GRADE ELEVEN LEARNERS' PARTICIPATION IN THE FUNCTIONS DISCOURSE: THE CASE OF A HYPERBOLA AND EXPONENTIAL FUNCTION

by

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ABSTRACT

The purpose of the study was to investigate the mathematical discourses of grade eleven learners related to the worded, numerical, tabular, and graphic asymptotes of the hyperbola and exponential functions. The theory of commognition was referred to, with particular emphasis on the characteristics of the mathematical discourse; that is, word use, visual mediators, endorsed narratives, and routines.

The study has fundamentally adopted an interview-based qualitative research design approach, with descriptive and interpretative elements complementing its data analysis processes. In addition, some quantitative aspects were administered by means of test-based activities. The study was conducted in four schools in the rural Mthatha district of the Eastern Cape Province of South Africa. Data was collected by means of a test administered to 112 respondents, and task-based interviews with 12 pairs of grade eleven students in those schools. In each school, about 30 learners participated in the test, and six from each school took part in the interviews. Data was analysed by means of the Discourse Profile of the Hyperbola and Exponential Function adapted from the Arithmetic Discourse Profile propounded by Ben-Yahuda and others.

The findings revealed that learners have learnt functions in class, and were all familiar with the asymptotes of the hyperbola and the exponential function. While learners could answer questions on functions, a rephrasing of the question changed their response. There were also challenges of linking different representations of a function to each other. Students would also work efficiently on procedure tasks, but struggled on action-oriented tasks.

ACKNOWLEDGEMENTS

ABBREVIATIONS AND ACRONYMS IN THE STUDY

ADP   Arithmetic Discourse Profile
ALD   Actual Level of Development
ANA   Annual National Assessment
CAPS   Curriculum and Assessment Policy Statement
CDE   Centre for Development and Enterprise
DBE   Department of Basic Education
DPH   Discourse Profile of the Hyperbola
DPHEF   Discourse Profile of the Hyperbola and Exponential Function
ECP   Eastern Cape Province
FAL   First Additional Language
FDP   Functions Discourse Profile
FET   Further Education and Training
FETC   Further Education and Training
GDP   Gross Domestic Product
HOD   Head of Department
KZN   KwaZulu-Natal
MKO   More Knowledgeable Other
NSC   National Senior Certificate
PLD   Potential Level of Development
UKZN   University of KwaZulu-Natal
ZPD   Zone of Proximal Development

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CHAPTER 1
OVERVIEW OF THE STUDY

1.1 Introduction

"South Africa is significantly underperforming in education in general, particularly in Mathematics teaching and learning. Mathematics teaching is often of poor quality, with teachers not able to answer questions in the curriculum they are teaching, one indicator of the challenge. Often, national testing is misleading as it does not show the major gaps at lower grade levels" (Minister of Basic Education, 2016). This statement by the Minister is an irrefutable indictment of the deteriorating state of Mathematics teaching and learning in the South African public school system. The Minister (Mrs A Motshekga) is not the only one to have reflected on the poor state of the teaching and learning of Mathematics in the country.

Researchers and practitioners such as Howie (2003) and Tachie and Chireshe (2013) have also reflected on the same issue, and found that several factors were attributable to the generally poor Mathematics results. These factors include lack of resources, inadequate learner preparedness in previous grade levels, the language of teaching, and the poor quality of teaching. Learners' utterances generally reflect what they have learnt, or what they have not yet learnt (Howie, 2003). The latter author also intimates that language was a hindrance in the learning of Mathematics, and becomes an issue in areas where learners and teachers mostly speak the same geographically-defined dominant language. In such situations, it is easy for lessons to be taught in the local language, which may not necessarily be the school's medium of instruction Setati, 2008). The use of local languages sometimes affects learners' language of literacy (Canagarajah, 2002; Vyncke, 2012). It is in this particular context that the present study does not consider language-related issues as peripheral to the investigation of learners' mathematical discourse on the hyperbola and exponential functions. To this effect, Nachliel and Tabach (2012) assert that words do help identify the mathematical objects they represent in a mathematical situation. Since Mathematics is an auto-poietic subject, the use of words brings life to abstract objects or concepts.

The importance of functions in the Mathematics curriculum is indicated by the amount of time allocated for the teaching of those functions in all the grade levels, from grade 10 to 12 in the Further Education and Training (FET) band of the official school curriculum in South Africa (Swarthout, Jones, Klespis & Cory, 2009). Of all the topics in the curriculum and assessment policy statement (CAPS), functions have the highest number of teaching weeks (Department of Basic Education/ DBE, 2011). Even the weighting of content in assessment, demonstrates that functions and their related components have at least 40 % of the marks in Paper 1 of the Mathematics high school examination. However, the Department of Basic Education's grade 12 reports prove that learners still experience difficulties in learning mathematical functions.

The focus on grade eleven was necessitated by the poor performance of learners in the learning of mathematical functions at this level. In terms of the curriculum expectations, most of the work in mathematical functions would have been studied by grade eleven. In fact, four of the five functions studied in the FET band are completed at grade eleven, and are not allocated any significant teaching time in grade 12 (DBE, 2011). The researcher deliberately decided to investigate the manner in which learners engaged with the hyperbola and the exponential function, mainly due to the maximum failure to realise the significance of the asymptotes that the functions tend to identify as they approach the extremes.

While there was sporadic use of formal language by some learners, the asymptotes were at the same time expressed more in colloquial terms. For instance, when learners expressed an asymptote as though it was a number. Learners seemed to be more comfortable with tasks they were familiar with, but struggled with those that were unfamiliar to them. For example, they would easily work with graphs of an exponential function written in algebraic form in terms of x and y, but experienced some degree of difficulty in the event that the variables changed to t and θ in an exponential function. Furthermore, the learners would identify the asymptote in an equation with variables x and y, but fail to identify the asymptote in the event that the variables are θ and t . In such instances, learners' narratives were based on what they could see. In this regard, they could explain that the asymptote should not intersect with the graph, but could not explain the reason for the non-intersection. Secondly, learners attributed their actions or a lack thereof, to the teacher's very own actions. They manifested this in expressions such as: *This is what we were taught*, or *This is how my teacher would do it*. The learner's various responses above emphasise that there are ritualised mathematical actions and non-mathematical routines. Generally, there were more of ritualised mathematical action routines.

1.2 The Research Problem and its Context

The teaching and learning of Mathematics in South Africa has been a cause for concern for the past twenty-four years of the post-apartheid democratic dispensation (DBE, 2013; Moutlana, 2007). While there has been some improvement in the number of learners in Mathematics at grade 12 from 224 635 in 2011, to 265 810 in 2016, these numbers are less when compared with those of Mathematical Literacy which grew from 275 380 to 361 948 for the same period (DBE, 2013; DBE, 2017). The slight increase in learners studying Mathematics, compared to those studying Mathematical Literacy at grade 12, is indicative of the self-defeating attitude that many learners have on Mathematics. The Centre for Development and Enterprise/ CDE (2013) suggests that the teaching of Mathematics in South African schools was among the worst in the world. In fact, Mogari (2014) laments this state of affairs, when compared to countries such as Zimbabwe and Zambia. In these countries, the GDP is not comparable to that of South Africa, yet more has been achieved despite the disparate levels of socio-economic

development. The poor state of teaching inevitably translates into a correspondingly poor state of learning (Tachie & Chireshe, 2013; Yi, 2006). In the period between 2011 and 2013, the Department of Basic Education administered the Annual National Assessment (ANA) in Mathematics for grades 3, 6 and 9. The results were of a sliding nature, with grade 3s producing far better results than grade 9s. The grade 9 ANA results during the afore-cited three-year period reached an average of about 12%, which is indicative of the unsatisfactory state of the learning of Mathematics.

Over the years, the National Senior Certificate (NSC) results have shown skewed patternss, with the majority of learners obtaining marks between 0% and 49.9% during the period 2013-2016, as shown in Figure 1.1 below.
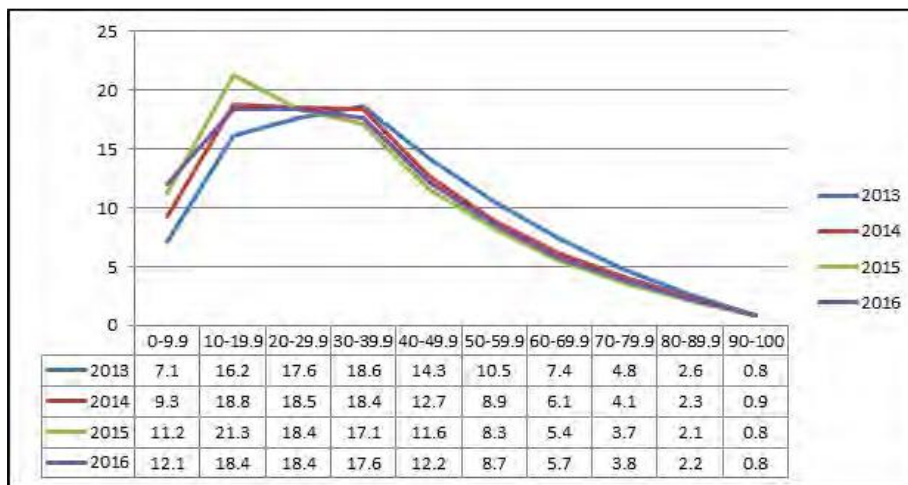
Figure 1.1: Mathematics National Senior Certificate results from 2013 to 2016

Figure 1.1 above illustrates that about 20% of the candidates attained more than 50% of the marks in the NSC Mathematics examinations. Additionally, most of these candidates had over the years obtained marks between 9% and 40%.

Functions and other related aspects constitute more than 40% of the NSC Mathematics Paper One examinations. A close scrutiny of the examination reports of the past five years indicates that the average mark for learners in functions have been less than 50%, as shown in Table 1.1 below.

Table 1.1: Average performance mark on questions involving functions

Year	2012	2013	2014	2015	2016	2017
Average Performance on Functions	41%	46%	43%	44%	36%	50%

Extrapolated from Table 1.1 above, it is clear that the period 2012-2017 was characterised by an average performance mark just above 40% for functions-related questions, with the highest

average of 50% for 2017; and the lowest average of 36% in 2016. Data from the 2017 Department of Basic Education's diagnostic report shows that 245 103 learners wrote their Mathematics examinations in 2017, less than 20 000 compared to 265 915 of those who wrote their Mathematics examinations in 2016. The 50% average performance on functions in 2017 may be an indicator of an improvement in the learning of functions.

1.2.1 Rationale of the study

The rationale of this study is premised on the researcher's motivation and justification of the main reason or reasons for undertaking the study (de Vos, Strydom, Fouche & Delport, 2011; Denscombe, 2007). Accordingly then, both the background/ context of the study and its rationale provide a coherent and logical affinity with both the aim and objectives of the study (Walliman, 2011) and its rationale.

Many of the interventions and investigations attempt to solve the challenges of poor learning of mathematics have focused on teachers in their investigation and possible solutions or recommendations. The idea is that once teachers 'get it right', it follows that the expected learning will follow on the part of the learners. In this study, the focus is more on learning by grade 11 (eleven) learners, than on the teachers as the providers of the expected learning. The specific focus on grade eleven was necessitated by the poor performance of learners in the learning of mathematical functions at this level.

Table 1.1 (see p. 3) does not specifically show an improvement in the learning of functions. It is this observable lack of improvement which inspired the researcher to investigate learners' discourse on functions in grade 11. In this regard, the researcher's decision to focus on functions was informed by their undeniable importance in Mathematics education in schools. The essence of functions in Mathematics is mostly underpinned by their linkage of Algebra to Calculus (Adler & Venkat, 2014). The Department of Basic Education's curriculum planners have shown their high regard of functions by allocating the highest teaching time in both grade 10 and grade 11, which in itself highlights the self-same high value the Department places on the indispensability of functions in Mathematics education in schools. Therefore, it is this trend of poor performance that prompted this researcher to investigate high school learners' mathematical discourse on functions (Swarthout et al., 2009).

1.3 Purpose of the Study

The terms 'purpose', 'aim', and 'goal' in research are used interchangeably to denote their synonymous nature, character, and interrelatedness (Babbie & Mouton, 2010; Henning, 2005). According to this perspective, these three interchangeable terms are distinguishable from the concept of 'study objectives'. Despite the distinction between 'purpose', 'aim', and 'goal' on the one hand, and 'study objectives' on the other, there exists a degree of complementarity in all of these nuances (Babbie & Mouton, 2010; Kumar, 2012).

In the light of the distinguishability of the research aim/ purpose/ goal and study objectives (explained further in section 1.4 below), the purpose, aim, or goal of the study refers to the researcher's more general, broader, or overall intentions for conducting the study (Babbie & Mouton, 2010; Walliman, 2011). In the context of this research, the overall intention or purpose was: ***To investigate learners' mathematical discourse on the hyperbola and exponential functions.*** Learners' mathematical utterances provide a clear reflection and context of what they have learnt, or what they have not yet learnt; that is, what they still have to learn (Ben-Yahuda et al., 2005; Howie, 2003). In this regard, both linguistic proficiency and mathematical knowledge of the hyperbola and exponential functions were inextricable aspects of the investigation.

1.4 Objectives of the Study

While the purpose, aim, or goal of the study is indicative of the more general aspects of the researcher's intentions, the objectives of the study are the writ large reduction, unbundling, or unpacking of the self-same research purpose (Babbie & Mouton, 2010; Katzenellenbogen & Joubert, 2007). By implication therefore, the study's objectives allocate a degree of immediacy or specificity and measurability, as they are premised and focused on the unambiguous and direct intentions for purposes of resolving the identified research problem. Hence the assertion by Kumar (2012) that a study's objectives rest on four pillars; that is, "to describe a situation, phenomenon, problem or issue (descriptive research); to establish or explore a relationship between two or more variables (correlational research); to explain why certain things happen the manner they do (explanatory research); and to examine the feasibility of conducting a study or exploring a subject area where nothing or little is known (exploratory research)" (p. 34). It is worth mentioning that the current study's number of objectives are four. That number was pre-determined by the researcher, and is independent of Kumar's four-fold characterisation of the nature of an objective as alluded to above. The current study was guided by the following four pre-determined objectives, namely:

- ***To understand the mathematical discourse of learners in a rural setting with regard to mathematical functions***

Functions are a very important component of Mathematics as they link algebra to calculus (Ben-Yahuda et al., 2005; Howie, 2003). Learners' mathematical discourse on functions would help identify challenges experienced by learners, such as the objectification of the functions discourse;

- ***To explore and identify the lexicon used by learners in expressing themselves in the context of mathematical functions***

It is in the use of words that learners' thought processes could be viewed as a window to the kind of Mathematics that they hold (Bradley, Campbell & McPetrie, 2013). For example, some

learners referred to the asymptote as a point, which is influenced by the extent of their reading and interpretation of the table of values in functions;

- *To explore, describe, and explain the mathematical discourse of learners on the four representations of a function, namely: functions expressed in words, as algebraic form, in the form of ordered pairs, and in graphical form*

One of the indications that learning has taken place is the learners' ability or capacity to link the four representations of a function with flexibility (Sfard, 2007; Sfard, 2016). The four representations are an important component of the curriculum requirements mainly due to the scoring or weight that functions and related topics carry in the examinations;

- *To explore and define the learners' experiences and perspectives pertaining to understanding the asymptotes of the functions*

Learning of Mathematics depends on previously learned mathematical objects (Brodie & Berger, 2010). Learning functions at high school level prepares learners for learning about rational functions. The kind of functions (e.g. hyperbola, exponential function and tangent function) do not show the characteristics of all the rational functions; thus contributing to learners' incomplete generalisation of the asymptote. The incomplete generalisation gives rise to contradictions as learners study rational functions further.

1.5 The Research Questions

By their nature, research questions are not framed in isolation of other critical units of analysis, such as the research problem, aim and objectives, the significance of the study; as well as the associated data collection and analysis processes of the study (Creswell, 2014; Kumar, 2012). As opposed to questions in a research instrument (e.g. questionnaires or interviews), the research questions listed below are fundamentally meant to guide the researcher throughout the research process (Babbie & Mouton, 2010). That is, these research questions are not meant to be responded to by the selected research participants or respondents. These questions therefore, serve as a form of checklist for the researcher during the entire research process. Therefore, this study was guided by the following four questions:

- What discourses do grade 11 learners in rural areas display when learning functions?
- How does the use of words (lexicon) afford or constrain the participation of learners in the mathematical functions discourse?
- What is the nature of grade 11 learners' mathematical discourse relating to the four different representations of functions in the context of the hyperbola and the exponential functions?
- What is the nature of learners' participation in mathematical discourse on the asymptote of the hyperbola and exponential function in grade 11?

1.6 Significance of the Study

The significance of the study provides a framework for the motivation, justification, or reasons advanced for the study's truth value in relation to "the complex relationship between the scientific environment and the real-life concerns" (Babbie & Mouton, 2010: 10). In essence, the complex relationship referred to above is emblematic of the utilitarian function of knowledge production and dissemination (Gcasamba, 2014). The utilitarian function itself is an illustration of the relationship between scientific research and the lived experiences or social reality of people in particular environments. Bunting (2002, p. 67) confirms that: "Knowledge is not regarded as something which is good in itself, and hence worth pursuing for its own sake. It follows that knowledge which could be used for a specific social, economic or political purpose would be the primary form pursued". From this premise, it is evident that a study could be either epistemological or socio-economic in its significance, value, or contribution (Vyncke, 2012).

The epistemological significance relates to the extent to which the study contributes to the body of knowledge in a particular field of study (such as Mathematics in this case), and identifies gaps (if any) between theory and practice (Kumar, 2012). It is envisaged that the current study will be of value in its contribution to the field of learners' mathematical discourses in the context of the worded, numerical, tabular, and graphic asymptotes of the hyperbola and exponential functions. In this regard, the researcher has developed and utilised an analytical tool termed the Discourse Profile of the Hyperbola and Exponential Function (DPHEF), which is an extension of the Discourse Profile of the Hyperbola (DPH) - an analytical tool used by the researcher for his Master's research project. The DPH was developed from the Arithmetic Discourse Profile (ADP) by Ben-Yahuda et al. (2005), which sought to distinguish mathematical communication that appeared similar at face value. The major additions of the DPHEF to the DPH is that the ritualised routines are divided into two parts with ritualised mathematical and ritualised non-mathematical categories. Ritualised routines are part of the growing mathematical discourses.

In the current study, clear distinctions have been made between the ritualised mathematical routines, in terms of which learners act in mathematically acceptable ways but cannot give mathematical reasons in support of the particular routine. In the case of the non-mathematical rituals, learners generally give wrong mathematical statements in relation to the particular mathematical routine. Ritualised mathematical routines show a growing mathematical discourse. Furthermore, ritualised mathematical routines are an indication that learners have gained something from their interaction with the interlocutor, whereas ritualised non-mathematical routines represent actions not taught in the classrooms, but learners trying to make meaning of the Mathematics presented to them (Sfard, 2012). Ritualised non-

mathematical routines are exemplified by learners failing to solve linear equations and in the conjoining of variables with numbers. Non-mathematical routines do not show a growing mathematical discourse as learners' routines. It is against this general state of affairs that the DPHEF is expected to contribute towards learners' knowledge, perceptions, and experiences in the learning of asymptotes of the hyperbola and exponential functions.

The study is of significant value insofar as both its methodological and socio-economic orientations are concerned. From a methodological perspective, the study makes a contribution insofar as centralising the 'voice' of learners providing their first-hand empirical/ experiential multiple realities from their own perspectives and in their own familiar environments (Canagarajah, 2002; Vyncke, 2012). In most research studies of this nature, learners' own perspectives are often neglected (Aljoundi, 2014). In this study, learners in a rural setting have been afforded an 'authorial voice' on the four representations of the two mathematical functions; that is, graphical, tabular, algebraic, and worded representations. A similar study was only held in 2015 in Johannesburg, arguably the economic hub of the country where learners have better resources and were most likely to perform better than those in rural areas. This 2015 study showed that there is not much difference between the mathematical discourses of the learners in rural areas when compared to their counterparts in urban areas (Tachie & Chireshe, 2013). The very same study also found that there was no significant margin of disparity in the urban-rural mathematical discourse among learners in grade eleven.

For policy development and implementation purposes in the realm of Mathematics education, the study is of particular importance to both the Department of Basic Education and the schools selected as the research sites of this study. Very few protracted studies have been conducted in rural areas in respect of the mathematical discourses of grade eleven learners related to the worded, numerical, tabular, and graphic asymptotes of the hyperbola and exponential functions. It is envisaged that both the findings and recommendations of the study will provide an evidence-based framework for the relevant policy makers and Mathematics teachers in the improvement of curriculum and teaching methodologies.

1.7 Scope/ Delimitations of the Study
The scope or delimitation of the study relates to the conceptual boundaries, frame of reference, or extent to which the investigation has been narrowed in order to address the most pertinent or core issues in the context of the research topic (Singh, 2006; Walliman, 2011). In this study, three symbiotically related factors established a framework of its scope or delimitations.

Firstly, the study was geographically confined to a rural setting in Mthatha, a rural town in the Eastern Cape Province (ECP). The province has been characterised by very high numbers of teacher-learner class ratios. In the last few years, the Eastern Cape schools have been either the

last or second last in the Mathematics matric results. For example, in 2017 the ECP schools achieved a pass rate of 42.3%, only 0.7% better than KwaZulu-Natal (KZN) Province (DBE, 2018). Against this background, the researcher then decided to investigate learners' mathematical discourse on functions in such a setting in order to understand the context of learning mathematical discourses by learners in typical rural or village schools. Furthermore, the rural milieu was chosen against the backdrop of the general adverse perceptions and opinions that rural schools account for the declining pass rates. In her announcement of the National Senior Certificate results for 2016, Minister Motshekga implied that rural schools were responsible for poor results.

Secondly, the study was restricted to the mathematical discourse on the hyperbola and exponential functions of grade eleven learners only. In terms of curriculum requirements, grade eleven learners are not required to define a function. Accordingly, as research participants, they were not subjected to defining the function, which is the curriculum domain of grade 12 Mathematics learners.

Lastly, the study was conducted in English, the language of power and access to economic opportunities (Canagarajah, 2002; Setati, 2008), in a predominantly isiXhosa-speaking geographic area of the country. To these learners, English is studied as a First Additional Language (FAL). Some of the language used by the participants may have been affected by the interference of the home language, and their responses to questions may have been affected by their understanding of the questions in the research instrument (questionnaire or interview) (Aljoundi, 2014). Given this state of affairs, the researcher selected participants who scored highly in the task-based interview tests, in accordance with the study's inclusion/ eligibility and exclusion/ ineligibility criteria as detailed in Chapter 4 (Research Design and Methods).

1.8 Chapter Outline
The layout of chapters in this study is premised on the thematic systematisation and logical sequencing of the core research variables (units of analysis) from the study's conceptualisation and its completion. Each of the eight chapters in the study is coherently linked with its preceding and subsequent core variables (Babbie & Mouton, 2010).

Chapter 1: Overview of the Study
This chapter presents and discusses an overview of the core units of analysis of the study in respect of the background/ context of the research problem; the purpose/ aim/ goal of the study and its objectives; the research questions; the study's significance/ value; as well as its scope (de Vos et al., 2011). From the perspective of the researcher, all of the afore-cited units of analysis are both interstitially and thematically linked with the research topic; which itself focuses on grade eleven learners' mathematical discourses in the context of worded, numerical, tabular, and graphic asymptotes of the hyperbola and exponential functions in Mathematics.

Chapter 2: Literature Review

The chapter specifically focuses on the diverse range of sources of information used in guiding the study from its conceptualisation, exploration, initiation, and ultimate completion. The most salient aspect of the literature review was its enabling of the researcher's familiarity with, and exposure to current trends in theory and practice; current methodological developments; as well as topical issues and challenges in the field of study being investigated (Babbie & Mouton, 2010). The researcher's approach to the search, consultation, review and analysis of pertinent information was directed by the research topic itself. Furthermore, the review of literature was a concurrent data collection and analysis process, as superfluous, redundant, and repetitive information could be identified and discarded. It is worth mentioning that the focus of the search and review of sources of information was more on the multiple scholarship perspectives in the field of learners' mathematical discourses; rather than on the mere listing and compilation of the sources of the information and data.

The theoretical and conceptual framework of commognition is also discussed in this chapter, with a focused description of the development of the mathematical discourse. Definitions of salient terms were also provided, such as commognition, objectification, meta-level and object-level learning; as well as the characterisation of the mathematical discourse in terms of use, visual mediators, narratives, and routines. The analytical tool, the DPHEF was also explained and discussed in this chapter.

Chapter 3: Theoretical/ Conceptual Framework

This chapter delves on the pertinent and inextricably associated concepts on which the philosophical foundations are premised, thus enabling a theoretical framework of the study's core mathematical phenomena under investigation (Hesse-Biber & Leavy, 2011). It is against this backdrop that commognition and socio-cultural learning are presented as two inter-related concepts which systematically help in explaining the association between and among critical variables in the asymptote and hyperbola environments (Mpofu & Pournara, 2018). The fundamental principles and tenets of commognition and social learning have been relevantly integrated such that these principles are shown to be applicable to the core variables of the research topic. The definition of key concepts in this chapter precedes the theories themselves, since concepts are conceived as the foundational pillars of theories (Ramenyi & Bannister, 2013).

Chapter 4: Research Design and Methods

In this chapter, the research approaches or perspectives and the rationale thereof are presented and discussed (Rajasekar, Philominathan, & Chinnathambi, 2013: 5). Also included in the chapter were the specific research instrumentation; data collection and analysis approaches; the research site, study population and sample size; the sampling techniques/ strategies and criteria;

as well as the ethical considerations.

Chapter 5: Representation of Funtions

The hyperbola and exponential functions constituted the most central tenet of this chapter. Visual presentations were made in the form of tables and graphs which explained the outcomes of tests conducted for purposes of determining the learners' mathematical discourses in respect of the hyperbola and exponential function (Mpofu & Pournara, 2018).

Chapter 6: The Asymptote

In this chapter, learners' mathematical discourse was presented and analysed with regard to the asymptote of the hyperbola and the exponential function. In essence, this chapter provides an analysis of the twenty-four research participants' task-based interviews based on identification of an asymptote from a graph and/ or formula (Flesher, 2003).

Chapter 7: Four Representations of a Function

The chapter focuses specifically on the data analysis pertaining to the four representations of a function. The Department of Basic Education's Mathematics curriculum requires that learners should work flexibly between the four representations of a function (DBE, 2011). These representations are the mathematical functions presented in numerical form; that is, in the form of ordered pairs usually presented as a table of value, graphical, verbal form as a function in the form of a story and algebraically in the form of a formulae or equation. Examinations, which largely influence the teaching and learning content at school, tend to focus on graphical and symbolic representations.

Chapter 8: Summary, Findings, Conclusions and Recommendations

This final chapter of the study presents and summarises the main findings accruing from the elicited responses of the research participants. This evidence served as the most authentic and reliable data from which the conclusions, recommendations and generalisability of the study could be supported (Greener, 2008; Ramenyi & Bannister, 2013).

Additionally, the chapter focused on the limitations of the study and their implications. Proposals for further study were discussed in order to guide the areas for improvements in studies of this nature.

1.9 Conclusion

The teaching and learning of learners' mathematical discourses relates to asymptotes of the hyperbola and exponential functions, and is an essential component of the Department of Basic Education's grade eleven curriculum requirement (DBE, 2011). Compared to their urban and more socio-economically advantaged counterparts, learners in rural settings are more prone to poor performance in this regard. The National Senior Certificate (NSC) Mathematics pass rate has been constant between 2014 and 2017, with only some slight and occasional t improvements (DBE, 2018). However, a closer analysis shows that it was only a few learners

who performed well, while the majority still obtained less than 40% in their NSC year-end examinations.

In the overview of the entire study, the critical units of analysis were mentioned, the most notable of which was the DPHEF analytical tool. The DPHEF was developed by the researcher in order to identify, describe, and analyse areas in which most challenges were encountered; as well as to provide possible solutions to these challenges. In the next chapter (Chapter 2), the review of literature is situated in the context of the learning of functions in general, as well as the National Senior Certificate's expected Mathematics curriculum requirements for learners in South African public schools.

CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

The review of pertinent literature in this study entailed the thematic and systematic search, consultation, synthesis, and analysis of multiple scholarship perspectives and sources of information and ideas relating to mathematical discourse and learning. In this regard, the review of literature provided both methodological and theoretical backgrounds to the researcher, including the identification of any gaps that may exist in the dominant mathematical discourse theories and practice (De Vos et al., 2011; Kumar, 2012). In addition, the review of literature illuminated on current trends, topical issues and innovations, challenges, policy implications, and any lessons that could be learnt from the international domain of mathematical discourse and literacy (Ramenyi & Banister, 2013). In addition to the empirically generated evidence, such lessons - if any - would be of immense benefit to the study' framework of recommendations (Babbie & Mouton, 2010). Given this brief introductory background of the literature review, it is therefore worth noting that the crux of this chapter is largely premised on two parameters; namely, mathematical discourse and its context of learning.

With regard to the learning context of mathematical discourse, the researcher distinguishes between object-level learning and meta-level learning in order to provide an uncluttered learning context of mathematical discourse (Sfard, 2007). With regard to the parameters of mathematical discourse, the researcher focuses on its development and also explains the four characteristics of mathematical discourse. It is from these four characteristics that the Discourse Profile of the Hyperbola and Exponential Functions (DPHEF) analytical tool is formulated.

2.2 Mathematical Discourse

Mathematical discourse is a sub-set of discourses, with its own key characteristics that distinguish it (mathematical discourse) from other discourses. These characteristics are what Berger (2013) refers to as "a range of permissible actions and reactions" (p. 2). A mathematical discourse is distinguished from other discourses by distinct characteristics of the commognition theory. The latter is discussed later in the chapter as an indication that Mathematics grows from within, and is said to be auto-poietic (Sfard, 2015; Nachlieli &Tabach, 2012). In concurrence with Berger's (2013) above-cited proposition, Sfard (2012, p. 2) further describes a discourse as a "specific type of communication". Communication is important in a discourse in that it establishes and enhances the conveying of commonly understood messages within a community the same participants. For instance, members of the same community would speak of a function in Mathematics as referring to an object that can be represented graphically, in words, as ordered pairs, or algebraically. This is opposed to the use of the word "function" in

everyday language, in which case it may refer to a purpose, task, use, or a role in which someone is engaged.

Mathematics is an interconnected subject where the absence of one component affects the success of the mathematical discourse that has to be learnt (Adler & Ronda, 2014). Mathematics is also built on previously established facts known as endorsed narratives. Against this backdrop, the development of mathematical discourse is then a result of a team effort in terms of which a community of mathematicians agrees on which new narratives to endorse, and which to reject, based on previously-endorsed narratives. The discourse expands as the need for new rules arises (Adler & Venkat, 2014). For example, some of the rules that learners hold of multiplication of natural numbers need to be altered in order to accommodate negative numbers and fractions. In natural numbers, the multiplication of two numbers results in an increased or expanded magnitude, whereas the multiplication of negative numbers and fractions will result in a lesser magnitude. Another example pertains to natural numbers, in which case addition and multiplication are not concerned with signs. Once negative numbers are introduced, the mathematical community agrees on those rules that should change, and those that should be retained. In this regard, mathematical discource entails an internal self-generation characteristic. Hence the assertion that Mathematics develops from within, with the community of mathematicians either endorsing or rejecting particular rules.

Discourses develop vertically when there is a combination of existing discourses to form new *meta-discourses* at a higher level (Berger, 2013). Algebra is an example of a vertical development in that it is a combination of arithmetic and numeric patterns. On the other hand, horizontal development is a result of a combination of separate discourses into a single new discourse. For example, the graphs of motion combine solutions to equations, rates and calculus. Functions would be classified as vertical development as they mostly emerge from algebra.

Discourses develop in an attempt to establish *compression*, the means by which worded and long mathematical statements are expressed in the shortest possible manner by expressing the same words or a list of numbers in symbolic form (Sfard, 2012). In this regard, an exponential function expressed as 2^x denotes a list of numbers that are growing exponentially, such as $...\frac{1}{8}; \frac{1}{4}; \frac{1}{2}$; 1; 2; 4; 8; 16; and so on. The exponential expression *(2^x)* may describe an exponential function, and may denote the shape produced by that function. The object *(2^x)* is key to the compression of the mathematical discourse on exponents and exponential functions. As a sub-set of development, learning enhances the ability to talk about, or use new mathematical objects. When new mathematical objects are used, there is a concomitant change in mathematical discourse, which could occur at either the meta- or object-level. In such a

context, the learner's individual involvement enhances benefit from the change arising in the mathematical discourse; be it at object- or meta-level. In the event that learners use new words, and make conjectures and generalisations, their mathematical discourse necessarily changes (Ben-Yahuda et al., 2005). Meta-level growth is mainly attained when there is an interlocutor who will help the learner with this change that has some discontinuity with previously accepted endorsed narratives.

The presence of an interlocutor (the more knowledgeable other) is important in the learning of mathematical discourse (Bradley et al., 2013). During the formal mathematical conversation, the interlocutor fulfils the role of an experienced discursant in the mathematical discourse, helping the learner to adjust some previously held informal or 'un-mathematical' language. The interlocutor begins the mathematical conversation from the learner's point of view, and gradually informs or alerts the learner of the different rules which now apply. For example, a linear function is denoted by a constant gradient, but in other functions learners only know that the gradient varies in a single function. The learners' conscious realisation of the inadequacy of their knowledge motivates them to participate in the mathematical discourse through their "thoughtful imitation" under the interlocutor's leadership (Bradley et al., 2013; Sfard, 2015).

The interlocutor or teacher mediates mathematical objects until the learner is able to do the work thoughtfully by himself/herself. Thoughtful or reflective imitation means that at first, the learner does what s/he sees the interlocutor doing. Some learners fail to make sense of the teacher's mathematical discourse, or fail to find the difference between the new mathematical object and the old. Learners in the latter category ultimately absorb ritualised routines, which are not mathematical. Another category of learners may mimic the teacher, thus applying unthoughtful or uncritical imitation with little, or no awareness for doing so. This group's routines are classified as ritualised mathematical, since they write or say the correct mathematical statements without understanding their meanings (Siyepu & Ralarala, 2014).

Mathematical discourse advances the course of communication and participation in mathematical practices, such as abstracting and generalising (Moschkovich, 2002; Sfard, 2012). In other words, participation in a mathematical discourse means focusing on objectification (conceptual understanding in acquisition of theories) rather than just ritualised routines (computational fluency). The participation is then individualised when there is talk with others or talk with oneself by means of thinking, which is expressed by discursive action (Sfard, 2012). Objectification then replaces 'talking about processes and actions' with 'talking about objects' (Sfard, 2008). Objectification is illustrated by learners referring to the asymptote, rather than to a line through which the graph does not pass. Objectification makes communication in the mathematical discourse more effective.

Objectification has a four-fold dimension (Nachlieli & Tabach, 2012). Firstly, the learners' *competence* in the realisation tree has to be established. A realisation tree is a signifier on which the mathematical object is organised or built (Nachlieli & Tabach, 2012). In the functions discourse, the realisation tree includes graphs, tables, equations, formulas and verbal expressions of the function. In the hyperbola, the realisation tree includes the curve and the table of values, in which the asymptote and the formula is clearly indicated in the form of $y = \frac{a}{x-p} + q$. Thus, learners should be able to draw curves, interpret the table of values, and work with equations and fractions. Key to objectification in the functions discourse is competence on the realisation tree, since the introduction of words such as *hyperbol*a, *parabola*, and *exponential functions* are new to learners. It is most unlikely that learners could know these terms prior to their introduction by the teacher in the classroom. Therefore, objectification is preceded by the competences in the realisation tree because Mathematics is auto-poietic, and its objects do not pre-exist the narrative routines of the self-same objects (Nachlieli & Tabach, 2012).

Secondly, objectification is characterised by *participation*. To this effect, Nachlieli and Tabach (2012, p. 17) assert that "participation in discourse is a precondition for the objectification of functions". Accordingly, the definition of "square" to a grade one learner would be different from that of a grade 12 learner. The grade 12 learner would have had continuous participation on the discourse such that his/her definition is not only guided by what s/he sees, but looks for other properties such as the size of angles and sides, parallel lines, and symmetry. Learners begin the objectification process by informally using examples and new words, rather than the formal discourse. A grade one learner would elementarily mention that a square has four sides. While this is true, the definition fits a number of trapeziums that are not squares; for instance, the rectangle, the rhombus, and kites. Participation in the discourse then grows gradually with the informal terms substituted by formal terms due to the contradictions of conjecture and generalisation. Normally, the next four-sided shape introduced after a square is a rectangle. Learners need to state differences they see between the two four-sided shapes, the square and the rectangle. Adler and Ronda (2014) describe this stage as non-mathematical, according to which routines are highly ritualised and based on visual routines, with learners describing mostly what they see. At this stage, learners may not realise that a square is a special kind of a rectangle. In such cases, the narrative is mainly colloquial, as the learners have not yet fully acquired and understood the mathematical lexicon to describe their objects. Learners may not use the terms 'exponential function' or 'hyperbola' at this stage (Adler & Venkat, 2014),.

Thirdly, objectificationis characterised by the gradual and gentle introduction of *formal discourse*, which includes the covariance of quantities, in which case the term 'function' is

largely used as a noun. In such cases, learners begin by using rules and talk about processes (Adler & Ronda, 2014). For example, a hyperbola would be a function with undefined values. While learners may draw continuous graphs using the table of values, they do not necessarily perceive the continuity in the table of values. Later in the third objectification stage, various ways of representing the function are introduced. For example, a point (2;5) on the table of values is the same as (2;5) on the curve and in the equation. When 2 (two) is substituted in the formula, the result is 5 (five), and the statement is true for all other points. Infinitely, many points would have the same relationship.

Fourthly, and finally, objectification is characterised by the principle that *reflection* on the object should take place. In practical terms, learners should be exposed to tasks which promote reflection on mathematical objects. The reflection is promoted by derivation of rules and equivalent representations, as well as the ability to explain the nature of relationships between, and among mathematical objects (Adler & Ronda, 2014). Learners should also ask questions which promote their growth in the discourse. For example, they should ask: How do the graph, algebraic representation, and table of values relate to each other? What kind of worded statement is a hyperbola graph? Since functions are governed by grounded principles and properties, the examples and tasks given to learners should be such that they lead learners to make conjectures and generalisations by working together with the interlocutor for the realisation of objectification. Ultimately, objectification is achieved when learners can express a function through each representation of that function by recognising the relationships between and among different representations of functions.

2.2.1 Key aspects of a discourse

Cognitive communication (commognition) constitutes a key aspect of promoting mathematical discourse. Sfard (2008) emphasises on the four characteristics of commognition that promote the development of the mathematical discourse. These characteristics are words and words use, the visual mediators, endorsed narratives and routines. It is from these characteristics of commognition that the Discourse Profile of the Hyperbola and Exponential Function (DPHEF) was formulated to analyse the development of the mathematical discourse of learners in this study.

2.2.1.1 Words and word use

Key words or signifiers, are mathematical words such as the equal sign, the function, the asymptote, the intercept, the axes, the coordinates, the graph, and so on. These words enhance learners' mathematical communication, and show a change in the development of their mathematical discourse. In Mathematics, key words are a crucial part of the formal discourse. These words are esoterically mathematical, and their peculiar usage is different from their everyday meaning and contexts (Kendall & Halliday, 2014). For example, "product" in

mathematical language is premised on multiplication. In everyday language, the same word would relate to measurable output based on work or something that has been done already. Other words such as *root, differentiation, function*, and many more,have a different mathematical denotation from their everyday use. The mathematical context of word use is sacrosanct as it unlocks the potential of learners to communicate effectively. Effective mathematical communication may be affected by the fact that some of the words used in Mathematics are not necessarily in English. Their provenance and roots is traceable from languages such as Greek and/ or Latin. These would include words such as *hypotenuse, numerator*, and *asymptote*s (Kendall & Halliday, 2014).

Learners' use of mathematical words show their level of participation in the mathematical discourse. Compression is quintessentially an expression of advanced levels of participation in mathematical discourse (Caspi & Sfard, 2012). By means of compresson, a few words are used to express statements that would have been wordy or verbose. In the functions discourse, the key words are *intercepts, domain, coordinates, turning points, minimum/maximum, gradient* and so on. In the general hyperbola discourse, the key words include *asymptotes*, *quadrants, axis of symmetry, intercepts, the vertex,* to name just a few. The exponential function would have words such as *vertical shift, exponent with a base greater than one, exponential growth*, and so on. Change in a discourse is evident in spoken or written communication. For example, instead of saying or writing: *A function whose value is a constant raised to the power of the argument*, compression is used and the sentence is mathematically translated to an exponential function.

2.2.1.2 Visual mediators

Key words are supported by visual mediators, which helps the participants to identify and coordinate the objects of their mathematical discourse (Adler & Venkat, 2014). Visual mediators themselves help to compress wordy statements. For instance, an exponential graph provides a synopsis for a lengthy explanation of a phenomenon and its manifestations. Visual mediators include concrete objects, iconic mediators and symbolic mediators. Concrete objects and their images are the usual mediators used in everyday life. In Mathematics, symbolic artefacts are often used to enhance communication. Concrete mediators are objects used to assist in explaining mathematical concepts. For example, a rectangular box that is opened in class to help illustrate the number of surfaces that are to be calculated, or a clock in the introduction of time. The hyperbola and exponential functions do not have concrete mediators as they are generally abstract.

Examples of visual mediators include diagrams, charts, drawings, graphs and symbols. Visual mediators can be drawn on paper or boards in the classroom, and it is possible for them to be developed mentally by thinking about them. Most of the written Mathematics uses the symbolic

syntactic mediators. These are the symbols used in mathematical manipulations. The functions notation and symbols used include the following: $y = a(x - p)^2 + q,$ for the parabola, $or\ f(x) = \frac{a}{x-p} + q, x \in \mathbb{R}, x \neq p$ for the hyperbola. Symbolic and iconic visual mediators are sometimes responsible for the challenges faced by learners in Mathematics as they first need to unlock these mediators before participating more fully in the discourse. In this study, some learners found it difficult to relate the symbolic representations to iconic visual mediators, which include graphs, tables of values or geometric representations of two-dimensional shapes. Iconic mediators are commonly used as pictorial diagrams that are used either in class discussion or in learners' writing tasks in order to assist learners' visualisation of an image of the function. For instance, a graph that represents the motion of a ball, which is an abstract object. Figure 2.1 below is a visualised illustration of the iconic mediator of the hyperbola, intended to help learners 'see' the image of a function.
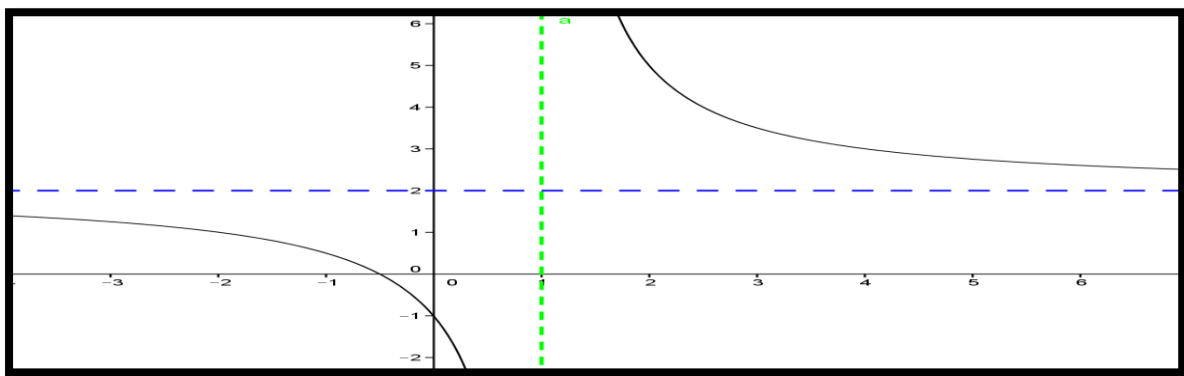
Figure 2.1: Iconic mediator of the hyperbola

The iconic mediator illuminates on the syntactic mediator by showing the intercepts, asymptotes, axes and the shape of the graph. It helps create an image that learners refer to when faced with the questions relating to the hyperbola. The DPHEF has four categories of the iconic visual mediators; the interpreted, not recognised, drawn and disallowed. The iconic visual mediator is regarded as "interpreted when a graph is used to find features of a function. When the learner is not able to interpret the graph his/her mathematical discourse is regarded as "not recognised". [Correspondingly] When learners successfully sketch functions, their mathematical discourse is regarded as "drawn" and if learners do not sketch the function, their action is classified as "disallowed" (Mpofu & Pournara, 2018, p. 6).

2.2.1.3 Endorsed narratives
The third key feature of mathematical discourse are the endorsed narratives, which are the rules that have been agreed upon by the mathematical community (Siyepu & Ralarala, 2014; Sfard, 2008). Endorsed narratives are a result of either object-level or meta-level learning (Sfard, 2012). Endorsed narratives include definitions, theorems, proofs and axioms. Change in development is realised when learners have arrived at new endorsed narratives. Endorsed narratives are therefore useful in the development of new discourses. Once a new discourse

such as a new function has been established, some basic narratives would have been endorsed. In the development of new discourses, mathematicians decide which narratives to keep and which to reject based on previously accepted mathematical rules. Learners engage with the mathematical objects at the object-level. Meta-level is governed by well-defined meta-rules, and learning can only happen through thoughtful imitation and perceiving the rules that govern the functions discourse. The endorsed narratives for a hyperbola includes a graph with two separate curves that are mirror images of each other, as well as a graph with two asymptotes that are perpendicular to each other. For the rectangular hyperbola, the equation $f(x) = \frac{a}{x-p} + q,$ leads to the equations of the asymptotes, $y=q$ and $x= p$. Mathematicians refer to the behaviour of the graph towards a linear function as the asymptote. In the function $(x) = \frac{1}{x}$, as x approaches zero from the right, *y* approaches infinity; and as x approaches zero from the left, *y* approaches negative infinity. The paths close to zero that the function approaches from both the left and the right are the asymptotes. Mathematicians had to endorse this narrative so that the narratives agreed upon earlier would not collapse. The interlocutors then bring these narratives to the fore in order that learners can own them. In such a context, the learner moves from the discourse of others, to the discourse of one-self (Ben-Zvi & Sfard, 2007).

In Mathematics, errors are narratives that are endorsed by learners, but universally disapproved or disowned by mathematicians and experienced participants (Brodie & Berger, 2010). In teaching and learning Mathematics, substantiations from learners make sense to them, but are not in agreement with what is universally accepted (Ben-Zvi & Sfard, 2007). For example, learners may ignore the asymptote and join all the points they see in a graph. When erroneously endorsed narratives have been identified, corrective measures should be taken. The researcher's self-developed DPHEF explains three categories of narratives: the substantiations, the memorisation and the authority narratives. Substantiations are endorsed narratives, while the memorosation and authority narratives represent some attempts by learners to explain their erroneous mathematical discourse.

2.2.1.4 Routines
Routines are well-defined patterns repeated over time, and are characteristic of a given discourse (Sfard, 2007; Swarthout et al., 2009). Routines include mathematical procedures used to perform mathematical tasks. Routines could either be exploratory or ritualised. Methods of proving and comparing graphs, searching for key words in questions, and making generalisations are examples of routines. It is within the routines that words, mediators and narratives meet. Exploratory routines add to the mathematical thinking of learners as they result in new endorsed narratives, make mathematical connections with previously endorsed routines, and further develop their mathematical discourse auto-poietically.

On the other hand, ritualised routines are those which are not mathematically classified as they lack exploration (Sfard, 2012). For purposes of this study, the researcher categorised ritualised routines into two groups. One is the ritualised mathematical, in which learners have the acceptable mathematical routines but do not explain why they do things in the manner they do. An example is found in Mpofu and Pournara (2018), according to which learners would sketch diagrams that exhibit growth in the mathematical discourse with asymptotes clearly shown. However, their narrative would imply that the diagrams did not have the asymptotes. The researcher has termed the second type of ritualised routines as non-mathematical routines. These routines do not make mathematical sense, as attested to by participants in a study by Berger (2013). These participants drew vertical asymptotes in the place of removable discontinuity. The procedures or explanations thereof do not make mathematical sense. For example, stating that a graph does not have the *x-axis*. Learners who exhibit mathematical ritualised routines are better placed for exploratory routines than those presenting non-mathematical routines.

The ritualised routines are divided into categories. Firstly, there are "correct" rituals, which stipulate that *what* the learners have written is correct. Secondly, incorrect ritualised routines refer to the learner's inability to understand what the interlocutor has done. In this regard, exploratory routines help determine the appropriate circumstances for taking particular individualised actions (Siyepu & Ralarala, 2014). Exploratory routines, rather than rituals, usually help learners to complete higher order and 'unseen' questions successfully. It is important to note that learning usually starts from ritualised routines, followed by the exploratory routines. Learning would have occurred when learners know *when* to take certain actions in their 'doing' of Mathematics

In this study, the researcher has termed the ritualised and exploratory routines as kinds of routines in the DPHEF; as opposed to applicability, flexibility and corrigibility, which he has termed 'the use of routines'. Flexibility is the use of more than a single routine so as to arrive at the same narrative and is mostly used to prove endorsed narratives (Ben-Yahuda et al., 2005). Flexibility is observed when learners use the table of values to show values approaching an asymptote and at the same time sketching a graph that shows points getting closer to the asymptote. Applicability routines are about the likelihood of a routine procedure to be used or applied in mathematical tasks (BenYahuda et al., 2005). Applicability routine is about *what* prompts learners to take certain actions. For example, learners will start looking for asymptotes when they see an equation with a fraction. Subsequently, the fraction triggers the idea of an asymptote. With regard to corrigibility, routines are used to check the correctness of answers arrived at. An example of corrigibility routine is the use of intercepts of a graph to solve an equation of a function equated to zero and algebra to check the correctness of the solution from

the graph. In DPHEF, applicability, flexibility and corrigibility routines are termed *the use of routines,* due to their explanation of the role of routines in the enhancement of the learning of Mathematics.

2.3 Mathematical Functions

In this section, the primary focus is on the hyperbola and the exponential function as the core areas of research as derived from the reviewed literature. The exponential function is discussed first, followed by the general hyperbola and the rectangular hyperbola. The asymptote, the general definition and the school definition are also discussed, followed by the importance of learning functions. This section further explores the literature review on the learning of functions, as well as the difficulties that learners encounter in their quest to objectify functions, which are essentially a relation where a set of inputs have exactly one output corresponding to it. A function is defined by a formula or algorithm that gives instructions on how to obtain an output from the given input. Sometimes, a function is represented as a graph or alternatively as a table with inputs and corresponding outputs. A function has a well-defined set of inputs, called the domain, and the set of outputs known as the co-domain, or the range (Mpofu & Pournara, 2018).

2.3.1 The exponential function

Exponential functions are a result of the work of Napier of Scotland and Burgi of Switzerland. Napier developed exponential functions from Algebra in 1614, while Burgi used geometry for the development of exponential functions in 1620. A function in the form of $f(x) = ab^x$ is an exponential function. In the latter function, the input x is an exponent. A general form of an exponential function and the parent function of an exponential function is expressed as $y = b^x$. One characteristic of this graph is that as the input values increase, the output grows exponentially. However, Mahlobo (2004) argues that the term "exponential function" is a misnomer. Mahlobo contends that the term does not resonate with naming of other functions as they are named according to the output value, rather than the input value. Mahlobo further gives examples of the quadratic and the cubic functions as deriving their names from the output value. Figure 2.2 below depicts the exponential function.
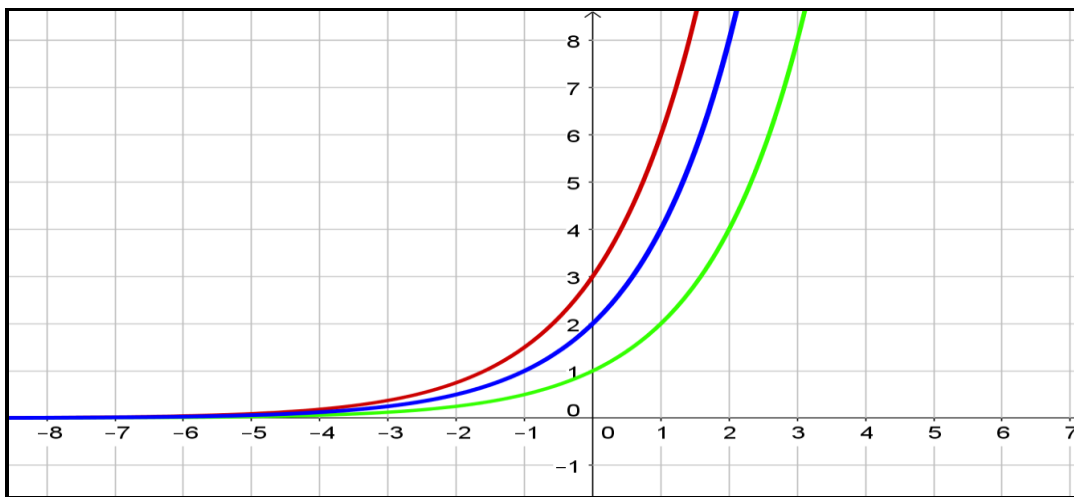
Figure 2.2: An exponential function with its transformation

Figure 2.2 above is a representation of an exponential function with part of its transformation. The value of **b** is greater than 1 (one). Characteristically, an exponential function has a single horizontal asymptote. Exponential functions decrease or increase depending on the value of **b,** for $b > 1$ there is growth and as the independent variable increases, the dependent variable also increases. A decay happens when $0 < b < 1$. As the independent variable increases, the dependent variable decreases. While **b** can never be negative, it is important for learners to be acquainted with the reasons why it is impossible for **b** to be negative. Objectification of the exponential function can only be achieved through the exploration of the mathematical object. The inverse of the exponential function is a logarithmic function. It is for this reason that an exponential function is sometimes referred to as the antilogarithm. However, there are challenges that learners face in the learning of exponential functions.

2.3.1.1 Learning of exponential functions

The learning of exponential functions is important in life. Not only do they help learners in their further education, they can be applied in many spheres of life as well. Webber (2002) argues that learning of exponential functions is important for learners' daily lives, and for those wishing to study further in Mathematics, especially in relation to topics such as calculus, differential equations, and complex analysis. Webber (2002) states further that exponential function teaching strategies should be improved, and suggests further that learning of the exponential function should come after a grounded learning of exponential expressions and equations has been achieved. Makgakga and Sepeng (2013) suggest that learning of exponential functions can be enhanced with transformation-oriented learning, notwithstanding that their research participants regarded the learning of exponential functions as difficult. Nonetheless, their study concluded that learning and transformation were symbiotic.

An understanding of exponential functions is very helpful for the interpretation of everyday real-world phenomena and sitiations. Exponential functions are useful in practical fields such as investment, modelling population growth, carbon dating of artefacts, and are used by

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coroners to determine time of death. Interpretations based on exponential functions enable various stakeholders to make informed decisions about the future. For example, a prediction of a growth in population sensitises the government to make future decision about planning for important programmes in housing, job creation, health care and many other decision-making initiatives.

2.3.2 The hyperbola

The hyperbola could be viewed from two perspectives, the general hyperbola and the rectangular or school hyperbola. The general hyperbola is fundamentally premised on disciplinary Mathematics, while the school Mathematics has been contextualised from disciplinary Mathematics through particular selections and emphases. The general hyperbola is beyond the school curriculum. It is the rectangular hyperbola that is largely found in the school curriculum.

2.3.2.1 The general hyperbola

A general hyperbola is a function with a smooth curve on a plane that satisfies a specific equation, and has points that are inversely proportional to each other. Furthermore, a hyperbola is a function which has two separate curved parts known as connected branches. The two curves are a mirror image of each other about a given line. A hyperbola is an open curve which does not have an end, but approaches a certain line from the left and the right; or from the top and the bottom - that is - from four directions. The point where the curve makes its sharpest turn is known as the *vertex*, and there are always two of these. Two asymptotes that are perpendicular to each other are not part of the diagram, but denote the path that the curve approaches as the graph tends to infinity or negative infinity. The asymptotes of the general hyperbola are not parallel to the axes as shown in Figure 2.2 (p. 28). In the school or rectangular hyperbola, the *x* asymptotes can easily be confused with axes since these asymptotes sometimes coincide with the axes, or are parallel to the axes. Figure 2.3 below represents an image of the general hyperbola.
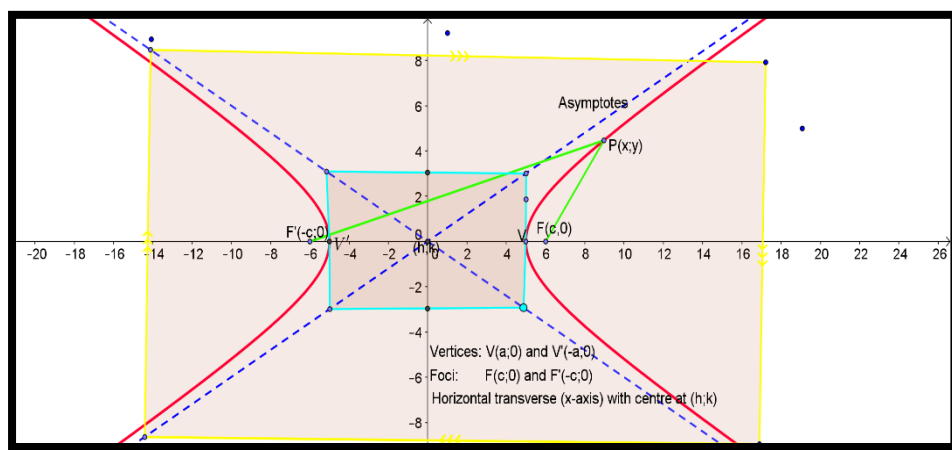
Figure 2.3: An image of the general hyperbola

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The mathematical objects presented in Figure 2.3 (p. 24) above illustrate the discourse of the mathematicians on the hyperbola.

