UNIVERSITY OF KWAZULU-NATAL

# SEMI-TETRAD DECOMPOSITION OF SPACETIME WITH CONFORMAL SYMMETRY

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# SEMI-TETRAD DECOMPOSITION OF SPACETIME WITH CONFORMAL SYMMETRY

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Submitted in fulfilment of the academic requirements for the degree of

Master of Science

to the

School of Mathematics, Statistics and Computer Science

College of Agriculture, Engineering and Science

University of KwaZulu-Natal

Durban

December 2018

As the candidate's supervisors, we have approved this thesis for submission.

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# Acknowledgements

- I owe my sincerest thanks to my supervisors, Prof. S D Maharaj and Dr R Goswami, for their valuable guidance and mentoring during the 2018 academic year. I always felt supported knowing their doors were open to me whenever I had a query which they would patiently hear out and then readily solve. I greatly appreciate their encouragement, dedication and energy which fueled my enthusiasm. I thank them for allowing me the opportunity to work with them. It has been a privilege.

- I have been a student of the University of KwaZulu-Natal since 2014 and express a deep gratitude to all my lecturers. The university has provided me with a conducive environment to conduct research. I am grateful for that and for all the friendly faces of staff and students who have helped me along this journey.

- I thank the National Research Foundation (NRF) for the financial support during my Masters.

- My parents, Sudan and Cynthia, have lovingly supported me through the years in all of my endeavours. Their belief in me motivates me to give off my best every day. I thank them for putting my needs ahead of theirs and taking care of literally everything else so that I was able to completely focus on my studies. I also thank my brother, Riaan, for always hearing me out and providing me with laughs through any stressful time. My family have made this journey possible. Special thanks to my friends and extended family for their concern and encouragement throughout this year. I love them all very much.

# Dedication

*To my dad,*

*Sudan Hansraj,*

*who continues to be my inspiration*

# Abstract

In this thesis, we study the kinematical and dynamical properties of a general spacetime that admits a conformal Killing vector. A 1+1+2 decomposition of the spacetime is performed using the fluid 4-velocity and a preferred spatial direction in the 3-space. The Lie derivatives of the 4-velocity vector and the preferred spatial direction vector are calculated and analyzed. We compare our results with the 1+3 decomposition of Maartens *et al* (1986), and find new results in the form of a scalar equation and constraint equation owing to the further decomposition. This provides new insights into the behaviour of the acceleration, expansion, shear and vorticity scalars which are not possible in the 1+3 decomposition. The general energy momentum tensor for an anisotropic fluid is considered and decomposed using the semi-tetrad covariant approach. We take the Lie derivative along the conformal Killing vector and apply to Einstein's field equations. This makes it possible to generate a set of constraint equations in the new geometrical variables. All the geometrical and thermodynamical quantities are written in terms of the 1+1+2 decomposition variables. This is a new result. We also find a system of equations that must be satisfied by the thermodynamical variables when a conformal symmetry exists applied to the perfect fluid case. We show that the conformal factor satisfies a damped wave equation with a potential.

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# Chapter 1

# Introduction

The theory of general relativity, developed by Einstein, describes the gravitational interaction between celestial bodies in the universe. In order to analyse the evolution of these astrophysical objects, such as galaxies and stars, we need to understand the nature and behaviour of their gravitational fields. This has to be done in the context of general relativity which requires Riemannian geometry. Comprehensive reviews on basic principles of general relativity can be found in Foster and Nightingale (1994), Narlikar (2002) and Stephani (2003).

In general relativity, the Einstein field equations relate matter to curvature and are represented by a system of nonlinear partial differential equations. Determining explicit solutions to the Einstein field equations, which are in general hard to solve, is necessary for astrophysical and cosmological applications. The most well known solutions of the Einstein field equations are the Schwarzschild exterior and interior solutions (Schwarzschild 1916a, 1916b) as well as the charged Reissner-Nordstrom solution (Nordstrom 1918, Reissner 1916). The exterior Schwarzschild solution is a spherically symmetric, static and vacuum solution. The interior Schwarzschild solution models the gravitational field when the energy density is constant. The Reissner-Nordstrom metric is a more general solution and describes the exterior spacetime of a spherical, non-rotating charged body.

Many methods exist to find exact solutions to the Einstein equations as detailed in Stephani *et al* (2009). These include the coordinate approach where a metric is specified, numerical methods, symmetry methods and tetrad formalisms. Examples of tetrad formalisms are the complex null tetrad of Newman and Penrose (1962) and the 1+3 covariant approach formally developed by Ehlers (1961) and Ellis (1971). The 1+3 covariant approach involves a full tetrad approach as well as a partial ‘covariant’ approach where only one timelike tetrad vector is chosen. Covariant methods, dating back to the work of Heckmann and Schucking (1955) and Raychaudhuri (1957), are advantageous because they describe the physics and geometry of the spacetime by tensor quantities and relations which remain valid in all coordinate systems. The 1+3 partial frame formalism was built on early cosmological perturbation work by Hawking (1966), Stewart and Walker (1974), Lyth and Mukherjee (1988) and Ellis and Bruni (1989), and developed further in covariant approaches by Ellis *et al* (1990), Bruni *et al* (1992) and Dunsby *et al* (1992). This formalism involves the splitting of the spacetime through a timelike vector into ‘time’ and ‘space’ where the 3-space is orthogonal to the timelike vector. Hence, the spacetime geometry and physics are described by scalars, 3-vectors and 3-tensors. All the important information in the system is captured in a set of kinematic and thermodynamic 1+3 variables that have a clear physical and geometrical meaning. A set of evolution and constraint equations, arising from the Bianchi and Ricci identities, relate the 1+3 variables. A closed system of equations results when an equation of state is chosen describing the matter. The 1+3 formalism has contributed greatly to the understanding of the physics behind the cosmic microwave background as shown in Dunsby (1997), Challinor and Lasenby (1998) and Maartens *et al* (1999). Examples of spacetimes where this formalism has generated useful results are dust spacetimes studied by Ellis (1967), locally rotationally symmetric spacetimes studied by Stewart and Ellis (1968) and Ellis and MacCallum

(1969), and Bianchi spacetimes studied by Krasinski *et al* (2003) and Bianchi (2001) (which is a republication of the original paper from 1889). We refer the reader to Ellis (2009) for a comprehensive review of the 1+3 formalism.

If we consider spacetimes that admit less symmetry, the resulting 1+3 equations are tensorial partial differential equations that are difficult to work with as in the case of inhomogeneous spacetimes. Hence a natural extension to the 1+3 covariant approach is the 1+1+2 covariant approach formally developed by Clarkson and Barrett (2003). A 1+1+2 decomposition of the spacetime is performed using the fluid 4-velocity and a preferred spatial direction in the 3-space. This semi-tetrad formalism is optimized for problems which have spherical symmetry, including the Schwarzschild solution and many classes of Bianchi models. It was first introduced by Greenberg (1970) and developed further by Tsamparlis and Mason (1983), van Elst (1996) and van Elst and Ellis (1996). The formalism was mainly used in the context of symmetric solutions of the Einstein field equations as shown in the works of Mason and Tsamparlis (1985), van Elst and Ellis (1996) and Zafiris (1997). In recent times, the 1+1+2 formalism has generated useful results in the analysis of: linear perturbations of the Schwarzschild spacetime studied by Clarkson and Barrett (2003); locally rotationally symmetric class II spacetimes investigated by Betschart and Clarkson (2004) in general relativity and Nzioki *et al* (2010) in $f(R)$ gravity; gravitational lensing studied by de Swardt *et al* (2010) and general locally rotationally symmetric spacetimes investigated by Singh *et al* (2017). We refer the reader to Clarkson (2007) for a comprehensive review of the 1+1+2 formalism.

An alternative method of determining solutions to Einstein's field equations is to assume the spacetime admits symmetry, e.g. a conformal symmetry. Such an assumption simplifies the field equations and makes them easier to integrate. Notably the Schwarzschild and Robertson-Walker models are spacetimes possessing high symmetry. Imposing conformal symmetry on the spacetime manifold means the manifold is invariant under the action of a group of conformal motions. Conformal symmetry is a widely researched area and has many applications in relativistic astrophysics. Dating back to the 1980's, Herrera *et al* (1984) and Herrera and Ponce de León (1985) used conformal motions in modeling an anisotropic relativistic sphere. Spherically symmetric cosmological models with vanishing shear admitting a conformal Killing vector were studied by Dyer *et al* (1987) and Maharaj *et al* (1991). Also in the context of spherically symmetric spacetimes in conformal geometry, the analysis of the kinematical and dynamical quantities was performed in many papers by Coley and Tupper (1990a, 1990b, 1990c, 1994). Kramer (1990) determined rigidly rotating or static perfect fluid solutions admitting conformal motion. Castejon-Amenedo and Coley (1992) and Hansraj *et al* (2005) considered applications of conformal symmetries in conformally related spacetimes. Conformal Killing vectors have been analyzed in the following spacetimes: Minkowski spacetime studied by Choquet-Bruhat *et al* (1977); Robertson-Walker spacetimes studied by Maartens and Maharaj (1986); *pp*-wave spacetimes studied by Maartens and Maharaj (1991) and extended by Keane and Tupper (2004) and static spherically symmetry spacetimes by Maharaj *et al* (1995). Recent developments were made in static spherically symmetric spacetimes by Manjonjo *et al* (2018), in shear-free spherically symmetric spacetimes by Moopanar and Maharaj (2013), in general spherically symmetric spacetimes by Moopanar and Maharaj (2010) and also, locally rotationally symmetric spacetimes studied by Singh *et al* (2018) using the 1+1+2 formalism. Clearly, considering conformal symmetry creates a huge advantage in analyzing geometrical properties of spacetimes.

In this thesis, we follow the work of Maartens *et al* (1986) who studied the kinematical and dynamical properties of conformal Killing vectors in anisotropic fluids. We attempt to write the results of Maartens *et al* (1986) completely in terms of the 1+1+2 formalism variables and perform a detailed analysis by considering a general spacetime that admits a conformal Killing vector.

A detailed outline of the thesis is as follows: In Chapter 2 we briefly outline concepts relating to curvature in general relativity necessary for this thesis. Chapter 3 contains a review of the 1+3 covariant approach which splits the spacetime using a timelike vector. Important derivatives and geometrical and thermodynamical variables are defined. The Weyl tensor and energy momentum tensor are decomposed and the evolution, propagation and constraint equations derived from the field equations are written down. In Chapter 4, by way of extension of the 1+3 covariant approach, we summarize the important equations pertaining to the 1+1+2 covariant approach. In this formalism, the spacetime is split further through a preferred spatial vector. The 1+3 kinematical and Weyl quantities are decomposed and important derivatives are specified. The evolution, propagation and constraint equations are written down completely in the 1+1+2 variables and analyzed. Chapter 5 is where we write the results of Maartens *et al* (1986) completely in terms of the 1+1+2 variables. We consider an arbitrary spacetime that admits a conformal Killing vector and consider the Lie derivatives of important quantities. A physical and geometrical analysis of the kinematical quantities is performed. In Chapter 6, we consider the dynamics of spacetime. We write down the Lie derivative of the Einstein field equations and expand it further using the 1+1+2 formalism. An analysis of the resulting equations is performed. In Chapter 7, we apply the resulting equations from Chapter 6 to a perfect fluid spacetime and determine the physical significance of our findings. In Chapter 8, we review the results obtained.

# Chapter 2

# Riemann curvature

We devote this brief chapter to outlining concepts relating to curvature in general relativity necessary for this thesis. We first introduce the concept of a manifold. Then we define the connection coefficients, the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor along with properties associated with them. For further reading, comprehensive reviews on the basic principles of general relativity and differential geometry can be found in Hawking and Ellis (1975), Wald (1984) and Straumann (2004).

In general relativity, spacetime $\mathcal{M}$ is taken to be a 4-dimensional pseudo-Riemannian manifold. Locally a pseudo-Riemannian space is similar to Euclidean space. Hence we can always find coordinate patches (subsets of the manifold) where local neighbourhoods of a pseudo-Riemannian space can be mapped to Euclidean space. However, we cannot perform this mapping globally. The manifold $\mathcal{M}$ is endowed with a symmetric, nonsingular metric tensor field $\boldsymbol{g}$ of signature $(-+++)$. A metric tensor is a bilinear map that assigns a real number to pairs of tangent vectors at each tangent space of the manifold. The properties of being symmetric and nondegenerate are necessary for a physically acceptable field. The points in $\mathcal{M}$ are labelled using the real coordinates $(x^a) = (x^0, x^1, x^2, x^3)$ where $x^0$ is timelike and $x^1, x^2, x^3$ are spacelike coordinates. Note that $x^0 = ct$ where $c$ is the speed of light in vacuum. We use units in which $c$ is unity.

The line element is denoted by

$$ds^2 = g_{ab}\, dx^a\, dx^b, \tag{2.1}$$

which defines the invariant distance between neighbouring points of a curve in the manifold. The connection $\Gamma$ is defined in terms of the metric tensor field $\boldsymbol{g}$. The coefficients of the metric connection $\Gamma$ are given by

$$\Gamma^a{}_{bc} = \frac{1}{2} g^{ad} \left( g_{cd,b} + g_{db,c} - g_{bc,d} \right), \tag{2.2}$$

which are also known as the Christoffel symbols of the second kind. The $\Gamma$'s are symmetric in their lower indices. The commas denote partial differentiation.

The Riemann curvature tensor $\boldsymbol{R}$ is a (1, 3) tensor field given by

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}, \tag{2.3}$$

which represents the curvature of the spacetime manifold. This tensor possesses the following symmetry properties

$$R_{abcd} = -R_{bacd}, \tag{2.4a}$$

$$R_{abcd} = -R_{abdc}, \tag{2.4b}$$

$$R_{abcd} = R_{cdab}, \tag{2.4c}$$

$$R_{abcd} + R_{acdb} + R_{adbc} = 0. \tag{2.4d}$$

In addition we have

$$R^a{}_{acd} = 0, \tag{2.5}$$

and

$$\nabla_{[e} R_{ab]cd} = 0, \tag{2.6}$$

which is the Bianchi identity. The derivative $\nabla_e$ represents covariant differentiation and square brackets on indices represents skew-symmetrization. Upon contraction of the Riemann tensor (2.3), in the first and third indices, we obtain the Ricci tensor given by

$$\begin{aligned} R_{ab} &= R^c{}_{acb} \\ &= \Gamma^d{}_{ab,d} - \Gamma^d{}_{ad,b} + \Gamma^e{}_{ab}\Gamma^d{}_{ed} - \Gamma^e{}_{ad}\Gamma^d{}_{eb}, \end{aligned} \tag{2.7}$$

which is symmetric. Contracting the Ricci tensor (2.7) results in the Ricci scalar as follows

$$R = g^{ab}R_{ab} = R^a{}_a. \tag{2.8}$$

Any given vector field $v^a$ defined on a manifold should obey the Ricci identity

$$2\nabla_{[a}\nabla_{b]}v_c = R^d{}_{abc}v_d. \tag{2.9}$$

Using the definitions for the Ricci tensor (2.7) and the Ricci scalar (2.8), we can construct the Einstein tensor $\boldsymbol{G}$ in the form

$$G^{ab} = R^{ab} - \frac{1}{2}Rg^{ab}, \tag{2.10}$$

where $\boldsymbol{G}$ is symmetric and has zero divergence:

$$\nabla_b G^{ab} = 0. \tag{2.11}$$

This property is known as the contracted Bianchi identity. Also note that applying a double contraction to (2.6) results in the twice-contracted Bianchi identity

$$\nabla_a R^a{}_c + \nabla_b R^b{}_c - \nabla_c R = 0 \qquad \Leftrightarrow \qquad \nabla^a G_{ab} = 0. \tag{2.12}$$

The distribution of matter is defined by the energy momentum tensor $\boldsymbol{T}$. Specific forms for $\boldsymbol{T}$ are considered in subsequent chapters. The Einstein field equations are given by

$$G^{ab} = R^{ab} - \frac{1}{2} R g^{ab} = T^{ab}, \tag{2.13}$$

which arises when the energy momentum tensor is coupled to the Einstein tensor (2.10). The coupling constant $k = \frac{8\pi G}{c^4}$ is set to unity. From the twice-contracted Bianchi identities (2.12), we know that the divergence of the left hand side of (2.13) is zero, making the divergence of the right hand side zero as well so that

$$\nabla_b G^{ab} = 0 \qquad \Longrightarrow \nabla_b T^{ab} = 0. \tag{2.14}$$

As a result the matter content is conserved.

The equations presented in this chapter are a brief outline of results that are required to build a foundation for later work.

# Chapter 3

# 1+3 formalism

## 3.1 Introduction

In this chapter, we review the 1+3 covariant approach developed by Ehlers (1961) and Ellis (1971). This decomposition of the manifold has proved to be useful in our understanding of spacetime structure, general relativity, and in particular, models in relativistic astrophysics and cosmology. All important information of the system is captured in a set of dynamic and kinematic 1+3 variables. A more detailed review of the formalism can be found in Ellis (2009). The application of the 1+3 covariant approach to general relativity is reviewed by Stephani *et al* (2009) in many spacetimes of physical interest. In particular we mention the application to dust spacetimes by Ellis (1967), locally rotationally symmetric spacetimes by Stewart and Ellis (1968) and Ellis and MacCallum (1969), and Bianchi spacetimes by Krasinski *et al* (2003) and Bianchi (2001) (which is a republication of the original paper from 1889). The geometrical and thermodynamical variables and their properties belonging to this formalism are defined in this chapter. Furthermore, the propagation, evolution and constraint equations for the 1+3 covariant variables are derived from the field equations and analyzed.

## 3.2 Kinematics

The nonintersecting timelike family of worldlines form a congruence in spacetime $(\mathcal{M}, \boldsymbol{g})$ representing the average motion of matter at each point. These worldlines are associated with fundamental observers comoving with the cosmological fluid. In each case, their 4-velocity is

$$u^a = \frac{dx^a}{d\tau}, \qquad u^a u_a = -1, \tag{3.1}$$

where $\tau$ is the proper time measured along the worldline of any fundamental observer. This unique vector field $u^a$ provides a timelike threading for the spacetime and represents the observers' congruence. Unique projection tensors are defined in terms of $u^a$ by

$$U^a{}_b = -u^a u_b, \tag{3.2}$$

$$h_{ab} = g_{ab} + u_a u_b, \tag{3.3}$$

where (3.2) projects parallel to $u^a$ and (3.3) projects onto the rest space of an observer moving with 4-velocity $u^a$. It follows that

$$U^a{}_c U^c{}_b = -U^a{}_b, \qquad U^a{}_b u^b = u^a, \qquad U^a{}_a = 1, \tag{3.4}$$

$$h_{ab} u^b = 0, \qquad h^a{}_c h^c{}_b = h^a{}_b, \qquad h^a{}_a = 3. \tag{3.5}$$

The effective volume element in the rest space of the comoving observer is defined as

$$\varepsilon_{abc} = \eta_{abcd} u^d, \qquad \text{where} \qquad \varepsilon_{abc} = \varepsilon_{[abc]} \qquad \text{and} \qquad \varepsilon_{abc} u^c = 0. \tag{3.6}$$

Here $\eta_{abcd}$ is the 4-dimensional volume element $\left(\eta_{abcd} = \sqrt{|\det g|}\, \delta^0{}_{[a} \delta^1{}_b \delta^2{}_c \delta^3{}_{d]}\right)$ so that

$$\eta_{abcd} = 2 u_{[a} \varepsilon_{b]cd} - 2 \varepsilon_{ab[c} u_{d]}. \tag{3.7}$$

and $\eta_{abcd} = \eta_{[abcd]}$. The following contractions hold

$$\begin{aligned}
\varepsilon_{abc}\,\varepsilon^{def} &= 3!\,h^d{}_{[a}\,h^e{}_b\,h^f{}_{c]}, \\
\varepsilon_{abc}\,\varepsilon^{dec} &= 2\,h^d{}_{[a}h^e{}_{b]}, \\
\varepsilon_{abc}\,\varepsilon^{dbc} &= 2\,h^d{}_a, \\
\varepsilon_{abc}\,\varepsilon^{abc} &= 3!,
\end{aligned} \tag{3.8}$$

since $\eta_{abcd}$ is totally skew-symmetric.

Furthermore, two derivatives which are useful can be defined. The covariant time derivative, denoted by ' $\dot{}$ ', along the observers' wordlines is defined, using the vector $u^a$, as

$$\dot{Z}^{a...b}{}_{c...d} = u^e\,\nabla_e\,Z^{a...b}{}_{c...d}, \tag{3.9}$$

for any tensor $Z^{a...b}{}_{c...d}$. The fully orthogonally projected covariant spatial derivative, denoted by 'D', is defined using the spatial projection tensor $h_{ab}$, as

$$\mathrm{D}_e\,Z^{a...b}{}_{c...d} = h^r{}_e\,h^p{}_c\ldots h^q{}_d\,h^a{}_f\ldots h^b{}_g\,\nabla_r\,Z^{f...g}{}_{p...q}, \tag{3.10}$$

with total projection on all the free indices.

Any spacetime 4-vector $v^a$ may be covariantly split into a scalar $V$ and a 3-vector $V^a$ as follows

$$v_a = -u_a\,V + V_a \qquad \text{where} \qquad V^a = h^a{}_b\,v^b \qquad \text{and} \qquad V = v_b\,u^b. \tag{3.11}$$

Here $V$ is the part of the vector parallel to $u^a$ and $V^a$ lies orthogonal to $u^a$. Any projected rank two tensor $S_{ab}$ can be split as

$$S_{ab} = S_{<ab>} + \frac{1}{3}S\,h_{ab} + S_{[ab]}. \tag{3.12}$$

In the above we have introduced

$$S = h_{ab}\,S^{ab}, \tag{3.13}$$

which is the spatial trace. $S_{<ab>}$ is the orthogonally projected symmetric trace-free part of the tensor defined as

$$S_{<ab>} = \left( h^c{}_{(a} h^d{}_{b)} - \frac{1}{3} h_{ab} h^{cd} \right) S_{cd}. \tag{3.14}$$

Lastly $S_{[ab]}$ is given by

$$S_{[ab]} = \varepsilon_{abc} S^c \quad \Leftrightarrow \quad S_a = \frac{1}{2} \varepsilon_{abc} S^{[bc]}, \tag{3.15}$$

which is the skew part of the rank two tensor that is spatially dual to the spatial vector $S^c$. We use angle brackets to represent the projected, symmetric and trace-free tensors. Additionally, we use the angle brackets to denote orthogonal projections of covariant time derivatives along $u^a$ as follows

$$\dot{V}^{<a>} = h^a{}_b \dot{V}^b, \qquad \dot{S}_{<ab>} = \left( h^c{}_{(a} h^d{}_{b)} - \frac{1}{3} h_{ab} h^{cd} \right) \dot{S}_{cd}. \tag{3.16}$$

Using the above definitions, we obtain the derivatives of the projection tensors and the 3-volume element

$$\mathrm{D}_a U_{bc} = \mathrm{D}_a h_{bc} = \mathrm{D}_a \varepsilon_{bc} = 0, \tag{3.17}$$

$$\dot{U}_{<ab>} = \dot{h}_{<ab>} = \dot{\varepsilon}_{<abc>} = 0, \tag{3.18}$$

$$\dot{h}_{ab} = 2u_{(a} \dot{u}_{b)}, \tag{3.19}$$

$$\dot{\varepsilon}_{abc} = 3\dot{u}^d \varepsilon_{d[ab} u_{c]}. \tag{3.20}$$

The covariant spatial divergence and curl for projected vectors and fully projected rank two tensors are given by

$$\begin{aligned} \text{div } V &= \mathrm{D}^a V_a, \\ (\text{div } S)_a &= \mathrm{D}^b S_{ab}, \\ \text{curl } V_a &= \varepsilon_{abc} \mathrm{D}^b V^c, \\ \text{curl } S_{ab} &= \varepsilon_{cd<a} \mathrm{D}^c S^d{}_{b>}, \end{aligned} \tag{3.21}$$

which generalises these Newtonian operators to curved spacetimes. We have followed the treatment of Maartens (1997). For a symmetric rank two tensor,

$$S_{ab} = S_{(ab)} \qquad \rightarrow \qquad \mathrm{curl}\ S_{ab} = \mathrm{curl}\ S_{<ab>}, \tag{3.22}$$

since curl $(k\,h_{ab}) = 0$ for any $k \epsilon \mathbb{R}$. We note that for vectors or rank two tensors, div curl *is not* in general zero, as in the Euclidean case.

