Computational and Analytical Modelling of Composite Structures Based on Exact and Higher Order Theories

Pavel Tabakov

Submitted in fulfillment of the academic requirements for the degree of Doctor of Philosophy in the Department of Mechanical Engineering at the University of Natal.

Durban, South Africa
November 1995

Abstract

The objective of the present study is the computational and analytical modelling of a stress and strain state of the composite laminated structures.

The exact three dimensional solution is derived for laminated anisotropic thick cylinders with both constant and variable material properties through the thickness of a layer. The governing differential equations are derived in a such form that to satisfy the stress functions and are given for layered cylindrical shell with open ends. The solution then extended to the laminated cylindrical shells with closed ends, that is to pressure vessels.

Based on the accurate three–dimensional stress analysis an approach for the optimal design of the thick pressure vessels is formulated. Cylindrical pressure vessels are optimised taking the fibre angle as a design variable to maximise the burst pressure. The effect of the axial force on the optimal design is investigated. Numerical results are given for both single and laminated (up to five layers) cylindrical shells. The maximum burst pressure is computed using the three–dimensional interactive Tsai–Wu failure criterion, which takes into account the influence of all stress components to the failure. Design optimisation of multilayered composite pressure vessels are based on the use of robust multidimensional methods which give fast convergence.

Transverse shear and normal deformation higher–order theory for the solution of dynamic problems of laminated plates and shells is studied. The theory developed is based on the kinematic hypotheses which are derived using iterative technique. Dynamic effects, such as forces of inertia and the direct influence of external loading on the stress and strain components are included at the initial stage of derivation where kinematic hypotheses are formulated. The proposed theory and solution methods provide a basis for theoretical and applied studies in the field of dynamics and statics of the laminated shells, plates and their systems, particularly for investigation of dynamic processes related to the highest vibration forms and wave propagation, for optimal design etc.

Geometrically nonlinear higher–order theory of laminated plates and shells with shear and normal deformation is derived. The theory takes into account both transverse shear and normal deformations. The number of numerical results are obtained based on the nonlinear theory developed. The results illustrate importance of the influence of geometrical nonlinearity, especially, at high levels of loading and in case when the laminae exhibit significant differences in their elastic properties.

Declaration

I declare that this dissertation is my own unaided work except where due acknowledgement is made to others. This dissertation is being submitted for the Degree of Doctor of Philosophy to the University of Natal, Durban, and has not been submitted previously for any other degree or examination.

Pavel Tabakov November 1995

Acknowledgements

I express my gratitude to my supervisors, Professors Sarp Adali and Viktor Verijenko, for their guidance and encouragement; to their collaborator, Professor V. G. Piskunov, for his contributions; and to my colleague Evan Summers for his support and encouragement.

Nomenclature

1. Anisotropic Thick Cylinders

r, θ, z	Cylindrical coordinates
E_r, E_θ, E_z	Moduli of elasticity
$\nu_{ij},\ (i, j = r, \theta, z)$	Poisson's ratios
$G_{r\theta}, G_{rz}, G_{\theta z}$	Shear moduli
$\sigma_r, \sigma_\theta, \sigma_z$	Normal stresses
$\tau_{r\theta}, \tau_{rz}, \tau_{\theta z}$	Shear stresses
$\epsilon_r, \epsilon_\theta, \epsilon_z$	Strain components
$\gamma_{r\theta}, \gamma_{rz}, \gamma_{\theta z}$	
u_r, u_θ, w	Displacements
R, Θ, Z	Components of body forces per unit of volume
$\alpha_{ij},\ (i, j = 1, 2, \ldots, 6)$	Elastic constants
φ	Angle of the fibre orientation
$\beta_{ij},\ (i, j = 1, 2, 4, 5, 6)$	Coefficients of the deformation
Φ_m, Ψ_m	Stress functions
nl	Number of layers
$m = 1, 2, \ldots, nl$	Layer number
$p_i,\ (i = 1, 2, \ldots, nl)$	Uniformly distributed load (pressure)
F	Axial force
C	Constant of integration
X_t, X_c, Y_t, Y_c, S	Material strengths
$\sigma_1, \sigma_2, \sigma_3, \tau_{12}$	Stress components in the material coordinate system
P_{cr}	Critical load

Nomenclature

2. Higher–order theory

x_1, x_2, z	Curvilinear orthogonal coordinates
k_{11}, k_{22}, k_{12}	Curvatures of a shell
$k = 1, \ldots, n$	Layer number
E_k, ν_k, G_k	Modulus of elasticity, Poisson's ratio and shear modulus in the k-th layer in the plane of isotropy
E'_k, ν'_k, G'_k	Modulus of elasticity, Poisson's ratio and shear modulus in the k-th layer in the transverse direction
$u_1(x), u_2(x), w(x)$	Displacements and deflection of the reference surface
$\sigma_{11}, \sigma_{12}, \sigma_{22}$	In–plane stresses
σ_{13}, σ_{23}	Transverse shear stresses
σ_{33}	Transverse normal stress
$E_{0k}, \nu_{0k},$ $A_{11k}, A_{12k}, A_{13k}, A_{33k}$	Stiffness parameters of the k-th layer
e_{ij}, e_{i3}, e_{33}	Components of the strain tensor
p^-, p^+	External load
ρ	Material density
$f_k(z), \overline{f}_k(z), f_{\rho k}(z)$	Distribution functions
$\varphi_{11}, \varphi_{12}, \varphi_{13}, \varphi_{14},$	Distribution functions
$B_{\xi i}, B, C_{\xi i}, C, C_i, D_{\xi i}, D$ $E_{\xi i}, E, E_i, K_{\xi i}, K_i, K, R_i$	Integrated stiffnesses of the shell
I, I_1, I_2, I_3	Integrated density characteristics

List of Figures

List of Tables

Contents

Chapter 1

Introduction

1.1 Overview

It has been obvious through history that the evolution of technology has been controlled by the materials available. It is increasingly so today when progress in aviation, space- and shipbuilding, and in many other mechanical and civil engineering applications crucially depends on the reasonable compromise between high strength, stiffness, corrosion resistance and other material properties on the one hand and reduced weight and cost on the other. The use of composite materials is therefore receiving ever wider attention in these and other areas of technology due to the inherent tailoring of the properties of these materials.

In modern times, it was the advent of composite materials such as concrete and then reinforced concrete that was a major breakthrough in construction. The development of new building materials is one of the most identifiable trends in today's technology. In a growing number of applications, composite materials have taken over the work that was previously done by metals and metal alloys.

It is a well–known fact that, given the same external conditions the response of composite material differs from that of a homogeneous isotropic material. Hence the large–scale introduction of composite materials and the wide diversity of them have created a need for further progress in such classical areas of mechanics as the theory of anisotropic and non-homogeneous deformable solids and the theory of optimisation.

The composite laminated plates and shells are widely used in modern engineering.

Their virtues related to environmental requirements may include structural integrity, weather–tightness, dimensional stability, sound or microwave absorbtions. From the point of view of structural performance, composite plates and shells offer an efficient structural design du to the great stiffness. This is achieved by different geometrical arrangements of the most highly stressed elements. Other advantages of composite structures may include high strength–to–weight ratios, increased fatigue life, endurance, low moisture permeability, electrical insulation, etc.

The investigation of anisotropic shells started in the 1920s and the first recorded paper on this subject was written by Shtayerman [82]. Most of the early literature on the subject was based on the classical laminated shell theory, incorporating the Kirchhoff–Love hypotheses for the entire shell Refs [2, 3, 6, 7, 17, 18, 50, 81]. It is well known that accurate prediction of the response and failure characteristics of shells made from modern advanced composites requires the use of refined higher–order theories which take into account transverse shear and normal deformations.

Several approaches have been proposed for the accurate stress–strain analysis of laminated plates and shells. All these approaches are based on principals which are similar to those used for homogeneous isotropic structures, namely, three–dimensional elasticity models, quasi–three–dimensional models, and various two–dimensional shear–deformable models. Exact and analytical solutions for cylindrical and spherical shells may be found in [16, 20, 25, 32, 57, 77, 84, 90, 91]. The possibility of using a three–dimensional theory is of limited use due to mathematical difficulties and the complexity of laminated systems. As a result, numerous higher–order theories of plates and shells have been formulated in recent years which approximate the three–dimensional solutions with reasonable accuracy. Nonclassical theories which include both transverse shear and normal deformations have been developed by Piskunov and Verijenko [62, 63, 64, 94, 95]. Several monographs have been written on different aspects of the higher–order theories [65, 66, 97, 9, 85, 11, 26, 46, 68, 79, 98].

There are many other topics in the field of composite materials, of much practical importance and in a very active state of development, that are not covered here but may be found discussed in detail in many other sources. Survey of different theoretical and computational models may be found in reviews [38, 19, 8, 59, 72].

1.2 Design of thick composite cylinders

A circular cylindrical shell is one of the most widespread models of thin and thick walled load bearing structures made of conventional or composite materials. Such shells are used as reservoirs, pressure vessels, chemical containers, pipes, aircraft and ship elements etc. The increased use of laminated cylindrical shells has led to a need for more accurate stress–strain analysis. The result is the creation of efficient applied theories and special methods for their analysis.

Along with stress–strain analysis the design optimisation of filament–wound pressure vessels is of considerable industrial interest. The most efficient configuration is the unidirectional composite. Obviously, the number of angles must be balanced between considerations of manufacturing and cost, and the requirement for stiffness and strength. In practice, many automated manufacturing processes lead to symmetric angle–ply composite laminates consisting of an even number of equivalent plies with material symmetry axes being alternately oriented at angles $+\varphi$ and $-\varphi$ to the cylindrical axis. It is this class of the structures is considered in the present study.

Because of the anisotropy in composites and the presence of curvature in shell structures, obtaining exact three–dimensional elasticity solutions for laminated cylinders is connected with the mathematical complexity. However, certain problems in which a three–dimensional approach can be used still exist. For example, load conditions specified as uniformly distributed load (pressure) considerably simplify the obtaining of the equations of exact three–dimensional theory. In the present study exact three–dimensional solution is obtained for these loading conditions. The governing differential equations are derived in a such form that to satisfy the stress functions and are given for layered cylindrical shell with open ends. This theory is extended to the laminated cylindrical shells with closed ends, that is to pressure vessels. It is evident that the stress analysis based on three–dimensional elasticity solution allow the burst pressure for both open ended (pipes) and closed ended (pressure vessels) cylinders to be predicted accurately. A reliable design method for thick cylinders should be also carried out with the use of an appropriate failure criterion which should include the contribution of all stress components to the failure.

In the present study the failure of the cylinders is predicted by using the quadratic three–dimensional interactive Tsai–Wu criterion. The use of maximum stress or strain criterion in three–dimensional stress or strain of anisotropic materials, in particular, gives rise to many vector equations. Moreover, neither of these two criterion includes the interactions among the stress or strain components, whereas the Tsai–Wu failure criterion yields only one scalar equation, and it also includes the interactions among the stress components.

1.3 High–order theory

Along with creating new types of laminated structures a demand arose for a rational use of traditional laminated systems. In evaluating the strength of these systems only the main carrying layer is taken into consideration while technological layers of monolithing, insulation are considered to be constructive. Taking into account their mutual work with carrying layers allows to reveal reserves of the strength of the entire system.

Improving theory and solution methods of laminated structures is a significant problem, the key feature of it lies in the fact that the hypotheses of straight inextensional normal or plane sections (for beams) and the classical theory based on these hypotheses is inapplicable. For solution of considered structures along with three–dimensional solutions, the use of approximate two–dimensional theories, which refine the classical theory by means of taking into account of deformations in the transverse direction, have received wide acceptance.

The refined theories are given the title refined or nonclassical. A number of different classifications can be made for these theories based on the nature of the approximations made in reducing the three–dimensional problem into–dimensional one. Global approximation approach is used in the present study where global through–the–thickness displacement approximation is introduced and laminated shell is replaced by an equivalent single layered anisotropic shell. Consequently, the order of the governing equations is independent of the number of layers. The theory is based on a nonlinear distribution of the displacements in the thickness direction. Such theories have come to be known as the higher–order theories.

Transverse shear and normal deformation higher–order theory for the solution of dynamic problems of laminated plates and shells is studied in Chapter 3. The theory developed is based on the kinematic hypotheses which are derived using iterative technique. Dynamic effects, such as forces of inertia and the direct influence of external loading on the components of stress and strain are included at the initial stage of derivation where kinematic hypotheses are formulated. The proposed theory and solution methods provide a basis for theoretical and applied studies in the field of dynamics and statics of the laminated shells, plates and their systems, particularly for investigation of dynamic processes related to the highest vibration forms and wave propagation, for optimal design etc.

Geometrically nonlinear higher-order theory of laminated plates and shells with shear and normal deformation is derived in Chapter 4. The theory takes into account both transverse shear and normal deformations. Modelling of the geometrical nonlinearity is especially important at high levels of loading, in case when the laminae exhibit significant differences in their elastic properties etc.

Both the above-mentioned theories are capable of treating plates and shells with an arbitrary number and sequence of layers which can differ significantly in their physical and mechanical properties including various loading and boundary conditions.

Chapter 2

Derivation of Three Dimensional Theory of Anisotropic Thick Composite Cylinders

2.1 Introduction

The objective of the present chapter is to derive a three-dimensional theory for anisotropic thick composite cylinders subjected to axisymmetrical load conditions. The analysis is based on the stress function approach. In order to illustrate the approach the brief overview of the general theory of elasticity of the cylindrical anisotropic body is given. The equations of equilibrium and the generalised Hooke's law are presented. It is shown how the transformation of the cylindrical coordinate system influences the value of the elastic constants. The governing differential equations of a body bounded by a cylindrical surface and possessing cylindrical anisotropy in which the stresses do not vary along the generator are presented. These equations are derived in a such form that to satisfy the stress functions and are given for a single layered cylindrical shell with open ends. This theory is extended to laminated pressure vessels, that is to cylindrical shells with closed ends.

First, the laminated cylindrical shell subjected to internal and external pressure as well as to axial forces is considered. The cylinder is constructed of filament-wound layers with a fibre orientation of $\pm\varphi°$. The layers are treated as anisotropic for winding angle $\varphi \neq 0, 90$ deg., otherwise they are orthotropic. Solution of the differential equations for this problem is given. The constants of integration and unknown interface normal forces are determined.

Second, using the same stress function approach the problem of continuously heterogeneous laminated cylinders, i.e. shells in which elastic constants are variable through the thickness of a layer, is investigated. Cylinder can be constructed of isotropic and/or orthotropic layers. This problem is also of interest because besides the internal and external applied pressure the pressure on the interfaces, as a result of shrink–fit, residual stresses, etc., can be taken into account. The problem is solved for an open–ended cylinder and then extended to the case of a closed–ended shell.

Finally, an optimum design approach is presented for laminated composite pressure vessels. The fibre orientations are taken as the design variables. The three–dimensional interactive Tsai–Wu criterion is employed. The multi–dimensional problem of maximisation of function is solved, where the function is the critical load for a given point within a thickness. With this aim in mind a search for the weakest point within the thickness is undertaken. In order to maximise the function three independent methods are employed, namely, the golden section method, iterative and gradient methods. Depending on particular problem the advantages and disadvantages of these methods are discussed.

2.2 Literature review

The increased use of laminated composite structures in many engineering applications has led to a need for more accurate stress–strain analysis. Thick–walled cylindrical shells with different layer properties are widely used in many branches of engineering. Along with stress–strain analysis the design optimisation of pressure vessels is of considerable industrial interest.

The problem of the stress distribution in anisotropic cylinders has been studied by several authors. The first comprehensive problem investigation of the stress distribution in a body with cylindrical anisotropy and study of the elastic equilibrium of a homogeneous cylinder with arbitrary anisotropy was made by Lekhnitskii in 1930's [42, 43]. Later on, the stress distribution in a thick–walled anisotropic tube under the influence of internal and external pressure was investigated by Mitinsky [53]. Chentsov [15] investigated certain aspects of the transformation of elastic constants of an orthotropic plate by rotation about an axis. This study has been serving as a basis for analysis of filament–wound composite structures till now. The most comprehensive study of a body possessing cylindrical anisotropy was done by Lekhnitskii in his book [44]. Solutions for the laminated composite curvilinear anisotropic ring, and for other specific problems can be found in another book written by Lekhnitskii [45]. More recently some analytical solutions were also offered for laminated composite cylinders. Elastoplastic analysis of cylindrically orthotropic composite thick–walled tube under uniform pressure is given by Zhou Ci-qing and Qiu Yi-yuan [99]. Investigation of the effects of a uniform temperature change on the stresses and deformations of reinforced composite tubes was done by Hyer *et al* [34, 35]. He also discussed the results of layer–by–layer analysis for cross–ply cylinders under external hydrostatic pressure [36]. The problem of thick–walled cylindrical shells buried underground was studied by Kuo-Yao and Bert [41]. Investigation of stress state in composite cylinders in which stresses and strains vary in the axial and radial directions due to imposed hygrothermal and mechanical loads was made by Kollár [40]. Chandrashekhara and Kumar [14] obtained exact solutions for a thick, transversely isotropic, simply supported circular cylindrical shell subjected to axisymmetrical load by using a displacement function approach. However, this solution in general is not applicable to the cross-ply laminated structures. Exact solutions for cross–ply laminated shells were obtained by Ren [76, 77]. Hung-Sying Jing and Kuan-Goang Tzeng [33] investigated the static response of the axisymmetrical problem of arbitrary laminated anisotropic cylindrical shells of finite length using three–dimensional elasticity equations. They used the differential equations with variable coefficients by choosing the solution composed of trigonometric functions along the axial direction. Calius and Springer [12] developed a comprehensive model of filament–wound thin cylinders made of a thermoset matrix composite. The model is applicable to cylinders for which the diameter is large compared to the wall thickness.

Optimisation problems of cylindrical shells were considered by several authors. Optimisation of the stress–strain state of a thick–walled pipe on the basis of Young's modulus of the material was made by Kalinnikov and Korlyakov [37]. Belingardi *et al* [5] studied optimisation of orthotropic multilayer cylinders and rotating disks using the maximum stress failure criterion. An analytical approach for predicting the probabilistic ultimate strength after initial failure of the carbon fiber helical–wound cylinders under internal pressure was made by Uemura and Fukunaga [89]. Fukunaga and Tsu-Wei Chou considered the use of simultaneous failure [21] and also optimum design of graphite/epoxy laminated composite pressure vessels under stiffness and strength constraints based upon the membrane theory [22]. The analysis based on the membrane theory of shells for laminated cylindrical pressure vessels under strength criterion was also done by Adali *et al* [1]. An efficient design method for thick composite cylinders was presented by Roy and Tsai [80]. The three-dimensional interactive Tsai–Wu failure criterion [87] forms the foundation of failure prediction in the most above mentioned works.

2.3 Basic equations

2.3.1 Stress–strain state of anisotropic cylindrical body

To provide some grounding in theory for an anisotropic body we should make some assumptions. The most important of them are

1. An elastic body is considered as a solid continuous medium. The state of stress at any given point is determined entirely by the components of stress in three mutually perpendicular planes which pass through the chosen point. In the following study the cylindrical coordinates will be used.
2. Relation between the strain components and projections of the displacements and their first derivatives with respect to the coordinates is a linear, that is we consider only small displacements and neglect the squares and products of the derivatives of the displacements.
3. There are linear relations between the stress and strain components, that is the generalised Hooke's law is valid for such a material. In addition, the coefficients of these relations can be constants (the case of uniform body) as well as variable functions of the coordinates, continuous or discontinuous (in case of nonuniform body).
4. We do not take into account initial stresses, i.e. the stresses which exist without the application of an external load.

We shall use a cylindrical coordinate system $r,\ \theta,\ z$. The components of stress acting on planes normal to the coordinate directions $r,\ \theta,\ z$ are denoted respectively by $\sigma_r,\ \tau_{\theta r},\ \tau_{zr};\ \sigma_\theta,\ \tau_{r\theta},\ \tau_{z\theta};\ \sigma_z,\ \tau_{rz},\ \tau_{\theta z}$ (see Figure 2.1). Here σ_i are normal stresses and $\tau_{ij} = \tau_{ji}$ are shear stresses. The corresponding strains are denoted as ϵ_i and γ_{ij}.

We denote the projections of the displacements at a given point on the coordinate axes $r,\ \theta,\ z$ as $u_r,\ u_\theta$ and w, respectively. The relations between the projections of displacements and strain components have the following form [44]

$$\epsilon_r = \frac{\partial u_r}{\partial r}; \qquad \epsilon_\theta = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}; \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{\theta z} = \frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}; \quad \gamma_{rz} = \frac{\partial w}{\partial r} + \frac{\partial u_r}{\partial z}; \quad \gamma_{r\theta} = \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \tag{2.1}$$

The components of stresses in a continuous body in equilibrium under the action of surface and body forces must satisfy three differential equations of equilibrium:

$$\begin{aligned}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r} + \Theta = 0 \\
\frac{\partial \tau_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} + Z = 0
\end{aligned} \tag{2.2}$$

where R, Θ and Z are the components of body forces per unit of volume in the coordinate directions r, θ, z.

2.3.2 The generalised Hooke's law

Taking the axis of anisotropy as the z-axis of the cylindrical coordinate system r, θ, z and arbitrarily directing the polar axis from which the angles θ are measured we obtain the equations of the generalised Hooke's law for the general case of cylindrical anisotropy as follows

$$\begin{aligned}
\epsilon_r &= \alpha_{11}\sigma_r + \alpha_{12}\sigma_\theta + \alpha_{13}\sigma_z + \alpha_{14}\tau_{\theta z} + \alpha_{15}\tau_{rz} + \alpha_{16}\tau_{r\theta} \\
\epsilon_\theta &= \alpha_{21}\sigma_r + \alpha_{22}\sigma_\theta + \dots\dots\dots\dots + \alpha_{26}\tau_{r\theta} \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
\gamma_{r\theta} &= \alpha_{61}\sigma_r + \alpha_{62}\sigma_\theta + \dots\dots\dots\dots + \alpha_{66}\tau_{r\theta}
\end{aligned} \tag{2.3}$$

where α_{ij} are elastic constants which are symmetrical with respect to their indices, i.e. $\alpha_{ij} = \alpha_{ji}$.

In this study we deal with cylindrical anisotropy. Let us assume that ξ is the axis of anisotropy which is associated with the body. It can pass either inside or outside the body. When in a homogeneous body the axis of anisotropy passes through the body, there must be some relations between coefficients α_{ij}. In fact, if the axis ξ coincides with the longitudinal axis z, there is not any difference on the axis z between radial r and tangential θ directions. Thus the following equalities take place

$$\alpha_{22} = \alpha_{11}, \quad \alpha_{23} = \alpha_{13}, \quad \alpha_{55} = \alpha_{44}, \quad \alpha_{26} = \alpha_{16}$$
$$\alpha_{35} = \alpha_{34}, \quad \alpha_{56} = \alpha_{46}, \quad \alpha_{24} = \alpha_{15}, \quad \alpha_{25} = \alpha_{14}$$

If a plane of elastic symmetry normal to ξ (or z) at each point, then

$$\alpha_{14} = \alpha_{24} = \alpha_{34} = \alpha_{46} = 0$$
$$\alpha_{15} = \alpha_{25} = \alpha_{35} = \alpha_{56} = 0$$

In case of an orthotropic body with cylindrical anisotropy when there two planes of elastic symmetry, radial and tangential, then in addition to above relations we have

$$\alpha_{16} = \alpha_{26} = \alpha_{36} = \alpha_{45} = 0$$

and non-zero coefficients are related with three equalities

$$\alpha_{22} = \alpha_{11}, \quad \alpha_{23} = \alpha_{13}, \quad \alpha_{55} = \alpha_{44}$$

The elastic coefficients α_{ij} form the so-called compliance matrix. Introducing technical constants (elastic characteristics), namely, the Young's moduli E_r, E_θ, E_z (for tension–compression); the Poisson's coefficients ν_{ij} $(i, j = r, \theta, z)$ (where eg. $\nu_{r\theta}$ characterises the compression in the direction θ for tension in the direction r) and the shear moduli $G_{r\theta}$, G_{rz} and $G_{\theta z}$ which characterise the variations of the angles in the directions r and θ, r and z and θ and z, the components of the compliance matrix (the case of an orthotropic body) may be defined as

$$\begin{aligned}
\alpha_{11} &= \frac{1}{E_r}; \quad \alpha_{12} = -\frac{\nu_{\theta r}}{E_\theta}; \quad \alpha_{13} = -\frac{\nu_{zr}}{E_z} \\
\alpha_{21} &= -\frac{\nu_{r\theta}}{E_r}; \quad \alpha_{22} = \frac{1}{E_\theta}; \quad \alpha_{23} = -\frac{\nu_{z\theta}}{E_z} \\
\alpha_{31} &= -\frac{\nu_{rz}}{E_r}; \quad \alpha_{32} = -\frac{\nu_{\theta z}}{E_\theta}; \quad \alpha_{33} = \frac{1}{E_z} \\
\alpha_{44} &= \frac{1}{G_{\theta z}}; \quad \alpha_{55} = \frac{1}{G_{rz}}; \quad \alpha_{66} = \frac{1}{G_{r\theta}}
\end{aligned} \tag{2.4}$$

2.3.3 The transformation of elastic constants under a transformation of the coordinate system

In case of an anisotropic body elastic constants depend on the direction of the axes of the coordinate system. If the direction of the axes varies, then the elastic constants vary. This is a very important feature of reinforced composite materials.

In an orthotropic cylindrical shell the new elastic constants α'_{ij} are defined by the formulas [44]

$$\alpha'_{11} = \frac{1}{E_r}$$

$$
\begin{aligned}
\alpha_{22}' &= \frac{\sin^4\varphi}{E_z} + \left(\frac{1}{G_{z\theta}} - \frac{2\nu_{z\theta}}{E_z}\right)\cos^2\varphi\sin^2\varphi + \frac{\cos^4\varphi}{E_\theta} \\
\alpha_{33}' &= \frac{\cos^4\varphi}{E_z} + \left(\frac{1}{G_{z\theta}} - \frac{2\nu_{z\theta}}{E_z}\right)\cos^2\varphi\sin^2\varphi + \frac{\sin^4\varphi}{E_\theta} \\
\alpha_{44}' &= \frac{\cos^2\varphi}{G_{\theta r}} + \frac{\sin^2\varphi}{G_{zr}} \\
\alpha_{55}' &= \frac{\sin^2\varphi}{G_{\theta r}} + \frac{\cos^2\varphi}{G_{zr}} \\
\alpha_{66}' &= 4\left(\frac{1}{E_z} + \frac{1}{E_\theta} + \frac{2\nu_{z\theta}}{E_z} - \frac{1}{G_{z\theta}}\right)\cos^2\varphi\sin^2\varphi + \frac{1}{G_{z\theta}} \\
\alpha_{12}' &= -\left(\frac{\nu_{\theta r}}{E_\theta}\cos^2\varphi + \frac{\nu_{zr}}{E_z}\sin^2\varphi\right) \\
\alpha_{13}' &= -\left(\frac{\nu_{\theta r}}{E_\theta}\sin^2\varphi + \frac{\nu_{zr}}{E_z}\cos^2\varphi\right) \\
\alpha_{16}' &= 2\left(\frac{\nu_{zr}}{E_z} - \frac{\nu_{\theta r}}{E_\theta}\right)\sin\varphi\cos\varphi \\
\alpha_{26}' &= \left[2\left(\frac{\cos^2\varphi}{E_\theta} - \frac{\sin^2\varphi}{E_z}\right) - \left(\frac{1}{G_{z\theta}} - \frac{2\nu_{z\theta}}{E_z}\right)(\cos^2\varphi - \sin^2\varphi)\right]\sin\varphi\cos\varphi \\
\alpha_{32}' &= \left(\frac{1}{E_z} + \frac{1}{E_\theta} + \frac{2\nu_{z\theta}}{E_z} - \frac{1}{G_{z\theta}}\right)\cos^2\varphi\sin^2\varphi - \frac{\nu_{z\theta}}{E_z} \\
\alpha_{36}' &= \left[2\left(\frac{\sin^2\varphi}{E_\theta} - \frac{\cos^2\varphi}{E_z}\right) + \left(\frac{1}{G_{z\theta}} - \frac{2\nu_{z\theta}}{E_z}\right)(\cos^2\varphi - \sin^2\varphi)\right]\sin\varphi\cos\varphi \\
\alpha_{45}' &= \left(\frac{1}{G_{\theta r}} - \frac{1}{G_{zr}}\right)\sin\varphi\cos\varphi
\end{aligned}
\tag{2.5}
$$

We shall indicate the most important invariants of this transformation, that is, the quantities which remain unchanged under rotation of the axes.

$$
\begin{aligned}
I_1 &= \alpha_{11}' + \alpha_{22}' + 2\alpha_{12}' = \frac{1}{E_z} + \frac{1}{E_\theta} - \frac{2\nu_{z\theta}}{E_z} \\
I_2 &= \alpha_{66}' - 4\alpha_{12}' = \frac{1}{G_{z\theta}} + \frac{4\nu_{z\theta}}{E_z} \\
I_3 &= \alpha_{44}' + \alpha_{55}' = \frac{1}{G_{zr}} + \frac{1}{G_{\theta r}} \\
I_4 &= \alpha_{13}' + \alpha_{23}' = -\left(\frac{\nu_{zr}}{E_z} + \frac{\nu_{\theta r}}{E_\theta}\right) = -\frac{\nu_{rz} + \nu_{r\theta}}{E_r}
\end{aligned}
\tag{2.6}
$$

When studying the stress-strain state of an anisotropic body bounded by a cylindrical surface, it is convenient to use the coefficients of deformation in the following form

$$
\beta_{ij} = \alpha_{ij} - \frac{\alpha_{i3}\alpha_{j3}}{\alpha_{33}}, \qquad (i, j = 1, 2, 4, 5, 6) \tag{2.7}
$$

Expressions for transformation of the coefficients β_{ij} may be obtained from the general expression (2.5) if we cancel all constants which contain the index 3, and instead of the remaining coefficients of deformation, we substitute the corresponding β_{ij}. The invariants of this transformation are given by the following expressions

$$
\begin{aligned}
I_1 &= \beta'_{11} + \beta'_{22} + 2\beta'_{12} = \beta_{11} + \beta_{22} + 2\beta_{12} \\
I_2 &= \beta'_{66} - 4\beta'_{12} = \beta_{66} - 4\beta_{12} \\
I_3 &= \beta'_{44} + \beta'_{55} = \beta_{44} + \beta_{55}
\end{aligned}
\tag{2.8}
$$

2.4 Method of solution

Before proceeding to particular problems let us examine our task in a general way. The structure under consideration is a cylindrical shell of finite length made from an anisotropic material.

Let us assume that the axis of anisotropy coincides with the axis of symmetry Oz of the cylinder and the stresses act on the planes normal to the generator and do not vary along the generator.

Let U, V and W be the functions which represent the displacement accompanied by elastic deformations, then

$$
\begin{aligned}
u_r &= U(r,\theta) + u_0 \cos\theta + v_0 \sin\theta \\
u_\theta &= V(r,\theta) - u_0 \sin\theta + v_0 \cos\theta + w_1 r \\
w &= W(r,\theta) + w_0
\end{aligned}
\tag{2.9}
$$

where u_0, v_0 and w_1 are the rigid displacements (without elastic deformations) in the plane of a given section, w_0 is the rigid shift in longitudinal direction.

The function U, V and W can be written in the following form

$$
\begin{aligned}
\frac{\partial U}{\partial r} &= \beta_{11}\sigma_r + \beta_{12}\sigma_\theta + \ldots + \beta_{16}\tau_{r\theta} \\
\frac{1}{r}\frac{\partial V}{\partial \theta} + \frac{U}{r} &= \beta_{12}\sigma_r + \beta_{22}\sigma_\theta + \ldots + \beta_{26}\tau_{r\theta} \\
\frac{1}{r}\frac{\partial W}{\partial \theta} &= \beta_{14}\sigma_r + \beta_{24}\sigma_\theta + \ldots + \beta_{46}\tau_{r\theta} \\
\frac{\partial W}{\partial r} &= \beta_{15}\sigma_r + \beta_{25}\sigma_\theta + \ldots + \beta_{56}\tau_{r\theta} \\
\frac{1}{r}\frac{\partial U}{\partial \theta} + \frac{\partial V}{\partial r} - \frac{V}{r} &= \beta_{16}\sigma_r + \beta_{26}\sigma_\theta + \ldots + \beta_{66}\tau_{r\theta}
\end{aligned}
\tag{2.10}
$$

The stresses can be expressed in terms of stress functions Φ and Ψ as [44]

$$\begin{aligned}
\sigma_r &= \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} + \overline{U}, \qquad \sigma_\theta = \frac{\partial^2 \Phi}{\partial r^2} + \overline{U} \\
\tau_{r\theta} &= \frac{\partial^2}{\partial r \partial \theta}\left(\frac{\Phi}{r}\right), \qquad \tau_{rz} = \frac{1}{r}\frac{\partial \Psi}{\partial \theta}, \qquad \tau_{\theta z} = -\frac{\partial \Psi}{\partial r}
\end{aligned} \tag{2.11}$$

where $\overline{U}$ is the potential of the body forces. The normal longitudinal stress can be expressed in terms of the other stresses as

$$\sigma_z = -\frac{1}{\alpha_{33}}(\alpha_{13}\sigma_r + \alpha_{23}\sigma_\theta + \alpha_{34}\tau_{\theta z} + \alpha_{35}\tau_{rz} + a_{36}\tau_{r\theta}) \tag{2.12}$$

Hereafter for simplicity we shall not use notation *"prime"* for the coefficients of the compliance matrix α'_{ij} or elastic constants β'_{ij}. It should be also noted that equation (2.12) is correct only for open–ended cylinders. Otherwise the additional constant must be added to it. For the special cases the evaluation of the unknown constant will be given below.

By eliminating U, V and W from equation (2.10) by means of differentiation, we obtain a system of two equations satisfied by the stress functions Φ and Ψ:

$$\begin{aligned}
L'_4\Phi + L'_3\Psi &= -(\beta_{12}+\beta_{22})\frac{\partial^2\overline{U}}{\partial r^2} + (\beta_{16}+\beta_{26})\frac{1}{r}\frac{\partial^2\overline{U}}{\partial r\partial\theta} - (\beta_{11}+\beta_{12})\frac{1}{r^2}\frac{\partial^2\overline{U}}{\partial\theta^2} \\
&\quad +(\beta_{11}-2\beta_{22}-\beta_{12})\frac{1}{r}\frac{\partial\overline{U}}{\partial r} + (\beta_{16}+\beta_{26})\frac{1}{r^2}\frac{\partial\overline{U}}{\partial\theta} \\
L''_3\Phi + L'_2\Psi &= (\beta_{14}+\beta_{24})\left(\frac{\partial\overline{U}}{\partial r} - \frac{\overline{U}}{r}\right) - (\beta_{15}+\beta_{25})\frac{1}{r}\frac{\partial\overline{U}}{\partial\theta}
\end{aligned} \tag{2.13}$$

where L'_4, L'_3, L''_3 and L'_2 are differential operators which are defined as follows

$$\begin{aligned}
L'_2 &= \beta_{44}\frac{\partial^2}{\partial r^2} - 2\beta_{45}\frac{1}{r}\frac{\partial^2}{\partial r\partial\theta} + \beta_{55}\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \beta_{44}\frac{1}{r}\frac{\partial}{\partial r} \\
L'_3 &= -\beta_{24}\frac{\partial^3}{\partial r^3} + (\beta_{25}+\beta_{46})\frac{1}{r}\frac{\partial^3}{\partial r^2\partial\theta} - (\beta_{14}+\beta_{56})\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2} \\
&\quad +\beta_{15}\frac{1}{r^3}\frac{\partial^3}{\partial\theta^3} + (\beta_{14}-2\beta_{24})\frac{1}{r}\frac{\partial^2}{\partial r^2} \\
&\quad +(\beta_{46}-\beta_{15})\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta} + \beta_{15}\frac{1}{r^3}\frac{\partial}{\partial\theta} \\
L''_3 &= -\beta_{24}\frac{\partial^3}{\partial r^3} + (\beta_{25}+\beta_{46})\frac{1}{r}\frac{\partial^3}{\partial r^2\partial\theta} - (\beta_{14}+\beta_{56})\frac{1}{r^2}\frac{\partial^3}{\partial r\partial\theta^2} \\
&\quad +\beta_{15}\frac{1}{r^3}\frac{\partial^3}{\partial\theta^3} - (\beta_{14}+\beta_{24})\frac{1}{r}\frac{\partial^2}{\partial r^2} + (\beta_{15}-\beta_{46})\frac{1}{r^2}\frac{\partial^2}{\partial r\partial\theta} \\
&\quad +(\beta_{14}+\beta_{56})\frac{1}{r^3}\frac{\partial^2}{\partial\theta^2} + \beta_{46}\frac{1}{r^3}\frac{\partial}{\partial\theta}
\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
L_4^{'} \;=\; & \beta_{22}\frac{\partial^4}{\partial r^4} - 2\beta_{26}\frac{1}{r}\frac{\partial^4}{\partial r^3\partial\theta} + (2\beta_{12}+\beta_{66})\frac{1}{r^2}\frac{\partial^4}{\partial r^2\partial\theta^2} - 2\beta_{16}\frac{1}{r^3}\frac{\partial^4}{\partial r\partial\theta^3} \\
& +\beta_{11}\frac{1}{r^4}\frac{\partial^4}{\partial\theta^4} + 2\beta_{22}\frac{1}{r}\frac{\partial^3}{\partial r^3} - (2\beta_{12}+\beta_{66})\frac{1}{r^3}\frac{\partial^3}{\partial r\partial\theta^2} + 2\beta_{16}\frac{1}{r^4}\frac{\partial^3}{\partial\theta^3} \\
& -\beta_{11}\frac{1}{r^2}\frac{\partial^2}{\partial r^2} - 2(\beta_{16}+\beta_{26})\frac{1}{r^3}\frac{\partial^2}{\partial r\partial\theta} + (2\beta_{11}+2\beta_{12}+\beta_{66})\frac{1}{r^4}\frac{\partial^2}{\partial\theta^2} \\
& +\beta_{11}\frac{1}{r^3}\frac{\partial}{\partial r} + 2(\beta_{16}+\beta_{26})\frac{1}{r^4}\frac{\partial}{\partial\theta}
\end{aligned}
$$

2.5 Analysis of laminated thick composite cylinder under axisymmetrical loading

2.5.1 Problem formulation

The structure under consideration is laminated close–ended cylinder of finite length made from an anisotropic material (see Figure 2.2). The cylinder is constructed of filament–wound layers with a fibre orientation of ($\pm\varphi^\circ$). There are no restrictions on the number of layers or their sequences. The layers can be made from different materials and have different thicknesses. The cylinder is subjected to axisymmetrical internal and external pressure as well as axial force. Let as assume that the axis of anisotropy coincides with the geometric axis of the shell. We also assume that the stresses which act on the end surfaces reduce to forces which are directed along the axis and to twisting moments. The equations of the generalised Hooke's law may be written as

$$
\begin{aligned}
\epsilon_r^{(m)} &= \alpha_{11}^{(m)}\sigma_r^{(m)} + \alpha_{12}^{(m)}\sigma_\theta^{(m)} + \alpha_{13}^{(m)}\sigma_z^{(m)} + \alpha_{14}^{(m)}\tau_{\theta z}^{(m)} \\
\epsilon_\theta^{(m)} &= \alpha_{21}^{(m)}\sigma_r^{(m)} + \alpha_{22}^{(m)}\sigma_\theta^{(m)} + \alpha_{23}^{(m)}\sigma_z^{(m)} + \alpha_{24}^{(m)}\tau_{\theta z}^{(m)} \\
\epsilon_z^{(m)} &= \alpha_{31}^{(m)}\sigma_r^{(m)} + \alpha_{32}^{(m)}\sigma_\theta^{(m)} + \alpha_{33}^{(m)}\sigma_z^{(m)} + \alpha_{34}^{(m)}\tau_{\theta z}^{(m)} \\
\gamma_{\theta z}^{(m)} &= \alpha_{41}^{(m)}\sigma_r^{(m)} + \alpha_{42}^{(m)}\sigma_\theta^{(m)} + \alpha_{43}^{(m)}\sigma_z^{(m)} + \alpha_{44}^{(m)}\tau_{\theta z}^{(m)}
\end{aligned}
\tag{2.15}
$$

where components of compliance matrix are given by (2.4).

It is obvious that the distribution of the external stresses will be identical in all cross sections and will depend only on the distance r from the axis. Therefore the stresses can be expressed in terms of stress functions $\Phi_m = \Phi_m(r)$, $\Psi_m = \Psi_m(r)$ as

$$
\sigma_r^{(m)} = \frac{1}{r}\frac{d\Phi_m}{dr}; \quad \sigma_\theta = \frac{d^2\Phi_m}{dr^2}; \quad \tau_{\theta z}^{(m)} = -\frac{d\Psi_m}{dr} \tag{2.16}
$$

and longitudinal stress

$$\sigma_z^{(m)} = C - \frac{1}{\alpha_{33}^{(m)}}(\alpha_{13}^{(m)}\sigma_r^{(m)} + \alpha_{23}^{(m)}\sigma_\theta^{(m)} + \alpha_{34}^{(m)}\tau_{\theta z}^{(m)}) \tag{2.17}$$

Moreover, due to symmetry

$$\tau_{rz}^{(m)} = \tau_{r\theta}^{(m)} = 0 \tag{2.18}$$

where index m denotes the *m-th* layer and $m = 1, 2, \ldots, nl$ with nl denoting the total number of layers.

The system (2.13) takes the following form

$$\begin{aligned}
&\beta_{22}^{(m)}\left(\frac{d^4\Phi_m}{dr^4} + \frac{2}{r}\frac{d^3\Phi_m}{dr^3}\right) + \beta_{11}^{(m)}\left(-\frac{1}{r^2}\frac{d^2\Phi_m}{dr^2} + \frac{1}{r^3}\frac{d\Phi_m}{dr}\right) \\
&-\beta_{24}^{(m)}\frac{d^3\Psi_m}{dr^3} + (\beta_{14}^{(m)} - 2\beta_{24}^{(m)})\frac{1}{r}\frac{d^2\Psi_m}{dr^2} = 0 \\
&-\beta_{24}^{(m)}\frac{d^3\Phi_m}{dr^3} - (\beta_{14}^{(m)} - \beta_{24}^{(m)})\frac{1}{r}\frac{d^2\Phi_m}{dr^2} \\
&+\beta_{44}^{(m)}\left(\frac{d^2\Psi_m}{dr^2} + \frac{1}{r}\frac{d\Psi_m}{dr}\right) - \frac{1}{r}Ca_{34}^{(m)} = 0
\end{aligned} \tag{2.19}$$

wherein $\beta_{ij}^{(m)}$ are aforementioned elastic constants given by

$$\beta_{ij}^{(m)} = \alpha_{ij}^{(m)} - \frac{\alpha_{i3}^{(m)}\alpha_{j3}^{(m)}}{\alpha_{33}^{(m)}}, \qquad i, j = 1, 2, 4 \tag{2.20}$$

2.5.2 Computation of stresses

The boundary conditions on the internal ($r = a_0$) and external ($r = a_{nl}$) surfaces specified as

$$\sigma_r^{(1)}(a_0) = -p_0; \qquad \sigma_r^{(nl)}(a_{nl}) = -p_{nl} \tag{2.21}$$

At the contact surfaces of adjacent layers we have the following conditions

$$\sigma_r^{(m)} = \sigma_r^{(m+1)}, \quad u_r^{(m)} = u_r^{(m+1)}, \quad u_\theta^{(m)} = u_\theta^{(m+1)} \tag{2.22}$$

The equilibrium of forces on the end surfaces gives

$$2\pi\sum_{m=1}^{nl}\int_{a_{m-1}}^{a_m}\sigma_z^{(m)} r\,dr = (p_0 - p_{nl})a_0^2 + F \tag{2.23}$$

where F is the applied axial force.

With regards to condition (2.23) and taking into account the assumptions about physical and geometrical properties assumed above, the general solution of the system (2.19) has the following form

$$
\begin{aligned}
\Phi_m &= C\zeta_1^{(m)}\frac{r^2}{2} + \frac{C_1}{1+k_m}r^{1+k_m} + \frac{C_2}{1-k_m}r^{1-k_m} \\
\Psi_m &= Cr\left(\frac{\alpha_{34}^{(m)}}{\beta_{44}^{(m)}} + \zeta_1^{(m)}g_1^{(m)}\right) + C_1\frac{1}{k_m}g_k^{(m)}r^{k_m} - C_2\frac{1}{k_m}g_{-k}^{(m)}r^{-k_m}
\end{aligned}
\tag{2.24}
$$

where $\zeta_1^{(m)}$, k_m, $g_1^{(m)}$, $g_k^{(m)}$ and $g_{-k}^{(m)}$ are β-dependent coefficients given by

$$
\zeta_1^{(m)} = \frac{(\alpha_{13}^{(m)} - \alpha_{23}^{(m)})\beta_{44}^{(m)} - \alpha_{34}^{(m)}(\beta_{14}^{(m)} - \beta_{24}^{(m)})}{\beta_{22}^{(m)}\beta_{44}^{(m)} - \beta_{24}^{(m)2} - (\beta_{11}^{(m)}\beta_{44}^{(m)} - \beta_{14}^{(m)2})}
$$

$$
k_m = \sqrt{\frac{\beta_{11}^{(m)}\beta_{44}^{(m)} - \beta_{14}^{(m)2}}{\beta_{22}^{(m)}\beta_{44}^{(m)} - \beta_{24}^{(m)2}}}, \quad g_1^{(m)} = \frac{\beta_{14}^{(m)} + \beta_{24}^{(m)}}{\beta_{44}^{(m)}}
$$

$$
g_k^{(m)} = \frac{\beta_{14}^{(m)} + k\beta_{24}^{(m)}}{\beta_{44}^{(m)}}, \quad g_{-k} = \frac{\beta_{14}^{(m)} - k_m\beta_{24}^{(m)}}{\beta_{44}^{(m)}}
$$

The stresses $\sigma_r^{(m)}$, $\sigma_\theta^{(m)}$ and $\tau_{\theta z}^{(m)}$ can be calculated from equations (2.16), viz.

$$
\begin{aligned}
\sigma_r^{(m)} &= C\zeta_1^{(m)} + C_1 r^{k_m-1} + C_2 r^{-k_m-1} \\
\sigma_\theta^{(m)} &= C\zeta_1^{(m)} + C_1 k_m r^{k_m-1} - C_2 k_m r^{-k_m-1} \\
\tau_{\theta z}^{(m)} &= -C\left(\frac{\alpha_{34}^{(m)}}{\beta_{44}^{(m)}} - \zeta_1^{(m)}g_1^{(m)}\right) - C_1 g_k^{(m)} r^{k_m-1} - C_2 g_{-k}^{(m)} r^{-k_m-1}
\end{aligned}
\tag{2.25}
$$

By satisfying the boundary conditions (2.21) and (2.22) the constants C_1 and C_2 can be expressed in terms of the constant C. Introducing notations

$$
c_m = \frac{a_{m-1}}{a_m}; \qquad \rho_m = \frac{r}{a_m} \qquad (c_m < 1,\ c_m \le \rho_m \le 1)
$$

the final expressions for the stresses can be written as

$$
\begin{aligned}
\sigma_r^{(m)} &= \frac{p_{m-1}c_m^{k+1} - p_m}{1 - c_m^{2k_m}}\rho_m^{k_m-1} + \frac{p_m c_m^{k_m-1} - p_{m-1}}{1 - c_m^{2k_m}}c_m^{k_m+1}\rho_m^{-k_m-1} \\
&+C\zeta_1^{(m)}\left(1 - \frac{1 - c_m^{k_m+1}}{1 - c_m^{2k(m)}}\rho_m^{k_m-1} - \frac{1 - c_m^{k_m-1}}{1 - c^{2k_m}}c_m^{k_m+1}\rho_m^{-k_m-1}\right) \\
\sigma_\theta^{(m)} &= \frac{p_{m-1}c_m^{k_m+1} - p_m}{1 - c_m^{2k_m}}k_m\rho_m^{k_m-1} - \frac{p_m c_m^{k_m-1} - p_{m-1}}{1 - c_m^{2k_m}}k_m c_m^{k_m+1}\rho_m^{-k_m-1} \\
&+C\zeta_1^{(m)}\left(1 - \frac{1 - c_m^{k_m+1}}{1 - c_m^{2k_m}}k_m\rho_m^{k_m-1} + \frac{1 - c_m^{k_m-1}}{1 - c_m^{2k_m}}k_m c_m^{k_m+1}\rho_m^{-k_m-1}\right)
\end{aligned}
\tag{2.26}
$$

$$\tau_{\theta z}^{(m)} = -\frac{p_{m-1}c_m^{k_m+1} - p_m}{1-c_m^{2k_m}} g_k^{(m)} \rho_m^{k_m-1} - \frac{p_m c_m^{k_m-1} - p_{m-1}}{1-c_m^{2k_m}} g_{-k}^{(m)} c_m^{k_m+1} \rho_m^{-k_m-1}$$
$$+C\left[-\zeta_2^{(m)} + \zeta_1^{(m)}\left(1 - \frac{1-c_m^{k_m+1}}{1-c_m^{2k_m}} g_k^{(m)} \rho_m^{k_m-1} + \frac{1-c_m^{k_m-1}}{1-c_m^{2k_m}} g_{-k}^{(m)} c_m^{k_m+1} \rho_m^{-k_m-1}\right)\right]$$

where

$$\zeta_2^{(m)} = \frac{(\alpha_{13}^{(m)} - \alpha_{23}^{(m)})(\beta_{14}^{(m)} + \beta_{24}^{(m)}) - \alpha_{34}^{(m)}(\beta_{11}^{(m)} - \beta_{22}^{(m)})}{\beta_{22}^{(m)}\beta_{44}^{(m)} - \beta_{24}^{(m)2} - (\beta_{11}^{(m)}\beta_{44}^{(m)} - \beta_{14}^{(m)2})}$$

In equation (2.26) we denoted p_{m-1} and p_m the normal forces acting on the internal and external surfaces of the *m-th* layer. The remaining unknown forces and constant C are determined from the boundary conditions (2.22) and (2.23) and are derived in the next section.

2.5.3 Evaluation of interface forces and constant of integration

First we derive the system of equations which will later allow calculation of unknown interface forces $p_1,\ p_2, \ldots, p_{nl-1}$. This system of equations is to be derived by satisfying the contact conditions of the interfaces. The equality of radial stresses on the layer interfaces ($\sigma_r^{(m)} = \sigma_r^{(m+1)}$) is seems to be simplest for this purpose. But, unfortunately, the use of this condition does not lead to the wanted result, since the left side of the equation $\sigma_r^{(m)} - \sigma_r^{(m+1)} = 0$ becomes already to be equal to zero at the stage of obtaining of the symbolic expression. The best plan is to use condition

$$\epsilon_\theta^{(m)} = \epsilon_\theta^{(m+1)} \qquad \text{at} \quad r = a_m \tag{2.27}$$

In effect, it is necessary to solve the following system of $nl-1$ equations for p_m

$$\epsilon_\theta^{(m)} - \epsilon_\theta^{(m+1)} = 0 \qquad m = 1, 2, \ldots, nl-1 \tag{2.28}$$

where

$$\epsilon_\theta^{(m)} = \alpha_{12}^{(m)}\sigma_r^{(m)} + \alpha_{22}^{(m)}\sigma_\theta^{(m)} + \alpha_{23}^{(m)}\sigma_z^{(m)} + \alpha_{24}^{(m)}\tau_{\theta z}^{(m)} \tag{2.29}$$

Let us introduce the following notations

$$\begin{aligned}
f_1^{(m)} &= c_m^{k_m+1} & f_2^{(m)} &= c_m^{k_m-1} \\
f_3^{(m)} &= \rho_m^{k_m-1} & f_4^{(m)} &= c_m^{k_m+1}\rho_m^{-k_m-1} \\
f_5^{(m)} &= k_m\rho_m^{k_m-1} & f_6^{(m)} &= k_m c_m^{k_m+1}\rho_m^{-k_m-1} \\
f_7^{(m)} &= g_k^{(m)}\rho_m^{k_m-1} & f_8^{(m)} &= g_{-k}^{(m)} c_m^{k_m+1}\rho_m^{-k_m-1}
\end{aligned}$$

$$
\begin{aligned}
W_1^{(m)} &= 1-\frac{1-c_m^{k_m+1}}{1-c_m^{2k(m)}}\rho_m^{k_m-1}-\frac{1-c_m^{k_m-1}}{1-c^{2k_m}}c_m^{k_m+1}\rho_m^{-k_m-1}\\
W_2^{(m)} &= 1-\frac{1-c_m^{k_m+1}}{1-c_m^{2k_m}}k_m\rho_m^{k_m-1}+\frac{1-c_m^{k_m-1}}{1-c_m^{2k_m}}k_m c_m^{k_m+1}\rho_m^{-k_m-1}\\
W_3^{(m)} &= -\zeta_2^{(m)}+\zeta_1^{(m)}\left(1-\frac{1-c_m^{k_m+1}}{1-c_m^{2k_m}}g_k^{(m)}\rho_m^{k_m-1}+\frac{1-c_m^{k_m-1}}{1-c_m^{2k_m}}g_{-k}^{(m)}c_m^{k_m+1}\rho_m^{-k_m-1}\right)
\end{aligned}
$$

Then the expressions for stresses (2.26) can be rewritten as

$$
\begin{aligned}
\sigma_r^{(m)} &= \frac{p_{m-1}f_1^{(m)}-p_m}{1-c_m^{2k_m}}f_3^{(m)}+\frac{p_m f_2^{(m)}-p_{m-1}}{1-c_m^{2k_m}}f_4^{(m)}+C\zeta_1^{(m)}W_1^{(m)}\\
\sigma_\theta^{(m)} &= \frac{p_{m-1}f_1^{(m)}-p_m}{1-c_m^{2k_m}}f_5^{(m)}-\frac{p_m f_2^{(m)}-p_{m-1}}{1-c_m^{2k_m}}f_6^{(m)}+C\zeta_1^{(m)}W_2^{(m)} \qquad (2.30)\\
\tau_{\theta z}^{(m)} &= -\frac{p_{m-1}f_1^{(m)}-p_m}{1-c_m^{2k_m}}f_7^{(m)}-\frac{p_m f_2^{(m)}-p_{m-1}}{1-c_m^{2k_m}}f_8^{(m)}+CW_3^{(m)}
\end{aligned}
$$

Substituting expressions (2.30) in equation (2.29) and then expression for the strains ϵ_θ in (2.28) the set of equations for unknown forces can be obtained as

$$
C(\Delta_1^{(m)}-\Delta_1^{(m+1)})+p_{m-1}\Delta_2^{(m)}+p_m(\Delta_3^{(m)}-\Delta_2^{(m+1)})+p_{m+1}\Delta_3^{(m+1)}=0 \qquad (2.31)
$$

where

$$
\begin{aligned}
\Delta_1^{(m)} &= \alpha_{23}^{(m)}+\zeta_1^{(m)}(W_1^{(m)}\beta_{12}^{(m)}+W_2^{(m)}\beta_{22}^{(m)})+W_3^{(m)}\beta_{24}^{(m)}\\
\Delta_2^{(m)} &= \frac{f_1^{(m)}f_3^{(m)}-f_4^{(m)}}{1-c_m^{2k_m}}\beta_{12}^{(m)}+\frac{f_1^{(m)}f_5^{(m)}+f_6^{(m)}}{1-c_m^{2k_m}}\beta_{22}^{(m)}\\
&\quad -\frac{f_1^{(m)}+f_7^{(m)}-f_8^{(m)}}{1-c_m^{2k_m}}\beta_{24}^{(m)} \qquad (2.32)\\
\Delta_3^{(m)} &= \frac{f_2^{(m)}f_4^{(m)}-f_3^{(m)}}{1-c_m^{2k_m}}\beta_{12}^{(m)}-\frac{f_2^{(m)}f_6^{(m)}+f_5^{(m)}}{1-c_m^{2k_m}}\beta_{22}^{(m)}\\
&\quad -\frac{f_2^{(m)}f_8^{(m)}-f_7^{(m)}}{1-c_m^{2k_m}}\beta_{24}^{(m)}
\end{aligned}
$$

The total number of unknown terms in the system of equations (2.31) is equal to the number of the layers nl. Therefore, in order to solve this system we need to add another equation, namely the equation (2.23). Having nl equations we nevertheless cannot solve this system as the system of linear algebraic equations because of the last equation which contains the piecewise integral. Therefore we shall consider both parts separately. First we evaluate the expressions for the unknown forces, wherein the constant of integration remains unknown, and then we substitute this expressions in equation (2.23).

Let us solve the system (2.31) for unknown interface forces in the following form

$$p_m = N^{(m)} + N_c^{(m)} C, \qquad m = 1, 2, \ldots, nl-1 \tag{2.33}$$

where $N^{(m)}$, $N_c^{(m)}$ are real numbers. To calculate them we shall rewrite the equations (2.31) in the matrix form

$$\{B\}\{P\} + \{S\} = 0 \tag{2.34}$$

or in expanded form

$$\{B\} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 & \vdots & \vdots \\ b_{21} & b_{22} & b_{23} & 0 & 0 & 0 & \vdots & \vdots \\ 0 & b_{32} & b_{33} & b_{34} & 0 & 0 & \vdots & \vdots \\ 0 & 0 & b_{43} & b_{44} & b_{45} & 0 & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \vdots & \vdots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & b_{nl-1,nl-2} & b_{nl-1,nl-1} \end{pmatrix} \tag{2.35}$$

$$\{P\} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ \vdots \\ p_{nl-1} \end{pmatrix} \qquad \{S\} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \\ \vdots \\ s_{nl-1} \end{pmatrix} \tag{2.36}$$

where b_{ij} and s_m are defined from equation (2.31). It should be noted that in general $b_{ij} \neq b_{ji}$ and to distinguish other terms in the system of equations we shall use additional indices l and r.

$$[l,\ r] \longrightarrow [m, m+1] \tag{2.37}$$

then coefficients b_{ij} and s_m can be defined as

$$\begin{aligned} b_{11} &= \Delta_{3,l}^{(1)} - \Delta_{2,r}^{(2)},\ b_{12} = \Delta_{3,r}^{(2)},\ b_{21} = \Delta_{2,l}^{(2)}, \ldots, b_{nl-1,nl-1} = \Delta_{3,l}^{(nl-1)} - \Delta_{2,r}^{(nl)} \\ s_1 &= C(\Delta_{1,l}^{(1)} - \Delta_{1,r}^{(2)}) + p_0 \Delta_{2,l}^{(1)},\ s_2 = C(\Delta_{1,l}^{(2)} - \Delta_{1,r}^{(3)}), \ldots, \\ & s_{nl-1} = C(\Delta_{1,l}^{(nl-1)} - \Delta_{1,r}^{(nl)}) + p_{nl} \Delta_{3,r}^{(nl)} \end{aligned} \tag{2.38}$$

Solution for coefficients $N^{(m)}$ and $N_c^{(m)}$ may be given in the following form

$$N^{(m)} = \frac{D_n^{(m)}}{D}; \qquad N_c^{(m)} = \frac{D_c^{(m)}}{D} \tag{2.39}$$

To illustrate, solution for a five layered cylinder gives

$$\begin{aligned}
D &= b_{12}b_{21}b_{34}b_{43} - b_{11}b_{22}b_{34}b_{43} - b_{11}b_{23}b_{32}b_{44} \\
&\quad -b_{12}b_{21}b_{33}b_{44} + b_{11}b_{22}b_{33}b_{44} \\
D_n^{(1)} &= p_0\Delta_{2,l}^{(1)}(b_{22}b_{34}b_{43} + b_{23}b_{32}b_{44} - b_{22}b_{33}b_{44}) + p_5\Delta_{3,r}^{(5)}b_{12}b_{23}b_{34} \\
D_n^{(2)} &= p_0\Delta_{2,l}^{(1)}(b_{21}b_{33}b_{44} - b_{21}b_{34}b_{43}) - p_5\Delta_{3,r}^{(5)}b_{11}b_{23}b_{34} \\
D_n^{(3)} &= -p_0\Delta_{2,l}^{(1)}b_{21}b_{32}b_{44} + p_5\Delta_{3,r}^{(5)}(b_{11}b_{22}b_{34} - b_{12}b_{21}b_{34}) \\
D_n^{(4)} &= p_0\Delta_{2,l}^{(1)}b_{21}b_{32}b_{43} + p_5\Delta_{3,r}^{(5)}(b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33} - b_{11}b_{22}b_{33}) \\
D_c^{(1)} &= (\Delta_{1,l}^{(1)} - \Delta_{1,r}^{(2)})(b_{22}b_{34}b_{43} + b_{23}b_{32}b_{44} - b_{22}b_{33}b_{44}) \\
&\quad -(\Delta_{1,l}^{(2)} - \Delta_{1,r}^{(3)})(b_{12}b_{34}b_{43} - b_{12}b_{33}b_{44}) \\
&\quad -(\Delta_{1,l}^{(3)} - \Delta_{1,r}^{(4)})b_{12}b_{23}b_{44} + (\Delta_{1,l}^{(4)} - \Delta_{1,r}^{(5)})b_{12}b_{23}b_{34} \\
D_c^{(2)} &= -(\Delta_{1,l}^{(1)} - \Delta_{1,r}^{(2)})(b_{21}b_{34}b_{43} - b_{21}b_{33}b_{44}) + (\Delta_{1,l}^{(2)} - \Delta_{1,r}^{(3)})(b_{11}b_{34}b_{43} - b_{11}b_{33}b_{44}) \\
&\quad +(\Delta_{1,l}^{(3)} - \Delta_{1,r}^{(4)})b_{11}b_{23}b_{44} - (\Delta_{1,l}^{(4)} - \Delta_{1,r}^{(5)})b_{11}b_{23}b_{34} \\
D_c^{(3)} &= -(\Delta_{1,l}^{(1)} - \Delta_{1,r}^{(2)})b_{21}b_{32}b_{44} + (\Delta_{1,l}^{(2)} - \Delta_{1,r}^{(3)})b_{11}b_{32}b_{44} \\
&\quad +(\Delta_{1,l}^{(3)} - \Delta_{1,r}^{(4)})(b_{12}b_{21}b_{44} - b_{11}b_{22}b_{44}) + (\Delta_{1,l}^{(4)} - \Delta_{1,r}^{(5)})(b_{11}b_{22}b_{34} - b_{12}b_{21}b_{34}) \\
D_c^{(4)} &= (\Delta_{1,l}^{(1)} - \Delta_{1,r}^{(2)})b_{21}b_{32}b_{43} - (\Delta_{1,l}^{(2)} - \Delta_{1,r}^{(3)})b_{11}b_{32}b_{43} \\
&\quad -(\Delta_{1,l}^{(3)} - \Delta_{1,r}^{(4)})(b_{12}b_{21}b_{43} - b_{11}b_{22}b_{43}) \\
&\quad +(\Delta_{1,l}^{(4)} - \Delta_{1,r}^{(5)})(b_{11}b_{23}b_{32} + b_{12}b_{21}b_{33} - b_{11}b_{22}b_{33})
\end{aligned} \tag{2.40}$$

Substituting $\sigma_z^{(m)}$ from equation (2.17) and the expressions for $\sigma_r^{(m)}$, $\sigma_\theta^{(m)}$ and $\tau_{\theta z}^{(m)}$ from equations (2.26) into equation (2.23) using (2.33), and performing the integration, the expression for constant C for a close-ended cylinder is given by

$$C = -\frac{Z_2^{(1)}p_0 + Z_3^{(nl)}p_{nl} + \sum_{m=1}^{nl-1} N^{(m)}(Z_2^{(m+1)} + Z_3^{(m)}) - L}{\sum_{m=1}^{nl} Z_1^{(m)} + \sum_{m=1}^{nl-1} N_c^{(m)}(Z_2^{(m+1)} + Z_3^{(m)})} \tag{2.41}$$

where

$$\begin{aligned}
Z_1^{(m)} &= \frac{1}{2}(a_m^2 - a_{m-1}^2)(1 - \frac{1}{\alpha_{33}^{(m)}}(\zeta_1^{(m)}(\alpha_{13}^{(m)} + \alpha_{23}^{(m)}) - \zeta_2^{(m)}\alpha_{34}^{(m)})) \\
&\quad + \frac{U_2^{(m)}c_m^{k_m+1}\zeta_1^{(m)}}{\alpha_{33}^{(m)}(1-k_m)}\lambda_1^{(m)}\lambda_3^{(m)} + \frac{U_1^{(m)}\zeta_1^{(m)}}{\alpha_{33}^{(m)}(1+k_m)}\lambda_2^{(m)}\lambda_4^{(m)} \\
Z_2^{(m)} &= \frac{c_m^{1+k_m}}{\alpha_{33}^{(m)}(1-c_m^{2k_m})(1-k_m)}\lambda_1^{(m)}\lambda_3^{(m)} - \frac{c_m^{1+k_m}}{\alpha_{33}^{(m)}(1-c_m^{2k_m})(1+k_m)}\lambda_2^{(m)}\lambda_4^{(m)}
\end{aligned}$$

$$
\begin{aligned}
Z_3^{(m)} &= \frac{c_m^{2k_m}}{\alpha_{33}^{(m)}(1-c_m^{2k_m})(1-k_m)}\lambda_1^{(m)}\lambda_5^{(m)} + \frac{1}{\alpha_{33}^{(m)}(1-c_m^{2k_m})(1+k_m)}\lambda_2^{(m)}\lambda_4^{(m)} \\
L &= \frac{1}{2}(p_0 - p_{nl})a_0^2 + F/2\pi \qquad (2.42) \\
\lambda_1^{(m)} &= a_m^2 - a_m^{k_m+1}a_{m-1}^{1-k_m} \qquad \lambda_2^{(m)} = a_m^2 - a_m^{1-k_m}a_{m-1}^{k_m+1} \\
\lambda_3^{(m)} &= \alpha_{13}^{(m)} - \alpha_{34}^{(m)}g_{-k}^{(m)} - \alpha_{23}^{(m)}k_m \qquad \lambda_4^{(m)} = \alpha_{13}^{(m)} - \alpha_{34}^{(m)}g_k^{(m)} + \alpha_{23}^{(m)}k_m \\
\lambda_5^{(m)} &= -\alpha_{13}^{(m)} + \alpha_{34}^{(m)}g_{-k}^{(m)} + \alpha_{23}^{(m)}k_m \\
U_1^{(m)} &= \frac{1-c_m^{k_m+1}}{1-c_m^{2k_m}}; \qquad U_2^{(m)} = \frac{1-c_m^{k_m-1}}{1-c_m^{2k_m}}
\end{aligned}
$$

In case of a single layered cylinder the coefficients $N^{(m)}$, $N_c^{(m)}$ equal zero and expression for constant of the integration takes the form

$$
C = \frac{Z_2p_0 + Z_3p_1 - L}{Z_1} \qquad (2.43)
$$

Finally, it should be noted that when winding angle $\varphi_m = 0°$ or $90°$ then we deal with an orthotropic layer with cylindrical anisotropy, which means that there are two planes of elastic symmetry, radial and tangential. Then $\alpha_{34}^{(m)} = \beta_{14}^{(m)} = \beta_{24}^{(m)} = g_k^{(m)} = g_{-k}^{(m)} = 0$ and tangential stresses $\tau_{\theta z}^{(m)}$ vanish. In the case when $\varphi_m = 0°$

$$
k_m = \sqrt{\frac{\beta_{11}^{(m)}}{\beta_{22}^{(m)}}} = 1 \qquad (2.44)
$$

and some denominators in expressions (2.42) become equal zero that leads to singularity. In actual computation, this difficulty can be overcome by assigning a very small number of φ_m (eg. $= 0.001°$) when $\varphi = 0°$.