2.3.2.2 The rectangular hyperbola

The rectangular hyperbola is also referred to as an equilateral or right hyperbola, one in which the asymptotes intersect at perpendicular angles to each other (Bradley et al., 2013). South African high school Mathematics textbooks do not define the hyperbola, but simply express the hyperbola formula. For example, one textbook (Bradley, 2013) defines the hyperbola, as a graph with the equation $xy = k$. Had the definition been stated in words, many learners would not experience the difficulty of not knowing that a hyperbola is actually an inverse proportion. This definition will have an impact on what a hyperbola is for school learners.

The rectangular hyperbola has a simple form in which the asymptotes coincide with axes. The current study focuses on this hyperbola, whose formula is reduced to $xy = \frac{a^2}{2}$, and is mostly written in the form of $xy = k$. For a positive k, the graph is in the first and third quadrants; while for the negative, it will be on the second and fourth quadrants. The asymptotes of any kind of rectangular hyperbola can be translated around the Cartesian plane, but the asymptotes remain parallel to the axes (Narasimhan, 2009).

The school mathematical discourse on the hyperbola focuses on the simple equations in which the coefficients of *x* and *y* are one. From the researcher's point of view, an explanation of the hyperbola is warranted, as it is prescribed in the Curriculum and Assessment Policy Statement (DBE, 2011). In terms of the CAPS, the curriculum does not expect grade 10 and grade 11 learners to define a function. In this study, however, some learners referred to the definition of the function only because the data collection period overlapped to some grade 12 some learners. Essentially, the definition of a function was not part of the tasks or question in this study. The curriculum expects learners to convert flexibly between four representations of a function: tables, graphs, words and formulae. In this study, the researcher further investigated learners' usage and application of words, tables, graphs and formulae as representations of the exponential functions and the hyperbola (Swarthout et al., 2009).

In grade 9, learners experience their first learning of the linear function. They use both the plotting of points and the use of the gradient in drawing and interpreting straight lines. When they reach grade 10, learners are faced with six more new functions in the form of the parabola, hyperbola, the exponential function, and the three trigonometric functions. To distinguish these functions from each other, learners are expected to first plot points of graphs until they are conversant with the shape of the graph; as well as relate the graph to the algebraic representation, table of values and words. It is only after there is fluidity between the four representations that learners can make generalisations by sketching the graphs, showing the

key points such as the intersection with the axes, as well as the shape of the graph and the line, which can be approached but not reached by the function. The curriculum refers to hyperbola functions as some rational functions which shows that the functions are not to be restricted to the hyperbola. In grade 10, learners learn about a vertical shift on all functions, and a horizontal shift in grade 11. Content clarification presents the rational functions as those that are in the form of $y = \frac{a}{x} + q$ in grade 10, and $y = \frac{a}{x-p} + q$ for grade 11. According to the curriculum, learners are expected to first plot the functions, and then make generalisations in order to sketch the graphs accordingly. The parameter q has the effect of moving the graph vertically. A positive q moves the graph upwards, while the negative moves it downwards. The parameter p moves the graph horizontally, either to the left or right. A positive p moves the graph to the right, while a negative moves it to the left. This movement resulting from these parameters moves the whole graph. (This means all points on the graph move by either q or p units). Figure 2.4 below depicts the hyperbola and its transformation in terms of the high school curriculum requirements.
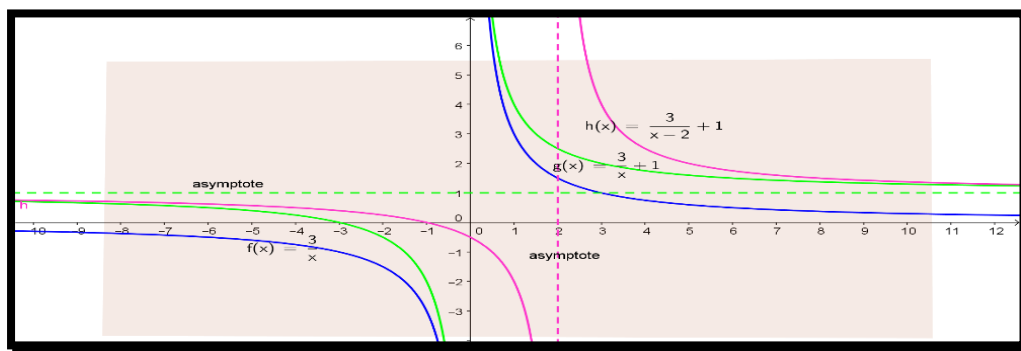
Figure 2.4: Transformation of the hyperbola

Figure 2.4 above illustrates the transformation of the hyperbola in the context of the curriculum's requirements, in terms of which learners are expected to master the equations in Figure 2.4 above and be able to sketch the graphs from given formulae, ordered points, or tables. Learners are also expected to find the equation of functions from the graphs and some points on the graph.

The learning of the hyperbola can be traced back to the inverse proportion relationship. In the South African school curriculum, this relationship is learnt only in arithmetic in the lower grades (DBE, 2011). Learners move from worded questions to arithmetic, and it is never expressed in algebraic form. The expression "y varies inversely to x" or "y is inversely proportional to x", written in symbolic form as $y = \frac{1}{x}$, means that $y = \frac{k}{x}$, where k is the constant of the variation. The difference is that in inverse variation, there are only positive values, since variation mainly addresses real objects. For example, the number of hours it takes to build a tower is inversely proportional to the number of workers. The asymptotes and the graph on the

positive side of the axis are the same, but with no negative values. Yet the same function in algebra will have the values (ordered pairs) in the first and third quadrants for positive k, since algebra addresses abstract objects (Caduri & Heyd-Metzuyanim 2015; Sfard, 2012). In the view of the researcher, this link is important in understanding the relevance of the hyperbola in the curriculum and in helping learners understand the path (shape) of the hyperbola.

In most South African schools textbooks, there are no application questions on the hyperbola and the exponential function. The textbooks mostly tend to provide a summary of each of these functions, as shown in Figure 2.4 (p. 26) above.

The exponential function: $y = a.b^x + q, b > 0, b \neq 1$

- The graphs of $y = a.b^x + q$ and $y = -(a.b^x + q)$ are reflections of each other about the x-axis.

- The graphs of $y = a.b^x + q$ for $b > 1$ and $y = a.b^x + q$ for $0 < b < 1$ are reflections of each other about the y-axis.

- The exponential function $y = a.b^x + q$ has a horizontal asymptote $y = q$ and cuts the y-axis at $(0; a + q)$.
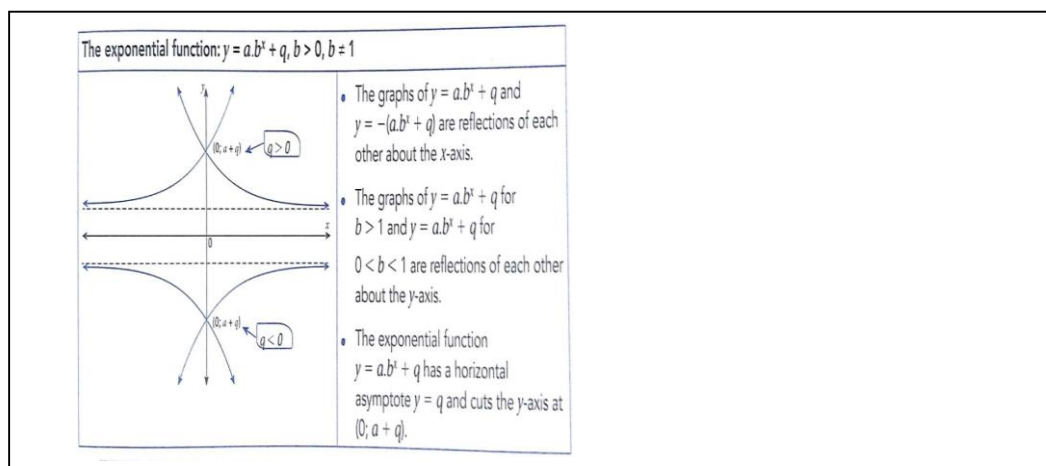
Figure 2.5: Examples of summaries for exponential function and hyperbola

In Figure 2.5 above, examples of the exponential function and the hyperbola are illustrated in summarised form. Thus, the summarised versions of most textbooks results in learners just following rules without understanding the reasons for the behaviour of the graphs. Another notable 'absentee' from the textbooks is the application questions. Application questions allocates purposefulness to learning of Mathematics, rather than rote learning for passing purposes only. The latter does not encourage learners's individualised exploration of mathematical objects; therefore, inimical to the acquisition of relevant knowledge in this regard.

The hyperbola is useful in practical real-life situations requiring some degree of mathematical calculations, such as in Astronomy, Physics, Architecture and Engineering. For instance, the cooling towers take the shape of a hyperbola, and they are used for taking out waste heat to the atmosphere. The hyperbolic structures of the cooling towers enhance their strength and capacity to withstand both wind and heat. The shapes of the curve of interference follow that of the hyperbola. A guitar's shape is another example of the hyperbolic shape in real-life. Its hollow exterior deep is in the form of a hyperbola, which helps echo the guitar sounds. Other real-life hyperbola-shaped objects include curved roofs, most potato chips, even the arch in buildings. In real-life application, an exponential function is represented also in Financial Mathematics to illustrate compound interest and the depreciation of cars. The depiction of

bacteria growth is another of many instances. The designers of questions need to think of these utilitarian and pragmatic value when choosing Mathematics textbooks. Sadly, teachers tend to use readily made examples from textbooks, and would rarely develop their own.

2.3.3 Learning of functions

The hyperbola best explains what happens in limits as x approaches an asymptote. Understanding of functions at school level establishes a solid base for further studies in Mathematics. However, there is agreement within the Mathematics research community that functions are not an easy concept for learners (Carlson, Oehrtman & Engelke, 2010; Clement, 2001; Yavuz, 2010; Moalosi, 2014). It is therefore not surprising, but also not acceptable that functions are poorly learnt and taught in schools. At the same time, when the reasons for the learning difficulty are laid bare, solutions can be found.

There are mainly two methods in which functions can be explored by learners, the point-wise and the global actions (Fraenkel, Wallen & Hyun, 2012). Point-wise actions refer to *what* learners do once they are first introduced to the function. At this stage, they are introduced to narratives that are associated with a particular function. Considering that all points of a linear function lie on a straight line, the point-wise actions then play a very important role in the explorations that produce such narratives. Hence the Curriculum and Assessment Policy Statement trajectory of the DBE's orientation that learners should begin by plotting points, followed by making generalisations based on their explorations (DBE, 2011). Global actions are characterised by the general behaviour of the function, such as the turning points, asymptotes, and shapes.

While the point-wise and global actions are equally important in the learning of functions, it is the point-wise actions that are supposed to lead learners to global actions (Essack (2015). In this regard, point-wise actions are generally related to the ritual routines, and explorations of the function should lead to more fluency in global actions. In some instances, the point-wise actions are neglected, with teaching and learning espousing global actions; which is detrimental to the learning of functions. In the event that point-wise actions have been neglected, learners tend to focus on the ritualised routine, rather than the mathematical object. In such a situation, it becomes difficult for learners to move individual point interpretation to a more holistic interpretation (Ronda, 2009). In most instances, learners would rather work with point-wise actions rather than the global actions. As such, learners should rather be helped to learn global actions in order to explore the mathematical objects and eventually generate endorsed narratives.

In the learning of functions, tasks assigned to learners should enhance their knowledge and understanding (Leinhardt, Zaslavsky & Stein, 1990 cited in Leshota, 2015). Tasks which

enhance the learning of functions include, but are not limited to interpretation and construction of functions. In the interpretation mode, the mathematical activities include identification of coordinates from a function and generalisation on the behaviour of the graph for certain equations. The movement of the graph in relation to the asymptote can be one of the foci on interpreting the graph. In the the construction mode, activities include drawing the graph using ordered pairs. The design of the tasks to be interpreted and constructed should be such that objectification is achieved.

Leshota (2015) illuminates further that the development of the functions discourse constitutes an integral aspect in the learning of functions. An adequate understanding of functions depends on the selection of examples and tasks that learners are exposed to. The success of the development of the functions discourse depends on what is made available for learners to learn (Adler & Ronda, 2014). Functions discourse can be developed using the four principles of objectification as explained in sub-section 2.3.1 (pp. 21-22) of this chapter. The development of the functions discourse is not easy to achieve, as learners have difficulties.

Moalosi (2014) suggests that the learning of functions should focus on relationship rather than on process. The formula or the equation is regarded by some learners as a process, since they regard the formula as a machine for producing the output. Leshota (2015) refers to the same scenario as the procedural action on which the learning of functions focuses on covariation rather than on correspondence. It is on the bais of the learning of functions that computation and plotting of points is emphasised without relating the ordered pair to each other.

Moalosi (2014) ascertains further that the learning of functions should focus on their multi-representations and terminology. The implication is that a function can be represented in different ways, thus reflecting a *translated* representation of the same function. A function expressed as a graph should be translated to an equation or table. The problem with the table of values is that learners sometimes choose input values for a hyperbola that are the same as those of a linear function. Correct terminology is important in the learning of functions. The definition of an asymptote, for example, has a bearing on the learners' understanding of the relationship between the asymptote and the graph.

2.3.4 The asymptote

As discussed earlier, endorsed narratives are agreed upon by the community of mathematicians, and the purpose is to have the same narratives applied in the community. There is a dilemma on the school-level definition of the asymptote. Mathematicians generally define an asymptote of a curve as a line that is tangent to a curve at infinity (Kuptsov, 2001). As the values of the curve increase, the curve and the asymptote approach each other. According to this definition, a curve may cross the horizontal asymptote. The vertical asymptote cannot intersect with the

curve. While some functions such as $f(x) = \frac{x^2-3x}{x-1}$ will have curves not crossing the asymptote, there are functions such as $g(x) = \frac{sinx}{x}$ in which the horizontal asymptote is crossed infinitely many times as the graph tends to infinity. The diagram below illustrates these curves and their asymptotes. Figure 2.6 and Figure 2.7 below illustrate the asymptotes in their curving and linearity contexts.
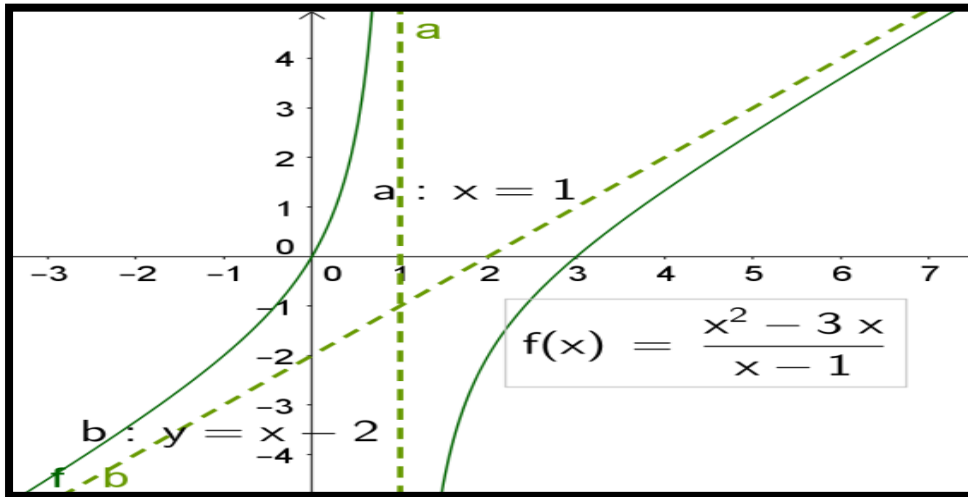
Figure 2.6: Illustration of curves and asymptotes
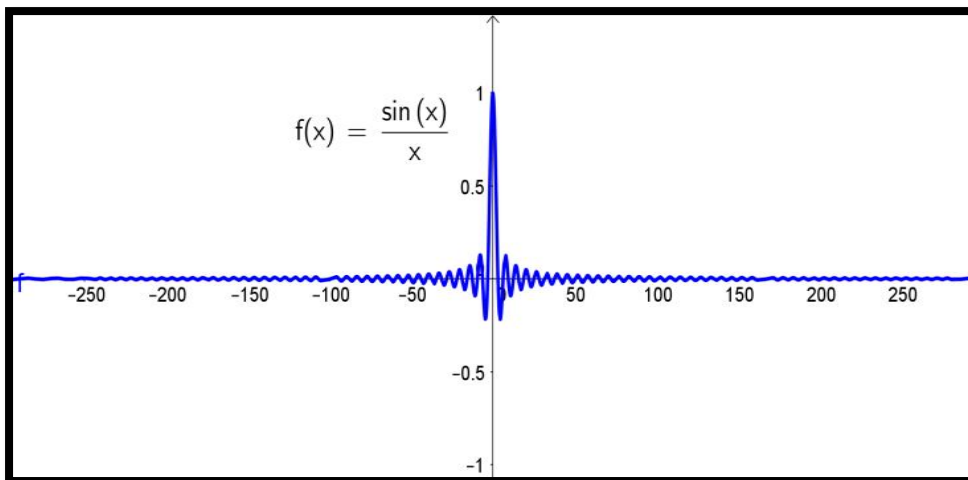
Figure 2.7: Illustration of asymptote linearity

In terms of Figure 2.6 and Figure 2.7 above, the asymptotes of the function $f(x) = \frac{x^2-3x}{x-1}$ are $x = 1$ and $y = x - 2$. The curve of the function f does not cross the asymptote. The asymptote for g is the x-axis, and the curve crosses the asymptote several times.

In terms of the school curriculum definition, the asymptote is a line whose distance from the curve tends to zero as they approach infinity. Grade 11 learners are not expected to learn limits. As such, the definition with limits would not be suitable for them. At the same time, the proposition that the curve should not cross the asymptote is not mathematically correct, and should therefore be avoided in the teaching and learning situation (Berger, 2013). As stated in the paragraphs above, the horizontal asymptote may, or may not cross the curve of a graph.

However, it should not mean that learners should be exposed to questions that show the asymptote crossing many times. What is emphasised here is for the community of mathematicians not to have contradictions.

There are generally three kinds of linear asymptotes, a vertical asymptote, a horizontal asymptote, and a removable discontinuity. In a vertical asymptote, the gradient is undefined and the asymptote is presented in the form of $x = a$. Furthermore, the vertical asymptote is parallel to, or coincides with the y-axis. In many instances, making a distinction between the vertical asymptote and the removable discontinuity poses a challenge to the learners (Berger, 2013). A function will have a vertical asymptote at the zero point of the denominator, provided the denominator is not a factor of the numerator. In the event that t the denominator is a factor of the numerator, the zero of the denominator is then the removable movable discontinuity.

The second type of an asymptote is horizontal, and is expressed in the form of the equation $y = b$. This asymptote is parallel to, or coincides with the *x-axis*. In the event that $ax^n + \ldots$ a polynomial to the nth degree and $bx^m + \ldots$ a polynomial to the *mth* degree, then in $f(x) = \frac{ax^n + \cdots}{bx^m + \cdots}$ then the following state of affairs unfolds:

1. If n < m, then the x-axis is the horizontal asymptote. An example is $f(x) = \frac{3}{x}$;
2. If n = m, then the horizontal asymptote is the line $y = \frac{a}{b}$. An example is $f(x) = \frac{3}{x-2} + 1$. Grade 11 learners are familiar with the above-mentioned formulae that translate to $f(x) = \frac{x+1}{x-2}$, showing that $n = m$ and the asymptote is y =*1*. It is rare to find the latter in the learning of Mathematics; and
3. If $n > m$, then there is no horizontal asymptote. The asymptote is an oblique one, or it will be a removable discontinuity. An example would be $f(x) = \frac{x^2 + 3x - 5}{x+1}$, in which case there is no horizontal asymptote but a slant asymptote: $y = x + 2$ and a vertical asymptote: x= -1. In $f(x) = \frac{x^2 - 1}{x+1}$, there is no asymptote altogether; instead, a removable discontinuity is presented.

The horizontal asymptote coincides with the curve when an oblique asymptote is expressed in the form of $y = mx + c$, where $m \neq 0$. The rectangular hyperbola, on the other hand, will always have the vertical and horizontal asymptotes. Figure 2.8 below (p. 32) provides a diagrammatic representation of the horizontal asymptote, the oblique asymptote, and the vertical asymptote.
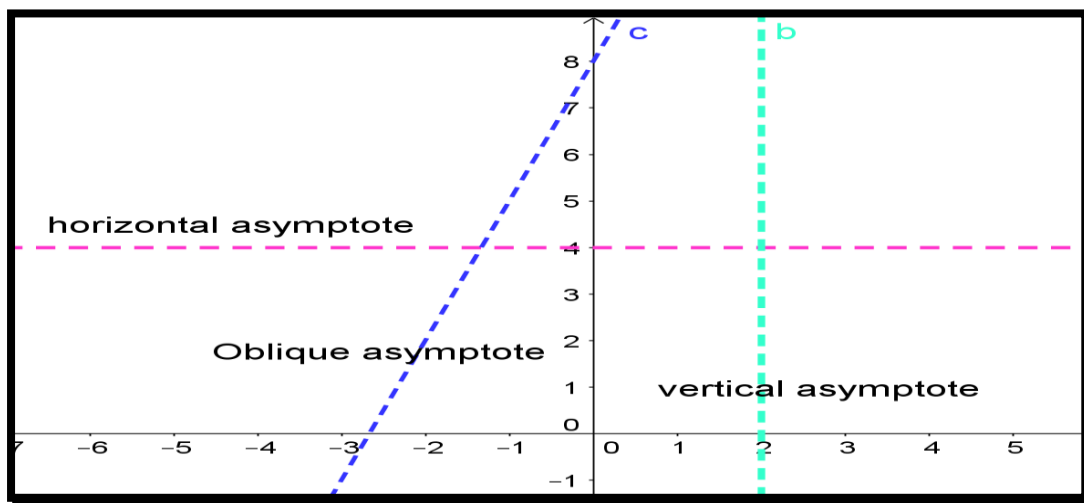
Figure 2.8: Graphical representations of kinds of asymptotes

Asymptotes are generally linear in nature, but there are instances in terms of which the distance between two curves tends to zero as they tend to infinity. This is known as a curvilinear asymptote. The curve $h(x) = \frac{x^3+2x^2+3x+4}{x}$ has an asymptote of $y = x^2 + 2x + 3$. Figure 2.9 illustrates the curvilinear asymptote, a curve in a green dotted line and the function h represented by the purple curve.
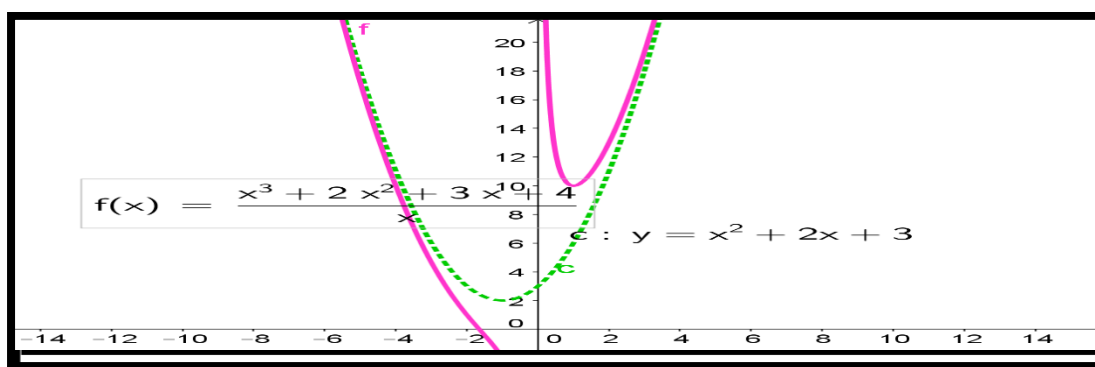
Figure 2.9: An example of a curvilinear asymptote

In essence then, there are four types of asymptotes, three of which are linear and one that is a curve. Asymptotes are not necessarily lines that are part of the graph. In Figure 2.9 above, the dotted line shows the location of the asymptote curve, and is drawn for that purpose. A graph that does not show an asymptote is mathematically correct, provided all the other requirements for that function are clearly represented (Ronda, 2009).

An asymptote and a removable discontinuity occur when the denominator is zero. The difference between the two is that an asymptote cannot be redefined to make the function continuous at that point. On the other hand, there is a way of defining a function in removable discontinuities such that they are continuous. In the event that a denominator of a rational function is zero at a certain value of x and the numerator is not zero, then there is a vertical asymptote at that point. In contradistinction, when both the numerator and the denominator are

zero, there is a removable discontinuity (Kuptsov, 2001). For example, $f(x) = \frac{4}{x-2}$ at $x=2$ would be $f(2) = \frac{4}{0}$; and $h(x) = \frac{4(x-4)^2}{x-4}$ at $x=4$ would result in $h(4) = \frac{0}{0}$. In this case, the function f has a vertical asymptote, while the function h has a removable discontinuity. At grade 11, the curriculum does not require knowledge of this difference.

2.3.4.1 Learners challenges concerning asymptotes

As stated earlier, the official high school Mathematics curriculum in South Africa presents learners with functions whose asymptotes are either parallel to the axes, or coincide with the axes. The hyperbola and the exponential functions have asymptotes at the zero points of the denominator, and there is no intersection between the horizontal asymptote and the graphs. The reviewed literature shows that learners tend to struggle with the concept of the asymptote (Mpofu & Pournara, 2018; Flesher, 2003, Kidron, 2011).

A study by Flesher (2003) found that learners' asymptote-related difficulties were universal. In her cognitively inclined study, Flesher (2003) focused on college learners who were expected to describe the meaning of graphical representations in their own words. Some of the findings of this study indicated that learners' mathematical discourse on the asymptote were at a granular stage. (Flesher's study was cognitive in nature, hence her usage of the term 'conceptual misunderstanding'). In the self-same study, some learners were of the impression that the asymptote was a number, and not a straight line. This will be most probably coming from the algebraic calculations rather than from the graphs. Some of the learners in the study struggled to position the asymptote, stating that the asymptote can only coincide with the axes. This view emanates from an asymptote of parent functions, in which there are no vertical or horizontal translations. Yet, some learners could not define the horizontal and and/or vertical asymptotes after correctly calculating the position of the asymptotes on the Cartesian plane.

Furthermore, learners generalise mathematical objects according to their observationse, whether such objects are a procedure or a representation. In her cognitive study, Kidron (2011) investigated learners' concept image on the horizontal asymptote. The learner in question struggled to understand that the horizontal asymptote can intersect with the function. The diagrams created some commognitive conflict in which the graph and the asymptote behaved in a manner that she (Kidron) did not expect. After checking for errors in her work, she finally accepted that the horizontal asymptote could intersect with the graph infinitlys. The lesson learnt from Kidron's study is that the current high school definition introduced to learners in the South African system of education will need to be changed, if learners are to continue studying rational functions. The horizontal asymptotes would intersect with the graph and in some instances, the asymptote would be a line in the form of $y = mx + c$.

In a study conducted by Yerusalemy (2003), learners defined the horizontal asymptote as a slope with a gradient of zero. Their initial thoughts on the vertical asymptotes were that it occurs on every zero point of the denominator. They did not distinguish the removable discontinuity from the asymptote. Some of these learners referred to the asymptote as a point, which demonstrates that their view of the asymptote was limited. Similar to the study by Nair (2010), learners in the study by Yerusalemy (2003) above also stated that asymptotes were invisible lines.

Some teachers too struggle in understanding the concept of an asymptote, as revealed in a study by Berger (2013), which revealed that teachers could not distinguish between the removable discontinuity and the asymptote. Two teachers drew a vertical dotted line that passed through the point where the function was undefined. For the two teachers, the zero points of the denominator meant that a vertical asymptote had to pass through that point. At the end of the above-mentioned study, teachers had developed new perspectives on how they viewed the vertical asymptote. Notwithstanding, this experience shows how deep-seated are the challenges in the learning of functions - especially the asymptotes.

2.4 Functions in the South African School Curriculum

In terms of the South African high school Mathematics curriculum, grade 10 learners are required to begin sketching a hyperbola and exponential functions. These functions are also learnt in grade 11 and examinable in grade 12. In grade 10, learners learn functions with a vertical shift only, and the horizontal shifts are only introduced in grade 11. Work on functions begins with the plotting of points, which enables the learning of the general shape of the function and the subsequent generalisations. Thereafter, learners are expected to sketch the graphs after mastering the key features of the function. In fact, generalisations on all transformations begin with the plotting of points in graphs. Once learners have understood the concept, they are then expected to generalise by sketching the graph showing only key points such as the intercepts and the asymptotes. In terms of the CAPS, learners should be able to progress flexibly between the four representations of a hyperbola and interpret the function from whichever representation (DBE, 2011). As Denbel (2015) noted, learners tend to work on the representations oblivious of relation between them. A review of the National Senior Certificate shows that questions are skewed towards the algebraic and graphical representations of the function. The recommendations of CAPS go further, requiring learners to make generalisations and prove them, especially with regard to the asymptote, intercepts and translations.

2.4.1 Learner challenges with functions

Learners' difficulties concerning functions is well documented in research (Clement, 2001). Carlson et al. (2010) attribute the over-reliance on procedures of learning functions as one of the reasons learners struggle to understand mathematical functions. On the other hand, Denbel (2015) attributes these difficulties concerning functions to the restricted image of functions that learners have. The latter author further noted that the different representations of a function were treated in isolation. The findings of the current study confirm the findings of Denbel's (2015) study - that learners experienced difficulties relating to the representation of functions among themselves

Carlsonet al. (2010) proposed that some of the difficulties associated with the learning of functions emanate from learners' failure to distinguish between a general equation and a function represented as an algebraic equation. An equation has a single indeterminate. For example, $2x = \frac{3}{x-4}$ and a function will have at least two indeterminates or variables. An example in this case would be $y = \frac{3}{x-4}$. While the two functions look similar to a novice, the equation and the formula are mathematically different. When solved, the equation will yield values of x which makes the equation to be true. In the context of the formula, the x values would have a corresponding y value for the statement to be true. In the context of the formula, there are many of the ordered pairs which satisfy the algebraic statement. Caspi and Sfard (2012) have noted unequivocally that learners' main difficulty was that they viewed functions as two expressions separated by the equal sign. Insufficient understanding of the meaning of the symbolic mediator, the equal sign, was the cause of such misinterpretation of the functions; which also translates to an under-developed realisation tree of a function makes. Clement (2001) mentions the following contributory factors to learners' limited understanding of functions. Learners are of the view that:

- a function emanates from a single rule; for example, functions should take the form $f(x) = ab^x + q$, nothing else would be acceptable;
- the piece-wise relation is not a function, but a mirror image of the above point;
- the graph of a function should be continuous; a graph with a discrete value, for example, number of men painting houses over time is not regarded as a function since men supposedly take integral values only;
- once there is a gap between the graph, then it ceases to be a function; for example, the hyperbola is not regarded as a function as it has two parts; and
- a function should be a one-on-one; $f(x) = 4$ is not regarded as a function. At the same time, the same statement [f(x) =4] drawn on paper is regarded as a function.

While learners are correct that an equation written as x is equal to a constant is not a function, the equivalent of the same drawn on a Cartesian plane is regarded as a function. It is also

difficult for learners to distinguish between *y* is equal to a constant and *x* is equal to a constant (*y =a* and *x=a*) (Clement, 2001).

There is a belief among learners that a function should be continuous (Carlson et al., 2010). Accordingly, learners would not classify as functions those graphs that are not continuous. The vertical test distinguishes a relation from a function. For a function, each input value should have a distinct output. The only instance a relation is regarded as a non-function is when an input has at least two output values. For example, in $y = \sqrt{x}$ for every value of *x*, there are two distinct y values for all values of x (except for $x = 0$). Learners regard all disjointed graphs as non-functions. A hyperbola would fall under this category.

Moalosi (2014) summarises that the difficulties experienced by learners are compounded by the requirement to define the function; the x and y intercepts; the effects of the parameters on the graph; interpreting the graph; translating between the representation of the graph; as well as the inability to distinguish between the various functions. In the context of the empirical observations of this study, learners' experienced difficulties with the effects of the parameters on the function, interpreting information on the table or graph; as well as the ability to translate the representations of the functions.

2.5 Conclusion

In this chapter, multiple perspectives were presented and discussed in the context of the hyperbola and asymptotes as emanating from the reviewed literature. It was on the basis of the background provided by the literature review that there was a deliberate emphasis on the difference between the object-level and meta-level development in this chapter. Accordingly, the four characteristics of communication and cognition were explained; namely, the use of words; the communication visual mediators; the routines; as well as the endorsed narratives. The general and the rectangular hyperbola were presented visually, with an explanation of the rectangular hyperbola process when the semi-major axis and the semi-minor axes are equal to each other. The asymptote and its various types were also described.

The aspect of learning as participation was also included, since the focus of this study is on the learning of the hyperbola. The chapter also made a concerted reference to the requirements of the Mathematics curriculum on the functions, in the context of the research topic; including the learning of functions and difficulties that learners face in learning these functions.

The next chapter (Chapter 3) focuses entirely on an inter-theoretical and conceptual framework of the study. The theory of commognition and the theory of socio-cultural learning constitute the most central aspect of the chapter, in addition to the definition of key concepts associated with the research topic and its attendant problem to be resolved.

CHAPTER 3
THEORETICAL/ CONCEPTUAL FRAMEWORK

3. 1 Introduction

The previous chapter provided a multifaceted review of the consulted and reviewed literature in the sphere of learners' mathematical discourses and functions. The researcher concurs with the assertion by Walliman (2011) in highlighting the role and value of the literature review process in the development of a theoretical or conceptual framework. The latter author asserts that: " The [literature] review can be used to show where you have gained inspiration to develop your ideas – and that does not just have to be only from academic sources. It should also demonstrate that you have a good understanding of the current conceptual frameworks in your subject, and that you can take a stance in placing your work within these" (Walliman, 2011, p. 57). The current chapter then centralises the inter-relatedness of mathematical discourse, mathematical functions and the learning of functions within the conceptual parameters or frameworks of commognition and social learning as the foundational guiding theories in the study in conjunction with the relevant philosophical assumptions (Knobloch, 2010).

Research studies are mainly based on some specific paradigms, frameworks or perspectives which necessarily establish the 'boundaries' for scientific investigation (Ramenyi & Bannister, 2013). Theoretical frameworks themselves are abstract in their nature, but systematically present generalisations which explain the association between and among a phenomenon's variables. In this study, mathematical discourse, mathematical functions and the learning of functions are posited as inter-related variables of commognition and social learning as specific theoretical frameworks. The latter coheres with the perspectives of authors such as Knobloch (2010), Ramenyi and Bannister (2013) and Saunders, Lewis, and Thornhill (2012); who emphasise that a theory is fundamentally a systematically and symbolically organised representation of perspectives pertaining to the reality of ideas, concepts or phenomena that are of interest to the researcher. For the purpose of this study, the theoretical framework or perspectives (paradigms) provided a context within which the key principles are defined and related to the practical domain of the study, in terms of which the research problem could also be conceptually relevant to the study's adopted philosophical assumptions (Kumar, 2012).

Philosophical assumptions are not peripheral to the researcher's own system of values, and mainly refer to the basic principles or paradigms (philosophically rooted points of view) which are subject to application with no need for proof or verification (Knobloch, 2010). In this regard, the assumptions guide the particular philosophical approach or "stance" adopted by the study in the observation of phenomena or critical research variables that are the subject of both observation and investigation (Marshall & Rossman, 2011). Additionally, assumptions (basic philosophical/ intellectual abstract ideas or concepts) could be ontological (assumptions based

on the nature ("being") of reality); epistemological (nature and construction of knowledge and its reality); methodological (the means by which knowledge of reality is acquired); axiological (extent to which detailed attention has been allocated to the various components and aspects of the whole study); and rhetorical (the researcher's art or capacity of written and oral persuasiveness). In this study, a hybrid approach was adopted in terms of the degree of applicability of different assumptions to different aspects of the research process.

3.2 Definition of Key Concepts

The definition of the key concepts is closely related with the conceptual and theoretical context and parameters of this study, since it is on the basis of such definitions that clarity and logic are accorded to the very thematically concepts which are necessarily embedded in the learners' mathematical discourse and functions, as well as their social learning (Narasimhan, 2009). As a result of the thematic coherence and logic-seeking orientation of the definitions of the below-cited key concepts, their alphabetic sequence does not necessarily signify any order of prioritisation or appearance in the study. The key concepts identified in this study are: asymptote, the Discourse Profile of the Hyperbola and Exponential Function (DPHEF), functions discourse, hyperbola, and mathematical discourse.

3.2.1 Asymptote

The term 'asymptote' derives from the Greek word literally meaning, "Not falling together" (Kupstov, 2001). Mathematically, it means that there were no points of intersection in a graph (Mpofu & Pournara, 2018). However, it is now common knowledge that the horizontal asymptotes sometimes intersect with the curve. Information about a curve is conveyed in an asymptote. In sketching functions, one of the important steps is to determine the asymptote. The mathematical definition of the asymptote, used by the community of mathematicians, differs from the school definition. The school definition is context-specific, as opposed to the purely mathematical definition. Throughout this study, the researcher has attempted to discuss the difference between the definitions of the asymptotes, and then showing the difference between the asymptote and the removable discontinuity.

An asymptote of a curve is a line constructed such that the distance between the curve and the line approaches zero as one or both x and y tends to infinity (Kidron, 2011). There are three types of asymptotes, the vertical, horizontal and oblique asymptotes. The horizontal asymptote may or may not intersect with the curve. The vertical asymptote does not intersect with the graph. An oblique asymptote is in the form of $y = mx + c$. Participants of this study only did functions with only the horizontal and vertical asymptotes that either coincide or are parallel with the axes. Often participants' definition of an asymptote was closely related to what they (participants) had seen in their learning.

3.2.2 Commognition

Communication is a mechanism for the facilitation and execution of mathematical discourse by combining and linking communication with cognition (Sfard (2008). In commognition, learners participate in a mathematical discourse by both talking and thinking. "Thinking is regarded as communication with oneself" (Sfard, 2012). Thinking is expressed in verbal or written form. In this regard, participation in mathematical discourse is seen in action, and development is seen in change of discourse (Sfard, 2012). The more fundamental aim of discussion is to enable learners to think and talk like mathematicians. In the event that the conversation in the discourse remains the same, then development has not taken place. Since learning is a sub-set of development, the focus is not on the change in the learner, but the change in the discourse. Development in the discourse is manifested by the learners' use of new rules in the mathematical discourse (Sfard, 2012). When learners use new rules, their communication changes. Therefore, learners would have developed in their commognition and mathematical discourse in the event that they are able to make a difference between forms of translation and show this change in their communication (Vyncke, 2012).

3.2.3 DPHEF (Discourse Profile of the Hyperbola and Exponential Function)

The DPHEF is an analytical tool used to analyse the mathematical discourse of the learners who participated in this study. The DPHEF was developed by the researcher from the original Discourse Arithmetic profile used by Ben-Yahuda et al (2005). The DPHEF was designed to identify and distinguish between the superficial and dispositional differences in learners' mathematical discourse. As an analytical tool, the DPHEF can be used for any mathematical function, and is not restricted to the hyperbola and the exponential function only. From the perspective (rather than assumption) of the researcher, the DPHEF it is envisaged that the DPHEF will contribute towards learners' knowledge, perceptions, and experiences in the learning of asymptotes of the hyperbola and exponential functions.

The DPHEF has two categories of word use, the mathematical and the colloquial. The mathematical use of words is defined as a form of communication about which all members of the community of mathematicians have the same interpretation (Tachie & Chireshe, 2013). An immediate example is: *An increasing exponential function has a base that is more than one*. The colloquial use of words relates to a mixture of mathematical and non-mathematical words. An example could be: *An increasing exponential function is the one that goes up*. The use of words is ambiguous and is prone to different interpretation by different communities of people.

3.2.4 Functions discourse

The functions discourse is grounded on the realisation and conceptualisation of Algebra as a branch of Mathematics (Caspi & Sfard, 2012; Nachlieli & Tabach, 2012). The words and symbols used for functions distinguish the functions discourse from other discourses. A

function can be represented in four various ways in commognitive terms. A function can be represented verbally by using words to describe a phenomenon; algebraically by using symbols to communicate; numerically by using ordered pairs mostly expressed in tables; and graphically by either sketching or drawing a graph. In this study, the participants mostly opted for graphical and algebraic representations of the functions. Figure 3.1 below illustrates the various developmental stages of a functions discourse, as well as the developmental linkage to routines.

Figure 3.1 Development of the functions discourse
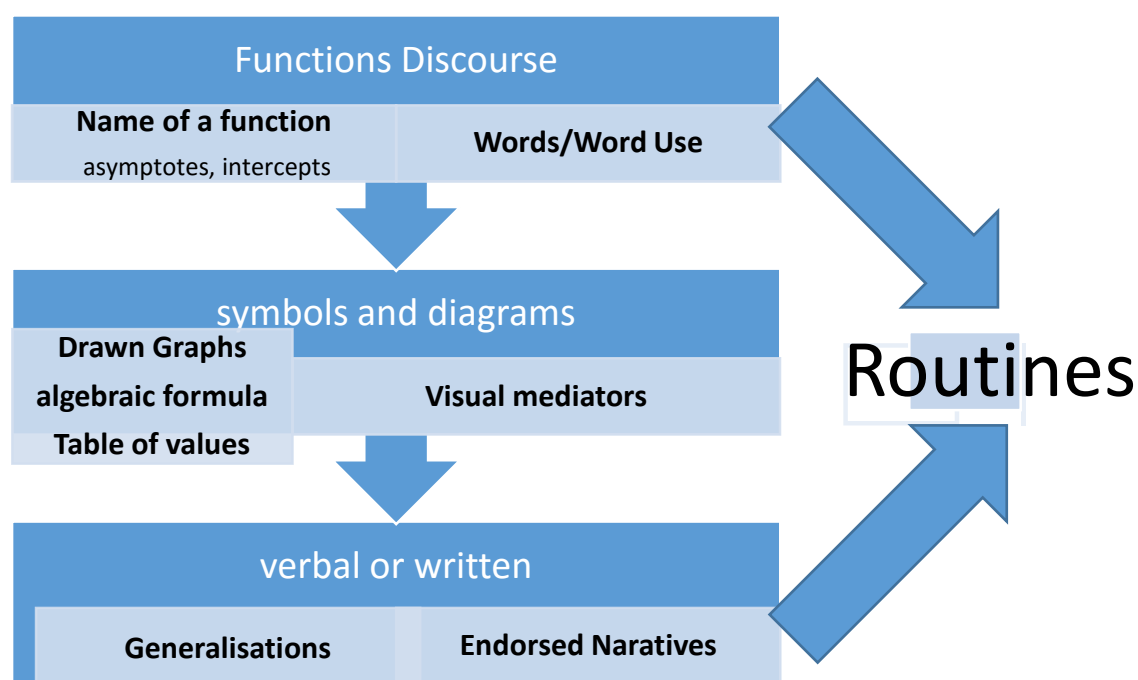
Source: Researcher's own initiative derived from the review of literature

Figure 3.1 above illustrates the centrality of words pertaining to the development of the functions discourse. These words include *asymptote, axes, intercept,* to name just a few. These words usually characterise visual mediators in the form of graphs, tables and formulae. Having had exposure to, and knowledge of these words, learners are then able to generalise based on their interpretation of the words and their attendant visual mediators. All three tenets of the functions discourse (word use, visual mediators and endorsed narratives) collectively determine whether learners function at the level of ritualised or exploratory routines. Generally, all learners begin from ritualised routines, and some proceed to exploratory routines while others take some time to move out of the ritualised routines. Competence in the hyperbola functions discourse comes after learners have had competence in each of the sub-discourses on inverse proportion, graphs, symbolic expressions and equations, and tables of values.

3.2.5 Hyperbola

A hyperbola is a function whose dual graphical representation of its two parts reflect each other's lines of symmetry (Nachlieli & Tabach, 2012). A hyperbola has two lines of symmetry that are given by the equation $y - q = \pm(x - p)$. For instance, the school or rectangular hyperbola has two asymptotes that are perpendicular to each other. Learners often fail to recognise the asymptote when it coincides with the axes (Mpofu and Pournara, 2018). The algebraic representation of a hyperbola is in the form of $y = \frac{a}{x-p} + q$. In this study, learners often viewed the parameters p and q as asymptotes, yet they only represent the horizontal and vertical movement, which is a display of ritualised routines.

There are basically two types of the hyperbola, the general and the rectangular hyperbola. Whereas the general hyperbola is a function with a smooth or open curve on a plane, the rectangular or equilateral hyperbola is one wherein the asymptotes intersect at perpendicular angles to each other. In addition, a hyperbola becomes rectangular when the semi-major and the semi-minor axes are equal. The semi-major axis is a line segment from the centre to the vertex of the hyperbola. The length of the semi-major is denoted by the letter a. A semi-minor axis is a line perpendicular to the semi-major axis and has a length of b. Therefore, in a rectangular hyperbola $a=b$, the eccentricity is equal to $\sqrt{2}$. The asymptotes of a rectangular hyperbola are generally represented by the equation $y = \pm x$ (provided there are no translations) since $a=b$ will yield 1 (one). The asymptotes make an angle of $\pm 45^o$ to the axis. The equation $x^2 - y^2 = a^2$ yields a rectangular hyperbola (Narasimhan, 2009).

A learner may plot points for a hyperbola and draw the asymptotes, but still continue to perform ritualised routines, due to an insufficient validation of the reasons for the steps to be followed in plotting the points of the hyperbola. When sketching a hyperbola, knowing how the graph relates to the asymptote is an exploratory routine, but just following the interlocutor's instructions is not. An action that may be an exploratory routine when undertaken by an interlocutor may be a ritualised routine when undertaken by a learner.

3.2.6 Mathematical discourse

Mathematical discourse is a type of communication and activity of generating the story of Mathematics. According to Sfard (2008, 2012, 2015), mathematical discourse is characterised by four key elements that help to illuminate the mathematical story of an individual. The elements of a mathematical discourse are words and word use, the visual mediators, endorsed narratives and the routines. These four elements are ontologically inextricable, and demonstrate the development of the mathematical discourse. In participatory learning theories, the textbook is regarded as an interlocutor, since learners are able develop their mathematical discourse from the textbooks. On a positive note, textbooks can help in the development of mathematical

discourses by providing opportunities for learners to explore mathematical objects; such as tasking learners to make a generalisation on the horizontal shift.

In this study, the learners' mathematical discourse is still in a state of developing. These learners could identify mathematical symbols and used a form of discourse they regarded as mathematical, but such discourse would not be endorsed by the community of mathematicians on the basis that the narratives of the learners were generally ritualised.

3.3 Theory of Commognition

The theory of commognition is cognate from the work of Wittgenstein, (1953) and Vygotsky (1978). All they were not contemporaries, these two luminaries espoused similar ideas on the nature and processes of learning. To them, learning and thinking are means by which one communicates mathematical concepts with others and oneself in an objectified manner (Daniels, 2001).

Commognition derives from one of the participation theories which posit learning as the process of individualising the mathematical discourse (Sfard, 2007; Sfard, 2014). It is in that context that commognition was developed from socio-cultural learning theory, because in its development. there is acknowledgement that learning of Mathematics is both societal and individual (Caduri & Heyd-Metzuyanim, 2015). The change in societal communication results in a change in communication with other individuals. In commognition, it is expected that participation progresses from ritualised routines to exploratory routines (Sfard, 2014).

As a concept, commognition is a combination of "communication" and "cognition". In the commognition perspective, talking and thinking are viewed as part of communication (Sfard, 2008); and thinking is viewed as an individualised form of communication activity with oneself (Sfard, 2007; Sfard, 2008). Thinking is largely expressed in the form of talking and through writing. Individuals change by being involved in various activities. For instance, participation of learners in the mathematical discourse is measured by their talking, gestures and writing in the particular sub-discourse they are engaged in. Therefore, learners participate by talking and bringing forth their reasoning in class or group discussion; or in solving problems in their writing.

According to the commognitive perspective, the aim of learning Mathematics is to enable learners' membership of the community of Mathematics (Nachlieli & Tabach, 2012). In such a context, learning is viewed as the process of becoming a competent participant in mathematical discourse or practice. This participation is evident in communication by a learner.

3.3.1 Foundational principles of commognition

Commognition has its foundation on three human activities and their constitutive principles (Sfard, 2014). Where communication changes in mathematical activities, there is a concomitant

transformation in the way we do things. Secondly, discourses function as propagators of innovation and repositories of complexity. Thirdly, learning happens at either object-level or meta-level. Object-level learning denotes accumulation of endorsed narratives, while meta-level learning is about a whole transformation of the endorsed narrative. The three foundational principles of commognition are discussed below.

3.3.1.1 The co-constitutive nature of discourses and other human activities

As an auto-poietic, Mathematics grows from within (Nachlieli & Tabach, 2012; Sfard, 2015). Mathematical activities and developments contribute to the growth of Mathematics as a field of study. As human beings engage in mathematical activities, a need for change arises. When humans were at ease with natural numbers, counting their wealth and so on, they arose a need of representing debts and the negative numbers were born out of that need. These, and a host of other human activities, are responsible for innovation within the field of Mathematics, while also generating new narratives and transforming other narratives as well. For example, the introduction of the hyperbola graph has changed the perception that a graph should always be continuous.

3.3.1.2 Discourses function as propagators of innovation and repositories of complexity

The ability to communicate enables the change and complexity to be preserved (Sfard, 2014). Verbal communication allocates a semblance of reflective communication. This communication is both intra-personal and interpersonal. The exchange of views among different mathematicians and reflective thought of individuals leads to the preservation and application of the changes and innovation to new situations. Change and innovations also produce compression, when symbols are used to minimise or reduce communication by saying less (Caspi & Sfard, 2012). Instead of writing a set of squares and listing all of them, we can write x^2; $x \in \mathbb{R}$. This then changes mathematical discourse from 'talk about process' to 'talk about objects'. In this regard, communication helps to say much with less, while preserving what would have been gained.

3.3.1.3 Two levels of discursive learning

In object-level development, there are new narratives. For example, when learners are introduced to exponential functions subsequent to exposure to the quadratic function, there are new objects for them to contend with. These objects include the formula/equation, intercepts, asymptote, ordered pairs and types of graphs to be produced. The graph is completed by drawing a line that joins all the points. The rules used in the exponential functions sub-discourse remain the same as those of plotting points on a quadratic, and the development is cumulative. At the introduction of exponential functions, learners are already familiar with the plotting of points. One major characteristic of object-level development is that it is possible for

learners to construct narratives without the interlocutor. This is facilitated by allowing learners to explore the mathematical object, as well as making conjectures and generalisations.

For grade 10 learners who have plotted points in earlier grades, it is possible for them to make generalisations on the linear function without the assistance of the interlocutor because the growth is cumulative. Points on linear functions always form a straight line. All linear functions have this characteristic, and it is not difficult for learner to make or prove a conjecture and make generalisations with less help from the interlocutor or teacher. Aother example of learning without the interlocutor relates to observing that the constant in a general function represents the *y-intercept*. The constant in functions is the value of the graph's intersection with the *y-axis*. Learners can observe the behaviour of the constant and expand their discourse. There are no contradictions between the rules of sketching a linear graph and those of plotting points. Learners can also realise that algebraically, when *x* is zero, the *y* value is always the constant. In object-level learning, it is possible for learners to generate endorsed narratives without the help of the interlocutor on condition that the previously endorsed narratives (realisation tree) are in place (Nachieli & Tabach, 2012).

For meta-level development rules, the interlocutor becomes the important factor. Meta-level development is characterised by a change in rules, and new objects change rules of endorsement. In meta-level development, there are apparent contradictions between the newly-introduced narratives and previously accepted narratives. The introduction of Calculus brings about meta-level development in that new objects, differential rules, the gradient function, and stationery points are introduced; just to name a few. There is no way that learners can use their previous knowledge to unlock the rules of differentiation. The interlocutor should explain the gradient function, the meaning of stationery points, and differential rules. In object-level development, learning is cumulative and there are no new rules, as is the case for meta-level development. The interlocutor comes in mainly to narrow the gap between meta-rules and the learners' current development. The purpose of narrowing the gap is to enable learners' advancement to the next possible level. This may be true for most of meta-level learning of mathematical objects. For example, the introduction of differential rules, stationery points, as well as the behaviour of the graph as it approaches horizontal and vertical infinite points (Sfard, 2015).

3.3.2 Applicability/ Relevance of commognition in this study

The special type of communication in Mathematics is guided by the characteristics of commognition. These are the words used, the interpretation of the visual mediators, the endorsed narratives and the routines. People engage in mathematical activities so that they may become a part of a community of mathematicians (Sfard, 2012). The use of words, interpretation of visual mediators and the narratives students endorse determine the routines

that learners operate on. In this study, has introduced an analytical tool known as the Discourse Profile of the Hyperbola and Exponential Function (DPHEF). The tool was useful in the analysis of learners' mathematical discourse, some of which was found to be ritualised since these learners based their mathematical actions on what they saw others (e.g. teachers) doing. On the other hand, other learners were able to lay claim to being part of the community of mathematicians by exhibiting exploratory routines by using words and interpreting visual mediators accordingly.

Sfard (2015) emphasises the centrality of the interlocutor in commognition, who facilitates reflective imitation. Accordingly, the learner does (imitates) what he or she sees the interlocutor doing. The more the learner becomes successful, the more s/he reflects carefully on the process s/he was involved in, and s/he comes up with conjectures and proves them. Given this state of affairs, what Sfard (2007) refers to as mathematical objects is what Vygotsky refers to as concepts. Mathematical objects are discursive constructs that replace 'talk about processes' with 'talk about things' (Sfard, 2015). Therefore, mathematical objects are used to say much with less. For example, instead of writing "the combination of two sets of marbles one with 6 (six) marbles and the other with 5 (five) is eleven marbles" we write "$6 + 5 = 11$". This form of mathematical configuration of learning in commognition theory is presented as the attainment of objectification, when the talk about objects replaces talk about processes (Sfard, 2008).

3.4 Theory of Socio-Cultural Learning

Vygotsky, a Russian psychologist, is credited with the development of the socio-cultural learning theory (Yasnitsky & van der Veer, 2015). Vygotsky believed that a new man, whom he termed *superman*, has the capacity to improve from what s/he is today, to what his/her environment shapes her/him to be. In this regard, everyone was the product of his or her environment. For Vygotsky, the society is responsible for producing individual *supermen*. Parents, caregivers and all members of society are responsible for the development of children to supermen by individualising societal norms. Although Vygotsky did not intend to develop an educational theory, his successors used the principles he was developing to generate the socio-learning theory, which gave birth to the social-cultural learning theory (Grigorenko, 2014).

Learning is a result of a wide range of activities that happen in society. According to Vygotsky (1931), culture is influential in the development of learning. Accordingly, a child grows and learns through the milieu of social interaction, which has a special role in cognitive and conceptual development. (Vygotsky, 1978). Cognitive development itself is reciprocally affected by the particular culture in which the child grows. The embeddedness of cognitive development within culture varies across cultures, and shapes children and learners' thinking

in the context of that particular environment. Daniels (2001) points out further that the context of learning also determines the kind of learning that the learner will undergo. In such a context or environment (milieu), language is the medium through which cultural practices are passed from one generation to the other. By their interaction with adults, children and learners acquire language, which then becomes a key factor of cognitive development (Canagarajah, 2002; Setati, 2008).

3.4.1 Foundational principles of socio-cultural learning theory

The socio-cultural learning theory has three broad principles, namely that human development and learning originate in cultural interactions; language mediates development of higher mental functions; and learning occurs within the Zone of Proximal Development.

3.4.1.1 Learning originate in social and cultural interactions

Learning in children happens twice, at the social level and later at the individual level. Learning happens between people and then happens within the child. Learners gain knowledge by working on a variety of tasks with other people. It is through the interactions with others that the learner gains the skills and strategies that s/he will use for solving problems (Scott & Palincsar, 2013). The participation is normally guided as there should be someone with higher skills that would help the student make meaning of the tasks. Development is about the change in activity rather that the transfer of knowledge (Matusov, 2015). Participation in cultural activities helps the learner acquire new skills that can be used to solve complex problems.

3.4.1.2 Language mediates development of higher mental functions

Language plays an important role in the learning process by facilitating the construction of knowledge. Language is the greatest tool in learning as it helps with construction of meaning and also communicating with others. Language is also a tool by means of which cultural practices are transmitted to the next generation. In addition, language skills are used to solve a variety of new problems (Scott & Palincsar, 2013).

3.4.1.3 Learning occurs within the zone of proximal development

According to Vygotsky (1978) learning happens within a zone of proximal development (ZPD). The ZPD is the distance between what the learner can do on his/her own and what s/he can do with the help of the teacher. Learning is depended on the level of development of the child rather than the age. Instruction should be structured in such a way that it takes into cognisance the previous learning. This calls for sequential arrangement of content so that learners may have building blocks or prior knowledge that can help learners to make connections and meaning on the content they learn.