The covariant decomposition of the derivative of a scalar $\Upsilon$ is given by

$$\nabla_a \Upsilon = -u_a \dot{\Upsilon} + \mathrm{D}_a \Upsilon. \tag{3.23}$$

Before we write down the exact form of the covariant decomposition of the derivatives of the 4-vector and then of the orthogonally projected rank two tensor, we introduce the algebraic terms $\Theta$, $\omega_{ab}$, $\sigma_{ab}$, $\dot{u}_a$. These terms are kinematic quantities arising from the relative motion of the comoving observers. The trace term is defined as

$$\Theta \;=\; \mathrm{D}^a u_a, \tag{3.24}$$

and is the rate of volume expansion scalar of the fluid. The shear tensor

$$\sigma_{ab} \;=\; \mathrm{D}_{<a} u_{b>}, \tag{3.25}$$

with properties

$$\begin{aligned} \sigma_{ab} &= \sigma_{(ab)}, \\ \sigma_{ab} u^b &= 0, \\ \sigma^a{}_a &= 0, \end{aligned} \tag{3.26}$$

is the trace-free part of the spatial change of $u_a$. This tensor describes the distortion in the matter flow, leaving the volume invariant. We can write down the shear magnitude as

$$\begin{aligned} \sigma^2 &\equiv \frac{1}{2}\sigma^{ab}\sigma_{ab} \geq 0, \\ \text{and} \qquad \sigma^2 = 0 &\Leftrightarrow \sigma_{ab} = 0. \end{aligned} \tag{3.27}$$

The anti-symmetric vorticity tensor

$$\omega_{ab} = \mathrm{D}_{[a} u_{b]}, \tag{3.28}$$

describes the rigid rotation of matter relative to a nonrotating frame with properties

$$\begin{aligned} \omega_{ab} &= \omega_{[ab]}, \\ \omega_{ab} u^b &= 0. \end{aligned} \tag{3.29}$$

We may also represent the vorticity tensor by the vorticity vector $\omega^a$ where

$$\begin{aligned} \omega^a &= \frac{1}{2}\eta^{abcd} u_d \omega_{bc} = \frac{1}{2}\varepsilon^{abc}\omega_{bc} = \frac{1}{2}\text{curl } u^a \qquad \Leftrightarrow \qquad \omega_{ab} = \varepsilon_{abc}\omega^c, \\ \omega^a u_a &= \omega^a \omega_{ab} = 0. \end{aligned} \tag{3.30}$$

The vorticity magnitude is expressed as

$$\begin{aligned} \omega^2 &= \frac{1}{2}\omega^a \omega_a = \omega^{ab}\omega_{ab} \geq 0, \\ \text{and} \qquad \omega &= 0 \qquad \Leftrightarrow \qquad \omega_a = 0 \qquad \Leftrightarrow \qquad \omega_{ab} = 0. \end{aligned} \tag{3.31}$$

Finally

$$\dot{u}_b = u^c \nabla_c u_b, \tag{3.32}$$

is the relativistic 4-acceleration vector that represents the degree to which matter moves under forces other than gravity and inertia. The acceleration vanishes for a free-falling observer, in a rest frame, meaning that the observer moves along geodesic curves.

Now we can define the exact form of the covariant decomposition of the derivative of the 4-vector (3.11) as

$$\begin{aligned} \nabla_a v_b &= -V\left(-u_a \dot{u}_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}\right) + u_b\left(\frac{1}{3}\Theta V_a + \sigma^c{}_a V_c + \omega^c{}_a V_c\right) \\ &\quad -u_a\left(\dot{V}_{<b>} + u_b \dot{u}_c V^c\right) + \frac{1}{3}(\text{div } V)\, h_{ab} + \mathrm{D}_{<a}V_{b>} + \frac{1}{2}\varepsilon_{abc}\text{curl } V^c \\ &\quad -u_b \nabla_a V. \end{aligned} \tag{3.33}$$

Furthermore the covariant decomposition of the derivative of the orthogonally projected rank two tensor (3.12) is given by

$$\nabla_c S_{ab} = -u_c\left(\dot{S}_{<ab>} + 2u_{(a}S_{b)d}\dot{u}^d\right) + 2u_{(a}\left(\frac{1}{3}\Theta S_{b)c} + S^d{}_{b)}\left(\sigma_{cd} - \varepsilon_{cde}w^e\right)\right)$$

$$+\frac{3}{5}\left(\text{div } S\right)_{<a}h_{b>c} - \frac{2}{3}\varepsilon_{dc(a}\text{curl } S^d{}_{b)} + \mathrm{D}_{<a}S_{bc>}. \tag{3.34}$$

Since the variation of velocity with position and time is of interest to us, we define the covariant derivative of the 4-velocity vector using (3.33) as

$$\nabla_a u_b = -u_a\dot{u}_b + \frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \omega_{ab}. \tag{3.35}$$

We further write down the covariant decomposition of the double derivative of a scalar $\Upsilon$:

$$\nabla^a\nabla^b\Upsilon = -\dot{\Upsilon}\left(\frac{1}{3}\Theta h^{ab} + \sigma^{ab} + \omega^{ab}\right)$$

$$+u^b\left(\frac{1}{3}\Theta \mathrm{D}^a\Upsilon + \sigma^{ac}\mathrm{D}_c\Upsilon + \omega^{ac}\mathrm{D}_c\Upsilon + u^a\ddot{\Upsilon} - \mathrm{D}^a\dot{\Upsilon}\right)$$

$$-u^a\left[h^{cb}\left(\mathrm{D}_c\Upsilon\right) + \dot{u}^c u^b\mathrm{D}_c\Upsilon - \dot{\Upsilon}\dot{u}^b\right] + \frac{1}{3}\left(\mathrm{D}^2\Upsilon\right)h^{ab}$$

$$+\mathrm{D}^{<a}\mathrm{D}^{b>}\Upsilon + \frac{1}{2}\varepsilon^{abc}\text{curl } \mathrm{D}_c\Upsilon, \tag{3.36}$$

which will be of use later.

## 3.3 The energy momentum tensor

The total energy momentum tensor $T_{ab}$, introduced in (2.13), can be decomposed, relative to $u^a$, by breaking it up into parts, that are parallel and orthogonal to $u^a$, as follows

$$T_{ab} = \mu u_a u_b + p h_{ab} + q_a u_b + u_a q_b + \pi_{ab}. \tag{3.37}$$

The total dynamic quantities are defined as follows: $\mu$ represents the effective energy density relative to $u^a$, $p$ is the isotropic pressure, $q_a$ represents the total energy flux relative to $u^a$ and lastly $\pi_{ab}$ is the projected symmetric trace-free anisotropic stress, such that

$$\mu = T_{ab} u^a u^b, \tag{3.38}$$

$$p = \frac{1}{3} T_{ab} h^{ab}, \tag{3.39}$$

$$q_a = -T_{bc} u^c h^b{}_a, \tag{3.40}$$

$$\pi_{ab} = T_{cd} h^c{}_{<a} h^d{}_{b>}. \tag{3.41}$$

For these quantities, the following properties:

$$\begin{aligned} q_a u^a &= 0, \\ q_a &= q_{<a>}, \\ \pi_{ab} u^b &= 0, \\ \pi_{ab} &= \pi_{(ab)}, \\ \pi^a{}_a &= 0, \end{aligned} \tag{3.42}$$

hold. Additionally we demand that the isentropic speed of sound

$$c_s^2 = (\partial p / \partial \mu)_{s = \text{constant}} , \tag{3.43}$$

obeys

$$0 \leq c_s^2 \leq 1 \qquad \Leftrightarrow \qquad 0 \leq (\partial p / \partial \mu)_{s = \text{constant}} \leq 1, \tag{3.44}$$

because this guarantees local stability of matter (lower bound) and causality (upper bound), respectively.

We note that we may write the field equations (2.13) in its trace-free reverse form as

$$R_{ab} = T_{ab} - \frac{1}{2} T g_{ab}. \tag{3.45}$$

Taking the trace of (3.45), we find the expression for the Ricci scalar in terms of the total thermodynamical quantities as follows

$$R = -T = \mu - 3p. \tag{3.46}$$

Also using (3.46) in (2.13), along with (3.37), we obtain an expression for the 1+3 split of the Ricci tensor $R_{ab}$ given by

$$R_{ab} = \frac{1}{2}\,(\mu + 3p)\,u_a u_b + \frac{1}{2}\,(\mu - p)\,h_{ab} + 2\,u_{(a}\,q_{b)} + \pi_{ab}. \tag{3.47}$$

## 3.4 Weyl curvature

The local free gravitational field is represented by the Weyl curvature tensor $\boldsymbol{C}$ given by

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2g^{[a}{}_{[c}\,R^{b]}{}_{d]} + \frac{1}{3}\,R\,g^{[a}{}_{[c}\,g^{b]}{}_{d]}, \tag{3.48}$$

which describes spacetime curvature that is not directly determined locally by the matter. The Weyl tensor possesses the same symmetry properties, given by (2.4), as the Riemann curvature tensor. An additional property is

$$C^{c}{}_{acb} = 0, \tag{3.49}$$

which indicates that the Weyl tensor is trace-free on all its indices. Hence we may think of the Ricci tensor $R_{ab}$ and the Weyl tensor $C_{abcd}$ as the trace and trace-free part of the Riemann curvature tensor $R_{abcd}$ respectively.

The Weyl tensor may be split relative to $u^a$ as

$$E_{ab} = C_{acbd}\,u^c u^d, \tag{3.50}$$

$$H_{ab} = \frac{1}{2}\,\varepsilon_{ade}\,C^{de}{}_{bc}\,u^c, \tag{3.51}$$

with properties

$$E^a{}_a = 0, \qquad E_{ab} = E_{(ab)}, \qquad E_{ab}\,u^b = 0, \tag{3.52}$$

$$H^a{}_a = 0, \qquad H_{ab} = H_{(ab)}, \qquad H_{ab}\,u^b = 0, \tag{3.53}$$

where $E_{ab}$ represents the electric part and $H_{ab}$ represents the magnetic part of Weyl curvature. The fully covariant 1+ 3 electromagnetic analogy for gravity was developed and applied by Maartens and Bassett (1998). Thus we can write $\boldsymbol{C}$ as follows

$$C_{abcd} = C^E_{abcd} + C^H_{abcd}, \tag{3.54}$$

where

$$C^E_{abcd} = \left(4 g_{a[p}\,g_{q]b}\,g_{c[r}\,g_{s]d} - \eta_{abpq}\,\eta_{cdrs}\right) u^p\,u^r\,E^{qs}, \tag{3.55}$$

$$C^H_{abcd} = 2\left(\eta_{abpq}\,g_{c[r}\,g_{s]d} + g_{a[p}\,g_{q]b}\,\eta_{cdrs}\right) u^p\,u^r\,H^{qs}. \tag{3.56}$$

The Bianchi identities (2.6), relating the Ricci tensor to the Weyl tensor, enable gravitational action at a distance of the gravitation field (such as tidal forces and gravitational waves) and influence the motion of matter and radiation through the geodesic deviation equation for timelike and null vectors. This relation can be seen in the works of Levi-Civita (1927), Szekeres (1965) and Szekeres (1966).

The vanishing of the Weyl tensor generates a conformally flat spacetime. By inserting equations (3.54), (3.47) and (3.46) into equation (3.48), we obtain the 1+3 completely decomposed form of the Riemann curvature tensor $\boldsymbol{R}$ as follows

$$R^{ab}{}_{cd} = R^{ab}_{P\,cd} + R^{ab}_{I\,cd} + R^{ab}_{E\,cd} + R^{ab}_{H\,cd}, \tag{3.57}$$

$$\begin{aligned}
R^{ab}_{P\,cd} &= \frac{2}{3}\,(\mu + 3p)\,u^{[a}\,u_{[c}\,h^{b]}{}_{d]} + \frac{2}{3}\mu\,h^{[a}{}_{[c}\,h^{b]}{}_{d]}, \\
R^{ab}_{I\,cd} &= -2\,u^{[a}\,h^{b]}{}_{[c}\,q_{d]} - 2\,u_{[c}\,h^{[a}{}_{d]}\,q^{b]} - 2\,u^{[a}\,u_{[c}\,\pi^{b]}{}_{d]} + 2\,h^{[a}{}_{[c}\,\pi^{b]}{}_{d]}, \\
R^{ab}_{E\,cd} &= 4\,u^{[a}\,u_{[c}\,E^{b]}{}_{d]} + 4\,h^{[a}{}_{[c}\,E^{b]}{}_{d]}, \\
R^{ab}_{H\,cd} &= 2\,\varepsilon^{abe}\,u_{[c}\,H_{d]e} + 2\,\varepsilon_{cde}\,u^{[a}\,H^{b]e},
\end{aligned} \tag{3.58}$$

where $P$ represents the perfect fluid part, $I$ represents the imperfect fluid part, $E$ describes the electric part and $H$ describes the magnetic part, respectively.

## 3.5 The field equations

We now consider the dynamical quantities in the 1+3 formalism of first order gravity for an arbitrary spacetime. The arbitrary spacetime may be completely characterised by the set of geometric quantities

$$\{\Theta, \dot{u}_a, \sigma_{ab}, \omega_{ab}, E_{ab}, H_{ab}\}\,, \tag{3.59}$$

as well as the set of thermodynamic variables

$$\{\mu, p, q_a, \pi_{ab}\}\,, \tag{3.60}$$

provided an equation of state is prescribed which relates the thermodynamic variables. We can obtain the propagation, evolution and constraint equations for the covariant variables, given by (3.59) and (3.60), from the field equations (2.13) and its related integrability conditions. This is discussed in detail in the following subsections.

### 3.5.1 The Ricci identities

We get the first set of propagation equations from the Ricci identities (2.9) for the timelike vector field $u^a$ given by

$$2\nabla_{[a}\nabla_{b]}u^c \;\;=\;\; R_{ab}{}^c{}_d u^d, \tag{3.61}$$

when substituting in from (3.35) and (3.57).

By contracting (3.61) with $u^a$ and separating out the orthogonally projected part into the trace, skew-symmetric and symmetric trace-free parts respectively, we obtain three propagation equations as follows:

1. The expansion propagation equation (the generalized Raychaudhuri (1955) equation)

$$\dot{\Theta} - \mathrm{D}_a \dot{u}^a = -\frac{1}{3}\Theta^2 + \dot{u}_a \dot{u}^a - \sigma_{ab}\sigma^{ab} + 2\omega_a \omega^a - \frac{1}{2}(\mu + 3p), \tag{3.62}$$

describes the nature of attraction of the matter present.

2. The vorticity propagation equation

$$\dot{\omega}^{<a>} - \frac{1}{2}\varepsilon^{abc}\mathrm{D}_b \dot{u}_c = -\frac{2}{3}\Theta\omega^a + \sigma^a{}_b\omega^b. \tag{3.63}$$

3. The shear propagation equation

$$\begin{aligned} \dot{\sigma}^{<ab>} - \mathrm{D}^{<a}\dot{u}^{b>} &= -\frac{2}{3}\Theta\sigma^{ab} + \dot{u}^{<a}\dot{u}^{b>} - \sigma^{<a}{}_c\sigma^{b>c} - \omega^{<a}\omega^{b>} \\ &\quad - \left(E^{ab} - \frac{1}{2}\pi^{ab}\right), \end{aligned} \tag{3.64}$$

indicates how the tidal gravitational field $E_{ab}$ directly induces shear which changes the nature of the fluid flow due to equations (3.62) and (3.63) being affected.

Three sets of constraint equations are obtained by first projecting (3.61) orthogonally to get:

1. The divergence equation for the rate of shear is obtained as

$$0 = (C_1)^a = \mathrm{D}_b\sigma^{ab} - \frac{2}{3}\mathrm{D}^a\Theta + \varepsilon^{abc}\left[\mathrm{D}_b\omega_c + 2\dot{u}_b\omega_c\right] + q^a, \tag{3.65}$$

by contracting over indices $b$ and $c$.

2. The divergence equation for vorticity is obtained as

$$0 = (C_2) = \mathrm{D}_a\omega^a - \dot{u}_a\omega^a, \tag{3.66}$$

by multiplying with $\varepsilon^{abc}$.

3. The magnetic constraint is obtained as

$$0 = (C_3)^{ab} = H^{ab} + 2\,\dot{u}^{<a}\omega^{b>} + \mathrm{D}^{<a}\omega^{b>} - \varepsilon^{cd<a}\mathrm{D}_c\sigma^{b>}{}_d, \tag{3.67}$$

by multiplying with $\varepsilon^{abc}$ and taking the projected symmetric trace-free part. Equation (3.67) characterizes $H_{ab}$ as being constructed from the vorticity 'distortion' and the 'curl' of the shear.

### 3.5.2 The Bianchi identities

From the twice contracted Bianchi identity (2.12), and definitions (3.37) and (3.35), we can rewrite (2.14) as

$$\dot{\mu} + \mathrm{D}_a q^a = -\Theta\,(\mu + p) - 2\,\dot{u}_a q^a - \sigma_{ab}\pi^{ab}, \tag{3.68}$$

by projecting parallel to $u^a$, and

$$\dot{q}^{<a>} + \mathrm{D}^a p + \mathrm{D}_b\pi^{ab} = -\frac{4}{3}\Theta q^a - \sigma^a{}_b q^b - (\mu + p)\,\dot{u}^a - \dot{u}_b\pi^{ab} - \varepsilon^{abc}\omega_b q_c, \tag{3.69}$$

by projecting orthogonal to $u^a$. Equations (3.68) and (3.69) are known as the energy conservation and momentum conservation equations, respectively.

By contracting the second Bianchi identity (2.6) once, we obtain another set of equations. The covariantly decomposed propagation equations are:

1. The gravito-electric $\dot{E}$ propagation equation is given by

$$\begin{aligned}
&\dot{E}^{<ab>} + \frac{1}{2}\dot{\pi}^{<ab>} - \varepsilon^{cd<a}\mathrm{D}_c H^{b>}{}_d + \frac{1}{2}\mathrm{D}^{<a}q^{b>} \\
&= -\frac{1}{2}(\mu + p)\,\sigma^{ab} - \Theta\left(E^{ab} + \frac{1}{6}\pi^{ab}\right) + 3\sigma^{<a}{}_c\left(E^{b>c} - \frac{1}{6}\pi^{b>c}\right) \\
&-\dot{u}^{<a}q^{b>} + \varepsilon^{cd<a}\left[2\,\dot{u}_c H^{b>}{}_d + \omega_c\left(E^{b>}{}_d + \frac{1}{2}\pi^{b>}{}_d\right)\right].
\end{aligned} \tag{3.70}$$

2. The gravito-magnetic $\dot{H}$ propagation equation is given by

$$\dot{H}^{ab} + \varepsilon^{cd<a}\,\mathrm{D}_c\left(E^{b>}{}_d - \frac{1}{2}\pi^{b>}{}_d\right) =$$

$$-\Theta\,H^{ab} + 3\sigma^{<a}{}_c\,H^{b>c} + \frac{3}{2}\omega^{<a}\,q^{b>}$$

$$-\varepsilon^{cd<a}\left[2\,\dot{u}_c\,E^{b>}{}_d - \frac{1}{2}\sigma^{b>}{}_c\,q_d - \omega_c\,H^{b>}{}_d\right]. \qquad (3.71)$$

Equations (3.70) and (3.71) describe gravitational radiation and when combined give a wave equation for $E_{ab}$ and also for $H_{ab}$.

From the once-contracted Bianchi identities, we obtain the constraint equations:

1. The gravito-electric (div $E$) divergence equation is given by

$$\begin{aligned} 0 = (C_4)^a &= \mathrm{D}_b\left(E^{ab} + \pi^{ab}\right) - \frac{1}{3}\mathrm{D}^a\,\mu + \frac{1}{3}\Theta\,q^a - \frac{1}{2}\sigma^a{}_b\,q^b \\ &\quad -3\omega_b\,H^{ab} - \varepsilon^{abc}\left(\sigma_{bd}\,H^d{}_c - \frac{3}{2}\omega_b\,q_c\right), \end{aligned} \qquad (3.72)$$

with the spatial gradient of the energy density as source.

2. The gravito-magnetic (div $H$) divergence equation is given by

$$\begin{aligned} 0 = (C_5)^a &= \mathrm{D}_b\,H^{ab} + (\mu + p)\,\omega^a + 3\omega_b\left(E^{ab} - \frac{1}{6}\pi^{ab}\right) \\ &+ \varepsilon^{abc}\left[\frac{1}{2}\mathrm{D}_b\,q_c + \sigma_{bd}\left(E^d{}_c + \frac{1}{2}\pi^d{}_c\right)\right], \end{aligned} \qquad (3.73)$$

with the fluid velocity as source.

Note that equations (3.65), (3.72) and (3.73) are not ‘real’ constraints due to the presence of spatial and temporal derivatives of the curvature in thermodynamic terms.

### 3.5.3 Evolving the constraints

The following system of equations arises by propagating the constraint equations (3.65), (3.66), (3.67), (3.72) and (3.73) along $u^a$:

$$\begin{aligned}
\left(\dot{C}_1\right)^{<a>} &= -\Theta\left(C_1\right)^a - \frac{3}{2}\sigma^a{}_b\left(C_1\right)^b + \frac{1}{2}\varepsilon^{abc}\omega_b\left(C_1\right)_c \\
&\quad -\frac{8}{3}\omega^a\left(C_2\right) - \varepsilon^{abc}\sigma_{bd}\left(C_3\right)_c{}^d - 3\omega_b\left(C_3\right)^{ab} \\
&\quad -\left(C_4\right)^a, && (3.74) \\
\left(\dot{C}_2\right) &= -\Theta\left(C_2\right), && (3.75) \\
\left(\dot{C}_3\right)^{<ab>} &= -\Theta\left(C_3\right)^{ab} + 3\sigma^{<a}{}_c\left(C_3\right)^{b>c} \\
&\quad +\varepsilon^{cd<a}\omega_c\left(C_3\right)^{b>}{}_d + \frac{1}{2}\varepsilon^{cd<a}\sigma^{b>}{}_c\left(C_1\right)_d \\
&\quad +\frac{3}{2}\omega^{<a}\left(C_1\right)^{b>}, && (3.76) \\
\left(\dot{C}_4\right)^{<a>} - \frac{1}{2}\varepsilon^{abc}\mathrm{D}_b\left(C_5\right)_c &= -\frac{4}{3}\Theta\left(C_4\right)^a + \frac{1}{2}\sigma^a{}_b\left(C_4\right)^b - \frac{1}{2}\varepsilon^{abc}\omega_b\left(C_4\right)_c \\
&\quad -\frac{1}{2}\left(\mu+p\right)\left(C_1\right)^a - \frac{1}{2}\pi^a{}_b\left(C_1\right)^b \\
&\quad +2\varepsilon^{abc}E_{bd}\left(C_3\right)_c{}^d + \frac{3}{2}\varepsilon^{abc}\dot{u}_b\left(C_5\right)_c, && (3.77) \\
\left(\dot{C}_5\right)^{<a>} + \frac{1}{2}\varepsilon^{abc}\mathrm{D}_b\left(C_4\right)_c &= -\frac{4}{3}\Theta\left(C_5\right)^a + \frac{1}{2}\sigma^a{}_b\left(C_5\right)^b - \frac{1}{2}\varepsilon^{abc}\omega_b\left(C_5\right)_c \\
&\quad -\frac{1}{2}\varepsilon^{abc}q_b\left(C_1\right)_c - \frac{2}{3}q^a\left(C_2\right) \\
&\quad +2\varepsilon^{abc}H_{bd}\left(C_3\right)_c{}^d - \frac{3}{2}\varepsilon^{abc}\dot{u}_b\left(C_4\right)_c. && (3.78)
\end{aligned}$$

More information on the above equations may be found in the treatments of van Elst (1996) and Maartens (1997). If the constraints are satisfied on the local 3-space surface at an initial instant, it follows from the above system of equations that the constraints vanish identically when propagated along $u^a$. Therefore, these constraints are satisfied for all time which verifies that under evolution, the constraint equations are preserved. Derivation of equations (3.74)–(3.78) involves application of the commutation relations which are defined in the last subsection of this chapter.

