
&\leq \left\|\sum_{i\in J}(\pi(u_i)P_i(a)) - P_i(a)\right\| + \left\|\sum_{i\in J} P_i(a) - a\right\| \\
&\leq \sum_{i\in J} \|\pi(u_i)P_i(a) - a\| + \|P_J(a) - a\| \\
&\sum_{i\in J} \|\pi(u_i)P_i(a) - P_i(a)\| + \|P_J(a) - a\| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

□

Proposition 3.3.20. [14] *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. If $\mathcal{A}$ is pseudo-amenable and*

$$\theta : \mathcal{A} \to \mathcal{B}$$

is a continuous epimorphism, then $\mathcal{B}$ is pseudo-amenable.

Proof. Since $\mathcal{A}$ is pseudo-amenable, then $\mathcal{A}$ has an apprimate diagonal and so there exists a net $(m_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$, such that

$$a.m_\alpha - m_\alpha.a \to 0, \ \ \pi(m_\alpha)a \to a.$$

Also since θ is an epimorphism, then for every $b \in \mathcal{A}$, there exists $a \in \mathcal{A}$ such that $\theta(a) = b$. Define a map

$$\theta \otimes \theta : \mathcal{A}\hat{\otimes}\mathcal{A} \to \mathcal{B}\hat{\otimes}\mathcal{B}.$$

Clearly, $\theta \otimes \theta$ is a Banach bimodule homomorphism. Let $m_\alpha = \sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)}$

for all α. Then for any $b \in \mathcal{B}$,

$$\begin{aligned}
\lim_\alpha b.(\theta \otimes \theta)(m_\alpha) &= \lim_\alpha \theta(a)(\theta \otimes \theta)\Big(\sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)}\Big) \\
&= \lim_\alpha \theta(a) \sum_k \theta(a_k^{(\alpha)}) \otimes \theta(b_k^{(\alpha)}) \\
&= \lim_\alpha \sum_k \theta(a a_k^{(\alpha)}) \otimes \theta(b_k^{(\alpha)}) \\
&= \lim_\alpha (\theta \otimes \theta)\Big(\sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)} \theta(a)\Big) \\
&= \lim_\alpha (\theta \otimes \theta)(m_\alpha)\theta(a) \\
&= \lim_\alpha (\theta \otimes \theta)(m_\alpha).b.
\end{aligned}$$

Also,

$$\begin{aligned}
\lim_\alpha \pi_{\mathcal{B}}((\theta \otimes \theta)(m_\alpha))b &= \lim_\alpha \pi_{\mathcal{B}}(\theta \otimes \theta)\Big(\sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)}\Big)\theta(a) \\
&= \lim_\alpha \pi_{\mathcal{B}}\Big(\sum_k \theta(a_k^{(\alpha)}) \otimes \theta(b_k^{(\alpha)})\theta(a)\Big) \\
&= \lim_\alpha \pi_{\mathcal{B}}\Big(\sum_k \theta(a_k^{(\alpha)}) \otimes \theta(b_k^{(\alpha)} a)\Big) \\
&= \lim_\alpha \theta\Big(\sum_k a_k^{(\alpha)} b_k^{(\alpha)} a\Big) \\
&= \lim_\alpha \theta(\pi_{\mathcal{A}}(m_\alpha)a) = \theta(a) = b.
\end{aligned}$$

Hence $((\theta \otimes \theta)(m_\alpha))$ is an approximate diagonal for $\mathcal{B}$. Therefore, $\mathcal{B}$ is pseudo-amenable. □

The following is another interesting hereditary property of pseudo-amenable Banach algebras.

Theorem 3.3.21. [14] *Let $\mathcal{A}$ be a pseudo-amenable Banach algebra, and I a two-sided closed ideal of $\mathcal{A}$. If I has an approximate identity, say (x_α) such that the associated left and right multiplication operators $L_\alpha : a \mapsto x_\alpha a$ and $R_\alpha : a \mapsto a x_\alpha$ from $\mathcal{A}$ into I are uniformly bounded, then I is pseudo-amenable.*

Proof. By the uniform boundedness of the left and right multiplication operators on (x_α), there exists a constant $K \geq 1$ such that $\|x_\alpha m\| \leq K\|m\|$

and $\|x_\alpha m\| \leq K\|m\|$ for all α and $m \in \mathcal{A}\hat{\otimes}\mathcal{A}$. Suppose $(m_\beta) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ is an approximate diagonal for $\mathcal{A}$. Then given $\epsilon > 0$ and finite set $F \subset I$, we choose β such that

$$\|am_\beta - m_\beta a\| \leq \frac{\epsilon}{2K^2}$$

and

$$\|\pi(m_\beta)a - a\| \leq \frac{\epsilon}{2K},$$

for $a \in F$. Then choose α such that

$$\|ax_\alpha - x_\alpha a\| < \frac{\epsilon}{4\|m_\beta\|K}, \;\; \|x_\alpha a - a\| < \frac{\epsilon}{4},$$

and

$$\|\pi(m_\beta)(x_\alpha a - a)\| < \frac{\epsilon}{4K}, \;\; (a \in F).$$

We then have that

$$\begin{aligned}
\|ax_\alpha m_\beta x_\alpha - x_\alpha m_\beta a\| &= \|ax_\alpha m_\beta x_\alpha - x_\alpha a m_\beta x_\alpha + x_\alpha a m_\beta x_\alpha - x_\alpha m_\beta x_\alpha a\| \\
&\leq \|ax_\alpha m_\beta x_\alpha - x_\alpha a m_\beta x_\alpha\| + \|x_\alpha a m_\beta x_\alpha - x_\alpha m_\beta x_\alpha a\| \\
&= \|ax_\alpha m_\beta x_\alpha - x_\alpha a m_\beta x_\alpha\| \\
&+ \|x_\alpha a m_\beta x_\alpha - x_\alpha m_\beta a x_\alpha + x_\alpha m_\beta a x_\alpha - x_\alpha m_\beta x_\alpha a\| \\
&\leq \|ax_\alpha m_\beta x_\alpha - x_\alpha a m_\beta a x_\alpha\| \\
&+ \|x_\alpha a m_\beta x_\alpha - x_\alpha m_\beta a x_\alpha\| + \|x_\alpha m_\beta a x_\alpha - x_\alpha m_\beta x_\alpha a\| \\
&= \|(ax_\alpha - x_\alpha) m_\beta x_\alpha\| \\
&+ \|x_\alpha (am_\beta - m_\beta a) x_\alpha\| + \|x_\alpha m_\beta (ax_\alpha - x_\alpha a)\| \\
&\leq 2\|ax_\alpha - x_\alpha a\| K \|m_\beta\| + \|x_\alpha m_\beta\| \|am_\beta - m_\beta a\| \\
&\leq 2\|ax_\alpha - x_\alpha a\| K \|m_\beta\| + \|am_\beta - m_\beta a\| K^2 \\
&< \frac{2\epsilon}{4\|m_\beta\|K} K \|m_\beta\| + \frac{\epsilon}{2K^2} K^2 = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
\|\pi(x_\alpha m_\beta x_\alpha)a - a\| &= \|\pi(x_\alpha m_\beta x_\alpha)a - \pi(x_\alpha m_\beta)a + \pi(x_\alpha m_\beta)a - x_\alpha a + x_\alpha a - a\| \\
&\leq \|\pi(x_\alpha m_\beta x_\alpha)a - \pi(x_\alpha m_\beta)a\| + \|\pi(x_\alpha m_\beta)a - x_\alpha a\| \\
&+ \|x_\alpha a - a\| \\
&\leq K\|\pi(m_\beta)(x_\alpha - a)\| + K\|\pi(m_\beta)a - a\| + \|x_\alpha a - a\| \\
&< K\frac{\epsilon}{4K} + K\frac{\epsilon}{2K} + \frac{\epsilon}{4} = \epsilon,
\end{aligned}$$

for all $a \in F$. It therefore follows that a subnet of $(x_\alpha m_\beta x_\alpha) \subset I\hat{\otimes}I$ is an approximate diagonal for I. Therefore, I is pseudo-amenable. $\square$

Corollary 3.3.22. [14] *Let $\mathcal{A}$ be a pseudo-amenable Banach algebra, and I a two-sided closed ideal of $\mathcal{A}$. If I has a bounded approximate identity, then I is pseudo-amenable.*

Proof. Let (x_α) be a bounded approximate identity for I. Clearly, $L_\alpha : a \mapsto ax_\alpha$ and $R_\alpha : a \mapsto x_\alpha a$ are uniformly bounded, therefore I is pseudo-amenable. □

Proposition 3.3.23. [16] *If $\mathcal{A}^{**}$ is pseudo-amenable, so is $\mathcal{A}$.*

Proof. The proof follows from **Theorem (3.3.14)**. Since $\mathcal{A}$ is a subset of $\mathcal{A}^{**}$ under the canonical embedding, we may choose (N_v) to be in $\mathcal{A}$ so that for any $a \in \mathcal{A}$,

$$a.N_v - N_v.a \xrightarrow{w^*} 0, \;\; \pi(N_v)a \xrightarrow{w^*} a.$$

By applying **Theorem (2.1.16(iii))**, we obtain another net $(\zeta_v) \subset \mathcal{A}$ such that

$$a.\zeta_v - \zeta_v.a \xrightarrow{\|\cdot\|} 0 \text{ and } \pi(\zeta_v)a \xrightarrow{\|\cdot\|} a.$$

□

3.3.3 Some relationships between notions of amenability of Banach algebras

The following results show some interesting relationships involving approximate amenability, pseudo-amenability and weak amenability.

Theorem 3.3.24. [38] *For a Banach algebra $\mathcal{A}$, the following statements are equivalent.*

(i) $\mathcal{A}$ is approximately amenable.

(ii) $\mathcal{A}$ is w^*-approximately amenable.

(iii) $\mathcal{A}$ is approximately contractible.

(iv) $\mathcal{A}^{\#}$ is pseudo-amenable.

Proof. $(i) \implies (ii)$ Obvious.
$(ii) \implies (iii)$ Suppose $\mathcal{A}$ is w^*-approximately amenable. Then $\mathcal{A}^{\#}$ is also w^*-approximately amenable. That is there exists a net $(M_v) \subset (\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^{**}$ such that for each $a \in \mathcal{A}$, $a.M_v - M_v.a \to 0$ and $\pi^{**}(M_v) \to e$ in the w^*-topology on $(\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^{**}$ and $(\mathcal{A}^{\#})^{**}$ respectively. Let $\epsilon > 0$ be given, we

take finite subsets $F \subset \mathcal{A}^{\#}$, $\Phi \subset (\mathcal{A}^{\#})^*$, and $N \subset (\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#})^{**}$. Then there exists v such that

$$|\langle a.f - f.a, M_v\rangle| = |\langle f, a.M_v - M_v.a\rangle| < \epsilon$$

and

$$|\langle \phi, \pi^{**}(M_v) - e\rangle| < \epsilon,$$

for all $a \in F$, $\phi \in \Phi$ and $f \in N$. By **Theorem (2.1.16(ii))**, and the w^*-continuity of π^{**}, there exists $m \in \mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$ such that

$$|\langle f, a.m - m.a\rangle| = |\langle a.f - f.a, m\rangle| < \epsilon$$

and

$$|\langle \phi, \pi(m) - e\rangle| < \epsilon,$$

for all $a \in F$, $\phi \in \Phi$ and $f \in N$. Thus we have a net $(m_\alpha) \subset \mathcal{A}\hat{\otimes}\mathcal{A}^{\#}$ such that for every $a \in \mathcal{A}^{\#}$, $a.m_\alpha - m_\alpha.a \to 0$ and $\pi(m_\alpha) \to e$ in the w-topology on $\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$ and $\mathcal{A}^{\#}$ respectively. Passing to convex combination and applying **Theorem (2.1.16(iii))**, we obtain a net $(m_\beta) \subset \mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$ such that $a.m_\beta - m_\beta.a \to 0$ and $\pi(m_\beta) \to e$ in the $\|.\|$-topology on $\mathcal{A}^{\#}\hat{\otimes}\mathcal{A}^{\#}$ and $\mathcal{A}^{\#}$ respectively. It then follows that (iii) holds.
$(iii) \implies (iv)$ This follows from **Theorem (3.3.10)**.
$(iv) \implies (i)$ This follows from **Theorem (3.3.7)**. □

Remark 3.3.25. Clearly, the result shows that approximate amenability as a concept is stronger than pseudo-amenability. In the case when the Banach algebra has a bounded approximate identity, then the Banach algebra being pseudo-amenable is equivalent to it being approximately amenable [50].