Vygotsky explains learning in the context of the Zone of Proximal Development (ZPD), in terms of which the learner is ready to learn new concepts and mathematical objects with the help of the More Knowledgeable Other. The ZPD is posited as the difference between what the

learner can do on his or her own, and what he or she can do with the help of the MKO. For example, the learner is able to plot points on the Cartesian plane and is supposed to be taught the drawing of linear functions. The level at which the learner is able to work without the help of the adult or MKO is known as the Actual Level of Development (ALD),which depicts the actual plotting of points on the Cartesian plane. Scaffolding is used to achieve this transition to the actual level of development. Scaffolding *per se* is referred to as reflective imitation in the commognitive perspective. The learner is being scaffolded in order to reach the Potential Level of Development (PLD), what the learner is able to do with the help of the adult (more knowledgeable other) as exemplified or demonstrated by the learner's drawing of linear functions. While Vygotsky clearly defines the role of the MKO, the role of the learner is not clearly defined. Learning in Vygotskian theory largely depends on the MKO, who is supposed to know the point at which the learner is ready to move to the next level. In commognition, the learner progresses to the next level of learning through reflective imitation by working together with the teacher to jointly construct knowledge. In commognition, the role of the learner in the learning process is clearly defined, and there is no ZPD equivalent.

3.4.2 Applicability/ Relevance of socio-cultural learning theory to the study

One of the important advantages of learning Mathematics is that it provides learners with the tools they can use for participation in everyday life (Moschkovich, 2002). These tools include interpreting, analysing, describing, making predictions, or even solving problems they may face in their schooling or adult life. Unfortunately, not all learners are able to access this knowledge or skills at the same time.

The study focused on rural learners in the Eastern Cape Province of South Africa, an area in which cultural and language factors were empirically noted to have some impact on the learning of Mathematics. The study found that language and the use of terms was similar as a result of the cultural artefacts that affected their learning. Secondly, the selected study participants were involved in the task-based interview in pairs, which enabled the collaborative construction of knowledge for shaping their mathematical lexicon and resolution of misconceptions. The study focused on the assumption that learners were taught all the work on functions by their teachers. The study did not focus much on the zone of proximal development, but it was also assumed that they had attained a certain level of development; which would allocate a degree of meaningfulness to their tasks.

3.5 Conclusion

In this chapter, the theoretical perspectives of both commognition and socio-cultural learning were presented and discussed in the context of their main principles and applicability or relevance to the study in conjunction with the definition of key concepts. The socio-cultural learning perspective was largely utilised to frame the data collection process, while the

commognitive perspective enhanced the data analysis process of the study. In the next chapter (Chapter 4), the research design and methods of the study are presented and discussed.

CHAPTER 4
RESEARCH DESIGN AND METHODS

4.1 Introduction

This chapter outlines the study's research design and the methods used to investigate learners' mathematical discourse on functions, with particular attention to the hyperbola and the exponential functions. While the chapter systematically focuses on the data collection methods and instruments, it is worth explicating the distinction between the research *processes* and research *instruments* employed in this study. For that reason, it is then necessary to briefly provide an outline of the difference between research design, research methodology, and research methods (Kumar, 2012; Maree, 2007). The rationale for such differentiation is necessitated by the assertion that these three research nuances and variables are viewed differently by different research scholars, professionals, and practitioners. For instance, some view these three concepts as synonymous (i.e. interchangeable *and* interrelated); while other schools of thought propound the view that these three terms are different from each other (*ergo*, interrelated *but* separate and distinct, and not interchangeable) (Babbie & Mouton, 2010).

4.1.1 Research design and methodology

The research design is essentially focused on the strategies or plans utilised to integrate and manage the various units and stages of the entire research process. To this effect, Yin (2014, p. 26) refers to the research design as "the overall integration and sequence of the research processes linking the critical units of analysis in order to enable the study reach its conclusions". In concurrence, and in addition to the strategic aspect of a research design, Kumar (2012, p. 96) further provides the operational perspective of the research design, stating that: "A research design is a [procedural] plan, structure and strategy of investigation so conceived as to obtain answers to research questions or problems. The plan is the complete scheme or programme of the research. It includes an outline of what the investigator will do from writing the hypotheses and their operational implications to the final analysis of data … A traditional research design is a blueprint or detailed plan for how a research study is to be completed - operationalizing [sic] variables so they can be measured, selecting a sample of interest to study, collecting data to be used as a basis for testing hypotheses, and analysing the results".

To the extent that the research methodology is viewed as the philosophical perspective or paradigm from which the overall plan/ strategy of the study is *operationalized* (Creswell, 2014; Hesse-Biber & Leavy, 2011; Yin, 2014), it (research methodology) functionally and purposefully conforms to the same intention as the research design. Based on the above argumentation by various scholarly research perspectives and traditions, this study adopts a perspective in terms of which research design and research methodology are viewed as

synonymous, interrelated, and interchangeable; but separate and distinct from research methods (Mouton, 2013: 55). It is the researcher's contention that such differentiation provides a cogent clarification of the *processes* (i.e. design and methodology) and the actual or specific research *instrumentation/ tools* opted for in the study in accordance with the researcher's well considered overall research management, integration, and execution plan or strategy (Kumar, 2012).

From both a strategic and philosophical paradigm, this study has encapsulated a combined qualitative and quantitative research design (methodology) approach. From a qualitative perspective, this study fundamentally incorporated descriptive and interpretive elements based on the theory of commognition as expounded in Chapter 3 of this study (Turner, 2010). One of the strengths of a descriptive-interpretive research design (methodology) is the focus between the goal (end) and the path (means) taken to achieve the stated goal (Babbie & Mouton, 2010); as well as the capacity to enhance the generalisability of the study's findings (Leady & Ormrod, 2013). The interpretive aspect was necessitated by the fact that the researcher needed to interpret the mathematical discourse of grade 11 Mathematics learners on functions, in addition to describing the processes leading to both the acquisition and analysis of the data.

Qualitative research is characterised by data that is not numerical in nature, and the findings are arrived at without the use of statistical procedures (Creswell, 2014). Hatch (2002) recognises the effect of qualitative research approaches, and how they afford the research participants the opportunities to construct their own meaning of their social experiences and realities. On the other hand, the partly quantitative (statistical and numerical) approach complemented the predominantly qualitative nature of the study. The combined qualitative-quantitative approach was beneficial to this study for purposes of complementing the articulation of explanations, predictions, descriptions, and association between variables in order to construct a credible basis for the generalisability of the study's findings (Creswell, 2014; Leady & Ormrod, 2013).

4.2 Data Collection
Having outlined the processes, strategies, and philosophical perspectives of the study (research design/ methodology), data collection focuses on the nature and type of the specific instruments/ tools utilised in the systematic accumulation of pertinent (qualitative and quantitative) data and information deemed by the researcher to be relevant for the achievement of the research objectives and the resolution of the identified research problem (Bless, Higson-Smth & Kagee, 2006; De Vos et al., 2011; Holtzhausen, 2007).

In addition to the review of literature, the qualitative data in this combined or method-triangulated study was collected by means of task-based interviews and documents that learners

had used during the data collection process. On the other hand, the partly quantitative data was derived from individually written tests by learners.

4.2.1 Qualitative data collection: semi-structured task-based interviews

Qualitative data was collected by means of task-based interviews with 12 pairs of grade 11 learners (a total of 24). Qualitative data collection entails the gathering of information in a prosaic (descriptive) and non-numerical (non-statistical) form in order to explore and/ or describe the characteristics of a phenomenon and its related variables(Kumar 2012; Walliman, 2011). As critical sources of primary data, the research participants of a qualitative study provide their perceptions and social realities, from which meaningful and intelligible conclusions could be arrived at by the researcher (Wiersa, 2000). As stated previously, task-based interviews constituted the core of the qualitative data collection method. Semi-structured task-based interviews were used in order to allow the voice of the participant to be heard more clearly (Assad, 2015). The researcher captured the task-based interviews by means of both video and (Opie, 2004) and audio recording as a back-up in case the video recording malfunctioned during the collection process. This was an attempt to optimally capture the participants in their natural settings (Opie, 2004).

Task-based interviews are typically an "elicitation of student thinking about problems and a belief that true understanding takes place when the student or the learner makes discoveries for themselves" (Assad, 2015, p. 1). The researcher was flexible in the administration of this research instrument, allowing the selected participants to use multiple methods of answering questions. Some preferred to answer questions on the chalkboard, some on paper, while others preferred verbal responses only. Participants were allowed to express themselves freely in their language of choice. While most of the participants responded in English, there were some instances of some learners choosing IsiXhosa (the local language widely used in the Eastern Cape) to express themselves. Allowing participants to use their language of choice was not intended to impose meaning to their perceptions, but rather to allow their mathematical discourse to be interpreted and contextually understood (Aljoundi, 2014; Banda, 2007). In instances where the researcher did not understand the meaning of their responses, probing questions were employed.

For purposes of enhancing "the authorial voice" of the learners (Vyncke, 2012, p. 21), the researcher selected questions for both test- and task-based interviews to be as close as possible to the learners' everyday classroom experiences. The "voice" of the learners refers to "the writer's [learner's] distinctive presence, the strength with which the writer comes over as the author of the text" (Vyncke, 2012, p. 21). In the context of this study, that 'voice' is the reflection of learners' own perspectives derived from their daily cognitive and narrative mathematical discourses. One of the advantages of the task-based interviews is their provision

of clarity-seeking opportunities to the researcher in situations of learners' ambiguous mathematical narrative (Turner, 2010). In task-based interviews, the interaction with the learners allowed for acquisition of a high quality and standard of informat, as the participants and the researcher had the opportunity to understand each other's perspectives. Probing was used when necessary, as it is also a powerful tool of prompting for more spontaneous learner interaction and responses (Jacob & Ferguson, 2012). Furthermore, task-based interviews were useful in helping learners to deepen their meaning-making, giving both solutions and reasons for their choices while working on mathe matical solutions (Goldin, 2000).

In the task-based environment, the identified task presents the interviewer with the opportunity to recognise and classify the mathematical discourse of the learners, which is not only about what learners say, but also their non-verbal communication and written work. Maher and Sigley (2014) ascertain that the type of tasks assigned provides an environment which will determine the kind and nature of the mathematical discourse the researcher will derive from learners. In this study, follow-up questions were not pre-planned, the interviewer encouraged learners to self-correct by further interrogating their answers. Some follow-up questions were sought for clarification in order to obtain the correct data from the learners. The other advantage of a task-based interview is that participants' mathematical discourse or communication is enhanced in the construction of the task (Assad, 2015). A well-constructed task is likely to be effective as a data collection tool than a poorly constructed one. For purposes of this study, the success factor of the qualitative data collection was entirely premised on the learners'/participants' *talking/ narrating* their perspectives. Accordingly, the researcher selected four tasks or sub-tasks close to the participants familiar environment. As such,the participants were able to explain themselves fully while the researcher was able to explore their responses optimally (Assad, 2015; Maher & Sigley, 2014). Appendix B (p. 195) contains the tasks in the form of the Interview Schedule.

4.2.1.1 Task 1: Naming of the exponential function and the hyperbola
In all the five tasks, the primary focus has been on the interrogation of learners' capacity to exhibit their mathematical discourse on different representations of the exponential function and the hyperbola. In the first task, there were two algebraic representations of an exponential function and a hyperbola. There were five sub-questions in this task, requiring learners to name each of the functions and thereafter explain each equation or formula relating to the parent function in the first sub-question. The second sub-question required learners to identify the asymptote of the exponential function and explain the reasons for their responses. The third sub-question was the same as the second, the only difference being that this was for a hyperbola. The fourth sub-section was premised on the identification of key points such as the asymptotes and intercepts on the table of values of both a hyperbola and an exponential

function. The final part of the tasks required a definition of an asymptote, as well as identification of the asymptote from a graph and an equation. This task afforded learners the opportunity to present their mathematical discourse from algebraic representation.

4.2.1.2 Task 2: Naming the functions and formulating the equation

The purpose of the second task was to engage learners in discussions which reveal their mathematical discourse on worded functions. In this second task, learners were presented with functions expressed in words. Learners were required to name these functions, formulate an equation using the word information supplied and sketch a graph of the function, and state restrictions to the functions where applicable. Lastly, learners were required to provide reasons in the event that graphs were expected to have asymptotes.

4.2.1.3 Task 3: Identifying unfamiliar equations of the exponential function and the hyperbola

In the third task, learners were presented with unfamiliar equations of both the exponential function and the hyperbola. The learners had to identify the function, express each function in standard form, and subsequently identify asymptotes from the graph. The main idea of the third task was to determine learners' degree of flexibility insofar as rearranging the given formulae to standard form is concerned.

4.2.1.4 Task 4: Identifying table of values of an exponential function

In the fourth task, the questions were based on the table of values of an exponential function. The task required learners to identify the function from the table of values, stating whether the function had an asymptote. The learners were also required to identify features such as the domain, the range, and the intercepts. Lastly, the learners had to formulate the equation from the self-same table of values. The curriculum requires learners to move flexibly among the four representations. The table of values is an iconic visual mediator and conveys information on the function that a representation does, but is rarely used beyond introducing learners to a particular function.

4.2.1.5 Task 5: Identifying the function from the graph

The fifth and final task was a graphical representation of the functions in which two graphs were drawn and the requirement was for learners to identify the function from the graph. The second part of the task required learners to explain the behaviour of the graph as it approaches both positive and negative infinity on both axes. While learners could draw and identify the key features of graphs, the researcher needed to establish whether learners could interpret the behaviour of these graphs.

4.2.2 Quantitative data collection: test-based questionnaires

While the above-cited qualitative data collection strategies were fundamentally descriptive/ prosaic and descriptive (discussion-oriented), the quantitative data collection variants are mainly non-descriptive, statistical/ numerical/, and measurable (de Vos et al., 2011). Additionally, the quantitative data collection process is objective, controlled, and not prone to manipulation by the researcher for purposes of 'aligninment' to some research interests or agendas (Creswell, 2014). Kumar (2012) asserts that the extent of the quantified data's controllability necessarily translates into standardisation of the concomitant data analysis process; precision, validity and accuracy of the research results; as well as the effective accomplishment of the study's objectives. In this study, 112 learners/ respondents were selected for participation in a test focusing on the tabular, algebraic, graphical and narrated representations of functions; with particular emphasis on the asymptotes of the hyperbola and the exponential functions.

4.2.2.1 Test-based data collection

The study was conducted in four schools in the rural Mthatha district of the Eastern Cape Province (ECP), South Africa. About 30 learners from each of the four schools took part in the test. From a possible total of 120 learners/ respondents, 8 (eight) withdrew from the tests, which were administered by the researcher in the afternoons after their scheduled lessons. Approximately one hour was allocated for the test administration. In each school, 30 learners were supposed to take part in the test under the researcher's guidance as an invigilator. Eventually, only 112 learners from the four schools participated in the study after the withdrawal of eight learners. The test was conducted under examination conditions, meaning that it was based on individual work. No discussion or passing of notes was allowed for the entire duration of the test, and the scheduled start and end times were strictly adhered to. Furthermore, the researcher supplied both the question paper and answering materials such as paper, ruler and pencils. Some learners opted to leave the venue at the time of writing and their limited input has been excluded in this study as participation was voluntary. Appendix A (p. 194) depicts all the questions of the grade 11 test referred to above.

A dual-purpose testing mechanism was used. Firstly, 6 (six) learners were selected for task-based pair interviews per school. Secondly, the test served the purpose of examining the mathematical discourse of grade 11 learners in the sphere of functions in general. Following is a description of the five questions of the test.

The first question required learners to draw a sketch of a hyperbola. This is a typical examination question. Firstly, two sub-questions served as the preparation for sketching the graph, identifying the equations of the asymptotes, and calculating the intercepts. The purpose of this question was to give confidence to the learners by including some exercises which were

not unfamiliar to the learners. Secondly, the question was intended to determine whether learners could sketch an iconic visual mediator from an equation. Generally, learners performed well in this regard, although the data analysis section 4.4 (see pp. 58-64) will discuss and provide more insight on some of the notable discourse of learners with regard to the intercepts and the difference between the algebraic and the iconic representation of the asymptote and the intercepts.

In the second question of the test, learners were required to use words to describe an equation and give reasons for the choice of their description. Furthermore, learners were to use words to interpret a symbolic visual mediator given in the form of an equation. The second part of this question required learners to name the asymptote of a transformed equation. Not many learners responded to this question successfully. In fact, few learners gave reasons for their description.

Concerning the third question, the researcher provided a contextual exponential function which excluded the usual variables, x and y instead t for time and θ for temperature. As a result (of this exclusion), many learners could not recognise that the function was exponential. The first part of the question required learners to complete a table of values. The second part required learners to draw a continuous graph using the coordinates from the table, after which the learners were required to name the graph from the equation or formula. The last part of this third question required learners to describe the meaning of the asymptote in real life.

The fourth question was a combination of a parabola and an exponential function. In this question, there was no scaffolding. Learners had to calculate the positions of the key features of these functions. The second part required learners to demonstrate their understanding of the points of intersection of graphs as representing the solution for the equations.

The fifth question generalised transformation of the hyperbola. Firstly, learners had to find the value of the constant, then proceed to a vertical transformation. Both horizontal and vertical transformation and the last question were based on compression. Most of the learners did not attempt any form of response to this question. The implication could be that the learners were not exposed to the generalised transformation of the hyperbola.

4.3 The Study Setting/ Research Site

The study setting or research site refers to the physical place at which the research was conducted (De Vos et al., 2011; Rajasekar et al., 2013). The current study was conducted at four rural high schools in the Mthata district of the Eastern Cape Province. Mthatha is a small township with a high population density. Education is affected by a variety of factors, including a high illiteracy rate of about 20%, and a low per household income as most residents rely on government grants and some households are child headed. Large classes averaging 60 learners in grade 12 also affect quality education (Mpofu & Pournara, 2018).

Generally, the National Senior Certificate results have not been encouraging for some time in the Eastern Cape. In fact, the Eastern Cape Province has been the worst performing province in NSC results for the past 3 years, with a 56.8% pass rate for 2015, a far cry from the provincial target of 70% (DBE, 2016). This challenge is compounded by a high rate of both unqualified and underqualified teachers in most of the schools (Tachie & Chireshe, 2013).

4.3.1 Study population and sample size

The study population refers to the larger representative group from which the study participants are to be selected (sampled) on the basis of their homogeneous characteristics, qualities, or attributes (De Vos et al., 2011). The core of the study population in this research consisted mainly of grade 11 learners at four high schools of the Mthatha district. In each school, 30 volunteering participants were invited to take part in the research, but only 112 learners from four secondary schools took part in the test as eight learners decided to withdraw in the middle of data collection. They felt the questions asked were too difficult for them. The test was conducted in the very schools the learners attended. The researcher chose their schools because they were familiar to learners. Such an environment was conducive for the learners to express themselves freely. The test was written under examination conditions, and was of 50 minutes' duration. The researcher invigilated the test in all the four (4) schools. The answer sheets were marked soon after the test, and 10 learners were selected for the task-based interview from each school. The researcher then gave general feedback to the Head of the Mathematics department at the school on learners' areas of strength and weakness.

4.3.1.1 Sample size and sampling method (strategy/ technique)

In total, 112 grade 11 learners were selected from the four Mthatha high schools for participation in the empirical (interview- and test-based) aspects of the study.

The researcher used the purposive sampling technique in this study on the basis of the selected research participants' possession of similar (homogeneous) representative qualities with the study population (Saunders et al., 2012). Purposive sampling is a sampling technique according to which the researcher relies on his/ her judgement and knowledge or maximum familiarity with the research milieu (environment or setting) to select the relevant research participants (Black, 2010). The other advantage of purposive sampling is that it is cost and time effective as judgement on selection is usually based on previously available data (Black, 2010).

Most of the learners do not perform well in functions and English is not their first language. They learn English at school as a second language, therefore selection focused on learners who were willing to participate in the study voluntarily. Learners who are not perfoming well in Mathematics tend not to answer questions accordingly, and the researcher did not wish to risk having many blank spaces in the test-based tasks.

4.3.2 Sampling criteria

As a concept, 'sampling criteria' refers to the researcher's own predetermined range of considerations or norm (standard) according to which the selected research participants were regarded as either relevant or unsuitable for inclusion in the empirical aspect of the research (De Vos et al., 2011; Kumar, 2012).

In the selection of learners for the study's task-based interviews, the researcher presented the names of learners who would take part in the interviews. Learners then discussed among themselves and recommended names of six volunteers for the next stage of the data collection. The other four (4) learners were placed as possible substitutes in the event that one of the selected participants did not come or decided to withdraw from participation. In each of the four schools, no selected participant withdrew from the task-based interviews, and three pairs of learners participated in the task-based interviews at their school after the designated teaching hours. Prior to the interviews, the researcher explained the expected level and nature of participation to the learners (Creswell, 2014; Saunders et al., 2014). Learners were also allowed to write some additional notes and bring them with if they desired to do so, in addition to being allowed to talk during the recorded interview sesssions. Only a pair was interviewed in the presence of the researcher.

The best performing 30 learners in grade 11 were selected from each of the four schools. The researcher and teachers in each school used the average of the marks obtained at the time of the study to select the best 40 learners to participate in the test. The researcher earmarked 30 learners and the other 10 (ten) were placed as substitutes in the case of learners who were not willing to participate in the study. In total, 112 learners participated in the test.

The researcher further selected 6 (six) best performing learners from each of the participating 4 (four) schools. These 24 best performers took part in a paired task-based interview. Only one selected learner dropped out because he changed schools, but was replaced by the learner ranked number seven at his school. The characteristics of these learners were such that they were the best in each school. In school A, the selected learners' marks ranged from 40% to 60%. In school B, the top six learners' marks ranged from 75% to 80%. In school C, the marks ranged from 85% to 90% for the best six; and at school D, the best six learners' marks ranged from 65% to 80%. Although 24 research participants were the best 6 from each school, their marks in the test ranged from 40% to 90%. Since this was a very wide range, the researcher did not expect their mathematical discourse to be the same. At face value, their mathematical discourse looks the same, but a closer analysis using the DPHEF exposes some differences that are discussed in the data analysis section below (Section 4.4).

4.4 Data Management and Analysis

Data management focuses on the treatment or preparation of the collected raw data and its preservation from contamination, thus ensuring its practical usability and translation for the resolution of the research problem; as well as for the maximum achievement of the stated research objectives (De Vos et al., 2011). The management and analysis of data are two sequentially and symbiotically interconnected processes (Marshall & Rossman, 2011). The latter authors allude to this inter-connectedness thus: "Data collection and analysis thus typically go hand in hand in order to build a coherent interpretation of the data. The researcher is guided by initial concepts and developing understanding, but shifts or modifies them as he collects and analyses the data" (p. 18). Therefore, despite their concurrence, data management precedes data analysis (Holloway & Wheeler, 2010; Marshall & Rossman, 2011). Following the treatment/ preparation and analysis of the qualitative and quantitative raw data, it was backed digitally with a master copy and kept in a secured place. Only the researcher has password access to the digitally secured information files. Hard copies are also kept safely at a secure location, where they will be shredded after a period of five years. In this regard, data management reflects and ensures a standardised monitoring and evaluation quality assurance mechanism.

In this study, the researcher personally collected the answer sheets that learners used during the interviews as evidence of their participation. Teachers in the selected four schools were only involved in ensuring the availability of fully furnished and conducive venues, and assisted with coordinating learners for tests and interviews. The data collection times differed from school to school. For instance, in some schools data was collected after official school hours when other learners had left for home. In other schools, data was collected during the study time after the daily lessons had ended. For effective quality assurance purposes of the interview, the teacher ensured that he was not drawn to into superfluous probing. The latter could inadvertently turn the interviews into teaching sessions and defeat the goal of these interview sessions.

Twenty (20) of the participants were video recorded with their informed consent, while four (4) participants did not agree to video recording; instead, they were only audio-recorded. The advantages of video recording was that it captured the original/ authenticated voice of the learners, their actions, their demeanour; as well as their verbal (e.g. talking) and non-verbal communication (e.g. writing, gestures and facial expressions). The researcher has realised that the video recording may have affected the manner in which some learners responded to the questions. Some of the learners were too aware of the presence of the recordings that happened. The four learners who opted for audio recording only did so as they were not comfortable with

their faces being recorded. For ethical reasons, both the video and audio recordings were strictly for the purpose of this study, and not for viewing by unauthorised persons.

4.4.1 The DPHEF data analytic tool: its relevance and application

In this study, data was analysed by means of the Discourse Profile of the Hyperbola and Exponential Function (DPHEF) as adapted from the Arithmetic Discourse Profile (ADP) by Ben-Yahuda et al (2005). The DPHEF itself was influenced by the work of Gcasamba (2014), and as an extension of Mpofu and Pournara (2018). As an analytical tool, the DPHEF was utilised in this study to analyse learners' mathematical discourse (both written text and verbal utterances).

The ADP framework was used for Arithmetic, but the researcher adapted it and termed it the discourse profile of the hyperbola and exponential function (DPHEF). The Functions Discourse Profile (FDP) by Gcasamba (2014) and Discourse Profile of the Hyperbola (DPH) by Mpofu and Pournara (2018) were very instrumental in the design of the DPHEF. The researcher opted for the DPHEF since the learners' mathematical discourse on functions in this study seemed to be the same, and the DPHEF was very useful in exposing the differences. At face value, learners seemed to be saying the same mathematical words, some of which were used inappropriately. The researcher for the DPHEF mainly because it explains the data analysis on the basis of the four characteristics of the mathematical discourse. These are words and their usage, the visual mediators, the routines, and the narratives. In the adaption of the DPHEF, some of the categories of the Arithmetic Discourse Profile (ADP) were excluded for two reasons. Firstly, some of the terms found in the ADP are not found in later works of Sfard – such as 'proficiency' 'routine closure' 'conditions'. Secondly, some categories - such as concrete mediators and derivative narratives were not investigated in this study; although they are still being used in Sfard's work.

In the DPHEF, the words and their use are divided into their colloquial and literal contexts. Colloquial words include all words which have a combination of mathematical and non-mathematical language. For example, when a learner says: *"My intercept is the line that is parallel to the y-axis"*. On the other hand, mathematically correct words are those which elicit the same meaning from the community of mathematicians, and will deduce the same meaning, such as*: "An intercept is where the graph intersects with an axis".* The researcher added a category of naming mathematical objects. In this category, the naming of mathematical and non-mathematical objects was included. When a learner interpreted a word statement and named it in a mathematically acceptable manner, the researcher classified the naming as mathematical. For example, in the event that a learner mentioned that an inverse proportional statement represented a hyperbola. The researcher named mathematical words, but used inappropriately as non-mathematical. Furthermore, when a learner named an asymptote as a

point. The following critical factors guided the utilisation of the DPHEF as the primary data analysis tool of this study:

- The researcher classified visual mediators into two groups, the iconic and symbolic mediators. Iconic mediators are pictorial. For instance, graphs and tables. The symbolic mediators are equations and other symbolic mediators, such as $y = 3^x$. The researcher has further sub-divided these into the unrecognised or interpreted iconic mediator. An iconic visual mediator is classified as interpreted in the event that the action taken is based on a presented iconic visual mediator;

- The researcher classified all iconic visual mediators as drawn, in the event that they resonated with the interpretations of the community of mathematicians;

- Routines were classified according to their types and properties, including ritualised, deeds and exploratory routines. A deed routine is characterised by the performance of practical mathematical actions, but failing to perform the same actions in abstract form (Berger, 2013; Sfard, 2007). An example of a deed routine is reflected in a street vendor easily multiplying the number of packets of tomatoes using price per packet, but failing do the same if given a number. If the packet of tomatoes costs R15.00 and the number of packets bought are 7 (seven), s/he knows the product should be R105.00 but is not able to multiply 7 and 15. This study did not pay much attention to deeds routines.

The use of routines includes applicability, flexibility and corrigibility. Applicability routines are applicable when certain routine procedures are likely to be produced (Ben-Yahuda et al., 2005). In this study, the applicability routine is exemplified by the following state of affairs:

- Solving problems: for example, calculation the intercepts or points that satisfy the graph; Sketching or drawing graphs and a table of values: The researcher termed this category as depicted routines;

- Use of a tables to identify key features of a hyperbola; Use of the key features (intercepts, asymptotes) to sketch a graph. Using a visual trigger, for example an asymptote signifying a vertical or horizontal translation.

- The researcher used the narratives to analyse the discourse of learners on the hyperbola. Narratives that resulted in the establishment of new theorems, definitions, and axioms were called derivations. In this study, no new narratives were derived as the learners involved in the study were not taught by the researcher. Therefore, the derivations narratives were not included in this analysis. The analysis will use substantiations, the actions on whose basis it is decided to endorse previously constructed narratives. The substantiations that govern school Mathematics are not as rigorous as those that govern the community of mathematicians. The justifications and reasons for actions are classified as substantiations.

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The community of mathematicians endorses some of these substantiations, while others are not;

- *Memorisation* and *authority* narratives were also used to analyse the learners' mathematicaldiscourses. Memorisation narratives are about remembering previously endorsed narratives. For example, $y = q$ is an equation of a horizontal asymptote. It is a memorisation narrative, and was endorsed for the grade 11 learners at the time of the study. Learners would simply remember the formula or the rule and apply it; and

- The researcher also included the *authority narrative*, which is the representation of the learner's own personal experiences and reality as the authority of justifying their actions. The teacher or the textbook could be referred to as the learners' authority, and is used as the reasons for their actions. For example, learners would state that the constant shows the value of the horizontal asymptote. When a question is posed: *How do you know this?,* they would cite the teacher as the authoritative source of the statement or information they referred to.

Table 4.1 below (p. 62) represents a summary of the DPHEF's usage in analysing the data, as was described in more detail in the preceding paragraphs of this sub-section (4.4.1, pp. 57-59).

Table 4.1: Application of the DPHEF

Four key characteristics of mathematical discourse		Classification	Description	Examples
Words/ words use	Words in sentences	Colloquial	Combination of literate and colloquial	An asymptote is an imaginary line that a graph can't pass
		Literate	The whole expression is mathematically accepted	A line whose distance to a given curve tends to zero. An asymptote may, or may not intersect its associated <u>curve.</u>
	Naming	Non mathematical	Ambiguous naming of mathematical objects	Naming an asymptote as a point
		Mathematical	Use of proper names for mathematical objects	Naming an asymptote as a line
Visual mediators		Iconic	Interpreted	Use of a hyperbola function to identify asymptotes
			Not recognised	The iconic visual mediator not used
			Drawn	Sketching a graph; table of values
			Disallowed	No attempt at drawing the graph/table or an incorrect iconic visual representation
		Symbolic	construe	Identification and interpretation of the asymptotes from the equation
			Not construed	Incorrect identification/use or no use of symbolic mediator
			Formulated/generated	Equations from a graph or table
Routines	Kinds of routines	Ritualised non-mathematical	Incorrect procedure/statement	Saying a function is linear yet the coordinates are not colinear
		Ritualised mathematical	Correct procedure no justification	Sketching a graph showing that the y-axis is an asymptote but talking as if there is no vertical asymptote
		Exploratory	Verification of narratives; Working with unfamiliar tasks	Choice of numbers on a table that show values moving towards a limit
	Use of routines	Applicability	Solving equations	Solving to find intercepts; asymptotes;
			Depicted	Hyperbola graph from an equation or table
			Use of a table of values	Identification of key features from a table of values
			Use of key features	Sketch a curve using intercepts, asymptotes and the equation
			Using visual trigger	An asymptote signifying a vertical or horizontal translation
		Corrigibility	Correction	Self-evaluate and correct
		Flexibility	Use of multiple routines	Using key features and/or table of values to sketch graphs
			translating	Being able to transform an equation to standard form
Narratives		Substantiation	Justifications and reasons	Justifications for actions, e.g. it is an asymptote because…
		Memorisation	Formula/rule	$y = \frac{a}{x-p} + q$ is a hyperbola
			Visual (justification based on what learners can see).	y = q is an asymptote
		Authority	Teacher/textbook	My teacher said … or the textbook says…..

Source: Researcher's own initiative

4.4.2 Exemplifying usage of the DPHEF

The researcher examined and followed-up on each research participant's evidence as obtained from their elicited responses. Learners were then grouped according to their responses. They were only shown how work was classified from two groups of learners. Learners were asked to discuss the asymptote of the exponential function. Their responses were then summarised in a table. Following is part of the table as an example of how data was analysed by means of the DPHEF. For ethical purposes, the actual names of the learners have been kept confidential.

Table 4.2: An example of the actual data analysis process

Name	Word Use	Mediator	K. Routine	Use Routines	Narratives
1. *Participant CC* *Participant M* *Participant HH,* *Participant KK* *Participant EE,* *Participant S* *Participant DD* *Participant JJ* *Participant AA* *Participant LL* *Participant BB*	Colloquial	Construed	Ritualised non-mathematical	Applicability	Memorisation
	Non-mathematical naming	Interpretation	Non-mathematical language	Visual trigger	Formula
2. *Participant R*	Literate	Construed	Ritualised mathematical	applicability	Memorisation
	Mathematical naming	Identification of an asymptote	Does not give a reason	Visual trigger	Visual

Table 4.2 above exemplifies the various phases of the analysis of the elicited responses. In the first category, the researcher utilised the response by **Participant LL** to the statement: *Name the asymptote of the function* $f(x) = 3^{x+1} - 9.$ In response, **Participant LL** stated that the asymptote was "negative nine (-9)". The researcher classified **Participant LL's** narrative discourse as colloquial because -9 is not a function, and it is impossible to located -9 on the graph. The naming of the asymptote was classified as non-mathematical, since asymptotes are functions named as formulas or equations.

Participant LL managed to interpret the algebraic representation of the function and identified the value that was generalised to an asymptote -9. The researcher classified this participant's interpretation of the function $f(x) = 3^{x+1} - 9$ as correct, because of her ability to identify the asymptote from the algebraic formula. While **Participant LL** could identify the asymptote from the formula, she could not express it in a mathematically accepted manner; where every member of the mathematical community would have the same interpretation. Furthermore, the researcher furthermore classified her kind of routines as ritualised non-mathematical because her narrative was ambiguous. Her use of routines was classified as 'applicability' due to her response being triggered by the constant on the formula.

When **Participant LL** was asked why she said her asymptote was -9, she responded by providing the mathematically accepted expression of an asymptote as *(y = -9)*. The researcher then asked her why she changed her response. **Participant LL** then said: *The teacher marks us wrong when we write negative nine but will mark it correct when it is y is equal to negative nine.* **Participant LL's** narrated responses were based on what the teacher said, but she did not know why the teacher marked her answers as wrong. The researcher proceeded to examine the mathematical discourse of **Participant R,** whose response to the question: *What is the asymptote of the $f(x) = 3^{x+1} - 9$?*; to which she responded: *The asymptote of f is y = -9.*

An asymptote of an exponential function is a horizontal line parallel to the *x-axis*. **Participant R** named the horizontal asymptote in a manner which resonates with that the discourse of community of mathematicians. The function $y = -9$ could be located on the Cartesian plane. When the researcher asked **Participant R** why she named the horizontal asymptote in the manner she did, she responded that she did not know the reason for that response. Her use of words was classified as "literate" and the naming of the asymptote as "mathematical", since her discourse was the same as that of the community of mathematicians.

Participant R identified the asymptote from the symbolic visual mediator. By naming the asymptote as $y = -9$, it showed that **Participant R** could interpret the symbolic visual mediator, and also that she could write the symbolic visual mediator in a mathematically acceptable manner. Her interpretation of the algebraic representation of the function was classified as "construed", because she did not only identify the asymptote, but also referred to it as a function.

As much as **Participant R** could interpret the algebraic representation of an exponential function and name the asymptote mathematically, she could not provide reasons for her responses. While **Participant R's** work is mathematical, she does not provide reasons for her actions. The researcher then classified her routines as "ritualised mathematical" because she expressed her narrative discourse mathematically. Her use of routines was classified as "applicability", because she had a visual trigger. She claimed to have recognised the constant, which was a trigger for her identification of the asymptote. Finally, the researcher classified **Participant R's** narratives as memorisation based on visuals because her justification for the asymptote was based on what she could see only. Data was analysed based on what learners said or did on paper or on the chalkboard.

4.5 Measures of Trustworthiness

The notion of 'trustworthiness' is a demonstration of "the soundness and adequacy" of the quality of the research methods opted for in the study (Holloway & Wheeler, 2010; Marshall & Rossman, 2011). On the other hand, measures of trustworthiness are the strategies or quality

assurance mechanisms designed to ensure the overall scientific rigour and to instill confidence in the findings of the study (Holloway & Wheeler, 2010; Kumar, 2012). The nature of the study necessitated that both the qualitative and quantitative measures of trustworthiness be indicated. Quantitative researchers often doubt the trustworthiness of the results of qualitative research (Silverman, 2001). These researchers do not trust the qualitative research methods due to the fact that validity and reliability are not addressed in the same manner. For this reason, qualitative researchers have developed terminology that seeks to establish the rigour in qualitative research.

Kumar (2012) asserts that there are generally four measures of trustworthiness, namely: credibility, transferability, dependability and confirmability, "and it is these four indicators that reflect validity and reliability in qualitative research" (p. 172). The below-mentioned measures of trustworthiness reflect both the qualitative and quantitative variants. For instance, instead of internal validity, the researcher discussed the elements that establish the study's credibility. Transferability was also discussed in the place of external validity by providing the context in which data was collected to allow for comparability of the findings in a different research milieu. The researcher also described the study's dependability by presenting those measures of trustworthiness which allow for the study's repeatability.

4.5.1 Credibility/ Internal validity

The credibility/ internal validity of the study is premised on the degree of accuracy and agreeability between the results and the quality the research instrument (Marshall & Rossman, 2011). Shenton (2004) reiterates that the credibility of the study is measured on account of its true reflection of the research process, particularly the empirical aspect.

All the data collected in this study has been kept as evidence and as a true record of the research process. Three forms of data were collected, from which this study's findings are based. Firstly, it is the test that was written by 112 learners in grade 11 at the time of the data collection from four different schools within the town of Mthatha. Secondly, it is the written notes that were used during the task based interviews. Thirdly, there are videos and audios that were captured during the time of the data collections. Some evidence of the collected data was also included in the analysis section through excerpts of transcripts and pictures of both written and drawn work of the participants.

Finally, the findings of the study confirm what has been found by other studies which were conducted previously (Makgakga & Sepeng, 2013). One such study is that conducted by Essack (2015), which came close to this study. For instance, Essack (2015) found that learners tend to have ritualised routines, and they struggle with questions that seek to interrogate reasons for their particular action. Learners did extremely well on procedures that could be classified as

'classroom type' and struggled with questions that were not normally attended to in everyday classroom situations. The researcher also allowed some participants to scrutinise the data of this project and ascertain that transcription was done correctly (member checking).

4.5.2 Transferability/ Generalisability

Transferability/ generalisability is viewed as the extent of the study's findings being useful or applicable to others in similar contexts as those that prevailed at the original site(s) where the study was conducted (Marshall & Rossman, 2011). In a qualitative research, the findings are generally affected by the sample size, and cannot be applied to a wider population (Turner, 2010). Shenton (2004) on the other hand, states that the context where data was collected should be described fully, in order to enable those who wish to apply the findings of the study to check the applicability of these findings to their own contexts. It is the responsibility of the researcher to explain the study context fully (Walliman, 2011). The study was conducted in the Eastern Cape Province of South Africa. In this regard, the study's generalisability was ensured by the consistency of the research instruments used (tests and interviews). Accordingly, the researcher ensured that all the questions focused on the mathematical discourse of the learners. No peripheral aspects of Mathematics were included in the repertoire of questions pertaining to learners' mathematical discourse of functions and the asymptote.

4.5.3 Dependability/ Reliability

Dependability/ reliability refers to the extent to which a research instrument consistently measures the characteristics of a research variable or construct. Joppe (2000) defines reliability in research as the consistency of results over a given population that could be reproduced under the same conditions that prevailed at the time of study. For purposes of ensuring the repeatability of the study, the researcher has detailed the planning and execution of the research by means of an audit trail. In this regard, the researcher has also described the geographic area and conditions under which the study was conducted. Accordingly, all documents relating to the entire research process were kept for any further research on a topic bearing some degree of verisimilitude with the current study. The purpose of an audit trail is essentially to enable interested researchers and readers to trace all the processes which unformed the researcher's decision-making throughout various stages of the entire research process (Holloway & Wheeler, 2010).

4.6 Ethical Considerations

Ethical considerations relate to the protocols, etiquette, and conduct which enhance the study's adherence to acceptable professional, legal, and/ or scientific norms and standards (Kendall & Halliday, 2014). There are basically two ethical categories: researcher-specific and participant-specific ethical norms and standards to be adhered by both parties (researcher and research participants). In this study, the researcher complied with both the stipulations and requirements

of the Research Ethics Policy of the University of KwaZulu-Natal (UKZN), which is the body entrusted with granting permission for the study to commence. Accordingly, the researcher applied for ethical clearance, which was eventually granted as **Protocol Reference Number: HSS/2009/016D.**

Under the participant-focused ethical considerations, the it was the researcher's responsibility to ensure that the research participants and respondents were treated with utmost human dignity as enshrined in the Constitution of the country (Act No. 108 of 1996). The researcher was legally and professionally bound to fully disclose the purpose of the study and how the results would be used. Therefore, all participants and their guardians (for minors) completed written consent forms indicating their voluntary participation, and that no financial inducements were used to lure them. The informed consent form also stated that they could withdraw from the study at any time if they felt that their human rights and dignity were violated by the researcher (Kendall & Halliday, 2014). In the event that the consent form's return slip was not brought back, the assumption was that the participant did not wish to take part in the study. Those who withdrew from the study were not prejudicially treated or penalised. All participants completed consent forms which stated that they participated in the research on their own will, and were free to withdraw at any time whenever they so desired on account of perceived or real violation of their human diugnu=ity.

4.6.1 Privacy, anonymity and confidentiality

The participants' privacy is an important aspect of empirical data collection (Hesse-Biber & Leavy, 2011). Privacy *per se* focuses on the participants' freedom to determine the circumstances, the time, and the degree to which their personal information or details could be shared. For instance, the research participants should be allowed the privacy to fill-in questionnaires and participate in interviews without any form of interference.

On the other hand, anonymity virtually refers to namelessness (Hesse-Biber & Leavy, 2011). Anonymity implies that the researcher should not disclose any of the participants' identities relating to their involvement in the research. The researcher should not even be able to link data to any particular participant. The researcher ensured the participants' anonymity by not revealing their names in both the tests and interviews. None of the participants' names nor their schools were named in any part of the study. Instead, pseudonyms were used, such as "Participant LL", "Participant C", and so on. When the study's preliminary results are published, the participants would have been formally informed prior to publication.

The ethical aspect of confidentiality implies that the data is not publicly available to any unauthorised persons who were not directly associated with the investigation (Kendall & Halliday, 2014). The researcher ensured that confidentiality was maintained by using

pseudonyms of participants and their schools. The researcher also ensured that there was an improved chance of participants being truthful and honest in their responses by stressing that their names and those of their schools were mentioned. The researcher used iterative questions to revise the scripts, checking for the truthfulness of the information by asking the same question differently. This step was necessary for eliminating any contradictions, while also ensuring that the participants' responses were clearly understood. Furthermore, all the data obtained from the tests, videos and audio-recorded interviews will be kept securely, with no one allowed any access as that would constitute a breach of confidentiality. A breach of confidentiality takes place in the event that the researcher permits access to documents to any unauthorised persons or revealing the identities of the participants (de Vos et al., 2011).

The researcher further enhanced confidence and confidentiality by means of the informed consent form. For instance, this form explained the purpose of the study, how long it will take, how learners' confidentiality would be protected, and the activities in which the school and learners were to take part in.

In the final research report, no description of the participants nor the precise geographical location of their school will be indicated. In addition, the researcher will ensure that all the data collected is not made available to other parties, including teachers or anyone else at the participating schools. The researcher will not discuss anything pertaining to participants, whether positive or negative with school authorities.

4.6.2 Feedback to schools

The researcher considers feedback to schools during, and after the empirical data collection phase as a sacrosanct area of the investigation. Other than for ethical reasons, such feedback ensures that trust and confidence prevail between the researcher, the gatekeepers, as well as the research participants and respondents themselves (Saunders et al., 2012; Walliman, 2011). After every stage of the data collection, the researcher summarised his findings and recommendations and submitted them to the schools of the participating learners in the form of a formal research report. The report was addressed to the Principal of the school, for the attention of the Mathematics Head of Department (HOD). At no stage was the researcher specific about the findings in terms of school or learners.

4.7 Conclusion

In this chapter, the researcher expounded on the preparation, collection, and analysis of the pertinent data of this study. The researcher ensured that the distinction between research design, research methodology, and research method was detailed and elaborated on. The DPHEF analytical tool was also introduced as a critical part of the data analysis processes in this research. The researcher also explained the relevance and applicability of the DPHEF as a relevant mathematical tool of analysis. In addition to the measures of trustworthiness and

ethical considerations, Chapter 4 is critical insofar as it presented the framework for the evidence base of the study, without which the need and significance of the study could be cast into doubt. In the next chapter (Chapter 5), the representation of functions is presented and discussed in the context of the data derived from the study participants themselves

CHAPTER 5
REPRESENTATION OF FUNCTIONS

5.1 Introduction

The major focus of this chapter is on the analysis of the tests written by the 112 participants in the four Mthatha high schools selected for participation in the study. The tests had a dual purpose. Firstly, to establish the mathematical discourse of the participants in the study; and secondly, to select participants that took part in the task-based interviews. Although the researcher envisaged the involvement of 30 best performing learners from each school, this did not happen as participation was voluntary. In addition, some participants did not complete the tests. The participants displayed various levels of mathematical ability, as shown by the results in Table 5.1. Participants scored highly in those routine questions which would normally appear in examinations, and not many of them performed well in the non-routine questions. In some instances, participants would write the correct answers and then write something that was mathematically unacceptable. The researcher contends that the participants exhibited ritualised routines and their word use is mostly between "literate" and "colloquial". Their interpretation of the visual mediators was mostly influenced by what they saw at the time, which was not consonant with the discourse of the community of mathematicians. It is also the researcher's intention to discuss participants' responses to each of the questions in a summarised form as indicated in Table 5.1 below.

5.2 Summary of Test Responses

In this section, a summary of the 120 learners' performance in the quantitatively-oriented functions test is provided. The table below further depicts the critical test variables and scores attained.

Table 5.1: Summary of learners' test-based performance

Test Variable	Total Mark	Number of Learners									
		0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Equations asymptote	2	12	13	87							
Intercepts	3	5	15	8	84						
Sketch graph	3	28	3	6	75						
Exponential graph	2	53	45	13							
Asymptote (exp graph)	1	61	51								
Table of values	2	38	0	74							
Plotting graph	2	41	2	69							
Naming function	1	77	45								
Equation of asymptote	1	81	31								
Meaning of asymptote	1	101	11								
Global method graph	9	19	7	3	15	7	5	10	5	5	49
Intersection	2	92		20							
Equation	1	33	79								
Translation	3	57	4	5	42						
Translation	3	45	3	3	43						
Stretch	2	86	0	3	21						

In terms of Table 5.1 (p. 70) above, the first column lists the question number. The second column locates the tested functions component, while the following column details the total marks allocated for the test. Finally, the last column shows the distribution of the marks per question. Table 5.1 (p. 70) above reveals further that the respondents performed well on the formal hyperbola questions. The test involved four functions, viz: the linear function, the hyperbola, the exponential function and the quadratic function; as well as their different representations. Exponential functions and hyperbola questions formed the majority of the tested items.

The test respondents experienced some difficulties in answering real-life questions relating to exponential functions. They struggled to state the reason they thought the function was increasing or decreasing, given an equation of a decreasing function. More than half (61) of the participants could not state the asymptote of a translated exponential function. However, most of the participants performed very well in completing the table of values.

Table 5.1 above (p. 70) shows the learners' performance in the test per sub-question. The participants correctly answered questions that had an algebraic manipulation bias. These questions also required learners to calculate the intercepts, state the asymptote, sketch the graphs, and complete a table of values. A significant number of the respondents struggled with questions that required justification of narratives and showing flexibility of routines. This kind of difficulty justifies the notion that most learners' routines are ritualised (Sfard, 2008). The learning of Mathematics seeks to engage participants of the mathematical discourse in such a way that they will be able to reach objectification (Nachlieli & Tabach, 2012). On reaching objectification, their mathematical discourse will not be only about procedures. They will also be able to explain their mathematical actions and arrive at different narratives by means of multiple routines. When a participant of the mathematical discourse has reached objectification, s/he will be able to address any question on the said discourse, including the unseen questions. Written work is one of the means by which participants communicate their thoughts to the outside world as evidenced by their test answer sheets in respect of sketching the hyperbola and decreasing the exponential function.

5.3 Sketching the Hyperbola

A hyperbola is sketched by using a point-by-point plotting mechanism known as a pointwise method; or by using the global method and its key features. Point plotting requires as much ordered pairs as possible to allow the shape of the graph to emerge. This is the beginning point for graph functions, and gives the idea to first-timers of the kind of a shape a function has. A global method is applied in the event that learners are more acquainted with different features of the function (DBE, 2011). For example, the points in a linear function form a straight line. Different functions have unique key features that distinguish them from each other. For

example, a novice cannot use just three ordered pairs to sketch a hyperbola, but can do so with a linear function. The assumption in this study is that participants have thoroughly worked with pointwise graphing, and are now familiar with the global method. Participants were required to sketch function f, in which question 1 (one) required them to state the asymptotes of $f(x) = \dfrac{3}{x-1} - 2$. There was no prescribed method in this question, and participants could choose any method they preferred. All the participants who responded to the question only stated the asymptote without showing any process or method. This is acceptable, but does not insightfully explain to the researcher how participants arrived at their responses. Following below are two participants' responses pertaining to a question on the equations of the asymptote (sub-section 5.3.1); and another on the calculation of the intercepts of the graph of f with the axes (sub-section 5.3.2).

5.3.1 Participants' responses to the equations of the asymptotes

The first part of Question 1 was: *Write the equations of the asymptotes of f*. Most of the participants performed well in this question. Over 75% expressed the asymptotes mathematically, that is, *x=1* and *y=-2*. However, about 25 participants (about 21%) did not express the asymptote in a mathematically acceptable manner. Four (4) participants did not respond to the question, and just left blank spaces. Twelve participants had all their responses incorrect, and could not name either of the asymptotes. Some of the responses will be discussed later in this chapter. Thirteen participants' responses (11%) were partially correct. One of the responses was correct on the only 1 (one) asymptote which was written. Most of the participants (75%) responded in a manner that is acceptable to the community of mathematicians by stating 2 (two) asymptotes that were perpendicular to each other. Below are some of the participants' demonstrated responses on asymptotes. Figure 5.1 below shows **Participant A's** responses in relation to the equation of the asymptotes of f.
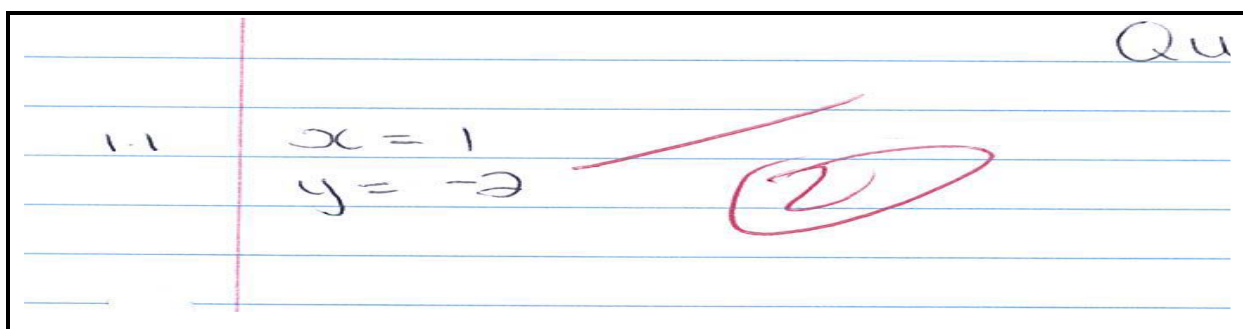
Figure 5.1: Participant A's response to the equations of the asymptotes of f

Figure 5.1 is in accord with the discourse of the community of mathematicians. In Figure 5.1 above, **Participant A** wrote the two asymptotes of $f(x) = \dfrac{3}{x-1} - 2$ mathematically because she expressed them as evidence of interpreting the symbolic mediator; that is, the function f. It

is important to write asymptotes in equation form, in order that anybody looking at **Participant A's** work should come to the same conclusion as **Participant A**. The latter's interpretation of the symbolic visual mediator is "construed" because she can interpret and write equations mathematically. The next figure (Figure 5.2) below is a reflection of some participants' non-mathematical responses as ritualised routines.

Participant B answered correctly, and then went on to write the coordinate (*1; -2*). The asymptote is regarded as a coordinate in the same manner that one would do for a turning point where the *x* and *y-values* are written as a coordinate.
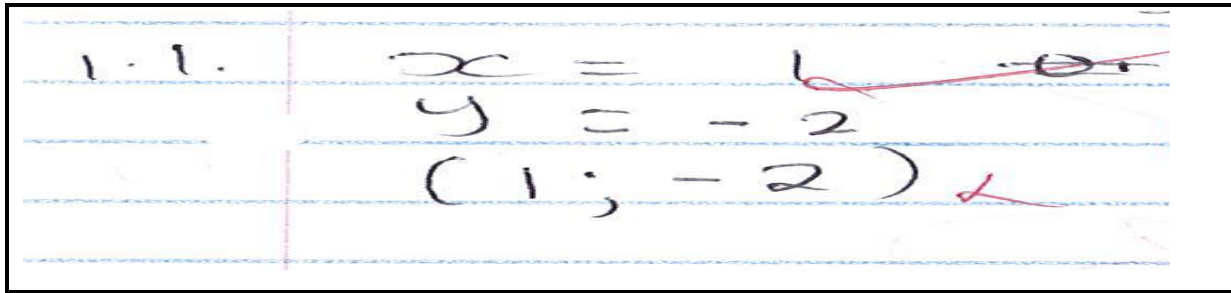
Figure 5.2: Participant B's response to the equations of the asymptotes of f

Figure 4.2 above is **Participant B's** response to question 1.1: *Write the equations of the asymptotes of f*, which sought the asymptote of $f(x) = \dfrac{3}{x-1} - 2$. **Participant B** identified the asymptotes as equations, and then went further to write the coordinates; which show that the *x* is from the vertical equation, and the *y* is taken from the horizontal equation. **Participant B's** equations are perfect and to the point. He and other four participants who presented the asymptote in the same manner did not take part in the task-based interviews. Nonetheless, the researcher could not make a follow-up on the reasons further than the required answer. However, it was clear that participants knew how to find the asymptote, but probably did not know what the asymptote was. **Participant B** could interpret the symbolic visual mediator $f(x) = \dfrac{3}{x-1} - 2$. which was classified according to the DPHEF analytical tool as "construed".

Participant B's routines were classified as "ritualised", because he showed little objectification on the mathematical object, the asymptote (Sfard, 2012).

The work of six participants indicated below, reflects the second type of mathematically unacceptable responses, presenting the equation in terms of the *p* and *q* parameters. **Participant C** presented his answer for the asymptote as *p = x =1* and *y = q = -2*. When a function is generalised, *p* and *q* are used to represent the vertical and horizontal shifts respectively. These parameters are used in almost all the functions in the Further Education and Training (FET) phase in South Africa. Learners then misconstrue mnemonics as part of the Cartesian plane (Adler & Ronda, 2014). The researcher's experience as a teacher has inculcated the notion of

simplifying Mathematics to learners by emphasising the parameters to such an extent that learners know the parameters to be the real asymptotes. In Figure, 5.3 below, **Participant C** wrote his equations and included the letters *p* and *q* together with the equations, which shows that he is able to interpret the symbolic visual mediator, as he is able to name asymptotes from the equation.

Figure 5.3: Participant C's response to the equations of the asymptotes of f

The researcher classified **Participant C's** interpretation of the asymptote as construed because he could identify the equations of the asymptote. However, the researcher classified his routines as "ritualised mathematical", as he included *p* and *q* in his equation. Although **Participant C** could identify the asymptote, he still had not yet objectified the mathematical object (Sfard, 2008). **Participant C** has not yet reached the stage of presenting the asymptote without including the mnemonics *p* and *q*.

The third type of response shows that participants had some knowledge of the asymptote of a hyperbola, which was demonstrated by the three respondents excluding the *y* and writing the equation as if it was a number. **Participant D** used inductive reasoning to arrive at her answer. Naming the asymptote as *-2* is mathematically incorrect, since *-2* cannot be located on the Cartesian plane. There is a tendency by some learners to think that *p* or *q* could replace *x* and *y* respectively, or that they were synonymous such that one could replace one with the other. Figure 5.4 below reflects **Participant D's** answer as *q* = asymptote, *q* = *-2*, in terms of which *asymptote = -2*.

Figure 5.4: Participant D's response to the equations of the asymptotes of f

Participant D first wrote a general equation $f(x) = \frac{a}{x-p} - q$, and had to be reminded of the asymptotes of the function. She went on to substitute the number 'three' (3) for a. She then equated q to the asymptote. Finally, she wrote that the asymptote was equal to *-2*. The object which **Participant D** wrote did not exist on the Cartesian plane. She did not do anything about the vertical asymptote. She also did not mention the vertical asymptote. The researcher classified her interpretation of the symbolic visual mediator as "not construed", since she could not identify asymptotes from the equation $f(x) = \frac{3}{x-1} - 2$. The researcher further classified her routines as "ritualised non-mathematical" because she made follow-up the routines of others as her own routines, which did not lead to mathematically endorsed narratives. Conclusively, **Participant D's** equations were not mathematical as they could not be located on the Cartesian plane.