2.6 Stress distribution in continuously heterogeneous laminated cylinders

2.6.1 Problem formulation

The problem of elastic equilibrium for a continuously heterogeneous hollow laminated cylinder is a more complicated than one considered in a previous section. Cylinder is made from a material with cylindrical anisotropy and can be subjected to internal, external and interlaminar pressure. We consider only the special case when the axis of anisotropy coincides with the axis of symmetry Oz of the cylinder. In this case at any point of the cylinder there exists a plane of elastic symmetry normal to this axis so that each layer may be considered orthotropic. However, elastic properties can vary through the thickness of a layer. Thus, cylinder is an orthotropic, the coefficients of deformation depend only on radial coordinate r and there are no body forces applied. Length of a cylinder is finite or infinite but with fixed ends (the case of an open-ended cylinder). The equations of the generalised Hooke's law for an orthotropic laminated cylinder may be written as

$$\begin{aligned}
\epsilon_r^{(m)} &= \alpha_{11}^{(m)}\sigma_r^{(m)} + \alpha_{12}^{(m)}\sigma_\theta^{(m)} + \alpha_{13}^{(m)}\sigma_z^{(m)} \\
\epsilon_\theta^{(m)} &= \alpha_{21}^{(m)}\sigma_r^{(m)} + \alpha_{22}^{(m)}\sigma_\theta^{(m)} + \alpha_{23}^{(m)}\sigma_z^{(m)} \\
\epsilon_z^{(m)} &= \alpha_{31}^{(m)}\sigma_r^{(m)} + \alpha_{32}^{(m)}\sigma_\theta^{(m)} + \alpha_{33}^{(m)}\sigma_z^{(m)}
\end{aligned} \tag{2.45}$$

The components of the compliance matrix $\alpha_{ij}^{(m)}$ are determined in (2.5).

Here and in the following expressions indices 1, 2 and 3 correspond to r, θ and z directions, respectively. It is noted that the compliance coefficients $\alpha_{ij}^{(k)}$ may be functions of the radial coordinate r.

Since the stress distribution in the cylinder is symmetrical with respect to the Oz axis, the stresses depend on the radial coordinate r only. The stresses $\sigma_r^{(m)}$ and $\sigma_\theta^{(m)}$ can be expressed in terms of a stress function $\Phi_m = \Phi_m(r)$ as

$$\sigma_r^{(m)} = \frac{1}{r}\frac{d\Phi_m}{dr}; \quad \sigma_\theta^{(m)} = \frac{d^2\Phi_m}{dr^2} \tag{2.46}$$

Moreover, due to symmetry

$$\tau_{r\theta}^{(m)} = \tau_{\theta z}^{(m)} = \tau_{rz}^{(m)} = 0; \tag{2.47}$$

By introducing the notation

$$\frac{d\Phi_m(r)}{dr} = \varpi_m(r)$$

the normal and circumferential stresses can be expressed as

$$\sigma_r^{(m)} = \varpi_m \cdot r^{-1}; \quad \sigma_\theta^{(m)} = \frac{d\varpi_m}{dr} \tag{2.48}$$

Next, a differential equation governing the equilibrium of open–ended cylinders is derived. Using equations (2.45) and noting that for an open–ended cylinder the longitudinal strain $\epsilon_z^{(m)} = 0$, stress–strain relations can be expressed as

$$\epsilon_r^{(m)} = \beta_{11}^{(m)}\,\sigma_r^{(m)} + \beta_{12}^{(m)}\,\sigma_\theta^{(m)}; \quad \epsilon_\theta^{(m)} = \beta_{21}^{(m)}\,\sigma_r^{(m)} + \beta_{22}^{(m)}\,\sigma_\theta^{(m)} \tag{2.49}$$

Let $u(r)$ denote the displacement in the r direction. Using strain–displacement relations

$$\epsilon_r^{(m)} = \frac{du_m}{dr}; \quad \epsilon_\theta^{(m)} = \frac{u_m}{r} \tag{2.50}$$

and the equations (2.48) and (2.49), the following expressions are obtained

$$\frac{du_m}{dr} = \beta_{11}^{(m)} \varpi_m r^{-1} + \beta_{12}^{(m)} \frac{d\varpi_m}{dr}; \quad u_m = \beta_{21}^{(m)} \varpi_m + \beta_{22}^{(m)} r \frac{d\varpi_m}{dr} \tag{2.51}$$

Eliminating the displacement from expressions in (2.51), a differential equation in terms of $\varpi_m(r)$ is derived, viz.

$$\frac{d}{dr}(\beta_{22}^{(m)} r \frac{d\varpi_m}{dr} + \beta_{12}^{(m)} \varpi_m) - \beta_{11}^{(m)} \varpi_m r^{-1} - \beta_{12}^{(m)} \frac{d\varpi_m}{dr} = 0 \tag{2.52}$$

where the coefficients β_{ij} are functions of the coordinate r only.

The differential equation (2.52) applies to the *m–th* layer and is to be solved subject to specified boundary and interface conditions. The boundary conditions on the internal ($r = a_0$) and external ($r = a_{nl}$) surfaces are given by

$$\sigma_r^{(1)}(a_0) = -p_0; \quad \sigma_r^{(nl)}(a_{nl}) = -p_{nl} \tag{2.53}$$

where p_0 and p_{nl} denote the internal and external pressure, respectively. At a given layer interface ($r = a_{m-1}$), the normal stress and the circumferential strain satisfy the continuity conditions

$$\sigma_r^{(m-1)} - \sigma_r^{(m)} = p_{m-1} \tag{2.54}$$

$$\epsilon_\theta^{(m-1)} = \epsilon_\theta^{(m)} \tag{2.55}$$

where p_{m-1} is the pressure between the layers $m-1$ and m, and may arise as a result of shrink–fit, residual stresses, etc.

The solution of equation (2.52) will contain two constants of integration A_m and B_m with the total number of unknown constants being $2nl$ for a cylinder with nl layers. The total number of equations available to compute $2nl$ constants follows from equations (2.53), (2.54) and (2.55) as

$$2 + 2(nl - 1) = 2nl.$$

Thus, the values of A_m, B_m, $m = 1, 2, \ldots, nl$ can be determined uniquely from the solution of the system of linear algebraic equations obtained using (2.53)–(2.55).

2.6.2 Method of solution

Let the general solution of equation (2.52) for the m–th layer be denoted by $\varpi_m(r)$ which is of the form

$$\varpi_m(r) = A_m\ f_m(r) + B_m\ g_m(r) \tag{2.56}$$

where $f_m(r)$ and $g_m(r)$ are known functions which depend on the coefficients $\beta_{ij}^{(m)}(r)$. The constants of integration A_m and B_m in equation (2.56) are determined from the boundary and interface conditions (2.53), (2.54) and (2.55). From equations (2.48) and (2.56), the stresses in m–th layer are computed as

$$\begin{aligned} \sigma_r^{(m)} &= (A_m\ f_m(r) + B_m\ g_m(r)) \cdot r^{-1}; \\ \sigma_\theta^{(m)} &= A_m\ \frac{df_m}{dr} + B_m\ \frac{dg_m}{dr} \end{aligned} \tag{2.57}$$

The circumferential deformation follows from equations (2.49) and (2.57) as

$$\epsilon_\theta^{(m)} = \beta_{21}^{(m)}(A_m\ f_m(r) + B_m\ g_m(r))\ r^{-1} + \beta_{22}^{(m)}\ (A_m\ \frac{df_m}{dr} + B_m\ \frac{dg_m}{dr}) \tag{2.58}$$

The implementation of the conditions (2.53)–(2.55) using equations (2.57) and (2.58) gives

$$A_1 f_1(a_0) + B_1\ g_1(a_0) = -p_0\ a_0;\ A_{nl} f_{nl}(a_{nl}) + B_{nl}\ g_{nl}(a_{nl}) = -p_{nl}\ a_{nl} \tag{2.59}$$

$$A_{m-1} f_{m-1}(a_{m-1}) + B_{m-1} g_{m-1}(a_{m-1}) - (A_m f_m(a_{m-1}) + B_m g_m(a_{m-1})) = -p_{m-1} a_{m-1}$$

$$\begin{aligned} &\beta_{22}^{(m-1)}(A_{m-1}\frac{df_{m-1}(a_{m-1})}{dr} + B_{m-1}\frac{dg_{m-1}(a_{m-1})}{dr}) + (\beta_{12}^{(m-1)} - \beta_{12}^{(m)})a_{m-1} \\ &\times(A_m f_m(a_{m-1}) + B_m g_m(a_{m-1})) - \beta_{22}^{(m)}(A_m\frac{df_m(a_{m-1})}{dr} + B_m\frac{dg_m(a_{m-1})}{dr}) = \\ &= -(\beta_{12}^{(m)} - \beta_{12}^{(m-1)})p_{m-1} \end{aligned} \tag{2.60}$$

For the open–ended cylinder the longitudinal stress σ_z is given by

$$\sigma_z^{(m)} = -\frac{1}{\alpha_{33}^{(m)}}(\alpha_{13}^{(m)}\sigma_r^{(m)} + \alpha_{23}^{(m)}\sigma_\theta^{(m)}) \tag{2.61}$$

The above solution is now extended to the case of a closed–ended cylinder under an axial load F. The cylinder is assumed to be long enough for the longitudinal bending deformation due to end closures to be limited to only small end portions of the cylinder compared to the overall length. Due to cylindrical orthotropy and axisymmetric loading and neglecting the longitudinal bending deformation due to end closures, the problem can be treated as a generalized plane strain problem.

Using Hooke's law, the expression for the longitudinal stress may be written as

$$\sigma_z^{(m)} = \frac{1}{\alpha_{33}^{(m)}}(\epsilon_z^{(m)} - \alpha_{13}^{(m)}\sigma_r^{(m)} - \alpha_{23}^{(m)}\sigma_\theta^{(m)}) \tag{2.62}$$

where $\epsilon_z^{(m)} = \epsilon_z$ is an unknown constant and is computed next.

The equilibrium of forces in the z direction gives

$$\sum_{m=1}^{nl} 2\pi \int_{a_{m-1}}^{a_m} \sigma_z^{(m)} r dr = \pi(p_0 - p_{nl})a_0^2 + F \tag{2.63}$$

Substituting equations (2.62) for $\sigma_z^{(m)}$ into equation (2.63) we obtain the expression for ϵ_z as

$$\epsilon_z = \frac{\sum\limits_{m-1}^{nl}\left(\int_{a_{m-1}}^{a_m} \frac{1}{\alpha_{33}^{(m)}}(\alpha_{13}^{(m)}\sigma_r^{(m)} + \alpha_{23}^{(m)}\sigma_\theta^{(m)})\, rdr\right) + (p_0 - p_{nl})a_0^2/2 + F/2\pi}{\sum\limits_{m-1}^{nl}\int_{a_{m-1}}^{a_m} \frac{1}{\alpha_{33}^{(m)}}\, rdr} \tag{2.64}$$

2.6.3 Example

The method of solution is illustrated by solving a specific example, namely, multilayered open-ended orthotropic cylinder with variable material properties. The simplest way to solve this problem when the coefficients $\alpha_{ij}^{(m)}(r)$ are proportional to some degree of the distance r. Let coefficients of deformation be assumed to be inversely proportional to the radial distance r such that

$$\alpha_{ij}^{(m)}(r) = \eta_{ij}^{(m)} r^{-n_m}; \quad \beta_{ij}^{(m)}(r) = \gamma_{ij}^{(m)} r^{-n_m} \tag{2.65}$$

where $\eta_{ij}^{(m)}$ are given coefficients for the m–th layer, n_m are given real numbers which characterize the rate of changes in elastic properties through the thickness and $\gamma_{ij}^{(m)}$ are constants given by

$$\gamma_{ij}^{(m)} = \eta_{ij}^{(m)} - \frac{\eta_{i3}^{(m)}\eta_{j3}^{(m)}}{\eta_{33}^{(m)}}$$

Substituting $\beta_{ij}^{(m)}$ from equation (2.65) in (2.52), the following differential equation is derived

$$\frac{d^2\varpi_m}{dr^2} + (1 - n_m)r^{-1}\frac{d\varpi_m}{dr} - \gamma_n^{(m)} r^{-2}\varpi_m = 0 \tag{2.66}$$

where

$$\gamma_n^{(m)} = \frac{\gamma_{11}^{(m)} + n_m\gamma_{12}^{(m)}}{\gamma_{22}^{(m)}}.$$

The corresponding equations for single–layered cylinders are given in [45].

Using the through-the-thickness material distribution assumed above for the coefficients $\alpha_{ij}^{(m)}(r)$, the expressions for functions $f_m(r)$ and $g_m(r)$ are obtained as

$$f_m(r) = r^{s_m}; \quad g_m(r) = r^{t_m} \tag{2.67}$$

where s_m and t_m are real numbers which are computed from equation (2.66) as

$$\left\{ \begin{array}{c} s_m \\ t_m \end{array} \right\} = \frac{1}{2}\left(n_m \pm \sqrt{n_m^2 - 4\gamma_n^{(m)}}\right) \tag{2.68}$$

The stresses and circumferential strain $\epsilon_\theta^{(m)}$ follow from equations (2.57) and (2.58) as

$$\begin{aligned} \sigma_r^{(m)} &= (A_m\ r^{s_m} + B_m\ r^{t_m}) \cdot r^{-1}; \\ \sigma_\theta^{(m)} &= (A_m\ s_m\ r^{s_m} + B_m\ t_m\ r^{t_m}) \cdot r^{-1}; \\ \epsilon_\theta^{(m)} &= \beta_{21}^{(m)}(A_m r^{s_m} + B_m r^{t_m}) \cdot r^{-1} + \beta_{22}^{(m)}(A_m s_m r^{s_m} + B_m t_m r^{t_m}) \cdot r^{-1} \end{aligned} \tag{2.69}$$

The implementation of the boundary and interface conditions (2.53)–(2.55) gives the system of equations

$$A_1 a_0^{s_1} + B_1 a_0^{t_1} = -p_0 a_0; \; A_{nl} a_{nl}^{s_{nl}} + B_{nl} a_{nl}^{t_{nl}} = -p_{nl} a_{nl} \tag{2.70}$$

$$\begin{aligned} A_{m-1}a_{m-1}^{s_{m-1}} + B_{m-1}a_{m-1}^{t_{m-1}} - (A_m a_{m-1}^{s_m} + B_m a_{m-1}^{t_m}) &= -p_{m-1}a_{m-1} \\ \beta_{22}^{(m-1)}(A_{m-1}s_{m-1}a_{m-1}^{s_{m-1}} + B_{m-1}t_{m-1}a_{m-1}^{t_{m-1}}) + (\beta_{12}^{(m-1)} - \beta_{12}^{(m)}) & \\ \times(A_m a_{m-1}^{s_m} + B_m a_{m-1}^{t_m}) - \beta_{22}^{(m)}(A_m s_m a_{m-1}^{s_m} + B_m t_m a_{m-1}^{t_m}) &= \\ = -(\beta_{12}^{(m)} - \beta_{12}^{(m-1)})p_{m-1}a_{m-1} & \end{aligned} \tag{2.71}$$

As an example, the system of equations for the case $nl = 4$ is given in Table 2.1.

Table 2.1. The system of equations to determine A_k and B_k with $n = 4$.

N/N	A_1	B_1	A_2	B_2	A_3	B_3	A_4	B_4	R.H.S
1	C_{11}	C_{12}	0	0	0	0	0	0	$-p_0a_0$
2	C_{21}	C_{22}	C_{23}	C_{24}	0	0	0	0	$-p_1a_1$
3	C_{31}	C_{32}	C_{33}	C_{34}	0	0	0	0	$-(\beta_{12}^{(2)} - \beta_{12}^{(1)})p_1a_1$
4	0	0	C_{43}	C_{44}	C_{45}	C_{46}	0	0	$-p_2a_2$
5	0	0	C_{53}	C_{54}	C_{55}	C_{56}	0	0	$-(\beta_{12}^{(3)} - \beta_{12}^{(2)})p_2a_2$
6	0	0	0	0	C_{65}	C_{66}	C_{67}	C_{68}	$-p_3a_3$
7	0	0	0	0	C_{75}	C_{76}	C_{77}	C_{78}	$-(\beta_{12}^{(4)} - \beta_{12}^{(3)})p_3a_3$
8	0	0	0	0	0	0	C_{87}	C_{88}	$-p_4a_4$

where

$C_{11} = a_0^{s_1}$, $\quad C_{12} = a_0^{t_1}$

$C_{21} = a_1^{s_1}$, $\quad C_{22} = a_1^{t_1}$

$C_{23} = -a_1^{s_2}$, $\quad C_{24} = -a_1^{t_2}$

$C_{31} = \beta_{22}^{(1)} s_1 a_1^{s_1}$, $\quad C_{32} = \beta_{22}^{(1)} t_1 a_1^{t_1}$

$C_{33} = a_1^{s_2}(\beta_{12}^{(1)} - \beta_{12}^{(2)} - \beta_{22}^{(2)} s_2)$, $\quad C_{34} = a_1^{t_2}(\beta_{12}^{(1)} - \beta_{12}^{(2)} - \beta_{22}^{(2)} t_2)$

$C_{43} = a_2^{s_2}$, $\quad C_{44} = a_2^{t_2}$

$C_{45} = -a_2^{s_3}$, $\quad C_{46} = -a_2^{t_3}$

$C_{53} = \beta_{22}^{(2)} s_2 a_2^{s_2}$, $\quad C_{54} = \beta_{22}^{(2)} t_2 a_2^{t_2}$

$C_{55} = a_2^{s_3}(\beta_{12}^{(2)} - \beta_{12}^{(3)} - \beta_{22}^{(3)} s_3)$, $\quad C_{56} = a_2^{t_3}(\beta_{12}^{(2)} - \beta_{12}^{(3)} - \beta_{22}^{(3)} t_3)$

$C_{65} = a_3^{s_3}$, $\quad C_{66} = a_3^{t_3}$

$C_{67} = -a_3^{s_4}$, $\quad C_{68} = -a_3^{t_4}$

$C_{75} = \beta_{22}^{(3)} s_3 a_3^{s_3}$, $\quad C_{76} = \beta_{22}^{(3)} t_3 a_3^{t_3}$

$C_{77} = a_3^{s_4}(\beta_{12}^{(3)} - \beta_{12}^{(4)} - \beta_{22}^{(4)} s_4)$, $\quad C_{78} = a_3^{t_4}(\beta_{12}^{(3)} - \beta_{12}^{(4)} - \beta_{22}^{(4)} t_4)$

$C_{87} = a_4^{s_4}$, $\quad C_{88} = a_4^{t_4}$

These results are now extended to closed–ended cylinders for which $\epsilon_z \neq 0$. With functions $f_m(r)$ and $g_m(r)$ given by (2.67), we can determine the expression for the longitudinal strain ϵ_z from equation (2.64). This calculation is performed using symbolic computation and gives

$$\epsilon_z = \left\{ F/\pi + (p_0 - p_{nl})a_0^2 + \sum_{k=1}^{n}(U_1^{(m)} + U_2^{(m)}) \right\} / \Delta \tag{2.72}$$

where

$$\begin{aligned} U_1^{(m)} &= 2A_m(a_m^{(1+s_m)} - a_{m-1}^{(1+s_m)})(\eta_{13}^{(m)} + \eta_{23}^{(m)} s_m)/\eta_{33}^{(m)}(1+s_m), \\ U_2^{(m)} &= 2B_m(a_m^{(1+t_m)} - a_{m-1}^{(1+t_m)})(\eta_{13}^{(m)} + \eta_{23}^{(m)} t_m)/\eta_{33}^{(m)}(1+t_m) \\ \Delta &= \sum_{m=1}^{nl} \frac{1}{\alpha_{33}^{(m)}}(a_m^2 - a_{m-1}^2). \end{aligned}$$

Let us now consider some particular cases of the above problem. If the coefficients of deformation for the layers are constant within each layer m, then the exponents in equation (2.65) are zero, i.e., $n_m = 0$. In this case

$$\gamma_n = \frac{\gamma_{11}^{(m)}}{\gamma_{22}^{(m)}}; \qquad \left\{ \begin{array}{c} s_m \\ t_m \end{array} \right\} = \pm\sqrt{\gamma_n} \tag{2.73}$$

The system of equations (2.59), (2.60) and (2.70), (2.71) remain the same.

For a single–ply cylinder ($nl = 1$), we have to take into account only conditions on the external and internal surfaces (2.59). In this case, a system of two equations needs to be solved which is given by equation (2.59) where $A_1 = A_{nl} = A,\ B_1 = B_{nl} = B$.

Final expressions for the stresses may be obtained in a form which is similar to that given in [45], namely,

$$\sigma_r = -\frac{pc}{\rho}\frac{\rho^s - \rho^t}{c^s - c^t} + \frac{q}{\rho}\frac{c^t\rho^s - c^s\rho^t}{c^s + c^t};$$

$$\sigma_\theta = -\frac{pc}{\rho}\frac{s\rho^s - t\rho^t}{c^s - c^t} + \frac{q}{\rho}\frac{sc^t\rho - tc^s\rho^t}{c^s - c^t} \tag{2.74}$$

where

$$c = \frac{a_0}{a_{nl}}; \quad \rho = \frac{r}{a_{nl}}$$

and p and q are internal and external pressure, respectively.

For a homogeneous single–ply isotropic cylinder ($n = 0;\ s = t = 1;\ \gamma_{11} = \gamma_{22};\ \gamma_n = 1$), formulae (2.74) reduce to the solution of Lame's problem, given in [45]

$$\left\{ \begin{array}{c} \sigma_r \\ \sigma_\theta \end{array} \right\} = \frac{pc^2 - q}{1 - c^2} \pm \frac{(q-p)c^2}{1 - c^2}\cdot\frac{1}{\rho^2}. \tag{2.75}$$

The solutions presented in this study are the exact elasticity solutions for a heterogeneous laminated cylinder subject to internal and external pressures as well as to pressures between the layers. This approach is capable of taking into account variable material properties as well as stresses at layer interfaces and allows the stress-strain state to be determined exactly.

2.6.4 Filament–wound cylinders

The problem set–up described above does not allow to calculate the shear stresses $\tau_{\theta z}$ which inevitably appear in filament–wound layers. But considering an angle–ply laminate $\pm\varphi$ we could treat such a layer as an orthotropic unit. Indeed, the change of the fibre angle sign remains the absolute value of shear stress $\tau_{\theta z}$ while the sign of the stress is changing that gives $\tau_{\theta z} = 0$ for the ply. In practice, many automated manufacturing processes lead to symmetric angle–ply composite laminates consisting of an even number of equivalent plies with material symmetry axes being alternately oriented at angles $+\varphi$ and $-\varphi$ to the element axis (namely, the z-axis). In calculations, such a system of plies is considered as one symmetrically reinforced layer. Such an approximation is a good description of real composite structure and is a substantial simplification of the stress–strain relationships: if each of symmetric plies is anisotropic then, working together, they form an orthotropic layer. Elastic constants are determined for such a layer in a different way than for a single ply, and may be computed in the following form

$$\begin{aligned} E_z &= A_{11} - \frac{A_{12}^2}{A_{22}}, \quad E_\theta = A_{22} - \frac{A_{12}^2}{A_{11}}, \quad G_{z\theta} = A_{33} \\ & \nu_{\theta z} = \frac{A_{12}}{A_{11}}, \qquad \nu_{z\theta} = \frac{A_{12}}{A_{22}} \end{aligned} \tag{2.76}$$

where

$$\begin{aligned} A_{11} &= \overline{E}_1 \cos^4\varphi + \overline{E}_2 \sin^4\varphi + 2\left(\overline{E}_1\nu_{12} + 2G_{12}\right)\sin^2\varphi\cos^2\varphi \\ A_{22} &= \overline{E}_1 \sin^4\varphi + \overline{E}_2 \cos^4\varphi + 2\left(\overline{E}_1\nu_{12} + 2G_{12}\right)\sin^2\varphi\cos^2\varphi \\ A_{12} &= \overline{E}_1\nu_{12} + \left[\overline{E}_1 + \overline{E}_2 - 2\left(\overline{E}_1\nu_{12} + 2G_{12}\right)\right]\sin^2\varphi\cos^2\varphi \\ A_{33} &= \left(\overline{E}_1 + \overline{E}_2 - 2\overline{E}_1\nu_{12}\right)\sin^2\varphi\cos^2\varphi + G_{12}\cos^2 2\varphi \end{aligned} \tag{2.77}$$

Here subscripts 1 and 2 correspond to the fibre and transversal directions of a material, respectively, and

$$\overline{E}_i = \frac{E_i}{1 - \nu_{12}\nu_{21}}, \qquad i = 1,2 \tag{2.78}$$

Components of the compliance matrix α_{ij} now can be obtained using relations of the Hooke's law (2.4).

It should be also noted that we assume that the material properties are the same in both plies and that the plies are perfectly bonded to one another, i.e. $\epsilon_\theta^\pm = \epsilon_\theta$.

Comparison study of equations (2.77) and (2.5) shows that, in the general case, a $\pm\varphi$ angle–ply laminate has a higher stiffness than the $+\varphi$ or $-\varphi$ laminate of the same thickness. The behaviour of the elastic constants of some composite materials as a function of angle φ is shown in Figure 2.3.

2.7 Optimisation of laminated composite cylinders under strength criterion

The strength of unidirectional layer is determined by the tensile and compressive strength along and across the fibers, and the in–plane shear strength. The carrying capacity of the material under longitudinal tensile stress is exhausted as a result of fiber breakage. Under compressive stresses material failure occurs because of fiber buckling or splitting parallel to the fibers. Material failure under transverse tension and shear is associated, as a rule, with failure of the matrix or with the separation of the matrix from the fibers. Failure of composite materials is a rather complicated process even under simple loading, and its theoretical description presents severe difficulties. The values of strengths are usually determined experimentally.

Therefore, engineers seek out an observational level of failure to which they can readily relate and feel comfortable with in description the appropriate failure mechanism. In this regard we generally attempt to specify lamina failure for anisotropic unidirectional composites or alternatively lamina as existing in composite orthotropic laminates. One of the earliest interactive failure criterion for anisotropic materials was initiated by Hill (1948). The plane stress results by Hill were simplified for the case of fiber reinforced composites by Azzi and Tsai (1965) [4] considering the composite to be transversely isotropic. A generalisation of this failure criterion to incorporate the effects of brittle materials was considered by Hoffman (1967) [31]. A generalisation of the Hoffman result to incorporate a more comprehensive definition for failure was later proposed by Tsai and Wu (1971) [87]. This criterion is best suited to the present study.

The basic assumption of the Tsai–Wu 3-dimensional failure criterion is that there exists a failure surface in the stress space in the following scalar form

$$f(\sigma_k) = F_i\sigma_i + F_{ij}\sigma_i\sigma_j = 1 \tag{2.79}$$

where $k,\ i,\ j = 1, 2, \ldots, 6$; F_i and F_{ij} are strength tensors of the second and forth rank, respectively. In case of laminated pressure vessel possessing cylindrical anisotropy equation (2.79) in expanded form can be written as

$$\begin{aligned} F_{11}^{(m)}\sigma_1^{(m)2} &+ F_{33}^{(m)}(\sigma_3^{(m)2} + \sigma_2^{(m)2}) + F_{44}^{(m)}\tau_{12}^{(m)2} \\ &+2F_{31}^{(m)}(\sigma_3^{(m)} + \sigma_2^{(m)})\sigma_1^{(m)} + 2F_{32}^{(m)}\sigma_3^{(m)}\sigma_2^{(m)} \\ &\quad +F_3^{(m)}(\sigma_3^{(m)} + \sigma_2^{(m)}) + F_1^{(m)}\sigma_1^{(m)} - 1 = 0 \end{aligned} \tag{2.80}$$

where

$$\begin{aligned} F_{11}^{(m)} &= \frac{1}{X_t^{(m)}X_c^{(m)}}, \quad F_{33}^{(m)} = \frac{1}{Y_t^{(m)}Y_c^{(m)}} \quad F_{44}^{(m)} = \frac{1}{S^{(m)2}} \\ F_3^{(m)} &= \frac{1}{Y_t^{(m)}} - \frac{1}{Y_c^{(m)}}, \qquad F_1^{(m)} = \frac{1}{X_t^{(m)}} - \frac{1}{X_c^{(m)}} \\ F_{31}^{(m)} &= -\frac{1}{2}\sqrt{F_{33}^{(m)}F_{11}^{(m)}}, \qquad F_{32}^{(m)} = -\frac{1}{2}F_{33}^{(m)} \end{aligned} \tag{2.81}$$

and X_t, X_c are longitudinal tensile and compressive strengths, respectively, Y_t, Y_c are those for transverse direction and S is the shear strength. It should be noted that the normal stresses σ_i, $i = 1,2,3$ and shear stress τ_{12} are stresses in material coordinates and can be computed as

$$\begin{aligned}
\sigma_1 &= \sigma_z \cos^2\varphi + \sigma_\theta \sin^2\varphi - \tau_{\theta z}\sin 2\varphi \\
\sigma_2 &= \sigma_z \sin^2\varphi + \sigma_\theta \cos^2\varphi + \tau_{\theta z}\sin 2\varphi \\
\sigma_3 &= \sigma_r \\
\tau_{12} &= (\sigma_\theta - \sigma_z)\sin\varphi\cos\varphi - \tau_{\theta z}\cos 2\varphi
\end{aligned} \tag{2.82}$$

2.7.1 Method of solution

The design objective is the maximisation of the burst pressure P_{cr} subject to the failure criterion (2.80). The design problem for a multilayered pressure vessel of a given thickness ratio b/a, number of layers and axial force F can be stated as

$$P_{max} \overset{\text{def}}{=} \max_{\theta} P_{cr}(\varphi, r) = \max_{\theta}\min_{r} P_{cr} \tag{2.83}$$

where $P_{cr}(\varphi, r)$ can be easily calculated from the quadratic equation

$$(F_{ij}\sigma_i\sigma_j)\, P_{cr}^2 + (F_i\sigma_i)\, P_{cr} - 1 = 0 \tag{2.84}$$

wherein the stresses are calculated for an applied unit pressure. Solution of the equation (2.84) gives

$$P_{cr} = -\left(\frac{\xi}{2\delta}\right) + \sqrt{\left(\frac{\xi}{2\delta}\right)^2 + \frac{1}{\delta}} \tag{2.85}$$

where

$$\begin{aligned}
\delta &= F_{33}^{(m)}(\sigma_3^{(m)2} + \sigma_2^{(m)2}) + 2F_{32}^{(m)}\sigma_3^{(m)}\sigma_2^{(m)} \\
&+ 2F_{31}^{(m)}(\sigma_3^{(m)} + \sigma_2^{(m)})\sigma_1^{(m)} + F_{11}^{(m)}\sigma_1^{(m)2} + F_{44}^{(m)}\tau_{12}^{(m)2} \\
\xi &= F_3^{(m)}(\sigma_3^{(m)} + \sigma_2^{(m)}) + F_1^{(m)}\sigma_1^{(m)}
\end{aligned}$$

The negative root for P_{cr} does not have any physical meaning and the positive value only must be taken into consideration.

The optimisation procedure involves the stages of iteratively improving $\varphi_{opt}^{(m)}$ in order to maximise P_{cr} for a given radius, thickness ratio and axial force.

The simplest case of the optimisation is a single layered cylinder where there is only one variable φ and so the one–dimensional problem is solved. For this purpose the golden section method in determine φ_{opt} is employed and it gives the fast convergence for this problem.

With increasing in number of layers the problem becomes more complicated and requires the use of reliable multi–dimensional optimisation methods. Depending on particular problem two methods are used, namely, iterative and gradient methods.

Let us consider the function $f(x_1, x_2, \ldots, x_n)$ which is to be optimised, i.e. it is necessary to find extremum of the function. Here $x_1, x_2, \ldots, x_n$ are functional variables, in our case they are angles. We begin from random values of $x_i^{[0]}$ $(i = 1, 2, \ldots, n)$ and build successively the approximations. In the case of the iterative technique they can be given as

$$x_i^{[j+1]} = x_i^{[j]} + \lambda^{[j]} v_i^{[j]}, \qquad (i = 1, 2, \ldots, n;\ j = 1, 2, \ldots) \tag{2.86}$$

which converge to some solution x_i when $j \to \infty$. In equation (2.86) $v_i^{[j]}$ characterises "direction" of a step and parameter $\lambda^{[j]}$ is a step which maximises the value $f(x_1^{[j+1]}, x_2^{[j+1]}, \ldots, x_n^{[j+1]})$ as a function of $\lambda^{[j]}$. On each step the only one argument x_i is consecutively changed in such a way as to decrease the biggest absolute value of the discrepancy. This approach is not too fast but quite reliable, especially when the extremum occurs within a region where slope of the functional surface changes rapidly.

On the other hand the gradient method is faster. In this method the functional variables x_i are changed simultaneously and "directions" are calculated as

$$v_i^{[j]} = \frac{\partial f}{\partial x_i^{[j]}} \tag{2.87}$$

In our case it is impossible to perform differentiation and all derivatives must be evaluated numerically in the following form

$$\frac{\partial f}{\partial x_i^{[j]}} = \frac{f(x_i^{[j]} + \lambda^{[j]}, \vec{x}^{[j]}) - f(\vec{x}_i^{[j]})}{\lambda^{[j]}} \tag{2.88}$$

where $\vec{x}^{[j]}$ are all arguments excluding $x_i^{[j]}$ and $\vec{x}_i^{[j]}$ is the vector which includes all arguments x_i. Then new coordinates of the point in the multi–dimensional space can be obtained by

$$x_i^{[j+1]} = S_c v_i^{[j]} + x_i^{[j]} \tag{2.89}$$

where S_c is a real number which characterises the rate of change of x_i.

This method can be very sensitive to the step λ. Usually $\lambda^{[j]}$ is decreased while j increases. The gradient method works very good when the functional surface is a rather smooth.

2.8 Numerical results

2.8.1 Stress–strain analysis

Let us consider some numerical results which are obtained on the basis of analytical solutions derived above. The results presented here are for both open–ended and closed cylinders.

Problem 2.1

The example involves an open–ended cylinder with a wall ratio $b/a = 1.25$ and internal radius of $a = 1m$. The cylinder consists of two orthotropic and three isotropic layers, the properties and thicknesses of which are given in Table 2.2.

Table 2.2. Material properties and thicknesses of layers.

Material	Thickness (m)	E_z (MPa)	E_θ (MPa)	$\nu_{\theta z}$	$\nu_{z\theta}$
Molybdenum	0.02	$3.4 \cdot 10^5$	$3.4 \cdot 10^5$	0.33	0.33
B(4)/5505 (Boron/Epoxy)	0.06	$2.04 \cdot 10^5$	$1.85 \cdot 10^4$	0.02086	0.23
Lead	0.095	$1.4 \cdot 10^4$	$1.4 \cdot 10^4$	0.45	0.45
Kevlar 49 (Aramid/Epoxy)	0.04	$7.6 \cdot 10^4$	$5.5 \cdot 10^3$	0.0246134	0.34
Steel 1008/1018	0.035	$2.07 \cdot 10^5$	$2.07 \cdot 10^5$	0.285	0.285

The exponent for the anisotropic layers is taken as $n = 1$ and $E_r = E_\theta$. The values of η_{ij} in equation (2.65) are obtained from equations (2.4) and (2.65) by setting $n = 0$ in equation (2.65). Three different types of loading conditions are considered, viz.

1. Cylinder under internal pressure ($p_0 = 1$ MPa).
2. Cylinder under the internal and external pressures ($p_0 = 5$ MPa, $p_5 = 2$ MPa). Moreover the residual stresses on the interfaces of the layers are taken into account by specifying $p_1 = 1.2$ MPa and $p_4 = -0.5$ MPa.
3. Cylinder under residual stresses on the layer interfaces with $p_1 = 1.2$ MPa and $p_4 = -0.5$ MPa.

Figure 2.4 shows the stress distribution through the thickness for the first case. Figures 2.5 and 2.6 show the corresponding results for the second and third loading cases.

Numerical results show that hoop and longitudinal stresses are considerably higher than the radial stresses. The highest stresses occur in the layers with higher moduli of elasticity. The distribution of the radial stresses σ_r is more uniform through the thickness of the cylinder. The discontinuities in radial stresses happen when the residual stresses on the layer interfaces are taken into account.

Let us now consider a closed–ended cylinder subjected to internal pressure $p_0 = 1$ MPa i.e., load case 1. Due to the ends being closed the longitudinal stresses σ_z are higher (see Fig. 2.7) than those of an open-ended cylinder (see Fig. 2.4).

Problem 2.2

In order to illustrate the behaviour of the normal stresses through the thickness of a cylindrical shell, depending on the material characteristics through the thickness, we consider a thick ring ($b/a = 2$) made of oak wood and subjected to only internal pressure p. Material properties are $E_r = 2 \cdot 10^6 \rho^n MPa$, $E_\theta = 0.95 \cdot 10^6 \rho^n MPa$ and $\nu_{r\theta} = 0$. Figure 2.8 shows the stress distribution of σ_θ and σ_r through the thickness for the different values of the exponent n. As is seen the circumferential stresses σ_θ remain almost unchanged on the mid-plane of the ring, while approaching the external surfaces the change in the stresses is more pronounced. The insert in Figure 2.8 shows the corresponding σ_r values.

2.8.2 Optimisation of the pressure vessels

Problem 2.3 Let us consider the optimisation of the composite single layered pressure vessels of different thickness ratio. If not mentioned otherwise, the composite material T300/Epoxy is used for the present study. All material data were taken from Ref [88] and are given in Table 2.3.

Figure 2.9 shows the failure surface with respect to radial direction and fibre orientation for $b/a = 1.25$ and $F = 0$. It is observed that the failure pressure reaches its maximum value at about the same angle for all $a \leq r \leq b$. Figure 2.10 indicates that the failure at a given angle ($\varphi = 45°$) may occur at $r = a$ or b depending the thickness ratio b/a. For lower values of b/a the location of failure is at $r = b$ while for higher values it is at $r = a$. For a given thickness ratio the location of the failure depends on the fibre orientation as shown in Figure 2.11 which shows the curves of P_{cr} plotted against the radial direction for $b/a = 1.25$. It is observed that the failure location is $r = a$ for low values of φ and $r = b$ for high values of φ.

Next the behaviour of the failure pressure with respect to fibre orientation is investigated. Figure 2.12 shows the curves of P_{cr} plotted against φ for various values of b/a with $F = 0$ and Figure 2.13 for various values of F with $b/a = 1.25$. In both figures P_{cr} is given at $r = a$. Figure 2.12 indicates that thickness has a marginal effect on the optimal φ and φ_{opt} is in the range 54°–57° for $1.025 \leq b/a \leq 1.5$. However, the axial load has a major effect on φ_{opt} and as F increases φ_{opt} decreases as the fibres align themselves with the axial load.

Optimisation results are given in Figure 2.14 which shows the curves of $P_{\max}$ versus b/a with $F = 0$ for various materials. The insert in Figure 2.14 shows the corresponding φ_{opt} values.

Problem 2.4 Next we consider pressure vessels of different thickness ratios and number of layers. For a multilayer CFRP T300/5208 cylinder, results in Table 2.4 were obtained for layers of equal thickness and a layer thickness ratio of $(b/a - 1)/nl$; nl is the number of layers. The last columns of the table shows that for $b/a = 1.1$ and 1.2 an increase in the number of layers increases the burst pressure by a significant amount. It is worth mentioning that the equal thickness consideration sometimes may not necessarily give the best burst pressure. In this case, for a given number of layers, along with the angle optimisation we should also optimise the layer thicknesses, instead of taking equal thickness, to obtain the best burst pressure.

All values of the burst pressure were calculated at the weakest points within the cylinder thickness. In the case of T300/5208 the failure point shifts from outside surface to the inside as the wall thickness increases. At about $b/a = 1.15$ this transition occurs.

Figure 2.15 and 2.16 show graphical representation (functional surface) of the two-dimensional optimising problem for the thin (b/a=1.1) and thick (b/a=1.5) two layered cylinders, respectively. As with the single layered cylinder the change in burst pressure is smoother for the case of the thick cylinder.

Table 2.3. Engineering constants and strength of the composite materials

Material	E_r, E_θ MPa	E_z MPa	$\nu_{\theta z}$	$G_{\theta z}$ MPa	X MPa	X' MPa	Y MPa	Y' MPa	S MPa
T300/5208	$1.03 \cdot 10^4$	$1.81 \cdot 10^5$	0.28	7170	1500	1500	40	246	68
Kev49/Epoxy	5500	$7.6 \cdot 10^4$	0.34	2300	1400	235	12	53	34
H-IM6/Epoxy	$1.12 \cdot 10^4$	$2.03 \cdot 10^5$	0.32	8400	3500	1540	56	150	98
E-glass/Epoxy	8270	$3.86 \cdot 10^4$	0.26	4140	1062	610	31	118	72

Table 2.4. Prediction of burst pressure of several multilayer vessels; material: T300/N5208

b/a	No of Layers nl	Optimum Angle Combination (inside to outside)	Burst Pressure (MPa)	Percent Change in Burst Pressure
1.01	1	54.2	4.49	—
	2	49.8/57.5	4.51	0.45
	3	47.2/62.7/48.6	4.68	3.77
	4	45.5/69.0/46.6/46.5	4.76	1.71
	5	44.0/77.8/44.8/44.7/44.7	4.92	3.36
1.1	1	54.5	47.90	—
	2	50.1/57.9	48.14	0.50
	3	45.9/45.5/66.7	56.38	17.12
	4	71.0/70.3/69.6/0.0	82.17	45.74
	5	70.8/70.2/69.7/69.2/0.0	91.99	11.95
1.2	1	54.7	104.27	—
	2	49.7/58.7	105.07	0.77
	3	46.0/46.5/67.9	126.46	20.36
	4	46.5/46.0/45.8/74.3	138.48	9.50
	5	0.0/70.0/68.6/67.7/67.1	183.97	32.85
1.3	1	56.3	150.10	—
	2	59.5/51.9	150.45	0.23
	3	58.2/58.6/50.9	150.47	0.013
	4	59.6/60.6/52.6/49.0	157.86	4.91
	5	68.0/56.8/51.4/48.4/46.6	162.48	2.93
1.5	1	57.3	181.00	—
	2	64.4/45.5	183.61	1.44
	3	68.9/50.2/43.7	184.08	0.26
	4	71.6/55.0/46.4/43.1	184.15	0.038
	5	72.3/60.3/49.3/45.0/42.9	184.14	-0.005

2.9 Conclusions

A three–dimensional theory for anisotropic thick composite cylinders subjected to axisymmetrical load conditions is derived. The exact analytical solution is obtained for multilayered anisotropic pressure vessels. There are no restrictions on the number of layers and their sequences since the thicknesses and elastic properties of the layers can differ considerably. The theory presented is capable of treating thick as well as thin cylindrical shells, and is in turn applicable to pressure vessels and open–ended cylinders, isotropic, orthotropic and anisotropic layers. The distinctive feature of this theory is that expressions for the radial, circumferential and shear stresses have been modified to include the effect of the closed ends. This is an advantageous feature not found in many available approaches. The proposed stress analysis allows an accurate prediction of the burst pressure.

In addition, the exact analytical solution for multilayered orthotropic cylinders with variable material properties is obtained. This theory also takes interlaminar stresses into account. The solution does not require the computation of the interface normal tractions since the system of equations *a priori* takes into account boundary and interface conditions. However, the radial and hoop stresses do not depend on whether the cylinder has closed or open ends. It should be noted that the influence of closed ends on the normal stresses does not have significant effect, but does cause a considerable in the shear stresses. The solutions are given for both open–ended and closed–ended cylinders.

The three–dimensional interactive Tsai–Wu failure criterion is employed in order to predict the maximum burst pressure. The optimisation of the pressure vessels shows that the stacking sequence can be employed effectively to maximize burst pressure. This problem involves certain mathematical difficulties including high sensitivity of the solution with respect to its parameters, and non-unimodality of the functional space. In this connection it should be noted that only very accurate stress analysis, namely three–dimensional elasticity solution, allows to predict the most efficient layer sequence. This is especially true for thick pressure vessels.

Mathematical background for maximisation of the function is presented. Three methods were used for this purpose, namely, golden section method as a one–dimensional method and iterative and gradient as multidimensional methods.

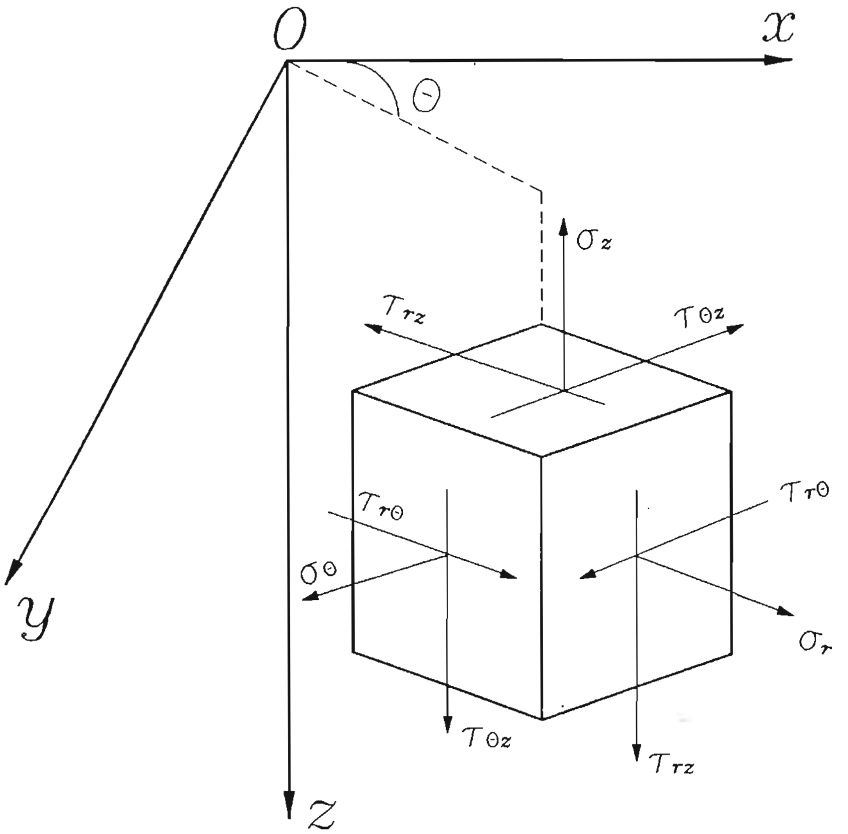


Figure 2.1 Stress components for the cylindrical coordinate system.

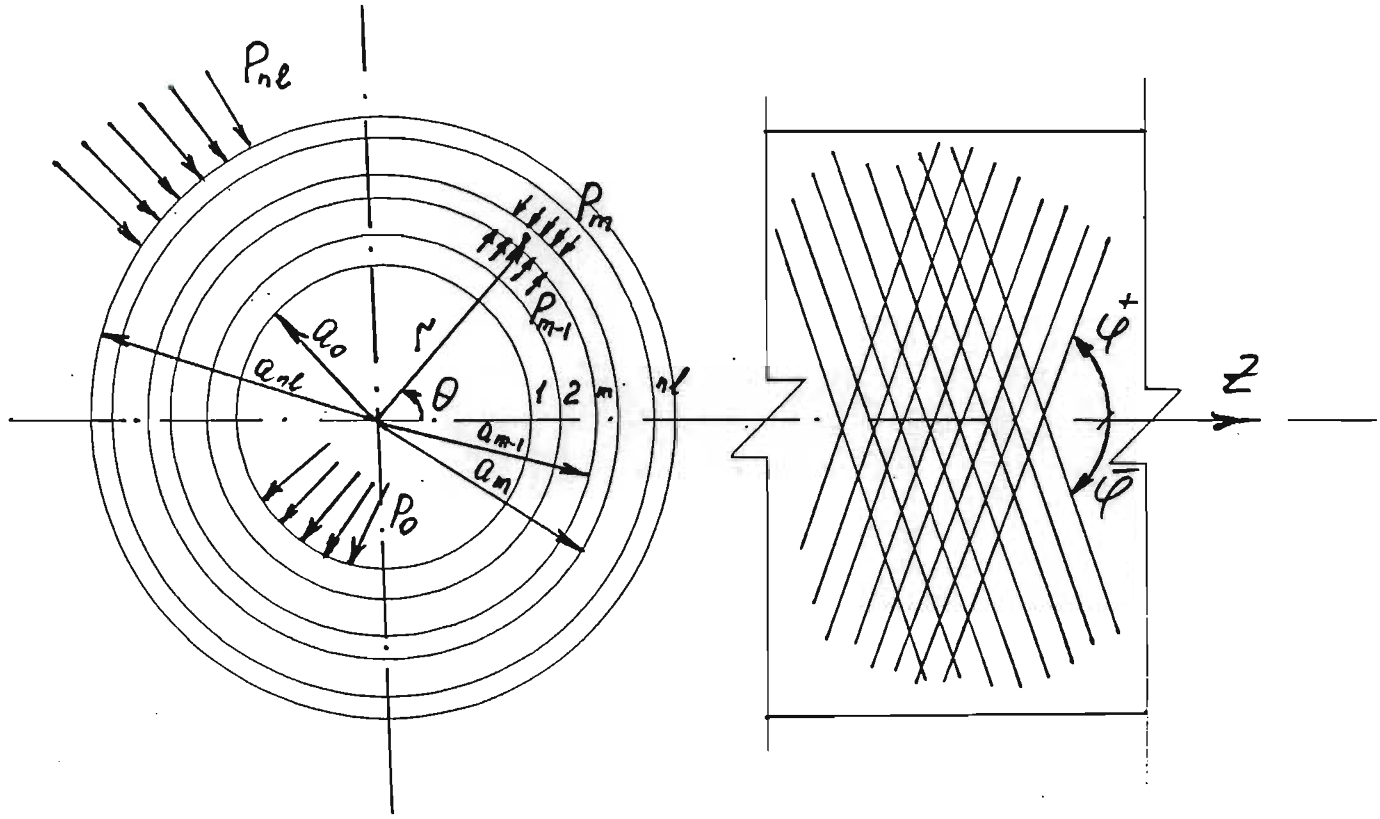


Figure 2.2 Configuration of thick anisotropic cylinder.

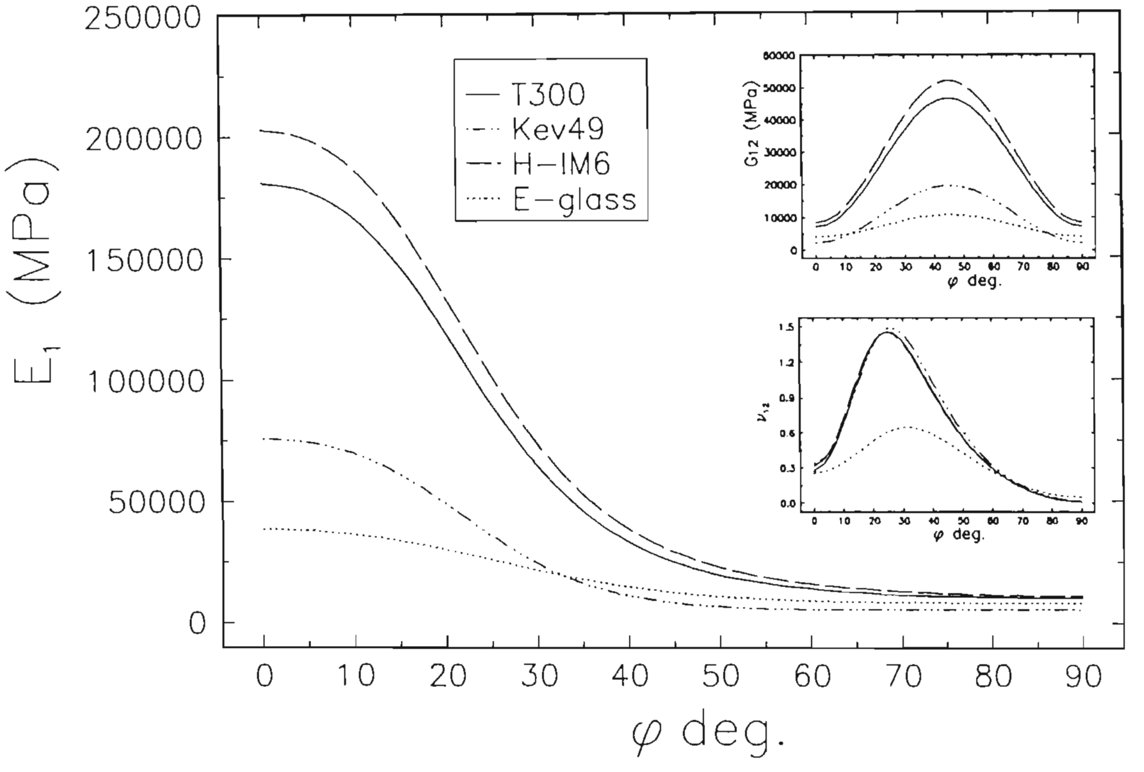


Figure 2.3 Elastic constants plotted against the angle φ.

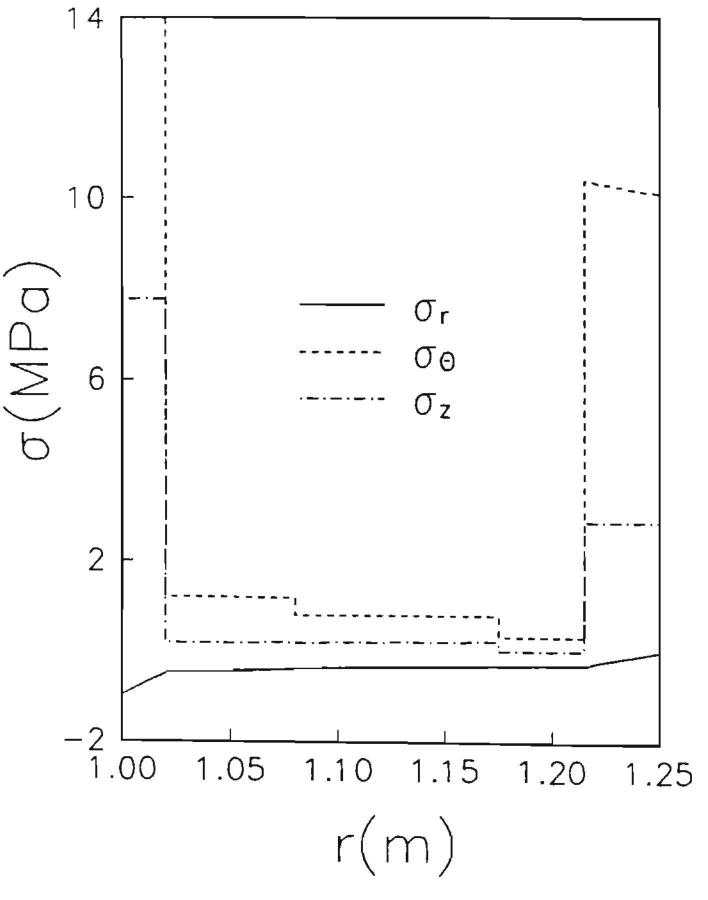


Figure 2.4 Stresses plotted against the radial distance r for a layered cylinder under the loading case 1.

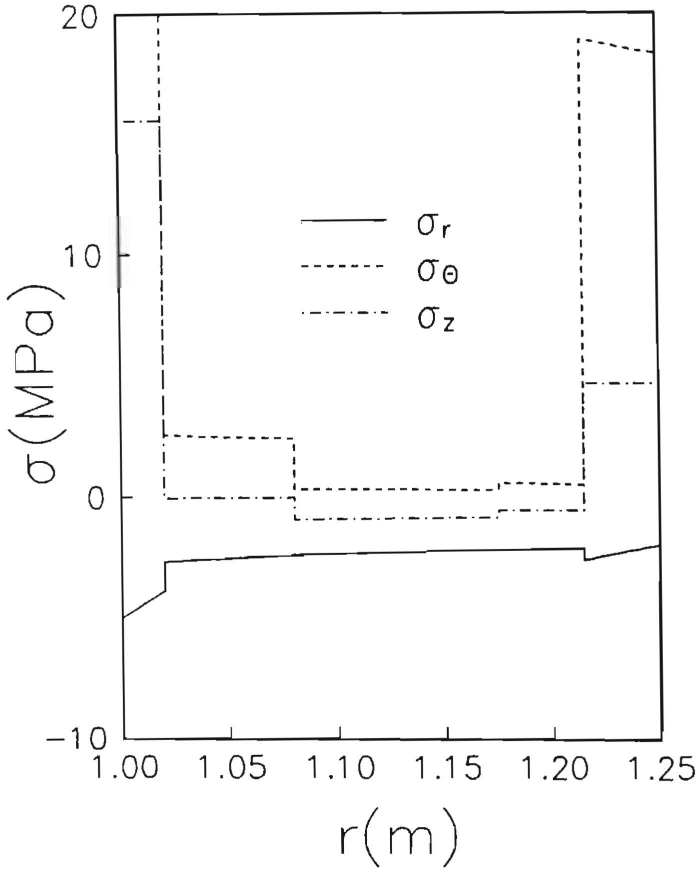


Figure 2.5 Stresses plotted against the radial distance r for a layered cylinder under the loading case 2.

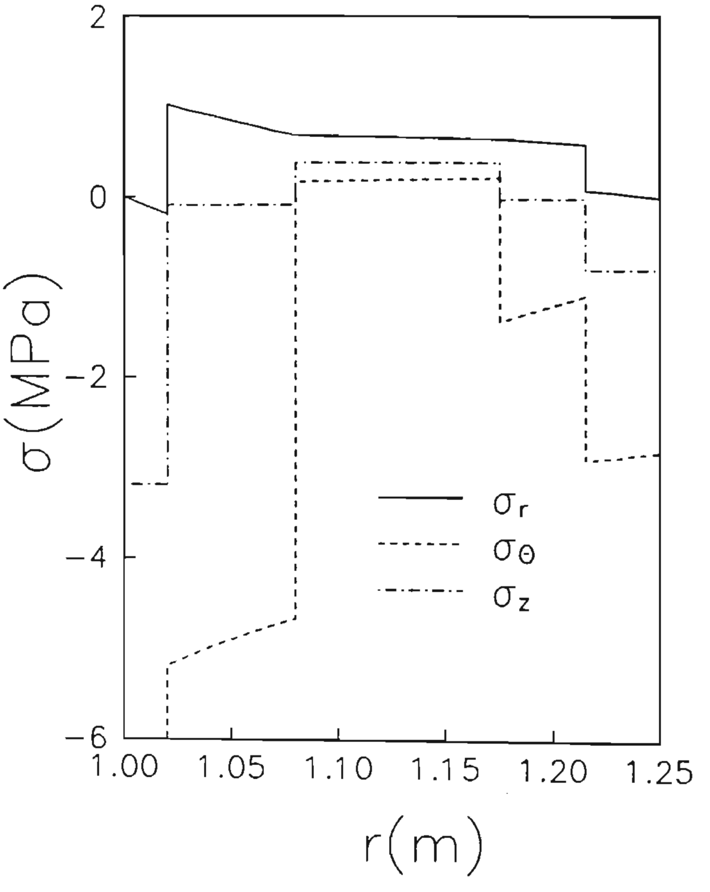


Figure 2.6 Stresses plotted against the radial distance r for a layered cylinder under the loading case 3.

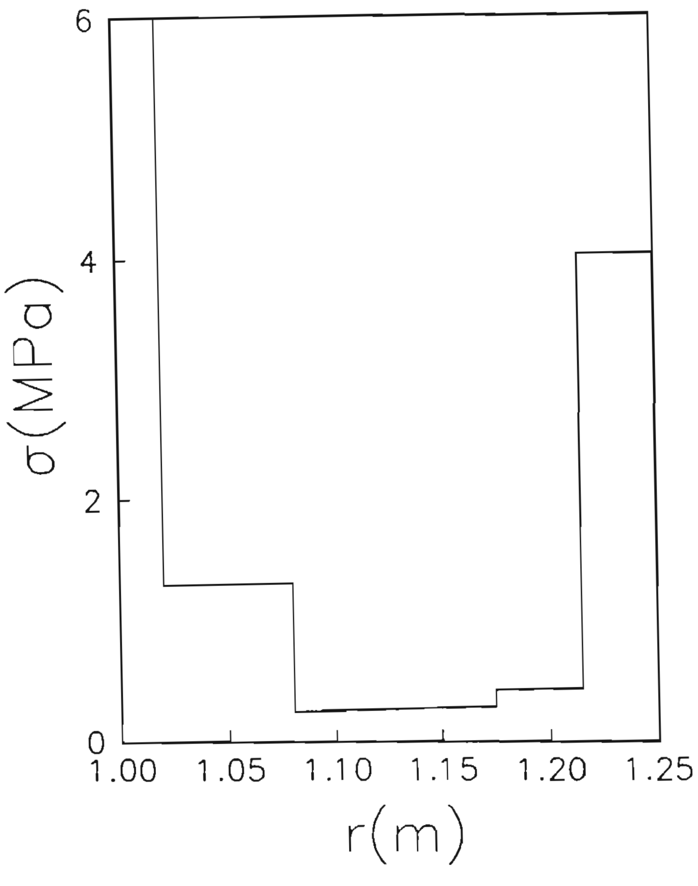


Figure 2.7 Stresses σ_z plotted against the radial distance r for a closed-ended layered cylinder under the loading case 1.