The following result is a direct application of **Theorem (3.3.24)**.

Proposition 3.3.26. [14] *For a Banach algebra $\mathcal{A}$, the following statements are equivalent.*

(i) $\mathcal{A}$ has an approximate diagonal $(m_\alpha)_{\alpha\in I} \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $(\pi(m_\alpha))_{\alpha\in I}$ is bounded.

(ii) $\mathcal{A}$ is pseudo-amenable and has a bounded approximate identity.

(iii) $\mathcal{A}$ is approximately amenable and has a bounded approximate identity.

Proof. $(i) \implies (ii)$ Obvious

$(ii) \implies (iii)$ From **Theorem (3.3.24)**, it suffices to show that $\mathcal{A}$ is w^*-approximately amenable. Let X be a Banach $\mathcal{A}$-bimodule. Since $\mathcal{A}$ has

a bounded approximate identity, we may assume without loss of generality that X is neo-unital. Let $D : \mathcal{A} \to X^*$ be a continuous derivation. We show that D is w^*-approximately inner. Consider the following.
Let (m_α) be an approximate diagonal for $\mathcal{A}$ and $\Phi : \mathcal{A}\hat{\otimes}\mathcal{A} \to X^*$ a bounded linear map determined by

$$\Phi(a \otimes b) = aDb.$$

Clearly,

$$\|\Phi(a \otimes b)\| = \|aDb\| \leq \|a\|\|b\|\|D\|,$$

which shows that

$$\|\Phi\| \leq \|D\|.$$

Let $m_\alpha = \sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)}$ for all α. Then

$$\begin{aligned}
a\Phi(m_\alpha) &= a\Phi\left(\sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)}\right) = a\sum_k \Phi(a_k^{(\alpha)} \otimes b_k^{(\alpha)}) \\
&= \sum_k aa_k^{(\alpha)} Db_k^{(\alpha)} - \sum_k a_k^{(\alpha)} D(b_k^{(\alpha)}a) + \sum_k a_k^{(\alpha)} D(b_k^{(\alpha)}a) \\
&= \sum_k (aa_k^{(\alpha)} Db_k^{(\alpha)} - a_k^{\alpha)} D(b_k^{(\alpha)}a)) + \sum_k a_k^{(\alpha)} b_k^{(\alpha)} Da + \sum_k a_k^{(\alpha)} D(b_k^{(\alpha)})a \\
&= \sum_k (\Phi(aa_k^{(\alpha)} \otimes b_k^{(\alpha)}) - \Phi(a_k^{(\alpha)} \otimes b_k^{(\alpha)}a)) + \sum_k a_k^{(\alpha)} b_k^{(\alpha)} Da \\
&+ \sum_k \Phi(a_k^{(\alpha)} \otimes b_k^{(\alpha)})a \\
&= \Phi\left(\sum_k aa_k^{(\alpha)} \otimes b_k^{(\alpha)} - \sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)} a\right) + \sum_k a_k^{(\alpha)} b_k^{(\alpha)} Da \\
&+ \Phi\left(\sum_k a_k^{(\alpha)} \otimes b_k^{(\alpha)}\right) a \\
&= \Phi(am_\alpha - m_\alpha a) + \pi(ma_\alpha)D(a) + \Phi(m_\alpha)a, \;\; (a \in \mathcal{A}).
\end{aligned}$$

That is,

$$\pi(m_\alpha)D(a) = (a\zeta_\alpha - \zeta_\alpha a) - \Phi(am_\alpha - m_\alpha a),$$

where $\zeta_\alpha = \Phi(m_\alpha)$. Since X is neo-unital, $D(a) = w^* - \lim_\alpha \pi(m_\alpha)D(a)$. Also

$$\|\Phi(am_\alpha - m_\alpha a)\| \leq \|\Phi\|\|am_\alpha - m_\alpha a\| \leq \|D\|\|am_\alpha - m_\alpha a\| \to 0.$$

Therefore, $D(a) = w^* - \lim_\alpha(a\zeta_\alpha - \zeta_\alpha a)$, $(a \in \mathcal{A})$.
$(iii) \implies (i)$ Let (e_β) be a bounded approximate approximate identity for $\mathcal{A}$. Since $\mathcal{A}$ is approximately amenable, then by **Theorem (3.3.24)**, $\mathcal{A}^{\#}$ is

pseudo-amenable. Let (M_α) be an approximate diagonal for $\mathcal{A}^\#$. Without loss of generality, let $\pi(M_\alpha) = e$.
We set

$$M_\alpha = u_\alpha + F_\alpha \otimes e + e \otimes G_\alpha + c_\alpha e \otimes e, u_\alpha \in \mathcal{A} \otimes \mathcal{A},\ F_\alpha, G_\alpha \in \mathcal{A},\ c_\alpha \in \mathbb{C},$$

for all α. Notice that

$$e = \pi(M_\alpha) = \pi(u_\alpha) + F_\alpha e + G_\alpha e + c_\alpha e = F_\alpha + G_\alpha + c_\alpha.$$

Since $e \notin \mathcal{A}$, $\pi(u_\alpha) + F_\alpha + G_\alpha \neq e$, so that $c_\alpha e = e$, $\implies$ $c_\alpha = 1$ for all α. It follows that $\pi(u_\alpha) + F_\alpha + G_\alpha = 0$. Also,

$$\begin{aligned} au_\alpha - u_\alpha a - F_\alpha \otimes a + a \otimes G_\alpha &= aM_\alpha - M_\alpha a + aF_\alpha \otimes e + c_\alpha ae \otimes e - e \otimes G_\alpha a \\ &\quad - e \otimes c_\alpha e \to aF_\alpha \otimes e + c_\alpha a \otimes e - e \otimes G_\alpha a - e \otimes c_\alpha a \\ &\quad \to 0, \end{aligned}$$

whenever $aF_\alpha \to -a,\ G_\alpha a \to -a$.
Set $m = m_{(\alpha,\beta)} = u_\alpha + F_\alpha \otimes e_\beta + e_\beta \otimes G_\alpha + e_\beta \otimes e_\beta$. Then for any $a \in \mathcal{A}$,

$$\begin{aligned} am - ma = {} & au_\alpha - u_\alpha a + ae_\beta \otimes G_\alpha - F_\alpha \otimes e_\beta a + aF_\alpha \otimes e_\beta + ae_\beta \otimes e_\beta \\ & - e_\beta \otimes G_\alpha a - e_\beta \otimes G_\alpha a - e_\beta \otimes_\beta \\ & \xrightarrow{\beta} au_\alpha \otimes u_\alpha a + a \otimes G_\alpha - F_\alpha \otimes a - a \otimes e_\beta + a \otimes e_\beta + e_\beta \otimes a - e_\beta \otimes a \\ & \xrightarrow{\alpha,\beta} 0. \end{aligned}$$

Also,

$$\begin{aligned} \pi(m) &= \pi(u_\alpha) + F_\alpha e_\beta + e_\beta G_\alpha + e_\beta^2 \\ &\xrightarrow{\beta} \pi(u_\alpha) + F_\alpha + G_\alpha + e_\beta \\ &= 0 + e_\beta = e_\beta. \end{aligned}$$

Then clearly, for any $a \in \mathcal{A}$, $\pi(m)a \to a$. That $(\pi(m_\alpha))$ is bounded readily follows from the fact that $\mathcal{A}$ has a bounded approximate identity. $\square$

Proposition 3.3.27. [14] *Let $\mathcal{A}$ be a Banach algebra with a central approximate identity. If $\mathcal{A}$ is approximately amenable, then it is pseudo-amenable.*

Proof. Let (e_α) be a central approximate identity for $\mathcal{A}$. Then given $\epsilon > 0$ and finite subset F of $\mathcal{A}$, there exists $e_{\alpha_1}, e_{\alpha_2} \in (e_\alpha)$ such that

$$\|e_{\alpha_1} a - a\| < \frac{\epsilon}{2},\ \ \|e_{\alpha_1} e_{\alpha_2} a - e_{\alpha_1} a\| < \frac{\epsilon}{2},\ a \in F.$$

Let $D : \mathcal{A} \to X = \ker \pi$ be a bounded derivation defined by

$$D(a) = ae_{\alpha_1} \otimes e_{\alpha_2} - e_{\alpha_1} \otimes e_{\alpha_2} a.$$

Since $\mathcal{A}$ is approximately amenable, then by **Theorem (3.3.24)**, it is approximately contractible, so that there exists a $u = u(e_{\alpha 1}, e_{\alpha_2}, \epsilon, F) \in X$, such that

$$\|D(a) - (au - ua)\| < \epsilon, \ (a \in F).$$

Set $M = e_{\alpha_1} \otimes e_{\alpha_2} - u$. Then clearly, $M \in \mathcal{A}\hat{\otimes}\mathcal{A}$. It follows that,

$$\begin{aligned}
\|aM - Ma\| &= \|a(e_{\alpha_1} \otimes e_{\alpha_2} - u) - (e_{\alpha_1} \otimes e_{\alpha_2} - u)a\| \\
&= \|ae_{\alpha_1} \otimes e_{\alpha_2} - au - e_{\alpha_1} \otimes e_{\alpha_2} a + ua\| \\
&= \|ae_{\alpha_1} \otimes e_{\alpha_2} - e_{\alpha_1} \otimes e_{\alpha_2} a - (au - ua)\| \\
&= \|D(a) - (au - ua)\| < \epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
\|\pi(M)a - a\| &= \|\pi(e_{\alpha_1} \otimes e_{\alpha_2} - u)a - a\| \\
&= \|\pi(e_{\alpha_1} \otimes e_{\alpha_2})a - \pi(u)a - a\| \\
&= \|e_{\alpha_1} e_{\alpha_2} a - a\| \\
&\leq \|e_{\alpha_1} e_{\alpha_2} a - e_{\alpha_1} a\| + \|e_{\alpha_a} a - a\| \\
&= \|e_{\alpha_2} e_{\alpha_1} a - e_{\alpha_1} a\| + \|e_{\alpha_a} a - a\| \\
&< \epsilon,
\end{aligned}$$

for $a \in F$. Hence M is an approximate diagonal for $\mathcal{A}$. $\square$

Corollary 3.3.28. [14] *Any approximately amenable commutative Banach algebra is pseudo-amenable.*

Proof. Let $b, c \in \mathcal{A}$ be fixed such that for any finite set F,

$$\|ba - a\| < \frac{\epsilon}{2}, \ \|cba - ba\| < \frac{\epsilon}{2}, \ \ (a \in F).$$

Let $D : \mathcal{A} \to X = \ker \pi$ be a bounded derivation determined by

$$D(a) = ab \otimes c - b \otimes ca.$$

Then the rest of the proof follows. $\square$

We state the following without proof.

Proposition 3.3.29. [13] *Any pseudo-amenable Banach algebra is weakly amenable. A pseudo-amenable Banach algebra with reflexive underlying space is permanently approximately weakly amenable.*

Chapter 4

Amenability property of $C(\mathcal{X})$

In this chapter, we discuss some interesting algebraic and topological properties of $C(\mathcal{X})$. In the case where $\mathcal{X}$ is a compact Hausdorff space, we give the proof of the amenability of $C(\mathcal{X})$ which is a result due to Seinberg. We further discuss the construction of a bounded approximate diagonal for $C(\mathcal{X})$, a result credited to Abtahi and Zhang.

4.1 The Banach algebra $C(\mathcal{X})$

Recall that for a non empty set S, $\mathbb{C}^S$ the collection of all complex valued functions on S is a commutative unital algebra with respect to pointwise product. Also recall that for a locally compact Hausdorff space $\mathcal{X}$, $C(\mathcal{X})$ is the algebra of all complex valued continuous functions over $\mathcal{X}$. We noted that $C(\mathcal{X})$ is a subalgebra of $\mathbb{C}^{\mathcal{X}}$ and is therefore commutative. We also defined $C^b(\mathcal{A}) \subset C(\mathcal{X})$ as the algebra of all bounded, continuous complex valued functions over $\mathcal{X}$ and $C^b(\mathcal{X})$ equipped with the norm

$$\|\phi\|_\infty = \sup_{t\in\mathcal{X}} |\phi(t)| \quad (\phi \in C^b(\mathcal{X})),$$

is a Banach algebra, where $\|.\|_\infty$ is the uniform norm. A function $\phi \in C(\mathcal{X})$ is said to vanish at infinity if given $\epsilon > 0$, there exists a compact set $M_\epsilon \subset \mathcal{X}$, such that $|\phi(t)| < \epsilon$, for every $t \in \mathcal{X} \setminus M_\epsilon$. We denote by $C_0(\mathcal{X})$, the collection of all $\phi \in C(\mathcal{X})$ that vanish at infinity. Clearly each $\phi \in C_0(\mathcal{X})$ is bounded, which implies that $C_0(\mathcal{X})$ is a subalgebra of $C^b(\mathcal{X})$. Notice further that the limit of a sequence of continuous functions that vanish at infinity also vanishes at infinity, it follows that $C_0(\mathcal{X})$ is closed and is therefore a Banach algebra when equipped with the uniform norm. Since the constant functions, the zero function excluded, do not satisfy this property, $C_0(\mathcal{X})$ is not unital.