### 3.5.4 Irrotational flow

According to the Frobenius theorem in general relativity, a vector field $\upsilon^a$ is hypersurface orthogonal if and only if

$$\upsilon_{[a}\nabla_b\upsilon_{c]} = 0. \tag{3.79}$$

A detailed explanation of the Frobenius theorem in general relativity can be found in Poisson (2004). If the fundamental vector $u^a$ is hypersurface orthogonal then it follows that

$$\omega_{ab} = 0 \qquad \Leftrightarrow \qquad 0 = u_{[a}\nabla_b u_{c]} = u_{[a}\mathrm{D}_b u_{c]} = u_{[a}\omega_{bc]}. \tag{3.80}$$

Thus the timelike congruence $u^a$ is irrotational. From the Frobenius theorem, we deduce that the distribution of the 3-vector rest spaces is integrable. These instantaneous rest spaces are defined at each point by $h_{ab}$ and 'combine' to make up 3-surfaces in the spacetime orthogonal to $u^a$.

The curvature tensor of the 3-spaces, denoted by ${}^{(3)}R_{abcd}$, is defined by the 3-dimensional version of the Ricci identity (2.9) given by

$$2\mathrm{D}_{[a}\mathrm{D}_{b]}V_c =^{(3)} R_{abc}{}^{d}V_d, \tag{3.81}$$

for any 3-vector $V_a$ on the 3-dimensional manifold. The Gauss equation relates the intrinsic 3-curvature tensor to the Riemann curvature tensor (2.3) and is given by

$${}^{(3)}R_{abcd} = (R_{abcd})_{\perp} - K_{ac}K_{bd} + K_{bc}K_{ad}, \tag{3.82}$$

where $\perp$ denotes projection with $h_{ab}$ with all indices. $K_{ab}$ is the extrinsic curvature defined as

$$K_{ab} = \mathrm{D}_a u_b = \frac{1}{3}\Theta h_{ab} + \sigma_{ab}. \tag{3.83}$$

The 1+3 decomposition of the Riemann tensor (3.57) takes the form

$$\left(R^{ab}{}_{cd}\right)_\perp = \frac{2}{3}\mu h^{[a}{}_{[c} h^{b]}{}_{d]} + 2 h^{[a}{}_{[c} \pi^{b]}{}_{d]} + 4 h^{[a}{}_{[c} E^{b]}{}_{d]}. \tag{3.84}$$

Inserting (3.84) into the Gauss equation (3.82) and contracting, we obtain an expression for the 3-Ricci tensor, denoted by ${}^{(3)}R_{ab}$, as follows

$${}^{(3)}R_{ab} = \left(\frac{2}{3}\mu - \frac{2}{9}\Theta^2\right) h_{ab} - \frac{1}{3}\Theta\omega_{ab} + E_{ab} + \frac{1}{2}\pi_{ab} + \sigma_{ac}\sigma^c{}_b. \tag{3.85}$$

Equation (3.85) can be divided into a trace and trace-free part as follows

$${}^{(3)}R_{ab} = {}^{(3)}S_{ab} + \frac{1}{3}{}^{(3)}R h_{ab}, \tag{3.86}$$

where ${}^{(3)}S_{ab}$ represents the trace-free part (which is essentially equivalent to $E_{ab}$) and $R$ is the 3-Ricci scalar derived by contracting (3.85), yielding

$${}^{(3)}R = 2\mu - \frac{2}{3}\Theta^2 + 2\sigma^2, \tag{3.87}$$

which is the generalized Friedmann equation. The trace and trace-free parts of ${}^{(3)}R_{ab}$ are related to each other by the Bianchi identities for 3-surfaces given by

$$\mathrm{D}_b {}^{(3)}S^b{}_a = \frac{1}{2}\mathrm{D}_a {}^{(3)}R, \tag{3.88}$$

which we note, due to (3.85), is equivalent to the constraint equation (3.72).

Lastly, we mention that the relation between the extrinsic curvature $K_{ab}$ and the 3-Ricci tensor (3.85) is given by the Codacci-Mainardi equation, that is

$$\mathrm{D}_a K^a{}_b - \mathrm{D}_b K^a{}_a = R_{cd} u^d h^c{}_b, \tag{3.89}$$

which we note is equivalent to the constraint equation (3.65) when the vorticity vanishes. For further reading on the Gauss equation and the Codacci-Mainardi equation in this context we refer to Hawking and Ellis (1975), Berger (2003) and the lecture notes of Gourgoulhon (2007).

## 3.6 Commutation relations

In general, the two derivatives - ‘ $\dot{}$ ’ and ‘D’ - do not commute which consequently gives rise to various commutator relations. This is due to the spacetime curvature which is derived from the Ricci identities for spacetime scalars $Z$, 3-vectors $V^a$ and rank two tensors $S^{ab}$ as follows

$$\nabla_{[a}\nabla_{b]}Z = 0, \tag{3.90}$$

$$2\nabla_{[a}\nabla_{b]}V_c = R_{abcd}V^d, \tag{3.91}$$

$$2\nabla_{[a}\nabla_{b]}S_{cd} = -R_{abec}S^e{}_d - R_{abed}S^e{}_c. \tag{3.92}$$

The 3-space commutator relations orthogonal to the congruence of $u^a$ are evaluated by writing out the 3-commutators explicitly and then using the Ricci identities (3.90), (3.91) and (3.92), the splitting of $\nabla_a u_b$ (3.35) and the generalized Gauss equation (3.82). The relations in this subsection can be found in Betschart (2005).

### 3.6.1 3-scalar derivatives

For any scalar function $Z$, the following holds

$$\mathrm{D}_{[a}\mathrm{D}_{b]}Z = \varepsilon_{abc}\omega^c\dot{Z} \qquad \Leftrightarrow \qquad \varepsilon^{abc}\mathrm{D}_b\mathrm{D}_c Z = 2\omega^a\dot{Z}, \tag{3.93}$$

$$\mathrm{D}_a\dot{Z} - (\mathrm{D}_a Z)\dot{}_{\perp} = -\dot{u}_a\dot{Z} + \left(\frac{1}{3}\Theta h_{ab} + \sigma_{ab} + \varepsilon_{abc}\omega^c\right)\mathrm{D}^b Z. \tag{3.94}$$

### 3.6.2 3-vector derivatives

For any 3-vector $V^a$, the following holds

$$
\begin{aligned}
\mathrm{D}_{[a}\mathrm{D}_{b]}V_c \;=\;& \left[\left(E_{c[a}+\frac{1}{2}\pi_{c[a}\right)-\frac{1}{3}\Theta\,\sigma_{c[a}+\frac{1}{3}\Theta\,\omega^d\,\varepsilon_{dc[a}+\omega_c\,\omega_{[a}\right.\\
&\left.+\frac{1}{3}\left(\mu-\frac{1}{3}\Theta^2-3\omega_d\,\omega^d\right)h_{c[a}\right]V_{b]}+\left[h_{c[a}\left(E_{b]d}+\frac{1}{2}\pi_{b]d}\right)\right.\\
&-\frac{1}{3}\Theta\,H_{c[a}\,\sigma_{b]d}-\sigma_{c[a}\,\sigma_{b]d}-\frac{1}{3}\Theta\,h_{c[a}\,\varepsilon_{b]de}\,\omega^e\\
&\left.-\sigma_{c[a}\,\varepsilon_{b]de}\,\omega^e+\sigma_{d[a}\,\varepsilon_{b]ce}\,\omega^e+h_{c[a}\,\omega_{b]}\,\omega_d\right]V^d\\
&+\varepsilon_{abd}\,\omega^d\,\dot{V}_{<c>}, \qquad (3.95)\\
\mathrm{D}_a\dot{V}_b-(\mathrm{D}_aV_b)^{\cdot}_{\perp} \;=\;& -\dot{u}_a\,\dot{V}_{<b>}+\left(\frac{1}{3}\Theta\,h_{ac}+\sigma_{ac}+\varepsilon_{acd}\,\omega^d\right)(\mathrm{D}^cV_b+V^c\,\dot{u}_b)\\
&-H_a{}^d\,\varepsilon_{dbc}\,V^c-\frac{1}{2}h_{ab}\,q_c\,V^c+\frac{1}{2}V_a\,q_b. \qquad (3.96)
\end{aligned}
$$

### 3.6.3 3-tensor derivatives

For any second rank 3-tensor $S_{ab}$, the following holds

$$
\begin{aligned}
\mathrm{D}_{[a}\mathrm{D}_{b]}S^{cd} \;=\;& 2\left[\left(E^c{}_a+\frac{1}{2}\pi^c{}_{[a}\right)-\frac{1}{3}\Theta\,\sigma^{(c}{}_{[a}+\frac{1}{3}\Theta\,\omega^e\,\varepsilon_{e[a}c+\omega^{(c}\,\omega_{[a}\right.\\
&\left.+\frac{1}{3}\left(\mu-\frac{1}{3}\Theta^2-3\omega_e\,\omega^e\right)h^c{}_{[a}\right]S^{d)}{}_{b]}\\
&+2\left[h^{(c}{}_{[a}\left(E_{b]e}+\frac{1}{2}\pi_{b]e}\right)-\frac{1}{3}\Theta\,h^{(c}{}_{[a}\,\sigma_{b]e}-\sigma^{(c}{}_{[a}\,\sigma_{b]e}\right.\\
&-\frac{1}{3}\Theta\,h^{(c}{}_{[a}\,\varepsilon_{b]ef}\,\omega^f-\sigma^{(c}{}_{[a}\,\varepsilon_{b]ef}\,\omega^f-\omega^f\,\varepsilon_{f[a}{}^{(c}\,\sigma_{b]e}\\
&\left.+h^c{}_{[a}\,\omega_{b]}\,\omega_e\right]S^{d)e}+\varepsilon_{abe}\,\omega^e\,\dot{S}^{<cd>}, \qquad (3.97)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{D}_a \dot{S}_{bc} - (\mathrm{D}_a S_{bc})^{\cdot}_{\perp} &= \left(\frac{1}{3}\Theta h_{ad} + \sigma_{ad} + \omega_{ad}\right)\left(\dot{u}_b S^d{}_c + \dot{u}_c S^d{}_b + \mathrm{D}^d S_{bc}\right) \\
&\quad - \dot{u}_a \left(\dot{S}_{bc}\right)_{\perp} \left(h_{a[e} q_{b]} - \varepsilon_{ebd} H^d{}_a\right) S^e{}_c \\
&\quad + \left(h_{a[e} q_{c]} - \varepsilon_{ecd} H^d{}_a\right) S^e{}_b. \qquad (3.98)
\end{aligned}
$$

## 3.7 Summary

In summary, in this chapter we have reviewed the 1+3 covariant approach which splits the spacetime using the timelike vector $u^a$. Two important derivatives, namely the covariant time derivative and the fully orthogonally projected covariant derivative, and their commutation relations were defined. Furthermore, the geometrical and thermodynamical variables and their properties were defined. The covariant derivative of $u^a$, the Weyl tensor and the energy momentum tensor were decomposed into their irreducible parts. Finally, we wrote down the evolution, propagation and constraint equations derived from the field equations which relate the set of 1+3 variables in the formalism.

# Chapter 4

# 1+1+2 formalism

## 4.1 Introduction

The 1+3 covariant approach has successfully been applied in general relativity. However, if we consider a spacetime that admits less symmetry, the resulting 1+3 equations are tensorial partial differential equations that are difficult to work with. We find that a further decomposition is useful. In this chapter, we review the 1+1+2 covariant approach developed by Clarkson and Barrett (2003). This formalism involves a further splitting of the 1+3 variables such that it isolates a specific spatial direction. Hence we obtain a set of variables that are advantageous to treat systems with one preferred spatial direction. For example, if we consider a spherically symmetric spacetime, the 1+1+2 approach is beneficial because we end up with scalar equations which are easier to work with than tensorial equations. The 1+1+2 formalism has generated useful results in the analysis of: linear perturbations of the Schwarzschild spacetime studied by Clarkson and Barrett (2003); locally rotationally symmetric class II spacetimes investigated by Betschart and Clarkson (2004) in general relativity and Nzioki *et al* (2010) in $f(R)$ gravity; gravitational lensing studied by de Swardt *et al* (2010) and general locally rotationally symmetric spacetimes investigated by Singh *et al* (2017). We include in this chapter the 1+1+2 kinematical and Weyl quantities and their properties by way of extension from the 1+3 formalism. The covariant derivatives of special quantities are given and the constraint, propagation and evolution equations are written down as per Clarkson (2007).

## 4.2 Kinematics

In the 1+3 formalism, the timelike unit vector $u^a$ is split in the form $R \otimes V$, where $R$ is the timeline along $u^a$ and $V$ is the 3-space perpendicular to $u^a$. We now split the 3-space $V$, by introducing the unit vector $e^a$ orthogonal to $u^a$ such that

$$e_a u^a = 0, \qquad e_a e^a = 1, \tag{4.1}$$

in the 1+1+2 covariant approach. The projection tensor

$$N_a{}^b \equiv h_a{}^b - e_a e^b = g_a{}^b + u_a u^b - e_a e^b, \tag{4.2}$$

projects vectors orthogonal to $e^a$ and $u^a$ onto 2-spaces referred to as sheets. It follows that

$$e^a N_{ab} = 0 = u^a N_{ab}, \qquad N^a{}_a = 2, \tag{4.3}$$

holds. Any spacetime 3-vector $\Phi^a$ can be irreducibly split into $\chi$, a scalar component along $e^a$ and a 2-vector, $\chi^a$, which is a sheet component orthogonal to $e^a$ as follows

$$\Phi^a = \chi e^a + \chi^a \quad \text{where} \quad \chi \equiv \Phi_a e^a \quad \text{and} \quad \chi^a \equiv N^{ab}\Phi_b \equiv \Phi^{\bar{a}}, \tag{4.4}$$

where the bar on a particular index denotes projection with $N_{ab}$ on that index such that the vector or tensor lies on the sheet.

Similarly we can split a projected, symmetric, trace-free tensor $\Phi_{ab}$ into scalar, 2-vector and 2-tensor parts as follows

$$\Phi_{ab} = \Phi_{<ab>} = \chi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\chi_{(a}e_{b)} + \chi_{ab}, \tag{4.5}$$

where the components

$$\chi \equiv e^a e^b \Phi_{ab} = -N^{ab}\Phi_{ab}, \tag{4.6}$$

$$\chi_a \equiv N_a{}^b e^c \Phi_{bc}, \tag{4.7}$$

$$\chi_{ab} \equiv \chi_{\{ab\}} = \left(N_{(a}{}^c N_{b)}{}^d - \frac{1}{2}N_{ab}N^{cd}\right)\Phi_{cd}, \tag{4.8}$$

are defined. The curly brackets denote the part of the tensor that is projected, symmetric and trace-free, with respect to $e^a$. We note also that

$$h_{\{ab\}} = 0 = N_{\{ab\}}, \qquad N_{<ab>} = -e_{<a}e_{b>} = N_{ab} - \frac{2}{3}h_{ab}. \tag{4.9}$$

The alternating Levi-Civita 2-tensor is defined as

$$\varepsilon_{ab} \equiv \varepsilon_{abc}e^c = \eta_{abcd}e^c u^d, \tag{4.10}$$

where $\varepsilon_{ab}$ is the natural 2-volume element carried by the sheet induced by $\varepsilon_{abc}$, the volume element of the 3-space. From the definition of $\varepsilon_{ab}$ and $N_{ab}$, the following relations:

$$\varepsilon_{ab}e^b = 0 = \varepsilon_{(ab)}, \tag{4.11}$$

$$\varepsilon_{abc} = e_a\varepsilon_{bc} - e_b\varepsilon_{ac} + e_c\varepsilon_{ab}, \tag{4.12}$$

$$\varepsilon_{ab}\varepsilon^{cd} = N_a{}^c N_b{}^d - N_a{}^d N_b{}^c, \tag{4.13}$$

$$\varepsilon_a{}^c\varepsilon_{bc} = N_{ab}, \tag{4.14}$$

$$\varepsilon^{ab}\varepsilon_{ab} = 2 \tag{4.15}$$

hold.

It follows that any object in the 1+1+2 formalism can be split into scalars, 2-vectors and projected, symmetric and trace-free 2-tensors where the latter two components are defined in the sheet. Apart from the 'time' (dot) derivative, defined along the timelike congruence $u^a$, of any object, we introduce two new derivatives for any object $\Phi_{a...b}{}^{c...d}$:

$$\hat{\Phi}_{a...b}{}^{c...d} \equiv e^f \nabla_f M_{a...b}{}^{c...d}, \tag{4.16}$$

$$\delta_f \Phi_{a...b}{}^{c...d} \equiv N_f{}^j N_a{}^l ... N_b{}^g N_h{}^c ... N_i{}^d \mathrm{D}_j \Phi_{l...g}{}^{h...i}, \tag{4.17}$$

defined by the congruence $e^a$. The hat-derivative '$\hat{\ }$' is the spatial derivative along the $e^a$ vector field in the surfaces orthogonal to $u^a$. Hence we observe the congruence $u^a$ retains its primary importance as in the 1+3 approach. The delta-derivative '$\delta$' is the projected spatial derivative onto the 2-sheet, with projection on every free index. Using the aforementioned definitions, we obtain the following relations for the derivatives of $N_{ab}$ and the sheet volume element $\varepsilon_{ab}$:

$$\dot{N}_{ab} = 2u_{(a}\dot{u}_{b)} - 2e_{(a}\dot{e}_{b)} = 2u_{(a}\varphi_{b)}, \tag{4.18}$$

$$\hat{N}_{ab} = -2e_{(a}a_{b)}, \tag{4.19}$$

$$\delta_c N_{ab} = 0, \tag{4.20}$$

$$\dot{\varepsilon}_{ab} = -2u_{[a}\varepsilon_{b]c}\mathcal{A}^c + 2e_{[a}\varepsilon_{b]c}\varphi^c, \tag{4.21}$$

$$\hat{\varepsilon}_{ab} = 2e_{[a}\varepsilon_{b]c}a^c, \tag{4.22}$$

$$\delta_c \varepsilon_{ab} = 0, \tag{4.23}$$

where $\mathcal{A}_a \equiv \dot{u}_{\bar{a}}, \qquad \varphi_a \equiv \dot{e}_{\bar{a}} \qquad$ and $\qquad a_a \equiv e^c \mathrm{D}_c e_a = \hat{e}_a.$

At this point, we take $e^a$ to be arbitrary and then split the 1+3 kinematical and Weyl quantities according to the decompositions (4.4) and (4.5), respectively. The 4-acceleration, vorticity, shear and electric and magnetic Weyl tensor quantities are split irreducibly as

$$\dot{u}^a = \mathcal{A} e^a + \mathcal{A}^a, \tag{4.24}$$

$$\omega^a = \Omega e^a + \Omega^a, \tag{4.25}$$

$$\sigma_{ab} = \Sigma \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 \Sigma_{(a} e_{b)} + \Sigma_{ab}, \tag{4.26}$$

$$E_{ab} = \mathcal{E} \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 \mathcal{E}_{(a} e_{b)} + \mathcal{E}_{ab}, \tag{4.27}$$

$$H_{ab} = \mathcal{H} \left( e_a e_b - \frac{1}{2} N_{ab} \right) + 2 \mathcal{H}_{(a} e_{b)} + \mathcal{H}_{ab}, \tag{4.28}$$

respectively. The expression

$$\sigma^2 = \frac{1}{2} \sigma_{ab} \sigma^{ab} = \frac{3}{4} \Sigma^2 + \Sigma_a \Sigma^a + \frac{1}{2} \Sigma_{ab} \Sigma^{ab}, \tag{4.29}$$

defines the form of $\sigma$, the shear scalar. Using equation (3.33), and the relations found in Appendix A, we obtain the exact form of the covariant decomposition of the derivative of the 3-vector (4.4) given by

$$\begin{aligned}
\nabla_a \Phi_b = & -u_a \left[ \left( \dot{\chi} - \chi_c \varphi^c \right) e_b + \chi \varphi_b + \dot{\chi}_{\bar{b}} \right] - u_a u_b \left( \mathcal{A} \chi + \mathcal{A}_c \chi^c \right) \\
& + u_b \left[ \left( \frac{1}{3} \Theta + \Sigma \right) \chi e_a + \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) \chi_a + \Sigma_a \chi + \Sigma^c \chi_c e_a \right. \\
& \left. + \Sigma_a{}^c \chi_c + \Omega \varepsilon_a{}^c \chi_c - \varepsilon_a{}^c \Omega_c \chi + e_a \varepsilon^{cd} \chi_c \Omega_d \right] \\
& + \frac{1}{3} \left( \hat{\chi} + \chi \phi - \chi_c a^c + \delta_c \chi^c \right) \left( N_{ab} + e_a e_b \right) \\
& + \frac{1}{3} \left( 2 \hat{\chi} - \phi \chi - 2 \chi_c a^c - \delta_c \chi^c \right) \left( e_a e_b - \frac{1}{2} N_{ab} \right) \\
& + \left[ \chi a_{(a} + \delta_{(a} \chi + \hat{\chi}_{(\bar{a}} - \frac{1}{2} \phi \chi_{(a} + \chi^c \left( \varsigma \varepsilon_{c(a} - \zeta_{c(a)} \right) \right] e_{b)} \\
& + \chi \zeta_{ab} + \delta_{\{a} \chi_{b\}} + \frac{1}{2} \varepsilon_{ab} \left( 2 \chi \varsigma + \varepsilon^{cd} \delta_c \chi_d \right) + e_{[a} \varepsilon_{b]c} \chi^c \varsigma
\end{aligned}$$

$$-e_{[a}\left(-\chi\, a_{b]} + \delta_{b]}\chi - \hat{\chi}_{b]} - \frac{1}{2}\phi\,\chi_{b]} - \zeta_{b]c}\chi^{c}\right), \tag{4.30}$$

where

$$\begin{aligned} \phi &\equiv \delta_a e^a, \\ a_a &\equiv e^c \mathrm{D}_c e_a = \hat{e}_a, \\ \mathcal{A} &\equiv e^a \dot{u}_a, \\ \varsigma &\equiv \frac{1}{2}\varepsilon^{ab}\delta_a e_b, \\ \zeta_{ab} &\equiv \delta_{\{a} e_{b\}}, \end{aligned} \tag{4.31}$$

are 1+1+2 kinematical variables that are fundamental objects in the spacetime, and their dynamics give us information about the spacetime geometry. Traveling along $e^a$, $\phi$ represents the sheet expansion, $a_a$ is the sheet acceleration, $\mathcal{A}$ is the radial component of the acceleration of $u^a$, $\varsigma$ represents the vorticity of $e^a$ (the twisting of the sheet) and $\zeta_{ab}$ represents the shear of $e^a$ (the distortion of the sheet). An analogous relation for rank two tensors holds by applying (3.34) and using the relations found in Appendix A.

Using (4.30), we define the full covariant derivative of $e^a$ in its irreducible form as

$$\begin{aligned} \nabla_a e_b &= -\mathcal{A} u_a u_b - u_a \varphi_b + \left(\frac{1}{3}\Theta + \Sigma\right) e_a u_b + \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right) u_b \\ &\quad + e_a a_b + \frac{1}{2}\phi N_{ab} + \varsigma\varepsilon_{ab} + \zeta_{ab}, \end{aligned} \tag{4.32}$$

from which we obtain the spatial derivative of $e^a$ given by

$$\mathrm{D}_a e_b = e_a a_b + \frac{1}{2}\phi N_{ab} + \varsigma\varepsilon_{ab} + \zeta_{ab}. \tag{4.33}$$

The other derivative of $e^a$ is

$$\dot{e}_a = \mathcal{A} u_a + \varphi_a, \tag{4.34}$$

which describes its change along $u^a$. We write down the 1+1+2 split of the full covariant derivative of $u^a$ given by

$$\begin{aligned}\nabla_a u_b &= -u_a\left(\mathcal{A}e_b + \mathcal{A}_b\right) + e_a e_b\left(\frac{1}{3}\Theta + \Sigma\right) + e_a\left(\Sigma_b + \varepsilon_{bc}\Omega^c\right)\\ &\quad + \left(\Sigma_a - \varepsilon_{ac}\Omega^c\right)e_b + N_{ab}\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right) + \Omega\varepsilon_{ab} + \Sigma_{ab}, \end{aligned} \tag{4.35}$$

which implies the useful relation

$$\hat{u}_a = \left(\frac{1}{3}\Theta + \Sigma\right)e_a + \Sigma_a + \varepsilon_{ab}\Omega^b, \tag{4.36}$$

for calculating the Ricci identities.