Figure 5.5: Participant D's corrected response to the equations of the asymptotes of f

In Figure 5.5 above, **Participant D** corrected herself. Initially, she erred on the *x-intercept,* but realised her mistake and tried to correct it, with little success. Not many of the participants could self-correct. The researcher classified **Participant D's** routine type as "corrigibility" (Ben-Yahuda et al., 2005), since she realised her mistake and attempted to correct it.

The fourth type of response was manifested with participants writing only 1 (one) asymptote. Four (4) participants only wrote the horizontal asymptote in the form of *y = - 2*. A hyperbola had 2 (two) asymptotes; namely, the horizontal and the vertical. The exponential function is the one with only 1 (one) asymptote.

Figure 5.6: Participant E's response to the equations of the asymptotes of f

Participant E was one of the four participants who only wrote the horizontal asymptote. The equation y = -2 was mathematical as the object could be located on the Cartesian plane, rendering **Participant E's** response incomplete. For that reason, his types of routines were classified as "ritualised" by the researcher due to the incompleteness of the response, and that nothing was written about the vertical asymptote. Some participants did not respond to the question and left a blank space as illustrated by **Participant E's** response below.
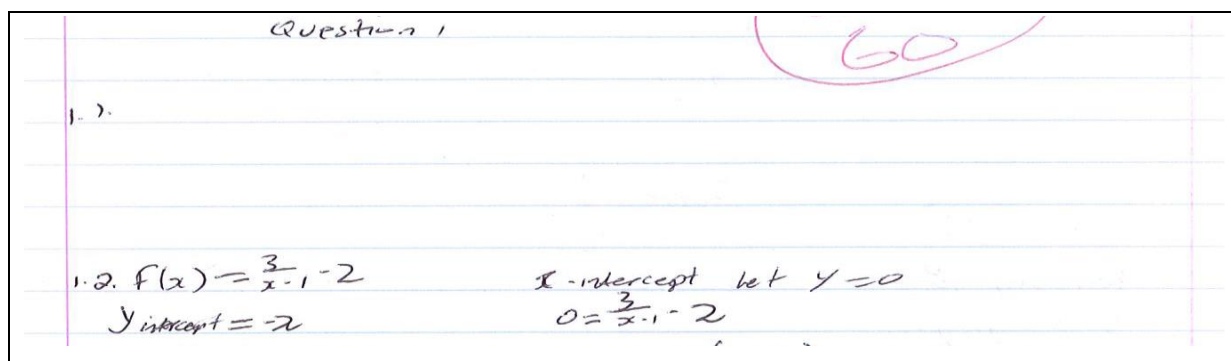
Figure 5.7: Participant E's non-response to the equations of the asymptotes of f

Participant E only acknowledged the question by writing '1.1' on the answer sheet. Like the other five participants, **Participant RR** also did not respond to the question. When participants do not respond to questions, it becomes very difficult for the researcher to speculate on the reasons for their actions. Based on that, the researcher then decided to take the top 20% of each of the four groups of 30 from each participating school. The second question in sub-section 5.3.2 below required the these respondents to calculatethe intercepts of a hyperbola.

5.3.2 Participants' responses to calculation of the intercepts of the graph

In terms of the question requiring participants' calculation of the graph's intercepts, 107 (95.5%) of the 112 participants obtained at least 1 (one) mark on this question. Although intercepts are part of the functions discourse, the success in responding to the question was mostly dependent on algebraic manipulation and solutions to equations. Key elements in solving the question was substitution, clearing of fractions and solution to linear equations. Participants who did not obtain full marks on this question either made an incorrect substitution or completely failed to manipulate the equation.

An intercept is a coordinate of the graph's intersection with the axes. An intercept is written in the form of (*a; 0*) for the *x-intercept,* and (*0; b*) for the *y-intercept*. For participants to calculate the coordinates of the intercepts, the learners needed to substitute zero (0) for *x* and *y* respectively. Eighty four (n=84, 75%) of the 120 (100%) participants calculated the intercepts and scored full marks. Figure 5.8 below indicates **Participant F's** response to question 2 (two), which required the participants to: *Calculate the intercepts of the graph/ hyperbola.*

Figure 5.8: Participant F's response to the calculation of the asymptote graph of f

Participant F showed her intention by calculating the *x-intercept* and also indicated the part of her response calculating the *y-intercept*. In the calculation of the *x-intercept*, her steps were identical to those of **Participant HH**, which showed the latter's developed mathematical discourse on the algebra sub-discourse. Her findings were well presented, and she clearly indicated that the calculation of the *x-intercept* should be premised on, and be directed by *y = 0*. **Participant F's** interpretation of the symbolic mediator was "construed", since she could work on the function by calculating the intercepts. She knew where, and what to substitute in the algebraic equation, which was an indication of the ease with which **Participant F** could calculate the symbolic visual mediator $f(x) = \frac{3}{x-1} = -2$. Although she did not explain, her method was evidence of calculation fluency. Her routines were in concurrence with those of the community of mathematicians. Therefore, **Participant F** showed she had objectified the mathematical discourse of finding the intercepts of a function (Nachlieli & Tabach, 2012). **Participant G** was one of those whose partial marks in this question (calculation of intercepts) could have obtained full marks, had it not been for the mistakes committed with regard to directed numbers. Figure 4.9 below (p. 77) shows **Participant G's** response to this question.

Figure 5.9: Participant G's response to the calculation of the asymptote graph of f

The response in Figure 4.9 above is an illustration of **Participant G's** familiarity with the calculation of the *x-intercept*. In the second part of her calculation of the *y-intercept*, she did everything well but erred on subtraction. One of the uses of routines was corrigibility (Ben-Yahuda et al., 2005), which is the routine about checking the correctness of work done. In that case, **Participant G** could have substituted her *y value* into the equation and checked whether the answer would be a zero. Self-correction was important in the development of the mathematical discourse. When comparing **Participant G's** work on calculating the *x-intercept* and the the *y-intercept*, the researcher discovered that calculating the *x-intercept* has a higher mathematical demand. Should **Participant G** had developed corrigibility routines, she could have realised her mistake. In the study, there are a few instances of participants checking their work and conducting some modicum ofself-reflection and self-correction.

There were some participants who knew how to calculate the intercepts, and there were others who did not know what to do. **Participant H** below is one of the five participants who knew that they should substitute the number 1 (one) of the variable by zero, but failed to do so in the correct variables. Figure 5.10 below reflects the latter state of affairs with **Participant H's** response to the question on the calculation of the intercepts.

Figure 5.10: Participant H's response to the calculation of the asymptote graph of f

In Figure 5.10 above, **Participant H** understood that in the calculation of the intercepts, either *x = 0* or *y = 0* but did not substitute at the correct place. **Participant H** substituted zero for the constant 1 (one) instead of substituting for *x*. **Participant H** substituted the constant instead of *y*. Such substitution indicates that **Participant H's** mathematical discourse was still developing (Sfard, 2012). **Participant H** knew there might be some substitutions, but did not know where exactly to substitute. In the next few paragraphs, the researcher discusses how the participants

drew the iconic visual mediator and the graph of a hyperbola.

A rectangular hyperbola graph is composed of two (2) parts which reflect each other and have two (2) asymptotes that are vertically opposite to each other (Yavuz, 2010). In the current study, thirty-seven participants could calculate the intercepts and identify the asymptotes, but struggled to sketch the iconic visual mediator. Twenty-eight participants could not obtain a single mark, three (3) obtained 1 (one) mark, and six (6) participants obtained 2 (two) marks. The majority of 75 participants answered all the questions correctly, which indicates a developed mathematical discourse of the hyperbola.

Participant I was one of the participants whose mathematical discourse on the hyperbola was developed, judging by the manner in which she drew the iconic visual mediator. Fihure 5. 11 below (p. 80) is Participant I's diagram of the hyperbola
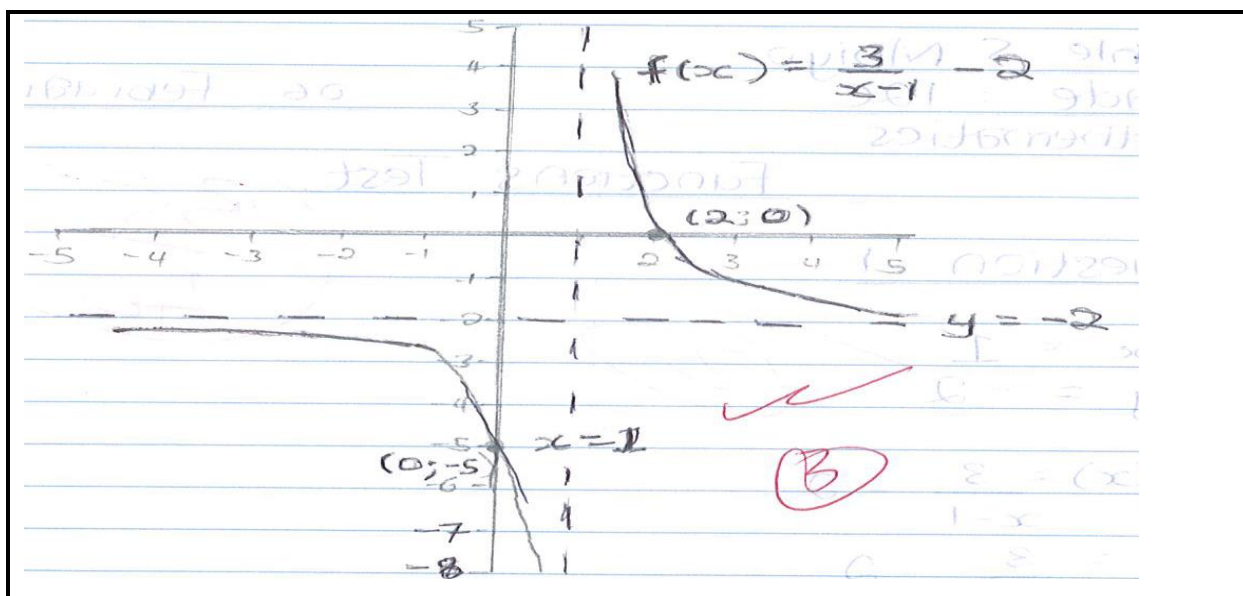
Figure 5.11: Participant I's developed response to calculation of the hyperbola

In Figure 5.11 above**, Participant I's** axes are clearly labelled together with intercepts and asymptotes, which demonstrates that **Participant I's** mathematical discourse is developed. The researcher classified her iconic visual mediator as "drawn" because she produced a diagram that was acceptable to the community of mathematicians. Most of the participants produced diagrams that were similar to that of **Participant I.**

Participant J is one of the participants in the study who managed to identify asymptotes and calculate the intercepts, but could not sketch a hyperbola. Intercepts and the asymptotes are the key features in sketching the graph using the global approach. Figure 5.12 below (p. 80) reflects **Participant J's** incorrect representation of the hyperbola.
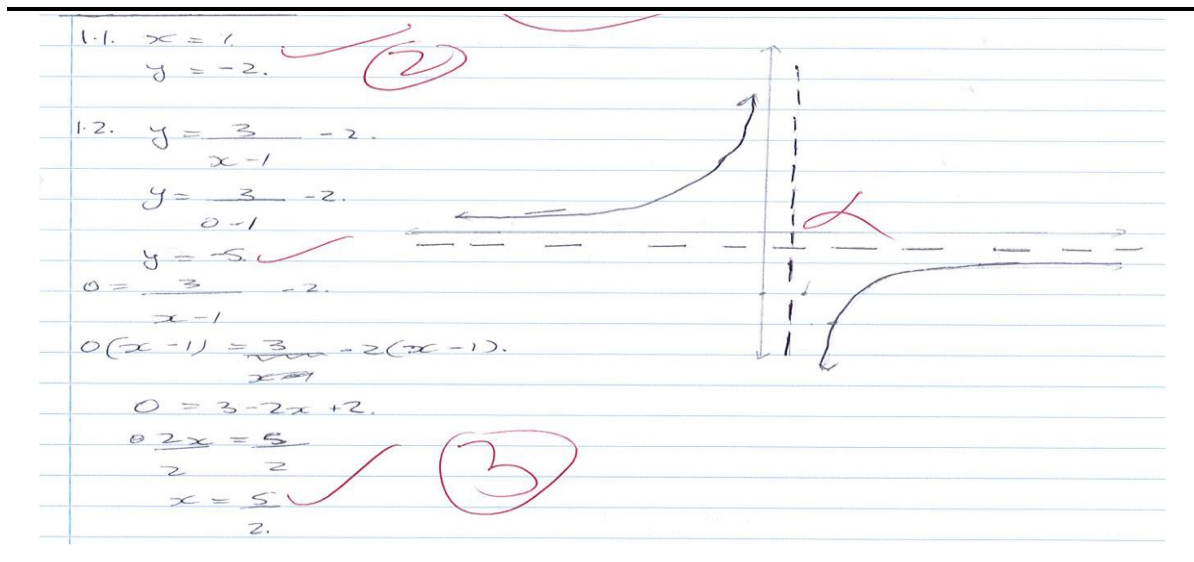
Figure 5.12 Participant J's representation of the hyperbola

In Figure 5.12 above, **Participant J** chose to sketch the graph using the global method, which applies in the event that key features of a graph have been used to sketch the graph. She had already identified the asymptotes, making her task somewhat easier; but could not sketch the correct diagram of the hyperbola. **Participant J** sketched the asymptotes of the hyperbola in the correct position, *x = 1* and *y = -2*. She also knew that the hyperbola is a function with two parts that reflect each other on the line $y - q = \pm(x - p)$. **Participant J's** mistake was on the plotting of *(0; -5)* on the *x-axis* instead of the *y-axis,* and $(\frac{5}{2}; 0)$ on the *y-axis* instead of the *x-axis*. **Participant J** would have realised that she made a mistake, had she used corrigibility routines by checking whether her answer made mathematical sense. In $f(x) = \frac{3}{x-1} - 2$, the value of three (3) is positive, which means that the two parts of the hyperbola are on the first and third quadrants of the asymptotes. Mathematical discourse develops when there is corrigibility. The researcher classified **Participant** J's routines as "ritualised non-mathematical" due to her failure to plot the correct point on the Cartesian plane and drawing the graph on the wrong quadrants.

Some participants did not draw hyperbolic functions, but drew some other diagrams that resembled other functions or undefined graphs. All the three participants in question recognised the intercepts, but were challenged by the shape of the hyperbola. This category of learners had not yet reached objectification, and their mathematical discourse was still developing. The researcher began by looking at their 'hyperbola'. The diagrams which **Participant K** and **Participant L** sketched below in Figure 5.13a and Figure 5.13b (p. 81) respectively are not a hyperbola, but a linear graph.
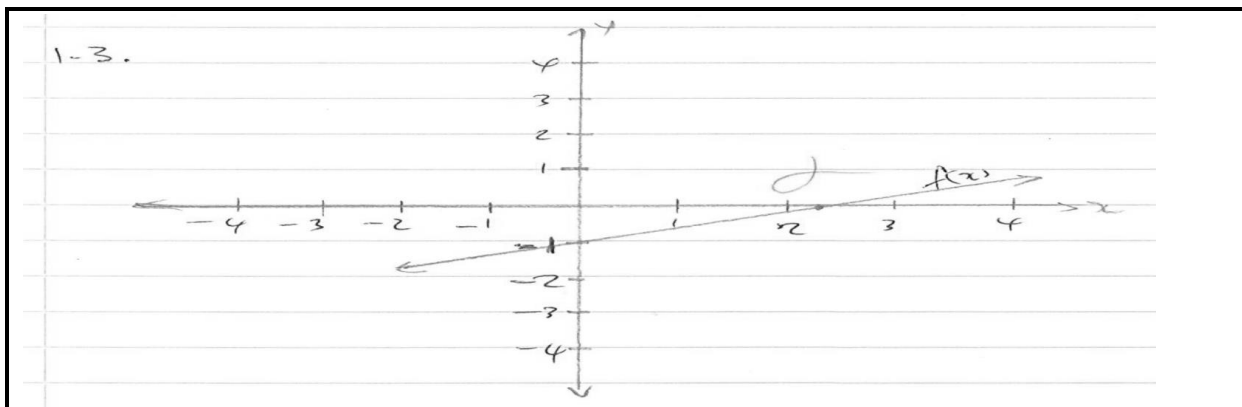
Figure 5.13a: Participant K's incorrect representation of the hyperbola
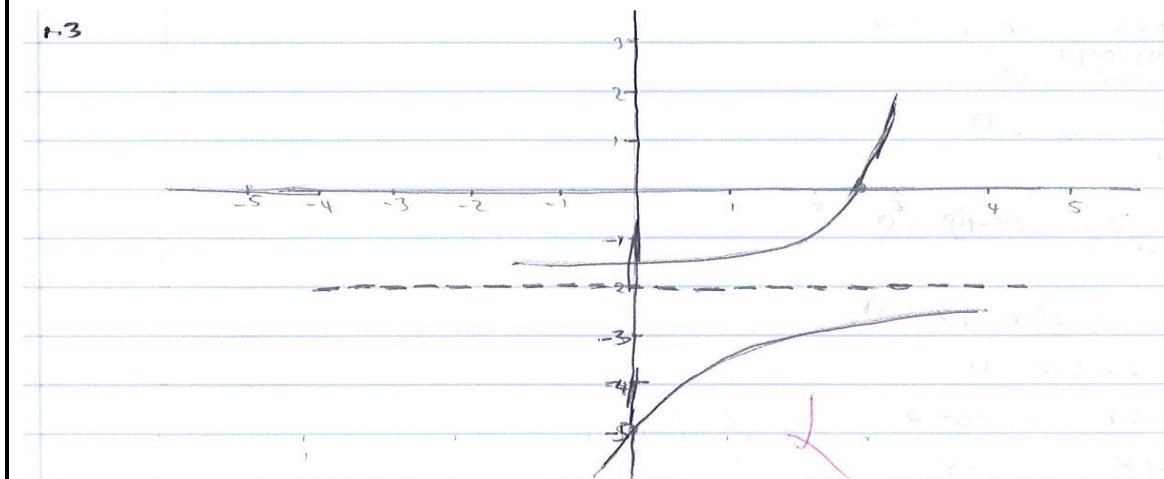
Figure 5.13b: Participant L

Figure 5.13b: Participant L's incorrect representation of the hyperbola

Participant K did not draw a hyperbola, but a linear function. **Participant K** also did not include the asymptotes in his diagram. The evidence from **Participant L's** diagram shows that it was possible he did not know how the hyperbola looked like because he sketched a linear graph. On the other hand, **Participant L** drew only 1 (one) asymptote and then drew two curved parts that passed through the intercepts. In such a situation, the routines of both **Participant K** and **Participant L** are "ritualised non-mathematical" since none of them could sketch the required graph.

While over 75% of the 112 participants could identify and calculate the asymptotes, only about 66% were able to complete the task successfully. The sketching of a hyperbola is not necessarily a difficult procedure since most of the participantsare supposed to have already undertaken and engaged for more than two years leading to grade 11.

5.4 Decreasing the Exponential Function

In this section, the researcher presents and discusses a question by means of which learners were asked to account for actions taken in resolving exponential functions. The results show that there was a noticeable degree of disparity pertaining to the learners' responses to these two questions:

Question 2.1: *Is f an increasing or decreasing function? Give a reason for your answer*; and

Question 2.2: *Write the equation of the asymptote of f (x) – 5.*

An exponential function is denoted by an exponential variable x. Furthermore, exponential functions either increase when the base is more than 1 (one), or decrease when the base is between 0 and 1. Mastering the decreasing functions needs grounding on exponential laws, and an acknowledgement that $f(x) = \left(\frac{1}{3}\right)^x$ and $g(x) = 3^{-x}$ are the same functions. A decreasing function is described as such; as the independent variable increases, the dependent variable also decreases correspondingly. Interpreting the base should show that as x increases, y would also decrease. Only 13 (11%) of the 120 participants obtained all the marks in the first part of the question. Almost half (58) of the research participants obtained at least a mark, meaning that they mentioned the decreasing factor of the graph. About 11% (n=13) of the 112 participants who answered the first question correctly also mentioned a mathematically acceptable reason for their choice. The second question (2.2) required participants to name the asymptote of a transformed function written as $f(x) - 5$. Fifty one (45.5%) of the participants answered correctly. The ensuing paragraphs are centrally premised on the function $f(x) = \left(\frac{1}{3}\right)^x$; as encapsulated in the following two questions:

Question 2.1: *Is f an increasing or decreasing function? Give a reason for your answer*; and

Question 2.2: *Write the equation of the asymptote of f (x) – 5.*

In relation to question 2.1 above, almost 10% (n=12) of the participants obtained 100% of the marks and provided reasons for their choice of answer. On the other hand, 45 participants (37.5%) managed to answer the first part of the question correctly, which required them to mention whether the function is increasing or decreasing. Almost half of the participants could not recognise that the graph was decreasing. One of the reasons for such a poor performance was that the question did not require participants to reproduce a routine. In the first question (2.1), the participants followed some well-known procedure, but the routine was different in the second question (2.2). The participants found it difficult to give reasons or justify their responses in the test and in the interview (Mpofu & Pournara, 2018).

Thirteen participants responded positively to the question (2.1) requiring them to state whether the exponential graph was increasing or decreasing. **Participant M** was one of the 13

participants who had a mathematically acceptable response to the question. The diagram below reflects her response to the question:
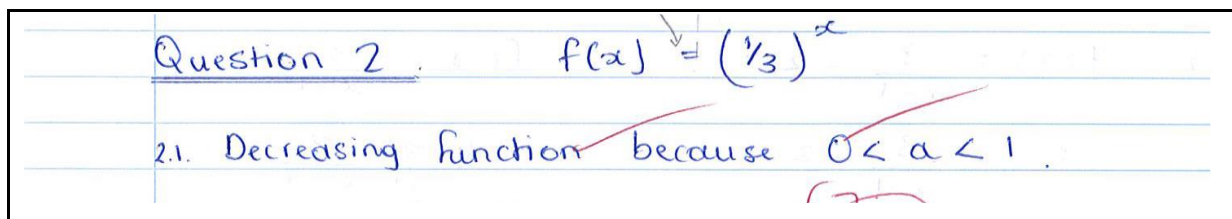
Figure 5.14: Participant M's response to decreasing the exponential graph

In a decreasing exponential function, the base is a fraction between zero (0) and 1 (one). The base can be written with a negative exponent. For example, as $f(x) = 3^{-x}$ or with the base as a fraction as in $f(x) = \left(\frac{1}{3}\right)^x$. **Participant M** was able to identify that the function was decreasing. Her reason was that base a is between 0 and 1. **Participant M's** mathematical discourse on this question was correct for the following reasons: The use of words was mathematical because she used the term 'decreasing' to describe the movement of the graph. Furthermore, she interpreted the symbolic visual mediator and could tell that the graph was decreasing. Therefore, her interpretation of the symbolic visual mediator is   classified as "construed", because she used "a" instead of the actual base $\frac{1}{3}$ for her choice. In this case, **Participant M's** response is characterised as "memorisation" based on rules. **Participant M** did not give a reason in her own words, but chose to use the textbook rule. In Figure 5.15 below, **Participant N's** response to the increase or decrease of the exponential graph shows that **Participant N** used mathematical words to describe the graph. In this regard, **Participant N** correctly interpreted the symbolic mediator $f(x) = \left(\frac{1}{3}\right)^x$. However; she could not provide a mathematically acceptable reason for the behaviour of the graph.
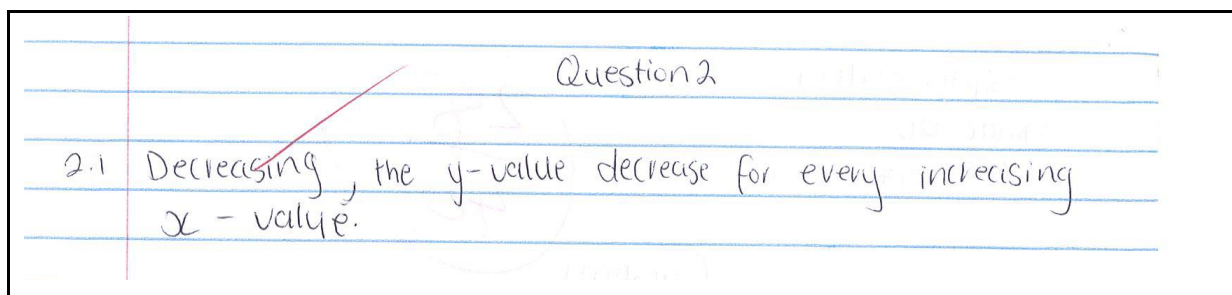
Figure 5.15: Participant N's response to decreasing the exponential graph

Participant N partially answered this question correctly. While it was true that the function is decreasing, the reason she provided was equally true for all functions, hence the partial correctness. In all functions, as x values increase, the y values decrease. **Participant N's** reasons are not necessarily incorrect, but her description could be applicable to any function. Words used in communication are meant to differentiate between objects. **Participant N** uses

words which do not make such differentiation. In addition, her use of words ascribed to her choice and reasons are colloquial, according to the DPHEF analytical tool. **Participant N** did not acknowledge the function *f* in her response. Nowhere in her response did she mention the base of the exponential function. On the latter basis, the researcher then classified her interpretation of the symbolic visual mediator $f(x) = \left(\frac{1}{3}\right)^x$ as "not construed" because she did not mention or refer to the mediator in her response. **Participant N's** response (that a decreasing graph *x* values increase as *y* values decrease) was a generalisation that suited all functions as stated above. In this regard, **Participant N's** routines were ritualised (Sfard, 2012) because she provided a general statement where a distinguishing statement was required.
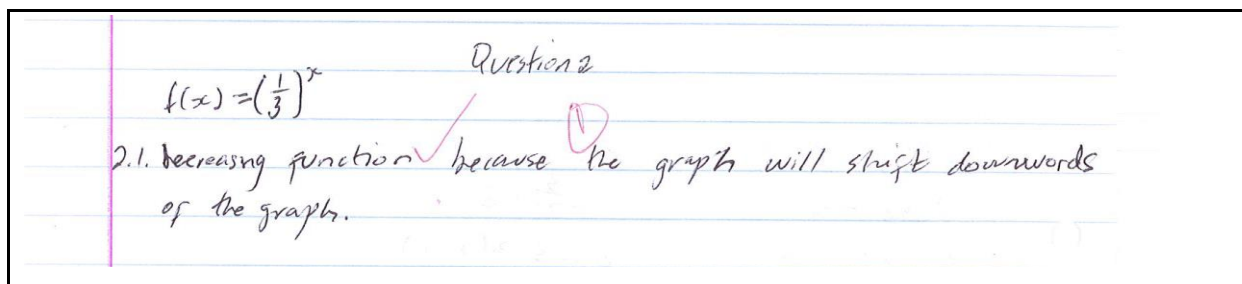
Figure 5.16: Participant O's response to decreasing the exponential graph

Participant O correctly answered that the function was decreasing, but his reasons for the decrease were incorrect for this graph. He attributed the graph's decrease to the vertical shift in the negative direction. The shifts, whether vertical or horizontal, do not affect the behaviour of the graph. The words used by **Participant O** were mathematical, but not appropriate for this question. The researcher classified her word use as "colloquial" due to her inappropriate use of mathematical words, meaning that she did not know where those words fitted in the functions discourse. **Participant O** could not interpret the symbolic visual mediator because she spoke of shifts when the function was a parent function. Furthermore, **Participant O's** mathematical discourse was similar to that of **Participant P**, because he (**Participant O**) only managed to successfully respond to the first question. In Figure 5.17 below, **Participant P** responded in a manner similar to that of **Participant O** by only answering the first part of the question correctly.
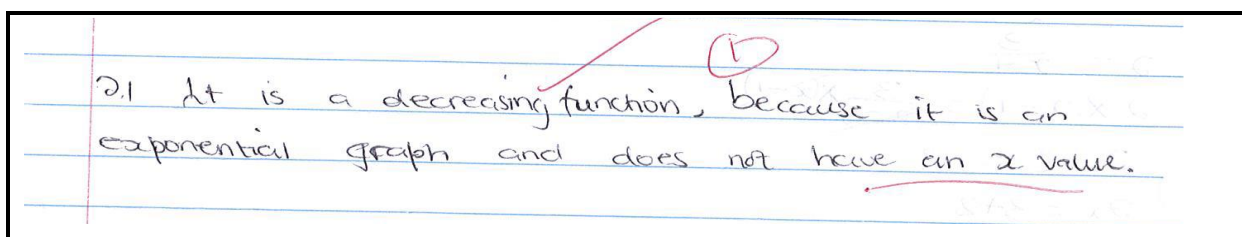
Figure 5.17: Participant P's response to decreasing the exponential graph

Participant P mentioned that the graph was decreasing on the basis that it was an exponential function, and implied that all exponential functions were decreasing; which was not the case. **Participant P** mentioned further that the function did not have the value of *x*. On an *x* and *y* Cartesian plane, it was not possible for a graph to have *x* values. Therefore, **Participant P's** statement that a function had no x-values was not mathematical. Based on that, the researcher classified her use and choice of words as "colloquial" (Ben-Yahuda et al., 2005).

In Figure 5.18 below, **Participant Q's** work is an example of some of those participants who did not score anything in the test. His choice and reasons were not in accordance with the norms (conventions) of the community of mathematicians.
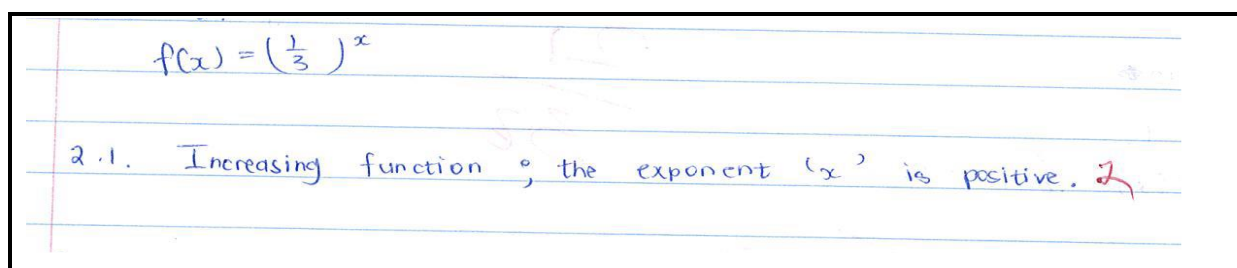
Figure 5.18 Participant Q response to decreasing the exponential graph

Figure 5.18 illustrates **Participant Q's** response to the question on the exponential function. Participant Q described the function as increasing, which showed that he did not know the difference between an increasing function and a decreasing one. An example relates to **Participant Q'**s reference to the exponent of *x* as positive. The exponent of $\frac{1}{3}$ appeared to be +*ve* (positive), whereas $\frac{1}{3}$ is $actually$ 3^{-1} in reality. The product of *-1* and *x* is *-ve* (negative). Therefore, the exponent is not +ve (positive), but -ve (negative). On the whole, the mathematical discourse of **Participant Q** was still far from the narratives and discourses of the community of mathematicians.

The overall responses to Question 2.1 (writing the equation of the asymptote of f (x) – 5) revealed that while participants could perform well on routine questions, they struggled with questions that sought reasons for their choices. They had to choose between two options, making it possible for the participants to arrive at the correct answer. The second part of the question became the determinant of how much learning of Mathematics had been learnt. Only 10% of the learners could give reasons for a decreasing function.

In question 1 (one) on the asymptote of a hyperbola, 100 participants got at least a mark correctly. However in a similar question with a minor modification, only 51 (45.5%) of the participants answered correctly. In Question 2.2 (*writing down the equation of the asymptote of f(x) – 5),* the asymptote of the translated exponential function should reflect the vertical movement that the function had undergone. That gave credence to the notion that participants'

routines were ritualised. The researcher used **Participant R** and **Participant W's** work to illustrate how different participants could respond to the same question successfully. Figure 5.19 below exemplifies **Participant R's** correct response to the asymptote.

Figure 5.19: Participant R's response to the equation of the asymptote of f (x) – 5

Participant R just wrote the answer without showing any working procedure or method. The answer was acceptable, as it was a horizontal asymptote. There was no ambiguity with her response, and the community of mathematicians would deduced the same meaning. **Participant R's** interpretation of the symbolic visual mediator was "construed," because her response showed a development in her communication. Her interpretation was endorsed in the mathematical discourse. In the ensuing Figure 5.20, the researcher describes **Participant W's** response to the question on the asymptote.

Figure 5.20: Participant W's response to the equation of the asymptote of f (x) – 5
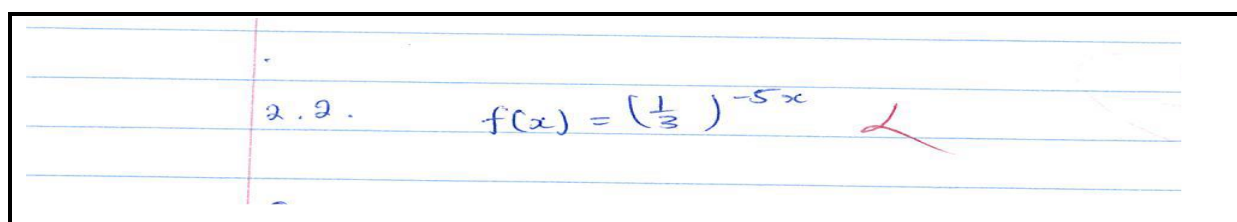
There were two outstanding features of **Participant W's** response. Firstly, she used the equal sign as another symbolic visual mediator appropriately in all instances. Some participants would have used the equal sign, but **Participant W** showed that she could communicate using the mathematical symbols. Secondly, **Participant W** showed that she understood the symbols $f(x)$ and $\left(\frac{1}{3}\right)^x$ to be the same. While **Participant R** and **Participant W** were examples of acceptable mathematical practice, there were participants who did not perform so well.

In Figure 5.21 below, **Participant SS** responded by writing an asymptote without showing any method or process, as did **Participant R** in Figure 5.19 above.

Figure 5.21: Participant SS's response to the equation of the asymptote of f (x) – 5

In Figure 5.21 (p. 86), it was difficult to understand the reasons for **Participant SS's** response, who wrote the equation as a vertical line. An asymptote of an exponential function is always horizontal. Furthermore, **Participant SS's** response did not recognise the vertical shift. The learners' confusion concerning an exponential equation lies in deciding whether the asymptote is horizontal or vertical. As opposed to a hyperbola, an asymptote has a horizontal and a vertical line. If asymptotes were arrived at through exploration of the function, then that seeming confusion would not prevail as participants would use the same explorations to arrive at narratives in case they had forgotten. When asked to give a reason for his answer, **Participant SS** stated: *It is because x does not have a shift*. **Participant SS** used the word "shift" to mean a horizontal translation. A show of a few steps would have made **Participant SS** realise his mistake, which is attributed to lack of exploratory routines and flexibility (Ben-Yahuda et al., 2005) that would have helped him to self-correct.

In Figure 5.22 below, **Participant TT** gave a different dimension to the asymptote of an exponential function.

Figure 5.22: Participant TT's response to the equation of the asymptote of f (x) – 5

In responding to the question requiring the participants to write the equation of the asymptote of f (x) − 5, **Participant TT** multiplied the constant *-5* with the exponent *x*. An asymptote is a linear function, but what **Participant TT** wrote was not a linear equation. **Participant TT** did not interpret the symbolic visual mediator $f(x) - 5$ in a mathematically acceptable manner. He confused $f(x) - 5$ with $f(5x)$. In fact, **Participant TT** only responded to the former $(f(x) - 5)$ by making substitutions to the equation, and did not respond to the question directly. He did not know what an asymptote was, and did not even know how it is represented as an equation.

In this section (4.4), the four categories of the participants' test responses were discussed. The first category is that of participants providing a correct answer without showing any process or method. The second category was that of participants whot provided the desired responses and also showed they processes leading to the answer. The third category of participants wrote a vertical line, instead of a horizontal asymptote. As a result, they misrepresented the asymptote of an exponential function by writing an equation showing a vertical line instead of a horizontal one. The last category consisted of a discordant depiction of responses which had nothing to do with the equation of an asymptote. The researcher proposes that the learning of Mathematics

should be enhanced by means of explorations, rather than learners merely following the rituals they saw from their teachers (interlocutors) or from other learners (Sfard, 2012).

5.4.1 The decreasing temperature

The third question of the functions test was a non-routine question which presented a practical decreasing exponential function. The question required participants to complete a table of values, as well as drawing and naming the graph. The phrase *exponential function*, did not appear in the question, but the equation was exponential. The participants were also supposed to state the asymptote of the graph and explain what the asymptote meant in real-life terms. Two-thirds of the participants (67%) successfully completed the table of values and plotted the corresponding graph. Just over a third of the participants (33%) named the function successfully. The number of participants that successfully named the asymptote went down to 26% (n=29) and the meaning of the asymptote in real life terms went down to 9% (n=10). When comparing the performance or success rate of the first and the third questions, a significant decline was observed with regard to participants' performance (response) in the context of the third question. The researcher attributes this decline to the general ritualisation of learning of functions. The difference in these two questions was that the familiar mathematical language was used in the first question, whereas practical language was used for the third question's responses.

The completion of a table of values is a routine task for participants in grade 11, especially that it is a requirement in earlier grades where learners are required complete a linear function's table of values. In the context of this study, the task given to the participants was not a daunting one. Notwithstanding, two factors were observed as contributing to the level of difficulty experienced by learners. Firstly, the question was not written in terms of x and y variables. Secondly, the exponent was in the form of a fraction. Figure 5.23 below illustrates **Participant W's** response to the completion of the table of values.

$$\theta = 60\left[2^{\frac{-t}{15}}\right] + 20$$

3.1 Complete the following table:

t	0	2	4	8	10	12	20	30
θ	80	74,70	69,87	61,46	57,80	54,46	43,81	35

(2)

Figure 5.23: Participant W's response to completion of the table of values

Participant W was one of the 74 participants who successfully completed the table. As stated earlier, the use of t for time and θ in the equation, posed a challenge to some participants. **Participant W** rounded off her output to two decimal places, which helped her to plot the

graph, unlike some participants who did not do so and could not even produce a smooth curve. **Participant W** showed some flexibility in her routines by not writing numbers as emanating from the calculator; but rounded them off in a manageable manner, especially for plotting points on the graph (Ben-Yahuda et al., 2005). That did not mean that those who did not round off their values for the output were incorrect in their responses.

Figure 5.24: Participant UU's response to completion of the table of values

In Figure 5.24 above, **Participant UU** rounded off her output values to the nearest whole number. While marking their work, the researcher awarded full marks to her although she produced a smooth curve due to the high degree of approximation. In Figure 5.25 below, the participants were required to draw a continuous graph of the data on the table provided.
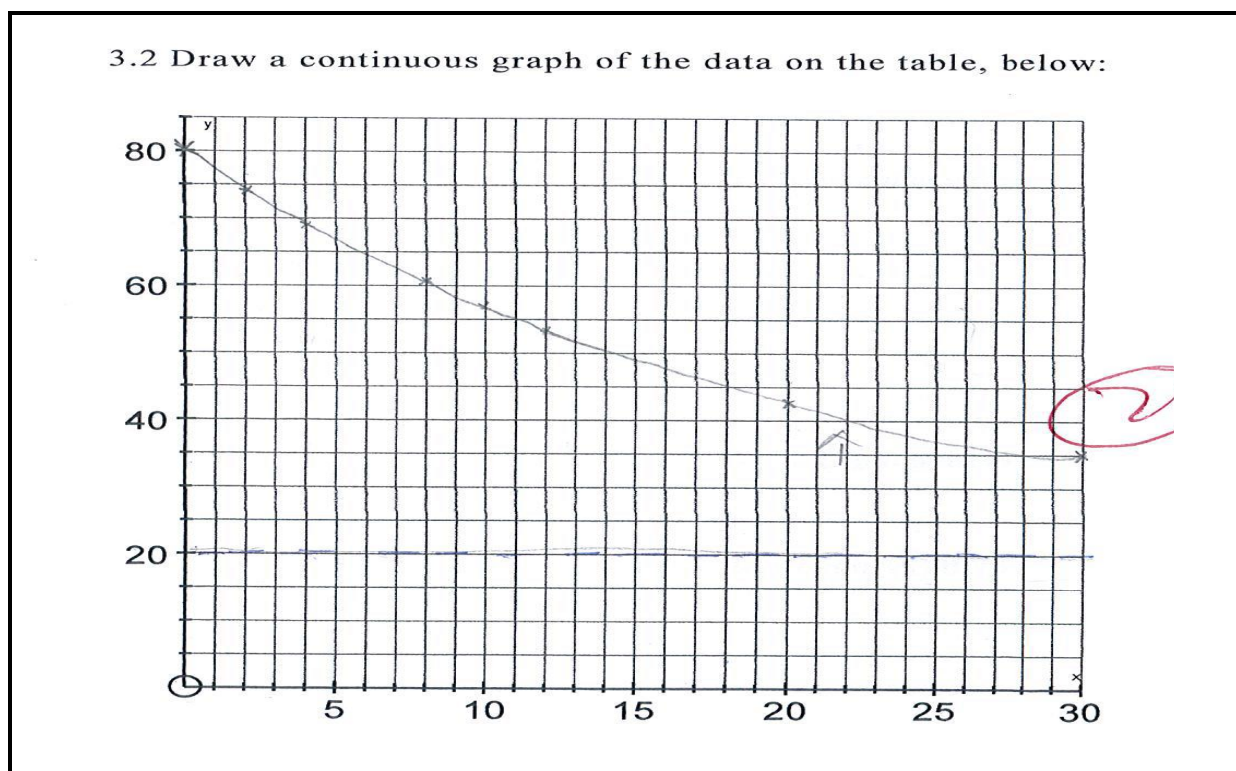
Figure 5.25: Participant W's response to **drawing a continuous graph**

In Figure 5.25 above, **Participant W,** drew a smooth decreasing exponential function. As indicated in the previous paragraph, rounding off the output coordinates contributed to the type of graph that **Participant W** drew and indicated the location of the asymptote of the graph on the Cartesian plane. Her graph and table of values did not show the asymptote of the function.

In Figure 5.26 below, **Participant UU's** graph showed that rounding off the output values to the whole number caused her graph not to be as smooth. The situation was compounded further as **Participant UU** chose to use writing paper instead of the grid that was provided; hence the faintness in the diagram.
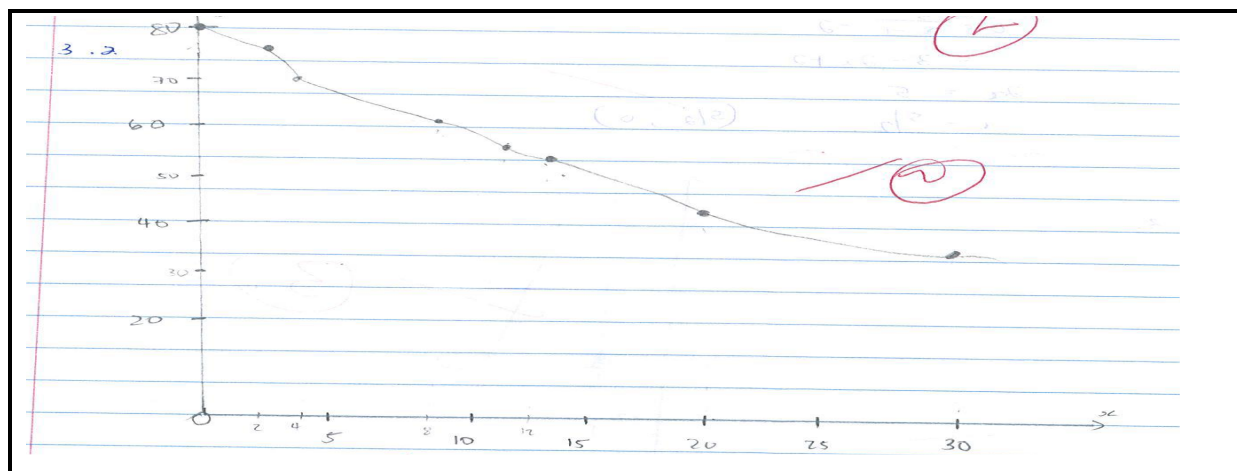
Figure 5.26 Participant UU's response to **drawing a continuous graph**

Although **Participant UU** thought this was a straight line, the diagram did not resemble a straight line at all. Rounding off to the nearest whole number and the choice of a scale on a writing pad affected **Participant UU's** accuracy. Some of the participants also thought the input and output values should be integral values. Such thinking was based on the fact that in most cases, integral values are used for convenience rather than as a rule when sketching or plotting a graph. Therefore, in the functions discourse, learners should not just sketch or draw a graph without realising that a graph is characteristicallycontinuous. This means that its coordinates are elements of real numbers. In this regard, **Participant UU's** diagram was then more of a sketch than a drawing.

Any kind of paper could be used for sketches. However, **Participant UU's** situation was compounded by the fact that the graph was drawn on paper of very poor quality, which resulted in its faintness. Nonetheless, there was a modicum of accuracy with the coordinates. In most instances, beginners' drawings involve many coordinates before the shape of the functions becomes clearly visible. **Participant UU's** sketch was more of an approximation, and accuracy of the scale was not important as only key features were used to show the path of the graph. Some participants did not complete the table of values accurately. Following are a few examples of those who did not complete the table of values during the tests.
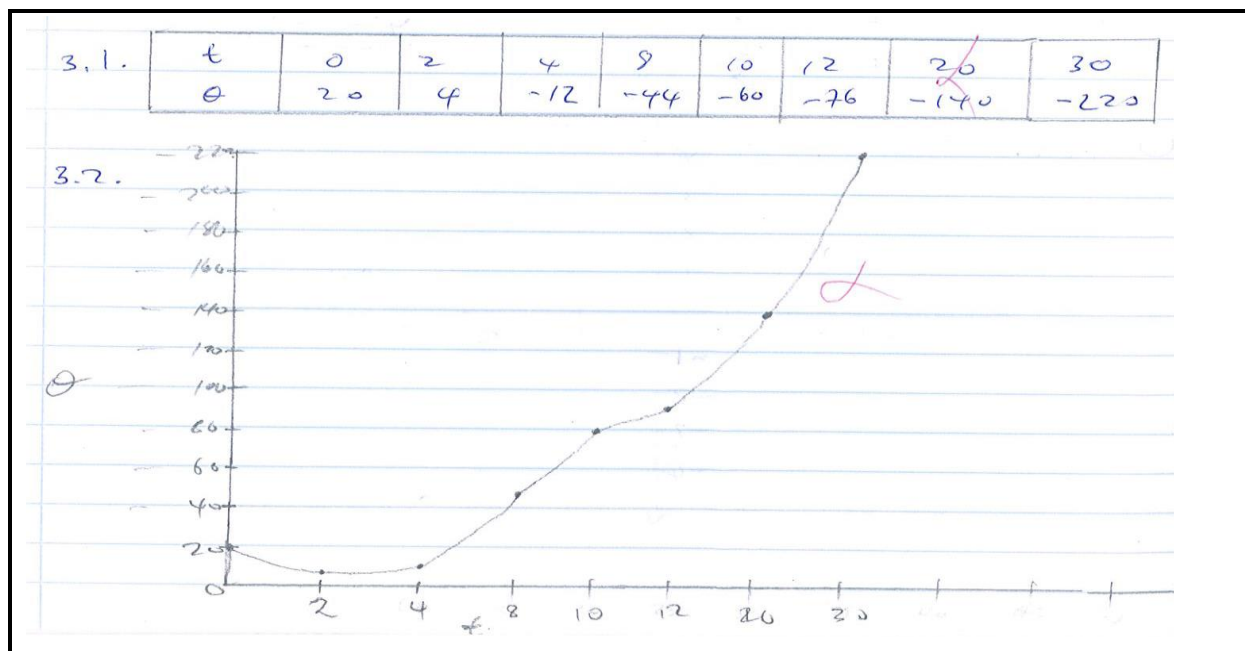
Figure 5.27: Participant VV's responses to completing the table and drawing a continuous graph

From Figure 5.27 above, it is clear that **Participant VV** failed to complete the table of values due to the negative values of the output from an exponential equation. His main challenge was primarily located on the failure to interpret the symbolic visual mediator (Ben-Zvi & Sfard, 2007). From the literature review, it is posited that mathematical symbols were important in communicating mathematical information (Bradley et al., 2013; Flesher, 2003). **Participant VV** did not recognise the *t* variable as an exponent. He worked on his answer as though the variable was part of the base, hence the output of negative numbers. Secondly, when drawing his graph, **Participant VV** mixed the -ve (negative) and the +ve (positive) numbers on the same side of the axis; resulting in positive (+24), negative (-12) and so on, on the same quadrant. Instead of his graph showing a decrease as the table of values suggests, it showed a decrease. This shows that in spite of **Participant VV** drawing a table of values and the axes of the graph, he did not understand their meaning. **Participant VV** understood the plotting of points as the diagram suggests, but his problem was with the axes and their meaning. It was difficult for **Participant VV** to draw any graph, because he did not understand how to place coordinates in the four quadrants.

Participants were presented with the equation, the table of values and the graph of the function in question 3.3, which required them to name the function. Forty-five participants (37.5%) could name the function as exponential. The most common response was that it was a line graph. In a line graph, a straight line joins different points. This means that there was no expectation of a defined function to be produced. Participants did not see a pattern emerging from either their table of values or the graph. Of the 74 participants (61.6%) who named the function as something other than the exponential function, 54 of them (45%) stated that it was

a line graph. Despite that some participants such as **Participant VV** had not drawn a line graph (as seen in Figure 5.27 (p. 91) above), they stated that the graph was linear. Figure 5.28 below shows **Participant UU's** naming of the type of graph.
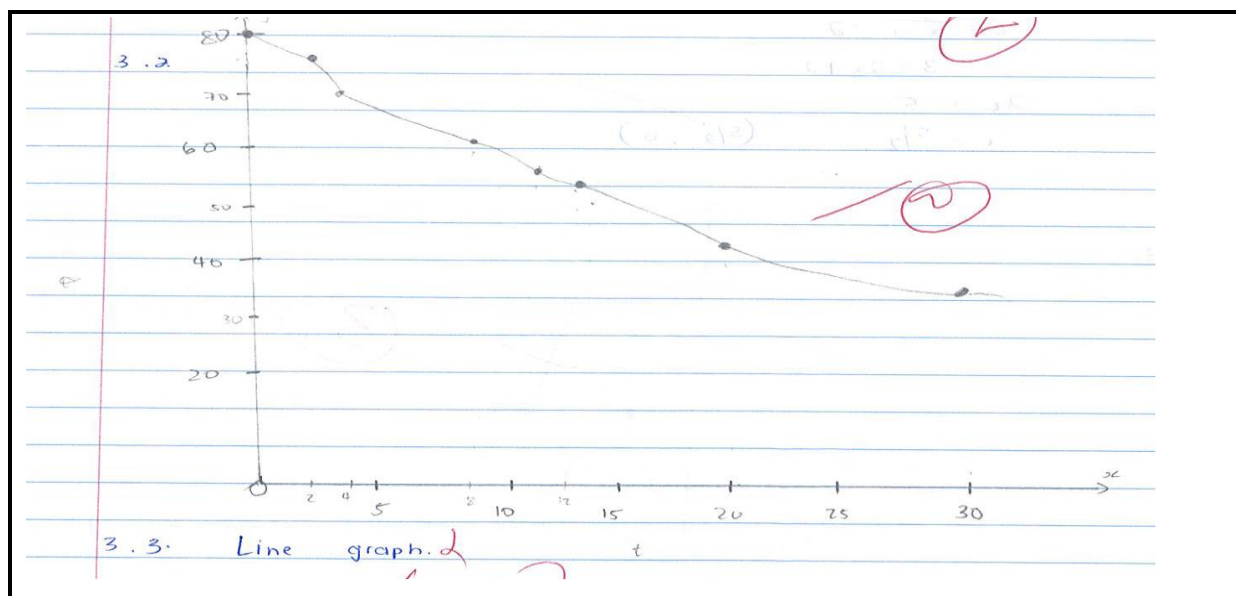
Figure 5.28: Participant UU's response to naming the type of graph

Participant UU's graph does not look like a line graph, although she names it as such. A possible reason for misnaming the graph was that she could not find any regular function which could fit the graph. Forty five participants (40%) named the function as exponential. In Figure 5.29 below, the graph was named correctly by **Participant W**.
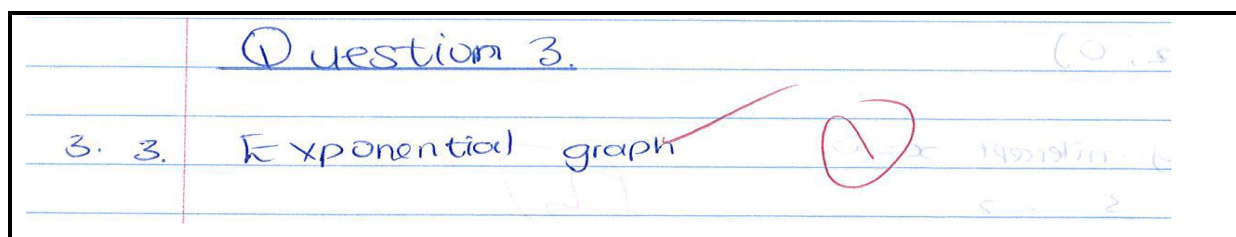
Figure 5.29: Participant W's response to naming the type of graph

While **Participant W's** response is acceptable in Figure 5.29 above, she did not give a reason for her particular response. It was very difficult to speculate as to why she decided to name the function as exponential. Only 31 (25.8%) of the participants named the asymptote correctly, while the rest of the participants (89, 74.2%) did not. The participants did not provide reasons for their choice of answers. The researcher noticed that the number of participants who responded positively decreased as the questions increased.

An asymptote is characterised by the graph's behaviour as the x or y values approach infinity. At infinity, the graph approached a straight line. In case of the cooling curve of coffee, an asymptote approached 20℃. As indicated by the wording in question 3, the cooling curve meant that as time increases, the temperature drops to 20°Celcius. Only 9% (n=11) of the

participants were able to provide an explanation that was close to an acceptable response. Figure 5.30 below is a representation of **Participant WW's** responses to question 3.5: *What is the real-life meaning of the asymptote?*
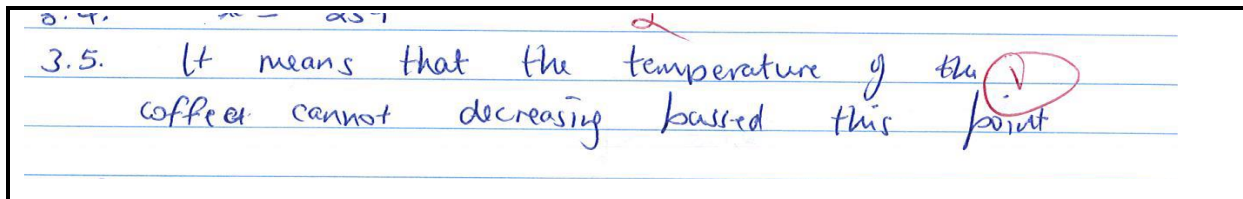
Figure 5.30: Participant WW's response to the real-life meaning of the asymptote

Participant WW indicated that the temperature of the coffee "should not pass this point", instead of stating that "the temperature of the coffee cannot be lower than the position of the asymptote". The point he referred to, was the asymptote. **Participant WW** spoke of an asymptote as though it was a point, when it was in fact a line. His view was that the closer the temperature was to the asymptote, the lesser the possibility of the temperature decreasing beyond that point. This shows further that **Participant WW** only understood the asymptote partially. While he knew that the graph would not pass through the asymptote, **Participant WW** wrote as though there was a point known as the asymptote. In the same vein as **Participant WW, Participant XX** also mentioned the word "point". Figure 5.31 below represents **Participant XX's** response to question 3.5 (the real life meaning of the asymptote).
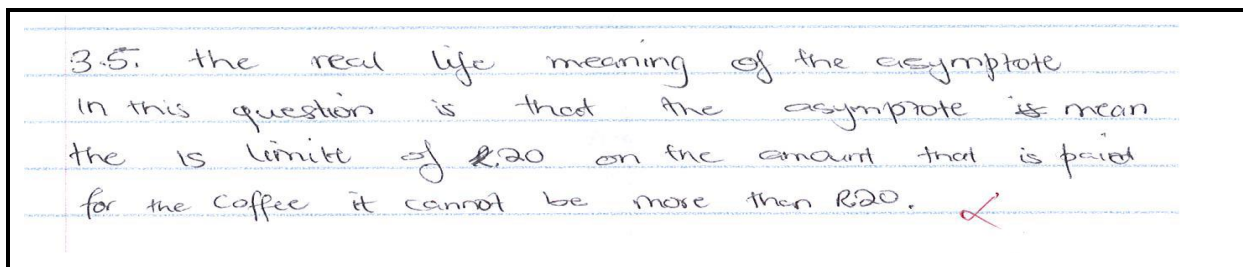
Figure 5.31: Participant XX's response to **the real life meaning of the asymptote**

An extrapolation of Figure 5.31 above indicates that **Participant XX** took θ to represent the amount of money to be paid for the coffee, instead of the temperature. **Participant XX** stated that the amount of money had a limit at R20. 00. In her explanation, **Participant XX** stated that the price of coffee should not be more than R20.00. **Participant XX** drew a graph with a decreasing exponential function, but she wrote the price of coffee as not exceeding R20. 00. This contradiction indicates ritualised routines, in terms of which participants acted without thorough meaning-making of what was actually happening (Gcasamba, 2014).