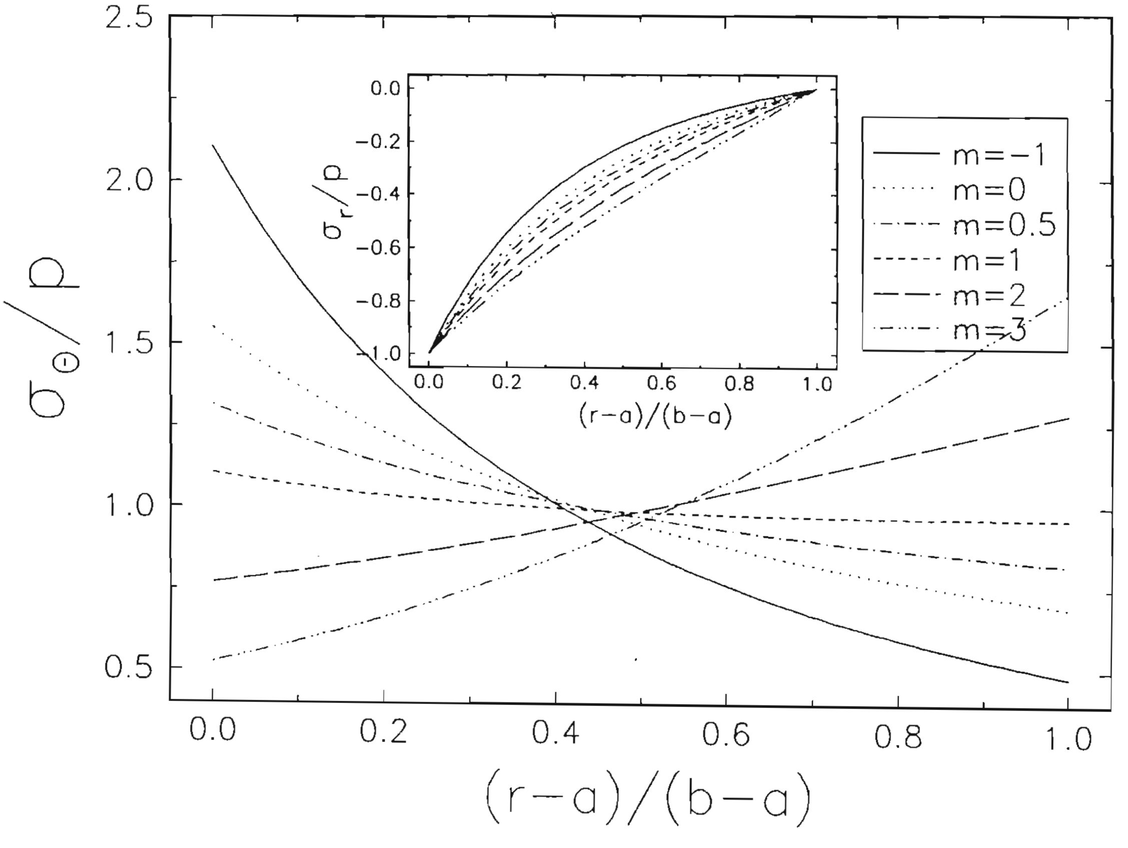


Figure 2.8 Stresses plotted against the radial distance for a homogeneous cylinder $b/a = 2$.

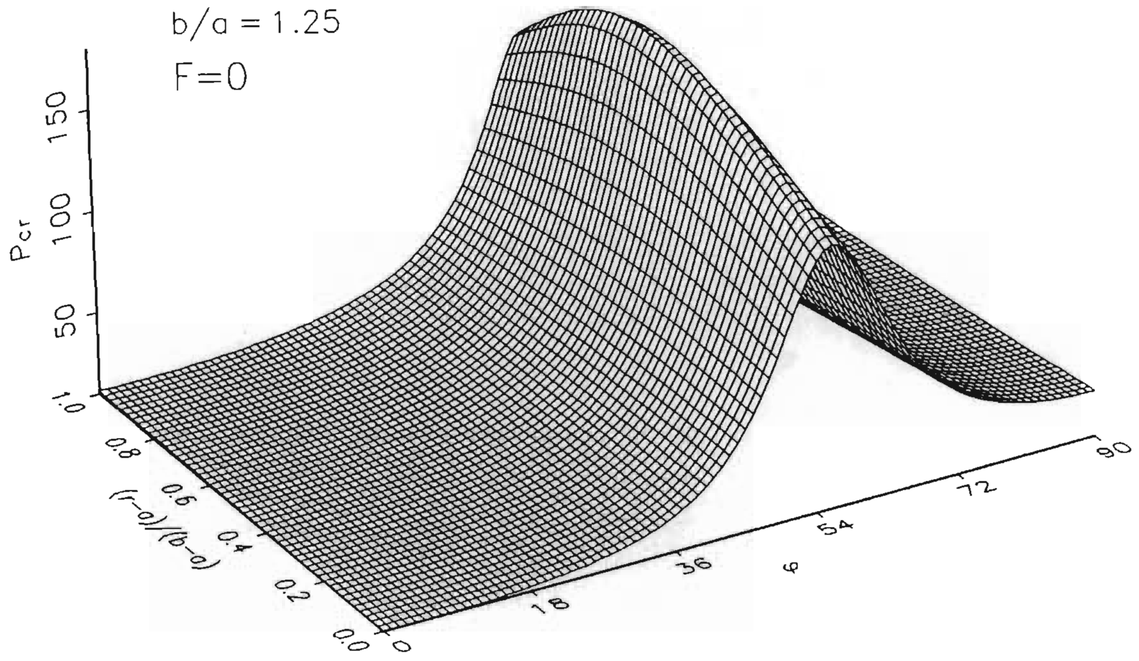


Figure 2.9 Failure surface plotted against the radial distance and the fibre orientation φ.

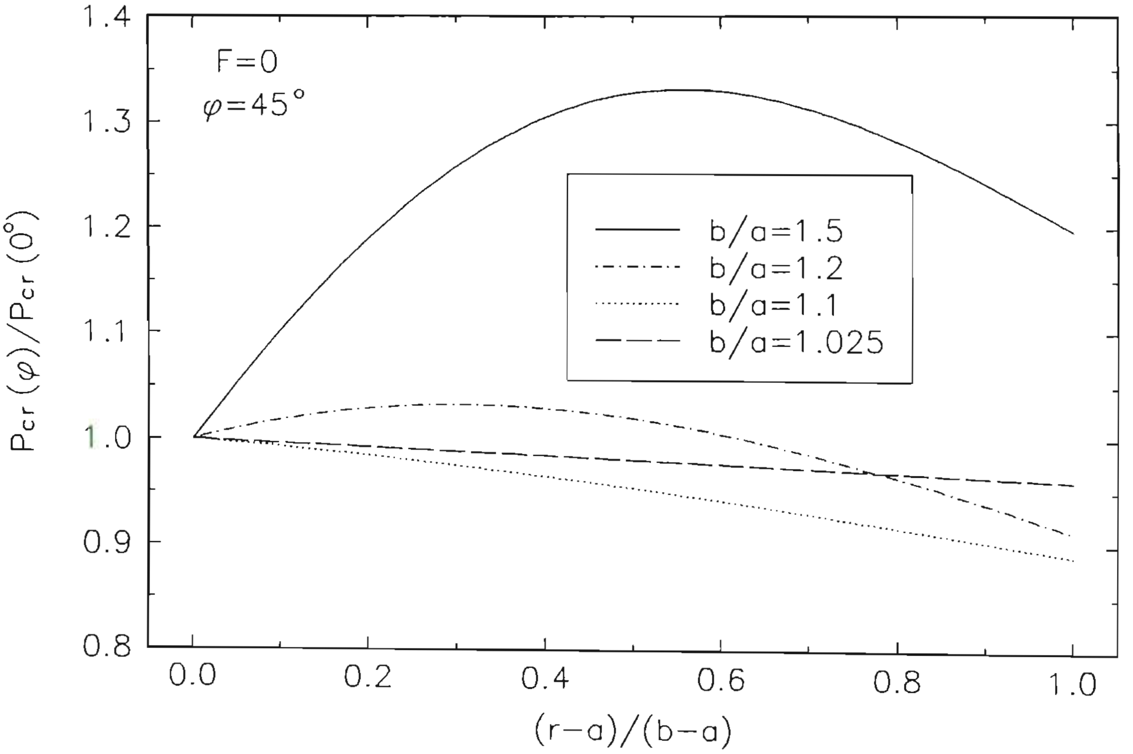


Figure 2.10 Distribution of the failure pressure through the thickness for a given angle $\varphi = 45°$.

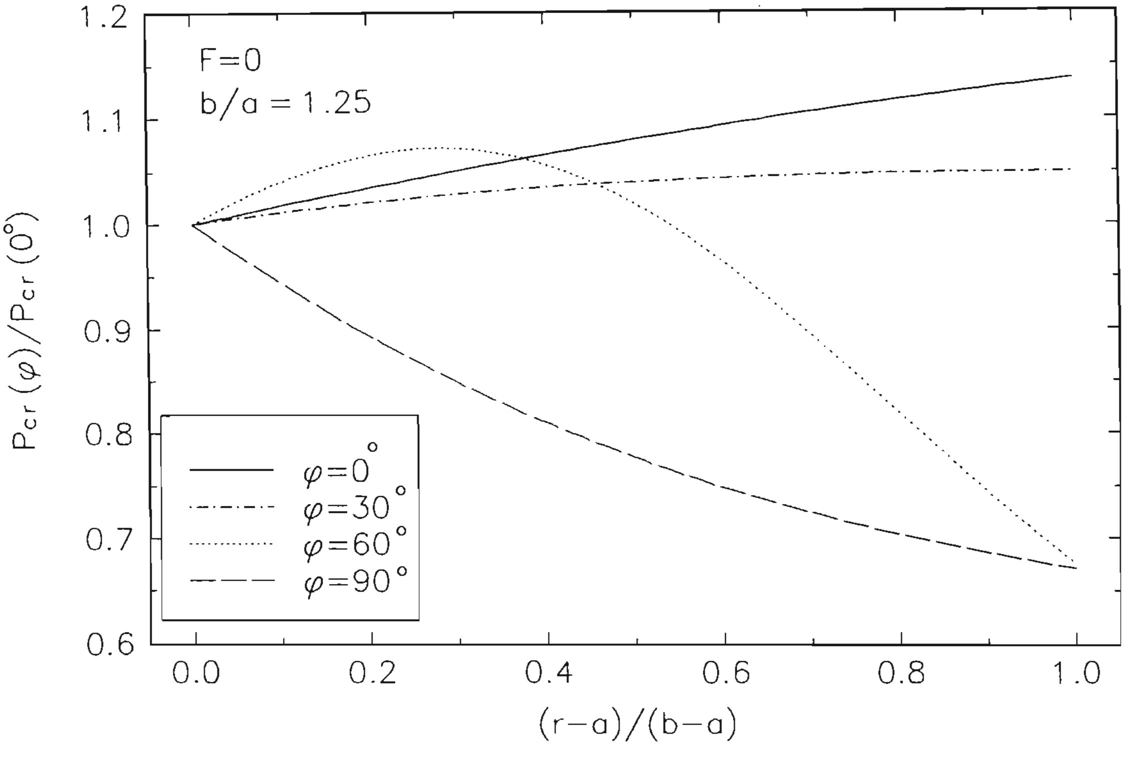


Figure 2.11 Failure pressure plotted against the radial direction for $b/a = 1.25$.

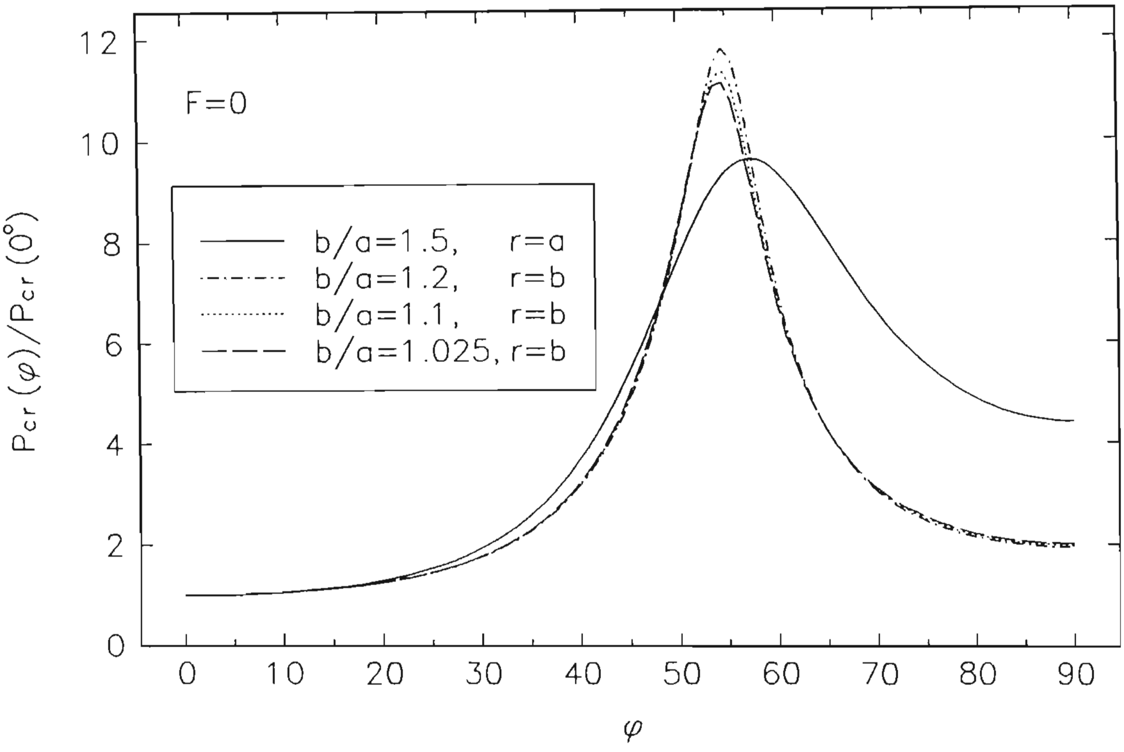


Figure 2.12 Failure pressure plotted against the angle φ for various ratios of b/a.

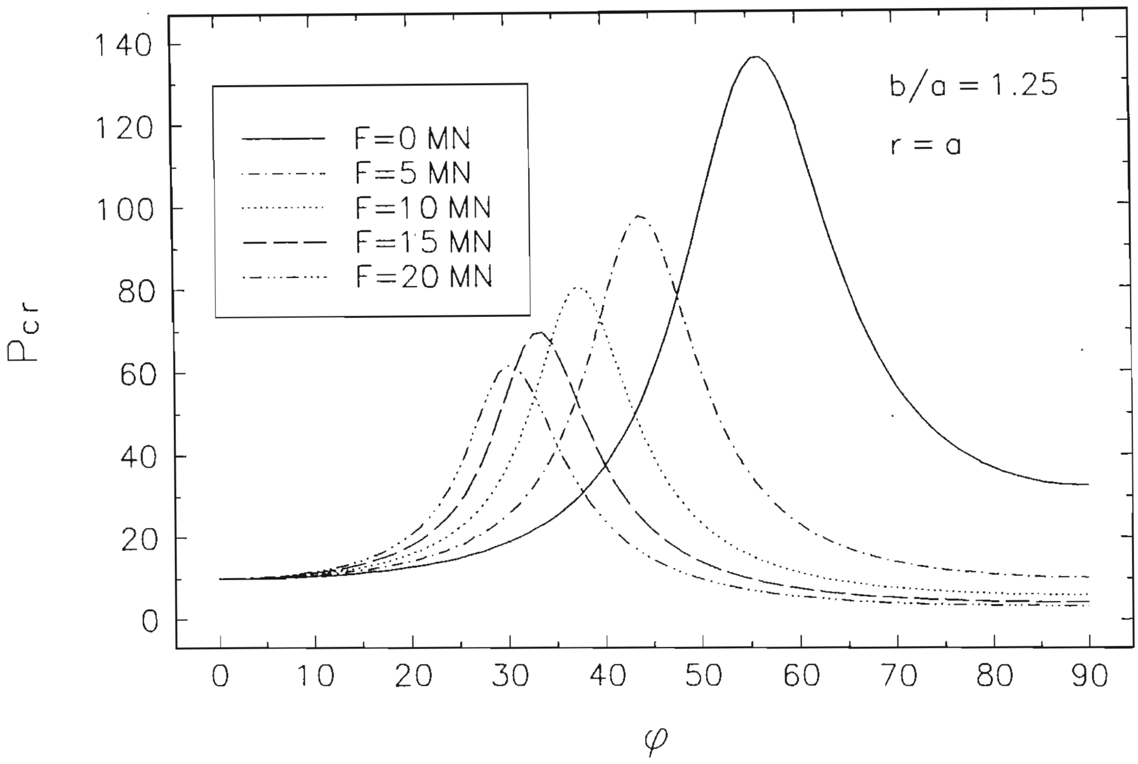


Figure 2.13 Failure pressure plotted against the angle φ for various values of F.

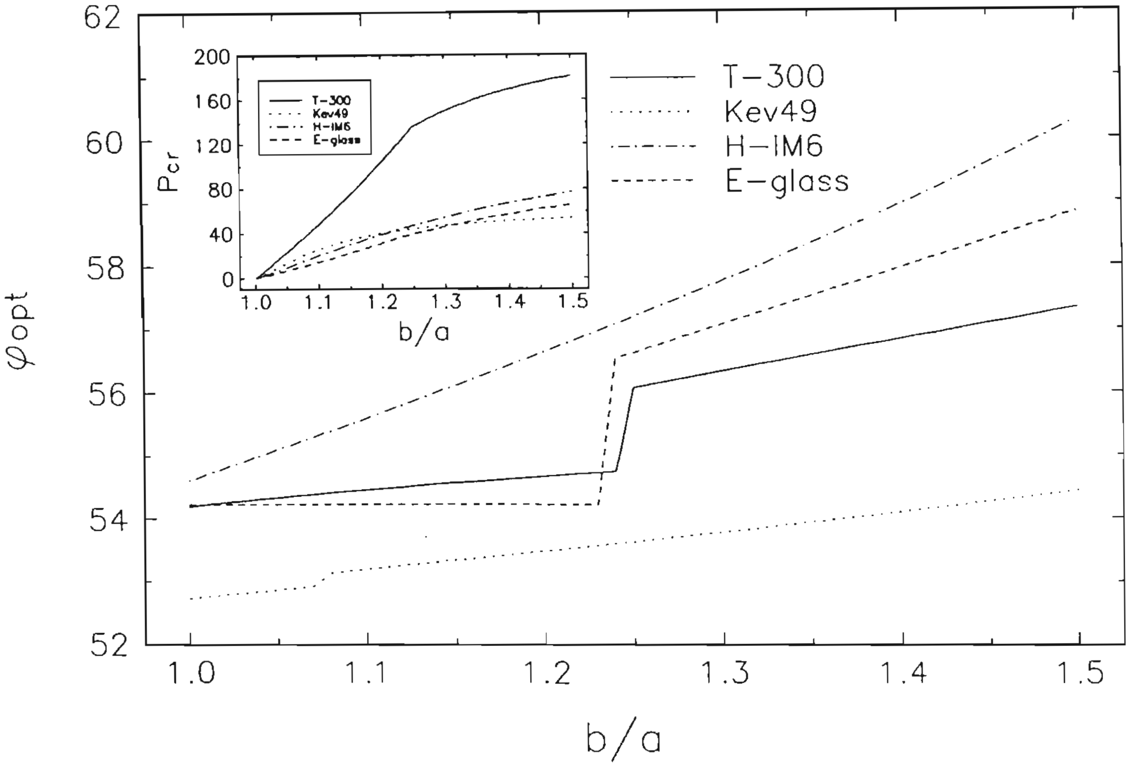


Figure 2.14 Optimal fibre orientation φ_{opt} and burst pressure versus b/a.

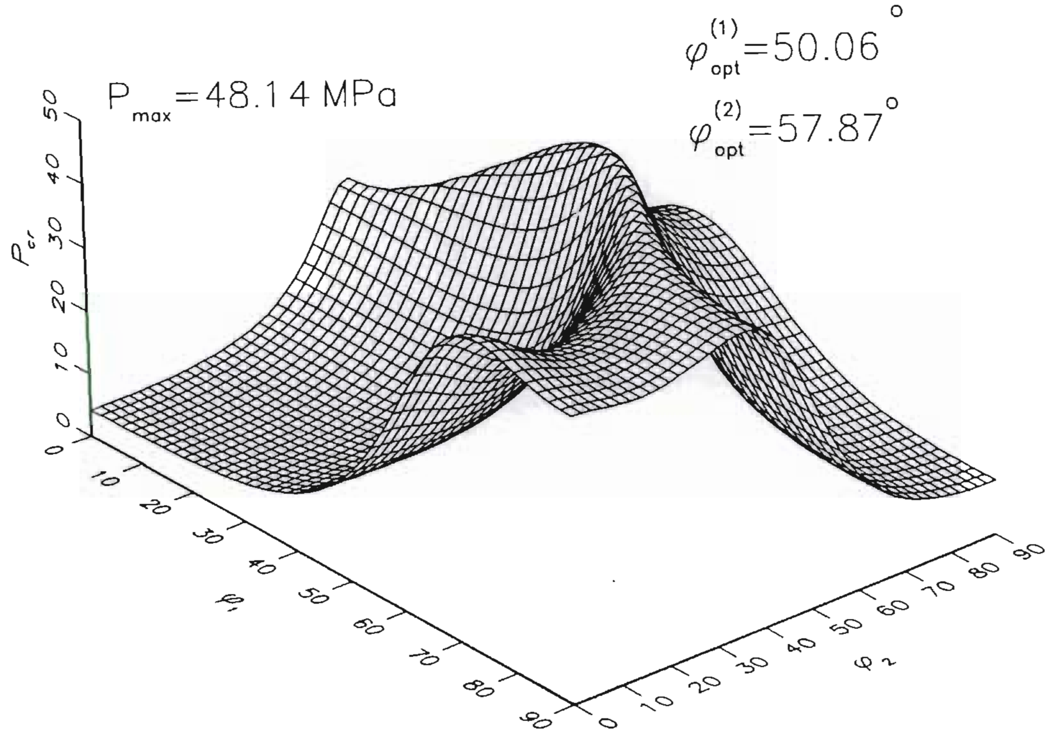


Figure 2.15 Burst pressure plotted against ply angles for two–layered pressure vessels with $b/a = 1.1$.

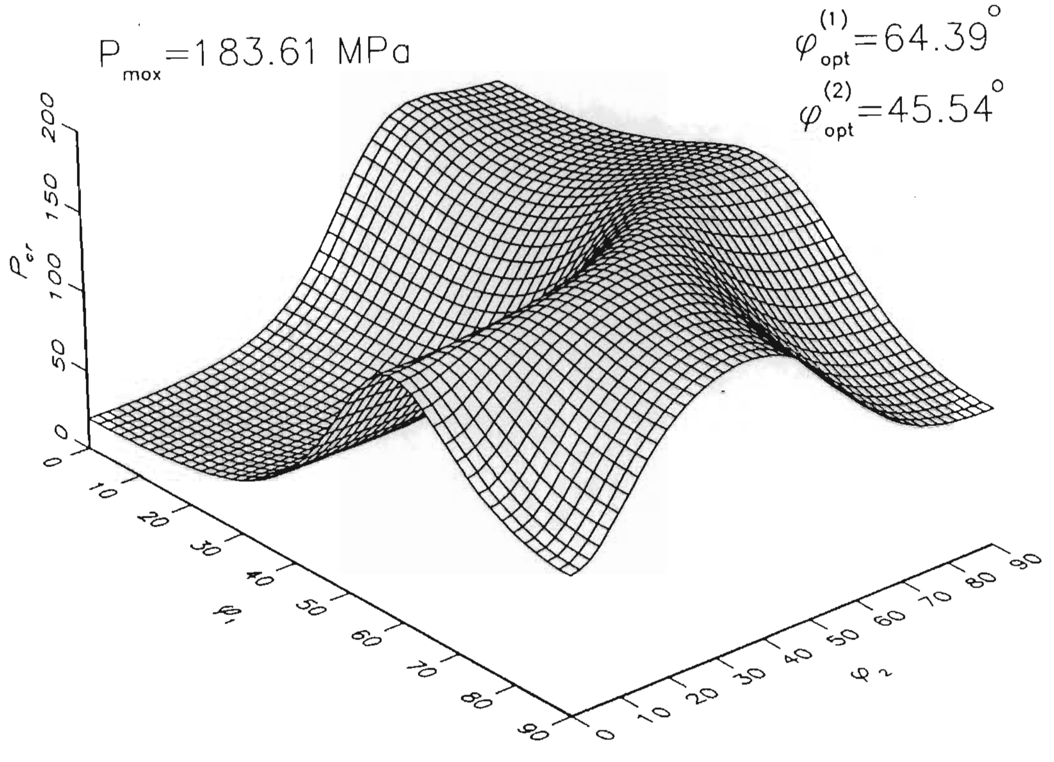


Figure 2.16 Burst pressure plotted against ply angles for two-layered pressure vessels with $b/a = 1.5$.

Chapter 3

Derivation of Higher–Order Theory for the Solution of Dynamic Problems of Laminated Plates and Shells

3.1 Introduction

The main objective of this chapter is to derive a comprehensive higher–order theory of laminated plates and shells which can accurately predict the dynamic behaviour of these structures under various loading conditions.

An improved transverse shear and normal deformation higher–order theory is developed for the solution of dynamic problems involving multilayered plates and shells with arbitrary number and sequence of transversely isotropic layers. The layers may differ significantly in their physical and mechanical properties. The theory developed is based on the kinematic hypotheses which are derived using iterative technique. Dynamic effects, such as forces of inertia, and the direct influence of external loading on the components of stress and strain are included in the initial stage of derivation where kinematic hypotheses are formulated. New variables which have clear physical meanings are introduced. The system of governing differential equations and the complete set of boundary conditions are derived. The closed form solutions are given for problems involving forced and natural vibrations.

Ren [77], and Verijenko *et al* [96]. In recent years, numerous refined approaches for the analysis of composite plates and shells have been formulated. Contributions by Ambartsumyan [3], Reddy [69, 71], Reddy and Phan [70], Librescu [47], Vasilyev [92] and Noor and Peters [57] should be mentioned. Survey of different theoretical and computational models may be found in reviews by Kant and Junghare [38], Dutchenko *et al* [19], Bert [8], Noor and Burton [58, 59] and Reddy [72]. Several monographs have also been written on the subject [9, 11, 26, 46, 63, 64, 68, 79, 98]. It is noted that the list of references is not intended to be a comprehensive one and the specific publications were referred to because of their relevance to the present chapter.

A study of the literature indicates that, in the case of dynamic analysis of laminated structures in which the layers may have significantly different physical characteristics, it is also necessary to consider the phenomenon of normal deformation. Moreover, most of the known dynamic higher–order theories are based on the hypotheses which are derived from a consideration of the quasi–static problem. In this case the kinematic hypotheses do not fully reflect the physical essence of the problem. Therefore, the study of the dynamic behaviour of laminated structures on the basis of improved higher–order theories will fill a gap in the analysis of thick composites under dynamic loads.

3.3 Basic assumptions and derivation of kinematic hypotheses

We consider shells with transversely isotropic layers which are weak in their resistance to transverse shear and normal deformation. No limitations are placed on the thickness, rigidity, density, number and/or sequence of the layers. The physical and mechanical characteristics of the layers may vary through the thickness. The assumption that the layers are perfectly bonded ensures their deformation as a single unit without delamination. Thus, the structure of the shell through the thickness is arbitrarily irregular and heterogeneous. The shell is represented by a curvilinear orthogonal coordinate system x_1Ox_2 which is parallel to the bounding surfaces and surfaces of contact between the layers (Fig. 3.1). The axes of the curvilinear coordinates $x_i = constant$ $(i = 1, 2)$ coincide with the principal lines of curvature and the coordinate $z = x_3$ is defined along the normal to the reference surface x_1Ox_2. It is assumed that the coefficients of the first quadratic form of a surface are close to unity, i.e., $A_1 \approx A_2 \approx 1$, and the main curvatures are constant, i.e.,

$k_{ij} = constant, \quad i,j = 1,2$. The total thickness of the shell is small in comparison to radii of the curvatures ($1 + k_{ij} \approx 1$). Dynamical loads are applied on the outer and inner surfaces of the laminate so that

$$p_s^{\pm}(x_i, t) = p^{\pm}; \quad s = 1,2,3 \tag{3.1}$$

where p^+ and p^- are loads applied on the outer and inner surfaces, respectively, and the subscript s denotes the corresponding coordinate axes. The reference surface x_1Ox_2 may be positioned arbitrarily through the thickness of the shell. It may be chosen within any layer, coincide with the interlaminar or external surfaces as dictated by the nature of the problem under consideration. The stress conditions on the external surfaces may be written as

$$\sigma_{s3}^{(1)} = -p_s^- \qquad \text{for } z = a_0 \ \ (k = 1) \tag{3.2}$$

$$\sigma_{s3}^{(n)} = +p_s^+ \qquad \text{for } z = a_n \ \ (k = n) \tag{3.3}$$

$$s = 1,2,3$$

where k denotes the layer number and n is the total number of layers.

Since the layers are assumed to be perfectly bonded, the continuity conditions for an arbitrary surface $z = a_{k-1}$ are given by

$$\sigma_{s3}^{(k)} = \sigma_{s3}^{(k-1)} \qquad \text{(static)} \tag{3.4}$$

$$u_s^{(k)} = u_s^{(k-1)} \qquad \text{(kinematic)} \tag{3.5}$$

In the following derivations, summation is assumed over subscripts $i,j = 1,2; \quad s,r, = 1,2,3$, and p,q,f,g. However no summation is implied over $k = 1,2,\ldots,m,\ldots,n$. A subscript after a comma denotes differentiation with respect to the variable following the comma and a superscript is expressed in brackets to distinguish it from an exponent.

Considering "small" bending [60] the strain components of the k–th layer may be expressed as

$$\begin{aligned} 2e_{ij}^{(k)} &= u_{i,j}^{(k)} + u_{j,i}^{(k)} + 2k_{ij}u_3^{(k)} \\ 2e_{i3}^{(k)} &= u_{i,3}^{(k)} + u_{3,i}^{(k)} \\ e_{33}^{(k)} &= u_{3,3}^{(k)} \end{aligned} \tag{3.6}$$

where $u_i^{(k)}(x_i, z, t)$ and $u_3^{(k)}(x_i, z, t)$ are displacements of the k–th layer in the tangential x_i $(i = 1,2)$ and normal $z = x_3$ directions, respectively, and k_{ij}'s are curvatures of the shell. The displacements of the reference surface ($z = 0, k = m$) may be expressed as

$$u_i^{(m)}(x_i, 0, t) = u_i; \; u_3^{(m)}(x_i, 0, t) = w \tag{3.7}$$

and the strains and the curvatures due to deformation as

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) + k_{ij}w; \; \kappa_{ij} = -w_{,ij} \tag{3.8}$$

which satisfy the well known relations [60]

$$\begin{aligned} &2\epsilon_{12,12} - \epsilon_{11,22} - \epsilon_{22,11} = k_{11}\kappa_{22} + k_{22}\kappa_{11} - 2k_{12}\kappa_{12} \\ &\kappa_{11,2} - \kappa_{12,1} = 0; \; \kappa_{22,1} - \kappa_{12,2} = 0 \end{aligned} \tag{3.9}$$

The generalized Hooke's law for a transversely isotropic layer k of the shell, where the surface of isotropy at any point (x_i, z) is orthogonal to the normal, may be expressed as [3]

$$\begin{aligned} e_{11}^{(k)} &= a_{11}^{(k)}\sigma_{11}^{(k)} + a_{12}^{(k)}\sigma_{22}^{(k)} + a_{13}^{(k)}\sigma_{33}^{(k)} \\ e_{22}^{(k)} &= a_{21}^{(k)}\sigma_{11}^{(k)} + a_{22}^{(k)}\sigma_{22}^{(k)} + a_{23}^{(k)}\sigma_{33}^{(k)} \\ e_{33}^{(k)} &= a_{31}^{(k)}\sigma_{11}^{(k)} + a_{32}^{(k)}\sigma_{22}^{(k)} + a_{33}^{(k)}\sigma_{33}^{(k)} \\ 2e_{23}^{(k)} &= a_{44}^{(k)}\sigma_{23}^{(k)}; \; 2e_{13}^{(k)} = a_{55}^{(k)}\sigma_{13}^{(k)}; \; 2e_{12}^{(k)} = a_{66}^{(k)}\sigma_{12}^{(k)} \end{aligned} \tag{3.10}$$

where $a_{ij}^{(k)}$ denotes the elastic compliance coefficients of the k–th layer. In equation (3.10) the compliance characteristics are given by

$$\begin{aligned} &a_{11}^{(k)} = a_{22}^{(k)} = \frac{1}{E_k}; \; a_{33}^{(k)} = \frac{1}{E'_k}; \; a_{12}^{(k)} = a_{21}^{(k)} = -\frac{\nu_k}{E_k} \\ &a_{13}^{(k)} = a_{31}^{(k)} = a_{23}^{(k)} = a_{32}^{(k)} = -\frac{\nu'_k}{E'_k}; \; a_{44}^{(k)} = a_{55}^{(k)} = \frac{1}{G'_k}; \; a_{66}^{(k)} = \frac{1}{G_k} \end{aligned}$$

where $E_k = E_k(z)$, $\nu_k = \nu_k(z)$, $G_k = G_k(z)$ are modulus of elasticity, Poisson's ratio and shear modulus in the plane of isotropy, respectively; $E'_k = E'_k(z)$, $G'_k = G'_k(z)$ are moduli of elasticity and shear in the transversal direction; $\nu'_k = \nu'_k(z)$ is Poisson's ratio, which characterizes reduction in the plane of isotropy when tension is applied in the transversal direction. All elastic properties of the k–th layer are assumed to be functions of the coordinate z $(a_{k-1} \leq z \leq a_k)$. The Hooke's law for specific cases of material can be obtained by specifying the material properties. For example if it is assumed that $\nu'_k = 0$, then $a_{13}^{(k)} = a_{31}^{(k)} = a_{23}^{(k)} = a_{32}^{(k)} = 0$ and we have a layer for which the influence of the normal stresses $\sigma_{33}^{(k)}$ on the tangential components $e_{11}^{(k)}, e_{22}^{(k)}$ of the strain tensor is excluded and, similarly, the influence of the tangential components $\sigma_{11}^{(k)}, \sigma_{22}^{(k)}$ of the stress tensor on the strain $e_{33}^{(k)}$ in the transversal direction is also excluded. Thus the normal deformation due to the Poisson's effect is excluded. If

$E'_k = \infty$, then $a_{13}^{(k)} = a_{31}^{(k)} = a_{23}^{(k)} = a_{32}^{(k)} = a_{33}^{(k)} = 0$ and we have an incompressible layer ($e_{33}^{(k)} = 0$) in which the strains $e_{11}^{(k)}, e_{22}^{(k)}$ are independent of the normal stress $\sigma_{33}^{(k)}$. This effect may be obtained by assuming $\sigma_{33}^{(k)} = 0$. Furthermore, if we assume $G'_k = \infty$, then the layer also becomes perfectly rigid under the transversal shear, i.e., $e_{13}^{(k)} = e_{23}^{(k)} = 0$. The above set of assumptions is equivalent to that used in the derivation of the classical theory which is a special case of the theory presented in this study.

3.3.1 Classical model

We first obtain the expressions of the classical model based on the Kirchhoff–Love hypotheses. We will subsequently make use of them for the derivation of kinematic hypotheses of the higher-order theory. In the classical model the following relations can be written for the *k–th* layer:

$$e_{i3}^{(k)} = 0; \; e_{33}^{(k)} = 0; \; \sigma_{33}^{(k)} = 0; \; i = 1, 2 \tag{3.11}$$

Substituting equations (3.6) into the first two hypotheses (3.11), and integrating the resulting expressions, the kinematic model of the shell may be obtained as

$$u_i^{(k)} = u_i - w_{,i} z; \; u_3^{(k)} = w \tag{3.12}$$

In these calculations, the continuity conditions (3.5) and relations (3.7) have been taken into account. The strains of the *k–th* layer in tangential directions can be obtained from equations (3.6) and (3.12) as

$$e_{ij}^{(k)} = \epsilon_{ij} + \kappa_{ij} z; \; i, j = 1, 2 \tag{3.13}$$

where $\epsilon_{12} = \epsilon_{21}, \kappa_{12} = \kappa_{21}, e_{12}^{(k)} = e_{21}^{(k)}$.

The normal stresses in the *k–th* layer may be determined from equations (3.10) in conjunction with the static hypothesis $\sigma_{33}^{(k)} = 0$, or from assumption $E'_k = \infty$ together with equation (3.13). These calculations give

$$\begin{aligned} \sigma_{11}^{(k)} &= E_{0k}[(\epsilon_{11} + \nu_k \epsilon_{22}) + (\kappa_{11} + \nu_k \kappa_{22}) z] \\ \sigma_{22}^{(k)} &\rightleftharpoons \sigma_{11}^{(k)}; \; \sigma_{12}^{(k)} = E_{0k}(1 - \nu_k)(\epsilon_{12} + \kappa_{12} z) \end{aligned} \tag{3.14}$$

where

$$E_{0k} = \frac{E_k}{1 - \nu_k^2}; \; 2G_k = \frac{E_k}{1 + \nu_k} = E_{ok}(1 - \nu_k)$$

In equation (3.14) the symbol $\rightleftharpoons$ indicates that the expression for $\sigma_{22}^{(k)}$ is of the same form as that for $\sigma_{11}^{(k)}$ with the provision that the subscript 11 is replaced with 22 and vice versa. The transverse shear and normal stresses cannot be found from the Hooke's law because of the hypotheses (3.11). In order to determine these stresses, we use the equations of motion of the shell, and for the *k–th* layer they may be written as [60]

$$\begin{aligned} &\sigma_{ij,j}^{(k)} + \sigma_{i3,3}^{(k)} = \rho_k \ddot{u}_i^{(k)} \\ &\sigma_{33,3}^{(k)} + \sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} = \rho_k \ddot{u}_3^{(k)};\ i,j = 1,2 \end{aligned} \tag{3.15}$$

where $\rho_k \ddot{u}_i^{(k)} = \rho_k u_{i,tt}^{(k)}, \rho_k \ddot{u}_3^{(k)} = \rho_k u_{3,tt}^{(k)}$ are the forces of inertia in the tangential and normal directions and $\rho_k = \rho_k(z)$ is the material density of the *k–th* layer. From the first expression in equation (3.15), we obtain the transverse shear stresses as

$$\sigma_{i3}^{(k)} = -\int_{a_{k-1}}^{z} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})dz + \Phi_{ik} \tag{3.16}$$

and using the second expression in (3.15) we derive the transverse normal stress as

$$\sigma_{33}^{(k)} = -\int_{a_{k-1}}^{z} (\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} - \rho_k \ddot{u}_3^{(k)})dz + \Phi_{3k} \tag{3.17}$$

where Φ_{ik} are the functions of integration for the *k–th* layer. The functions of integrations Φ_{ik} are determined using the condition (3.2) by setting $s = i$ so that

$$\Phi_{ik} = -p_i^- - \int_{a_0}^{a_{k-1}} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})dz \tag{3.18}$$

Here the following rules of integration for piecewise functions and for integration with variable upper limits have been used:

$$\int_{a_0}^{a_{k-1}} (...)^{(k)} dz = \sum_{r=1}^{k-1} \int_{a_{r-1}}^{a_r} (...)^{(r)} dz$$

$$\int_{a_0}^{z} (...)^k dz = \int_{a_{k-1}}^{z} (...)^k dz + \sum_{r=1}^{k-1} \int_{a_{r-1}}^{a_r} (...)^r dz$$

Substituting equation (3.18) into equation (3.16) we obtain the transverse shear stresses

$$\sigma_{i3}^{(k)} = -p_i^- - \int_{a_0}^{z} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})dz \tag{3.19}$$

The expression for the external loading may be found using the condition (3.3) in equation (3.19) with $s = i$:

$$p_i^- + p_i^+ = -\int_{a_0}^{a_n} (\sigma_{ij,j}^{(k)} - \rho_k \ddot{u}_i^{(k)})dz \tag{3.20}$$

The tangential forces and forces of inertia may now be determined as

$$N_{ij} = \int_{a_0}^{a_n} \sigma_{ij}^{(k)} dz;\ i,j = 1,2 \tag{3.21}$$

$$T_i = \int_{a_0}^{a_n} \rho_k \, \ddot{u}_i^{(k)} \, dz; \; i = 1, 2 \tag{3.22}$$

From equation (3.20) we may obtain the well known equations of motion for the classical theory of shells as

$$N_{ij,j} - T_i + p_i^+ + p_i^- = 0; \; i, j = 1, 2 \tag{3.23}$$

The transverse normal stresses may be found from equation (3.17) in conjunction with condition (3.2) and are given by

$$\sigma_{33}^{(k)} = -p_3^- - \int_{a_0}^{z} [\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} - \rho_k \, \ddot{u}_3^{(k)}] dz \tag{3.24}$$

Moreover,

$$p_3^- + p_3^+ = - \int_{a_0}^{a_n} [\sigma_{i3,i}^{(k)} - k_{ij}\sigma_{ij}^{(k)} - \rho_k \, \ddot{u}_3^{(k)}] dz \tag{3.25}$$

The third equation of motion for the classical theory of shells may now be written as

$$M_{ij,ij} - k_{ij}N_{ij} - (T_{ii} + T_3) + (p_3^+ + p_3^-) + (p_{i,i}^+ \, a_n + p_{i,i}^- \, a_0) = 0; \; i, j = 1, 2 \tag{3.26}$$

where

$$M_{ij} = \int_{a_0}^{a_n} \sigma_{ij}^{(k)} z \, dz; \; i, j = 1, 2 \tag{3.27}$$

are the moments of the internal forces. The moments T_{ij} due to the forces of inertia which are acting in the tangential directions are given by

$$T_{ij} = \int_{a_0}^{a_n} \rho_k \, \ddot{u}_{i,j}^{(k)} \, z \, dz; \;\; i = 1, 2; \;\; j = i \tag{3.28}$$

and the force of inertia in the normal direction is given as

$$T_3 = \int_{a_0}^{a_n} \rho_k \, \ddot{u}_3^{(k)} \, dz \tag{3.29}$$

The equations (3.23) and (3.26) represent the system of equations of motion for the classical theory of laminated shells. Using (3.21), (3.27) and expressions (3.12)–(3.14) we can rewrite this system in terms of displacements u_i and w. Next we use the above expressions in the derivation of the kinematic hypotheses of the higher-order theory.

3.3.2 Transverse shear stresses

Using equations (3.13), (3.14) in conjunction with equations (3.19), (3.20) we can obtain the following relations for the case $i = 1$:

$$\sigma_{13}^{(k)} \;\; = \;\; -p_1^- - \int_{a_0}^{z} (\sigma_{11,1}^{(k)} + \sigma_{12,2}^{(k)} - \rho_k \, \ddot{u}_1^{(k)}) dz =$$

$$
\begin{aligned}
&-p_1^- - [(\epsilon_{11,1}+\epsilon_{12,2})\int_{a_0}^{z} E_{0k}dz + (\epsilon_{22,1}-\epsilon_{12,2})\int_{a_0}^{z} E_{0k}\nu_k dz] \\
&-[(\kappa_{11,1}+\kappa_{12,2})\int_{a_0}^{z} E_{0k}zdz + (\kappa_{22,1}-\kappa_{12,2})\int_{a_0}^{z} E_{0k}\nu_k zdz] \\
&+(\ddot{u}_1\int_{a_0}^{z}\rho_k dz - \ddot{w}_{,1}\int_{a_0}^{z}\rho_k zdz)
\end{aligned} \tag{3.30}
$$

$$
\begin{aligned}
p_1^+ + p_1^- &= -[(\epsilon_{11,1}+\epsilon_{12,2})\int_{a_0}^{a_n} E_{0k}dz + (\epsilon_{22,1}-\epsilon_{12,2})\int_{a_0}^{a_n} E_{0k}\nu_k dz] \\
&\quad -[(\kappa_{11,1}+\kappa_{12,2})\int_{a_0}^{a_n} E_{0k}zdz + (\kappa_{22,1}-\kappa_{12,2})\int_{a_0}^{a_n} E_{0k}\nu_k zdz] \\
&\quad +(\ddot{u}_1\int_{a_0}^{a_n}\rho_k dz - \ddot{w}_{,1}\int_{a_0}^{a_n}\rho_k zdz)
\end{aligned} \tag{3.31}
$$

We introduce through–the–thickness distribution functions given by

$$
\begin{aligned}
f_k(z) &= \int_{a_0}^{z} E_{0k}dz;\ f_{\nu k} = \int_{a_0}^{z} E_{0k}\nu_k dz; \\
\overline{f}_k(z) &= \int_{a_0}^{z} E_{0k}zdz;\ \overline{f}_{\nu k}(z) = \int_{a_0}^{z} E_{0k}\nu_k zdz; \\
f_{\rho k}(z) &= \int_{a_0}^{z}\rho_k dz;\ \overline{f}_{\rho k}(z) = \int_{a_0}^{z}\rho_k zdz
\end{aligned} \tag{3.32}
$$

Similarly we introduce the constants

$$
\begin{aligned}
&B = \int_{a_0}^{a_n} E_{0k}dz = \sum_{k=1}^{n}\int_{a_{k-1}}^{a_k} E_{0k}dz;\ B_\nu = \int_{a_0}^{a_n} E_{0k}\nu_k dz; \\
&B_1 = \int_{a_0}^{a_n} E_{0k}zdz;\ B_{1\nu} = \int_{a_0}^{a_n} E_{0k}\nu_k zdz; \\
&B_\rho = \int_{a_0}^{a_n}\rho_k dz;\ B_{1\rho} = \int_{a_0}^{a_n}\rho_k zdz
\end{aligned} \tag{3.33}
$$

We note the following relations in equations (3.30) and (3.31):

$$
\begin{aligned}
\kappa_{11,1}+\kappa_{12,2} &= -(w_{,11}+w_{,22})_{,1} = -\Delta w_{,1}; \\
\kappa_{22,1}-\kappa_{12,2} &= -(w_{,22}-w_{,22})_{,1} = 0
\end{aligned} \tag{3.34}
$$

Then, substituting (3.32)-(3.34) into (3.30) and (3.31) gives

$$
\begin{aligned}
\sigma_{13}^{(k)} &= -p_1 - [(\epsilon_{11,1}+\epsilon_{12,2})f_k + (\epsilon_{22,1}-\epsilon_{12,2})f_{\nu k}] \\
&\quad +\Delta w_{,1}\overline{f}_k + \ddot{u}_1 f_{\rho k} - \ddot{w}_{,1}\overline{f}_{\rho k}
\end{aligned} \tag{3.35}
$$

$$
\begin{aligned}
p_1^+ + p_1^- &= -[(\epsilon_{11,1}+\epsilon_{12,2})B + (\epsilon_{22,1}-\epsilon_{12,2})B_\nu] \\
&\quad +\Delta w_{,1}B_1 + \ddot{u}_1 B_\rho - \ddot{w}_{,1} B_{1\rho}
\end{aligned} \tag{3.36}
$$

Now we eliminate from equation (3.35) those terms which contain the tangential strains $\epsilon_{11}, \epsilon_{22}$ and ϵ_{12}. In order to do this we rewrite equation (3.36) as

$$[(\epsilon_{11,1} + \epsilon_{12,2}) + (\epsilon_{22,1} - \epsilon_{12,2})\nu] =$$

$$-\frac{p_1^- + p_1^+}{B} + \Delta w_{,1}\frac{B_1}{B} + \ddot{u}_1 \frac{B_\rho}{B} - \ddot{w}_{,1} \frac{B_{1\rho}}{B} \tag{3.37}$$

where ν is a generalized Poisson's ratio for the entire shell thickness which is taken to be equal for each layer, that is,

$$\nu = \nu_k = \frac{B_\nu}{B}; \tag{3.38}$$

Here ν represents an average Poisson's ratio for the reference surface. For individual layers the exact values of the Poisson's ratios are used as shown in equation (3.33). By doing this the error introduced by taking an average ν for the reference surface has minimal effect on the overall results. Then, taking into account that $f_{\nu k} = \nu f_k$ we can write

$$(\epsilon_{11,1} + \epsilon_{12,2})f_k + (\epsilon_{22,1} - \epsilon_{12,2})f_{\nu k} = [(\epsilon_{11,1} + \epsilon_{12,2}) + (\epsilon_{22,1} - \epsilon_{12,2})\nu]f_k \tag{3.39}$$

Substituting equation (3.39) into equation (3.35) and taking into account equation (3.37) we obtain the transverse shear stresses as

$$\begin{aligned} \sigma_{13}^{(k)} &= \Delta w_{,1}\left(\overline{f}_k - \frac{B_1}{B}f_k\right) + p_1^-\left(\frac{f_k}{B} - 1\right) + p_1^+\frac{f_k}{B} \\ &\quad + \ddot{u}_1\left(f_{\rho k} - \frac{B_\rho}{B}f_k\right) - \ddot{w}_{,1}\left(\overline{f}_{\rho k} - \frac{B_{1\rho}}{B}f_k\right) \end{aligned} \tag{3.40}$$

The expression for $\sigma_{23}^{(k)}$ can be obtained in a similar manner. The general expression for the transverse shear stresses $(i = 1, 2)$ may be written as

$$\sigma_{i3}^{(k)} = \Delta w_{,i} f_{1k} + p_i^- f_{2k} + p_i^+ f_{3k} + \ddot{u}_i\, f_{4k} - \ddot{w}_{,i}\, f_{5k} \tag{3.41}$$

where the distribution functions are given by

$$f_{1k}(z) = \overline{f}_k - \frac{B_1}{B}f_k;\ f_{2k}(z) = \frac{f_k}{B} - 1;\ f_{3k}(z) = \frac{f_k}{B};$$

$$f_{4k}(z) = f_{\rho k} - \frac{B_\rho}{B}f_k;\ f_{5k}(z) = \overline{f}_{\rho k} - \frac{B_{1\rho}}{B}f_k \tag{3.42}$$

The expression (3.41) differs from those given in Refs [64, 65] as it contains terms which take into account the influence of the tangential forces of inertia and rotary inertia.

3.3.3 Transverse normal stresses

Substituting equation (3.12) in (3.24) and using expression (3.41) we obtain the transverse normal stresses in the following form

$$\begin{aligned}\sigma_{33}^{(k)} &= -p_3^- - \Delta w_{,ii} \int_{a_0}^{z} f_{1k} dz - p_{i,i}^- \int_{a_0}^{z} f_{2k} dz - p_{i,i}^+ \int_{a_0}^{z} f_{3k} dz \\ &- \ddot{u}_{i,i} \int_{a_0}^{z} f_{4k} dz + \ddot{w}_{,ii} \int_{a_0}^{z} f_{5k} dz + \ddot{w}\, f_{\rho k} + k_{ij} \int_{a_0}^{z} \sigma_{ij}^{(k)} dz; \\ & i,j = 1,2\end{aligned} \tag{3.43}$$

Using equation (3.14) we may rewrite the last term in equation (3.43) as

$$k_{ij} \int_{a_0}^{z} \sigma_{ij}^{(k)} dz = k_\epsilon f_k + k_\kappa \overline{f}_k \tag{3.44}$$

where

$$\begin{aligned} k_\epsilon &= k_{ij}\epsilon_{ij} + \nu(k_{11}\epsilon_{22} - 2k_{12}\epsilon_{12} + k_{22}\epsilon_{11}) \\ k_\kappa &= k_{ij}\kappa_{ij} + \nu(k_{11}\kappa_{22} - 2k_{12}\kappa_{12} + k_{22}\kappa_{11}) \end{aligned} \tag{3.45}$$

Let us also rewrite equation (3.25) as

$$\begin{aligned} p_3^- + p_3^+ &= -\Delta w_{,ii} D_1 - p_{i,i}^- D_2 - p_{i,i}^+ D_3 \\ &- \ddot{u}_{i,i}\, D_4 - \ddot{w}_{,ii}\, D_5 + \ddot{w}\, D_6 + k_\epsilon D_7 + k_\kappa D_8 \end{aligned} \tag{3.46}$$

where the following integral stiffnesses were introduced:

$$D_r = \int_{a_0}^{a_n} f_{rk} dz; \; r = 1, ..., 5; \;\; D_6 = B_\rho; \; D_7 = B; \; D_8 = B_1 \tag{3.47}$$

The equation (3.46) may be solved for the operator $\Delta w_{,ii} = \Delta^2 w$, and using the result thus obtained we can eliminate this operator from equation (3.43). Then the transverse normal stress may be written in the final form as

$$\begin{aligned} \sigma_{33}^{(k)} &= p_3^- F_{1k} + p_3^+ F_{2k} + p_{i,i}^- F_{3k} + p_{i,i}^+ F_{4k} \\ &+ \ddot{u}_{i,i}\, F_{5k} - \Delta\ddot{w}\, F_{6k} - \ddot{w}\, F_{7k} - k_\epsilon F_{8k} - k_\kappa F_{9k} \end{aligned} \tag{3.48}$$

where

$$F_k = \int_{a_0}^{z} f_{1k} dz; \;\; F_{1k} = \frac{F_k}{D_1} - 1; \;\; F_{2k} = \frac{F_k}{D_1}; \tag{3.49}$$

$$\begin{aligned} F_{(r+1)k} &= \frac{D_r}{D_1} F_k - \int_{a_0}^{z} f_{rk} dz; \; r = 2, ..., 5; \\ F_{7k} &= \frac{D_6}{D_1} F_k - f_{\rho k}; \;\; F_{8k} = \frac{D_7}{D_1} F_k - \int_{a_0}^{z} f_k dz; \\ F_{9k} &= \frac{D_8}{D_1} F_k - \int_{a_0}^{z} \overline{f}_k dz \end{aligned} \tag{3.50}$$

The expression for the transverse normal stresses includes terms which take into account the influence of the inertia forces.

3.3.4 Transverse deformations

Using equation (3.41) the transverse shear strains can be obtained from the Hooke's law as

$$2e_{i3}^{(k)} = \frac{\sigma_{i3}^{(k)}}{G'_k} = \Delta w_{,i}\overline{f}_{1k} + p_i^-\overline{f}_{2k} + p_i^+\overline{f}_{3k} + \ddot{u}_i\,\overline{f}_{4k} - \ddot{w}_{,i}\,\overline{f}_{5k} \qquad (3.51)$$

where the through–the–thickness distribution functions of the transverse shear are defined as

$$\overline{f}_{rk}(z) = \frac{f_{rk}(z)}{G'_k};\ r = 1, ..., 5 \qquad (3.52)$$

Using equations (3.10) and (3.48) we can find the normal strains as

$$\begin{aligned} e_{33}^{(k)} &= -\frac{\nu'_k}{E'_k}\left(\sigma_{11}^{(k)} + \sigma_{22}^{(k)}\right) + \frac{\sigma_{33}^{(k)}}{E'_k} = \\ &= \nu'_{0k} z\Delta w - \nu'_{0k} u_{i,i} + p_3^-\overline{F}_{1k} + p_3^+\overline{F}_{2k} + p_{i,i}^-\overline{F}_{3k} + p_{i,i}^+\overline{F}_{4k} \\ &\quad + \ddot{u}_{i,i}\,\overline{F}_{5k} - \Delta\ddot{w}\,\overline{F}_{6k} - \ddot{w}\,\overline{F}_{7k} - k_\epsilon \overline{F}_{8k} - k_\kappa \overline{F}_{9k} \end{aligned} \qquad (3.53)$$

where the generalized Poisson's ratio of the material of the *k–th* layer is defined as

$$\nu'_{0k} = \frac{E_k \nu'_k}{E'_k(1 - \nu'_k)} \qquad (3.54)$$

and we also define

$$\overline{F}_{rk}(z) = \frac{F_{rk}}{E'_k};\ r = 1, ..., 9 \qquad (3.55)$$

The above expressions for transverse shear and normal strains as well as expressions for corresponding stresses are not relevant in the classical theory since they only demonstrate the contradictions in this theory. However, they are important for the derivation of the higher–order theory which follows.

3.4 Derivation of the higher–order theory

3.4.1 Hypotheses

In deriving a higher–order theory we assume that the transverse shear and normal strains as well as the transverse normal stresses are not equal to zero, that is

$$e_{i3}^{(k)} \neq 0;\ e_{33}^{(k)} \neq 0;\ \sigma_{33}^{(k)} \neq 0 \qquad (3.56)$$

These quantities can be expressed using equations (3.48), (3.51) and (3.53). Using the expressions for the strains and the strain–displacement relations (3.6), we can find the more accurate components of the displacement vector which constitutes the next stage of the derivation.

3.4.2 Normal displacements

Integrating the third equation in (3.6) we obtain

$$u_3^{(k)} = w + \int_0^z e_{33}^{(k)} dz \tag{3.57}$$

where $w = u_3^{(m)}(x_i, 0, t)$ is the normal displacement of the reference surface positioned arbitrarily through the thickness of layers ($k = m$). Substituting expression (3.53) for $e_{33}^{(k)}$ into (3.57) and satisfying conditions (3.5) with $s = 3$, we introduce the following distribution functions of the normal component of the displacement vector:

$$\begin{aligned}
&\varphi_{11}^{(k)} = 1; \; \varphi_{21}^{(k)} = \int_0^z \nu'_{0k} z dz; \; \varphi_{31}^{(k)} = -\int_0^z \nu'_{0k} dz; \\
&\varphi_{12}^{(k)} = -\int_0^z \overline{F}_{7k} dz; \; \varphi_{22}^{(k)} = -\int_0^z \overline{F}_{6k} dz; \; \varphi_{32}^{(k)} = \int_0^z \overline{F}_{5k} dz; \\
&\varphi_{13}^{(k)} = \int_0^z \overline{F}_{1k} dz; \; \varphi_{23}^{(k)} = \int_0^z \overline{F}_{3k} dz; \varphi_{33}^{(k)} = -\int_0^z \overline{F}_{8k} dz; \\
&\varphi_{14}^{(k)} = \int_0^z \overline{F}_{2k} dz; \; \varphi_{24}^{(k)} = \int_0^z \overline{F}_{4k} dz; \; \varphi_{34}^{(k)} = -\int_0^z \overline{F}_{9k} dz
\end{aligned} \tag{3.58}$$

The equation for the normal displacements may now be written as

$$\begin{aligned}
u_3^{(k)} \;=\; & w\varphi_{11}^{(k)} + \Delta w \varphi_{21}^{(k)} + u_{i,i}\varphi_{31}^{(k)} + \ddot{w}\, \varphi_{12}^{(k)} + \Delta \ddot{w}\, \varphi_{22}^{(k)} + \ddot{u}_{i,i}\, \varphi_{32}^{(k)} \\
& + p_3^- \varphi_{13}^{(k)} + p_{i,i}^- \varphi_{23}^{(k)} + k_\epsilon \varphi_{33}^{(k)} + p_3^+ \varphi_{14}^{(k)} + p_{i,i}^+ \varphi_{24}^{(k)} + k_\kappa \varphi_{34}^{(k)} \\
& i = 1, 2
\end{aligned} \tag{3.59}$$

The distribution functions of the normal displacement in equation (3.59) allow to satisfy the continuity conditions on the layer interfaces for the normal displacement when the reference surface is positioned arbitrarily through the thickness of layers.

3.4.3 Tangential displacements

From the second expression in equation (3.6) we obtain

$$u_{i,3}^{(k)} = 2e_{i3}^{(k)} - u_{3,i}^{(k)} \tag{3.60}$$

and integrating this relation we have

$$u_i^{(k)} = u_i + \int_0^z (2e_{i3}^{(k)} - u_{3,i}^{(k)}) dz \tag{3.61}$$

where $u_i = u_i^{(m)}(x_i, 0, t)$ are the tangential displacements of the reference surface. We introduce the distribution functions given by

$$\begin{aligned}
&\psi_1^{(k)} = 1;\ \psi_{11}^{(k)} = \int_0^z \varphi_{11}^{(k)} dz; \\
&\psi_{21}^{(k)} = \int_0^z (\varphi_{21}^{(k)} - \overline{f}_{1k}) dz; \psi_{31}^{(k)} = \int_0^z \varphi_{31}^{(k)} dz; \\
&\psi_2^{(k)} = \int_0^z \overline{f}_{4k} dz;\ \psi_{12}^{(k)} = \int_0^z (\varphi_{12}^{(k)} + \overline{f}_{5k}) dz; \\
&\psi_{22}^{(k)} = \int_0^z \varphi_{22}^{(k)} dz;\ \psi_{32}^{(k)} = \int_0^z \varphi_{32}^{(k)} dz; \\
&\psi_3^{(k)} = -\int_0^z \overline{f}_{2k} dz;\ \psi_4^{(k)} = -\int_0^z \overline{f}_{3k} dz;\ \psi_{pg}^{(k)} = \int_0^z \varphi_{pg}^{(k)} dz; \\
&\qquad p = 1,2,3;\ g = 3,4
\end{aligned} \tag{3.62}$$

Substituting equation (3.51) in (3.61) and satisfying conditions (3.5) ($s = i = 1,2$), and using the functions defined in equation (3.62), the expression for tangential displacements may be written as

$$\begin{aligned}
u_i^{(k)} &= u_i\psi_1^{(k)} - w_{,i}\psi_{11}^{(k)} - \Delta w_{,i}\psi_{21}^{(k)} - u_{i,ij}\psi_{31}^{(k)} \\
&+ \ddot{u}_i\ \psi_2^{(k)} - \ddot{w}_{,i}\ \psi_{12}^{(k)} - \Delta\ \ddot{w}_{,i}\ \psi_{22}^{(k)} - \ddot{u}_{i,ij}\ \psi_{32}^{(k)} \\
&+ p_i^-\psi_3^{(k)} - p_{3,i}^-\psi_{13}^{(k)} - p_{i,ij}^-\psi_{23}^{(k)} - k_\epsilon\psi_{24}^{(k)} \\
&+ p_i^+\psi_4^{(k)} - p_{3,i}^+\psi_{14}^{(k)} - p_{i,ij}^+\psi_{24}^{(k)} - k_\kappa\psi_{34}^{(k)};\ i,j = 1,2
\end{aligned} \tag{3.63}$$

As was the case for normal displacements, the distribution functions defined in equation (3.62) allow to satisfy the continuity conditions in between the layers for the tangential displacements when the reference surface is positioned arbitrarily through the thickness of the shell.

3.4.4 Relations for the higher–order theory

Expressions (3.59) and (3.63) for the displacements are written in terms of the unknown functions u_i and w of the classical theory. In order to derive a nonclassical higher–order theory, we introduce new unknown functions of the reference surface using the following irreversible relations:

$$[u_i; w; \Delta w; u_{i,j}] \ \rightarrow\ [v_{i1}; \chi_{11}; \chi_{21}; \chi_{31}]; \tag{3.64a}$$

$$[\ddot{u}_i; \ddot{w}; \Delta\ddot{w}; \ddot{u}_{i,j}] \ \rightarrow\ [v_{i2}; \chi_{12}; \chi_{22}; \chi_{32}];\quad i,j = 1,2 \tag{3.64b}$$

The physical meanings of the new unknown functions defined in equation (3.64) may be deduced from equations (3.59) and (3.63) and were explained in detail in [65].

Briefly, these functions describe behaviour of the normal which is distorted due to the influence of the transverse shear and the normal deformation. We call them shear and compression functions. Complementary to the functions in equation (3.64) which were introduced in [65], in the case of a dynamic problem we also have dynamic shear and dynamic compression functions which are given by expression (3.64).

Let us introduce the following relations for the functions of the given external loading conditions, their derivatives and also components which depend on the curvature of the shell. These relations are defined as

$$\begin{aligned}
[p_i^-; p_3^-; p_{i,j}^-; k_\epsilon] &= [v_{i3}; \chi_{13}; \chi_{23}; \chi_{33}]; \\
[p_i^+; p_3^+; p_{i,j}^+; k_\kappa] &= [v_{i4}; \chi_{14}; \chi_{24}; \chi_{34}]; \quad i,j = 1,2
\end{aligned} \tag{3.65}$$

These functions are known and they are determined from the solution obtained using the classical theory and from the given loadings p^- and p^+ on the external surfaces and, therefore, take into account directly the effects of the transverse shear and normal deformation due to the loading conditions.

Replacing the functions in equations (3.59) and (3.63) in accordance with the relationships defined by equations (3.64) and (3.65), the expressions for the components of the displacement vector may be written as

$$u_i^{(k)} = v_{ig}\psi_{0g}^{(k)} - \chi_{pg,i}\psi_{pg}^{(k)} \tag{3.66}$$

$$u_3^{(k)} = \chi_{pg}\varphi_{pg}^{(k)}; \quad i = 1,2; \; p = 1,2,3; \; g = 1,2,3,4 \tag{3.67}$$

The expressions (3.66) and (3.67) will be used as kinematic hypotheses for the derivation of the higher–order theory.