The spatial covariant derivative of a scalar $\kappa$ is defined as

$$\mathrm{D}_a\kappa = \hat{\kappa}e_a + \delta_a\kappa, \tag{4.37}$$

and for any vector $\kappa^a$ that lies in the sheet, orthogonal to both $u^a$ and $e^a$, the different parts of its spatial derivative are decomposed as follows

$$\mathrm{D}_a\kappa_b = -e_a e_b \kappa_c a^c + e_a\hat{\kappa}_{\bar{b}} - e_b\left[\frac{1}{2}\phi\kappa_a + \left(\varphi\varepsilon_{ac} + \zeta_{ac}\right)\kappa^c\right] + \delta_a\kappa_b. \tag{4.38}$$

Similarly for a projected, symmetric and trace-free 2-tensor $\Psi_{ab}$

$$\mathrm{D}_a\kappa_{bc} = -2e_a e_{(b}\kappa_{c)d}a^d + e_a\hat{\kappa}_{bc} - 2e_{(b}\left[\frac{1}{2}\phi\kappa_{c)a} + \kappa_{c)}{}^d\left(\varphi\varepsilon_{ad} + \zeta_{ad}\right)\right] + \delta_a\kappa_{bc}, \tag{4.39}$$

where $\kappa_{ab} = \kappa_{\{ab\}}$. Finally, we write down the 1+1+2 double derivative expression for the scalar $\kappa$ as

$$\begin{aligned}
\nabla^a \nabla^b \kappa &= -\dot{\kappa}\left\{\frac{1}{3}\Theta\left(N^{ab}+e^a e^b\right)+\Sigma\left(e^a e^b-\frac{1}{2}N_{ab}\right)+2\Sigma^{(a}e^{b)}+\Sigma^{ab}\right. \\
&\left.+e^a\varepsilon^{bc}\Omega_c-e^b\varepsilon^{ac}\Omega_c+\varepsilon^{ab}\Omega\right\}+u^b\left\{\frac{1}{3}\Theta\left(\hat{\kappa}e^a+\delta^a\kappa\right)\right. \\
&+\left[\Sigma\left(e^a e^c-\frac{1}{2}N^{ac}\right)+2\Sigma^{(a}e^{c)}+\Sigma^{ac}\right]\left(\hat{\kappa}e_c+\delta_c\kappa\right) \\
&\left.+\left[e^a\varepsilon^{cd}\Omega_d-e^c\varepsilon^{ad}\Omega_d+\varepsilon^{ac}\Omega\right]\left(\hat{\kappa}e_c+\delta_c\kappa\right)+u^a\ddot{\kappa}-\left(\hat{\dot{\kappa}}e^a+\delta^a\dot{\kappa}\right)\right\} \\
&-u^a\left\{\left(N^{cb}+e^c e^b\right)\left(\hat{\kappa}e_c+\delta_c\kappa\right)+u^b\left(\mathcal{A}e^c+\mathcal{A}^c\right)\left(\hat{\kappa}e_c+\delta_c\kappa\right)\right. \\
&\left.-\dot{\kappa}\left(\mathcal{A}e^b+\mathcal{A}^b\right)\right\} \\
&+\frac{1}{3}\left\{\hat{\hat{\kappa}}+\phi\hat{\kappa}-\delta^c\kappa a_c+\delta^c\delta_c\kappa\right\}\left(N^{ab}+e^a e^b\right) \\
&+\frac{1}{3}\left\{2\hat{\hat{\kappa}}-\phi\hat{\kappa}-2\delta^c\kappa a_c-\delta^c\delta_c\kappa\right\}\left(e^a e^b-\frac{1}{2}N^{ab}\right) \\
&+\left\{2\delta^{(a}\hat{\kappa}-\left(\Sigma^{(a}+\Omega_c\varepsilon^{c(a}\right)\dot{\kappa}-\phi\delta^{(a}\kappa+2\delta_c\kappa\left(\varsigma\varepsilon^{c(a}-\zeta^{c(a)}\right)\right\}e^{b)} \\
&+\hat{\kappa}\zeta^{ab}+\delta^{\{a}\delta^{b\}} \\
&+\frac{1}{2}\left(e^a\varepsilon^{bc}-e^b\varepsilon^{ac}+e^c\varepsilon^{ab}\right)\left\{\left(2\varsigma\hat{\kappa}+\varepsilon_{mn}\delta^m\delta^n\kappa\right)e_c+\varsigma\delta_c\kappa\right. \\
&\left.+\varepsilon_{cm}\left(\Sigma^m\dot{\kappa}-\varepsilon^{mc}\Omega_c\dot{\kappa}+\varepsilon^{mc}\varsigma\delta_c\kappa\right)\right\}. \qquad (4.40)
\end{aligned}$$

## 4.3 The energy momentum tensor

We split the anisotropic fluid variables $q_a$ and $\pi_{ab}$ as follows

$$q_a = Q e_a + Q_a, \qquad (4.41)$$

$$\pi_{ab} = \Pi\left(e_a e_b-\frac{1}{2}N_{ab}\right)+2\Pi_{(a}e_{b)}+\Pi_{ab}, \qquad (4.42)$$

and hence we can write down the total energy momentum tensor as

$$T_{ab} = \mu u_a u_b + p h_{ab} + 2u_{(a}\left[Q e_{b)} + Q_{b)}\right] + \Pi\left(e_a e_b - \frac{1}{2}N_{ab}\right) + 2\Pi_{(a} e_{b)} + \Pi_{ab}, \quad (4.43)$$

in terms of the 1+1+2 variables. The thermodynamic quantities found in (4.43) are representative of the total combination of standard matter and curvature quantities.

## 4.4 Derivatives and commutators

In general the dot '˙', the hat '^' and the delta '$\delta_a$' derivatives do not commute. The commutation relations for any scalar $\kappa$ are

$$\hat{\dot{\kappa}} - \dot{\hat{\kappa}} = -\mathcal{A}\dot{\kappa} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\kappa} + \left(\Sigma_a + \varepsilon_{ab}\Omega^b - \varphi_a\right)\delta^a\kappa, \quad (4.44)$$

$$\delta_a\dot{\kappa} - (\delta_a\kappa)^{\cdot}_{\perp} = -\mathcal{A}_a\dot{\kappa} + \left(\varphi_a + \Sigma_a - \varepsilon_{ab}\Omega^b\right)\hat{\kappa} + \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\delta_a\kappa + \left(\Sigma_{ab} + \Omega\varepsilon_{ab}\right)\delta^b\kappa, \quad (4.45)$$

$$\delta_a\hat{\kappa} - (\delta_a\kappa)^{\hat{}}_{\perp} = -2\varepsilon_{ab}\Omega^b\dot{\kappa} + a_a\hat{\kappa} + \frac{1}{2}\phi\delta_a\kappa + \left(\zeta_{ab} + \varsigma\varepsilon_{ab}\right)\delta^b\kappa, \quad (4.46)$$

$$\delta_{[a}\delta_{b]}\kappa = \varepsilon_{ab}\left(\Omega\dot{\kappa} - \varphi\hat{\kappa}\right), \quad (4.47)$$

where we reintroduce the symbol $\perp$ which now denotes projection onto the sheet. From equations (4.44) and (4.47), we note that the 2-sheet will be a genuine 2-surface, instead of being a collection of tangent planes, if and only if

- The sheet derivatives commute. Specifically the delta derivative will be a true covariant derivative on the surface. This occurs when $\varsigma = \Omega = a^a = 0$.
- The commutator of the time and hat derivative does not depend on any component. This occurs when Greenberg's (1970) vector

$$\Sigma_a + \varepsilon_{ab}\Omega^b - \varphi_a, \quad (4.48)$$

vanishes. Thus the two vector fields $u^a$ and $e^a$ are 2-surface forming.

The commutation relations for any 2-vector $\kappa_a$ are

$$\begin{aligned}
\hat{\dot{\kappa}}_{\bar{a}} - \dot{\hat{\kappa}}_{\bar{a}} &= -\mathcal{A}\,\dot{\kappa}_{\bar{a}} + \left(\frac{1}{3}\Theta + \Sigma\right)\hat{\kappa}_{\bar{a}} + \left(\Sigma_b + \varepsilon_{bc}\,\Omega^c - \varphi_b\right)\delta^b\,\kappa_a \\
&\quad + \mathcal{A}_a\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right)\kappa^b + \mathcal{H}\,\varepsilon_{ab}\,\kappa^b, \qquad (4.49)\\
\delta_a\,\dot{\kappa}_b - \left(\delta_a\,\kappa_b\right)^{\cdot}_{\perp} &= -\mathcal{A}_a\,\dot{\kappa}_b + \left(\varphi_a + \Sigma_a - \varepsilon_{ac}\,\Omega^c\right)\hat{\kappa}_{\bar{b}} \\
&\quad + \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\delta_a\,\kappa_b + \kappa_a\,\mathcal{A}_b\right) \\
&\quad + \left(\Sigma_{ac} + \Omega\,\varepsilon_{ac}\right)\left(\delta^c\,\kappa_b + \kappa^c\,\mathcal{A}_b\right) + \frac{1}{2}\left(\kappa_a\,Q_b - N_{ab}\,\kappa^c\,Q_c\right) \\
&\quad - \left(\frac{1}{2}\phi\,N_{ac} + \varsigma\,\varepsilon_{ac} + \zeta_{ac}\right)\kappa^c\,\varphi_b + \mathcal{H}_a\,\varepsilon_{bc}\,\kappa^c, \qquad (4.50)\\
\delta_a\,\hat{\kappa}_b - \left(\delta_a\,\kappa_b\right)^{\hat{}}_{\perp} &= -2\,\varepsilon_{ac}\,\Omega^c\,\dot{\kappa}_{\bar{b}} + \frac{1}{2}\phi\left(\delta_a\,\kappa_b - \kappa_a\,a_b\right) \\
&\quad + \left(\zeta_{ac} + \varsigma\,\varepsilon_{ac}\right)\left(\delta^c\,\kappa_b - \kappa^c\,a_b\right) \\
&\quad - 2\left(\Omega\,\varepsilon_{a[b} + \Sigma_{a[b}\right)\left(\Sigma_{c]} + \varepsilon_{c]d}\,\Omega^d\right)\kappa^c \\
&\quad - \kappa_a\left[\left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\left(\Sigma_b + \varepsilon_{bc}\,\Omega^c\right) + \frac{1}{2}\Pi_b + \mathcal{E}_b\right] \\
&\quad + N_{ab}\left[\left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\left(\Sigma_c + \varepsilon_{cd}\,\Omega^d\right) + \frac{1}{2}\Pi_c + \mathcal{E}_c\right]\kappa^c, \qquad (4.51)\\
\delta_{[a}\,\delta_{b]}\,\kappa^c &= \left[\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)^2 - \frac{1}{4}\phi^2 + \frac{1}{2}\Pi + \mathcal{E} - \frac{1}{3}\mu\right]\kappa_{[a}\,N_{b]}{}^c \\
&\quad - \kappa_{[a}\left[-\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\Sigma_{b]}{}^c + \Omega\,\varepsilon_{b]}{}^c\right) + \frac{1}{2}\phi\left(\zeta_b{}^c + \varsigma\,\varepsilon_{b]}{}^c\right)\right. \\
&\quad \left. + \frac{1}{2}\Pi_{b]}{}^c + \mathcal{E}_{b]}{}^c\right] + N_{[a}{}^c\left[-\left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\Sigma_{b]d} + \Omega\,\varepsilon_{b]d}\right)\right. \\
&\quad \left. + \frac{1}{2}\phi\left(\zeta_{b]d} + \varsigma\,\varepsilon_{b]d}\right) + \frac{1}{2}\Pi_{b]d} + \mathcal{E}_{b]d}\right]\kappa^d
\end{aligned}$$

$$- \left(\Sigma_{[a}{}^{c} + \Omega\,\varepsilon_{[a}{}^{c}\right)\left(\Sigma_{b]d} + \Omega\,\varepsilon_{b]d}\right)\kappa^{d}$$

$$+ \left(\zeta_{[a}{}^{c} + \varsigma\,\varepsilon_{[a}{}^{c}\right)\left(\zeta_{b]d} + \varsigma\,\varepsilon_{b]d}\right)\kappa^{d} + \varepsilon_{ab}\left(\Omega\,\dot{\kappa}^{\bar{c}} - \varsigma\hat{\kappa}^{\bar{c}}\right). \qquad (4.52)$$

## 4.5 The field equations

The irreducible set of geometric variables

$$\{\Theta, \mathcal{A}, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \phi, \varsigma, \mathcal{A}_a, \Omega_a, \Sigma_a, \varphi_a, a_a, \mathcal{E}_a, \mathcal{H}_a, \Sigma_{ab}, \zeta_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}\}, \qquad (4.53)$$

together with the irreducible set of thermodynamic variables

$$\{\mu, p, Q, \Pi, Q_a, \Pi_a, \Pi_{ab}\}, \qquad (4.54)$$

make up the key variables in the 1+1+2 formalism of first order gravity for a given equation of state. The full 1+1+2 equations for the above covariant variables are obtained by applying the 1+1+2 decomposition procedure to the 1+3 equations, outlined in Appendix A, and also by covariantly splitting the Ricci identities for $e^a$ as follows

$$R_{abc} \equiv 2\,\nabla_{[a}\nabla_{b]}\,e_c - R_{abcd}\,e^d = 0, \qquad (4.55)$$

where $R_{abcd}$ is the Riemann curvature tensor (2.3). Splitting this rank three tensor using $u^a$ and $e^a$, we obtain the evolution equations (along $u^a$) and the propagation equations (along $e^a$) for $\varphi_a$, $a_a$, $\phi$, $\varsigma$, $\zeta_{ab}$. In the subsections below, we write down the full set of 1+1+2 equations for arbitrary spacetime produced and analysed by Clarkson (2007).

### 4.5.1 The evolution equations

We obtain the evolution equations for $\phi$, $\varsigma$, $\zeta_{ab}$ from the projection of $u^a\, R_{abc}$ as follows:
$u^a\, N^{bc}\, R_{abc}$:

$$\begin{aligned}\dot{\phi} &= \left(\frac{2}{3}\Theta - \Sigma\right)\left(\mathcal{A} - \frac{1}{2}\phi\right) + 2\varsigma\,\Omega + \delta_a\,\varphi^a + \mathcal{A}^a\left(\varphi_a - a_a\right) \\ &\quad + \left(a^a - \mathcal{A}^a\right)\left(\Sigma_a - \varepsilon_{ab}\,\Omega^b\right) - \zeta^{ab}\,\Sigma_{ab} + Q. \end{aligned} \tag{4.56}$$

$u^a\,\varepsilon^{bc}\,R_{abc}$:

$$\begin{aligned}\dot{\varsigma} &= \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\varsigma + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Omega + \frac{1}{2}\left(a^a + \mathcal{A}^a\right)\left[\Omega_a + \varepsilon_{ab}\left(\varphi^b + \Sigma^b\right)\right] \\ &\quad + \frac{1}{2}\varepsilon_{ab}\,\delta^a\,\varphi^b - \frac{1}{2}\varepsilon_{ca}\,\zeta_b{}^c\,\Sigma^{ab} + \frac{1}{2}\mathcal{H}. \end{aligned} \tag{4.57}$$

$u^c\, R_{c\{ab\}}$:

$$\begin{aligned}\dot{\zeta}_{\{ab\}} &= \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right)\zeta_{ab} + \Omega\,\varepsilon_{c\{a}\,\zeta_{b\}}{}^c + \left(\mathcal{A} - \frac{1}{2}\phi\right)\Sigma_{ab} - \varsigma\,\varepsilon_{c\{a}\,\Sigma_{b\}}{}^c \\ &\quad - \zeta_{c\{a}\,\Sigma_{b\}}{}^c + \delta_{\{a}\,\varphi_{b\}} + \left(\mathcal{A}_{\{a} - a_{\{a}\right)\varphi_{b\}} \\ &\quad - \left(\mathcal{A}_{\{a} + a_{\{a}\right)\left(\Sigma_{b\}} - \varepsilon_{b\}d}\,\Omega^d\right) - \varepsilon_{c\{a}\,\mathcal{H}_{b\}}{}^c. \end{aligned} \tag{4.58}$$

We note that not all information needed to determine the complete 1+1+2 equations is contained in $R_{abc}$. Hence we use the 1+1+2 decomposition of the standard 1+3 equations to obtain the remaining evolution equations given below.
Vorticity evolution equation:

$$\dot{\Omega} = \frac{1}{2}\varepsilon_{ab}\,\delta^a\,\mathcal{A}^b + \mathcal{A}\varsigma + \Omega\left(\Sigma - \frac{2}{3}\Theta\right) + \Omega_a\left(\Sigma^a + \varphi^a\right). \tag{4.59}$$

Shear evolution equation:

$$\begin{aligned}\dot{\Sigma}_{ab} &= \delta_{\{a}\mathcal{A}_{b\}} + \mathcal{A}_{\{a}\mathcal{A}_{b\}} - \Sigma_{\{a}\left(\Sigma_{b\}} + 2\varphi_{b\}}\right) - \Omega_{\{a}\Omega_{b\}} + \mathcal{A}\zeta_{ab} \\ &\quad -\left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Sigma_{ab} - \Sigma_{c\{a}\Sigma_{b\}}{}^{c} - \mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}. \end{aligned} \tag{4.60}$$

### 4.5.2 Mixture of propagation and evolution equations

In this subsection, we write down a mixture of propagation and evolution equations either by projecting $R_{abc}$ (as indicated) or as a further decomposition of the 1+3 equations.

$u^a e^b R_{ab\bar{c}} = e^a u^b R_{ab\bar{c}}$:

$$\begin{aligned}\hat{\varphi}_{\bar{a}} - \dot{a}_{\bar{a}} &= -\left(\frac{1}{2}\phi + \mathcal{A}\right)\varphi_a - \varsigma\varepsilon_{ab}\varphi^b + \left(\frac{1}{3}\Theta + \Sigma\right)(\mathcal{A}_a - a_a) \\ &\quad +\left(\frac{1}{2}\phi - \mathcal{A}\right)\left(\Sigma_a + \varepsilon_{ab}\Omega^b\right) - \varsigma\left(\varepsilon_{ab}\Sigma^b - \Omega_a\right) \\ &\quad +\zeta_{ab}\left(-\varphi^b + \Sigma^b + \varepsilon^{bc}\Omega_c\right) + \frac{1}{2}Q_a - \varepsilon_{ab}\mathcal{H}^b. \end{aligned} \tag{4.61}$$

$u^a e^b u^c R_{abc} = -e^a u^b u^c R_{abc}$:

$$\begin{aligned}\hat{\mathcal{A}} - \frac{1}{3}\dot{\Theta} - \dot{\Sigma} &= -\mathcal{A}^2 + \left(\frac{1}{3}\Theta + \Sigma\right)^2 - 2\varphi_a\Sigma^a + \Sigma_a\Sigma^a - \Omega_a\Omega^a - a_a\mathcal{A}^a \\ &\quad +\varepsilon_{ab}\varphi^a\Omega^b + \frac{1}{6}(\mu + 3p) + \mathcal{E} - \frac{1}{2}\Pi. \end{aligned} \tag{4.62}$$

Raychaudhuri equation:

$$\begin{aligned}\hat{\mathcal{A}} - \dot{\Theta} &= -\delta_a\mathcal{A}^a - (\mathcal{A} + \phi)\mathcal{A} + (a_a - \mathcal{A}_a)\mathcal{A}^a + \frac{1}{3}\Theta^2 + \frac{3}{2}\Sigma^2 - 2\Omega^2 \\ &\quad +2\Sigma_a\Sigma^a - 2\Omega_a\Omega^a + \Sigma_{ab}\Sigma^{ab} + \frac{1}{2}(\mu + 3p). \end{aligned} \tag{4.63}$$

Vorticity evolution equation:

$$
\begin{aligned}
\dot{\Omega}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{A}}^b &= -\left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Omega_a + \frac{1}{2}\varepsilon_{ab}\left(\delta^b\mathcal{A} - \mathcal{A}a^b - \frac{1}{2}\phi\mathcal{A}^b\right) \\
&\quad + \Omega\left(\Sigma_a - \varphi_a\right) + \frac{1}{2}\varsigma\mathcal{A}_a - \frac{1}{2}\varepsilon_{ab}\zeta^{bc}\mathcal{A}_c + \Sigma_{ab}\Omega^b. \qquad (4.64)
\end{aligned}
$$

Shear evolution equation:

$$
\begin{aligned}
\dot{\Sigma} - \frac{2}{3}\hat{\mathcal{A}} &= \frac{1}{3}\left(2\mathcal{A} - \phi\right)\mathcal{A} - \left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Sigma - \frac{2}{3}\Omega^2 + \Sigma_a\left(2\varphi^a - \frac{1}{3}\Sigma^a\right) \\
&\quad - \frac{1}{3}\delta_a\mathcal{A}^a - \frac{1}{3}\mathcal{A}_a\left(2a^a - \mathcal{A}^a\right) + \frac{1}{3}\Omega_a\Omega^a + \frac{1}{3}\Sigma_{ab}\Sigma^{ab} - \mathcal{E} \\
&\quad + \frac{1}{2}\Pi. \qquad (4.65)
\end{aligned}
$$

$$
\begin{aligned}
\dot{\Sigma}_{\bar{a}} - \frac{1}{2}\hat{\mathcal{A}}_{\bar{a}} &= \frac{1}{2}\delta_a\mathcal{A} + \left(\mathcal{A} - \frac{1}{4}\phi\right)\mathcal{A}_a - \left(\frac{2}{3}\Theta + \frac{1}{2}\Sigma\right)\Sigma_a + \frac{1}{2}\mathcal{A}a_a \\
&\quad - \frac{3}{2}\Sigma\varphi_a - \Omega\Omega_a - \frac{1}{2}\left(\varsigma\varepsilon_{ab} + \zeta_{ab}\right)\mathcal{A}^b + \Sigma_{ab}\left(\varphi^b - \Sigma^b\right) - \mathcal{E}_a \\
&\quad + \frac{1}{2}\Pi_a. \qquad (4.66)
\end{aligned}
$$

Additionally the conservation equations and the magnetic and electric Weyl evolution equations are listed below.