The following figure (Figure 5.32, p. 94) further indicates participants' deviation from the asymptote. In this regard, **Participant GG's** response to the meaning of the asymptote in the context of cooling coffee had nothing to do with the asymptote

Figure 5.32: Participant GG's response to the real-life meaning of the asymptote

Participant GG described the phenomenon of what happened to the coffee over time, which had no direct bearing or effect on the asymptote. This particular participant provided the correct asymptote for question 2.2 (writing the equation of the asymptote) and drew relevant asymptotes for question 1.3 (sketching the asymptote of the graph), as reflected in Figure 5.33) below.
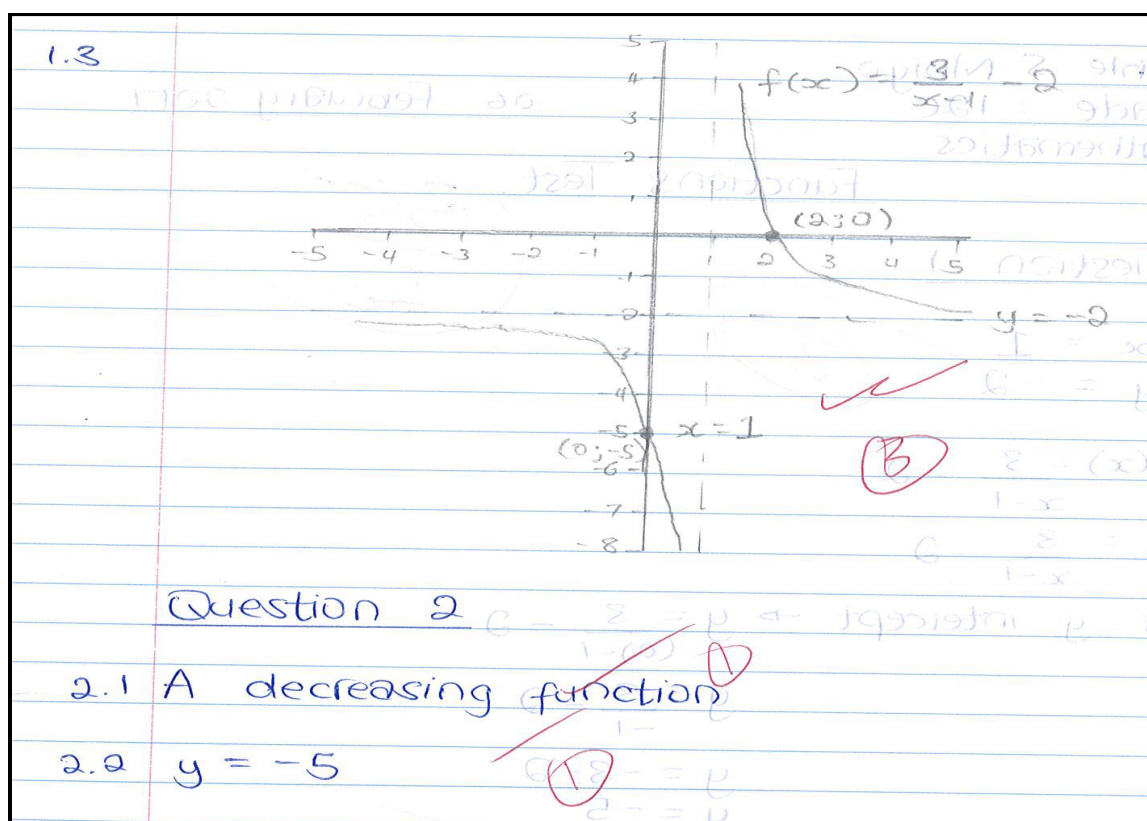
Figure 5.33: Participant GG's response to sketching the asymptotes

In Figure 5.33 above, **Participant GG** could sketch asymptotes and state their equation in the hyperbola and the exponential function. Questions 1.3 (sketching the graph) and question 2.2 (writing the equation of the asymptote) represented the daily mathematical realities that participants were exposed; while question 3.5 (the real-life meaning/ implications of the asymptote) represents an unseen question. Asking learners to explain the meaning of an asymptote from a contextual situation was not a regular occurrence. Failure to respond to application questions reinforced the view that participants' mathematical discourse was mostly ritualised routines. This question had the least number of positive responses of all the questions in the test.

5.4.2 The intersection of two graphs

Question 4 of the test required participants to sketch both an exponential and a quadratic function on the same set of axes by means of the global method. When using the global method, sketching the graph is undertaken by calculating key features such as intercepts, turning points and asymptotes. A global method is applicable when participants have been grounded in the plotting of graphs. In addition they should knowing the kind of shape to expect from each equation. In this case, the question required the sketching of the graph by using key features. About 60% (n=72) of the participants successfully responded to the questions. The quadratic function was better responded to than the exponential function was.

While participants could sketch the two graphs (the parabola and the hyperbola), the interpretation thereof was not as impressive. Only 20 (16.7%) of the 120 participants could interpret the intersection of the two graphs as the solution to the equation $-x^2 + 4x + 12 = \left(\dfrac{1}{2}\right)^x - 1$. Most of the participants tried to solve it algebraically, but failed due to their inability to simplify the exponent x. The question required participants to mark the solution using the letters A and B. Figure 5.34 below represents **Participant R's** sketching of the exponential and quadratic functions. **Participant R's** work was an example of participants who succeeded in completing the task of sketching the two graphs and showing the location of the solution to the equations. Although the graphs do not look smooth, they conveyed the essential message.
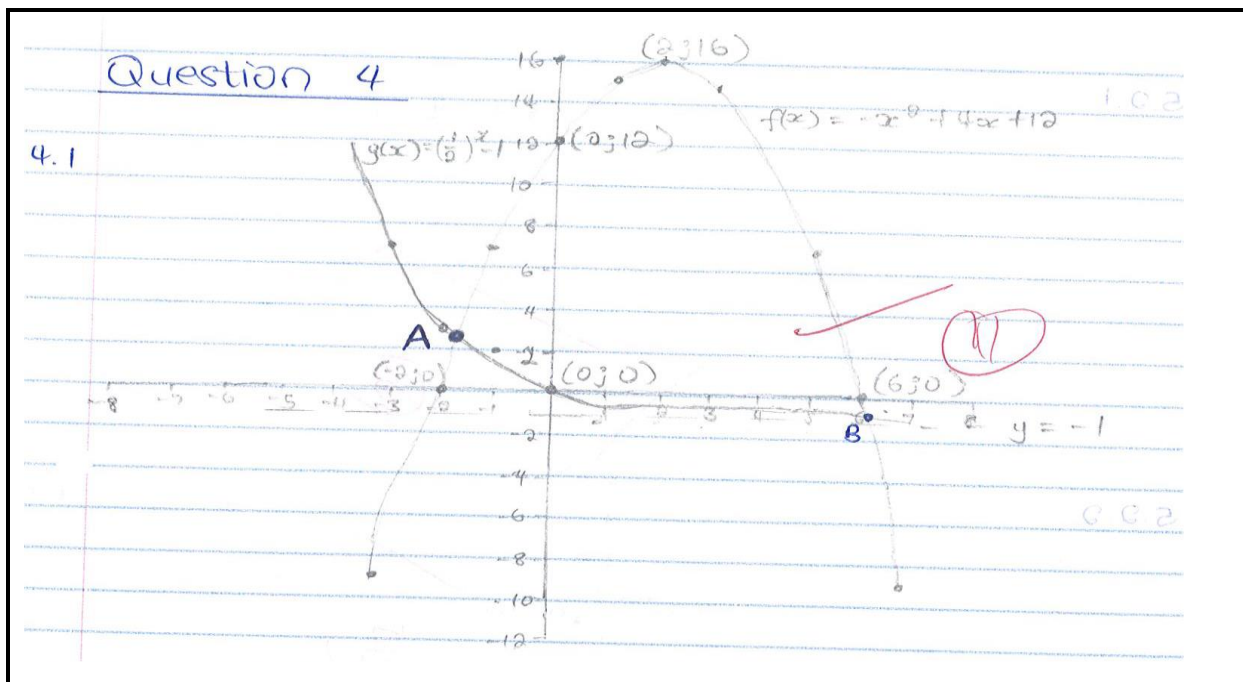
Figure 5.34: Participant R's sketch of 2 graphs and location of the equation

Figure 5.34 above indicates that **Participant R** had many coordinates on her graph. This reflects a state of uncertainty regarding the shape of the graph. The interpretation of the graphs

by the community of mathematicians was the same. Using the DPHEF analytical tool, the iconic visual mediator was considered as represented, since **Participant R** showed all the key features of the two graphs; viz: the turning points, the intercepts, the asymptotes, and the shape of each graph. The researcher also classified her use of routines as "applicability", based on her correct use of the key features when producing the two graphs (Ben-Yahuda et al., 2005; Kidron, 2011).

The solution to the equation $-x^2 + 4x + 12 = \left(\dfrac{1}{2}\right)^x - 1$ is at the intersection of the two graphs.

The point of intersection was the solution because at that point, the x and y values of the two graphs coincide. **Participant R** marked the point of intersection of the two graphs in bold letters. Her iconic visual mediator was "construed" because she could tell the point of intersection of these two graphs was the solution to the two equations. **Participant R** displayed flexibility routines because she demonstrated there was more than one way of solving equations (Ben-Yahuda et al., 2005). Other participants in the study did not respond to the question in the manner that **Participant R** did. Figure 5.35 below illustrates an almost similar response of a participant whose graph is well sketched, but with few details.
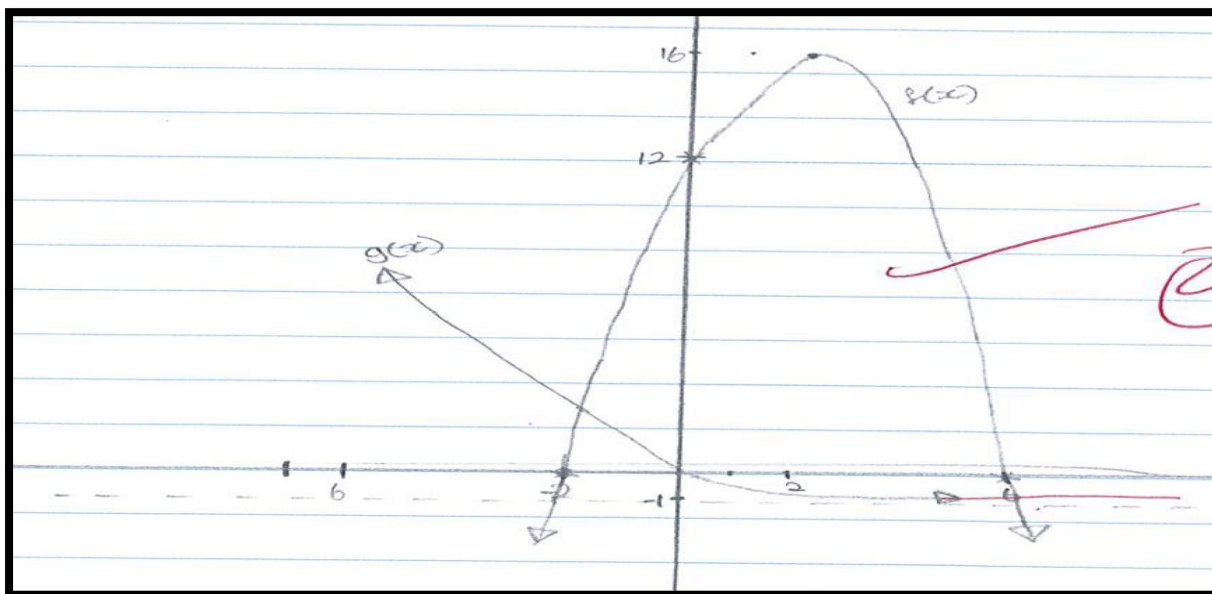
Figure 5.35: Participant W's sketch of 2 graphs and location of the equation

In Figure 5.35 above, **Participant W's** response is almost the same as that of **Participant R** in Figure 5.34 (p. 95). The difference between these two participants is only that **Participant W's** graph was not as detailed as **Participant W's**, whose graphs elaborately showed the key features of the quadratic and the exponential functions. Unlike **Participant R** who showed each of the key coordinates of the graph, **Participant W** further located her points on the axes. **Participant W's** diagram is an example of good sketching, because her graph was not overly crowded or cluttered. The graph also demonstrates that she knew the shapes of the two

functions and did not rely on plotting coordinates. Using the DPHEF analytic tool, **Participant W's** mathematical discourse on sketching of the graph was characterised as "depicted", based on her competent sketching of the graph. To the extent that he did not rely on plotting points for her shapes, displaying confidence and knowledge, Partipant W's routines are then characterised as "routine".

While **Participant W** exhibited flexibility in sketching her graph, she did not indicate the solution to the point of intersection of the two graphs. During the interview, when the researcher asked her why she did not respond to question 4.2, she responded that she did not know how to solve the equation. She was able to draw the graphs, but was unable to explain that the interpretation of the graphs indicated ritualised routines. **Participant W's** routines were "ritualised", because she did not realise that the point of intersection guided the solution to the graphs.

Participant YY was one of the few participants who sketched the parabola well, but experienced difficulties regarding the exponential function. Figure 5.36 below is an illustration of **Participant YY's** sketch of the parabola and the exponential function.
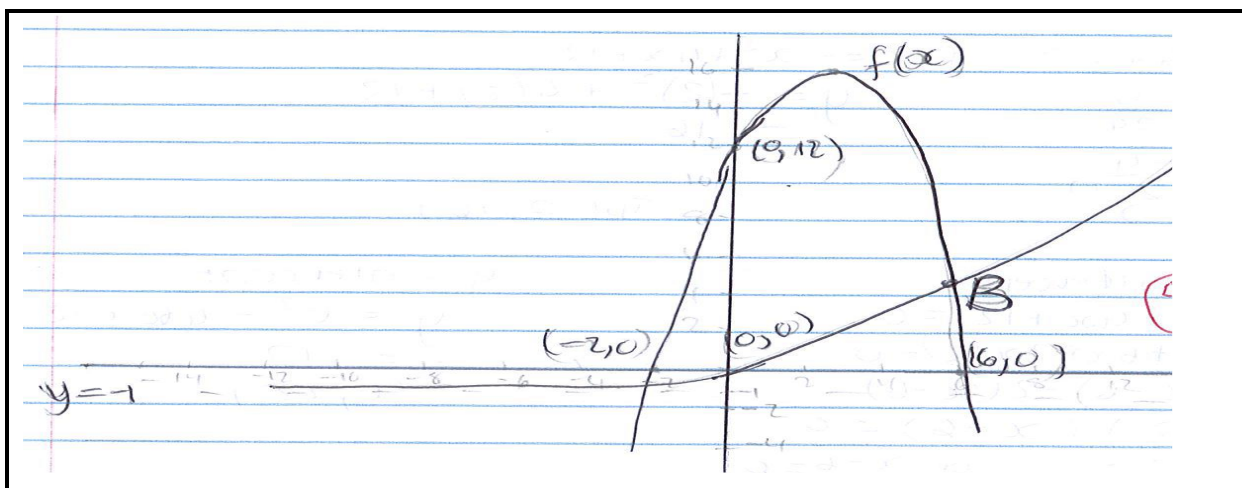
Figure 5.36: Participant YY's sketch of 2 graphs and location of the equation

In Figure 5.36 above, **Participant YY** sketched the parabola showing the intercepts and the turning point, although she did not show the coordinates of the turning point. **Participant YY** identified and drew the asymptote of the exponential function $y = -1$. She also marked the intercept with the axes, which was the point of origin. **Participant YY** went on to draw an increasing function. She did not check the correctness of her work by finding another point on the graph that would have acted as a guide. Another point would have helped her recognise that the graph was decreasing. The action of not checking the correctness of the answer (or lack of it) indicates lack of flexibility in her routines.

In Figure 5.36, **Participant YY** marked one of the points of intersection B, which indicates that she knew the point of intersection of the two graphs was the solution to the algebraic

equation of these two graphs. However, **Participant YY** did not mark the other point of intersection. When the researcher asked for the reason, she stated that she forgot to write it. She further pointed to the solution on the graph, indicating that she knew how to find the solution for the equation. In Figure 5.37 below, **Participant YY's** calculation of the intercepts is reflected.

Figure 5.37 Participant YY's calculation of the intercepts

In Figure 5.37 above, there are two calculations for the intercepts of the exponential function. **Participant YY's** calculation was correct, but there were some evident elements of ritualised routines. After calculating the *x-intercept* and getting a zero, **Participant YY** should have known that the coordinate (0; 0) is both the *y* and *x* intercept. The ritualised aspects of **Participant YY's** work would have shown some flexibility if she had checked the correctness of her answer (Ben-Yahuda et al., 2005). In the next few paragraphs, the researcher shows examples of participants who tried to solve the equation $-x^2 + 4x + 12 = \left(\frac{1}{2}\right)^x - 1$ algebraically, as shown in Figure 5.38 below.

Figure 5.38: Participant AA's response to intersection of graphs

In Figure 5.38 above, **Participant AA** tried to solve the equation $-x^2 + 4x + 12 = \left(\frac{1}{2}\right)^x - 1$ algebraically, yet the instruction in the question had clearly stated they should mark the solution on the graph. **Participant AA** eventually had the exponent x as a denominator. There is no mathematical explanation to support his action. In short, **Participant AA** moved from an exponential function to a fraction, and then concluded with a linear equation. In this regard, **Participant AA's** routines were non-mathematically ritualised since he just wrote mathematical statements that were neither coordinated nor supported by mathematically endorsed narratives.

On the whole, the collective responses indicate that participants could perform routines that were practiced in the classroom - such as sketching graphs - but experienced challenges with 'unseen' questions. It was not common in textbooks to solve equations using the graphical method. Most of the equations were algebraically solved, hence participants like **Participant AA** opted for the algebraic method. It was also observed that participants did not read instructions carefully. The questions did not require the participants to solve the equation, but rather to mark points at which they expected the solution to be on the graph. On the whole, while participants performed well in the sketching of graphs, they were not as efficient and able in transforming them.

5.5 Conclusion

In this chapter, the researcher discussed the mathematical discourse of participants in the functions test using the Discourse Profile of the Hyperbola and the Exponential Function (DPHEF). The researcher also examined the selected participants' mathematical discourse on the transformation of identified graphs. The South African Mathematics curriculum in Further Education and Training requires the learning of transforming functions. Notwithstanding the grade 11 curriculum requirements, participants in this research generally displayed a sense of developed mathematical discourse on the graphical, numerical and algebraic representation of familiar questions, but experienced difficulties with regard to questions requiring interpretation of the said representations.

During the interview sessions, the grade 11 research participants often described the functions or some aspects thereof in terms of the translations. Participants' mathematical discourse on the hyperbola was more objectified than that of the exponential function. Participants could calculate and identify the intercepts and asymptotes of the hyperbola respectively, yet less than 45% of the participants (n=50) could identify the asymptote of an exponential function. Although most of the research participants referred to the shifts when presented with questions on translations, only a third (33%, n=37) responded positively. About half of the participants did not respond to the questions on translation. One of the reasons for this slow response trend

is that the question had three unknowns, two variables and *k* as numerator to the fraction. For most of the questions to which the participants responded, there would be only two unknowns, *x* and *y*. Only a sixth (16%, n=19) of the participants responded positively to the question on the sketch. While two thirds (67%, n=75 of the participants correctly calculated and found the value of *k*, they still worked on their responses with reference to *k,* even though they equated the value of *k* to 3 (three).

While two thirds (67%, n=75) of the participants sketched the hyperbola, the exponential function and the parabola, the interpretation of components of the graphs was not as high. About 10% (n=11) of the participants could explain the meaning of the asymptote in a real-life problem. About 60% (n=72) of the participants sketched intersecting graphs, but only 18% (n=22) could interpret the meaning of the point of intersection.

Interpretation of both the algebraic and graphical functions discourse showed that the mathematical discourse of participants had not yet reached objectification levels. The participants' unfamiliarity with some of the terms or expressions used, tended to somewhat pose a challenge to their correct interpretation of some questions' requirements. The participants' routines were generally "ritualised", since they worked well with familiar questions. Where a twist was introduced, some of the participants could not respond in a manner they did in respect of familiar questions.

In the next chapter (Chapter 6), the researcher explores and describes the participants' mathematical discourses in the context of the asymptote of a hyperbola and the exponential function.

CHAPTER 6
REPRESENTATION OF THE ASYMPTOTES OF A HYPERBOLA
AND EXPONENTIAL FUNCTIONS

6.1 Introduction

In this chapter, the researcher discusses the use of the word 'asymptote' as represented from the perspectives of the 24 participants in this study. Participants' test and written responses from the task-based interviews showed that they could identify an asymptote from a graph and/or formula with ease. Their asymptote narratives did not necessarily indicate the objectification of this mathematical object (the asymptote). The researcher categorised the use of the term asymptote into four sections.

Firstly, the researcher discusses *how* the participants *named* the asymptote. Their naming of the asymptote was facilitated through a linear equation. While there was evidence of identifying an asymptote from the equation and the graph, some participants' naming of the asymptote was difficult to locate on the Cartesian plane. Objects on the Cartesian plane were identified by means of either their coordinates or the equation of the line defined.

Secondly, the researcher discusses the participants' *representation* of the asymptote on the Cartesian plane. On the graph, an asymptote was represented as a linear function, yet some of the participants spoke as though the asymptote was a point.

Thirdly, the asymptote was *viewed as a boundary* that blocked the graph from passing through the Cartesian plane. The asymptote is a line approached by the graph of a function as *x* or *y* tends to infinity (Denbel, 2015). The asymptote does not block the graph. It is the behaviour of the graph that results in some graphs having an asymptote. In some rational functions, there is an intersection between the asymptote and the graph. Therefore, the notion that the asymptote would never interact with the graph is not always true for all functions.

Fourthly, the term 'asymptote' was *used synonymously* with the notion of 'undefined'. At the zeros (0s) of the denominator, rational functions are undefined (Denbel, 2015; Gcasamba, 2014). Some of the study participants expressed that whenever the output in a function was undefined, the graph would then always have an asymptote. In some rational functions, it would be a removable discontinuity. In this study, participants tended to describe graphs in terms of what they saw from the very graphs. There was little evidence of a concerted exploration of the mathematical objects. The overall study results provide an indication that participants' mathematical discourse was *in status nascendi*; that is, still in a state of development. Correspondingly, their mathematical communication (discourse/ narration) was not yet at the level which could justifiably be endorsed by the community of mathematicians.

6.2 Naming of Asymptotes

An asymptote is a graph towards which another graph tends as *x* or *y* approaches positive or negative infinity (Denbel, 2015; Gcasamba, 2014). In this study, the asymptote is represented as a linear graph coinciding with, or parallel to either of the axes. In naming the asymptote, an equation depicting a linear function is the only acceptable way of representing an asymptote in algebraic form (Flesher, 2003). The naming of an asymptote as a number has stood out in every task performed by the selected participants during the task-based interviews. The tasks in this study involved functions in algebraic form, in tabular form, in word form, and in the form of the graphs themselves. In all these instances, participants used the algebraic form and identified the asymptote as though it was a number. For example, they would state that the asymptote was negative nine (-9). While some of the participants would correct themselves and name the asymptote in the form of an equation and name it as *y = -9*, most of them named the asymptote as a number. In this section, the researcher used task 1 (one) of question 2 (two), which reads: *What is the asymptote of $f(x) = 3^{x+1} - 9$? How do you know this?"* as an illustration of how the asymptote was named by participants in this study.

All participants identified the asymptote from an equation $(x) = 3^{x+1} - 9$. The 24 participants who took part in the interviews represented the five best performing participants who wrote the functions test. The interviewed participants consisted of the top six learners in each school who took part in the study. Six of the participants (25%) named the asymptote as *y* is equal to negative nine $(y = -9)$, which is an equation. Eighteen participants (33.3%) named the asymptote as though it was a number. For example, they would mention that "the asymptote is negative nine". When an asymptote is named as as a number, it becomes difficult to identify the object referred to on the Cartesian plane.

The researcher has included Table 5.1 (see. p. 70) as a summary of learners' responses to the question: *What is the asymptote of $f(x) = 3^{x+1} - 9$. How do you know this*? While all participants responded to the first part of the question (in which they had to name the asymptote of the exponential function), the same could not be said for the second part of the question (in which they explained the narratives of arriving at the asymptote).

In Table 5.1, the first column represents the names of the participants, who were grouped according to their answers to questions. For example, the first eleven participants responded to the question in the same manner as they did in all the five parts of the DPHEF analytical tool. The second column represents the classification of how words were used. The word use is classified as "literate" in the event that it is universally accepted in the community of mathematicians; for example, when the asymptote is represented as" *x* is equal to 2 $(x = 2)$". In the event that the words used are prone to a different interpretation by the community of

mathematicians, then the particular words used are classified as "colloquial"; for example, in the event that the response is: "the asymptote is negative nine (-9)". The mediator could be either symbolic or iconic. The symbolic mediator includes numbers written in symbols, equations, signs and so on. On the other hand, the iconic mediators are diagrammatic representations incorporating graphs and tables.

The researcher used the term "construed" to depict a situation in which the mediator is correctly interpreted. In the event that a participant responded that the asymptote is negative nine (-9), or that the asymptote is ($y = -9$), the researcher classified both as "construed". In the event that the participant's response did not show a correct interpretation of the mediator, it was classified as "not construed" by the researcher. In such cases, the participants would have responded that the asymptote x is equal to 1 ($x = 1$), or the asymptote is 1 (one). The term "not construed" represents situations in which the mathematical object is interpreted incorrectly. On the fourth column of Table 5.1, the kinds of routines are represented, which are repetitive steps that are performed on the mathematical object. They include procedures for solving mathematical problems.

Routines are categorised into a typology of three, viz: exploratory routines, ritualised mathematical routines, and the ritualised non-mathematical routines. Exploratory routines refer to the participants' ability to go beyond procedures (Howie, 2003). These routines also explain and verify endorsed narratives. An example would be: *What is the asymptote of $f(x) = 3^{x+1} - 9$?. How do you know this*? This can also be represented to as: "The asymptote is y is equal to negative nine ($y = -9$)". As the *x-values* approach infinity, the graph tends to y is equal to negative nine.

The second kind of routines are the ritualised mathematical routines. In the second category, participants correctly identified the mathematically accepted responses, but could not explain or provide a reason for their particular responses. They only knew the *what* part, as opposed to the *why* (Sfard, 2008). For example, in the event that a participant responded by only stating that the asymptote is y is equal to negative nine without providing reasons for it. The third category is the ritualised non-mathematical routines, which occur in the event that the response is not mathematically acceptable. For example, in the event the participants mention that the asymptote is y is equal to 1 (one) for the function $f(x) = 3^{x+1} - 9$.

According to Ben-Yahuda et al. (2005), properties of routines include applicability, flexibility and corrigibility. In this study, the researcher refers to applicability, flexibility and corrigibility as the use of routines, because these properties of routines show the approach which participants use to engage with mathematical problems. Applicability routines occur when

certain routines are likely to be produced (Mpofu & Pournara, 2018).). The most dominant applicability routine in this study has been the visual trigger. Compared to any other use of routines, participants used what they saw in order to respond to mathematical situations. Flexibility routines occur when either one or more procedures are produced for a solution. The second and subsequent procedures are for checking the authenticity of the first routine by arriving at the same endorsed narrative. Flexibility routines are also noticed in unfamiliar tasks where participants work on tasks they have not seen before; but can use previously endorsed narratives to solve a problem. Flexibility routines can be communicated by using different representations or visual mediators to explain endorsed narratives.

The last column of Table 5.1 represents the narratives, which are exemplified by substantiations, memorisation, and authority categories. Authority narratives are those in which the reasons for actions taken are attributed to the teacher or the textbook as the authoritative source. For instance, when **Participant S** stated: "the teacher marks it wrong when we don't write it like that", or "that is how it is written in the textbook". Only a few participants specifically referred to the authority narratives. Memorisation narratives represent generalisations which are based on memory. There are two branches of the memorisation narratives. These are reference to the formula or rule and visuals. An example of reference to a formula is represented in the statement by **Participant S**: *The asymptote is negative nine because it is the q in the equation*. For visuals, the participants would refer to the graph not intersecting the asymptote. When participants' narratives match those of the community of mathematicians, the narratives were classified as "substantiations" by the researcher. Table 6.1 below represents a summary of the naming of the asymptote of an exponential function.

Table 6.1: Summary of the naming of the asymptote

Name	Word Use	Mediator	K. Routine	Use Routines	Narratives
1 *Participant CC*, *Participant M*, *Participant HH*, *Participant KK Participant EE Participant S Participant DD Participant JJ Participant AA Participant QQ Participant BB*	Colloquial	Construed	Ritualised non-mathematical	Applicability	Memorisation
	Non-mathematical naming	Interpretation	Non-mathematical language	Visual trigger	Formula
2 *Participant R*	Literate	Construed	Ritualised mathematical	Applicability	Memorisation
	Mathematical naming	Identification of an asymptote	Does not give a reason	Visual trigger	Visual

Name	Word Use	Mediator	K. Routine	Use Routines	Narratives
3. *Participant Y* *Participant II*	literate	Not construed	Ritualised non-mathematical	Applicability	Memorisation
	Mathematical naming	Non-mathematical interpretation	Non-mathematical interpretation	Visual trigger	Rule
4 *Participant GG* *Participant W*	Colloquial	Not construed	Ritualised non-mathematical	Corrigibility	Memorisation
	Non-mathematical naming	Non-mathematical interpretation	Non-mathematical naming and mathematical language	A bit of corrigibility with prompts	Rule
5 *Participant OO*	Colloquial	Construed	Ritualised non-mathematical	Applicability	Memorisation
	Non-mathematical naming	Interpretation	Non-mathematical language	Visual trigger	Visual
	Does not name	Wrong interpretation	Does not respond to the question	Description of visuals	Visuals
6 *Participant FF*	Colloquial	Construed	Ritualised non-mathematical	Applicability	Memorisation
	Non-mathematical naming	Interpretation	Non-mathematical language	Visual trigger	Visuals
7 *Participant U, Participant Y, Participant NN*	Literate	Construed	Ritualised mathematical	Applicability	Memorisation
	Mathematical naming	Interpretation	Explains the process	Visual	Rule
8 *Participant Z*	Literate	Construed	Exploratory	Flexibility	Substantiations
	Mathematical Naming	interpretation	representation	Uses another form	Uses graph as justification
	Mathematical naming	Interpretation	Mathematical	Visual	Formula

Of the six (6) participants who named the asymptote mathematically, 4 (four) expressed their responses in a manner acceptable to the community of mathematicians. These four participants identified the asymptote from the equation $f(x) = 3^{x+1} - 9$, and then expressed the asymptote in such a manner that it could be located on the Cartesian plane. **Participant R's** response to the above equation was unambiguous. On being asked about the asymptote of an exponential function, **Participant R** responded thus: *The asymptote of f is y = -9.*

An asymptote of an exponential function is a straight line. **Participant R's** naming of the asymptote is acceptable to the community of mathematicians because the linear equation could

be located on the Cartesian plane. Therefore, the linear equation *y* is equal to negative nine $(y = -9)$ is a true identification of the asymptote as an equation. In this regard, the linear equation *y* is equal to nine has a graphical equivalent representation, a horizontal line parallel to the *y*-axis and passing through *-9* on the *y* axis. The word use is classified as "literate" due to the interpretation of **Participant R's** equation of the asymptote being universally accepted by the mathematical community in the meaning of that statement. This statement could be expressed in different representations of the functions. The question asked is in the form of a symbolic visual mediator, which is a mathematical representation that is in the form of a symbol (Adler & Ronda, 2014). **Participant R** identified the asymptote from $f(x) = 3^{x+1} - 9$.

Participant R interpreted the symbolic visual mediator of the equation of the exponential function in the same manner that would be acceptable to the community of mathematicians (Flesher, 2003). According to the DPHEF analytical tool, the visual mediator is classified as "construed" when participants' interpretation of that visual mediator is the same as that of the community of mathematicians. **Participant R's** response is also classified as "literate", as she could relate a symbolic visual mediator to the iconic visual mediator. An iconic visual mediator is a mathematical representation that is in the form of a diagram, such as graphs and tables of values. **Participant R** responded that: *The asymptote never touches the graph never touches the asymptote and it does not pass.*

Instead of **Participant R** explaining her response according to the formula or the equation, she referred to the graph. Using different representations of a functions illustrates **Participant R's** developing mathematical discourse of functions. When asked to provide a reason for her answer, she responded: "the asymptote never touches the …graph". To state that the asymptote never touches the graph is not mathematical, because this definition would change in the event that participants are exposed to functions in which the horizontal asymptote intersects as stated in the literature review (Kuptsov, 2001). Horizontal asymptotes could intersect with the graph in some functions. Her assertion that the graph and the asymptote would never intersect is based on empirical evidence, rather than a result of mathematical explorations. An asymptote is a result of the behaviour of the graph at extremes. It is to be noted that it is not the asymptote that controls how the graph behaves (Mpofu & Pournara, 2018). Her routines are classified as "ritualised". They are based on what she sees rather than on mathematical reasons.

Ritualised routines are performed mathematical procedures that are based on what other people do (Ben-Yahuda et al, 2005). While it is true that the graph and the asymptote will not intersect in exponential functions, it is not true of all functions. The South African FET functions do not include functions in which the horizontal asymptote intersects with the graph. Applicability

routines are premised on the use of mathematical decisions taken to arrive at a procedure. In other words, applicability routines are based on the likelihood of certain routines being produced. The visual trigger means that the mathematical procedure was influenced mainly by what **Participant S** saw. In this case, **Participant R's** use of routines was classified as "applicability" because she only used the visuals and not procedures at her disposal to explain her answer. She could have used other representations such as the table of values to explain the behaviour of the functions as *x* becomes smaller and approaches negative infinity.

According to the DPHEF, memorisation narratives are generalisations that are based on what the participant remembers. **Participant R** may not have seen graphs intersecting with the asymptote, consequently assuming that all graphs would not intersect with the asymptote. **Participant R's** narratives then become memorisations based on visuals, because she remembered that there were no intersections between the asymptote and all the graphs she worked on. Endorsed narratives are based on mathematical explorations (Mpofu & Pournara, 2018). They are arrived at after conjectures and proofs, as well as disproving those conjectures. She mentions that the asymptote would never touch the graph, but does not give reasons for her statement. As a result, the researcher classified her narratives as "memorisation based on visuals". Furthermore, **Participant R** was not specific in her generalisation. In general, an asymptote may, or may not intersect with the graph.

The other learners who named the asymptote mathematically were **Participant U, Participant Y,** and **Participant NN**. The researcher grouped these participants because their utterances were similar when using the DPHEF analytical tool. The common factor in their responses was that they used the rule relating the asymptote to the vertical shift. When functions are expressed in the standard form of $f(x) = a^{x+p} + q$, the parameters p and q (which show the vertical and/or horizontal movement from the parent function movement), would have their magnitude equal to either the horizontal or the vertical axis. For an exponential function and the hyperbola, the asymptotes of the parent function coincide with the axes. Some participants tended to associate the asymptote with the horizontal or vertical movement of the graph. They explained the presence of the asymptote in terms of vertical or horizontal shifts, rather than the behaviour of the graph.

While the three participants (**Participant U, Participant Y** and **Participant NN**) eventually named the asymptote mathematically, they vacillated between mathematical language and colloquial terms. **Participant U** referred to an asymptote of x in an exponential function, which gave the impression that **Participant U** was referring to the vertical asymptote; when he had in essence confused the horizontal asymptote and the vertical. When asked by the researcher to show that on a diagram, he pointed to the horizontal line, which was an indication that he did not mean there was a vertical asymptote. In an exponential function with base greater than

1 (one) as x approaches positive infinite, the graph approaches infinity and it grows exponentially. In a situation where the base of the exponential function is between 0 (zero) and 1 (one) as x gets smaller (approaching negative infinite), the graph also approaches negative infinity. This means there is no vertical asymptote in an exponential function. The only asymptote in any exponential function is horizontal. The asymptote is named a vertical asymptote rather than the asymptote of x. In this regard, **Participant U** stated: *From the asymptote of 'x' here is zero, y is equal to negative nine ($y = -9$). It is the values of 'q' in the equation of the graph of f*.

While **Participant U** names the asymptote mathematically, the narrative is that of an x asymptote. There is an *x-coordinate,* but there is nothing known as the x asymptote. His narrative seems to suggest that there is a vertical asymptote. Further probing revealed that **Participant U** did not mean a vertical asymptote, but rather that there was no asymptote at all. **Participant U** responded to the question: *Why are you stating the asymptote of x is zero?* by stating: *It is zero because there is no asymptote*. The function in question is exponential, as explained earlier. Only 1 (one) asymptote was expected. For **Participant U**, there seemed to be no difference between an asymptote in which x or y are equal to zero, and a situation in which there is no asymptote at all. An exponential function has only 1 (one) asymptote, which is horizontal. While his naming of the asymptote was mathematical (y= -9) his utterances were not mathematical. Using the DPHEF analytical tool, the naming of the asymptote was classified as "literate", but his reference to the asymptote of x as zero was colloquial because there is nothing known as "the asymptote of x". The second characteristic of an exponential function is that it has a horizontal asymptote. The work of **Participant U** is referred to in this context.

Participant U could interpret the symbolic mediator $f(x) = 3^{x+1} - 9$. He could identify the asymptote from the equation or formula. Accordingly, the researcher classified his interpretation of the symbolic visual mediator as "construed", because the community of mathematicians would have come to the same conclusion as his. **Participant U** identified function f as an exponential function whose asymptote is y is equal to negative nine ($y = -9$).

Participant U's routines are ritualised, despite his identification of the asymptotes of f. He also gave the impression that the exponential function has two asymptotes. **Participant U** refers to the asymptote of x as zero. The asymptote could not be an asymptote of x or y. It is the asymptote of a function, and not of the axes or part of a coordinate. The asymptote is related to a function, it is not an asymptote of a coordinate or of an axis (Kuptsov, 2001). Although he managed to identify the horizontal asymptote*, Participant U* also gave the impression that there was an expectation of the vertical asymptote. Using the DPHEF, the researcher classified **Participant U's** routines as "ritualised", as he did not explain the presence of the asymptote in terms of the behaviour of the graph, but as a consequence of the vertical shift. His use of

routines was again classified as "applicability based on visual trigger", because he named the asymptote mainly from what he saw. **Participant U** did not explain the behaviour of the graph as x approaches negative infinity. He only refers to shifts. His routines were not based on any mathematical procedures, but only on what he saw.

In his explanation of the asymptote, *Participant U* refers to the y asymptote and the graph not having values. **Participant U's** first language is isi-Xhosa. Speaking in English was somewhat problematic. The medium of communication and instruction is instrumental in the participant's knowledge and understanding of routines. In this regard, the table of values, the ordered pairs, the input and output values, as well as the *x* and y coordinates necessitate expressions that are accepted by the community of mathematicians. Coordinates are a set of values which show an exact position. Therefore it is impossible for a graph not to have coordinates. Following are **Participant U's** responses to the table of values.

Participant U: *I was going to state sir, the graph sir, when it doesn't have the values but when you are given the format of the graph, it is given that it is a to the power of x (a^x) and then n-q. so that 'q' is the value of the y asymptote.*
Interviewer: *What do you mean by value or values?*
Participant U: *Values are these…the x and y values.*

Participant U stated further that *q* was the value of the asymptote. A value in Mathematics carries several meanings. The value may refer to a number in a variable or mathematical object, or it may refer to a result of a mathematical computation (Ben-Yahuda et al., 2005). The mathematical object which **Participant U** refers to as the value or values is not clear. According to **Participant U's** knowledge of the asymptote and the function, value is misplaced. Further probing revealed that by "values", **Participant U** meant the coordinates. Graphs consist of many coordinates, otherwise the object being referred to would not be regarded as a graph. When **Participant U** mentions that the graph has no values, he refers to the part on the table of values of which the x-coordinate has no corresponding y-coordinate. According to **Participant U,** no values are related to the asymptote. Interpretation of the visual mediators lends itself to explanation of all phenomenon without adequate exploration or investigation. For **Participant U,** the constant represents an asymptote.

In an equation, the value of the parameter *q* represents a vertical shift from the parent function (Mpofu & Pournara, 2018). Explorations of the mathematical objects with asymptotes (i.e. the exponential function and the hyperbola) show that the equation of the asymptote is equal to the value of the constant (Flesher, 2003). In the event that the value of the constant is 6 (six), for instance, then the horizontal asymptote of the function will be *y* is equal to six *(y=6).* It is worth noting this does not mean the constant is an asymptote. The explanation for the presence of the

asymptote would still be the behaviour of the graph as *x-coordinates* approach either positive or negative infinity for the exponential function and both for the hyperbola. **Participant U's** narratives on the horizontal asymptote of an exponential function were classified as memorisation based on rules.

Participant U refers to rules which he followed to identify the horizontal asymptote. The prevalence of the constant means that there is an asymptote. Even though the answer may be correct and **Participant U** may score full marks in a test, his narratives are not based on proven mathematically acceptable facts or procedures. In addition, **Participant U** did not demonstrate beyond reasonable doubt that his reference of the horizontal asymptote to the constant of the exponential function was based on mathematical explorations. To a great extent, his narratives are an example of memorisation based on the rule of the constant, based on the horizontal asymptote of a hyperbola and an exponential function. He further states that there is a *y-asymptote,* instead of the horizontal asymptote.

Participant X is one of the participants who named the asymptote of an exponential function in a mathematically acceptable manner. She stated that the asymptote of $f(x) = 3^{x+1} - 9$ is $y = -9$. However, she was unsure of the number of asymptotes in an exponential function. The excerpt below illustrates **Participant X's** response to the naming of the asymptotes of $f(x) = 3^{x+1} - 9$.

Participant X: *Y is equal to negative nine (y=-9) ... it's always a point that you keep in mind for the original equation of an exponential or any type of function, you always need to keep it in mind, and then immediately. There is only 1 asymptote if am not mistaken.*

Interviewer: *How about the asymptote for the parent function y is equal to three to the exponent x ($y=3^x$)*

Participant X: *For two (2) or three to the power x (3^x) , there are no asymptote, because you do not have any shift*

Participant X names the asymptote mathematically. The researcher classified her utterances as mathematically based only on the naming of the asymptote. The implication is that although **Participant X** named the asymptote mathematically, her discourse on the mathematical object was still developing. **Participant X** is not sure of the number of the asymptote in an exponential function. Her own words show lack of confidence on the asymptote of an exponential function. She hesitantly states an exponential function has only 1 (one) asymptote. In trying to explain the relationship between the parent function *y=3^x* and $f(x) = 3^{x+1} - 9$, **Participant X** states the parent function has no asymptote. **Participant X's** reason for stating there were no asymptotes was that "you do not have any shift", suggesting that shifts or parameters were associated with the asymptotes.

Parameters are the units through which a function is transformed (Nair, 2010). For example, $f(x) = 3^{x+1} - 9$ x = -1 and y = - 9 are the parameters of the function. These parameters show the number of units horizontally or vertically from the parent function. She then states the "absence" of the parameters implies the absence of the asymptotes. An exponential function or a hyperbola with no parameters has asymptotes, because an asymptote in a function is determined by the behaviour of the graph (Nair, 2010. In a parent exponential function (function without shifts), the asymptote is *y* is equal to zero ($y = 0$). In a parent hyperbola, the asymptotes would be *y* is equal to zero ($y = 0$) and ($x = 0$). **Participant** X takes the parameters themselves to be the asymptotes. In fact, when the parameters are not written, that does not mean the parameters are not present, but just that they are equal to zero. Associating the asymptote with the parameters is problematic. It results in asymptotes not being noticed when the parameters are not written, or when there is a compound of shifts. The asymptote is a consequence of the behaviour of the graph, rather than a response to the shifts or parameters. For this reason, her word use is classified as "mathematical". The researcher classified her interpretation of the symbolic visual mediator as "not construed", because she interpreted the lack of parameters as implying that there were no asymptotes.

Participant X identified the asymptote from the symbolic mediator appropriately. She was able to interpret the symbolic mediator and its translation. However, **Participant X's** challenges show in the event that the exponential function is a parent function. She interprets it as a function without asymptotes. Her statement implies that asymptotes are the actual parameters. Her mathematical discourse on the asymptote of an exponential function is at a developing stage. She has not yet reached a stage where her discourse is at the same level as that of the community of mathematicians. While her interpretation of the symbolic visual mediator of an exponential function could have been classified as "construed", her statement on the interpretation on the parent exponential function renders her discourse as "not construed". Therefore, her routines are classified as mathematically ritualised with applicability (Ben-Yahuda et al., 2005). While her response on the original question was mathematically acceptable, further probing on the asymptote of the parent function of the exponential function revealed that she regarded the parameters as the asymptote. In essence, parameters are not the asymptotes as stated earlier, but may help in identifying the position of the asymptote on the Cartesian plane. Over-reliance on the parameters is a problem for participants because there is no consistency. In a hyperbola, both the vertical and the horizontal parameters are a hint of the asymptotes. In an exponential function, it is the vertical shift only. The mathematically acceptable answer is a result of aligning her routines with those of others, probably the teacher or what she has read from a textbook. In the event her mathematically

acceptable response emerged from explorations, she would have been able to recognise the asymptote in a parent exponential function.

Participant X could have realised that functions with parameters were a result of translating the parent function. She used the visual trigger to name the asymptote. Visual triggers are the hints that participants use to identify mathematical objects (Adler & Ronda, 2014). The parameters in this situation are the visual triggers which help participants to identify the position or equation of the asymptote. Visual triggers which are not grounded on mathematical explorations result in participants failing to identify mathematical objects when there is a change of form, such as when the function does not have shifts. When an exponential function is translated, she named the asymptote. She was not able to do so in a parent function. The researcher then classified her routines as ritualised, and the use of routines as applicability based on visuals.

According to **Participant X**, a transformed exponential function has an asymptote, but a parent exponential function does not have any asymptotes. There is evidence that **Participant X** could talk of the asymptote of a transformed exponential function, her generalisation was not based on endorsed narratives but on memorising what she has seen done in the community of mathematicians. **Participant X**'s narratives are memorisation based on rules. She could use the rules to obtain mathematically correct answers, but could not transfer that knowledge in a similar situation in a different representation of the same mathematical object.

Participant Y and **Participant II** named the asymptote mathematically, but their interpretation of the function was not acceptable within the community of mathematicians. Both participants used the mnemonics which helped them identify the asymptotes from the hyperbola and the exponential function. In a hyperbola represented by $f(x) = \frac{a}{x-p} + q$, the asymptotes are $y = q \ and \ x = p$, and in $g(x) = a^{x+p} + q$, the asymptote is $y = q$. **Participant Y** and **Participant II** confused these mnemonics and used p instead of q, which is the vertical shift. Instead of using the constant as the hint for a horizontal asymptote, they used the horizontal shift. **Participant Y** named the asymptote of $f(x) = 3^{x+1} - 9$ as $y = -1$, and Participant II named the asymptote as $x = -1$. When asked to explain his response, **Participant II** stated: *I can state that the asymptote can be any x variable that is added or subtracted from the function within that x variable. Going back to the formula, 'q' is always y, and ok 'q' is y and y is the asymptote, that means that the graph will not touch there, so... yeah I think that's it.*

In his explanation, **Participant II** described an asymptote as a variable. An asymptote is a linear graph, and is presented in algebraic form as y is equal to a ($y = a, x = b$) for the hyperbola and exponential function. The definition or description of an asymptote should at least fit any of these two categories (i.e. linearity and algebraic compatibility). **Participant II**

stated that an asymptote was a variable. From the perspective of the researcher, language was a barrier for **Participant II** insofar as communicating his ideas on the asymptote was concerned. He named an asymptote as an equation, and not as an expression. Further probing by the researcher revealed that by "variable", **Participant II** was actually referring to the equation. The question was: *Can you give me an example of a variable?*

Participant II: *For example, you have f(x)=2^x +9, yes, the 9 will be the asymptote. No, y= 9 is the asymptote.* **Participant II** recovered from the previous mistake of using the horizontal shift as a mnemonic for the asymptote, and used the vertical shift when given an example of a function and its asymptote. Furthermore, **Participant II** could formulate an exponential function and state the asymptote mathematically, which is an indication that **Participant II's** mathematical discourse is still developing (Sfard, 2012) because he could not identify and distinguish mathematical objects of the functions discourse. In spite of **Participant II's** exhibition of mathematical discourse, his word use was classified as "colloquial", because he referred to an equation a variable. At the same time, his naming of the asymptote was classified as "literate", because all members of the community of mathematicians can identify the mathematical object that he named. **Participant II's** symbolic visual mediator is "construed", because he could formulate an exponential function and name its asymptote mathematically. The researcher regarded his mention of $x = 1$ as a mistake from which he recovered.

The researcher classified **Participant II's** kind of routines as "near exploratory", because he was able to formulate and give examples of the mathematical object. **Participant II** is showing progress in the mathematical discourse by exhibiting some form of objectification, which illustrates his knowledge of the exponential function using examples. The use of an example is one of the ways of showing flexibility. When **Participant II** provides an example, it is evidence of his developing mathematical discourse. Accordingly, the researcher classified **Participant II's** use of routines as flexible due to his use of examples.

By refering to an asymptote as a variable, **Participant II's** generalisation is not aligned to the narratives of the community of mathematicians. Endorsed narratives are generalisations, theorems and proofs that the community of mathematicians endorse for use in the practice of Mathematics. **Participant II's** narratives on the asymptote of an exponential function are not endorsed by the community of mathematicians, but is acceptable at the level of **Participant II.** At grade 11, there is no expectation for learners to establish the intersection of the asymptote with the graph. **Participant II**'s narratives were primarily based on what he has observed from the graphs. Therefore, his narratives were classified as "endorsed" from the perspective of the community of mathematicians.

6.2.1 Asymptote as a Number

Mathematics is a discourse. Like any other discourse, it has rules that govern it. Proper use of mathematical language helps in the development of the discourse. In a discourse, rules help to advance the discourse participants' common interpretation of the object being discussed. In the functions discourse, an asymptote is a function. Since an asymptote is a function, it is expressed in the form of a graph or an equation. In this study, there was a tendency by participants to misname the asymptote as a number. More than half (15) of the participants who took part in this study, did exactly so, naming the asymptote as though it was a number. While 13 (%??) of them could identify the negative nine (-9), 2 (two, %??)) participants could not identify the asymptote. One possibility for this misnaming of asymptotes is attributed to the emphasis on the parameters during learning. Emphasis on the procedures only results in participants knowing *how* without knowing *what* and *why*. When asked for an asymptote verbally, some of the participants would provide a number. When asked to write the number on paper, the participants would instead write an equation in the form of $y = a$ or $x = b$. **Participant LL's** responses reveal a tendency in which there is no understanding of the reasons for expressing the asymptote as an equation. The only reason they would express the asymptote in equation form, was for them attain full marks. The researcher asked **Participant LL** to provide a reason for expressing the asymptote as negative nine (-9), or as *y* is equal to negative nine *(y= -9)*. Her response was:

The teacher marks us wrong when we write negative nine but will mark it correct when it is y is equal to negative nine.

Participant LL's response above evinces ritualised routines. Responses could be mathematically acceptable in the event that the participant acts in the mathematical discourse. **Participant LL** has not explored the mathematical object and discovered for herself the difference in expressing an asymptote as a number and as an equation.

Parameters which transform the function of the parent function are often used as mnemonics in helping participants in the mathematical discourse with the mathematical objects; such as the asymptote or the turning point in some functions reference). The writ large emphasis on the parameters sometimes entices participants into believing that the parameters were the real asymptote. In the ensuing discussion, there is an indication that participants justified their utterances on the parameters. Some even went as far as stating that *q* represents an asymptote.

In the following excerpt, **Participant OO's** utterances illustrate the association of the asymptote of an exponential function with vertical and horizontal translations.

Participant OO: *The asymptote of f is negative nine (-9), because if you see in exponential graph, the, the, in exponential graph, we have 1 asymptote which is q, so their x, we ignore it because in*

exponential graph, it is shifted in a vertical way in a vertical not horizontal. So we see but they have q which is shifted in a vertical way

In her response above, **Participant OO** states mathematically that an exponential function will have 1 (one) asymptote, but his reasons are not mathematical. The asymptote does not depend on the translations, but on the behaviour of the graph. **Participant OO** states further that there is only 1 (one) asymptote because the exponential function can be shifted vertically only. In The opposite is in fact a mathematical truism. It is axiomatic that exponential functions can be shifted both vertically and horizontally, and the shift or translation does not result in an asymptote, as **Participant OO** suggests. Parent functions also have asymptotes because of the behaviour of the graph. **Participant OO** states that the asymptote is q. This may be a reason for regarding the asymptote as a number, because further probing did not produce a response from him. When **Participant OO** states, "1 asymptote which is q", he substitutes the number represented by q and refers to it an asymptote. For **Participant OO,** the asymptote is not a line, but a number generalised as q. **Participant OO** mentions further that x is ignored because it is is an exponential function. According to **Participant OO,** the reason x should be ignored is that, "an exponential graph is shifted in a vertical way and not in a horizontal way". All functions can be translated both horizontally and vertically. Axiomatically, the parameter q shows a vertical translation, but does not necessarily mean that the constant will be an asymptote in all functions.

Despite his failure to express the function mathematically, **Participant OO** could still identify the asymptote of the function. Using the DPHEF, the researcher classified the word use as "colloquial", since an asymptote is a graph whose algebraic representation is an equation and not a number. Furthermore, asymptotes do not depend on translations but on the behaviour of the graph.

Participant OO evinces the ability to interpret the symbolic visual mediator, $f(x) = 3^{x+1} - 9$. He knows that the function has been translated nine units vertically. He also knows that the vertical shift of the exponential function is associated with the asymptote. The researcher then classified his symbolic visual mediator as being "construed" on the basis that he could explain the function.

Participant OO has no full objectification of the asymptote. He translates and names the asymptote in the realm of a number. Although **Participant OO** could interpret the function, he has not yet fully developed the asymptote as a mathematical object. **Participant OO** mistakes the constant for the asymptote, whereas the constant only provides a clue for locating the asymptote. His use of routines was classified as "applicability based on visuals" because his explanations do not transcend what he sees (the visual). **Participant OO** uses a rule premised

on the parameters showing the distance being moved by all the coordinates. In a parent exponential function, the asymptote coincides with the x-axis. After a shift of *q* units, the asymptote necessarily shifts with the graph. **Participant OO** then mistakes the vertical movement of the graph for an asymptote. He generalises and assumes the number showing the distance from the parent function is the asymptote. Therefore, his narratives were classified as "memorisation based on rules". From the above statements, it could be conclude that **Participant OO's** mathematical discourse is not yet objectified.

Of the 24 participants, **Participant W** responded differently from all others. **Participant W** focused on the table of values into the function. The table of values is an array of ordered pairs. A table of values is one of the representations that participants learn and use in the functions discourse. Depending on the choice of the input values, it is possible for *an undefined* to appear on the output. The undefined may, or may not indicate that the asymptote passes through the input value. In the event that the degree of the numerator is more than that of the denominator, the undefined may not be an asymptote. It is possible for the chosen values of the input not to have undefined as a corresponding value. This does not mean that the graph does not have an asymptote. A table of values is a very difficult representation to use for an asymptote. It is easier to identify a vertical asymptote from a table of values than a horizontal asymptote, due to the difficulty of obtaining an undefined value on the input. The only way for a horizontal asymptote to be visible is when the input values are carefully chosen and include no integral values. In this case, **Participant W** refers to the undefined values of a horizontal asymptote, which clearly indicates ritualised routines. She is referring to something that is impossible in a table of values.

Participant W continues to describe an asymptote algebraically. Such an orientation posits the constant as a mnemonic for an asymptote. **Participant W** explains further how she will recognise the asymptote even in circumstances where the constant of the algebraic form of an exponential function is zero. She refers to both the table of values and algebraic representation of the graph in the following manner: *If you don't get an undefined value on your graph, you look at the algebraic representation of the graph and then where there is supposed to be q in the exponential graph is where there is an asymptote. So if there is no q then it means that the asymptote is equal to zero.*

In the above response, **Participant W** refers to agraph in the manner of a table of values. She states that an asymptote is identified by an undefined. This probably means a function expressed in the form of a table of values, in terms of which one of the values of the input does not have an output. As discussed above, it is almost impossible to obtain undefined on the input of the table of values. **Participant W** further mentions that the asymptote could be identified from the equation. This is corroborated in the part of her response stating that *q* will be the

asymptote. **Participant W** could link two representations of a function by referring to a table of values and an algebraic representation of the function. A value cannot be undefined. It will still not have a corresponding output. **Participant W** misnames asymptotes for numbers. It is for this reason that the researcher classified the words by **Participant W** as" colloquial", because she mixes and inappropriately uses mathematical terms; stating that "if there is no q" then it means the asymptote is equal to zero.

In one of the tasks requiring **Participant W** to explain her identification of the asymptote by using 2 (two) representations of a function, her response indicated knowledge of an undefined in a table of values as signifying an asymptote. She was also aware that in some instances, the undefined may not appear in a table of values. In such situations, it does not necessarily mean there is no asymptote. Where the table of values does not indicate the presence of the asymptote, an examination of the algebraic form of the function may help in identifying the asymptote. Using the DPHEF, the researcher was able to classify **Participant W's** interpretation of the mediator as construed, due to her awareness of what each representation of a function offers.