Let $\tilde{Z}(\phi) = \{t \in \mathcal{X} : \phi(t) = 0\}$, the support of ϕ, $\operatorname{supp} \phi = \overline{\mathcal{X} \setminus \tilde{Z}(\phi)}$. The collection of all $\phi \in C(\mathcal{X})$ with compact support is denoted by $C_c(\mathcal{X})$. It should be noted that $C_c(\mathcal{X})$ is not necessarily a Banach algebra and that $C_c(\mathcal{X})$ is dense in $C_0(\mathcal{X})$. It follows that, $C_0(\mathcal{X})$ is referred to as the completion of $C_c(\mathcal{X})$. In [43], it is shown that a subalgebra $\mathbb{H}$ of $\mathbb{C}^{\mathcal{X}}$ seperates the points of $\mathcal{X}$ if for every distinct points $s, t \in \mathcal{X}$, there exists $\phi \in \mathbb{H}$ such that $\phi(s) \neq \phi(t)$. The subalgebra $\mathbb{H}$ seperates strongly the points of $\mathcal{X}$ if it seperates the points of $\mathcal{X}$, and for every $t \in \mathcal{X}$ there exists $\varphi \in \mathbb{H}$ such that $\varphi(t) \neq 0$.

Proposition 4.1.1. [43] *Let $\mathcal{X}$ be a non empty locally compact space and let $\mathcal{A}$ be a subalgebra of $C_0(\mathcal{X})$ which seperates strongly the points of $\mathcal{X}$. Then $\mathcal{A}$ is a function algebra on $\mathcal{X}$.*

For a completely regular topological space, say Ω, the following is an interesting property of $C^b(\Omega)$.

Lemma 4.1.2. [9]

(i) Let U be an open neighboorhood of a compact set K, there exists a $\phi \in C^b(\Omega)$ such that $0 \leq \phi \leq 1$ and $\operatorname{supp} \phi \subset U$, if $\varphi \in C^b(K)$, there exists a $\tilde{\varphi} \in C^b(\Omega)$ such that $\tilde{\varphi}|_K = \varphi$ and $\operatorname{supp} \tilde{\varphi} \subset U$.

(ii) If $\{U_i, i = 1, 2, ..., n\}$ is an open cover of K, then there exist $h_i \in C^b(\Omega)$, $i = 1, 2, ..., n$, such that $0 \leq h_i \leq 1$, $\operatorname{supp} h_i \subset U_i$, and $\sum_{i=1}^{n} h_i(t) = 1$ for all $t \in K$.

Remark 4.1.3. Since a locally compact Hausdorff space is completely regular, we may choose $\mathcal{X} = \Omega$.

Remark 4.1.4. The collection $\{h_1, h_2, ..., h_n\}$ is referred to as the partition of unity.

Another interesting subalgebra of $C(\mathcal{X})$ is $C_{\mathbb{R}}(\mathcal{X})$, the algebra of all continuous real valued functions on $\mathcal{X}$. Notice that since $\mathbb{R}$ is a Banach space, it then follows that $C_{\mathbb{R}}(\mathcal{X})$ is indeed a Banach algebra. It should be noted that in the case when $\mathcal{X}$ is compact, $C_c(\mathcal{X}) = C_0(\mathcal{X}) = C^b(\mathcal{X}) = C(\mathcal{X})$.

4.2 Amenability of $C(\mathcal{X})$

The Gel'fand representation theorem shows that any commutative C*-algebra $\mathcal{A}$ is isometrically isomorphic to $C_0(\Phi_{\mathcal{A}})$, where $\Phi_{\mathcal{A}}$ is the character space of $\mathcal{A}$. Recall that $\Phi_{\mathcal{A}}$ is a locally compact Hausdorff space with respect to the

Gel'fand topology described earlier. Also recall that $\Phi_{\mathcal{A}}$ has a one point compactification $\Phi_{\mathcal{A}} \cup \varphi_\infty$ where φ_∞ is the zero functional. In general, for a locally compact Hausdorff space $\mathcal{X}$, the unitization of $C_0(\mathcal{X})$ is $C(\mathcal{Y})$, where $\mathcal{Y}$ is the one point compactification of $\mathcal{X}$. Furthermore, **Corollary (3.1.31)** shows that if the unitization of a Banach algebra is amenable, so is the Banach algebra. It then follows that studying the amenability properties of $C_0(\mathcal{X})$ gives a rich insight into the amenability properties of a rich collection of Banach algebras.

The proof of the amenability of the Banach algebra $C(\mathcal{X})$ by Seinberg depends substantially on the version of Stone - Weierstass theorem given below.

Theorem 4.2.1 (Stone - Weierstrass)**.** *Let $\mathcal{X}$ be a locally compact Hausdorff space and $\mathcal{B}$ a subalgebra of $C_0(\mathcal{X})$. $\mathcal{B}$ is dense if it is closed under complex conjugation, seperates points of $\mathcal{X}$ and does not varnish identically at any point of $\mathcal{X}$.*

We give the proof of the result due to Seinberg below.

Theorem 4.2.2. [48] *Let $\mathcal{X}$ be a compact Hausdorff space, then $C(\mathcal{X})$ is amenable.*

Proof. Let $G = C_{\mathbb{R}}(\mathcal{X})$. Clearly, G is an additive abelian group and is therefore amenable as a group. This further shows that the group algebra $l^1(G)$ is amenable as a Banach algebra. Notice that each $f \in l^1(G)$ is of the form $f = \sum_{h\in G} a_h\delta_h$, where δ_h is the characteristic function of (h), $a_h \in \mathbb{R}$ and $\sum_{h\in G} |a_h| < \infty$.

Let

$$\theta : l^1(G) \mapsto C(\mathcal{X})$$

be determined by

$$\theta\left(\sum_{h\in G} a_h\delta_h\right) = \sum_{h\in G} a_h \exp(ih)$$

Clearly,

$$\begin{aligned}\|\theta(f)\|_G = \left\|\theta\left(\sum_{h\in G} a_h\delta_h\right)\right\| = \left\|\sum_{h\in G} a_h \exp(ih)\right\| \leq \sum_{h\in G} |a_h| \|\exp(ih)\| \leq \sum_{h\in G} |a_h| \\ \leq \left\|\left(\sum_{h\in G} a_h\delta_h\right)\right\|_G = \|f\|_G\end{aligned}$$

Also,

$$\theta\bigg(\sum_{h_1\in G} a_{h_1}\delta_{h_1} \star \sum_{h_2\in G} a_{h_2}\delta_{h_2}\bigg) = \theta\bigg(\sum_{h_1+h_2=h_3} a_{h_1}a_{h_2}\delta_{h_3}\bigg)$$
$$= \sum_{h_3\in G} a_{h_1}a_{h_2}\exp(ih_3)$$
$$= \sum_{h_3\in G} a_{h_1}a_{h_2}\exp(ih_1)\exp(ih_2)$$
$$= \sum_{h_1\in G} a_{h_1}\exp(ih_1)\sum_{h_2\in G} a_{h_2}\exp(ih_2)$$
$$= \theta\bigg(\sum_{h_1\in G} a_{h_1}\delta_{h_1}\bigg)\theta\bigg(\sum_{h_2\in G} a_{h_2}\delta_{h_2}\bigg).$$

This shows that θ is a norm decreasing homomorphism and is therefore continuous. Notice that $\theta(l^1(G))$ contains the identity element in $C(\mathcal{X})$. Furthermore, $\theta(l^1(G))$ seperates points of $\mathcal{X}$. Notice that for any $h \in C_{\mathbb{R}}(\mathcal{X})$, $\overline{\exp(ih)} = \exp(\overline{ih}) = \exp(-ih) \in C(\mathcal{X})$, so that for every $f \in l^1(G)$,

$$\overline{\theta(f)} = \overline{\theta\bigg(\sum_{hc\in G} a_h\delta_h\bigg)} = \overline{\sum_{h\in G} a_h\exp(ih)} = \sum_{h\in G} a_h\exp(-ih).$$

It follows

$$\overline{\theta(f)} = \theta(\overline{f}) \in \theta(l^1(G)),$$

so that $\theta(l^1(G))$ is closed under complex conjugation. By applying **Theorem (4.2.1)**, $\theta(l^1(G))$ is dense in $C(\mathcal{X})$. It therefore follows from **Theorem (3.1.24)** that $C(\mathcal{X})$ is amenable. □

Although Seinberg already gave a concise proof of the amenability of $C(\mathcal{X})$ for a compact Hausdorff space $\mathcal{X}$, the abstractness of the proof given by Seinberg does not allow for a genaralisation of the amenability property of $C(\mathcal{X})$ to the Banach algebra $C(\mathcal{X}, \mathcal{A})$ for a non commutative Banach algebra $\mathcal{A}$. Recall the famous result of Johnson, which showed that a Banach algebra is amenable if and only if it has a bounded approximate diagonal. Abtahi and Zhang [2] in 2010, constructed a bounded approximate diagonal for $C(\mathcal{X})$ for a compact Hausdorff space $\mathcal{X}$ making use of the following result due to Helemskii.

Lemma 4.2.3. [25] *For Banach algebras $\mathcal{A}$ and $\mathcal{B}$, if $u = \sum_{i=1}^{n} u_i \otimes v_i \in \mathcal{A}\hat{\otimes}\mathcal{B}$, then the projective tensor norm of u,*

$$\|u\|_p \leq \frac{1}{n}\sum_{k=1}^{n}\bigg\|\sum_{l=1}^{n} u_l\zeta^{kl}\bigg\|\bigg\|\sum_{j=1}^{n} v_j\zeta^{-kj}\bigg\|,$$

where $\zeta = e^{2\frac{\pi i}{n}}$.

The following lemma by Abtahi and Zhang is a direct consequence of **Lemma (4.2.3)**.

Lemma 4.2.4. [2] *Let* $n \in \mathbb{N}$ *and* $z_k, w_k \in \mathbb{C}, k \in \mathbb{N}_n$. *Let* $\zeta = e^{i\theta}$, *where* $\theta = \frac{2\pi}{n}$. *Then*

$$\frac{1}{n}\sum_{k=1}^{n}\left|\sum_{l=1}^{n} z_l\zeta^{kl}\right|\left|\sum_{j=1}^{n} w_j\zeta^{-kj}\right| \leq \frac{1}{2}\left(\sum_{l=1}^{n}|z_i|^2 + \sum_{j=1}^{n}|w_j|^2\right).$$

Proof. Let $1 \leq k \leq n$, let $\alpha_k = \left|\sum\limits_{l=1}^{n} z_l\zeta^{kl}\right|$ and $\beta_k = \left|\sum\limits_{j=1}^{n} w_j\zeta^{-kj}\right|$. If

$$A = \frac{1}{n}\sum_{k=1}^{n}\alpha_k\beta_k,$$

then clearly,

$$A \leq \frac{1}{2n}\sum_{k=1}^{n}({\alpha_k}^2 + {\beta_k}^2).$$

Notice that

$$\alpha_k^2 = \left(\sum_{l=1}^{n} z_l\zeta^{kl}\right)\left(\overline{\sum_{l=1}^{n} z_l\zeta^{kl}}\right) = \sum_{l=1}^{n} z_l\zeta^{kl}\sum_{l}\bar{z}_l\zeta^{-kl} = \sum_{l=1}^{n}|z_l|^2 + 2\operatorname{Re}\sum_{j<l} z_j\bar{z}_j\zeta^{k(l-j)}.$$