Energy conservation equation:

$$
\begin{aligned}
\dot{\mu} + \hat{Q} &= -\Theta\left(\mu + p\right) - \left(\phi + 2\mathcal{A}\right)Q - \frac{3}{2}\Sigma\Pi + \left(a_a - 2\mathcal{A}_a\right)Q^a \\
&\quad - \delta_a Q^a - 2\Sigma_a\Pi^a - \Sigma_{ab}\Pi^{ab}. \qquad (4.67)
\end{aligned}
$$

Momentum conservation equation:

$$
\begin{aligned}
\dot{Q}+\hat{p}+\hat{\Pi} &= -\delta_a\Pi^a-\left(\frac{3}{2}\phi+\mathcal{A}\right)\Pi-\left(\frac{4}{3}\Theta+\Sigma\right)Q-(\mu+p)\,\mathcal{A} \\
&\quad+\left(\varphi_a-\Sigma_a+\varepsilon_{ab}\Omega^b\right)Q^a+(2a_a-\mathcal{A}_a)\,\Pi^a+\zeta_{ab}\Pi^{ab}. \qquad (4.68) \\
\dot{Q}_{\bar{a}}+\hat{\Pi}_{\bar{a}} &= -\delta_a p+\frac{1}{2}\delta_a\Pi-\delta^b\Pi_{ab}-Q\left(\varphi_a+\Sigma_a+\varepsilon_{ab}\Omega^b\right)-\frac{3}{2}\Pi\, a_a \\
&\quad-\left(\frac{4}{3}\Theta-\frac{1}{2}\Sigma\right)Q_a+\Omega\,\varepsilon_{ab}Q^b-\left(\frac{3}{2}\phi+\mathcal{A}\right)\Pi_a+\varsigma\,\varepsilon_{ab}\Pi^b \\
&\quad-\left(\mu+p-\frac{1}{2}\Pi\right)\mathcal{A}_a-\Sigma_{ab}Q^b-\zeta_{ab}\Pi^b+\Pi_{ab}\left(a^b-\mathcal{A}^b\right). \qquad (4.69)
\end{aligned}
$$

Electric Weyl evolution equation:

$$
\begin{aligned}
\dot{\mathcal{E}}+\frac{1}{2}\dot{\Pi}+\frac{1}{3}\hat{Q} &= \varepsilon_{ab}\delta^a\mathcal{H}^b+\frac{1}{6}\delta_a Q^a+\left(\frac{3}{2}\Sigma-\Theta\right)\mathcal{E}-\frac{1}{2}\left(\frac{1}{3}\Theta+\frac{1}{2}\Sigma\right)\Pi \\
&\quad+\frac{1}{3}\left(\frac{1}{2}\phi-2\mathcal{A}\right)Q+3\varsigma\mathcal{H}-\frac{1}{2}(\mu+p)\,\Sigma \\
&\quad+\frac{1}{3}(a_a+\mathcal{A}_a)\,Q^a+\left(2\varphi_a+\Sigma_a-\varepsilon_{ab}\Omega^b\right)\mathcal{E}^a \\
&\quad+\left(\varphi_a-\frac{1}{6}\Sigma_a-\frac{1}{2}\varepsilon_{ab}\Omega^b\right)\Pi^a+2\varepsilon_{ab}\mathcal{A}^a\mathcal{H}^b \\
&\quad-\Sigma_{ab}\left(\mathcal{E}^{ab}+\frac{1}{2}\Pi^{ab}\right)+\varepsilon_{ab}\mathcal{H}^{bc}\zeta^a{}_c. \qquad (4.70)
\end{aligned}
$$

$$
\begin{aligned}
&\dot{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\varepsilon_{ab}\hat{\mathcal{H}}^b + \frac{1}{2}\dot{\Pi}_{\bar{a}} + \frac{1}{4}\hat{Q}_{\bar{a}} = \\
&\frac{3}{4}\varepsilon_{ab}\delta^b\mathcal{H} + \frac{1}{2}\varepsilon_{bc}\delta^b\mathcal{H}^c{}_a - \frac{1}{4}\delta_a Q + \frac{3}{4}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\varepsilon_{ab}\Omega^b \\
&-\frac{1}{2}\left(\mu + p - \frac{3}{2}\mathcal{E} + \frac{1}{4}\Pi\right)\Sigma_a - \frac{1}{2}Q\mathcal{A}_a + \frac{3}{2}\mathcal{H}\varepsilon_{ab}\mathcal{A}^b \\
&-\frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\varphi_a - \frac{1}{4}Q\,a_a - \frac{3}{4}\mathcal{H}\varepsilon_{ab}a^b - \frac{1}{2}\Omega\varepsilon_{ab}\mathcal{E}^b \\
&+\left(\frac{3}{4}\Sigma - \Theta\right)\mathcal{E}_a + \frac{5}{2}\varsigma\mathcal{H}_a - \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\mathcal{H}^b + \frac{1}{4}\varsigma\varepsilon_{ab}Q^b \\
&+\frac{1}{2}\left(\frac{1}{4}\phi - \mathcal{A}\right)Q_a - \frac{1}{2}\left(\frac{1}{3}\Theta + \frac{1}{4}\Sigma\right)\Pi_a - \frac{1}{4}\Omega\varepsilon_{ab}\Pi^b \\
&+\frac{1}{2}\Sigma_{ab}\left(3\mathcal{E}^b - \frac{1}{2}\Pi^b\right) + \frac{1}{2}\left(3\mathcal{E}_{ab} - \frac{1}{2}\Pi_{ab}\right)\Sigma^b - \mathcal{H}_{ab}\varepsilon^{bc}\mathcal{A}_c \\
&-\left(\mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\right)\left(\varphi^b + \frac{1}{2}\varepsilon^{bc}\Omega_c\right) + \frac{1}{2}\zeta_{ab}\left(\varepsilon^{bc}\mathcal{H}_c + Q^b\right).
\end{aligned}
\tag{4.71}
$$

$$
\begin{aligned}
&\dot{\mathcal{E}}_{\{ab\}} - \varepsilon_{c\{a}\hat{\mathcal{H}}_{b\}}{}^c + \frac{1}{2}\dot{\Pi}_{\{ab\}} = \\
&-\varepsilon_{c\{a}\delta^c\mathcal{H}_{b\}} - \frac{1}{2}\delta_{\{a}Q_{b\}} - \frac{1}{2}\left(\mu + p + 3\mathcal{E} - \frac{1}{2}\Pi\right)\Sigma_{ab} \\
&-\frac{1}{2}Q\zeta_{ab} - \frac{3}{2}\mathcal{H}\varepsilon_{c\{a}\zeta_{b\}}{}^c - \left(\Theta + \frac{3}{2}\Sigma\right)\mathcal{E}_{ab} + \Omega\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^c \\
&-\left(\frac{1}{6}\Theta - \frac{1}{4}\Sigma\right)\Pi_{ab} + \frac{1}{2}\Omega\varepsilon_{c\{a}\Pi_{b\}}{}^c + \varsigma\mathcal{H}_{ab} \\
&+\left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c - \mathcal{A}_{\{a}Q_{b\}} + 2\varepsilon_{c\{a}\mathcal{H}_{b\}}\left(a^c - \mathcal{A}_c\right) \\
&-\left(\varphi_{\{a} + \frac{1}{2}\varepsilon_{c\{a}\Omega^c\right)\left(2\mathcal{E}_{b\}} + \Pi_{b\}}\right) + \Sigma_{\{a}\left(3\mathcal{E}_{b\}} - \frac{1}{2}\Pi_{b\}}\right) \\
&+\Sigma_{c\{a}\left(3\mathcal{E}_{b\}}{}^c - \frac{1}{2}\Pi_{b\}}{}^c\right) + \varepsilon_{c\{a}\mathcal{H}_{b\}d}\zeta^{cd}.
\end{aligned}
\tag{4.72}
$$

Magnetic Weyl evolution equation:

$$\begin{aligned}
\dot{\mathcal{H}} &= -\varepsilon_{ab}\,\delta^a\,\mathcal{E}^b + \frac{1}{2}\varepsilon_{ab}\,\delta^a\,\Pi^b - 3\varsigma\,\mathcal{E} + \left(\Theta + \frac{3}{2}\Sigma\right)\mathcal{H} + \Omega\,Q + \frac{3}{2}\varsigma\,\Pi \\
&\quad -2\,\varepsilon_{ab}\,\mathcal{A}^a\,\mathcal{E}^b + \left(2\varphi_a + \Sigma_a - \varepsilon_{ab}\,\Omega^b\right)\mathcal{H}^a - \frac{1}{2}\left(\Omega_a + \varepsilon_{ab}\,\Sigma^b\right)Q^a \\
&\quad -\Sigma_{ab}\,\mathcal{H}^{ab} - \frac{1}{2}\varepsilon_{ab}\,\mathcal{E}^{bc}\,\zeta^a{}_c. \qquad (4.73)
\end{aligned}$$

$$\begin{aligned}
&\dot{\mathcal{H}}_{\bar{a}} - \frac{1}{2}\varepsilon_{ab}\,\hat{\mathcal{E}}^b + \frac{1}{4}\varepsilon_{ab}\,\hat{\Pi}^b = \\
&-\frac{3}{4}\varepsilon_{ab}\,\delta^b\,\mathcal{E} + \frac{3}{8}\varepsilon_{ab}\,\delta^b\,\Pi - \frac{1}{2}\varepsilon_{bc}\,\delta^b\,\mathcal{E}^c{}_a + \frac{1}{4}\varepsilon_{bc}\,\delta^b\,\Pi^c{}_a + \frac{3}{4}\mathcal{H}\,\Sigma_a \\
&+\frac{1}{3}Q\,\varepsilon_{ab}\,\Sigma^b + \frac{3}{4}Q\,\Omega_a + \frac{3}{4}\mathcal{H}\,\varepsilon_{ab}\,\Omega^b - \frac{3}{2}\mathcal{E}\,\varepsilon_{ab}\,\mathcal{A}^b - \frac{3}{2}\mathcal{H}\,\varphi_a \\
&+\frac{3}{4}\left(\mathcal{E} - \frac{1}{2}\Pi\right)\varepsilon_{ab}\,a^b - \frac{5}{2}\varsigma\,\mathcal{E}_a + \left(\frac{1}{4}\phi + \mathcal{A}\right)\varepsilon_{ab}\,\mathcal{E}^b - \frac{1}{8}\phi\,\varepsilon_{ab}\,\Pi^b \\
&+\left(\frac{3}{4}\Sigma - \Theta\right)\mathcal{H}_a - \frac{1}{2}\Omega\,\varepsilon_{ab}\,\mathcal{H}^b + \frac{3}{4}\Omega\,Q_a - \frac{3}{8}\Sigma\,\varepsilon_{ab}\,Q^b + \frac{5}{4}\varsigma\Pi_a \\
&+\Sigma_{ab}\left(\frac{3}{2}\mathcal{H}^b + \frac{1}{4}\varepsilon^{bc}\,Q_c\right) + \frac{3}{2}\varepsilon_{ab}\,\zeta^{bc}\left(\mathcal{E}_c - \frac{1}{2}\Pi_c + \frac{2}{3}\mathcal{A}_c\right) \\
&+\mathcal{H}_{ab}\left(\varphi^b + \frac{3}{2}\Sigma^b - \frac{1}{2}\varepsilon^{bc}\,\Omega_c\right). \qquad (4.74)
\end{aligned}$$

$$
\begin{aligned}
&\dot{\mathcal{H}}_{\{ab\}} + \varepsilon_{c\{a}\hat{\mathcal{E}}_{b\}}{}^{c} - \frac{1}{2}\varepsilon_{c\{a}\hat{\Pi}_{b\}}{}^{c} = \\
&\varepsilon_{c\{a}\delta^{c}\mathcal{E}_{b\}} - \frac{1}{2}\varepsilon_{c\{a}\delta^{c}\Pi_{b\}} - \frac{3}{2}\mathcal{H}\Sigma_{ab} + \frac{1}{2}Q\varepsilon_{c\{a}\Sigma_{b\}}{}^{c} \\
&+\frac{3}{2}\left(\mathcal{E} - \frac{1}{2}\Pi\right)\varepsilon_{c\{a}\zeta_{b\}}{}^{c} - \varsigma\mathcal{E}_{ab} - \left(\Theta + \frac{3}{2}\Sigma\right)\mathcal{H}_{ab} \\
&-\left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{c\{a}\mathcal{E}_{b\}}{}^{c} - \Omega\varepsilon_{c\{a}\mathcal{H}_{b\}}{}^{c} + \frac{1}{2}\varsigma\Pi_{ab} \\
&+\frac{1}{4}\phi\varepsilon_{c\{a}\Pi_{b\}}{}^{c} + \Sigma_{\{a}\left(3\mathcal{H}_{b\}} - \varepsilon_{b\}c}Q^{c}\right) - 2\varphi_{\{a}\mathcal{H}_{b\}} \\
&+\Omega_{\{a}\left(\frac{3}{2}Q_{b\}} - \varepsilon_{b\}c}\mathcal{H}^{c}\right) + \mathcal{E}_{\{a}2\varepsilon_{b\}c}\left(a^{c} + \mathcal{A}^{c}\right) \\
&-\Pi_{\{a}\varepsilon_{b\}c}a^{c} + 3\Sigma_{c\{a}\mathcal{H}_{b\}}{}^{c} - \varepsilon_{c\{a}\zeta^{cd}\left(\mathcal{E}_{b\}d} - \frac{1}{2}\Pi_{b\}d}\right).
\end{aligned}
\tag{4.75}
$$

### 4.5.3 The propagation equations

Following a similar procedure, the propagation and constraint equations are derived by either projecting $R_{abc}$ as shown in this subsection, or from projections of the 1+3 constraint equations in Section 3.5.

$e^{a}N^{bc}R_{abc}$:

$$
\begin{aligned}
\hat{\phi} &= -\frac{1}{2}\phi^{2} + 2\varsigma^{2} + \left(\frac{1}{3}\Theta + \Sigma\right)\left(\frac{2}{3}\Theta - \Sigma\right) + \delta_{a}a^{a} - a_{a}a^{a} \\
&\quad -\zeta_{ab}\zeta^{ab} + 2\varepsilon_{ab}\varphi^{a}\Omega^{b} - \Sigma_{a}\Sigma^{a} + \Omega_{a}\Omega^{a} - \frac{2}{3}\mu - \frac{1}{2}\Pi - \mathcal{E}.
\end{aligned}
\tag{4.76}
$$

$e^{a}\varepsilon^{bc}R_{abc}$:

$$
\hat{\varsigma} = -\phi\varsigma + \left(\frac{1}{3}\Theta + \Sigma\right)\Omega + \frac{1}{2}\varepsilon_{ab}\delta^{a}a^{b} + \frac{1}{2}\varepsilon_{ab}\Sigma^{a}a^{b} + \left(\frac{1}{2}a_{a} + \varphi_{a}\right)\Omega^{a}. \tag{4.77}
$$

$e^a\,R_{a\{bc\}}$:

$$
\begin{aligned}
\hat{\zeta}_{\{ab\}} \;\; &= \;\; -\phi\zeta_{ab} - \zeta^c{}_{\{a}\zeta_{b\}c} + \delta_{\{a}a_{b\}} - a_{\{a}a_{b\}} + 2\varphi_{\{a}\varepsilon_{b\}c}\Omega^c - \Omega_{\{a}\Omega_{b\}} \\
&\quad -\Sigma_{\{a}\Sigma_{b\}} + \left(\frac{1}{3}\Theta + \Sigma\right)\Sigma_{ab} - \frac{1}{2}\Pi_{ab} - \mathcal{E}_{ab}. \qquad (4.78)
\end{aligned}
$$

Additionally the divergence equations for the shear, vorticity and the electric and magnetic Weyl parts are written below.

Shear divergence equation $(C_1)^a\,e_a$:

$$
\begin{aligned}
\hat{\Sigma} - \frac{2}{3}\hat{\Theta} \;\; &= \;\; -\frac{3}{2}\phi\Sigma - 2\varsigma\Omega - \delta_a\Sigma^a - \varepsilon_{ab}\delta^a\Omega^b + 2\Sigma_a a^a - 2\varepsilon_{ab}\mathcal{A}^a\Omega^b \\
&\quad +\Sigma_{ab}\zeta^{ab} - Q. \qquad (4.79)
\end{aligned}
$$

$(C_1)_{\bar{a}}$:

$$
\begin{aligned}
\hat{\Sigma}_{\bar{a}} - \varepsilon_{ab}\hat{\Omega}^b \;\; &= \;\; \frac{1}{2}\delta_a\Sigma + \frac{2}{3}\delta_a\Theta - \varepsilon_{ab}\delta^b\Omega - \frac{3}{2}\phi\Sigma_a + \varsigma\varepsilon_{ab}\Sigma^b - \varsigma\Omega_a - \frac{3}{2}\Sigma a_a \\
&\quad + \left(\frac{1}{2}\phi + 2\mathcal{A}\right)\varepsilon_{ab}\Omega^b + \Omega\varepsilon_{ab}\left(a^b - 2\mathcal{A}^b\right) - \delta^b\Sigma_{ab} - \zeta_{ab}\Sigma^b \\
&\quad +\Sigma_{ab}a^b + \varepsilon_{ab}\zeta^{bc}\Omega_c - Q_a. \qquad (4.80)
\end{aligned}
$$

Vorticity divergence equation $(C_2)$:

$$
\hat{\Omega} \;\; = \;\; -\delta_a\Omega^a + (\mathcal{A} - \phi)\,\Omega + (a_a + \mathcal{A}_a)\,\Omega^a. \qquad (4.81)
$$

$(C_3)_{\{ab\}}$:

$$
\begin{aligned}
\hat{\Sigma}_{\{ab\}} \;\; &= \;\; \delta_{\{a}\Sigma_{b\}} - \varepsilon_{c\{a}\delta^c\Omega_{b\}} - \frac{1}{2}\phi\Sigma_{ab} + \varsigma\varepsilon_{c\{a}\Sigma_{b\}}{}^c + \frac{3}{2}\Sigma\zeta_{ab} - \Omega\varepsilon_{c\{a}\zeta_{b\}}{}^c \\
&\quad -2\Sigma_{\{a}a_{b\}} - 2\varepsilon_{c\{a}\mathcal{A}^c\Omega_{b\}} - \Sigma_{c\{a}\zeta_{b\}}{}^c - \varepsilon_{c\{a}\mathcal{H}_{b\}}{}^c. \qquad (4.82)
\end{aligned}
$$

Electric Weyl Divergence equation $(C_4)^a\, e_a$:

$$\begin{aligned}
\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} + \frac{1}{2}\hat{\Pi} &= -\delta_a \mathcal{E}^a - \frac{1}{2}\delta_a \Pi^a - \frac{3}{2}\phi\left(\mathcal{E} + \frac{1}{2}\Pi\right) + \left(\frac{1}{2}\Sigma - \frac{1}{3}\Theta\right) Q \\
&\quad +3\Omega\mathcal{H} + (2\mathcal{E}_a + \Pi_a)\, a^a + \frac{1}{2}\Sigma_a Q^a + 3\Omega_a \mathcal{H}^a \\
&\quad -\frac{3}{2}\varepsilon_{ab}\Omega^a Q^b + \varepsilon_{ab}\Sigma^{ac}\mathcal{H}_c{}^b + \left(\mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\right)\zeta^{ab}.
\end{aligned} \tag{4.83}$$

$(C_4)_{\bar{a}}$:

$$\begin{aligned}
\hat{\mathcal{E}}_{\bar{a}} + \frac{1}{2}\hat{\Pi}_{\bar{a}} &= \frac{1}{2}\delta_a \mathcal{E} + \frac{1}{3}\delta_a \mu + \frac{1}{4}\delta_a \Pi - \delta^b \mathcal{E}_{ab} - \frac{1}{2}\delta^b \Pi_{ab} + \frac{1}{2} Q \Sigma_a + \mathcal{H}\varepsilon_{ab}\Sigma^b \\
&\quad -\frac{3}{2}\mathcal{H}\Omega_a - \frac{3}{2} Q \varepsilon_{ab}\Omega^b - \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right) a_a - \frac{3}{2}\phi\left(\mathcal{E}_a + \frac{1}{2}\Pi_a\right) \\
&\quad +\frac{3}{2}\Omega\varepsilon_{ab} Q^b + \varsigma\varepsilon_{ab}\left(\mathcal{E}^b + \frac{1}{2}\Pi^b\right) + 3\Omega\mathcal{H}_a - \Sigma\varepsilon_{ab}\mathcal{H}^b \\
&\quad -\left(\frac{1}{3}\Theta + \frac{1}{4}\Sigma\right) Q_a + \frac{1}{2}\Sigma_{ab} Q^b - \zeta_{ab}\left(\mathcal{E}^b + \frac{1}{2}\Pi^b\right) \\
&\quad +\left(\mathcal{E}_{ab} + \frac{1}{2}\Pi_{ab}\right) a^b + 3\mathcal{H}_{ab}\Omega^b.
\end{aligned} \tag{4.84}$$

Magnetic Weyl divergence equation $(C_5)^a\, e_a$:

$$\begin{aligned}
\hat{\mathcal{H}} &= -\delta_a \mathcal{H}^a - \frac{1}{2}\varepsilon_{ab}\delta^a Q^b - \frac{3}{2}\phi\mathcal{H} - \left(3\mathcal{E} + \mu + p - \frac{1}{2}\Pi\right)\Omega - Q\varsigma \\
&\quad +2\mathcal{H}_a a^a - 3\Omega_a\left(\mathcal{E}^a - \frac{1}{6}\Pi^a\right) + \zeta_{ab}\mathcal{H}^{ab} - \varepsilon_{ab}\Sigma^a{}_c\left(\mathcal{E}^{bc} + \frac{1}{2}\Pi^{bc}\right).
\end{aligned} \tag{4.85}$$

$(C_5)_{\bar{a}}$:

$$
\begin{aligned}
\hat{\mathcal{H}}_{\bar{a}} - \frac{1}{2}\varepsilon_{ab}\hat{Q}^b &= \frac{1}{2}\delta_a\mathcal{H} - \delta^b\mathcal{H}_{ab} - \frac{1}{2}\varepsilon_{ab}\delta^b Q - \frac{3}{2}\left(\mathcal{E} + \frac{1}{2}\Pi\right)\varepsilon_{ab}\Sigma^b - \frac{3}{2}\phi\mathcal{H}_a \\
&\quad - \left(-\frac{3}{2}\mathcal{E} + \mu + p + \frac{1}{4}\Pi\right)\Omega_a - \frac{3}{2}\mathcal{H}a_a + \frac{1}{2}Q\varepsilon_{ab}a^b - 3\Omega\mathcal{E}_a \\
&\quad + \frac{3}{2}\Sigma\varepsilon_{ab}\mathcal{E}^b + \varsigma\varepsilon_{ab}\mathcal{H}^b - \frac{1}{2}\varsigma Q_a + \frac{1}{4}\phi\varepsilon_{ab}Q^b + \frac{1}{2}\Omega\Pi_a \\
&\quad + \frac{3}{4}\Sigma\varepsilon_{ab}\Pi^b + \mathcal{H}_{ab}a^b - \zeta_{ab}\mathcal{H}^b - 3\left(\mathcal{E}_{ab} - \frac{1}{6}\Pi_{ab}\right)\Omega^b \\
&\quad + \frac{1}{2}\varepsilon_{ab}\zeta^{bc}Q_c. \qquad (4.86)
\end{aligned}
$$

### 4.5.4 Constraint equations

Lastly, we write down the 1+1+2 constraint equations and analyze the system of equations.

$\varepsilon^{ab}u^c R_{abc}$:

$$
\delta_a\Omega^a + \varepsilon_{ab}\delta^a\Sigma^b = (2\mathcal{A} - \phi)\Omega - 3\varsigma\Sigma + \varepsilon_{ab}\zeta^{ac}\Sigma^b{}_c + \mathcal{H}. \qquad (4.87)
$$

$N^{bc}R_{\bar{a}bc}$:

$$
\begin{aligned}
&\frac{1}{2}\delta_a\phi - \varepsilon_{ab}\delta^b\varsigma - \delta^b\zeta_{ab} = \\
&-\Omega\left(\Omega_a + \varepsilon_{ab}\Sigma^b - 2\varepsilon_{ab}\varphi^b\right) - \left(\frac{1}{3}\Theta - \frac{1}{2}\Sigma\right)\left(\Sigma_a - \varepsilon_{ab}\Omega^b\right) \\
&-2\varsigma\varepsilon_{ab}a^b - \left(\Sigma^b - \varepsilon^{bc}\Omega_c\right)\Sigma_{ab} - \frac{1}{2}\Pi_a - \mathcal{E}_a. \qquad (4.88)
\end{aligned}
$$

From $(C_3)_{ab}\, e^b$ and $(C_1)_{\bar{a}}$ or $e^a\, u^c\, R_{a\bar{b}c}$:

$$\delta_a \Sigma - \frac{2}{3}\delta_a \Theta + 2\varepsilon_{ab}\,\delta^b\,\Omega + 2\,\delta^b\,\Sigma_{ab} =$$
$$-\phi\left(\Sigma_a - \varepsilon_{ab}\,\Omega^b\right) - 2\varsigma\left(\Omega_a - 3\,\varepsilon_{ab}\,\Sigma^b\right) - 4\,\Omega\,\varepsilon_{ab}\,\mathcal{A}^b$$
$$+2\zeta_{ab}\,\Sigma^b + 2\,\varepsilon_{ab}\,\zeta^{bc}\,\Omega_c + \Sigma_{ab}\,a^b - 2\,\varepsilon_{ab}\,\mathcal{H}^b - Q_a. \tag{4.89}$$

We note a few things:

– Equations (4.88) and (4.89) are not actual constraints in the proper sense because of the presence of curvature thermodynamic terms that have spatial and temporal derivatives of the curvature.

– The equation formed when considering $(C_3)_{ab}\, e^a\, e^b$ is equivalent to (4.87).

– Equation (4.62) can be written in terms of (4.63) and (4.65) with the combination: (4.62) = $\frac{1}{3}$(4.63) – (4.65).

– The redundancy in the field equations occurs because some of the information contained in $R_{abc}$ is already contained in the 1+3 equations.

– Finally, there are no evolution equations for $\mathcal{A}$, $\mathcal{A}_a$, $\varphi_a$ and no propagation equation for $a_a$ written down. These are determined when we choose a particular frame.

## 4.6 Summary

In summary, in this chapter we have presented an overview of the 1+1+2 covariant approach. In this formalism, the spacetime is further split through a preferred spatial vector $e^a$ which is orthogonal to $u^a$. The hat and delta derivatives and their commutation relations were defined. The 1+3 kinematical and Weyl quantities were decomposed irreducibly and the covariant derivatives of $u^a$ and $e^a$ were specified in the context of this formalism. Finally, the evolution, propagation and constraint equations were written down and analyzed.