In a table of values, it is not expected to find an undefined on the input because input is in the form of real values. **Participant W** failed to recognise that in a table of values, the undefined is only for a vertical asymptote. **Participant W** also names the asymptote as though it were a number. She mentions that the asymptote is negative nine (-9), and that the asymptote is zero in the absence of q. The objects she named could not be located on the Cartesian plane. Accordingly, the researcher classified her kind of routines as "ritualised non-mathematical", because she used function terms without exploring them. In addition, **Participant W's** use of routines was classified as "flexible", largely due to her ability to use more than one representation to explain endorsed narratives.

Furthermore, **Participant W** acknowledges that there is a possibility of asymptotes going unnoticed (not recognised). Her alternative (to the non-recognition of asymptotes) is to rather focus on the algebraic representation. Her generalisations are that the constant, which she terms q, is where the asymptote is to be found. The constant is mainly a hint, clue or mnemonic, but certainly not the asymptote in the sense of **Participant W**'s erroneous reference of the constant as an asymptote; as if the q is not there. All algebraic representations have a constant. Sometimes, the constant is not written since it would have been zero. **Participant W** noticed and concluded that the absence of a constant meant that the horizontal asymptote would be y = 0. In addition, **Participant W** generalises according to what she sees (her visual perception), which informs her usage of terms such as "q" instead of the constant. The researcher then classified her narratives as "memorisation" based on visuals, because she bases her reason for

the asymptotes on memory, and does not use narratives as would be endorsed by the community of mathematicians.

While there is evidence of the participants' ability to interpret a formula or equation of an exponential function, their word use was mostly colloquial. It was difficult to identify the mathematical objects she named. Most of the participants were able to identify the asymptote mathematically from the algebraic representation using the parameters. The use of routines was mostly classified as "applicability", since the participants tended to use the visual triggers; that is, the vertical and/or horizontal translations, to identify the asymptote. The particpants' routines were mostly classified as "ritual", as they could identify the asymptote but named it as a number. Furthermore, their narratives were classified mostly as "memorisation", as they were based on rules or authority of the teacher. In the sub-section 6.2.2, the researcher presents and discusses the participants' description of an asymptote as a point.

6.2.2 Asymptote as a Point

A point is a specific position on a Cartesian plane, while a line is defined as a set of points whose coordinates satisfy a given linear equation (Flesher, 2003). In this study, the participants defined the asymptote as a point or points at least 14 times. An asymptote is a graph defined by an equation. In this study, an asymptote is referred to as a straight line. The participants' test- and interview-based diagrams reflect an asymptote as a (broken) line, yet they refer to an asymptote as a point. In some instances, the participants used the terms "point" and "equation" as though they were synonyms. Table 6.1 (p. 119) is an illustration of participants' sketching of a hyperbola or exponential functions with an asymptote shown as a line. Participants such as **Participant BB** referred to the asymptote as "the graph not touching that point". Fourteen (%//) of the twenty-four participants used the term "point" incongruously from its acceptable usage in the community of mathematicians.
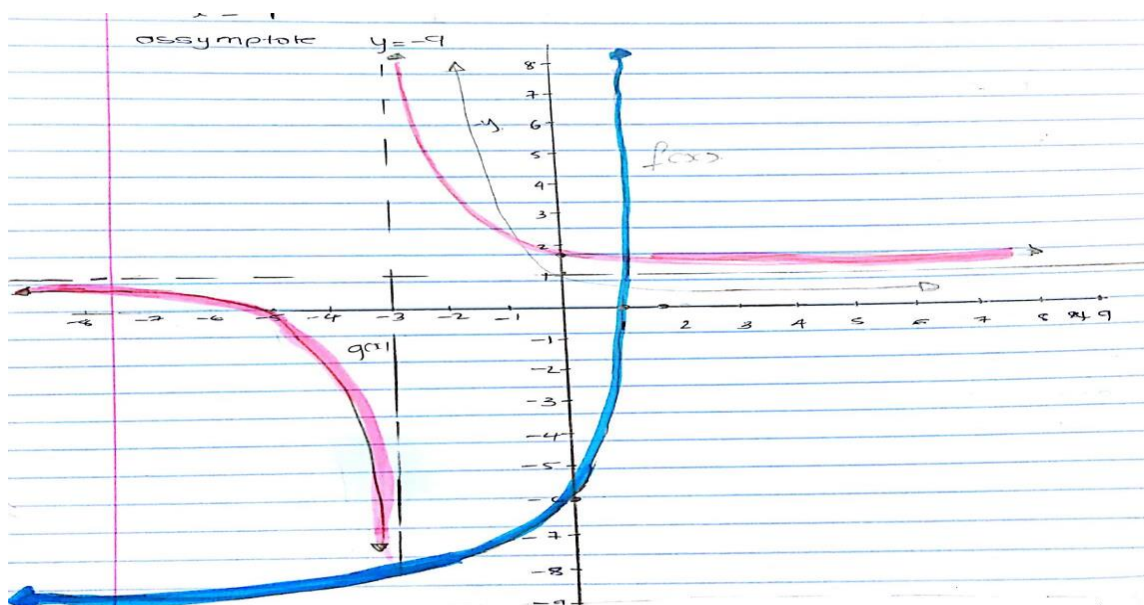
Figure 6.1: Participant GG's diagram showing asymptotes

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While **Participant GG** and **Participant NN** used the term "point" as referring to an asymptote, their explanations indicate their knowledge of an asymptote as a line, and not a point. Although **Participant GG** refers to the asymptotes as the points at which the graph is not defined, she corrected herself and subsequently mentioned that an asymptote could not be a point but a line. On the other hand, **Participant NN** provided an explanation that was similar to that of **Participant GG,** but explained further that the asymptote passes through the undefined point identified from the table of values. However, in realistic terms this is not a point, since a point should have *x* and *y* values in order to form a coordinate. Therefore, **Participant NN's** and **Participant GG's** responses to the identification of the asymptote from the table of values are similar. The researcher then chose to focus on the work of **Participant GG** because her language was easier to follow than that of **Participant NN**. **Participant GG** responded to the question: *Explain how you would identify key features like the intercepts and asymptotes in a table of values for the above functions f and g.*

Participant GG: *For the asymptotes, you have to look at the points where the graph is undefined. The line where the graph is undefined to be specific. Using the table of values, you substitute by but like when you try and substitute by like a certain, like a domain, you could give ... a certain, like a domain, you can substitute like those values and then if u find that err the graph at a specific point, like when you substitute with the specific point and the graph you don't get an out then you can ,,, you can actually conclude that that line is points ... you draw a line through that point and then ... it is the asymptote*

According to **Participant GG**, a table of values is a set of separate coordinates with no relationship whatsoever; therefore, these ordered pairs are supposedly stand-alone points. This is observable in her statement that: *You can substitute like those values and then if you find that error the graph at a specific point*. Mathematically so, the ordered pairs are the points, and they represent coordinates through which a specific function is defined on a Cartesian plane. **Participant GG** implies that "undefined" on the table of values means the asymptote passes through this coordinate. This perspective is reflected in her statement that: *You draw a line through that point… and then it is an asymptote*. As discussed in the literature review, an undefined in a function is not always an asymptote. Sometimes it is a removable discontinuity (Berger, 2013).

Participant GG vacillated in and out of mathematical language in her explanation of the determination of an asymptote from a table of values. The graph is defined on the coordinates through which it passes on the Cartesian plane, rather than by means of points through which it does not pass. **Participant GG** also defined the asymptote as a line where the graph is not defined. Her function is not defined on numerous points of the Cartesian plane, and all these

points are not the asymptote of the functions. Using the DPHEF analytical tool, the researcher classified her use of words as "near literate", due to her definition of an asymptote as a line despite the difficulty of locating the asymptote on the Cartesian plane using her definition. **Participant GG** can visualise the table of values and make an inference on it. For that reason, the researcher then classified her use of iconic visual mediator as "construed". Her routines are further classified as "ritualised non-mathematical", due to the difficulty of locating the asymptote on the Cartesian plane from her utterances. The researcher also classified her use of routines as "corrigibility" because she could self-correct. For instance, she had earlier referred to an asymptote as a point, but later realised it was not possible for an asymptote to become a point, but a line. Accordingly, the researcher classified her narratives as "memorisation based on rules" due to her statements such as: "An asymptote cannot be a point but a line…". Additionally, **Participant GG** relies on rules in her identification of the asymptote from the table of values. In the next paragraph, the researcher discusses **Participant R's** reference to an asymptote as a point.

Participant R identified $g(x) = \frac{2}{x+3} + 1$ as a hyperbola with asymptotes $x = -3$, and $y = 1$. She mentioned further that the graph would not touch these two points. There are no two points referred to except the two equations stated above, which are $x = -3$ and $y = 1$. The only point that the two equations (x = -3 and y =1) have in common is their point of intersection. It is not clear what mathematical objects **Participant R** referred to when mentioning that the graph did not touch these two points. What is clear is, however, is that she was referring to the asymptote and the function. The following excerpts show how **Participant R** identified the asymptote of *g,* as well as the reasons for her response.

Qquestion 3, Task 1: *What is the asymptote of g? How do you know this?*

Participant R: *S,o g is a hyperbola graph and therefore it has 2 asymptotes which is x is equal to three (x=-3) and y is equal to 1 (y= 1). We know this because the graph never touches these 2 points. The graph never touches these points.*

Using the DPHEF analytical tool, the researcher categorised **Participant R's** first sentence as "mathematical", having stated equations of the asymptote for the hyperbola. However, when providing her reasons for the asymptote, she mentioned that the graph would not touch these two points. She is being specific when she refers to the two points, meaning that each equation is a point rather than that the lines representing a set of points. The use of words in the second sentence is classified as "colloquial", because lines and points are two different mathematical objects.

Participant R construed the symbolic mediator $g(x) = \frac{2}{x+3} + 1$, having identified the function and mathematically stated the asymptotes. However, the interpretation of the iconic visual mediators is not mathematical, because an asymptote is neither a graph nor a point. Instead of stating that the graph never touches these two points, **Participant R** could have stated that the graph never touches those lines. In this context, **Participant R's** mathematical discourse on the asymptote is classified as "ritualised non-mathematical", since she associates the asymptote with a point. She named the asymptote as an equation, but then refers to an asymptote as if it was a point. Furthermore, **Participant R** did not view the asymptote as a graph, but as a point. It is against this background that the researcher classified her use of routines as "applicability" based on the use of the table of values, because she referred to points that were found in a table of values. She could write the mathematical statement on the asymptotes, but the subsequent explanation showed her deficient understanding of an asymptote. Writing the two equations representing the asymptote did not necessarily mean that **Participant R** objectified the mathematical object.

The following discussion depicts **Participant V's** responses to the question: *Explain how you would identify key features like the intercepts and asymptotes in a table of values for the above functions f and g*. **Participant V** responded in a manner similar to that of **Participant R**.

Participant V: *For asymptote, from the exponential graph it is where you look for at which value. For the exponential graph, it is the point or it is the number that the Y values don't reach. On a table, you will see the number or the value that the Y.... you see the value of the number that Y never reaches. Let state for instance in a decreasing function or an increasing function, on your table.*

A graph is a diagrammatic representation of a function (Kidron, 2011). **Participant V** used the term "graph" synonymously with "function". A table of values is not a graph, but a representation of the function in ordered pairs in as much as a graph is a function in diagrammatical form. **Participant V** used the terms "point" and "number" as though they meant the same stating: "It is a point or number that the y-values will not reach". As stated in earlier paragraphs, an asymptote is not a point, but a well-defined line or curve. There is some contradiction in her statement that: "…you see the value of the number that y never reaches…" A logical question (from the researcher's perspective) would then be: "How can that number be seen when it is never reached?" That is the contradiction, when *y-values* do not reach a certain number. This participant explains the presence of an asymptote in a table of values using his knowledge of the other iconic visual mediator, the graph. In a graphical representation, the asymptote and the graph do not intersect for the graphs as participants have done (Kidron, 2011).

Participant V's description of an asymptote from a table of values has a background of a graph. It is from the perspective of a graph that **Participant V** visualises the graph not intersecting with the asymptote. Using the DPHEF analytical tool, the researcher classified **Participant V's** word use as "colloquial", because of the use of the non-mathematical words. For example, stating that an asymptote is a point identified in a table of values by not "reaching y-values". His interpretation of the table of values is classified as "not construed", because he tries to explain an iconic (a table of values) using the properties of another iconic visual mediator (the graph). In this regard, **Participant V's** narratives are classified as "memorisation with a visual justification", largely because his explanations are based on what he sees. He uses the words "look" and "see" several times in response to the question requiring the identification of an asymptote. Accordingly, his identification of the asymptote from the table of values is classified as "ritualised non-mathematical", because an asymptote is not a point. In this regard, he has interpreted the table of values according to what he sees.

The researcher then asked participants to define an asymptote and explain how they (participants) could identify it as an equation in a graph. Five participants responded to the question: *What is an asymptote and how would you identify it in (i) a graph (ii) equation?* Table 6.2 below shows the classification of participants' responses using the DPHEF analytical tool.

Table 6.2: Classification of participants' responses on the asymptote

Name	Word Use	Mediator	Kinds of routines	Use of routines	Narratives
Participant CC 1.5 there is a point it doesn't touch	Colloquial	Iconic	Ritualised non-mathematical	Applicability	Memorisation
	Mixture of literate and colloquial	Not construed	Use the word point as if its synonyms with line	Key features	Rule
Participant V 1.5 an asymptote is a point the graph will be restricted	Colloquial	Iconic	Ritualised non-mathematical	Applicability	Memorisation
	Talks as if asymptote is a point	Construed	An asymptote is not a line	Visual trigger	Based on visuals
Participant DD 1.5 it is a point on the graph, the graph does not exist beyond that point	colloquial	Iconic	Ritualised non-mathematical	Applicability	Memorisation
	An asymptote on the graph and asymptote is a point	Not construed	Asymptote on the graph and	Visual trigger	Visual justification
Participant M 1.5 a specific point that the graph cannot touch, so we draw the line	colloquial	Iconic	Ritualised non-mathematical	Applicability	Memorisation
	Specific point where the graph never touches	Construed	Asymptote is a point	Visual trigger	Visual trigger
	Colloquial	Iconic	Ritualised mathematical	Applicability	memorisation

Name	Word Use	Mediator	Kinds of routines	Use of routines	Narratives
Participant BB 1.5 the graph does not touch that point	Value that x and y values cannot touch	Construed	Use of words not mathematical	Visual	Visual

In terms of Table 6.2 above, **Participant CC, Participant M** and **Participant BB** described an asymptote as a point that the graph did not touch. For **Participant V,** the graph is restricted at the point. **Participant DD** described the asymptote as a point that the graph did not touch. Following below is a discussion of how each of these learners described the asymptote.

Participant CC vaccillated between an asymptote as a line and an asymptote as a point. For him, the graph moved next to the asymptote, but did not touch it. He continued to state that the asymptote is not drawn on the graph. This is a reflection of a linguistic challenge in this instance (Nair, 2010). What **Participant CC** means is that the graph and the asymptote do not intersect. His definition of the asymptote is confined to the graphical representation only. He goes on to describe the asymptote as "a point where the graph does not touch up to infinity". It seems that the word "point" is used synonymously with "line".

Participant CC's word use was classified as "colloquial", because of the difficulty of understanding the meaning of his statements. For instance, he states that the graph moves next to the asymptote, or that an asymptote is "a point where the graph does not touch up to infinity". **Participant CC** has not yet construed the iconic visual mediator. To him, there seems to be no difference between a point and a line. His routines were then classified as "ritualised non-mathematical", because a point is not a line. His use of routines is "applicability" with prominent key features, due to the emphasis on the graph not touching up to infinity. This shows that he is aware of the key features in a graph with an asymptote. **Participant CC's** narratives are also memorisation based on rules. Five times in four sentences, he mentions that the graph and the asymptote will not touch.

Participant M defines the asymptote as a line or point. She goes further to state that the line should be drawn. **Participant M** seems to know that an asymptote is a line as there are several sentences where she refers to a line rather than a point. Below is a reflection of **Participant M's** definition of the asymptote:

Participant M: *If it cuts or touches the axes and there is no dotted line drawn, then there will be a specific like line or a point where the graph will never touch. And the line must be drawn. We usually draw it all the time.*

Participant M seems to state that the asymptote is not drawn as dotted lines in the event it coincides with the axes. An asymptote does pass through axes. There is no mathematical reason why it should not. **Participant M's** description suggests that she meant that in the case where

the asymptote and the axes intersect, there is no need to draw the asymptote. This is probably how she has seen the graphs with asymptotes on the axes being drawn, possibly because of the use of the same colour of chalk or pencil. However, this is not a rule of thumb. If different colours are used, the asymptote can be visible even on the axes. **Participant M** uses the terms "line" and "point" as if they were synonymous. For her, the evidence of an asymptote is a line drawn. An asymptote is premised on the behaviour of the graph rather than the dotted line (Nair, 2010).

Participant M's use of words is classified as "colloquial", because her use of terms is not mathematical. An asymptote is not a point. Further asymptotes are not defined by their "not intersecting" with the graph. The researcher classified **Participant M's** interpretation of the iconic visual mediator as construed because from her utterances, she has an image of an asymptote and it relation to the function despite her discordant language. Her routines were classified as "ritualised non-mathematical", because an asymptote is not a point. She also states that there is no dotted line drawn when the asymptote and the axes intersect. This led the researcher to classify her use of routines as "applicability based visual trigger", confirming that **Participant M** describes what she sees. An asymptote is not "visible" (not shown as dotted lines) when it coincides with the axes. She refers to an asymptote as a point probably because of her interpretation of the table of values. For that reason, the researcher classified her use of routines as "visually triggered". For this same reason again, the researcher classified her narratives as "memorisation", based on visuals.

Participant BB also defined the asymptote as a line, but mentioned a point that was not touched by the graph. There is a contradiction in her narrative, which reflects a line. She once more mentions a point that is not touched by the graph, stating: *In a graph, it's a dotted line. You will see, if it's a dotted line. The graph goes straight but it doesn't touch that point. that value.*

For **Participant BB,** a dotted line denotes an asymptote. According to her, a dotted line is a way of expressing the asymptote. The dotted line is drawn to distinguish the function and its asymptote. When drawing functions using geogebra (an interactive online platform used for drawing graphs and other mathematical objects like angles, statistics etc.), the asymptote is not shown because it is not part of the function. The dotted line shows the path of the asymptote and is not necessarily part of the function. An asymptote is a line or a curve, and not a point.

The researcher classified **Participant BB's** use of words as "colloquial", because she refers to a value that the graph never touches. Such utterances are not mathematical. In addition, **Participant BB** displays recognition of the asymptote. She has observed that for the graphs she has seen so far, none intersected with the graph. Accordingly, the researcher classified her

interpretation of the iconic visual mediator as "construed", because she was able to locate the asymptote on the Cartesian plane despite challenges imposed by her linguistic 'interferences' (isiXhosa and English). The researcher then classified he routines as "ritualised mathematical", because she understood that an asymptote was a line**. Participant BB's** narratives have also been classified as "memorisation", due to her attempt to remember what she has seen in class. She does not give mathematical reasons to her responses, but explains what she has seen by stating that the asymptote will not touch that point.

In responding to a sub-task that required him to state whether a function had asymptotes, **Participant DD** mentioned that the coordinate *y* could be a point, stating: *You have two asymptotes; you have the X as well as the y so you have to be specific and state X or Y asymptote is. Otherwise it's not clear. If given Y=3, Y it can be a point, it can also be an asymptote.*

A hyperbola has two asymptotes that are perpendicular to each other. These are the horizontal and the vertical asymptotes. **Participant DD** refers to the *x* and *y* asymptotes by naming them the *x* and *y* asymptotes, since they are presented in the form of $x = a$ and $y = b$. **Participant DD** is of the view that the naming of the asymptotes in the form of equations enhances clarity. In the event that asymptotes are expressed in any other way, there is no clarity as to which of the two asymptotes are being referred to. While **Participant DD** is able to explain how asymptotes should be expressed by means of $y = 3$ as an example, he then implies *y* could be a point or an asymptote. This suggests that a point and an asymptote are synonymous. As explained above, an asymptote is a line and cannot be a point. The community of mathematicians agrees on the definition of terms, and distinguished a point from a line: thus distinguishing between an asymptote and a point.

Participant DD further referes to the need for specificity when naming asymptotes, but at the same time refers to the asymptote as of coordinates, and narrates an asymptote and a point as being synonymous (Nair, 2010). His use of words has been classified as "colloquial", because it does not pass the test of the community of mathematicians. **Participant DD's** call for specificity in naming the asymptote justified the researcher's classification of his interpretation of the iconic mediator as "construed"; which demonstrates that his mathematical discourse is still developing. He vacillates between mathematical and non-mathematical utterances. For this reason, the researcher then classified his routines as "ritualised non-mathematical". **Participant DD's** narratives were further classified as "memorisation based on rules". While he referes to *x* and *y* asymptotes, he also mentions that *y* can be both a point and an asymptote.

6.2.3 Asymptote as a Boundary

In this study, participants interpreted the asymptote as a boundary, or a wall preventing the graph from passing through it. From the literature review, a horizontal asymptote of a graph can intersect a graph. Unfortunately, the type of functions to which participants were exposed show that an asymptote does not intersect the graph. Learners' definition of an asymptote is based mostly on unproven conjectures based only on their observations. However, CAPS does not necessarily restrict the learning of functions to cases where the asymptote does not intersect with the graph. Of the 24 participants, 20 (83%) described an asymptote as a boundary which should not be breached. The horizontal asymptote in some rational functions could intersect with the asymptote. **Participant BB** is one of the 20 participants who used the term "asymptote" as if it is a boundary. The following excerpt demonstrates **Participant BB's** reference to an asymptote as a boundary: *An asymptote is a form of a boundary, it's a value that both Y, if it's a horizontal asymptote the Y -values cannot touch or come cross that line. If it's a vertical line, it's a vertical asymptote, the X values cannot touch or go through that line. So it's a boundary but you can't touch it, even though it's a boundary, you can't touch it. So your graph goes like this*.

Participant BB describes an asymptote as a boundary that cannot be touched or intersected by the graph. While emphasising that there is no intersection on the asymptotes, **Participant BB** also states that: "The x values cannot touch or go through that line". What **Participant BB** meant by *x-values* not crossing was essentially that there is no expectation of the function to have a coordinate that intersects with the asymptote.

Unlike **Participant BB's** reference to the asymptote as a boundary, his primary conceptualisation of an asymptote is that it does not cut through the graph. Unlike other learners who emphasised on the graph not touching, **Participant R** focused on the graph not cutting and not going through the graph, as depicted in the following response: S*o, we've said an asymptote is the line in the graph that is not cut by the graph or the graph does not pass through that line. so that's is how I will identify it in a graph.*

Participant R describes what she sees in a graph. The assumption that she and other participants have is that the reason for the graph not crossing or cuttting the asymptote is due to the position of the asymptote. **Participant GG** and **Participant NN** also described the asymptote in the same manner, as a line through which the graph does not go.

Participant BB's words are "colloquial", because an asymptote is not a boundary that should not be touched (Mpofu & Pournara, 2018). An asymptote shows the path that the function takes as the graph tends to infinity. The researcher classified **Participant BB's** interpretation of the iconic visual mediator as "not construed", as she could not recognise that an asymptote was

not a boundary. Therefore, **Participant BB's** routines were classified as "ritualised mathematical", because her narratives indicate she could sketch a mathematical diagram, yet she interprets the asymptote based only on what she can see. It is in this context that her use of routines were classified as "applicability with visual trigger". Her narratives were further classified as "memorisation based on visuals", because she sees the asymptote as a boundary.

Participant Z also has the same view as other learners (seeing an asymptote as a boundary). Yet, she mentions that there are no asymptotes in a parent function, as demonstrated in the following response: *In a graph the asymptote is illustrated, it's a dotted line. When you don't have an asymptote, it is there but it's ok for example if we take a hyperbola both asymptote of yours will be on the X axis and on the Y axis, so you won't see a dotted line. On a graph you identify an asymptote by looking for a dotted line. You are supposed to show it when we draw, if you don't then it is incomplete.*

For **Participant Z,** a dotted line denotes the presence of an asymptote. The latter is a practice and norm that has been accepted as a norm in their learning of Mathematics. This norm has become a rule of thumb to them. A dotted line shows the path of the asymptote, and is not part of the function (Flesher, 2003). The 'normalisation' is demonstrated by **Participant Z's** statement: *When we don't have an asymptote, it is there….*

The response above appears to be a contradiction. **Participant Z** ascertains the absence of asymptotes on the basis of presumption that the asymptotesonly coincided with the axes. In essence, the graph will show by its behaviour indicates that there is an asymptote. When the asymptotes coincide with the axes, it becomes impossible to make a dotted line visible on a solid line, especially when using the same colour pen. The asymptotes will not be visible because the colour used for the graph and asymptote is the same. The dilemma facing **Participant Z** is that the asymptote must be shown. What should be done when it coincides with the axes? It is this dilemma which causes her to suggest that when the asymptote is not there, it is still there. **Participant Z** further implies that the graph is rendered incomplete mainly by the absence of an asymptote. Linguistic interferences and terminological opacity could have posed a hindrance to **Participant Z's** articulation of her thoughts (Aljoundi, 2014).

Using the DPHEF analytic tool, the researcher classified **Participant Z's** word use as "nearly mathematical", because it is accepted that an asymptote is represented using a dotted line. **Participant Z** understands that despite the visual or symbolic mediators not "showing", she nonetheless understands the presence of the asymptote to be there. Her use of iconic and symbolic visual mediators was then classified as "construed". **Participant Z** mentions nothing on the behaviour of the graph in her description of the asymptote. For this reason, the researcher

classified her routines as "ritualised mathematical". Based on her reference to what she sees, **Participant Z's** narratives were classified as "applicability with visual bias".

Unlike **Participant Z, Participant OO's** interpretation of the intersection of the asymptote and the function is different. For **Participant OO**, a function with asymptotes must have dotted lines drawn on its graph, and the equation or formula of that function should have the parameters *p* and *q*. **Participant OO** responded to the question pertaining to the definition and identification of an asymptote as a graph and as an equation as follows: *An asymptote is a dotted line represented in a graph. If there is no dotted line, it will mean there is no asymptote. In an equation, you will see the inclusive of p and q.*

In a graph, an asymptote is identified by a dotted line, without which there is no asymptote. The asymptote is a stand-alone with no relation to the graph (Nair, 2010). The behaviour of the graph determines the asymptote. With or without the dotted line, there would be an asymptote based on the type of graph in question (Mpofu & Pournara, 2018). **Participant OO** further states that an equation or formula could only have an asymptote in the event that there are *p* and *q* parameters. **Participant OO** sees the asymptotes in these parameters. **Participant OO** derives this notion from observing and listening to the interlocutor (teacher) who often stresses that learners should focus on the parameters of the asymptote. Using the DPHEF, the researcher convincingly classified **Participant OO's** use of words as "colloquial", because an asymptote is not just a dotted line. The dotted line only serves to distinguish between the graph and the asymptote. While **Participant OO** mathematically observes graphs with asymptotes, he seems not to know the purpose served by the asymptotes in a graph. The researcher then classified her interpretation of iconic and symbolic visual mediators as "not construed". Furthermore, her routines were classified as "ritualised non-mathematical", since an asymptote is defined by the behaviour of the graph, rather than the presence or absence of the dotted line. When the parameters *p* and *q* are zero, there will still be asymptotes. His use of routines was classified as "applicability with trigger visual", because he interprets the asymptote according to what he sees. **Participant OO's** narratives are characterised by memorisation based on visuals (Mpofu & Pournara, 2018) because he uses what he sees to justify his responses.

6.2.4 Asymptote as "Undefined"

In general, many learners face the challenges of distinguishng a vertical asymptote from a removable discontinuity, especially that both are calculated at the zeros of the denominator. It is impossible for learners to get an undefined on the horizontal asymptote because it is normally an output, rather than an input (for vertical asymptote). Fourteen participants (58%) in this study used the word "undefined" 72 times. To these participants, the term "undefined" signifies an asymptote. While this may be true for the rational functions that they have been exposed to, it is not necessarily true for all the functions. For further clarity on the undefined asymptote,

the researcher deemed it necessary to explore the participants' responses to responses to the question: *Explain how you would identify key features like the asymptotes in a table of values for the above functions* $f = 3^{x+1} - 9$ *and* $g = \frac{2}{x+3} + 1$.

The researcher selected four participants and interrogated their use of the term "undefined" in relation to the asymptote. **Participant Z** is one of those participants who used the word "undefined" more than four times in three phrases. For **Participant Z** "undefined" and "asymptote" are used interchangeably, because she thinks that "undefined" means "asymptote". Following is an extract confirming **Participant Z's** interchangeable usage: *Undefined means asymptote because it states the graph never touches the asymptote because if it touches it, it becomes undefined, so conclude and state undefined is the asymptote. In a tangent function, for example in y=t and x, there is an asymptote. You get the undefined when you punch it on the calculator.*

According to **Participant Z**, "undefined" means "asymptote". When **Participant Z** sees "undefined" on the table of values, she immediately knows that there must be an asymptote. "Undefined" in this case is a result of a zero on the denominator for certain values. It is not always the case that an asymptote is present wnenever there is a zero. In some instances, an asymptote will be a removable discontinuity, which happens when the denominator is a factor of the numerator. In such a case, there is 'undefined' at the zero of the denominator, but the asymptote will not pass through that coordinate. Undefined can only be used to determine a vertical asymptote, and not a horizontal one. It is not mathematical to then conclude the presence of an asymptote whenever there is an undefined. **Participant Z** further explains 'undefined' on the basis of a graph touching the asymptote. The function is only defined in terms of an input and output, otherwise it is undefined in all other instances. The asymptote does not necessarily show where the graph is undefined; but explains the behaviour of the graph as the function tends to infinity. **Participant Z** relates the question of the asymptotes to trigonometry, where there is a vertical asymptote for the tangent function. **Participant Z** further shows the universality of the asymptote at the zeros of the denominator. **Participant Z** shows that it is not only with the iconic visual mediator that an undefined will result in an asymptote, but also in symbolic visual mediator when she narrates the calculator. She explains and puts it beyond any reasonable doubt that for her, "undefined" means "asymptote".

Based on **Participant Z's** utterances, the researcher classified her use of words for this question as "colloquial", because it is not mathematical to always associate "undefined" with "asymptote" as synonymous, which is based on what she has seen. The researcher then classified her interpretation of the visual mediator routines as "ritualised mathematical", because her utterances clearly demonstrate both the ability to draw mathematically acceptable

graphs and further interpret the table of values and formulae. Since **Participant Z** relates the table of values or ordered pairs to graphs to formulae, the researcher then classified her use of routines as "flexible" substantiations that are not approved by the community of mathematicians. She justifies her statements by relating the three representations of a function.

Participant KK is one of those participants to whom the word "undefined" was synonymous with asymptote. He uses the term "undefined" eleven times in response to the question pertaining to his identification of an asymptote from a table of values. The following extract represents his response: *The graph will never touch this line because when it touches this line it will be undefined. If it is undefined, it means that it does not touch this line. This line gives you a… then it becomes, our what, our asymptote. I will state asymptote means undefined because, on undefined, the graph, the graph there is not undefined, the graph does not have values. I will state it means undefined. If you present a time graph, you know a tan graph, you have something like this, a time graph, this is 0, come to this line to 180. You find out that in 45, you have this number and when you press it on your calculator, it is what. It is undefined. But this graph, it will go on the same manner like this but it will not. So that's why I'm stating, it is undefined.*

Participant KK responded further thus: *If you have a tan graph, tan, tan it begins it 0, and this is step 45, Yeah, this will become a graph. A tan graph actually it is 0, 45, 90, it's from 35, 180. it ends here. Then, you will find in your 90 is what is n. You draw this line and there is what, there is error undefined. and in your 180, in your 270 you can add and your graph, will have it in this manner and your graph, never touches this line. Because in this line it is what, it is error meaning it is equal, it is undefined. So this line becomes the asymptote. So I'm still stating undefined means it is the asymptote.*

According to **Participant KK,** in the event that the graph and the asymptote intersect, the graph then becomes either invalid or undefined, based on the graph touching the asymptote line. Furthermore, **Participant KK** mentioned that 'asymptote' and "undefined" were synonymous, mainly due to the graph not having undefined values. What he actually means is that there is no corresponding *y-value* at the zero of the graph. Unfortunately, the language barrier induces a narrative which suggests there are many more coordinates, yet he speaks of only one *x-value;* which does not even have a corresponding *y-value*. To buttress his point, **Participant KK** uses another function, a tangent, to illustrate that the function is undefined at the zero of the denominator. He does not show the relationship between undefined and the asymptote. **Participant KK** uses what he sees to interpret mathematical actions. When performing calculations, he commits an error on the calculator. According to him, there is no corresponding y-value at the point of calculation. **Participant KK** realises that when proceeding from the

ordered pairs to the graph, the disproportionate part of the x-value and the y- value is then sketched as a dotted line and named the asymptote. According to **Participant KK** still, whenever there is an undefined on the table of values, we automatically have an asymptote. What **Participant KK** does not explain is how he arrived at the horizontal asymptote.

The researcher classified **Participant KK's** word use as "colloquial", because undefined does not necessarily mean there is an asymptote. Sometimes, it might be a removable discontinuity. **Participant KK** did not explain how that undefined translated to an undefined. His interpretation of the iconic visual mediator is then classified as "not construed". His routines are ritualised because he could perform all the mathematical processes without showing some understanding of the reasons for the mathematical process. He has not objectified the mathematical object. Accordingly, the researcher classified his use of routines as "flexible", because he is able to use the tangent function to explain the hyperbola. Using the DPHEF analytic tool, the researcher further classified Participant KK's narratives as "memorisation based on visuals" due to his memorisation ofwhat he has seen in both the table of values and the graphs.

A mathematical error occurs on a calculator when the calculator does not recognise the mathematical manipulation performed on it. For example, some calculators show an error on the screen when a square root of a negative number is written. The square root of negative nine is three I (3i), but the calculator may display an error. This means that an error on a calculator is all about the programming of thecalculator, than a mathematical error *per se.* At the zeros of denominators of rational functions, calculatotors normally display an erro*r.* **Participant KK** and **Participant Y** are examples of learners who used the word *error* for an asymptote. The following excerpt by **Participant Y** used the word *error* to explain the presence of the asymptote: *For asymptotes, we will see where there is an error. For asymptote, that's what's the graph states, if there is an error in Y, then the X values will be the asymptote.*

As explained in the above paragraph, an *error* does not necessarily mean the prevalence of an asymptote passing through that coordinate. **Participant Y** states that the identification of the table of values should be shown by an error. Undefined is usually written on the table of values not as an error. As explained earlier, undefined does not necessarily mean the graph has an asymptote. **Participant Y's** response shows that she relied most on what she saw on the calculator without necessarily making explorations on the mathematical object. An error does not necessarily mean that the asymptote will pass through that particular point.

Using the DPHEF analytical tool, the researcher classified **Participant Y's** word use as "colloquial", because an asymptote is not identified by an error on the graph; but the behaviour of the graph as the function tends to infinity. The iconic visual mediator is not construed

because a calculator is supposed to be a calculation aid. What is found on the calculator should be used without processing. When a calculator uses words such as *error*, participants should be able to interpret their meaning and use mathematical language. Given her responses, the researcher classified her routines as "ritualised non-mathematical", because on a table of values, the word "error" should not be written or found. The researcher accordingly classified her use of routines as "applicability with a visual trigger". Participant Y justifies her answers based on what she sees. On that account, her narratives are classified as "memorisation based on visuals".

According to CAPS, learners should be able to work flexibly between four representations of the function. These representations are verbal (words), equations (formulae), numeric (ordered pairs), and diagrammatic (graph). These four representations of the function are the same phenomenon expressed in different formats. Participants should be able to identify key features of the function from whichever representation. A table of values or ordered pairs also represents a function. The choice of the input values for the vertical asymptote determines whether key features such as intercepts, turning points, or asymptotes could be identified. Three of the participants stated that the asymptote could not be identified from the table of values. These participants (**Participant MM, Participant FF** and **Participant OO)** indicated that an asymptote could not be identified from the table of values. Their statements were virtually the same, suggesting that an asymptote cannot be identified from a table of values. Since the four representations of an asymptote are an expression of the same function, there should be a way of expressing the table of values such that key features such as the intercepts and asymptotes are visible. Based on the nearly similar statements of the participants, the researcher classified their word use as "colloquial", because it is possible to identify an asymptote from the table of values; especially when non-integral values are used for the input (Moalosi, 2014). Furthermore, the researcher classified the interpretation of the iconic visual mediator as "not construed", due to their failure to recognise that the asymptote could be construed from the table of values. These types of routines are "ritualised mathematical". While participants are able to draw tables of values, they are unable to interpret these tables fully.

6.3 Conclusion

In this chapter, the researcher demonstrated that the mathematical discourse of the participants was still developing and characterised mostly by colloquial and ritualised routines, as well as narratives that were mainly based on what participants could see, rather than the mathematical explorations (Mpofu & Pournara, 2018). The researcher utilised the DPHEF analytic tool and discussed participants' representation of the term "asymptote". Language posed a barrier for most of the participants. Such a state of affairs is inimical to their capacity to fully represent

their knowledge, understanding and experiences. Inadvertently, this leads to learners referring to an asymptote as a point.

While most of the words used by learners were mathematical, there were instances of their confusing communication as accepted in the community of mathematicians; for example, when they named an asymptote as a number, an asymptote of an exponential function, or as a vertical line. Learners' visual mediators were generally construed, meaning that they were able to interpret both the iconic and symbolic mediators of the two functions. It was difficult to obtain reasons for these mathematical actions performed by the participants, which are also largely premised on what they saw and/or what they have been taught. Learners deduced meaning from what they saw from symbolic and iconic visual mediators.

The lerners'/ particpants' representation of the asymptote of the hyperbola and exponential functions in this chapter was of critical importance to the broader domain of the study's objectives. Similarly, the next chapter (Chapter 7) provides a conclusive link to learners' mathematical discourses and the community of mathematicians. Such a link is necessary, since the learning Mathematics is not a end in itself, neither is such learning peripheral to learners connection with their daily realities (Bradley et al., 2013; Caduri et al, 2015)

CHAPTER 7
FOUR REPRESENTATIONS OF A FUNCTION

7.1 Introduction

In Chapter 6, the researcher discussed all participants' responses to the interview questions pertaining to the hyperbola and exponential functions. The South African high school Mathematics curriculum requires grade 11 learners to work flexibly between the four representations of a function (DBE, 2011). In the current chapter (Chapter 7), the researcher analyses the mathematical discourse of the 24 participants using the DPHEF. From the researcher's observation, participants were more comfortable with interpreting functions that were expressed in algebraic and graphical forms. Although the participants' mathematical discourse was largely visual and based on what they could see, there was evidence of their developed mathematical discourse on the function sub-discourse.

An algebraic formula with an input and an output is one of the four representations of a function. An equation with at least two variables represents a function in algebraic form (Mahlobo, 2004). In such an equation, there is an input and output value. The algebraic representation of functions illustrates the continuity of the graphs because any real values of the input produce an output, except in cases where there is an asymptote or a removable discontinuity (Berger, 2013). Not all equations produce a function. For example, the inverse of a quadratic function and the equation of a circle are not functions. For purposes of this study, the researcher focused on the exponential function and the hyperbola. An exponential function is characterised by a fixed base, and an exponential function has a variable. The researcher expected participants to describe or identify an exponential function as one in which the variable x is an exponent. Such description and identification was expected to be central to discussing the algebraic representation of the exponential function in a hyperbola as a function, where x and *y-values* are generally inversely proportional to each other. As x values increase, the *y-values* decrease proportionally. Participants rarely described the algebraic representations of the two functions in the manner described above; that is, in terms of inverse proportionality.

7.2 Translation of Functions

In its simplest form, a parent function is a function in a family of functions that bears the characteristics of that function in terms of shape and other properties of the entire family (Webber, 2002). For example, in a hyperbola, the parent function would be the two asymptotes which are perpendicular to each other in a graph which has two parts that are mirror images of each other. In essence, a parent function could be conceived as a function without any form of transformation. In fact, all other transformations on functions are performed on it. All functions have a parent function and the description of the function should have a relationship with the parent function. It is against this background that the study participants expressed the hyperbola

and the exponential functions in terms of their vertical and/or horizontal translations. Most of these participants explained the relationship of the graph and its parent function from the transformed function to the parent function. No participant referred to the parent function by its name. Words such as "standard function" and "original function" were often used.

Furthermore, all participants identified the two functions expressed in the form of the algebraic equations by name. The researcher identified six categories of the participants' explanation of the relationship between the parent function and the transformed function. Ten participants (42%) either named the parent function as the original function, or as the standard function. Five participants (21%) described the translation of the function as a way of identifying self-same functions. In a way, the participants did not state how they identified the function because all functions can be translated. Two participants (8%) saw an index and concluded that the function was exponential. Four learners saw a denominator and then concluded that the function was a hyperbola. **Participant OO** did not identify the function, but only spoke of an asymptote. In the following paragraphs, the researcher discusses how each participant identified the functions.

The first task in the first question assigned to the learners was: $f(x) = 3^{x+1} - 9$ and $g(x) = \frac{2}{x+3} + 1$. Participants were required to name these two functions and relate them to the parent function. The researcher then subsequently focused on an analysis of the participants' responses.

Five participants (**Participant HH**, **Participant M, Participant II, Participant SS** and **Participant V**) named the functions as the exponential function and the hyperbola. They also described the relationship between each function and its parent function in terms of the translation between the parent function and the transformed function.

Participant HH and **Participant II** described the translation from the parent function to the translated function. That was how they responded to the question that required them to show the relationship between $y = 3^x$ and $f(x) = 3^{x+1} - 9$. **Participant HH** and **Participant II** responed thus**:** *I think f(x) is shifted 9 units downwards and 1unit upwards to the left and 9 units downwards.*

Participant HH's and **Participant II's** descriptions show that the translated functions originated from the parent function. These two participants used mathematically appropriate words. For that reason alone, it was not difficult to understand how their reasoning of the function *f* was arrived at, by shifting the parent function 9 (nine) units down and 1 (one) unit to the left. Using the DPHEF analytic tool, the researcher classified their interpretation of the algebraic representation of the function as "construed" because of their ability to relate the

parent function to the transformed function. **Participant II's** and **Participant HH's** routines were also visually triggered, because they based their descriptions of the function on the parameters that showed the translations (Ben-Yahuda et al., 2005). The two participants described the relationship between the parent and translated functions in a reverse order, where there was a suggestion of a translated function preceding the parent function. **Participant V** explained the translation thus: *The first function is an exponential graph, the graph shifts 1 unit to the right and 9 units upwards.*

Interviewer: *Why do you say that?*

Participant V: *Because for you to get* $y = 3^x$ *you shift the graph 1 unit to the right and 9 units upwards.*

The responses above indicate that they did not recognise $y = 3^x$ as the parent function from which other functions were translated. Functions were translated from the parent function and all descriptions of transformations should have reflected that they emanated from the parent function. **Participant V** used mathematical words, which showed a developing mathematical discourse. Although they reversed the description of the translation, there was evidence of active participation in mathematical discourse.

Both **Participant V** and **Participant M** could interpret the mathematical object by naming and describing the mathematical object. These two participants' routines of naming the function $f(x) = 3^{x+1} - 9$ were classified as "ritualised mathematical" because they could name the function but did not recognise its parent function.

While **Participant M's** mathematical discourse was the same as that of **Participant V**, she attempted to explain the process by translating functions horizontally, which shows that most of her narratives were teacher-driven; rather than from explorations or investigations from her own calculations (Sfard, 2012). Below is the conversation between **Participant M** and the researcher (interviewer):

Participant M: *It is going to the right because of the signs. We can. Let's say here in* $f(x) = 3^{x+1}$. *We can state negative one, which is the shift, negative one. Then one and one will cancel, then you will be left with* 3^x. *So when the graph is shifting to the right, we are saying it is negative.*

Interviewer: *Now I want us to talk about that if it is positive one, it means it is going to the left. Where do you get that?*

Participant M: *It depends on the x-axis or the y-axis. If it is positive that it means it is going up, if it is negative, which means it is going down.*

Interviewer: What tells you that?

Participant S: *That is what we were taught.*

Interviewer: *That is how you were taught?*

Participant S and **Participant M:** *Yes*

Interviewer: *Did you try to investigate and see.*

Participant M: *We did not try that*

Just as **Participant V** did, **Participant M** explained the relationship from the transformed function to the parent function. **Participant M** focused on trying to explain the horizontal translation of the exponential function. Meanwhile, the researcher then asked **Participant M** to elaborate further on the rules of horizontal translations. These rules state: if $y = f(x)$, then $y = f(x - h)$ yields a horizontal translation of the graph of f. The translation was h units to the left, because h was greater than zero; and h units to the right for h was less than zero, where h was an element of real numbers. **Participant M's** explanation was unclear. Although one could tell that she was familiar with the applicable rule, she could not articulate it.

Participant M: *When the graph is shifting to the right, we are saying it is negative.*

In the above response, **Participant M** implies that for negative h, the graph moves to the right. The confusion arose when she used the word "negative". It was not clear whether she meant a translation in the mould of $y = f(x - h)$, or $y = f(x + h)$. A negative h in $y = f(x - h)$ moves the graph to the left rather than to the right. When the researcher probed **Participant M** on horizontal translation, she spoke of the vertical translations, stating: *It depends on the x-axis or the y-axis. If it is positive that means it is going up. If it is negative, it means it is going down.* **Participant M's** statement vacillates from the horizontal translation to the vertical one. She speaks of the graph either going down or going up. The researcher was then interested in obtaining her mathematical discourse on translations. For that reason, the researcher let her continue with the vertical translation. In this regard, **Participant M** spoke of the x and y-axes as the basis of translation, instead of the horizontal or the vertical translation.

The researcher classified **Participant M's** word use as "colloquial", because of the discordant use from the perspective of the community of mathematicians. Although she used mathematical terms, she did not use them appropriately. There was no translation in the *x* or *y-axes*. Furthermore, **Participant M's** interpretation of the symbolic visual mediators, $f(x) = 3^{x+1} - 9$ and $y = 3^x$ was "construed", because she realised the relationship between the two algebraic representations of functions to be translations. Although it was not easy for her to fully describe the relationship, there was evidence of a developing mathematical discourse (Sfard, 2012).

In addition, **Participant M's** routines were ritualised. Although she could tell that a translation was performed, she could not fully state the translation which had taken place. Lastly, their narrative was based on authority. When asked to provide reasons for their narratives,

Participant M and **Participant S** attributed their knowledge to the teacher rather than to what they investigated themselves, emphasising that it was how they were taught.

Participant Y recognises that the function was a translated exponential function, but she could not describe the change mathematically. In that regard, **Participant Y** responded to the question and named the function *f* and described how it related to $y = 3^x$.

Participant Y: *F(x) in this graph is an exponential graph and the graph relates to 3^x. You can say that $y = 3^x$... and it will be shifted.*
Interviewer: *How will it be shifted?*
Participant Y: *The asymptote is one unit down.*
Interviewer: *How do you know this?*
Participant Y: *The exponential graph does not shift horizontally, it shifts up or down.*

Question 1.1 presented a function in the algebraic form: $f(x) = 3^{x+1} - 9$. Although **Participant Y** used mathematical terms, they were used in a wrong place, and she did not make a distinction between different representations of a function. An algebraic representation cannot be referred to as a graph, because a graph is diagrammatic representation of a function (Pinter, 2014).

Participant Y went on to narrate the shifts, which are mathematically referred to as translations. She realised that $f(x) = 3^{x+1} - 9$ was a translated function. Such realisation shows that her mathematical discourse was advanced. However, she moved away from talking about shifts and talked more about an asymptote. In responding to the question on the translation of function *f*, **Participant Y** spoke of an asymptote moving one unit down. In a way, she associated translations with the asymptotes. The translation itself indicated ritualised routines because the algebraic representation of the function showed the translation and the position of the asymptotes were the consequence of the function being translated. According to **Participant Y,** the function should have been translated one unit down, because she stated that the asymptote had been shifted one unit down. She went on to state that the exponential function did not shift horizontally, but that it was translated vertically up and down in terms of which all functions could be translated. In this regard, **Participant Y** based her argument on the asymptote, stating that: *With the asymptote, if you notice, it cuts up or down. It doesn't have a vertical asymptote because it cuts on the y-axis.*

From the above statement, it could be deduced that **Participant Y** largely associated translations with the asymptotes. While it was true that the exponential function does not have a vertical asymptote, it was not correct to state that a function does not have a horizontal translation only because of the absence of the asymptote. **Participant Y's** utterances brought to the fore the challenge of translating the vertical and horizontal asymptotes. When asked how

she knew that the graph did not have a horizontal shift, **Participant Y** responded thus: *You inspect the graph to know that it does not have a vertical shift.*

Participant Y then erred by stating that there was no vertical shift on the graph. She performed well in the functions test by obtaining 85% of the marks. However, she did not show any significant degree of competence on the translations, due to her association of the asymptote with translations. In addition, she explained the translation of the hyperbola from the perspective of the asymptote. In the hyperbola, she mentioned both the horizontal and the vertical translations.

Although **Participant Y** used mathematical words, she was saming (Nachlieli & Tabach, 2012) translations with asymptotes. Saming refers to the treatment of two different objects as though they were similar. Where she could identify the asymptote, she did so in the context of a translation. Where she did not find an asymptote, she was unable to notice the translation. Consequently, the researcher classified her interpretation of the algebraic symbolic mediator as not "construed". Since she could respond to mathematical questions on functions, but could not explain herself, **Participant Y's** routines were then classified as "ritualised". Her use of routines was mostly visual, basing her utterances on what she could see. In addition, her narratives were classified as "memorisations based on visuals", because she explained all her actions according to what she could see (Mpofu & Pournara, 2018).

In this study, the researcher has elaborately explained the relationship between the parent function and the transformed function using the translations from the participants' perspectives. All five participants used the term "shifts" while also identifying and naming the functions. However, their description of the function was translated differently. Two participants explained the translation from the parent function to the transformed function. Another two spoke of the "shifts" as though they were from the transformed function to the parent function. The fifth participant could not distinguish a translation from an asymptote at all.

7.3 Parental Function

A parent function is the simplest function in its family (Kuang & Gilman, 2011). It is from this parent function that other functions are 'born' from translations, reflections and other transformations. Sometimes, the parent function is referred to as the basic function. In this study, the researcher concentrated only on the vertical and horizontal translations. In exponential functions, the parent function is expressed as $f(x) = b^x$, where b is a positive rational (b > 0; b≠ 1). With regard to the hyperbola, the parent function is expressed as $g(x) = \frac{a}{x}$. Twelve participants (50%) described the parent function as either a standard or original function, but none of them used the term "parent function". A standard quadratic function is

written in the form of $f(x) = a(x - p)^2 + q,$ while its parent function is $f(x) = x^2$. In the next few paragraphs, the researcher describes how participants discussed the parent function.

Participant R and **Participant W** used the term "basic graph" to describe the parent function. The use of the term "basic graph" indicates that the learners were able to describe the parent function. These two participants were presented with a graph in the form of an algebraic representation of a function. They could have referred to the graph as an equation. They named the algebraic representation of an exponential function as follows:

Participant W: *So, f is a exponential and y is a basic graph but f has obtain a shift in y 1unit to the left and then g(x) is a hyperbola graph and y is a basic form of a hyperbola but g(x) is obtain by shifting y 3 units left and 1 unit upwards.*
Interviewer: *What do you think?*
Participant R: *It is the same thing.*

While **Participant W** posits function *f* as an exponential function, she did not state for which function was it the basic graph. She only explained the horizontal shift of the function and said nothing of the horizontal for both functions. She just agreed with **Participant W**, and they both did not provide more information relating to their mathematical discourse. However, both participants used mathematical words. In the meantime, **Participant Z** used both the standard and original functions as though they were synonymous. She named the functions as exponential function and the hyperbola. She went on to describe the relationship between the parent function and the transformed function. She described the relationship between the algebraic representations $y = 3^x$ and $f(x) = 3^{x+1} - 9$ as follows:

Participant Z: *The question is given as okay. y^2, $y = 3^x$, is the standard form. And f (x) which is y to the, which is three to the power x plus one minus nine $(f(x) = 3^{x+1} - 9)$. Here it will do a shifting one: a shifting or a translation both vertical and horizontal translation*
Interviewer: *What do you mean by standard form.*
Participant Z: *Standard form means it is not shifted. Just original. , it not shifted. where x , there is no translation*

Participant Z's response above suggests that she was able to described the relationship between the parent function and the transformed function as a translation. Therefore, her use of the word "translation" was mathematical. It was unlike some participants who did not use the term "translation" at all. She also used the term "standard form". When asked for the meaning of "standard form", she stated that it" means it is not shifted, just original". What she meant was that $y = 3^x$, was not translated, but $f(x) = 3^{x+1} - 9$ has been transformed. Language use might have affected how **Participant Z** expressed herself, but her intentions were clear. For her, a standard form was the parent function. She did not use words

appropriately, as indicated by her use of "original" and "standard form" as though they were synonymous. A standard function is not the same as the parent function, nor is it known as an original function.

Participant AA, who was a partner of **Participant Z** during the interviews, spoke of translations as **Participant OO** spoke of eliminations. However, **Participant AA** used the "translations" in his response. The question was asked: *Explain how $f(x) = 3^{x+1} - 9$ relates to $y = 3^x$ and $g(x) = \frac{2}{x+3} + 1$ relates to $= \frac{2}{x}$*, to which he responded as follows:

Participant AA: *So now ,what we did here is, we saw that $y = 3^x$ we saw that the -9 and the +1 were **Participant OO** eliminated from the whole equation so we thought, see it here, y to the power 3^x is the standard form of an exponential function and f is translated graph of y = 3^x is obtained by shifting y 9 units down and 1 unit to the left.*
Interviewer: *Ok, go on.*
Participant AA: *And the hyperbola $y = \frac{2}{x}$ is the standard form of a hyperbola and g is obtain by shifting the standard graph 3 units to the left and 1unit up.*

To some extent, **Participant AA** explained the relationship from the transformed function to the parent function, yet all functions were transformed from the parent function. However, he self-corrected and used a proper mathematical term, "translation". He explained correctly that *f* was a translation of the parent function, and went on to explain the hyperbola accordingly.

Although **Participant AA** could explain the relationship between the parent function and its offspring, he still used "standard form" instead of "parent function". His use of words was somewhere between "colloquial" and "mathematical". At the same time, **Participant AA's** interpretation of the symbolic visual mediator was classified as "construed", since he was able to show the relationship between the two functions, as well as to explain the translation. There was evidence of a developing mathematical discourse in the functions discourse because **Participant AA** could talk of the relationship between the parent function and its offspring. **Participant BB** and **Participant CC** both used the term "original function" instead of "parent function". These two participants did not show or state how the two functions were related. Following is the trajectory of the interview with the two participants.

Participant BB: *So $f(x) = 3^{x+1} - 9$. It is an exponential function and it is related to $y = 3^x$ because $y = 3^x$.It is the original graph of the function above and also $g(x) = \frac{2}{x+3} + 1$ is a hyperbolic function and it is related to $y = \frac{2}{x}$ since $y = \frac{2}{x}$ it an original function.*
Interviewer: *What do you think?*
Participant CC: *Yes. f(x), f is an exponential graph because it goes $y = 3^x$ I can see it is the original graph of this graph*

Participant CC and **Participant BB** agreed on the use of the term "original graph". From their statements, it was clear that they meant the parent graph. The representation that they were referring to, is the algebraic representation of a function. The two participants did not explain how they related the two functions. The researcher classified their use of words as "colloquial", because they referred to the parent function as an original graph, and to the algebraic representation of the function as a graph. They both interpreted the two pairs of symbolic representations and visual mediator mathematically, and the researcher classified this interpretation as "construed". In this context, these two participants exhibit a growing mathematical discourse.

Participant X and **Participant DD** also spoke of the relationship between $y = 3^x$ and $f(x) = 3^{x+1} - 9$ as a vertical stretch. Stretching the graph vertically moves it away from the *x-axis*. In a stretch, there is a stretch factor greater than 1 (one) that multiplies the function. One (1) cannot be a stretch factor as it is a multiplication identity. Both participants stated that the relationship between the two functions was a vertical stretch, yet the multiplicand is 1 (one). Following is the trajectory of their responses during the interview.

Participant X: *I said that f(x) is an exponential function and g(x) is a hyperbola. f relates to y=3 exponent x ($y = 3^x$) because it is an exponential graph or function without shift and g relates to $y = \frac{2}{x}$ because it is a hyperbola without any vertical or horizontal shift.*

At first, **Participant X** identified the parent function by stating that it did not undergo any transformation. At the same time, she did not explain how the functions are related to each other. She realised that $y = \frac{2}{x}$ is a parent function, but did not name it as such. The researcher followed up on both participants, as indicated below.

Interviewer: *So, if you look at $y = 3^x$ and f(x), how do they relate?*
Participant DD: *Ok. from my answer I say f(x) is an exponential function. g(x) is a hyperbola. f(x) graph and $y = 3^x$ are related such that they have the same vertical stretch. Their 'a' is the same. That is what I'm trying to say.*
Interviewer: *Do you agree?*
Participant X: *Yes, I agree.*

Instead of pursuing the idea of a parent function, **Participant DD** digressed, and spoke of the stretch. The value of *a* was the same for both equations, implying that there was no stretch. **Participant DD** acknowledged that there was no stretch, stating that, "the value of *a* is the same". **Participant DD** was implying that both functions were stretched, but did not give a definitive stretch factor that would have clearly indicated that a stretch had occurred. The use of terms by both these participants show that their mathematical discourse was still developing.