For $1 \leq j < l \leq n$, ζ^{l-j} is clearly an nth root of unity and $\zeta^{l-j} \neq 1$, so that

$$\sum_{l=1}^{n}\zeta^{k(l-j)} = 0.$$

Therefore,

$$\alpha_k^2 = \left(\sum_{l=1}^{n} z_l\zeta^{kl}\right)\left(\overline{\sum_{l=1}^{n} z_l\zeta^{kl}}\right) = \sum_{l=1}^{n} z_l\zeta^{kl}\sum_{l=1}^{n}\bar{z}_l\zeta^{-kl} = \sum_{l=1}^{n}|z_l|^2 + 2\operatorname{Re}\sum_{j<l} z_j\bar{z}_j\zeta^{k(l-j)} = \sum_{l=1}^{n}|z_l|^2.$$

In a similar manner, we obtain

$$\beta_k^2 = \sum_{j=1}^{n} |w_j|^2.$$

Hence the proof follows. $\square$

Corollary 4.2.5. [2] *Let $\mathcal{X}$ be a compact space, let $u = \sum_{k=1}^{n} u_k \otimes v_k \in C(\mathcal{X})\hat{\otimes}C(\mathcal{X})$. Then,*

$$\|u\|_p \leq \frac{1}{2}\bigg(\bigg\|\sum_{i=1}^{n} |u_i|^2\bigg\|_\infty + \bigg\|\sum_{j=1}^{n} |v_j|^2\bigg\|_\infty\bigg).$$

Proof. Let $u = \sum_k u_k \otimes v_k \in C(\mathcal{X}) \otimes C(\mathcal{X})$. By combining **Lemma (4.2.3)** and **Lemma (4.2.4)**, we have that

$$\begin{aligned}\|u\|_p \leq \frac{1}{n}\sum_{k}^{n}\bigg\|\sum_{i=1}^{n} u_i\zeta^{ki}\bigg\|\bigg\|\sum_{j=1}^{n} v_j\zeta^{-kj}\bigg\| &\leq \frac{1}{2}\bigg(\sum_{i=1}^{n} |u_i|^2 + \sum_{j=1}^{n} |v_j|^2\bigg) \\ &\leq \frac{1}{2}\bigg(\bigg\|\sum_{i=1}^{n} |u_i|^2\bigg\| + \bigg\|\sum_{j=1}^{n} |v_j|^2\bigg\|\bigg).\end{aligned}$$

$\square$

We now give the constructive proof of the amenability of $C(\mathcal{X})$ for compact Hausdorff space $\mathcal{X}$.

Theorem 4.2.6. [2] *Let $\mathcal{X}$ be a compact Haursdorff space, then $C(\mathcal{X})$ has a bounded approximate diagonal and is therefore amenable.*

Proof. Let F be a finite subset of $C(\mathcal{X})$ and $\epsilon > 0$. Given $t \in \mathcal{X}$, there exists a neighbourhood V_t of t such that, if $t_* \in V_t$ and $\phi \in F$, then $|\phi(t_*) - \phi(t)| < \frac{\epsilon}{2}$. Since $\mathcal{X}$ is compact, it has a finite cover. Without loss of generality, we choose open subsets $V_1, V_2, ..., V_n \subset \mathcal{X}$, such that $\mathcal{X} = \bigcup_{i=1}^{n} V_i$. Each V_i for $i = 1, 2, ..., n$ is chosen such that $V_i = V_{t_i}$ for each i. That is each V_i is a neighbourhood of $t_i \in \mathcal{X}, i = 1, 2, ..., n$. Hence by **Lemma (4.1.2(ii))**, there exists a partition of unity $\{h_1, h_2, ..., h_n\}$ such that $\text{supp}(h_k) \subset V_k, 1 \leq k \leq n$, and $\sum_{k=1}^{n} h_k = 1$ on $\mathcal{X}$. Let $u_k = \sqrt{h_k}, u = \sum_{k=1}^{n} u_k \otimes u_k, k = 1, 2, ..., n$. Then

$$\pi(u) = \pi\bigg(\sum_{k=1}^{n} u_k \otimes u_k\bigg) = \sum_{k=1}^{n} \pi(u_k \otimes u_k) = \sum_{k=1}^{n} u_k^2 = \sum_{k=1}^{n} h_k = 1,$$

for every $t \in \mathcal{X}$. It suffices to show that:

(i) $\|u\|_p \leq 1$,

(ii) $\|\phi u - u\phi\|_p < \epsilon$ for all $\phi \in F$.

By **Corollary (4.2.5)**,

$$\begin{aligned}\|u\|_p &\leq \frac{1}{2}\left(\left\|\sum_{k=1}^{n}|u_k|^2\right\| + \left\|\sum_{k=1}^{n}|u_k|^2\right\|\right) \leq \frac{1}{2}\left(\left\|\sum_{k=1}^{n}h_k\right\| + \left\|\sum_{k=1}^{n}h_k\right\|\right)\\ &= \frac{1}{2}(1+1) = 1.\end{aligned}$$

In addition, let $\phi \in F$, we choose each ϕ_k such that $\phi_k = \phi - \phi(x_k)$. Notice that for any $s \in V_k$,

$$|\phi_k(s)| = |\phi(s) - \phi(t_k)| \leq \frac{\epsilon}{2}.$$

Also,

$$\begin{aligned}\phi u - u\phi &= \sum_{k=1}^{n}(\phi u_k \otimes u_k - u_k \otimes u_k\phi)\\ &= \sum_{k=1}^{n}(\phi u_k \otimes u_k - u_k \otimes \phi u_k)\\ &= \sum_{k=1}^{n}(\phi u_k \otimes u_k - \phi(t_k)u_k \otimes u_k + \phi(t_k)u_k \otimes u_k - u_k \otimes u_k)\\ &= \sum_{k=1}^{n}(\phi u_k \otimes u_k - \phi(t_k)u_k \otimes u_k + u_k \otimes \phi(t_k)u_k - u_k \otimes \phi u_k)\\ &= \sum_{k=1}^{n}\Big((\phi - \phi(t_k))u_k \otimes u_k - u_k \otimes (\phi - \phi(t_k)u_k)\Big)\\ &= \sum_{k=1}^{n}\phi_k u_k \otimes u_k - \sum_{k=1}^{n}u_k \otimes \phi_k u_k.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\phi u - u\phi\|_p &= \left\|\sum_{k=1}^{n}\phi_k u_k \otimes u_k - \sum_{k=1}^{n}u_k \otimes \phi_k u_k\right\|_p\\ &\leq \left\|\sum_{k=1}^{n}\phi_k u_k \otimes u_k\right\|_p + \left\|\sum_{k=1}^{n}u_k \otimes \phi_k u_k\right\|_p.\end{aligned}$$

We set

$$\delta = \frac{\epsilon}{2}.$$

It then follows that

$$\sum_{k=1}^{n} \phi_k u_k \otimes u_k = \frac{\sqrt{\delta}}{\sqrt{\delta}} \sum_{k=1}^{n} \phi_k u_k \otimes u_k = \sum_{k=1}^{n} \frac{1}{\sqrt{\delta}} \phi_k u_k \otimes \sqrt{\delta} u_k.$$

By applying **Corollary (4.2.5)**,

$$\begin{aligned}
\left\| \sum_{k=1}^{n} \phi_k u_k \otimes u_k \right\|_p &\leq \frac{1}{2} \left(\left\| \sum_{k=1}^{n} \frac{1}{\delta} |\phi_k u_k|^2 \right\|_\infty + \left\| \sum_{k=1}^{n} \delta |u_k|^2 \right\|_\infty \right) \\
&= \frac{1}{2} \left(\left\| \sum_{k=1}^{n} \frac{1}{\delta} |\phi_k|^2 |u_k|^2 \right\|_\infty + \left\| \sum_{k=1}^{n} \delta |u_k|^2 \right\|_\infty \right) \\
&= \frac{1}{2} \left(\left\| \sum_{k=1}^{n} \frac{1}{\delta} |\phi_k|^2 h_k \right\|_\infty + \left\| \sum_{k=1}^{n} \delta h_k \right\|_\infty \right) \\
&= \frac{1}{2} \left(\frac{1}{\delta} \left\| \sum_{k=1}^{n} |\phi_k|^2 h_k \right\|_\infty + \delta \left\| \sum_{k=1}^{n} h_k \right\|_\infty \right) \\
&\leq \frac{1}{2} \left(\frac{2}{\epsilon} \frac{\epsilon^2}{4} + \frac{\epsilon}{2} \right) = \frac{1}{2} \left(\frac{\epsilon}{2} + \frac{\epsilon}{2} \right) = \frac{\epsilon}{2}.
\end{aligned}$$

In a similar manner, we obtain

$$\left\| \sum_{k=1}^{n} u_k \otimes \phi_k u_k \right\|_p < \frac{\epsilon}{2}.$$

Hence,

$$\|\phi u - u\phi\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

Chapter 5

Some notions of amenability of $C(\mathcal{X}, \mathcal{A})$

Let $\mathcal{X}$ be a compact Hausdorff space and $\mathcal{A}$ a Banach algebra. In this chapter, we discuss some important properties of the Banach algebra $C(\mathcal{X}, \mathcal{A})$. We also give the construction of a bounded approximate diagonal for $C(\mathcal{X}, \mathcal{A})$ and show that $C(\mathcal{X}, \mathcal{A})$ is amenable if and only if $\mathcal{A}$ is amenable. This result is credited to Ghamarshoushtari and Zhang, see [17]. We further discuss the weak amenability of $C(\mathcal{X}, \mathcal{A})$ and show that if $\mathcal{A}$ is commutative and weakly amenable, then $C(\mathcal{X}, \mathcal{A})$ is also weakly amenable. This result can be found in [49] and is due to Zhang.

5.1 The Banach algebra $C(\mathcal{X}, \mathcal{A})$

Let $\mathcal{X}$ be a compact Hausdorff space and $\mathcal{A}$ a Banach algebra. The collection of all $\mathcal{A}$-valued continuous functions on $\mathcal{X}$ is denoted by $C(\mathcal{X}, \mathcal{A})$. If we define a pointwise product and the uniform norm:

$$\|f\|_\infty = \sup_{t\in\mathcal{X}} \|f(t)\|_{\mathcal{A}} \quad (f \in C(\mathcal{X}, \mathcal{A}))$$

on $C(\mathcal{X}, \mathcal{A})$, where $\|.\|_{\mathcal{A}}$ is the norm on $\mathcal{A}$, then $C(\mathcal{X}, \mathcal{A})$ is a Banach algebra. It should be noted that $C(\mathcal{X}, \mathcal{A})$ is not in general a commutative Banach algebra, and is commutative if $\mathcal{A}$ is also commutative. Let I be a closed two sided ideal of $\mathcal{A}$. Notice that for any $f \in C(\mathcal{X}, \mathcal{A})$, $g \in C(\mathcal{X}, I)$, $fg(x) = f(x)g(x) \in I$ and $gf(x) = g(x)f(x) \in I$, for all $x \in \mathcal{X}$. This shows that $C(\mathcal{X}, I)$ is a closed two sided ideal of $C(\mathcal{X}, \mathcal{A})$. In particular, $C(\mathcal{X}, \mathcal{A})$ is a closed two sided ideal of $C(\mathcal{X}, \mathcal{A}^\#)$. As a matter of fact, the algebraic structure of $C(\mathcal{X}, \mathcal{A})$ is determined by the Banach algebra $\mathcal{A}$.

The following is an important result by Hausner, which serves as a basis for the proofs of some of the results in this chapter.

Lemma 5.1.1. [24] *Let $f \in C(\mathcal{X}, \mathcal{A})$ and $\epsilon > 0$. Then there exist $a_i \in \mathcal{A}$, and $\phi_i \in C(\mathcal{X})$ such that:*

$$\left\| f - \sum_{i=1}^{n} \phi_i a_i \right\|_\infty < \epsilon.$$

Remark 5.1.2. The result above shows that $\overline{C(\mathcal{X})\mathcal{A}} = C(\mathcal{X}, \mathcal{A})$. That is, $\mathrm{lin}\{\phi a : \phi \in C(\mathcal{X}), a \in \mathcal{A}\}$ is dense in $C(\mathcal{X}, \mathcal{A})$.