Let us now obtain the components of the strain tensor for the *k–th* layer. Taking into account the kinematic hypotheses (3.66) and (3.67), the tangential components may be written as

$$\begin{aligned}
e_{ij}^{(k)} &= \frac{1}{2}(u_{i,j}^{(k)} + u_{j,i}^{(k)}) + k_{ij}u_3^{(k)} = \\
&= \frac{1}{2}[(v_{ig,j} + v_{jg,i})\psi_g^{(k)} - (\chi_{pg,ij} + \chi_{pg,ji})\psi_{pg}^{(k)}] + k_{ij}\chi_{pg}\varphi_{pg}^{(k)};
\end{aligned} \tag{3.68}$$

$$i,j = 1,2; \; p = 1,2,3; \; g = 1,2,3,4$$

and the transverse shear strains as

$$2e_{i3}^{(k)} = u_{i,3}^{(k)} + u_{3,i}^{(k)} = v_{ig}\alpha_g^{(k)} + \chi_{pg,i}\alpha_{pg}^{(k)}; \tag{3.69}$$

$$i = 1,2; \; p = 1,2,3; \; g = 1,2,3,4$$

where the following notations are employed:

$$\alpha_g^{(k)} = \psi_{g,3}^{(k)}; \ \alpha_{pg}^{(k)} = \varphi_{pg}^{(k)} - \psi_{pg,3}^{(k)} \tag{3.70}$$

The strains due to normal compression are given by

$$e_{33}^{(k)} = u_{3,3}^{(k)} = \chi_{pg}^{(k)}\beta_{pg}^{(k)}; \quad p = 1,2,3; \ g = 1,2,3,4. \tag{3.71}$$

where

$$\beta_{pg}^{(k)} = \varphi_{pg,3}^{(k)} \tag{3.72}$$

The components of the stress tensor can be determined using the Hooke's law for a transversely isotropic material which are

$$\begin{aligned} \sigma_{11}^{(k)} &= A_{11}^{(k)}e_{11}^{(k)} + A_{12}^{(k)}e_{22}^{(k)} + A_{13}^{(k)}e_{33}^{(k)} \\ \sigma_{22}^{(k)} &= A_{21}^{(k)}e_{11}^{(k)} + A_{22}^{(k)}e_{22}^{(k)} + A_{23}^{(k)}e_{23}^{(k)} \\ \sigma_{33}^{(k)} &= A_{31}^{(k)}e_{11}^{(k)} + A_{32}^{(k)}e_{22}^{(k)} + A_{33}^{(k)}e_{33}^{(k)} \\ \sigma_{23}^{(k)} &= 2A_{44}^{(k)}e_{23}^{(k)}; \ \sigma_{13}^{(k)} = 2A_{55}e_{13}^{(k)}; \ \sigma_{12}^{(k)} = 2A_{66}e_{12}^{(k)} \end{aligned} \tag{3.73}$$

where the elastic constants of the k–th layer are

$$\begin{gathered} A_{11}^{(k)} = A_{22}^{(k)} = \frac{\Delta_{11}^{(k)}}{\Delta_k}; \ A_{12}^{(k)} = A_{21}^{(k)} = \frac{\Delta_{12}^{(k)}}{\Delta_k}; \\ A_{13}^{(k)} = A_{31}^{(k)} = A_{23}^{(k)} = A_{32}^{(k)} = \frac{\Delta_{13}^{(k)}}{\Delta_k}; \ A_{33}^{(k)} = \frac{\Delta_{33}^{(k)}}{\Delta_k}; \\ \Delta_k = \frac{(1+\nu_k)[1-\nu_k-2(\nu_k')^2E_k/E_k']}{E_k^2E_k'}; \ \Delta_{11}^{(k)} = \frac{1-(\nu_k')^2E_k/E_k'}{E_kE_k'}; \\ \Delta_{12}^{(k)} = \frac{\nu_k+(\nu_k')^2E_k/E_k'}{E_kE_k'}; \ \Delta_{13}^{(k)} = \frac{\nu_k'(1+\nu_k)}{E_kE_k'}; \ \Delta_{33}^{(k)} = \frac{1-\nu_k^2}{E_k^2}; \\ A_{44}^{(k)} = G_{23}^{(k)} = G_k'; \ A_{55}^{(k)} = G_{13}^{(k)} = G_k'; \ A_{66}^{(k)} = G_{12}^{(k)} = G_k \end{gathered} \tag{3.74}$$

Substituting the strains (3.68), (3.69) and (3.71) into (3.73) we obtain the tangential stresses

$$\begin{aligned} \sigma_{11}^{(k)} &= A_{1i}^{(k)}e_{ir}^{(k)} + A_{i3}^{(k)}e_{33}^{(k)} = \\ &= A_{1i}^{(k)}(v_{ig,r}\psi_g^{(k)} - \chi_{pg,ir}\psi_{pg}^{(k)} + k_{ir}\chi_{pg}\varphi_{pg}^{(k)}) + A_{13}^{(k)}\chi_{pg}\beta_{pg}^{(k)}; \\ \sigma_{22}^{(k)} &= A_{2i}^{(k)}e_{ir}^{(k)} + A_{i3}^{(k)}e_{33} = \\ &= A_{2i}^{(k)}(v_{ig,r}\psi_g^{(k)} - \chi_{pg,ir}\psi_{pg}^{(k)} + k_{ir}\chi_{pg}\varphi_{pg}^{(k)}) + A_{23}\chi_{pg}\beta_{pg}^{(k)}; \\ & i = 1,2; \ r = i; \ p = 1,2,3; \ g = 1,2,3,4 \end{aligned} \tag{3.75}$$

$$\begin{aligned} \sigma_{12}^{(k)} &= 2A_{66}^{(k)}e_{12}^{(k)} = \\ &= G_{12}^{(k)}[(v_{1g,2}+v_{2g,1})\psi_g^{(k)} - 2\chi_{pg,12}\psi_{pg}^{(k)} + 2k_{12}\chi_{pg}\varphi_{pg}^{(k)}]; \end{aligned} \tag{3.76}$$

$$p = 1,2,3; \; g = 1,2,3,4$$

and the stresses in the transverse direction

$$\begin{aligned}
\sigma_{13}^{(k)} &= 2A_{55}^{(k)} e_{13}^{(k)} = 2G_{13}^{(k)} (v_{1g} \alpha_g^{(k)} + \chi_{pg,1} \alpha_{pg}^{(k)}); \\
\sigma_{23}^{(k)} &= 2A_{44}^{(k)} e_{23}^{(k)} = 2G_{23}^{(k)} (v_{2g} \alpha_g^{(k)} + \chi_{pg,2} \alpha_{pg}^{(k)}); \qquad (3.77)\\
\sigma_{33}^{(k)} &= A_{3i}^{(k)} e_{ir}^{(k)} + A_{33}^{(k)} e_{33}^{(k)} = \\
&= A_{3i}^{(k)} (v_{ig,r} \psi_g^{(k)} - \chi_{pg,ir} \psi_{pg}^{(k)} + k_{ir} \chi_{pg} \varphi_{pg}^{(k)}) + A_{33}^{(k)} \chi_{pg} \beta_{pg}^{(k)}; \qquad (3.78)
\end{aligned}$$

$$i = 1,2; \; r = i; \; p = 1,2,3; \; g = 1,2,3,4$$

The transverse normal stresses $\sigma_{33}^{(k)}$ given in equation (3.78) does not satisfy the conditions (3.2)–(3.4). In order to satisfy these conditions we have to use the transverse normal stresses in the form of equation (3.48) as

$$\sigma_{33}^{(k)} = \chi_{pg} F_{pg}^{(k)}; \; p = 1,2,3; \; g = 3,4 \qquad (3.79)$$

where

$$F_{pg}^{(k)} = E_k' \beta_{pg}^{(k)}$$

In this case equations (3.15) will be satisfied exactly when $\nu' = 0$, otherwise they will be satisfied integrally for the whole thickness of the laminate.

The equations which are given above define all the components of the displacement vector and the stress–strain tensor at an arbitrary point in the *k–th* layer and they form the nonclassical higher–order model of the stress and strain state of a dynamically loaded laminated shell which takes into account transverse shear and normal deformation. The refined model includes the independent unknown functions of the reference surface v_{ig}, χ_{pg} $(i = 1,2; \; p = 1,2,3; \; g = 1,2)$, the known functions v_{ig}, χ_{pg} $(i = 1,2; \; p = 1,2,3; \; g = 3,4)$ which depend on the deformations of the reference surface obtained using the classical theory and on the loading conditions on the external surfaces, and the known functions of the normal z which involve through–the–thickness distribution functions. The distribution functions are defined in a form which facilitates the satisfaction of the conditions on the external surfaces and the continuity conditions in between the layers when the reference surface is positioned arbitrarily through the thickness of the shell. Clearly, the governing equations are independent of the thicknesses, stiffnesses and other properties of the layers. Moreover, using this model we can consider layers with elastic characteristics that are constant or variable through the thickness and thus the model is comprehensive with respect to the properties of the layers.

The important feature of the present model is the inclusion of the dynamic factors such as forces of inertia and rotary inertia at the initial stage of derivation when the kinematic hypotheses are formulated.

3.5 Variational equation, equations of motion and boundary conditions

3.5.1 Variational equation

The equations of motion and the boundary conditions may be determined using the Reissner variational principle

$$\int_{t_1}^{t_2} [(\delta R - \delta K) - \delta H] dt = 0 \tag{3.80}$$

where δR is the variation of the Reissner functional, δK the variation of the kinetic energy, and δH the variation of the work done by the external forces.

For a laminated shell the variation of the Reissner functional has the following form [74]

$$\begin{aligned} \delta R \;=\; & \iiint_V [\sigma_{rs}^{(k)} \delta e_{rs}^{(k)} + \delta\sigma_{11}^{(k)}(e_{11}^{(k)} - a_{1s}^{(k)}\sigma_{ts}^{(k)}) + \delta\sigma_{22}^{(k)}(e_{22}^{(k)} - a_{2s}^{(k)}\sigma_{ts}^{(k)}) \\ & + \delta\sigma_{33}^{(k)}(e_{33}^{(k)} - a_{3s}^{(k)}\sigma_{ts}^{(k)}) + \delta\sigma_{23}^{(k)}(2e_{23}^{(k)} - a_{44}^{(k)}\sigma_{23}) \\ & + \delta\sigma_{13}^{(k)}(2e_{13}^{(k)} - a_{55}^{(k)}\sigma_{13}^{(k)}) + \delta\sigma_{12}^{(k)}(2e_{12}^{(k)} - a_{66}^{(k)}\sigma_{12}^{(k)})] dV; \end{aligned} \tag{3.81}$$

$$r, s = 1, 2, 3\ ;\ t = s$$

Firstly, let us consider the implications of the variation of this functional being zero, i.e., $\delta R = 0$. From the basic lemma of calculus of variations it follows that in this case each term in the variational equation (3.81) is equal to zero. Substituting expressions for the components of the stress and strain tensors into the multipliers of the variations, we are able to ascertain that the stresses $\sigma_{ij}^{(k)}$, $\sigma_{i3}^{(k)}$ $(i, j = 1, 2)$ are zero. This implies that the equations of the Hooke's law for the strains $e_{ij}^{(k)}$, $e_{i3}^{(k)}$ $(i, j = 1, 2)$ are satisfied exactly.

The constitutive equations can be derived from the variational equation (3.80) in the form given by equation (3.10). For the strains $e_{33}^{(k)}$, the constitutive equations are satisfied "integrally" (in the sense that the integral corresponding to this equation equals zero over the domain of the shell) since $\delta\sigma_{33}^{(k)} = 0$.

As long as all the terms in (3.81), excluding the first, are identically equal to zero, the variation of the Reissner functional is equivalent to the variation of the potential energy of the deformation, viz.

$$\delta R \rightarrow \delta U = \iiint_V \sigma_{rs}^{(k)} \delta e_{rs}^{(k)} dV; \quad r, s = 1,2,3 \tag{3.82}$$

3.5.2 Variation of the potential energy

We now consider the tangential and normal components of the stress and strain tensors in equation (3.82) given by

$$\delta U = \iiint_V \left[\sigma_{ij}^{(k)} \delta e_{ij}^{(k)} + 2\sigma_{i3}^{(k)} \delta e_{i3}^{(k)} + \sigma_{33} \delta e_{33}^{(k)}\right] dV; \quad i, j = 1,2 \tag{3.83}$$

Substituting the strains from equations (3.68), (3.69) and (3.71) into equation (3.83), we can express the variation of the potential energy in terms of the displacements given by equations (3.66) and (3.67). Through–the–thickness integral of the shell can be expressed as

$$\begin{aligned} \delta U &= \iint_S \{ \int_{a_0}^{a_n} [\sigma_{ij}^{(k)} (\psi_g^{(k)} \delta v_{ig,j} - \psi_{pg}^{(k)} \delta \chi_{pg,ij} + k_{ij} \varphi_{pg}^{(k)} \delta \chi_{pg}) \\ &\quad + \sigma_{i3}^{(k)} (\alpha_g^{(k)} \delta v_{ig} + \alpha_{pg}^{(k)} \delta \chi_{pg,i}) + \sigma_{33}^{(k)} (\beta_{pg}^{(k)} \delta \chi_{pg})] dz \} dS; \end{aligned} \tag{3.84}$$

$$i, j = 1,2 \; ; \; p = 1,2,3 \; ; \; g = 1,2,3,4$$

where S is the two–dimensional domain of the shell surface. It is noted that the variations of the functions with subscripts $g = 3,4$ are equal to zero. Using a notation similar to that of the classical theory, we may now consider the integral characteristics of the stresses due to internal forces and moments, viz. generalized forces and moments which are defined as

$$\begin{gathered} \left[N_{ij}^{(q)}; M_{ij}^{(fq)}; N_{ij}^{(fq)}\right] = \int_{a_0}^{a_n} \sigma_{ij}^{(k)} \left[\psi_q^{(k)}; \psi_{fq}^{(k)}; \varphi_{fq}^{(k)}\right] dz \\ \left[Q_i^{(q)}; Q_i^{(fq)}\right] = \int_{a_0}^{a_n} \sigma_{i3}^{(k)} \left[\alpha_q^{(k)}; \alpha_{fq}^{(k)}\right] dz \; ; \; Q_3^{(fq)} = \int_{a_0}^{a_n} \sigma_{33} \beta_{fq}^{(k)} dz \; ; \\ i, j = 1,2 \; ; \; f = 1,2,3 \; ; \; q = 1,2 \end{gathered} \tag{3.85}$$

Substituting equation (3.85) into equation (3.84), and using Ostrogradsky–Gauss theorem, we obtain the following expression for the variation of the potential energy:

$$\begin{aligned} \delta U &= -\iint_S [(N_{ij,j}^{(q)} - Q_i^{(q)}) \delta v_{iq} + (M_{ij,ij}^{(fq)} - k_{ij} N_{ij}^{(fq)} + Q_{i,i}^{(fq)} - Q_3^{(fq)}) \delta \chi_{fq}] dS \\ &\quad + \int_L [(N_{hh}^{(q)} \delta v_{hq} + N_{hl}^{(q)} \delta v_{lq}) + (M_{hh,h}^{(fq)} + 2M_{hl,l}^{(fq)} + Q_h^{(fq)}) \delta \chi_{fq} \\ &\quad - M_{hh}^{(fq)} \delta \chi_{fq,h}] dL + \left[M_{hl}^{(fq)} \delta \chi_{fq}\right]_{L_1}^{L_2} \end{aligned} \tag{3.86}$$

where h and l are normal and tangent to the boundary L of the shell, respectively. For the forces on the boundary of the shell it was assumed that h and l are equivalent to i and j in equation (3.85).

3.5.3 Variation of the kinetic energy

The expression for the variation of the kinetic energy may be written as

$$\begin{aligned}\delta K &= -\iiint_V \rho_k\, \ddot{u}_s^{(k)}\, \delta u_s^{(k)} dV = -\iiint_V \rho_k \left[\ddot{u}_i^{(k)}\, \delta u_i^{(k)} + \ddot{u}_3^{(k)}\, \delta u_3^{(k)}\right] dV = \\ &-\iiint_V \rho_k \left[\ddot{u}_i^{(k)}\, (\psi_q^{(k)} \delta v_{iq} - \psi_{fq}^{(k)} \delta \chi_{fq,i}) + \ddot{u}_3^{(k)}\, (\varphi_{fq}^{(k)} \delta \chi_{fq})\right] dV; \qquad (3.87)\\ &i = 1,2; \quad f = 1,2,3; \quad q = 1,2\end{aligned}$$

Let us introduce the integral characteristics of the inertia forces in the shell, i.e., the generalized inertia forces which are defined as

$$\left[T_i^{(q)}; T_i^{(fq)}\right] = \int_{a_0}^{a_n} \rho_k\, \ddot{u}_i^{(k)} \left[\psi_q^{(k)}; \psi_{fq}^{(k)}\right] dz;$$

$$T_3^{(fq)} = \int_{a_0}^{a_n} \rho_k\, \ddot{u}_3^{(k)}\, \varphi_{fq}^{(k)} dz; \quad i = 1,2 \qquad (3.88)$$

Substituting equations (3.88) into equation (3.87), we can rewrite the expression for the kinetic energy in the following form

$$\begin{aligned}\delta K &= -\iint_S [T_i^{(q)} \delta v_{iq} - T_i^{(fq)} \delta \chi_{fq,i} + T_3^{(fq)} \delta \chi_{fq}] dS = \\ &= \iint_S [T_i^{(q)} \delta v_{iq} + (T_{i,i}^{(fq)} + T_3^{(fq)}) \delta \chi_{fq}] dS - \int_L (T_h^{(fq)} \delta \chi_{fq}) dL; \qquad (3.89)\\ &i = 1,2; \quad h = i; \quad f = 1,2,3; \quad q = 1,2\end{aligned}$$

3.5.4 Variation of the work of the external loading

The variation of the work of the external loading is

$$\delta H = \delta H_1 + \delta H_2 \qquad (3.90)$$

which consists of the work done by forces H_1 on the inner and outer surfaces and by the boundary forces H_2. Therefore, using the relations (3.66) and (3.67) and introducing the notation

$$\begin{aligned}p_i^{(q)} &= p_i^- \psi_q^{(1)}(a_0) + p_i^+ \psi_q^{(n)}(a_n)\ ; \ p_h^{(fq)} = p_h^- \varphi_{fq}^{(1)}(a_0) + p_h^+ \varphi_{fq}^{(n)}(a_n);\\ p_3^{(fq)} &= \left[p_{i,i}^- \psi_{fq}^{(1)}(a_0) + p_3^- \varphi_{fq}^{(1)}(a_0) + p_{i,i}^+ \psi_{fq}^{(n)}(a_n) + p_3^+ \varphi_{fq}^{(n)}(a_n)\right]\end{aligned} \qquad (3.91)$$

for the generalized loading, we obtain the variation of the work of the load H_1 as

$$
\begin{aligned}
\delta H_1 &= \iint_S (p_s^- \delta u_s^{(1)} + p_s^+ \delta u_s^{(n)}) dS = \\
&= \iint_S [p_i^- \delta u_i^{(1)} + p_i^+ \delta u_i^{(n)} + p_3^- \delta u_3^{(1)} + p_3^+ \delta u_3^{(n)}] dS = \\
&= \iint_S \{p_i^- [\psi_q^{(1)}(a_0)\delta v_{iq} - \psi_{fq}^{(1)}(a_0)\delta\chi_{fq,i}] + p_3^- [\varphi_{fq}^{(1)}(a_0)\delta\chi_{fq}] \\
&\quad + p_i^+ [\psi_q^{(n)}(a_n)\delta v_{iq} - \psi_{fq}^{(n)}(a_n)\delta\chi_{fq,i}] + p_3^+ [\varphi_{fq}^{(h)}(a_n)\delta\chi_{fq}]\} dS = \\
&= \iint_S (p_i^{(q)}\delta v_{iq} + p_3^{(fq)}\chi_{fq}) dS - \int_L (p_h^{(fq)}\delta\chi_{fq}) dL;
\end{aligned}
\tag{3.92}
$$

$$i = 1,2; \quad f = 1,2,3; \quad q = 1,2$$

3.5.5 Variation of the work of the boundary forces

The corresponding expression for the boundary forces has the form

$$\delta H_2 = \int_L \left\{ \int_{a_0}^{a_n} \left[\sigma_{hh}^{(k)} \delta u_h^{(k)} + \sigma_{h3}^{(k)} \delta u_3^{(k)} + \sigma_{hl}^{(k)} \delta u_l^{(k)} \right] dz \right\} dL \tag{3.93}$$

where $\sigma_{hh}^{(k)}, \sigma_{h3}^{(k)}, \sigma_{hl}^{(k)}$ are components of the stress tensor and $u_h^{(k)}, u_3^{(k)}, u_l^{(k)}$ are components of the displacement vector at an arbitrary point of the *k-th* layer on the boundary L of the shell. Taking into account that $i = h$ or l in the expressions (3.66) and (3.67) for the tangential and normal displacements and substituting (3.66) and (3.67) into equation (3.93) we obtain

$$
\begin{aligned}
\delta H_2 &= \int_L \{ \int_{a_0}^{a_n} [\sigma_{hh}^{(k)}(\varphi_q^{(k)}\delta v_{hq} - \psi_{fq}^{(k)}\delta\chi_{fq,h}) + \sigma_{hl}^{(k)}(\psi_q^{(k)}\delta v_{lq} - \psi_{fq}^{(k)}\delta\chi_{fq,l}) \\
&\qquad + \sigma_{h3}^{(k)}(\varphi_{fq}^{(k)}\delta\chi_{fq})] dz \} dL = \\
&= \int_L [\overset{*}{N}{}_{hh}^{(q)} \delta v_{hq} + \overset{*}{N}{}_{hl}^{(q)} \delta v_{lq} - \overset{*}{M}{}_{hh}^{(fq)} \delta\chi_{fq,h} + (\overset{*}{M}{}_{hl,l}^{(fq)} + \overset{*}{Q}{}_{h3}^{(fq)})\delta\chi_{fq}] dL \\
&\quad - \left[\overset{*}{M}{}_{hl}^{(fq)} \delta\chi_{fq} \right]_{L_1}^{L_2}; \quad f = 1,2,3; \quad q = 1,2
\end{aligned}
\tag{3.94}
$$

where an asterisk denotes the forces acting on the boundary of the shell which may be expressed by equations (3.85). Also in equation (3.94) we have

$$\overset{*}{Q}{}_{h3}^{(fq)} = \int_{a_0}^{a_n} \sigma_{h3}^{(k)} \varphi_{fq}^{(k)} dz \tag{3.95}$$

3.5.6 Equation of motion and boundary conditions

Substituting the variations (3.81), (3.86), (3.89), (3.92), (3.94) into (3.80), we derive the following variational equation:

$$\begin{aligned}
&\int_{t_1}^{t_2}\iint_S\{(N_{ij,j}^{(q)} - Q_i^{(q)} - T_i^{(q)} + p_i^{(q)})\delta v_{iq} + [M_{ij,ij}^{(fq)} - k_{ij}N_{ij}^{(fq)} + Q_{i,i}^{(fq)} - Q_3^{(fq)} \\
&-(T_{i,i}^{(fq)} + T_3^{(fq)}) + p_3^{(fq)}]\delta\chi_{fq}\}dSdt - \int_{t_1}^{t_2}\int_L\{(N_{hh}^{(q)} - \overset{*}{N}{}_{hh}^{(q)})\delta v_{hq} + (N_{hl}^{(q)} - \overset{*}{N}{}_{hl}^{(q)})\delta v_{lq} \\
&+[M_{hh,h}^{(fq)} + 2M_{hl,l}^{(fq)} + Q_h^{(fq)} + T_h^{(fq)} + p_h^{(fq)} - (\overset{*}{M}{}_{hl,l}^{(fq)} + \overset{*}{Q}{}_{h3}^{(fq)})]\delta\chi_{fq} \\
&-(M_{hh}^{(fq)} - \overset{*}{M}{}_{hh}^{(fq)})\delta\chi_{fq,h}\}dLdt - \left[(M_{hl}^{(fq)} - \overset{*}{M}{}_{hl}^{(fq)})\delta\chi_{fq}\right]_{L_1}^{L_2} = 0
\end{aligned} \tag{3.96}$$

$$i,j = 1,2; \quad f = 1,2,3; \quad q = 1,2$$

where t_1 and t_2 denote the initial and final time, respectively. The variations of the independent functions v_{iq} and χ_{fq}, which determine the displacements in the shell, have arbitrary values everywhere over the domain of the shell, excluding the boundary, and, consequently they cannot be equal to zero. Equating the multipliers of the variations in the first integral of equation (3.96) to zero, we obtain the system of equations of motion of the shell as

$$\begin{aligned}
&N_{ij,j}^{(q)} - Q_i^{(q)} - T_i^{(q)} + p_i^{(q)} = 0; \\
&M_{ij,ij}^{(fq)} - k_{ij}N_{ij}^{(fq)} + Q_{i,i}^{(fq)} - Q_3^{(fq)} - (T_{i,i}^{(fq)} + T_3^{(fq)}) + p_3^{(fq)} = 0; \\
&\quad i,j = 1,2; \quad f = 1,2,3; \quad q = 1,2
\end{aligned} \tag{3.97}$$

The boundary conditions follow from the boundary integral in equation (3.96) and they may be written as

$$\begin{aligned}
&(N_{hh}^{(q)} - \overset{*}{N}{}_{hh}^{(q)})\delta v_{hq} = 0; \quad (N_{hl}^{(q)} - \overset{*}{N}{}_{hl}^{(q)})\delta v_{lq} = 0; \\
&[M_{hh,h}^{(fq)} + 2M_{hl,l}^{(fq)} + Q_h^{(fq)} + T_h^{(fq)} + p_h^{(fq)} - (\overset{*}{M}{}_{hl,l}^{(fq)} + \overset{*}{Q}{}_{h3}^{(fq)})]\delta\chi_{fq} = 0; \\
&(M_{hh}^{(fq)} - \overset{*}{M}{}_{hh}^{(fq)})\delta\chi_{fq,h} = 0; \quad f = 1,2,3; \quad q = 1,2
\end{aligned} \tag{3.98}$$

There are 16 boundary conditions, which is the same as the order of the system of equations (3.97). Detailed interpretation of the boundary conditions is given in [64, 65].

3.6 Generalized forces and moments and the system of governing equations

Let us rewrite the generalized forces introduced earlier in equations (3.85) and use the expressions for the stresses given by equations (3.75), (3.77) and (3.78). Then we have for the tangential forces

$$\begin{aligned} N_{11}^{(q)} &= B_{1i}^{(gq)} v_{ig,r} - B_{1i}^{(pgq)} \chi_{pg,ir} + (C_{1i}^{(pgq)} k_{ir} + C_1^{(pgq)}) \chi_{pg}; \\ N_{22}^{(q)} &= B_{2i}^{(gq)} v_{ig,r} - B_{2i}^{(pgq)} \chi_{pg,ir} + (C_{2i}^{(pgq)} k_{ir} + C_2^{(pgq)}) \chi_{pg}; \\ N_{12}^{(q)} &= B^{(gq)} (v_{1g,2} + v_{2g,1}) - 2B^{(pgq)} \chi_{pg,12} + 2C^{(pgq)} k_{12} \chi_{pg}; \\ & i = 1,2, \ r = i \end{aligned} \tag{3.99}$$

for the moments

$$\begin{aligned} M_{11}^{(fq)} &= \overline{B}_{1i}^{(gfq)} v_{ig,r} - D_{1i}^{(pgfq)} \chi_{pg,ir} + (E_{1i}^{(pgfq)} k_{ir} + E_1^{(pgfq)}) \chi_{pg}; \\ M_{22}^{(fq)} &= \overline{B}_{2i}^{(gfq)} v_{ig,r} - D_{2i}^{(pgfq)} \chi_{pg,ir} + (E_{2i}^{(pgfq)} k_{ir} + E_2^{(pgfq)}) \chi_{pg}; \\ M_{12}^{(fq)} &= \overline{B}^{(gfq)} (v_{1g,2} + v_{2g,1}) - 2D^{(pgfq)} \chi_{pg,12} + 2E^{(pgfq)} k_{12} \chi_{pg}; \\ & i = 1,2, \ r = i \end{aligned} \tag{3.100}$$

for the higher-order tangential forces

$$\begin{aligned} N_{11}^{(fq)} &= \overline{C}_{1i}^{(gfq)} v_{ig,r} - \overline{E}_{1i}^{(pgfq)} \chi_{pg,ir} + (K_{1i}^{(pgfq)} k_{ir} + K_1^{(pgfq)}) \chi_{pg}; \\ N_{22}^{(fq)} &= \overline{C}_{2i}^{(gfq)} v_{ig,r} - \overline{E}_{1i}^{(pgfq)} \chi_{pg,ir} + (K_{2i}^{(pgfq)} k_{ir} + K_2^{(pgfq)}) \chi_{pg}; \\ N_{12}^{(fq)} &= \overline{C}^{(gfq)} (v_{1g,2} + v_{2g,1}) - 2\overline{E}^{(pgfq)} \chi_{pg,12} + 2K^{(pgfq)} k_{12} \chi_{pg}; \\ & i = 1,2, \ r = i \end{aligned} \tag{3.101}$$

and for the shear forces

$$\begin{aligned} Q_1^{(q)} &= 2R_1^{(gq)} v_{1g} + 2R_1^{(pgq)} \chi_{pg,1}; \quad Q_2^{(q)} = 2R_2^{(gq)} v_{2g} + 2R_2^{(pgq)} \chi_{pg,2}; \\ Q_1^{(fq)} &= 2\overline{R}_1^{(gfq)} v_{1g} + 2R_1^{(pgfq)} \chi_{pg,1}; \quad Q_2^{(fq)} = 2\overline{R}_2^{(gfq)} v_{2g} + 2R_2^{(pgfq)} \chi_{pg,2}; \\ Q_3^{(fq)} &= \overline{C}_i^{(gfq)} v_{ig,r} - C_{3i}^{(pgfq)} \chi_{pg,ir} + (\overline{C}_i^{(pgfq)} k_{ir} + C_3^{(pgfq)}) \chi_{pg}; \\ & i = 1,2, \ r = i \end{aligned} \tag{3.102}$$

In equations (3.99)-(3.102) we have $q = 1,2$; $f,p = 1,2,3$; $g = 1,2,3,4$ and we also assume summation over i,p and g. The equations for the generalized forces and moments include the integrated stiffnesses of laminated shell given by

$$B_{\xi i}^{(gq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_g^{(k)} \psi_q^{(k)} dz; \quad B_{\xi i}^{(pgq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{pg}^{(k)} \psi_q^{(k)} dz;$$

$$B^{(gq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_g^{(k)} \psi_q^{(k)} dz; \quad B^{(pgq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{pg}^{(k)} \psi_q^{(k)} dz;$$

$$\overline{B}_{\xi i}^{(gfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_g^{(k)} \psi_{fq}^{(k)} dz; \quad \overline{B}^{(gfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_g^{(k)} \psi_{fq}^{(k)} dz;$$

$$C_{\xi i}^{(pgq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \varphi_{pg}^{(k)} \psi_q^{(k)} dz; \quad C^{(pgq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \varphi_{pg}^{(k)} \psi_q^{(k)} dz;$$
$$\overline{C}_{\xi i}^{(gfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_g^{(k)} \varphi_{fq}^{(k)} dz; \quad \overline{C}^{(gfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_g^{(k)} \varphi_{fq}^{(k)} dz;$$
$$C_i^{(pgq)} = \int_{a_0}^{a_n} A_{i3}^{(k)} \beta_{pg}^{(k)} \psi_q^{(k)} dz; \quad \overline{C}_i^{(gfq)} = \int_{a_0}^{a_n} A_{3i}^{(k)} \psi_g^{(k)} \beta_{fq}^{(k)} dz;$$
$$D_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{pg}^{(k)} \psi_{fq}^{(k)} dz; \quad D^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{pg}^{(k)} \psi_{fq}^{(k)} dz;$$
$$E_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \varphi_{pg}^{(k)} \psi_{fq}^{(k)} dz; \quad E^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \varphi_{pg}^{(k)} \psi_{fq}^{(k)} dz; \tag{3.103}$$
$$\overline{E}^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \psi_{pg}^{(k)} \varphi_{fq}^{(k)} dz; \quad \overline{E}_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \psi_{pg}^{(k)} \varphi_{fq}^{(k)} dz;$$
$$E_i^{(pgfq)} = \int_{a_0}^{a_n} A_{i3}^{(k)} \beta_{pg}^{(k)} \psi_{fq}^{(k)} dz; \quad K_{\xi i}^{(pgfq)} = \int_{a_0}^{a_n} A_{\xi i}^{(k)} \varphi_{pg}^{(k)} \varphi_{fq}^{(k)} dz;$$
$$K_i^{(pgfq)} = \int_{a_0}^{a_n} A_{i3}^{(k)} \beta_{pg}^{(k)} \varphi_{fq}^{(k)} dz; \quad K^{(pgfq)} = \int_{a_0}^{a_n} G_{12}^{(k)} \alpha_{pg}^{(k)} \alpha_{fq}^{(k)} dz;$$
$$R_i^{(gq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_g^{(k)} \alpha_q^{(k)} dz; \quad R_i^{(pgq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_{pg}^{(k)} \alpha_q^{(k)} dz;$$
$$R_i^{(pgfq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_{pg}^{(k)} \alpha_{fq}^{(k)} dz; \quad \overline{R}_i^{(gfq)} = \int_{a_0}^{a_n} G_{i3}^{(k)} \alpha_g^{(k)} \alpha_{fq}^{(k)} dz;$$
$$C_{3i}^{(pgfq)} = \int_{a_0}^{a_n} A_{3i}^{(k)} \psi_{pg}^{(k)} \beta_{fq}^{(k)} dz; \quad C_i^{(pgfq)} = \int_{a_0}^{a_n} A_{3i}^{(k)} \varphi_{pg}^{(k)} \beta_{fq}^{(k)} dz;$$
$$C_3^{(pgfq)} = \int_{a_0}^{a_n} A_{33}^{(k)} \beta_{pg}^{(k)} \beta_{fq}^{(k)} dz;$$
$$i, q = 1,2; \quad p, f = 1,2,3; \quad g = 1,2,3,4; \quad \xi = 1,2$$

The generalized forces of inertia (3.88) may be rewritten in the form

$$\begin{gathered} T_i^{(q)} = I^{(gq)}\, \ddot{v}_{ig} - I_1^{(pgq)}\, \ddot{\chi}_{pg,i}; \\ T_i^{(fq)} = \overline{I}_1^{(gfq)}\, \ddot{v}_{ig} - I_2^{(pgfq)}\, \ddot{\chi}_{pg,i}; \\ i = j = 1,2; \\ T_3^{(fq)} = I_3^{(pgfq)}\, \ddot{\chi}_{pg}; \end{gathered} \tag{3.104}$$

The equations for generalized forces of inertia include integrated density characteristics of the shell given by

$$I^{(gq)} = \int_{a_0}^{a_n} \rho_k \psi_g^{(k)} \psi_q^{(k)} dz; \quad I_1^{(pgq)} = \int_{a_0}^{a_n} \rho_k \psi_{pg}^{(k)} \psi_q^{(k)} dz;$$
$$I_2^{(pgfq)} = \int_{a_0}^{a_n} \rho_k \psi_{pg}^{(k)} \psi_{fq}^{(k)} dz; \quad I_3^{(pgfq)} = \int_{a_0}^{a_n} \rho_k \varphi_{pg}^{(k)} \psi_{fq}^{(k)} dz; \tag{3.105}$$
$$\overline{I}_1^{(gfq)} = \int_{a_0}^{a_n} \rho_k \psi_g^{(k)} \psi_{fq}^{(k)} dz;$$
$$q = 1,2; \quad p, f = 1,2,3; \quad g = 1,2,3,4$$

The equations of the higher–order theory given above are in the form of a system of differential equations. This system may be written in the following matrix form:

$$[D]\{V\} + [I]\{\ddot{V}\} = [F]\{p\} \tag{3.106}$$

where $[D]$ is the matrix of differential operators over the vector of unknown functions of the reference surface and time given by

$$\{V\} = \{v_{ig}; \chi_{pg}\}^T; \;\; i = 1,2; \;\; p = 1,2,3; \;\; g = 1,2,3,4 \tag{3.107}$$

$[I]$ is the matrix of the differential operators over the acceleration vector of these functions defined as

$$\{\ddot{V}\} = \{\ddot{v}_{ig} \; ; \; \ddot{v}_{pg}\}^T; \;\; i = 1,2; \;\; p = 1,2,3; \;\; g = 1,2,3,4 \tag{3.108}$$

and $[F]$ is the matrix of differential operators over the vector of given loads which is

$$\{p\} = \{p_i^{\pm}; p_3^{\pm}\}^T; \;\; i = 1,2 \tag{3.109}$$

Finally the matrices and the corresponding vectors may be written in the form

$$[D]\{V\} = \begin{array}{|c|c|c|} \hline v_{1g} & v_{2g} & \chi_{pg} \\ \hline A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \\ \hline \end{array} \tag{3.110}$$

where

$$\begin{aligned}
[A_{11}] &= B_{11}^{(gq)}(...)_{,11} + B^{(gq)}(...)_{,22} - 2R_1^{(gq)}(...) \\
[A_{21}] &= (B_{21}^{(gq)} + B^{(gq)})(...)_{,21} \\
[A_{31}] &= \{\overline{B}_{11}^{(gfq)}(...)_{,11} + (\overline{B}_{21}^{(gfq)} + 2\overline{B}^{(gfq)})(...)_{,22} - [(2\overline{R}_1^{(gfq)} - \overline{C}_1^{(gfq)}) \\
&\quad -(\overline{C}_{11}^{(gfq)}k_{11} + \overline{C}_{21}^{(gfq)}k_{22})](...)\}_{,1} - 2\overline{C}^{(gfq)}k_{21}(...)_{,2} \\
[A_{12}] &= (B_{12}^{(gq)} + B^{(gq)})(...)_{,12} \\
[A_{22}] &= (B_{22}^{(gq)}(...)_{,22} + B^{(gq)}(...)_{,11} - 2R_2^{(gq)}(...) \\
[A_{32}] &= \{\overline{B}_{22}^{(gfq)}(...)_{,22} + (\overline{B}_{12}^{(gfq)} + 2\overline{B}^{(gfq)})(...)_{,11} + [(2\overline{R}_2^{(gfq)} - \overline{C}_2^{(gfq)}) \\
&\quad -(\overline{C}_{22}^{(gfq)}k_{22} + \overline{C}_{12}^{(gfq)}k_{11})](..)\}_{,2} + \overline{C}^{(gfq)}k_{21}(...)_{,1}
\end{aligned}$$

$$[A_{13}] = -\{B_{11}^{(pgq)}(...)_{,11} + (B_{12}^{(pgq)} + 2B^{(pgq)})(...)_{,22} + [(2R_1^{(pgq)} - C_1^{(pgq)}) - (C_{11}^{(pgq)}k_{11} + C_{12}^{(pgq)}k_{22})](...)\}_{,1} + 2C^{(pgq)}k_{12}(...)_{,2}$$

$$[A_{23}] = -\{B_{22}^{(pgq)}(...)_{,22} + (B_{21}^{(pgq)} + 2B^{(pgq)})(...)_{,11} + [(2R_2^{(pgq)} - C_2^{(pgq)}) - (C_{22}^{(pgq)}k_{22} + C_{21}^{(pgq)}k_{11})](...)\}_{,2} + 2C^{(pgq)}k_{21}(...)_{,1}$$

$$\begin{aligned}[A_{33}] = & -\{[D_{11}^{(pgfq)}(...)_{,11} + (D_{12}^{(pgfq)} + 2D^{(pgfq)})(...)_{,22}]_{,11} \\ & +[D_{22}^{(pgfq)}(...)_{,22} + (D_{21}^{(pgfq)} + 2D^{(pgfq)})(...)_{,11}]_{,22} \\ & -[2R_1^{(pgfq)} + (E_{11}^{(pgfq)} + \overline{E}_{11}^{(pgfq)})k_{11} + (E_{12}^{(pgfq)} + \overline{E}_{11}^{(pgfq)})k_{22} \\ & +(C_{31}^{(pgfq)} + E_1^{(pgfq)})](...)_{,11} + [2R_2^{(pgfq)} + (E_{22}^{(pgfq)} + \overline{E}_{22}^{(pgfq)})k_{22} \\ & +(E_{21}^{(pgfq)} + \overline{E}_{21}^{(pgfq)})k_{11} + (C_{32}^{(pgfq)} + E_2^{(pgfq)})](...)_{,22} \\ & +4(E^{(pgfq)} + \overline{E}^{(pgfq)})k_{12}(...)_{,12} - [C_3^{(pgfq)} + 4K^{(pgfq)}k_{12} \\ & +k_{11}(C_1^{(pgfq)} + K_{11}^{(pgfq)}k_{11} + K_{12}^{(pgfq)}k_{22} + K_1^{(pgfq)}) \\ & +k_{22}(C_2^{(pgfq)} + K_{22}^{(pgfq)}k_{22} + K_{21}^{(pgfq)}k_{11} + K_2^{(pgfq)})](...)\end{aligned}$$

Moreover

$[I]\{\ddot{V}\} =$ (3.111)

$\ddot{v}_{1g}$	$\ddot{v}_{2g}$	$\ddot{\chi}_{1g}$
$-I^{gq}$	$-$	$I_1^{(pgq)}(...)_{,1}$
$-$	$-I^{(gq)}(...)$	$I_1^{(pgq)}(...)_{,2}$
$-\overline{I}_1^{(pgq)}(...)_{,1}$	$-\overline{I}_1^{(pgq)}(...)_{,2}$	$(I_2^{(pgfq)} \nabla - I_3^{(pgfq)})(...)$

where ∇ is the Laplace operator.

$[F^-]\{p^-\} =$ (3.112)

p_1^-	p_2^-	p_3^-
$\psi_q^{(1)}(a_0)(...)$	$-$	$-$
$-$	$\psi_q^{(1)}(a_0)(...)$	$-$
$\psi_{fq}^{(1)}(a_0)(...)_{,1}$	$\psi_q^{(1)}(a_0)(...)_{,2}$	$\varphi_{fq}^{(1)}(a_0)(...)$

$[F^+]\{p^+\} =$ (3.113)

p_1^+	p_2^+	p_3^+
$\psi_q^{(n)}(a_n)(...)$	$-$	$-$
$-$	$\psi_q^{(n)}(a_n)(...)$	$-$
$\psi_{fq}^{(n)}(a_n)(...)_{,1}$	$\psi_{fq}^{(n)}(a_n)(...)_{,2}$	$\varphi_{fq}^{(n)}(a_n)(...)$

The total number of equations in the system (3.106) is equal to 10. All particular cases of the general system of equations (3.106) can be obtained by making assumptions about the properties of the layers and these cases are shown in Table 3.1.

Considering some features of the present higher–order theory, one must note that the normal deformations depend on two factors which involve the influence of the Poisson's effect in the transverse direction, which can be ignored by assuming $\nu'_k = 0$, and the influence of the external loading and forces of inertia which can be excluded by assuming $E'_k = \infty$.

The higher–order theory developed in the present study is considerably different from those in which the equations of motion are obtained on the basis of the quasi–static approach when the influence of the forces of inertia is not taken into account in the hypotheses.

Table 3.1. Some particular cases of the system of equations of motion for the laminated shell.

	Present setting of problem (including inertia forces)			Quasi–static setting of problem (excluding inertia forces)		
Material assumptions	Number of equations	Order of equations	Number of boundary conditions	Number of equations	Order of equations	Number of boundary conditions
None	10	32	16	5	16	8
Excluding Poisson effect $\nu'_k = 0$	9	28	14	4	12	6
Excluding transverse shear $G'_k = \infty$	8	28	14	5	16	8
Excluding normal compression $E'_k = \infty$	7	20	10	4	12	6
Excluding shear and compression $G'_k = \infty, E'_k = \infty$	3	8	4	3	8	4

3.7 The generalized symmetric eigenvalue problem

Solving the system of equations (3.106) in order to find the vibration frequencies we come up against the generalized eigenvalue problem. As this problem presents a real challenge to many engineers, next the efficient algorithm is given.

In the generalized eigenvalue problem

$$(\boldsymbol{A} - \lambda \boldsymbol{B})x = 0 \tag{3.114}$$

The matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are symmetric, and $\boldsymbol{B}$ is also positive definite. Under these conditions the problem can be reduced to a special eigenvalue problem with symmetric matrix $\boldsymbol{C}$

$$(\boldsymbol{C} - \lambda \boldsymbol{I})y = 0 \tag{3.115}$$

For this purpose let us introduce the matrices $\boldsymbol{D}$ and $\boldsymbol{U}$, where $\boldsymbol{D}$ is a diagonal matrix defined as

$$\boldsymbol{D} = \left| \begin{array}{cccc} 1/\sqrt{\lambda_b^{(1)}} & 0 & \vdots & 0 \\ 0 & 1/\sqrt{\lambda_b^{(2)}} & \vdots & 0 \\ \dots & \dots & \vdots & \dots \\ 0 & 0 & \vdots & 1/\sqrt{\lambda_b^{(n)}} \end{array} \right| \tag{3.116}$$

and matrix $\boldsymbol{U}$ is given by

$$\boldsymbol{U} = \boldsymbol{S}_b \boldsymbol{D} \boldsymbol{S}_b \tag{3.117}$$

In equations (3.116) and (3.117) the following notations are used: index n denotes the order of the matrix, $\lambda_b^{(i)}$ are the eigenvalues of the matrix $\boldsymbol{B}$, and $\boldsymbol{S}_b$ are eigenvectors of $\boldsymbol{B}$.

Then, matrix $\boldsymbol{C}$ can be determined as

$$\boldsymbol{C} = \boldsymbol{U}\boldsymbol{A}\boldsymbol{U} \tag{3.118}$$

The eigenvalues λ of $\boldsymbol{C}$ agree with those of the generalized eigenvalue problem, and the interrelationship of the eigenvectors of y and x are given by

$$x = \boldsymbol{U}y \tag{3.119}$$

The special eigenvalue problem is solved by Jacobi's method. Since we need to determine all eigenvalues, this method is the most efficient and the safest, it always yields a system of orthonormal vectors which approximate the eigenvectors in columns of the orthogonal matrix.

3.8 Some analytical solutions and results

The analytical solution of the system of differential equations (3.106) is possible only for some particular cases. Let us consider the case of a simply supported shell with a rectangular plan view. The boundary conditions are specified as (e.g. for $x_1 = const$)

$$v_{2g} = 0; \quad N_{11}^{(q)} = 0; \quad \chi_{fg} = 0; \quad M_{11}^{(fq)} = 0; \tag{3.120}$$

$$q = 1,2; \quad f = 1,2,3; \quad g = 1,2,3,4.$$

The solution may be obtained in a manner similar to the procedure used in the classical theory due to the fact that the equations of the higher–order theory have a mathematical structure which is similar to that of the classical shallow shell theory.

Let the loading be expressed by the trigonometric series

$$\begin{aligned} p_1^{\pm} &= \sum_m \sum_n a_{mn}^{\pm} \cos \lambda_m x_1 \sin \gamma_n x_2 e^{-i\Omega_{mn} t} \\ p_2^{\pm} &= \sum_m \sum_n b_{mn}^{\pm} \sin \lambda_m x_1 \cos \gamma_n x_2 e^{-i\Omega_{mn} t} \\ p_3^{\pm} &= \sum_m \sum_n c_{mn}^{\pm} \sin \lambda_m x_1 \sin \gamma_n x_2 e^{-i\Omega_{mn} t} \end{aligned} \tag{3.121}$$

where

$$\lambda_m = \frac{m\pi}{a_1}; \quad \lambda_n = \frac{n\pi}{a_2}$$

Now the unknown functions can be expanded as

$$\begin{aligned} v_{1g} &= \sum_m \sum_n A_{mn}^{(g)} \cos \lambda_m x_1 \sin \gamma_n x_2 e^{-i\omega_{mn} t} \\ v_{2g} &= \sum_m \sum_n B_{mn}^{(g)} \sin \lambda_m x_1 \cos \gamma_n x_2 e^{-i\omega_{mn} t} \\ \chi_{pg} &= \sum \sum C_{mn}^{(pg)} \sin \lambda_m x_1 \sin \gamma_n x_2 e^{-i\omega_{mn} t} \end{aligned} \tag{3.122}$$

where a_1 and a_2 are the dimensions of the shell in the x_1 and x_2 directions, respectively; $a_{mn}^{\pm}, b_{mn}^{\pm}, c_{mn}^{\pm}$ are Fourier coefficients for the entries of the load vector; $A_{mn}^{(g)}, B_{mn}^{(g)}, C_{mn}^{(pg)}$ are amplitudes of the unknown functions; Ω_{mn} and ω_{mn} are the frequencies of the excitation load and free vibrations of the shell, respectively.

The solution of the forced vibration problem reduces to the solution of the system of linear algebraic equations

$$[D_0 - I_0\omega_{mn}^2]\{A_{mn}\} = [F_0\Omega_{mn}^2]\{a_{mn}\} \tag{3.123}$$

where the vector of the amplitudes of the unknown functions is given by

$$\{A_{mn}\} = \{A_{mn}^{(g)};\ B_{mn}^{(g)};\ C_{mn}^{(pg)}\}^T; \quad p = 1,2,3; \quad g = 1,2 \tag{3.124}$$

and the vector of the given load amplitudes by

$$\{a_{mn}\} = \{a_{mn}^{\pm};\ b_{mn}^{\pm};\ c_{mn}^{\pm}\}^T \tag{3.125}$$

The solution for the natural frequencies reduces to the solution of the generalized eigenvalue problem which may be expressed by the following characteristic determinant equation for the spectrum of the frequencies ω_{mn}:

$$\left|D_0 - I_0\omega_{mn}^2\right| = \begin{array}{|c|c|c|} \hline A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \\ \hline \end{array} = 0 \tag{3.126}$$

where

$$\begin{aligned}
[A_{11}] &= -\left[(B_{11}^{(gq)}\lambda_m^2 + B^{(gq)}\gamma_n^2) - 2R_1^{(gq)}\right] - I^{(gq)}\omega_{mn}^2 \\
[A_{21}] &= -\left(B_{21}^{(gq)} + B^{(gq)}\right)\lambda_m\gamma_n \\
[A_{31}] &= \{\overline{B}_{11}^{(gfq)}\lambda_m^2 + (\overline{B}_{21}^{(gfq)} + 2\overline{B}^{(gfq)}\gamma_n^2 - [(2\overline{R}_1^{(gfq)} - \overline{C}_1^{(gfq)}) \\
&\quad -(\overline{C}_{11}^{(gfq)}k_{11} + \overline{C}_{21}^{(gfq)}k_{22})]\lambda_m\} + \overline{I}_1^{(gfq)}\lambda_m\omega_{mn}^2 \\
[A_{12}] &= -(B_{12}^{(gq)} + B^{(gq)})\lambda_m\gamma_n \\
[A_{22}] &= -[(B_{22}^{(gq)}\lambda_m^2 + B^{(gq)}\gamma_n^2) - 2R_2^{(gq)}] - I^{(gq)}\omega_{mn}^2 \\
[A_{32}] &= \{\overline{B}_{22}^{(gfq)}\gamma_n^2 + (\overline{B}_{12}^{(gfq)} + 2\overline{B}^{(gfq)})\lambda_m^2 - [(2\overline{R}_2^{(gfq)} - \overline{C}_2^{(gfq)}) \\
&\quad -(\overline{C}_{22}^{(gfq)}k_{22} + \overline{C}_{12}^{(gfq)}k_{11})]\gamma_n\} + \overline{I}_1^{(gfq)}\gamma_n\omega_{mn}^2 \\
[A_{13}] &= \{B_{11}^{(pgq)}\lambda_m^2 + (B_{12}^{(pgq)} + 2B^{(pgq)})\gamma_n^2 - [(2R_1^{(pgq)} - C_1^{(pgq)}) \\
&\quad -(C_{11}^{(pgq)}k_{11} + C_{12}^{(pgq)}k_{22})]\lambda_m\} + I_1^{(pgq)}\lambda_m\omega_{mn}^2 \\
[A_{23}] &= \{B_{22}^{(pgq)}\gamma_n^2 + (B_{21}^{(pgq)} + 2B^{(pgq)})\lambda_m^2 - [(2R_2^{(pgq)} - C_2^{(pgq)}) \\
&\quad -(C_{22}^{(pgq)}k_{22} + C_{21}^{(pgq)}k_{11})]\gamma_n\} + I_1^{(pgq)}\gamma_n\omega_{mn}^2 \\
[A_{33}] &= -\{[D_{11}^{(pgfq)}\lambda_m^2 + (D_{12}^{(pgfq)} + 2D^{(pgfq)})\gamma_n^2]\lambda_m^2 \\
&\quad +[D_{22}^{(pgfq)}\gamma_n^2 + (D_{21}^{(pgfq)} + 2D^{(pgfq)})\lambda_m^2]\gamma_n^2\} \\
&\quad -2[R_1^{(pgfq)} + (E_{11}^{(pgfq)} + \overline{E}_{11}^{(pgfq)})k_{11} + (E_{12}^{(pgfq)} + \overline{E}_{12}^{(pgfq)})k_{22} \\
&\quad +(C_{31}^{(pgfq)} + E_1^{(pgfq)})]\lambda_m^2 - [2R_2^{(pgfq)} + (E_{22}^{(pgfq)} + \overline{E}_{22}^{(pgfq)})k_{22} \\
&\quad +(E_{21}^{(pgfq)} + \overline{E}_{21}^{(pgfq)})k_{11} + (C_{32}^{(pgfq)} + E_2^{(pgfq)})]\gamma_n^2 \\
&\quad -[C_3^{(pgfq)} + k_{11}(C_1^{(pgfq)} + K_{11}^{(pgfq)}k_{11} + K_{12}^{(pgfq)}k_{22} + K_1^{(pgfq)}) \\
&\quad +k_{22}(C_2^{(pgfq)} + K_{22}^{(pgfq)}k_{22} + K_{21}^{(pgfq)}k_{11} + K_2^{(pgfq)})] \\
&\quad -[I_2^{(pgfq)}(\lambda_m^2 + \gamma_n^2) + I_3^{(pgfq)}]\omega_{mn}^2
\end{aligned}$$

Special cases of the general problem may be obtained by specifying the material properties. In these cases the order of the determinant is reduced and the corresponding number of equations for each case is given in Table 3.1. Next we consider some specific problems.

Problem 3.1

Let us consider the free vibrations of a homogeneous square plate ($a_1 = a_2 = a$) with boundary conditions given by equations (3.120). Results presented in Table 3.2 are obtained for frequencies which are equal to the half–wavelength l of the vibration mode in the orthogonal directions ($l = a/m$). Results are presented for various ratios of half–wavelength l and thickness of the plate h where the side length is taken as $a = 40h$. The results agree very closely with the exact three–dimensional solution in the interval $40 \geq l/h \geq 1$. Thus the higher–order theory offered allows the determination of frequencies for which the half–wavelength is equal to the thickness of the homogeneous plate ($l/h = 1$). The classical theory is acceptable only when $l/h > 8$.

Table 3.2 Free vibration frequencies of the homogeneous plate.

Problem data: $\theta = 10^2 \frac{\omega l}{\pi}\sqrt{\rho/2G}$; $l = a/m$; $a = 40h$; $\nu = 0.3$					
Theory	$l/h = 40$	8	4	2	1
3–D	5.408	25.74	45.62	68.25	84.22
Present HOT	5.408	25.74	45.62	68.29	84.41
Classical	5.419	27.10	54.19	–	–

Problem 3.2

Let us consider the problem of free vibrations of a square sandwich plate made of isotropic layers with $a_1 = a_2 = a$. The following characteristics are used:

$$2h_1 + h_2 = h; \quad h_2 = 18h_1; \quad a = 40h;$$

$$G_1 = 10^3 G_2; \quad \rho_1 = 10\rho_2; \quad \nu_1 = 0.3; \quad \nu_2 = 0$$

where h_1 and h_2 are the thicknesses of the surface and core layers, respectively. Figure 3.2 shows the curves of the normalized parameter

$$\theta^2 = 10^4 \frac{\omega^2 l^2}{\pi^2} \cdot \frac{\rho_2}{2G_2}$$

plotted against h_2/h_1 where ω is the fundamental vibration frequency. Solutions are given for the following theories:

1. Reissner–Timoshenko theory
2. present higher–order theory
3. three-dimensional theory for the core and classical theory for the external layers
4. exact three–dimensional theory

The results of the higher–order theory coincide very closely with those of the three–dimensional theory on the interval $40 \geq l/h \geq 1$. When $l/h = 1$ the discrepancy is 5% for the parameter θ. The solution based on the Reissner–Timoshenko theory cannot be used for the investigation of large frequencies when $l/h > 4$.

The limitations on the use of the higher–order theory for sandwich plates depend on three parameters: l/h ratio between the half–wavelength and the thickness; h_2/h_1 ratio between the thicknesses of the core and external layers and G_1/G_2 ratio between the shear moduli of the external and core layers. The regions of these ratios for which the higher–order theory coincides with the three–dimensional theory are shown in Fig. 3.3. Domain where the higher–order theory is applicable lies under the corresponding curve. It is observed that when $l/h = 40$ the results obtained on the basis of the higher–order theory occupy almost the entire region within the following bounds

$$0 \leq \log(G_1/G_2) \leq 5; \quad 0 \leq h_2/h_1 \leq 50.$$

If $l/h \leq 4$ the bounds are

$$0 \leq \log(G_1/G_2) \leq 4; \quad 10 \leq h_2/h_1 \leq 50;$$

and for $l/h = 1$

$$0 \leq \log(G_1/G_2) \leq 3; \quad 10 \leq h_2/h_1 \leq 50.$$

Within $0 \leq h_2/h_1 \leq 10$ the domain of the values $\log(G_1/G_2)$ is reduced on the average by one order. The highest reduction is reached when the thickness ratio is $h_2/h_1 = 0.6$. For this ratio we observe the highest influence of the transverse shear deformations for the sandwich plate.

Problem 3.3

In this problem the results obtained for the fundamental frequency of a sandwich beam are compared with the experimental data obtained by Khatua and Cheng [39] and Osternik [61]. The results obtained using the higher–order theory are in good agreement with the experimental results as shown in Table 3.3. The classical theory is inapplicable when we either decrease the relative length of the span l/h or the shear modulus of the core.

Table 3.3 Comparison with experimental results for sandwich plates.

h_2/h_1	l/h	$\log(G_1/G_2)$	Experim. ω, s^{-1}	HOT ω, sec^{-1}	Δ, %	Classical ω, sec^{-1}	Δ, %
7.8	42.4	1.43	70.2	70.7	0.7	72.6	3.4
5.3	54.9	2.65	80.1	78.3	-2.2	86.6	8.1
17.0	14.7	2.82	123	120	-2.3	214	74
7.9	11.8	2.82	161	151	-6.0	390	143

Problem 3.4

Figure 3.4 shows the results of the comparison of natural frequencies for simply supported sandwich plates using different theories. In the figure ω denotes the fundamental frequencies obtained using the general solution (3.126) and $\overline{\omega}$ are the frequencies obtained using particular cases given in Table 3.1. Results are given for parameters

$$l/h = 4,\ \rho_1/\rho_2 = 10^2 \text{ and } h_2/h_1 = 18.$$

In the case when major influence on the frequencies is due to the effect of the transverse shear deformations ($\log(G_1/G_2) \geq 3$), all results agree very closely. The influence of the normal deformations increases as the shear modulus of the core material becomes larger ($\log(G_1/G_2) < 3$). The solutions become inaccurate for the case when dynamic factors in the hypotheses are excluded by setting $\ddot{u}_i = 0$ (quasi–static problem) and for the case when the normal deformation is not taken into account ($E'_k = \infty$).

Problem 3.5

Let us consider various plates and shallow shells with different sequences of the strong and weak layers through the thickness. The total thickness and mass of the shell for each case is kept constant. The fundamental frequencies ω_0 and their ratio to the frequencies ω, which are obtained using the classical theory, are given in Table 3.4. It is observed that the influence of the transverse shear becomes significant in structures where the strong layers are separated by weak layers. The influence of transverse shear and normal deformation increases as the strong material is redistributed more closely to the external surfaces, and also as the half–wavelength of the vibration is reduced. For single–layer plates, the results are very close to those given by the classical theory.

Problem 3.6

Next we consider sandwich plates and spherical shells with different boundary conditions and different shapes in plan. A number of analytical and numerical methods of solution is used to obtain the numerical results. Table 3.5 shows the fundamental vibration frequencies $\omega/2\pi, s^{-1}$ in the sandwich structures which are circumscribed by circles of equal radii in plan. The influence of the transverse shear deformation (ratio ω_0/ω) becomes more pronounced as the area enclosed by the structure becomes smaller. This influence is also more pronounced for plates with clamped boundaries as compared to plates with simply supported or free boundaries. It is also observed that this influence decreases as the curvature of the shell gets larger and increases as the acute angle of the oblique–angled plate decreases.

Tables 3.4 and 3.5 indicate that as the influence of the transverse shear and normal deformation increases, the difference in the vibration frequencies of plates and shells with different geometrical and mechanical properties of layers and different boundary conditions decreases and the frequency spectrum broadens.

Table 3.4 Free vibration frequencies of plates of equal mass.

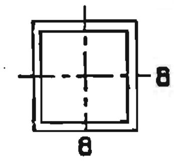
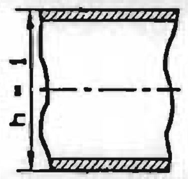
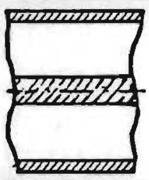
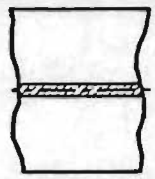
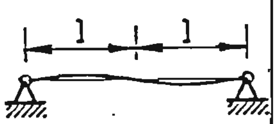
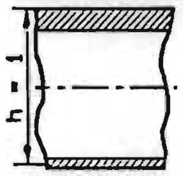
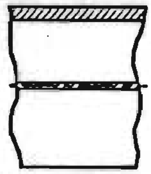
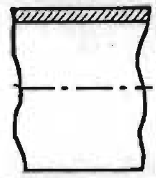
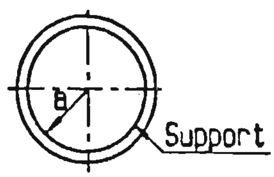
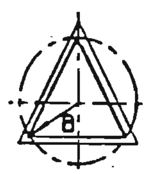
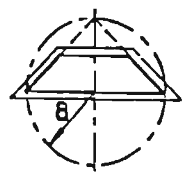
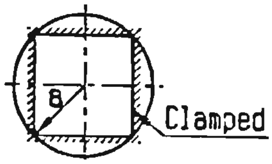
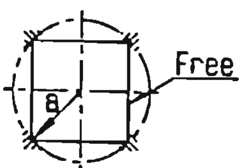
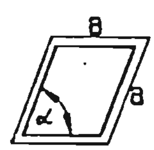
Problem data: $a = 40h$; $G_1 = 10^3 G_2$; $\rho_1 = 10\rho_2$; $\nu_1 = 0.3$; $\nu_2 = 0.4$							
Frequencies: $\omega/2\pi$, sec^{-1}.				Shear influence: ω_0/ω			
Simply supported square plates of equal mass							
			0.025 0.9 0.025		0.025 0.45 0.05 0.45 0.025		0.45 0.1 0.45
l/h	40	260	1.36	216	1.20	31.1	1.0
	20	677	2.10	627	1.65	124	1.0
	10	1481	3.84	1421	2.81	488	1.02
			0.075 0.9 0.025		0.075 0.45 0.025 0.45		0.1 0.9
l/h	40	241	1.19	145	1.15	46.4	1.01
	20	657	1.96	444	1.51	183	1.02
	10	1461	3.38	1069	2.48	692	1.08

Table 3.5 The fundamental frequencies of sandwich plates and spherical shells.

								
r/a	∞	129	1.17	398	1.59		552	1.87
	4	529	1.01	649	1.25		753	1.52
	2	1034	1.00	1100	1.08		1164	1.25
								
r/a	∞	452	1.44	154	1.26	α°, 60	310	1.44
	4	684	1.21	525	1.03	45	396	1.59
	2	1120	1.08	1037	1.00	30	595	1.95

3.9 Summary and conclusions

A higher–order theory of laminated plates and shells which takes into account transverse shear and normal deformation is developed for the solution of dynamic problems. The proposed theory is capable of treating plates and shells with an arbitrary number and sequences of layers which may differ significantly in their physical and mechanical properties. The elastic characteristics may be constant or variable through the thickness of each layer. The kinematic hypotheses are derived using an iterative technique where the classical theory is used as a first approximation. The important feature of the model is that the dynamic factors such as forces of inertia and rotary inertia are included in the model at the initial stage when the kinematic hypotheses are formulated. This procedure leads to a number of new unknown functions and subsequently to a number of additional higher–order equations of motion. The new variables which are introduced have clear physical meanings. The direct influence of the loading conditions on the transverse shear and normal deformation of the shell is also incorporated into the model. It is shown that the results obtained using a quasi–static theory in which the forces of inertia and rotary inertia are neglected in the kinematic hypotheses are not as accurate as the results obtained on the basis of the present approach. The level of accuracy of a given theory depends on several factors which are discussed and elucidated in the context of example problems.

The equations of motion and the complete set of boundary conditions are derived using a variational formulation. Different particular cases are also studied as special cases of the general theory.