Although they did not use mathematical words appropriately, there was evidence of their familiarity with mathematical terms on functions such as "stretch" and "shifts". Both participants may have used mathematical terms, but they did not grasp the full import of the meanings of those terms. In this regard, they have not yet fully objectified the mathematical object known as a function (Sfard, 2012). After naming the algebraic representations of functions as an exponential function and a hyperbola, **Participant EE** and **Participant FF** spoke of a standard equation when actually referring to the parent functions of the respective functions named above. They also spoke of a hyperbola as a function with a fraction. As stated in the literature review, not all fractions are equations (Larson, 2016). For example, $f(x) = \frac{1}{2}x$ is not a hyperbola. In this regard, both participants' responses were as indicated below.

Participant EE: *Yeah the standard equation has an exponent and that is how we are taught because you can see the exponent there, so it is an exponential graph because the name states so and then ...*

Participant FF: *It is a standard equation*

These two participants name the algebraic expression of a function as a standard equation. A standard equation refers more to equations in general, rather than to functions. They could have spoken of the standard form of a function. The terms used by these to participants made communication difficult within the community of mathematicians, as it would be hard to identify the mathematical object of their narrative. **Participant EE** went on to present an exponential function as exponential, merely because the name suggests so. He also alluded that the exponent on the function resulted in the function being named as such.

One other important utterance from **Participant EE** has to do with his narratives. This participant attributed his narrative to what he had been taught, which is an authority perspective. In terms of the DPHEF, **Participant EE's** routines are classified as "ritualised". This classification was based on the fact that he did and said what he saw and heard others state on the functions discourse. **Participant EE's** routines were not a result of explorations, but rather premised on the procedures and processes of others.

The community of mathematicians does not use the term "original function". Therefore, it is not mathematical. A standard function is one defined on an interval of R that is obtained by a finite sequence of standard operations beginning from any combination of three basic functions (Larson, 2016).

7.4 Functions in Word Form

Teaching and learning is sometimes influenced by the emphasis placed on content of the examinations, more than on the content of the curriculum. Verbal representations of the functions formed part of examples which were rarely examined in the curriculum. As a result, these representations may not have been given much attention in teaching and learning, as the evidence of this study suggests. In the next paragraphs of this section, the researcher discusses how grade 11 learners participated in the hyperbola represented in word form.

7.4.1 Hyperbola expressed in a verbal form

A rectangular hyperbola (also known as a school hyperbola) is an example of an inversely proportional function (Kuang & Gilman, 2011). If all other variables in the function were kept constant, the size of one variable increased as the other decreased. That could also be true for a decreasing exponential function, but the difference was that the product of any corresponding points of a hyperbola was constant. That aspect was noticed in a parent hyperbola. Participants were presented with a verbal parent hyperbola in which the number of workers increased as the time decreased. The statement to which they were required to responded, was: *A group of workers are planning to paint 10 houses in a complex. It takes 80 hours for 1 (one) person to paint all the houses, 40 hours for 2 workers, 20 hours for 4 workers and so on.*

Only 2 (two, 8%?) of the 24 participants recognised the function as a hyperbola. Fourteen participants (58%??) thought that the function was exponential; 4 (four, %?) participants stated it was linear; and a further 4 (four, 16%) stated they could not respond to the question. The researcher then interrogated the 2 (two) participants' identification of the verbal expression as a hyperbola**. Participant GG** was the only one who exhibited a developed mathematical discourse on the hyperbola expressed in words.

Interviewer: *What do you think it is?*

Participant GG: *I think it's a hyperbolic function.*

Interviewer: *Hmmm why do you say so?*

Participant GG: *I think if we look at the statement at the end. It says, it takes eighty hours for 1 person to paint all the houses. Forty hours for work.....eighty...forty hours for two workers and twenty hours for four workers, so we gonna multiply the number of workers, and the number of workers yes by the number of hours so we gonna take eighty multiplied by 1 equal to eighty, eighty multiplied by two is equal to...oh no... And then forty multiplied by two because it takes forty hours for two workers, equal to eighty. And twenty hours times four workers is equal to eighty. So now, if we multiply the x-values by the y-values we get a constant number that is eighty. So.... so x times y, we can make x the number of hours because number of people are independent of time. Time depend on the number on the number of workers that are painting the house, so we gonna say x times y equal to k, k is for this constant number and then*

if you divided by x, you divide by x on this side, y is gonna be equal to $\frac{k}{x}$, this is the general graph of a hyperbola.

Participant GG exhibited an objectified mathematical discourse on the verbal representation of the hyperbola. She realised that the product of any corresponding points resulted in a constant. In this question, the constant was 80. **Participant GG** went on to generalise and wrote the function in terms of x, y and k. Her use of words was mathematical. Furthermore, she showed that the product of the independent and dependent variables resulted in a constant. In this regard, she was abled to identify the function in question as a hyperbola.

Participant GG was also able to interpret the symbolic visual mediator. The graph was expressed in word form and she managed to move from words to algebraic form. Subsequent to showing a pattern in the product of the independent and dependent variables, she went on to generalise and stated the algebraic form of the function. Using the analytical tool the DPHEF, her interpretation of the verbal representation was classified as "formulated" or "generated", because **Participant GG** came up with a generalised algebraic function derived from a worded statement. Her ability to move from words to arithmetic statements and then to algebraic equation, showed that her routines were more of exploratory. In addition, **Participant GG's** use of routines was classified as "flexible", because she could relate various forms of the representation of the hyperbola (Ben-Yahuda et al., 2005). The researcher then classified her narratives as "substantiations", because of her clear justification of her worded statement as a representation of a hyperbola.

Participant GG was the only one of the 24 participants who explained the relationship and justified it. **Participant HH** on the other ahnd, only named the function as a hyperbola but could not show or state why she said so in a mathematically acceptable manner. The following excerpt shows **Participant HH's** responded to a question requiring her to name the function from a worded statement. The statement was: *A group of workers are planning to paint 10 houses in a complex. It takes 80 hours for 1 person to paint all the houses, 40 hours for 2 workers, 20 hours for 4 workers and so on.*

Interviewer: *And what did you say it was?*

Participant HH: *A hyperbola.*

Interviewer: *You are saying is a hyperbola, how do you know?*

Participant HH: *Because when you divide forty by two you get the same answer as when you divide twenty hours by four and so on.*

Interviewer: *What is the same answer? Divide forty by two what did you get.*

Participant HH: *Twenty*

Interviewer: *You divide twenty by four, what did you get?*

Participant HH: *You get three comma something no, you get a comma something.*

Interviewer: *Twenty by four?*

Participant HH: *A five.*

Interviewer: *Ok, you get a five. So now I am interested in the reasons why you are saying it's a hyperbola.*

Participant HH: *No response.*

While **Participant HH** named the function in written words as a hyperbola, she could not provide mathematically acceptable reasons for stating so. **Participant HH** further stated that the division of the ordered pairs resulted in the same answer. If that was true, then the function was most likely to be linear with an intercept of zero. When the researcher asked her to justify her statement, this showed that her division was not up to the expected level. She could not divide 4 (four) into 20. After realising that division of the ordered pairs was not constant, she could not provide any further cogent reasons. Her use of the phrase "same answer" was not mathematical. In the context of **Participant HH's** narrative, "same answer" would have meant that the division of the number of hours by the number of people should yield the same number. Her use of words in this instance was classified as" colloquial", because she did not obtain the same answer as she purported to. Her use of symbolic mediator necessitated that she be classified as 'not construed". She could not interpret the verbal representation of the function. Her routines were classified as "ritualised incorrect", because she experienced challenges of division (Sfard, 2012). Her narratives were mostly memorisation as she just stated what she could remember. In the next few paragraphs, the researcher discusses the following verbal response by participants, as representing a linear function: *A group of workers are planning to paint 10 houses in a complex. It takes 80 hours for 1 person to paint all the houses, 40 hours for 2 workers, 20 hours for 4 workers and so on.*

7.4.2 Linear function

A linear function is denoted by a common gradient and coordinates which form a straight line (Stewart, 2012). One way of testing the linearity of a function is to check whether the points were collinear or not. Corrigibility routines are mostly about checking one's assertions on various mathematical narratives (Ben-Yahuda et al, 2005). Four participants stated that the verbal representation of the function was linear, but none of them checked whether or not the points were collinear by calculating the gradient.

Participant II stated that the function was linear, but did not demonstrate the collinearity of the points. Instead, he stated that there was a common factor of two for both the dependent and the independent variable. This is how the interview proceeded:

Interviewer: *What did you say it was? There is the question there.*

Participant II: *I said that a linear function.*

Interviewer: *Why?*

Participant II: *Because the relationship between the number of hours and number of workers is the same. Number of hours only decrease by a factor of two, and then the number of workers increase by a factor of two.*

Interviewer: *So did you prove that it is a straight line?*

Participant II: *Yes*

Interviewer: *How did you do it?*

Participant II: *For the number of hours, I used it as an independent variable so I will used the y-value to represent the number of hours and then x has to represent the number of workers.*

According to **Participant II**, the relationship between the number of hours and the number of workers were the same. He went on to talk about the factors of decrease. What the researcher was not sure of, was how these relate to the straight-line graph. Hence, the question the researcher asked him was based on how he proved that it was a straight-line graph. **Participant II** responded by stating how he ordered his variables, but said nothing about the characteristics of the linear function. His use of words was "colloquial", because the words did not support the characteristics of a linear function. **Participant II** could not interpret the visual mediator presented to him, because he could not substantiate his claim of a linear graph. In this regard, his routines were ritualised non-mathematical, because he spoke of a linear function at the same time he spoke of ratios. In this regard, his narratives were dissimilar to those of the community of mathematicians. His "proof"of the function's linearity was premised on merely stating which variables were taken to be *x* and *y*.

Participant CC also named the verbal function as linear. At the same time, he may have confused a line graph and a linear graph because he said that he plotted points and the graph was linear. He responded to the question which required him to identify the graph in the following manner:

Participant CC: *I could say it is a linear function.*

Interviewer: *Why?*

Participant CC: *It is because you see; let me put it like in a graph form. It is 80 then it is 1 worker, then it goes on and say 40, let's say let us put the person, the workers as our X intercept and the hours as out Y intercept. If the graph will move, if we plot the graph, the graph will be a straight line.*

Interviewer: *Did you did you if you say it is a linear, did you try to find the gradient.*

Participant CC: *Yes. The gradient.*

Interviewer: *Did you?*

Participant CC: *Yes sir.*

Interviewer: *Now if you found the gradient then how do you test that the function is linear, what do you do? How do you test using a gradient that a function is linear?*

Participant CC: *Using the gradient.*

Interviewer: *Yes. How do show that. How does the gradient for a linear. How do you know from the gradient that this one is linear?*

Participant CC claimed to have plotted the points and found that the graph was a linear function. However, he did not have the plotted graph on whose basis the researcher could ask him further questions. The researcher then shifted attention to the properties of a linear function. Furthermore, the researcher asked him whether he had tried to use the gradient as a way of showing that the function was a straight line. In his response, **Participant CC** initially stated he had found the gradient. When asked how he had used the gradient to show that the function was linear, he did not respond. Although he used mathematical terms such as linear function and gradient, he did not show much of his mathematical discourse. He did not produce a graph or some procedure to elaborate on his responses, which is an indication that he may not have developed the mathematical discourse on functions because he could not give reasons for his assertion on the function's linearity, and also did not produce evidence supporting what he purported.

Participant JJ and **Participant KK** also thought that the worded function was linear. They had a different reason from the other two participants above. **Participant JJ** and **Participant KK** link functions to sequences in their explanation of their 'stance' on the verbally expressed function. The discussion proceeded as indicated below.

Participant JJ: *I said it is the linear function because it has the first constant difference. Yes, it has the first constant.*

Interviewer: *First constant difference?*

Participant JJ: *Yes, as the. It is it is.*

Interviewer: *Yeah is that same reason or different reason.*

Participant KK: *It is the same reason, it has a constant, common difference.*

Interviewer: *What is the common difference?*

Participant KK: *After forty hours, you have twenty hours, the difference between here and there is negative twenty, so this one.*

Interviewer: *But it began at eighty isn't it.*

Participant JJ: *Participant KK: Yes.*

Interviewer: *What is the difference between eighty and forty it is twenty?*

Participant JJ: *The difference between eighty and forty.*

Interviewer: *Yes.*

Participant JJ: *Participant KK: It takes eighty hours for 1 person to build the houses, forty hours for two people.*

Participant JJ: *I am saying stating, one, and this will refer to one person, two, three, four.*

The two participants above stated that the function was linear, based on what they noticed to be a common difference within the sequence. Relating functions to sequence showed some flexibility in their use of routines (Ben-Yahuda et al., 2005). While it is true that a sequence with a common difference is an arithmetic sequence when sketched results in a linear function, **Participant JJ** and **Participant KK** did not show how they had a linear sequence. These two participants did not show that the graph was a straight line by using the gradient to prove that the points were collinear. They rather focused on 1 (one) variable and ignored the other. The coordinates in the function were given as (80; 1); (40; 2) and (20; 4). They focused on men and assumed that the number of men would increase from 1, to 2, 3 and 4, thus forming an arithmetic sequence. **Participant JJ** confirmed this assumption by stating: "*I am saying, one, and this will refer to one person, two, three, four*". The researcher asked them whether they had tried checking the gradients in order to ascertain that the graph's linearity. Both participants responded by stating they did not do that.

By implication, both **Participant JJ** and **Participant KK** could not show convincingly why they thought the function was linear. While these two participants may have had reason for thinking that the function was linear, they did not use their previous knowledge to check whether their assertions were correct or not. The use of routines could have been beneficial in that they could have checked whether their reasoning was correct. This would have been achieved by sketching the graph or trying to find an equation and use of characteristics of the graph they could have identified. In the next few paragraphs, the researcher discusses the choice of the majority of participants, most of whom responded by stating that the function represented an exponential function.

There were four major categories of responses from participants who stated the function was exponential. Four participants stated that it was an exponential function based on the inverse proportionality of one variable increasing and the other decreasing. Another 4 (four) participants stated that the verbal function was an exponential because they noticed a common ratio. For instance, **Participant X** stated that the verbal function was an inverse proportional, but also went on to mention that it was an exponential function. On the other hand, **Participant BB** removed the zeros and said it was an exponential function. **Participant NN** initially stated that the function was an exponential, but later changed her mind and stated that it was a hyperbola.

7.4.3 A decreasing exponential function

Four participants responded to the following question with reference to a decreasing exponential function: *A group of workers are planning to paint 10 houses in a complex. It takes 80 hours for 1 person to paint all the houses, 40 hours for 2 workers, 20 hours for 4 workers and so on. Name the function represented by the above information. Please explain your answer.*

Both **Participant Y** and **Participant LL** stated that they first sketched the graph, and found that the function was an exponential function. When the researcher probed them further to explain why they thought the function was an exponential, they responded by stating that the function could neither be linear nor a parabola. Their responses were as follows:

Interviewer: *Why do you say it is an exponential function? Why do you say it is an exponential?*

Participant LL: *Because the ... I first say if 80 hours for 1 person, then I made the sketch, the hours I entered x values and the number of people and then that is why I, as the number of workers there are increase, then the hours be less.*

Interviewer: *Ok.*

Participant Y: *I said it an exponential function because as the number of workers increase, as the workers increase, the hour decrease. So that means, if we plot it in a graph, if the x, the y increases, the y decreases in exponential function. When you plot the number of workers and the hours, we see, we get a graph like this that means it decreases.*

Participant LL: *And it is not a straight because the number of hours is not the same as the number of workers so it snot the straight*

Interviewer: *Yeah, but I think the straight line is not the only graph in a function*

Participant LL: *Nor the parabola.*

Interviewer: *Ok.*

Participant Y: *It is not a hyperbola because in a hyperbola, we get two graphs*

Interviewer: *What if I say it is a hyperbola. What will you says?*

Participant LL: *You can never have maybe 1 and a half people*

Both **Participant Y** and **Participant LL** said that they sketched the graph and found that it was an exponential function. They based their argument on that the function was decreasing function. Several graphs fit into their description of a decreasing graph. When the researcher asked them, how they made the distinction, **Participant LL** in particular stated that a linear function and a parabola did not fit into that category. When the researcher suggested the hyperbola, **Participant Y** did not affirm that a hyperbola could fit into the verbally expressed function because a hyperbola has two graphs. As a mathematical fact, a hyperbola does not have two graphs as such, but has two parts that are a reflection of each other. What **Participant**

Y did not recognise was that with restrictions on the graph, it was possible to have one part of the hyperbola emerging. For **Participant LL**, the graph could not be a hyperbola because we cannot never have one-and-a-half people. **Participant LL** recognised that while time could be in fractions, the number of men could only take integral values.

While **Participant LL** and **Participant Y** could not identify the function as a hyperbola, they were able to show their knowledge of the functions in that they could tell that the function was neither linear nor quadratic. They also acknowledged that a parabola had two parts although their expression was that it was two graphs. Both participants show their developing mathematical discourse because they were able to use another form of representation, the graphical method. They were not successful in this endeavour, but they exhibited that their mathematical discourse was developing by sketching the graph as a way of trying to find what function it was. What **Participant LL** and **Participant Y** did not do, was to check on the correctness of their assertion that the function was exponential by trying to find the equation of that function. On the other hand, not only did **Participant DD** state that the hyperbola had two graphs. He also mentioned that there were two hyperbola graphs. The structure of the hyperbola seemed to pose challenges for participants, to whom the graph was drawn on paper. In a hyperbola, they saw two different graphs, rather than a graph with two parts:

Participant DD: *I saw it in the same way but if it is a hyperbola, obviously there will have to be two hyperbola, there will be two graphs, so that is how I saw it, so that is how I decided that it will be an exponential graph but mentioning the restrictions but unfortunately here there are no restrictions.*

Participant DD decided that the function was exponential because a hyperbola had two graphs. To **Participant DD** for a function to be a hyperbola, it must have the two parts. The interview took place when **Participant DD** was in grade 12, which was the reason he spoke of restrictions. It would seem **Participant DD** was now confusing the inverse of the parabola and the hyperbola. At the same time, restrictions could have induced restrictions on the hyperbola as well. In the statement of painting a fixed number of the houses, the restrictions on the presented variables implied they could not have negative time and painting men to be integral values. The graph itself did not show that it was a hyperbola except for a close examination of the ordered pairs whose product was (80) eighty and was a hyperbola. The difficulties experienced with the identification of the function gave credence to participants' learning and objectification of the four representations of a function objectification (Nachlieli & Tabach, 2012). In the next few paragraphs, the researcher discusses how participants tried to use sequences to identify the function.

Participant U, Participant V, Participant S and **Participant T** related the exponential function to the sequences. They noticed the common ratio and the exponential function. The common ratio was related to the base of the exponential function. A geometric sequence produced the same graph as the exponential function. Following are the two respective participants' statements.

Participant V: *It is an exponential function because it has a common ratio.*
Interviewer: *What do you think?*
Participant U: *Yes, it has a common ratio.*
Interviewer: *What is the common ratio?*
Participant V: *Sir?*
Interviewer: *What is the common ratio?*
Participant V: *The common ratio is the ratio where you*
Interviewer: *What is it there?*
Participant V: *The common ratio.*
Interviewer: *Yes*
Participant V: *It is half (1/2).*

Both **Participant U** and **Participant V** stated that the function was an exponential. They claimed to have noticed a common ratio. A common ratio in a geometric sequence is the same as the base in an exponential function. In this instance, there was no common ratio referred to in the given statement. The number of men were 1; 2 and 4 so there was no common ratio as supposed by these participants.

Participant U and **Participant V** erred by considering the third pair of coordinates as though it was on the third position. They showed some flexibility because they could relate exponential functions to geometric sequence. Furthermore, they used mathematical terms such as 'common ratio' appropriately.

Like **Participant S** and **Participant T, Participant U** and **Participant V** also thought the function was exponential. They also related the exponential function to the geometric sequence even though **Participant S** and **Participant T** did not use the term *geometric sequence*. **Participant T** and **Participant S** took the *x-coordinates* as numbers in a geometric sequence, but they did not match them with the *y-coordinates* because they would have seen that 20 was the "fourth term" rather than the third, thus rendering the sequence to be invalid. This is how the participants responded in the interview:

Participant S: *The function that are represented by the above equation, you can say it is an exponential function. Because from the exponential pattern, number of pattern, we know, the… it has to be a common ratio. 'r' is the common ratio.*

Interviewer: *So what is your common ratio?*

Participant S: *So here, our common ratio will be 40 over 20, which will be two. So here, we get ...*

Interviewer: *It is what?*

Participant T and Participant S: *It is two.*

Interviewer: *How do you know that it is two? How do you know that it is correct?*

Participant S: *How do you know that it is correct, because it will be, we can find common ratio by stating second term divided by first term (t2/t1). So the first term is 80, it says 80, the first term in this case, when we were doing our own number 80.*

Participant S introduced terms such as "exponential pattern", whose common name is "geometric sequence. It is true that the geometric pattern has exponential growth, but in Mathematics it is not named as such. **Participant S** showed some flexibility by relating her knowledge of functions to geometric sequences. While her flexibility was commendable, she did not notice that the third term was missing. She also did not use her sequencing appropriately as she was supposed to have a decay and her common ratio should have been between zero and 1 (one). Her common ratio was stated as two (2) instead of 1/2 (half), according to her own assumptions. Corrigibility routines are not evident in **Participant S's** utterances because she did not check whether the common ratio was indeed two (2) or not, by using other numbers on her presumptuous sequence (Ben-Yahuda et al., 2005). Her utterances showed that she understood the calculation of the common ratio. This is demonstrated by her ability to state the formula for calculating the common ratio. She even mentions 80 as the first term, but still went on to mention 2 (two) as the common ratio. She also did not check whether the numbers formed a geometric sequence as she purported. A second look would have indicated to her that the ordered pairs were not for an exponential function. **Participant T**, who had earlier agreed with **Participant S** that the common ratio was 2 (two), eventually corrected herself as indicated below.

Participant T: *The common ratio is half.*

Interviewer: *Then. What about. How do you know that, it is correct?*

When the researcher asked her how she knew that she was correct, she responded that she used the formula. According to that formula, the second term should be divided by the first term. The researcher then suggested that both these participants could possibility be wrong, as shown below.

Interviewer: *If I say there is no common ratio, will you say that I am wrong?*

Participant S: *There is no common ratio?*

Participant T: *You will have to specific why you are saying that there is no common ratio. You must have your reasons.*

While **Participant S** exclaimed that there was no common ratio, **Participant T** responded that there should be reasons for stating so. In that regard, **Participant T** was exhibiting growth in her mathematical discourse, because mathematical narratives should have verifiable justification. Furthermore, **Participant T** knew that only mathematical reasons which they did not realise, could annul their assertion of an exponential function. On the whole, both **Participant S** and **Participant T** show a developing mathematical discourse, because they are able to relate functions to other mathematical aspects such as sequences.

In addition to the responses by **Participant S** and **Participant T, Participant AA** stated that since they had been taught geometric sequences, they realised that it related to the exponential functions. In **Participant AA's** own words, he realised that the coordinate (20; 4) was not the third term, but continued with the notion of a geometric sequence.

Participant AA: *Oh, ok. OK so sir what we saw, since we have been taught geometric sequences, we saw that an exponential function is somehow related to a geometric sequence equation. So we did here ok, it is said that if you want the common ratio of the ex ....ok in this case the common ratio ok.*

In terms of the above excerpt, **Participant AA's** narratives were based on authority. The participants had been taught geometric sequence, and they managed to make connections which showed their relationship. The relationship between sequences and functions may have prompted learners' assumptions that they were addressing exponential functions.

Participant AA: *Ok, so it is said here. In this case [drew a 2x3 table as he explains that 80 is 1, the 2 is 40, then 3 oh it is 4 it is 20]. So we took these as ok we took these as our positions as Tn [circling n in Tn and pointing to 1, 2, and 4 on the table] the T1 = 80, then T2= 40, then T3= 20.*

Participant AA contradicted himself on the table of values he sketched. He realised that the third coordinate was not the third term. He wrote it as (20; 4), but went on to identify the coordinate as though it represented the third term. This participant adhered to his notion of an exponential function. Accordingly, he then ignored the sign which indicated deviation of the function from a geometric function. Therefore, it was not an exponential function.

In this section, the researcher discussed the participants' knowledge of geometric sequences in relation to the exponential function. While it was true that there was a relationship between exponential functions and geometric sequence, the coordinates did not follow a geometric pattern as the third term was not given, but the fourth. In the latter case, participants could not

show that the coordinates were for an exponential function. They exhibited flexibility routines by being able to relate the two aspects of Mathematics. At the same time, if they had used corrigibility instead of routines to verify the correctness of their narratives, they could have realised that the function was a hyperbola rather than an exponential function. Both flexibility and corrigibility routines are important in the development of the mathematical discourse (Ben-Yahuda et al., 2005). Participants struggled with word or verbal representations of the hyperbola. It would be incorrect to suggest the learners struggled in all verbal functions. More than half of the participants correctly identified the exponential function in a verbal growth representation. In the next few paragraphs, the researcher discusses how participants responded to the verbal function on exponential function.

7.4.4 Exponential growth

An exponential function is the steepest graph (Webber, 2002). Furthermore, the exponential function is mostly useful in practical situations such as representing growth in infections, births or businesses. In sequence and series, the exponential function is a counterpart of geometric sequences. It is characterised by a fixed base and a variable exponent, hence the name "exponential function". In this study, the research participants were required to use ordered pairs for purposes of calculating the fixed base, or to sketch the graph using the points. In this study, all participants opted for calculation of the fixed base. In this sub-question, there were more positive responses than in the previous question on the hyperbola. While only 1 (one) participant identified and gave a mathematical explanation on the hyperbola, approximately 13 participants (54%) managed to deduce that the verbal statement was an exponential function. Two participants (8%) thought it was a linear function, while another two (%) thought it was a hyperbola, and 1 (one) was of the view that it was a quadratic. Six participants (25%) did not respond to the question below.

Given this scenario: Mr Mkhize, a chicken farmer, begins his poultry business with 200 chicken, in the second year, he plans to have 400 chicken, increasing them to 800 in the third, 1600 chicken in the fourth. He plans to continue growing the number of chicken in that manner for a long time.

Based on the above scenario: Name the function represented by the above information, why do you say so?

There were two main categories of responses from the participants. One group calculated the equation of the function, and the other used the geometric sequence of the common ratio. Three (3) participants (13%) formulated the equation using the statement presented earlier of chicken that increased yearly. Participants' utterances were similar in structure. **Participant MM's** response are referred to below due to the level of detail entailed.

Participant MM: *Ok. Here. Ok in this graph, this function is an exponential function.*

Interviewer: *Why do you say so?*

Participant MM: *Because of, when I calculated it using the scenario given which is above, then I have calculated it. I noticed that there is. Ok. Firstly, exponential function, says it is equation says y is equal to a times b to the power of x plus or negative p, negative q or positive p, ($y = ab^{x-p} + q$) then here, we are given, here our, Ok. When I have calculated it, our 'a' was 20, then which we multiply 2 to the power x, then we have 200. 200 is made up of 20 times5, oh 20 times 2 to the power of 3 that is 200. Then 400 here it is 20 times 2 to the power of 4, then we have 800 which is equal to 20 times 2. It is 25 not 20. 25. It is 25*

An exponent denotes algebraic representation of the exponential function (Webber, 2002. Subsequent to realising that the function was exponential, **Participant MM** used a general equation of an exponential function ($y = ab^{x-p} + q$), to show that she was correct in her assertion. She managed to calculate the base of the exponent and then tried to prove her equation was correct. **Participant MM** did not use the explicit name for the base. She then attempted to show that her algebraic equation was correct by making substitutions into her formula. She used appropriate mathematical language, because she named the function correctly and used terms such as "power" for "an exponent". She also showed some corrigibility by show the correctness of her equation by means of substitutions into the formula. **Participant MM** also showed that her mathematical discourse is developing because she was able to move from verbal expression of a function to an algebraic form of expression. Most of the participants (n=12, 50%) decided to use the geometric sequence to show that the function was exponential.

Of the 12 participants that used the geometric sequence formula to identify the verbally expressed function, the researcher discusses the responses of the 2 (two) participants who correctly stated that there was a common ratio. A geometric series was provided in the form of $f(x) = ar^{n-1}$, where *a* was the first term; *r* was the common ratio; and n the position of the term on the sequence. A number of participants concentrated on the common ratio and did not overly focus on the other component of the geometric sequence. In her explanation, **Participant HH** showed all the components of the geometric sequence.

Interviewer: *Yes, what is the equation?*

Participant HH: *Y is equal to 100 multiplied by 2 raise to the exponent x ($y = 100.2^x$)*

Interviewer: *How did you find it?*

Participant HH: *I said when you times a hundred by 2, you get 200 to the power 1, which is what you get , then you get the, and then you make x=1, when you want to work for the first 200 chickens and then when you say, and I then say 100 to the power 2, then 2 to the power 2*

which gives you a 400, and then you say 100 times 2 then we put a 3, which gives you a 800, and then you say 100 times 2 to the power 4, which gives you 1600.

Participant HH calculated the equation of the function, which was an indication that she could proceed from one representation of the algebraic function to the other with ease. She thus exhibits flexibility by not only naming the function as an exponential function, but also stating how she could prove it by showing the equation of the function. She also exhibited corrigibility by showing how the equation satisfied the statement given for this function. In addition, she also demonstrated some exploratory routines by not only stating the equation, but also proving that the equation works. **Participant HH** also demonstrates a developing mathematical discourse, as she has objectified the mathematical object, the exponential function. Contrary to **Participant HH's** mathematical discourse, **Participant KK's** utterances showed some semblance of growth, but not at par with Participant HH's.

Although **Participant KK** made a number of unmathematical statements, there was evidence of his familiarity with geometric sequences. While **Participant KK's** counterpart thought the function was a parabola because of a "common second difference", **Participant KK** maintained his stance and insisted that the function was exponential, as indicated below.

Interviewer: *How did you respond to the question?*

Participant KK: *I said it is a what, it is an exponential graph because my first term which is 200, when he began his farm and then they multiply by 2 and get what? 400. And in the next year, you multiply also by 2 and get what? You get 800. Then you multiply and so on and so on. meaning each year, each year this chicken farmer, what he is going to do is his chicken are multiply by 2, so if I can say the common difference here it is, common ratio, it is 2, it is 2, it is 2. Then I can reverse that by saying such as this the nth term or y is equal to a times plus x plus q (Tn= or $y = a.b^x + q$), and then I don't have my q then you take, then my first is 400 times two to the power x (400.2^x).*

Participant KK clearly used the geometric sequence as a base for his contention that the verbal function was exponential. He noted that 2 (two) multiplied the next term, thereby noting that the sequence has a common ratio. Participant KK also stated the general formula for the geometric sequence. When he tried to apply his formula, he did not succeed because he had a first term as 200, and it was highly improbable that he could have 200 from 400.2^x. While **Participant KK** has some knowledge of the relation between the geometric sequence and the exponential function, he could not perform the algebraic manipulation. He was able to interpret the symbolic mediator expressed in the form of words because he recognised that the worded statement represented an exponential function. He also managed to recognise that the equation used to find the *nth* term of the geometric sequence was similar to the equation by stating the

general formula and its counterpart 400.2^x. However, he could not perform correct algebraic manipulation because 400.2^x does not represent the worded statement. On that basis, the classified his routines as ritualised. As much as he had developed his mathematical discourse, his algebraic manipulation was not fluid (Sfard, 2012).

When interpreting the symbolic visual mediator $y = a.b^x + q$, **Participant KK** stated that there was no q in $400(2)^x$. That the q value was not written, does not mean the q value is absent. It only means that there is no vertical translation. If the function is not translated vertically from the parent function, then the value of q is zero and does not need to be written. **Participant KK** did not interpret the symbolic visual mediator mathematically (Sfard, 2012). He continued to explain himself as captured below.

Participant KK: *I say my nth term is same as a times b to the power n minus 1 (ab^{n-1})*
Participant JJ: *Yeah*
Participant KK: *Then I will, if I can change this to be in the form of y is equal to a times b to the exponent x minus 1 (ab^{x-1}), I get my y is equal to 200 times 2 to the exponent x minus 1 ($y = 200.2^{x-1}$). Because if I can say 200 times two divided 1, therefore I get my first term which is 200 and then saying, 200 times two divided by two minus 1, I get my 400, which is my second term, and saying 200 times two to the power three I get my second which will be 800.*

Participant KK emphasised the relationship between the geometric sequence and the exponential function by stating that the nth term is the same as that of an exponential function, to which his counterpart **Participant JJ** agreed to. **Participant KK** then recovered from his earlier utterances where he had written an incorrect formula. The researcher could not attribute this change to corrigibility routines as there was no evidence of something he did to recover. He used the unsimplified geometric sequence formula. The algebraic manipulation of $y = 200.2^{x-1}$ yields $y = 100.2^x$. **Participant KK** went further to prove that his formula was correct when used to show the number of chickens in each successive year. Participant KK in this instance showed some flexibility and corrigibility routines. Flexibility routines were shown by taking the worded statement and expressing it in algebraic form. Corrigibility was also displayed by showing that his equation was correct using three of the four coordinates provided in the worded statement (Ben-Yahuda et al., 2005). **Participant KK** demonstrates that he was moving towards objectification of exponential function by expressing the worded function as an equation and proving how the ordered pairs were calculated. He further demonstrates exploratory routines by formulating the equation and showing how it works.

During the interviews **Participant KK's** partner (Participant JJ) was of the view that the function was quadratic, due to a common second difference. This showed that for the participants, there is a strong link in learners' mathematical discourse between sequences and

functions. Participant JJ did not have convincing reasons for stating that the function was a parabola. Following is how Participant JJ argued her case.

Participant JJ: *I said, it is the quadratic function or a parabola.*

Interviewer: *Why? I am interested in the reason.*

Participant JJ: *... because it has the second common difference.*

Interviewer: *Second common difference, ok*

Participant JJ: *Yes …*

Participant JJ's assertion of the second difference indicates that the function is quadratic in nature. It also shows that her narratives could be classified as substantiation because they are the same as those of the community of mathematicians. However, **Participant JJ** made an unproven assertion earlier. She could not prove that there was a common second difference. The researcher gave her the opportunity to prove her assertion, but she could not. This attests to **Participant JJ's** challenges in arithmetic manipulation as her utterances show below.

Participant JJ: *Then. Press it on the calculator. Then first, he got here, we got negative 200, then it is 400, then it is 600.*

Interviewer: *Are you sure.*

Participant JJ: *Yes, I am sure sir.*

Interviewer: *What is 1600 – 800?*

Participant JJ knew that a sequence with a common second difference is quadratic. As shown above, her main challenges were located in the area of subtraction. It is not that she could not subtract, but she did not check the correctness of her answers. This has been a challenge noticed throughout the study, in terms of which participants made assertions without checking for the correctness of what they thought the functions presented. Therefore, **Participant JJ** is not the only participant whose mathematical discourse is at granular stage, but has room for further development (Caspi & Sfard, 2012). In the next few paragraphs, the researcher discusses the respective participants' responses to the question on exponential growth.

Participant NN and **Participant GG** were of the view that the function was linear, but they did not have corroborating evidence for their assertion. After their conversations with the researcher, these two above-mentioned changed their stance to adopt the stance that the function was exponential.

7.5 Table of Values
A table of values presents the *x* and *y* values that satisfy a given equation. The *x-values* represent the input and the *y-values* the output. All these values need to satisfy an equation that represents a particular function. All 24 participants identified the graph as an exponential function, although their justifications were different. The responses of 8 (eight, 33 %)

participants focused on the asymptote; 10 (42%) sketched the graph to prove that the table of values represented the exponential function; and the last 6 (six, 25%) spoke of the proportionality of the numbers and the common ratio. These participants were responding to the question below:

1.1 Name the function represented by the above table of values, state reasons for your answer
1.2 How would you identify the asymptote from the table of values?

The table below shows the ordered pairs for a certain function: $x \in \mathbb{R}$

Table 7.1: Ordered pairs of the function $x \in \mathbb{R}$

x	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	-0,998	-0,996	-0,99	-0,98	-0,97	-0,94	-0,88	-0,75	-0,5	0	1	3	7	15	31	63

The participants expected the asymptotes on the table of values to denote the asymptotes by an error or undefined on the output section of the table of values. In the study, the participants were presented with an exponential function whose asymptote was horizontal and required an undefined on the input. For example when **Participant LL** was asked how she would identify an asymptote from the table of values, she responded as follows:

Participant LL: *It maybe an error*
Interviewer: *Only that? If there is no error or undefined, then there is no asymptote?*
Participant LL: *No sir, that what's the graphs, if there is an error in y, then the y values will be the asymptote.*

The intention of the question was for participants to state how they would identify the asymptotes from a table in general. The error which **Participant LL** refers to, appears in a table of values only for the vertical asymptote when the undefined part of the denominator is not a zero of the numerator. Otherwise, it will be just a removable discontinuity. **Participant LL's** assertion that an error always denotes an asymptote is partly correct. It is not always a certainty that in instances of discontinuity, there would be an asymptote and exhibits ritualised routines. Accordingly, the researcher then classified **Participant LL's** narratives as "memorisation based on visuals", because her assertions were based on what she had seen at face value; rather than from her investigations and explorations on the mathematical object.

Participant Y, who is **Participant LL's** interview partner identified the table of values above as an exponential function. When the researcher asked for her reasons, she responded that the function had only one asymptote. Further probing by the researcher resulted in the following response by **Participant Y**:

Interviewer: *Why do you say exponential?*
Participant Y: *Because it has only 1(one) asymptote.*

Interviewer: *What is the asymptote?*

Participant Y: *It is zero.*

Interviewer: What is it?

Interviewer: And where did you get your zero?

Participant Y: *From the .... From negative three (-3), when x is negative three (-3) or negative one (-1), it is negative one (-1). Because the y values, when you sketch it, it decreases, it will never touch negative one (-1). And by looking, when you look at the positive side, the graph is increasing...it not the same as this one and when it gets to negative one (-1), when it gets close to negative one (-1) it becomes -0.9998, as it decreases, it does not get to the actual number negative one (-1).*

Emanating from the statements above, it is axiomatic that **Participant Y** was able to identify the function as representing the table of values, but could not justify herself for stating so. At first, **Participant Y** justified her statement on the basis that the function had only one asymptote. When she was probed further to explain how she identified that asymptote from the table of values, she mentioned another representation of the function. She then explained that the graph she sketched had an asymptote of -1 (negative one) and not the 0 (zero) she talked about. **Participant Y's** use of words was nearly mathematical because she did not talk of the relationship between the *x* and *y* coordinates, but referred to a decrease on the negative side and an increase on the positive side. An exponential function either increases or decreases, but couldn't be both increasing and decreasing on the same function. Her interpretation of the exponential function was not construed because it did not fit the description of an exponential function. The researcher then classified her routines as ritualised mathematical, because she knew that an exponential function has one asymptote despite her failure to explain how she could identify it from the table of values. She could not give a mathematically acceptable explanation of the function's movement. Her narratives were based on what she saw. There was a side of the graph increasing, but the coordinates were not both negative. For **Participant Y**, it meant the graph was decreasing, and for the part of the positive coordinates, she mistook it to be an increase. She used a table of values as a means of arriving at a graph, rather than seeing it as a representation of a function in its own right.

Participant Y was not the only participant who identified the function as an exponential but could not provide mathematical reasons for stating so. For instance, **Participant OO** also stated that the table of values represented an exponential function. When asked to explain how he knew that the function was exponential, **Participant OO** spoke of the graph not touching 1 (one). While an asymptote is related to an exponential function, there was no need for **Participant OO** to use it in that instance. Below is the researcher's depiction of **Participant OO's** interpretation of the table of values.

Participant OO: *It is an exponential*

Interviewer: *Yes exponential. However, why do you say so?*

Participant OO: *If we represent this data on a graph. The graph will not touch one. It will come close to it. You can see that it means the asymptote is one.*

Participant OO also used the table of values as a means of drawing a graph rather than a representation of a function. His explanation shows that he did not regard a table of values as one of the representations of a function which could be interpreted such that the features of a function could be brought to the fore. From the table of values, one could derive such features as the intercepts, asymptotes, the point at which the graph was increasing or decreasing, points of discontinuity, and the name of the graph. This is the curriculum's underlying intentions for learners to be able to convert the four representations of a function flexibly (DBE, 2011).

Participant OO used the presence of an asymptote as the means by which the presence of the asymptote is shown. The presence of the asymptote does not mean that the function is exponential as there are other functions which have asymptotes as part of their features. Mentioning the presence of an asymptote is not the most convincing reason. Furthermore, **Participant OO** stated that the asymptote was 1 (one), yet an exponential function y is equal to one (y= 1) features in the middle of the function and cannot be the asymptote.

Participant OO's use of words was non-mathematical. He mentioned that the graph would not touch 1 (one) instead of stating that the graph would not intersect with the asymptote or line y was equal to 1 (one) (y=1). **Participant OO** mentioned an asymptote as though it were a number by stating that it is 1 (one). The researcher, therefore classified his use of words as "non-mathematical". **Participant OO** had challenges in interpreting the table of values, to such an extent that she had to sketch a graph in order to obtain a picture of what function it was. The researcher then classified her interpretation of the table of values as "not construed", because he interpreted the function using another representation. **Participant OO's** narratives were classified as "memorisation based on visuals" because his talk was based on what he saw rather than on generalisations he found through the exploration of the table of values. Therefore, the researcher classified his routines as ritualised.

As one of the four representations of a function, the table of values needed to be interpreted with some key features of that function being recognised. **Participant AA** tried to explain the function from the table of values themselves, rather than using other representations such as the graph. This is demonstrated in the following explanation by **Participant AA**:

Participant AA: *It is exponential*

Interviewer: *Why do you say so?*

Participant AA: *You can tell that the graph is approaching the asymptote from the table by because we can see, you can see the difference between these terms it's not the way that it is eemm between the y- axis, between the y terms it is decreasing. The difference between the terms it is decreasing. At first, Ok the first term let us say the first position x value it is 5, then you can have the y value is 63, then the second term it is 4, second term of x is 4, then the y term or the y value it is 31 then the difference between them is 32. Then moving between ok the difference. Ok the fourth. OK the third term which means x on x=3, y=15, then the difference between the y values is 16, therefore the decrease shows that it is an exponential function.*

While **Participant AA** could not clearly articulate himself, it was not difficult to understand his ideas. At first, he spoke of the function approaching the asymptote. He went on to show that the differences between the successive output values had a pattern, which was provided with the number that had exponents of 2 (two). He mentioned 32, and then the next output value was 16. He then concluded that the difference showed that the graph was exponential.

When **Participant AA** mentioned that the "difference between the terms was decreasing", it was due to his interpretation of the graph from the right to the left. He mentioned the differences from the largest numbers to the smallest. He also referred to the last term as the first, but he moves in the opposite direction. When reading the table of values backwards, he named some terms first when they were far from being the first. He also alluded to the y-values as decreasing, yet the graph was increasing. The researcher therefore classified his use of words as colloquial. **Participant AA** explained the table of values as if the function moved from right to left. He described the function as decreasing and because of that; the researcher classified her interpretation of the iconic visual mediator as being not construed. His narratives were memorisation based on what she could see. Therefore, his routines were ritualised non-mathematical.

In this section, the researcher discussed how participants interpreted the table of values that represented an exponential function. The participants attempted to explain the table of values using the asymptote. Their use of words was predominantly "colloquial". Their interpretation of the table of values was largely "not construed", and their routines were mostly "ritualised mathematical". In the next few paragraphs, the researcher discusses the mathematical discourse of participants who sketched the graph as a way of identifying the function represented by the table of values.

While **Participant U** and **Participant V** claimed to have drawn the graph represented by the table of values, they did not have the evidence to show that they skethched the graph. The other reason was that **Participant U** and **Participant V** did not answer follow-up questions directly,

and that made it difficult for their exposure to mathematical discourse. The following responses show how **Participant U** and **Participant V** responded to the question*: What function was represented by the table of values given above.*

Participant V: *It is an exponential function*
Interviewer: *What is the reason for saying so?*
Participant V: *Because it also has a common ratio*
Interviewer: *Where is the ratio, what is the ratio?*

Participant V mentioned that the function was an exponential because it had a common ratio. While it was true that an exponential function had a common ratio, **Participant V** did not explain or state what the common ratio was. That made it difficult to examine the mathematical discourse of the participant. He did not respond to the researcher's question concerning the ratio and **Participant U** continued to respond in the following manner:

Participant U: *Besides the common ratio, since you have the values, the x values and the y values, when you draw the graph taking the values, it will show you that it is an exponential*
Interviewer: *Oh, you drew the graph*
Participant U: *No, I did not draw it. I drew it on a rough paper.*

Instead of responding to the question on the common ratio, **Participant U** shifted the attention to sketching of the graph. It was from that graph that he obtained the idea of the graph as an exponential function. When asked to show evidence of the graph he sketched, he responded by stating that he did it on a rough paper and could not bring the required evidence attesting that the graph was an exponential function. As a result, it was difficult to obtain the mathematical discourse of **Participant U** on the table of values.

Participant U referred to the coordinates as the x and y values. The coordinates consisted of x and y values. It is worth noting that the appropriate name is coordinates, instead of values. As values, the naming of the coordinates was not the most appropriate way of expressing coordinates. The researcher then classified **Participant U's** usage of words as "colloquial", because he referred to coordinates as values. In the community of mathematicians and the public at large, the term "values" means something different from "coordinates". Hence, the researcher classified his use of words as "colloquial" (Sfard, 2008).

Participant NN and **Participant GG** also spoke about sketching of the graph. They initially tried to check whether the graph was linear in terms of finding the collinearity of the function. When they discovered that the function was not linear, they decided to draw the graph using the coordinates on the table of values provided.

Participant NN: *There is 63, 31, and 15*

Participant GG: *OK, Let us draw the graph*

Participant NN: *Let us draw the graph for from 0 to*

Participant GG: *Ok, let us begin.*

Participant GG: *From the first quadrant. OK it is minus... I was telling you this is the first quadrant. One, OK 1, 2,3,4,5. OK the y values are from 3,7,15, ok then 1. I think this should be something such as; OK 1 goes with this one, and then seven [writing]. It is going to be something such as this [pointing]? Straight? No not straight. Exponential. I think it is an exponential. OK.*

Participant GG and **Participant NN** sketched the graph from the positive side of the x-axis. Sketching the graph from the y-axis indicates that they did not have to plot all the coordinates to determine the shape of the function in order to identify the function represented by information on the table of values. What **Participant NN** and **Participant GG** indicated that they could not identify the function from the table of values. In other words, **Participant GG** and **Participant NN** could only identify exponential growth from the graph or the algebraic representation of the function. These two participants showed that the function was not linear by calculating the gradient. When they could not determine the kind of graph they were engaged with, they agreed to sketch the graph in spite of using the word "draw". Such exploratory actions from learners of Mathematics enable their mathematical discourse to move towards objectification.

Participant JJ and **Participant KK** also plotted the coordinates of the table of values so that they could identify the function. **Participant JJ** and **Participant KK** thought there was no other way of plotting the graph. The following is a reflection of their response to the question on the function on the table of values.

Participant JJ: *Name the function represented by the above table of values, state reasons for your answer. I said it is an exponential*

Interviewer: *Why. Why is it exponential*

Participant JJ: *it is because, it is an exponential because, when you plot.*

Interviewer: *OK. What did you do to see that it is exponential?*

Participant JJ: *Because when you plot x and y-values you get your graph.*

Interviewer: *Did you sketch it as well*

Participant KK: *Yes sir. You get a cap*

Once more, **Participant JJ** and **Participant KK** correctly named the function as an exponential function, but did so only after sketching the graph. **Participant JJ** emphasised that the plotting of points produces a graph, and that only happens when there is a special relationship between the coordinates. Both **Participant JJ** and **Participant KK** did not

respond to questions that required them to show their reasons for taking action. When the researcher asked **Participant JJ** for reasons of the function being exponential, she responded repeatedly that coordinates were plotted. While **Participant JJ** could identify the exponential function from the graph, she could not state what made the exponential function different from other functions. **Participant JJ** could also sketch the graph using the ordered pairs from the table of values, but she could not explain why the graph was exponential as an indication of ritualised routines. **Participant JJ** plotted the points, and decided that the graph was exponential, but could not explain why the self-same graph was known as an exponential function.

Participants in the section above identified the exponential function by sketching the graph. The table of values was used only as a means of plotting points so that they could identify the function. The table of values was not taken to be a representation of a function. Otherwise, there could have been attempts to identify the function from a table of values. In the next few paragraphs, the researcher discusses how some participants spoke of the common ratio as a way of identifying the function represented by a table of values.

A common ratio is a component of a geometric sequence which follows an exponential function. Therefore, it was no far-fetched for participants to seek for a common ratio in an exponential function. **Participant S** stated she was somehow disturbed by the decimals on the table of values. She then decided that the function was not exponential because of the decimals and failure to calculate and finding the common ratio. **Participant S's** statements below are indicative of her responses to the table of values.

Interviewer: *What kind of a function is that?*

Participant S: *This. I think we did not write the question because we were so much disturbed by these decimals.*

Interviewer: *The decimal*

Participant S: *Yes.*

Interviewer: *What is wrong with the decimals?*

Participant S: *Because at first I thought it was an exponential but when I tried to look for a common ratio, it did not come out. Therefore, I did not try for it*

Interviewer: *So in an exponential there is a common ratio every time*

Participant T: *In exponential graph?*

Interviewer: *Yes, that is*

Participant S: *In an exponential graph that has a common ratio which a pattern represent a function. That is what we were taught.*

Interviewer: *That is what you were taught?*

Learner 2: *We were taught*

Participant S and **Participant T** did not respond to the question on the table of values because of the presence of the decimal numbers. **Participant S** attributed their non-response to being disturbed by the decimals. When the interviewer further asked how the decimals disturbed the interpretation of the graph, it was revealed that the response had nothing to do with the decimals. **Participant S's** response suggested that decimals could not be part of the table of values, yet the continuous graph suggested that all real numbers were part of the function. The choice of the input values that were predetermined to produce integral output values was mostly responsible for the notion that said decimal fraction cannot be part of the graph. Decimals are part of the number system and therefore, part of the table of values.

Participant S further stated that she could not find the common ratio from the table of values. It was less onerous to calculate the common ratio from data where the function was a parent with no transformation. The common ratio in an exponential function will be the base of the exponent. In the question, the base was not easy to notice because of the vertical shift of -1 unit. **Participant S** gave up on the attempt to investigate the table of values because she could not find the common ratio. She immediately attributed her narrative to the teacher. While it was true that all exponential functions were actually the geometric series, it did not follow that when learners fail to calculate the common ratio then the graph would not be an exponential function. **Participant S's** assertion gave credence to the view that she was actually taught that every exponential function has to be identified by a common ratio. Such a state of affairs demonstrates that her routines were ritualised mathematically (Sfard, 2012). She had the correct mathematical concept, but had not yet owned it. She needed to give her own reasons for whatever mathematical ideas she held rather than mention, "This is how we were taught".

Participant S's interpretation of the mathematical symbols was not mathematical. As axiomatic as it is, decimals are part of the numbering system. In fact, all continuous graphs have the decimals somewhere within them. The challenge that **Participant S** faced was that she had only seen the table of values where only integral values were used for the convenience of calculation. Therefore, the researcher classified **Participant S's** interpretation of the table of values as not construed because she interpreted the table of values as being representative of only integral values. This indicated that when she drew/sketched a graph, she did not notice that the graph passed through some coordinates that were not integers. Using the DPHEF, the researcher classified **Participant S's** routines as ritualised non-mathematical. While she could draw graphs but interpreted a table of values as being composed of only integers. The researcher further classified her narratives as authority based, because she attributed her

mathematical discourse to what she had been taught rather than what she explored and discovered herself (Moalosi, 2014).

7.6 Graphical Representation

When some learners talked of functions, they are referring to the graphical representation. This was brought about by the fact that when functions were learnt, these functions culminated with the drawing or interpretation of the graph. As expected, 21 (87%) of the 24 learners who participated in this study positively identified the two graphs, namely the hyperbola and the exponential function. The three (13%) participants who did not positively identify the graphs did so because they did not respond to the questions posed to them. In this part of the analysis, the researcher specifically discusses participants' mathematical discourse on a statement which required learners to "Explain what happens to the function as *x* approaches infinite".

Functions are distinguished and named according to their behaviour, which is conspicuous through the graphical representation of the function. For instance, the characteristics of hyperbolas are such that they had two parts which are a reflection of each other and asymptotes which are perpendicular to each other. As stated in the previous paragraph, participants could identify the two (hyperbola and asymptote). However, 13 participants (54%) had challenges describing the behaviour of the graph as the graph approached the infinite from either side. In the next paragraphs, the researcher discusses to the question: "W*hat happens to the graph as x or y approaches both positive and negative infinite"?*

Participants encountered problems when describing the behaviour of the graph as it became closer to infinity. Eleven participants stated that the graph would approach the asymptotes for the hyperbola. The researcher analysed **Participant GG's** and **Participant NN's** responses to the question on the behaviour of the graph because they came close to explaining the behaviour of the graph at the required instances. The following statements reflect **Participant GG's** and **Participant NN's** response to the question: *Which is the asymptote, explain what happens as function x approaches infinity.*

Participant GG: *In the hyperbola as x is approaching the infinity, there are going to touch the asymptote sir*

Participant GG: *And Participant NN: Closer*

Participant GG: *Yes, it will be closer; I think it is supposed to be closer. Yes.*

Participant GG: *Ok, and then the exponential, same applies.*

Participant NN: *It will get closer to the asymptote.*

Participant GG: *Yeah closer.*

Participant GG had initially stated that the hyperbola graph was going to touch the asymptote, but then **Participant NN** interjected to mention that it was closer rather than a touch. When

Participant GG mentioned, "They are going to touch the asymptote", she actually meant that the asymptote and the graph would intersect. As stated by these participants already, the graph became closer to the asymptote as the *x* values tend to infinite. The following diagrams (Figure 7.1) from geogebra indicates that the asymptote and the graph appear to intersect.
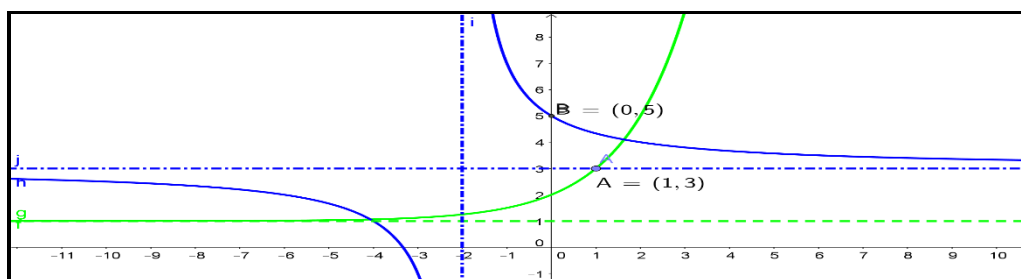
Figure 7.1: Representation of the hyperbola and the exponential function

In Figure 7.1 above, the exponential function and its asymptote appear to be coinciding. In actual fact, they do not. This diagram may have affected **Participant GG's** utterances, but she was corrected quickly and accepted the correction. This situation shows that her narratives on the behaviour of the graph was affected by what she saw.

After **Participant NN** corrected **Participant GG**, she went on to state that the same applied to the exponential function. As *x* approaches infinity in a hyperbola, the graph approached the asymptote. However, the graph appears exponentially large in an exponential function. Accordingly, **Participant GG's** routines were classified as ritualised. While her utterances were correct for a hyperbola, the same was not true for an exponential function. This meant that **Participant GG's** routines on functions appeared to be ritualised since she did not distinguish the behaviour of the hyperbola graph from that of an exponential function. What **Participant GG** did was common in the learning of Mathematics. Learners exhibit objectification of the mathematical object only on one aspect. When the researcher asked her to examine her utterances, **Participant GG** described the behaviour of the graph as x approached negative infinity. This shows that **Participant GG** conducted some self-correction. Initially, she said both graphs would become closer to the asymptote as *x* approached infinity. The dialogue below illustrates that the two learners' mathematical discourse was ritualised.

Participant GG: *So now am talking about exponential. As x approaches the negative infinity, the graph will get closer to the asymptote sir.*
Interviewer: *Ok fine.*
Participant GG: *And then it approaches the positive infinity, it will go away from the.*
Participant GG: *And Participant NN: It will move away from the x-axis.*
Participant NN: *Closer to the y.*
Participant GG: *Yes*

Participant GG clarified that the behaviour of the graph x approached negative infinity rather than the positive side. In explaining how the exponential graph would behave as x approaches infinity, **Participant GG** and **Participant NN** mentioned that the graph would move away from the *x-axis*. It did not show how the graph behaved. Both these participants also agreed that the graph would be closer to y. Any graph is composed of the x and y-coordinates. It was difficult following her statement that the graph would be close to the y. At first, **Participant GG** and **Participant NN** seemed to have the mathematically acceptable narratives on the behaviour of the hyperbola graph, but their utterances indicated that their routines were ritualised. These two participants' words were not mathematical as they used utterances such as: "The graph moves away from the x-axis", or "The graph would be closer to y". It is unlikely that the community of mathematicians agrees with the two learners' explanations of the behaviour of the exponential function. **Participant GG's** and **Participant NN's** explanation of the behaviour of the hyperbola as x approaches infinity was classified as "construed". As *x* approaches infinity, the graph gets closer to the asymptote. At the same time, when explaining the same for an exponential function, the two participants' responses contradicted the earlier classification and interpretation of "construed". This led to the researcher classifying their routines as "ritualised" because they did not objectify the mathematical object – the function. These two participants successfully described the behaviour of the hyperbola, but could not do the same with the exponential function. They were not the only participants who struggled to explain the behaviour of the exponential function.