The following result shows that the Banach algebra $C(\mathcal{X}, \mathcal{A})$ is identified with the injective tensor product $C(\mathcal{X})\check{\otimes}\mathcal{A}$, where the relationship is determined by the map;

$$(\phi \otimes a)(x) = \phi(x)a, \quad x \in \mathcal{X}, \;\; \phi \in C(\mathcal{X}), \;\; a \in \mathcal{A}.$$

Theorem 5.1.3. [43] *Let $\mathcal{X}$ be a locally compact Hausdorff space and $\mathcal{A}$ a Banach algebra. Then the map defined above induces an isometric algebra isomorphism of $C_0(\mathcal{X})\check{\otimes}\mathcal{A}$ onto $C_0(\mathcal{X}, \mathcal{A})$.*

Remark 5.1.4. The proof of **Theorem (5.1.3)** shows that the image of $C_0(\mathcal{X})\check{\otimes}\mathcal{A}$ is closed under complex conjugation and seperates points of $\mathcal{X}$ and is therefore dense in $C_0(\mathcal{X}, \mathcal{A})$ by **Theorem (4.2.1)**. It should be noted that since a compact space is also locally compact, the result also holds for $C(\mathcal{X}, \mathcal{A})$ when the Hausdorff space $\mathcal{X}$ is compact.

5.2 Amenability of $C(\mathcal{X}, \mathcal{A})$

It is known that the algebraic properties of $C(\mathcal{X}, \mathcal{A})$ derives from those of the range Banach algebra $\mathcal{A}$. We are often interested in finding out if this further implies that the amenability properties of $C(\mathcal{X}, \mathcal{A})$ can be inferred directly from that of $\mathcal{A}$. Ghamarshoushtari and Zhang [17] recently gave an all important answer to this question. They showed that $C(\mathcal{X}, \mathcal{A})$ being amenable is equivalent to $\mathcal{A}$ being amenable. It should be noted that the result is a generalisation of the result due to Abtahi and Zhang [2], which was the focus of our study in the previous chapter and that it relies heavily on the important inequality due to Grothendieck which is given below.

Theorem 5.2.1 (Grothendieck)**.** *Let $\mathcal{X}_1, \mathcal{X}_2$ be compact Hausdorff spaces, and let Φ be a bounded scalar valued bilinear form on $C(\mathcal{X}_1) \otimes C(\mathcal{X}_2)$. Then there exists probability measures μ_1, μ_2 on $\mathcal{X}_1, \mathcal{X}_2$ respectively, such that*

$$|\Phi(\varphi_1, \varphi_2)| \leq K\|\Phi\|\bigg(\int_{\mathcal{X}_1} |\varphi_1|^2 d\mu_1 \int_{\mathcal{X}_2} |\varphi_2|^2 d\mu_2\bigg)^{\frac{1}{2}}, (\varphi_1 \in C(\mathcal{X}_1), \varphi_2 \in C(\mathcal{X}_2)).$$

Remark 5.2.2. The smallest of such K is referred to as the Grothendieck constant denoted by $\mathcal{K}_G^{\mathbb{C}}$. In [23], we see that $\frac{4}{\pi} \leq \mathcal{K}_G^{\mathbb{C}} < 1.405$.

Corollary 5.2.3. [17] *Let $\mathcal{X}_1, \mathcal{X}_2$ be compact Hausdorff spaces, let $u = \sum_{k=1} \phi_i \otimes \varphi_i \in C(\mathcal{X}_1) \otimes C(\mathcal{X}_2)$. Then,*

$$\|u\|_p \leq c\bigg(\bigg\|\sum_{k=1}^{n} |\phi_i|^2\bigg\|_\infty + \bigg\|\sum_{k=1}^{n} |\varphi_i|^2\bigg\|_\infty\bigg),$$

where $c = \frac{1}{2}\mathcal{K}_G^{\mathbb{C}}$

Proof. Let $\phi \otimes \varphi$ be an elemental tensor in $C(\mathcal{X}_1) \otimes C(\mathcal{X}_2)$. Then by **Theorem (5.2.1)**,

$$\begin{aligned} \|\phi \otimes \varphi\|_p &= \sup_{\Phi \in \mathcal{B}^2(C(\mathcal{X}_1), C(\mathcal{X}_2))} |\Phi(\phi, \varphi)| \\ &\leq \mathcal{K}_G^{\mathbb{C}}\bigg(\int |\phi|^2 d\mu_1 \int |\varphi|^2 d\mu_2\bigg)^{\frac{1}{2}} \leq c\bigg(\int |\phi|^2 d\mu_1 + \int |\varphi|^2 d\mu_2\bigg). \end{aligned}$$

It then follows that for any $u = \sum_{i=1}^{n} \phi_i \otimes \varphi_i$,

$$\begin{aligned} \|u\|_p &\leq c\bigg(\int \sum_{i=1}^{n} |\phi_i|^2 d\mu_1 + \int \sum_{i=1}^{n} |\varphi_i|^2 d\mu_2\bigg) \\ &\leq c\bigg(\bigg\|\sum_{i=1}^{n} |\phi_i|^2\bigg\|_\infty + \bigg\|\sum_{i=1}^{n} |\varphi_i|^2\bigg\|_\infty\bigg). \end{aligned}$$

□

We now give the proof of the result due to Ghamarshoushtari and Zhang.

Theorem 5.2.4. [17] *Let X be a compact Hausdorff space and $\mathcal{A}$ a Banach algebra, $C(\mathcal{X}, \mathcal{A})$ has a bounded approximate diagonal if and only if $\mathcal{A}$ has a bounded approximate diagonal.*

Proof. Let $u = \sum_i u_i \otimes v_i \in C(\mathcal{X})\hat{\otimes}C(\mathcal{X})$, $\alpha = \sum_j \alpha_j \otimes \beta_j \in \mathcal{A}\hat{\otimes}\mathcal{A}$. We define a map,

$$T : \Big(C(\mathcal{X})\hat{\otimes}C(\mathcal{X}), \mathcal{A}\hat{\otimes}\mathcal{A} \Big) \to C(\mathcal{X}, \mathcal{A})\hat{\otimes}C(\mathcal{X}, \mathcal{A}),$$

by

$$T(u, \alpha) = \sum_{i,j} u_i\alpha_j \otimes v_i\beta_j.$$

Clearly,

$$\|T(u, \alpha)\| = \Big\| \sum_{i,j} u_i\alpha_j \otimes v_i\beta_j \Big\| \leq \sum_{i,j} \|u_i\alpha_j \otimes v_i\beta_j\|_p \leq \|u\|_p \|\alpha\|_p.$$

This shows that T is bounded if u and α are bounded. Suppose $(\alpha_\lambda) \subset \mathcal{A}\hat{\otimes}\mathcal{A}$ is a bounded approximate diagonal for $\mathcal{A}$, then, there exists $K > 0$ such that $\|\alpha_\lambda\|_p \leq K$, for all λ. Let $F \subset C(\mathcal{X}, \mathcal{A})$ be a finite set, our aim is to show that given $\epsilon > 0$, there exists $U = U_{(F,\epsilon)} \subset C(\mathcal{X}, \mathcal{A})\otimes C(\mathcal{X}, \mathcal{A})$ and a constant $K_1 > 0$ such that for any $f \in F$,

(i) $\|U\|_p \leq K_1$,

(ii) $\|f.U - U.f\|_p < \epsilon$,

(iii) $\|\pi(U)f - f\| < \epsilon$.

Recall that for any $\phi \in C(\mathcal{X})$ and $a \in \mathcal{A}$, $\phi a \in C(\mathcal{X}, \mathcal{A})$. Also by **Lemma (5.1.1)**, $V = \mathrm{lin}\{\phi a : \phi \in C(\mathcal{X}),\ a \in \mathcal{A}\}$ is dense in $C(\mathcal{X}, \mathcal{A})$. Hence naturally we have two cases for the nature of the finite set F.

Case 1: Suppose each $f \in F$ is of the form $\sum_k \phi_k a_k$, $\phi_k \in C(\mathcal{X})$, $a_k \in \mathcal{A}$. Clearly the collection of the a_ks in the finite set F form a finite set $F_\mathcal{A} \subset \mathcal{A}$ and the collection of the ϕ_ks form a finite set $F_C \subset C(\mathcal{X})$. Let $L > 0$ be such that $\|b\|_\mathcal{A} \leq L, b \in F_\mathcal{A}$, $\|\phi\|_\infty \leq L, \phi \in F_C$. By the compactness of $\mathcal{X}$, there exist a finite cover $V_1, ..., V_n$ of $\mathcal{X}$ such that $\mathcal{X} = \bigcup_{i=1}^{n} V_i$ and

$$|\phi(s) - \phi(t)| < \frac{\epsilon}{8cNL\|\alpha\|_p}, \quad \phi \in F_C,\ s, t \in V_i,\ i = 1, ..., n.$$

By applying **Lemma (4.1.2(ii))**, we obtain continuous functions $h_1, ..., h_n \in C(\mathcal{X})$ such that $\mathrm{supp}(h_i) \subset V_i$, $\sum_{i=1}^{n} h_i = 1$ on $\mathcal{X}$.

Let $u = \sum_{i=1}^{n} u_i \otimes u_i \in C(\mathcal{X}) \otimes C(\mathcal{X})$, where $u_i = \sqrt{h_i}$. Clearly $\|u\|_p \leq 2c$ and

$$\|\phi.u - u.\phi\|_P \leq \Big\| \sum_i (\phi - \phi(t_i)) u_i \otimes u_i \Big\|_p + \Big\| \sum_i u_i \otimes (\phi - \phi(t_i)) u_i \Big\|_p < \frac{\epsilon}{2LN\|\alpha\|_p}.$$

In addition, for any $\alpha \in (\alpha_\lambda)$,

$$\|b.\alpha - \alpha.b\|_p < \frac{\epsilon}{4cNL}, \quad \|\pi(\alpha)b - b\|_{\mathcal{A}} < \frac{\epsilon}{4NL}.$$

We therefore have that,

$$\|U\|_p = \|T(u,\alpha)\|_p \leq \|u\|_p \|\alpha\|_p \leq 2Kc$$

for all $\alpha \in (\alpha_\lambda)$. We set $K_1 = 2Kc$. Notice that

$$\begin{aligned} f.U &= \sum_k \phi_k a_k T(u,\alpha) = \sum_k \phi_k a_k \sum_{i,j} u_i \alpha_j \otimes v_i \beta_j \\ &= \sum_k \sum_{i,j} \phi_k u_i a_k \alpha_j \otimes v_i \beta_j = \sum_k T(\phi_k u, a_k \alpha). \end{aligned}$$

Also,

$$\begin{aligned} U.f &= T(u,\alpha) \sum_k \phi_k a_k = \sum_{i,j} u_i \alpha_j \otimes v_i \beta_j \sum_k \phi_k a_k \\ &= \sum_k \sum_{i,j} u_i \alpha_j \otimes v_i \phi_k \beta_j a_k = \sum_k T(u\phi_k, \alpha a_k). \end{aligned}$$

Notice that ,

$$\begin{aligned} T(\phi_k u, a_k\alpha - \alpha a_k) &= \sum_{i,j} \phi_k u_i a_k \alpha_j \otimes v_i \beta_j a_k - \sum_{i,j} \phi_k u_i \alpha_j \otimes v_i \beta_j a_k \\ &= T(\phi_k u, \alpha a_k) - T(\phi_k u, \alpha a_k). \end{aligned}$$

And,

$$\begin{aligned} T(\phi_k u - u\phi_k, \alpha a_k) &= \sum_{i,j} \phi_k u_i \alpha_j \otimes v_i \beta_j a_k - \sum_{i,j} u_i \alpha_j \otimes v_i \phi_k \beta_j a_k \\ &= T(\phi_k u, \alpha a_k) - T(u\phi_k, \alpha a_k). \end{aligned}$$

We have that,

$$
\begin{aligned}
\|f.U - U.f\|_p &= \left\| \sum_k T(\phi_k u, a_k \alpha) - \sum_k T(u\phi_k, \alpha a_k) \right\|_p \\
&\leq \left\| \sum_k T(\phi_k u, a_k\alpha - \alpha a_k) - \sum_k T(\phi_k u - u\phi_k, \alpha a_k) \right\|_p \\
&\leq \sum_k \|T(\phi_k u, a_k\alpha - \alpha a_k)\|_p + \sum_k \|T(\phi_k u - u\phi_k, \alpha a_k)\|_p \\
&\leq \sum_k \|u\|_p \|\phi_k\|_\infty \|a_k\alpha - \alpha a_k\|_p + \sum_k \|\phi_k u - u\phi_k\|_p \|\alpha\|_p \|a_k\|_{\mathcal{A}} \\
&\leq 2Lc\frac{\epsilon}{4cNL} \sum_k 1 + \|\alpha\|_p L \frac{\epsilon}{2\|\alpha\|_p LN} \sum_k 1 \\
&\leq NL\left(2c\frac{\epsilon}{4cNL} + \|\alpha\|_p \frac{\epsilon}{2\|\alpha\|_p LN} \right) = \epsilon.
\end{aligned}
$$