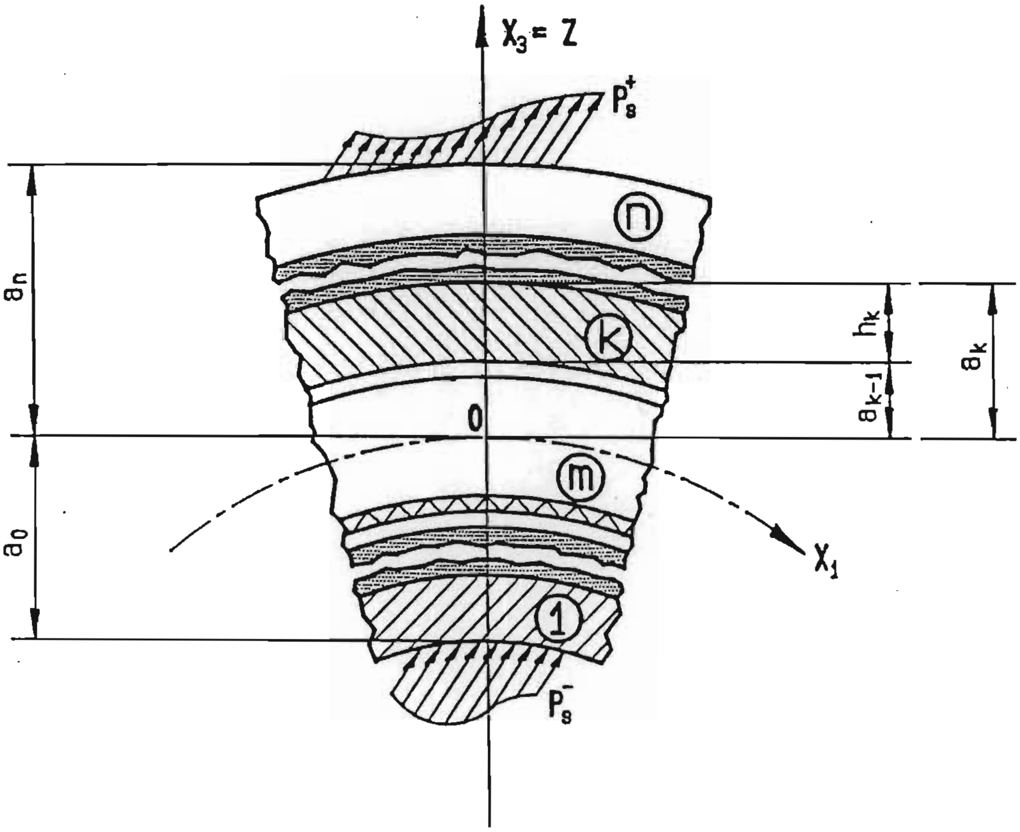


Figure 3.1 Geometry of a laminated shell.

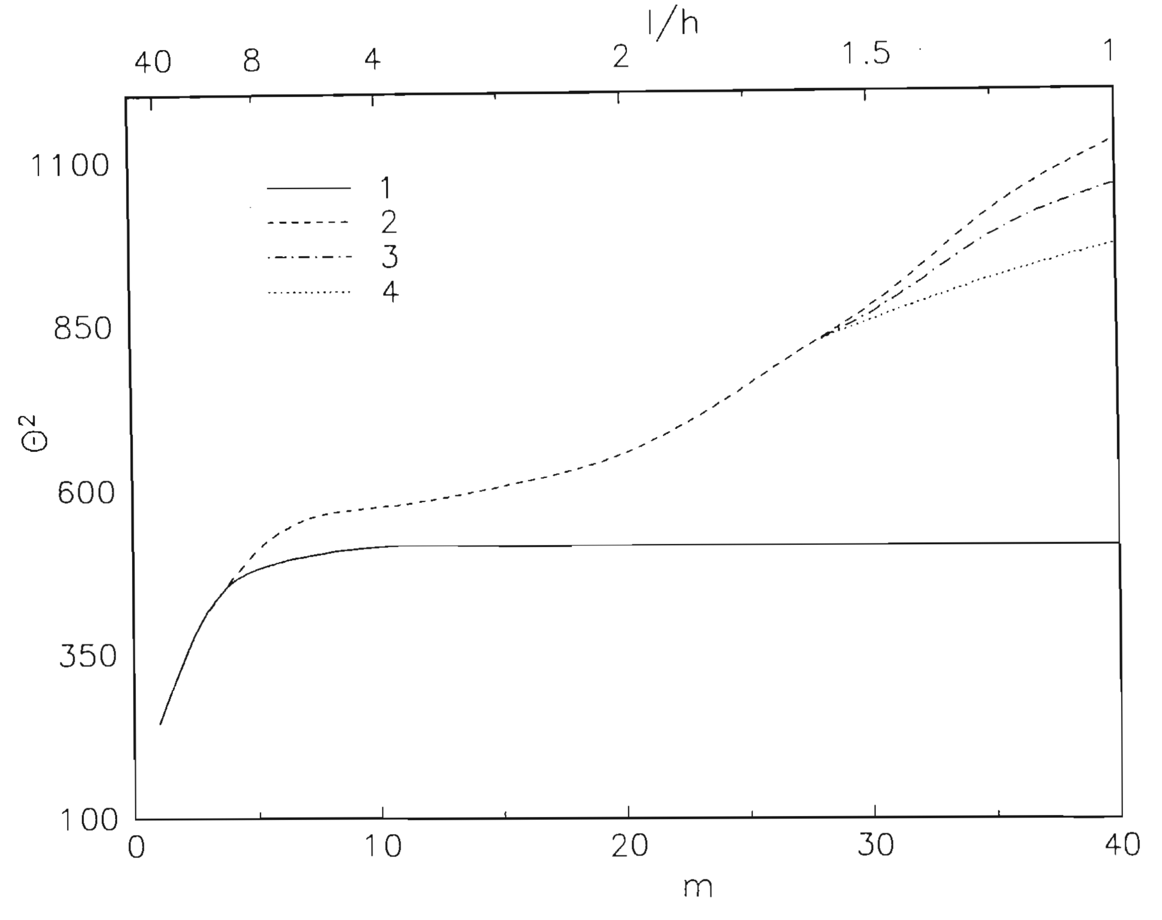


Figure 3.2 Comparison of the solutions for the sandwich plates: (1) Reissner–Timoshenko theory; (2) present theory; (3) first–order theory; (4) exact three–dimensional solution.

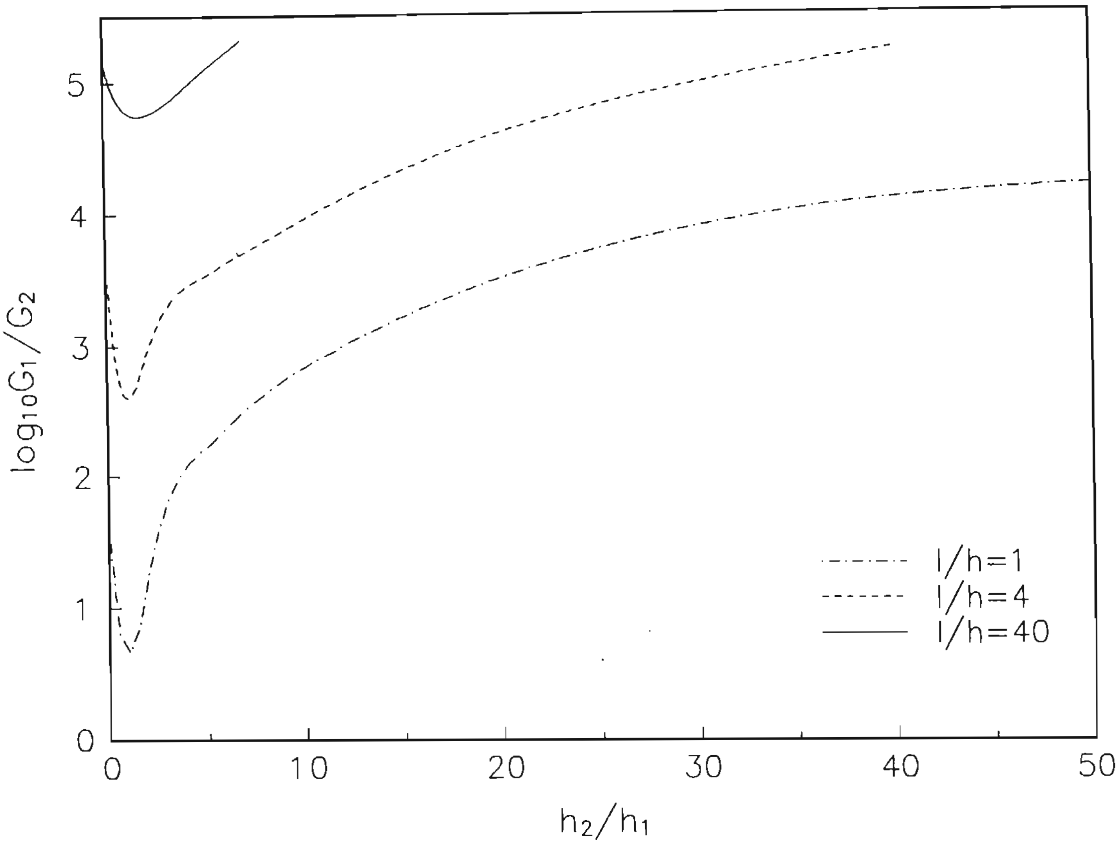


Figure 3.3 Domains in which the higher–order and three–dimensional theory coincide. (This domain lies under each curve).

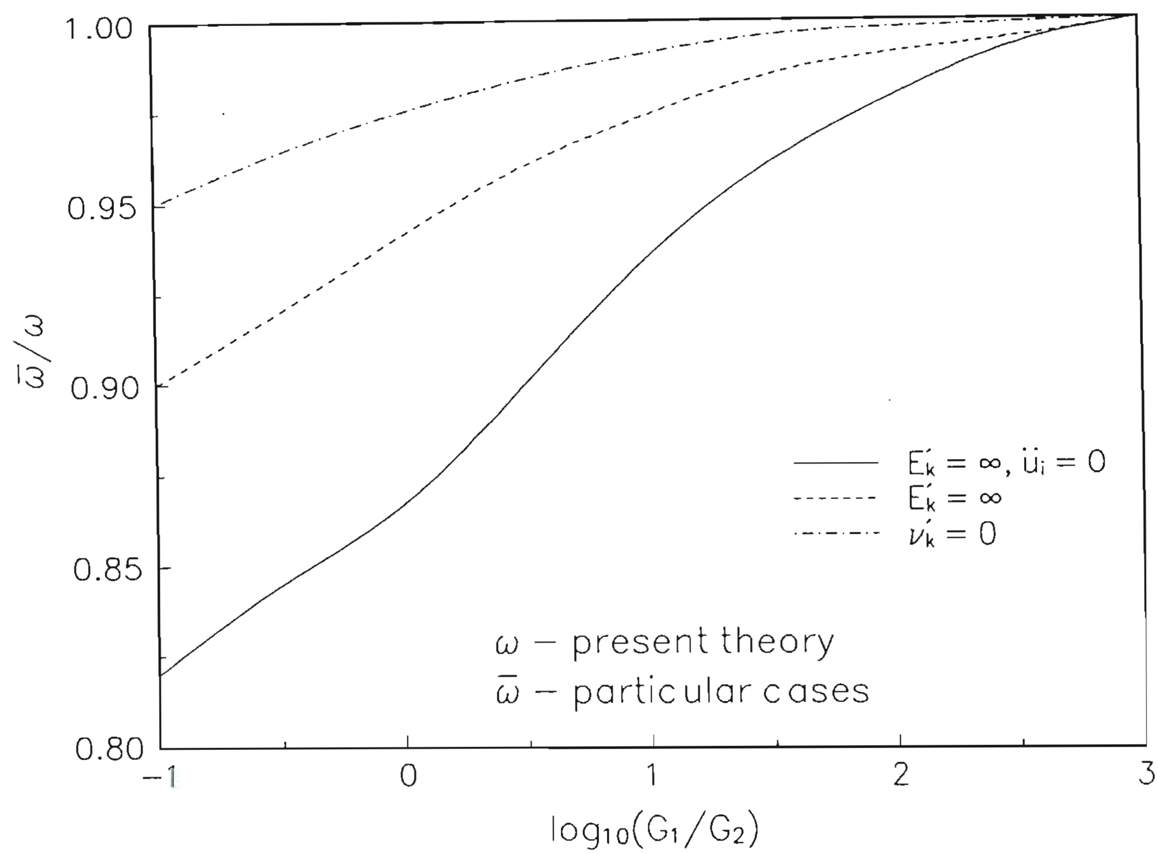


Figure 3.4 Comparison of the present theory with theories neglecting the effects of some parameters.

Chapter 4

Geometrically Nonlinear Higher Order Theory of Multilayered Orthotropic Shells

4.1 Introduction

The main objective of this chapter is to derive a comprehensive geometrically nonlinear theory of laminated plates and shells which can accurately determine the behaviour of such structures under various loading and boundary conditions.

A geometrically nonlinear higher order theory of laminated composite plates and shells which takes into account both transverse shear and normal deformation is presented. The theory is based on the kinematic hypotheses which are not assumed *a priori* but are derived on the basis of an iterative technique. The closed form solutions for some laminated plates and shells are given to illustrate, first, the importance of modelling the geometrical nonlinearity especially at high levels of loading, and second, the importance of including both transverse shear and normal deformation in the stress/strain analysis of composite structures.

4.2 Basic equations and assumptions

A study of the reviews mentioned earlier reveals that although numerous approaches have been formulated for refined classical theories of laminated anisotropic plates and shells, the geometrical nonlinear analysis of such structural elements based on a higher order theory has not been developed sufficiently enough. Moreover, most of available geometrically nonlinear higher order theories are based on the kinematic hypotheses which are derived from a consideration of linear or quasi–linear problems and therefore, these hypotheses do not fully reflect the physical essence of the problem. For the plates and shells in which the layers have significantly different physical characteristics, it is necessary to consider not only transverse shear but also normal deformation [65].

Let us consider shells with orthotropic layers of varying thickness and stiffness which are weak in their resistance against transverse shear and normal deformation. The physical and mechanical characteristics of the layers (which are assumed to be perfectly bonded) may vary through the thickness. It is also assumed that although the deformations of the shell are small, the Von Karman geometrical nonlinearity must be taken into account.

4.2.1 Deformations in the shell

Let us represent the shell in a curvilinear coordinate system x_1Ox_2 which is parallel to the bounding surfaces and the surfaces of contact between the layers (Fig. 3.1). The axes of the curvilinear coordinates $x_i = constant$ $(i = 1, 2)$ coincide with the principle lines of curvature and the coordinate $z = x_3$ is defined along the normal to the reference surface x_1Ox_2. It is assumed that the coefficients of the first quadratic form of a surface are close to unity, i.e. $A_1 \approx A_2 \approx 1$ and the main curvatures k_{ij} are constant, i.e. $k_{ij} = constant$ $(i, j = 1, 2)$. The shell is taken as a structure with small curvatures relative to its thickness.

The loads are applied on the outer (q^+) and inner (q^-) surfaces, respectively, so that

$$q^+ = q_s^+(x_i); \quad q^- = q_s^-(x_i); \quad s = 1, 2, 3; \quad i = 1, 2 \tag{4.1}$$

In the following derivations a subscript after a comma denotes differentiation with respect to the variable following the comma and the superscript k refers to the *k-th* layer $(k = 1, 2, ..., n)$.

The conditions on the external surfaces may be written as

$$\begin{aligned} \sigma_{s3}^{(1)} &= -q_s^- \qquad \text{for} \quad z = c_0; \quad k = 1 \\ \sigma_{s3}^{(n)} &= q_s^- \qquad \text{for} \quad z = c_n; \quad k = n \end{aligned} \tag{4.2}$$

Continuity conditions for the perfectly bonded layers at an arbitrary surface $z = c_{k-1}$ are given by

$$\begin{aligned} \sigma_{s3}^{(k)} &= \sigma_{s3}^{(k-1)} \text{ (static)} \\ u_{s3}^{(k)} &= u_{s3}^{(k-1)} \text{ (kinematic)} \end{aligned} \tag{4.3}$$

Taking into account geometrical nonlinearity the deformations in the *k-th* layer may be expressed in terms of the components of the displacement vector $\{u\}_k = \{u_i^{(k)}, u_i^{(k)}\}^T$ as [55]

$$\begin{aligned} 2e_{ij}^{(k)} &= u_{i,j}^{(k)} + u_{j,i}^{(k)} + 2k_{ij}u_3^{(k)} + u_{3,i}^{(k)}\, u_{3,j}^{(k)} \\ 2e_{i3}^{(k)} &= u_{i,3}^{(k)} + u_{3,i}^{(k)} + u_{3,i}^{(k)}\, u_{3,3}^{(k)} \\ 2e_{33}^{(k)} &= 2u_{3,3}^{(k)} + u_{3,3}^{(k)}\, u_{3,3}^{(k)} \end{aligned} \tag{4.4}$$

4.2.2 Constitutive equations

We assume that each of n layers of the shell ($k = 1, \ldots, n$) is an orthotropic.

Then the constitutive equations for the orthotropic *k-th* layer may be written as

$$e_{ij}^{(k)} = J_{ijtl}^{(k)}\, \sigma_{tl}^{(k)} \tag{4.5}$$

where $[J^{(k)}]$ is a tensor of elastic characteristics given by

$$[J^{(k)}] = \left| \begin{array}{cccccc} 1/E_1^{(k)} & -\nu_{21}^{(k)}/E_2^{(k)} & -\nu_{31}^{(k)}/E_3^{(k)} & 0 & 0 & 0 \\ -\nu_{12}^{(k)}/E_1^{(k)} & 1/E_2^{(k)} & \nu_{32}^{(k)}/E_3^{(k)} & 0 & 0 & 0 \\ \nu_{13}^{(k)}/E_2^{(k)} & \nu_{23}^{(k)}/E_2^{(k)} & 1/E_3^{(k)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2G_{12}^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2G_{13}^{(k)} & 0 \\ 0 & 0 & & 0 & 0 & 1/2G_{23}^{(k)} \end{array} \right| \tag{4.6}$$

In equation (4.6), $E_s^{(k)}, G_{is}^{(k)}$ are the moduli of elasticity and shear moduli, respectively, and $\nu_{is}^{(k)}$, $\nu_{si}^{(k)}$ $(i \neq s)$ are Poisson's ratios. It should be also noted that the all elastic properties are variable through the thickness of a layer h_k, i.e. they are functions of the coordinate z_k.

In order to formulate the kinematic hypothesis for the derivation of a higher-order theory, let us implement an iterative technique proposed by Ambartsumian [3] in which the theory based on the classical hypothesis of Kirchoff-Love is considered as a first iteration. Firstly, we assume that $\nu_{i3}^{(k)} = 0$, $E_3^{(k)} = \infty$ and $G_{i3} = \infty$ in equation (4.5) which then becomes

$$
\begin{aligned}
e_{11}^{(k)} &= \frac{1}{E_1^{(k)}}\sigma_{11}^{(k)} - \frac{\nu_{21}^{(k)}}{E_2^{(k)}}\sigma_{22}^{(k)} \\
e_{22}^{(k)} &= \frac{1}{E_2^{(k)}}\sigma_{22}^{(k)} - \frac{\nu_{12}^{(k)}}{E_1^{(k)}}\sigma_{11}^{(k)} \\
e_{12}^{(k)} &= \frac{\sigma_{12}^{(k)}}{2G_{12}^{(k)}}
\end{aligned}
\tag{4.7}
$$

The tangential components of the stress tensor in this case may be written as

$$
\begin{aligned}
\sigma_{11}^{(k)} &= E_{01}^{(k)} e_{11}^{(k)} + E_{0\nu}^{(k)} e_{22}^{(k)} \\
\sigma_{22}^{(k)} &= E_{0\nu}^{(k)} e_{11}^{(k)} + E_{02}^{(k)} e_{22}^{(k)} \\
\sigma_{12}^{(k)} &= 2G_{12}^{(k)} e_{12}^{(k)}
\end{aligned}
\tag{4.8}
$$

where

$$
E_{0i}^{(k)} = \frac{E_i^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}; \quad E_{0\nu}^{(k)} = \frac{E_i^{(k)}\nu_{gp}^{(k)}}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}}
$$

$$
i = 1,2; \quad p = i \neq g
$$

Then

$$
\begin{aligned}
\sigma_{11}^{(k)} &= E_{01}^{(k)} e_{11}^{(k)} + E_{0\nu}^{(k)} e_{22}^{(k)} \\
\sigma_{22}^{(k)} &= E_{0\nu}^{(k)} e_{11}^{(k)} + E_{02}^{(k)} e_{22}^{(k)} \\
\sigma_{12}^{(k)} &= 2G_{12}^{(k)} e_{12}^{(k)}
\end{aligned}
\tag{4.9}
$$

4.2.3 Classical model

The assumptions which have been made are equivalent to the Kirchoff-Love hypotheses, viz.

$$
e_{i3}^{(k)} = 0; \quad e_{33}^{(k)} = 0; \quad \sigma_{33}^{(k)} = 0 \tag{4.10}
$$

Substituting equations (4.4) into the first two of these hypotheses and integrating, the following kinematic model can be obtained

$$
\begin{aligned}
u_i^{(k)} &= u_i - (z - h_0)w_{,i} \\
u_3^{(k)} &= w
\end{aligned}
\tag{4.11}
$$

where $u_i(x_1, x_2)$, $w = w(x_1, x_2)$ are the displacements of the reference surface and h_0 is the distance from the reference surface to the surface on which tangential displacements are equal to zero. We note that the reference surface may be positioned arbitrarily through the thickness of the shell. In fact, it may be chosen within any layer, coincide with the interlaminar or external surfaces as dictated by the character of the problem under consideration.

Substituting equations (4.11) into (4.4) we obtain the deformations of the *k-th* layer in the tangential directions as

$$e_{ij}^{(k)} = \epsilon_{ij} + z\kappa_{ij} \tag{4.12}$$

where the strains and curvatures due to deformations of the reference surface are given by

$$\begin{aligned} \epsilon_{ij} &= \frac{1}{2}\left[(u_{i,j} + u_{j,i}) + w_{,i}w_{,j}\right] + k_{ij}w \\ \kappa_{ij} &= -w_{,ij} \end{aligned} \tag{4.13}$$

Equations (4.13) satisfy the well-known relations [60]

$$\begin{aligned} 2\epsilon_{12,12} - \epsilon_{11,22} - \epsilon_{22,11} &= \kappa_{11}\kappa_{22} - \kappa_{12}^2 + k_{11}\kappa_{22} + k_{22}\kappa_{11} - 2k_{12}\kappa_{12} \\ \kappa_{11,2} - \kappa_{12,1} &= 0 \\ \kappa_{22,1} - \kappa_{21,2} &= 0 \end{aligned} \tag{4.14}$$

Substituting equation (4.13) in (4.9) the tangential components of the stress tensor may now be obtained as

$$\begin{aligned} \sigma_{11}^{(k)} &= E_{01}^{(k)}(\epsilon_{11} + \kappa_{11}z) + E_{0\nu}^{(k)}(\epsilon_{22} + \kappa_{22}z) \\ \sigma_{22}^{(k)} &\rightleftharpoons \sigma_{11}^{(k)} \\ \sigma_{12}^{(k)} &= 2G_{12}^{(k)}(\epsilon_{12} + \kappa_{12}z) \end{aligned} \tag{4.15}$$

The symbol $\rightleftharpoons$ indicates that the expression for σ_{22} is of the same form as that for σ_{11} with the provision that the subscript 11 is replaced with 22 and vice versa.

Hence, the physical and geometrical relations of the classical model of laminated orthotropic shells have been obtained, and they are taken as basic relations for the further derivations of the theory.

4.3 Transverse stresses and strains

4.3.1 Equations of equilibrium

Other components of the stress tensor cannot be found using Hooke's law because of the hypothesis (4.10). For their derivation we employ the equations of equilibrium for a shell [3], which for the *k-th* layer may be expressed as

$$\sigma_{ij,j}^{(k)} + \sigma_{i3,3}^{(k)} = 0; \;\; \sigma_{i3,i}^{(k)} - (k_{ij} + \kappa_{ij})\,\sigma_{ij}^{(k)} + \sigma_{33,3}^{(k)} = 0 \tag{4.16}$$

Using the first equation in (4.15) we derive the transverse shear stresses as

$$\sigma_{i3}^{(k)} = -\int_{c_{k-1}}^{z} \sigma_{ij,j}^{(k)} dz + \Phi_{ik} \tag{4.17}$$

where Φ_{ik} is a function of integration which may be obtained satisfying conditions (4.2) and (4.3) and can be written in the following form

$$\Phi_{ik} = -q_i^- - \int_{c_0}^{c_{k-1}} \sigma_{ijj}^{(k)} dz, \quad i = 1,2 \tag{4.18}$$

Hereafter the following rule of integration for piecewise function is used

$$\sum_{r=1}^{k-1} \int_{c_{r-1}}^{c_r} (\ldots)^{(r)} dz = \int_{c_0}^{c_{k-1}} (\ldots)^{(k)} dz \tag{4.19}$$

Taking into account (4.18) into (4.17) the transverse shear stresses can be obtained

$$\sigma_{i3}^{(k)} = -q_i^- - \int_{c_0}^{z} \sigma_{ij,j}^{(k)} dz \tag{4.20}$$

Henceforward, according to (4.19), the following rule for integration with variable upper limits is used

$$\int_{c_0}^{z} (\ldots)^{(k)} dz = \int_{c_{k-1}}^{z} (\ldots)^{(k)} dz + \sum_{r=1}^{k-1} \int_{c_{r-1}}^{c_r} (\ldots)^{(r)} dz \tag{4.21}$$

Taking conditions (4.2) on the surface $z = c_n$ into account in (4.20) the following relations can be found

$$q_i^+ + q_i^- = -\int_{c_0}^{c_n} \sigma_{ij,j}^{(k)} dz \tag{4.22}$$

Introducing the tangential forces

$$N_{ij} = \int_{c_0}^{c_n} \sigma_{ij}^{(k)} dz, \quad i,j = 1,2 \tag{4.23}$$

we obtain from (4.22) the well-known equilibrium equations of the shell

$$N_{ij,j} + q_i^+ + q_i^- = 0 \tag{4.24}$$

Using equations (4.16), we can also derive the transverse normal stresses as

$$\sigma_{33}^{(k)} = -\int_{c_{k-1}}^{z} \left[\sigma_{i3,i}^{(k)} - (k_{ij} + \kappa_{ij})\,\sigma_{ij}^{(k)}\right] dz + \Phi_{3k} \tag{4.25}$$

Satisfying conditions (4.2) and (4.3) (for $s = 3$) the following two expressions can be obtained from (4.25)

$$\sigma_{33}^{(k)} = q_3^- - \int_{c_0}^{z} \left[\sigma_{i3,i}^{(k)} - (k_{ij} + \kappa_{ij})\sigma_{ij}^{(k)}\right] dz \tag{4.26}$$

$$q_3^+ + q_3^- = -\int_{c_0}^{c_n} \left[\sigma_{i3,i}^{(k)} - (k_{ij} + \kappa_{ij})\sigma_{ij}^{(k)}\right] dz \tag{4.27}$$

Equation (4.27) corresponds to the equilibrium equation of the classical theory of shells

$$M_{ij,ij} - (k_{ij} + \kappa_{ij})N_{ij} + (q_3^+ + q_3^-) + (q_{i,i}^+ c_n + q_{i,i}^- c_0) = 0 \tag{4.28}$$

where

$$M_{ij} = \int_{c_0}^{c_n} \sigma_{ij}^{(k)} z dz, \quad i,j = 1,2 \tag{4.29}$$

The equilibrium equations (4.24) and (4.28) are well-known equations of the classical theory of shallow shells. Using expressions for internal forces (4.23) and moments (4.29), it is possible to obtain the system of nonlinear equations of the classical theory of multilayered shells in terms of the displacements u_i, w.

4.3.2 Transverse shear stresses

Using relations (4.15) in (4.20) and (4.22) we can derive the following expressions

$$\begin{aligned} \sigma_{i3}^{(k)} &= -q_i^- - \Big(\epsilon_{pp,i} \int_{c_0}^{z} E_{0p}^{(k)} dz + \epsilon_{gg,i} \int_{c_0}^{z} E_{0\nu}^{(k)} dz + 2\epsilon_{12,j} \int_{c_0}^{z} G_{12}^{(k)} dz \\ &+ \kappa_{pp,i} \int_{c_0}^{z} E_{0p}^{(k)} z dz + \kappa_{gg,i} \int_{c_0}^{z} E_{0\nu}^{(k)} z dz + 2\kappa_{12,j} \int_{c_0}^{z} G_{12}^{(k)} z dz\Big) \end{aligned} \tag{4.30}$$

$$\begin{aligned} q_i^+ + q_i^- &= -(\epsilon_{pp,i} A_{p1} + \epsilon_{gg,i} A_2 + \epsilon_{12,i} A_3 + \kappa_{pp,i} B_{p1} + \kappa_{gg,i} B_2 + \kappa_{12,j} B_3) \\ & \quad i \neq j = 1,2; \quad p = i; \quad g = j \end{aligned} \tag{4.31}$$

It should be noted that there is no summation over the indices p and g. From (4.31) we have

$$\begin{aligned} &\left(\epsilon_{pp,i} + \epsilon_{gg,i} \frac{A_2}{A_{p1}} + \epsilon_{12,j} \frac{A_3}{A_{p1}}\right) \\ &= -\frac{1}{A_{p1}} \left[(q_i^+ + q_i^-) + (\kappa_{pp,i} B_{p1} + \kappa_{gg,i} B_2 + \kappa_{12,j} B_3)\right] \end{aligned} \tag{4.32}$$

where the following stiffness characteristics have been used

$$A_{i1} = A_{p1} = \int_{c_0}^{c_n} E_{0i}^{(k)} dz,\ A_2 = \int_{c_0}^{c_n} E_{0\nu} dz,\ A_3 = \int_{c_0}^{c_n} 2G_{12}^{(k)} dz$$

$$B_{i1} = B_{p1} = \int_{c_0}^{c_n} E_{0i}^{(k)} zdz,\ B_2 = \int_{c_0}^{c_n} E_{0\nu}^{(k)} zdz,\ B_3 = \int_{c_0}^{c_n} 2G_{12}^{(k)} zdz \qquad (4.33)$$

Let us introduce the following relations

$$\begin{aligned}\epsilon_{pp,i}\int_{c_0}^{z} E_{0i}^{(k)} dz &+ \epsilon_{gg,i}\int_{c_0}^{z} E_{0\nu}^{(k)} dz + \epsilon_{12,j}\int_{c_0}^{z} 2G_{12}^{(k)} dz \\ &\approx (\epsilon_{pp,i} + \epsilon_{gg,i}\frac{A_2}{A_{p1}} + \epsilon_{12,j}\frac{A_3}{A_{p1}})\int_{c_0}^{z} E_{0p}^{(k)} dz \qquad (4.34)\end{aligned}$$

$$\begin{aligned}\nu_i^{(k)} &= \frac{E_{0\nu}^{(k)}}{E_{0i}^{(k)}} \approx \frac{A_2}{A_{i1}} = \frac{A_2}{A_{p1}} = \nu_{i2} \\ \frac{1-\nu_{12}\nu_{22}}{1+\nu_{i3}} &\approx \frac{A_3}{A_i1} = \frac{A_3}{A_{p1}}, \quad \nu_{i3} \approx \nu_{i3}^{(k)} \qquad (4.35) \\ G_{12}^{(k)} = \frac{E_i^{(k)}}{2(1+\nu_{i3}^{(k)})} &\approx \frac{E_i^{(k)}}{2(1+\nu_{i3})}, \quad i = p = 1,2\end{aligned}$$

Substituting relation (4.34) in (4.30) we obtain transverse shear stresses as

$$\begin{aligned}\sigma_{i3}^{(k)} &= -q_i^- + \frac{q_i^+ + q_i^-}{A_{p1}}\int_{c_0}^{z} E_{0p}^{(k)} dz \\ &+ \kappa_{pp,i}(\frac{B_{p1}}{A_{p1}}\int_{c_0}^{z} E_{0p}^{(k)} dz - \int_{c_0}^{z} E_{0p}^{(k)} zdz) \\ &+ \kappa_{gg,i}(\frac{B_2}{A_{p1}}\int_{c_0}^{z} E_{0p}^{(k)} dz - \int_{c_0}^{z} E_{0\nu}^{(k)} zdz) \qquad (4.36) \\ &+ \kappa_{12,j}(\frac{B_3}{A_{p1}}\int_{c_0}^{z} E_{0p}^{(k)} dz - \int_{c_0}^{z} 2G_{12}^{(k)} zdz)\end{aligned}$$

Taking into account relations (4.14), equation (4.36) can be expressed as follows

$$\begin{aligned}\sigma_{i3}^{(k)} &= \kappa_{pp,i} f_{pp}^{(k)} + \kappa_{gg,i} f_{pg}^{(k)} - q_i^- f_{p3}^{(k)} - q_i^+ f_{p4}^{(k)} \\ &\quad i = 1,2; \quad p = i; \quad g \neq i \qquad (4.37)\end{aligned}$$

where

$$\begin{aligned}f_{pp}^{(k)} &= \frac{B_{p1}}{A_{p1}} f_p^{(k)} - \int_{c_0}^{z} E_{0p}^{(k)} zdz; \quad f_p^{(k)} = \int_{c_0}^{z} E_{0p}^{(k)} dz \\ f_{pg}^{(k)} &= \frac{B_2 + B_3}{A_{p1}} f_p^{(k)} - \int_{c_0}^{z} (E_{0\nu}^{(k)} + 2G_{12}^{(k)}) zdz \qquad (4.38) \\ f_{p3}^{(k)} &= 1 - \frac{f_p^{(k)}}{A_{p1}}; \quad f_{p4}^{(k)} = -\frac{f_p^{(k)}}{A_{p1}}\end{aligned}$$

are the distribution functions which enable the given conditions on the external surfaces of the shell to be satisfied once the reference surface is positioned arbitrarily through the thickness of the layers. These functions also take into account the influence of the elastic properties of each layer on the distribution of the transverse shear stresses through the thickness of the laminated shell.

With allowance made for (4.35) it is possible to write

$$f_{pg}^{(k)} \approx \nu_p f_{pp}^{(k)}, \quad \nu_p = \frac{A_2 + A_3}{A_{p1}} \tag{4.39}$$

Then equation (4.37) can be expressed in the following form

$$\begin{aligned} \sigma_{i3}^{(k)} &= (\kappa_{pp} + \nu_p \kappa_{gg})_{,i} f_{pp}^{(k)} - q_i^- f_{p3}^{(k)} - q_i^+ f_{p4}^{(k)} \\ & \quad i = p \neq g = 1,2 \end{aligned} \tag{4.40}$$

4.3.3 Transverse normal stresses

The transverse normal stresses can be obtained using (4.26) and taking into account equation (4.37) as follows

$$\begin{aligned} \sigma_{33}^{(k)} &= -q_3^- + \int_{c_0}^{z} (q_i^- f_{p3}^{(k)} + q_i^+ f_{p4}^{(k)})_{,i} dz \\ &- \int_{c_0}^{z} (\kappa_{pp} f_{pp}^{(k)} + \kappa_{gg} f_{pg}^{(k)})_{,ii} dz + (k_{ij} + \kappa_{ij}) \int_{c_0}^{z} \sigma_{ij}^{(k)} dz \end{aligned} \tag{4.41}$$

$$\begin{aligned} &= q_3^- + q_{i,i}^- f_3^{(k)} + q_{i,i}^+ f_4^{(k)} - (\kappa_{pp} + \kappa_{gg})_{,ii} f_2^{(k)} + B_\sigma^{(k)} \\ & \quad i,j = 1,2; \;\; i = p \neq g \end{aligned} \tag{4.42}$$

In addition, the following distribution functions are introduced

$$\begin{aligned} f_t^{(k)} &= \frac{1}{2} \int_{c_0}^{z} (f_{1t}^{(k)} + f_{2t}^{(k)}) dz; \qquad t = 3,4 \\ f_2^{(k)} &= \frac{1}{4} \int_{c_0}^{z} (f_{11}^{(k)} + f_{12}^{(k)} + f_{21}^{(k)} + f_{22}^{(k)}) dz \end{aligned} \tag{4.43}$$

and also the generalised function of the tangential forces

$$B_\sigma^{(k)} = (k_{ij} + \kappa_{ij}) \int_{c_0}^{z} \sigma_{ij}^{(k)} dz \tag{4.44}$$

From (4.27) we obtain

$$q_3^+ + q_3^- \approx q_{i,i}^- f_3(c_n) + q_{i,i}^+ f_4(c_n) - (\kappa_{pp} + \kappa_{gg})_{,ii} f_2(c_n) + B_\sigma(c_n) \tag{4.45}$$

where

$$
\begin{aligned}
f_t(c_n) &= \frac{1}{2}\int_{c_0}^{c_n}(f_{1t}^{(k)} + f_{2t}^{(k)})dz, \qquad t=3,4 \\
f_2(c_n) &= \frac{1}{4}\int_{c_0}^{c_n}(f_{11}^{(k)} + f_{12}^{(k)} + f_{21}^{(k)} + f_{22}^{(k)})dz \\
B_\sigma(c_n) &= (k_{ij} + \kappa_{ij})\int_{c_0}^{c_n}\sigma_{ij}^{(k)}dz
\end{aligned}
$$

Using (4.45) in (4.41) the transverse normal stresses can be obtained. This computation gives

$$
\sigma_{33}^{(k)} = q_3^- F_1^{(k)} + q_3^+ F_2^{(k)} + q_{i,i}^- F_3^{(k)} + q_{i,i}^+ F_4^{(k)} + F_\sigma^{(k)} \tag{4.46}
$$

wherein

$$
\begin{aligned}
F_2^{(k)} &= \frac{f_2^{(k)}}{f_2^{(k)}(c_n)}, \qquad F_1^{(k)} = F_2^{(k)} - 1 \\
F_t^{(k)} &= f_t^{(k)} - \frac{f_t(c_n)}{f_2(c_n)}f_2^{(k)}, \qquad t=3,4 \\
F_\sigma^{(k)} &= B_\sigma^{(k)} + \frac{B_\sigma(c_n)}{f_2(c_n)}f_2^{(k)}
\end{aligned}
\tag{4.47}
$$

4.3.4 Normal deformations and displacements

Now using the constitutional equations (4.5) we can obtain transverse shear and normal deformations which were assumed equal to zero during the first iteration when the model based on the classical hypothesis was derived. Transverse shear deformations may be expressed as

$$
2e_{i3}^{(k)} = \frac{\sigma_{i3}^{(k)}}{G_{p3}^{(k)}} = \kappa_{pp,i}\,\varphi_{pp}^{(k)} + \kappa_{gg,i}\,\varphi_{pg}^{(k)} - q_i^-\,\varphi_{p3}^{(k)} - q_i^+\,\varphi_{p4}^{(k)} \tag{4.48}
$$

where distribution functions are given by

$$
\begin{aligned}
\varphi_{pp}^{(k)} &= \frac{f_{pp}^{(k)}}{G_{p3}^{(k)}}; \quad \varphi_{pg}^{(k)} = \frac{f_{pg}^{(k)}}{G_{p3}^{(k)}} \\
\varphi_{p3}^{(k)} &= \frac{f_{p3}^{(k)}}{G_{p3}^{(k)}}; \quad \varphi_{p4}^{(k)} = \frac{f_{p4}^{(k)}}{G_{p3}^{(k)}}
\end{aligned}
\tag{4.49}
$$

For the derivation of the normal deformations we can use constitutional equations in the following form

$$
\sigma_{ij}^{(k)} = C_{ijtl}^{(k)}\,e_{tl}^{(k)} \tag{4.50}
$$

where $C_{ijtl}^{(k)} \equiv [\lambda_k]$ and

$$[\lambda_k] = \begin{vmatrix} \lambda_1^{(k)} & \lambda_4^{(k)} & \lambda_6^{(k)} & 0 & 0 & 0 \\ 0 & \lambda_2^{(k)} & \lambda_5^{(k)} & 0 & 0 & 0 \\ 0 & 0 & \lambda_3^{(k)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda_7^{(k)} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda_9^{(k)} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\lambda_8^{(k)} \end{vmatrix} \tag{4.51}$$

Omitting indices k, the elements of the matrix (4.51) are given by

$$\begin{aligned}
\lambda_1^{(k)} &= \frac{1}{E_2 A}\left(\frac{1}{E_3} - \frac{\nu_{23}^2}{E_2}\right); \ \lambda_2^{(k)} = \frac{1}{E_3 A}\left(\frac{1}{E_1} - \frac{\nu_{31}^2}{E_3}\right) \\
\lambda_3^{(k)} &= \frac{1}{E_1 A}\left(\frac{1}{E_2} - \frac{\nu_{12}^2}{E_1}\right); \ \lambda_4^{(k)} = \frac{1}{E_3 A}\left(\frac{\nu_{31}\nu_{23}}{E_2} + \frac{\nu_{12}}{E_1}\right) \\
\lambda_5^{(k)} &= \frac{1}{E_1 A}\left(\frac{\nu_{12}\nu_{31}}{E_3} + \frac{\nu_{23}}{E_2}\right); \ \lambda_6^{(k)} = \frac{1}{E_2 A}\left(\frac{\nu_{12}\nu_{23}}{E_1} + \frac{\nu_{31}}{E_3}\right) \\
\lambda_7^{(k)} &= G_{12}; \ \lambda_8^{(k)} = G_{23}; \ \lambda_9^{(k)} = G_{31} \\
A^{(k)} &= \frac{1}{E_1 E_2 E_3}\left(1 - 2\nu_{12}\nu_{23}\nu_{31} - \frac{E_1}{E_3}\nu_{31}^2 - \frac{E_2}{E_1}\nu_{12}^2 - \frac{E_3}{E_2}\nu_{23}^2\right)
\end{aligned} \tag{4.52}$$

With respect to (4.50) and the following relations, and also (4.12), we obtain

$$\begin{aligned}
e_{33}^{(k)} &= \frac{1}{\lambda_3^{(k)}}(\sigma_{33}^{(k)} - \lambda_6^{(k)} e_{11}^{(k)} - \lambda_5^{(k)} e_{22}^{(k)}) \\
&= \frac{1}{\lambda_3^{(k)}}\left[\sigma_{33}^{(k)} - (\epsilon_{11}\lambda_6^{(k)} + \epsilon_{22}\lambda_5^{(k)}) - (\kappa_{11}\lambda_6^{(k)} + \kappa_{22}\lambda_5^{(k)})z\right]
\end{aligned} \tag{4.53}$$

Using stresses $\sigma_{33}^{(k)}$ in a form of (4.46) the normal deformations can be now written as

$$\begin{aligned}
e_{33}^{(k)} &= -\left(\kappa_{11} F_1^{'(k)} + \kappa_{22} F_2^{'(k)}\right) + \epsilon_{11} F_3^{'(k)} + \epsilon_{22} F_4^{'(k)} \\
&\quad + q_3^- F_5^{'(k)} + q_3^+ F_6^{'(k)} + q_{i,i}^- F_7^{'(k)} + q_{i,i}^+ F_8^{'(k)} + F_\sigma^{'(k)}
\end{aligned} \tag{4.54}$$

where

$$\begin{aligned}
F_t^{'(k)}(z) &= \frac{\lambda_{(7-t)}^{(k)}}{\lambda_3^{(k)}} z; \ F_{(t+2)}^{'(k)}(z) = -\frac{\lambda_{(7-t)}^{(k)}}{\lambda_3^{(k)}}; \ t = 1,2 \\
F_{(r+4)}^{'(k)}(z) &= \frac{F_r^{(k)}}{\lambda_3^{(k)}}; \ r = 1, ..., 4 \\
F_\sigma^{'(k)}(z) &= \frac{F_\sigma^{(k)}}{\lambda_3^{(k)}}
\end{aligned} \tag{4.55}$$

Using equation (4.54) we can obtain the refined expression for the normal displacement given by

$$\begin{aligned}
u_3^{(k)} &= w + \int_{h_0}^{z} e_{33}^{(k)} dz = w - \kappa_{11}\varphi_1^{(k)} - \kappa_{22}\varphi_2^{(k)} + \epsilon_{11}\varphi_3^{(k)} \\
&\quad + \epsilon_{22}\varphi_4^{(k)} + q_3^-\varphi_5^{(k)} + q_3^+\varphi_6^{(k)} + q_{i,i}^-\varphi_7^{(k)} + q_{i,i}^+\varphi_8^{(k)} + \varphi_\sigma^{(k)}
\end{aligned} \tag{4.56}$$

where

$$\varphi_t^{(k)} = \int_{h_0}^{z} F_t'^{(k)} dz; \;\; \varphi_\sigma^{(k)} = \int_{h_0}^{z} F_\sigma'^{(k)} dz; \;\; t = 1, ..., 8 \tag{4.57}$$

The above refined expressions for $e_{i3}^{(k)}$, $e_{33}^{(k)}$ and for $\sigma_{i3}^{(k)}$, $\sigma_{33}^{(k)}$ are not relevant to the model based on the classical Kirchoff-Love hypotheses since they only demonstrate the contradictions in this model, but are important for the derivation of a higher-order theory.

4.4 Derivation of a higher–order theory

4.4.1 Hypotheses

The derivation of the higher-order theory will be based on the assumption that transverse shear deformations, transverse normal deformations and transverse normal stresses are not equal to zero $e_{i3}^{(k)} \neq 0, e_{33}^{(k)} \neq 0, \sigma_{33}^{(k)} \neq 0$. The transverse shear deformations are expressed as

$$\begin{aligned} 2e_{i3}^{(k)} &= -\chi_{p,i}\varphi_{pp}^{(k)} - \chi_{g,i}\varphi_{pg}^{(k)} - q_i^- \varphi_{p3}^{(k)} - q_i^+ \varphi_{p4}^{(k)} = \\ &= -\chi_{s,i}\varphi_{ps}^{(k)} + 2e_{i3}^{(0k)} = \theta_{si}\varphi_{ps}^{(k)} + 2e_{i3}^{(0k)} \\ & \quad i = p \neq g = 1,2; \;\; s = p, g \end{aligned} \tag{4.58}$$

where the following notations have been introduced

$$\theta_{pi} = -\chi_{p,i}; \;\; \theta_{gi} = -\chi_{g,i} \tag{4.59}$$

In the equation (4.58), $e_{i3}^{(0k)}$ is the part of the deformations which may be determined directly from the given external loading as

$$2e_{i3}^{(0k)} = -q_i^- \varphi_{p3}^{(k)} - q_i^+ \varphi_{p4}^{(k)} \tag{4.60}$$

The transverse normal deformation is taken in the following form

$$\begin{aligned} e_{33}^{(k)} &= \chi_1 F_1'^{(k)} + \chi_2 F_2'^{(k)} + \epsilon_{11}^{(0)} F_3'^{(k)} + \epsilon_{22}^{(0)} F_4'^{(k)} + q_3^- F_5'^{(k)} + q_3^+ F_6'^{(k)} \\ & +q_{i,i}^- F_7'^{(k)} + q_{i,i}^+ F_8'(k) + F_\sigma'^{(k)} = \chi_s F_s'^{(k)} + e_{33}^{(0k)} \\ & \quad s = p, g; \;\; p \neq g = 1,2 \end{aligned} \tag{4.61}$$

where the part of deformation which may be determined directly from the given external loading is presented as

$$e_{33}^{(0k)} = \epsilon_{11}^{(0)} F_3'^{(k)} + \epsilon_{22}^{(0)} F_4'^{(k)} + q_3^- F_5'^{(k)} + q_3^+ F_6'^{(k)} + q_{i,i}^- F_7'^{(k)} + q_{i,i}^+ F_8'^{(k)} + F_\sigma'^{(k)} \tag{4.62}$$

In equation (4.62), $\epsilon_{11}^{(0)}$ and $\epsilon_{22}^{(0)}$ are tangential deformations of the reference surface defined by equation (4.13) and they include nonlinear terms. The transverse normal stresses may be assigned as given stresses determined directly from the external loading as a result of the first iteration when the problem was considered on the basis of the classical model. The σ_{33} stresses are given in the form of equation (4.46) which must satisfy both the conditions on the external surfaces and the interlaminar conditions.

4.4.2 Displacement vector

Now the components of the displacement vector of the higher-order theory can be obtained. Let us firstly present the equation for the transverse normal deformation in the following form

$$2e_{33}^{(k)} = 2\bar{e}_{33}^{(k)} + 2\bar{\bar{e}}_{33}^{(k)} \tag{4.63}$$

where the first term is a linear part of deformation which is unknown, viz.

$$2\bar{e}_{33}^{(k)} = 2u_{3,3}^{(k)} \tag{4.64}$$

and the second nonlinear term is taken as a given deformation defined by equation (4.56), viz.

$$2\bar{\bar{e}}_{33}^{(k)} = \left(u_{3,3}^{(0k)}\right)^2 \tag{4.65}$$

where

$$\begin{aligned} u_{3,3}^{(0k)} &= -\kappa_{11}^{(0)}\varphi_{1,3}^{(k)} - \kappa_{22}^{(0)}\varphi_{2,3}^{(k)} + \epsilon_{11}^{(0)}\varphi_{3,3}^{(k)} + \epsilon_{22}^{(0)}\varphi_{4,3}^{(k)} \\ &\quad + q_3^-\varphi_{5,3}^{(k)} + q_3^+\varphi_{6,3}^{(k)} + q_{i,i}^-\varphi_{7,3}^{(k)} + q_{i,i}^+\varphi_{8,3}^{(k)} + \varphi_{\sigma,3}^{(k)} \end{aligned} \tag{4.66}$$

The superscript 0 in the above expressions identify terms which are considered as known terms defined as a result of the solution of the problem on the basis of the classical model. Such an iterative approach was offered by Ambartsumian [3] and in the case of a geometrically nonlinear problem considered in this chapter we apply this approach only for the refined definition of the nonlinear part of the deformations. Substituting equation (4.64) into equation (4.63) and integrating we obtain

$$u_3^{(k)} = w + \int_{c_{k-1}}^{z} \left(e_{33}^{(k)} - \bar{\bar{e}}_{33}^{(k)}\right) dz + C_{3k} \tag{4.67}$$

The constants of integration C_{3k} may be calculated satisfying kinematic conditions (4.3) and the conditions on the reference surface given as

$$u_3^{(m)}(x_1, x_2, h_0) = w \qquad \text{for} \qquad z = h_0 \;\; (k = m) \tag{4.68}$$

Then we can write

$$u_3^{(k)} = w + \int_{h_0}^{z} e_{33}^{(k)} dz + U_3^{(0k)} \tag{4.69}$$

where the known nonlinear part of the displacement defined as a result of first iteration is given by

$$U_3^{(0k)} = -\int_{h_0}^{z} \bar{\bar{e}}_{33}^{(k)} dz = -\frac{1}{2}\int_{h_0}^{z} \left(u_{3,3}^{(0k)}\right)^2 dz \tag{4.70}$$

Taking into account hypothesis (4.61) finally we obtain the following expression for the normal displacement:

$$\begin{aligned} u_3^{(k)} &= w + \chi_1\varphi_1^{(k)} + \chi_2\varphi_2^{(k)} + U_3^{(k)} = \chi_s\varphi_s^{(k)} + U_3^{(k)} \\ & s = 0, p, g: \quad p \neq g = 1, 2 \end{aligned} \tag{4.71}$$

where

$$U_3^{(k)} = \epsilon_{11}^{(0)}\varphi_3^{(k)} + \epsilon_{22}^{(0)}\varphi_4^{(k)} + q_3^{-}\varphi_5^{(k)} + q_3^{+}\varphi_6^{(k)} + q_{i,i}^{-}\varphi_7^{(k)} + q_{i,i}^{+}\varphi_8^{(k)} + \varphi_\sigma^{(k)} + U_3^{(0k)} \tag{4.72}$$

In equations (4.71) and (4.72) we have $\chi_0 = w$ and also distribution functions which are given by equation (4.57). Moreover the following notations are introduced

$$\varphi_0^{(k)} = 1\,;\ \varphi_p^{(k)} = \varphi_1^{(k)}\,;\ \varphi_g^{(k)} = \varphi_2^{(k)}$$

In order to obtain tangential components of the displacement vector let us present transverse shear deformations as

$$2e_{i3}^{(k)} = 2\bar{e}_{i3}^{(k)} + 2\bar{\bar{e}}_{i3}^{(k)} \tag{4.73}$$

where the unknown part of the deformations is given by

$$2\bar{e}_{i3}^{(k)} = u_{i,3}^{(k)} + u_{3,i}^{(k)} \tag{4.74}$$

and the known part of the deformations is expressed as

$$2\bar{\bar{e}}_{i3}^{(k)} = u_{3,i}^{(0k)} u_{3,3}^{(0k)} \tag{4.75}$$

In equation (4.75) $u_{3,3}^{(0k)}$ is defined by equation (4.66) and, moreover, we have

$$\begin{aligned} u_{3,i}^{(0k)} &= w_{0,i} - \kappa_{11,i}^{(0)}\varphi_1^{(k)} - \kappa_{22,i}^{(0)}\varphi_2^{(k)} + \epsilon_{11,i}^{(0)}\varphi_3^{(k)} + \epsilon_{22,i}\varphi_4^{(k)} + \\ & + q_{3,i}^{-}\varphi_5^{(k)} + q_{3,i}^{+}\varphi_6^{(k)} + q_{j,ji}^{-}\varphi_7^{(k)} + q_{j,ji}^{+}\varphi_8^{(k)} + \varphi_{\sigma,i}^{(k)} \\ & i = 1, 2, \quad j = i \end{aligned} \tag{4.76}$$

Substituting equations (4.74) and (4.75) into equation (4.73) and integrating we obtain

$$u_i^{(k)} = u_i + \int_{h_o}^{z} \left(2e_{i3}^{(k)} - u_{3,i}^{(k)}\right) dz + U_i^{(0k)} \tag{4.77}$$

where the known nonlinear part of the displacements is defined as

$$U_i^{(0k)} = -\int_{h_0}^{z} 2\overline{\overline{e}}_{i3}^{(k)} dz = -\int_{h_0}^{z} u_{3,i}^{(0k)} u_{3,3}^{(0k)} dz \tag{4.78}$$

Taking into account hypothesis (4.58), the equation for the normal displacement (4.71) and expression (4.75) we obtain the tangential components of the displacement vector as

$$\begin{aligned}
u_i^{(k)} &= u_i - w_{,i}\psi_0^{(k)} - \chi_{p,i}\left(\psi_p^{(k)} + \psi_{pp}^{(k)}\right) - \chi_{g,i}\left(\psi_g^{(k)} + \psi_{pg}^{(k)}\right) + U_i^{(k)} \\
&= u_i - \chi_{s,i}\Phi_{ps}^{(k)} + U_i^{(k)} \\
& \qquad i = p \neq g = 1,2; \quad s = 0, p, g; \quad i \rightleftharpoons j \to p \rightleftharpoons g
\end{aligned} \tag{4.79}$$

where the distribution functions of the displacements through the thickness of the laminated shell are defined as follows

$$\begin{aligned}
\psi_0^{(k)} &= \int_{h_0}^{z} dz; \quad \psi_2^{(k)} = \int_{h_0}^{z} \varphi_r^{(k)} dz; \quad \psi_{pp}^{(k)} = \int_{h_0}^{z} \varphi_{pp}^{(k)} dz \\
\psi_{pg}^{(k)} &= \int_{h_0}^{z} \varphi_{pg}^{(k)} dz; \quad \psi_{pt}^{(k)} = \int_{h_0}^{z} \varphi_{pt}^{(k)} dz; \quad \psi_{\sigma i}^{(k)} = \int_{h_0}^{z} \varphi_{\sigma,i}^{(k)} dz \\
& \qquad r = 1, 2, ..., 8; \quad t = 3, 4 \\
\Phi_{p0}^{(k)} &= \Phi_{g0}^{(k)} = \psi_0^{(k)}; \quad \Phi_{pp}^{(k)} = \psi_p^{(k)} = \psi_{pp}^{(k)} \\
\Phi_{pg}^{(k)} &= \psi_g^{(k)} = \psi_{pg}^{(k)}; \quad \varphi_0^{(k)} = 1 \\
\varphi_p^{(k)} &= \varphi_1^{(k)}; \quad \varphi_g^{(k)} = \varphi_2^{(k)}
\end{aligned} \tag{4.80}$$

The given part of the displacements in equation (4.79) can be expressed as

$$\begin{aligned}
U_i^{(k)} = & -q_i^- \psi_{p3}^{(k)} - q_i^+ \psi_{p4}^{(k)} - \epsilon_{11,i}^{(0)} - \psi_3^{(k)} \\
& -\epsilon_{22,i}^{(0)} \psi_4^{(k)} - q_{3,i}^- \psi_5^{(k)} - q_{3,i}^+ \psi_6^{(k)} \\
& -q_{j,ji}^- \psi_7^{(k)} - q_{j,ji}^+ \psi_8^{(k)} - \psi_{\sigma i}^{(k)} + U_i^{(0k)}
\end{aligned} \tag{4.81}$$

4.4.3 Stress and strain tensors

Since the components of the displacement vector have been obtained we can now define the tangential components of the strain tensor. Taking into account the expressions for the displacements (4.71) and (4.79) we present these deformations as a sum of the unknown linear and nonlinear parts and also the given part, which must be defined from the results obtained on the basis of the classical model. Then we can write

$$e_{ij}^{(k)} = \overline{e}_{ij}^{(k)} + \tilde{e}_{ij}^{(k)} + \overline{\overline{e}}_{ij}^{(k)}; \quad i, j = 1, 2 \tag{4.82}$$

where these parts are given respectively as

$$\begin{aligned}
\overline{e}_{pp}^{(k)} &= u_{p,p} - \chi_{s,pp}\Phi_{ps}^{(k)} + k_{pp}\chi_s\varphi_s^{(k)} \\
\overline{e}_{pg}^{(k)} &= \frac{1}{2}[(u_{p,g} + u_{g,p}) - (\chi_{s,pg}\Phi_{ps}^{(k)} + \chi_{s,gp}\Phi_{gs}^{(k)}) + 2k_{pg}\chi_s\varphi_s^{(k)}] \\
\tilde{e}_{ij}^{(k)} &= \frac{1}{2}[(\chi_{s,i}\varphi_s^{(k)}\chi_{t,j}\varphi_t^{(k)} + (\chi_{s,i}U_{3,j}^{(k)} + \chi_{s,j}U_{3,i}^{(k)})\varphi_s^{(k)}] \qquad (4.83) \\
\overline{\overline{e}}_{ij}^{(k)} &= \frac{1}{2}[(U_{i,j} + U_{j,i}) + 2k_{ij}U_3^{(k)} + U_{3,i}^{(k)}U_{3,j}^{(k)}] \\
& \qquad s,t = 0,p,g; \quad p \neq g = 1,2
\end{aligned}$$

The components of the stress tensor may be expressed similar to those of the components of the strain tensor as

$$\sigma_{ij}^{(k)} = \overline{\sigma}_{ij}^{(k)} + \tilde{\sigma}_{ij}^{(k)} + \overline{\overline{\sigma}}_{ij}^{(k)} \; ; \; i,j = 1,2 \qquad (4.84)$$

where

$$\begin{aligned}
\overline{\sigma}_{pp}^{(k)} &= \lambda_p^{(k)}\overline{e}_{pp}^{(k)} + \lambda_4^{(k)}\overline{e}_{gg}^{(k)} + \lambda_q^{(k)}\chi_s F_s^{(k)} = \\
&= \lambda_p^{(k)}u_{p,p} + \lambda_4^{(k)}u_{g,g} - (\lambda_p^{(k)}\chi_{s,pp}\Phi_{ps}^{(k)} + \lambda_4^{(k)}\chi_{s,gg}\Phi_{gs}^{(k)}) \\
&\quad + k_{pp}\lambda_g^{(k)} + k_{gg}\lambda_4^{(k)})\chi_s\varphi_s^{(k)} + \lambda_q^{(k)}\chi_s F_s^{(k)} \qquad (4.85)
\end{aligned}$$

$$\begin{aligned}
\tilde{\sigma}_{pp}^{(k)} &= \lambda_p^{(k)}\tilde{e}_{pp}^{(k)} + \lambda_4^{(k)}\tilde{e}_{gg}^{(k)} = \\
&= \frac{1}{2}[\lambda_p^{(k)}(\chi_{s,p}\varphi_s^{(k)})(\chi_{t,p}\varphi_t^{(k)}) + \lambda_4^{(k)}(\chi_{s,g}\varphi_s^{(k)})(\chi_{t,g}\varphi_t^{(k)}) \qquad (4.86) \\
&\quad + 2(\lambda_p\chi_{s,p}U_{3,p}^{(k)} + \lambda_4\chi_{s,g}U_{3,g}^{(k)})\varphi_s^{(k)}]
\end{aligned}$$

$$\begin{aligned}
\overline{\overline{\sigma}}_{pp}^{(k)} &= \lambda_p^{(k)}\,\overline{\overline{e}}_{pp}^{(k)} + \lambda_4^{(k)}\,\overline{\overline{e}}_{gg}^{(k)} + \lambda_q^{(k)}e_{33}^{(0k)} = \\
&= \lambda_p^{(k)}U_{p,p}^{(k)} + \lambda_4^{(k)}U_{g,g}^{(k)} + (k_{pp}\lambda_p^{(k)} + k_{gg}\lambda_{gg}\lambda_4^{(k)})U_3^{(k)} \qquad (4.87) \\
&\quad + \frac{1}{2}[\lambda_p^{(k)}(U_{3,p}^{(k)})^2 + \lambda_4^{(k)}U_{3,g}^{(k)}] + \lambda_q^{(k)}e_{33}^{(0k)} \\
& \quad s,t = 0,p,g; \; p = i \neq g; \; q = 7-p
\end{aligned}$$

The $\sigma_{22}^{(k)}$ stress can be obtained from the expression for $\sigma_{11}^{(k)}$ by making the following substitutions:

$$\lambda_1^{(k)} \to \lambda_4^{(k)}, \quad \lambda_4^{(k)} \to \lambda_2^{(k)}, \quad \lambda_6^{(k)} \to \lambda_5^{(k)}$$

Moreover, we can obtain

$$\sigma_{pg}^{(k)} = \overline{\sigma}_{pg}^{(k)} + \tilde{\sigma}_{pg}^{(k)} + \overline{\overline{\sigma}}_{pg}^{(k)} \qquad (4.88)$$

where

$$\begin{aligned}
\overline{\sigma}_{pg}^{(k)} &= 2G_{pg}\overline{e}_{pg}^{(k)} = G_{pg}^{(k)}[u_{p,g} - u_{g,p} \\
&- (\chi_{s,pg}\Phi_{ps}^{(k)} + \chi_{s,gp}\Phi_{gs}^{(k)}) + 2k_{pg}\chi_s\varphi_s^{(k)}] \qquad (4.89)
\end{aligned}$$

$$\tilde{\sigma}_{pg}^{(k)} = 2G_{pg}^{(k)}\tilde{e}_{pg}^{(k)} = G_{pg}^{(k)}(\chi_{s,p}\chi_{t,g}\varphi_t^{(k)} + \chi_{s,p}U_{3,g}^{(k)} + \chi_{s,g}U_{3,p}^{(k)})\varphi_s^{(k)} \tag{4.90}$$

$$\overline{\overline{\sigma}}_{pg}^{(k)} = G_{pg}^{(k)}[(U_{p,g}^{(k)} + U_{g,p}^{(k)}) + 2k_{pg}U_3^{(k)} + U_{3,p}^{(k)}U_{3,g}^{(k)}]; \quad s,t = 0,p,g \tag{4.91}$$

The transverse shear stresses can be obtained from the constitutive equations (4.50) in which hypothesis (4.58) for the transverse shear deformations must be taken into account. Then we can write

$$\sigma_{i3}^{(k)} = 2e_{i3}G_{p3}^{(k)} = -\chi_{s,i}f_{ps}^{(k)} + \sigma_{i3}^{(0k)} = \theta_{si}f_{ps}^{(k)} + \sigma_{i3}^{(0k)} \tag{4.92}$$
$$i = p = 1,2; \quad s = p,g; \quad g \neq p$$

where

$$\sigma_{i3}^{(0k)} = -(q_i^- f_{p3}^{(k)} + q_i^+ f_{p4}^{(k)}) \tag{4.93}$$

Thus all components of the stress and strain tensors as well as components of the displacement vector have been defined for an arbitrary point in the *k-th* layer.

4.5 Analysis of the nonlinear higher order theory

For better visualization of the special features of the nonlinear higher order theory, its distinction from classical and other refined theories, let us consider some special cases and carry out the analysis of the kinematic model (4.71), (4.79).