# Chapter 5

# Conformal symmetry: Kinematics

## 5.1 Introduction

Conformal symmetry is a current topic that has been widely studied in the context of general relativity. It possesses the geometric property of preserving the structure of the null cone by mapping null geodesics to null geodesics. These symmetries are physically significant as they generate constants of the motion along null geodesics for massless particles. Conformal symmetry has been applied to cosmology in many different spacetimes. Recent advances were made in static spherically symmetric spacetimes by Manjonjo *et al* (2018), in shear-free spherically symmetric spacetimes by Moopanar and Maharaj (2013) and in general spherically symmetric spacetimes by Moopanar and Maharaj (2010). We also mention its application to relativistic stars analyzed by Kileba Matondo *et al* (2018). Of particular interest to us is the study of Maartens *et al* (1986) where the kinematic and dynamic properties of conformal Killing vectors in anisotropic fluids were investigated. In this chapter, we extend the Lie derivative kinematic results of Maartens *et al* (1986) completely in terms of the 1+1+2 decomposition variables for a general spacetime. This process is conducted in order to transparently bring out the behaviours of certain scalars which was not possible before. We perform an analysis of the findings thereafter.

## 5.2 Kinematics

We begin by defining a conformal Killing vector (CKV) field in general by the relation

$$\mathcal{L}_{\mathbf{X}}\, g_{ab} \;=\; 2\,\Psi\, g_{ab}, \tag{5.1}$$

where $\mathcal{L}_{\mathbf{X}}$ represents the Lie derivative along the CKV $\mathbf{X}$ and $\Psi\,(x^c)$ is the conformal factor. The set of all CKVs generates a Lie algebra with basis $\{\mathbf{X}_I\}$. The elements of the basis are related by

$$[\mathbf{X}_I, \mathbf{X}_J] = C^K{}_{IJ}\,\mathbf{X}_K, \tag{5.2}$$

where $C^K{}_{IJ}$ are the structure constants of the group that satisfy the following conditions

$$C^K{}_{IJ} = -C^K{}_{JI} \qquad \text{[Anti-symmetry]}, \tag{5.3}$$

$$C^K{}_{LM}\,C^M{}_{IJ} + C^K{}_{IM}\,C^M{}_{JL} + C^K{}_{JM}\,C^M{}_{LI} = 0 \qquad \text{[Lie identity]}. \tag{5.4}$$

The integrability condition for the existence of the conformal vector (5.1) is

$$\mathcal{L}_{\mathbf{X}}\, C^a{}_{bcd} = 0, \tag{5.5}$$

as given by Hall and Steele (1991).

Now suppose an anisotropic fluid spacetime admits a CKV $\boldsymbol{\xi}$ such that

$$\mathcal{L}_{\boldsymbol{\xi}}\, g_{ab} \;=\; 2\,\Psi\, g_{ab}, \tag{5.6}$$

in accordance with (5.1). The CKV $\xi^a$ can be timelike or spacelike. Considering the fluid 4-velocity $u^a$, we note

$$u^a\,u_a = -1 \qquad \Longrightarrow \qquad \mathcal{L}_{\boldsymbol{\xi}}\,(u^a\,u_a) = 0. \tag{5.7}$$

We can define $\mathcal{L}_{\boldsymbol{\xi}}\, u^a$ in general as

$$\mathcal{L}_{\boldsymbol{\xi}}\, u^a = A\, u^a + B\, e^a + C^{<a>}, \tag{5.8}$$

where $C^{<a>}$ is a 2-vector and $u^a\, C_{<a>} = 0 = e^a\, C_{<a>}$. Substituting (5.8) into (5.7) and expanding we get

$$\begin{aligned} u_a\, \mathcal{L}_{\boldsymbol{\xi}}\, u^a + u^a\, \mathcal{L}_{\boldsymbol{\xi}}\, u_a &= 0, \\ u_a \left(A\, u^a + B\, e^a + C^{<a>}\right) + u^a\, \mathcal{L}_{\boldsymbol{\xi}} \left(g_{ab}\, u^b\right) &= 0, \\ -A + u^a \left(g_{ab}\, \mathcal{L}_{\boldsymbol{\xi}}\, u^b + u^b\, \mathcal{L}_{\boldsymbol{\xi}}\, g_{ab}\right) &= 0, \\ -A + u_b\, \mathcal{L}_{\boldsymbol{\xi}}\, u^b + u^a\, u^b \left(2\, \Psi\, g_{ab}\right) &= 0, \\ -A + u_b \left(A\, u^b + B\, e^b + C^{<b>}\right) - 2\, \Psi &= 0, \\ A + \Psi &= 0. \end{aligned} \tag{5.9}$$

We substitute (5.9) into the general expression (5.8) to obtain

$$\mathcal{L}_{\boldsymbol{\xi}}\, u^a = -\Psi\, u^a + B\, e^a + C^{<a>}, \qquad \text{and} \tag{5.10}$$

$$\mathcal{L}_{\boldsymbol{\xi}}\, u_a = \Psi\, u_a + B\, e_a + C_{<a>}, \tag{5.11}$$

similarly. We can define $\mathcal{L}_{\boldsymbol{\xi}}\, e^a$ in general as

$$\mathcal{L}_{\boldsymbol{\xi}}\, e^a = D\, e^a + E\, u^a + F^{<a>}, \tag{5.12}$$

where $F^{<a>}$ is a 2-vector and $u^a\, F_{<a>} = 0 = e^a\, F_{<a>}$. Following a similar procedure, detailed above, we conclude

$$D + \Psi = 0. \tag{5.13}$$

Substituting (5.13) into the general expression (5.12), we get

$$\mathcal{L}_{\boldsymbol{\xi}}\, e^a = -\Psi\, e^a + E\, u^a + F^{<a>}, \qquad \text{and} \tag{5.14}$$

$$\mathcal{L}_{\boldsymbol{\xi}}\, e_a = \Psi\, e_a + E\, u_a + F_{<a>}, \tag{5.15}$$

similarly.

We observe we can always write $\xi^a$ as

$$\xi^a = \alpha u^a + \beta e^a + \nu^{<a>}, \tag{5.16}$$

where $u^a \nu_{<a>} = 0 = e^a \nu_{<a>}$, $\alpha = -\xi_a u^a$ and $\beta = \xi_a e^a$. Therefore

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} u_a &= \xi^b \nabla_b u_a + u_b \nabla_a \xi^b \\
&= \left(\alpha u^b + \beta e^b + \nu^{<b>}\right) \nabla_b u_a + u_b \nabla_a \left(\alpha u^b + \beta e^b + \nu^{<b>}\right) \\
&= \alpha u^b \nabla_b u_a + \beta e^b \left(\nabla_b u_a\right) + \nu^{<b>} \left(\nabla_b u_a\right) \\
&\quad + u_b \left(\nabla_a \alpha\right) u^b + \alpha u_b \left(\nabla_a u^b\right) + u_b \left(\nabla_a \beta\right) e^b \\
&\quad + \beta u_b \left(\nabla_a e^b\right) + u_b \left(\nabla_a \nu^{<b>}\right).
\end{aligned} \tag{5.17}$$

Now substituting the definitions of the 1+1+2 covariant derivatives of vector fields $u_a$ (see (4.35)) and $e_a$ (see (4.32)) along with $u_b \left(\nabla_a \nu^{<b>}\right) = -\nu^{<b>} \left(\nabla_a u_b\right)$, we progress further and obtain

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} u_a &= \alpha \left(\mathcal{A} e_a + \mathcal{A}_a\right) + \beta \left(\mathcal{A} u_a + 2 \varepsilon_{ac} \Omega^c\right) \\
&\quad + 2 \nu^{<b>} \left(\frac{1}{2} u_a \mathcal{A}_b - e_a \varepsilon_{bc} \Omega^c - \Omega \varepsilon_{ab}\right) + \left(\dot{\alpha} u_a - \hat{\alpha} e_a - \delta_a \alpha\right) \\
&= \left(\beta \mathcal{A} + \mathcal{A}_b \nu^{<b>} + \dot{\alpha}\right) u_a \\
&\quad + \left(\alpha \mathcal{A} - 2 \nu^{<b>} \varepsilon_{bc} \Omega^c - \hat{\alpha}\right) e_a \\
&\quad + \alpha \mathcal{A}_a + 2 \beta \varepsilon_{ac} \Omega^c - 2 \nu^{<b>} \Omega \varepsilon_{ab} - \delta_a \alpha,
\end{aligned} \tag{5.18}$$

after some general simplification. Thus we can evaluate the components of $\mathcal{L}_{\boldsymbol{\xi}} u_a$ in (5.11) as

$$\Psi = \beta \mathcal{A} + \mathcal{A}_b \xi^b + \dot{\alpha}, \tag{5.19}$$

$$B = \alpha \mathcal{A} - 2 \xi^b \varepsilon_{bc} \Omega^c - \hat{\alpha}, \tag{5.20}$$

$$C_{<a>} = \alpha \mathcal{A}_a + 2 \beta \varepsilon_{ac} \Omega^c - 2 \xi^b \Omega \varepsilon_{ab} - \delta_a \alpha, \tag{5.21}$$

where we have substituted $\xi^a$ in the place of $\nu^{<a>}$ according to the definition given by equation (5.16).

We consider two special cases. The first case involves assuming $\xi_a u^a = 0 = -\alpha$. Then expressions (5.19), (5.20) and (5.21) reduce to

$$\Psi = \beta\mathcal{A} + \mathcal{A}_b\xi^b, \tag{5.22}$$

$$B = -2\xi^b\varepsilon_{bc}\Omega^c, \tag{5.23}$$

$$C_{<a>} = 2\beta\varepsilon_{ac}\Omega^c - 2\xi^b\Omega\varepsilon_{ab}. \tag{5.24}$$

From (5.22) it follows that a CKV $\xi^a$ orthogonal to $u^a$ is necessarily a Killing vector if $\xi^a$ is also orthogonal to $\dot{u}^a \equiv (\mathcal{A}e^a + \mathcal{A}^a)$ or if we have a geodesic flow. This is because $\xi_a\dot{u}^a \equiv \xi_a(\mathcal{A}e^a + \mathcal{A}^a) = \mathcal{A}\beta + \mathcal{A}^a\xi_a = 0$ implies either $\xi^a \perp (\mathcal{A}e^a + \mathcal{A}^a)$ or $(\mathcal{A}e^a + \mathcal{A}^a) = 0$. The latter case corresponds to a geodesic form. Considering both (5.23) and (5.24), we have

$$\xi_a u^a = 0 \qquad \Longrightarrow \qquad \mathcal{L}_{\boldsymbol{\xi}} u^a = -\Psi u^a - 2\xi^b\varepsilon_{bc}\Omega^c e^a + 2\left(\beta\varepsilon^{ac}\Omega_c - \xi_b\Omega\varepsilon^{ab}\right), \tag{5.25}$$

and therefore if $\xi^a u_a = 0$ then

$$\mathcal{L}_{\boldsymbol{\xi}} u^a = -\Psi u^a \qquad \Leftrightarrow \qquad -\xi^b\varepsilon_{bc}\Omega^c e^a + \beta\varepsilon^{ac}\Omega_c - \xi_b\Omega\varepsilon^{ab} = 0. \tag{5.26}$$

This result shows that the condition $\xi^a u_a$ places a restriction on the vorticity vector $\omega^a \equiv \Omega e^a + \Omega^a$ on the 2-sheet. Furthermore if $\xi^a u_a = 0$ and the vorticity $\omega \neq 0$ then

$$-\xi^b\varepsilon_{bc}\Omega^c e^a + \beta\varepsilon^{ac}\Omega_c - \xi_b\Omega\varepsilon^{ab} = 0 \qquad \Leftrightarrow \qquad \xi^a \parallel \omega^a \equiv (\Omega e^a + \Omega^a). \tag{5.27}$$

If the fluid is irrotational then

$$\mathcal{L}_{\boldsymbol{\xi}} u^a = -\Psi u^a, \tag{5.28}$$

is satisfied by any CKV orthogonal to $u^a$. However if the fluid is rotational then (5.28) is satisfied by a CKV orthogonal to $u^a$ if and only if $\xi^a$ is parallel to $(\Omega e^a + \Omega^a)$.

The second case involves assuming $\xi^a$ is parallel to $u^a$ then (5.28) is clearly satisfied so $B e^a + C^{<a>} = 0$. Since $\xi^a = \alpha u^a$ we have

$$
\begin{aligned}
B e_a + C_{<a>} &= (\alpha \mathcal{A} - \hat{\alpha})\, e_a + \alpha \mathcal{A}_a + 2\beta \varepsilon_{ac} \Omega^c - \delta_a \alpha = 0, && (5.29)\\
\implies \dot{u}_a \equiv \mathcal{A} e_a + \mathcal{A}_a &= \frac{-\hat{\alpha}\, e_a + 2\beta \varepsilon_{ac} \Omega^c - \delta_a \alpha}{\alpha}, && (5.30)
\end{aligned}
$$

which gives an expression for the acceleration. We note here that contracting (5.30) with $e^a$, we get

$$
\begin{aligned}
\mathcal{A} &= -\frac{\hat{\alpha}}{\alpha},\\
&= -\widehat{\log(\alpha)}. && (5.31)
\end{aligned}
$$

This is the scalar version of the equation

$$
\dot{u}^a = -\left(\log \alpha^{-1}\right)_{,b} h^b{}_a, \qquad (5.32)
$$

presented in the work by Maartens *et al* (1986). The result (5.31) indicates that the acceleration scalar depends on the acceleration vector along the preferred direction.

Using the definition of $\xi^a$ in (5.16), we can write down the form for $\mathcal{L}_{\boldsymbol{\xi}} e_a$ as follows:

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} e_a &= \xi^b \left(\nabla_b e_a\right) + e_b \nabla_a \xi^b\\
&= \alpha u^b \nabla_b e_a + \beta e^b \left(\nabla_b e_a\right) + \nu^{<b>} \left(\nabla_b e_a\right)\\
&\quad + e_b \nabla_a(\alpha) u^b + \alpha e_b \left(\nabla_a u^b\right) + e_b \nabla_a(\beta) e^b\\
&\quad + \beta e_b \left(\nabla_a e^b\right) + e_b \left(\nabla_a \nu^{<b>}\right), && (5.33)
\end{aligned}
$$

and thereafter substituting the definitions of the 1+1+2 covariant derivatives of vector fields $u_a$ (see (4.35)) and $e_a$ (see (4.32)) along with $e_b \left(\nabla_a \nu^{<b>}\right) = -\nu^{<b>} \left(\nabla_a e_b\right)$, we progress further and obtain

$$
\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}} e_a &= \left[\alpha\left(\frac{1}{3}\Theta+\Sigma\right)+a_b\nu^{<b>}+\hat{\beta}\right]e_a \\
&+\left[\beta\left(\Sigma+\frac{1}{3}\Theta\right)+\varphi_b\nu^{<b>}+\Sigma_b\nu^{<b>}-\varepsilon_{bc}\Omega^c\nu^{<b>}-\dot{\beta}\right]u_a \\
&+\alpha\varphi_a+\beta a_a-2\varsigma\varepsilon_{ab}\nu^{<b>}+(\Sigma_a-\varepsilon_{ac}\Omega^c)+\delta_a\beta, \qquad (5.34)
\end{aligned}
$$

after some general simplification. Hence we may evaluate the components of $\mathcal{L}_{\boldsymbol{\xi}} e_a$ (5.15) as

$$\Psi = \alpha\left(\frac{1}{3}\Theta+\Sigma\right)+a_b\xi^b+\hat{\beta}, \qquad (5.35)$$

$$E = \beta\left(\Sigma+\frac{1}{3}\Theta\right)+\varphi_b\xi^b+\Sigma_b\xi^b-\varepsilon_{bc}\Omega^c\xi^b-\dot{\beta}, \qquad (5.36)$$

$$F_{<a>} = \alpha\varphi_a+\beta a_a-2\varsigma\varepsilon_{ab}\xi^b+\Sigma_a-\varepsilon_{ac}\Omega^c+\delta_a\beta, \qquad (5.37)$$

where again we have substituted $\xi^a$ in the place of $\nu^{<a>}$ according to the definition given by equation (5.16).

We note identities (5.14) and (5.15) hold whether or not $e_a u^a = 0$. However if $e_a u^a = 0$ then

$$e_a\mathcal{L}_{\boldsymbol{\xi}} u^a + u^a\mathcal{L}_{\boldsymbol{\xi}} e_a = 0. \qquad (5.38)$$

Expanding (5.38) using (5.10) and (5.15) results in the equation

$$B - E = 0. \qquad (5.39)$$

Substituting the expressions of $B$ (from (5.20)) and $E$ (from (5.36)) into (5.39), we obtain the following new constraint equation

$$\alpha\mathcal{A}-\xi^b\varepsilon_{bc}\Omega^c-\hat{\alpha}-\left[\beta\left(\Sigma+\frac{1}{3}\Theta\right)+\varphi_b\xi^b+\Sigma_b\xi^b-\dot{\beta}\right]=0, \qquad (5.40)$$

which must be satisfied if the spacetime admits the CKV $\xi^a$. We note that equation (5.40) relates the kinematical quantities $\mathcal{A}$, $\Theta$, $\Sigma_b$ and $\Omega^c$. Also, following (5.40) we can write

$$\mathcal{L}_{\boldsymbol{\xi}}\, e^a = -\Psi\, e^a + B\, u^a + F^{<a>}, \qquad \text{and} \tag{5.41}$$

$$\mathcal{L}_{\boldsymbol{\xi}}\, e_a = \Psi\, e_a + B\, u_a + F_{<a>}, \tag{5.42}$$

which are the definitions we will use henceforth in our calculations.

## 5.3 Geometrical and physical application

We consider geometrical and physical applications of equation (5.28). First, we consider the Lie derivative of $h_{ab} = e_a\, e_b + N_{ab}$ given by

$$\mathcal{L}_{\boldsymbol{\xi}}\, h_{ab} = 2\,\Psi\,(e_a\, e_b + N_{ab}) + 2\left[B\, u_{(a}\, e_{b)} + u_{(a}\, C_{<b>)}\right], \tag{5.43}$$

which is derived by simple calculations using the definition of $N_{ab}$ (see (4.2)) and equations (5.11) and (5.15). Now since $u^a\, e_a = 0 = u^a\, C_{<a>}$ we have

$$\mathcal{L}_{\boldsymbol{\xi}}\, u^a = -\Psi\, u^a \qquad \Leftrightarrow \qquad \mathcal{L}_{\boldsymbol{\xi}}\, h_{ab} = 2\,\Psi\,(e_a\, e_b + N_{ab}) = 2\,\Psi\, h_{ab}. \tag{5.44}$$

Hence (5.28) is satisfied if and only if $\xi^a$ is a conformal motion of the fluid projection tensor $h_{ab}$. In an irrotational fluid, the rest spaces orthogonal to $u^a$ at each point form global spacelike hypersurfaces orthogonal to $u^a$ with $h_{ab}$ as its intrinsic metric tensor. If $\xi^a$ is also orthogonal to $u^a$ then $\xi^a$ lies in these hypersurfaces and when $\omega = 0$, we know from (5.25) that (5.28) is satisfied. Therefore $\mathcal{L}_{\boldsymbol{\xi}}\, h_{ab} = 2\,\Psi\,(e_a\, e_b + N_{ab}) = 2\,\Psi\, h_{ab}$ also holds and $\xi^a$ must be an intrinsic CKV of the hypersurfaces.

Another geometrical interpretation of (5.28) is that $\xi^a$ maps integral curves of $u^a$ into integral curves of $u^a$. When $\Psi \neq 0$, the mapping involves a rescaling of $u^a$ by a change of parameter but the entire family of integral curves of $u^a$ is mapped onto itself.

Therefore $\xi^a$ is said to be a dynamical symmetry of the fluid flow. This means that new constants of the fluid motion may be generated from existing constants. Suppose that $f$ is a constant of the fluid motion then

$$u^a \nabla_a f \equiv \dot{f} = 0. \tag{5.45}$$

Then $\xi^a \nabla_a f$ is also a constant of the fluid motion if (5.28) is satisfied as follows

$$u^a \nabla_a \left(\xi^b \nabla_b f\right) = \left[u^a \nabla_a \left(\xi^b \nabla_b\right) - \xi^b \nabla_b \left(u^a \nabla_a\right)\right] f + \xi^b \nabla_b \left(u^a \nabla_a f\right), \tag{5.46}$$

and after simplification, equation (5.46) can be written as

$$\Psi \dot{f} + \xi^a \nabla_a \left(\dot{f}\right) = 0. \tag{5.47}$$

A physical interpretation of (5.28) involves the presence of material curves which are curves that are made up of the same fluid particles at all times, and therefore they move with the fluid as the fluid evolves. The integral curves of $\xi^a$ are therefore material curves in the fluid if (5.28) is satisfied. An important special case of material curves occurs when $\xi^a u_a = 0$. If $\omega \neq 0$ and (5.28) is satisfied then the CKV $\xi^a$ must be parallel to $(\Omega e^a + \Omega^a)$. The integral curves of $\xi^a$ are therefore vortex lines that will be material curves in the fluid. This is due to the symmetry property of the flow and not due to the physical nature of the fluid. Conversely, if $\xi^a$ is a CKV orthogonal to $u^a$ and if the integral curves of $\xi^a$ are material curves then if $\omega \neq 0$, they must be vortex lines. We note $-\xi^b \varepsilon_{bc} \Omega^c e^a + \beta \varepsilon^{ac} \Omega_c - \xi_b \Omega \varepsilon^{ab}$ is orthogonal to both $u^a$ and $\xi^a$ and therefore it follows from (5.25) that $-\xi^b \varepsilon_{bc} \Omega^c e^a + \beta \varepsilon^{ac} \Omega_c - \xi_b \Omega \varepsilon^{ab} = 0$. Otherwise, the integral curves of $\xi^a$ would not move with the fluid. Hence $\xi^a$ must be parallel to $(\Omega e^a + \Omega^a)$.

## 5.4 Summary

In this chapter we extended the Lie derivative kinematic results of Maartens *et al* (1986) completely in terms of the 1+1+2 decomposition variables for a general spacetime. We calculated the components of the Lie derivatives of $u^a$ and $e^a$ and performed analysis by considering two special cases. Thereafter geometrical and physical applications involving the Lie derivative of $u^a$ were considered. The new results in this chapter are given by the scalar equation (5.31) and constraint equation (5.40). The result (5.31) indicates that the acceleration scalar $\mathcal{A}$ (in $\dot{u}^a \equiv \mathcal{A}e^a + \mathcal{A}^a$) depends on the acceleration vector along the preferred direction. The result (5.40) shows that the existence of a CKV constrains the kinematical quantities $\mathcal{A}$, $\Theta$, $\Sigma_b$ and $\Omega^c$. These results arise directly from the 1+1+2 decomposition of spacetime.

# Chapter 6

# Conformal symmetry: Dynamics

## 6.1 Introduction

In this chapter we analyze the dynamics of a general fluid spacetime with conformal symmetry by considering Einstein's field equations. We make initial assumptions on the most general form of the energy momentum tensor, and write the decomposed form according to the 1+1+2 formalism. Then we find the 1+1+2 decomposed form of the Lie derivative of Einstein's field equations. Following a similar procedure of Maartens *et al* (1986), we perform contractions using combinations of $u^a, e^a$ and $N^{ab}$ and obtain new general results after detailed simplification. As mentioned previously particular analyses have been performed on selected spacetimes. Recently Singh *et al* (2018) investigated conformal symmetries in locally rotationally symmetric spacetimes using the semi-tetrad covariant formalism, and followed a similar Lie derivative and contraction approach. However, to our knowledge, such investigations have not been done in general for an arbitrary spacetime admitting conformal symmetry, using the 1+1+2 formalism. Our new results are therefore applicable to *all* spacetimes.