Meanwhile, **Participant FF** admitted that it was difficult describing what happened to the exponential graph as *x* increased towards infinity. It would seem that little attention was paid to the behaviour of the graph, especially on the extremes when learning about the functions. **Participant EE** did not show a distinction between an exponential function and a hyperbola in her utterances. Both **Participant EE** and **Participant FF** demonstrated some degree of a growing mathematical discourse by partly explaining the behaviour of the hyperbola. Below is an excerpt of their response to questions on the behaviour of the hyperbola and an exponential function.

Interviewer: *What happens to the graph as x approaches infinity?*
Participant FF: *As x approaches infinity? Which graph is x?*
Interviewer: *No, x in any graph. What happens to the graph as x approaches infinity? Let us begin maybe with an exponential function. As x approaches infinity, what happens to the graph?*
Participant FF: *These ones are tough*

Participant FF seemed not to understand the question. For her, *x* was supposed to be a graph. In functions, it is axiomatic that x is a coordinate, and it cannot be confused with a graph. In fact, **Participant FF** did not respond to the question. She only stated that the question was tough. While she was able to identify the graphs as the hyperbola and the exponential function, she could not explain the behaviour of the graph towards the extremes. As a result, the researcher classified her word use as "not mathematical", because she thought that *x* was a graph. Her interpretation of the graph was "not construed ",because she could not describe the behaviour of the graphs that she identified. Additionally, her narratives were not in concord with those of the community of mathematicians. She thought *x* was a graph. While **Participant FF** could name the graphs, she did not only describe the behaviour of the graph but also showed some confusion with some components of the graph as she referred to *x* as a graph.

Participant EE: *Ok so the exponential is such as so and then we increasing function so it could go higher. I think*

Interviewer: *The x -values will go higher or y values will go higher or the graph will go*

Participant EE: *Sorry, the exponential*

Participant EE's mathematical lexicon may have hindered her from freely articulating his thoughts on the behaviour of the exponential function. He gestured with his hands as he indicated that the function would increase exponentially. Evidently, language became a barrier for **Participant EE** as he tried to explain himself. He stated, "…so it could go higher", referring to the exponential function. Any function could get higher. His ambiguous description could fit in any function. When it came to the hyperbola, both these participants were more comfortable in their narratives, and found the appropriate mathematical language to express themselves.

Interviewer: *Ok, its fine, it fine, what about the hyperbola.*

Participant FF: *Ok sir, so as x approaches the infinity in the hyperbola, it goes closer and closer to the asymptote even though it will never touches it, but it will go closer and closer.*

Interviewer: *Ok and the. Yes as x approaches negative infinity.*

Participant FF: *In the hyperbola*

Interviewer: *Yes in all of them, hyperbola and exponential*

Participant FF: *It does the same thing but in the negative side*

Interviewer: *Where do you get that?*

Participant EE: *It is what we were taught.*

Participant FF: *Textbook.*

Participant FF described the behaviour of the hyperbola as it approaches the asymptote. She emphasised that the graph would approach the asymptote but would not come into contact with

it. While that was true that the the exponential function and the hyperbola graph did not intersect the asymptote, it was not true for all asymptote functions. **Participant FF** mentioned that the same would happen as x approaches negative infinity. When the interviewer asked them to justify their narratives, they attributed their utterances to authority. **Participant FF** explained that she was taught that graphs behaved that way, while **Participant EE** attributed his narratives to the authority of the textbooks. When learners rely on the textbook or teacher for their narratives, it indicates that they have not yet explored the mathematical object. As a result, their routines are ritualised (Sfard, 2012).

7.7 Conclusion

The empiricalluy-based discussions in this chapter revealed that learners' mathematical discourse on the hyperbola and exponential function was still developing. There were signs of growth in their mathematical discourse because they could communicate mathematically, although they could not properly name some of the objects.

While learners could identify the two algebraic representations of the functions, they could not explain the relationship between the parent function and the transformed function. None of the learners mentioned the translations in their explanations. Instead, they explained the translations in terms of asymptotes.

Learners often justified their actions by imitating their communication from their teachers, and attributed their reasons for their talk to the very teachers. The learners did not give reason for their action, except that it was how they were taught. That gave credence to the view that learners' narratives were based on authority. In the event that the teacher gave the rule, that rule had to be followed. The discourse of learners attributing their actions to their teachers still shows a discourse of others instead of a discourse of their own.

Learners managed to show that the equation of a geometric series was the same as the equation of the exponential function. When learners are able to recognise the relationship between two separate topics in Mathematics, it shows that their mathematical discourse is still developing. Notwithstanding the latter, learners still struggled with the hyperbola expressed in words. Once more, the challenge was premised on the parent function. Most of the learners thought that the hyperbola was also an exponential function, posing the problem of saming two distinct functions.

While learners' routines were generally ritualised, their narratives were mathematical. Most of what learners wrote or said would earn them marks in an examination. However, the challenge would prevail in the event that reasons were required for why they wrote what they have written. This is an area in which learners were found most wanting. In questions that are more than just recall of facts, learners' mathematical discourse was found to be lacking and needed

some improvement. There were few instances in which flexibility and corrigibility were noticed from learners' mathematical discourse.

In the next chapter (Chapter 8), the researcher concludes the study and discusses some relevant and critical recommendations accruing from the findings of the study.

CHAPTER 8

SUMMARY OF MAIN CONCLUSIONS, FINDINGS, AND RECOMMENDATIONS

8.1 Introduction

Researchers such as Tachie and Chireshe (2008) have documented inordinate research studies on poor learner attainments in Mathematics in rural areas such as Mthatha in the Eastern Cape Province. Various reasons have been advanced for this lack of achievement, including lack of both human and materials resources, poor teacher attendance, lack of innovative teaching techniques, as well as teacher absenteeism. In the current study, the researcher found that one of the reasons for the poor learning of Mathematics was that learners often did not progress beyond ritualised learning of Mathematics, and that mathematical objects in learning of functions were rarely explored. The learning of Mathematics rarely progressed beyond the context of authority (teacher, textbook or any other sources presented to learners).

The main aim of the study was to investigate the mathematical discourse of learners in a rural area in the learning of functions with particular attention to the asymptote of hyperbola and exponential functions. Rural schools are generally characterised by poor infrastructure, unqualified and underqualified teachers and a high rate of absenteeism by both teachers and learners (Lewis, 2007; Tachie & Chireshe, 2008). Using the DPHEF analytical tool, the researcher discovered that rural school learners proved that they have learnt the functions, and their routines were generally ritualised mathematical, as opposed to the ritualised non-mathematical variant. In ritualised mathematical routines, learners did what their teachers taught them and consequently thought that was acceptable in Mathematics. In the context of non-mathematical rituals, the learners' thinking is often not mathematical.

The researcher investigated learners's word use in functions, as well as their communication regarding the asymptotes of the hyperbola and the exponential function. While learners could draw the asymptotes on a graph, that did not necessarily mean they have objectified the notion of an asymptote. When learners spoke of the asymptotes, they exposed a gap between their mathematical discourse and the mathematical discourse of the community of mathematicians. The researcher has already addressed learners' views on asymptotes in Chapter 6 of this study.

The researcher also investigated the learners' mathematical discourse in the four representations of a function with particular attention on the hyperbola and the exponential functions as prescribed in the CAPS curriculum of the department of Basic Education's requirements. The learning of functions begins with ordered pairs. These order pairs satisfy a certain algebraic formula and culminate with the characterisation of a certain group of functions because of their behaviour. In the learning of the functions, all four representations of the functions are important and their interconnections need to be flexible for learners to objectify the mathematical object known as a function. The researcher found out that while learners

could work with the representations of a function in isolation it became difficult for learners to show some inter-connectedness.

8.2 Main Conclusions

The study's main conclusions reflect primarily on those critical methodological and theoretical aspects that advanced the researcher's intentions for undertaking this investigation (de Vos et al, 2011; Leady & Ormrod, 2013). In this sub-section, the researcher presents conclusions derived from the utilisation of the DPHEF analytical tool. In general, there was not very much of a difference between what is said about learners' learning on functions from DBE reports, and what the researcher discovered in this study. While there is evidence of teaching taking place in the schools, the learning of Mathematics could still be enhanced using the routines, especially flexibility and corrigibility routines as they are partly responsible for moving learners from ritualised routines to exploratory routines. The researcher has summarised the findings in Figure 8.1 below.
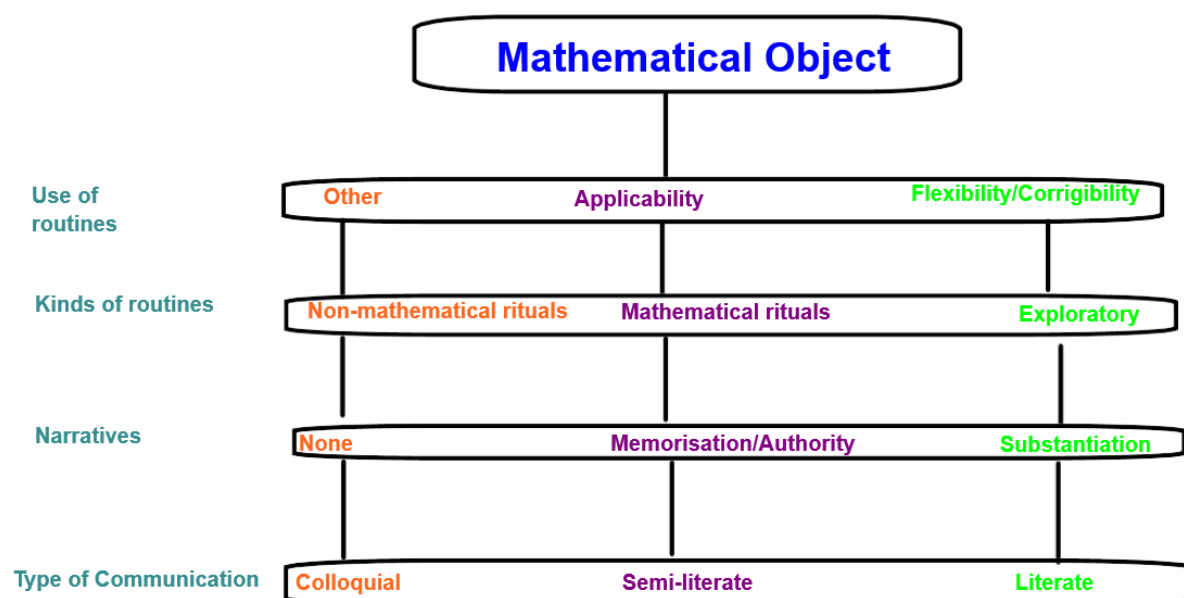
Figure 8.1: Researcher's summary of findings

Using the DPHEF, the researcher was able to categorise the types of mathematical discourse into three groups and demonstrated how different characteristics of the mathematical discourse linked to each other. For learners to communicate in a literate manner, they needed both flexibility and corrigibility routines as dominant routines. Flexibility relates to the ability to use more than one method in solving mathematical problems (Ben-Yahuda et al, 2005). For example, learners' mathematical discourse on exponential functions showed signs of literate communication, because they (learners) related exponential functions to geometric sequences. The ability to use more than one approach improved learners' chances of acquiring exploratory routines. The usage of both routines became the foundation for literate discourses and narratives because learners tended to objectify the mathematical object by using the literate

language in the endorsement of routines. Mathematical discourses developed when learners were encouraged to use flexibility and corrigibility routines in their development of exploratory routines. In this regard, learners' language, both written and verbal, improves with flexibility and corrigibility routines.

The applicability routines were explored intensely in this study. The importance of the applicability routines is premised on the fact that they are foundational to the development of exploratory routines. Furthermore, applicability routines gave rise to ritualised mathematical routines in terms of which learners tended to do and say what was said and done by their teachers. Accordingly, learners' mathematical reasons were based on what they were taught by their teachers, leading to ritualised mathematical routines characterised by communication that was not backed by cogent reasons. Consequently, the applicability routines dominated learners' mathematical discourse, thus rendering the classification (categorisation) of their communication between "colloquial", "literate" and "memorisation" dominated by learners' overwhelming reference to rules. Learners with dominant applicability routines also referred to the authority as the 'prescriptive' reason for their mathematical actions. While applicability routines were necessary for learners' mathematical development, learners should not settle on these routines alone. They need to add flexibility and corrigibility routines in order that their mathematical discourse is viewed as congruous with that of the community of mathematicians.

There are instances where there is no use of routines. In such contexts, the communication will not be mathematical. The routines were known to be ritualised according to the commognition theory. To distinguish the mathematical ritualised routines from those that are incorrect, the researcher termed the unmathematical routines as non-mathematical ritualised routines, which were characterised by colloquial communication.

The learning of Mathematics as reflected and measured by public examinations, is more than just covering the topics in the prescribed curriculum. There is a great need to inculcate learners' flexibility between all the four representations of mathematical functions. Various aspects of the study's empirically generated evidence attest to the conclusion that teachers and learners should purposefully work towards the intensification of learners' word use from colloquial to literate mathematical discourse. This could be achieved by asking learners to define routinely used terms in the Mathematics classes. Learners should also be encouraged to verbalise their thinking by explaining their mathematically induced actions.

8.3 Main Findings of the Study
The findings of any study highlight its significance to the broader research community, while also giving an indication of the extent to which the self-same study was able (or not able) to accomplish its objectives, resolve the investigated problem or phenomenon, and answer the

research questions (Babbie & Mouton, 2010; Kumar, 2012; Yin, 2018). In this regard then, the study's findings become the practical application and justification of its relevance and usefulness. All of the researcher's observations cited in Section 8.1 bear reference to the study's broader intention of investigating learners' mathematical discourse on the hyperbola and exponential functions. To this end, the study was guided by the following research questions:

What discourses do grade 11 learners in rural areas display when learning functions?

How does the use of words (lexicon or mathematical discourse) afford or constrain the participation of learners in the functions discourse?

What is the nature of grade 11 learners' mathematical discourse relating to the four different representations of functions related to the hyperbola and the exponential functions?

What is the nature of learners' participation in mathematical discourse on the asymptote of the hyperbola and exponential function in Grade 11?

The following below-mentioned research objectives are a response to the above-cited research questions, clearly indicating the symbiotic link between the research questions themselves, the research objectives, as well as the extent to which the study achieved (or did not achieve) its stated objectives (Babbie & Mouton, 2010; de Vos et al., 2011):

- *To understand the mathematical discourse of learners in a rural setting with regard to mathematical functions*
- *To explore and identify the lexicon used by learners in expressing themselves in the context of mathematical functions*
- *To explore, describe, and explain the mathematical discourse of learners on the four representations of a function, namely: functions expressed in words, as algebraic form, in the form of ordered pairs, and in graphical form*
- *To explore and define the learners' experiences and perspectives pertaining to understanding the asymptotes of the functions*

8.3.1 Findings Relating to Learning of Functions

In this study, the realm of the findings specifically focused on two categories of findings, namely: the findings relating to learning of functions, as well as findings in relation to previous studies on grade learners; mathematical learning. The findings relating to the learning of functions are further categorised into four specific areas, namely: grade 11 mathematical discourse in rural settings; the use of words in mathematical discourse; the hyperbola and exponential functions in the context of four different representations; as well as learners' participation in the context of the asymptote of the hyperbola and exponential function.

8.3.1.1 Discourses used by learners when learning functions

What learners communicated in word or in text form, translates into their thinking as well (Assad, 2015; Howie, 2003). Learners could easily work with familiar questions, which are those that they learnt in class every day. For example, 90% (n=100) of the 112 learners could write an equation of the asymptote from a formula. On the other hand, 95% (n=106) of the learners calculated the intercepts of the hyperbola expressed in a familiar form; for example, $f(x) = \dfrac{3}{x-1} - 2$. When the same question was asked in a different format, learners tended to struggle, as only 45% (n=50) of the learners were able to express the asymptote of an exponential function written in the form of $f(x) - 5$. The difference in the format of the question resulted in half of the learners (those who were able to write the asymptote) failing to express themselves mathematically.

While learners could sketch graphs, there seemed to be little understanding of what the graphs meant. For example, learners sketched $f(x) = -x^2 + 4x + 12$ and $g(x) = (\frac{1}{2})^x - 1$, but only 20 (18%) of 112 learners could interpret the meaning of the point of intersection by marking A and B as the solution to the $-x^2 + 4x + 12 = \left(\dfrac{1}{2}\right)^x - 1$. This is an indication of learners' uninformed mathematical meanings of graphs, despite their noted ability to mechanically draw those graphs. While most of the learners could interpret mathematical words such as "sketch" "symbolic visual mediators" and "the graph of f and g", they displayed inadequate knowledge of the relationship between the point of intersection of the graphs and the equation. In such instances, the researcher concluded that the learners' mathematical discourse was largely ritualised.

8.3.1.2 The use of words in the functions discourse

Learners defined the asymptote as a line which the graph approaches without cutting through it (the graph). Such a definition of an asymptote is not entirely mathematically acceptable, since there are instances of the horizontal asymptote intersecting with the graph several times. While their definition will affect their learning on functions in the future, it is acceptable at grade 11 level. In their sketching of graphs, some learners still did not clarify whether the asymptote was a point or a line. In this regard, there was both ambiguity and synonimity on the asymptote and the removable discontinuity. The table of values was construed as points that were discontinuous, rather than coordinates representing the path through which the graphs pass through. Therefore, they referred to the asymptote as a point.

Furthermore, some learners did not see the asymptote as an equation or a graph, but a number. They often stated that: "The asymptote is two". Some learners mentioned that they wrote

asymptote as a function for fear of losing marks. They intimated that teachers marked them down in the event that they did not express the asymptote as an equation.

8.3.1.3 The mathematical discourse of learners in relation to the four representations of the hyperbola and the exponential functions

The curriculum requires learners to move flexibly between the four representations of the functions (DBE, 2011). Learners' mathematical discourse of the algebraic and graphical representations of the two functions was more developed than that of worded and tabular representations. Worded representations are usually tested in the examinations more than the tabular form; hence, the emphasis on algebraic representation and the graphs in learning and teaching. Learners' examination-orientedness induced a state of affairs in which they seemed more at ease with the identification of algebraic representations of the hyperbola and exponential functions. However, when asked to explain the relationship between the parent function and the translated functions, they could not explain themselves. In word use, some learners referred to the formula as a graph. The usage of the same name for two distinct mathematical objects, displays a deficiency in the unsaming of terms (Nachlieli & Tabach, 2012).

While learners could draw and work with aspects of the functions such as the intercepts and asymptotes, they hardly explained what happened with the translations, which some learners explain without using the word the "translations" of what they wrote and what they knew. The latter view justifies the perspective that their routines were generally ritualised, because they gave "authority" as the reason for their actions asked to substantiate on their responses.

The ability of the learners to relate exponential functions to geometric sequences demonstrates their developed flexibility in mathematical discourse on functions, because they could relate to mathematical topics. When learners are able to relate topics in Mathematics, they consequently improve on their chances to reach exploratory routines (Ben-Yahuda et al., 2005).

Regarding the table of values, there is an indication that there is a disjuncture between the table itself and the graphs derived from the self-sametable of values. Learners often called the asymptote a point, showing that they did not see the order pairs as part of a continuous graph.

8.3.1.4 The nature of learners' participation in mathematical discourses of the asymptote of the hyperbola and exponential functions

The study found that generally, the mathematical discourse of learners is comprised of ritualised and applicability routines. Learners would correctly name a mathematical object, but struggle to explain how the function relates to the parent function. The path to exploratory routines passes through the ritualised routines. The mathematically ritualised routines are a

necessity for all learners that are to succeed in the learning of Mathematics. Ritualised routines, be they mathematical or non-mathematical, are part of the learning process.

The study also found that in their interpretation of the table of values, some learners named the asymptote as a point and not as a line on a Cartesian plane. Such an erroneous interpretation shows some disjuncture between learners' spoken narratives and their written engagement with graphs.

Furthermore, the study revealed the extent of learners' misreading or misinterpretation of the asymptote as a boundary which did not allow graphs to pass through. The implication is that learners were not seeing the graph moving towards the asymptote as the graph moved towards infinity at extremes. They viewed it as an 'opaque' wall through which the graph was prevented from passing through. This is largely attributed to their description of the asymptote according to what they saw, rather than from their exploration of the graph and its asymptote.

It was established that most of the learners exhibited ritualised routines. While there were instances in which learners showed some forms of exploration by explaining the relationship between the asymptote and the graphs, most of the learners would write a mathematically acceptable answer, but talk incongruently. For example, learners spoke and wrote of vertical asymptotes of an exponential function by stating that the asymptote is $x = -9$ for a function $f(x) = 3^{x-1} - 9$. Learners whose routines were categorised as non-mathematical, are still far from growing in discourses and cannot imitate the interlocutor. Such rituals are a cause for concern, as it is difficult to move such learners towards the growing discourses.

On the use of routines, the researcher found that most of these were applicability based on what learners viewed. Learners' descriptions were based on what they saw as the parameters for either a vertical or horizontal translation of the asymptotes.

8.3.1.5 The mathematical value of the DPHEF
The DPHEF is an analytical tool used to analyse the mathematical discourse of the learners who participated in this study. From the researcher's point of view, the analytic tool, the Discourse Profile of the Hyperbola and Exponential Function (DPHEF) is uniquely credited with the systematic unbundling of learners' mathematical discourses in the context of the worded, numerical, tabulated, and graphic asymptotes of the hyperbola and exponential functions. The DPHEF was **personally** developed by the researcher as an extension of the Discourse Profile of the Hyperbola (DPH). The latter analytical tool was used by the researcher for his Master's degree research project. The DPH was developed from the Arithmetic Discourse Profile (ADP) of Ben-Yahuda et al (2005). The major additions of the DPHEF to the DPH is that the ritualised routines are divided into ritualised mathematical and ritualised non-mathematical categories.

The utilitarian value of the DPHEF translated into the researcher's efficient separation of the ritualised mathematical routines from the ritualised non-mathematical actions. terms of the ritualised mathematical routines, learners act in mathematically acceptable ways but cannot provide mathematical reasons for the particular routine. In the case of the non-mathematical routines, learners generally provide mathematically incorrect statements in relation to the particular mathematical routine.

The DPHEF was designed to separate the differences in learners' mathematical discourse, and can be utilised for any mathematical function. The DPHEF is not confined to the hyperbola and the exponential function only. It is the researcher's contention that the DPHEF will effectively contribute to the knowledge, perceptions, and experiences of the learners in the learning of asymptotes of the hyperbola and exponential functions. The DPHEF has two categories of word use, the mathematical and the colloquial. The mathematical use of words is defined as a form of communication about which all members of the community of mathematicians have the same interpretation (Tachie & Chireshe, 2013). The colloquial usage, on the other hand, premises on everyday and non-mathematical discourse, or a combination of both.

In the context of this study, Table 4.1 (p. 62) in this study is the quintessential example of the application of the DPHEF's mathematical value by its eclectic integration of the four key characteristics of the mathematical discourse. In this regard, examples, descriptions, and classification of mathematical discourses were systematically analysed as indicated in sub-section 4.4.1 (pp. 59-61) in this study. On the whole, the DPHEF is mostly credited with the systematisation of analysing multiple sets of data to establish a credible framework of the study's evidence-based approach to data collection and usefulnesss of that same data.

8.3.2 Findings in Relation to Previous Research

While it has been an expectation that learners' mathematical discourse would largely be ritualised, the researcher did not expect most of their routines to be mathematical. The study confirms what other researchers found in the past. Learners' mathematical discourse was mostly visual. They described what they saw, thus corroborating the findings by Mpofu and Pournara (2018). In this study, learners would refer to the asymptote in a table of values as a point because they "saw" distinct coordinates instead of points that were a representation of a function.

In this study, the researcher found out that learners experienced difficulties with functions. Such difficulties included word problems, especially word problems involving the hyperbola. The learners struggled with the interpretation of the table of values of an exponential function, and could not generate the formula of the graph even though they had a number of ordered

pairs from the self-same graph. This is not a surprising. Previous research by Berger (2013), Kidron (2011) and Flesher (2003) found that learners and teachers struggle with functions.

Furthermore, the researcher found that there was a mismatch between what learners wrote and what they said. Learners would write an asymptote as $y = -9$, but speak of the asymptote as -9. In Mpofu and Pournara (2018), learners drew diagrams with asymptotes and spoke as though there was no asymptote. The mismatch is a cause for concern because it shows that learners' mathematical discourse has not developed enough. The mismatches are evidence of ritualised routines. Notwithstanding the evidence of teaching, the mismatches show that learning does not always happen in teaching. Explorations enhanced by means of reflective imitation are the only way that will improve learner achievements. The above findings confirm that there are difficulties in the learning of functions in particular and Mathematics in general (Fraenkel et al., 2012).

8.4 Recommendations

In research, recommendations are propositions developed by the researcher in order to validate the authentic relevance and contribution of the study (Rajasekar et al., 2013; Ramenyi & Bannister, 2013). In addition, the recommendations are a reflection of the study's juxtaposition of the research problem and research objectives on the one hand; as well as the data collection and analysis processes and the findings of the study, on the other (Kumar, 2012; Walliman, 2011). On the whole, recommendations are further intended to benefit the study as a reliable evidence-based point of reference for improvements in the teaching and learning of high school mathematical discourses.

8.4.1 Recommendations pertaining to teaching and learning

To a greater or lesser degree, the findings of this study provide an explanation for the Mathematics results of the National Senior Certificate producing less than three percent (3%) of learners who score more than 80% of the marks (DBE, 2018). In fact, the 2017 National Senior Certificate examinations diagnostic report alludes to 61.1% of learners obtaining marks between 0% and 49% in Mathematics. For learners to benefit from participating in the mathematical discourse, their scores should be in the region of 60%. The ritualised learning of Mathematics is not sufficient for learners that seek to further their education and take careers that need Mathematics.

In this study, there was evidence of teaching. There is no single learner that made a positive contribution, be it in the test or during the task-based interviews. The challenge is that the participants did not go beyond ritualised learning. In as much as reflective imitation encourages explorative learning, teachers need to create any environment that is conducive to producing explorations. Explorations moves learners from discourse of others to their own discourses.

Often, learners attributed their reasons for action to the teacher. They would often say, "This is how we were taught", or "If I wrote it the way I say it, my teacher will mark me wrong". Teaching should help learners explore the mathematical object before generalising. For example, on asymptotes, they should not just be told that the asymptotes are the parameters in a standard equation. That should come as a generalisation after learners have worked with several graphs and discovered and linked algebra to the graph. Explorations are improved by the types of tasks given to learners. Tasks should go beyond the basic of introducing features and rules that govern functions. Learners should be asked for reasons pertaining to their responses to questions. In addition learners should be able to explain generalisations on functions and perform unfamiliar tasks.

Mathematical language including symbols is part of mathematical discourse. Learners tend to use the language that is "tolerated" or allowed in Mathematics classes. Unmathematical language use inhibits the learning of Mathematics. In this study, learners regularly used expressions such as "the asymptote is four"; or they will say, "When you see undefined, then you know there is an asymptote". Learners do not see anything wrong with the mentioned language and would be surprised when the teachers mark them down for communicating in that manner. Learners often referred to the parameters "p" and "q" as the asymptotes. Given this state of affairs, learners will not have a correct image of an asymptote. They become confused in the event that an asymptote is a function or a number.

More emphasis should be placed on the connections between the four representations of a function. While learners could work with basic algebraic and graphical representations of functions, they could not show the same efficiency with functions in tabular form and those expressed in words. In some instances, the disconnect between the representations were apparent. Learners would draw a function with an asymptote represented as a linear function, but talk as though an asymptote was a number. When expressing the asymptote in algebraic terms, learners would also talk of an asymptote as though it were a point. The exploratory learning of Mathematics could be seen in how learners relate the different representations of the functions with flexibility.

8.4.2 Recommendations for future research

Future research is definitely needed in the improvement of learners' mathematical discourses. Such research could be repeated with learners engaged in learning during lessons in various demographically representative context. The study would help unearth the source of the unmathematical language that learners use. The study will further bring to the fore, teacher and learner habits which lead to the failure of using mathematical language. The study may also give researchers an opportunity to know the kind of task and exemplification that happens in the classroom, and how it promotes or stifles the development of the mathematical discourse.

Subsequent research should also focus on teachers' mathematical discourse on functions. interlocutors make mathematical discourse on functions available to learners. This makes teachers important role players in the learning of functions. The study may focus on exemplification and task choices by teachers during their teaching as examples of tasks which determine the types of routines that learners will have as they engage in mathematical problems.

8.5 Limitations of the Study

The limitation of the study determined the "conceptual/ theoretical, operational, empirical, and methodological constraints" that have the potential "to weaken or reduce" the "scientific worth" of research (Rajasekar et al., 2013: 598). It is important to note that (potential or actual) limitations, restrictions, or shortcomings do not necessarily compromise the value of the study itself. Instead, the limitations cited below are an indication of the study to contribute significantly towards the improvement of Mathematics teaching and learning in South African schools.

The findings of this study are generally limited to the 112 learners who were selected from four schools in the Mthatha District of the Eastern Cape. Such a geographic confinement has the potential to limit the generalisability of the findings to a wider or broader regional, provincial, or South African context. The numerical representativity of the study may be construed as not reflecting a broader grade 11 learner cohort. The number of participants who took part in the study were about 30 high performing learners who participated voluntarily. The researcher had intended to have 120 learners participating in the test in the initial task of this study but could only obtain 112 learners. The researcher then intentionally had task-based interviews with six high performing learners from each of the four schools. The study reflects their views and a different group may probably produce a different outcome. What has been found in this study is a reflection of the participants, and may not be applicable to another group.

English is the language of learning and teaching in all the schools that took part in this study. Textbooks are written in English, and learners' assessments are also conducted in English. In an area where one language is dominant, it is not far-fetched to assume that the local language is used sporadically or occasionally. In this study, the interviews were conducted in English. In some instances, however, learners used their local isiXhosa language.

8.6 Reflections on the Study

This study helped the researcher to grow both as a Mathematics teacher and as a researcher. The focus on learning helped he researcher realise the difficulties that both teachers and learners experience in the teaching and learning of Mathematics. The theory of Commognition helped to distinguish mathematical learning that is predominantly ritualised against the

expected goal of having exploratory routines. At face value learners appeared to meet the requirements for learning Mathematics. The DPHEF analytical tool was designed from commognitive theory by the researcher. This tool helped to generate the realisation that the mathematical discourse of learners was not adequate to make them achieve objectification. Teachers need to work together with their learners in order for them to reach a level in which better results could be attained.

Mathematical mediators used in the classrooms should not just be used without their explanations. For example, when learners view a table of values it should be clear that the ordered pairs on the table of values are not the only coordinates for the functions as most functions are composed of real numbers although the table of values will only have integral values coming into the fore. Sometimes learners looked at the table of values and thought that the coordinates were arbitrary, but sketched continuous graphs. As a teacher, the researcher has learnt that there are gaps in the learning of Mathematics which teachers take for granted; such as the interpretation of the table of values, and hope that learners will learn them because from practice. The researcher also thought of how mathematical information is made available to learners. ***Does the teacher bring the information to learners refined and ready for use or should learners discover information for themselves?*** In this study, learners often referred their reason for actions to what their teachers said or did. The discourse should move from being the discourse of the teacher, to being the discourse of learners. The researcher has further learnt that the reason learners do not perform at exceptionally good levels is the discourse of others.

One of the lessons learnt is that language is important in the learning Mathematics. The researcher was very enthused by listening to learners using their language to try to explain mathematical concepts. However, their home language posed difficulties, as the mathematical language was lost in translation. The use of the local language allowed some confidence in their speech. The researcher decided that it was conducive for learners to express themselves in their home language, but teachers need to provide them with literate language. The balance has to be struck between the use of the local language and the language of teaching and learning. When teachers use literate language, learners should then be expected to try practising the language of teaching and learning as this would become the means by which entry into the literate discourse is accomplished.

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Appendix A: UKZN Ethical Clearance Letter

UNIVERSITY OF ™
KWAZULU-NATAL

INYUVESI
YAKWAZULU-NATALI

22 February 2017

Mr Sihlobosenkosi Mpofu 216076925
School of Education
Edgewood Campus

Dear Mr Mpofu

Protocol Reference Number: HSS/2009/016D
Project title: Grade 11 learners' participation in the functions discourse: The case of a hyperbola and exponential function

Full Approval – Expedited Application

In response to your application received 15 November 2016, the Humanities & Social Sciences Research Ethics Committee has considered the abovementioned application and the protocol has been granted **FULL APPROVAL**.

Any alteration/s to the approved research protocol i.e. Questionnaire/Interview Schedule, Informed Consent Form, Title of the Project, Location of the Study, Research Approach and Methods must be reviewed and approved through the amendment /modification prior to its implementation. In case you have further queries, please quote the above reference number.

PLEASE NOTE: Research data should be securely stored in the discipline/department for a period of 5 years.

The ethical clearance certificate is only valid for a period of 3 years from the date of issue. Thereafter Recertification must be applied for on an annual basis.

I take this opportunity of wishing you everything of the best with your study.

Yours faithfully

....
Dr Shenuka Singh (Chair)
Humanities & Social Sciences Research Ethics Committee

/pm

cc Supervisor: Dr V Mudaly
cc. Academic Leader Research: Dr SB Khoza
cc. School Administrator: Ms Tyzer Khumalo

Total Marks 40

Time: 50 mins

Instructions

- This test shall be used for research purposes only
- Attempt all questions
- Show your working and explanations on the answer sheet provided
- The test is not for marks and you under no obligation to take part and you are free to stop writing whenever you feel like doing so.
- Your participation in this study is highly appreciated

QUESTION 1

Consider the function $f(x) = \dfrac{3}{x-1} - 2.$

1.1   Write down the equations of the asymptotes of f.   (2)

1.2   Calculate the intercepts of the graph of f with the axes.   (3)

1.3   Sketch the graph of f.   (3)

[8]

QUESTION 2

Consider the function $f(x) = \left(\dfrac{1}{3}\right)^x.$

2.1   Is f an increasing or decreasing function? Give a reason for your answer.   (2)

2.2   Write down the equation of the asymptote of $f(x) - 5$.   (1)

[3]

QUESTION 3

John and Nash went to a coffee shop. They read the following on the notice board in the shop:

When coffee is poured out, its temperature θ (in degrees centigrade), t minutes after pouring, is given by the formula:

$$\theta = 60\left[2^{\frac{-t}{15}}\right] + 20$$
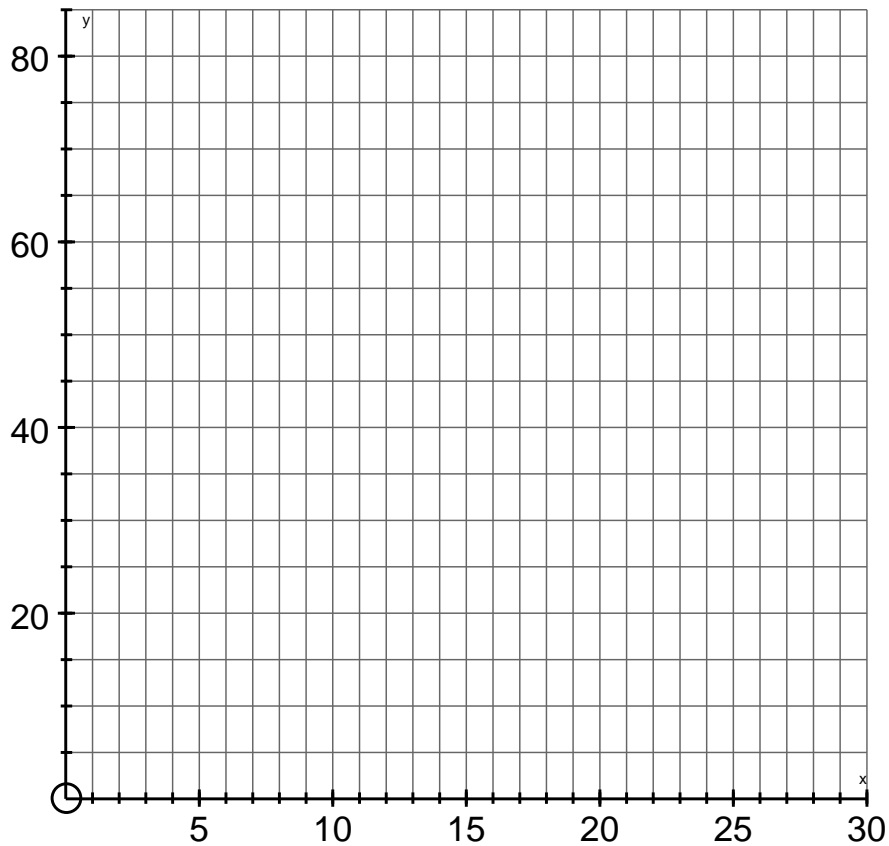
3.1 Complete the following table:

t	0	2	4	8	10	12	20	30

θ								

(2)

3.2 Draw a continuous graph of the data on the table, below: (2)

3.3 What type of graph is this? (1)
3.4 Write down the equation of the asymptote. (1)

3.5 What do you think the real life meaning of the asymptote is in this question? (1)

[7]

QUESTION 4

4.1 On the same set of axes, draw sketch graphs of the following functions: (Clearly show all intercepts with axes, turning points and asymptotes.)

$$f(x) = -x^2 + 4x + 12$$

$$g(x) = \left(\frac{1}{2}\right)^x - 1$$

(9)

4.2 Show with A, B etc. where you will read the solution to the equation:
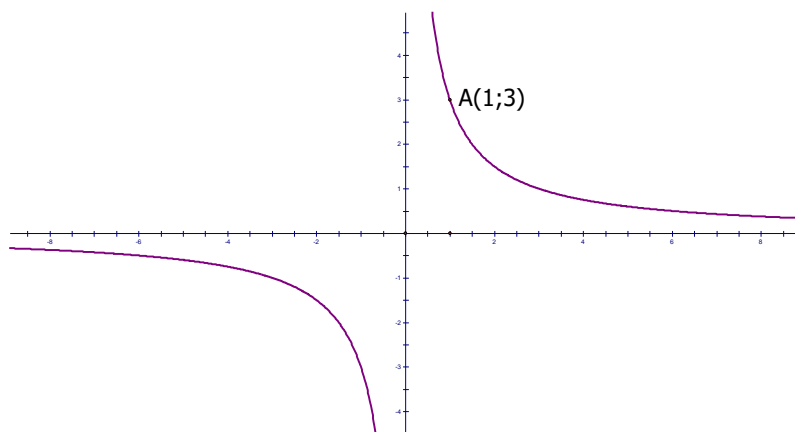
$$-x^2 + 4x + 12 = \left(\frac{1}{2}\right)^x - 1$$ (2)

[11]

QUESTION 5

Given the sketch of $g(x) = \dfrac{k}{x}$

5.1 If A is the point (1;3), determine the value of k.   (1)

5.2 Use the sketch of $g(x) = \dfrac{k}{x}$ to draw sketches, each on its own set of axes, of the following, showing intercepts with the axes as well as one other point on the graph. Draw and label the asymptotes where necessary.

$\qquad$ 5.2.1 $y = \dfrac{k}{x} - 1$ (3)

$\qquad$ 5.2.2 $y = \dfrac{k}{x-2} + 2$ (3)

$\qquad$ 5.2.3 $y = 2.\dfrac{k}{x}$ (2)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ **[9]**

Task-based Interview Schedule

Name	Response
Task 1: Given: $f(x) = 3^{x+1} - 9$ and $g(x) = \frac{2}{x+3} + 1$	
1. Name each of the functions given above and explain how f relates to $y = 3^x$ and g relates to $y = \frac{2}{x}$	
2. What is the asymptote of f? How do you know this?	
3. What is the asymptote of g? How do you know this?.	
4. Explain how you would identify key features like the intercepts and asymptotes in a table of values for the above functions f and g.	
5. What is an asymptote and how would you identify it in (i) a graph (ii) equation?	

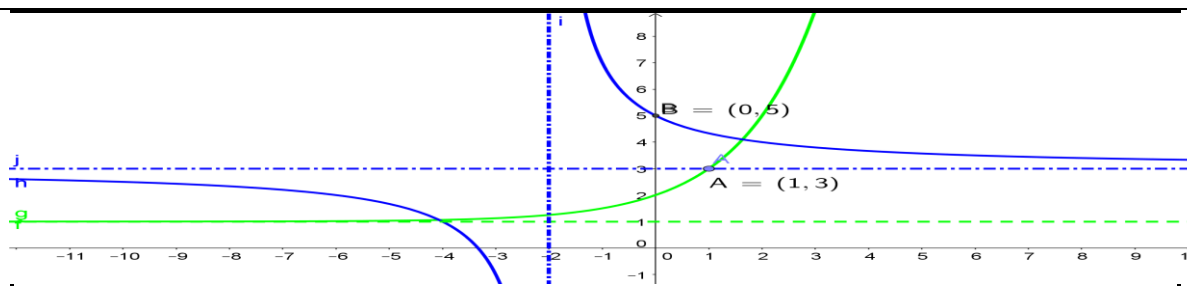| Name	
Task 2	
1. Given this scenario: A group of workers are planning to paint 10 houses in a complex. It takes 80 hours for one person to paint all the houses, 40 hours for 2 workers, 20 hours for 4 workers and so on.	
1.1 Name the function represented by the above information, why do you say so?	
1.2 What equation fits the above information and why do you say so?	
1.3 How would you express the above information as a graph?	
1.4 Are there any restrictions? Why do you think so?	
1.5 Write other ordered pairs for the above information which will help illustrate the behaviour of the graph, at least 3 points	
1.6 Are there asymptotes for this function? What reasons can you give for your answer?	
Given this scenario: Mr Mkhize, a chicken farmer, starts his poultry business with 200 chicken, in the second year he plans to have 400 chicken, increasing them to 800 in the third, 1600 chicken in the fourth. He plans to continue growing the number of chicken in that manner for a long time	
2.1 Name the function represented by the above information, why do you say so?	
2.2 What equation fits the above information and why do you say so?	

2.3 Express the above information as a graph?	
2.4 Are there any restrictions? Why do you think so?	
2.5 Write other ordered pairs for the above information, at least 3 points	
2.6 Are there asymptotes for this function? What reasons can you give for your answer?	

| Name | |

Task 3: Use the 2 following equations to answer questions:

a) $\dfrac{y-2}{2^{x+4}} = 3$ b) $(y-3)(x+2) = 4$

1.1 Identify, with reasons, the functions given above	
1.2 How would you express each of these equations in standard form?	
1.3 Name asymptotes, if any, from the above functions and give reasons for your answers	

Task 4 Grade 11

Task 4:

1. The table below shows the ordered pairs for a certain function: $x \in \mathbb{R}$

	-10	-9	-8	-7	-6	-5	-4	-3						
	-0,998	-0,996	-0,99	-0,98	-0,97	-0,94	-0,88	-0,75						

1.1 Name the function represented by the above table of values, state reasons for your answer	
1.2 Does this function have an asymptote? Why do you say so?	
1.3 Identify key features of this function (Intercepts, domain, range, etc.)	
1.4 What is the algebraic representation of this function?	

| Name: | |
| Task 5 | |

The graph below shows 2 functions.

1. Name the functions shown in the above diagram	
2. Explain what happens as to the functions as a) x approaches infinite. b) x approaches negative infinite c) y approaches infinite d) y approaches negative infinite	

Informed Consent Letter to School Principal

Dear Principal

Ms/Mrs/Mr/Dr……………………………………

Name of school …………………………………..

Re: Permission to conduct a research study in your school

I am writing to request your permission to conduct a research study in your school. This research study is entitled:

Grade 11 learners' participation in the functions discourse: The case of a hyperbola and exponential function

My name is **Sihlobosenkosi Mpofu** and I am currently studying towards a PhD Degree at the University of KwaZulu-Natal (UKZN). As part of the requirements of this degree, I am required to complete a research thesis. This study focuses on learning of functions by Grade 11 learners.

I require 30 Grade 11 Mathematics learners of mixed gender to participate in this research. I would be very grateful if you would consent to these learners participating in this study. They will be selected from your school.

If you agree to this, they will be invited to a test, task-based pair interviews and a text review of their exercise books.

All discussions, interviews and dialogues with participants will be video and audio recorded using a video and voice recorder, and thereafter transcribed verbatim to produce transcriptions. This research information (data) is required for the analysis of data and completion of the actual write up of the thesis. Collecting research information for this study will take approximately an hour for a period of about 6 weeks. All tests, task-based interview, and text analysis will take place on the school premises, with your permission. Times and dates will be discussed and arranged with you and the participants at a later stage. I will try to ensure that this takes place during their lunch breaks and free periods, in an attempt to avoid any disruptions during lessons. Participants will also be encouraged to eat their lunch during discussions, interviews and activities, as well as make use of the school toilet should the need arise. I will not deprive them of these opportunities, especially since I intend to use some of their free time in order to collect sufficient data for my study.

Data generation activities will also take place in one of the classrooms with your consent. If I am unable to collect my data during school hours, I will make arrangements with your consent and that of my participants' parents/guardians, to perhaps do this after school hours, on days when school closes early or during weekends. I will also provide transport for some of my participants to return home, should the need arise.

Please note:

* Times and dates of this data generation process will be at your sole discretion. I have merely presented you with an outline of what I intend to do, however you are free to make any changes and suggestions, if necessary.

* Participation is completely voluntary and participants have the right to withdraw from this study at any time. They will not be penalised if they choose to do so.

* Confidentiality and anonymity will be maintained at all times. The identity of your school and all participants will not be revealed at any time, as pseudonyms (different names) will be used to protect everyone's right to privacy.

* Any information provided by the participants will not be used against them, or against the school, and will be used for purposes of this research only.

* Participation in this study will not result in any cost to your school or the participants.

*Neither the participants nor your school will receive financial remuneration. However costs incurred by participants as a result of their involvement in this project will be covered.

* This study does not intend to harm the participants in any way.

* Both parents/guardians as well as participants will be handed letters of consent which they will have to carefully read and sign, before I begin data collection.

I may be contacted at:

Email address: sihlobosenkosi@gmail.com

Tel: 0792364875

My supervisor's contact details are:

Email address: Mudalyv@ukzn.ac.za

Tel: 031 2603682/0829770577

You may also contact the Research Office through:

Mariette Snyman

HSSREC Research Office,

Tel: 031 260 8350 E-mail: snymanm@ukzn.ac.za

If you would like any further information or if you are unclear about anything, please feel free to contact me at any time. Your co-operation and consent will be greatly appreciated.

If you grant permission to conduct this research at your school, please complete the form below and return to me.

Warm regards

Sihlobosenkosi Mpofu

Appendix E: School Principal's Declaration Letter

I …………………………………………………………… (full name/s of school principal) of …………………………………….….. (name of school) hereby confirm that I understand the contents of this document and the nature of this research project, and I consent to the participants participating in this research project. I also grant permission for my school to be used as the research site.

Additional consent

I understand that interviews will be audio-recorded and I grant permission for this.   YES/NO

I understand that the participants and the school are free to withdraw from the research project at any time   YES/NO

SIGNATURE OF SCHOOL PRINCIPAL: …………………..……… DATE……………..………

Appendix F: Participants' Information Sheet

DATE: …………………………..

Dear Learner

My name is **Sihlobosenkosi Mpofu** and I am a **PhD Learner** in the School of Education at the University of KwaZulu-Natal.

I am doing research on **learning of functions.**

My investigation involves giving a test individual participants for a period not more than an hour and a task to six participants working in pairs that takes about 40 minutes followed by an interview that may take about 45 minutes. All material needed for the project shall be provided.

I was wondering whether you would mind if I asked you to be part of this research.

I need your help with writing a test, a task and participating in an interview. During the interview a pair of participants shall discuss their answers with a few questions coming from me. The discussion shall take about 45 minutes.   I will do a video recording but none of your names shall be mentioned as my interest would be what you say and do on paper. I shall have an audio recorder as a backup in case something goes wrong to the video recorder. Photos will be taken of the working on functions you shall do. None of your faces shall appear on these photos.

Remember, this test is for research purposes only, it is not for marks and it is voluntary, which means that you don't have to do it. Also, if you decide halfway through that you prefer to stop, this is completely your choice and will not affect you negatively in any way.

I am inviting you to take part in this research.

I will not be using your own name but I will make one up so no one can identify you. All information about you will be kept confidential in all my writing about the study. Also, all collected information will be stored safely and destroyed after 3 years of the completion of the project.

Your parents have also been given an information sheet and consent form, but at the end of the day it is your decision to join us in the study.

I look forward to working with you!

Please feel free to contact me if you have any questions.

Thank you

SIGNATURE

NAME: Sihlobosenkosi Mpofu

ADDRESS: 23 Ebony Street, Primrose

EMAIL: sihlobosenkosi@gmail.com

TELEPHONE NUMBERS: 0792364875

Appendix G: Learners' Consent Form

Please fill in the reply slip below if you agree to participate in my study called: Grade 11 **learners' participation in the functions discourse: The case of a hyperbola and exponential function**

My name is: _____

Permission for my written work to be used for the study	Circle one
I agree that answers to a task during an interview can be used in this study only	YES/NO

Permission for test and task

I agree to write a test and a task during an interview for this study.	YES/NO

Permission to be audiotaped

I agree to be audiotaped during the interview	YES/NO
I know that the audiotapes will be used in this study only.	YES/NO

Permission to be interviewed

I would like to be interviewed for this study.	YES/NO

I know that I can stop the interview at any time and don't have to answer all the questions asked. YES/NO

Permission to be photographed

I know that I can stop this permission at any time.	YES/NO
I know that the photos will be used in this study only.	YES/NO

Permission to be videotaped

I agree to be videotaped during the interview.	YES/NO

I know that the videotapes will be used in this study only.   YES/NO

Informed Consent

I understand that:

- my name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- I do not have to answer every question and can withdraw from the study at any time.
- I can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed 3 years after the completion of the research project.

Sign_____ Date_____

<div align="right">DATE:</div>

Dear Parent

My name is **Sihlobosenkosi Mpofu** and I am a **PhD learner** in the School of Education at the University of KwaZulu-Natal.

I am conducting research on **learning of functions**.

My research involves giving a test and a task to participants and then ask them to discuss the answers where I would interject by asking for clarification in the statements they say. I shall video record all the proceedings.

The reason why I have chosen your child is because he/she is keen learner of mathematics.

I was wondering whether you would mind allowing your child to take part in the research by writing the test and task and taking part in the discussion that would follow, where video recording shall take place. I shall also audio record the proceedings as a way of backing up. I shall also take photos of the work they would have done. None of the participants' names shall be used in the report, but names shall be assigned to them. The test shall take a period not more than an hour while the task shall take about 40 minutes and the discussion thereafter about 45 minutes.

Your child will not be advantaged or disadvantaged in any way. S/he will be reassured that s/he can withdraw her/his permission at any time during this project without any penalty. There are no foreseeable risks in participating and your child will not be paid for this study.

I am inviting you to be part of this research by allowing your child to participate.

Your child's name and identity will be kept confidential at all times and in all academic writing about the study. His/her individual privacy will be maintained in all published and written data resulting from the study.

All research data will be destroyed 3 years after the completion of the research project.

Please let me know if you require any further information.

Thank you very much for your help.

Yours sincerely,

SIGNATURE

NAME: Sihlobosenkosi Mpofu
ADDRESS: 23 Ebony Street, Primrose
EMAIL: sihlobosenkosi@gmail.com
TELEPHONE NUMBERS: 0972364875

Appendix I: Parents' Consent Form

Please fill in and return the reply slip below indicating your willingness to allow your child to participate in the research project called: Grade 11 **learners' participation in the functions discourse: the case of a hyperbola and exponential function.**

I, _____ the parent of _____

	Circle one
Permission for written work to be used for this study	
I agree that my child's task can be used in this study only.	YES/NO
Permission for task	
I agree that my child may write a task for this study.	YES/NO
Permission to be audiotaped	
I agree that my child may be audiotaped during interview.	YES/NO
I know that the audiotapes will be used in this study only	YES/NO
Permission to be interviewed	
I agree that my child may be interviewed for this study.	YES/NO
I know that he/she can stop the interview at any time and doesn't have to	
answer all the questions asked.	YES/NO
Permission to be photographed	
I agree that my child's work may be photographed during the study.	YES/NO
I know that I can stop this permission at any time.	YES/NO
I know that the photos will be used in this study only.	YES/NO
Permission to be videotaped	
I agree my child may be videotaped during an interview.	YES/NO

I know that the videotapes will be used in this study only.   YES/NO

Informed Consent

I understand that:

- my child's name and information will be kept confidential and safe and that my name and the name of my school will not be revealed.
- he/she does not have to answer every question and can withdraw from the study at any time.
- he/she can ask not to be audiotaped, photographed and/or videotape
- all the data collected during this study will be destroyed after 3 years after completion of the project.

Sign_____ Date_____

Video 154344

INTERVIEWER = I

LEARNER 1 (UNATHI)=U

LEARNER 2 (ALLU) =A

LEARNER 1 & 2 (Allu & Unathi) = AU

[00:00:03] I: Ok Welcome Allu and Unathi to this interview.aaaahh may you start by. I will start with ee with Unathi, may you start by telling us what you've done for Question number 1, Task 1. Question number 1. Which says name each

[00:00:26] U: Name each of the functions given above and explain how f relates to y=3^x and g relates to y=2/x.

[00:00:38] U: Firstly, the graph of f(x) is the exponential function and g(x) is the hyperbolic function. And there is that questions that says how it relates to y=3^x. This graph 3^x has been shifted 1unit to the left and 9units down.

[00:01:07] I:Ok

[00:01:10] U: And then this one g(x) has been shifted 3units to the left and 1unit opposite.

[00:01:20] I: Alright. So can you take us to the next question?

[00:01:29] A: ok. What is the asymptote of f? How do you know this?

[00:01:33] A: er The asymptote of f(x). F(x) is an exponential, so an exponential has 1 asymptote, which is the -9, no it's actually 1.

[00:01:51] U: it's 1

[00:01:53] A: No it's -1.

[00:01:54] U: No this is a horizontal shift. Utini.

[00:01:57] A: Wait

[00:02:01] I: Ok. Yes. Talk.

[00:02:03] A: OK 9. So X+1=0, it equate back to 0.Right.

[00:02:09] U: Yes

[00:02:12] A: That's whereby the function should not cross the line. And X is equal to 9

[00:02:17] U: No an exponential function has

[00:02:21] I: Ok you can speak up it's fine. No wrong answer discussed, talk.

[00:02:24] AU: ok ok

[00:02:26] U: The exponential function has only one asymptote series , this is the only asymptote that f(x) has.ok This one is just the horizontal shift.

[00:02:32] A: Ohh Yes

[00:02:33] U: and this one is the verticeal shifting

[00:02:35] U: and then This one, this -9 is the horizontal asymptote

[00:02:38] A: Ok, ok, yes

[00:02:41] I: Why do you say it is the horizontal asymptote

[00:02:44] U:horizontal asymptote,

[00:02:45] I: Yes

[00:02:47] U:it is the line that the function must not go through or touch. you see.

[00:02:52] A: on the y-axis

[00:02:53] I: On the y-axis

[00:02:55] AU: Yes

[00:02:57] A:must not go

[00:02:58] I: hmm

[00:03:00] I: Ok, ok, am trying to understand you. right. So you are saying that the asymptote of f is

[00:03:09] U: minus, Y = -9.

[00:03:11] I: so is there a difference if I say Y= -9 and if I say -9. It is the same thing or its different

[00:03:18] A & U: If you just say -9, you are not specific, it can be x= -9 or y= -9. So it's better you say that Y=-9.

[00:03:27] I: so we agree that

[00:03:29] U: to specify that it must not touch the line y= -9.

[00:03:33] I: are you sure of that

[00:03:35] U: Yes, yes, sir.

[00:03:36] I: ehhheheh

[00:03:38] A: Ha Ezo confuse

[00:03:39] I: am not confusing you.so you are saying that it is an asymptote because it must not touch

EDITOR'S LETTER

This letter is proof of editing services provided to **Mr Sihlobosenkosi Mpofu** (Student Number: 216076925) in respect of his Doctoral research with the University of KwaZulu-Natal. This service was provided for purposes of enhancing both the academic integrity and professional acceptability of his study entitled:

Grade 11 Learners' Participation in the Functions Discourse: The Case of a Hyperbola and Exponential Function

As an independent academic editor, I further confirm that Mr S Mpofu's study was subjected to comprehensive (substantive) academic editing and review, language control, and technical compliance in accordance with the rigorous intensity expected in postgraduate research studies.

Mr S Mpofu's preliminary research report and final drafts were subjected to the editor's manual Internet-based plagiarism check, and yielded no significant ethical impropriety concerns.

I further provided editorial support in respect of his academic supervisor's suggested corrections and recommendations in compliance with acceptable practices in research methodology.

In compliance with conventional ethical principles of research, I have undertaken to keep all aspects of **Mr S Mpofu's** study confidential, and as his own individual initiative.

Sincerely.

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Signed: _____   Date: _31 July 2018_
Dr TJ Mkhonto   dd/mm/yyyy