4.5.1 Special cases

Special cases of the presented model may be classified under a few types:

a) type of elastic symmetry (transverse isotropy, when $\lambda_1^{(k)} = \lambda_2^{(k)} = \lambda_4^{(k)} + 2\lambda_7^{(k)}$; $\lambda_5^{(k)} = \lambda_6^{(k)}$; $\lambda_8^{(k)} = \lambda_9^{(k)}$; isotropy where, in addition to above relations, $\lambda_6^{(k)} = \lambda_4^{(k)} = \lambda^{(k)}$; $\lambda_9^{(k)} = \lambda_7^{(k)} \equiv \mu = G$; $\lambda_3^{(k)} = \lambda_4^{(k)} + 2\lambda_7^{(k)} = \lambda + 2\mu$);

b) character of the layer rigidity (absolutely rigid under normal compression: $E_3^{(k)} = \infty$ and $\nu_{i3} = 0$; under transverse shear: $G_{is} = \infty$);

c) geometrically linear models (these do not contain quadratic terms in expressions for the deformations and in other relations);

d) models of shell or/and plate which are continuously heterogeneous through the thickness, or homogeneous;

e) total number of the unknown functions which determine the kinematics of the refined model (degrees of freedom which are related to physical properties of a material).

Some of these special cases are related to each other. Let us consider items a) and e). The first one is a "natural" special case, and second is an artificial way for decreasing the number of degrees of freedom. This statement can be explained by means of the following example.

Example 1. Let us consider simplification of the distribution function in case of transverse isotropy.

For a transverse–isotropic (or isotropic) material, according to item a), we have from (4.57)

$$\varphi_1^{(k)} = \varphi_2^{(k)} = \int_{h_0}^{z} F_2^{(k)} z dz = \int_{h_0}^{z} \frac{\lambda_5^{(k)}}{\lambda_3^{(k)}} z dz = \varphi^{(k)}$$

Taking into account that $E_1^{(k)} = E_2^{(k)} = E_k$, $\nu_{12}^{(k)} = \nu_{21}^{(k)} = \nu_k$, $G_{13}^{(k)} = G_{23}^{(k)} = G_k'$, constant (4.33) may be obtained as follows

$$\begin{aligned} A_{1p} &= \int_{c_0}^{c_n} \frac{E_k}{1-\nu_k^2} dz, \quad A_2 + A_3 = \int_{c_0}^{c_n} \frac{E\nu}{1-\nu_k^2} dz + \int_{c_0}^{c_n} 2G_{12} dz \\ &= \int_{c_0}^{c_n} \left[\frac{E_k \nu_k}{1-\nu_k^2} + \frac{E_k(1-\nu_k)}{(1+\nu_k)(1-\nu_k)} \right] dz = \int_{c_0}^{c_n} \frac{E_k}{1-\nu_k^2} dz \end{aligned}$$

In a similar manner, the sum of the constants

$$B_2 + B_3 = \int_{c_0}^{c_n} \frac{E_k z}{1-\nu_k^2} dz$$

and the function

$$\int_{c_0}^{z} (E_{0\nu}^{(k)} + 2G_{12}^{(k)}) z dz = \int_{c_0}^{z} \frac{E_k z}{1-\nu_k^2} dz$$

can be obtained. Thus, it follows from (4.38) that $f_{pp}^{(k)} = f_{pg}^{(k)}$ and further, from (4.49), (4.80) we have $\varphi_{pp}^{(k)} = \varphi_{pg}^{(k)}$; $\psi_{pp}^{(k)} = \psi_{pg}^{(k)}$; $\Phi_{pp}^{(k)} = \Phi_{pg}^{(k)} = \Phi^{(k)}$.

As it can be seen from the example, the distribution functions for the orthogonal directions are equal in case of transformation from the orthotropic model to transverse–isotropic one. The corresponded independent unknown functions χ_p, χ_g, $(p \neq g)$ have been united into their sum. Thus, instead of (4.71) and (4.79) we have the following model

$$\begin{aligned} u_i^{(k)} &= u_i - w_{,i}\psi_0^{(k)} - (\chi_p + \chi_g)_{,i}\Phi^{(k)} + U_i^{(k)} \\ u_3^{(k)} &= w + (\chi_p + \chi_g)\varphi^{(k)} + U_3^{(k)}, \quad i = 1,2; \; p = i \neq g \end{aligned} \tag{4.94}$$

Here the following replacement is obvious

$$\chi_p + \chi_g \equiv \chi$$

By this means the number of "shear" kinematic degrees of freedom for the transverse–isotropic (isotropic) model decreases from four in (4.79) to two in (4.94). The reason is that the model of orthotropic shell has been derived in such a way that the degrees of freedom $\chi_{p,i}$ and $\chi_{g,i}$, $(i = p \neq g = 1, 2)$ correspond not only to the orthogonal directions x_1 and x_2, but are in complete conformity with physical and mechanical properties of the material. In the orthotropic material these properties are represented by given four generalised distribution functions of the normal $f_{pp}^{(k)}$, $f_{pg}^{(k)}$, $p \neq g = 1, 2$.

Reasoning from item e), in order to artificially decrease the number of the degrees of freedom in the model, such intimacy is abandoned. For this purpose we introduce relation (4.39) $f_{pg}^{(k)} \approx \nu_p f_{pp}^{(k)}$ that leads to the two generalised normal functions. It allows the use of the transverse shear stresses in a form (4.40) in order to derive the refined model. Then, assuming in (4.40) that

$$(\kappa_{pp} + \nu_p \kappa_{gg})_{,i} \equiv \chi_{,i}$$

we also reduce to one function χ. By this means we again obtain an expression of the model (4.94). However, using this artificial method, the conformity of the unknown functions with the properties of orthotropy is lost. The model becomes "quasi–isotropic". In fact, for the case of transverse isotropy (isotropy), from (4.39) we obtain

$$\nu_p = \frac{A_2 + A_3}{A_{p1}} = 1$$

and in (4.40)

$$(\kappa_{pp} + \kappa_{gg})_{,i} \equiv \chi_{,i}$$

we have again only one function χ, i.e. the model (4.94) which is equivalent to case a).

It should be noted that most of known refined models of the orthotropic shells are developed using an artificial way described in d), which requires averaging either strains [67] or material properties [3]. The sought functions, which characterise the shear in orthogonal directions are chosen in such a way that only one function $\alpha(x_1, x_2)$ corresponds to each orthotropic direction x_i [67, 3], or the shear deformations are characterised by partial derivatives $\chi_{,i}$ with respect to one sought function [62, 78].

Let us consider the special type of higher order theory described in item b). Making assumption that $E_3^{(k)} = \infty$, $\nu_{i3}^{(k)} = 0$ as a result we have $\lambda_3^{(k)} = \infty$. Then according to (4.55), (4.57) and (4.80) $\varphi_r^{(k)} = \varphi_i^{(k)} = 0$, $(r = 1 \dots 8)$, $\varphi_\sigma^{(k)} = \varphi_{\sigma i}^{(k)} = 0$. From (4.71) and (4.79) we obtain the following kinematic model

$$
\begin{aligned}
u_i^{(k)} &= u_i - w_{,i}\psi_0^{(k)} - \chi_{p,i}\psi_{pp}^{(k)} - \chi_{g,i}\psi_{pg}^{(k)} - q_i^- \psi_{p3}^{(k)} - q_i^+ \psi_{p4}^{(k)} \\
u_3^{(k)} &= w \qquad p = i \neq g = 1,2
\end{aligned} \tag{4.95}
$$

where $z = h_0$ is a surface on which the tangential displacements equal u_i.

The linear models of type c) follow from the general model and its variants when the quadratic terms are neglected in all the relations. As this takes place, firstly, the strains $\epsilon_{11}^{(0)}$, $\epsilon_{22}^{(0)}$ are simplified (they are known functions in (4.62)) and the hypothesis (4.61) therewith looses the influence of the geometrical nonlinearity. Secondly, the nonlinear strain $\tilde{e}_{\alpha\beta}^{(k)}$ and stress $\tilde{\sigma}_{\alpha\beta}^{(k)}$ $(\alpha = \beta = 1, 2, 3)$ terms are excluded.

The cases of type d) are obvious since the functions of the continuous (nonlinear or linear) and constant characteristics through the thickness of the shell follow from the continuously piecewise normal functions which, for a given model, are the elasticity properties of the layers. It also should be noted that no limitation are placed on the number of the layers. Thus there is always a possibility of approximating the continuous function "layer–by–layer", i.e. using the continuously piecewise function when the shell is constructed from sufficiently large number of layers. By this means the model allows both constant and variable physical and mechanical characteristics through the layer and whole layer package to be taken into account. Moreover, a layer package can be constructed from both orthotropic and isotropic layers, and there is no limitation on sequences and thicknesses of the layers. Thus the versatility of the model with respect to the structure through the thickness is provided.

4.5.2 Reference surface

Let us note one more feature of the higher order theory developed. The hypotheses which form the basis of the model and, also, components of the displacement vector exactly satisfy the interface conditions and conditions on the external surfaces. It arises from natural (on a basis of the relations of the classical theory) rather than *a priori* choice of the distribution functions of all components of the stress–strain state. In relation to this, it should be noted that the reference surface can be chosen arbitrarily, it can be any surface $(c_0 \leq z \leq c_n)$ which is equidistant to external ones. The choice can be made from the different considerations. For example, if a plate is on an elastic foundation, then for the uniqueness of the unknown displacements of the foundation surface and plate surface which contacts with it, it is convenient to chose a later one as a reference surface. To illustrate a freedom in choosing the reference surface let us consider the following example.

Example 2. The structure under consideration is a homogeneous shell (plate) subject to the normal load ($q_i^\pm = 0$) and has the thickness h. Let us show that in the correspondent points the value of the functions which define the change in the transverse shear stresses $\sigma_{13}^{(k)}$ is the same when the reference surface coincides with the external bottom and mean surfaces. At given conditions, according to (4.92) and taking for clarity $i = 1$, we obtain

$$\sigma_{13} = -\chi_{1,1}f_{11} - \chi_{2,1}f_{21}$$

where the normal functions (4.38) can be written as

$$f_{11} = \frac{B_{11}}{A_{11}}f_1 - \int_{c_0}^{z} E_{01}zdz; \; f_1 = \int_{c_0}^{z} E_{01}dz; \; f_{12} = \frac{B_2 + B_3}{A_{11}}f_1 - \int_{c_0}^{z}(E_{0\nu} + 2G_{12})zdz$$

With the reference and bottom surfaces coincident, the coordinates of the external surfaces are $c_0 = 0$ and $c_n = h$. Then

$$\begin{aligned}
B_{11} &= \int_0^h E_{01}zdz = \frac{E_{01}h^2}{2}; \; A_{11} = \int_0^h E_{01}dz = E_{01}h \\
\frac{B_{11}}{A_{11}} &= \frac{h}{2}; \; B_2 = \int_0^h E_{0\nu}zdz = \frac{E_{0\nu}h^2}{2} \\
B_3 &= \int_0^h 2G_{12}zdz = G_{12}h^2; \; \frac{B_2 + B_3}{A_1 1} = \frac{(E_{0\nu} + 2G_{12})h}{2E_{01}} \\
f_{11} &= \frac{h}{2}\int_0^z E_{01}dz - \int_0^z E_{01}zdz = \frac{E_{01}}{2}(hz - z^2) \\
f_{12} &= \frac{(E_{0\nu} + 2G_{12})h}{2E_{01}}\int_0^z E_{01}dz - \int_0^z (E_{0\nu} + 2G_{12})zdz = \frac{E_{0\nu} + 2G_{12}}{2}(hz - z^2)
\end{aligned}$$

Value of the functions on the external and mean surfaces are

$$f_{11}(0,h) = f_{12}(0,h) = 0; \; f_{11}(\tfrac{h}{2}) = \frac{E_{01}h^2}{8}; \; f_{12}(\tfrac{h}{2}) = \frac{(E_{0\nu} + 2G_{12})h^2}{8}$$

When the reference and mean surfaces coincide we have $c_0 = -\frac{h}{2}$, $c_n = \frac{h}{2}$ and, respectively, $B_{11} = B_2 = B_3 = 0$. Then

$$\begin{aligned}
f_{11} &= -\int_{-h/2}^{z} E_{01}zdz = -\frac{E_{01}}{2}(z^2 - \frac{h^2}{4}) \\
f_{12} &= -\int_{-h/2}^{z}(E_{0\nu} + 2G_{12})zdz = -\frac{E_{0\nu} + 2G_{12}}{2}(z^2 - \frac{h^2}{4})
\end{aligned}$$

We now have the desired result for the functions:

$$f_{11}(-\tfrac{h}{2},\tfrac{h}{2})=0;\ \ f_{11}(0)=\frac{E_{01}h^2}{8};\ \ f_{12}(0)=\frac{(E_{0\nu}+2G_{12})h^2}{8}$$

Hence the result is the same for these surfaces.

All relations of the model developed are invariant, irrespective of the position of the reference surface within the thickness. It is not typical for some theories of the laminated structures and introduces the additional limitations to them.

4.5.3 Tangential loads

In the higher order theory developed the law of the change of the transverse shear stresses through the thickness of the shell is related directly to the given tangential forces on the external surfaces. In many known references, for example in [3, 68], the linear law of the change through the thickness is *a priori* assumed for these stresses and is given as

$$\sigma_{i3}^{(0k)}=q_i^-\frac{c_0-z}{h}+q_i^+\frac{c_0+z}{h} \tag{4.96}$$

In the proposed model, according to (4.92) and (4.38), we have

$$\sigma_{i3}^{(0k)}=-(q_i^-f_{p3}^{(k)}+q_i^+f_{p4}^{(k)})=q_i^-\left(\frac{f_p^{(k)}}{A_{1p}}-1\right)+q_i^+\frac{f_p^{(k)}}{A_{1p}} \tag{4.97}$$

where

$$f_p^{(k)}/A_{1p}=\int_{c_0}^{z}E_{0p}^{(k)}dz\Big/\int_{c_0}^{c_n}E_{0p}^{(k)}dz;\quad i=p=1,2$$

By this means distribution of the transverse shear stresses depends on material properties of the layers. Let us demonstrate this effect using the following example.

Example 3. Let us build the law of change for the stresses $\sigma_{i3}^{(0k)}$ under the influence of load $q_1^+=q$ for the three–layered package with the following characteristics

$$\begin{aligned}h_1&=0.2h;\quad h_2=0.5h;\quad h_3=0.3h;\quad h=1\\ E_{01}^{(1)}&=10^2\cdot q;\quad E_{01}^{(2)}=q;\quad E_{01}^{(3)}=10\cdot q\end{aligned}$$

The reference surface coincides with the bottom surface of the package ($c_0=0$). Using (4.97) we obtain

$$\begin{aligned}f_1^{(k)}&=E_{01}^{(k)}(z-c_{k-1})+\sum_{r=1}^{k-1}E_{0r}h_r\\ A_{11}&=\sum_{k=1}^{3}E_{01}^{(k)}h_k=(0.2\cdot10^2+0.5\cdot1+0.3\cdot10)q=23.5\cdot q\\ \sigma_{13}^{(0k)}&=\frac{f_1^{(k)}}{23.5}\end{aligned}$$

and from (4.96)

$$\sigma_{13}^{(0k)} = \frac{z}{h}$$

Diagrams of the stresses for this example are given in Figure 4.1.

4.5.4 Normal compression

The analysis of the higher order theory developed shows that this theory takes into account the change in length of the normal element during the process of the shell deformation, that is taking the normal compression into account. To illustrate this let us consider the following case.

Example 4. For homogeneous orthotropic plate we shall express the linear part of the displacements (4.71), (4.79) in polynomial form.

$$\begin{aligned} u_3^{(k)} &= b_0 + b_1 z + b_2 z^2 + \ldots = b_{n-1} z^{n-1} \\ u_i^{(k)} &= a_0 + a_1 z + a_2 z^2 + \ldots = a_{n-1} z^{n-1} \end{aligned} \tag{4.98}$$

The summation is assumed over index n. After transformation of the normal functions, which are included in the linear terms of the displacements, we obtain the following coefficients of polynomials:

$$\begin{aligned}
b_0 &= w; \qquad a_0 = u_i \\
b_1 &= -\frac{1}{\lambda_3}\left[(\epsilon_{11}^{(0)}\lambda_6 + \epsilon_{22}^{(0)}\lambda_5) - \frac{1}{2}(q_3^+ - q_3^-) - \frac{h}{8}(q_{i,i}^+ + q_{i,i}^-)\right] \\
a_1 &= -w_{,i} - \frac{1}{2G_{p3}}\left\{\frac{h^2}{4}\left[\chi_{p,i}E_{0p} + \chi_{g,i}(E_{0\nu} + 2G_{pg})\right] - (q_i^+ - q_i^-)\right\} \\
b_2 &= \frac{1}{2\lambda_3}\left[(\chi_p\lambda_6 + \chi_g\lambda_5) + (q_3^+ + q_3^-)\frac{3}{2h} + \frac{1}{4}(q_{i,i}^+ - q_{i,i}^-)\right] \\
a_2 &= \frac{1}{2hG_{p3}}(q_i^+ + q_i^-) \\
&+ \frac{1}{2\lambda_3}\left[(\epsilon_{11,i}^{(0)}\lambda_6 + \epsilon_{22,i}^{(0)}\lambda_5) - \frac{1}{2}(q_{3,i}^+ - q_{3,i}^-) - \frac{h}{8}(q_{i,ij}^+ + q_{i,ij}^-)\right] \\
b_3 &= -\frac{1}{6h\lambda_3}(q_{i,i}^+ + q_{i,i}^-) \\
a_3 &= \frac{1}{6}\left[\chi_{p,i}\left(\frac{E_{0p}}{G_{p3}} - \frac{\lambda_6}{\lambda_3}\right) + \chi_{g,i}\left(\frac{E_{0\nu} + 2G_{pg}}{G_{p3}} - \frac{\lambda_5}{\lambda_3}\right)\right] \\
&- \frac{1}{4\lambda_3}\left[\frac{1}{h}(q_{3,i}^+ + q_{3,i}^-) - \frac{1}{6}(q_{i,ij}^+ - q_{i,ij}^-)\right] \\
b_4 &= -\frac{1}{2h^2\lambda_3}\left[\frac{1}{2}(q_{i,i}^+ - q_{i,i}^-) + \frac{1}{h}(q_3^+ + q_3^-)\right]
\end{aligned}$$

$$
\begin{aligned}
a_4 &= \frac{1}{24h\lambda_3}(q^+_{i,ij} + q^-_{i,ij}); \qquad b_5 = 0 \\
a_5 &= \frac{1}{10h^2\lambda_3}\left[(q^+_{3,i} + q^-_{3,i})\frac{1}{h} + \frac{1}{2}(q^+_{i,ij} - q^-_{i,ij})\right]
\end{aligned}
$$

Thus, taking the normal compression into account, the normal and tangential displacements are expanded into a fourth and fifth order polynomials, respectively. Let us note that for the orthotropic layers made from composite materials, the stiffness in the normal direction to the layer is determined by the matrix properties and usually is considerably lower than in transverse directions, i.e. the stiffness moduli of type λ_3 are comparatively small values. In this connection, as is seen from expressions for the coefficients, the influence of terms containing the stiffness λ_3 is quite important. It should be noted that these terms account for the direct influence of the external load on the change of the displacements through the thickness.

Attention is drawn to the fact that in many refined theories of the laminated structures the change in displacements through the thickness due to effect of the normal compression is ignored. For instance, consideration of this effect in [68, 78] is related only with the Poisson's effect. It is equivalent to the following values of the coefficients for u_3:

$$
b_0 = w; \qquad b_2 = \frac{1}{2\lambda_3}(\chi_p\lambda_6 + \chi_g\lambda_5)
$$

or in special case of transverse isotropic material from (4.94)

$$
b_2 = \chi\frac{E\nu'}{2E'(1-\nu)}
$$

Remaining coefficients of the polynomial are equal to zero. In this case the distribution character of the normal displacements through the thickness does not correspond to the real behaviour. If one of the external surfaces is loaded, then the displacements u_3 turn out to be symmetrical about the mean surface.

Setting $E_3 = \infty$, $\nu_{i3} = \nu_{3i} = 0$ and, respectively, $\lambda_3 = \lambda_5 = \infty$, then in expressions for expansion we obtain only coefficients which correspond to the shear model (4.95). They are

$$
\begin{aligned}
a_0 &= u_i; \qquad b_0 = w \\
a_1 &= -w_{,i} - \frac{1}{2G_{p3}}\left\{\frac{h^2}{4}\left[\chi_{p,i}E_{0p} + \chi_{g,i}(E_{0\nu} + 2G_{pg})\right] + (q^+_i - q^-_i)\right\} \\
a_2 &= \frac{1}{2hG_{p3}}(q^+_i + q^-_i) \\
a_3 &= \frac{1}{6G_{p3}}\left[\chi_{p,i}E_{0p} + \chi_{g,i}(E_{0\nu} + 2G_{pg})\right]
\end{aligned}
$$

For transverse isotropic material these coefficients take the form:

$$
\begin{aligned}
a_0 &= u_i; \qquad b_0 = w \\
a_1 &= -w_{,i} - \chi_{,i}\frac{Eh^2}{2G'(1-\nu^2)} - \frac{1}{2G'_k}(q_i^- - q_i^+) \\
a_2 &= \frac{1}{2G'h}(q_i^- + q_i^+); \quad a_3 = \chi_{,i}\frac{E}{6G'(1-\nu^2)}
\end{aligned}
$$

In this case the tangential displacements are expressed by a third order polynomial.

Let us note that the sought function χ (functions χ_p, χ_g in general model) is associated in expansion coefficients both with shear G_{p3} and compression $E_3(\lambda_3)$ moduli. Thus, this function describes both indicated effects, namely, transverse shear and normal compression. According to notations set up in [62, 78], we shall call functions χ or χ_p, χ_g the shear functions.

4.6 Variational equation, equations of equilibrium and boundary conditions

4.6.1 Variational equation

The equations of equilibrium and the boundary conditions may be determined using Reissner's variational principal

$$\delta(R - A) = 0 \tag{4.99}$$

where R is Reissner's functional and A is the work of the external forces.

Variational of the functional R has the following form

$$
\begin{aligned}
\delta R &= \iiint_V \left[\sigma_{\alpha\beta}^{(k)} \delta e_{\alpha\beta}^{(k)} + \delta\sigma_{11}^{(k)} \left(e_{11}^{(k)} - \frac{\sigma_{11}^{(k)}}{E_1^{(k)}} + \frac{\nu_{21}^{(k)}}{E_2^{(k)}}\sigma_{22}^{(k)} + \frac{\nu_{31}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)} \right) \right. \\
&+ \delta\sigma_{22}^{(k)} \left(e_{22}^{(k)} - \frac{\sigma_{22}^{(k)}}{E_2^{(k)}} + \frac{\nu_{12}^{(k)}}{E_1^{(k)}}\sigma_{11}^{(k)} + \frac{\nu_{32}^{(k)}}{E_3^{(k)}}\sigma_{33}^{(k)} \right) \\
&+ \delta\sigma_{33}^{(k)} \left(e_{33}^{(k)} - \frac{\sigma_{33}^{(k)}}{E_3^{(k)}} + \frac{\nu_{13}^{(k)}}{E_1^{(k)}}\sigma_{11}^{(k)} + \frac{\nu_{23}^{(k)}}{E_2^{(k)}}\sigma_{22}^{(k)} \right) \\
&+ \delta\sigma_{12}^{(k)} \left(2e_{12}^{(k)} - \frac{\sigma_{12}^{(k)}}{G_{12}^{(k)}} \right) + \delta\sigma_{13}^{(k)} \left(2e_{13}^{(k)} - \frac{\sigma_{13}^{(k)}}{G_{13}^{(k)}} \right)
\end{aligned} \tag{4.100}
$$

$$+ \quad \delta\sigma_{23}^{(k)} \left(2e_{23}^{(k)} - \frac{\sigma_{23}^{(k)}}{G_{23}^{(k)}} \right) \Bigg] dV; \qquad \alpha,\ \beta = 1, 2, 3$$

Variational of the work of the external loading is

$$\begin{aligned} \delta A &= \iint_S \left[q_\alpha^+ \delta u_\alpha(c_n) + q_\alpha^- \delta u_\alpha(c_0) \right] dS \\ &+ \iint_L P_\alpha^{(k)} u_\alpha^{(k)}(l) dL; \qquad \alpha = 1, 2, 3 \end{aligned} \tag{4.101}$$

where S is domain of the external surfaces of the shell, L is boundary of the domain, $q_\alpha^\pm$ are external loads $z = c_n$ and $z = c_0$, P_α is load on the end surfaces (domain boundary) and u_α are displacements which complying with the considered surfaces.

Substituting (4.100), (4.101) in (4.99) and considering the variational equation using the components of the stress–strain tensor derived in section 4.3 let us ensure that variational coefficients of the stresses $\sigma_{ij}^{(k)}$, $\sigma_{i3}^{(k)}$, $(i, j = 1, 2)$ are identically equal to zero. This implies that the equations of the Hooke's law for the strains $e_{ij}^{(k)}$, $e_{i3}^{(k)}$ are satisfied exactly. For the strains $e_{33}^{(k)}$, the constitutive equations are satisfied "integrally" (in the sense that the integral corresponding to this equation equals zero over the domain of the shell) since the stress $\sigma_{33}^{(k)}$ assumed in the form given by equation (4.46) contains only the given functions and $\delta\sigma_{33}^{(k)} = 0$. On this basis the variational equation (4.99) can be written in the following form

$$\delta\Pi - \delta A = 0 \tag{4.102}$$

where

$$\delta\Pi = \iiint_V \sigma_{\alpha\beta}^{(k)} \delta e_{\alpha\beta}^{(k)} dV; \qquad \alpha, \beta = 1, 2, 3 \tag{4.103}$$

4.6.2 Variation of the potential energy

Let us represent $\delta\Pi$ as the sum of the linear, nonlinear and "zero" parts.

$$\delta\Pi = \delta\overline{\Pi} + \delta\tilde{\Pi} + \delta\overline{\overline{\Pi}} \tag{4.104}$$

where

$$\begin{aligned} \delta\overline{\Pi} &= \iiint_V \sigma_{\alpha\beta}^{(k)} \delta\bar{e}_{\alpha\beta}^{(k)} dV \\ \delta\tilde{\Pi} &= \iiint_V \sigma_{\alpha\beta}^{(k)} \delta\tilde{e}_{\alpha\beta}^{(k)} dV \\ \delta\overline{\overline{\Pi}} &= \iiint_V \sigma_{\alpha\beta}^{(k)} \delta e_{\alpha\beta}^{(0k)} dV = 0 \end{aligned} \tag{4.105}$$

Last relation is obvious since $\delta e_{\alpha\beta}^{(0k)} = 0$ as variation of the given strains. Substituting expressions for the strains from section 4.3 into (4.105) we obtain

$$
\begin{aligned}
\delta\overline{\Pi} &= \iint_S \left\{ \int_{c_0}^{c_n} \left[\sigma_{ij}^{(k)} \delta\overline{e}_{ij}^{(k)} + 2\sigma_{i3}^{(k)} \delta\overline{e}_{i3}^{(k)} + \sigma_{33}^{(k)} \delta\overline{e}_{33}^{(k)} \right] dz \right\} dS \\
&= \iint_S \left\{ \int_{c_0}^{c_n} \left[\sigma_{pp}^{(k)} \delta\overline{e}_{pp}^{(k)} + 2\sigma_{pg}^{(k)} \delta\overline{e}_{pg}^{(k)} + 2\sigma_{i3}^{(k)} \delta\overline{e}_{i3}^{(k)} + \sigma_{33}^{(k)} \delta\overline{e}_{33}^{(k)} \right] dz \right\} dS \\
&= \iint_S \left\{ \int_{c_0}^{c_n} \left[\sigma_{pp}^{(k)} (\delta u_{p,p} - \Phi_{ps}^{(k)} \delta\chi_{s,pp} + k_{pp}\varphi_s^{(k)} \delta\chi_s) \right.\right. \qquad (4.106) \\
&+ \sigma_{pg}^{(k)} (\delta u_{p,g} + \delta u_{g,p} - \Phi_{ps} \delta\chi_{s,pg} - \Phi_{gs}^{(k)} \delta\chi_{s,gp} + 2k_{pg}\varphi_s^{(k)} \delta\chi_s) \\
&- \left.\left. \sigma_{i3}^{(k)} \varphi_{ps}^{(k)} \delta\chi_{s,i} + \sigma_{33}^{(k)} F_s^{\prime(k)} \delta\chi_s \right] dz \right\} dS; \quad i,j = 1,2; \; p = i; \; g \neq i; \; s = 0,p,g
\end{aligned}
$$

$$
\begin{aligned}
\delta\tilde{\Pi} &= \iint_S \left(\int_{c_0}^{c_n} \sigma_{ij}^{(k)} \delta\tilde{e}_{ij}^{(k)} dz \right) dS \\
&= \iint_S \left\{ \int_{c_0}^{c_n} \sigma_{ij}^{(k)} \frac{1}{2} [\varphi_s^{(k)} \varphi_t^{(k)} \delta(\chi_{s,t}\chi_{t,j}) + \varphi_s^{(k)} U_{3,i}^{(k)} \delta\chi_{s,j}] dz \right\} dS \qquad (4.107) \\
&\qquad i,j = 1,2; \quad s,t = 0,p,g
\end{aligned}
$$

Integral characteristics of the stresses in the shell (generalised forces) are given by

$$
\begin{aligned}
[N_{ij}^{(s)}; \quad M_{ij}^{(ps)}; \quad & T_{ij}^{(st)}; \; H_{ij}^{(is)}] = \\
&= \int_{c_0}^{c_n} \sigma_{ij}^{(k)} [\varphi_s^{(k)} \; ; \; \Phi_{ps}^{(k)}; \; (\varphi_s^{(k)} \varphi_t^{(k)}) \; ; \; (\varphi_s^{(k)} U_{3,i}^{(k)})] dz \\
Q_i^{(ps)} &= \int_{c_0}^{c_n} \sigma_{i3}^{(k)} \varphi_{ps}^{(k)} dz; \quad Q_3^{(s)} = \int_{c_0}^{c_n} \sigma_{33}^{(k)} F_s^{\prime(k)} dz; \qquad (4.108) \\
&\qquad i,j = 1,2; \quad s,t = 0,p,g
\end{aligned}
$$

wherein $\varphi_{p0}^{(k)} = 0$, as it follows from (4.58).

Taking into account the generalised forces given above, linear and nonlinear part of the variation of the potential energy of the strains may be defined as

$$
\begin{aligned}
\delta\overline{\Pi} &= \iint_S (N_{ij}^{(0)} \delta u_{i,j} - M_{ij}^{(ps)} \delta\chi_{s,ij} k_{ij} \delta\chi_s + Q_1^{(ps)} \delta\chi_{s,i} + Q_3^{(s)} \delta\chi_s) dS \\
&= -\iint_S \left[N_{ij,ij}^{(0)} \delta u_i + (M_{ij,ij}^{(ps)} - N_{ij}^{(s)} k_{ij} + Q_{i,i}^{(ps)} - Q_3^{(s)}) \delta\chi_s \right] dS \qquad (4.109) \\
&+ \int_L \left[(N_{hh}^{(0)} \delta u_h + N_{hl}^{(0)} \delta u_l) + (M_{hh,h}^{(hs)} + M_{hl,l}^{(hs)} + M_{hl,l}^{(ls)} + Q_h^{(hs)}) \delta\chi_s \right. \\
&- \left. M_{hh}^{hs} \delta\chi_{s,h} \right] dL + \left[M_{hl}^{hs} \delta\chi_s \right]_{L_1}^{L_2}
\end{aligned}
$$

$$
\begin{aligned}
\delta\tilde{\Pi} &= \iint_S \left[\frac{1}{2} T_{ij}^{(st)} \delta(\chi_{s,i}\chi_{t,j}) + H_{ij}^{(is)} \delta\chi_{s,j} \right] dS \\
&= \iint_S \left[(T_{ij}^{(st)} \chi_{t,j})_{,i} + H_{ij,j}^{(is)} \right] \delta\chi_s dS \qquad (4.110) \\
&+ \int_L \left[(T_{hh}^{(st)} \chi_{t,h} + T_{hl}^{(st)} \chi_{t,l}) \delta\chi_s + H_{hl}^{(hs)} \delta\chi_s \right] dL
\end{aligned}
$$

4.6.3 Variation of the external load

By taking into account relations for the strains the work of the external load on the surfaces $z = c_0,\ c_n$ takes the following form

$$\begin{aligned}
\delta A_1 &= \iint_S \Big\{[q_3^+\varphi_s^{(n)}(c_n) + q_3^-\varphi_s^{(1)}(c_0)]\delta\chi_s \\
&+ q_i^+[\delta u_i - \Phi_{ps}^{(n)}(c_n)\delta\chi_{s,i}] + q_i^-[\delta u_i - \Phi_{ps}^{(1)}(c_0)\delta\chi_{s,i}]\Big\}\,dS \\
&= \iint_S \Big\{[q_3^+\varphi_s^{(n)} + q_3^-\varphi_s^{(1)}(c_0)]\delta\chi_s + (q_i^+ + q_i^-)\ \delta u_i \\
&+ [q_{i,p}^+\Phi_{is}^{(n)}(c_n) + q_{i,p}^-\Phi_{is}^{(1)}(c_0)]\delta\chi_s\Big\}\,dS \\
&- \int_L [q_h^+\Phi_{hs}^{(n)}(c_n) + q_h^-\Phi_{hs}^{(1)}(c_0)]\delta\chi_s dL; \quad p = i;\ s = 0, p, g
\end{aligned} \tag{4.111}$$

Introducing the generalised loads

$$\begin{aligned}
q_i^{(0)} &= q_i^+ + q_i^- \\
q_3^{(s)} &= q_3^+\varphi_s^{(n)} + q_3^-\varphi_s^{(1)}(c_0) + q_{i,p}^+\Phi_{is}^{(n)}(c_n) + q_{i,p}^-\Phi_{is}^{(1)}(c_0) \\
q_h^{(s)} &= q_h^+\Phi_{hs}^{(n)} + q_h^-\Phi_{hs}^{(1)}(c_0); \quad p = i;\ s = 0, p, g
\end{aligned} \tag{4.112}$$

we obtain from (4.111)

$$\delta A_1 = \iint_S (q_i^{(0)}\delta u_i + q_3^{(s)}\delta\chi_s)dS - \int_L q_h^{(s)}\delta\chi_s dL; \quad s = 0, p, g \tag{4.113}$$

The corresponding expression for the boundary forces has the form

$$\delta A_2 = \int \Big\{\int_{c_0}^{c_n} [\sigma_{hh}^{(k)}\delta u_h^{(k)} + \sigma_{hl}^{(k)}\delta u_l^{(k)} + \sigma_{h3}^{(k)}\delta u_3^{(k)}]dz\Big\}\,dL \tag{4.114}$$

where $\sigma_{h\alpha}^{(k)}$, $(\alpha = h, l, 3)$ are components of the stress tensor and $u_\alpha^{(k)}$ are components of the displacement vector at an arbitrary point of the *k-th* layer on the boundary L of the shell. Using expressions for the tangential deflection (4.79), where $i = h, l$, and for the normal deflections (4.71) in (4.114) we derive

$$\begin{aligned}
\delta A_2 &= \int_L \Big\{\int_{c_0}^{c_n} [\sigma_{hh}^{(k)}(\delta u_h - \Phi_{ps}^{(k)}\delta\chi_{s,h}) + \sigma_{hl}^{(k)}(\delta u_l - \Phi_{ps}^{(k)}\delta\chi_{s,l}) + \sigma_{h3}^{(k)}\varphi_s^{(k)}\delta\chi_s]dz\Big\}\,dL \\
&= \int_L [\overset{*}{N}{}_{hh}^{(0)}\ \delta u_h + \overset{*}{N}{}_{hl}^{(0)}\ \delta u_l - \overset{*}{M}{}_{hh}^{(hs)}\ \delta\chi_{s,h} \\
&+ (\overset{*}{M}{}_{hl,l}^{(hs)} + \overset{*}{Q}{}_h^{(s)})\delta\chi_s]dL - \Big[\overset{*}{M}{}_{hl}^{(hs)}\ \delta\chi_s\Big]_{L_1}^{L_2}; \quad s = 0, p, g
\end{aligned} \tag{4.115}$$

where the asterisk $(*)$ denotes the forces acting on the boundary of the shell which may be expressed by equation (4.108), when $h, l = i, j$, and in addition

$$\overset{*}{Q}{}_h^{(s)} = \int_{c_0}^{c_n} \sigma_{h3}^{(k)}\varphi_s^{(k)}dz \tag{4.116}$$

4.6.4 Equations of equilibrium and boundary conditions

Substituting the variations (4.109), (4.110), (4.113), (4.115) into (4.99) we derive the following variational equation

$$
\begin{aligned}
&\iint_S \{ (N^{(0)}_{ij,j} + q^{(0)}_i)\delta u_i + [M^{(ps)}_{ij,ij} - N^{(s)}_{ij} k_{ij} + Q^{(ps)}_{i,i} - Q^{(s)}_3 \\
+ \;& (T^{(st)}_{ij}\chi_{t,j})_{,i} + H^{(is)}_{ij,j} + q^{(s)}_3]\delta\chi_s \} \, dS \qquad (4.117)\\
- \;& \int_L \Big[(N^{(0)}_{hh} - \overset{*}{N}{}^{(0)}_{hh})\delta u_h + (N^{(0)}_{hl} - \overset{*}{N}{}^{(0)}_{hl})\delta u_l + (M^{(hs)}_{hh,h} + M^{(hs)}_{hl,l} + M^{(ls)}_{lh,l} + Q^{(hs)}_h \\
+ \;& H^{(hs)}_{hl} + T^{(st)}_{hh}\chi_{t,h} + T^{(st)}_{hl}\chi_{t,l} + q^{(s)}_h - \overset{*}{M}{}^{(hs)}_{hl,l} - \overset{*}{Q}{}^{(s)}_h)\delta\chi_s - (M^{(hs)}_{hh} - \overset{*}{M}{}^{(hs)}_{hh})\delta\chi_{s,h} \Big] dL \\
- \;& \Big[M^{(hs)}_{hl} - \overset{*}{M}{}^{(hs)}_{hl} \Big]^{L_2}_{L_1} = 0; \qquad i,j = 1,2; \;\; p = i; \;\; s,t = 0,p,g
\end{aligned}
$$

Using the above variational principle the equations of equilibrium can be obtained as

$$
\begin{aligned}
N^{(0)}_{ij,j} + q^{(0)}_i &= 0 \\
M^{(ps)}_{ij,ij} - N^{(s)}_{ij} k_{ij} + Q^{(ps)}_{i,i} - Q^{(s)}_3 + H^{(is)}_{ij,i}& \\
+(T^{(st)}_{ij}\chi_{t,j})_{,i} + q^{(s)}_3 &= 0 \qquad (4.118)\\
i,j = 1,2; \;\; s,t = 0,p,g; \;\; p = i \neq g&
\end{aligned}
$$

and the boundary conditions as

$$
\begin{aligned}
&(N^{(0)}_{hh} - \overset{*}{N}{}^{(0)}_{hh})\delta u_h = 0 \\
&(N^{(0)}_{hl} - \overset{*}{N}{}^{(0)}_{hl})\delta u_l = 0 \\
&(M^{(hs)}_{hh,h} + M^{(hs)}_{hl,l} + M^{(ls)}_{lh,l} + Q^{(hs)}_h + H^{(hs)}_{hl} \\
&+T^{(st)}_{hh}\chi_{t,h} + T^{(st)}_{hl}\chi_{t,l} + q^{(s)}_h \qquad (4.119)\\
&- \overset{*}{M}{}^{(hs)}_{hl,l} - \overset{*}{Q}{}^{(s)}_h)\delta\chi_r = 0 \\
&(M^{(hs)}_{hh} - \overset{*}{M}{}^{(hs)}_{hh})\delta\chi_{r,h} = 0 \\
&\qquad s,t = 0,p,g; \;\; r = s
\end{aligned}
$$

where the asterisk ($*$) denotes the forces on the boundary of the shell, h and l are the normal and tangential directions to the boundary of the shell.

In the following, the equilibrium equations will form the basis of the governing differential equations and corresponding boundary conditions of the geometrically nonlinear theory of the laminated shells.

4.7 Elasticity relations

The generalised forces of the shell are given in the form of integrals (4.108). They are statical equivalent of the stresses acting through the thickness in directions orthogonal to the coordinate axes x_i. Substituting the components of the stress tensor (see Section 4.3) in equations (4.108) allows to obtain the elasticity relations, which, as in case of the stresses, we shall express as the sum of the linear, nonlinear and given parts. Such presentation is a formal since the forces $T_{ij}^{(st)}$, $H_{ij}^{(is)}$ belong only to the nonlinear part of the variation of the potential energy.

The tangential forces can be obtained as the follows sum

$$N_{ij}^{(s)} = \overline{N}_{ij}^{(s)} + \overline{\overline{N}}_{ij}^{(s)}; \quad s = 0, p, g; \quad p = i \neq g; \quad i, j = 1, 2 \tag{4.120}$$

where the linear parts are defined by

$$\begin{aligned} \overline{N}_{pp}^{(s)} &= \overline{C}_{pp}^{(s)} u_{p,p} + \overline{C}_{g4}^{(s)} u_{g,g} - \overline{D}_{pp}^{(st)} \chi_{t,pp} - \overline{D}_{g4}^{(st)} \chi_{t,gg} + \overline{K}_{pp}^{(st)} \chi_t \\ \overline{N}_{pg}^{(s)} &= \overline{C}_{pg}^{(s)} (u_{p,g} + u_{g,p}) - \overline{D}_{pg}^{(st)} \chi_{t,pg} - \overline{D}_{gp}^{(st)} \chi_{t,gp} + \overline{K}_{pg}^{(st)} \chi_t \end{aligned} \tag{4.121}$$

Here the following stiffness characteristics have been introduced (index k is omitted)

$$\begin{aligned} \overline{C}_{pp}^{(s)} &= \int_{c_0}^{c_n} \lambda_p \varphi_s dz; \qquad \overline{C}_{g4}^{(s)} = \int_{c_0}^{c_n} \lambda_4 \varphi_s dz \\ \overline{D}_{pp}^{(st)} &= \int_{c_0}^{c_n} \lambda_p \varphi_s \Phi_{pt} dz; \quad \overline{D}_{g4}^{(st)} = \int_{c_0}^{c_n} \lambda_4 \varphi_s \Phi_{gt} dz \\ \overline{K}_{pp}^{(st)} &= \int_{c_0}^{c_n} [(k_{pp}\lambda_p + k_{gg}\lambda_4)\varphi_t + \lambda_q F_t] \varphi_s dz \\ \overline{C}_{pg}^{(s)} &= \int_{c_0}^{c_n} G_{pg} \varphi_s dz; \qquad \overline{D}_{pg}^{(st)} = \int_{c_0}^{c_n} G_{pg} \varphi_s \Phi_{pt} dz \\ \overline{D}_{gp}^{(st)} &= \int_{c_0}^{c_n} G_{gp} \varphi_s \Phi_{gt} dz; \quad \overline{K}_{pg}^{(st)} = 2k_{pg} \int_{c_0}^{c_n} G_{pg} \varphi_s \varphi_t dz \\ & s, t = 0, p, g; \quad q = 7 - p \end{aligned} \tag{4.122}$$

The nonlinear parts of the forces (4.120) are given by

$$\begin{aligned} \tilde{N}_{pp}^{(s)} &= \tilde{C}_{pp}^{(str)} \chi_{r,p} \chi_{t,p} + \tilde{C}_{g4}^{(str)} \chi_{r,g} \chi_{t,g} + \tilde{D}_{pp}^{(st)} \chi_{t,p} + \tilde{D}_{g4}^{(st)} \chi_{t,g} \\ \tilde{N}_{pg}^{(s)} &= \tilde{C}_{pg}^{(str)} \chi_{r,p} \chi_{t,g} + \tilde{D}_{pg}^{(st)} \chi_{t,p} \chi_{t,g} + \tilde{D}_{gp}^{(st)} \chi_{t,g} \end{aligned} \tag{4.123}$$

where the stiffness characteristics are (index k is omitted)

$$\begin{aligned} \tilde{C}_{pp}^{(str)} &= \frac{1}{2} \int_{c_0}^{c_n} \lambda_p \varphi_s \varphi_t \varphi_r dz; \quad \tilde{C}_{g4}^{(str)} = \frac{1}{2} \int_{c_0}^{c_n} \lambda_4 \varphi_s \varphi_t \varphi_r dz \\ \tilde{D}_{pp}^{(st)} &= \int_{c_0}^{c_n} \lambda_p \varphi_s \varphi_t U_{3,p} dz; \quad \tilde{D}_{g4}^{(st)} = \int_{c_0}^{c_n} \lambda_4 \varphi_s \varphi_t U_{3,p} dz \end{aligned}$$

$$\tilde{D}_{pg}^{(st)} = \int_{c0}^{c_n} G_{pg}\varphi_s\varphi_t U_{3,g} dz; \quad \tilde{D}_{gp}^{(st)} = \int_{c0}^{c_n} G_{gp}\varphi_s\varphi_t U_{3,p} dz \tag{4.124}$$

$$\tilde{C}_{pg}^{(str)} = \int_{c0}^{c_n} G_{pg}\varphi_s\varphi_t\varphi_r dz; \qquad s,t,r = 0,p,g$$

The given part of the tangential forces are determined by the general expression

$$\overline{\overline{N}}_{ij}^{(s)} = \int_{c0}^{c_n} \overline{\overline{\sigma}}_{ij}^{(k)} \varphi_s^{(k)} dz \tag{4.125}$$

where $\overline{\overline{\sigma}}_{ij}^{(k)}$ are known stresses obtained from (4.87), (4.91) on the base of the classical theory. Then, according to (4.108) we have forces which comply with the nonlinear part of the strains.

$$T_{ij}^{(st)} = \overline{T}_{ij}^{(st)} + \tilde{T}_{ij}^{(st)} + \overline{\overline{T}}_{ij}^{(st)}; \quad s,t = 0,p,g; \quad p = i \neq g; \quad i,j = 1,2 \tag{4.126}$$

For the linear parts we obtain

$$\begin{aligned} \overline{T}_{pp}^{(st)} &= \overline{A}_{pp}^{(st)} u_{p,p} + \overline{A}_{g4}^{(st)} u_{g,g} - \overline{L}_{pp}^{(str)} \chi_{r,pp} - \overline{L}_{g4}^{(str)} \chi_{r,gg} + \overline{S}_{pp}^{(str)} \chi_r \\ \overline{T}_{pg}^{(st)} &= \overline{A}_{pg}^{(st)} (u_{p,g} + u_{g,p}) - \overline{L}_{pg}^{(str)} \chi_{r,pg} - \overline{L}_{gp}^{(str)} \chi_{r,gp} + \overline{E}_{pg}^{(str)} \chi_r \end{aligned} \tag{4.127}$$

and the stiffness characteristics are

$$\begin{aligned} \overline{A}_{pp}^{(st)} &= \int_{c0}^{c_n} \lambda_p \varphi_s\varphi_t dz; \qquad \overline{A}_{g4}^{(st)} = \int_{c0}^{c_n} \lambda_4 \varphi_s\varphi_t dz \\ \overline{L}_{pp}^{(str)} &= \int_{c0}^{c_n} \lambda_p \varphi_s\varphi_t \Phi_{pr} dz; \quad \overline{L}_{g4}^{(str)} = \int_{c0}^{c_n} \lambda_4 \varphi_s\varphi_t \Phi_{gr} dz \\ \overline{S}_{pp}^{(str)} &= \int_{c0}^{c_n} \left[(k_{pp}\lambda_p + k_{gg}\lambda_4)\varphi_r + \lambda_q F_r' \right] \varphi_s\varphi_t dz \\ \overline{A}_{pg}^{(st)} &= \int_{c0}^{c_n} G_{pg}\varphi_s\varphi_t dz; \qquad \overline{L}_{pg}^{(str)} = \int_{c0}^{c_n} G_{pg}\varphi_s\varphi_t \Phi_{pr} dz \\ \overline{L}_{gp}^{(str)} &= \int_{c0}^{c_n} G_{gp}\varphi_s\varphi_t \Phi_{gr} dz; \quad \overline{E}_{pg}^{(str)} = 2k_{pg} \int_{c0}^{c_n} G_{pg}\varphi_s\varphi_t\varphi_r dz \end{aligned} \tag{4.128}$$

The nonlinear parts are defined as follows

$$\begin{aligned} \tilde{T}_{pp}^{(st)} &= \tilde{A}_{pp}^{(strm)} \chi_{r,p}\chi_{m,p} + \tilde{A}_{g4}^{(strm)} \chi_{r,g}\chi_{m,g} + \tilde{L}_{pp}^{(str)} \chi_{r,p} + \tilde{L}_{g4}^{(str)} \chi_{r,g} \\ \tilde{T}_{pg}^{(st)} &= \tilde{L}_{pg}^{(str)} \chi_{r,p} + \tilde{L}_{gp}^{(str)} \chi_{r,g} + \tilde{A}_{pg}^{(strm)} \chi_{r,p}\chi_{m,g} \end{aligned} \tag{4.129}$$

The corresponded stiffness characteristics are

$$\begin{aligned} \tilde{A}_{pp}^{(strm)} &= \frac{1}{2}\int_{c0}^{c_n} \lambda_p \varphi_s\varphi_t\varphi_r\varphi_m dz; \quad \tilde{A}_{g4}^{(strm)} = \frac{1}{2}\int_{c0}^{c_n} \lambda_4 \varphi_s\varphi_t\varphi_r\varphi_m dz \\ \tilde{L}_{pp}^{(str)} &= \int_{c0}^{c_n} \lambda_p \varphi_s\varphi_t\varphi_r U_{3,p} dz; \quad \tilde{L}_{g4}^{(str)} = \int_{c0}^{c_n} \lambda_4 \varphi_s\varphi_t\varphi_r U_{3,g} dz \\ \tilde{L}_{pg}^{(str)} &= \int_{c0}^{c_n} G_{pg} \varphi_s\varphi_t\varphi_r U_{3,g} dz; \quad \tilde{L}_{gp}^{(str)} = \int_{c0}^{c_n} G_{gp} \varphi_s\varphi_t\varphi_r U_{3,p} dz \\ &\tilde{A}_{pg}^{(strm)} = \int_{c0}^{c_n} G_{pg} \varphi_s\varphi_t\varphi_r\varphi_m dz \end{aligned} \tag{4.130}$$

And the given parts of the forces can be written as

$$\overline{\overline{T}}_{ij}^{(s)} = \int_{c0}^{c_n} \overline{\overline{\sigma}}_{ij}^{(k)} \varphi_s^{(k)} \varphi_t^{(k)} dz \tag{4.131}$$

The moments may be expressed in the following form

$$M_{ij}^{(ps)} = \overline{M}_{ij}^{(ps)} + \tilde{M}_{ij}^{(ps)} + \overline{\overline{M}}_{ij}^{(ps)}; \quad s = 0, p, g; \;\; p = i \neq g; \;\; i, j = 1, 2 \tag{4.132}$$

The linear parts of the moments are given by

$$\begin{aligned} \overline{M}_{pp}^{(ps)} &= \overline{C}_{pp}^{(ps)} u_{p,p} + \overline{C}_{g4}^{(ps)} u_{g,g} - \overline{D}_{pp}^{(pst)} \chi_{t,pp} - \overline{D}_{g4}^{(pst)} \chi_{t,gg} + \overline{K}_{pp}^{(pst)} \chi_t \\ \overline{M}_{pg}^{(ps)} &= \overline{C}_{pg}^{(ps)} (u_{p,g} + u_{g,p}) - \overline{D}_{pg}^{(pst)} \chi_{t,pg} - \overline{D}_{gp}^{(pst)} \chi_{t,gp} + \overline{K}_{pg}^{(pst)} \chi_t \end{aligned} \tag{4.133}$$

and the stiffness characteristics are defined as

$$\begin{aligned} \overline{C}_{pp}^{(ps)} &= \int_{c0}^{c_n} \lambda_p \Phi_{ps} dz; \qquad \overline{C}_{g4}^{(ps)} = \int_{c0}^{c_n} \lambda_4 \Phi_{ps} dz \\ \overline{D}_{pp}^{(pst)} &= \int_{c0}^{c_n} \lambda_p \Phi_{ps} \Phi_{pt} dz; \quad \overline{D}_{g4}^{(pst)} = \int_{c0}^{c_n} \lambda_4 \Phi_{ps} \Phi_{gt} dz \\ \overline{K}_{pp}^{(pst)} &= \int_{c0}^{c_n} \left[(k_{pp}\lambda_p + k_{gg}\lambda_4)\varphi_t + \lambda_q F' \right] \Phi_{ps} dz \\ \overline{C}_{pg}^{(ps)} &= \int_{c0}^{c_n} G_{pg} \Phi_{ps} dz; \qquad \overline{D}_{pg}^{(pst)} = \int_{c0}^{c_n} G_{pg} \Phi_{ps} \Phi_{pt} dz \\ \overline{D}_{gp}^{(pst)} &= \int_{c0}^{c_n} G_{gp} \Phi_{ps} \Phi_{gt} dz; \quad \overline{K}_{pg}^{(pst)} = 2k_{pg} \int_{c0}^{c_n} G_{gp} \varphi_t \Phi_{ps} dz \\ & \qquad s, t = 0, p, g; \qquad q = 7 - p \end{aligned} \tag{4.134}$$

The nonlinear parts of the moments are given as follows

$$\begin{aligned} \tilde{M}_{pp}^{(ps)} &= \tilde{C}_{pp}^{(pstr)} \chi_{r,p} \chi_{t,p} + \tilde{C}_{g4}^{(pstr)} \chi_{r,g} \chi_{t,g} + \tilde{D}_{pp}^{(pst)} \chi_{t,p} + \tilde{D}_{g4}^{(pst)} \chi_{t,g} \\ \tilde{M}_{pg}^{(ps)} &= \tilde{D}_{pg}^{(pst)} \chi_{t,p} + \tilde{D}_{gp}^{(pst)} \chi_{t,g} + \tilde{C}_{pg}^{(pstr)} \chi_{r,p} \end{aligned} \tag{4.135}$$

and for the stiffness characteristics we obtain

$$\begin{aligned} \tilde{C}_{pp}^{(pstr)} &= \frac{1}{2} \int_{c0}^{c_n} \lambda_p \varphi_t \varphi_r \Phi_{ps} dz; \quad \tilde{C}_{g4}^{(pstr)} = \frac{1}{2} \int_{c0}^{c_n} \lambda_4 \varphi_t \varphi_r \Phi_{ps} dz \\ \tilde{D}_{pp}^{(pst)} &= \int_{c0}^{c_n} \lambda_p \varphi_t U_{3,p} \Phi_{ps} dz; \quad \tilde{D}_{g4}^{(pst)} = \int_{c0}^{c_n} \lambda_4 \varphi_t U_{3,g} \Phi_{ps} dz \\ \tilde{D}_{pg}^{(pst)} &= \int_{c0}^{c_n} G_{pg} \varphi_t U_{3,g} \Phi_{ps} dz; \quad \tilde{D}_{gp}^{(pst)} = \int_{c0}^{c_n} G_{gp} \varphi_t U_{3,p} \Phi_{ps} dz \\ \tilde{C}_{pg}^{(pstr)} &= \int_{c0}^{c_n} G_{pg} \varphi_r \varphi_t \Phi_{ps} dz; \qquad s, t = 0, p, g \end{aligned} \tag{4.136}$$

The given parts of the moments are determined with the following integrals

$$\overline{\overline{M}}_{ij}^{(ps)} = \int_{c0}^{c_n} \overline{\overline{\sigma}}_{ij}^{(k)} \Phi_{ps}^{(k)} dz \tag{4.137}$$

Let us write the forces which can be considered as the moments related to the nonlinear part of the strains,

$$H_{ij}^{(is)} = \overline{H}_{ij}^{(is)} + \tilde{H}_{ij}^{(is)} + \overline{\overline{H}}_{ij}^{(is)}; \quad i,j = 1,2; \quad s = 0,p,g; \quad p = i \neq g \tag{4.138}$$

Each of the forces (4.138) is in fact the combined characteristics since there is an agreement about summation over the index $i = 1,2$. For instance

$$\overline{H}_{ij}^{(is)} = \overline{H}_{1j}^{(1s)} + \overline{H}_{2j}^{(2s)} \tag{4.139}$$

The separate items are determined by the following expressions

$$\begin{aligned}
\overline{H}_{pp}^{(ps)} &= \overline{B}_{pp}^{(ps)} u_{p,p} + \overline{B}_{g4}^{(ps)} u_{g,g} - \overline{F}_{pp}^{(pst)} \chi_{t,pp} - \overline{F}_{g4}^{(pst)} \chi_{t,gg} + \overline{R}_{pp}^{(pst)} \chi_t \\
\overline{H}_{pg}^{(ps)} &= \overline{B}_{pg}^{(ps)} (u_{p,g} + u_{g,p}) - \overline{F}_{pg}^{(pst)} \chi_{t,pg} - \overline{F}_{gp}^{(pst)} \chi_{t,gp} + \overline{R}_{pg}^{(pst)} \chi_t
\end{aligned} \tag{4.140}$$

where the stiffness characteristics are given by

$$\begin{aligned}
\overline{B}_{pp}^{(ps)} &= \int_{c0}^{c_n} \lambda_p \varphi_s U_{3,p} dz; \qquad \overline{B}_{g4}^{(ps)} = \int_{c0}^{c_n} \lambda_4 \varphi_s U_{3,p} dz \\
\overline{F}_{pp}^{(pst)} &= \int_{c0}^{c_n} \lambda_p \Phi_{pt} \varphi_s U_{3,p} dz; \quad \overline{F}_{g4}^{(pst)} = \int_{c0}^{c_n} \lambda_4 \Phi_{gt} \varphi_s U_{3,p} dz \\
\overline{R}_{pp}^{(pst)} &= \int_{c0}^{c_n} \left[(k_{pp} \lambda_p + k_{gg} \lambda_4) \varphi_t + \lambda_q F_t' \right] \varphi_s U_{3,p} dz \\
\overline{B}_{pg}^{(ps)} &= \int_{c0}^{c_n} G_{pg} \varphi_s U_{3,p} dz; \qquad \overline{F}_{pg}^{(pst)} = \int_{c0}^{c_n} G_{pg} \varphi_s U_{3,p} \Phi_{pt} dz \\
\overline{F}_{gp}^{(pst)} &= \int_{c0}^{c_n} G_{pg} \varphi_s U_{3,p} \Phi_{gt} dz; \qquad \overline{R}_{pg}^{(pst)} = 2k_{pg} \int_{c0}^{c_n} G_{pg} \varphi_t \varphi_s U_{3,p} dz \\
& \qquad s,t = 0,p,g; \quad q = 7 - p
\end{aligned} \tag{4.141}$$

The nonlinear parts of the forces are

$$\begin{aligned}
\tilde{H}_{pp}^{(ps)} &= \tilde{B}_{pp}^{(pstr)} \chi_{r,p} \chi_{t,p} + \tilde{B}_{g4}^{(pstr)} \chi_{r,g} \chi_{t,g} + \tilde{F}_{pp}^{(pst)} \chi_{t,p} + \tilde{F}_{g4}^{(pst)} \chi_{t,g} \\
\tilde{H}_{pg}^{(ps)} &= \tilde{F}_{pg}^{(pst)} \chi_{t,p} + \tilde{F}_{gp}^{(pst)} \chi_{t,g} + \tilde{B}_{pg}^{(pstr)} \chi_{r,p} \chi_{t,g}
\end{aligned} \tag{4.142}$$

and the related stiffness characteristics are

$$\begin{aligned}
\tilde{B}_{pp}^{(pstr)} &= \frac{1}{2} \int_{c0}^{c_n} \lambda_p \varphi_s \varphi_t \varphi_r U_{3,p} dz; \quad \tilde{B}_{g4}^{(pstr)} = \frac{1}{2} \int_{c0}^{c_n} \lambda_4 \varphi_s \varphi_t \varphi_r U_{3,p} dz \\
\tilde{F}_{pp}^{(pst)} &= \int_{c0}^{c_n} \lambda_p \varphi_s \varphi_t U_{3,p} dz; \quad \tilde{F}_{g4}^{(pst)} = \int_{c0}^{c_n} \lambda_4 \varphi_s \varphi_t U_{3,p} dz \\
\tilde{F}_{pg}^{(pst)} &= \int_{c0}^{c_n} G_{pg} \varphi_s \varphi_t U_{3,p} U_{3,g} dz; \quad \tilde{F}_{gp}^{(pst)} = \int_{c0}^{c_n} G_{gp} \varphi_s \varphi_t U_{3,p}^2 dz \\
\tilde{B}_{pg}^{(pstr)} &= \int_{c0}^{c_n} G_{pg} \varphi_s \varphi_t \varphi_r U_{3,p} dz
\end{aligned} \tag{4.143}$$

The given parts of the considered moments may be defined as

$$\begin{aligned}
\overline{\overline{H}}_{pp}^{(ps)} &= \int_{c_0}^{c_n} \overline{\overline{\sigma}}_{pp}^{(k)} \varphi_s^{(k)} U_{3,p}^{(k)} dz \\
\overline{\overline{H}}_{pg}^{(ps)} &= \int_{c_0}^{c_n} \overline{\overline{\sigma}}_{pg}^{(k)} \varphi_s^{(k)} U_{3,p}^{(k)} dz
\end{aligned} \tag{4.144}$$

The transverse forces can be obtained from (4.108) and (4.92) as

$$Q_i^{(ps)} = \overline{Q}_i^{(ps)} + \overline{\overline{Q}}_i^{(ps)}; \qquad i = 1,2; \;\; p = i \neq g; \;\; s = 0,p,g \tag{4.145}$$

Here the sought part is

$$\overline{Q}^{(ps)} = -\overline{D}_p^{(pst)} \chi_{t,i} \tag{4.146}$$

where

$$\overline{D}_p^{(pst)} = \int_{c_0}^{c_n} \varphi_{ps}^{(k)} f_{pt}^{(k)} dz; \qquad s,t = 0,p,g \tag{4.147}$$

and the given part is

$$\overline{\overline{Q}}_i^{(ps)} = \int_{c_0}^{c_n} \sigma_{i3}^{(0k)} \varphi_{ps}^{(0k)} dz \tag{4.148}$$

Moreover, there are also the normal forces which are connected with the normal stresses $\sigma_{33}^{(k)}$. Using equations (4.108) and expressions for the stresses and strains we can express the considered forces as the sum of the linear, nonlinear and given parts.

$$Q_3^{(s)} = \overline{Q}_3^{(s)} + \tilde{Q}_3^{(s)} + \overline{\overline{Q}}_3^{(s)} \tag{4.149}$$

where the linear part is defined as

$$\overline{Q}_3^{(s)} = \overline{C}_{3p}^{(s)} u_{p,p} + \overline{C}_{3g}^{(s)} u_{g,g} - \overline{D}_{3p}^{(st)} \chi_{t,pp} - \overline{D}_{3g}^{(st)} \chi_{t,gg} + \overline{K}_{3p}^{(st)} \chi_t \tag{4.150}$$

Here the following stiffness characteristics have been introduced

$$\begin{aligned}
\overline{C}_{3p}^{(s)} &= \int_{c_0}^{c_n} \lambda_6 F_s' dz; \quad \overline{C}_{3g}^{(s)} = \int_{c_0}^{c_n} \lambda_5 F_s' dz \\
\overline{D}_{3p}^{(st)} &= \int_{c_0}^{c_n} \lambda_6 \Phi_{pt} F_s' dz; \quad \overline{D}_{3g}^{(st)} = \int_{c_0}^{c_n} \lambda_5 \Phi_{gt} F_s' dz \\
\overline{K}_{3p}^{(st)} &= \int_{c_0}^{c_n} \left[(k_{pp}\lambda_6 + k_{gg}\lambda_4)\varphi_t + \lambda_3 F_t' \right] F_s' dz; \qquad s,t = 0,p,g
\end{aligned} \tag{4.151}$$

The nonlinear part of the force is given by

$$\tilde{Q}_3^{(s)} = \tilde{C}_{3p}^{(str)} \chi_{r,p}\chi_{t,p} + \tilde{C}_{3g}^{(str)} \chi_{r,g}\chi_{t,g} + \tilde{D}_{3p}^{(st)} \chi_{t,p} + \tilde{D}_{g3}^{(st)} \chi_{t,g} \tag{4.152}$$

where we have

$$\begin{aligned}
\tilde{C}_{3p}^{(str)} &= \frac{1}{2} \int_{c_0}^{c_n} \lambda_6 \varphi_t \varphi_r F_s' dz; \quad \tilde{C}_{3g}^{(str)} = \frac{1}{2} \int_{c_0}^{c_n} \lambda_5 \varphi_t \varphi_r F_s' dz \\
\tilde{D}_{3p}^{(st)} &= \int_{c_0}^{c_n} \lambda_6 \varphi_t U_{3,p} F_s' dz; \quad \tilde{D}_{3g}^{(st)} = \int_{c_0}^{c_n} \lambda_5 \varphi_t U_{3,g} F_s' dz; \\
& s,t,r = 0,p,g
\end{aligned} \tag{4.153}$$

And the given part of the force

$$\overline{\overline{Q}}_3^{(s)} = \int_{c_0}^{c_n} [\lambda_6 \overline{\overline{e}}_{pp}^{(0k)} + \lambda_4 \overline{\overline{e}}_{gg}^{(0k)} + \lambda_3 e_{33}^{(0k)}] F_s^{\prime(k)} dz \tag{4.154}$$

where strains are given in the Section 4.3.