Further more,

$$
\begin{aligned}
\|\pi(U)f - f\| &= \|\pi(u)\pi(\alpha)f - f\| \\
&= \|\pi(\alpha)f - f\| = \left\| \pi(\alpha) \sum_k \phi_k a_k - \sum_k \phi_k a_k \right\|_\infty \\
&= \left\| \sum_k \phi_k(\pi(\alpha)a_k - a_k) \right\|_\infty \leq \sum_k \|\phi_k(\pi(\alpha)a_k - a_k)\|_\infty \\
&= \sum_k \|\phi_k\|_\infty \|\pi(\alpha)a_k - a_k\|_{\mathcal{A}} \leq \sum_k L\|\pi(\alpha)a_k - a_k\|_{\mathcal{A}} \\
&< L\frac{\epsilon}{NL} \sum_k 1 \leq NL\frac{\epsilon}{NL} = \epsilon.
\end{aligned}
$$

Case 2:

Let F be any finite set in $C(\mathcal{X}, \mathcal{A})$, notice that every $f \in C(\mathcal{X}, \mathcal{A})$ is approximately equal to $f_\epsilon = \sum_{k=1}^n \phi_k a_k$, where $\phi_k \in C(\mathcal{X})$, $a_k \in \mathcal{A}$. In addition, since $\mathcal{X}$ is a compact set, there exists $X_1, ..., X_n \subset \mathcal{X}$, such that $\mathcal{X} = \bigcup_{k=1}^n X_k$ and $\|f(x) - f(y)\|_{\mathcal{A}} < \epsilon$, for $x, y \in X_k$. Let $a_k = f(x_k)$ for each $x_k \in X_k$. From **Lemma (4.1.2(ii))**, there are $\phi_k \in C(\mathcal{X})$, such that $\operatorname{supp}(\phi_k) \subset \mathcal{X}_k$ for all $k = 1, ..., n$, $0 \leq \phi_k(x) \leq 1, x \in \mathcal{X}$, and $\sum_k \phi_k = 1$ on $\mathcal{X}$. Clearly, $f_\epsilon = \sum_k \phi_k a_k$ satisfies the requirement.
We may choose each $f_\epsilon = \sum_k \phi_k a_k$ such that for any $f \in F$,

$$\|f - f_\epsilon\|_\infty < \min\left\{ \frac{\epsilon}{4K}, \frac{\epsilon}{8Kc} \right\}.$$

It follows that $F_\epsilon = \{f_\epsilon, \ \ f \in F\}$ is a finite subset of $C(\mathcal{X}, \mathcal{A})$ satisfying all the conditions in **Case 1**, so that there exists a $U = T(u, \alpha) \in C(\mathcal{X}, \mathcal{A})\hat{\otimes}C(\mathcal{X}, \mathcal{A})$, such that,

$$\|U\|_p \leq 2Kc, \quad \|f_\epsilon.U - U.f_\epsilon\|_p < \frac{\epsilon}{2}, \ \ \|\pi(U)f_\epsilon - f_\epsilon\|_{\mathcal{A}} < \frac{\epsilon}{2}, \quad f_\epsilon \in F_\epsilon.$$

Hence for any $f \in F$,

$$\begin{aligned}
\|f.U - U.f\|_p &= \|(f - f_\epsilon + f_\epsilon).U - U.(f - f_\epsilon + f_\epsilon)\|_p \\
&= \|f.U - f_\epsilon.U + f_\epsilon.U - U.f + U.f_\epsilon - U.f_\epsilon\|_p \\
&= \|(f - f_\epsilon).U + (f_\epsilon - f).U + f_\epsilon.U - U.f_\epsilon\|_p \\
&\leq \|(f - f_\epsilon).U\|_p + \|(f_\epsilon - f).U\|_p + \|f_\epsilon.U - U.f_\epsilon\|_p \\
&\leq \|f - f_\epsilon\|_\infty \|U\|_p + \|f_\epsilon - f\|_\infty \|U\|_p + \|f_\epsilon.U - U.f_\epsilon\|_p \\
&< 2\frac{\epsilon}{8Kc}2Kc + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
\|\pi(U)f - f\| &= \|\pi(U)(f - f_\epsilon + f_\epsilon) - (f - f_\epsilon + f_\epsilon)\| \\
&= \|\pi(U)f - \pi(U)f_\epsilon + \pi(U)f_\epsilon - f + f_\epsilon - f_\epsilon\| \\
&= \|\pi(U)(f - f_\epsilon) + f_\epsilon - f + \pi(U)f_\epsilon - f_\epsilon\| \\
&\leq \|\pi(U)(f - f_\epsilon)\| + \|f_\epsilon - f\| + \|\pi(U)f_\epsilon - f_\epsilon\| \\
&\leq \|\pi(U)\|\|f - f_\epsilon\|_\infty + \|f_\epsilon - f\|_\infty + \|\pi(U)f_\epsilon - f_\epsilon\| \\
&= \|\pi(u)\pi(\alpha)\|\|f - f_\epsilon\|_\infty + \|f_\epsilon - f\|_\infty + \|\pi(U)f_\epsilon - f_\epsilon\| \\
&= \|\pi(\alpha)\|\|f - f_\epsilon\|_\infty + \|f_\epsilon - f\|_\infty + \|\pi(U)f_\epsilon - f_\epsilon\| \\
&\leq \|\alpha\|_p\|f - f_\epsilon\|_\infty + \|f_\epsilon - f\|_\infty + \|\pi(U)f_\epsilon - f_\epsilon\|_{\mathcal{A}} \\
&< K\frac{\epsilon}{4K} + \frac{\epsilon}{4} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

This shows that $C(\mathcal{X}, \mathcal{A})$ has a bounded approximate diagonal.
Conversely, Suppose $C(\mathcal{X}, \mathcal{A})$ has a bounded approximate diagonal, then it is amenable. We define a linear map

$$\theta : C(\mathcal{X}, \mathcal{A}) \to \mathcal{A}, \ \ f \mapsto f(t_0) \ \ (f \in C(\mathcal{X}, \mathcal{A})),$$

where t_0 is fixed in $\mathcal{X}$. Clearly, θ is an epimorphism so that by **Theorem (3.1.24)**, $\mathcal{A}$ has a bounded approximate diagonal.

□

Remark 5.2.5. Notice that since $\mathcal{A}$ is asummed to be amenable then it has a bounded approximate identity. Without loss of generality, we may suppose that $(\pi(\alpha_\lambda))$ is a bounded approximate identity for $\mathcal{A}$. Recall that a bounded approximate identity for $\mathcal{A}$ is also a bounded approximate identity for $C(\mathcal{X},\mathcal{A})$, it then follows that given $\epsilon > 0, \alpha \in (\alpha_\lambda),\ f \in F \subset C(\mathcal{X},\mathcal{A})$,

$$\|\pi(U)f - f\| = \|\pi(u)\pi(\alpha)f - f\| = \|\pi(\alpha)f - f\| < \epsilon.$$

This is an alternative proof of part (iii) of the requirements for (U_λ) to be a bounded approximate diagonal for $C(\mathcal{X},\mathcal{A})$.

Remark 5.2.6. By virtue of **Theorem (5.1.3)**, an alternative way to show that $C(\mathcal{X},\mathcal{A})$ is amenable whenever $\mathcal{A}$ is amenable is to determine if $C(\mathcal{X})\breve{\otimes}\mathcal{A}$ is amenable. It also serves as an abstract proof of the afore-stated theorem.

5.3 Weak amenability of $C(\mathcal{X},\mathcal{A})$

Let $\mathcal{X}$ be a compact Hausdorff space. In the section above, we showed that we are often interested in determining if the amenability properties of $C(\mathcal{X},\mathcal{A})$ can be derived from that of its range algebra $\mathcal{A}$. The case of the weak amenability of $C(\mathcal{X},\mathcal{A})$ is no different. Recently, Zhang [49] showed that for a commutative Banach algebra $\mathcal{A}$, the implication actually holds. This section will be entirely concerned with discussing the proof by Zhang. Consider the following.

Theorem 5.3.1. [17] *Let $\mathcal{X}$ be a compact Hausdorff space and $\mathcal{A}$ be a commutative Banach algebra with a bounded approximate identity. If $\mathcal{A}$ is weakly amenable, then so is $C(\mathcal{X},\mathcal{A})$.*

Proof. Since $\mathcal{A}$ is commutative, so is $C(\mathcal{X},\mathcal{A})$. Also, $\mathcal{A}$ is a closed subalgebra of $C(\mathcal{X},\mathcal{A})$. It follows that for any $a \in \mathcal{A},\ g \in C(\mathcal{X},\mathcal{A}),\ ag,\ ga \in C(\mathcal{X},\mathcal{A})$, and;

$$\|ag\|_\infty \le \|a\|\|g\|_\infty,\ \ \|ga\|_\infty \le \|a\|\|g\|_\infty,$$

which shows that $C(\mathcal{X},\mathcal{A})$ is a Banach $\mathcal{A}$-bimodule. Also note that $C(\mathcal{X},\mathcal{A})$ is a Banach $C(\mathcal{X})$-bimodule. By virtue of **Definition (3.2.1)**, we are to show that every continuous derivation from $C(\mathcal{X},\mathcal{A})$ into $C(\mathcal{X},\mathcal{A})^*$ is trivial. Let

$$D : C(\mathcal{X},\mathcal{A}) \to C(\mathcal{X},\mathcal{A})^*$$

be a continuous derivation. Since $\mathcal{A}$ is a closed subalgebra of $C(\mathcal{X},\mathcal{A})$,

$$D|_{\mathcal{A}} : \mathcal{A} \to C(\mathcal{X},\mathcal{A})^*$$

is a continuous derivation. But $\mathcal{A}$ is commutative and weakly amenable, it then follows that $D|_{\mathcal{A}} \equiv 0$. Let (e_λ) be a bounded approximate identity for $\mathcal{A}$. Then clearly for any $g \in C(\mathcal{X}, \mathcal{A})$,

$$\|ge_\lambda - g\|_\infty,\ \|e_\lambda g - g\|_\infty \to 0,$$

for all λ. That is (e_λ) is indeed a bounded approximate identity for $C(\mathcal{X}, \mathcal{A})$. Since $D|_{\mathcal{A}} \equiv 0$, it follows that $w^* - \lim_\lambda D(e_\lambda) = 0$ for all λ.
Claim: For each $\phi \in C(\mathcal{X})$, $w^* - \lim_\lambda D(\phi e_\lambda)$ exists.
To prove our claim, we show that all weak* subnets of $(D(\phi e_\lambda))$ converge to the same limit. Let (e_i), (e_j) be convergent subnets of (e_λ). Suppose $w^* - \lim_i D(\phi e_i)$ and $w^* - \lim_j D(\phi e_j)$ exists. Then

$$\begin{aligned} D(\phi e_i) = w^* - \lim_j D(\phi e_j e_i) &= \lim_j (\phi e_j) D(e_i) + w^* - D(\phi e_j) e_i \\ &= \phi D(e_i) + w^* - \lim_j D(\phi e_j) e_i. \end{aligned}$$

It follows that

$$\begin{aligned} w^* - \lim_i D(\phi e_i) &= \lim_i \phi D(e_i) + w^* - \lim_i D(\phi e_j) e_i \\ &= 0 + w^* - \lim_j D(\phi e_j) = w^* - \lim_j D(\phi e_j). \end{aligned}$$

That is, our claim holds. It then follows that the continuous linear map: $\tilde{D} : C(\mathcal{X}) \to C(\mathcal{X}, \mathcal{A})^*$, given by $\tilde{D}(\phi) = w^* - \lim_\lambda D(\phi e_\lambda)$ is well defined. Notice that for $\phi_1, \phi_2 \in C(\mathcal{X})$,

$$\begin{aligned} \tilde{D}(\phi_1\phi_2) &= w^* - \lim_j D(\phi_1\phi_2 e_j) = w^* - \lim_j (\lim_i D(\phi_1 e_i \phi_2 e_j)) \\ &= w^* - \lim_j \phi_1 D(\phi_2 e_j) + w^* - \lim_i D(\phi_1 e_i)\phi_2 = \phi_1 \tilde{D}(\phi_2) + \tilde{D}(\phi_1)\phi_2. \end{aligned}$$

Hence, $\tilde{D}$ is indeed a derivation. Since $C(\mathcal{X})$ is commutative and amenable, $\tilde{D} \equiv 0$. For any $\phi \in C(\mathcal{X})$, $a \in \mathcal{A}$,

$$D(\phi a) = \tilde{D}(\phi).a + \phi.D_{\mathcal{A}}(a) = 0.$$

So that $D = 0$ on $\text{lin}\{\phi a : \phi \in C(\mathcal{X}),\ a \in \mathcal{A}\}$. But $\text{lin}\{\phi a : \phi \in C(\mathcal{X}),\ a \in \mathcal{A}\}$ is dense in $C(\mathcal{X}, \mathcal{A})$, as is the case in **Theorem (5.2.4)**. Therefore, $D \equiv 0$ on $C(\mathcal{X}, \mathcal{A})$. Hence, $C(\mathcal{X}, \mathcal{A})$ is weakly amenable. □

The following is a more general form of the result stated above.