## 6.2 Dynamics

We begin by considering the most general expression of the energy momentum tensor in the 1+1+2 formalism given by equation (4.43). Here we assume

$$\begin{aligned} Q^a &= 0, \\ \Pi^a &= 0, \\ \Pi^{ab} &= 0, \end{aligned} \tag{6.1}$$

which gives a new form for expressions (4.41) and (4.42) as follows

$$q^a = Q\,e^a, \tag{6.2}$$

$$\pi^{ab} = \Pi\left(e^a e^b - \frac{1}{2}N^{ab}\right). \tag{6.3}$$

Thus the energy momentum tensor has the form

$$T_{ab} = \mu\,u_a\,u_b + p\,(e_a\,e_b + N_{ab}) + 2Q\,u_{(a}\,e_{b)} + \Pi\left(e_a\,e_b - \frac{1}{2}N_{ab}\right). \tag{6.4}$$

Taking the Lie derivative of $T_{ab}$ in (6.4), we obtain

$$\begin{aligned} \mathcal{L}_{\boldsymbol{\xi}} T_{ab} &= (\mathcal{L}_{\boldsymbol{\xi}}\mu)\,u_a\,u_b + \mu\,\mathcal{L}_{\boldsymbol{\xi}}\,(u_a\,u_b) + (\mathcal{L}_{\boldsymbol{\xi}}\,p)\,e_a\,e_b + p\,\mathcal{L}_{\boldsymbol{\xi}}\,(e_a\,e_b) \\ &\quad + (\mathcal{L}_{\boldsymbol{\xi}}\,p)\,N_{ab} + p\,\mathcal{L}_{\boldsymbol{\xi}}\,(N_{ab}) + 2\,[\mathcal{L}_{\boldsymbol{\xi}}\,Q]\,u_{(a}\,e_{b)} + 2Q\mathcal{L}_{\boldsymbol{\xi}}\left[u_{(a}\,e_{b)}\right] \\ &\quad + (\mathcal{L}_{\boldsymbol{\xi}}\,\Pi)\,e_a\,e_b + \Pi\,\mathcal{L}_{\boldsymbol{\xi}}\,(e_a\,e_b) - \frac{1}{2}(\mathcal{L}_{\boldsymbol{\xi}}\,\Pi)\,N_{ab} - \frac{1}{2}\Pi\,\mathcal{L}_{\boldsymbol{\xi}}\,(N_{ab}) \qquad (6.5) \\ &= [\mathcal{L}_{\boldsymbol{\xi}}\,\mu + 2\,(\Psi\,\mu + BQ)]\,u_a\,u_b \\ &\quad + [\mathcal{L}_{\boldsymbol{\xi}}\,p + \mathcal{L}_{\boldsymbol{\xi}}\,\Pi + 2BQ + 2\,\Psi\,(p + \Pi)]\,e_a\,e_b \\ &\quad + \left[\mathcal{L}_{\boldsymbol{\xi}}\,p - \frac{1}{2}\mathcal{L}_{\boldsymbol{\xi}}\,\Pi + \Psi\,(2p - \Pi)\right] N_{ab} \\ &\quad + 2\,[\mathcal{L}_{\boldsymbol{\xi}}\,Q + B\,(\mu + p + \Pi) + 2\,\Psi\,Q]\,u_{(a}\,e_{b)} \end{aligned}$$

$$+\left[2\left(\mu+p\right)-\Pi\right]u_{(a}C_{<b>)}+3\Pi\,e_{(a}F_{<b>)}$$
$$+2Q\left[u_{(a}F_{<b>)}+e_{(a}C_{<b>)}\right], \qquad (6.6)$$

after using the definitions of $\mathcal{L}_{\boldsymbol{\xi}}\,u_a$ (see (5.11)) and $\mathcal{L}_{\boldsymbol{\xi}}\,e_a$ (see (5.42)), and some general simplification.

The Lie derivative along a CKV $\xi^a$ of Einstein's field equations was first evaluated by Herrera *et al* (1984). In the 1+3 decomposition we obtain

$$\mathcal{L}_{\boldsymbol{\xi}}\,G_{ab} = 2\Box\Psi\,g_{ab}-2\left(\nabla_a\nabla_b\Psi\right), \qquad (6.7)$$

where $\Box = g^{ab}\left(\nabla_a\nabla_b\Psi\right)$. In the 1+1+2 formalism, the expression (6.7) equates to

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\,G_{ab} = {}& 2\left[-\Theta\dot{\Psi}-\ddot{\Psi}+\hat{\Psi}\left(\mathcal{A}+\phi\right)+\hat{\hat{\Psi}}-\delta^c\delta_c\Psi\right]\left[N_{ab}-u_a u_b+e_a e_b\right] \\
& -2\left[-\dot{\Psi}\left\{\frac{1}{3}\Theta\left(N_{ab}+e_a e_b\right)+\Sigma\left(e_a e_b-\frac{1}{2}N_{ab}\right)+2\Sigma_{(a}e_{b)}\right.\right. \\
& \left.+\Sigma_{ab}+e_a\varepsilon_{bc}\Omega^c-e_b\varepsilon_{ac}\Omega^c+\varepsilon_{ab}\Omega\right\}+u_b\left\{\frac{1}{3}\Theta\left(\hat{\Psi}e_a+\delta_a\Psi\right)\right. \\
& +\left[\Sigma\left(e_a e_c-\frac{1}{2}N_{ac}\right)+2\Sigma_{(a}e_{c)}+\Sigma_{ac}\right]\left(\hat{\Psi}e^c+\delta^c\Psi\right) \\
& +\left[e_a\varepsilon_{cd}\Omega^d-e_c\varepsilon_{ad}\Omega^d+\varepsilon_{ac}\Omega\right]\left(\hat{\Psi}e^c+\delta^c\Psi\right)+u_a\ddot{\Psi} \\
& \left.-\left(\hat{\dot{\Psi}}e_a+\delta_a\dot{\Psi}\right)\right\}-u_a\left\{\left(N_{cb}+e_c e_b\right)\left(\hat{\Psi}e^c+\delta^c\Psi\right)\right. \\
& \left.+u_b\left(\mathcal{A}e_c+\mathcal{A}_c\right)\left(\hat{\Psi}e^c+\delta^c\Psi\right)-\dot{\Psi}\left(\mathcal{A}e_b+\mathcal{A}_b\right)\right\} \\
& +\frac{1}{3}\left\{\hat{\hat{\Psi}}+\phi\hat{\Psi}-\delta^c\Psi\,a_c+\delta^c\delta_c\Psi\right\}\left(N_{ab}+e_a e_b\right) \\
& +\frac{1}{3}\left\{2\hat{\hat{\Psi}}-\phi\hat{\Psi}-2\delta^c\Psi\,a_c-\delta^c\delta_c\Psi\right\}\left(e_a e_b-\frac{1}{2}N_{ab}\right) \\
& +\left\{2\delta_{(a}\hat{\Psi}-\left(\Sigma_{(a}+\Omega^c\varepsilon_{c(a)}\right)\dot{\Psi}-\phi\,\delta_{(a}\Psi+2\delta^c\Psi\left(\varsigma\,\varepsilon_{c(a}-\zeta_{c(a)}\right)\right\}e_{b)}
\end{aligned}$$

$$+\hat{\Psi}\,\zeta_{ab}+\delta_{\{a}\delta_{b\}}$$

$$+\frac{1}{2}\left(e_a\,\varepsilon_{bc}-e_b\,\varepsilon_{ac}+e_c\,\varepsilon_{ab}\right)\left\{\left(2\varsigma\,\hat{\Psi}+\varepsilon_{mn}\,\delta^m\,\delta^n\,\Psi\right)e^c+\varsigma\,\delta^c\,\Psi\right.$$

$$\left.+\varepsilon^{cm}\left(\Sigma_m\,\dot{\Psi}-\varepsilon_{mc}\,\Omega^c\,\dot{\Psi}+\varepsilon_{mc}\,\varsigma\,\delta^c\,\Psi\right)\right\}\Big], \tag{6.8}$$

using our definition for the double derivative of a scalar given by (4.40).

According to the Einstein field equations (2.13), we equate (6.6) and (6.7) (the condensed version of equation (6.8)) to obtain

$$\begin{aligned}
&2\Box\Psi\left(N_{ab}-u_a\,u_b+e_a\,e_b\right)-2\left(\nabla_a\nabla_b\Psi\right)=\\
&\left[\mathcal{L}_{\boldsymbol{\xi}}\,\mu+2\left(\Psi\,\mu+BQ\right)\right]u_a\,u_b\\
&+\left[\mathcal{L}_{\boldsymbol{\xi}}\,p+\mathcal{L}_{\boldsymbol{\xi}}\,\Pi+2BQ+2\,\Psi\left(p+\Pi\right)\right]e_a\,e_b\\
&\left[\mathcal{L}_{\boldsymbol{\xi}}\,p-\frac{1}{2}\mathcal{L}_{\boldsymbol{\xi}}\,\Pi+\Psi\left(2p-\Pi\right)\right]N_{ab}\\
&+2\left[\mathcal{L}_{\boldsymbol{\xi}}\,Q+B\left(\mu+p+\Pi\right)+2\,\Psi\,Q\right]u_{(a}\,e_{b)}\\
&+\left[2\left(\mu+p\right)-\Pi\right]u_{(a}\,C_{<b>)}+3\,\Pi\,e_{(a}\,F_{<b>)}\\
&+2Q\left[u_{(a}\,F_{<b>)}+e_{(a}\,C_{<b>)}\right].
\end{aligned} \tag{6.9}$$

Contracting the left hand side and right hand side of equation (6.9) with $u^a\,u^b$, $u^a\,e^b$, $u^a\,N^{bf}$, $e^a\,e^b$, $e^a\,N^{bf}$, $N^{ab}$ and $N^{af}\,N^{bk}-\frac{1}{2}N^{ab}\,N^{fk}$, we derive the following seven constraint equations given below

$u^a\,u^b$:

$$-2\Box\Psi-2\left(\nabla_a\nabla_b\Psi\right)u^a\,u^b \;=\; \mathcal{L}_{\boldsymbol{\xi}}\,\mu+2\left(\Psi\,\mu+B\,Q\right), \tag{6.10}$$

$u^a\,e^b$:

$$2\left(\nabla_a\nabla_b\Psi\right)u^a\,e^b \;=\; \mathcal{L}_{\boldsymbol{\xi}}\,Q+2\,\Psi\,Q+B\left(\mu+p+\Pi\right), \tag{6.11}$$

$u^a N^{bf}$:

$$2\left(\nabla_a \nabla_b \Psi\right) u^a N^{bf} \quad = \quad \left(\mu + p - \frac{1}{2}\Pi\right) C^{<f>} + Q F^{<f>}, \tag{6.12}$$

$e^a e^b$:

$$2\Box\Psi - 2\left(\nabla_a \nabla_b \Psi\right) e^a e^b \quad = \quad \mathcal{L}_{\boldsymbol{\xi}} p + \mathcal{L}_{\boldsymbol{\xi}} \Pi + 2\Psi\left(p + \Pi\right) + 2BQ, \tag{6.13}$$

$e^a N^{bf}$:

$$-2\left(\nabla_a \nabla_b \Psi\right) e^a N^{bf} \quad = \quad \frac{3}{2}\Pi F^{<f>} + Q C^{<f>}, \tag{6.14}$$

$N^{ab}$:

$$2\Box\Psi - \left(\nabla_a \nabla_b \Psi\right) N^{ab} \quad = \quad \mathcal{L}_{\boldsymbol{\xi}} p - \frac{1}{2}\mathcal{L}_{\boldsymbol{\xi}} \Pi + \Psi\left(2p - \Pi\right), \tag{6.15}$$

$N^{af} N^{bk} - \frac{1}{2} N^{ab} N^{fk}$:

$$\nabla_a \nabla_b \Psi\left(N^{af} N^{bk} - \frac{1}{2} N^{ab} N^{fk}\right) \quad = \quad 0. \tag{6.16}$$

Expanding the left hand side of the above seven constraint equations using the definition of the double derivative of a scalar (4.40), we obtain

$$2\left(\Theta\dot{\Psi} - \hat{\hat{\Psi}} - \phi\hat{\Psi} + \delta_c \delta^c \Psi + \mathcal{A}_c \delta^c \Psi\right) = \mathcal{L}_{\boldsymbol{\xi}} \mu + 2\left(\Psi \mu + BQ\right), \tag{6.17}$$

$$2\left(\hat{\dot{\Psi}} - \mathcal{A}\dot{\Psi}\right) = \mathcal{L}_{\boldsymbol{\xi}} Q + 2\Psi Q + B\left(\mu + p + \Pi\right), \tag{6.18}$$

$$\left(\mu + p - \frac{1}{2}\Pi\right) C^{<f>} + Q F^{<f>} = 2\left(\delta^f \Psi - \dot{\Psi}\mathcal{A}^f\right), \tag{6.19}$$

$$2\left[\left(\Sigma - \frac{2}{3}\Theta\right)\dot{\Psi} + \left(\mathcal{A} + \phi\right)\hat{\Psi} - \ddot{\Psi} - \delta_c \delta^c \Psi + \delta_c \Psi a^c\right] =$$

$$\mathcal{L}_{\boldsymbol{\xi}} p + \mathcal{L}_{\boldsymbol{\xi}} \Pi + 2BQ + 2\Psi\left(p + \Pi\right), \tag{6.20}$$

$$4\dot{\Psi}\Sigma^{f} - 2\delta^{f}\hat{\Psi} + \varepsilon_{c}{}^{f}\Omega^{c}\dot{\Psi} + \phi\delta^{f}\Psi - \delta_{c}\Psi\left(\varsigma\varepsilon_{c}{}^{f} - 2\zeta_{c}{}^{f}\right) =$$

$$\frac{3}{2}\Pi F^{<f>} + QC^{<f>}, \tag{6.21}$$

$$-\left(\frac{4}{3}\Theta + \Sigma\right)\dot{\Psi} + (2\mathcal{A} + \phi)\hat{\Psi} + 2\hat{\hat{\Psi}} - 2\ddot{\Psi} - 3\delta_{c}\delta^{c}\Psi =$$

$$\mathcal{L}_{\boldsymbol{\xi}}p - \frac{1}{2}\mathcal{L}_{\boldsymbol{\xi}}\Pi + \Psi(2p - \Pi), \tag{6.22}$$

$$\dot{\Psi}\Sigma^{fk} - \hat{\Psi}\zeta^{fk} - \delta^{\{f}\delta^{k\}}\Psi = 0, \tag{6.23}$$

using properties of $\varepsilon_{ab}$ to simplify.

Manipulating equations (6.17), (6.18), (6.20) and (6.22), we can derive expressions for the Lie derivatives of scalars $\mu, p, Q$ and $\Pi$ along the CKV $\xi^{a}$ as follows

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}\mu &= 2\left(\Theta\dot{\Psi} - \hat{\hat{\Psi}} - \phi\hat{\Psi} + \delta_{c}\delta^{c}\Psi + \mathcal{A}_{c}\delta^{c}\Psi - BQ - \Psi\mu\right), && (6.24)\\
\mathcal{L}_{\boldsymbol{\xi}}p &= \frac{4}{3}\left(\phi\hat{\Psi} + \hat{\hat{\Psi}} - \Theta\dot{\Psi}\right) - 2\left(\ddot{\Psi} - \mathcal{A}\hat{\Psi} + \Psi p\right) - \frac{8}{3}\delta_{c}\delta^{c}\Psi \\
&\quad + \frac{2}{3}\left(\delta_{c}\Psi a^{c} - BQ\right), && (6.25)\\
\mathcal{L}_{\boldsymbol{\xi}}Q &= 2\left(\hat{\Psi} - \dot{\Psi}\mathcal{A} - \Psi Q\right) - B(\mu + p + \Pi), && (6.26)\\
\mathcal{L}_{\boldsymbol{\xi}}\Pi &= 2\left(\Sigma\dot{\Psi} - \Psi\Pi\right) + \frac{2}{3}\left(\phi\hat{\Psi} + \delta_{c}\delta^{c}\Psi\right) + \frac{4}{3}\left(\delta_{c}\Psi a^{c} - \hat{\hat{\Psi}} - BQ\right). && (6.27)
\end{aligned}$$

We note that we can write $\mathcal{L}_{\boldsymbol{\xi}}\mu$ as

$$\mathcal{L}_{\boldsymbol{\xi}}\mu = \alpha\dot{\mu} + \beta\hat{\mu} + \nu^{b}\delta_{b}\mu, \tag{6.28}$$

and similarly

$$\begin{aligned}
\mathcal{L}_{\boldsymbol{\xi}}p &= \alpha\dot{p} + \beta\hat{p} + \nu^{b}\delta_{b}p, && (6.29)\\
\mathcal{L}_{\boldsymbol{\xi}}Q &= \alpha\dot{Q} + \beta\hat{Q} + \nu^{b}\delta_{b}Q, && (6.30)\\
\mathcal{L}_{\boldsymbol{\xi}}\Pi &= \alpha\dot{\Pi} + \beta\hat{\Pi} + \nu^{b}\delta_{b}\Pi. && (6.31)
\end{aligned}$$

Also, the energy conservation equation (4.67) now has the form

$$\dot{\mu} + \hat{Q} = -\Theta\left(\mu + p\right) - Q\left(\phi + 2\mathcal{A}\right) - \frac{3}{2}\Sigma\Pi, \tag{6.32}$$

and the momentum conservation equations given by (4.68) and (4.69) have the form

$$\dot{Q} + \hat{p} + \hat{\Pi} = -\Pi\left(\frac{3}{2}\phi + \mathcal{A}\right) - Q\left(\frac{4}{3}\Theta + \Sigma\right) - \mathcal{A}\left(\mu + p\right), \tag{6.33}$$

$$-\delta_a p + \frac{1}{2}\delta_a \Pi - Q\left(\varphi_a + \Sigma_a + \varepsilon_{ab}\Omega^b\right) - \frac{3}{2}\Pi a_a - \left(\mu + p - \frac{1}{2}\Pi\right)\mathcal{A}_a = 0, \tag{6.34}$$

after using the assumptions in (6.1).

Now equating the expressions (6.24)–(6.27) with (6.28)–(6.31) respectively and expanding further with equations (6.32) and (6.33), we obtain

$$2\left(\Theta\dot{\Psi} - \hat{\dot{\Psi}} - \phi\hat{\Psi} + \delta_c\delta^c\Psi + \mathcal{A}_c\delta^c\Psi\right) + Q\left(\alpha\phi + 2\alpha\mathcal{A} - 2B\right)$$

$$+\mu\left(\alpha\Theta - 2\Psi\right) + \alpha\left(p\Theta + \frac{3}{2}\Sigma\Pi + \hat{Q}\right) - \beta\hat{\mu} - \nu^b\delta_b\mu = 0, \tag{6.35}$$

$$\frac{4}{3}\left(\hat{\dot{\Psi}} - \Theta\dot{\Psi}\right) - 2\left(\ddot{\Psi} + \Psi p\right) - \frac{8}{3}\delta_c\delta^c\Psi + \frac{2}{3}\delta_c\Psi a^c - \alpha\dot{p}$$

$$+Q\left(\frac{4}{3}\Theta\beta + \Sigma\beta - \frac{2}{3}B\right) + \mathcal{A}\left(\Pi\beta + p\beta + \mu\beta + 2\hat{\Psi}\right)$$

$$+\phi\left(\frac{4}{3}\hat{\Psi} + \frac{3}{2}\Pi\beta\right) + \beta\left(\dot{Q} + \hat{\Pi}\right) - \nu^b\delta_b p = 0, \tag{6.36}$$

$$2\left(\hat{\Psi} - \dot{\Psi}\mathcal{A}\right) - B\left(\mu + p + \Pi\right) + Q\left[\alpha\left(\frac{4}{3}\Theta\Sigma\right) + \beta\left(\phi + 2\mathcal{A}\right) - 2\Psi\right]$$

$$+\Pi\left[\alpha\left(\frac{3}{2}\phi + \mathcal{A}\right) + \frac{3}{2}\Sigma\beta\right] + \left(\alpha\mathcal{A} + \beta\Theta\right)\left(\mu + p\right) + \alpha\left(\hat{p} + \hat{\Pi}\right)$$

$$+\beta\dot{\mu} - \nu^b\delta_b Q = 0, \tag{6.37}$$

$$
\begin{aligned}
&2\left(\Sigma\dot{\Psi}-\Psi\Pi\right)+\frac{2}{3}\delta_c\delta^c\Psi+\frac{4}{3}\left(\delta_c\Psi a^c-\hat{\hat{\Psi}}\right)-\alpha\dot{\Pi}\\
&+Q\left(\frac{4}{3}\Theta\beta+\Sigma\beta-\frac{4}{3}B\right)+\phi\left(\frac{2}{3}\hat{\Psi}+\frac{3}{2}\Pi\beta\right)+\mathcal{A}\beta\left(\mu+p+\Pi\right)\\
&+\beta\left(\dot{Q}+\hat{p}\right)-\nu^b\delta_b\Pi=0,
\end{aligned}
\tag{6.38}
$$

where $B=\alpha\mathcal{A}-2\xi^b\varepsilon_{bc}\Omega^c-\hat{\alpha}$ according to (5.20). These are constraints that must be satisfied if the spacetime is to admit the CKV $\xi^a$.

## 6.3 Summary

In this chapter we found the Lie derivative of the 1+1+2 decomposed form of the total energy momentum tensor. This enabled us to write down the Lie derivative of Einstein's field equations. We performed contractions on the resulting equation, expanded and simplified extensively. This process resulted in seven constraint equations. We note that for an arbitrary fluid spacetime admitting a CKV $\xi^a$ it is necessary that eight constraints, given by equations (6.19), (6.21), (6.23), (6.35)–(6.38) and (5.40) (found in the previous chapter) need to be satisfied. We emphasize that these results are general as we have not specified the line element. These constraint equations can be applied to a number of spacetimes in general relativity and physically significant results can occur. We demonstrate this in the next chapter.

# Chapter 7

# Perfect fluids

## 7.1 Introduction

A perfect fluid spacetime is characterized by its rest frame energy density and isotropic pressure. This fluid can be thought of as a smoothed out approximation to the matter in the universe which makes it a more realistic fluid model. Notably perfect fluids have no shear stresses, viscosity or heat conduction. Such a fluid appears to be a good description of the observed universe on a large scale. The absence of the aforementioned quantities is a great advantage as the relativistic equations become simpler and better analysis can be performed which could lead to physically significant results. This is shown in the works of Herrera *et al* (1984), Maartens *et al* (1986), Ellis and van Elst (1998) in the 1+3 formalism and Clarkson (2007) in the 1+1+2 formalism. Results for perfect fluids with a particular form of the CKV are contained in the works of Coley and Tupper (1990a, 1990b, 1990c, 1994). In this chapter we apply the seven constraint equations, found in the final part of the previous chapter, to a perfect fluid spacetime which admits the CKV $\xi^a$ in (5.16).

## 7.2 General equations

Due to the description of a perfect fluid spacetime, the anisotropic fluid variables (4.41) and (4.42) vanish, i.e.

$$\Pi = 0,$$
$$Q = 0, \tag{7.1}$$

where the initial assumptions (6.1) still hold. Thus the energy conservation equation (6.32) now has the form

$$\dot{\mu} \;=\; -\Theta\,(\mu + p)\,, \tag{7.2}$$

and the momentum conservation equations given by (6.33) and (6.34) become

$$\hat{p} \;=\; -\mathcal{A}\,(\mu + p)\,, \tag{7.3}$$
$$\delta_a p \;=\; -\mathcal{A}_a\,(\mu + p)\,. \tag{7.4}$$

Substituting (7.1) into the constraint equations (6.35), (6.36), (6.37), (6.38), (6.19), (6.21) and (6.23) we obtain

$$2\left(\Theta\,\dot{\Psi} - \hat{\hat{\Psi}} - \phi\,\hat{\Psi} + \delta_c\delta^c\,\Psi + \mathcal{A}_c\,\delta^c\,\Psi - \Psi\,\mu\right) + \alpha\,\Theta\,(\mu + p) - \beta\,\hat{\mu}$$
$$-\nu^b\,\delta_b\,\mu = 0, \tag{7.5}$$
$$\frac{4}{3}\left(\phi\,\hat{\Psi} + \hat{\dot{\Psi}} - \Theta\,\dot{\Psi}\right) - 2\left(\ddot{\Psi} - \mathcal{A}\,\hat{\Psi} + \Psi\,p\right) - \frac{8}{3}\delta_c\delta^c\,\Psi + \frac{2}{3}\delta_c\,\Psi\,a^c - \alpha\,\dot{p}$$
$$+\beta\,\mathcal{A}\,(\mu + p) - \nu^b\,\delta_b\,p = 0, \tag{7.6}$$
$$2\left(\hat{\dot{\Psi}} - \dot{\Psi}\,\mathcal{A}\right) - B\,(\mu + p) = 0, \tag{7.7}$$
$$2\,\Sigma\,\dot{\Psi} + \frac{2}{3}\left(\phi\,\hat{\Psi} + \delta_c\delta^c\,\Psi\right) + \frac{4}{3}\left(\delta_c\,\Psi\,a^c - \hat{\hat{\Psi}}\right) = 0, \tag{7.8}$$
$$(\mu + p)\,C^{<f>} - 2\left(\delta^f\,\Psi - \mathcal{A}^f\,\dot{\Psi}\right) = 0, \tag{7.9}$$

$$4\dot{\Psi}\Sigma^f - 2\delta^f\hat{\Psi} + \varepsilon_c{}^f\Omega^c\dot{\Psi} + \phi\delta^f\Psi - \delta^c\Psi\left(\varsigma\varepsilon_c{}^f - 2\zeta_c{}^f\right) = 0, \tag{7.10}$$

$$\dot{\Psi}\Sigma^{fk} - \hat{\Psi}\zeta^{fk} - \delta^{\{f}\delta^{k\}}\Psi = 0, \tag{7.11}$$

for the perfect fluid case. These are the constraints, along with (5.40), to be satisfied if the perfect fluid spacetime is to admit the CKV $\xi^a$. We note (7.11) has remained unchanged.