Thus the all elasticity relations (constitutive equations) forming the system of the forces in the shell have been obtained. The forces and their parts, which are unknown, are given in the expanded form by the sought functions and generalised characteristics of the stiffnesses. The given forces or parts of the forces are presented in the form of integrals with respect to the given functions over the thickness of the shell.

4.8 The system of governing differential equations.

On the basis of the system of equations for the forces (4.118) and expressions for the forces obtained in the previous section, the general system of governing differential equations of the geometrically nonlinear theory of the laminated orthotropic shallow shells can be formulated.

4.8.1 General structure of the system of differential equations

The system of governing differential equations may be given in matrix form as

$$(\overline{D} + \tilde{D})\{V\} = \{Q\} + \overline{\overline{D}}\{N\} \tag{4.155}$$

where $\bar{D}$, $\tilde{D}$ and $\overline{\overline{D}}$ are the matrices of differential operators which correspond to the linear, nonlinear and given parts of the constitutional equations, respectively. The vector of the unknown functions is given by

$$\{V\} = \{u_1, u_2, \chi_t\}^T; \;\; t = 0, 1, 2 \tag{4.156}$$

and the vector of given part of the stress and moment resultants by

$$\{N\} = \{\overline{\overline{N}}_{1j}^{(0)}, \overline{\overline{N}}_{2j}^{(0)}, \overline{\overline{M}}_{ij}^{(ps)}, \overline{\overline{N}}_{ij}^{(s)}, \overline{\overline{Q}}_i^{(ps)}, \overline{\overline{Q}}_3^{(s)}, \overline{\overline{H}}_{ij}^{(is)}, \overline{\overline{T}}_{ij}^{(st)}\}^T \tag{4.157}$$
$$i, j = 1, 2; \;\; p = i \neq g; \;\; s, t = 0, p, g$$

The system of differential equations (4.155) also incorporates the vector of external loads

$$\{Q\} = \{q_1^{(0)},\ q_2^{(0)},\ q_3^{(s)}\}^T; \quad s = 0, 1, 2 \tag{4.158}$$

The structure of the matrices of the differential operators is quite complicated and does not allow the system (4.155) to be represented in the visible expanded form. For convenience we shall present the matrices $\overline{D}$ and $\tilde{D}$ as the sums of sub-matrices, namely

$$\overline{D} = \overline{A} + \overline{B} + \overline{C}; \qquad \tilde{D} = \tilde{A} + \tilde{B} + \tilde{C} \tag{4.159}$$

and in turn the sub-matrices (4.159) have the following structure

$$\begin{aligned}
\overline{A} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{A}_{31}^{(s)} & \overline{A}_{32}^{(s)} & \overline{A}_{33}^{(st)} \end{bmatrix}; \quad \overline{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \overline{B}_{31}^{(s)} & \overline{B}_{32}^{(s)} & \overline{B}_{33}^{(st)} \end{bmatrix} \\
\overline{C} &= \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13}^{(t)} \\ \overline{C}_{21} & \overline{C}_{22} & \overline{C}_{23}^{(t)} \\ \overline{C}_{31} & \overline{C}_{32} & \overline{C}_{33}^{(st)} \end{bmatrix}; \quad \tilde{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{A}_{33}^{(st)} \end{bmatrix} \\
\tilde{B} &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{B}_{33}^{(st)} \end{bmatrix}; \quad \tilde{C} = \begin{bmatrix} 0 & 0 & \tilde{C}_{13}^{(t)} \\ 0 & 0 & \tilde{C}_{23}^{(t)} \\ 0 & 0 & \tilde{C}_{33}^{(st)} \end{bmatrix}
\end{aligned} \tag{4.160}$$

4.8.2 Structure of the linear part

Let us consider the symmetrical matrix $\overline{C}$. It corresponds to the linear problem, where we have

$$\overline{D} = \overline{C}; \qquad \overline{A} = \overline{B}; \qquad \tilde{D} = 0 \tag{4.161}$$

Matrix $\overline{C}$ may be thought of as forms the fundamental base of the system of equations (4.155). Its operators have the following structure:

$$\begin{aligned}
\overline{C}_{11} &= \overline{C}_{11}^{(0)}(\ldots)_{,11} + \overline{C}_{12}^{(0)}(\ldots)_{,22}; \quad \overline{C}_{12} = (\overline{C}_{12}^{(0)} + \overline{C}_{24}^{(0)})(\ldots)_{,12} \\
\overline{C}_{13}^{(t)} &= \left[\overline{D}_{11}^{(0t)}(\ldots)_{,11} + (\overline{D}_{12}^{(0t)} + \overline{D}_{21}^{(0t)} + \overline{D}_{24}^{(0t)})(\ldots)_{,22}\right]_{,1} \\
&+ \overline{K}_{11}^{(0t)}(\ldots)_{,1} + \overline{K}_{12}^{(0t)}(\ldots)_{,2} \\
\overline{C}_{22} &= \overline{C}_{22}^{(0)}(\ldots)_{,2} + \overline{C}_{21}^{(0)}(\ldots)_{,11} \\
\overline{C}_{23}^{(t)} &= \left[\overline{D}_{22}^{(0t)}(\ldots)_{,22} + (\overline{D}_{21}^{(0t)} + \overline{D}_{12}^{(0t)} + \overline{D}_{14}^{(0t)})(\ldots)_{,11}\right]_{,2}
\end{aligned}$$

$$
\begin{aligned}
& + \quad \overline{K}_{22}^{(0t)}(\ldots)_{,2} + \overline{K}_{21}^{(0t)}(\ldots)_{,1} && (4.162)\\
\overline{C}_{33}^{(st)} &= \left[\overline{D}_{11}^{(1st)}(\ldots)_{,11} + (\overline{D}_{12}^{(1st)} + \overline{D}_{12}^{(2st)} + \overline{D}_{14}^{(2st)})(\ldots)_{,22}\right]_{,11}\\
&+ \left[\overline{D}_{22}^{(2st)}(\ldots)_{,22} + (\overline{D}_{21}^{(2st)} + \overline{D}_{21}^{(1st)} + \overline{D}_{24}^{(1st)})(\ldots)_{,11}\right]_{,22}\\
&- (\overline{K}_{11}^{(1st)} - \overline{D}_{1}^{(1st)} + \overline{D}_{11}^{(st)}k_{11} + \overline{D}_{14}^{(st)}k_{22} + \overline{D}_{31}^{(st)})(\ldots)_{,11}\\
&- (\overline{K}_{22}^{(2st)} - \overline{D}_{2}^{(2st)} + \overline{D}_{22}^{(st)}k_{22} + \overline{D}_{24}^{(st)}k_{11} + \overline{D}_{32}^{(st)})(\ldots)_{,22}\\
&- \left[\overline{K}_{12}^{(1st)} + \overline{K}_{21}^{(2st)} + 2(\overline{D}_{12}^{(st)} + \overline{D}_{21}^{(st)})k_{12}\right](\ldots)_{,12}\\
&+ (\overline{K}_{11}^{(st)}k_{11} + 2\overline{K}_{12}^{(st)}k_{12} + \overline{K}_{22}^{(st)}k_{22} + \overline{K}_{31}^{(st)})(\ldots); \quad s,t = 0,1,2
\end{aligned}
$$

Matrices $\overline{A}$ and $\overline{B}$ follow from the linear parts of the elasticity relations, which, however, correspond to the nonlinear part of the variation of the potential energy (4.110). Therefore these sub-matrices are unique to the nonlinear problem. The nonzero elements of the matrix $\overline{B}$ have the following structure

$$
\begin{aligned}
\overline{B}_{31}^{(s)} &= -\Big[\overline{B}_{11}^{(1s)}(\ldots)_{,11} + (\overline{B}_{14}^{(2s)})(\ldots)_{,12} + \overline{B}_{12}^{(1s)}(\ldots)_{,22}\\
&+ (\overline{B}_{11,1}^{(1s)} + \overline{B}_{14,2}^{(2s)})(\ldots)_{,1} + (\overline{B}_{12,2}^{(1s)} + \overline{B}_{21,1}^{(2s)})(\ldots)_{,2}\Big]\\
\overline{B}_{32}^{(s)} &= -[\overline{B}_{22}^{(2s)}(\ldots)_{,22} + (\overline{B}_{24}^{(1s)} + \overline{B}_{12}^{(1s)})(\ldots)_{,12} + \overline{B}_{21}^{(2s)}(\ldots)_{,11}\\
&+ (\overline{B}_{22,2}^{(2s)} + \overline{B}_{24,1}^{(1s)})(\ldots)_{,2} + (\overline{B}_{12,2}^{(1s)} + \overline{B}_{21,1}^{(2s)})(\ldots)_{,1}]\\
\overline{B}_{33}^{(s)} &= \overline{F}_{11}^{(1st)}(\ldots)_{,111} + \overline{F}_{22}^{(2st)}(\ldots)_{,222} + (\overline{F}_{14}^{(2st)} + \overline{F}_{21}^{(2st)} + \overline{F}_{12}^{(2st)})(\ldots)_{,112}\\
&+ (\overline{F}_{24}^{(1st)} + \overline{F}_{12}^{(1st)} + \overline{F}_{21}^{(1st)})(\ldots)_{,221} && (4.163)\\
&- [(\overline{R}_{11}^{(1st)} + \overline{R}_{21}^{(2st)})(\ldots)_{,1} + (\overline{R}_{22}^{(2st)} + \overline{R}_{12}^{(1st)})(\ldots)_{,2})]\\
&+ (\overline{F}_{11,1}^{(1st)} + \overline{F}_{14,2}^{(2st)})(\ldots)_{,11} + (\overline{F}_{22,2}^{(2st)} + \overline{F}_{24,1}^{(1st)})(\ldots)_{,22}\\
&+ (\overline{F}_{12,2}^{(1st)} + \overline{F}_{21,2}^{(1st)} + \overline{F}_{21,1}^{(2st)} + \overline{F}_{12,1}^{(1st)})(\ldots)_{,12}\\
&- [(\overline{R}_{11}^{(1st)} + \overline{R}_{21}^{(2st)})_{,1} + (\overline{R}_{22}^{(2st)} + \overline{R}_{12}^{(1st)})_{,2}](\ldots); \quad s,t = 0,1,2
\end{aligned}
$$

The stiffness characteristics from (4.163) are variable values since according to (4.141) they contain the given functions of the normal displacements $U_{3,p}^{(k)}$. Let us write also the nonzero elements of the matrix $\overline{A}$

$$
\begin{aligned}
\overline{A}_{31}^{(s)} &= -\Big\{(\overline{A}_{11}^{(st)}\chi_{t,11} + \overline{A}_{14}^{(st)}\chi_{t,22})(\ldots)_{,1} + 2\overline{A}_{12}^{(st)}\chi_{t,12}(\ldots)_{,2}\\
&+ [\overline{A}_{11}^{(st)}(\ldots)_{,11} + \overline{A}_{12}^{(st)}(\ldots)_{,22}]\chi_{t,1} + (\overline{A}_{12}^{(st)} + \overline{A}_{14}^{(st)})\chi_{t,2}\Big\}\\
\overline{A}_{32}^{(s)} &= \Big\{(\overline{A}_{22}^{(st)}\chi_{t,22} + \overline{A}_{24}^{(st)}\chi_{t,11})(\ldots)_{,2} + 2\overline{A}_{21}^{(st)}\chi_{t,21}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad [\overline{A}^{(st)}_{22}(\ldots)_{,22} + \overline{A}^{(st)}_{21}(\ldots)_{,11}]\chi_{t,2} + (\overline{A}^{(st)}_{21} + \overline{A}^{(st)}_{24})\chi_{t,1}(\ldots)_{,21}\big\}
\end{aligned}
$$

$$
\begin{aligned}
\overline{A}^{(st)}_{33} &= (\overline{L}^{(str)}_{11}\chi_{r,11} + \overline{L}^{(str)}_{24}\chi_{r,22} - \overline{R}^{(str)}_{11}\chi_r)(\ldots)_{,11} \\
&+ (\overline{L}^{(str)}_{22}\chi_{r,22} + \overline{L}^{(str)}_{14}\chi_{r,11} - \overline{R}^{(str)}_{22}\chi_r)(\ldots)_{,22} \\
&+ [2(\overline{L}^{(str)}_{12} + \overline{L}^{(str)}_{21})\chi_{r,12} - (\overline{E}^{(str)}_{12} + \overline{E}^{(str)}_{21})\chi_r](\ldots)_{,12} \\
&+ (\overline{L}^{(str)}_{11}\chi_{r,11} + \overline{L}^{(str)}_{24}\chi_{r,22} - \overline{R}^{(str)}_{11}\chi_r)_{,1}(\ldots)_{,1} \\
&+ (\overline{L}^{(str)}_{22}\chi_{r,22} + \overline{L}^{(str)}_{14}\chi_{r,11} - \overline{R}^{(str)}_{22}\chi_r)_{,2}(\ldots)_{,2} \\
&+ [(\overline{L}^{(str)}_{12} + \overline{L}^{(str)}_{21})\chi_{r,12} - \overline{E}^{(str)}_{12}\chi_r]_{,1}(\ldots)_{,2} \\
&+ [(\overline{L}^{(str)}_{21} + \overline{L}^{(str)}_{12})\chi_{r,21} - \overline{E}^{(str)}_{21}\chi_r]_{,2}(\ldots)_{,1}; \quad s,t,r = 0,1,2
\end{aligned}
\tag{4.164}
$$

4.8.3 Structure of the nonlinear part

Let us consider the sub-matrices which form the matrix $\tilde{D}$. The matrix $\tilde{C}$ is formed by the nonlinear parts of the forces connected with the variation of the linear part of the potential energy (4.109). Elements of this matrix are given by

$$
\begin{aligned}
\tilde{C}^{(t)}_{13} &= \tilde{C}^{(0tr)}_{11}[\chi_{3,1}(\ldots)_{,1}]_{,1} + \tilde{C}^{(0tr)}_{24}[\chi_{r,2}(\ldots)_{,2}]_{,1} + \tilde{C}^{(0tr)}_{12}[\chi_{r,1}(\ldots)_{,2}]_{,2} \\
&+ [\tilde{D}^{(0t)}_{11}(\ldots)_{,1} + \tilde{D}^{(0t)}_{24}(\ldots)_{,2}\tilde{D}^{(0t)}_{21}(\ldots)_{,2}]_{,1} + [\tilde{D}^{(0t)}_{12}(\ldots)_{,1}]_{,2}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}^{(t)}_{23} &= \tilde{C}^{(0tr)}_{22}[\chi_{r,2}(\ldots)_{,2}]_{,2} + \tilde{C}^{(0tr)}_{14}[\chi_{r,1}(\ldots)_{,1}]_{,2} + \tilde{C}^{(0tr)}_{21}[\chi_{r,2}(\ldots)_{,1}]_{,1} \\
&+ [\tilde{D}^{(0t)}_{22}(\ldots)_{,2} + \tilde{D}^{(0t)}_{14}(\ldots)_{,1} + \tilde{D}^{(0t)}_{12}(\ldots)_{,1}]_{,2} + [\tilde{D}^{(0t)}_{21}(\ldots)_{,2}]_{,1}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}^{(st)}_{33} &= -\big\{[\tilde{C}^{(1str)}_{11}\chi_{r,1}(\ldots)_{,1} + \tilde{C}^{(1str)}_{24}\chi_{r,2}(\ldots)_{,2} + \tilde{D}^{(1st)}_{11}(\ldots)_{,1} + \tilde{D}^{(1st)}_{24}(\ldots)_{,2}]_{,11} \\
&+ [\tilde{C}^{(2str)}_{22}\chi_{r,2}(\ldots)_{,2} + \tilde{C}^{(2str)}_{14}\chi_{r,1}(\ldots)_{,1} + \tilde{D}^{(2st)}_{22}(\ldots)_{,2}\tilde{D}^{(2st)}_{14}(\ldots)_{,1}]_{,22} \\
&+ [\tilde{C}^{(1str)}_{12}\chi_{r,1}(\ldots)_{,2} + \tilde{C}^{(2str)}_{21}\chi_{r,2}(\ldots)_{,1} + \tilde{D}^{(1st)}_{12}(\ldots)_{,1} \\
&+ \tilde{D}^{(1st)}_{21}(\ldots)_{,2} + \tilde{D}^{(2st)}_{21}(\ldots)_{,2} + \tilde{D}^{(2st)}_{12}(\ldots)_{,1}]_{,12}\big\} \\
&+ [k_{11}(\tilde{C}^{(str)}_{11}\chi_{r,1} + \tilde{D}^{(st)}_{11}) + k_{22}(\tilde{C}^{(str)}_{14}\chi_{r,1} + \tilde{D}^{(st)}_{14}) + k_{12}(\tilde{C}^{(str)}_{21}\chi_{r,2} + 2\tilde{D}^{(st)}_{12}) \\
&+ (\tilde{C}^{(str)}_{13}\chi_{r,1} + \tilde{D}^{(st)}_{13})](\ldots)_{,1} + [k_{11}(\tilde{C}^{(str)}_{24}\chi_{r,2} + \tilde{D}^{(st)}_{24}) + k_{22}(\tilde{C}^{(str)}_{22}\chi_{r,2} + \tilde{D}^{(st)}_{22}) \\
&+ k_{21}(\tilde{C}^{(str)}_{12}\chi_{r,1} + 2\tilde{D}^{(st)}_{21}) + (\tilde{C}^{(str)}_{23}\chi_{r,2} + \tilde{D}^{(st)}_{23})](\ldots)_{,2}; \qquad s,t,r = 0,1,2
\end{aligned}
\tag{4.165}
$$

The sub-matrix $\tilde{B}$ follows from the nonlinear parts of the elasticity relations and corresponds to the nonlinear part of the variation of the potential energy (4.110). The nonzero element of it is

$$
\tilde{B}^{(st)}_{33} = -\big\{[(\tilde{B}^{(1str)}_{11}\chi_{r,1} + \tilde{F}^{(1st)}_{11})(\ldots)_{,1} + (\tilde{B}^{(1str)}_{24}\chi_{r,2} + \tilde{F}^{(1st)}_{24})(\ldots)_{,2}
$$

$$
\begin{aligned}
&+ \quad (\tilde{B}_{21}^{(2str)}\chi_{r,2} + \tilde{F}_{12}^{(2st)})(\ldots)_{,1} + \tilde{F}_{21}^{(2st)}(\ldots)_{,2}]_{,1} \qquad (4.166)\\
&+ \quad [(\tilde{B}_{22}^{(2str)}\chi_{r,2} + \tilde{F}_{22}^{(2st)})(\ldots)_{,2} + (\tilde{B}_{14}^{(2str)}\chi_{r,1} + \tilde{F}_{14}^{(2st)})(\ldots)_{,1}\\
&+ \quad (\tilde{B}_{12}^{(1str)}\chi_{r,1} + \tilde{F}_{24}^{(1st)})(\ldots)_{,2} + \tilde{F}_{12}^{(1st)}(\ldots)_{,1}]_{,2}\Big\}; \quad s,t,r = 0,1,2
\end{aligned}
$$

The matrix $\tilde{A}$ has also strictly nonlinear character, the nonzero part of it has the following form

$$
\begin{aligned}
\tilde{A}_{33}^{(st)} \;=\; & -\Big\{\tilde{A}_{11}^{(strm)}[\chi_{r,1}\chi_{m,1}(\ldots)_{,1}]_{,1} + \tilde{A}_{24}^{(strm)}[\chi_{r,2}\chi_{m,2}(\ldots)_{,1}]_{,1}\\
&+ \tilde{A}_{12}^{(strm)}[\chi_{r,1}\chi_{m,2}(\ldots)_{,2}]_{,1} + +\tilde{A}_{21}^{(strm)}[\chi_{r,2}\chi_{m,1}(\ldots)_{,1}]_{,2}\\
&+ \tilde{A}_{22}^{(strm)}[\chi_{r,2}\chi_{m,2}(\ldots)_{,2}]_{,2} + +\tilde{A}_{14}^{(strm)}[\chi_{r,1}\chi_{m,1}(\ldots)_{,2}]_{,2} \qquad (4.167)\\
&+ [(\tilde{L}_{11}^{(str)}\chi_{r,1} + \tilde{L}_{24}^{(str)}\chi_{r,2})(\ldots)_{,1} + (\tilde{L}_{12}^{(str)}\chi_{r,1} + \tilde{L}_{21}^{(str)}\chi_{r,2})(\ldots)_{,2}]_{,1}\\
&+ [(\tilde{L}_{22}^{(str)}\chi_{r,2} + \tilde{L}_{14}^{(str)}\chi_{r,1})(\ldots)_{,2} + (\tilde{L}_{21}^{(str)}\chi_{r,2} + \tilde{L}_{12}^{(str)}\chi_{r,1})(\ldots)_{,1}]_{,2}\Big\}\\
&\qquad s,t,r,m = 0,1,2
\end{aligned}
$$

4.8.4 Structure of the given part

It is also necessary to write the matrix $\overline{\overline{D}}$, which is the operator matrix over the vector of the given forces $\{N\}$. This matrix can be written as follows

$$
\overline{\overline{D}} = \begin{bmatrix}
\overline{\overline{N}}_{1j}^{(0)} & \overline{\overline{N}}_{2j}^{(0)} & \overline{\overline{M}}_{ij}^{(ps)} & \overline{\overline{N}}_{ij}^{(ps)} & \overline{\overline{Q}}_{i}^{(ps)} & \overline{\overline{Q}}_{3}^{(s)} & \overline{\overline{H}}_{ij}^{(is)} & \overline{\overline{T}}_{ij}^{(st)}\\
-(\ldots)_{,j} & - & - & - & - & - & - & -\\
- & -(\ldots)_{,j} & - & - & - & - & - & -\\
- & - & (\ldots)_{,ij} & -(\ldots)k_{ij} & (\ldots)_{,i} & -(\ldots) & (\ldots)_{,j} & [(\ldots)\chi_{t,j}]_{,i}
\end{bmatrix} \qquad (4.168)
$$

Hence the system of governing differential equations of the geometrically nonlinear nonclassical theory of the laminated orthotropic shallow shells has been formulated. The order of the general system of differential equations is equal to 16 and therefore 8 boundary conditions have to be satisfied on each edge of the shell.

4.9 Some analytical solutions

Let us consider the derivation of the closed form solutions for some particular cases of the system of governing equations of geometrically nonlinear higher order theory. Firstly, we consider cylindrical bending of a laminated plate under uniformly distributed load $q_3^+ = q$. We assume that the plate is symmetrical. Initially we consider a solution based on the classical theory of the plates which can be obtained from (4.155) assuming $G_{13}^{(k)} = \infty$ and $k_1 = 0$ as

$$\begin{aligned} &\overline{C}_{11}^{(0)} u_{1,11} + \tilde{C}_{11}^{(0)} (w_{,1}^2)_{,1} = 0 \\ &\overline{D}_{11}^{(100)} w_{,1111} - \overline{C}_{11}^{(0)} (w_{,11} u_{1,1} + w_{,1} u_{1,11}) \\ - \; &\tilde{A}_{11}^{(0)} [w_{,1}^2 w_{,11} + (w_{,1}^2)_{,1} w_{,1}] = q \end{aligned} \tag{4.169}$$

where

$$\begin{aligned} \overline{C}_{11}^{(0)} &= C = \int_{c_0}^{c_n} \lambda_1 dz = \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} \lambda_1^{(k)} dz \\ \overline{D}_{11}^{(100)} &= D = \int_{c_0}^{c_n} \lambda_1 z^2 dz = \sum_{k=1}^{n} \int_{c_{k-1}}^{c_k} \lambda_1^{(k)} z^2 dz, \end{aligned} \tag{4.170}$$

and also $u_1 = u$. In equation (4.169), the quantities C and D define the rigidity of the laminated plate in which moduli $\lambda_1^{(k)}$ might be variable through the thickness.

Let us consider now simply supported plates for which the boundary conditions are

$$w = 0; \quad M_{11}^{(0)} \equiv w_{,11} = 0; \quad n = 0 \text{ when } x = \pm a \tag{4.171}$$

We also assume that the boundaries of the plate do not approach each other as a result of deformation of the plate, e.g.

$$\Delta = \int_{-a}^{a} u_{,1} dx = u(a) - u(-a) = 0 \tag{4.172}$$

and we can rewrite the first equation of the system (4.169) as

$$C(u_{,1} + \frac{1}{2} w_{,1}^2)_{,1} = N_{11,1} = 0 \tag{4.173}$$

Integrating we obtain

$$N_{11} = C(u_{,1} + \frac{1}{2} w_{,1}^2) = A \tag{4.174}$$

Let us introduce the notation

$$\lambda^2 = N_{11}/D \quad \text{for } N_{11} > 0$$

and rewrite the second equation of the system (4.169) as

$$w_{,1111} - \lambda^2 w_{,11} = \frac{q}{D} \tag{4.175}$$

The general solution of this equation may now be obtained and is given by

$$w = C_1 + C_2 x + C_3 \sinh \lambda x + C_4 \cosh \lambda x - \frac{q w^2}{2 D \lambda^2} \tag{4.176}$$

where the constants of integration can be found from boundary conditions (4.171) as

$$C_1 = \frac{q(a^2\lambda^2 - 2)}{2D\lambda^4}; \quad C_2 = C_3 = 0$$
$$C_4 = \frac{q}{D\lambda^4} \cdot \frac{1}{\cosh \lambda a} \tag{4.177}$$

Substituting (4.177) into (4.176) we can rewrite expression for the deflection in the following form

$$w = \left[2\left(\frac{\cosh \lambda x}{\cosh \lambda a} - 1\right) + \lambda^2(a^2 - x^2) \right] \frac{q}{2D\lambda^4} \tag{4.178}$$

The unknown parameter λ can be defined from the condition (4.172) which is given by

$$\begin{aligned} \Delta &= \int_{-a}^{a} u_{,1} dx = \int_{-a}^{a} \left(\frac{D}{C}\lambda^2 - \frac{1}{2}w_{,1}^2 \right) dx = \frac{D}{C} 2a\lambda^2 \\ &- \frac{q^2}{D^2\lambda^7}\left(\frac{5}{2}\tanh \lambda a - \frac{5}{2}\lambda a + \frac{\lambda a}{2}\tanh^2 \lambda a + \frac{\lambda^3 a^3}{3} \right) = 0 \end{aligned} \tag{4.179}$$

Introducing the following dimensionless parameters

$$\mu = \lambda a; \ \beta = \frac{a}{h}; \ \gamma = \frac{w}{h}; \ \eta = \frac{x}{a}; \ p = \frac{qa^4}{Dh}$$

we can rewrite equation (4.179) as

$$\mu^9 - p^2(15 \tanh \mu - 15\mu + 3\mu \tanh^2 \mu + 2\mu^3) = 0 \tag{4.180}$$

from which the dimensionless deflection may be found as

$$\gamma = \frac{p}{2\mu^4}\left[2\left(\frac{\cosh \mu\eta}{\cosh \mu} - 1 \right) + \mu^2(1 - \eta^2) \right]; \ \ (-1 \geq \eta \leq 1) \tag{4.181}$$

Let us now introduce a dimensionless stress in *k-th* layer

$$\sigma_1^{(k)} = \sigma_{11}^{(k)} \frac{Ca^2}{3\lambda_1^{(k)} D} \tag{4.182}$$

Then we can write

$$\sigma_1^{(k)} = \frac{1}{3\mu^2}\left[\mu^4 - \frac{PhC}{D} z \left(\frac{\cosh \mu\eta}{\cosh \mu} - 1 \right) \right] \tag{4.183}$$

The extremum stresses in laminated plate now may be obtained, viz.

$$\sigma_1^{\pm} = \sigma_m \pm \sigma_b = \frac{\mu^2}{3} \pm \frac{Ph^2C}{6D\mu^2}\left(1 - \frac{1}{\cosh \mu} \right) \tag{4.184}$$

$$(\text{for } z = \pm\frac{h}{2}; \ \eta = 0)$$

where σ_m and σ_b are membrane and bending components of the stresses, respectively.

For the case of the clamped plates ($w = 0$; $w_{,1} = 0$; $u_1 = 0$ when $x = \pm a$) dimensionless deflection can be obtained as

$$\gamma = \frac{P}{2\mu^2}\left[1 - \eta^2 + \frac{2(\cosh\mu\eta - \cosh\mu)}{\mu\sinh\mu}\right] \tag{4.185}$$

and stresses as

$$\sigma_1^{(k)} = \frac{1}{3\mu^2}\left[\mu^4\frac{PhC}{D}z\left(\mu\frac{\cosh\mu\eta}{\sinh\mu} - 1\right)\right] \tag{4.186}$$

$$\sigma_1^{\pm} = \sigma_m \pm \sigma_b = \frac{\mu^2}{3} \pm \frac{Ph^2C}{6D\mu^2}\left(1 - \frac{\mu}{\sinh\mu}\right) \text{ (for } z = \pm\frac{h}{2};\ \eta = 0) \tag{4.187}$$

Following the procedure described above we can extend the solution obtained for the case of cylindrical bending of laminated plates which includes influence of transverse shear. Then for the case of simply supported plates the dimensionless deflection is given as

$$\gamma = \frac{P^*}{2\mu^{*4}}\left[2\left(\frac{\cosh\mu^*\eta}{\cosh\mu^*} - 1\right) + \mu^{*2}(1 - \eta^2)\right] \tag{4.188}$$

where

$$P^* = \frac{qa}{D^*h} = \frac{P}{1 + \lambda^2\frac{D_1}{D}};\ \mu^* = \lambda^* a = \frac{\lambda a}{\sqrt{1 + \lambda^2\frac{D_1}{D}}} = \frac{\mu}{\sqrt{1 + \lambda^2\frac{D_1}{D}}}$$

$$D^* = D\left(1 + \lambda^2\frac{D_1}{D}\right);\ D_1 = \sum_{k=1}^{n}\int_{c_{k-1}}^{c_k}\lambda_1^{(k)} z\psi_{11}^{(k)}dz \tag{4.189}$$

For the determination of the parameter μ we can use equation (4.180).

Introducing dimensionless stress in the *k-th* layer as

$$\sigma_1^{(k)} = \sigma_{11}^{(k)}\frac{Ca^2}{3\lambda_1^{(k)}D^*} \tag{4.190}$$

we can obtain stress in the following form

$$\sigma_1^{(k)} = \frac{1}{3\mu^{*2}}\left(\mu^{*4} - \frac{P^*hC}{D^*}z^*\right) \tag{4.191}$$

where z^* is the dimensionless coordinate through the thickness coordinate given by

$$\begin{aligned} z^* &= z\Phi^{(k)}; \\ \Phi^{(k)} &= \left[\left(1 + \frac{\mu^2}{a^2}\frac{\psi_{11(k)}}{z}\right)\left(\frac{\cosh\mu^*\eta}{\cosh\mu^*} - 1\right) + \left(1 + \frac{\mu^2}{a^2}\frac{D_1}{D}\right)\frac{\psi_{11}^{(k)}}{z}\frac{\mu^2}{a^2}\right] \end{aligned} \tag{4.192}$$

Extremum stresses $z = \pm\frac{h}{2}$, $\eta = 0$ are given as

$$\begin{aligned}
\sigma_1^{(\pm)} &= \sigma_m \pm \sigma_{bs} = \frac{\mu^{*2}}{3} \pm \frac{P^* h^2 C}{6D^* \mu^{*2}} \Phi_1^{(\pm)} \\
\Phi^{\pm} &= \left[\left(1 \pm 2\frac{\mu^2}{a^2}\frac{\psi_{11}^{(\pm)}}{h}\right)\left(1 - \frac{1}{\cosh\mu^*}\right) \pm 2\left(1 + \frac{\mu^2}{a^2}\frac{D_1}{D}\right)\frac{\psi_{11}^{(\pm)}}{h}\frac{\mu^{*2}}{a^2}\right]
\end{aligned} \tag{4.193}$$

where σ_{bs} is a component of the stresses which takes into account bending and shear. It must be emphasised that in both components of the stresses (σ_m and σ_{bs}) the influence of the transverse shear is taken into account.

In the case of the cylindrical bending of the clamped plates we have the following solution

$$\gamma = \frac{P^*}{2\mu^{*2}}\left[1 - \eta^2 + \frac{2(\cosh\mu^*\eta - \cosh\mu^*)}{\mu^* \sinh\mu^*}\right] \tag{4.194}$$

$$\mu^6 - 12P^2\left(\frac{1}{\mu\tanh\mu} - \frac{1}{\tanh^2\mu} + \frac{4}{\mu^2} + \frac{5}{3}\right) = 0 \tag{4.195}$$

$$z^* = z\left[\left(1 + \frac{\mu^2}{a^2}\frac{\psi_{11}^{(k)}}{z}\right)\left(\mu^*\frac{\cosh\mu^*\eta}{\sinh\mu^*} - 1\right) + \left(1 + \frac{\mu^2}{a^2}\frac{D_1}{D}\right)\frac{\psi_{11}^{(k)}}{z}\frac{\mu^{*2}}{a^2}\right] = z\Phi_1^{(k)} \tag{4.196}$$

$$\Phi_1^{(\pm)} = \left[\left(1 \pm 2\frac{\mu^2}{a^2}\frac{\psi_{11}^{(\pm)}}{h}\right)\left(1 - \mu^*\frac{1}{\sinh\mu^*}\right) \pm 2\left(1 + \frac{\mu^2}{a^2}\frac{D_1}{D}\right)\frac{\psi_{11}^{(\pm)}}{h}\frac{\mu^{*2}}{a^2}\right] \tag{4.197}$$

Let us now consider deformation of the cylindrical panel with the curvature k_{11} under a uniformly distributed loading $q_3^+ = q$. Then the system of differential equations (4.155) for the case of transverse shear model may be written as

$$\begin{aligned}
&C\left[u_{,1} + k_{11}w + \frac{1}{2}(w_{,1})^2\right]_{,1} = 0 \\
&Dw_{,1111} + D_1\chi_{,1111} = C\left[u_{,1} + k_{11}w + \frac{1}{2}(w_{,1})^2\right](w_{,11} - k_{11}) + q \\
&D_1 w_{,1111} + D_2\chi_{,1111} = D_1\chi_{,11}
\end{aligned} \tag{4.198}$$

where

$$D_2 = \sum_{k=1}^{n}\int_{c_{k-1}}^{c_k} \lambda_1^{(k)}[\psi_{11}^{(k)}]^2 dz \tag{4.199}$$

For the simply supported cylindrical panel the dimensionless deflection can be obtained in the following form

$$\gamma = \frac{(P^* - k\mu^{*2})}{\mu^{*2}}\left[\left(1 - \frac{\cos\mu^*\eta}{\cos\mu^*}\right)\frac{1}{\mu^{*2}} + \frac{1 - \eta^2}{2}\right] \tag{4.200}$$

where

$$k = k_{11}\frac{a^2}{h}$$

Equation for the computation of the parameter μ^* is given as

$$\left(\frac{P^* + \mu^{*2}k}{\mu^{*2}k}\right)^2 \left(\tan^2\mu^* - \frac{5\tan\mu^*}{4\mu^*} + \frac{1}{6}\mu^{*2} + \frac{5}{4}\right)$$
$$-\left(\frac{P^* + \mu^{*2}k}{\mu^{*2}k}\right)\left(\frac{\tan\mu^*}{\mu^*} - \frac{1}{3}\mu^{*2} - 1\right) + \frac{\mu^{*4}D}{\cosh^2 k^2} = 0 \tag{4.201}$$

Let us consider now an application of the model for which transverse shear and normal deformation are taken into account. For the case of the cylindrical bending of the laminated plates the system of differential equations (4.155) may be written as

$$\begin{aligned} Cu_{1,11} &= -(Cw_{,1}w_{,11} + \overline{\overline{N}}^{(0)}_{11,1}) \\ Dw_{,1111} &= C(u_{,1} + \frac{1}{2}w_{,1}^2) + \overline{\overline{N}}^{(0)}_{11,1} + w_{,11} - \overline{\overline{M}}^{(0)}_{11,11} \end{aligned} \tag{4.202}$$

Following the procedure described above we can rewrite the second equation of this system as

$$w_{,1111} - \lambda^2 w_{,11} = \frac{\overline{\overline{q}}}{D} \tag{4.203}$$

where $\overline{\overline{q}}$ is a quasi load given as

$$\overline{\overline{q}} = q - \overline{\overline{M}}^{(0)}_{11,11} \tag{4.204}$$

In equation (4.204), the bending moment $\overline{\overline{M}}^{(0)}_{11}$ may be considered as a given part of the bending moment, viz.

$$\overline{\overline{M}}^{(0)}_{11} = -w_{0,11}qQ \tag{4.205}$$

where

$$Q = \int_{c_0}^{c_n} \lambda_1^{(k)}\varphi_6^{(k)} z dz$$

Due to the iterative nature of the theory developed, the solution of the equation (4.203) can be found in two steps. For the first step problem must be solved with the load q. Then we obtain

$$w_{0,11} = \frac{q}{D}\frac{\cosh\lambda_0 x}{\cosh\lambda_0 a} = \frac{q}{D}\frac{\cosh\mu_0\eta}{\cosh\mu_0} \tag{4.206}$$

where w_0, λ_0, μ_0 are the parameters of the first iteration. For the second step we have to define quasi-load $\overline{\overline{q}}$ which is given as

$$\overline{\overline{q}} = q\left(1 + \frac{qQ}{D}\frac{\cosh\mu_0\eta}{\cosh\mu_0}\right) \tag{4.207}$$

and then a new value of the load P

$$P = \frac{\overline{\overline{q}}a^4}{Dh} \tag{4.208}$$

for which the problem must be solved once more. Finally dimensionless deflection and stress can be found using equations (4.181) and (4.184), respectively. In the case of the clamped laminated plate the expression for the quasi-load may be written as

$$\bar{\bar{q}} = q\left(1 + \frac{qQ}{D}\frac{\mu_0 \cosh \mu_0 h}{\sinh \mu_0}\right) \tag{4.209}$$

and the final solution can be found using equations (4.185) and (4.187).

4.10 Numerical results

Let us consider some numerical results which are obtained on the basis of closed form solutions derived in the previous section.

Problem 4.1

We consider the problem of cylindrical bending of the laminated plates under uniformly distributed load $q_3^+ = q$ using shear model. The stiffness parameters are taken as $D_1/D = 0,\ 1,\ 10$, where ratio $D_1/D = 0$ corresponds to the case of the classical theory. Plates are assumed to be symmetrically laminated.

The influence of transverse shear and geometrical nonlinearity is shown in Fig. 4.2a, 4.2b for simply supported plates and in Fig. 4.3a, 4.3b for clamped plates, respectively.

It is observed that for a fixed value of dimensionless deflection membrane components of the stresses are much higher than those of bending components due to the geometrical nonlinearity. The influence of the transverse shear is quite substantial and it is even more pronounce for the case of clamped plates.

Problem 4.2

Using transverse shear model let us consider the deformation of the cylindrical panel with the curvature k_{11} under an uniformly distributed load $q_3^+ = q$. The panel has curvature parameter $k = k_{11}a^2/h = 0.75$ and stiffness parameters $D_1/D = 0,\ 5,\ 10$, where $D_1/D = 0$ corresponds to the classical theory.

The effect of the transverse shear on the load/displacement curve is shown in Fig. 4.4 where dimensionless displacement γ plotted against load P.

It is observed that the presence of transverse shear in the geometrically nonlinear analysis of laminated cylindrical panels contributes towards the increase of the global stiffness of the structure.

Problem 4.3

We consider cylindrical bending of the clamped laminated plate for which we will take into account normal deformation. Let us now estimate the influence of the normal deformation due to direct loading $q_3^+ = q$. For this purpose we assume that $\nu_{13}^{(k)} = \nu_{31}^{(k)} = 0$, $\lambda_5^{(k)} = \lambda_6^{(k)} = 0$ and also that the influence of transverse shear is negligible ($G_{13}^{(k)} = \infty$). The influence of the elastic moduli on σ_{bs} is shown in Fig. 4.5 where $E_1/E_3 = \lambda_1(1-\nu^2)/\lambda_3$.

It is observed that dimensionless deflections γ and the bending component of the stresses σ_{bs} increase as the ratio E_1/E_3 increases.

4.11 Conclusions

A new geometrically non–linear higher order theory of laminated plates and shells which takes into account transverse shear and normal deformation is formulated. This theory is based on kinematic hypotheses which are not assumed *a priori* but are derived on the basis of an iterative technique. Geometrical nonlinearity is included at the initial stage of the derivation of the theory when the kinematic hypotheses are formulated.

The proposed theory is capable of treating plates and shells with an arbitrary number and sequence of layers which can differ significantly in their physical and mechanical properties. Various loading and boundary conditions are considered with transverse shear and normal deformation fully taken into account.

The closed form solutions for some particular cases of the system of governing differential equations are obtained. Numerical results illustrate that if the laminae exhibit significant differences in their elastic properties, it is essential to take into account geometrically nonlinear behaviour together with the effect of transverse shear and normal deformation.

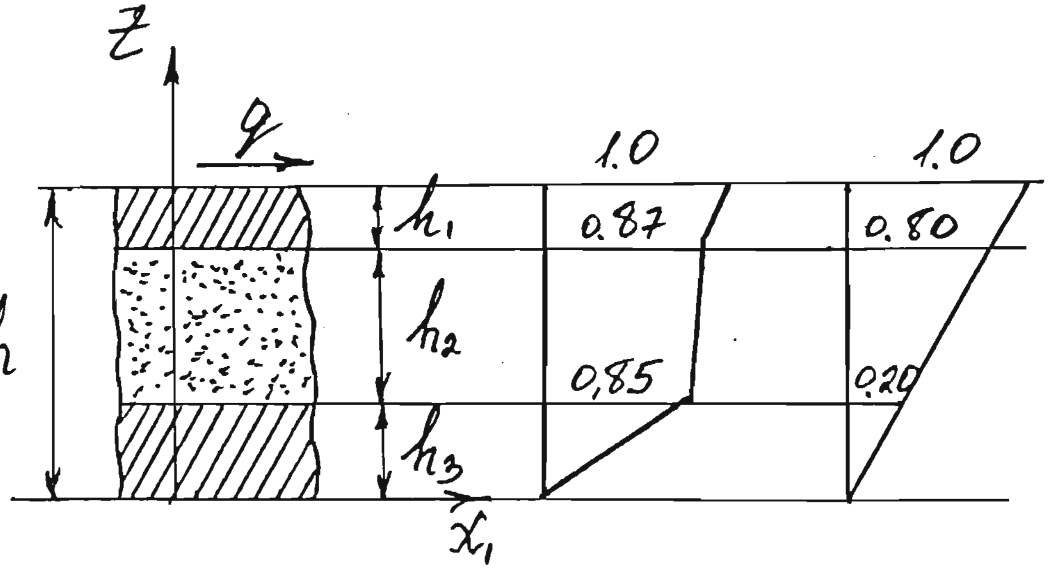
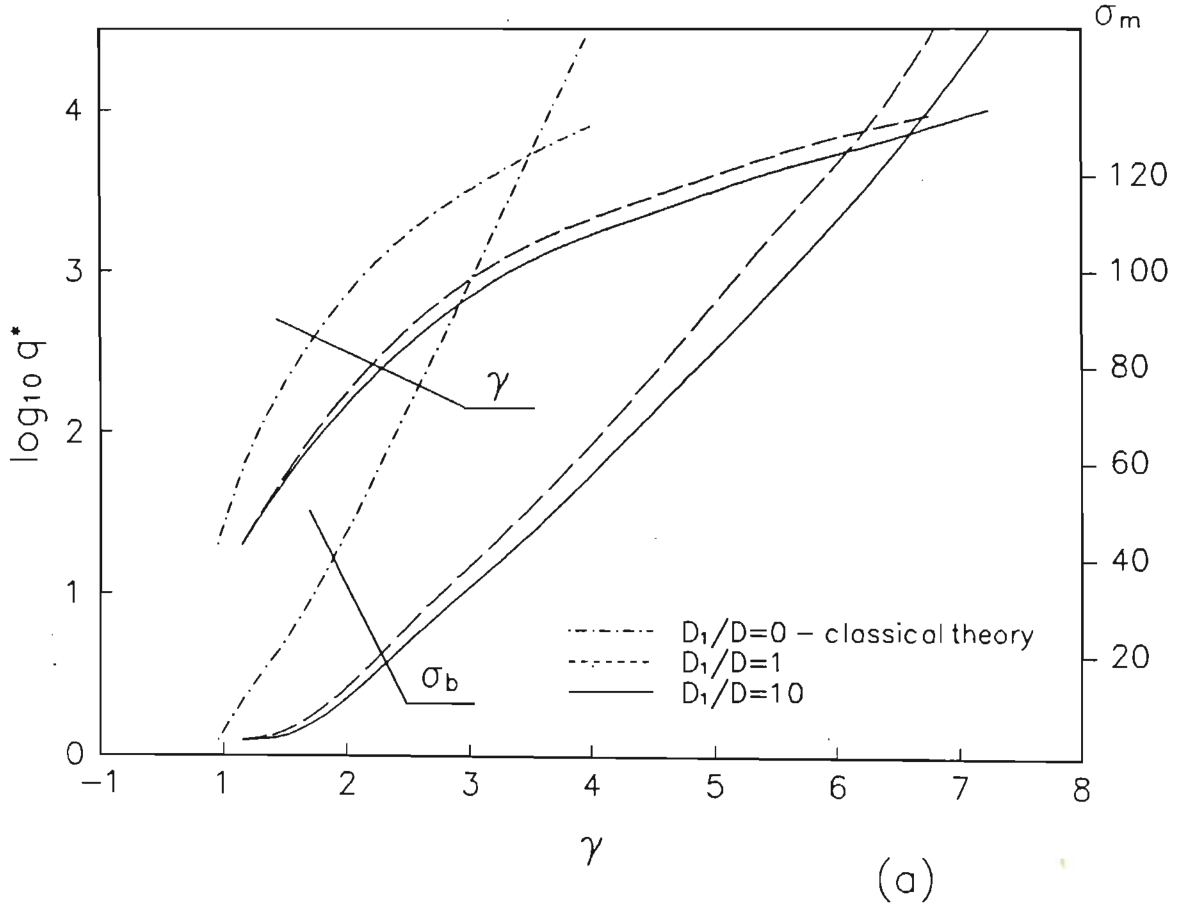


Figure 4.1 Diagram $\sigma_{13}^{(k)}/q$.



σ_m
$\log_{10} q^*$
γ
σ_b
D₁/D=0 – classical theory
D₁/D=1
D₁/D=10
(a)

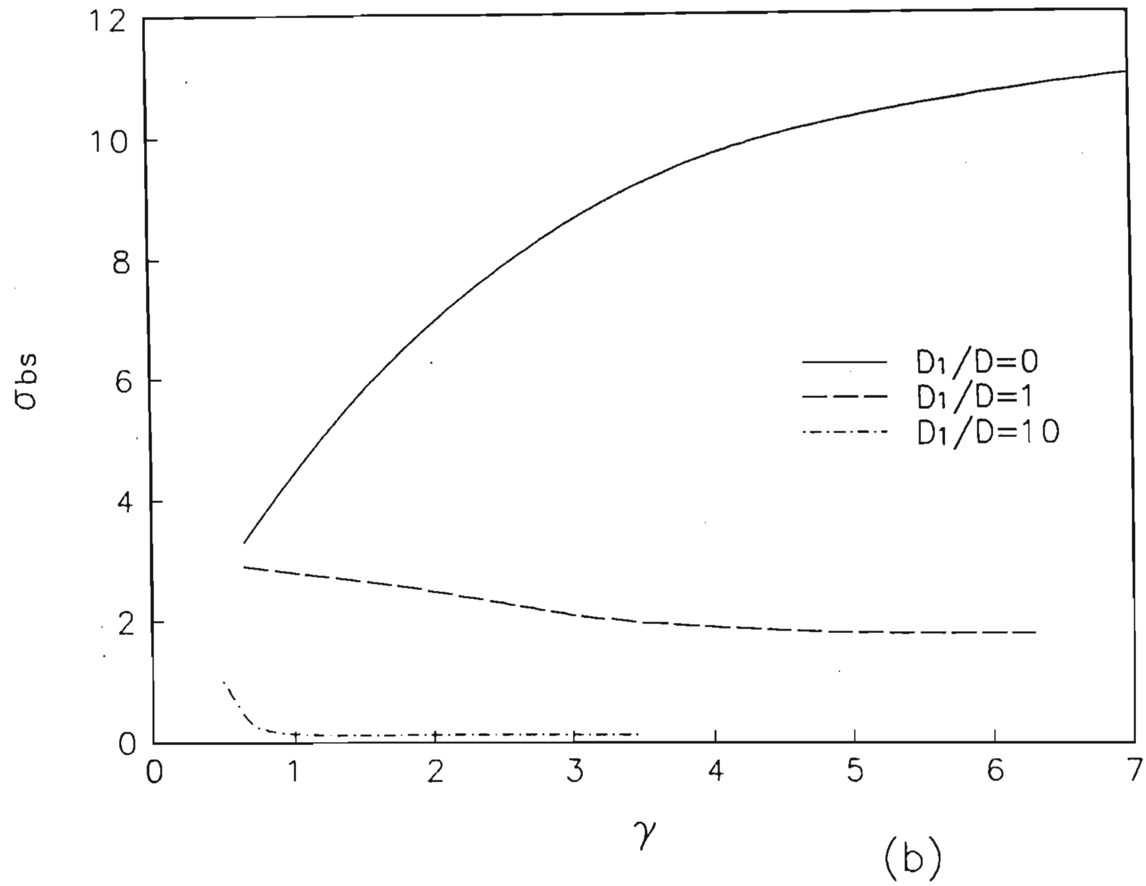
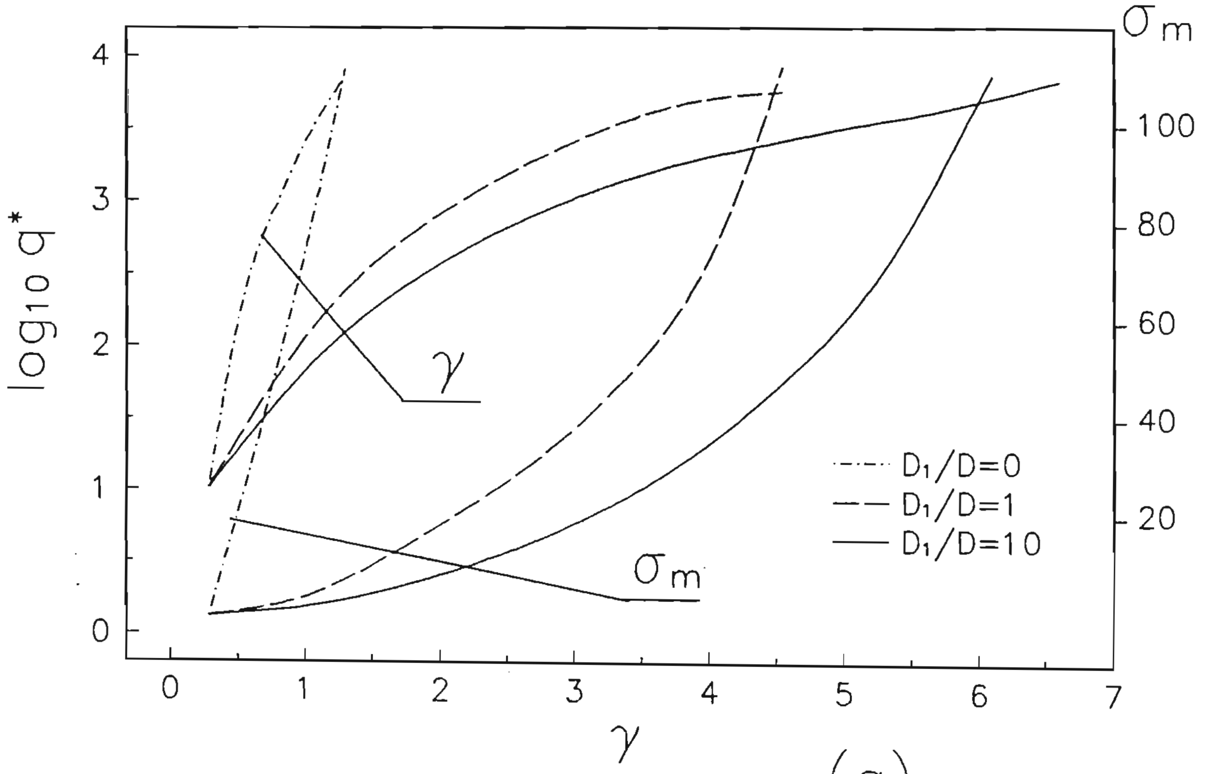


Figure 4.2 Results for simply supported plates with different D_1/D ratios: (a) curves of load and membrane stress components versus deflection; (b) curves of bending–shear stress component versus deflection.



$\log_{10} q^*$
0
1
2
3
4
σ_m
20
40
60
80
100
γ
0
1
2
3
4
5
6
7
γ
σ_m
$D_1/D=0$
$D_1/D=1$
$D_1/D=10$

(a)

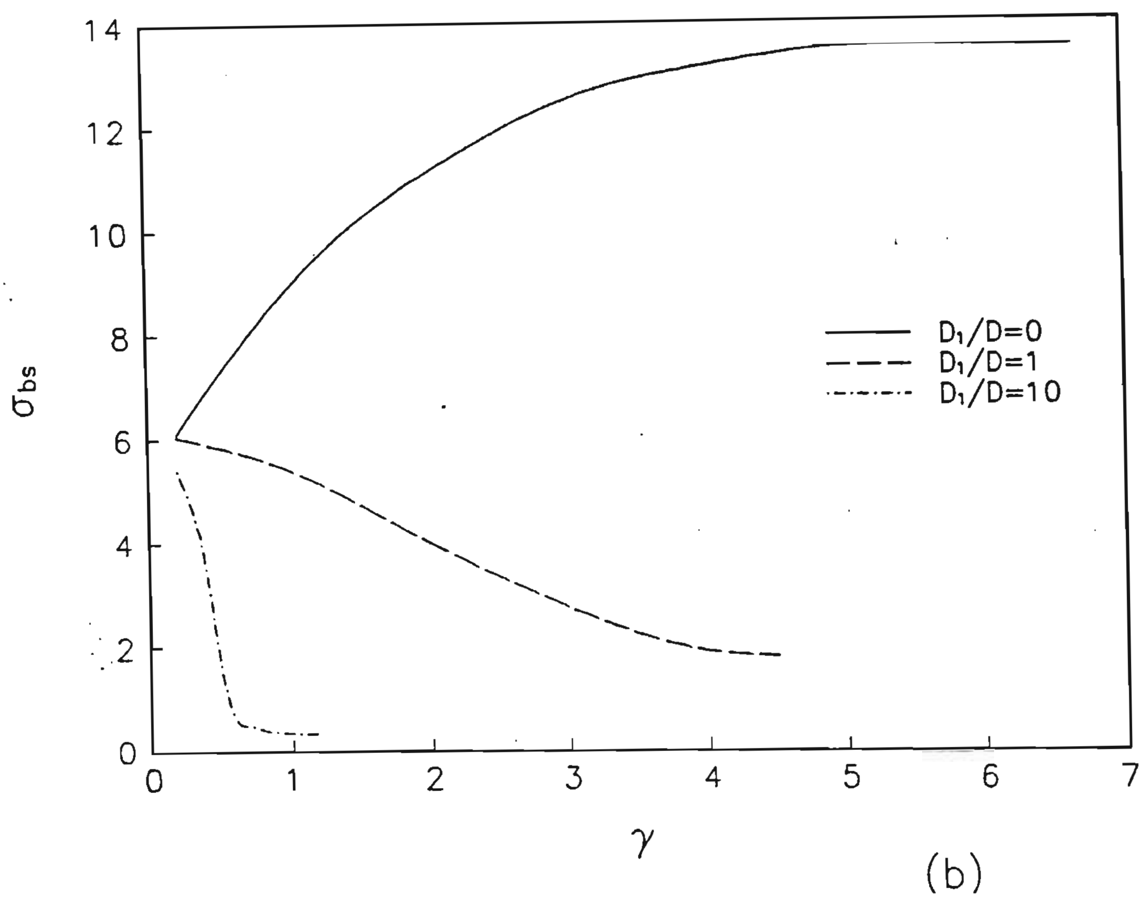


Figure 4.3 Results for clamped plates with different D_1/D ratios: (a) curves of load and membrane stress components versus deflection; (b) curves of bending–shear stress component versus deflection.

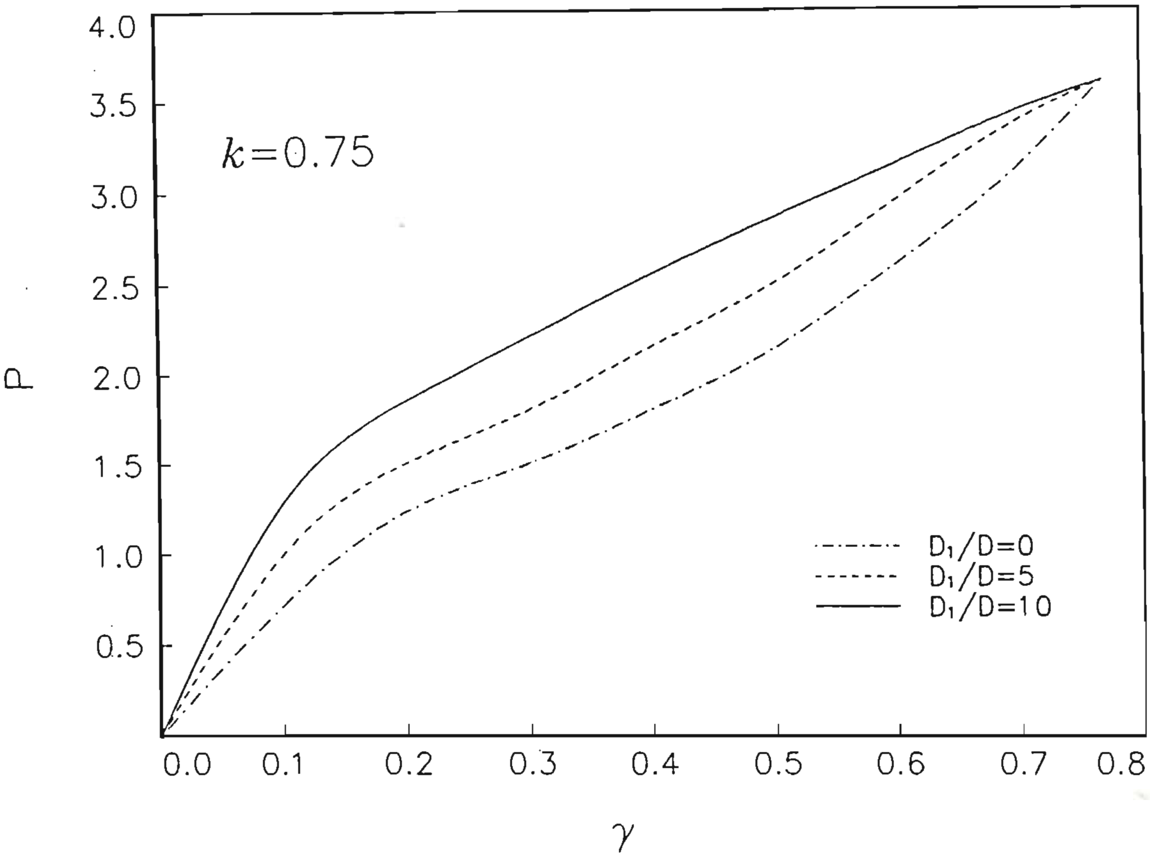


Figure 4.4 Load–deflection curves for simply supported cylindrical panels with different D_1/D ratios.

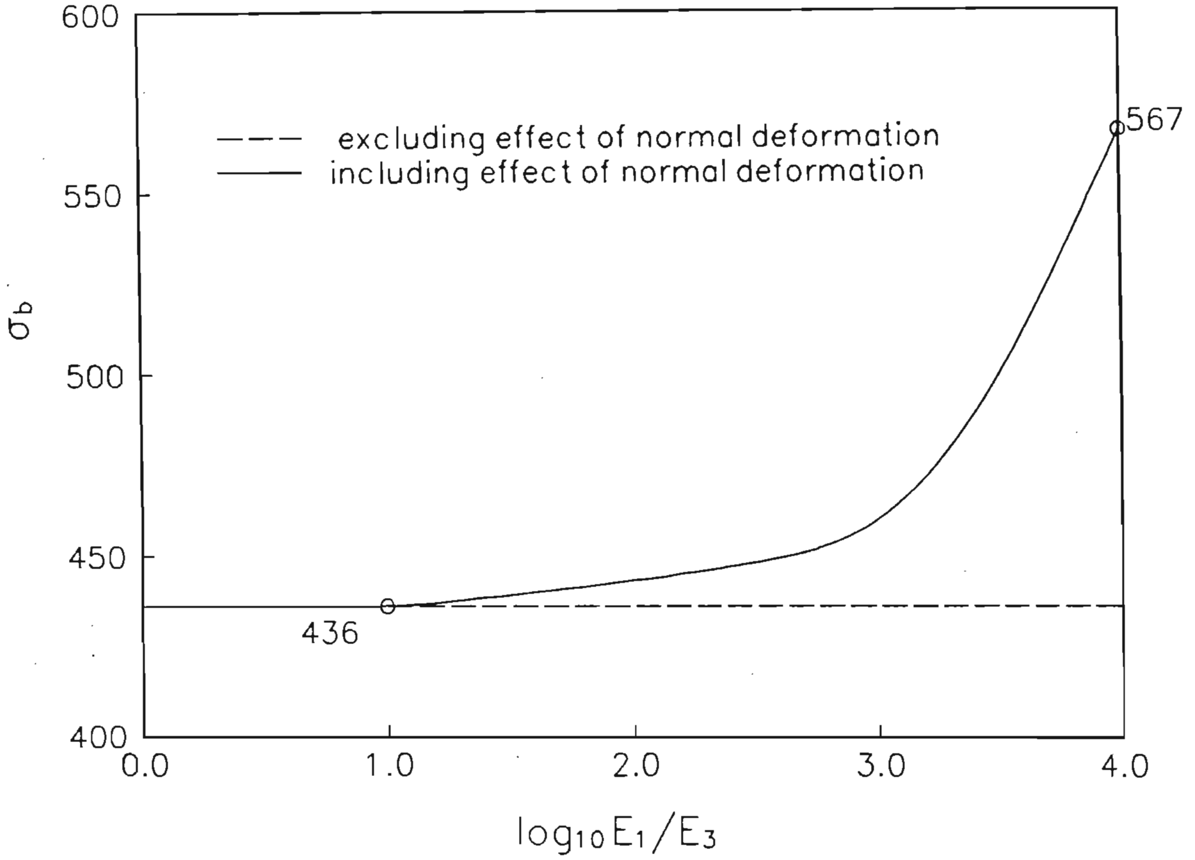


Figure 4.5 Curves of bending stress versus elastic moduli for laminated plates undergoing cylindrical bending.

Chapter 5

Conclusions

5.1 Overview

In the present study exact three–dimensional elasticity theory and higher–order theory for multilayered composite structures are derived. In the first instance, thick laminated cylindrical pressure vessels are considered. For this purpose three–dimensional elasticity solution is derived using stress function approach, where the radial, circumferential and shear stresses are refined with respect to the close ends of the cylindrical shell. In addition, continuously heterogeneous thick laminated cylinders are studied. This theory also takes interlaminar stresses into account.

Based on an accurate three–dimensional stress analysis an approach for the optimal design of the thick pressure vessels is given. Cylindrical pressure vessels are optimised taking the fibre angles as design variables to maximise the burst pressure. The effect of the axial force on the optimal designs is investigated. Numerical results are given for both single and laminated (up to five layers) cylindrical shells. The maximum burst pressure is computed using the three–dimensional interactive Tsai–Wu failure criterion. Design optimisation of multilayered composite pressure vessels are based on the use of robust multidimensional methods which give fast convergence. Three methods are used to determine the optimum ply angles, namely, golden section method, iterative and gradient methods. A brief mathematical analysis is provided for better understanding of the optimisation problem in the multidimensional space.

It must be emphasized that the solutions derived above allow the stress–strain state of the cylindrical shell to be obtained with a high degree of accuracy. Therewith, the generation of an algorithm for the problem is comparatively easy and the computation time required does not exceed a few seconds even for cumbersome tasks. Accurate stress analysis allows the optimal value of the critical load to be obtained. Mathematical setting up of the problem allows other variables of the P_{cr}–function to be easily implemented, for example, layer thicknesses, material properties, number of layers, etc. in various combinations.