Theorem 5.3.2. [49] *Let $\mathcal{X}$ be a compact Hausdorff space and $\mathcal{A}$ a commutative Banach algebra. Then $C(\mathcal{X}, \mathcal{A})$ is weakly amenable if and only if $\mathcal{A}$ is weakly amenable.*

Proof. Suppose $\mathcal{A}$ is weakly amenable, then by **Corollary (3.2.10)**, $\mathcal{A}^{\#}$ is weakly amenable. Since $\mathcal{A}^{\#}$ is unital, it must have a bounded approximate identity. By **Theorem (5.3.1)**, $C(\mathcal{X}, \mathcal{A}^{\#})$ is weakly amenable. Recall that $C(\mathcal{X}, \mathcal{A})$ is a closed ideal of $C(\mathcal{X}, \mathcal{A}^{\#})$ and that $\mathcal{A}$ being commutative implies $C(\mathcal{X}, \mathcal{A})$ is commutative. Hence by **Theorem (3.2.9)**, it suffices to show that $C(\mathcal{X}, \mathcal{A})^2$ is dense in $C(\mathcal{X}, \mathcal{A})$. Consider the following.
Recall that $\mathrm{lin}\{\phi a : \phi \in C(\mathcal{X}), a \in \mathcal{A}\}$ is dense in $C(\mathcal{X}, \mathcal{A})$. By **Proposition (3.2.4)**, the weak amenability of $\mathcal{A}$ implies $\mathcal{A}^2$ is dense in $\mathcal{A}$, so that $\mathrm{lin}\{ha : h \in V, a \in \mathcal{A}\}$ is dense in $C(\mathcal{X}, \mathcal{A})$, where $V = \mathrm{lin}\{\phi a : \phi \in C(\mathcal{X}), a \in \mathcal{A}\}$. It therefore follows that $C(\mathcal{X}, \mathcal{A})^2$ is dense in $C(\mathcal{X}, \mathcal{A})$. $\square$

Chapter 6

Pseudo-amenability of $C(\mathcal{X}, \mathcal{A})$

In this chapter, we give results obtained from our study. These results serve as our contribution to knowledge. In particular, for a Banach algebra $\mathcal{A}$ with a bounded approximate identity such that $\mathcal{A}\hat{\otimes}\mathcal{A}$ is norm irregular, we show that if $\mathcal{A}$ has an aproximate diagonal which is bounded with respect to the multiplier norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$, then $C(\mathcal{X}, \mathcal{A})$ has an aproximate diagonal. This result provides a partial answer to the question of pseudo-amenability of $C(\mathcal{X}, \mathcal{A})$ which follows from the work of Ghamarshoushtari and Zhang in [17].

6.1 Norm irregularity of $C(\mathcal{X}, \mathcal{A})$

For a Banach algebra $\mathcal{A}$, the multiplier semi-norm on $\mathcal{A}$ is defined as;

$$\|a\|_M = \sup_{b\in\mathcal{A},\ \|b\|\le 1}\{\|ab\|,\ \|ba\|\},\ \ (a \in \mathcal{A}).$$

Clearly, $\max\{\|ab\|,\ \|ba\|\} \le \|a\|_M\|b\|$ for all $a, b \in \mathcal{A}$, so that $\|a\|_M \le \|a\|$. Recall that the annihilator ideal of $\mathcal{A}$ denoted by $\mathrm{ann}(\mathcal{A})$ is defined as

$$\mathrm{ann}(\mathcal{A}) = \{a \in \mathcal{A} : ab = ba = 0,\ \ b\ \in \mathcal{A}\}.$$

If $\mathrm{ann}(\mathcal{A}) = \{0\}$, then $\|.\|_M$ is indeed an algebra norm on $\mathcal{A}$ called the multiplier norm. If the Banach algebra $\mathcal{A}$ is norm irregular, in the sense that $\|.\|_M$ does not coincide with and is strictly weaker than $\|.\|$, then $\|.\|_M$ is not necessarily a complete norm on $\mathcal{A}$, that is, $(\mathcal{A}, \|.\|_M)$ is not a Banach algebra. The completion of $\mathcal{A}$ with respect to the multiplier norm is denoted by $\tilde{\mathcal{A}}$. Notice that for a locally compact Hausdorff space, $\mathrm{ann}(C_0(\mathcal{X}, \mathcal{A})) = \{0\}$. It then follows that $(\|.\|_\infty)_M$ is a norm on $C_0(\mathcal{X}, \mathcal{A})$. The following result shows that the multiplier norm on $C_0(\mathcal{X}, \mathcal{A})$ is determined by the multiplier norm on $\mathcal{A}$. Here, we make use of the notation;

$$\|f\|_\infty^{(M)} = \sup_{t\in\mathcal{X}} \|f(t)\|_M, \ \ (f \in C_0(\mathcal{X}, \mathcal{A})).$$

Proposition 6.1.1. [40] *Let $\mathcal{X}$ be a locally compact Hausdorff space and $\mathcal{A}$ a Banach algebra. Then, the multiplier norm on $C_0(\mathcal{X}, \mathcal{A})$ satisfies*

$$(\|f\|_\infty)_M = \|f\|_\infty^{(M)} \ \ (f \in C_0(\mathcal{X}, \mathcal{A})).$$

Remark 6.1.2. Since a compact space is also locally compact, the result above holds for $C(\mathcal{X}, \mathcal{A})$, where $\mathcal{X}$ is a compact Hausdorff space.

6.2 Results

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ a compact Hausdorff space. In the previous chapter, we gave the construction of a bounded approximate diagonal for $C(\mathcal{X}, \mathcal{A})$ and showed that $C(\mathcal{X}, \mathcal{A})$ has a bounded approximate diagonal if and only if $\mathcal{A}$ has a bounded approximate diagonal. It should be noted that the construction relies heavily on the norm boundedness condition on the approximate diagonal for $\mathcal{A}$. Recall that $\mathcal{A}$ is pseudo-amenable if it has an approximate diagonal, which need not be bounded. Hence, the construction of an approximate diagonal for $C(\mathcal{X}, \mathcal{A})$ given earlier fails if we remove the condition that the approximate diagonal for $\mathcal{A}$ is bounded with respect to the projective norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$. In other words, it is unknown if the possession of an approximate diagonal by $C(\mathcal{X}, \mathcal{A})$ follows from the possession from an approximate diagonal by $\mathcal{A}$. Our main result shows that under certain restrictions, if $\mathcal{A}$ has an approximate diagonal, then so does $C(\mathcal{X}, \mathcal{A})$.

Recall that for a norm irregular Banach algebra, say $(\mathcal{A}, \|.\|)$, the multiplier norm on $\mathcal{A}$ denoted by $\|.\|_M$ is strictly weaker than $\|.\|$, so that the normed algebra $(\mathcal{A}, \|.\|_M)$ is not necessarily complete. Let $\alpha \in \mathcal{A}\hat{\otimes}\mathcal{A}$ such that $\alpha = \sum_i a_i \otimes b_i$, the multipler norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$ assumes the form

$$\begin{aligned}(\|\alpha\|_p)_M &= \sup_{\|\beta\|_p \leq 1} \{\|\alpha\beta\|_p, \ \|\beta\alpha\|_p\} \\ &= \sup_{\inf \sum_j \|\alpha_j\|\|\beta_j\| \leq 1} \left\{ \inf \sum_{i,j} \|a_i\alpha_j\|\|b_i\beta_j\|, \ \inf \sum_{i,j} \|\alpha_j a_i\|\|\beta_j b_i\| \right\},\end{aligned}$$

where the infimum is taken over all finite representations of α and $\beta = \sum_j \alpha_j \otimes \beta_j$. Since the projective tensor norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$ depends on the norm on $\mathcal{A}$, it then follows that if $\mathcal{A}\hat{\otimes}\mathcal{A}$ is norm irregular, so is $\mathcal{A}$.

The following lemma is an important component of our main result.

Lemma 6.2.1. *Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and let $(\|.\|_p)_M$ be the multiplier semi-norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$. Then there exists $K > 0$ such that*

$$\|a\beta\|_p \leq K(\|\beta\|_p)_M\|a\|,$$

for all $a \in \mathcal{A}$, $\beta \in \mathcal{A}\hat{\otimes}\mathcal{A}$.

Proof. Let (e_α) be a bounded approximate identity for $\mathcal{A}$. Then there exists a $C > 0$ such that $\|e_\alpha\| \leq C$ for all α.
Case 1: Suppose $1 \leq \|e_\alpha\| \leq C$, for all α. Then for any $a \in \mathcal{A}$ and $\beta \in \mathcal{A}\hat{\otimes}\mathcal{A}$,

$$\begin{aligned}\|a\beta\|_p &\leq \|(a \otimes e_\alpha)\beta\|_p \\ &\leq (\|\beta\|_p)_M\|a\|\|e_\alpha\| \\ &\leq K(\|\beta\|_p)_M\|a\|.\end{aligned}$$

Here, we chose $K = C$.
Case 2: Suppose $C < 1$, then $\|e_\alpha\| < 1$ for all α. It then follows that there exists $L > 1$ such that $L\|e_\alpha\| \geq 1$ for all α. For any $a \in \mathcal{A}$ and $\beta \in \mathcal{A}\hat{\otimes}\mathcal{A}$,

$$\begin{aligned}\|a\beta\|_p &\leq L\|(a \otimes e_\alpha)\beta\|_p \\ &\leq L(\|\beta\|_p)_M\|a\|\|e_\alpha\| \\ &\leq LC(\|\beta\|_p)_M\|a\| \\ &= K(\|\beta\|_p)_M\|a\|,\end{aligned}$$

where $K = LC$. □

We now give our main result.

Theorem 6.2.2. *Let $\mathcal{X}$ be a compact Hausdorff space and $(\mathcal{A}, \|.\|)$ be a Banach algebra with a bounded approximate identity such that $\mathcal{A}\hat{\otimes}\mathcal{A}$ is norm irregular. If $\mathcal{A}$ has an approximate diagonal which is bounded with respect to the multiplier norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$, then $C(\mathcal{X}, \mathcal{A})$ has an approximate diagonal.*

Proof. We define a linear map

$$T : (C(\mathcal{X})\hat{\otimes}C(\mathcal{X}), \mathcal{A}\hat{\otimes}\mathcal{A}) \to C(\mathcal{X}, \mathcal{A})\hat{\otimes}C(\mathcal{X}, \mathcal{A}),$$

determined by

$$T(v, \beta) = \sum_{i,j} u_i\alpha_j \otimes v_i\beta_j,$$

where $v = \sum_i u_i \otimes v_i \in C(\mathcal{X}) \otimes C(\mathcal{X})$ and $\beta = \sum_j \alpha_j \otimes \beta_j$. Notice that

$$\sum_{i,j} \|u_i\alpha_j\|\|v_j\beta_j\| \leq \sum_i \|u_i\|\|v_i\| \sum_j \|\alpha_j\|\|\beta_j\|.$$

It follows that $\|T(v,\beta)\|_p \leq \|v\|_p \|\beta\|_p$, where $\|.\|_p$ is the projective tensor norm. By a similar argument in the proof of **Proposition (3.3.19)**, we show that for every $\epsilon > 0$ and finite set $F \subset C(\mathcal{X},\mathcal{A})$ there exists $U = U_{(F,\epsilon)} \in C(\mathcal{X},\mathcal{A})\hat{\otimes}C(\mathcal{X},\mathcal{A})$ such that

(i) $\|g.U - U.g\|_p < \epsilon$, and

(ii) $\|\pi(U)g - g\| < \epsilon$, for all $g \in F$.