## 7.3 Equation of state: $p = p(\mu)$

We now choose an equation of state

$$p = p(\mu), \tag{7.12}$$

where the isotropic pressure $p$ is a function of $\mu$, the effective energy density. Considering the definition of the isentropic speed of sound (3.43), we note that we can write

$$\hat{\mu} = \frac{\hat{p}}{c_s^2}, \tag{7.13}$$

$$\dot{p} = c_s^2\dot{\mu}, \tag{7.14}$$

$$\delta_b p = c_s^2\delta_b\mu. \tag{7.15}$$

Substituting these expressions into equations (7.5) and (7.6) we obtain

$$2\left(\Theta\dot{\Psi} - \hat{\hat{\Psi}} - \phi\hat{\Psi} + \delta_c\delta^c\Psi + \mathcal{A}_c\delta^c\Psi - \Psi\mu\right) + \alpha\Theta(\mu+p) + \frac{\beta}{c_s^2}\mathcal{A}(\mu+p)$$
$$-\nu^b\delta_b\mu = 0, \tag{7.16}$$

$$2\left(\mathcal{A}\hat{\Psi} - \ddot{\Psi} - \Psi p\right) - \frac{8}{3}\delta_c\delta^c\Psi + \frac{4}{3}\left(\phi\hat{\Psi} + \hat{\hat{\Psi}} - \Theta\dot{\Psi}\right) + \frac{2}{3}\delta_c\Psi a^c$$
$$+\alpha\Theta c_s^2(\mu+p) + \beta\mathcal{A}(\mu+p) - \nu^b c_s^2\delta_b\mu = 0, \tag{7.17}$$

after simplification. We have also incorporated the energy and momentum conservation equations given by (7.2)–(7.4) to arrive at equations (7.16) and (7.17). Manipulating the above equations results in

$$-2\ddot{\Psi} - \frac{4}{3}\hat{\hat{\Psi}} + \frac{4}{3}\Theta\dot{\Psi} + 2\mathcal{A}\hat{\Psi} + \frac{2}{3}\delta_c\Psi a^c + \frac{8}{3}\mathcal{A}_c\delta^c\Psi - \Psi\left(\phi + 2p + \frac{8}{3}\mu\right)$$
$$+ (\mu + p)\left[\alpha\Theta\left(\frac{4}{3} + c_s^2\right) + \left(\beta\mathcal{A} + \nu^b\mathcal{A}_b\right)\left(\frac{4}{3c_s^2} + 1\right)\right] = 0, \tag{7.18}$$

which is a damped wave equation in $\Psi$ with a

$$\text{potential} = \left(\phi + 2p + \frac{8}{3}\mu\right). \tag{7.19}$$

The forcing term is generated by $(\mu + p)$.

## 7.4 Summary

In this chapter we have found a system of equations that must be satisfied by the thermodynamical variables for perfect fluids when a conformal symmetry exists. We have shown that the conformal factor satisfies a damped wave equation with a potential. This proves that the semi-tetrad decomposition is useful in bringing out physically significant results that was not possible before.

# Chapter 8

# Conclusion

In this thesis we followed the procedure of Maartens *et al* (1986) who considered the kinematic and dynamic properties of an anisotropic fluid spacetime admitting a conformal Killing vector using a 1+3 approach. Our main goal in this thesis was to perform a semi-tetrad decomposition of a general spacetime that admits a conformal Killing vector. This process was done in order to further investigate the kinematics and dynamics of spacetimes admitting conformal symmetry. We found that the 1+1+2 decomposition leads to new results. In Chapter 2 we outlined concepts relating to curvature in general relativity necessary for this thesis. We first introduced the concept of a manifold and then defined the connection coefficients, the Riemann tensor, the Ricci tensor, the Ricci scalar and the Einstein tensor along with properties associated with them.

Next we wrote down the 1+3 covariant approach equations in Chapter 3 which built a foundation for the 1+1+2 equations that followed. The 1+3 equations, found in Ellis (2009), the fluid 4-velocity unit vector $u^a$ and related projection tensors were introduced. Two important derivatives, namely the covariant time derivative and the fully orthogonally projected covariant spatial derivative, were defined and decomposed. Kinematic quantities arising from the relative motion of the comoving observers and their properties were specified. Of high relevance to this thesis was writing down the covariant decomposition of the 4-velocity vector. The total energy momentum tensor was decomposed relative to $u^a$ and each of the dynamical quantities and their properties were defined. The Weyl tensor was also split relative to $u^a$ and the electric and magnetic parts were specified according to Maartens and Bassett (1998). A set of propagation, evolution and constraint equations arising from the Bianchi and Ricci identities were generated. Lastly, the commutation relations for the two derivatives were given explicitly.

In Chapter 4, we extended the 1+3 equations and wrote down the 1+1+2 covariant approach equations, given by Clarkson and Barrett (2003). The spacetime was split further using a preferred spatial direction in the 3-space. The unit vector $e^a$, orthogonal to $u^a$, and the related projection tensor were introduced. Two derivatives, their properties and commutation relations were defined. They were the spatial derivative along $e^a$ in the surfaces orthogonal to $u^a$ and the projected spatial derivative onto the 2-sheet. The 1+3 kinematical and Weyl quantities were split irreducibly and new 1+1+2 kinematical variables introduced. The 1+1+2 split of the full covariant derivatives of $u^a$ and $e^a$ were defined. We obtained the evolution and propagation equations, given by Clarkson (2007), by applying the 1+1+2 decomposition procedure to the 1+3 equations, and also by covariantly splitting the Ricci identities for $e^a$.

In Chapter 5 we wrote the kinematic results of Maartens *et al* (1986) completely in terms of the 1+1+2 variables. We considered an arbitrary spacetime that admits a conformal Killing vector (CKV) $\xi^a$ in terms of new vectors. The Lie derivatives of $u^a$ and $e^a$ were calculated explicitly in terms of the 1+1+2 variables. Our analysis is consistent with the findings of Maartens *et al* (1986). However, in the case of assuming the CKV $\xi^a$ parallel to $u^a$, we obtained a new scalar version of the acceleration equation presented by Maartens *et al* (1986). The scalar equation is given by

$$\begin{aligned} \mathcal{A} &= -\frac{\hat{\alpha}}{\alpha}, \\ &= -\widehat{\log(\alpha)}. \end{aligned} \tag{8.1}$$

Furthermore since $e^a$ is orthogonal to $u^a$, taking the Lie derivative of both quantities produced a new constraint equation given by

$$\alpha\mathcal{A} - \xi^b\varepsilon_{bc}\Omega^c - \hat{\alpha} - \left[\beta\left(\Sigma + \frac{1}{3}\Theta\right) + \varphi_b\xi^b + \Sigma_b\xi^b - \dot{\beta}\right] = 0, \tag{8.2}$$

which must be satisfied if the spacetime admits the CKV $\xi^a$. Equation (8.2) shows that the existence of a CKV constrains the kinematical quantities $\mathcal{A}$, $\Theta$, $\Sigma_b$ and $\Omega^c$. We note here that the 1+1+2 decomposition allowed us to yield new results in an area previously analyzed.

In Chapter 6 we investigated the dynamics of a general spacetime admitting the CKV $\xi^a$. We considered the 1+1+2 decomposed form of the total energy momentum tensor in general and found the Lie derivative of the total energy momentum tensor. This enabled us to write down the Lie derivative of Einstein's field equations. We performed contractions on the resulting equation, expanded and simplified extensively. This process resulted in seven more constraint equations given by

$$\begin{aligned} &2\left(\Theta\dot{\Psi} - \hat{\dot{\Psi}} - \phi\hat{\Psi} + \delta_c\delta^c\Psi + \mathcal{A}_c\delta^c\Psi\right) + Q\left(\alpha\phi + 2\alpha\mathcal{A} - 2B\right) \\ &+\mu\left(\alpha\Theta - 2\Psi\right) + \alpha\left(p\Theta + \frac{3}{2}\Sigma\Pi + \hat{Q}\right) - \beta\hat{\mu} - \nu^b\delta_b\mu = 0, \end{aligned} \tag{8.3}$$

$$\begin{aligned} &\frac{4}{3}\left(\hat{\dot{\Psi}} - \Theta\dot{\Psi}\right) - 2\left(\ddot{\Psi} + \Psi p\right) - \frac{8}{3}\delta_c\delta^c\Psi + \frac{2}{3}\delta_c\Psi a^c - \alpha\dot{p} \\ &+Q\left(\frac{4}{3}\Theta\beta + \Sigma\beta - \frac{2}{3}B\right) + \mathcal{A}\left(\Pi\beta + p\beta + \mu\beta + 2\hat{\Psi}\right) \\ &+\phi\left(\frac{4}{3}\hat{\Psi} + \frac{3}{2}\Pi\beta\right) + \beta\left(\dot{Q} + \hat{\Pi}\right) - \nu^b\delta_b p = 0, \end{aligned} \tag{8.4}$$

$$2\left(\hat{\Psi}-\dot{\Psi}\mathcal{A}\right)-B\left(\mu+p+\Pi\right)+Q\left[\alpha\left(\frac{4}{3}\Theta\Sigma\right)+\beta\left(\phi+2\mathcal{A}\right)-2\Psi\right]$$

$$+\Pi\left[\alpha\left(\frac{3}{2}\phi+\mathcal{A}\right)+\frac{3}{2}\Sigma\beta\right]+\left(\alpha\mathcal{A}+\beta\Theta\right)\left(\mu+p\right)+\alpha\left(\hat{p}+\hat{\Pi}\right)$$

$$+\beta\dot{\mu}-\nu^{b}\delta_{b}Q=0, \tag{8.5}$$

$$2\left(\Sigma\dot{\Psi}-\Psi\Pi\right)+\frac{2}{3}\delta_{c}\delta^{c}\Psi+\frac{4}{3}\left(\delta_{c}\Psi a^{c}-\hat{\hat{\Psi}}\right)-\alpha\dot{\Pi}$$

$$+Q\left(\frac{4}{3}\Theta\beta+\Sigma\beta-\frac{4}{3}B\right)+\phi\left(\frac{2}{3}\hat{\Psi}+\frac{3}{2}\Pi\beta\right)+\mathcal{A}\beta\left(\mu+p+\Pi\right)$$

$$+\beta\left(\dot{Q}+\hat{p}\right)-\nu^{b}\delta_{b}\Pi=0, \tag{8.6}$$

$$\left(\mu+p-\frac{1}{2}\Pi\right)C^{<f>}+QF^{<f>}=2\left(\delta^{f}\Psi-\dot{\Psi}\mathcal{A}^{f}\right), \tag{8.7}$$

$$4\dot{\Psi}\Sigma^{f}-2\delta^{f}\hat{\Psi}+\varepsilon_{c}{}^{f}\Omega^{c}\dot{\Psi}+\phi\delta^{f}\Psi-\delta_{c}\Psi\left(\varsigma\varepsilon_{c}{}^{f}-2\zeta_{c}{}^{f}\right)=$$

$$\frac{3}{2}\Pi F^{<f>}+QC^{<f>}, \tag{8.8}$$

$$\dot{\Psi}\Sigma^{fk}-\hat{\Psi}\zeta^{fk}-\delta^{\{f}\delta^{k\}}\Psi=0, \tag{8.9}$$

where $B=\alpha\mathcal{A}-2\xi^{b}\varepsilon_{bc}\Omega^{c}-\hat{\alpha}$. These constraints, and (8.2), must be satisfied if the spacetime is to admit the CKV $\xi^{a}$.

In Chapter 7, we applied the eight constraint equations above to a perfect fluid spacetime where the anisotropic stress and heat flux vanish. Thus, we wrote down the new forms of the energy and momentum conservation equations. This led to the simplified system

$$2\left(\Theta\dot{\Psi}-\hat{\hat{\Psi}}-\phi\hat{\Psi}+\delta_{c}\delta^{c}\Psi+\mathcal{A}_{c}\delta^{c}\Psi-\Psi\mu\right)+\alpha\Theta\left(\mu+p\right)-\beta\hat{\mu}$$

$$-\nu^{b}\delta_{b}\mu=0, \tag{8.10}$$

$$\frac{4}{3}\left(\phi\,\hat{\Psi}+\hat{\hat{\Psi}}-\Theta\,\dot{\Psi}\right)-2\left(\ddot{\Psi}-\mathcal{A}\,\hat{\Psi}+\Psi\,p\right)-\frac{8}{3}\delta_c\delta^c\Psi+\frac{2}{3}\delta_c\Psi\,a^c-\alpha\,\dot{p}$$

$$+\beta\,\mathcal{A}\,(\mu+p)-\nu^b\,\delta_b p=0, \tag{8.11}$$

$$2\left(\hat{\Psi}-\dot{\Psi}\,\mathcal{A}\right)-B\,(\mu+p)=0, \tag{8.12}$$

$$2\,\Sigma\dot{\Psi}+\frac{2}{3}\left(\phi\,\hat{\Psi}+\delta_c\delta^c\Psi\right)+\frac{4}{3}\left(\delta_c\Psi\,a^c-\hat{\hat{\Psi}}\right)=0, \tag{8.13}$$

$$(\mu+p)\,C^{<f>}-2\left(\delta^f\,\Psi-\mathcal{A}^f\,\dot{\Psi}\right)=0, \tag{8.14}$$

$$4\,\dot{\Psi}\,\Sigma^f-2\,\delta^f\,\hat{\Psi}+\varepsilon_c{}^f\,\Omega^c\,\dot{\Psi}+\phi\,\delta^f\,\Psi-\delta^c\,\Psi\left(\varsigma\,\varepsilon_c{}^f-2\zeta_c{}^f\right)=0, \tag{8.15}$$

$$\dot{\Psi}\,\Sigma^{fk}-\hat{\Psi}\,\zeta^{fk}-\delta^{\{f}\,\delta^{k\}}\,\Psi=0 \tag{8.16}$$

along with (8.2) which remains unchanged. We then chose the equation of state

$$p=p\,(\mu)\,, \tag{8.17}$$

and applied this relation to equations (8.10) and (8.11). After some simplification, involving the energy and momentum conservation equations, we arrived at

$$-2\,\ddot{\Psi}-\frac{4}{3}\,\hat{\hat{\Psi}}+\frac{4}{3}\,\Theta\,\dot{\Psi}+2\,\mathcal{A}\,\hat{\Psi}+\frac{2}{3}\,\delta_c\,\Psi\,a^c+\frac{8}{3}\,\mathcal{A}_c\,\delta^c\,\Psi-\Psi\left(\phi+2\,p+\frac{8}{3}\,\mu\right)$$

$$+\,(\mu+p)\left[\alpha\,\Theta\left(\frac{4}{3}+c_s^2\right)+\left(\beta\,\mathcal{A}+\nu^b\,\mathcal{A}_b\right)\left(\frac{4}{3\,c_s^2}+1\right)\right]=0, \tag{8.18}$$

which is a damped wave equation in $\Psi$ with a

$$\text{potential}=\left(\phi+2p+\frac{8}{3}\,\mu\right). \tag{8.19}$$

The forcing term is generated by $(\mu+p)$.

In conclusion, we note that in performing the 1+1+2 decomposition, we were able to obtain new results in Chapter 5 applicable to a general spacetime. This further proves that the semi-tetrad decomposition is useful in bringing out the behaviour of certain geometrical and dynamical quantities that was not possible before. We further stress that the results presented in Chapter 6 are for an arbitrary spacetime so these constraint equations can be applied to particular metrics. The existence of a conformal Killing vector has led to new constraint equations on the scalars associated with the kinematics and dynamics. The properties of specific spacetimes will simplify some of the equations and could lead to physically important results. We look forward to extending our research in this area of semi-tetrad decomposition of spacetime with conformal symmetry to specific spacetimes.

# Appendix A

# Key relations to aid decomposition

Given any quantity in the 1+3 formalism, the following relations can be used to aid decomposition into 1+1+2 variables as given by Clarkson (2007). Chapter 4, in particular, makes use of these relations to derive certain expressions. Note that 1+3 space 3-vectors, $x^a$, $y^a$ and projected, symmetric and trace-free 3-tensors $\psi_{ab}$ and $\phi_{ab}$ may be decomposed as follows

$$x^a = X\,e^a + X^a, \tag{A.1}$$

$$y^a = Y\,e^a + Y^a, \tag{A.2}$$

$$\psi_{ab} = \psi_{<ab>} = \Psi\left(e_a\,e_b - \frac{1}{2}N_{ab}\right) + 2\,\Psi_{(a}\,e_{b)} + \Psi_{ab}, \tag{A.3}$$

$$\phi_{ab} = \phi_{<ab>} = \Phi\left(e_a\,e_b - \frac{1}{2}N_{ab}\right) + 2\,\Phi_{(a}\,e_{b)} + \Phi_{ab}. \tag{A.4}$$

The following expansions from 1+3 quantities to 1+1+2 variables may be performed

$$x_a\,x^a = X^2 + X_a\,X^a, \tag{A.5}$$

$$\eta_{abc}\,x^b\,y^c = \left(\varepsilon_{bc}\,X^b\,Y^c\right)e_a + \varepsilon_{ab}\left(Y\,X^b - X\,Y^b\right), \tag{A.6}$$

$$x_{<a}\,y_{b>} = \frac{1}{3}\left(2XY - X_c\,Y^c\right)\left(e_a\,e_b - \frac{1}{2}N_{ab}\right) +$$

$$\left[X\,Y_{(a}+Y\,X_{(a}\right]e_{b)}+X_{\{a}Y_{b\}}, \tag{A.7}$$

$$\psi_{ab}\,x^b \;=\; \left(X\,\Psi+X_b\,\Psi^b\right)e_a-\frac{1}{2}\,\Psi\,X_a+X\,\Psi_a+\Psi_{ab}\,X^b, \tag{A.8}$$

$$\begin{aligned}\eta_{cd<a}\,x^c\,\psi_{b>}{}^d \;&=\; \varepsilon_{cd}\,X^c\,\Psi^d\left(e_a\,e_b-\frac{1}{2}\,N_{ab}\right)+X\,\varepsilon_{c\{a}\,\Psi_{b\}}{}^c-X^c\,\varepsilon_{c\{a}\,\Psi_{b\}}\\ &\quad+\left[\left(X\,\Psi^c-\frac{3}{2}\,\Psi\,X^c\right)\varepsilon_{c(a}+\varepsilon_{cd}\,X^c\,\Psi^d{}_{(a}\right]e_{b)},\end{aligned} \tag{A.9}$$

$$\psi_{ab}\,\psi^{ab} \;=\; \frac{3}{2}\,\Psi^2+2\Psi_a\,\Psi^a+\Psi_{ab}\,\Psi^{ab}, \tag{A.10}$$

$$\begin{aligned}\psi_{c<a}\,\phi_{b>}{}^c \;&=\; \left(\frac{1}{2}\,\Psi\,\Phi+\frac{1}{3}\,\Psi_c\,\Phi^c-\frac{1}{3}\,\Psi_{cd}\,\Phi^{cd}\right)\left(e_a\,e_b-\frac{1}{2}\,N_{ab}\right)\\ &\quad+\left[\frac{1}{2}\,\Psi\,\Phi_{(a}+\frac{1}{2}\,\Phi\,\Psi_{(a}+\Psi^c\,\Phi_{c(a}+\Phi^c\,\Psi_{c(a}\right]e_{b)}\\ &\quad-\frac{1}{2}\,\Psi\,\Phi_{ab}-\frac{1}{2}\,\Phi\,\Psi_{ab}+\Psi_{\{a}\,\Phi_{b\}}+\Psi_{c\{a}\,\Phi_{b\}}{}^c,\end{aligned} \tag{A.11}$$

$$\eta_{abc}\,\psi^b{}_d\,\phi^{dc} \;=\; e_a\,\varepsilon_{bc}\,\Psi^b{}_d\,\Phi^{dc}+\frac{3}{2}\,\varepsilon_{ab}\left(\Phi\,\Psi^b-\Psi\,\Phi^b\right). \tag{A.12}$$

For the 1+3 covariant time dot derivative '˙' and the fully orthogonally projected covariant spatial derivative 'D' we have

$$\dot{x}_{<a>} \;=\; \left(\dot{X}-X_b\,\varphi^b\right)e_a+X\,\varphi_a+\dot{X}_{\bar{a}}, \tag{A.13}$$

$$\begin{aligned}\dot{\psi}_{<ab>} \;&=\; \left(\dot{\Psi}-2\,\Psi_c\,\varphi^c\right)e_a\,e_b+\left[3\,\Psi\,\varphi_{(a}+2\,\dot{\Psi}_{(\bar{a}}-2\,\varphi^c\,\Psi_{c(a}\right]e_{b)}\\ &\quad-\frac{1}{2}\,\dot{\Psi}\,N_{ab}+2\,\Psi_{(a}\,\varphi_{b)}+\dot{\Psi}_{\{ab\}},\end{aligned} \tag{A.14}$$

$$\mathrm{D}_a\,x^a \;=\; \hat{X}+X\,\phi-X_a\,a^a+\delta_a\,X^a, \tag{A.15}$$

$$\begin{aligned}\eta_{abc}\,\mathrm{D}^b\,x^c \;&=\; \left(2\,X\,\varsigma+\varepsilon_{bc}\,\delta^b\,X^c\right)e_a+\varsigma\,X_a\\ &\quad+\varepsilon_{ab}\left[-X\,a^b+\delta^b\,X-\hat{X}^b-\frac{1}{2}\,\phi\,X^b-\zeta^{bc}\,X_c\right],\end{aligned} \tag{A.16}$$

$$
\begin{aligned}
\mathrm{D}_{<a} x_{b>} \;\; &= \;\; \frac{1}{3}\left[2\hat{X} - \phi X - 2X_c a^c - \delta_c X^c\right]\left(e_a e_b - \frac{1}{2}N_{ab}\right) + X\zeta_{ab} \\
&+ \left[X a_{(a} + \delta_{(a} X + \hat{X}_{(\bar{a}} - \frac{1}{2}\phi X_{(a} + X^c\left(\varsigma\varepsilon_{c(a} - \zeta_{c(a)}\right)\right] e_{b)} \\
&+ \delta_{\{a} X_{b\}}, \qquad \text{(A.17)} \\
\mathrm{D}^b \psi_{ab} \;\; &= \;\; \left(\hat{\Psi} + \frac{3}{2}\phi\Psi - 2\Psi_b a^b + \delta_b \Psi^b - \Psi_{bc}\zeta^{bc}\right) e_a + \hat{\Psi}_{\bar{a}} \\
&+ \frac{3}{2}\phi\Psi_a + \frac{3}{2}\Psi a_a - \frac{1}{2}\delta_a\Psi - \Psi_{ab}a^b + \left(-\varsigma\varepsilon_{ab} + \zeta_{ab}\right)\Psi^b \\
&+ \delta^b \Psi_{ab}, \qquad \text{(A.18)} \\
\eta_{cd<a}\mathrm{D}^c \psi_{b>}{}^d \;\; &= \;\; \left[3\varsigma\Psi + \varepsilon_{cd}\delta^c\Psi^d - \varepsilon_{cd}\Psi^{de}\zeta^c{}_e\right]\left(e_a e_b - \frac{1}{2}N_{ab}\right) \\
&+ \left(-\frac{3}{2}\delta^c\Psi + \frac{3}{2}\Psi a^c + \hat{\Psi}^c + \frac{1}{2}\phi\Psi^c + 2\Psi_d\zeta^{cd}\right)\varepsilon_{c(a} e_{b)} \\
&+ \left[5\varsigma\Psi_{(a} + \varepsilon^{cd}\left(\Psi_d\zeta_{c(a} + \delta_c\Psi_{d(a)}\right)\right] e_{b)} - \varepsilon_{c\{a}\delta^c\Psi_{a\}} \\
&+ 2\varepsilon_{c\{a}a^c\Psi_{b\}} + \varepsilon_{c\{a}\hat{\Psi}^c{}_{b\}} + \frac{1}{2}\phi\varepsilon_{c\{a}\Psi^c{}_{b\}} - \frac{3}{2}\Psi\varepsilon_{c\{a}\zeta^c{}_{b\}} \\
&+ \varsigma\Psi_{ab} + \varepsilon_{c\{a}\Psi_{b\}d}\zeta^{cd}. \qquad \text{(A.19)}
\end{aligned}
$$

These relations may be substituted directly into 1+3 equations to aid decomposition.

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