Two transverse shear and normal deformation higher order theories for the solution of dynamic and geometrically nonlinear problems are developed. The theories developed are capable of treating multilayered plates and shells with an arbitrary number and sequences of transversely isotropic layers where the layers may differ significantly in their physical and mechanical properties. The higher order theories are based on the kinematic hypotheses which are not assumed *a priori* but are derived on the basis of an iterative technique.

The higher order theory for the solution of dynamic problems differs significantly from other known theories of laminated shells. The direct influence of external loading on the components of stress–strain state are included in the initial stage of the derivation where kinematic hypotheses are formulated. These components are polynomials of higher degree than those in theories where the normal compression is not taken into account or only associated with the Poisson's effect. For transversely isotropic shells the given factors, which account for the three–dimensional character of stress–strain state, assume a significance in connection with the weakness of the layers in the transverse direction. A distinguishing feature of all relations of the model is also mathematical analogy of the terms which are related to the deformation of pure bending, transverse shear and normal compression. Based on the derived system of governing differential equations the model allows to study a broad spectrum of both static and dynamic problems including high–frequency transverse vibrations and stress–strain state of thick laminated plates and shells.

A geometrically nonlinear high–order model of the stress–strain state of laminated orthotropic structures is developed. The distinctive feature of the model is that the finite normal displacements, which are characterised by a kinematic nonlinearity, are taken into consideration not only when the main relations of stress–strain state are derived but also on the stage when hypotheses of the model are formulated.

Based on the model developed the geometrically nonlinear theory of multilayered orthotropic shallow shells is derived. The theory includes the elasticity relations for generalised internal forces which account for geometrically character of the shell deformation. The system of governing equations is represented in a matrix form having linear and nonlinear vector operators of the sought generalised displacements.

The equations derived are suitable for special cases of the theory: transverse–shear laminated shells, thin orthotropic laminated shells (shear and "quasi–orthotropic" models), orthotropic laminated elongated plates (cylindrical bending).

Analytical solutions obtained for some equations of the theory developed offer a clearer view of the qualitative character of the influence of geometrical and kinematical nonlinearity on the stress–strain state of the laminated systems. The new quantitative assessments of the taking into account transverse shear deformation and normal compression depending on physical and mechanical characteristics of layers and load intensity have been obtained. It shows that if the laminae exhibit significant differences in their elastic properties it is essential to take into account geometrically nonlinear behaviour together with the effect of transverse shear and normal deformation.

5.2 Future directions for research

A new generalized nonlinear higher–order theory of laminated composite structures is to be developed. This research is aimed at formulating new mathematical models for the nonlinear stress and strain state of heterogeneous laminated plates and shells which are subject to shear and normal deformation. The main objective of this research is to improve the accuracy of linear and nonlinear modelling of composite materials. As a step toward achieving this goal, the following studies and extensions are proposed:

— accurate estimation of the range of applicability of nonlinear models;

— accurate modelling of interlaminar stresses in order to predict delamination;

— suitability for accurate numerical analysis;

— suitability for thermo–elastic, thermo–plastic and nonlinear dynamic problems.

Development of reliable optimisation methods is essential due to the expanded use of composite shells in high-tech industries. Therefore, there is a need to improve the current optimisation methods and approaches. Development of practical numerical techniques for predicting, in measurable and controllable parameters, the failure initiation and crack propagation in composite shells subjected to different loading conditions is a logical sequel of the design optimization and of paramount importance in studies of the structures.

Implementation of symbolic computation for the derivation of analytical solutions on the basis of higher–order theories is highly recommended. This research could be directed towards obtaining analytical results for the analysis of composite structures and estimating the range of applicability of the generalized higher order theory to be developed. The use of symbolic computation facilitates the implementation of the theory which in turn makes the treatment of a number of related problems (including optimization) possible. Several problems which are cumbersome or exceedingly difficult to solve using conventional techniques could be treated using symbolic computation.

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Appendix

Routines for Multidimensional Optimisation

```
* * * * * * * * * * * * * * * * * * * * * * * * * * *
*  Exact 3-D solution of laminated pressure vessels *
*  made of anisotropic material, stress - strain    *
*  analysis and design optimisation using 3-D     *
*  failure criterion.     *
*     *
* P. Tabakov, University of Natal, Durban     *
*      June, 1994      *
* * * * * * * * * * * * * * * * * * * * * * * * * * *
PROGRAM OPTIM
IMPLICIT none
COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #    P(0:5),X(5),Xpr(5),
     #    Y(5),Ypr(5),SS(5),F,KR(5),NL
COMMON/ANL/THETA(5),NA
C  COMMON/OPT/Xmin
 COMMON/RHML/RHO,ML
REAL*8 THETA,AX,BX,CX,TOL,XMIN,RATIO,Pmax1,Pmax2,PI,RES,
     #        DELT,GOLDEN,H,DH(5),RADIUS(5,25),RHO,Pcr(5,26),PMAX
REAL*8 E1,E2,E3,G12,V12,Rad,P,X,Xpr,Y,Ypr,SS,F,ANGLE_opt(5)
INTEGER*2 NA,KR,NL,J,ML,layer,point,m
        CHARACTER*1 Z
    EXTERNAL ratio

PARAMETER (PI=3.1415926535897932385)
* 1 - Z; 2 - THETA; 3 - RAD.
$DEBUG
          Z = CHAR(7)
          OPEN(6,FILE='CON')
```

```
  CALL DATA
*******************************************************************
***        R A D I I       ***
*******************************************************************

 do m=1,nl
 H=RAD(m)-RAD(m-1)
 dh(m)=H/kr(m)
    end do

  do m=1,nl
     do j=1,kr(m)+1
     radius(m,j)=RAD(m-1)+dh(m)*(j-1)
     end do
     end do

***********************************************************************
 ax=0.00001
 bx=0.872664626
 cx=pi/2.
TOL=0.00001
***********************************************************************

      IF (NL.EQ.1) THEN
       Pmax=1.e09
    do mL=1, nl
       do j=1, kr(ml)+1
   rho = radius(mL,j)/RAD(mL)
 THETA(1)=PI/4.

 NA=1
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
      Pcr(ml,j)=RATIO(XMIN)*(-1.)
     if(Pcr(ml,j).lt.Pmax) then
     Pmax=Pcr(ml,j)
     layer=ml
     point=j
     angle_opt(1)=xmin
     end if
      PRINT 12, radius(ml,j), Pcr(ml,j),THETA(1)*180./PI
  END DO
```

```
  print*, '  '
  END DO
  print 11, Pmax,layer, point
  print 13, angle_opt(1)*180./pi
11 format(1x,'Critical load is ',g14.6,', layer: ',i1,'(',i2,')')
13 format(1x,'Optimum angle:',g14.6/)
        WRITE(6,*) Z,Z,Z
      STOP
      END IF
***************  T W O L A Y E R S  *****************************
      IF (NL.EQ.2) THEN
      Pmax=1.e09
    do mL=1, nl
       do j=1, kr(ml)+1
   rho = radius(mL,j)/RAD(mL)
 THETA(1)=PI/4.
 THETA(2)=PI/4.

222  NA=1
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
      Pmax1=RATIO(XMIN)*(-1.)

 NA=2
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
      Pmax2=RATIO(XMIN)*(-1.)

      DELT=ABS(PMAX2-PMAX1)

    IF(DELT.LE.0.00000001) then
     Pcr(ml,j)=Pmax2
     goto 123
     else
     GOTO 222
     end if
123 continue
    PRINT 12, radius(ml,j), Pcr(ml,j),THETA(1)*180./PI,
     #       THETA(2)*180./PI

     if(Pcr(ml,j).lt.Pmax) then
     Pmax=Pcr(ml,j)
     angle_opt(1)=theta(1)*180./pi
```

```
     angle_opt(2)=theta(2)*180./pi
     layer=ml
     point=j
     end if
    end do
    print*, ' '
    end do
  print 11, Pmax,layer, point
  print 15, angle_opt(1),angle_opt(2)
15 format(1x,'Optimum angles:',2g14.6/)
          WRITE(6,*) Z,Z,Z
     STOP
     END IF
*****************  T H R E E   L A Y E R S***************************
       IF (NL.EQ.3) THEN
       Pmax=1.e09
    do mL=1, nl
       do j=1, kr(ml)+1
   rho = radius(mL,j)/RAD(mL)
 THETA(1)=PI/4.
 THETA(2)=PI/4.
 THETA(3)=PI/4.
333  NA=1
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
      Pmax1=RATIO(XMIN)*(-1.)

 NA=2
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)

 NA=3
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
      Pmax2=RATIO(XMIN)*(-1.)

      DELT=ABS(PMAX2-PMAX1)
    IF(DELT.LE.0.00000001) then
     Pcr(ml,j)=Pmax2
     goto 124
     else
     GOTO 333
     end if
124 continue
```

```
  PRINT 12, radius(ml,j), Pcr(ml,j),THETA(1)*180./PI,
   #       THETA(2)*180./PI,THETA(3)*180./PI
   if(Pcr(ml,j).lt.Pmax) then
   Pmax=Pcr(ml,j)
   angle_opt(1)=theta(1)*180./pi
   angle_opt(2)=theta(2)*180./pi
   angle_opt(3)=theta(3)*180./pi
   layer=ml
   point=j
   end if
  end do
  print*, ' '
  end do
 print 11, Pmax,layer, point
 print 16, angle_opt(1),angle_opt(2),angle_opt(3)
16 format(1x,'Optimum angles:',3g14.6/)
        WRITE(6,*) Z,Z,Z
    STOP
    END IF
******************  F O U R  L A Y E R S  ************************
      IF (NL.EQ.4) THEN
      Pmax=1.e09
   do mL=1, nl
      do j=1, kr(ml)+1
  rho = radius(mL,j)/RAD(mL)
 THETA(1)=PI/4.
 THETA(2)=PI/4.
 THETA(3)=PI/4.
 THETA(4)=PI/4.
444  NA=1
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
      Pmax1=RATIO(XMIN)*(-1.)

 NA=2
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)

 NA=3
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)

 NA=4
      RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
```

```
      Pmax2=RATIO(XMIN)*(-1.)

      DELT=ABS(PMAX2-PMAX1)
    IF(DELT.LE.0.00000001) then
     Pcr(ml,j)=Pmax2
     goto 125
     else
     GOTO 444
     end if
125 continue
    PRINT 12, radius(ml,j), Pcr(ml,j),THETA(1)*180./PI,
     #     THETA(2)*180./PI,THETA(3)*180./PI,THETA(4)*180./PI
     if(Pcr(ml,j).lt.Pmax) then
     Pmax=Pcr(ml,j)
     angle_opt(1)=theta(1)*180./pi
     angle_opt(2)=theta(2)*180./pi
     angle_opt(3)=theta(3)*180./pi
     angle_opt(4)=theta(4)*180./pi
     layer=ml
     point=j
     end if
    end do
    print*, ' '
    end do
  print 11, Pmax,layer, point
  print 17, angle_opt(1),angle_opt(2),angle_opt(3),angle_opt(4)
17 format(1x,'Optimum angles:',4g14.6/)
        WRITE(6,*) Z,Z,Z
     STOP
     END IF
****************  F I V E  L A Y E R S *************************
      IF (NL.EQ.5) THEN
      Pmax=1.e09
    do mL=1, nl
       do j=1, kr(ml)+1
   rho = radius(mL,j)/RAD(mL)
         THETA(1)=PI/4.
         THETA(2)=PI/4.
         THETA(3)=PI/4.
         THETA(4)=PI/4.
         THETA(5)=PI/4.
```

```
555  NA=1
       RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
       Pmax1=RATIO(XMIN)*(-1.)

  NA=2
       RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)

  NA=3
       RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)

  NA=4
       RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)

  NA=5
       RES=GOLDEN(AX,BX,CX,RATIO,TOL,XMIN)
       Pmax2=RATIO(XMIN)*(-1.)

       DELT=ABS(PMAX2-PMAX1)
     IF(DELT.LE.0.00000001) then
      Pcr(ml,j)=Pmax2
      goto 127
      else
      GOTO 555
      end if
127 continue
     PRINT 12, radius(ml,j), Pcr(ml,j),THETA(1)*180./PI,
      #      THETA(2)*180./PI,THETA(3)*180./PI,THETA(4)*180./PI,
      #      THETA(5)*180./PI
      if(Pcr(ml,j).lt.Pmax) then
      Pmax=Pcr(ml,j)
      angle_opt(1)=theta(1)*180./pi
      angle_opt(2)=theta(2)*180./pi
      angle_opt(3)=theta(3)*180./pi
      angle_opt(4)=theta(4)*180./pi
      angle_opt(5)=theta(5)*180./pi
      layer=ml
      point=j
      end if
     end do
     print*, ' '
     end do
```

```
 print 11, Pmax,layer, point
 print 18,angle_opt(1),angle_opt(2),angle_opt(3),angle_opt(4),
   #    angle_opt(5)
18 format(1x,'Optimum angles:',5g14.6/)
        WRITE(6,*) Z,Z,Z
     STOP
     END IF
12 FORMAT(1X,10G14.6)
 stop
 end
*********************************************************************

        Function ratio(ANGLE)
        IMPLICIT NONE
        COMMON/RT/CRITICAL
COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #            P(0:5),X(5),Xpr(5),
     #            Y(5),Ypr(5),SS(5),F,KR(5),NL
COMMON/COMPLEX/A(5,4,4),BETA11(5),BETA22(5),BETA14(5),
     #   BETA44(5),BETA24(5),BETA12(5),K(5),G1(5),
     #   GK(5),G_K(5),C(5),KAPPA1(5),KAPPA2(5)
COMMON/COMPL_SYST_CF/CM(5),U1(5),U2(5),F1(5),F2(5)
        COMMON/SY/DELTA1m,DELTA2m,LAMBDAm,LAMBDAm1,
     #           DELTA1m1,DELTA2m1
COMMON/Z_CF/Z1(5),Z2(5),Z3(5),L
COMMON/INT_CN/ CONST
COMMON/N_COEF/N1(4),Nc(4)
        COMMON/ANL/THETA(5),NA
C        COMMON/OPT/Xmin
        REAL*8 E1,E2,E3,G12,V12,RAD,P,THETA,X,Xpr,Y,Ypr,
     #         SS,F,A,BETA11,BETA22,BETA44,
     #        BETA14,BETA24,K,G1,GK,G_K,C,KAPPA1,KAPPA2,beta12
        REAL*8 CM,U1,U2,F1,F2,Z1,Z2,Z3,L,DELTA1m(4),DELTA2m(4),
     #          LAMBDAm(4),DELTA1m1(4),DELTA2m1(4),LAMBDAm1(4),
     #          CONST,N1,Nc,ANGLE,RATIO,CRITICAL,STEP,XTOL,FTOL
        INTEGER*2 KR,NL,NA,I,NLIM
        EXTERNAL FAILURE

 $DEBUG
************************************************************************
         theta(NA)=angle
```

```
************************************************************************
          CALL COMPL_CF

****************************************************************
*** Iterative procedure, Muller's method.
*** External function is failure criterion.

        p(0) = 500.0
        STEP = 490.0
        XTOL = 0.000001
        FTOL = 0.000001
        NLIM = 100
        I = 0

        CALL MULLR(Failure,p(0),STEP,XTOL,FTOL,NLIM,I)
****************************************************************
            RATIO=p(0)*(-1.)
                return
END

****************************************************************

        SUBROUTINE COMPL_CF
        IMPLICIT NONE
        COMMON/ANL/THETA(5),FLAG
        COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #           P(0:5),X(5),Xpr(5),
     #           Y(5),Ypr(5),SS(5),F,KR(5),NL
        COMMON/COMPLEX/A(5,4,4),BETA11(5),BETA22(5),BETA14(5),
     #           BETA44(5),BETA24(5),BETA12(5),K(5),G1(5),
     #           GK(5),G_K(5),C(5),KAPPA1(5),KAPPA2(5)
        COMMON/COMPL_SYST_CF/CM(5),U1(5),U2(5),F1(5),F2(5)
        REAL*8 E1,E2,E3,G12,V12,RAD,P,THETA,X,Xpr,Y,Ypr,
     #       SS,F,CN,S,C2,C3,C4,S2,S3,S4,A,BETA11,BETA22,BETA44,
     #       BETA14,BETA24,K,G1,GK,G_K,C,KAPPA1,KAPPA2,pi,beta12
        REAL*8 CM,U1,U2,F1,F2

        INTEGER*2 KR,NL,M,flag
        PARAMETER (PI=3.1415926535897932385)
$DEBUG
* 1 - Z; 2 - THETA; 3 - RAD.
```

```
$DEBUG


         DO M =1, NL
***     COMPONENTS OF COMPLIANCE MATRIX   ***
           cn=cos(THETA(M))
           s=sin(THETA(M))
           c2=cn*cn
           c3=c2*cn
           c4=c2*c2
           s2=s*s
           s3=s2*s
           s4=s2*s2
           a(m,3,3)=c4/E3(M)+(1./G12(M)-2.*v12(m)/E3(M))*s2*c2+s4/E2(M)
           a(m,2,2)=s4/E3(M)+(1./G12(M)-2.*v12(m)/E3(M))*s2*c2+c4/E2(M)
           a(m,2,3)=(1./E3(M) +1./E2(M)+2.*v12(m)/E3(M)-
     #              1./G12(M))*s2*c2-v12(m)/E3(M)
           a(m,4,4)=4.*(1./E3(M)+1./E2(M)+2.*v12(m)/E3(M)-
     #              1./G12(M))*s2*c2+1./G12(M)
           a(m,3,4)=(2.*(s2/E2(M)-c2/E3(M))+(1./G12(M)-
     #              2.*v12(m)/E3(M))*(c2-s2))*s*cn
           a(m,2,4)=(2.*(c2/E2(M)-s2/E3(M))-(1./G12(M)-
     #              2.*v12(m)/E3(M))*(c2-s2))*s*cn
           a(m,1,3)=-(v12(m)*s2/E2(M)+v12(m)*c2/E3(M))
           a(m,1,2)=-(v12(m)*c2/E2(M)+v12(m)*s2/E3(M))
           a(m,1,1)=1./E1(M)
           a(m,1,4)=2.*(v12(m)/E3(M)-v12(m)/E2(M))*s*cn
           a(m,3,1)=a(m,1,3)
           a(m,2,1)=a(m,1,2)
           a(m,4,1)=a(m,1,4)
           a(m,4,2)=a(m,2,4)
           a(m,4,3)=a(m,3,4)
           a(m,3,2)=a(m,2,3)
*************************************************************
         beta11(m)=a(m,1,1)-(a(m,1,3)*a(m,1,3))/a(m,3,3)
         beta44(m)=a(m,4,4)-(a(m,3,4)*a(m,3,4))/a(m,3,3)
         beta14(m)=a(m,1,4)-(a(m,1,3)*a(m,3,4))/a(m,3,3)
         beta22(m)=a(m,2,2)-(a(m,2,3)*a(m,2,3))/a(m,3,3)
         beta24(m)=a(m,2,4)-(a(m,2,3)*a(m,3,4))/a(m,3,3)
         beta12(m)=a(m,1,2)-(a(m,1,3)*a(m,2,3))/a(m,3,3)
         k(m)=SQRT((beta11(m)*beta44(m)-beta14(m)*beta14(m))/
```

```
     #        (beta22(m)*beta44(m)-beta24(m)*beta24(m)))
        g1(m)=(beta14(m)+beta24(m))/beta44(m)
        gk(m)=(beta14(m)+k(m)*beta24(m))/beta44(m)
        g_k(m)=(beta14(m)-k(m)*beta24(m))/beta44(m)
        c(m)=Rad(m-1)/Rad(m)
***********************************************************
       kappa1(m)=((a(m,1,3)-a(m,2,3))*beta44(m)-a(m,3,4)*(beta14(m)-
     #  beta24(m)))/(beta22(m)*beta44(m)-beta24(m)**2-(beta11(m)*
     #               beta44(m)-beta14(m)**2))

       kappa2(m)=((a(m,1,3)-a(m,2,3))*(beta14(m)+beta24(m))-a(m,3,4)*
     #          (beta11(m)-beta22(m)))/(beta22(m)*beta44(m)-
     #           beta24(m)**2-(beta11(m)*beta44(m)-beta14(m)**2))
********************************************************************
        CM(m) = 1.-c(m)**(2.*k(m))
        U1(m)=(1.-c(m)**(k(m)+1.))/CM(m)
        U2(m)=(1.-c(m)**(k(m)-1.))/CM(m)
        f1(m) =  c(m)**(k(m)+1.)
        f2(m) =  c(m)**(k(m)-1.)
       END DO
        RETURN
        END
*********************************************************************

        SUBROUTINE CONST1
        IMPLICIT NONE
        COMMON/INT_CN/ CONST
        COMMON/Z_CF/Z1(5),Z2(5),Z3(5),L
        COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #           P(0:5),X(5),Xpr(5),
     #           Y(5),Ypr(5),SS(5),F,KR(5),NL
        REAL*8 E1,E2,E3,G12,V12,Rad,P,THETA,X,Xpr,
     #       Y,Ypr,SS,F,Z1,Z2,Z3,L
        REAL*8 CONST
        INTEGER*2 NL,KR

        CONST= -((-L+Z2(1)*P(0)+Z3(1)*P(1))/Z1(1))

         RETURN
         END
************************************************************************
```

```
      SUBROUTINE CONST_CF
      IMPLICIT NONE
      COMMON/COMPLEX/A(5,4,4),BETA11(5),BETA22(5),BETA14(5),
     #           BETA44(5),BETA24(5),BETA12(5),K(5),G1(5),
     #           GK(5),G_K(5),C(5),KAPPA1(5),KAPPA2(5)
      COMMON/COMPL_SYST_CF/CM(5),U1(5),U2(5),F1(5),F2(5)
      COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #           P(0:5),X(5),Xpr(5),
     #           Y(5),Ypr(5),SS(5),F,KR(5),NL
      COMMON/Z_CF/Z1(5),Z2(5),Z3(5),L
      REAL*8 A,BETA11,BETA22,BETA14,BETA44,BETA24,BETA12,K,G1,
     #       GK,G_K,C,KAPPA1,KAPPA2,X,Xpr
      REAL*8 A1,A2,B1,B2,B3,Z1,Z2,Z3,L,PI,RAD,P,E1,E2,E3,G12,
     #       V12,Y,Ypr,SS,F,U1,U2,F1,F2,CM
      INTEGER*2 KR,NL,M
      PARAMETER (PI=3.1415926535897932385)

         DO m=1,NL
         A1=rad(m)**2-rad(m)**(k(m)+1.)*rad(m-1)**(1.-k(m))
         A2=rad(m)**2-rad(m)**(1.-k(m))*rad(m-1)**(k(m)+1.)

         B1=a(m,1,3)-a(m,3,4)*g_k(m)-a(m,2,3)*k(m)
         B2=a(m,1,3)-a(m,3,4)*gk(m)+a(m,2,3)*k(m)
         B3=-a(m,1,3)+a(m,3,4)*g_k(m)+a(m,2,3)*k(m)

         Z1(m)=0.5*(rad(m)**2-rad(m-1)**2)*(1-(1./a(m,3,3))*
     #          (kappa1(m)*(a(m,1,3)+a(m,2,3))-kappa2(m)*a(m,3,4)))+
     $   ((U2(m)*c(m)**(k(m)+1.)*kappa1(m))/(a(m,3,3)*(1.-k(m))))*
     $   A1*B1+((U1(m)*kappa1(m))/(a(m,3,3)*(1.+k(m))))*
     $   A2*B2

         Z2(m)=((c(m)**(1.+k(m)))/(CM(m)*a(m,3,3)*(1.-k(m))))*
     $   A1*B1-((c(m)**(1.+k(m)))/(CM(m)*a(m,3,3)*(1.+k(m))))*
     $   A2*B2

         Z3(m)=((c(m)**(2.*k(m)))/(CM(m)*a(m,3,3)*(1.-k(m))))*
     $     A1*B3+(1/(CM(m)*a(m,3,3)*(1.+k(m))))*A2*B2
          end do

         L= 0.5*(p(0)-p(nl))*rad(0)**2+F/(2.*Pi)
```

```
      RETURN
      END
*********************************************************************

      SUBROUTINE CONST_ML
      IMPLICIT NONE
      COMMON/N_COEF/N1(4),Nc(4)
      COMMON/INT_CN/ CONST
      COMMON/Z_CF/Z1(5),Z2(5),Z3(5),L
      COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #            P(0:5),X(5),Xpr(5),
     #            Y(5),Ypr(5),SS(5),F,KR(5),NL
      REAL*8 E1,E2,E3,G12,V12,Rad,P,X,Xpr,
     #        Y,Ypr,SS,F,Z1,Z2,Z3,L
      REAL*8 CONST,N1,Nc,S2,SZ1,S1
      INTEGER*2 NL,INX,KR,M

      S2=0.
      SZ1=0.
      S1=0.
c       print*,n1(1)
c       print*,nc(1)
c       pause
      DO M=1,NL-1
      S1=S1+N1(m)*(Z2(M+1)+Z3(M))
      S2=S2+NC(M)*(Z2(M+1)+Z3(M))
      END DO

      DO M=1,NL
      SZ1=SZ1+Z1(m)
      END DO
*************************************************************
      CONST=-((-L+Z2(1)*p(0)+Z3(NL)*p(NL)+S1)/
     /        (SZ1+S2))
*************************************************************
       RETURN
       END
*************************************************************

      INTERFACE TO FUNCTION SYSTEM[C] (STRING)
```

```
      INTEGER*2 SYSTEM
      CHARACTER*1 STRING[REFERENCE]
      END
      SUBROUTINE DATA
**********************************************************************
*      NL - NUMBER OF LAYERS                                           *
*      E1 - MODULUS OF ELASTICITY IN R-DIRECTION                       *
*      E2 - MODULUS OF ELASTICITY IN THETA-DIRECTION                   *
*      E3 - MODULUS OF ELASTICITY IN Z-DIRECTION                       *
*      V12 - POISSON'S RATIO                                           *
*      A - RADII OF A CYLINDER                                         *
*      P - PRESURE                                                     *
*      F -AXIAL FORCE                                                  *
*      THETA - INITIAL ANGLE                                           *
*      D_THETA - DELTA_THETA                                           *
**********************************************************************
              IMPLICIT NONE
      COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),A(0:5),
     #           P(0:5),X(5),Xpr(5),
     #           Y(5),Ypr(5),S(5),F,KR(5),NL
      REAL*8 E1,E2,E3,G12,v12,A,P,F,X,Xpr,Y,Ypr,S
      INTEGER*2 SYSTEM,CLEAR,NL,I,KR
      CHARACTER*62 fname
$debug
      CLEAR = SYSTEM('CLS'C)
      WRITE(*,900)
900   FORMAT(/////10X,' INPUT FILE: '\)
      READ (*,910) fname
      CLEAR = SYSTEM('CLS'C)
910   FORMAT(A)

      OPEN(unit = 6,FILE= FNAME, status='OLD')
            READ(6,'(/I4)')NL
      if(nl.lt.1.or.nl.gt.5) then
       print*,' You''re wrong about number of layers !'
      stop
      end if
      READ(6,'(/3G15.6)')  (E1(I),E2(I),E3(I),I=1,NL)
      READ(6,'(/10G15.6)') (G12(I),I=1,NL)
      READ(6,'(/10G15.6)') (v12(I),I=1,NL)
      READ(6,'(/11G12.5)') (A(I),I=0,NL)
```

```
      READ(6,'(/11G12.5)') (P(I),I=0,NL)
      READ(6,'(/G12.5)') F
      READ(6,'(/10I4)') (KR(I),I=1,NL)
      READ(6,'(/10G12.5)') (X(I),I=1,NL)
      READ(6,'(/10G12.5)') (Xpr(I),I=1,NL)
      READ(6,'(/10G12.5)') (Y(I),I=1,NL)
      READ(6,'(/10G12.5)') (Ypr(I),I=1,NL)
      READ(6,'(/10G12.5)') (S(I),I=1,NL)
      CLOSE(unit=6,status='KEEP')
      RETURN
      END
*********************************************************************

      REAL*8 FUNCTION FAILURE(xP)
      IMPLICIT NONE
      COMMON/RHML/RHO,M
      COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),A(0:5),
     #           P(0:5),X(5),Xpr(5),
     #           Y(5),Ypr(5),S(5),F,KR(5),NL
************************************************************************
      COMMON/FCNST/Frr(5),Fzz(5),Fss(5),Fr(5),Fz(5),Frz(5),Frt(5)
      COMMON/STRTR/Sgr,Sg1,Sg2,Tau12
************************************************************************
      REAL*8 Frr,Fzz,Fss,Fr,Fz,Frz,Frt,SGR,SG1,SG2,TAU12,RHO
      real*8 p,e1,e2,e3,g12,v12,a,x,xpr,y,ypr,s,f,kr,xp
      INTEGER*2 M,nl
      p(0)=xp
$DEBUG

  CALL CONST_CF
        IF (NL.EQ.1) THEN
        CALL CONST1
        ELSE
        CALL SYST_CF
        CALL N_CF
  CALL CONST_ML
         END IF
         CALL STRESS
****************************************************************************
      FAILURE=(Fzz(M)*SG1*SG1+Frr(M)*(SGR*SGR+SG2*SG2)+
     2           Fss(M)*TAU12*TAU12+2.*Frz(M)*(SGR+SG2)*SG1+
```

```
     3  2.*Frt(M)*SGR*SG2) +(Fr(M)*(SGR+SG2)+Fz(M)*SG1)-1.
***************************************************************************
                RETURN
END
*******************************************************************************

      FUNCTION GOLDEN(AX,BX,CX,F,TOL,XMIN)
       IMPLICIT NONE
       REAL*8 AX,BX,CX,F,TOL,XMIN,X1,X2,R,C,X0,X3,F1,F2,F0,F3
      PARAMETER (R=.61803399,C=.38196602)
      X0=AX
      X3=CX
      IF(ABS(CX-BX).GT.ABS(BX-AX))THEN
        X1=BX
        X2=BX+C*(CX-BX)
      ELSE
        X2=BX
        X1=BX-C*(BX-AX)
      ENDIF
      F1=F(X1)
      F2=F(X2)
1     IF(ABS(X3-X0).GT.TOL*(ABS(X1)+ABS(X2)))THEN
        IF(F2.LT.F1)THEN
          X0=X1
          X1=X2
          X2=R*X1+C*X3
          F0=F1
          F1=F2
          F2=F(X2)
        ELSE
          X3=X2
          X2=X1
          X1=R*X2+C*X0
          F3=F2
          F2=F1
          F1=F(X1)
        ENDIF
      GOTO 1
      ENDIF
      IF(F1.LT.F2)THEN
        GOLDEN=F1
```

```
        XMIN=X1
      ELSE
        GOLDEN=F2
        XMIN=X2
      ENDIF
      RETURN
      END
*********************************************************************
C -------------------------------------------------------------
C
        SUBROUTINE MULLR(FCN,XR,H,XTOL,FTOL,NLIM,I)
        IMPLICIT NONE
C-------------------------------------------------------------
C
C       SUBROUTINE MULLR:
C                      This subroutine finds the root of F(X) = 0 by
C                      quadratic interpolation on three points -
C                      Muller's method.
C-----------------------------------------------------------------
C       PARAMETERS ARE:
C
C        FCN -      Function that computes values for f(x), must be
c                   declared external in calling program. it has one
c                   argument, X.
C        XR  -      Initial approximation to the root. Used to begin
C                   iterations. also returns the value of the root.
C        H   -      Displacement from x used to begin calculations.
C                   The first quadratic is fitted at F(X),F(X+H),F(X-H).
C        XTOL,FTOL -Tolerance values for X, F(X) to terminate iterations.
C         I   -     A signal for how routine terminated.
C         I = 1   Meets tolerance for x values.
C         I = 2   Meets tolerance for F(X).
C         I = -1  NLIM exceeded.
C
C    When subroutine is called, the value of I indicates whether to
C    print each value or not. I=0 means print them, I.NE.0 means don't.
C
C-----------------------------------------------------------------
C
        REAL*8 FCN,XR,H,XTOL,FTOL
```

```
      INTEGER*2 NLIM,I,J
      REAL*8 X0,X1,X2,F0,F1,F2,H1,H2,G,A,B,C,DISC,FR,DELX
      CHARACTER*1 Z

      Z = CHAR(7)
      open(6,file='con')
C
C-----------------------------------------------------------------
C
C     Set initial values
C
$DEBUG
      X0 = XR
      X1 = XR + H
      X2 = XR - H
c      print*,'x0=',x0
c      print*,'x1=',x1,' x2=',x2
          F1 = FCN(X1)
          F2 = FCN(X2)
C          print*,'f1 @ f2',f1,f2
C          pause
      IF((F2 .LT. 0.0) .AND. (F1 .GT. 0.0)) THEN
      CONTINUE
          ELSE
          WRITE(6,*) Z,Z,Z
           STOP ' Function has the same sign at X1 & X2! '
      END IF
C
C     Begin iterations
C
      DO 20 J=1,NLIM
         F0 = FCN(X0)
         F1 = FCN(X1)
         F2 = FCN(X2)
         H1 = X1 - X0
         H2 = X0 - X2
         G = H2/H1
         A = (F1*G - F0*(1.0+G) + F2)/(G*H1*H1*(1.0 + G))
         B = (F1 - F0 - A*H1*H1)/H1
         C = F0
         DISC = SQRT(B*B - 4.0*A*C)
```

```
          IF (B .LT. 0.0) DISC = -DISC
C
C      Find root of quadratic: A * V^2 + B*V + C = 0
C
       DELX =  (2.0*C)/(B + DISC)
C
C      UPDATE XR
C
       XR = X0 - DELX
       FR = FCN(XR)
c       IF (I .EQ. 0) PRINT 199, J,XR,FR
C
C-----------------------------------------------------------------
C
C      Check stopping criteria
C
       IF (ABS(DELX) .LE. XTOL) THEN
       I=1
c       PRINT 202, J,XR,FR
       RETURN
       END IF
C
       IF (ABS(FR) .LE. FTOL) THEN
       I = 2
c       PRINT 203, J,XR,FR
       RETURN
       END IF
C
C---------------------------------------------------------------------
C
C  Select the three points for the next iteration. When XR .GT. X0, choose
C  X0, X1 & XR, BUT WHEN XR .LT. 0 CHOOSE X0, X2, & XR.
C
       IF (XR .LT. X0) THEN
       X1 = X0
       X0 = XR
        ELSE IF (XR .GT. X0) THEN
        X2 = X0
        X0 = XR
       END IF
20     CONTINUE
```

```
C
C-------------------------------------------------------------------
C
C   When loop is normally terminated, NLIM is exceeded
C
        I = -1
        PRINT 200, NLIM,XR,FR
        RETURN
C
C---------------------------------------------------------------------
C
199     FORMAT(' At iteration ',I3,3X,' X =',E12.5,4X,'F(X) =',E12.5)
200     FORMAT(/' Tolerance not met after ',I4,' iterations X = ',
     +         E12.5,' F(X) =',E12.5)
202     FORMAT(/' Tolerance met in ',I2, ' iterations X = ',E12.5,
     +        '  F(X) =' E12.5)
203     FORMAT(/' F tolerance met in ',I4,' iterations X =',E12.5,
     +         ' F(X) =' E12.5)
        END
*******************************************************************

        SUBROUTINE N_CF
        IMPLICIT NONE
        COMMON/N_COEF/N1(4),Nc(4)
        COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #            P(0:5),X(5),Xpr(5),
     #            Y(5),Ypr(5),SS(5),F,KR(5),NL
        COMMON/SY/DELTA1m(4),DELTA2m(4),LAMBDAm(4),LAMBDAm1(4),
     #           DELTA1m1(4),DELTA2m1(4)
        REAL*8 N1,Nc,E1,E2,E3,G12,V12,Rad,P,X,Xpr,
     #        Y,YPR,SS,F,DELTA1m,DELTA2m,LAMBDAm,
     $        DELTA1m1,DELTA2m1,LAMBDAm1,b(4,4),Det
        INTEGER*2 KR,NL
****************  2  L A Y E R S  *************************************
         IF (NL.EQ.2) THEN
         N1(1)= -((LAMBDAm(1)*p(0)+LAMBDAm1(2)*p(2))/
     /         (DELTA2m(1)+DELTA2m1(2)))
         Nc(1)=-((DELTA1m(1)-DELTA1m1(2))/(DELTA2m(1)+DELTA2m1(2)))
         END IF
****************  3  L A Y E R S  *************************************
        IF (NL.EQ.3) THEN
```

```
      b(1,1)=delta2m(1)+delta2m1(2)
      b(1,2)=lambdam1(2)
      b(2,1)=lambdam(2)
      b(2,2)=delta2m(2)+delta2m1(3)
      Det=b(1,1)*b(2,2)-b(1,2)*b(2,1)
      N1(1)=(-p(0)*lambdam(1)*b(2,2)+p(3)*lambdam1(3)*b(1,2))/Det
      Nc(1)=(b(1,2)*(delta1m(2)-delta1m1(3))-b(2,2)*(delta1m(1)-
     #        delta1m1(2)))/Det
      N1(2)=(-p(3)*lambdam1(3)*b(1,1)+p(0)*lambdam(1)*b(2,1))/Det
      Nc(2)=(b(2,1)*(delta1m(1)-delta1m1(2))-b(1,1)*(delta1m(2)-
     #        delta1m1(3)))/Det
      END IF
**************** 4 L A Y E R S ************************************
      IF (NL.EQ.4) THEN
      b(1,1)=delta2m(1)+delta2m1(2)
      b(1,2)=lambdam1(2)
      b(2,1)=lambdam(2)
      b(2,2)=delta2m(2)+delta2m1(3)
      b(2,3)=lambdam1(3)
      b(3,2)=lambdam(3)
      b(3,3)= delta2m(3)+delta2m1(4)
      Det=b(1,1)*b(2,2)*b(3,3)-b(3,2)*b(2,3)*b(1,1)-
     *     b(2,1)*b(1,2)*b(3,3)

      N1(1)=(p(0)*lambdam(1)*(b(2,3)*b(3,2)-b(2,2)*b(3,3))-
     *       p(4)*lambdam1(4)*b(1,2)*b(2,3))/Det
      Nc(1)=(-(delta1m(3)-delta1m1(4))*b(1,2)*b(2,3)+
     *       (delta1m(1)-delta1m1(2))*(b(2,3)*b(3,2)-b(2,2)*b(3,3))+
     *       (delta1m(2)-delta1m1(3))*b(1,2)*b(3,3))/Det

      N1(2)=(p(4)*lambdam1(4)*b(1,1)*b(2,3)+
     *        p(0)*lambdam(1)*b(2,1)*b(3,3))/Det
      Nc(2)=((delta1m(3)-delta1m1(4))*b(1,1)*b(2,3)-
     *       (delta1m(2)-delta1m1(3))*b(1,1)*b(3,3)+
     *       (delta1m(1)-delta1m1(2))*b(2,1)*b(3,3))/Det

      N1(3)=(-p(0)*lambdam(1)*b(2,1)*b(3,2)+
     *       p(4)*lambdam1(4)*(b(1,2)*b(2,1)-b(1,1)*b(2,2)))/Det
      Nc(3)=(-(delta1m(1)-delta1m1(2))*b(2,1)*b(3,2)+
     *       (delta1m(3)-delta1m1(4))*(b(1,2)*b(2,1)-b(1,1)*b(2,2))+
     *       (delta1m(2)-delta1m1(3))*b(1,1)*b(3,2))/Det
```

```
      END IF
**************** 5 L A Y E R S **************************************
      IF (NL.EQ.5) THEN
      b(1,1)=delta2m(1)+delta2m1(2)
      b(1,2)=lambdam1(2)
      b(2,1)=lambdam(2)
      b(2,2)=delta2m(2)+delta2m1(3)
      b(2,3)=lambdam1(3)
      b(3,2)=lambdam(3)
      b(3,3)=delta2m(3)+delta2m1(4)
      b(3,4)=lambdam1(4)
      b(4,3)=lambdam(4)
      b(4,4)=delta2m(4)+delta2m1(5)
      Det=b(1,2)*b(2,1)*b(3,4)*b(4,3)-b(1,1)*b(2,2)*b(3,4)*b(4,3)-
     *     b(1,1)*b(2,3)*b(3,2)*b(4,4)-b(1,2)*b(2,1)*b(3,3)*b(4,4)+
     *     b(1,1)*b(2,2)*b(3,3)*b(4,4)
           N1(1)=(p(5)*lambdam1(5)*b(1,2)*b(2,3)*b(3,4)+
     *              p(0)*lambdam(1)*(b(2,2)*b(3,4)*b(4,3)+
     *              b(2,3)*b(3,2)*b(4,4)-b(2,2)*b(3,3)*b(4,4)))/Det
           Nc(1)=((delta1m(4)-delta1m1(5))*b(1,2)*b(2,3)*b(3,4)-
     *       (delta1m(2)-delta1m1(3))*(b(1,2)*b(3,4)*b(4,3)-
     *      b(1,2)*b(3,3)*b(4,4))+(delta1m(1)-delta1m1(2))*
     *       (b(2,2)*b(3,4)*b(4,3)+b(2,3)*b(3,2)*b(4,4)-
     *        b(2,2)*b(3,3)*b(4,4))-(delta1m(3)-delta1m1(4))*
     *        b(1,2)*b(2,3)*b(4,4))/Det

           N1(2)=(-p(5)*lambdam1(5)*b(1,1)*b(2,3)*b(3,4)+
     *              p(0)*lambdam(1)*(b(2,1)*b(3,3)*b(4,4)-
     *              b(2,1)*b(3,4)*b(4,3)))/Det
           Nc(2)=(-(delta1m(4)-delta1m1(5))*b(1,1)*b(2,3)*b(3,4)+
     *       (delta1m(2)-delta1m1(3))*(b(1,1)*b(3,4)*b(4,3)-
     *      b(1,1)*b(3,3)*b(4,4))+(delta1m(3)-delta1m1(4))*
     *      b(1,1)*b(2,3)*b(4,4)-(delta1m(1)-delta1m1(2))*
     *       (b(2,1)*b(3,4)*b(4,3)-b(2,1)*b(3,3)*b(4,4)))/Det

           N1(3)=(p(5)*lambdam1(5)*
     *      (b(1,1)*b(2,2)*b(3,4)-b(1,2)*b(2,1)*b(3,4))-
     *              p(0)*lambdam(1)*b(2,1)*b(3,2)*b(4,4))/Det
           Nc(3)=((delta1m(4)-delta1m1(5))*(b(1,1)*b(2,2)*b(3,4)-
     *        b(1,2)*b(2,1)*b(3,4))+
```

```
     *        (deltalm(2)-deltalm1(3))*b(1,1)*b(3,2)*b(4,4)-
     *        (deltalm(1)-deltalm1(2))*b(2,1)*b(3,2)*b(4,4)+
     *        (deltalm(3)-deltalm1(4))*
     *        (b(1,2)*b(2,1)*b(4,4)-b(1,1)*b(2,2)*b(4,4)))/Det

          N1(4)=(p(5)*lambdam1(5)*(b(1,1)*b(2,3)*b(3,2)+
     *            b(1,2)*b(2,1)*b(3,3)-b(1,1)*b(2,2)*b(3,3))+
     *            p(0)*lambdam(1)*b(2,1)*b(3,2)*b(4,3))/Det
          Nc(4)=((deltalm(4)-deltalm1(5))*(b(1,1)*b(2,3)*b(3,2)+
     *            b(1,2)*b(2,1)*b(3,3)-b(1,1)*b(2,2)*b(3,3))-
     *        (deltalm(2)-deltalm1(3))*b(1,1)*b(3,2)*b(4,3)+
     *        (deltalm(1)-deltalm1(2))*
     *         b(2,1)*b(3,2)*b(4,3)-(deltalm(3)-deltalm1(4))*
     *         (b(1,2)*b(2,1)*b(4,3)-b(1,1)*b(2,2)*b(4,3)))/Det

      END IF

      RETURN
      END
*****************************************************************

      SUBROUTINE STRESS
      IMPLICIT NONE
      COMMON/RT/CRITICAL
      COMMON/ANL/THETA(5),FLAG
       COMMON/RHML/RHO,ML
      COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
     #           P(0:5),X(5),Xpr(5),
     #           Y(5),Ypr(5),SS(5),F,KR(5),NL
      COMMON/N_COEF/N1(4),Nc(4)
      COMMON/INT_CN/ CONST
      COMMON/COMPL_SYST_CF/CM(5),U1(5),U2(5),F1(5),F2(5)
      COMMON/COMPLEX/A(5,4,4),BETA11(5),BETA22(5),BETA14(5),
     #               BETA44(5),BETA24(5),BETA12(5),K(5),G1(5),
     #               GK(5),G_K(5),C(5),KAPPA1(5),KAPPA2(5)
*****************************************************************
      COMMON/FCNST/Frr(5),Fzz(5),Fss(5),Fr(5),Fz(5),Frz(5),Frt(5)
      COMMON/STRTR/Sgr,Sg1,Sg2,Tau12
*****************************************************************
      REAL*8     E1,E2,E3,G12,V12,Rad,P,THETA,X,Xpr,CRITICAL,
     #           Y,Ypr,SS,F,H,DH(5),RADIUS(5,22),N1,NC,CONST,Frr,
```

```
#              Fzz,Fss,Fr,Fz,Frz,Frt,rho,DISPL
      REAL*8  A,BETA11,BETA22,BETA14,BETA44,BETA24,BETA12,K,G1,
#             GK,G_K,C,KAPPA1,KAPPA2,CM,U1,U2,F1,F2,SGR,SGT,SGZ,
#             TAUTZ,SG1,SG2,TAU12,RH,ETA,Y1(5),Y2(5),RESULT
      REAL*8 XR,STEP,XTOL,FTOL,FAILURE
      INTEGER*2 ,KR,NL,M,J,flag,MMM,ML,I,NLIM
      EXTERNAL FAILURE
*****************************************************************
***                     F O R C E S                           ***
*****************************************************************

             DO M=1,NL-1
             p(M)=N1(M)+Nc(M)*Const
             END DO

*****************************************************************
***                   S T R E S S E S                         ***
*****************************************************************
           do m=1, nl
          Frr(m) = 1./(Y(m)*Ypr(m))
          Fzz(m) = 1./(X(m)*Xpr(m))
          Fss(m) = 1./(SS(m)*SS(m))
          Fr(m)  = 1./Y(m) - 1./Ypr(m)
          Fz(m)  = 1./X(m) - 1./Xpr(m)
          Frz(m) = -0.5*Sqrt(Frr(m)*Fzz(m))
          Frt(m) = -0.5*Frr(m)
      Y1(m) = (p(m-1)*c(m)**(k(m)+1.)-p(m))/(1.-c(m)**(2.*k(m)))
      Y2(m) = (p(m)*c(m)**(k(m)-1.)-p(m-1))/(1.-c(m)**(2.*k(m)))
            end do

       Sgr=Y1(mL)*rho**(k(mL)-1.)+Y2(mL)*c(mL)**(k(mL)+1.)*
#          rho**(-k(mL)-1.)+Const*kappa1(mL)*
#          (1.-U1(mL)*rho**(k(mL)-1.)-
#          U2(mL)*c(mL)**(k(mL)+1.)*rho**(-k(mL)-1.))

       Sgt = Y1(mL)*k(mL)*rho**(k(mL)-1.)-Y2(mL)*k(mL)*
#            c(mL)**(k(mL)+1.)*rho**(-k(mL)-1.)+
#            Const*kappa1(mL)*(1.-U1(mL)*k(mL)*rho**(k(mL)-1.)+
#            U2(mL)*k(mL)*c(mL)**(k(mL)+1.)*rho**(-k(mL)-1.))

       Tautz= -Y1(mL)*gk(mL)*rho**(k(mL)-1.)-Y2(mL)*g_k(mL)*
```

```
#           c(mL)**(k(mL)+1.)*rho**(-k(mL)-1.)+
#      Const*(-kappa2(mL)+kappa1(mL)*(U1(mL)*
#             gk(mL)*rho**(k(mL)-1.)+
#      U2(mL)*g_k(mL)*c(mL)**(k(mL)+1.)*rho**(-k(mL)-1.)))

      Sgz= Const-(1./a(ML,3,3))*(a(ML,1,3)*Sgr+a(ML,2,3)*Sgt+
#       a(ML,3,4)*Tautz)

C        displ=A(ML,4,1)*SGR+A(ML,4,2)*SGT+A(ML,4,3)*SGZ+A(ML,4,4)*TAUTZ
C        PRINT*,'DISPL=',DISPL

       Sg1 = Sgz*Cos(theta(mL))**2+Sgt*Sin(theta(mL))**2-
#           Tautz*Sin(2.*theta(mL))

       Sg2 = Sgz*Sin(theta(mL))**2+Sgt*Cos(theta(mL))**2+
#           Tautz*Sin(2.*theta(mL))

       Tau12= (Sgt-Sgz)*Sin(theta(mL))*Cos(theta(mL))-
#        Tautz*Cos(2.*theta(mL))

      RETURN
      END
*******************************************************************

        SUBROUTINE SYST_CF
        IMPLICIT NONE
      COMMON /DT/E1(5),E2(5),E3(5),G12(5),V12(5),Rad(0:5),
#              P(0:5),X(5),Xpr(5),
#              Y(5),Ypr(5),SS(5),F,KR(5),NL
      COMMON/COMPLEX/A(5,4,4),BETA11(5),BETA22(5),BETA14(5),
#              BETA44(5),BETA24(5),BETA12(5),K(5),G1(5),
#              GK(5),G_K(5),C(5),KAPPA1(5),KAPPA2(5)
      COMMON/COMPL_SYST_CF/CM(5),U1(5),U2(5),F1(5),F2(5)
      COMMON/SY/DELTA1m(4),DELTA2m(4),LAMBDAm(4),LAMBDAm1(4),
#             DELTA1m1(4),DELTA2m1(4)

      REAL*8 A,BETA11,BETA22,BETA14,BETA44,BETA24,K,G1,GK,G_K,C,
#          KAPPA1,KAPPA2,CM,U1,U2,DELTA1m,DELTA2m,
$          DELTA1m1,
#          DELTA2m1,f1,f2,f3m(4),f4m(4),f5m(4),
#          f6m(4),f7m(4),f8m(4),f3m1(4),f4m1(4),LAMBDAm1,
```

```
#        f5m1(4),f6m1(4),f7m1(4),f8m1(4),LAMBDAm

      REAL*8 W1m(4),W2m(4),W3m(4), W1m1(4),W2m1(4),W3m1(4),
#        GAMMA24m(4),GAMMA24m1(4),E1,E2,E3,G12,V12,RAD,P,
#        X,XPR,Y,YPR,SS,F,rho(5),beta12

      INTEGER*2 M,NL,KR

       DO M=1,NL-1
       rho(m)=1.
      f3m(m) =  rho(m)**(k(m)-1.)
      f4m(m) =  c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)
      f5m(m) =  k(m)*rho(m)**(k(m)-1.)
      f6m(m) =  k(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)
      f7m(m) =  gk(m)*rho(m)**(k(m)-1.)
      f8m(m) =  g_k(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)

      W1m(m) = 1.-U1(m)*rho(m)**(k(m)-1.)-
#         U2(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)

      W2m(m) = 1.-U1(m)*k(m)*rho(m)**(k(m)-1.)+
#         U2(m)*k(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)

      W3m(m) = -kappa2(m)+kappa1(m)*(U1(m)*gk(m)*
#          rho(m)**(k(m)-1.)+U2(m)*g_k(m)*c(m)**(k(m)+1.)*
#          rho(m)**(-k(m)-1.))

      gamma24m(m) = a(m,2,4)-(a(m,2,3)*a(m,3,4))/a(m,3,3)

      delta1m(m)=a(m,2,3)+W3m(m)*gamma24m(m)+kappa1(m)*
#      (W1m(m)*beta12(m)+W2m(m)*beta22(m))
      delta2m(m)=beta12(m)*((f2(m)*f4m(m)-f3m(m))/CM(m))-
#           beta22(m)*((f2(m)*f6m(m)+f5m(m))/CM(m))+
#           gamma24m(m)*((f7m(m)-f2(m)*f8m(m))/CM(m))

        lambdam(m)=beta12(m)*((f1(m)*f3m(m)-f4m(m))/CM(m))+
$             beta22(m)*((f1(m)*f5m(m)+f6m(m))/CM(m))-
$             gamma24m(m)*((f1(m)*f7m(m)-f8m(m))/CM(m))
        END DO
*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*!*
       DO M=2,NL
```

```
      rho(m)=rad(m-1)/rad(m)
      f3m1(m) =  rho(m)**(k(m)-1.)
      f4m1(m) =  c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)
      f5m1(m) =  k(m)*rho(m)**(k(m)-1.)
      f6m1(m) =  k(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)
      f7m1(m) =  gk(m)*rho(m)**(k(m)-1.)
      f8m1(m) =  g_k(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)

      W1m1(m) = 1.-U1(m)*rho(m)**(k(m)-1.)-
     #        U2(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)

      W2m1(m) = 1.-U1(m)*k(m)*rho(m)**(k(m)-1.)+
     #        U2(m)*k(m)*c(m)**(k(m)+1.)*rho(m)**(-k(m)-1.)

      W3m1(m) = -kappa2(m)+kappa1(m)*(U1(m)*gk(m)*
     #         rho(m)**(k(m)-1.)+U2(m)*g_k(m)*c(m)**(k(m)+1.)*
     #         rho(m)**(-k(m)-1.))

      gamma24m1(m) = a(m,2,4)-(a(m,2,3)*
     #                  a(m,3,4))/a(m,3,3)

      delta1m1(m)=a(m,2,3)+W3m1(m)*gamma24m1(m)+kappa1(m)*
     #  (W1m1(m)*beta12(m)+W2m1(m)*beta22(m))

      delta2m1(m)=beta12(m)*((f4m1(m)-f1(m)*f3m1(m))/CM(m))-
     #           beta22(m)*((f1(m)*f5m1(m)+f6m1(m))/CM(m))+
     #           gamma24m1(m)*((f1(m)*f7m1(m)-f8m1(m))/CM(m))

        lambdam1(m)=beta12(m)*((f3m1(m)-f2(m)*f4m1(m))/CM(m))+
     $           beta22(m)*((f5m1(m)+f2(m)*f6m1(m))/CM(m))-
     $           gamma24m1(m)*((f7m1(m)-f2(m)*f8m1(m))/CM(m))
        END DO
           RETURN
           END
```

Routines for Continuosly Heterogeneous Cylinder

```
          PROGRAMM SHELL
         COMMON /DT/E(3,10),NU(6,10),M(10),A(0:10),P(0:10),er,N,KR(10)
         COMMON /mat/mater,NAME

         REAL*8 C(20,20),alf(3,3,10),gam(2,2,10),gam_k(10),s(10),
     #         t(10),bet22(10),bet12(10),C1(420),H(10),
     #         R(10,0:12),SGr(10,0:12),SGt(10,0:12),SGz(10,0:12),DH(10),
     #         A,NU,M,P,E,RK(10),btk12(10),btk22(10),ar1,ar2,ar3,er,D,
     #         U1,U2,Delta,epsilon_z
         character*62 fname
         CHARACTER*32 mater(10)
$debug
                   CALL INPUT
         DO 1000 K=1,N
             DO 1 I=1,3
                  alf(i,i,k)=1/e(i,k)
1            CONTINUE
             alf(1,2,k)= -nu(1,k)/e(2,k)
             alf(1,3,k)= -nu(2,k)/e(3,k)
             alf(2,1,k)= -nu(3,k)/e(1,k)
             alf(2,3,k)= -nu(4,k)/e(3,k)
             alf(3,1,k)= -nu(5,k)/e(1,k)
             alf(3,2,k)= -nu(6,k)/e(2,k)
         ar1 = abs(alf(1,2,k)-alf(2,1,k))
         ar2 = abs(alf(1,3,k)-alf(3,1,k))
         ar3 = abs(alf(3,2,k)-alf(2,3,k))
         if(abs(ar1).gt.er) GOTO 963
         if(abs(ar2).gt.er) GOTO 963
         if(abs(ar3).gt.er) GOTO 963
         GOTO 964
963      PRINT*,'LAYER- ',K
         print*,'  НЕПРАВИЛЬНО ВВЕДЕНЫ ФИЗИЧЕСКИЕ ХАРАКТЕРИСТИКИ !!!'
964      CONTINUE
             DO 2 i=1,2
               DO 2 j=1,2
               gam(i,j,k)= alf(i,j,k) - alf(i,3,k)*alf(j,3,k)/alf(3,3,k)
2        CONTINUE
            gam_k(k) = (gam(1,1,k)+m(k)*gam(1,2,k))/gam(2,2,k)
          s(k) = .5*(m(k)+dsqrt(ABS(m(k)*m(k)+4*gam_k(k))))
```

```
        t(k) = .5*(m(k)-dsqrt(ABS(m(k)*m(k)+4*gam_k(k))))
            bet22(k)=gam(2,2,k)*a(k-1)**(-m(k))
            bet12(k)=gam(1,2,k)*a(k-1)**(-m(k))
1000   CONTINUE
       do 161 k=2,n
       btk12(k-1)=gam(1,2,k-1)*a(k-1)**(-m(k-1))
       btk22(k-1)=gam(2,2,k-1)*a(k-1)**(-m(k-1))
161    continue
**************************************************************

*** ФОРМИРОВАНИЕ СИСТЕМЫ УРАВНЕНИЙ ***
       do 162 i=1,2*n
           do 162 j=1,2*n
162              c(i,j)=0.0
       do 163 i=1,j*j+j
163    c1(i) = 0.0
        c(1,1)=a(0)**s(1)
        c(1,2)=a(0)**t(1)
        c(2*n,2*n-1)=a(n)**s(n)
        c(2*n,2*n)=a(n)**t(n)
        k=2
        i=2
        j=1
        do 2000 jk=1,n-1
        c(i,j)=a(k-1)**s(k-1)
        c(i,j+1)=a(k-1)**t(k-1)
        c(i,j+2)= -a(k-1)**s(k)
        c(i,j+3)= -a(k-1)**t(k)
        c(i+1,j)=a(k-1)**s(k-1)*btk22(k-1)*s(k-1)
        c(i+1,j+1)=a(k-1)**t(k-1)*btk22(k-1)*t(k-1)
        c(i+1,j+2)=a(k-1)**s(k)*(btk12(k-1)-bet12(k)-bet22(k)*s(k))
        c(i+1,j+3)=a(k-1)**t(k)*(btk12(k-1)-bet12(k)-bet22(k)*t(k))
        k=k+1
        i=i+2
        j=j+2
2000   CONTINUE
*** ПРЕОБРАЗОВАНИЕ В ОДНОМЕРНЫЙ МАССИВ ***
        kl=1
        do 165 j=1,2*n
           do 166 i=1,2*n
              c1(kl)=c(i,j)
```

```
            kl=kl+1
 166      continue
 165      continue
***    ПРАВАЯ ЧАСТЬ  ***

          nl=4*n*n
          c1(nl+1) = -p(0)*a(0)
          c1(nl+2*n) = -p(n)*a(n)
         nm=nl+2
           do 167 k=2,n
              c1(nm) = -p(k-1)*a(k-1)
              c1(nm+1) = -(bet12(k)-btk12(k-1))*p(k-1)*a(k-1)
         nm = nm + 2
 167     continue

*** РЕШЕНИЕ СИСТЕМЫ УРАВНЕНИЙ ***
          n1=2*n
          call rsudm(c1,n1,1,ner,d,ied)
          do 168 jk=1,2*n
          c1(jk)=c1(4*n*n+jk)
 168      continue
*** ФОРМИРОВАНИЕ МАССИВА ТОЛЩИН И РАДИУСОВ ***
          do 169 k=1,n
          R(k,0)=A(k-1)
          RK(k)=DREAL(KR(k))
 169      continue
           DO 170 K=1,N
          H(K)  = A(K) - A(K-1)
          DH(K) = H(K)/RK(k)
 170      continue
           DO 22 K=1,N
             DO 22 J =1,KR(k)
 22          R(k,J)=A(k-1)+dh(k)*j
          jn=1
          U1=0.
          U2=0.
          Delta=0.
          do k=1,n
          U1=U1+2.*c1(jn)*(a(k)**(1.+s(k))-a(k-1)**(1.+s(k)))*
     #      (alf(1,3,k)+alf(2,3,k)*s(k))/(alf(3,3,k)*(1.+s(k)))
          U2=U2+2.*c1(jn+1)*(a(k)**(1.+t(k))-a(k-1)**(1.+t(k)))*
```

```
     #     (alf(1,3,k)+alf(2,3,k)*t(k))/(alf(3,3,k)*(1.+t(k)))
        Delta=Delta+(a(k)**2-a(k-1)**2)/alf(3,3,k)
        jn=jn+2
        end do
****
        epsilon_z= ((p(0)-p(n))*a(0)**2+U1+U2)/Delta
       print*,epsilon_z
       pause 12

*** ВЫЧИСЛЕНИЕ НАПРЯЖЕНИЙ ***
        ju=1
        do 800 k=1,n
           do 700 i=0,kr(k)
           SGr(k,i)=c1(ju)*r(k,i)**(s(k)-1)+c1(ju+1)*r(k,i)**(t(k)-1)
           SGt(k,i)=c1(ju)*s(k)*r(k,i)**(s(k)-1)+c1(ju+1)*t(k)*r(k,i)**
     #                (t(k)-1)
           SGz(k,i)=(1/alf(3,3,k))*(epsilon_z-alf(1,3,k)*SGr(k,i)-
     #              alf(2,3,k)*SGt(k,i))
700     continue
        ju=ju+2
800     continue
***  ПЕЧАТЬ ПОЛУЧЕННЫХ НАПРЯЖЕНИЙ ***
        WRITE(*,900)
900     FORMAT(/////10X,'  OUTPUT FILE: '\)
        READ (*,910) fname
910     FORMAT(A)
        OPEN(unit = 6,FILE=fname)
        WRITE(6,109) NER
109     FORMAT(1X,'Error index =',I2)
        write(6,'(//15x,16h S T R E S S E S/)')
        do 171 k=1,n
           write(6,100) k
        if (name.eq.1)then
          write(6,130) mater(k)
        endif
           write(6,140) H(k)
           write(6,101)
        do 172 i=0,kr(k)
           write(6,102) r(k,i),SGr(k,i),SGt(k,i),SGz(k,i)
172     continue
171     continue
```

```
100       format(/15x,'L A Y E R -',i2/)
130       format(5x,'Material - ',A)
140       format(5x,'Thickness -',G12.4/)
101       format(5x,'Rad',9x,'SGr',9x,'SGt',9x,'SGz'/)
102       format(1x,4g12.4)
          stop
          end
**********************************************************************

          SUBROUTINE RSUDM(A,N1,N2,NER,DET,IED)
        DIMENSION IND(75)
        REAL*8 A(1),PT,SW,DET
$debug
        DETER=1.0
        IED=0
        N=N1
        MAT=N+N2
        IM=N1
        NM=N-1
        IVC=1-IM
        DO 11 MA=1,N
        PT=0.0D+0
        IVC=IVC+IM
        IV1=IVC+MA-1
        IV2=IVC+NM
        DO 2 I=IV1,IV2
        IF(DABS(A(I))-DABS(PT))2,2,1
      1 PT=A(I)
        LP=I
      2 CONTINUE
        IF(PT)3,15,3
      3 IC=LP-IVC+1
        IND(MA)=IC
        IF(IC-MA)6,6,4
      4 DETER=-DETER
        IC=IC-IM
        I3=MA-IM
        DO 5 I=1,MAT
        IC=IC+IM
        I3=I3+IM
        SW=A(I3)
```

```
      A(I3)=A(IC)
    5 A(IC)=SW
    6 DETER=DETER*PT
      IE=INT(ALOG10(ABS(DETER)+1.E-20)+20)-20
      DETER=DETER/10.**IE
      IED=IED+IE
      PT=1./PT
      I3=IVC+NM
      DO 7 I=IVC,I3
    7 A(I)=-A(I)*PT
      A(IV1)=PT
      I1=MA-IM
      IC=1-IM
      DO 10 I=1,MAT
      IC=IC+IM
      I1=I1+IM
      IF(I-MA)8,10,8
    8 JC=IC+NM
      SW=A(I1)
      I3=IVC-1
      DO 9 I2=IC,JC
      I3=I3+1
    9 A(I2)=A(I2)+SW*A(I3)
      A(I1)=SW*PT
   10 CONTINUE
   11 CONTINUE
      DO 14 I1=1,N
      MA=N+1-I1
      LP=IND(MA)
      IF(LP-MA)12,14,12
   12 IC=(LP-1)*IM+1
      JC=IC+NM
      IVC=(MA-1)*IM+1-IC
      DO 13 I2=IC,JC
      I3=I2+IVC
      SW=A(I2)
      A(I2)=A(I3)
   13 A(I3)=SW
   14 CONTINUE
      DET=DETER
      NER=0
```

```
      RETURN
   15 NER=-1
      DET=DETER
      WRITE(5,17)MA
   17 FORMAT('    ZERO COLUMN  ',I4)
      RETURN
      END
**************************************************************
```