Let (α_λ) be an approximate diagonal for $\mathcal{A}$ bounded with respect to $(\|.\|_p)_M$ on $\mathcal{A}\hat{\otimes}\mathcal{A}$. Then there exists $K_1 > 0$ such that $(\|\alpha_\lambda\|_p)_M \leq K_1$ for all λ. Let $\epsilon > 0$ and let $F \subset C(\mathcal{X},\mathcal{A})$ be a fixed finite subset. For fixed finite subsets $F_C, F_{\mathcal{A}}$ in $C(\mathcal{X})$ and $\mathcal{A}$ respectively, $\phi_j s \in F_C$ and $a_j s \in F_{\mathcal{A}}$, we see that $\sum_j \phi_j a_j \in F$, where the sum is finite. Let L be a positive real number such that

$$\|a\| < L,\ \|\phi\| < L,$$

for every $a \in F_{\mathcal{A}}$ and $\phi \in F_C$. Since (α_λ) is an approximate diagonal for $\mathcal{A}$, it then follows that for any $b \in F_{\mathcal{A}}$,

$$\|b.\alpha - \alpha.b\| < \frac{\epsilon}{8cNL},\ \ \|\pi(\alpha)b - b\| < \frac{\epsilon}{4cNL}\ \text{ for some } \alpha \in (\alpha_\lambda),$$

where N is a positive integer chosen to be no less than the number of terms in the finite sums $\sum_j \phi_j a_j$, and c is the constant in **Theorem (5.2.3)**. Since $\mathcal{X}$ is compact, there exist finite open sets $V_i \subset \mathcal{X},\ \ i = 1,2,..,n$ such that $\mathcal{X} = \bigcup_{i=1}^{n} V_i$ and

$$|\phi(x) - \phi(y)| < \frac{\epsilon}{16KK_1NL}\ \ (\phi \in F_C,\ x,y \in V_i).$$

By applying **Lemma (4.1.2(ii))**, we obtain continuous functions $h_i \in C(\mathcal{X})$, $i = 1,2,...,n$ such that $\mathrm{supp}(h_i) \subset V_i,\ \ h_i \in [0,1]$ for all i and $\sum_i h_i = 1$ on $\mathcal{X}$. Let $u_i = \sqrt{h_i}$ and $u = \sum_i^n u_i \otimes u_i$. Clearly, $\pi(u) = 1$ on $\mathcal{X}$, $\|u\|_p \leq 2c$ and for every $\phi \in F_C$,

$$\|\phi.u - u.\phi\|_p \leq \left\| \sum_i (\phi - \phi(t_i))u_i \otimes u_i \right\|_p + \left\| \sum_i u_i \otimes (\phi - \phi(t_i))u_i \right\|_p < \frac{\epsilon}{4KK_1NL},$$

where K is the constant in **Lemma (6.2.1)**. Since $\mathrm{lin}\{\phi a : \phi \in C(\mathcal{X}), a \in \mathcal{A}\}$ is dense in $C(\mathcal{X},\mathcal{A})$, it then follows that for any $g \in F$,

$$\left\| g - \sum_j \phi_j a_j \right\|_\infty < \frac{\epsilon}{8\|\alpha\|_p c}.$$

Thus for any $\alpha \in (\alpha_\lambda)$,

$$\begin{aligned}
\left\| \sum_i \phi_j a_j T(u,\alpha) - T(u,\alpha) \sum_j \phi_j a_j \right\| &= \left\| \sum_j \phi_j a_j T(u,\alpha) - \sum_j T(u,\alpha) \phi_j a_j \right\| \\
&= \left\| \sum_j T(\phi_j u, a_j \alpha) - \sum_j T(u\phi_j, \alpha a_j) \right\| \\
&\leq \sum_j \|T(\phi_j u, a_j\alpha) - T(u\phi_j, \alpha a_j)\| \\
&\leq \sum_j \|T(\phi_j u, a_j\alpha) - T(u\phi_j, a_j\alpha)\| \\
&+ \sum_j \|T(u\phi_j, a_j\alpha) - T(u\phi_j, \alpha a_j)\| \\
&\leq \sum_j \|\phi_j u - u\phi_j\|_p \|a_j\alpha\|_p \\
&+ \sum_j \|u\phi_j\| \|a_j\alpha - \alpha a_j\|_p.
\end{aligned}$$

Since $\mathcal{A}$ has a bounded approximate identity, we apply **Lemma (6.2.1)** and obtain

$$\begin{aligned}
\sum_j \|\phi_j u - u\phi_j\|_p \|a_j\alpha\|_p + \sum_j \|u\phi_j\| \|a_j\alpha - \alpha a_j\|_p & \\
&\leq \sum_j \|\phi_j u - u\phi_j\|_p K \|a_j\| (\|\alpha\|_p)_M \\
&+ \sum_j \|u\|_p \|\phi_j\|_\infty \|a_j\alpha - \alpha a_j\|_p \\
&= K(\|\alpha\|_p)_M \sum_j \|\phi_j u - u\phi_j\|_p \|a_j\| \\
&+ \|u\|_p \sum_j \|\phi_j\|_\infty \|a_j\alpha - \alpha a_j\|_p \\
&< KK_1 L \frac{\epsilon}{4KK_1NL} \sum_j 1 + 2cL \frac{\epsilon}{8cNL} \sum_j 1 \\
&\leq KK_1 L \frac{\epsilon}{4KK_1NL} N + 2cL \frac{\epsilon}{8cNL} N \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Set $f_N = \sum_j \phi_j a_j$, so that for any $g \in F$,

$$\begin{aligned}\|gT(u,\alpha) - T(u,\alpha)g\| &= \|(g - f_N + f_N)T(u,\alpha) - T(u,\alpha)(g - f_N + f_N)\| \\ &\leq \|(g - f_N)T(u,\alpha)\| + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\ &+ \|T(u,\alpha)(g - f_N)\| \\ &\leq 2\|T(u,\alpha)\|\|g - f_N\|_\infty + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\ &\leq 2\|u\|_p\|\alpha\|_p\|g - f_N\|_\infty + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\ &\leq 4c\|g - f_N\|_\infty\|\alpha\|_p + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\ &< 4c\frac{\epsilon}{8\|\alpha\|_p c}\|\alpha\|_p + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

Also,

$$\begin{aligned}\left\|\pi(T(u,\alpha))\sum_j \phi_j a_j - \sum_j \phi_j a_j\right\| &= \left\|\pi(u)\pi(\alpha)\sum_j \phi_j a_j - \sum_j \phi_j a_j\right\| \\ &= \left\|\sum_j \phi_j(\pi(\alpha)a_j - a_j)\right\| \\ &\leq \sum_j \|\phi_j\|_\infty\|\pi(\alpha)a_j - a_j\| \\ &< NL\frac{\epsilon}{4cNL} < \frac{\epsilon}{2},\ (2c = \mathcal{K}_G^{\mathbb{C}} > 1).\end{aligned}$$

Let f_N be as defined earlier. Without loss of generality, we may suppose $\|\alpha\|_p \geq 1$. Then for any $g \in F$,

$$\begin{aligned}\|\pi(T(u,\alpha))g - g\| &= \|\pi(T(u,\alpha))(g - f_N + f_N) - (g - f_N + f_N)\| \\ &= \|\pi(T(u,\alpha))(g - f_N) + \pi(T(u,\alpha))f_N - f_N - (g - f_N)\| \\ &\leq \|\pi(\alpha)(g - f_N)\| + \|\pi(T(u,\alpha))f_N - f_N\| + \|g - f_N\| \\ &\leq \|\pi(\alpha)\|_M\|g - f_N\|_\infty + \|\pi(T(u,\alpha))f_N - f_N\| + \|g - f_N\|_\infty \\ &\leq \|\alpha\|_p\|g - f_N\|_\infty + \|\pi(T(u,\alpha))f_N - f_N\| + \|g - f_N\|_\infty \\ &\leq 2\|\alpha\|_p\|g - f_N\|_\infty + \|\pi(T(u,\alpha))f_N - f_N\| \\ &< 2\|\alpha\|_p\frac{\epsilon}{8\|\alpha\|_p} + \frac{\epsilon}{2} < \epsilon.\end{aligned}$$

We set $T(u,\alpha) = U$, the natural partial order $(F_1, \epsilon_1) \prec (F_2, \epsilon_2)$ if and only if $F_1 \subset F_2$, $\epsilon_1 \geq \epsilon_2$, ensures we obtain a net $(U_{(F,\epsilon)})$, which is the desired approximate diagonal for $C(\mathcal{X}, \mathcal{A})$. $\square$

Remark 6.2.3. Since $\mathcal{A}$ has a bounded approximate identity, by applying **Proposition (3.3.26)**, we may assume that $(\pi(\alpha_\lambda))$ is a bounded approximate identity for $\mathcal{A}$. This further implies that $(\pi(\alpha_\lambda))$ is a bounded approximate identity for $C(\mathcal{X}, \mathcal{A})$, so that for any $g \in C(\mathcal{X}, \mathcal{A})$, $\|\pi(T(u,\alpha)g - g\|_\infty = \|\pi(\alpha)g - g\|_\infty \to 0$, which serves as another proof for condition (ii).

Remark 6.2.4. Clearly, the map $C(\mathcal{X}, \mathcal{A}) \to \mathcal{A}$, $f \mapsto f(t_0)$, where t_0 is fixed in $\mathcal{X}$ is a continuous epimorphism. This shows that the converse of **Theorem (6.2.2)** clearly holds, that is, if $C(\mathcal{X}, \mathcal{A})$ has an aproximate diagonal, then $\mathcal{A}$ has an approximate diagonal.

We define $\|\alpha\|_p^{(M)} = \inf \sum_i \|a_i\|_M \|b_i\|_M$, where $\alpha = \sum_i a_i \otimes b_i$ and

$$\sum_i \|a_i\|_M \|b_i\|_M =$$

$$\sup_{\sum_j \|\alpha_j\|, \sum_j \|\beta_j\| \le 1} \Big\{ \sum_{i,j} \|a_i\alpha_j\|\|b_i\beta_j\|, \sum_{i,j} \|\alpha_j a_i\|\|b_i\beta_j\|,$$
$$\sum_{i,j} \|a_i\alpha_j\|\|\beta_j b_i\|, \sum_{i,j} \|\alpha_j a_i\|\|\beta_j b_i\| \Big\},$$

each of the sums being finite.

Proposition 6.2.5. *The multiplier norm on $\mathcal{A}\hat{\otimes}\mathcal{A}$ satisfies*

$$(\|\alpha\|_p)_M \le \|\alpha\|_p^{(M)}, \alpha \in \mathcal{A}\hat{\otimes}\mathcal{A}.$$

Proof. Let $\{\alpha_j\}, \{\beta_j\}$ be finite collections of elements in $\mathcal{A}_{[1]}$ chosen such that $\|\alpha_j\|, \|\beta_j\| \le \frac{1}{\sqrt{N}}$, where $N > 1$ is an integer chosen such that $\sqrt{N}$ is no less than the number of terms in any finite representation of $\beta = \sum_j \alpha_j \otimes \beta_j$. Clearly, $\sum_j \|\alpha_j\|, \sum_j \|\beta_j\| \le 1$, and

$$\|\beta\|_p = \inf \sum_j \|\alpha_j\|\|\beta_j\| \le \inf \sum_j \|\alpha_j\| \sum_j \|\beta_j\| \le 1.$$

This shows that $\beta \in (\mathcal{A}\hat{\otimes}\mathcal{A})_{[1]}$. It then follows that for any $\alpha = \sum_i a_i \otimes b_i \in \mathcal{A}\hat{\otimes}\mathcal{A}$,

$$\|\alpha\beta\|_p = \inf \sum_{i,j} \|a_i\alpha_j\|\|b_i\beta_j\| \le \inf \sum_{i,j} \|a_i\|_M \|\alpha_j\|\|b_i\|_M \|\beta_j\|$$
$$\le \inf \sum_i \|a_i\|_M \|b_i\|_M \inf \sum_j \|\alpha_j\|\|\beta_j\| \le \inf \sum_i \|a_i\|_M \|b_i\|_M.$$

In a similar manner, we obtain

$$\|\beta\alpha\|_p \le \inf \sum_i \|a_i\|_M \|b_i\|_M.$$

Therefore, $(\|\alpha\|_p)_M \le \inf \sum_i \|a_i\|_M \|b_i\|_M = \|\alpha\|_p^{(M)}$.

□

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